Local properties of Richardson varieties in symplectic and orthogonal Grassmannians

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

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by

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August, 2022

Dedicated To My Parents

Late Shri. Pramatha Ranjan Ray

&

Smt. Sulekha Ray

Declaration

I do hereby declare that this thesis entitled **Local properties of Richardson varieties in symplectic and orthogonal Grassmannians** is a presentation of my original research work done under the supervision of **Dr. Shyamashree Upadhyay**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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Certificate

It is to certify that the work contained in this thesis entitled **Local properties of Richardson varieties in symplectic and orthogonal Grassmannians** has been carried out by **Papi Ray**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, under my supervision for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

August, 2022
Guwahati, India

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August, 2022 **Papi Ray**

ABSTRACT

In a paper by Kodiyalam and Raghavan, they provided an explicit combinatorial description of the Hilbert function of the tangent cone at any point on a Schubert variety in the Grassmannian, by giving a certain "degree-preserving" bijection between a set of monomials defined by an initial ideal and a "standard monomial basis". In this thesis, we have proved that this bijection is in fact a bounded RSK correspondence. As an application, we have proved that the bijection given in a paper of Ghorpade and Raghavan (for the symplectic Grassmannian) is also a bounded RSK correspondence. In the PhD thesis of Kreiman, he had given a bijection between the same two combinatorially defined sets as in the paper of Kodiyalam and Raghavan. In this thesis, we have proved that the bijection given in Kreiman's thesis and the bijection given in the paper of Kodiyalam and Raghavan are equivalent. Using the above results, we have given an explicit Gröbner basis for the ideal of the tangent cone at any T-fixed point of a Richardson variety in the symplectic Grassmannian. In this thesis, we have also provided formulae for the multiplicity at any T-fixed point of a Richardson variety in the symplectic as well as the orthogonal Grassmannians; together with an interpretation of the multiplicity in terms of certain non-intersecting lattice paths.

LIST OF FIGURES

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Introduction

The work presented in this thesis arose as an extension of the work of Kreiman (see [Kre08]) to the case of the symplectic Grassmannian. In [Kre08], Kreiman provided an explicit Gröbner basis for the ideal of the tangent cone at any torus-fixed point of a Richardson variety in the ordinary Grassmannian. In the same paper ([Kre08]), Kreiman also used the Gröbner basis result to deduce a formula which computes the multiplicity of a Richardson variety (in the ordinary Grassmannian) at any torus-fixed point by counting families of certain non-intersecting lattice paths. In this thesis, we have extended the above mentioned work of Kreiman to Richardson varieties in the symplectic Grassmannian. This work generalizes the work done in [GR06] by Ghorpade and Raghavan, where they considered similar problems for Schubert varieties. In this thesis, we have also provided a formula for the multiplicity of a Richardson variety in the orthogonal Grassmannian at any torus-fixed point, and an interpretation of the multiplicity in terms of certain non-intersecting lattice paths. This work has generalized the work done in [RU10] by Raghavan and Upadhyay, where they considered similar problems for Schubert varieties.

In [Kre08], the proof of the main theorem is based on a generalization of the Robinson-Schensted-Knuth (RSK) correspondence, which Kreiman called the bounded RSK (BRSK). The work done in [Kre08] is a generalization of the work done in [KR03] by Kodiyalam and Raghavan. In [KR03], Kodiyalam and Raghavan provided an explicit combinatorial description of the Hilbert function of the tangent cone at any point on a Schubert variety in the ordinary Grassmannian, by giving a certain "degree-preserving" bijection between a set of monomials defined by an initial ideal and a "standard monomial basis" (see [LR08] for the definition of a standard monomial basis). In Chapter 5 of this thesis, we have proved that this bijection, is in fact, a bounded RSK correspondence. As an application, we have also proved here that, the bijection given in [GR06] by Ghorpade and Raghavan (for the symplectic Grassmannian) is also a bounded RSK correspondence.

In [Kre03], Kreiman and Lakshmibai had given a bijection between the same two combinatorially defined sets as in [KR03]. There is a comment in [Kre08] to the effect that the bijections constructed in [KR03], [Kre08], and [Kre03] are all the same (although there is no proof). This thesis has provided a proof of this fact. In some sense, the work presented in this thesis provides more evidence of the ubiquity of the RSK correspondence. Sturmfels [Stu90] and Herzog-Trung [HT92] also proved results on a class of determinantal varieties which are equivalent to the results of [KR03], [Kre03], and [Kre08] for the case of Schubert varieties at the torus-fixed point e_{id} (where id denotes the "identity coset"). The key to their proofs was to use a version of the RSK correspondence (see [Ful97] for the classical RSK) in order to establish a "degree-preserving" bijection between a set of monomials defined by an initial ideal and a "standard monomial basis".

Briefly speaking, the symplectic Grassmannian means the variety of isotropic subspaces of maximum possible dimension of an even dimensional vector space endowed with a skew-symmetric non-degenerate bilinear form. The definition of the orthogonal Grassmannian is similar, the only differences being that the bilinear form is symmetric (plus non-degenerate), and the underlying field has characteristic $\neq 2$. These Grassmannians have been defined in §2.1 and §2.2 of this thesis in details. A Richardson variety X_α^γ in any such Grassmannian is defined to be the intersection of a Schubert variety X^γ and an opposite Schubert variety X_α therein.

The study of Schubert varieties in different kinds of Grassmannians has a long and rich history. Richardson varieties are a natural generalization of Schubert varieties. We are interested in Richardson varieties in the symplectic and orthogonal Grassmannians. In this thesis, we have considered initial ideals of tangent cones to Richardson varieties in the symplectic Grassmannian. We have provided an explicit Gröbner basis for the ideal of the tangent cone at any torus-fixed point of a Richardson variety in the symplectic Grassmannian. Similar work for the orthogonal Grassmannian was done in [Upa13] by Upadhyay. In [KR03], Kodiyalam and Raghavan provided (with respect to certain conveniently chosen term orders) an explicit Gröbner basis for the ideal of the tangent cone at any torus-fixed point of a Schubert variety in the ordinary Grassmannian, thereby proving the conjectures of Kreiman and Lakshmibai (made in [KL04a]). Then in [GR06], Ghorpade and Raghavan did the analogous work for Schubert varieties in the symplectic Grassmannian. And finally in [RU09, RU10], Raghavan and Upadhyay did the analogous work for Schubert varieties in the orthogonal Grassmannian.

The above results on Schubert varieties do not admit a straight forward generalization to Richardson varieties. The local properties of Schubert varieties at any torus-fixed point determine the local properties at all other points, because of the B-action; but this does not extend to Richardson varieties, since Richardson varieties only have a T-action. However, in [Kre08], Kreiman had extended the results of Kodiyalam and Raghavan to Richardson varieties in the ordinary Grassmannian. The analogous work for the orthogonal Grassmannian was done by Upadhyay in [Upa13]. The case of Richardson varieties in the symplectic Grassmannian has been addressed in this thesis.

We are motivated by a work of Knutson, Woo and Yong ([KWY13]), where they gave a short proof of the fact that essentially all questions concerning singularities of Richardson varieties reduce to corresponding questions about Schubert varieties. We are also motivated by the method used by Kreiman (in [Kre08]) to compute an explicit Gröbner basis for the ideal of the tangent cone at any torus-fixed point of a Richardson variety in the ordinary Grassmannian. Our motivation from [KWY13] allows us to look at [GR06], where Ghorpade and Raghavan had proved that in the case of Schubert varieties in the symplectic Grassmannian, certain objects called "good admissible pairs" give rise to a Gröbner basis for the ideal of the tangent cone at any torus-fixed point. In this thesis, we have defined "good admissible pairs" as a natural extension of the "good admissible pairs" of [GR06]. Thereafter, we have followed the techniques used by Kreiman in [Kre08] to obtain an explicit Gröbner basis in our case. In this thesis, we have used the map BRSK of [Kre08] to obtain our desired result about an explicit Gröbner basis. The way in which the map BRSK of [Kre08] has been used here to obtain an explicit Gröbner basis has been explained in §7.5.1 of this thesis.

In the study of singularities of Schubert varieties, Woo and Yong investigated Kazhdan-Lusztig ideals (see [WY08]). These ideals encode coordinates and equations for neighborhoods of type A Schubert varieties at torus-fixed points. In [WY12], Woo and Yong provided a Gröbner basis for the Kazhdan-Lusztig ideals. Also, in [BC12], the authors discussed three natural generalizations of Richardson varieties which they called projection varieties, intersection varieties, and rank varieties. In [BC12], they studied the singularities of each type of generalization. In [KLS14], Knutson, Lam, and Speyer had shown that many of the geometric properties of Richardson varieties hold more generally for projected Richardson varieties (by a projected Richardson variety, they had meant the projection of a Richardson variety in G/B to G/P); they are normal, Cohen-Macaulay, have rational resolutions, and are compatibly Frobenius split with respect to the standard splitting. One combinatorial analogue of a Richardson variety is the order complex of the corresponding Bruhat interval in the Weyl group W; this complex is known to be an EL-shellable ball. In [KLS14], Knutson, Lam and Speyer had proved the projection of such a complex into the order complex of the Bruhat order on W/W_P is given a shellable ball. In the case that G/P is minuscule (e.g. a Grassmannian), they had shown that its Gröbner degeneration takes each projected Richardson variety to the Stanley-Reisner scheme of its corresponding ball. In [GK15], Graham and Kreiman gave combinatorial descriptions of the restrictions to torus-fixed points of the classes of structure sheaves of Schubert varieties in the T-equivariant K-theory of Grassmannians and of maximal isotropic Grassmannians of orthogonal and symplectic types. Graham and Kreiman also gave formulas for the Hilbert series and Hilbert polynomials at torus-fixed points of the corresponding Schubert varieties. These descriptions and formulas are given in terms of two equivalent combinatorial models: excited Young diagrams and set-valued tableaux.

The problem of finding the Hilbert function of the tangent cone to a Schubert variety in the ordinary Grassmannian at any point x of it was first considered by Kreiman and Lakshmibai in [KL04a]. When x is the "identity coset", Kreiman and Lakshmibai had deduced an expression for the Hilbert function in terms of the combinatorics of the Weyl group (using well known results of Hodge-Pedoe [HP54] and Musili [Mus72]). From this, they recovered the interpretation of the multiplicity, due to Herzog and Trung ([HT92]), as the cardinality of a certain set of non-intersecting lattice paths. For points other than the "identity coset", they conjectured expressions for the Hilbert function and the multiplicity. These conjectures were proved by Kodiyalam and Raghavan (see [KR03]), and independently by Kreiman in his PhD thesis (see [Kre03] and also [KL04b]) for Schubert varieties in the ordinary Grassmannian. Krattenthaler [Kra05] had also proved the multiplicity conjecture of [KL04a] independently. In the case of the symplectic and the orthogonal Grassmannians, these conjectures were proved in [GR06] and [RU10] respectively.

An interpretation of multiplicity in terms of counting certain non-intersecting lattice paths was first given by Krattenthaler in [Kra01] for points on Schubert varieties in Grassmannians. The work of Krattenthaler in [Kra01] was an interpretation of the closed-form determinantal formula for the multiplicity, given by Rosenthal and Zelevinsky in [RZ01], for any point on Schubert varieties in Grassmannians. This closed-form determinantal formula by Rosenthal and Zelevinsky, was in turn obtained from certain recursive formulae given by Lakshmibai and Weyman in [LW90] for the multiplicity (at any point, not just at the identity coset) and the Hilbert function in case of a minuscule G/P (and also in the case of symplectic Grassmannians). An explicit closed-form formula for the multiplicity at the point corresponding to the identity coset on any Schubert variety in the Grassmannian was also given by Lakshmibai and Weyman in [LW90].

In [GR06], Ghorpade and Raghavan computed the multiplicity at any point on a Schubert variety in the symplectic Grassmannian. They also provided an interpretation of the multiplicity in terms of certain non-intersecting lattice paths. Similar work for Schubert varieties in the ordinary Grassmannian was done in [KR03] by Kodiyalam and Raghavan. In Chapter 8 of this thesis, we have provided a generalization of the work of Ghorpade and Raghavan [GR06] to Richardson varieties. However, we have provided the multiplicity result for Richardson varieties in the symplectic Grassmannian only at torus-fixed points (not at all points). In Chapter 8 of this thesis, we have applied techniques similar to [Kre08] for obtaining the multiplicity result. In [Bal13], Michaël Balan has shown that in cominuscule partial flag variety G/P, the multiplicity of an arbitrary point on a Richardson variety X_v^w is the product of its multiplicities on the Schubert variety X^w and opposite Schubert variety X_v.

The work done in this thesis for the symplectic Grassmannian is also available as pre-prints on the arxiv (see [RU20], [RU19], and [RU22]). In addition to this, we have also provided a formula in Chapter 9 of this thesis, which computes the multiplicity of a Richardson variety at any torus-fixed point in the orthogonal Grassmannian. In the same chapter, we have provided an interpretation of this multiplicity formula in terms of certain non-intersecting lattice paths. The techniques that we have used for this work are analogous to those in [RU10]. At this point, it is worth mentioning that in [Upa13], Upadhyay had provided an explicit Gröbner basis for the ideal of the tangent cone at any torus-fixed point of a Richardson variety in the orthogonal Grassmannian. But in [Upa13], there is no result on the multiplicity. In Chapter 9 of this thesis, we have provided the corresponding results for the multiplicity.

In a nutshell, the work presented in this thesis is giving the solutions to the following 5 problems:

- Proving that the map $\tilde{\pi}$ of [KR03] is the same as the map BRSK of [Kre08].
- Proving that the map $\tilde{\pi}$ of [KR03] is the same as the bijection Φ of [Kre03].
- Providing an explicit Gröbner basis for the ideal of the tangent cone at any torus-fixed point of a Richardson variety in the symplectic Grassmannian.
- Providing a formula for the multiplicity of a Richardson variety at any torus-fixed point in the symplectic Grassmannian in terms of certain non-intersecting lattice paths.
- Providing a formula for the multiplicity of a Richardson variety at any torus-fixed point in the orthogonal Grassmannian, and its interpretation in terms of certain non-intersecting lattice paths.

The table of contents indicates how the work in this thesis has been organized. We have also provided a brief description of the organization of the thesis here.

1.1.1 Organization of the thesis

Here we have provided a chapterwise description of the thesis. To begin with, Chapter 2 has described our basic objects of interest, namely, the symplectic and the orthogonal Grassmannians, and Richardson varieties in them. In Chapters 3 and 4, we have described the bounded RSK correspondence (given by Kreiman in [Kre08]) and the maps of Kodiyalam-Raghavan (given in [KR03]) respectively. Then in Chapter 5, we have proved that the maps BRSK (of [Kre08]) and $\tilde{\pi}$ (of [KR03]) are equivalent. In Chapter 6, we have proved that the map $\tilde{\pi}$ (of [KR03]) and the map Φ of Kreiman's thesis ([Kre03]) are also equivalent. Chapter 7 has provided an explicit Gröbner basis for the ideal of the tangent cone at any torus-fixed point of a Richardson variety in the symplectic Grassmannian. In Chapter 7, we have used the main result of Chapter 5 for computing the explicit Gröbner basis. In Chapters 8 and 9, we have provided a formula for the multiplicity of a Richardson variety at any torus-fixed point in the symplectic and the orthogonal Grassmannians respectively.

1.2 A comment about the figures

In the pdf file of this document, some figures might not appear correctly on a computer screen. But all the figures appear correctly in a print-out.

CHAPTER 2

THE SYMPLECTIC AND ORTHOGONAL GRASSMANNIANS, AND RICHARDSON VARIETIES IN THEM

The following definitions and notation are written in the same way as in [GR06].
Given any positive integer n, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. Given positive integers r and n with $r \leq n$, we denote by $I(r, n)$ the set of all r-element subsets of $[n]$. Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in I(r, n)$, where $1 \leq \alpha_1 < \ldots < \alpha_r \leq n$. If $\beta = (\beta_1, \ldots, \beta_r) \in I(r, n)$ be such that $1 \leq \beta_1 < \ldots < \beta_r \leq n$, then we say that $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, r$. Clearly, $\leq$ defines a partial order on $I(r, n)$.

We write $I(d, 2d)$ for the set of all d-element subsets of $\{1, \cdots, 2d\}$. There is a natural partial order on $I(d, 2d)$: $v = (v_1 < \cdots < v_d) \leq w = (w_1 < \cdots < w_d)$ if and only if $v_1 \leq w_1, \cdots, v_d \leq w_d$. For $\mu = \{\mu_1, \cdots, \mu_d\} \in I(d, 2d)$, $\mu_1 < \cdots < \mu_d$, define the **complement** of μ as $\{1, \cdots, 2d\} \setminus \mu$ and denote it by $\bar{\mu}$.

2.1 The symplectic Grassmannian

The following definitions and notation are written in the same way as in [GR06].
A positive integer d will be kept fixed throughout this thesis. For $j \in [2d]$, set $j^* := 2d + 1 - j$. Let $I(d)$ denote the set of all d-element subsets v of $[2d]$ with the property that exactly one of j, j^* belongs to v for every $j \in [d]$. Clearly, $I(d) \subseteq I(d, 2d)$. In particular, we have the partial order $\leq$ on $I(d)$ induced from $I(d, 2d)$.

Fix a vector space V of dimension $2d$ over an algebraically closed field K of arbitrary characteristic. A bilinear form $\langle\ ,\ \rangle : V \times V \to K$ is called **skew-symmetric** if $\langle x, x\rangle = 0\ \forall\ x \in V$. Note that if $\text{char}(K) \neq 2$, then the above definition implies that $\langle x, y\rangle = -\langle y, x\rangle\ \forall\ x, y \in V$. Now, the bilinear form $\langle\ ,\ \rangle$ is called **non-degenerate** if $\langle x, y\rangle =$

$0 \ \forall \ y \in V \Rightarrow x = 0$. Fix a **non-degenerate skew-symmetric bilinear form** $\langle \ , \ \rangle$ on V.

Lemma 2.1.1. *There exists a basis* $e_1, \ldots, e_{2d}$ *of* V *such that*

$$\langle e_i, e_j \rangle = \begin{cases} 1 \text{ if } i = j^* \text{ and } i < j \\ -1 \text{ if } i = j^* \text{ and } i > j \\ 0 \text{ otherwise.} \end{cases}$$

Proof. We will prove this theorem by induction on the dimension of V. Suppose V is a vector space with dimension $2d$ and $\langle \ , \ \rangle$ is a non-degenerate, skew-symmetric bilinear form on V. By definition of skew-symmetry, for any vector e_1 in V, we have $\langle e_1, e_1 \rangle = 0$. Again, by definition of non-degeneracy of $\langle \ , \ \rangle$, it follows that there exists a non-zero vector, call it e_{2d} such that $\langle e_1, e_{2d} \rangle = 1$ (taking $e_1 \neq 0$). Let W be a subspace of V given by $(\text{span}\{e_1, e_{2d}\})^{\perp}$. That is,

$$W = \{v \in V | \langle e_1, v \rangle = \langle e_{2d}, v \rangle = 0\}.$$

Now, we know that for any vector space V and for any subspace W of it, $W \oplus W^{\perp} = V$ and $\dim(W) + \dim W^{\perp} = \dim(V)$. So $\dim(W) = 2d - 2 \lneqq 2d$.
Fact: The bilinear form $\langle \ , \ \rangle$ restricted to W is non-degenerate.
Proof of the fact: Suppose for some $w_0 \in W$,

$$\langle w_0, w \rangle = 0 \ \forall \ w \in W \tag{2.1.0.1}$$

To show $w_0 = 0$.
Now, definition of W and (2.1.0.1) together imply $\langle w_0, v \rangle = 0 \ \forall \ v \in V$, and $\langle \ , \ \rangle$ is non-degenerate on $V \Rightarrow w_0 = 0$. Hence, the fact is proved.
Since $\dim(W) < 2d = \dim(V)$, so by induction hypothesis on $\dim(V)$, there exists a basis $\{e_2, e_3, \cdots, e_{2d-1}\}$ of W such that

$$\langle e_i, e_j \rangle = \begin{cases} 1 \text{ if } i = j^* \text{ and } i < j \\ -1 \text{ if } i = j^* \text{ and } i > j \\ 0 \text{ otherwise.} \end{cases}$$

Again, as W is a subspace of V, so the bilinear form $\langle \ , \ \rangle$ restricted to W is skew-symmetric. Extend the basis $\{e_2, e_3, \cdots, e_{2d-1}\}$ of W to a basis of V by adding e_1 and e_{2d}. Hence the proof. □

A linear subspace W of V is said to be **isotropic** if the form $\langle \ , \ \rangle$ **vanishes identically** on it. Let $Sp(V)$ denote the group of all linear automorphisms of V that preserve $\langle \ , \ \rangle$.

Lemma 2.1.2. *Every maximal isotropic subspace of* V *has dimension* d.

Proof. Let W be a maximal isotropic subspace of V. Since W is isotropic, we have $W \subseteq W^{\perp}$.
Claim: $W = W^{\perp}$.
Suppose not. Then there exists $w_0 \in W^{\perp} \setminus W$. Observe that $W \oplus \text{span}\{w_0\}$ is an isotropic subspace of V which contains W properly, a contradiction to the maximality of W. Hence $W = W^{\perp}$. This proves the claim.
Now, we know that, $\dim W + \dim W^{\perp} = \dim V$, so $2 \dim W = \dim V$. Hence $\dim W = d$. □

Let

$$G_d(V) = \text{ the Grassmannian of all } d\text{-dimensional subspaces of } V$$

and

$$\mathfrak{M}_d(V) = \text{ the set of all maximal isotropic subspaces of } V.$$

Then $\mathfrak{M}_d(V)$ is a closed subvariety of $G_d(V)$ and is called the **symplectic Grassmannian**.

2.2 The orthogonal Grassmannian

The following definitions and notation are written in the same way as in [Upa13].
Fix a natural number d. For k an integer such that $1 \leq k \leq 2d$, set $k^* := 2d + 1 - k$. Let $\mathfrak{O}I(d)$ denote the set of all d-element subsets of $\{1, \cdots, 2d\}$ of cardinality d satisfying the following two conditions:

- for each k, $1 \leq k \leq 2d$, the subset contains exactly one of k, k^*, and
- the number of elements in the subset that exceed d is even.

Clearly, $\mathfrak{O}I(d) \subseteq I(d, 2d)$. In particular, we have the partial order $\leq$ on $\mathfrak{O}I(d)$ induced from $I(d, 2d)$.

Fix an algebraically closed field K of characteristic not equal to 2. Fix a vector space V of dimension $2d$ over K and a **non-degenerate symmetric bilinear form** $\langle \ , \ \rangle$ on V. As in the case of symplectic Grassmannian, we can fix a basis $e_1, \ldots, e_{2d}$ of V such that

$$\langle e_i, e_k \rangle = \begin{cases} 1 \text{ if } i = k^* \\ 0 \text{ otherwise.} \end{cases}$$

Denote by $SO(V)$, the group of linear automorphisms of V that preserve the bilinear form $\langle , \rangle$ and also the volume form. A linear subspace of V is said to be **isotropic** if the bilinear form $\langle \ , \ \rangle$ vanishes identically on it. As in the case of symplectic Grassmannian, we can show that the dimension of a maximal isotropic subspace of V is d. Denote by $\mathfrak{M}_d(V)'$ the closed sub-variety of the Grassmannian of d-dimensional subspaces consisting of the points corresponding to maximal isotropic subspaces. The action of $SO(V)$ on V induces an action on $\mathfrak{M}_d(V)'$.

There are two orbits for this action. These orbits are isomorphic: acting by a linear automorphism that preserves the form but not the volume form gives an isomorphism. We denote by $\mathfrak{M}_d(V)$ the orbit of the span of $e_1, \cdots, e_d$ and call it the **even orthogonal Grassmannian**. One can define the orthogonal Grassmannian in the case when the dimension of V is not necessarily even. But it is enough to consider the case when the dimension of V is even, the reason being the following: Suppose that the dimension of V is odd, say dimension of $V = 2d+1$. Let $\widetilde{n} := 2d+2$ and $\widetilde{V}$ be a vector space of dimension $\widetilde{n}$ with a non-degenerate symmetric form. Let $\widetilde{e}_1, \cdots, \widetilde{e}_{\widetilde{n}}$ be a basis of $\widetilde{V}$ such that

$$\langle \widetilde{e}_i, \widetilde{e}_k \rangle = \begin{cases} 1 \text{ if } i = k^* \\ 0 \text{ otherwise.} \end{cases}$$

Put $e := \widetilde{e}_{d+1}$ and $f := \widetilde{e}_{d+2}$. Take λ to be an element of the field such that $\lambda^2 = 1/2$. We can take V to be the subspace of $\widetilde{V}$ spanned by the vectors $\widetilde{e}_1, \cdots, \widetilde{e}_d, \lambda e + \lambda f, \widetilde{e}_{d+3}, \cdots, \widetilde{e}_{\widetilde{n}}$, and a basis of V to be these vectors in that order.

There is a natural map from $\mathfrak{M}_{d+1}(\widetilde{V})'$ to $\mathfrak{M}_d(V)$: Intersecting with V an isotropic subspace of $\widetilde{V}$ of dimension $d+1$ gives an isotropic subspace of V of dimension d, we denote this map by $\cap$. The map $\cap$ is onto, for every isotropic subspace of $\widetilde{V}$ (and hence of V) is contained in an isotropic subspace of $\widetilde{V}$ of dimension $d+1$. In fact, more is true: the map $\cap$ is two-to-one. the map $\cap$ being two-to-one, it is also elementary to see that the two points in any fiber lie one in each component of $\mathfrak{M}_{d+1}(\widetilde{V})'$. We therefore get a natural isomorphism between $\mathfrak{M}_{d+1}(\widetilde{V})$ and $\mathfrak{M}_d(V)$. Therefore, now onwards we call the even orthogonal Grassmannian $\mathfrak{M}_d(V)$ (as defined above for a $2d$ dimensional vector space V) the **orthogonal Grassmannian**.

Let $\mathfrak{M}_d(V) \subseteq G_d(V) \hookrightarrow \mathbb{P}(\bigwedge^d V)$ be the Plüker embedding ($G_d(V)$ denotes the Grassmannian of all d-dimensional, subspaces of V). Thus $\mathfrak{M}_d(V)$ is closed sub variety of the projective variety $G_d(V)$, and hence $\mathfrak{M}_d(V)$ inherits the structure of a projective variety.

2.3 Richardson varieties in the orthogonal (symplectic) Grassmannian

Richardson varieties were first defined by Richardson in [Ric92]. Here we recall Richardson variety in the orthogonal (resp. symplectic) Grassmannian.

Let B^+ (resp. B^-) be the subgroup of $SO(V)$ (resp. $Sp(V)$) consisting of those elements that are upper triangular (resp. lower triangular) with respect to the basis $e_1, \cdots, e_{2d}$, and the subgroup T of $SO(V)$ (resp. $Sp(V)$) consisting of those elements that are diagonal with respect to $e_1, \cdots, e_{2d}$. It can be easily checked that T is a maximal torus of $SO(V)$ (resp. $Sp(V)$); B and B^- are Borel subgroups of $SO(V)$ (resp. $Sp(V)$) which contain T.

Theorem 2.3.1. *The group $SO(V)$ (resp. $Sp(V)$) acts transitively on $\mathfrak{M}_d(V)$.*

To prove Theorem 2.3.1 we need Witt's theorem (see [Lan84]).

Proof. If $SO(V)$ (resp. $Sp(V)$) acts transitively on $\mathfrak{M}_d(V)$, then for any two maximal isotropic subspaces $U_1, U_2 \in \mathfrak{M}_d(V)$ there exists $T \in SO(V)$ (resp. $Sp(V)$) such that

$$T.U_1 = U_2 \tag{2.3.0.1}$$

So, to prove Theorem 2.3.1, it is sufficient to prove (2.3.0.1). Since $U_1, U_2 \in \mathfrak{M}_d(V)$, therefore any linear isomorphism $\tilde{T} : U_1 \mapsto U_2$ is an isometry, as for any $u_1', u_1'' \in U_1$,

$$\langle u_1', u_1'' \rangle = \langle \tilde{T}u_1', \tilde{T}u_1'' \rangle = 0.$$

So, using Witt's theorem, we can extend $\tilde{T}$ to the whole of V. Let T be the extension of $\tilde{T}$ to V. Since T is an isometry on V, so it preserves $\langle \ , \ \rangle$. Hence $T \in SO(V)$ (resp. $Sp(V)$) and T restricted to U_1 is $\tilde{T}$, which is a mapping from U_1 onto U_2. So $T.U_1 = U_2$. Hence the proof. □

The T-fixed points of $\mathfrak{M}_d(V)$ under this action are easily seen to be of the form $\langle e_{i_1}, \cdots e_{i_d} \rangle$ for $(i_1, \cdots, i_d)$ in $\mathfrak{O}I(d)$ (resp. in $I(d)$).

The B^+-orbits (as well as B^--orbits) of $\mathfrak{M}_d(V)$ are naturally indexed by its T-fixed points: Each B^+-orbit (as well as B^--orbit) contains one and only one such point. Let $\alpha \in \mathfrak{O}I(d)$ (resp. in $I(d)$) be arbitrary and let e^α denote the corresponding T-fixed point of $\mathfrak{M}_d(V)$. The Zariski closure of the B^+-orbit (resp. B^--orbit) through e^α, with canonical reduced scheme structure, is called a Schubert variety (resp. opposite Schubert variety), and denoted by X^α (resp. X_α). For $\alpha, \gamma \in \mathfrak{O}I(d)$ (resp. in $I(d)$), the scheme theoretic intersection $X_\alpha^\gamma = X_\alpha \cap X^\gamma$ is called a **Richardson variety**. Each B^+-orbit (resp. B^--orbit) being irreducible and open in its closure, it follows that B^+-orbit closures (resp. B^--orbit closures) are indexed by the B^+-orbits (resp. B^--orbits). Thus the set $\mathfrak{O}I(d)$ (resp. $I(d)$) becomes an indexing set for Schubert varieties in $\mathfrak{M}_d(V)$, and the set consisting of all pairs of elements of $\mathfrak{O}I(d)$ (resp. $I(d)$) becomes an indexing set for Richardson varieties in $\mathfrak{M}_d(V)$. It can be shown that X_α^γ is nonempty if and only if $\alpha \leq \gamma$; and that for $\beta \in \mathfrak{O}I(d)$ (resp. in $I(d)$), $e^\beta \in X_\alpha^\gamma$ if and only if $\alpha \leq \beta \leq \gamma$. For the rest of this thesis let us fix the elements α, β, and γ of $I(d)$ in the case of symplectic Grassmannian and of $\mathfrak{O}I(d)$ for orthogonal Grassmannian such that $\alpha \leq \beta \leq \gamma$.

CHAPTER 3

THE BOUNDED RSK CORRESPONDENCE

In this chapter we recall the map $BRSK$ of [Kre08]. But for that, we first need to recall a lot of definitions and lemmas from [Kre08]. Most of the sections in this chapter are taken from [Kre08] except §3.2.2.

3.1 Multisets on $\mathbb{N}$ and $\mathbb{N}^2$

Let S be any set. A **multiset** E on S is defined to be a function $E : S \to \{0, 1, 2, \ldots\}$. One should think of E as consisting of the set S of elements, but with each $s \in S$ occurring $E(s)$ times. Note that a set is a special type of multiset in which each element occurs exactly once. We call $E(s)$ the **degree** or **multiplicity** of s in E. The **degree** of a multiset E on S is defined to be the sum of $E(s)$ for all $s \in S$. Define the multiset $E\dot{\cup}F$ as follows:

$$(E\dot{\cup}F)(s) = E(s) + F(s), \;\; s \in S \tag{3.1.0.1}$$

Example 3.1.1. *Let* $E = \{1, 1, 2, 2, 2, 5, 6\}$, $F = \{1, 1, 3, 5, 5\}$. *Then*

$$E\dot{\cup}F = \{1, 1, 1, 1, 2, 2, 2, 2, 3, 5, 5, 5, 6\}.$$

Multisets on $\mathbb{N}$:

Let $\mathbb{N}$ denote the positive integers. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be two multisets on $\mathbb{N}$ of the same degree, with $a_i \leq a_{i+1}$, $b_i \leq b_{i+1}$, for all i. We say that A is less than or equal to B in the **termwise order** if $a_i \leq b_i$ for all i. We denote this by $A \leq B$. We say that A is less than B in the **strict termwise order** if $a_i < b_i$ for all i. We denote this by $A \lessdot B$.

If A, B, C, and D are multisets on $\mathbb{N}$ such that $|A\dot{\cup}D| = |B\dot{\cup}C|$, where $|.|$ defines the cardinality. Then we write

$$A - C \leq B - D \text{ to indicate that } A \dot{\cup} D \leq B \dot{\cup} C. \tag{3.1.0.2}$$

Note that, $A - B \leq C - D$ is a transitive relation.

In general no meaning is attached to the expression $A - C$ by itself. However, if A and C are both sets, then

$$A - C := A \dot{\cup} (\mathbb{N} \setminus C). \tag{3.1.0.3}$$

In addition, if $A \subset \bar{\beta}$ and $C \subset \beta$, then

$$A - C := A \dot{\cup} (\beta \setminus C). \tag{3.1.0.4}$$

Multisets on $\mathbb{N}^2$:
Let $U = \{(e_1, f_1), (e_2, f_2), \ldots\}$ be a multiset on $\mathbb{N}^2$. Define $U_{(1)}$ and $U_{(2)}$ to be the multisets $\{e_1, e_2, \ldots\}$ and $\{f_1, f_2, \ldots\}$ respectively on $\mathbb{N}$. Define the **nonvanishing**, **negative**, and **positive parts** of U to be the following multisets:

$$\begin{aligned} U^{\neq 0} &= \{(e_i, f_i) \in U \mid e_i - f_i \neq 0\}, \\ U^{-} &= \{(e_i, f_i) \in U \mid e_i - f_i < 0\}, \\ U^{+} &= \{(e_i, f_i) \in U \mid e_i - f_i > 0\}. \end{aligned}$$

We say that U is **nonvanishing** if $U \subset (\mathbb{N}^2)^{\neq 0}$, **negative** if $U \subset (\mathbb{N}^2)^{-}$, and **positive** if $U \subset (\mathbb{N}^2)^{+}$. Impose the following transitive relation on multisets on $\mathbb{N}^2$:

$$U \leq V \iff U_{(1)} - U_{(2)} \leq V_{(1)} - V_{(2)}. \tag{3.1.0.5}$$

A **chain** in $\mathbb{N}^2$ is a subset $C = \{(e_1, f_1), \ldots, (e_m, f_m)\}$ of $\mathbb{N}^2$ such that $e_1 < \cdots < e_m$ and $f_1 > \cdots > f_m$. Let T and W be negative and positive subsets of $\mathbb{N}^2$ respectively. A nonvanishing multiset U on $\mathbb{N}^2$ is said to be **bounded by** T, W if for every chain C which is contained in the underlying set of U, we have

$$T \leq C \leq W.$$

Definition 3.1.2. *Let ι be the map on multisets on $\mathbb{N}^2$ defined by,*

$$\iota(\{(e_1, f_1), (e_2, f_2), \ldots\}) := \{(f_1, e_1), (f_2, e_2), \ldots\}.$$

Then ι is an ***involution*** *and it maps negative multisets on $\mathbb{N}^2$ to positive ones and vice-versa.*

3.2 The RSK correspondence

3.2.1 Young tableau and ordinary insertion

Definition 3.2.1. *A* ***Young diagram*** *is a collection of boxes arranged into a left and top justified array. The* ***empty Young diagram*** *is the Young diagram with no boxes. A Young diagram can not contain rows with no boxes, unless it is the empty Young diagram. A* ***Young tableau*** *is a filling of the boxes of a Young diagram with positive integers. The* ***empty Young tableau*** *is the Young tableau with no boxes. Let P a Young tableau. We say that P is* ***row strict*** *if the entries of any row of P strictly increase as one moves to the right. Again, the Young tableau P is* ***semistandard*** *if it is row strict and the entries of any column weakly increase as one moves down.*

Example 3.2.2. *A semistandard Young tableau P,*

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 & 5 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}.$$

Let us now recall the **ordinary Schensted insertion** process from §3 of [Kre08]. It is an algorithm which takes as input a semistandard Young tableau P and a positive integer a, and produces as output a new semistandard Young tableau with the same shape as P plus one extra box, and with the same entries as P (possibly in different locations) plus one additional entry, namely a. To begin, insert a into the first row of P, as follows. If a is strictly bigger than all entries in the first row of P, then place a in a new box on the right end of the first row, and the insertion process terminates. Otherwise, find the smallest entry of the first row of P which is greater than or equal to a, and replace that number with a. We say that the number which was replaced, was "bumped" from the first row. Insert the bumped number into the second row in precisely the same way as a was inserted into the first row. This process continues down the rows until, at some point, a number is placed in a new box on the right end of some row, at which point the process terminates and we denote this process by $P \leftarrow a$.

Example 3.2.3. *Let* $P = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 6 \\ \hline 1 & 3 & 4 & 7 & 8 \\ \hline 2 & 3 & 5 \\ \cline{1-3} 2 & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2}\end{array}$ *and $a = 6$. So after inserting a, P will be looking* *like,* $P \leftarrow 6 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 6 \\ \hline 1 & 3 & 4 & 6 & 8 \\ \hline 2 & 3 & 5 & 7 \\ \cline{1-4} 2 & 6 \\ \cline{1-2} 2 & 7 \\ \cline{1-2} \end{array}$.

3.2.2 Ordinary RSK correspondence

Definition 3.2.4. *A* ***notched diagram*** *is a collection of boxes arranged into left justified rows. The notched diagram may contain rows with no boxes. A* ***notched tableau*** *is a filling of the boxes of a notched diagram with positive integers.*

Definition 3.2.5. *A* ***notched bitableau*** *is a pair (P, Q) of notched tableaux of the same shape (i.e., the same number of rows and the same number of boxes in each row).*

Definition 3.2.6. *The ordinary RSK correspondence, RSK is a function which maps negative multisets on $\mathbb{N}^2$ to the set of all notched bitableaux (P, Q) such that*
(i) the number of boxes in P (or Q) is equal to the number of elements in the multiset, and
(ii) P is a semistandard Young tableau.

Definition 3.2.7. *Let $U = \{(a_1, b_1), \ldots, (a_t, b_t)\}$ be a negative multiset on $\mathbb{N}^2$. We say that the entries of U are listed in* ***lexicographic order****, if (i) $b_1 \geq \cdots \geq b_t$ and (ii) if $b_i = b_{i+1}$, for any $i \in \{1, \cdots, t-1\}$, then $a_i \geq a_{i+1}$.*

We inductively form a sequence of notched bitableaux $(P^{(0)}, Q^{(0)}), \cdots, (P^{(t)}, Q^{(t)})$, as follows:
Let $(P^{(0)}, Q^{(0)}) = (\emptyset, \emptyset)$. Assume inductively that we have formed $(P^{(i)}, Q^{(i)})$ using ordinary Schensted insertion process. Define $P^{(i+1)} = P^{(i)} \leftarrow a_{i+1}$. Let j be the row number of the new box of this insertion. Define $Q^{(i+1)}$ to be the tableau obtained by placing b_{i+1} on the left end of row j of $Q^{(i)}$ (and shifting all other entries of $Q^{(i)}$ to the right end box). Clearly, $P^{(i+1)}$ and $Q^{(i+1)}$ both have same shape. Also, P contains the elements a_i and Q contains the elements b_i.
Note: If (P, Q) is a notched bitableau, we define $\iota(P, Q)$ to be the notched bitableau obtained by reversing the order of the rows of (Q, P). If U is a positive multiset on $\mathbb{N}^2$, then $RSK(U)$ is defined to be $\iota(RSK(\iota(U)))$.
Example 3.2.8 below gives an illustration of the map RSK.

Example 3.2.8. *Let $U = \{(2, 1), (5, 4), (6, 3), (6, 9), (8, 13), (11, 13)\}$ be a multiset on $\mathbb{N}^2$. Now,*

$$\{(2, 1), (5, 4), (6, 3)\} \subset (\mathbb{N}^2)^+$$

and

$$\{(6, 9), (8, 13), (11, 13)\} \subset (\mathbb{N}^2)^-.$$

Let $U^+ = \{(2,1),(5,4),(6,3)\}$ *and* $U^- = \{(6,9),(8,13),(11,13)\}$*. So* $U = U^+ \cup U^-$*. Now, after arranging* $\iota(U^+)$ *in lexicographic order, we have*

$$\iota(U^+) = \{(3,6),(4,5),(1,2)\}.$$

Let us first apply the map RSK *on* $\iota(U^+)$*. Then*

$$P^{(0)} = \emptyset \qquad Q^{(0)} = \emptyset$$

$$P^{(1)} = \emptyset \leftarrow 3 = \begin{array}{|c|}\hline 3 \\ \hline\end{array} \qquad Q^{(1)} = \begin{array}{|c|}\hline 6 \\ \hline\end{array}$$

$$P^{(2)} = \begin{array}{|c|}\hline 3 \\ \hline\end{array} \leftarrow 4 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array} \qquad Q^{(2)} = \begin{array}{|c|c|}\hline 5 & 6 \\ \hline\end{array}$$

$$P^{(3)} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array} \leftarrow 1 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad Q^{(3)} = \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Therefore,

$$RSK(U^+) = \iota(RSK(\iota(U^+))) = \left(\begin{array}{|c|c}\cline{1-1} 2 \\ \hline 5 & \multicolumn{1}{c|}{6} \\ \hline\end{array}, \begin{array}{|c|c}\cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline\end{array} \right).$$

After arranging U^- *in lexicographic order, we have* $U^- = \{(11,13),(8,13),(6,9)\}$*. Let us now apply the map* RSK *on* U^-*. Then*

$$P^{(0)} = \emptyset \qquad Q^{(0)} = \emptyset$$

$$P^{(1)} = \emptyset \leftarrow 11 = \begin{array}{|c|}\hline 11 \\ \hline\end{array} \qquad Q^{(1)} = \begin{array}{|c|}\hline 13 \\ \hline\end{array}$$

$$P^{(2)} = \begin{array}{|c|}\hline 11 \\ \hline\end{array} \leftarrow 8 = \begin{array}{|c|}\hline 8 \\ \hline 11 \\ \hline\end{array} \qquad Q^{(2)} = \begin{array}{|c|}\hline 13 \\ \hline 13 \\ \hline\end{array}$$

$$P^{(3)} = \begin{array}{|c|}\hline 8 \\ \hline 11 \\ \hline\end{array} \leftarrow 6 = \begin{array}{|c|}\hline 6 \\ \hline 8 \\ \hline 11 \\ \hline\end{array} \qquad Q^{(3)} = \begin{array}{|c|}\hline 13 \\ \hline 13 \\ \hline 9 \\ \hline\end{array}$$

Therefore,

$$RSK(U^-) = \left(\begin{array}{|c|}\hline 6 \\ \hline 8 \\ \hline 11 \\ \hline\end{array}, \begin{array}{|c|}\hline 13 \\ \hline 13 \\ \hline 9 \\ \hline\end{array} \right) \text{ and } RSK(U) = \left(\begin{array}{|c|c}\cline{1-1} 6 \\ \cline{1-1} 8 \\ \cline{1-1} 11 \\ \cline{1-1} 2 \\ \hline 5 & \multicolumn{1}{c|}{6} \\ \hline\end{array}, \begin{array}{|c|c}\cline{1-1} 13 \\ \cline{1-1} 13 \\ \cline{1-1} 9 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline\end{array} \right).$$

3.3 The bounded RSK correspondence

3.3.1 Notched tableaux

Definition 3.3.1. *Let P be a notched tableau, we say that P is* ***row strict*** *if the entries of any row of P strictly increase as one moves to the right.*

Example 3.3.2. *A row strict notched tableau P,*

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 & & \\ \cline{1-3} 5 & 6 & 7 & \\ \hline 4 & 5 & 7 & 8 \\ \hline \end{array} .$$

Let P be a row strict notched tableau, and b be a positive integer. Since P is row strict, its entries which are greater than or equal to b are right justified in each row. If we remove these entries (which are greater than or equal to b) from P, then we are left with a row strict notched tableau, which we denote by $P^{<b}$. We say that P is **semistandard on** b if $P^{<b}$ is a semistandard Young tableau.

Example 3.3.3. *For the row strict notched tableau P in Example 3.3.2, and $b = 4$, we have*

$$P^{<b} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} .$$

However, for the same P, if we take $b = 6$, then

$$P^{<b} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} \end{array} .$$

Hence, P is semistandard on 4, but not on 6.

3.3.2 The bounded insertion

We now describe the **bounded insertion algorithm**, which takes as input a positive integer b, a notched tableau P which is semistandard on b, and a positive integer $a < b$, and produces as output a notched tableau which is semistandard on b, which we denote by $P \overset{b}{\leftarrow} a$.

Bounded Insertion

Step 1. *Remove all entries of P which are greater than or equal to b from P, resulting in the semistandard Young tableau $P^{<b}$.*

Step 2. *Insert a into $P^{<b}$ using the ordinary Schensted insertion process (as described above).*

Step 3. *Place the entries of P which were removed when forming $P^{<b}$ in Step 1 back into the Young tableau resulting from Step 2, in the same rows from which they were removed.*

This insertion process is effectively the ordinary Schensted insertion of a into P, but acting only on the part of P which is "bounded" by b. The fact that bounded insertion preserves the property of being semistandard on b follows immediately from the fact that ordinary Schensted insertion preserves the property of being semistandard.

Example 3.3.4. *Let* $P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 \\ \cline{1-3} 5 & 6 & 7 \\ \hline 4 & 5 & 7 & 8 \\ \hline \end{array}$,

$a = 3$, and $b = 4$. We compute $P \overset{b}{\leftarrow} a$.

Step 1. Remove all entries of P which are greater than or equal to b, resulting in

$$P^{<b} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}.$$

Step 2. Insert a into $P^{<b}$ using the Schensted insertion process: a bumps 3 from the first row, which is placed in a new box on the right end of the second row, to form

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}.$$

Step 3. Place the entries removed from P in Step 1 back into the Young tableau resulting from Step 2, in the same rows from which they were removed, to obtain

$$P \overset{b}{\leftarrow} a = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 5 & 6 & 7 \\ \hline 4 & 5 & 7 & 8 \\ \hline \end{array}.$$

We define the **bumping route** of the bounded insertion algorithm to be the sequence of boxes in P, from which entries are bumped in Step 2, together with the **new box** which is added at the end of Step 2.

3.3.3 Semistandard notched bitableaux

Recall the definition of a notched bitableau from Definition 3.2.5.

Definition 3.3.5. *The **degree** of (P, Q) is the number of boxes in P (or Q). A notched bitableau (P, Q) is said to be **row strict** if both P and Q are row strict. A row strict* *notched bitableau (P, Q) is said to be* ***semistandard*** *if*

$$P_1 - Q_1 \leq \cdots \leq P_r - Q_r, \tag{3.3.3.1}$$

where r is the total number of rows in P (or Q) and for each $i \in \{1, \ldots, r\}$, P_i (resp. Q_i) denotes the i-th row (from the top) of P (resp. Q). A row strict notched bitableau (P, Q) is said to be ***negative*** *if $P_i \lessdot Q_i$, $i = 1, \ldots, r$,* ***positive*** *if $P_i \gtrdot Q_i$, $i = 1, \ldots, r$, and* ***nonvanishing*** *if either*

$$P_i \lessdot Q_i \quad \text{or} \quad P_i \gtrdot Q_i, \tag{3.3.3.2}$$

for each $i = 1, \ldots, r$.

Example 3.3.6. *Consider the notched bitableau*

$$(P, Q) = \left(\begin{array}{|c|c|c|c} \hline 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \hline 4 & 5 & 6 & \multicolumn{1}{c|}{7} \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|c|c} \hline 7 & 8 & 9 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 4 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \right).$$

We have that

1. *(P, Q) is row strict.*

2. *$P_1 \dot{\cup} Q_2 = \{1, 2, 2, 3, 3, 4, 5\} \leq \{4, 5, 6, 7, 7, 8, 9\} = P_2 \dot{\cup} Q_1$. Therefore, $P_1 - Q_1 \leq P_2 - Q_2$. Thus (P, Q) is semistandard.*

3. *$P_1 \lessdot Q_1$, and $P_2 \gtrdot Q_2$. Thus (P, Q) is nonvanishing.*

Let (P, Q) be a semistandard notched bitableau. If, for subsets T and W of $\mathbb{N}^2$,

$$T_1 - T_{(2)} \leq P_{(1)} - Q_{(1)} \text{ and } P_{(r)} - Q_{(r)} \leq W_{(1)} - W_{(2)}, \tag{3.3.3.3}$$

then we say that (P, Q) is **bounded by T,W**. Note that 3.3.3.3 combined with 3.3.3.1 implies that

$$T_{(1)} - T_{(2)} \leq P_{(1)} - P_{(2)} \leq \ldots \leq P_{(r)} - Q_{(r)} \leq W_{(1)} - W_{(2)}.$$

3.3.4 The bounded RSK correspondence

Definition 3.3.7. *The* ***bounded RSK correspondence****, $BRSK$, a function which maps negative multisets on $\mathbb{N}^2$ to negative semistandard notched bitableaux.*

Let

$$U = \{(a_1, b_1), \ldots, (a_t, b_t)\}$$

be a negative multiset on $\mathbb{N}^2$, whose entries we assume are listed in **lexicographic order**. We inductively form a sequence of notched bitableaux

$$(P^{(0)}, Q^{(0)}),\ (P^{(1)}, Q^{(1)}), \cdots, (P^{(t)}, Q^{(t)}),$$

such that $P^{(i)}$ is semistandard on b_i, $i = 1, \cdots, t$, as follows:

> Let $(P^{(0)}, Q^{(0)}) = (\emptyset, \emptyset)$ and let $b_0 = b_1$. Assume inductively that we have formed $(P^{(i)}, Q^{(i)})$, such that $P^{(i)}$ is semistandard on b_i, and thus on b_{i+1}, since $b_{i+1} \leq b_i$. Define $P^{(i+1)} = P^{(i)} \overset{b_{i+1}}{\leftarrow} a_{i+1}$. Since bounded insertion preserves semistandardness on b_{i+1}, $P^{(i+1)}$ is also semistandard on b_{i+1}. Let j be the row number of the new box of this bounded insertion. Define $Q^{(i+1)}$ to be the notched tableau obtained by placing b_{i+1} on the left end of row j of $Q^{(i)}$ (and shifting all other entries of $Q^{(i)}$ to the right one box). Clearly, $P^{(i+1)}$ and $Q^{(i+1)}$ have the same shape.

Then $BRSK(U)$ is defined to be $(P^{(t)}, Q^{(t)})$. In the process above, we write

$$(P^{(i+1)}, Q^{(i+1)}) = (P^{(i)}, Q^{(i)}) \overset{b_{i+1}}{\leftarrow} a_{i+1}.$$

In terms of this notation,

$$BRSK(U) = ((\emptyset, \emptyset) \overset{b_1}{\leftarrow} a_1) \cdots \overset{b_t}{\leftarrow} a_t.$$

Note: If (P, Q) is a nonvanishing semistandard notched bitableaux, we define $\iota(P, Q)$ to be the notched bitableaux obtained by reversing the order of the rows of (Q, P). If U is a positive multiset on $\mathbb{N}^2$, then $BRSK(U)$ is defined to be $\iota(BRSK(\iota(U)))$.
Example 3.3.8 below gives an illustration of the map $BRSK$.

Example 3.3.8. *Let $U = \{(2,1), (5,3), (6,4), (6,9), (8,13), (11,13)\}$ be a multiset on $\mathbb{N}^2$. Now,*

$$\{(2,1), (5,3), (6,4)\} \subset (\mathbb{N}^2)^+$$

and

$$\{(6,9), (8,13), (11,13)\} \subset (\mathbb{N}^2)^-.$$

Let $U^+ = \{(2,1), (5,3), (6,4)\}$ and $U^- = \{(6,9), (8,13), (11,13)\}$. So $U = U^+ \cup U^-$. After arranging $\iota(U^+)$ in lexicographic order, we have

$$\iota(U^+) = \{(4,6), (3,5), (1,2)\}.$$

Let us first apply the map $BRSK$ on $\iota(U^+)$. Then

$$
\begin{aligned}
P^{(0)} &= \emptyset & \qquad Q^{(0)} &= \emptyset \\
P^{(1)} &= \emptyset \xleftarrow{6} 4 = \begin{array}{|c|}\hline 4 \\ \hline\end{array} & \qquad Q^{(1)} &= \begin{array}{|c|}\hline 6 \\ \hline\end{array} \\
P^{(2)} &= \begin{array}{|c|}\hline 4 \\ \hline\end{array} \xleftarrow{5} 3 = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array} & \qquad Q^{(2)} &= \begin{array}{|c|}\hline 6 \\ \hline 5 \\ \hline\end{array} \\
P^{(3)} &= \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array} \xleftarrow{2} 1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} & \qquad Q^{(3)} &= \begin{array}{|c|c|}\hline 2 & 6 \\ \hline 5 \\ \cline{1-1}\end{array}
\end{aligned}
$$

Therefore,

$$
BRSK(U^+) = \iota(BRSK(\iota(U^+))) = \left(\begin{array}{|c|c|}\cline{1-1} 5 \\ \hline 2 & 6 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 4 \\ \hline 1 & 3 \\ \hline\end{array} \right).
$$

After arranging U^- in lexicographic order, we have $U^- = \{(11, 13), (8, 13), (6, 9)\}$. Let us now apply the map $BRSK$ on U^-. Then

$$
\begin{aligned}
P^{(0)} &= \emptyset & \qquad Q^{(0)} &= \emptyset \\
P^{(1)} &= \emptyset \xleftarrow{13} 11 = \begin{array}{|c|}\hline 11 \\ \hline\end{array} & \qquad Q^{(1)} &= \begin{array}{|c|}\hline 13 \\ \hline\end{array} \\
P^{(2)} &= \begin{array}{|c|}\hline 11 \\ \hline\end{array} \xleftarrow{13} 8 = \begin{array}{|c|}\hline 8 \\ \hline 11 \\ \hline\end{array} & \qquad Q^{(2)} &= \begin{array}{|c|}\hline 13 \\ \hline 13 \\ \hline\end{array} \\
P^{(3)} &= \begin{array}{|c|}\hline 8 \\ \hline 11 \\ \hline\end{array} \xleftarrow{9} 6 = \begin{array}{|c|c|}\cline{1-1} 6 \\ \hline 8 & 11 \\ \hline\end{array} & \qquad Q^{(3)} &= \begin{array}{|c|c|}\cline{1-1} 13 \\ \hline 9 & 13 \\ \hline\end{array}
\end{aligned}
$$

Therefore,

$$
BRSK(U^-) = \left(\begin{array}{|c|c|}\cline{1-1} 6 \\ \hline 8 & 11 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 13 \\ \hline 9 & 13 \\ \hline\end{array} \right) \text{ and } BRSK(U) = \left(\begin{array}{|c|c|}\cline{1-1} 6 \\ \hline 8 & 11 \\ \hline 5 \\ \hline 2 & 6 \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 13 \\ \hline 9 & 13 \\ \hline 4 \\ \hline 1 & 3 \\ \hline\end{array} \right).
$$

Now, we will state [Kre08, Lemma 6.2], which is stated below as Lemma 3.3.9.

Lemma 3.3.9. *If U is a negative multiset on $\mathbb{N}^2$, then $BRSK(U)$ is a negative semistandard notched bitableaux.*

In [Kre08], Kreiman had defined the inverse of the map $BRSK$, which he called **reverse** of $BRSK$, or $RBRSK$. First we will recall $RBRSK$ from [Kre08].

The map $RBRSK$: The bounded insertion used to form $(P^{(i+1)}, Q^{(i+1)})$ from $(P^{(i)}, Q^{(i)})$,

$i = 1, \cdots, t-1$, is reversible. In other words, by knowing only $(P^{(i+1)}, Q^{(i+1)})$, one can retrieve $(P^{(i)}, Q^{(i)}), a_{i+1}$, and b_{i+1}. First Kreiman had obtained b_{i+1}; it is the minimum entry of Q^{i+1}. Then, it is the lowest row in which b_{i+1} appears in $Q^{(i+1)}$, select the greatest entry of $P^{(i+1)}$ which is less than b_{i+1}. This entry was the new box on the bounded insertion. Kreiman began reverse bounded insertion with this entry, and he retrieved $P^{(i)}$ and a_{i+1}. Finally, $Q^{(i)}$ was retrieve from $Q^{(i+1)}$ by removing the lowest occurrence of b_{i+1} appearing in $Q^{(i+1)}$. This occurrence must be on the left end of some row. All other entries of that row should be moved one box of the left, thus eliminating the empty box vacated by b_{i+1}.

It follows that one can reverse the entire sequence used to define $BRSK$ by reversing each step in the sequence. If someone generates $(P^{(t)}, Q^{(t)})$ via $BRSK$, then he or she can retrieve the negative multiset U using this procedure. Kreiman called the process of obtaining $(P^{(i-1)}, Q^{(i-1)}), a_i$, and b_i from $(P^{(i)}, Q^{(i)})$ described in the paragraph above, a **reverse step** and denote it by $(P^{(i-1)}, Q^{(i-1)}) = (P^{(i)}, Q^{(i)}) \xrightarrow{b_i} a_i$. Kreiman called the process of applying all the reverse steps sequentially to retrieve U from $(P^{(t)}, Q^{(t)})$, the **reverse of** $BRSK$, or $RBRSK$.

If U is an arbitrary multiset on $\mathbb{N}^2$, then in Example 3.3.8, we have seen that how we can apply the map $BRSK$ on U. Similarly, someone can apply reverse $BRSK$ on any nonvanishing semistandard notched bitableaux.

Now, we will state [Kre08, Proposition 6.4], which is stated below as Proposition 3.3.10.

Proposition 3.3.10. *The map $BRSK$ is a degree-preserving bijection from the set of nonvanishing (resp. negative, positive) multisets of $\mathbb{N}^2$ to the set of nonvanishing (resp. negative, positive) semistandard notched bitableaux.*

CHAPTER 4

THE MAPS OF KODIYALAM AND RAGHAVAN

In this chapter, we will describe two important maps π and ϕ from [KR03]. But before going to the description let us first fix some notation and state some definitions and results from [KR03].

4.1 Basic notation

For this subsection, let us fix an arbitrary element v of $I(d, N)$ (defined in Chapter 2), where d and N are positive integers such that $1 \leq d \leq N$. We will be dealing extensively with ordered pairs (r, c), $1 \leq r, c \leq N$, such that r is not and c is an entry of v. Let $\mathfrak{R}(v)$ denote the set of all such ordered pairs, that is,

$$\mathfrak{R}(v) = \{(r, c) \mid r \in \{1, \ldots, 2d\} \setminus v, c \in v\}.$$

Set

$$\mathfrak{N}(v) := \{(r, c) \in \mathfrak{R}(v) \mid r > c\},$$

$$\mathfrak{R}(v) \setminus \mathfrak{N}(v) := \{(r, c) \in \mathfrak{R}(v) \mid r < c\},$$

$$\mathfrak{O}\mathfrak{R}(v) := \{(r, c) \in \mathfrak{R}(v) \mid r < c^*\},$$

$$\widetilde{\mathfrak{O}}\mathfrak{R}(v) := \{(r, c) \in \mathfrak{R}(v) \mid r > c^*\},$$

$$\mathfrak{O}\mathfrak{N}(v) := \{(r, c) \in \mathfrak{R}(v) \mid r > c, r < c^*\} = \mathfrak{O}\mathfrak{R}(v) \cap \mathfrak{N}(v),$$

$$\widetilde{\mathfrak{O}}\mathfrak{N}(v) := \{(r, c) \in \mathfrak{R}(v) \mid r > c, r > c^*\} = \widetilde{\mathfrak{O}}\mathfrak{R}(v) \cap \mathfrak{N}(v),$$

$$\mathfrak{d}(v) := \{(r, c) \in \mathfrak{R}(v) \mid r = c^*\},$$

$$\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v)) := \{(r, c) \in \mathfrak{R}(v) \mid r < c, r > c^*\}.$$

We will refer to $\mathfrak{d}(v)$ as the **diagonal**. Figure 4.1 in Example 4.1.1 below gives a pictorial look of the above sets.

Example 4.1.1. *Let* $d = 7$*,* $N = 14$*, and* $v = (1, 3, 4, 7, 9, 10, 13)$*.*

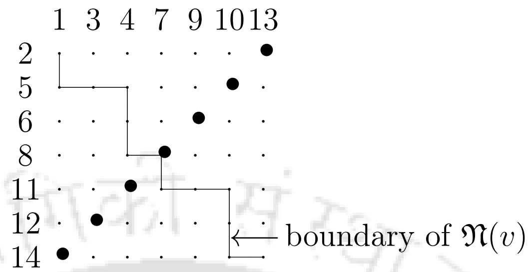


Figure 4.1: The grid representing $\mathfrak{R}(v)$

The points (including the dark circles) of the above grid represent the set $\mathfrak{R}(v)$ *for* $v = (1, 3, 4, 7, 9, 10, 13)$*. The path sketched on the grid by some piecewise line segments denote the boundary of* $\mathfrak{N}(v)$*. The points on this grid which lie on the boundary of* $\mathfrak{N}(v)$ *or to the left of it belong to the set* $\mathfrak{N}(v)$*. The dark circles denote the diagonal elements. Points above the diagonal belong to the set* $\mathfrak{D}\mathfrak{R}(v)$*. Again, points which are towards the left of the boundary of* $\mathfrak{N}(v)$*, and which also lie above the diagonal, are the points of* $\mathfrak{D}\mathfrak{N}(v)$*. Points towards the right of the boundary of* $\mathfrak{N}(v)$ *belong to the set* $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$*. Points below the diagonal belong to the set* $\widetilde{\mathfrak{D}}\mathfrak{R}(v)$*. Points below the diagonal and towards the left of the boundary of* $\mathfrak{N}(v)$ *belong to the set* $\widetilde{\mathfrak{D}}\mathfrak{N}(v)$*. Points towards the right of the boundary of* $\mathfrak{N}(v)$ *and below the diagonal belong to the set* $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$*.*

We will be considering *monomials* in some of these sets.

Definition 4.1.2. *A* ***monomial****, as usual, is a subset with each member being allowed a multiplicity (the multiplicity taking values in the non-negative integers). The* ***degree*** *of a monomial also has the usual meaning: consider the underlying set of the monomial and look at the multiplicity with which each element of this underlying set appears in the monomial, the degree of the monomial is the sum of these multiplicities.*

Let $mon\mathfrak{R}(v)$ denote the set of all monomials in $\mathfrak{R}(v)$ and T^v denote the set of all monomials in $\mathfrak{N}(v)$.

Definition 4.1.3. *The* ϵ*-**degree** (where* $\epsilon = (1, \cdots, d)$*) of an element* x *of* $I(d)$ *is the cardinality of* $x \setminus [d]$ *or equivalently that of* $[d] \setminus x$*. More generally, given any* $v \in I(d)$*, the* v*-**degree** of an element* x *of* $I(d)$ *is the cardinality of* $x \setminus v$ *or equivalently that of* $v \setminus x$*.*

Example 4.1.4. *Let* $d = 5$*. So* $2d = 10$ *and* $\epsilon = (1, 2, 3, 4, 5)$*. Let* $x = (1, 2, 4, 6, 8)$*. Hence in this case, the* ϵ*-degree of* x *is* 2*. Again let* $v = (1, 2, 3, 4, 6)$*. So according to the definition,* v*-degree of* x *is* 1*.*

Definition 4.1.5. *A **standard monomial** in $I(d,N)$ is a totally ordered sequence $\theta_1 \geq \ldots \geq \theta_t$ of elements of $I(d,N)$. Such a standard monomial is called v-**compatible** if each θ_j is comparable to v (with respect to the partial order $\leq$) but no θ_j equals v; it is **anti-dominated** by v if $\theta_t \geq v$. Let $\widetilde{SM}^{v,v}$ denote the set of all v-**compatible standard monomials in** $I(d,N)$ **anti-dominated by** v.*

Example 4.1.6. *Let $d = 6$ and $N = 13$. Then the totally ordered sequence $\theta_1 = (4,7,8,11,12,13) \geq \theta_2 = (3,5,6,9,10,11) \geq \theta_3 = (1,2,3,6,7,10)$ is a standard monomial in $I(6,13)$. Let $v = (1,2,4,7,8,11)$. Then the above standard monomial (namely, $\theta_1 \geq \theta_2 \geq \theta_3$) is v-compatible because each θ_j $(j = 1,2,3)$ is comparable to v but no θ_j equals v. In fact, in this case, we have $\theta_1 \geq \theta_2 \geq v$, $v \geq \theta_3$ and none of the θ_i $(i = 1,2,3)$ equals v. Moreover, the standard monomial $\theta_1 \geq \theta_2$ is anti-dominated by v because $\theta_2 \geq v$. The standard monomial $\theta_1 \geq \theta_2$ is an element of $\widetilde{SM}^{v,v}$.*

Definition 4.1.7. *Given any $\beta_1 = (r_1,c_1)$, $\beta_2 = (r_2,c_2)$ in $\mathfrak{N}(v)$, we say that $\beta_1 = (r_1,c_1) > \beta_2 = (r_2,c_2)$ if $r_1 > r_2$ and $c_1 < c_2$. A sequence $\beta_1 > \ldots > \beta_t$ of elements of $\mathfrak{N}(v)$ is called a v-**chain**. Given a v-chain $\beta_1 = (r_1,c_1) > \ldots > \beta_t = (r_t,c_t)$, we define*

$$s_{\beta_1} \ldots s_{\beta_t} v := (\{v_1,\ldots,v_d\} \setminus \{c_1,\ldots,c_t\}) \cup \{r_1,\ldots,r_t\}.$$

*We say that an element w of $I(d,N)$ **dominates** the v-chain $\beta_1 > \ldots > \beta_t$ if $w \geq s_{\beta_1} \ldots s_{\beta_t} v$.*

Figure 4.2 in Example 4.1.8 shows a v-chain for $v = (1,2,4,7,8,11)$.

Example 4.1.8. *Let $d = 6$, $N = 13$, and $v = (1,2,4,7,8,11)$.*

The four dark circles in this grid denote a v-chain given by $\beta_1 > \beta_2 > \beta_3 > \beta_4$, where

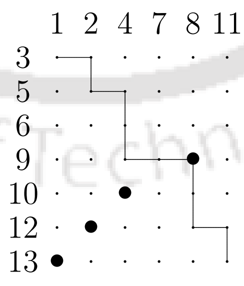


Figure 4.2: A v-chain

$\beta_1 = (13,1)$, $\beta_2 = (12,2)$, $\beta_3 = (10,4)$, $\beta_4 = (9,8)$. If we take $w = (8,9,10,11,12,13)$, then this v-chain is dominated by w because $s_{\beta_1} \ldots s_{\beta_4} v = (7,9,10,11,12,13) \leq w$.

Definition 4.1.9. *Two elements α and β in $\mathfrak{N}(v)$ are said to be **comparable** if $\alpha > \beta$ or $\beta > \alpha$.*

Definition 4.1.10. *Let $\mathfrak{S}$ be a monomial in $\mathfrak{R}(v)$. By a v-chain in $\mathfrak{S}$, we mean a sequence $\beta_1 > \ldots > \beta_t$ of elements of $\mathfrak{S} \cap \mathfrak{N}(v)$. We say that w **dominates** $\mathfrak{S}$ if w dominates every v-chain in $\mathfrak{S}$.*

Definition 4.1.11. *A subset $\mathfrak{S}$ of $\mathfrak{N}(v)$ is called **distinguished** if it satisfies the following conditions:*
(A) for $(r, c) \neq (r', c')$ in $\mathfrak{S}$, we have $r \neq r'$ and $c \neq c'$,
(B) if $\mathfrak{S} = \{(r_1, c_1), \ldots, (r_p, c_p)\}$ with $r_1 < r_2 < \ldots < r_p$, then for j, $1 \leq j \leq p-1$, we have either $c_j > c_{j+1}$ or $r_j < c_{j+1}$.
Condition (B) can be restated as follows:
(B *) for (r, c), (R, C) in $\mathfrak{S}$ with $r < R$, either $C < c$ or $r < C$.

Example 4.1.12. *For $v = (1, 2, 4, 7, 8, 11)$, the subset $\mathfrak{S}$ of $\mathfrak{N}(v)$ given by $\mathfrak{S} = \{(3, 2), (5, 1), (9, 8), (12, 7)\}$ is distinguished.*

Proposition 4.1.13 and Remark 4.1.14 below are taken in the same way as in [KR03, Proposition 4.3] and [KR03, Remark 4.4].

Proposition 4.1.13. *There exists a bijection between elements w of $I(d, N)$ satisfying $w \geq v$ on the one hand and distinguished subsets of $\mathfrak{N}(v)$ on the other hand. We denote this bijective correspondence by $w \leftrightarrow \mathfrak{S}_w$.*

Proof. Given $w \geq v$, consider the sets $\{v_1, \cdots, v_d\} \setminus \{w_1, \cdots, w_d\}$ and $\{w_1, \cdots, w_d\} \setminus \{v_1, \cdots, v_d\}$. Both these have the same cardinality $t = v$-degree(w) (stated in Definition 4.1.3). The first set provides us with the column indices of elements of $\mathfrak{S}_w$, the second with the row indices. If we arrange the row indices in an decreasing order, say $r_1 < \cdots < r_t$, then there is a unique way to arrange the column indices such that condition (B*) stated above is satisfied: proceed by induction, and if $c_1, \cdots, c_j$ have been chosen, choose c_{j+1} to be the maximum among the remaining column indices that are less than r_{j+1}. This defines the map $w \mapsto \mathfrak{S}_w$. It is clear that cardinality of $\mathfrak{S}_w$ equals v-degree(w).

For the converse part, given $\mathfrak{S}$, to obtain w, start with $v = (v_1, \cdots, v_d)$, delete those entries that occur as the column indices in $\mathfrak{S}$, add those that occur as row indices in $\mathfrak{S}$, and finally arrange the entries in increasing order. It is readily seen that the two maps are inverses to each other. □

Remark 4.1.14. *1. Subsets of distinguished monomials are themselves distinguished.*

2. If $\mathfrak{S}$ is a subset of $\mathfrak{N}(v)$ satisfying the condition (A), then we can still define a corresponding element w of $I(d, N)$ as in the proof of the Proposition 4.1.13. If $\mathfrak{S} \subset \tilde{\mathfrak{S}}$ are subsets of $\mathfrak{N}(v)$ satisfying condition (A) stated above, w and $\tilde{w}$ being the corresponding elements of $I(d, N)$, then $w \leq \tilde{w}$.

Definition 4.1.15. *Let $\mathfrak{S}$ be a non-empty monomial in $\mathfrak{N}(v)$. If $\beta_1 > \cdots > \beta_t$ is a v-chain in $\mathfrak{S}$, then we call β_1 the **head** of the v-chain and β_t its **tail**. We call t to be the **length** of the v-chain. We say that an element β of $\mathfrak{S}$ is t-**deep** in $\mathfrak{S}$ (where t is a positive integer) if β is the tail of a v-chain in $\mathfrak{S}$ of length t. The **depth** of β in $\mathfrak{S}$ is defined to be t if β is t-deep in $\mathfrak{S}$, but not $(t+1)$-deep in $\mathfrak{S}$.*

Example 4.1.16. *Let $v = (1, 2, 4, 7, 8, 11)$ and $\mathfrak{S} = \{(9, 1), (6, 2), (5, 4), (13, 8), (12, 11)\}$ be a monomial in $\mathfrak{N}(v)$. Let $\beta = (5, 4)$. Then it is easy to see that β is 1-deep, 2-deep, and 3-deep in $\mathfrak{S}$. But β is not 4-deep in $\mathfrak{S}$. In fact, $(9, 1) > (6, 2) > (5, 4)$ is a v-chain in $\mathfrak{S}$ and this is a v-chain in $\mathfrak{S}$ of maximum length having $\beta = (5, 4)$ as its tail. Hence, the depth of β in $\mathfrak{S}$ is 3 here.*

Proposition 4.1.17. *No two elements of the same depth in $\mathfrak{N}(v)$ are comparable.*

Proof. Let α and β be two elements in $\mathfrak{N}(v)$ of the same depth k. Without loss of generality, suppose $\alpha > \beta$. Since the depth of α is k, so there exists a v-chain $(r_1, c_1) > \cdots > (r_{k-1}, c_{k-1}) > \alpha$, that is α is the tail of the v-chain of length k. Then $(r_1, c_1) > \cdots > (r_{k-1}, c_{k-1}) > \alpha > \beta$ will be the v-chain of length $k+1$ and β is the tail of that v-chain, a contradiction as the depth of β is k.

□

We will now recall the map π of [KR03].

4.2 Description of the map π

Let v be an element of $I(d, N)$. Now, the map π is a function from T^v to $I(d, N) \times T^v$ (where T^v denotes the set of all monomials in $\mathfrak{N}(v)$). For any monomial $\mathfrak{S}$ in T^v, set

$$\pi(\mathfrak{S}) = (w, \mathfrak{S}^{(1)}),$$

where w and $\mathfrak{S}^{(1)}$ are described after Definition 4.2.2. This map enjoys the following good properties:

Proposition 4.2.1.

1. *$w \geq v$.*
2. *v-degree (w) + degree$(\mathfrak{S}^{(1)})$ = degree$(\mathfrak{S})$.*
3. *w dominates $\mathfrak{S}^{(1)}$.*
4. *w is the least element of $I(d, N)$ that dominates $\mathfrak{S}$.*

Proof. The proof is given in [KR03, §4.3]. Hence we omit the proof. □

Now, let us describe the map π.

Let $\mathfrak{S}$ be a non-empty monomial in the elements of $\mathfrak{N}(v)$. We partition $\mathfrak{S}$ in two stages. First we partition $\mathfrak{S}$ into subsets $\mathfrak{S}_1, \cdots, \mathfrak{S}_k$, where k is the largest length of a v-chain in $\mathfrak{S}$: $\beta \in \mathfrak{S}$ belongs to $\mathfrak{S}_j$ if it is j-deep but not $(j+1)$-deep.

Now, we partition each $\mathfrak{S}_j$ into subsets called **blocks** as follows. We arrange the elements of $\mathfrak{S}_j$ in non-decreasing order of their row numbers (all arrangements are from left to right; and elements occur with their respective multiplicities). Among those with the same row number, the arrangement is by non-decreasing order of column numbers. Two consecutive members (r, c), (R, C) in this arrangement are said to be **related** (that is they are in same block) if $r > C$.

Definition 4.2.2. *The **blocks** are the equivalence classes of the smallest equivalence relation containing the above relations. Let $\{(r_1, c_1), \cdots, (r_n, c_n)\}$ be a single block, where $r_1 \leq r_2 \leq \cdots \leq r_n$ and $c_1 \leq c_2 \leq \cdots \leq c_n$. Then (r_1, c_1) is called the **first element** of the block and (r_n, c_n) is called the **last element** of the block. By a **singleton block**, we mean blocks which contain only one element.*

Let $\mathfrak{B}$ be a single block of some $\mathfrak{S}_j$. Let

$$(r_1, c_1), \cdots, (r_p, c_p)$$

be the elements of $\mathfrak{B}$ written in non-decreasing order of both row and column numbers (in such an arrangement, the elements occur with their respective multiplicities). We set $w(\mathfrak{B}) := (r_p, c_1)$ and $\mathfrak{B}'$ to be the monomial

$$\{(r_1, c_2), (r_2, c_3), \ldots, (r_{p-2}, c_{p-1}), (r_{p-1}, c_p)\}.$$

Set $\mathfrak{S}_j^{(1)} := \cup_{\mathfrak{B}} \mathfrak{B}'$ (where the index $\mathfrak{B}$ runs over all blocks of $\mathfrak{S}_j$) and $\mathfrak{S}^{(1)} := \cup_{j=1}^{k} \mathfrak{S}_j^{(1)}$. It follows from [KR03, Corollary 4.13] that the set

$$\{w(\mathfrak{B}) \mid \mathfrak{B} \text{ is a block of } \mathfrak{S}\}$$

is a distinguished subset of $\mathfrak{N}(v)$. Let w be the corresponding element of $I(d, N)$ (under the correspondence given in Remark 4.1.13). Set

$$\pi(\mathfrak{S}) := (w, \mathfrak{S}^{(1)}).$$

This finishes the description of the map π of [KR03].

Example 4.2.3 below gives a detailed illustration of the map π of [KR03].

Example 4.2.3. *Let $d = 6$, $N = 13$, and $v = (1, 2, 4, 7, 8, 11)$. The dark circles in the* *grid in Figure 4.3 represent a monomial $\mathfrak{S}$ in $\mathfrak{N}(v)$, where*

$$\mathfrak{S} = \{(3,2),(5,4),(6,2),(9,1),(9,1),(10,7),(10,7),(10,7),(10,8),(12,1),(13,4)\}.$$

The numbers written near the dark circles denote the multiplicities of these elements in the monomial $\mathfrak{S}$. For this monomial $\mathfrak{S}$, we have

$$\mathfrak{S}_1 = \{(9,1),(9,1),(12,1),(13,4)\},$$

$$\mathfrak{S}_2 = \{(3,2),(6,2),(10,7),(10,7),(10,7),(10,8)\},$$

$$\textit{and } \mathfrak{S}_3 = \{(5,4)\}.$$

Here $\mathfrak{S}_1$ and $\mathfrak{S}_3$ are single blocks. And $\mathfrak{S}_2$ has two blocks given by $\{(3,2),(6,2)\}$ and $\{(10,7),(10,7),(10,7),(10,8)\}$. The dark line segments on the grid show the block decomposition of the monomial $\mathfrak{S}$. The set

$$\{w(\mathfrak{B}) | \mathfrak{B} \textit{ is a block of } \mathfrak{S}\} = \{(13,1),(6,2),(10,7),(5,4)\}.$$

Therefore, $w = (5,6,8,10,11,13)$ and

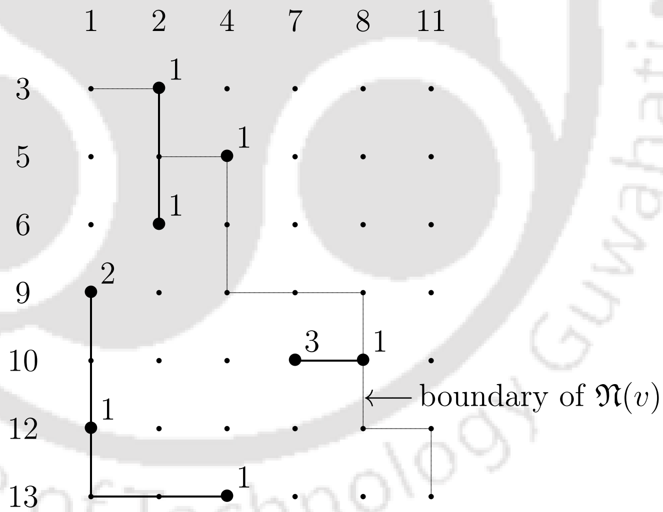


Figure 4.3: A monomial $\mathfrak{S}$

$$\mathfrak{S}^{(1)} = \{(9,1),(9,1),(12,4),(3,2),(10,7),(10,7),(10,8)\}.$$

Using π, we now recall the map $\tilde{\pi}$ of [KR03] from T^v to $\widetilde{SM^{v,v}}$. Proceed by induction on the degree of an element $\mathfrak{S}$ of T^v. The image of the empty monomial under $\tilde{\pi}$ is taken to be the empty monomial. Let $\mathfrak{S}$ be non-empty and suppose that $\pi(\mathfrak{S}) = (w, \mathfrak{S}^{(1)})$. By (1) and (2) of Proposition 4.2.1, the degree of $\mathfrak{S}^{(1)}$ is strictly less than that of $\mathfrak{S}$, and so by induction, $\tilde{\pi}(\mathfrak{S}^{(1)})$ is defined. Suppose that $\tilde{\pi}(\mathfrak{S}^{(1)}) = w' \geq \ldots$. By induction we also know that, the degree of $\mathfrak{S}^{(1)}$ is the same as that of $w' \geq \ldots$ and that w' is the least element of $I(d,N)$ that dominates $\mathfrak{S}^{(1)}$. By (3) of Proposition 4.2.1, we have $w \geq w'$, and we set $\tilde{\pi}(\mathfrak{S}) := w \geq \tilde{\pi}(\mathfrak{S}^{(1)})$. This finishes the description of the map $\tilde{\pi}$ of [KR03].

Example 4.2.4. *For the monomial $\mathfrak{S}$ in Example 4.2.3 above, we have* $\tilde{\pi}(\mathfrak{S}) = (5,6,8,10,11,13) \geq (3,4,8,10,11,12) \geq (2,4,7,8,10,11) \geq (1,2,7,8,10,11) \geq (1,2,4,8,9,11) \geq (1,2,4,7,9,11)$.

4.2.1 Some necessary results

Before going to the description of the map ϕ of [KR03] let us first state some results from [KR03] (the proofs of which can be found in [KR03]), which are needed to describe the map ϕ of [KR03].

Lemma 4.2.5. *Let $\beta_1 = (r_1, c_1) > \cdots > \beta_t = (r_t, c_t)$ be a v-chain, w an element of $I(d,N)$ with $w \geq v$, and $\mathfrak{S}_w$ the distinguished subset of $\mathfrak{N}(v)$ associated to w. Then w dominates $\beta_1 > \cdots > \beta_t$ if and only if there exists a v-chain $\alpha_1 = (R_1, C_1) > \cdots > \alpha_t = (R_t, C_t)$ in $\mathfrak{S}_w$ such that $C_j \leq c_j$ and $R_j \geq r_j$, for $1 \leq j \leq t$.*

Lemma 4.2.6. *Let $\beta = (r,c)$ be an element of $\mathfrak{N}(v)$ and w an element of $I(d,N)$ such that $w \geq s_\beta v$. Then there exists an element (R,C) in the distinguished monomial $\mathfrak{S}_w$ associated to w such that $C \leq c$ and $r \leq R$.*

Corollary 4.2.7. *Let w be an element of $I(d,N)$ and $\mathfrak{S}_w$ the corresponding distinguished subset of $\mathfrak{N}(v)$. For a positive integer j, let $\mathfrak{S}_w^j$ denote the subset of $\mathfrak{S}_w$ of those elements that are j-deep, and w^j the corresponding element of $I(d,N)$. Let $\beta_1 = (r_1, c_1) > \cdots > \beta_t = (r_t, c_t)$ be a v-chain.*

1. *If w^k dominates $\beta_1 > \cdots > \beta_t$, then w^{k+1} dominates $\beta_2 > \cdots > \beta_t$, w^{k+2} dominates $\beta_3 > \cdots > \beta_t$, and so on.*
2. *If for integers $l > k$, there exists (R,C) in $\mathfrak{S}_w^l$ such that $C \leq c_1$ and $r_1 \leq R$, and w^{k+1} does not dominate $\beta_1 > \cdots > \beta_t$, then w^{k+2} does not dominate $\beta_2 > \cdots > \beta_t$, w^{k+3} does not dominate $\beta_3 > \cdots > \beta_t$, and so on until, finally, w^{l+1} does not dominate $\beta_{l-k+1} > \cdots > \beta_t$.*

4.3 Description of the map ϕ

Let w be an element of $I(d,N)$ with $w > v$. Now, before going to the description of the map ϕ we need to fix a notation.
Notation: Let $mon_{resw}\mathfrak{N}(v)$ denote the **set of all monomials in $\mathfrak{N}(v)$ which are dominated by w.**

Now, ϕ is a mapping from $\{w\} \times mon_{resw}\mathfrak{N}(v)$ to T^v. It takes a pair $(w, \mathfrak{T})$ as input (where $\mathfrak{T}$ is a monomial in $mon_{resw}\mathfrak{N}(v)$) and produces an element $\mathfrak{T}'$ in T^v as output. That is

$$\phi(w, \mathfrak{T}) = \mathfrak{T}'.$$

Let $\mathfrak{S}_w$ be a monomial in $\mathfrak{N}(v)$ associated to w as in Proposition 4.1.13, and k be the maximum length of a v-chain in $\mathfrak{S}_w$. For a positive integer j, $1 \le j \le k$, let $\mathfrak{S}_w^j$ be the subset of elements of $\mathfrak{S}_w$ that are j-deep. Let w^j be the element associated by Proposition 4.1.13 to $\mathfrak{S}_w^j$, we have, $w = w^1 \ge \cdots \ge w^k \ge v$ (see Remark 4.1.14). Clearly, $\mathfrak{S}_w = \mathfrak{S}_w^1 \supseteq \cdots \supseteq \mathfrak{S}_w^k$.

For a positive integer j, $1 \le j \le k$, let $\mathfrak{T}_w^j$ be the subset of $\mathfrak{T}$ of elements β such that β is the head of a v-chain in $\mathfrak{T}$, dominated by w^j but not by w^{j+1} (we set $w^{k+1} = v$), and every v-chain in $\mathfrak{T}$ with head β is dominated by w^j. Then $\mathfrak{T}_w^j$ forms a partition of $\mathfrak{T}$ (some of the $\mathfrak{T}_w^j$ could be empty). Two distinct elements belonging to the same $\mathfrak{T}_w^j$ are not comparable: let β, β' both are elements of $\mathfrak{T}_w^j$ such that $\beta > \beta'$. Since β' belongs to $\mathfrak{T}_w^j$ so there exists a v-chain $\beta' > \cdots$ in $\mathfrak{T}$ that is not dominated by w^{j+1}, so by Condition (1) of Corollary 4.2.7, the v-chain $\beta > \beta' > \cdots$ is not dominated by w^j.

We now further partition each $\mathfrak{T}_w^j$ into subsets called pieces as follows:
let $\mathfrak{S}_{w,j}$ denote the set of all elements of $\mathfrak{S}_w$ that are j-deep but not $(j+1)$-deep. Clearly, no two distinct elements of $\mathfrak{S}_{w,j}$ are comparable.

Now, we will state [KR03, Lemma 4.1.7], which is stated below as Lemma 4.3.1, and the proof can be found in [KR03].

Lemma 4.3.1. *For an element (r, c) of $\mathfrak{T}_w^j$, there exists a unique element (R, C) of $\mathfrak{S}_{w,j}$ such that $C \le c$ and $r \le R$.*

We will index the pieces of $\mathfrak{T}_w^j$ by elements of $\mathfrak{S}_{w,j}$. Let $\beta = (R, C)$ be an element of $\mathfrak{S}_{w,j}$. Then the corresponding piece p_β of $\mathfrak{T}_w^j$ consists of all those (r, c) in $\mathfrak{T}_w^j$ with $r \le R$ and $C \le c$. Of course, some of these pieces could be empty. Let us arrange the elements of p_β in non-decreasing order of the row entries; among those with equal row entries the arrangement is by non-decreasing order of column entries. Since no two distinct elements of $\mathfrak{T}_w^j$ are comparable, the column entries are also in non-decreasing order. Suppose the arrangement is

$$(r_1, c_1), (r_2, c_2), \cdots, (r_p, c_p).$$

Note that $C \le c_1$ and $r_p \le R$. Let $p_\beta^\star$ denote the monomial

$$\{(r_1, C), (r_2, c_1), \cdots, (r_p, c_{p-1}), (R, c_p)\}.$$

Clearly, elements of $p^\star_\beta$ also belong to $\mathfrak{N}(v)$. Set

$$(\mathfrak{T}^j_w)^\star := \bigcup_{\beta \in \mathfrak{S}_{w,j}} p^\star_\beta,$$

$$\phi(w, \mathfrak{T}) := \bigcup_j (\mathfrak{T}^j_w)^\star.$$

This finishes the description of ϕ.

Now, we will state [KR03, Proposition 4.2], which is stated below as Proposition 4.3.2, and the proof can be found in [KR03].

Proposition 4.3.2. *The maps π and ϕ are inverses to each other.*

CHAPTER 5

RELATION BETWEEN THE MAPS $BRSK$ AND $\widetilde{\pi}$

5.1 Relation between the maps $BRSK$ and $\widetilde{\pi}$ for a monomial in $\mathfrak{N}(v)$

5.1.1 Statement of the main theorem

The main results of this chapter are Theorem 5.1.1 and Corollary 5.1.3 below.

Theorem 5.1.1. *Let U be a finite monomial in $\mathfrak{N}(v)$. Let $\pi(U) = (w_0, U^{(1)})$, $\pi(U^{(1)}) = (w_1, U^{(2)}), \cdots$ and so on till $\pi(U^{(m)}) = (w_m, \emptyset)$, where $\emptyset$ is the empty monomial . Then for each $r \in \{0, 1, \cdots, m\}$, the following holds :*
(i) all the row numbers of the distinguished subset $\mathfrak{S}_{w_r}$ (corresponding to w_r) consist of the $((m+1)-r)$-th row entries of the left hand notched tableau of $BRSK(U)$.
(ii) All the column numbers of $\mathfrak{S}_{w_r}$ comprise of the $((m+1)-r)$-th row entries of the right hand notched tableau of $BRSK(U)$.

Example 5.1.2 below illustrates Theorem 5.1.1.

Example 5.1.2. *Let $d = 6$, $N = 13$, and $v = (1, 2, 4, 7, 8, 11)$. Let U be a finite monomial in $\mathfrak{N}(v)$ given by*

$$U = \{(3, 2), (5, 4), (6, 2), (9, 1), (9, 1), (10, 7), (10, 7), (10, 7), (10, 8), (12, 1), (13, 4)\}.$$

Then $\pi(U) = (w_0, U^{(1)})$, $\pi(U^{(1)}) = (w_1, U^{(2)})$, $\cdots$, and so on till $\pi(U^{(5)}) = (w_5, \emptyset)$, where

$$w_0 = (5, 6, 8, 10, 11, 13), w_1 = (3, 4, 8, 10, 11, 12), w_2 = (2, 4, 7, 8, 10, 11),$$

$$w_3 = (1, 2, 7, 8, 10, 11), w_4 = (1, 2, 4, 8, 9, 11), w_5 = (1, 2, 4, 7, 9, 11)$$

and

$$U^{(1)} = \{(9,1),(9,1),(12,4),(3,2),(10,7),(10,7),(10,8)\},$$

$$U^{(2)} = \{(9,1),(9,4),(10,7),(10,8)\}, U^{(3)} = \{(9,4),(9,7),(10,8)\},$$

$$U^{(4)} = \{(9,7),(9,8)\}, U^{(5)} = \{(9,8)\}.$$

Here $m = 5$ *and* $BRSK(U) = \left(\begin{array}{|c|c|c|c|} \hline 9 \\ \cline{1-1} 9 \\ \cline{1-1} 10 \\ \cline{1-1} 10 \\ \cline{1-1} 3 & 10 & 12 \\ \cline{1-3} 5 & 6 & 10 & 13 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline 8 \\ \cline{1-1} 7 \\ \cline{1-1} 4 \\ \cline{1-1} 1 \\ \cline{1-1} 1 & 2 & 7 \\ \cline{1-3} 1 & 2 & 4 & 7 \\ \hline \end{array} \right).$

Also, $\mathfrak{S}_{w_0} = \{(13,1),(6,2),(10,7),(5,4)\}$, $\mathfrak{S}_{w_1} = \{(12,1),(3,2),(10,7)\}$, $\mathfrak{S}_{w_2} = \{(10,1)\}$, $\mathfrak{S}_{w_3} = \{(10,4)\}$, $\mathfrak{S}_{w_4} = \{(9,7)\}$, *and* $\mathfrak{S}_{w_5} = \{(9,8)\}$. *For each* $r \in \{0,1,\cdots,5\}$, *conditions (i) and (ii) of the above theorem hold true. For example, if we take* $r = 1$, *then all the row numbers of the distinguished subset* $\mathfrak{S}_{w_1}$ *are given by* $\{3,10,12\}$, *which are the* $((5+1)-1)$*-th row entries of the left hand notched tableau of* $BRSK(U)$. *Similarly, for the column numbers of* $\mathfrak{S}_{w_1}$, *which comprise of the* $((5+1)-1)$*-th row entries of the right hand notched tableau of* $BRSK(U)$.

Corollary 5.1.3. *For any finite monomial* U *in* $\mathfrak{N}(v)$, $\tilde{\pi}(U) = BRSK(U)$.

Before we go into a proof of Corollary 5.1.3, we have an important remark (namely, Remark 5.1.4), and an example (namely, Example 5.1.5) below, which together illustrate Corollary 5.1.3.

Remark 5.1.4. *In Corollary 5.1.3 above, when we say that* $\tilde{\pi}(U) = BRSK(U)$, *we mean the following: let* $BRSK(U) = (P,Q)$. *Let* $P_1,\cdots,P_r$ *(resp.* $Q_1,\cdots,Q_r$*) denote all the rows of* P *(resp.* Q*) from top to bottom. Then we mean that* $\tilde{\pi}(U)$ *equals the standard monomial* $P_r - Q_r \geq \cdots \geq P_1 - Q_1$ *(where* $P_i - Q_i$ *has to be interpreted as* $P_i \cup (v \setminus Q_i)$ *in the sense of §3.1).*

Example 5.1.5. *For the monomial* U *in Example 5.1.2, we have* $\tilde{\pi}(U) = w_0 \geq w_1 \geq \cdots \geq w_5$, *where* $w_0,\ldots,w_5$ *are as given in Example 5.1.2. It is now easy to verify (using Remark 5.1.4 above) that* $\tilde{\pi}(U) = BRSK(U)$. $BRSK(U)$ *is given in Example 5.1.2 above. Let* $P_1,\ldots,P_6$ *denote the rows (from top to bottom) of the left hand notched tableau of* $BRSK(U)$. *Similarly, let* $Q_1,\cdots,Q_6$ *denote the rows (from top to bottom) of the right hand notched tableau of* $BRSK(U)$. *Then observe that* $w_0 = P_6 - Q_6, w_1 = P_5 - Q_5, w_2 = P_4 - Q_4, w_3 = P_3 - Q_3, w_4 = P_2 - Q_2$, *and* $w_5 = P_1 - Q_1$. *Hence* $\tilde{\pi}(U) = BRSK(U)$.

Proof of Corollary 5.1.3: According to the notation used in the statement of Theorem 5.1.1, we have $\tilde{\pi}(U) = w_0 \geq w_1 \geq \cdots \geq w_m$. We need to show that $\tilde{\pi}(U) = BRSK(U)$ for any finite monomial U in $\mathfrak{N}(v)$.

Let $BRSK(U) = (P, Q)$. In view of Remark 5.1.4 and the notation used in the statement of Theorem 5.1.1, it suffices to show that for each $r \in \{0, 1, \cdots, m\}$, we have:

$$P_{(m+1)-r} - Q_{(m+1)-r} = w_r,$$

where $P_{(m+1)-r}$ (respectively $Q_{(m+1)-r}$) denotes the $((m+1)-r)$-th row of P (respectively Q) from the top. But we can interpret $P_{(m+1)-r} - Q_{(m+1)-r}$ as $P_{(m+1)-r} \uplus (v \setminus Q_{(m+1)-r})$ (in the sense of §4 of [Kre08]). So in other words, it suffices to show that

$$P_{(m+1)-r} \uplus (v \setminus Q_{(m+1)-r}) = w_r.$$

But from the statement of Theorem 5.1.1, we have that all row numbers of the distinguished subset $\mathfrak{S}_{w_r}$ (corresponding to w_r) consist of the entries of $P_{(m+1)-r}$, and all column numbers of $\mathfrak{S}_{w_r}$ consist of the entries of $Q_{(m+1)-r}$. Now, if we look at the proof of Remark 4.1.13 (the second paragraph of the proof, to be more precise), it follows from the procedure given there that

$$(\text{the row numbers of } \mathfrak{S}_{w_r}) \uplus (v \setminus \text{the column numbers of } \mathfrak{S}_{w_r}) = w_r.$$

But observe that the row numbers of $\mathfrak{S}_{w_r}$ are nothing but the entries of $P_{(m+1)-r}$ and the column numbers of $\mathfrak{S}_{w_r}$ are the entries of $Q_{(m+1)-r}$. □

Definition 5.1.6. *Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_l$ be all the blocks of $\mathfrak{S}_j$, where $\mathfrak{B}_1$ contains the element(s) of $\mathfrak{S}_j$ having the least row and column number. Then $\mathfrak{B}_1$ is called the **topmost block** of $\mathfrak{S}$ of depth j.*

5.1.2 The strategy and the proof of Theorem 5.1.1

In this subsection, we will frequently use Definitions 4.2.2, and 5.1.6. The reader is urged to recall these definitions.

Let n be a positive integer. Let U be a finite monomial in $\mathfrak{N}(v)$ of cardinality n. Then U can be regarded as a *positive multiset* on $\mathbb{N}^2$ of degree n . Now, we know from the note of §3.2.7 that $BRSK(U) = \iota(BRSK(\iota(U)))$ (where ι has stated in Definition 3.1.2). Let $(P^{(n)}, Q^{(n)})$ denote $BRSK(\iota(U))$. It then follows from the definition of ι that for proving Theorem 5.1.1, it suffices to prove the following:

Theorem 5.1.7. *With notation as in the statement of Theorem 5.1.1, the following two statements hold true for each $r \in \{0, 1, \cdots, m\}$:*
(i) the column numbers of the elements of $\mathfrak{S}_{w_r}$ consist of the $(r+1)$-th row entries of $P^{(n)}$.
(ii) The row numbers of the elements of $\mathfrak{S}_{w_r}$ consist of the $(r+1)$-th row entries of $Q^{(n)}$.

Observe now (with the notation used in Theorem 5.1.1) that for each $r \in \{0, 1, \cdots, m\}$, we have $\pi(U^{(r)}) = (w_r, U^{(r+1)})$, and $\mathfrak{S}_{w_r}$ is the distinguished monomial in $\mathfrak{N}(v)$ corresponding to w_r. It now follows from the description of the map π (on any finite monomial in $\mathfrak{N}(v)$, as given in §4.2) that it suffices to prove the following:

Theorem 5.1.8. *With notation as in the statement of Theorem 5.1.1, the following two statements hold true for each $r \in \{0, 1, \cdots, m\}$:*
(i) the least column numbers of all blocks of $U^{(r)}$ consist of the $(r+1)$-th row entries of $P^{(n)}$.
(ii) The largest row numbers of all blocks of $U^{(r)}$ consist of the $(r+1)$-th row entries of $Q^{(n)}$.

We will prove Theorem 5.1.8 by induction on the cardinality n of U.

The induction hypothesis: Theorem 5.1.8 is true for all finite monomials U in $\mathfrak{N}(v)$ having cardinality $\leq n-1$.

If $n = 1$, the theorem is obvious. So we assume that $n > 1$. Let ι be the involution map which was described in Definition 3.1.2. Let U be a monomial in $\mathfrak{N}(v)$ of cardinality n. Arrange $\iota(U)$ in lexicographic order (which is stated in Definition 3.2.7), say, $\iota(U) = \{(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)\}$. Let $F = \{(b_1, a_1), (b_2, a_2), \cdots, (b_{n-1}, a_{n-1})\}$. Then $b_n \leq b_{n-1}$ and if $b_n = b_{n-1}$, we have $a_{n-1} \geq a_n$ (by the definition of lexicographic order). The element (b_n, a_n) enters into F to make it U. Let $BRSK(\iota(F)) = (P^{(n-1)}, Q^{(n-1)})$. Then we have:

$$(P^{(n)}, Q^{(n)}) = BRSK(\iota(U)) = (P^{(n-1)}, Q^{(n-1)}) \overset{b_n}{\leftarrow} a_n.$$

To explain the strategy and the proof further, first we need a definition (see Definition 5.1.9 below) and a Lemma (see Lemma 5.1.11 below).

Definition 5.1.9. *Let $\{(R_1, C_1), (R_2, C_2), \cdots, (R_p, C_p)\}$ be a block $\mathfrak{B}$ of some finite monomial $\mathfrak{S}$ of $\mathfrak{N}(v)$. Let $b_n \leq R_1$ be such that $b_n \in [N] \setminus v$ and $b_n > C_1$. Let $a_n \in v$ be such that $a_n \leq C_1$. Then we say that $\{(b_n, a_n), (R_1, C_1), \cdots, (R_p, C_p)\}$ is a* ***left concatenation*** *of $\mathfrak{B}$ by (b_n, a_n).*

Example 5.1.10 below illustrates the above definition of left concatenation of a block.

Example 5.1.10. *Let $d = 9$, $N = 17$, and $v = (1, 2, 4, 6, 7, 11, 12, 13, 15)$. Let $\mathfrak{S}$ be the finite monomial in $\mathfrak{N}(v)$ given by*

$$\mathfrak{S} = \{(8,2), (8,2), (8,6), (8,6), (8,7), (9,2), (10,1), (14,13), (16,12), (16,13), (17,11)\}.$$

Figure 5.1 shows the monomial $\mathfrak{S}$ and its block decomposition. The dark circles denote the elements of the monomial $\mathfrak{S}$, and the numbers written near these dark circles denote the *multiplicities with which these elements occur in the monomial $\mathfrak{S}$. The dark line segments (in case of blocks consisting of more than one distinct elements) and the dark circles (in case of blocks consisting of a single element possibly appearing more than once) are the blocks of $\mathfrak{S}$.*

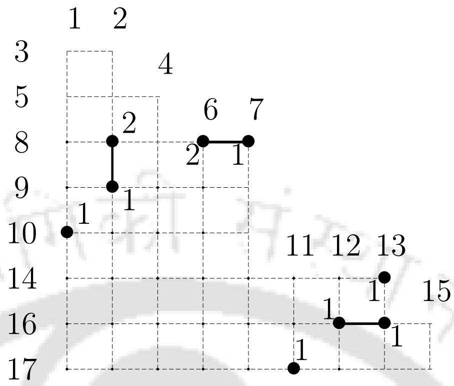


Figure 5.1: The monomial $\mathfrak{S}$ and its block decomposition

Consider the block $\mathfrak{B}$ of $\mathfrak{S}$ given by $\mathfrak{B} = \{(8,6),(8,6),(8,7)\}$. Then $p = 3$, $(R_1, C_1) = (R_2, C_2) = (8,6)$, and $(R_3, C_3) = (8,7)$. Let $(b_n, a_n) = (8,4)$. Then $b_n \leq R_1$, $b_n \in [N] \setminus v$, and $b_n > C_1$. Also, $a_n \in v$ and $a_n \leq C_1$. Observe that $\{(8,4),(8,6),(8,6),(8,7)\}$ is a left concatenation of the block $\mathfrak{B}$ by $(b_n, a_n) = (8,4)$ (see Figure 5.2 for an illustration). The dark circles in Figure 5.2 denote the elements of the monomial $\mathfrak{S} \cup \{(8,4)\}$, and the numbers written near these dark circles denote the multiplicities with which these elements occur in the monomial $\mathfrak{S} \cup \{(8,4)\}$. The dark line segments (in case of blocks consisting of more than one distinct elements) and the dark circles (in case of blocks consisting of a single element possibly appearing more than once) are the blocks of $\mathfrak{S} \cup \{(8,4)\}$.

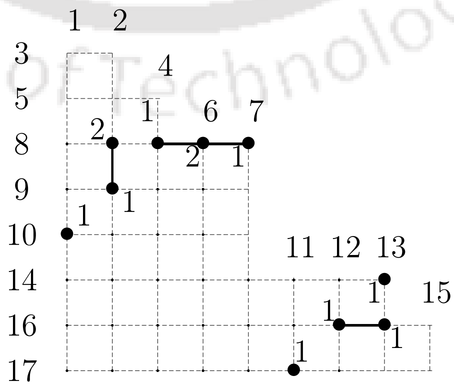


Figure 5.2: Left concatenation of a block

Lemma 5.1.11. *Let* $U = \{(b_1, a_1), (b_2, a_2), \cdots, (b_n, a_n)\}$ *be a finite monomial in* $\mathfrak{N}(v)$ *such that* $\iota(U)$ *is in lexicographic order (which is stated in Definition 3.2.7). Let* $F = \{(b_1, a_1), (b_2, a_2), \cdots, (b_{n-1}, a_{n-1})\}$. *The element* (b_n, a_n) *enters into* F *to make it* U. *Then either the singleton set* $\{(b_n, a_n)\}$ *is itself a block in* U *or the element* (b_n, a_n) *left concatenates a block of* F.

Proof. Suppose the singleton set $\{(b_n, a_n)\}$ is not a block in U and the element (b_n, a_n) does not left concatenate any block of F as well.

Then there exists a block $\mathfrak{B}$ of F such that in the process of forming the monomial U from the monomial F, the element (b_n, a_n) gets added to the block $\mathfrak{B}$ of F, but not at the leftmost end. Observe now that b_n is the least possible row number among all elements of U. All these facts put together imply that there exists an element $(b_n, p) \in \mathfrak{B}$ such that $p < a_n$. This in turn implies that there exist(s) element(s) in F having row number b_n. So, we must have $b_{n-1} = b_n$. But then since $(b_n, p) \in F$, it follows from the definition of lexicographic order that $a_n \leq p$. This contradicts the fact that $p < a_n$. □

Recall that our goal is to prove Theorem 5.1.8 above. Let F and U be as in the statement of Lemma 5.1.11 above. Then we can easily see (from the statement of Lemma 5.1.11) that we can divide the proof of Theorem 5.1.8 into 2 cases:

Case I: when the singleton set $\{(b_n, a_n)\}$ is itself a block in U.
Case II: when the element (b_n, a_n) left concatenates a block of F.

Case I

In this case, we can easily see that all the blocks of U other than the block $\{(b_n, a_n)\}$ are same as all the blocks of F. In fact, the block $\{(b_n, a_n)\}$ is the topmost block of U of some depth (say, k). This is simply because b_n is the least possible row number among all elements of U.

Let $U^{(0)} = U$ and $\pi(U^{(r)}) = (w_r, U^{(r+1)})$ for all $r \in \{0, 1, \cdots, m\}$ (where the integer m is as given in the statement of Theorem 5.1.1). Similarly, it also holds true for F. Then it is easy to see that for each $r \in \{0, 1, \cdots, m\}$, all the blocks of $U^{(r+1)}$ are going to be the same as all the blocks of $F^{(r+1)}$.

Clearly, in this case, (b_n, a_n) becomes an element of $\mathfrak{S}_{w_0}$. Hence we must have that a_n is an entry in the $(0+1)$-th row of $P^{(n)}$ and b_n is an entry in the $(0+1)$-th row of $Q^{(n)}$. All other entries of $(P^{(n)}, Q^{(n)})$ remain the same as in $(P^{(n-1)}, Q^{(n-1)})$ (in the same rows). So, to prove Theorem 5.1.8, all that we need to show is the following:

- during the process $P^{(n-1)} \overset{b_n}{\leftarrow} a_n$ of bounded insertion, the element a_n gets placed in the first row of $P^{(n-1)<b_n}$ and a_n bumps nothing.

This is achieved by Lemma 5.2.3 below and Lemma 5.2.1 is used in the proof of Lemma 5.2.3. Lemma 5.2.1 is stated and proved in §5.2.1 below and Lemma 5.2.3 is stated and proved in §5.2.2 below. This finishes the proof of Theorem 5.1.8 in this case.

Case II

The proof in this case follows directly from the statement of Lemma 5.2.7 together with the statement of Lemma 5.2.5. Lemma 5.2.5 provides a precise description of what happens (at each stage) to the blocks of a finite monomial F in $\mathfrak{N}(v)$ (when the element (b_n, a_n) enters into it to make it U) under the procedure given by the map $\tilde{\pi}$ of §4.2. Again, Lemma 5.2.7 provides the analog of Lemma 5.2.5 in terms of the map $BRSK$ of Definition 3.2.7. The interrelation between Lemma 5.2.5 and Lemma 5.2.7 is the following:

with notation as in Lemma 5.2.5, a block of $F^{(t)}$ is **left concatenated** by $(b_n, \star)$ (during the process of transition from F to U) $\iff$ there is a **bumping** at the $(t+1)$-th row of $P^{(n-1)}$, where $t \in \{0, 1, \cdots, k'\}$.

This interrelation is proved inside the proof of Lemma 5.2.7, which finishes the proof of Theorem 5.1.8 in this case.

5.2 Lemmas needed to prove Theorem 5.1.8

In this section, we will frequently use Definitions 4.1.15, 4.2.2, and 5.1.6. The reader is urged to recall these definitions.

5.2.1 A general lemma

In this subsection, we are going to prove a lemma, which is crucial in the proof of Lemma 5.2.3, which in turn is used to prove Theorem 5.1.8 in Case I.

Let k be a positive integer such that elements of depth k exist in the monomial F. Let $\{(r_1, c_1), \cdots, (r_p, c_p)\}$ denote the topmost block of F of depth k, where the elements of the block are written in non-decreasing order of both row and column indices. It then follows from the induction hypothesis that c_1 is an entry in the first row of $P^{(n-1)}$.

Lemma 5.2.1. *The entries in the first row of $P^{(n-1)}$ which are strictly less than c_1, are the column numbers of the first elements of some blocks (here we consider the blocks in the sense of Definition 4.2.2) of F of depth $< k$ (here we consider the depth in the sense of Definition 4.1.15).*

Proof. Suppose not. Say, some $p_{1j'}(< c_1)$ is the column number of the first element of some block $\mathfrak{B}$ of F of depth greater than or equal to k. Say, the first element of the block $\mathfrak{B}$ is $(R, p_{1j'})$.

Case (i) : $R > r_1$

If $R > r_1$, then we will have $(R, p_{1j'}) > (r_1, c_1)$. As depth of $(R, p_{1j'}) \geq k$, so in this case, (r_1, c_1) will have depth $> k$ in F, a contradiction.

Case (ii) : $R = r_1$

As $R = r_1$ and $p_{1j'} < c_1$, so if the depth of $(R, p_{1j'}) > k$, then depth of $(r_1, c_1) > k$ (a contradiction because the depth of $(r_1, c_1) = k$). Now, if depth of $(R, p_{1j'}) = k$, then the depth of $(r_1, c_1) \geq k$. For the case when the depth of (r_1, c_1) is greater than k, again we get a contradiction. But if depth of $(r_1, c_1) = k$ then (r_1, c_1) cannot be the first element of the topmost block of F of depth k (because we have $p_{1j'} < c_1$ and we have arranged the row and column indices of the topmost block of F in non-decreasing order), again a contradiction.

Case (iii) : $R < r_1$

If $(R, p_{1,j'})$ is of depth k, then clearly, (r_1, c_1) cannot be the first element of the topmost block of F of depth k (as we have taken $p_{1j'} < c_1$ and we have arranged the row and column indices of the topmost block of F in non-decreasing order). If $(R, p_{1j'})$ has depth $s > k$, then there exists a v-chain of length s having tail $(R, p_{1j'})$. Say, the v-chain is $(e_1, f_1) > (e_2, f_2) > \cdots > (e_s, f_s) = (R, p_{1j'})$. Then (e_k, f_k) will have depth k in F. If $e_k \leq r_1$, then actually (r_1, c_1) is not the first element of the topmost block of F of depth k, a contradiction. If $e_k > r_1$, then $(e_k, f_k) > (r_1, c_1)$, a contradiction to the depth of (r_1, c_1) in F (as both (r_1, c_1) and (e_k, f_k) are of depth k and by Proposition 4.1.17, no two elements of same depth are comparable). □

Example 5.2.2 below illustrates Lemma 5.2.1.

Example 5.2.2. *Let* $d = 9$*,* $N = 17$*, and* $v = (1, 2, 4, 6, 7, 11, 12, 13, 15)$*. Let* F *be the finite monomial in* $\mathfrak{N}(v)$ *given by*

$$F = \{(8,2), (8,2), (8,6), (8,6), (8,7), (9,2), (10,1), (14,13), (16,12), (16,13), (17,11)\}.$$

Figure 5.3 shows the monomial F *and its block decomposition. The dark circles denote the elements of the monomial* F*, and the numbers written near these dark circles denote the multiplicities with which these elements occur in the monomial* F*. The dark line segments (in case of blocks consisting of more than one distinct elements) and the dark circles (in case of blocks consisting of a single element possibly appearing more than once) are the blocks of* F*. Let* $BRSK(\iota(F)) = (P^{(n-1)}, Q^{(n-1)})$*.*

Then

$$(P^{(n-1)}, Q^{(n-1)}) = \left(\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 11 & 12 & 13 \\ \hline 2 & 13 \\ \cline{1-2} 2 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|c|c|c|c|} \hline 8 & 9 & 10 & 14 & 16 & 17 \\ \hline 8 & 16 \\ \cline{1-2} 8 \\ \cline{1-1} 8 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \right).$$

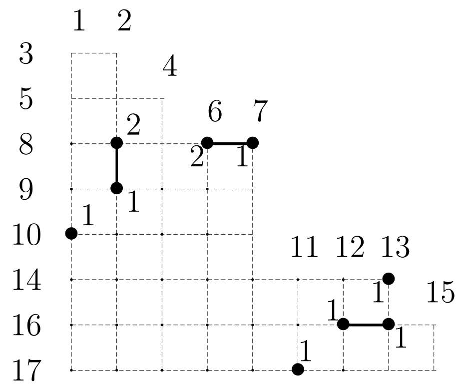


Figure 5.3: The monomial F and its block decomposition

Let $k = 3$. Then $\{(r_1, c_1), (r_2, c_2), (r_3, c_3)\}$ is the topmost block of F of depth k, where $(r_1, c_1) = (r_2, c_2) = (8, 6)$, and $(r_3, c_3) = (8, 7)$. Here $c_1 = 6$. Observe that, $c_1 = 6$ is an entry in the first row of $P^{(n-1)}$. Also Observe that, the entries in the first row of $P^{(n-1)}$ which are strictly less than c_1 are 1 and 2. Note that 1 and 2 are the column numbers of the first elements of some blocks of F of depth $< k = 3$. In fact, 1 is the column number of the first element of the block $\{(10, 1)\}$ of F, which is of depth 1. Similarly, 2 is the column number of the first element of the block $\{(8, 2), (8, 2), (9, 2)\}$ of F, which is of depth 2.

5.2.2 Lemma for Case I

Lemma 5.2.3. *Let (b_n, a_n) enter into the monomial F to make it U in such a way that the singleton set $\{(b_n, a_n)\}$ becomes the topmost block of U of depth k, and the next block of U of depth k from the top being $\{(r_1, c_1), (r_2, c_2), \cdots, (r_p, c_p)\}$. Then the entries in the first row of $P^{(n-1)}$ which are strictly less than b_n are all strictly less than a_n. Furthermore, the largest row numbers of all blocks of F are all strictly bigger than b_n.*

Proof. It follows from the hypothesis of this lemma that $b_n < c_1$ (as (b_n, a_n) is an entry of the first block of depth k and (r_1, c_1) is an entry of the second block of depth k), $b_n < r_1 \leq r_2 \leq \cdots \leq r_p$ and $a_n < c_1 \leq c_2 \leq \cdots \leq c_p$ (as the second block of depth k lies in the below and right side than the first block of depth k). It also follows from the induction hypothesis that the first row of $P^{(n-1)}$ contains the smallest column numbers of each block of F. In particular, it contains the entry c_1.

We will prove this lemma by method of contradiction. Suppose the conclusion of this lemma does not hold. Say, some entry p_{1j_0} of the first row of $P^{(n-1)}$ which is strictly less than b_n is greater than or equal to a_n. Then we have $a_n \leq p_{1j_0} < b_n < c_1$. By induction hypothesis, we know that p_{1j_0} is the smallest column number of some block of F, say block $\mathfrak{D}$. Since $p_{1j_0} < c_1$, it follows (from Lemma 5.2.1) that the block $\mathfrak{D}$ has depth $< k$. Say, $\mathfrak{D}$ is a block of depth $s(< k)$. Now, since (b_n, a_n) is of depth k in U, there exists an element (R, C) in U of depth s $(< k)$ such that (R, C) and (b_n, a_n) form a v-chain. That is, $b_n < R$ and $C < a_n$. Say, (R, C) lies in the block $\mathfrak{B}$ of U. Clearly then, $\mathfrak{B} \neq \mathfrak{D}$ (since (R, C) lies in the block $\mathfrak{B}$ of U, which is of depth s and $C < a_n$, it follows that the smallest column number of the block $\mathfrak{B}$ is less than a_n; but the smallest column number of the block $\mathfrak{D}$ is p_{1j_0} which is greater than or equal to a_n.). Let $(\widehat{R}, \widehat{C})$ be the last element of the block $\mathfrak{B}$. Then $R \leq \widehat{R}$ and $C \leq \widehat{C}$.

Hence, we have $\widehat{R} \geq R > b_n > p_{1j_0} \Rightarrow \widehat{R} > p_{1j_0}$. That is, $\mathfrak{B}$ and $\mathfrak{D}$ are not two different blocks of depth s, a contradiction.

It now remains to prove the last assertion of this lemma. We will prove it by the method of contradiction. Suppose not, say there exists a block $\mathfrak{B}$ of F whose largest row number R is less than or equal to b_n.

Since b_n is the least possible row number among all elements of U, it follows that $R = b_n$ and that all the elements of the block $\mathfrak{B}$ of F will have row number b_n. Let (b_n, p) denote the first element of $\mathfrak{B}$. Since there exists an element in F having row number b_n (namely, (b_n, p)), it follows that $b_{n-1} = b_n$. Then from the definition of lexicographic order, it follows that $a_n \leq p$.

Since the singleton set $\{(b_n, a_n)\}$ itself is a block in U, it follows that (b_n, a_n) and (b_n, p) cannot have the same depth in U. Let k be the depth of (b_n, a_n) in U and s be the depth of (b_n, p) in U. We have $s \neq k$.

If $s < k$, then there exists an element (e, f) in F (and hence in U) such that (e, f) is of depth s in F and (e, f) forms a v-chain with (b_n, a_n). But then, since $a_n \leq p$, we can easily see that (e, f) will form a v-chain with (b_n, p) as well. This contradicts Proposition 4.1.17.

If $s > k$, then by some similar arguments, we will either arrive at a contradiction to the fact that no two elements of the same depth are comparable or arrive at a contradiction to the fact that the singleton set $\{(b_n, a_n)\}$ is a block in U. $\square$

Example 5.2.4 below illustrates Lemma 5.2.3.

Example 5.2.4. *Let d, N, v, F be as given in Example 5.2.2 above. Then $(P^{(n-1)}, Q^{(n-1)})$ is also given in Example 5.2.2 above. Let $(b_n, a_n) = (5, 4)$ and $k = 3$. Then (b_n, a_n) enters into the monomial F to make it U in such a way that the singleton set $\{(b_n, a_n)\}$ becomes the topmost block of U of depth k. Observe that, the entries in the first row of $P^{(n-1)}$ which are strictly less than $b_n = 5$ are 1 and 2. Both 1 and 2 are strictly less than $a_n = 4$ as well. Also observe that, the last assertion of Lemma 5.2.3 also holds true in this example.*

5.2.3 Lemmas for Case II

Lemma 5.2.5. *Let $\{(r_1, c_1), \cdots, (r_p, c_p)\}$ be the topmost block of F of depth k. Let (b_n, a_n) enter into the monomial F to make it U in such a way that $\{(b_n, a_n), (r_1, c_1), \cdots, (r_p, c_p)\}$ becomes the topmost block of U of depth k. Let m be the positive integer as given in the statement of Theorem 5.1.1. Let $U^{(0)} := U$, then there exists an integer k' where $0 \leq k' \leq m-1$ such that:*
(i) *for each $t \in \{0, 1, \cdots, k'\}$, all the blocks of $U^{(t)}$ except one are the same as the blocks of $F^{(t)}$. The one block of $U^{(t)}$ that is different, is, in fact, a left concatenation of a block of $F^{(t)}$ by $(b_n, \star)$, where $\star$ is some entry of v which is $\geq a_n$.*
(ii) *The set of all blocks of $U^{(k'+1)}$ is equal to the set of all blocks of $F^{(k'+1)}$ union one more block, which is of the form $\{(b_n, \star)\}$, where $\star$ is some entry of v which is $\geq a_n$.*
(iii) *For each $t \in \{k'+2, \cdots, m\}$, the set of all blocks of $U^{(t)}$ is the same as the set of all blocks of $F^{(t)}$.*

Proof. Clearly, all blocks of U except one (namely, $\{(b_n, a_n), (r_1, c_1), \ldots, (r_p, c_p)\}$) are the same as all blocks of F, and the block $\{(b_n, a_n), (r_1, c_1), \ldots, (r_p, c_p)\}$ is a left concatenation of the block $\{(r_1, c_1), \ldots, (r_p, c_p)\}$ of F by (b_n, a_n).
Let $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_{n_0}$ denote all the other blocks of U or F (they are the same!). $U^{(1)}$ contains the elements $(b_n, c_1), (r_1, c_2), \cdots, (r_{p-1}, c_p)$, and all other elements of $U^{(1)}$ are given by $\mathfrak{B}'_1, \cdots, \mathfrak{B}'_{n_0}$. If $\{(b_n, c_1)\}$ is a singleton block in $U^{(1)}$, then $k' = 0$.

Suppose now, $\{(b_n, c_1)\}$ is not a singleton block in $U^{(1)}$.
Claim: There exists a block in $U^{(1)}$, which is a left concatenation of a block of $F^{(1)}$ by (b_n, c_1).
Proof of claim: Suppose not. Then (b_n, c_1) is not the first element of any block of $U^{(1)}$. Since $(b_n, c_1) \in U^{(1)}$, there exists some block (say, $\mathfrak{D}$) of $U^{(1)}$ such that $(b_n, c_1) \in \mathfrak{D}$, $\mathfrak{D} = \{(\widehat{r_1}, \widehat{c_1}), \cdots, (\widehat{r_k}, \widehat{c_k})\}$, and $(b_n, c_1) = (\widehat{r_j}, \widehat{c_j})$ for some $j \in \{2, \cdots, k\}$. Since $b_n \leq b_i$ for all $i \in \{1, \cdots, n-1\}$ (we have taken b_n in such a way), we have $\widehat{r_1} = \cdots = \widehat{r_j} = b_n$. Also, since $\{(\widehat{r_1}, \widehat{c_1}), \cdots, (\widehat{r_k}, \widehat{c_k})\}$ is a block (of $U^{(1)}$), we have $\widehat{c_1} \leq \cdots \leq \widehat{c_j} = c_1$. Since $\widehat{r_1} = \cdots = \widehat{r_j} = b_n$ and (b_n, c_1) is not the first element of any block of $U^{(1)}$, we must have $\widehat{c_1} < c_1$. Observe that the element $(\widehat{r_1}, \widehat{c_1})$ actually belongs to $F^{(1)}$ (because $U^{(1)}$ and $F^{(1)}$ differ only by one element, namely (b_n, c_1) and $\widehat{c_1} < c_1$). This implies that there had existed an element $(b_n, \widehat{p})$ in F such that $\widehat{p} \leq \widehat{c_1}$. Clearly, $\widehat{p} \geq a_n$ (because a_n is the least column number of any element in U having row number b_n). So we have,

$$a_n \leq \widehat{p} \leq \widehat{c_1} < c_1.$$

In particular, we have $a_n < c_1$.
Case (I): If $\widehat{p} = a_n$.
In this case, we have $(b_n, \widehat{p}) = (b_n, a_n) \in F$. Clearly then, $b_n = b_{n-1}$ and $a_n = a_{n-1}$.

Since $b_n \leq b_i$ for all $i \in \{1, \cdots, n-1\}$ and a_{n-1} $(= a_n)$ is the least column number of any element in F having row number b_{n-1} $(= b_n)$, we must have that $(b_n, \widehat{p}) = (b_n, a_n)$ is the first element of the topmost block of F of some depth. So, when (b_n, a_n) enters into the monomial F to make it U, it gets added to the same block of F to which $(b_n, \widehat{p}) = (b_n, a_n)$ belongs. Therefore, the block $\{(r_1, c_1), \cdots, (r_p, c_p)\}$ of F, to which (b_n, a_n) gets added, must have the property that $(r_1, c_1) = (b_n, a_n)$. This implies that $a_n = c_1$, which contradicts the fact that $a_n < c_1$.
Case (II): If $\widehat{p} > a_n$.
In this case, we have $a_n < \widehat{p} < c_1$. Since $\{(b_n, a_n), (r_1, c_1), \cdots, (r_p, c_p)\}$ is a block of U, we have $b_n \leq r_1$ and $b_n > c_1$. Now, since $(b_n, \widehat{p}) \in F$, $b_n \leq r_1$, $\widehat{p} < c_1$ and $b_n > c_1$, we must have that either $(b_n, \widehat{p})$ belongs to the same block (of depth k) of F as (r_1, c_1) or the depth of $(b_n, \widehat{p})$ in F is not equal to k.

If $(b_n, \widehat{p})$ belongs to the same block (of depth k) of F as (r_1, c_1), then since $b_n \leq r_1$ and $\widehat{p} < c_1$, the element $(b_n, \widehat{p})$ must come before the element (r_1, c_1) in that block of F. But this is a contradiction because we already know that (r_1, c_1) is the first element of some block of F.

If the depth of $(b_n, \widehat{p})$ in F is less than k, then let s $(< k)$ be the depth of $(b_n, \widehat{p})$ in F. Since (b_n, a_n) is the first element of the topmost block of U of depth k, there must exist an element $(g, h) \in F$ such that $(g, h) > (b_n, a_n)$ and the depth of (g, h) in F is s. Then $g > b_n$ and $h < a_n$. This implies that $g > b_n$ and $h < \widehat{p}$, which in turn implies that $(g, h) > (b_n, \widehat{p})$. This is a contradiction to Proposition 4.1.17.

If the depth of $(b_n, \widehat{p})$ in F is greater than k, then there must exist an element (e, f) in F of depth k such that $(e, f) > (b_n, \widehat{p})$. So, we have $e > b_n$ and $f < \widehat{p}$. Since $f < \widehat{p}$ and $\widehat{p} < c_1$, we have $(e, f) \neq (r_1, c_1)$. Also, $b_n > c_1 > \widehat{p} > f$. So, (b_n, a_n) and (e, f) belong to the same block of U of depth k and (b_n, a_n) comes before (e, f) in that block of U. Note that if $e > r_1$, then we will have $(e, f) > (r_1, c_1)$, which is impossible. Hence $e \leq r_1$. Since $e \leq r_1$ and $f < c_1$, the element (e, f) comes before the element (r_1, c_1) in the block of U of depth k containing (r_1, c_1). The above statement implies that (e, f) lies strictly between (b_n, a_n) and (r_1, c_1) in this block of U, which is not possible. So, we arrive at a contradiction.

This proves our claim.

So, the blocks of $U^{(1)}$ and $F^{(1)}$ are essentially the same except one block, which is a left concatenation of a block of $F^{(1)}$ by (b_n, c_1). Let that block of $U^{(1)}$ be $\{(b_n, c_1), (\widetilde{r_1}, \widetilde{c_2}), \cdots, (\widetilde{r_{l-1}}, \widetilde{c_l})\} = \mathfrak{C}$. Then $\mathfrak{C}' = \{(b_n, \widetilde{c_2}), (\widetilde{r_1}, \widetilde{c_3}), \cdots, (\widetilde{r_{l-2}}, \widetilde{c_l})\}$, where $c_1 \leq \widetilde{c_2}$. Now, $U^{(2)}$ contains the elements of $\mathfrak{C}'$. If $\{(b_n, \widetilde{c_2})\}$ is a singleton block in $U^{(2)}$, take $k' = 1$. Otherwise proceed similarly. This process will stop at a stage $k' \leq m - 1$, because U is a finite monomial, and at some stage, we will surely get a block consisting of a single element $\{(b_n, \star)\}$, where $\star$ is some entry of v which is greater than or equal to a_n. After the $(k'+1)$-th stage, again the blocks of $U^{(t)}$ and $F^{(t)}$ will remain same. □

Example 5.2.6 below illustrates Lemma 5.2.5.

Example 5.2.6. *Let d, N, v, F be as in Example 5.2.2 above. Let $k = 2$. Then $\{(8,2),(8,2),(9,2)\}$ is the topmost block of F of depth k. Let $(b_n, a_n) = (5,2)$. Then $(b_n, a_n) = (5,2)$ enters into the monomial F to make it U in such a way that $\{(5,2),(8,2),(8,2),(9,2)\}$ becomes the topmost block of U of depth k. Observe that $\pi(U) = (w_0, U^{(1)}), \pi(U^{(1)}) = (w_1, U^{(2)}), \pi(U^{(2)}) = (w_2, U^{(3)}), \pi(U^{(3)}) = (w_3, U^{(4)})$, and $\pi(U^{(4)}) = (w_4, \emptyset)$, where w_0, w_1, w_2, w_3, w_4 are some elements of $I(9,17)$, $U^{(1)} = \{(5,2),(8,2),(8,2),(16,13),(8,6),(8,7)\}$, $U^{(2)} = \{(5,2),(8,2),(8,6),(8,7)\}$, $U^{(3)} = \{(5,2),(8,6),(8,7)\}$, and $U^{(4)} = \{(8,7)\}$. So, $m = 4$ here.*

Observe that, there exists an integer $k' = 2 \in \{0,1,2,3(= m-1)\}$ such that the following statements hold true:
(i) all the blocks of $U^{(0)}(= U)$ except one are the same as the blocks of $F^{(0)}(= F)$. The one block of $U^{(0)}$ that is different, is, in fact, a left concatenation of the block $\{(8,2),(8,2),(9,2)\}$ of $F^{(0)}$ by $(5,2)$. Similarly, all the blocks of $U^{(1)}$ except one are the same as the blocks of $F^{(1)}$. The one block of $U^{(1)}$ that is different, is, in fact, a left concatenation of the block $\{(8,2),(8,2),(8,6),(8,7)\}$ of $F^{(1)}$ by $(5,2)$. Finally, all the blocks of $U^{(2)}$ except one are the same as the blocks of $F^{(2)}$. The one block of $U^{(2)}$ that is different, is, in fact, a left concatenation of the block $\{(8,2),(8,6),(8,7)\}$ of $F^{(2)}$ by $(5,2)$.
(ii) The set of all blocks of $U^{(3)}$ is equal to the set of all blocks of $F^{(3)}$ plus one more block, which is $\{(5,2)\}$.
(iii) The set of all blocks of $U^{(4)}$ is same as the set of all blocks of $F^{(4)}$.

Lemma 5.2.7. *With the same notation as Lemma 5.2.5, bumping in the process $(P^{(n-1)}, Q^{(n-1)}) \overset{b_n}{\leftarrow} a_n$ of bounded insertion happens upto the $(k'+1)$-th row of $P^{(n-1)}$. At the $(k'+2)$-th row of $P^{(n-1)}$, the bumping stops.*

Proof. Let k' be the integer as given in the statement of Lemma 5.2.5. The statement of part (i) of Lemma 5.2.5 implies that during the process of transition from F to U (by adding the element (b_n, a_n)), for each $t \in \{0, 1, \cdots, k'\}$, the least column numbers of all blocks of $F^{(t)}$ (except one block) remain the same as before. Also, the least column number of the block for which it is different, is in fact, less than or equal to the least column number of the corresponding block of $F^{(t)}$ as well as it is greater than or equal to a_n.

Now, from the induction hypothesis, we know that the least column numbers of all blocks of $F^{(t)}$ consist of the $(t+1)$-th row entries of $P^{(n-1)}$. Similarly, the largest row numbers of all blocks of $F^{(t)}$ consist of the $(t+1)$-th row entries of $Q^{(n-1)}$. In the language of $BRSK$, this precisely means that during the process $(P^{(n-1)}, Q^{(n-1)}) \overset{b_n}{\leftarrow} a_n$ of bounded insertion, a **bumping** happens upto the $(k'+1)$-th row of $P^{(n-1)}$.

It follows from parts (ii) and (iii) of Lemma 5.2.5 and the induction hypothesis that the **bumping stops** at the $(k'+2)$-th row of $P^{(n-1)}$ and b_n is placed in a new box at the leftmost end of the $(k'+2)$-th row of $Q^{(n-1)}$. From the $(k'+3)$-th row onwards, there is **no bumping**. All the entries of $P^{(n)}$ and $P^{(n-1)}$ from the $(k'+3)$-th row onwards are the same. Similarly, all the entries of $Q^{(n)}$ and $Q^{(n-1)}$ from the $(k'+3)$-th row onwards are the same. $\square$

Example 5.2.8 below illustrates Lemma 5.2.7.

Example 5.2.8. *Let d, N, v, F be as in Example 5.2.6 above. Let $BRSK(\iota(F)) = (P^{(n-1)}, Q^{(n-1)})$. Then we know from Example 5.2.2 that*

$$(P^{(n-1)}, Q^{(n-1)}) = \left(\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 11 & 12 & 13 \\ \hline 2 & 13 \\ \cline{1-2} 2 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|c|c|c|c|} \hline 8 & 9 & 10 & 14 & 16 & 17 \\ \hline 8 & 16 \\ \cline{1-2} 8 \\ \cline{1-1} 8 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \right).$$

Let $(b_n, a_n) = (5, 2)$ be as in Example 5.2.6. The element $(b_n, a_n) = (5, 2)$ enters the monomial F to make it U. We can see from Example 5.2.6 that $k' = 2$ here. Let $BRSK(\iota(U)) = (P^{(n)}, Q^{(n)}) = (P^{(n-1)}, Q^{(n-1)}) \xleftarrow{b_n} a_n$. Then one can easily see that

$$(P^{(n)}, Q^{(n)}) = \left(\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 11 & 12 & 13 \\ \hline 2 & 13 \\ \cline{1-2} 2 \\ \cline{1-2} 2 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|c|c|c|c|} \hline 8 & 9 & 10 & 14 & 16 & 17 \\ \hline 8 & 16 \\ \cline{1-2} 8 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} \right).$$

Observe that, in the process $(P^{(n-1)}, Q^{(n-1)}) \xleftarrow{b_n} a_n$ of obtaining $(P^{(n)}, Q^{(n)})$ from $(P^{(n-1)}, Q^{(n-1)})$ and (b_n, a_n), there is bumping upto the third row of $P^{(n-1)}$, and at the fourth row of $P^{(n-1)}$, the bumping stops. Here $k' = 2$. Therefore, $3 = k'+1$ and $4 = k'+2$.

5.2.4 An illustration of Lemmas 5.2.5 and 5.2.7 for the case $k' = 0$

With the same notation as in Lemma 5.2.5, suppose $k' = 0$. Then we must have the following statements hold true:

1. a_n is an entry in the first row of $P^{(n)}$ and r_p is an entry in the first row of $Q^{(n)}$. a_n bumps c_1 from the first row of $P^{(n-1)}$, and c_1 is placed in the second row of $P^{(n-1)}$.

2. c_1 is the least column number of some block of $U^{(1)}$.

3. In the process $(P^{(n-1)}, Q^{(n-1)}) \overset{b_n}{\leftarrow} a_n$, if c_1 does not bump anything from the second row of $P^{(n-1)}$, then the singleton set $\{(b_n, c_1)\}$ is a block in $U^{(1)}$.

4. If the singleton set $\{(b_n, c_1)\}$ is a block in $U^{(1)}$, then the largest row numbers of all blocks of $F^{(1)}$ are strictly bigger than b_n.

5. If the singleton set $\{(b_n, c_1)\}$ is a block in $U^{(1)}$, then the least column numbers of all blocks of $F^{(1)}$ which are strictly less than b_n, are in fact, strictly less than c_1.

Now, we will provide a proof of all these 5 statements now.

Proof of (1): By induction hypothesis, the first row of $P^{(n-1)}$ contains the smallest column number of each block of F. In particular, it contains the entry c_1, and the first row of $Q^{(n-1)}$ contains the entry r_p. In the process $P^{(n-1)} \overset{b_n}{\leftarrow} a_n$ of bounded insertion, firstly all entries of $P^{(n-1)}$ which are greater than or equal to b_n are removed. Since $c_1 < b_n$, c_1 is not removed.

Claim: a_n bumps c_1 from the first row of $P^{(n-1)}$.

Proof of claim: Clearly, $a_n \leq c_1$. It suffices to show that all the entries in the first row of $P^{(n-1)}$ which are less than c_1 are also strictly less than a_n.

Suppose not. Say, p_{1j_0} is an entry in the first row of $P^{(n-1)}$ such that $a_n \leq p_{1j_0} < c_1$. By Lemma 5.2.1, it follows that p_{1j_0} is the column number of the first element of some block $\mathfrak{C}$ of F of depth $s(< k)$. Say, the first element of the block $\mathfrak{C}$ is (R, p_{1j_0}). Now, since (b_n, a_n) is an element of depth k in U and $s < k$, there exists an element (e_s, f_s) in U of depth s such that $(e_s, f_s) > (b_n, a_n)$. Clearly then, $e_s > b_n$ and $f_s < a_n \leq p_{1j_0}$. Also, $e_s \leq R$, because if $e_s > R$, then $(e_s, f_s) > (R, p_{1j_0})$, a contradiction to the depth of (R, p_{1j_0}) in U.

Hence, we have $b_n < e_s \leq R$ and $f_s < p_{1j_0}$. Now, (e_s, f_s) and (R, p_{1j_0}) are two elements in U of depth s. Also, $e_s > b_n > c_1 > p_{1j_0}$. Which implies $e_s > p_{1j_0}$, that is, (e_s, f_s) belongs to the same block $\mathfrak{C}$ as (R, p_{1j_0}). Since $e_s \leq R$ and $f_s < p_{1j_0}$, we get a contradiction to the fact that (R, p_{1j_0}) is the first element of the block $\mathfrak{C}$.

This proves the above claim.

Now, since a_n bumps c_1 from the first row of $P^{(n-1)}$, it follows that a_n is placed at the position of c_1 in the first row of $P^{(n)}$. Also, c_1 gets inserted in the 2nd row of $P^{(n-1)}$. $\square$

Proof of (2): Suppose not. Say, all the blocks of $U^{(1)}$ are having their least column numbers different from c_1. Then the element (b_n, c_1) of $U^{(1)}$ lies in a block of $U^{(1)}$, which has another element in it (to the left of (b_n, c_1)) of the form (b_n, p) where $p < c_1$. This element (b_n, p) must have come from $F^{(1)}$. That means, there had existed an element of the form (b_n, β) in F such that $\beta \leq p < c_1$, and the block of F in which (b_n, β) lies, is different from the block $\{(r_1, c_1), \cdots, (r_p, c_p)\}$ of F. Say, (b_n, β) lies in the block $\mathfrak{B}$ of F.

Now, (b_n, a_n) enters the monomial F to make it U in such a way that $\{(b_n, a_n), (r_1, c_1), \cdots, (r_p, c_p)\}$ is the topmost block of U of depth k. It follows (from the fact that (b_n, β) lies in F) that $b_{n-1} = b_n$ and hence by the definition of lexicographic order, we have $a_n \leq \beta$. It then follows from the definition of blocks that $\mathfrak{B}$ and $\{(b_n, a_n), (r_1, c_1), \cdots, (r_p, c_p)\}$ are two blocks in U of different depths. The depth of the block $\{(b_n, a_n), (r_1, c_1), \cdots, (r_p, c_p)\}$ in U is k. Let s denote the depth of the block $\mathfrak{B}$ in U (or in F). So, we have $s \neq k$.

If $s < k$, then there exists an element $(\hat{R}, \hat{C})$ in U of depth s such that $(\hat{R}, \hat{C})$ forms a v-chain with (b_n, a_n). But then one can easily verify that $(\hat{R}, \hat{C})$ and (b_n, β) are comparable. This is a contradiction to Proposition 4.1.17.

If $s > k$, then there exists an element $(\hat{\hat{R}}, \hat{\hat{C}})$ in U of depth k such that it forms a v-chain with (b_n, β). Since no two elements of the same depth in U are comparable (by Proposition 4.1.17), so $(\hat{\hat{R}}, \hat{\hat{C}})$ does not form a v-chain with (b_n, a_n). In fact, we have $\hat{\hat{R}} > b_n$ and $a_n \leq \hat{\hat{C}} < \beta < c_1$. Now, $(b_n, \beta) \in \widetilde{\mathfrak{N}}^v$ implies that $b_n > \beta$, and this implies that $b_n > \beta > \hat{\hat{C}}$. This in turn implies that (b_n, a_n) and $(\hat{\hat{R}}, \hat{\hat{C}})$ are two elements in U in the same block of depth k, and $(\hat{\hat{R}}, \hat{\hat{C}})$ comes to the right of (b_n, a_n) in that block. But $\hat{\hat{C}} < c_1$ and $\hat{\hat{R}} > b_n$. This is a contradiction since the element $(\hat{\hat{R}}, \hat{\hat{C}})$ is distinct from (r_1, c_1) and (b_n, a_n), and it comes right in between the two elements (b_n, a_n) and (r_1, c_1) of the block $\{(b_n, a_n), (r_1, c_1), \ldots, (r_p, c_p)\}$ in U of depth k. □

Proof of (3): We know from part (2) of this illustration that c_1 is the least column number of some block of $U^{(1)}$. Let $\mathfrak{B}$ be that block.

Claim: b_n is the least row number of the block $\mathfrak{B}$.

Proof of Claim: Suppose not. Say, R_1 is the least row number of the block $\mathfrak{B}$ and $b_n < R_1$. Then (R_1, c_1) is an element of the block $\mathfrak{B}$ of $U^{(1)}$. Since (b_n, c_1) also belongs to $U^{(1)}$, it follows that (b_n, c_1) and the elements of the block $\mathfrak{B}$ of $U^{(1)}$ have different depths. Say, the depth of (b_n, c_1) in $U^{(1)}$ is $\hat{k}$ and that of (R_1, c_1) is $\hat{s}$.

If $\hat{s} < \hat{k}$, then there exists an element (a, b) in $U^{(1)}$ of depth $\hat{s}$ such that (a, b) forms a v-chain with (b_n, c_1). Since no two elements of the same depth are comparable (by Proposition 4.1.17), we must have $a \leq R_1$. Since $a > b_n$ and $b_n > c_1$, we have $a > c_1$. It then follows that (a, b) and (R_1, c_1) are in the same block $\mathfrak{B}$. But since $a \leq R_1$ and $b < c_1$, we have that (a, b) and (R_1, c_1) are two distinct elements of the same block $\mathfrak{B}$. Also, $b < c_1$ implies that the least column number of the block $\mathfrak{B}$ is less than or equal to b (which is less than c_1), a contradiction.

If $\hat{s} > \hat{k}$, then there exists an element (e, f) in $U^{(1)}$ of depth $\hat{k}$ such that (e, f) forms a v-chain with (R_1, c_1). Clearly then, (e, f) also forms a v-chain with (b_n, c_1). This is a contradiction to the fact that no two elements of the same depth are comparable (Proposition 4.1.17).

This proves the claim.

It follows from the above claim that (b_n, c_1) is the first element of some block of $U^{(1)}$. Hence, c_1 is the least column number of that block of $U^{(1)}$. So, c_1 must be placed in the second row of $P^{(n-1)}$ during the process $(P^{(n-1)}, Q^{(n-1)}) \xleftarrow{b_n} a_n$ of bounded insertion.

We know from our hypothesis that c_1 does not bump anything from the second row of $P^{(n-1)}$. It hence follows that b_n must be placed in the second row of $Q^{(n-1)}$ at the extreme left. This implies that the set of all largest row numbers of all blocks of $U^{(1)}$ must contain the element b_n. That is, b_n must be the largest row number of some block of $U^{(1)}$. But there is no smaller row number than b_n in the entire monomial U. This implies that b_n is the only row number of some block of $U^{(1)}$ (say, block $\mathfrak{D}$).

Claim: $\mathfrak{D}$ must be the block containing the single element (b_n, c_1).

Proof of claim: Suppose not. Then there exists at least one element of the form (b_n, p) in $\mathfrak{D}$, where $p \neq c_1$, and there are no elements in $\mathfrak{D}$ which lie between (b_n, p) and (b_n, c_1).

If $p < c_1$, then we arrive at a contradiction to the fact that (b_n, c_1) is the first element of some block of $U^{(1)}$. If $p > c_1$, then the element (b_n, p) must have come from $F^{(1)}$. Say, (b_n, p) lies in the block $\mathfrak{C}$ of $F^{(1)}$. Say, (b_n, α) is the first element of the block $\mathfrak{C}$ of $F^{(1)}$. Clearly then, α lies in the second row of $P^{(n-1)}$. If $\alpha \geq c_1$, then there is a bumping by c_1 in the process $(P^{(n-1)}, Q^{(n-1)}) \overset{b_n}{\leftarrow} a_n$ of bounded insertion, a contradiction. Hence, we must have $\alpha < c_1$.

Now, since (b_n, c_1) is the first element of some block of $U^{(1)}$, it follows that (b_n, α) and (b_n, c_1) lie in two blocks of different depths in $U^{(1)}$. Let $\tilde{k}$ and $\tilde{s}$ denote the depths of (b_n, c_1) and (b_n, α) in $U^{(1)}$ respectively.

If $\tilde{k} < \tilde{s}$, then there exists an element $(\tilde{x}, \tilde{y})$ of depth $\tilde{k}$ in $U^{(1)}$ such that $(\tilde{x}, \tilde{y})$ forms a v-chain with (b_n, α). Clearly then, $(\tilde{x}, \tilde{y})$ also forms a v-chain with (b_n, c_1). This contradicts Proposition 4.1.17.

If $\tilde{s} < \tilde{k}$, then there exists an element (x, y) of depth $\tilde{s}$ in $U^{(1)}$ such that (x, y) forms a v-chain with (b_n, c_1). Since no two elements of the same depth are comparable (by Proposition 4.1.17), we must have $y \geq \alpha$. Observe that $b_n > y$, because $b_n > c_1$ and $c_1 > y$. Since $b_n > y$, it follows that (b_n, α) and (x, y) are in the same block (of depth $\tilde{s}$) in $U^{(1)}$. Now, (b_n, α) lies in $F^{(1)}$ and (x, y) also lies in $F^{(1)}$ (this is because $y < c_1$ and (b_n, c_1) is the only extra element in $U^{(1)}$, which is not in $F^{(1)}$). This implies that $(b_n, \alpha), (x, y)$ and (b_n, p) lie in the same block $\mathfrak{C}$ of $F^{(1)}$. This is not possible since $x > b_n$ and $\alpha \leq y < c_1 < p$.

This proves our claim and this claim completes the proof. □

Proof of (4): Suppose not. Say, there exists a block $\mathfrak{B}$ in $F^{(1)}$ whose largest row number is less than or equal to b_n (hence $= b_n$, since b_n is the smallest row number in U). Then all the elements of that block $\mathfrak{B}$ of $F^{(1)}$ are having row number b_n. Say, $\mathfrak{B} = \{(b_n, p_1), (b_n, p_2), \ldots, (b_n, p_l)\}$.

Now, since the singleton set $\{(b_n, c_1)\}$ is a block in $U^{(1)}$, we must have that $\mathfrak{B}$ and $\{(b_n, c_1)\}$ are of different depths in $U^{(1)}$. Let $\hat{k}$ and $\hat{s}$ denote the depths of (b_n, c_1) and $\mathfrak{B}$ in $U^{(1)}$ respectively.

If $\hat{k} > \hat{s}$, then there exists an element $(\hat{R}, \hat{C})$ in $U^{(1)}$ of depth $\hat{s}$ such that $(\hat{R}, \hat{C})$ forms a v-chain with (b_n, c_1). That is, $\hat{R} > b_n$ and $\hat{C} < c_1$. But then since $b_n > c_1$, we get that

$b_n > \hat{C}$. This implies that $(\hat{R}, \hat{C})$ belongs to $\mathfrak{B}$. This is a contradiction because $\hat{R} > b_n$ and all elements in $\mathfrak{B}$ have row number b_n.

If $\hat{k} < \hat{s}$, then there exists an element $(\tilde{R}, \tilde{(C)})$ in $U^{(1)}$ of depth $\hat{k}$ such that $(\tilde{R}, \tilde{C})$ forms a v-chain with (b_n, p_l). That is, $\tilde{R} > b_n$ and $\tilde{C} < p_l$. If $c_1 > \tilde{C}$, then $(\tilde{R}, \tilde{C})$ forms a v-chain with (b_n, c_1), which is a contradiction to Proposition 4.1.17. If $c_1 \leq \tilde{C}$, then $b_n > \tilde{C}$ because $b_n > p_l$ and $p_l > \tilde{C}$. Hence, (b_n, c_1) and $(\tilde{R}, \tilde{C})$ are in the same block of $U^{(1)}$. This contradicts the fact that the singleton set $\{(b_n, c_1)\}$ is a block in $U^{(1)}$. □

Proof of (5): Suppose not. Say, there exists a block $\mathfrak{B}$ of $F^{(1)}$ whose least column number p is $< b_n$, and p is also $\geq c_1$. Let $\hat{k}$ be the depth of (b_n, c_1) in $U^{(1)}$. Let $\hat{s}$ denote the depth of the block $\mathfrak{B}$ in $U^{(1)}$. Let (R_1, p) be the first element of the block $\mathfrak{B}$. If $\hat{k} = \hat{s}$, then a contradiction to the fact that the singleton set $\{(b_n, c_1)\}$ is a block in $U^{(1)}$, because $b_n > p$.

If $\hat{k} < \hat{s}$, then there exists an element (R_0, C_0) in $U^{(1)}$ of depth $\hat{k}$ such that (R_0, C_0) forms a v-chain with (R_1, p). Clearly then, $R_1 \geq b_n, p \geq c_1, C_0 < p$ and $R_0 > R_1$. Since $R_0 > R_1 \geq b_n$, therefore we must have $C_0 \geq c_1$, for otherwise, no distinct elements in $U^{(1)}$ of the same depth will become comparable. On the other hand, $b_n > C_0$ (because $b_n > p$ and $p > C_0$). Hence, (b_n, c_1) and (R_0, C_0) are in the same block of $U^{(1)}$ of depth $\hat{k}$. Since $b_n < R_0$, therefore (b_n, c_1) and (R_0, C_0) are two distinct elements of the same block of $U^{(1)}$ of depth $\hat{k}$. This contradicts our assumption that the singleton set $\{(b_n, c_1)\}$ is a block in $U^{(1)}$.

If $\hat{k} > \hat{s}$, then there exists an element (R_{00}, C_{00}) in $U^{(1)}$ of depth $\hat{s}$ such that (R_{00}, C_{00}) forms a v-chain with (b_n, c_1). If $R_{00} > R_1$, then it is easy to see that (R_{00}, C_{00}) forms a v-chain with (R_1, p), which contradicts the fact that no two distinct elements in $U^{(1)}$ of the same depth are comparable. If $R_{00} \leq R_1$, then (since $R_{00} > b_n$ and $b_n > p$) we have $R_{00} > p$. This (together with the facts that $R_{00} \leq R_1$ and $C_{00} < p$) implies that (R_{00}, C_{00}) and (R_1, p) are in the same block $\mathfrak{B}$ of $U^{(1)}$ of depth $\hat{s}$. But $C_{00} < p$ implies that the element (R_{00}, C_{00}) is distinct from the element (R_1, p), and (R_{00}, C_{00}) comes to the left of (R_1, p) in that block $\mathfrak{B}$. So, (R_1, p) cannot be the first element of the block $\mathfrak{B}$, a contradiction. □

Example 5.2.9 illustrates the statements (1) – (5) mentioned in this subsection.

Example 5.2.9. *Let $d = 7, N = 13,$ and $v = (1, 2, 4, 5, 7, 8, 9)$. Let*

$$F = \{(10, 4), (10, 9), (11, 1), (11, 2), (11, 7), (11, 8), (12, 2), (13, 8)\}.$$

Then

$$BRSK(\iota(F)) = (P^{(n-1)}, Q^{(n-1)}) = \left(\begin{array}{lll} 1 & 4 & 9 \\ 2 & 7 & \\ 2 & & \\ 8 & & \\ 8 & & \end{array} , \begin{array}{lll} 10 & 11 & 13 \\ 10 & 12 & \\ 11 & & \\ 11 & & \\ 11 & & \end{array} \right). \tag{5.2.4.1}$$

Figure 5.4 gives the block decomposition of F*. Let* $k = 2$ *(here* k *is as in Lemma 5.2.5). Observe that* $\{(10,4),(11,7),(11,8)\}$ *is the topmost block of* F *of depth* $k = 2$*. Then (according to the notation used in Lemma 5.2.5) we have* $(r_1, c_1) = (10,4), (r_2, c_2) = (11,7)$ *and* $(r_p, c_p) = (r_3, c_3) = (11,8)$.

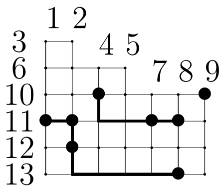


Figure 5.4: The block decomposition of the monomial F

Here $F^{(1)} = \{(11,2),(11,2),(12,8),(10,7),(11,8)\}$*. Figure 5.5 gives the block decomposition of* $F^{(1)}$.

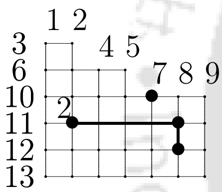


Figure 5.5: The block decomposition of the monomial $F^{(1)}$

Now, let $(b_n, a_n) = (6,2)$*. Then*

$$U = \{(6,2),(10,4),(10,9),(11,1),(11,2),(11,7),(11,8),(12,2),(13,8)\},$$

and $\{(6,2),(10,4),(11,7),(11,8)\}$ *becomes the topmost block of* U *of depth* 2*. Now,*

$$BRSK(\iota(U)) = (P^{(n)}, Q^{(n)}) = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 9 \\ \hline 2 & 4 & 7 \\ \hline 2 \\ \cline{1-1} 8 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|} \hline 10 & 11 & 13 \\ \hline 6 & 10 & 12 \\ \hline 11 \\ \cline{1-1} 11 \\ \cline{1-1} 11 \\ \cline{1-1} \end{array} \right). \tag{5.2.4.2}$$

Figure 5.6 gives the block decomposition of U*.*

So, $U^{(1)} = \{(11,2),(11,2),(12,8),(6,4),(10,7),(11,8)\}$*. Figure 5.7 gives the block decomposition of* $U^{(1)}$*.*

Observe that

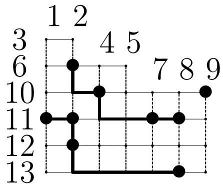


Figure 5.6: The block decomposition of the monomial U

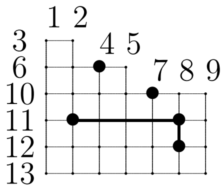


Figure 5.7: The block decomposition of the monomial $U^{(1)}$

- *$a_n = 2$ is an entry in the first row of $P^{(n)}$ and $r_p = r_3 = 11$ is an entry in the first row of $Q^{(n)}$. $a_n = 2$ bumps $c_1 = 4$ from the first row of $P^{(n-1)}$, and $c_1 = 4$ is placed in the second row of $P^{(n-1)}$.*

- *$c_1 = 4$ is the least column number of some block of $U^{(1)}$.*

- *Since $c_1 = 4$ does not bump anything from the second row of $P^{(n-1)}$ during the process $(P^{(n-1)}, Q^{(n-1)}) \overset{b_n}{\leftarrow} a_n$ of bounded insertion, it follows that the singleton set $\{(b_n, c_1)\} = \{(6, 4)\}$ is a block in $U^{(1)}$.*

- *Since the singleton set $\{(b_n, c_1)\} = \{(6, 4)\}$ is a block in $U^{(1)}$, it follows that the largest row numbers of all blocks of $F^{(1)}$ are strictly bigger than $b_n = 6$.*

- *Since the singleton set $\{(b_n, c_1)\} = \{(6, 4)\}$ is a block in $U^{(1)}$, it follows that the least column numbers of all blocks of $F^{(1)}$ which are strictly less than b_n $(= 6)$, are in fact, strictly less than $c_1 = 4$.*

5.2.5 For any monomial U in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$, $\tilde{\pi}(U) = BRSK(U)$

Corollary 5.2.19 below states that for any monomial U in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$, $\tilde{\pi}(U) = BRSK(U)$. This is a corollary to Theorem 5.2.17. But before we state Theorem 5.2.17 and Corollary 5.2.19, we need to define the map $\tilde{\pi}$ on monomials in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. But for doing so, we need some preparation.

Definition 5.2.10. *Let $v = (v_1, \cdots, v_d)$. Given $\beta_1 = (r_1, c_1)$ and $\beta_2 = (r_2, c_2)$ in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$, we say that $\beta_1 > \beta_2$ if $r_1 < r_2$ and $c_2 < c_1$. A sequence $\beta_1 > \cdots > \beta_t$ of elements of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ is called an **anti-v-chain**. Given an anti-v-chain $\beta_1 = (r_1, c_1) > \cdots > \beta_t = (r_t, c_t)$, we define*

$$s_{\beta_1} \cdots s_{\beta_t} v := (\{v_1, \cdots, v_d\} \setminus \{c_1, \cdots, c_t\}) \cup \{r_1, \cdots, r_t\}.$$

*We say that an element w of $I(d,N)$ **anti-dominates** the anti-v-chain $\beta_1 > \cdots > \beta_t$ if $w \leq s_{\beta_1} \cdots s_{\beta_t} v$. Let $\mathfrak{S}$ be a monomial in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. We say that w **anti-dominates** $\mathfrak{S}$ if w anti-dominates every anti-v-chain in $\mathfrak{S}$.*

Definition 5.2.11. *We call **distinguished** the subsets $\mathfrak{S}$ of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ satisfying the following conditions:*
(A) for $(r,c) \neq (r',c')$ in $\mathfrak{S}$, we have $r \neq r'$ and $c \neq c'$.
(B) If $\mathfrak{S} = \{(r_1,c_1), \cdots, (r_p,c_p)\}$ with $r_1 > r_2 > \cdots > r_p$, then for j, $1 \leq j \leq p-1$, we have either $c_j < c_{j+1}$ or $r_j > c_{j+1}$.
Condition (B) can be restated as follows:
*(B *) for $(r,c),(R,C)$ in $\mathfrak{S}$ with $r > R$, either $c < C$ or $r > C$.*

Remark 5.2.12. *It can be proved similarly as in Proposition 4.1.13 that there exists a bijection between elements w of $I(d,N)$ satisfying $w \leq v$ on the one hand and distinguished subsets of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ on the other hand. We denote this bijective correspondence by $w \leftrightarrow \mathfrak{S}_w$.*

Proof. Given $w \leq v$, consider the sets $\{v_1, \cdots, v_d\} \setminus \{w_1, \cdots, w_d\}$ and $\{w_1, \cdots, w_d\} \setminus \{v_1, \cdots, v_d\}$. Both these have the same cardinality $t = v$-degree(w) (which is stated in Definition 4.1.3). the first set provides us with the column indices of elements of $\mathfrak{S}_w$, the second with the row indices. If we arrange the row indices in an decreasing order, say $r_1 > \cdots > r_t$, then there is a unique way to arrange the column indices such that condition (B*) above is satisfied: proceed by induction, and if $c_1, \cdots, c_j$ have been chosen, choose c_{j+1} to be the minimum among the remaining column indices that are greater than r_{j+1}. This defines the map $w \mapsto \mathfrak{S}_w$. It is clear that cardinality of $\mathfrak{S}_w$ equals v-degree(w).

For the converse part, given $\mathfrak{S}$, to obtain w, start with $v = (v_1, \cdots, v_d)$, delete those entries that occur as the column indices in $\mathfrak{S}$, add those that occur as row indices in $\mathfrak{S}$, and finally arrange the entries in increasing order. It is readily seen that the two maps are inverses to each other. □

Example 5.2.13. Let $d = 7$, so $2d = 14$. Let $v = (4.6, 7, 10, 12, 13, 14)$ and $w = (1, 2, 3, 4, 5, 8, 9)$. Clearly, v and w belongs to $I(d, 2d)$. Now, $w \setminus v = \{1, 2, 3, 5, 8, 9\}$ and $v \setminus w = \{6, 7, 10, 12, 13, 14\}$. So,

$$\mathfrak{S}_w = \{(9,10), (8,12), (5,6), (3,7), (2,13), (1,14)\}.$$

Again, for the above v and $\mathfrak{S}_w$ the corresponding element w of $I(d, 2d)$ will be

$$w = (\{4,6,7,10,12,13,14\} \setminus \{6,7,10,12,13,14\}) \cup \{1,2,3,5,8,9\} = (1,2,3,4,5,8,9).$$

Definition 5.2.14. *Let $\mathfrak{S}$ be a non-empty monomial in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. If $\beta_1 > \cdots > \beta_t$ is an anti-v-chain in $\mathfrak{S}$, then we call β_1 the **head** of the anti-v-chain and β_t its **tail**. We* *call t to be the **length** of the anti-v-chain. We say that an element β of $\mathfrak{S}$ is t-**deep** in $\mathfrak{S}$ (where t is a positive integer) if β is the tail of an anti-v-chain in $\mathfrak{S}$ of length t. The **depth** of β in $\mathfrak{S}$ is defined to be t if β is t-deep in $\mathfrak{S}$ but not $(t+1)$-deep in $\mathfrak{S}$.*

We will now define the map π on any monomial in $\mathfrak{R}(v)\setminus\mathfrak{N}(v)$. Let $\mathfrak{S}$ be a non-empty monomial in the elements of $\mathfrak{R}(v)\setminus\mathfrak{N}(v)$. We partition $\mathfrak{S}$ in two stages. First we partition $\mathfrak{S}$ into subsets $\mathfrak{S}_1, \cdots, \mathfrak{S}_k$, where k is the largest length of an anti-v-chain in $\mathfrak{S}$: $\beta \in \mathfrak{S}$ belongs to $\mathfrak{S}_j$ if it is j-deep but not $(j+1)$-deep.

Now, we partition each $\mathfrak{S}_j$ into subsets called **blocks** as follows: We arrange the elements of $\mathfrak{S}_j$ in non-increasing order of their row numbers (where elements occur with their respective multiplicities). Among those with the same row number, the arrangement is by non-increasing order of column numbers. Two consecutive members $(r, c), (R, C)$ in this arrangement are said to be **related** if $r < C$.

Definition 5.2.15. *The **blocks** are the equivalence classes of the smallest equivalence relation containing the above relations.*

Let $\mathfrak{B}$ be a single block of some $\mathfrak{S}_j$. Let

$$(r_1, c_1), \cdots, (r_p, c_p)$$

be the elements of $\mathfrak{B}$ written in non-increasing order of both row and column numbers (in such an arrangement, the elements occur with their respective multiplicities). We set $w(\mathfrak{B}) := (r_p, c_1)$ and $\mathfrak{B}'$ to be the monomial

$$\{(r_1, c_2), (r_2, c_3), \cdots, (r_{p-2}, c_{p-1}), (r_{p-1}, c_p)\}.$$

Set $\mathfrak{S}_j^{(1)} := \cup_{\mathfrak{B}} \mathfrak{B}'$ (where the index $\mathfrak{B}$ runs over all blocks of $\mathfrak{S}_j$) and $\mathfrak{S}^{(1)} := \cup_{j=1}^{k} \mathfrak{S}_j^{(1)}$. It follows (similarly as in [KR03, Corollary 4.13]) that the set

$$\{w(\mathfrak{B}) \mid \mathfrak{B} \text{ is a block of } \mathfrak{S}\}$$

is a distinguished subset of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. Let w be the corresponding element of $I(d, N)$ (under the correspondence given in Remark 5.2.12). Set

$$\pi(\mathfrak{S}) := (w, \mathfrak{S}^{(1)}).$$

This finishes the description of the map π.

Definition 5.2.16. *A standard monomial $\theta_1 \geq \cdots \geq \theta_t$ in $I(d, N)$ is called **dominated** by v if $v \geq \theta_1$. Let $\widetilde{SM}_v^v$ denote the set of all v-**compatible standard monomials in** $I(d, N)$ **dominated by** v.*

Using π, we now define the map $\tilde{\pi}$ from the set of all monomials in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ to $\widetilde{SM}^v_v$. We proceed by induction on the degree of a monomial $\mathfrak{S}$ in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. The image of the empty monomial under $\tilde{\pi}$ is taken to be the empty monomial. Let $\mathfrak{S}$ be non-empty and suppose that $\pi(\mathfrak{S}) = (w, \mathfrak{S}^{(1)})$. It can be shown (similarly as in (1) and (2) of Proposition 4.2.1) that the degree of $\mathfrak{S}^{(1)}$ is strictly less than that of $\mathfrak{S}$, and so by induction $\tilde{\pi}(\mathfrak{S}^{(1)})$ is defined. Suppose that $\tilde{\pi}(\mathfrak{S}^{(1)}) = w' \leq \cdots$. It can be shown (similarly as in (3) of Proposition 4.2.1) that $w \leq w'$. We set $\tilde{\pi}(\mathfrak{S}) := w \leq \tilde{\pi}(\mathfrak{S}^{(1)})$. This finishes the description of the map $\tilde{\pi}$ on the set of all monomials in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$.

The proof of Theorem 5.2.17 below is similar to the proof of Theorem 5.1.1. Hence we omit the proof.

Theorem 5.2.17. *Let U be a finite monomial in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. Let $\pi(U) = (w_0, U^{(1)})$, $\pi(U^{(1)}) = (w_1, U^{(2)})$, ... and so on till $\pi(U^{(k)}) = (w_k, \emptyset)$, where $\emptyset$ is the empty monomial. Then for each $r \in \{0, 1, \ldots, k\}$, the following holds :*
(i) all the row numbers of the distinguished subset $\mathfrak{S}_{w_r}$ (corresponding to w_r) consist of the $(r+1)$-th row entries of the left hand notched tableau of $BRSK(U)$.
(ii) All the column numbers of $\mathfrak{S}_{w_r}$ comprise of the $(r+1)$-th row entries of the right hand notched tableau of $BRSK(U)$.

Example 5.2.18 below illustrates Theorem 5.2.17.

Example 5.2.18. *Let $d = 5, N = 11,$ and $v = (3, 6, 8, 10, 11)$. Let*

$$U = \{(9, 11), (4, 11), (7, 10), (5, 10), (7, 8), (1, 8), (4, 6)\}$$

be a finite monomial in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. Figure 5.8 shows the monomial U and its block

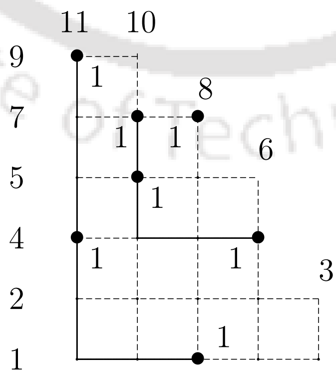


Figure 5.8: The monomial U in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ and its block decomposition

decomposition. The dark circles in the figure represent points in the monomial U with their respective multiplicities (the multiplicity of each point in the monomial U is 1 here, *which is written near those points in the grid). The dark line segments (together with the point $(7,8)$) denote the blocks of U.*

For this monomial U, we have $\pi(U) = (w_0, U^{(1)})$, where $w_0 = (1,3,4,6,7)$ and $U^{(1)} = \{(9,11),(4,8),(7,10),(5,6)\}$. Then $\pi(U^{(1)}) = (w_1, U^{(2)})$, where $w_1 = (3,4,5,8,10)$ and $U^{(2)} = \{(9,10),(7,8)\}$. Finally, $\pi(U^{(2)}) = (w_2, \emptyset)$, where $w_2 = (3,6,7,9,11)$.

So we have $k = 2$, $\mathfrak{S}_{w_0} = \{(1,11),(4,10),(7,8)\}$, $\mathfrak{S}_{w_1} = \{(4,11),(5,6)\}$ and $\mathfrak{S}_{w_2} = \{(9,10),(7,8)\}$. It is also easy to check that

$$BRSK(U) = \left(\begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 4 & 5 & \\ \cline{1-2} 7 & 9 & \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|} \hline 8 & 10 & 11 \\ \hline 6 & 11 & \\ \cline{1-2} 8 & 10 & \\ \cline{1-2} \end{array} \right).$$

Note that, for $r = 0$, all the row numbers of $\mathfrak{S}_{w_0}$ are given by $\{1,4,7\}$, which consist of the first row entries of the left hand notched tableau of $BRSK(U)$. Also all the column numbers of $\mathfrak{S}_{w_0}$ are given by $\{8,10,11\}$, which consist of the first row entries of the right hand notched tableau of $BRSK(U)$. Similarly for $r = 1$ and $r = 2$.

Corollary 5.2.19. *For any monomial U in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$, $\tilde{\pi}(U) = BRSK(U)$.*

Corollary 5.2.20. *For any monomial U in $\mathfrak{R}(v)$, $\tilde{\pi}(U) = BRSK(U)$.*

Proof. Follows from Corollary 5.1.1 and Corollary 5.2.17. □

5.3 An application of the main theorem

In this subsection, we will provide an application of Theorem 5.1.1.

5.3.1 Some necessary definitions and notation

The following definitions and notation are written in the same way as given in [GR06] and [KR03].

Definition 5.3.1. *An ordered pair $\mathfrak{w} = (x,y)$ of elements of $I(d)$ is called an* ***admissible pair*** *if $x \geq y$ and the ϵ-degrees of x and y are equal. Sometimes x and y are referred as the top and bottom of $\mathfrak{w}$ and written as top($\mathfrak{w}$) for x and bot($\mathfrak{w}$) for y. Given an admissible pair $\mathfrak{w} = (x,y)$, we define the v-**degree** of $\mathfrak{w}$ by v-degree$(\mathfrak{w}) := \frac{1}{2}(|x \setminus v| + |y \setminus v|)$.*

Definition 5.3.2. *Given any two admissible pairs $\mathfrak{w} = (x,y)$ and $\mathfrak{w}' = (x',y')$, we say that $\mathfrak{w} \geq \mathfrak{w}'$ if $y \geq x'$, that is , if $x \geq y \geq x' \geq y'$. An ordered sequence $(\mathfrak{w}_1 \ldots, \mathfrak{w}_t)$ of admissible pairs is called a* ***standard sequence*** *if $\mathfrak{w}_i \geq \mathfrak{w}_{i+1}$ for $1 \leq i < t$. Sometimes $\mathfrak{w}_1 \geq \ldots \geq \mathfrak{w}_t$ is written to denote a standard sequence $(\mathfrak{w}_1 \ldots, \mathfrak{w}_t)$. Given any $w \in I(d)$, we say that a standard sequence $\mathfrak{w}_1 \geq \ldots \geq \mathfrak{w}_t$ is w-**dominated** if $w \geq$ top$(\mathfrak{w}_1)$. Given* *any $v \in I(d)$, we say that the standard sequence $\mathfrak{w}_1 \geq \ldots \geq \mathfrak{w}_t$ is v-**compatible** if for each $\mathfrak{w}_i$, either $v \geq top(\mathfrak{w}_i)$ or $bot(\mathfrak{w}_i) \geq v$, and $\mathfrak{w}_i \neq (v, v)$. Given v and w in $I(d)$, we denote by SM_w^v the set of all w-**dominated** v-**compatible standard sequences**. For any positive integer m, let $SM_w^v(m)$ denote the set of all w-**dominated** v-**compatible standard sequences of degree** m, where the **degree of a standard sequence** $\mathfrak{w}_1 \geq \ldots \geq \mathfrak{w}_t$ is defined to be the sum of the v-degrees of $\mathfrak{w}_1, \ldots, \mathfrak{w}_t$. Let $SM^{v,v}$ denote the set of all v-**compatible standard sequences that are anti-dominated by** v: a standard sequence $\mathfrak{w}_1 \geq \ldots \geq \mathfrak{w}_t$ is called **anti-dominated** by v if $bot(\mathfrak{w}_t) \geq v$.*

Example 5.3.3. *Let $d = 5$. So, $2d = 10$ and $\epsilon = (1, 2, 3, 4, 5)$. Let $x_1 = (3, 4, 6, 9, 10)$, and $y_1 = (2, 4, 6, 8, 10)$. Clearly, the ϵ-degrees of x_1 and y_1 are equal (which is 3), and $x_1 \geq y_1$. Hence, $\mathfrak{w}_1 = (x_1, y_1)$ is an admissible pair.*

Let $\mathfrak{w}_2 = (x_2, y_2)$, where $x_2 = (1, 3, 6, 7, 9)$, $y_2 = (1, 2, 6, 7, 8)$. Let $\mathfrak{w}_3 = (x_3, y_3)$, where $x_3 = (1, 2, 5, 7, 8)$, $y_3 = (1, 2, 4, 6, 8)$. Then $\mathfrak{w}_2$ and $\mathfrak{w}_3$ are both admissible pairs such that $\mathfrak{w}_1 \geq \mathfrak{w}_2 \geq \mathfrak{w}_3$. Hence, $\mathfrak{w}_1 \geq \mathfrak{w}_2 \geq \mathfrak{w}_3$ is a standard sequence.

Let $w = (4, 5, 8, 9, 10)$. Then clearly, $w \geq top(\mathfrak{w}_1)$. Hence, $\mathfrak{w}_1 \geq \mathfrak{w}_2 \geq \mathfrak{w}_3$ is dominated by w. Again, for $v = (1, 2, 3, 4, 6) \in I(d)$, $bot(\mathfrak{w}_i) \geq v$ for all $i \in \{1, 2, 3\}$, and $\mathfrak{w}_i \neq (v, v)$ for all $i \in \{1, 2, 3\}$. So the above standard sequence is v-compatible. Hence, it belongs to SM_w^v.

Also, $bot(\mathfrak{w}_3) \geq v$, so $\mathfrak{w}_1 \geq \mathfrak{w}_2 \geq \mathfrak{w}_3$ is anti-dominated by v. So, the standard sequence belongs to $SM^{v,v}$.

Fix an element v of $I(d)$. Define

- $\mathfrak{R}^v := \{(r, c) \in [2d] \setminus v \times v : r \leq c^*\}$.
- S^v denote the set of all monomials in $\mathfrak{R}^v$.
- T^v the set of all monomials in $\mathfrak{N}(v)$.
- w be another element of $I(d)$ such that $v \leq w$.
- S_w^v denote the set of all w-dominated monomials in $\mathfrak{R}^v$ (where w-domination of a monomial is defined as in Chapter 4), and (for any positive integer m) let $S_w^v(m)$ denote the set of such monomials of degree m.

Example 5.3.4. *Let $d = 5$ and $v = (1, 2, 4, 6, 8)$. Clearly, v is in $I(d)$. Let $\mathfrak{S} = \{(3, 4)^2, (3, 6), (5, 4), (7, 2), (9, 1)\}$. So, $\mathfrak{S}$ is a monomial in $\mathfrak{R}^v$. Now, degree of $\mathfrak{S}$ is 6, and $\beta_1 = (9, 1) > \beta_2 = (7, 2) > \beta_3 = (5, 4)$ is a v-chain in $\mathfrak{S}$. Again, $s_{\beta_1} \cdots s_{\beta_3} v = (\{1, 2, 4, 6, 8\} \setminus \{1, 2, 4\}) \cup \{5, 7, 9\} = (5, 6, 7, 8, 9)$. Let $w = (5, 7, 8, 9, 10)$. The v-chain $\beta_1 > \beta_2 > \beta_3$ is dominated by w. Similarly, we can check that any other v-chain in $\mathfrak{S}$ is also dominated by w. Hence, $\mathfrak{S}$ is dominated by w. So, $\mathfrak{S}$ is an element of $S_w^v(6)$.*

Definition 5.3.5. *For an element $\alpha = (r, c)$ in $\mathfrak{R}(v)$, define $\alpha^\# := (c^\star, r^\star)$, and for a monomial $\mathfrak{S} \in \mathfrak{R}(v)$, let $\mathfrak{S}^\#$ denote the set $\{(c^*, r^*)|(r, c) \in \mathfrak{S}\}$. Again, $\mathfrak{S}$ is **symmetric** if $\mathfrak{S} = \mathfrak{S}^\#$.*

Example 5.3.6. *Let $d = 7$, $v = (1, 3, 4, 7, 9, 10, 13)$. Let $\mathfrak{S} = \{(2, 1), (6, 4)^2, (5, 10)\}$. Then $\mathfrak{S}$ is a monomial in $\mathfrak{R}(v)$. The underlying set of the monomial $\mathfrak{S}$ is*

$$\{(2, 1), (6, 4), (5, 10)\}.$$

The degree of $\mathfrak{S}$ is 4. The multiplicity of $(2, 1)$, $(6, 4)$, and $(5, 10)$ are respectively 1, 2, and 1. Also, for the monomial $\mathfrak{S}$, $\mathfrak{S}^\# = \{(14, 13), (11, 9)^2, (5, 10)\}$.

Elements of the form $(r, r^\star)$ of $\mathfrak{N}(v)$ are referred to as belonging to the "diagonal", and the set of all diagonal elements of $\mathfrak{N}(v)$ is denoted by $\mathfrak{d}(v)$.

Definition 5.3.7. *A monomial $\mathfrak{S}$ of T^v is **special** if*
(1) $\mathfrak{S} = \mathfrak{S}^\#$ and
(2) the multiplicity of every diagonal element in $\mathfrak{S}$ is even.
Equivalently, $\mathfrak{S}$ is special if there exists $\mathfrak{T}$ in T^v with $\mathfrak{S} = \mathfrak{T} \cup \mathfrak{T}^\#$. The set of all special monomials is denoted by $\mathfrak{E}$.

Example 5.3.8. *Let $d = 5$, and $v = (1, 2, 4, 6, 8)$. Let $\mathfrak{T} = \{(7, 3), (6, 5)\}$. So, $\mathfrak{T}^\# = \{(8, 4), (6, 5)\}$. Hence, $\mathfrak{S} = \mathfrak{T} \cup \mathfrak{T}^\# = \{(6, 5)^2, (7, 3), (8, 4)\}$ is a special monomial.*

5.3.2 The result of Ghorpade and Raghavan

Theorem 5.3.9. *Let v, w be elements of $I(d)$ with $v \leq w$. Let X^w be the Schubert variety corresponding to w, e^v the T-fixed point in X^w corresponding to v, and R be the coordinate ring of the tangent cone to X^w at the point e^v. Then the dimension as a vector space of the m^{th} graded piece $R(m)$ of R equals the cardinality of $S^v_w(m)$.*

5.3.3 The application

Before going to the application, we first need to state [GR06, Proposition 4.1], which is stated below as Proposition 5.3.10.

Proposition 5.3.10. *There is a bijection between $SM^{v,v}$ and T^v that respects domination and degree.*

The proof of Theorem 5.3.9 (as given in [GR06]) relies on a bijection between the two combinatorially defined sets $SM^v_w(m)$ and $S^v_w(m)$. And this bijection in turn, relies upon a bijection between $SM^{v,v}$ and T^v, which is stated in Proposition 5.3.10 above. The application of the main theorem here is to prove that the bijection between $SM^{v,v}$ and T^v (as mentioned in Proposition 5.3.10 above) is a bounded RSK correspondence.

5.3.4 Proof of the application

For the proof of the application, we need a definition, which we state first.

Definition 5.3.11. *Let v be an element of $I(d, 2d)$. A* ***standard monomial*** *in $I(d, 2d)$ is a totally ordered sequence $\theta_1 \geq \ldots \geq \theta_t$ of elements of $I(d, 2d)$. Such a monomial is called v-**compatible** if each θ_j is comparable to v but no θ_j equals v; it is* ***anti-dominated*** *by v if $\theta_t \geq v$. Let $\widetilde{SM}^{v,v}$ denote the set of all v-**compatible standard monomials** in $I(d, 2d)$ anti-dominated by v.*

Proof of the application: There is a natural injection $f : SM^{v,v} \to \widetilde{SM}^{v,v}$ given by

$$f(\mathfrak{w}_1 \geq \ldots \geq \mathfrak{w}_t) := top(\mathfrak{w}_1) \geq bot(\mathfrak{w}_1) \geq \ldots \geq top(\mathfrak{w}_t) \geq bot(\mathfrak{w}_t).$$

Composing this map f with the bijection $\tilde{\phi}$ (it is the inverse map of $\tilde{\pi}$ given in [KR03, §4]) from $\widetilde{SM}^{v,v} \to T^v$, we get an injection of $SM^{v,v}$ into T^v. It then follows from [GR06, Lemma 4.5] that under this composition, the image of $SM^{v,v}$ in T^v is the set $\mathfrak{E}$ of all **special monomials**. On the other hand, there is a bijective map (call it g) from the set $\mathfrak{E}$ of all special monomials to T^v as given in [GR06, §4.1]: given any $\mathfrak{S}$ in $\mathfrak{E}$, to get $g(\mathfrak{S})$, replace those (r, c) of $\mathfrak{S}$ with $r > c^*$ by (c^*, r^*) and then take the (positive) square root. The composition $\eta := g \circ \tilde{\phi} \circ f$ is the required bijection from $SM^{v,v}$ to T^v.

Therefore, to prove that the composition map η is a bounded RSK correspondence, it suffices to show that the map $\tilde{\phi}$ (of [KR03, §4]) is a bounded RSK correspondence. But the maps $\tilde{\pi}$ and $\tilde{\phi}$ (of [KR03]) are inverses of each other. Hence, it now suffices to show that the map $\tilde{\pi}$ of [KR03] is equal to the map $BRSK$ of [Kre08]. This fact has been proved in Corollary 5.1.3 above.

CHAPTER 6

THE BIJECTION OF KREIMAN'S THESIS

In [Kre03], Kreiman and Lakshmibai provided a bijection between the same two combinatorially defined sets as in [KR03]. In this chapter, we prove that the bijections of [Kre03] and [KR03] are same. But for proving this, one needs to first observe that in [Kre03], τ plays the role of v (of [KR03]), and we need to state some definitions and notation from [Kre03].
Let N and d be positive integers such that $1 \leq d < N$. Let

$$G = SL_N(K) \text{ and } P_d = \left\{ A \in G \mid A = \begin{pmatrix} \star & \star \\ 0_{(N-d)\times d} & \star \end{pmatrix} \right\},$$

where K is an algebraically closed field of arbitrary characteristic, $SL_N(K)$ is the set of all $N \times N$ matrices over the field K with determinant 1, $0_{(N-d)\times N}$ is $(N-d) \times d$ zero matrix, and $\star$ can be any matrix with appropriate size. Let T be the subgroup of diagonal matrices in G, B^+ the subgroup of upper triangular matrices in G, and B^- the subgroup of lower triangular matrices in G. Let W be the Weyl group of G relative to T, and W_{P_d} the Weyl group of P_d. Note that $W = S_N$, the group of permutations of a set of N elements, and $W_{P_d} = S_d \times S_{N-d}$. Also note that, $I(d, N)$ (where the notation $I(d, N)$ has been mentioned in Chapter 2) can be identified with W/W_{P_d}. In [Kre03], Kreiman has identified $I(d, N)$ with the set of "minimal representatives" of W/W_{P_d} in S_N; to be precise, a d-tuple $\alpha = (i_1, \cdots, i_d) \in I(d, N)$, is identified with the element $(i_1, \cdots, i_d; j_1, \cdots, j_{N-d}) \in \mathcal{S}_N$, where $\{j_1, \cdots, j_{N-d}\}$ is the complement of $\{i_1, \cdots, i_d\}$ in $\{1, \cdots, N\}$ arranged in increasing order, and $\mathcal{S}_N$ denotes the set of all arrangements of the integers in the set $\{1, \ldots, N\}$. Kreiman has denoted the set of such minimal representatives of S_N by W^{P_d}.

Let R denote the root system of G relative to T, and R^+ the set of positive roots relative to B^+. Let R_{P_d} denote the root system of P_d, and $R^+_{P_d}$ the set of positive roots.

In [Kre03, §2.3], Kreiman has described that $R^+ \setminus R^+_{P_d}$ can be identified with the set of transpositions in S_N of the form $s_{i,j}$, $1 \leq j \leq d < i \leq N$. Kreiman has denoted this set by $\mathcal{R}$. It is easily seen that, for $s_{i,j},\ s_{i,'j'} \in \mathcal{R},\ s_{i,j} \geq s_{i',j'}$ (in the Bruhat order) $\Longleftrightarrow i \geq i'$ and $j \leq j'$. Thus $\mathcal{R}$ is an $(N-d)\times d$ distributive lattice. Kreiman defined the partial order $\succ$ on $\mathcal{R}$ by $s \succ t$ if $s > t$ and $st = ts,\ s,t \in \mathcal{R}$. Recall that $s_{i,j}s_{i',j'} = s_{i',j'}s_{i,j} \Longleftrightarrow i \neq i'$ and $j \neq j'$. Thus $s_{i,j} \succ s_{i',j'} \Longleftrightarrow i > i'$ and $j < j'$.

For the rest of this chapter, fix $w, \tau \in W/W_{P_d}$, $w \geq \tau$.

Definition 6.0.1. *A* ***chain of commuting reflections*** *is a nonempty subset $\{s_1, \cdots, s_t\}$ of $\mathcal{R}$ such that $s_1 \succ \cdots \succ s_t$; t is referred to as the* ***length*** *of the chain. Again, an* ***antichain of commuting reflections*** *is a nonempty subset of $\mathcal{R}$ whose elements are pairwise incomparable and pairwise commuting.*

Example 6.0.2. *Let $d = 6$ and $N = 13$. In Figure 6.1, the dots represent the lattice of reflections $s_{i,j}$, $1 \leq j \leq 6 < i \leq 13$. The dark circles form a chain of commuting reflections, and the stars form an antichain of commuting reflections.*

	1	2	3	4	5	6
7	⋆	·	·	·	·	●
8	·	·	●	·	·	·
9	·	●	·	·	·	·
10	·	·	·	·	·	·
11	·	·	·	⋆	·	·
12	●	·	·	·	⋆	·
13	·	·	·	·	·	⋆

Figure 6.1: Chain and antichain

Definition 6.0.3. *A* ***multiset*** *is similar to a set, but with repetitions of entries allowed. Define the* ***cardinality*** *(or* ***order****) $|S|$ of a multiset S to be the number of elements in S, including repetitions. If $M,\ S$ are multisets, then by $M \subset S$ we mean that every element of M is also in S (although possibly a different number of times).*

Example 6.0.4. *Let $d = 6$ and $N = 13$. Then for $1 \leq j \leq 6 < i \leq 13$, $S = \{s_{8,1}, s_{8,1}, s_{9,3}, s_{10,2}, s_{10,2}, s_{10,2}, s_{13,6}\}$ is a multiset with cardinality seven. Again, if we consider the multiset $M = \{s_{8,1}, s_{9,3}, s_{10,2}, s_{10,2}, s_{13,6}\}$, then $M \subset S$.*

Notation: Let $S = \{s_1, \cdots, s_t\}$ be a chain of commuting reflections in a multiset M and $\tau = (\tau_1, \cdots, \tau_d; \tau_{d+1}, \cdots, \tau_N)$. Let $s_i := s_{p_i,q_i}$. Then define

$$\tau s_1 \cdots s_t := \{\tau_1, \cdots, \tau_d\} \dot{\cup} \{\tau_{p_1}, \cdots, \tau_{p_t}\} \setminus \{\tau_{q_1}, \cdots, \tau_{q_t}\}, \tag{6.0.0.1}$$

where $1 \leq q_i \leq d < p_i \leq N$.

Example 6.0.5. *Let, $d, N,$ and M be as in Example 6.0.4. Also, let $\tau = (1, 3, 6, 9, 10, 11; 2, 4, 5, 7, 8, 12, 13)$. Now, $s_1 = s_{10,2} \succ s_2 = s_{9,3}$ is a chain of commuting reflections in M, and for this chain of commuting reflections*

$$\tau s_1 s_2 = \{1, 3, 6, 9, 10, 11\} \dot{\cup} \{5, 7\} \setminus \{6, 3\} = \{1, 5, 7, 9, 10, 11\}.$$

Fix $\tau = (\tau_1, \cdots \tau_d; \tau_{d+1}, \cdots, \tau_N) \in W^{P_d}$, where $1 \leq \tau_1 < \cdots < \tau_d \leq N$, and $1 \leq \tau_{d+1} < \cdots < \tau_N \leq N$. For $j = 1, \cdots, d$ define $f_\tau(j) := d + \tau_j$-j+0.5. Kreiman has defined the τ-line to be the path connecting the points $(f_\tau(j), j)$, $j = 1, \cdots, d$, as in Figure 6.2. Note that the points $(f_\tau(j), j)$ do not lie on lattice points; they lie halfway between them. Let $d, N,$ and τ be as in Example 6.0.5. Then Figure 6.2 shows the corresponding τ-line.

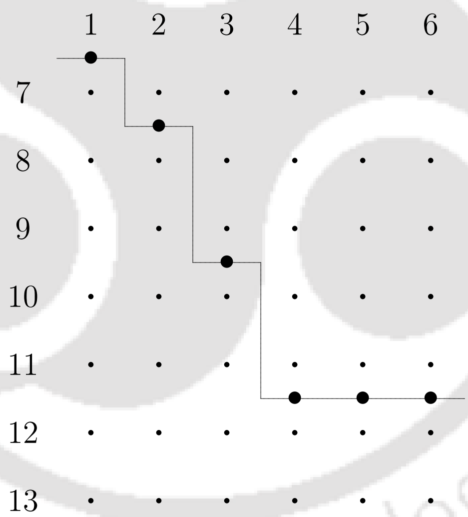


Figure 6.2: A τ-line

The τ-line divides $\mathcal{R}$ into two sets: $\mathcal{R}_\tau^+ := \{s_{i,j} \mid i > f_\tau(j)\}$ and $\mathcal{R}_\tau^- := \{s_{i,j} \mid i < f_\tau(j)\}$. Note that $f_\tau(j)$ is a non decreasing function of j. This implies that for $s, s' \in \mathcal{R}_\tau^+$, $s \vee s' \in \mathcal{R}_\tau^+$, and for $t, t' \in \mathcal{R}_\tau^-$, $t \wedge t' \in \mathcal{R}_\tau^-$.

Now, we will state and prove [Kre03, Lemma 3.2.1], which is stated below as Lemma 6.0.6.

Lemma 6.0.6. $\mathcal{R} = \mathcal{R}_\tau^+ \dot{\cup} \mathcal{R}_\tau^-$, *where $\tau s_{i,j} > \tau$ if $s_{i,j} \in \mathcal{R}_\tau^+$, and $\tau s_{i,j} < \tau$ if $s_{i,j} \in \mathcal{R}_\tau^-$.*

Proof. Writing $\tau = (\tau_1, \cdots, \tau_d; \tau_{d+1}, \cdots, \tau_N) \in W^{P_d}$, we have that $\tau s_{i,j}$ swaps τ_i with τ_j. Thus $\tau s_{i,j} > \tau$ if $\tau_i > \tau_j$ and $\tau s_{i,j} < \tau$ if $\tau_i < \tau_j$.

Fix j. The number of positive integers less than τ_j is $\tau_j - 1$, and the number of entries among $\tau_1, \cdots, \tau_d$ less than τ_j is $(\tau_j - 1) - (j-1) = \tau_j - j$. Thus $\tau_{d+1}, \cdots, \tau_{d+\tau_j - j}$ are $< \tau_j$ and $\tau_{d+\tau_j-j+1}, \cdots, \tau_N$ are $> \tau_j$. Thus $s_{i,j} \in \mathcal{R}_\tau^+ \iff i > d + \tau_j - j \iff \tau_i > \tau_j \iff \tau_{s_{i,j}} > \tau$. □

Next we will recall **separably ordered set** from [Kre03].

Definition 6.0.7. *For $s \in \mathcal{R}_\tau^+$, define the ball $B_s := \{t \in \mathcal{R}_\tau^+ \mid t < s\}$. Then $s, t \in \mathcal{R}_\tau^+$ are **separated** if $s \wedge t \in \mathcal{R}_\tau^-$, or equivalently if B_s and B_t are disjoint. The multisets $M, M' \subset \mathcal{R}_\tau^+$ are **separated** if any $s \in M$, $t \in M'$ are separated, or equivalently if there exist nonempty balls A, B such that $M \subset A$, $M' \subset B$, and $A \cap B = \emptyset$. In this case $A \cap M$, $B \cap M'$ is said to be a **separation** of $M \cup M'$. If a multiset $M \subset \mathcal{R}_\tau^+$ has no separation then it is called **connected**. Again, a maximal connected submultiset M' of M is called a **connected component** of M. It can be easily seen that M can be uniquely written as the disjoint union of its connected components. A pairwise commuting subset M of $\mathcal{R}_\tau^+$ is said to be a **separably ordered set of commuting reflections** if for any $s, t \in M$, either s and t are comparable or they are separated.*

Notation: Let S_w denote the unique separably ordered set $\{t_1, \cdots, t_p\}$ such that $\tau t_1 \cdots t_p = w$. (The existence and uniqueness of such a separably ordered set follows from Lemma 3.3.1 of [Kre03].)

Next we will recall the definitions of chainlength and depth from [Kre03].

Definition 6.0.8. *The **chainlength** of a multiset in $\mathcal{R}$ is the maximum length of a chain of commuting reflections in the multiset. For $M \subset \mathcal{R}_\tau^+$ a multiset and $m \in M$, define $depth_M(m)$ to be the chainlength of $\{m' \in M \mid m' \succeq m\}$. If $S \subset \mathcal{R}_\tau^+$ is a multiset, then $S \geq M$ if for every $m \in M$, there exists $s \in S$ such that $s \geq m$, and $depth_S(s) \geq depth_M(m)$. This defines a partial order among the collection of multisets in $\mathcal{R}_\tau^+$.*

Example 6.0.9. *Let $d = 6, N = 13$, and $M = \{s_{13,1}, s_{11,2}, s_{10,3}, s_{9,6}, s_{8,6}, s_{7,5}\}$. Notice that $s_{13,1} \succ s_{11,2} \succ s_{10,3} \succ s_{9,6}$, $s_{13,1} \succ s_{11,2} \succ s_{10,3} \succ s_{8,6}$, and $s_{13,1} \succ s_{11,2} \succ s_{10,3} \succ s_{7,5}$ are all the chains of commuting reflections in M of maximum length 4. So, chainlength of M is 4. Again, $depth_M(10,3) = 3$. Now, if we consider the set $S = \{s_{13,1}, s_{12,5}, s_{11,2}, s_{10,3}, s_{9,4}, s_{8,6}, s_{8,5}, s_{7,4}\}$, then clearly $S \geq M$.*

Definition 6.0.10. *Define $SM_{w,\tau}^\tau$ to be the set of sequences $\theta_1 \geq \cdots \geq \theta_t$, $\theta_i \in W^{P_d}$, such that $w \geq \theta_1$ and $\theta_t > \tau$, plus the singleton set $\{\tau\}$. If $S = \theta_1 \geq \cdots \geq \theta_t$ is an element of $SM_{w,\tau}^\tau$, then we define $|S|$ to be the sum of the τ-degrees of $\theta_1, \ldots, \theta_t$, where for any $\theta \in W^{P_d}$, we define the τ-degree of θ to be the cardinality of $\theta \setminus \tau$. Define*

$$SM_{w,\tau}^\tau(m) := \{S \in SM_{w,\tau}^\tau \ : \ |S| = m\}.$$

Example 6.0.11. *Let $d = 6$ and $N = 13$. Let $w = (2, 6, 8, 11, 12, 13; 1, 3, 4, 5, 7, 9, 10), \theta_1 = (2, 4, 6, 8, 10, 12; 1, 3, 5, 7, 9, 11, 13), \theta_2 = (2, 3, 5, 7, 9, 11; 1, 4, 6, 8, 10, 12, 13), \theta_3 = (1, 3, 4, 6, 8, 11; 2, 5, 7, 9, 10, 12, 13)$, and $\tau = (1, 3, 4, 5, 7, 10; 2, 6, 8, 9, 11, 12, 13)$. Clearly, all of $w, \theta_1, \theta_2, \theta_3$, and τ are in W^{P_d}. Also, $(2, 6, 8, 11, 12, 13) > (2, 4, 6, 8, 10, 12) > (2, 3, 5, 7, 9, 11) > (1, 3, 4, 6, 8, 11) > (1, 3, 4, 5, 7, 10)$. Hence, $\theta_1 \geq \theta_2 \geq \theta_3$ is an element of $SM^{\tau}_{w,\tau}$.*

Definition 6.0.12. *Define $S^{\tau}_{w,\tau}$ to be the set of multisets S of $\mathcal{R}$, such that for every chain of commuting reflections $s_1 \succ \cdots \succ s_t$, $s_i \in S$, we have that $w \geq \tau s_1 \cdots s_t \geq \tau$ (note that the empty multiset is included); define*

$$S^{\tau}_{w,\tau}(m) := \{S \in S^{\tau}_{w,\tau} \ : \ |S| = m\}$$

Example 6.0.13. *Let $d = 6$ and $N = 13$. Let $w = (2, 6, 8, 11, 12, 13; 1, 3, 4, 5, 7, 9, 10), \tau = (1, 3, 4, 5, 7, 10; 2, 6, 8, 9, 11, 12, 13)$, and $S = \{s_{11,2}, s_{10,3}, s_{9,6}, s_{7,3}\}$. Let $s_1 = s_{11,2}, s_2 = s_{10,3}, s_3 = s_{9,6}, s_4 = s_{7,3}$. Now, (i) s_1, (ii) s_2, (iii) s_3, (iv) s_4, (v) $s_1 \succ s_2$, (vi) $s_1 \succ s_3$, (vii) $s_1 \succ s_4$, (viii) $s_2 \succ s_3$, and (ix) $s_1 \succ s_2 \succ s_3$ are all the chains of commuting reflections in S. Again, for the chain of commuting reflections $s_{11,2} \succ s_{10,3} \succ s_{9,6}$ we have, $(1, 3, 4, 5, 7, 10) \leq \tau s_1 s_2 s_3 = \{1, 3, 4, 5, 7, 10\} \dot{\cup} \{8, 9, 11\} \setminus \{3, 4, 10\} = \{1, 5, 7, 8, 9, 11\} \leq \{2, 6, 8, 11, 12, 13\}$. Similarly, we can check that the above condition is true for all the chains of commuting reflections in S. So, S is an element of $S^{\tau}_{w,\tau}$.*

Now, using Lemma 6.0.6 and Definition 6.0.12, we can say the Remark 6.0.14.

Remark 6.0.14. *Any reflection s in a multiset S of $S^{\tau}_{w,\tau}$ must satisfy $w \geq \tau s \geq \tau$ (from the definition of $S^{\tau}_{w,\tau}$); in particular, $s \in \mathcal{R}^{+}_{\tau}$. Thus for any $S \in S^{\tau}_{w,\tau} \Rightarrow S \subset \mathcal{R}^{+}_{\tau}$. Similarly, $S \in S^{v}_{\tau,\tau} \Rightarrow S \subset \mathcal{R}^{-}_{\tau}$.*

The following definition is needed to describe the map Φ below in §6.1.

Definition 6.0.15. *If $Q = \{s_{i_k,j_k} | k = 1, \ldots, t\}$ is a multiset of reflections, then we define the ordered multisets $Row(Q) := \{i_k | k = 1, \ldots, t\}$ and $Col(Q) := \{j_k | k = 1, \ldots, t\}$, each listed in non decreasing order (with possible repetitions of entries).*

Next we are going to recall the map Φ from [Kre03] which is going to be very important for rest of this chapter.

6.1 The Map $\Phi : S^{\tau}_{w,\tau}(m) \mapsto SM^{\tau}_{w,\tau}(m)$

To describe Φ, Kreiman began by defining a map ϕ, which assigns to a multiset M two multisets: M^F and S_M. In [Kre03, Lemma 3.4,1], Kreiman has proved that S_M is a separably ordered set. The map ϕ is used to define the map Φ.

Let $M \subset \mathcal{R}_\tau^+$ be a multiset. Let $D_M :=$ Max{depth$(m) \mid m \in M\}$. For $1 \leq p \leq D_M$, let $M_p := \{m \in M \mid \text{depth}(m) = p\}$, and let $M_{p,1}, \cdots, M_{p,q_p}$ be the **connected components** of M_p. Let $\mathcal{M}$ be the collection of all multisets $M_{p,q}$. Note that each $M_{p,q} \in \mathcal{M}$ is a multipath (where a multipath is defined to be a multiset in $\mathcal{R}$ of chainlength 1). For some p, q, if $\{i_k,\ k = 1, \cdots, t\} :=$ Row$(M_{p,q})$ and $\{j_k,\ k = 1, \cdots, t\} :=$ Col$(M_{p,q})$, define $s_{p,q} := s_{i_t,j_1}$. Define $S_M := \{s_{p,q} \mid M_{p,q} \in \mathcal{M}\}$. Define $M_{p,q}^F$ to be the multipath $\{s_{i_k,j_{k+1}},\ k = 1, \cdots, t-1\}$. Note that, $M_{p,q}^F \subset \mathcal{R}_\tau^+$ since $M_{p,q}$ is connected. Define $M^F := \dot{\bigcup}_{M_{p,q} \in \mathcal{M}} M_{p,q}^F$. Define ϕ to be the map which produces from M the multiset M^F and separably ordered set S_M.

The map Φ assigns to a multiset $M \in S_{w,\tau}^\tau(m)$ a sequence of decreasing separably ordered sets $S_M \geq S_{M^F} \geq S_{(M^F)^F} \geq \cdots \geq S_{(((M^F)^F)\cdots)^F}$ in $SM_{w,\tau}^\tau(m)$. Next we will recall the map Φ from [Kre03] in detail.

To define Φ, the map ϕ is first applied to the multiset M, producing multiset M^F and separably ordered set S_M (S_M is separably ordered set by [Kre03, Lemma 3.4.1]). Then ϕ is applied to M^F, producing multiset $(M^F)^F$ and separably ordered set S_{M^F}. This process is continued, applying ϕ in that way until $(((M^F)^F)\cdots)^F$ is separably ordered set, and at this point, ϕ is applied once more to $(((M^F)^F)\cdots)^F$, producing multiset $(((M^F)^F)\cdots)^F = \emptyset$ and separably ordered set $S_{(((M^F)^F)\cdots)^F} = (((M^F)^F)\cdots)^F$. Here the process terminates. $\Phi(M)$ is the sequence of separably ordered sets $S_M \geq S_{M^F} \geq S_{(M^F)^F} \geq \cdots S_{(((M^F)^F)\cdots)^F}$. This sequence is descending by [Kre03, Lemma 3.4.2]. Also, since $M \in S_{w,\tau}^\tau$, $S_w \geq M$; thus $S_w \geq S_M$. Thus $S_M \geq S_{M^F} \geq S_{(M^F)^F} \geq S_{(((M^F)^F)\cdots)^F}$ is an element of $SM_{w,\tau}^\tau(m)$.

6.1.1 Proof of the equivalence of the bijections of Kreiman's Thesis ([Kre03]) and Kodiyalam-Raghavan ([KR03])

Observe first that in [Kre03], τ plays the role of v (of [KR03]). After recalling necessary definitions and notation from [Kre03], we can say that to prove the equivalence of the bijections in [Kre03] and [KR03], one needs to show the following:

1. The set $\mathcal{R}_\tau^+$ of [Kre03] (which is also defined at the starting of this chapter) is equivalent to the set $\mathfrak{N}(v)$ of [KR03] ($\mathfrak{N}(v)$ is also defined in §4.1 of this thesis).

2. A chain of commuting reflections in $\mathcal{R}_\tau^+$ of [Kre03] (also mentioned in Definition 6.0.1 of this thesis) is equivalent to a v-chain in $\mathfrak{N}(v)$ of [KR03] (also mentioned in Definition 4.1.7 of this thesis).

3. $S_{w,\tau}^\tau(m)$ of [Kre03] (also mentioned in Definition 6.0.12 of this thesis) is equivalent to $S_w^v(m)$ of [KR03] (also mentioned in §5.3.1 of this thesis).

4. $SM_{w,\tau}^\tau(m)$ of [Kre03] (also mentioned in Definition 6.0.10 of this thesis) is same as $SM_w^v(m)$ of [KR03].

5. A multiset M in $\mathcal{R}^+_\tau$ of [Kre03] (also mentioned in Definition 6.0.3 of this thesis) is equivalent to a monomial $\mathfrak{S}$ in $\mathfrak{N}(v)$ of [KR03] (also mentioned in Definition 4.1.2 of this thesis).

6. The depth of an element m in a multiset M of $\mathcal{R}^+_\tau$ of [Kre03](also mentioned in Definition 6.0.8 of this thesis) is equivalent to the depth (in the sense of [KR03]) of an element β in a monomial $\mathfrak{S}$ of $\mathfrak{N}(v)$ (also mentioned in Definition 4.1.15 of this thesis).

7. The notation $D_M = \text{Max}\{\text{depth}(m) | m \in M\}$ in [Kre03] (also mentioned in §6.1 of this thesis) is analogous to the largest length of a v-chain in a monomial $\mathfrak{S}$ in $\mathfrak{N}(v)$ of [KR03] (also mentioned in Definition 4.1.15 of this thesis).

8. For $1 \le p \le D_M$, the notation M_p of [Kre03] (also mentioned in §6.1 of this thesis) is analogous to the notation $\mathfrak{S}_j$ of [KR03] (also mentioned in §4.2 of this thesis).

9. The connected components $M_{p,q}$ of M_p of [Kre03] (also mentioned in §6.1 of this thesis) are equivalent to the blocks of $\mathfrak{S}_j$ of [KR03] (also mentioned in Definition 4.2.2 of this thesis).

10. The notation M^F_{pq} of [Kre03] (also mentioned in §6.1 of this thesis) is equivalent to the notation $\mathfrak{B}'$ of [KR03] (also mentioned in §4.2 of this thesis). Similarly, the notation M^F of [Kre03] (also mentioned in §6.1 of this thesis) is equivalent to the notation $\mathfrak{S}^{(1)}$ of [KR03] (also mentioned in §4.2 of this thesis).

11. The concept of a "separably ordered set" of [Kre03] (also mentioned in Definition 6.0.7 of this thesis) is equivalent to the concept of a distinguished subset on [KR03] (also mentioned in Definition 4.1.11 of this thesis).

12. The notation S_M for a multiset M of [Kre03] (also mentioned in §6.1 of this thesis) is equivalent to the notation $\mathfrak{S}_w$ for the distinguished monomial corresponding to w (where $\pi(\mathfrak{S}) = (w, \mathfrak{S}^{(1)})$) of [KR03] (also mentioned in the proof of the Proposition 4.1.13 of this thesis).

13. The map ϕ of [Kre03] (also mentioned in §6.1 of this thesis) plays the role of the map π of [KR03] (also mentioned in §4.2 of this thesis).

14. The map Φ of [Kre03] (also mentioned in §6.1 of this thesis) plays the role of the map $\tilde{\pi}$ of [KR03] (also mentioned in §4.2 of this thesis).

Proof. 1. Recall that,

$$\mathcal{R}^+_\tau := \{s_{i,j} \mid i > f_\tau(j),\ 1 \le j \le d < i \le n\},$$

$$\text{where } f_\tau(j) := d + \tau_j - j + 0.5.$$

So, $i > f_\tau(j) \iff i > d + \tau_j - j + 0.5 \iff i - d > \tau_j - j$.
Again we have,

$$\mathfrak{N}(\tau) := \{(\tau_p, \tau_q) \mid 1 \le q \le d < p \le n,\ \tau_p > \tau_q\}.$$

Now, the proof follows from the proof of Lemma 6.0.6. Hence, $\mathcal{R}^+_\tau$ and $\mathfrak{N}(\tau)$ are equivalent.

2. Let $s_1 \succ \cdots \succ s_t$ be a chain of commuting reflections in $\mathcal{R}^+_\tau$. Let $s_i = s_{p_i,q_i}$, $1 \le i \le t$. We know from the beginning of this section that

$$s_i \succ s_j \iff s_{p_i,q_i} \succ s_{p_j,q_j} \iff p_i > p_j,\ q_i < q_j. \tag{6.1.1.1}$$

Observe that, $p_i, p_j \in \{d+1, \cdots, N\}$ and $q_i, q_j \in \{1, \cdots, d\}$. Also observe that, $p_i > p_j \iff \tau_{p_i} > \tau_{p_j}$ and $q_i < q_j \iff \tau_{q_i} < \tau_{q_j}$. Therefore, (6.1.1.1) gives

$$s_i \succ s_j \iff \tau_{p_i} > \tau_{p_j} \text{ and } \tau_{q_i} < \tau_{q_j}. \tag{6.1.1.2}$$

This precisely implies that $s_1 \succ \cdots \succ s_t \iff (\tau_{p_1}, \tau_{q_1}) > \cdots > (\tau_{p_t}, \tau_{q_t})$ is a τ-chain in $\mathfrak{N}(\tau)$ in the sense of Definition 4.1.7.

3. Observe that any reflection s in a multiset S of $S^\tau_{w,\tau}$ must satisfy $w \ge \tau s \ge \tau$. This in particular implies that $s \in \mathcal{R}^+_\tau$, that is, $S \subseteq \mathcal{R}^+_\tau$.
Now, let S be an element of $S^\tau_{w,\tau}(m)$ and $s_1 \succ \cdots \succ s_t$ (where $s_i \in S$) be a chain of commuting reflections. For each $i \in \{1, \ldots, t\}$, let $s_i = s_{p_i,q_i}$, where $1 \le q_i \le d < p_i \le n$. Let $\beta_i = (\tau_{p_i}, \tau_{q_i})$. It is now easy to see that $\tau s_1 \cdots s_t$ is nothing but the element $s_{\beta_1} \cdots s_{\beta_t}$ of $I(d, N)$ (in the sense of Definition 4.1.7). The rest of the proof now follows from the definition of $S^\tau_w(m)$ (in the sense of §5.3.1).

4. Obvious from definitions.

5. Obvious from definitions.

6. The proof follows from the simple observation that $\text{depth}_M(m)$ (in the sense of Definition 6.0.8) is precisely the same as the maximum length of a τ-chain in the monomial $M(\subseteq \mathfrak{N}(\tau))$ (in the sense of Definition 4.1.15).

7. Follows from (6) and definitions.

8. Follows from (6) and definitions.

9. The proof follows from the observation that two consecutive elements (r, c) and (R, C) in $\mathfrak{S}_j$ are said to be in different blocks (in the sense of §4.2) if and only if $r < C$, that is, if and only if (r, C) belongs to $\mathcal{R}^{-}_{\tau}$. If we let $s_{i,j}$ and $s_{i',j'}$ denote the elements (r, c) and (R, C) respectively, then (r, C) is nothing but $s_{i,j} \wedge s_{i',j'}$. The rest of the proof follows from definitions.

10. We know from (9) that a block $\mathfrak{B}$ of a monomial $\mathfrak{S}$ in $\mathfrak{N}(\tau)$ (in the sense of Definition 4.2.2) is equivalent to a connected component $M_{p,q}$ of a multiset M in $\mathcal{R}^{+}_{\tau}$. Now, if $\{(r_1, c_1), \ldots, (r_t, c_t)\}$ is an arrangement of the elements of $\mathfrak{B}$ with $r_1 \leq \cdots \leq r_t$ and $c_1 \leq \cdots \leq c_t$, and if the corresponding connected component $M_{p,q}$ equals $\{s_{i_k,j_k} | 1 \leq i \leq t\}$ (where $i_1 \leq \cdots \leq i_t$ and $j_1 \leq \cdots \leq j_t$), then the element (r_t, c_1) of $\mathfrak{N}(\tau)$ is same as the element $s_{p,q}$ $(= s_{i_t,j_1})$ of $\mathcal{R}^{+}_{\tau}$ (as mentioned in §6.1 of this paper).
With the help of the above facts together with some related definitions, the rest of the proof follows easily.

11. For $s_{i,j}$ and $s_{i',j'}$ in $\mathcal{R}^{+}_{\tau}$ (where $i \leq i'$), they are comparable if and only if $i < i'$ and $j > j'$. Also, they are separated if and only if $s_{i,j'} = s_{i,j} \wedge s_{i',j'}$ belongs to $\mathcal{R}^{-}_{\tau}$, that is, if and only if, $i < j'$.
Now, $i < i'$ and $j > j'$ if and only if $\tau_i < \tau_{i'}$ and $\tau_j > \tau_{j'}$. Also, $i < j'$ if and only if $\tau_i < \tau_{j'}$.
Therefore, if $s_{i,j}$ and $s_{i',j'}$ are either comparable or separated, and if $\tau_i < \tau_{i'}$, then we have either $\tau_j > \tau_{j'}$ or $\tau_i < \tau_{j'}$. This is precisely the same as condition (B^*) in the definition of a distinguished subset of $\mathfrak{N}(\tau)$ (in the sense of Definition 4.1.11).

12. Obvious from definitions.

13. Obvious from definitions.

14. Obvious from definitions.

□

CHAPTER 7

INITIAL IDEALS OF TANGENT CONES TO RICHARDSON VARIETIES IN THE SYMPLECTIC GRASSMANNIAN

7.1 Term order

Let F be a field. A **term order** on $F[x_1.\cdots,x_n]$ is a total order $\succ$ on the set of all monomials $x^a = x_1^{a_1}\cdots x_n^{a_n}$ which has the following two properties:

1. It is multiplicative; that is, $x^a \succ x^b$ implies $x^{a+c} \succ x^{b+c}$ for all $a, b, c \in \mathbb{N}^n$.

2. The constant monomial is the smallest; that is, $x^a \succ 1$ for all $a \in \mathbb{N}^n \setminus \{0\}$.

Here are some significant examples of term order. For the example we will take $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ for multi indices, and set $p = x^a$, and $q = x^b$. By renaming the variables, we may always achieve $x_1 \succ x_2 \succ \cdots \succ x_n$, and we will only describe orders with this property.

Example 7.1.1. ***Lexicographic order:*** *$p \succ_{lex} q$ if and only if $a_i > b_i$ for the first index i with $a_i \neq b_i$. For $n = 2$,*

$$\cdots \succ x_1^3 \succ x_1^2x_2 \succ x_1^2 \succ x_1x_2 \succ x_1 \succ x_2 \succ 1.$$

Example 7.1.2. ***Homogeneous lexicographic order:*** *$p \succ_{hlex} q$ if and only if deg $p >$ deg q or deg $p =$ deg q and $a_i > b_i$ for the first index i with $a_i \neq b_i$.*

Example 7.1.3. ***Reverse lexicographic order:*** *$p \succ_{rlex} q$ if and only if deg $p >$ deg q or deg $p =$ deg q, and $a_i < b_i$ for the last index i with $a_i \neq b_i$.*
It is clear that,

$$x_1^3 \succ_{hlex} x_1^2 \succ_{hlex} x_1x_2 \text{ and}$$

$$x_2^2 \succ_{rlex} x_1x_2.$$

7.2 Gröbner basis

Fix a term order $\succ$ on $F[x_1, \cdots, x_n]$, then every polynomial f has a unique **initial term** $\text{in}_\succ(f)$. This is the $\succ$ largest monomial which occurs with nonzero coefficient in the expansion of f. We write the terms of f in $\succ$ decreasing order.

Suppose now, I is an ideal in $F[x_1, \cdots, x_n]$. Then its initial ideal $\text{in}_\succ(I)$ is the ideal generated by the initial terms of all polynomial in I:

$$\text{in}_\succ(I) = \langle \text{in}_\succ(f) \mid f \in I \rangle.$$

Definition 7.2.1. *A finite subset $\mathcal{G}$ of I is a* ***Gröbner*** *basis with respect to the term order $\succ$ if the initial terms of the elements in $\mathcal{G}$ suffice to generate the initial ideal:*

$$in_\succ(I) = \langle in_\succ(g) \mid g \in \mathcal{G} \rangle.$$

7.3 Ideals of tangent cones to Richardson varieties

Let β be the element of $I(d)$, which was fixed at the end of the Chapter 2. Consider the matrix of size $2d \times d$ whose columns are numbered by the entries of β, the rows by $\{1, \ldots, 2d\}$, the rows corresponding to the entries of β form the $d \times d$ identity matrix, and the remaining d rows form a matrix whose entries are $X_{(r,c)}$ such that $(r, c) \in \mathfrak{R}(\beta)$, where $X_{(r,c)} = -X_{(c^*,r^*)}$ if either $r > d$ and $c^* < d$ or $r < d$ and $c^* > d$, and $X_{(r,c)} = X_{(c^*,r^*)}$ otherwise.

For $d = 4$, $\beta = (1, 2, 5, 6)$, the $2d \times d$ matrix is given in below:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_{31} & x_{32} & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{45} & x_{35} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ x_{71} & x_{72} & x_{42} & -x_{32} \\ x_{81} & x_{71} & x_{41} & -x_{31} \end{pmatrix} \tag{7.3.0.1}$$

Let $\mathfrak{M}_d(V) \subseteq G_d(V) \hookrightarrow \mathbb{P}(\bigwedge^d V)$ be the Plücker embedding (where $G_d(V)$ denotes the Grassmannian of all d-dimensional subspaces of V). For θ in $I(d, 2d)$, let p_θ denote the corresponding Plücker coordinate. Consider the affine patch $\mathbb{A}$ of $\mathbb{P}(\bigwedge^d V)$ given by $p_\beta \neq 0$. The affine patch $\mathbb{A}^\beta := \mathfrak{M}_d(V) \cap \mathbb{A}$ of the symplectic Grassmannian $\mathfrak{M}_d(V)$ is an affine space whose coordinate ring can be taken to be the polynomial ring in variables of the form $X_{(r,c)}$ with $(r,c) \in \mathfrak{OR}(\beta)$ (mentioned in §4.1).

For $\theta \in I(d, 2d)$, consider the submatrix of the above mentioned matrix given by the rows numbered $\theta \setminus \beta$ and columns numbered $\beta \setminus \theta$. Let $f_{\theta,\beta}$ denote the determinant of this submatrix. Clearly, $f_{\theta,\beta}$ is a homogeneous polynomial in the variables $X_{(r,c)}$, where $(r,c) \in \mathfrak{OR}(\beta)$.

Example 7.3.1 below gives an illustration of $f_{\theta,\beta}$.

Example 7.3.1. *Let $d = 4$, $\beta = (1, 2, 5, 6)$, and $\theta = (1, 3, 4, 5)$. So, $\theta \in I(4, 8)$. Now, $f_{\theta,\beta}$ is the determinant of the submatrix whose rows are numbered by $\theta \setminus \beta = \{3, 4\}$ and columns are numbered by $\beta \setminus \theta = \{2, 6\}$, that is*

$$f_{\theta,\beta} = \begin{vmatrix} x_{32} & x_{36} \\ x_{42} & x_{35} \end{vmatrix} = x_{32}x_{35} - x_{36}x_{42}. \tag{7.3.0.2}$$

Clearly, $f_{\theta,\beta}$ is a homogeneous polynomial in the variables $X_{(r,c)}$, where $(r,c) \in \mathfrak{OR}(\beta)$.

The ϵ-**degree** of an element x of $I(d)$ is defined as the cardinality of $x \setminus [d]$ or equivalently that of $[d]\setminus x$. An ordered pair $\mathfrak{w} = (x, y)$ of elements of $I(d)$ is called an **admissible pair** if $x \geq y$ and the ϵ-degrees of x and y are equal. We refer to x and y as the **top** and the **bottom** of $\mathfrak{w}$ and write top$(\mathfrak{w})$ for x and bot$(\mathfrak{w})$ for y. Given any admissible pairs $\mathfrak{w} = (x, y)$ and $\mathfrak{w}' = (x', y')$, we say that $\mathfrak{w} \geq \mathfrak{w}'$ if $y \geq x'$, that is, if $x \geq y \geq x' \geq y'$. Let $\mathfrak{w} = (x, y)$ be an admissible pair. Let θ be the element $(x \cap [d]) \cup (y \cap [d]^c)$ of $I(d, 2d)$ (as mentioned in [GR06, Proposition 3.4]). For any admissible pair $\mathfrak{w}$, let us denote the polynomial $f_{\theta,\beta}$ by $f_{\mathfrak{w},\beta}$.

Example 7.3.2 below gives an illustration of the admissible pairs.

Example 7.3.2. *Let $d = 4$, so $\epsilon = (1, 2, 3, 4)$. Let $x = (1, 4, 6, 7)$. The ϵ-degree of x is 2. Let $y = (1, 3, 5, 7)$. Clearly, the ϵ-degree of y is also 2 and $x, y \in I(d)$ with $x \geq y$. Hence, $\mathfrak{w} = (x, y)$ is an admissible pair. Also, here* top$(\mathfrak{w}) = (1, 4, 6, 7)$ *and* bot$(\mathfrak{w}) = (1, 3, 5, 7)$. *Again, let $x' = (1, 3, 5, 7)$ and $y' = (1, 2, 4, 6)$. Then the ϵ-degrees of both x' and y' are 1, and $x', y' \in I(d)$ with $x' \geq y'$. So, $\mathfrak{w}' = (x', y')$ is also an admissible pair. As $x \geq y \geq x' \geq y'$, so $\mathfrak{w} \geq \mathfrak{w}'$. Again, for the above $\mathfrak{w}$, $\theta = (1, 4, 5, 7)$. So, for this θ and for $\beta = (1, 2, 5, 6)$, (using the 8×4 matrix of (7.3.0.1)) we have,*

$$f_{\mathfrak{w},\beta} = f_{\theta,\beta} = -x_{32}x_{42} - x_{35}x_{72}.$$

Set $Y_\alpha^\gamma(\beta) := X_\alpha^\gamma \cap \mathbb{A}^\beta$ (where X_α^γ is mentioned in §2.3). From [LMS79] we can deduce a set of generators for the ideal $I_{\alpha,\beta}^\gamma$ of functions on $\mathbb{A}^\beta$ vanishing on $Y_\alpha^\gamma(\beta)$. The following equation gives the generators:

$$I_{\alpha,\beta}^{\gamma} = (f_{\mathfrak{w},\beta} \mid \mathfrak{w} = (x, y) \text{ is an admissible pair}, \alpha \not\leq y \text{ or } x \not\leq \gamma)\,. \quad (7.3.0.3)$$

We are interested in the tangent cone to X_α^γ at e^β or, what is the same, the tangent cone to $Y_\alpha^\gamma(\beta)$ at the origin. Observe that $f_{\mathfrak{w},\beta}$ is a homogeneous polynomial. Because of this, $Y_\alpha^\gamma(\beta)$ itself is a cone and equal to its tangent cone at the origin. Therefore, the ideal of the tangent cone to X_α^γ at e^β is the ideal $I_{\alpha,\beta}^\gamma$ in (7.3.0.3).

7.4 Extended β-chains

Let β be the element of $I(d)$, which was fixed at the end of the Chapter 2. For elements $\lambda = (R, C), \mu = (r, c)$ of $\mathfrak{R}(\beta)$, we write $\lambda > \mu$, if $R > r$ and $C < c$ (note that these are strict inequalities). A sequence $\lambda_1 > \cdots > \lambda_k$ of elements of $\mathfrak{R}(\beta)$ is called an **extended β-chain**. Note that an extended β-chain can also be empty. Letting C to be an extended β-chain, we define $C^+ := C \cap \mathfrak{N}(\beta)$ and $C^- := C \cap (\mathfrak{R}(\beta) \setminus \mathfrak{N}(\beta))$. We call C^+ (resp. C^-) the **positive** (resp. **negative**) **part** of the extended β-chain C. We call an extended β-chain C **positive** (resp. **negative**) if $C = C^+$ (resp. $C = C^-$). The extended β-chain C is called **nonvanishing** if at least one of its positive or negative part is non-empty. Then clearly, every non-empty extended β-chain is nonvanishing.

An extended β-chain that lies completely in $\mathfrak{O}\mathfrak{R}(\beta)$ is called an **extended upper β-chain**. We similarly define **extended upper positive** and **extended upper negative** β-chains.

Example 7.4.1 below illustrates an extended β-chain and an extended upper β-chain.

Example 7.4.1. *Let $d = 7$ and $\beta = (1, 3, 4, 7, 9, 10, 13)$. Let $\lambda_1 = (14, 1)$, $\lambda_2 = (12, 3)$, $\lambda_3 = (6, 7)$, and $\lambda_4 = (5, 13)$. Then clearly, $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. So, this is an extended β-chain in $\mathfrak{R}(\beta)$. If we denote the above β-chain by C, then $C^+ = \lambda_1 > \lambda_2$ and $C^- = \lambda_3 > \lambda_4$. Again, $\lambda_1 > \lambda_2 > \lambda_3$ is an extended upper β-chain.*

Definition 7.4.2. *Let $A \subset \bar{\beta}$ (where $\bar{\beta}$, the complement of β, which is defined in Chapter 2) and $B \subset \beta$. We define $A - B$ as the set $A \cup (\beta \setminus B)$.*

Definition 7.4.3. *Let C be an extended upper β-chain. Let $(P^C, Q^C) = BRSK(C \cup C^\#)$. Let (P_1^C, Q_1^C) denote the **topmost row** of (P^C, Q^C) and (P_r^C, Q_r^C) denote the **bottom-most row** of (P^C, Q^C). Let $\operatorname{top}(C^+)$ denote the element $P_r^C - Q_r^C$ of $I(d, 2d)$ and $\operatorname{bot}(C^-)$ denote the element $P_1^C - Q_1^C$ of $I(d, 2d)$, where $P_r^C - Q_r^C$ and $P_1^C - Q_1^C$ are having the meaning as given in Definition 7.4.2.*

Notation (A): Let $u = (u_1, \ldots, u_d)$ be an element of $I(d, 2d)$. Define $u^\star := (u_d^\star, \ldots, u_1^\star)$ and $u^\# := [2d] \setminus u^\star$.

It is clear that, $u = u^{\#}$ if and only if $u \in I(d)$.

Before going to the Theorem 7.4.5 below, we will state [GR06, Proposition 5.6], which is going to be used in the proof of Theorem 7.4.5. [GR06, Proposition 5.6] is stated below as Proposition 7.4.4.

Proposition 7.4.4. *The map π respects $\#$ (the map π is defined in §4.2). More precisely, if $\pi(\mathfrak{S}) = (w, \mathfrak{S}^{(1)})$, then $\pi(\mathfrak{S}^{\#}) = (w^{\#}, \mathfrak{S}^{(1)\#})$ (for a monomial $\mathfrak{S}$, $\mathfrak{S}^{\#}$ is mentioned in Definition 5.3.5).*

Theorem 7.4.5. *Given any extended upper β-chain C, the elements $\mathrm{top}(C^+)$ and $\mathrm{bot}(C^-)$ of $I(d, 2d)$ in fact belong to $I(d)$.*

Proof. Let $(C \cup C^{\#})^+$ and $(C \cup C^{\#})^-$ denote respectively the positive and negative parts of the multiset $C \cup C^{\#}$. We know that, $BRSK(C \cup C^{\#})$ is equal to the notched bitableau obtained by placing the notched bitableau $BRSK((C \cup C^{\#})^-)$ on top of the notched bitableau $BRSK((C \cup C^{\#})^+)$.

Recall the map $\tilde{\pi}$ from §4.2 of Chapter 4. We know from Corollary 5.2.20 of Chapter 5, that $BRSK((C \cup C^{\#})^+) = \tilde{\pi}((C \cup C^{\#})^+)$. Also, $(C \cup C^{\#})^+ = ((C \cup C^{\#})^+)^{\#}$. It hence follows from Proposition 7.4.4 and the above Notation (A), that all the elements of $I(d, 2d)$ corresponding to all the rows of $BRSK((C \cup C^{\#})^+)$ in fact belong to $I(d)$. In particular, the element $P_r^C - Q_r^C = \mathrm{top}(C^+)$ also belongs to $I(d)$. The proof of the fact that $\mathrm{bot}(C^-)$ belongs to $I(d)$ is similar (we omit the proof here because it involves proving that the maps $BRSK$ and $\tilde{\pi}$ are equal on negative multisets, and this proof is similar to that in Chapter 5). □

The example below illustrates Theorem 7.4.5.

Example 7.4.6. *Let $d = 7$ and $\beta = (1, 3, 4, 7, 9, 10, 13)$. Clearly, $\beta \in I(d)$. Now, $\bar{\beta} = (2, 5, 6, 8, 11, 12, 14)$. Consider the upper extended β-chain*

$$C = \{(12, 1), (11, 3), (8, 4), (6, 7), (5, 9), (2, 10)\}.$$

According to Chapter 5,

$$BRSK(C \cup C^{\#}) = \left(\begin{array}{|c|c|c|} \hline 2 & 5 & 6 \\ \hline 5 & 6 & 8 \\ \hline 8 & 11 & 12 \\ \hline 11 & 12 & 14 \\ \hline \end{array} \;,\; \begin{array}{|c|c|c|} \hline 9 & 10 & 13 \\ \hline 7 & 9 & 10 \\ \hline 3 & 4 & 7 \\ \hline 1 & 3 & 4 \\ \hline \end{array} \right). \tag{7.4.0.1}$$

Now, $\mathrm{top}(C^+) = P_r^C - Q_r^C = \{11, 12, 14\} \cup (\beta \smallsetminus \{1, 3, 4\}) = (7, 9, 10, 11, 12, 13, 14)$, *and* $\mathrm{bot}(C^-) = P_1^C - Q_1^C = \{2, 5, 6\} \cup (\beta \smallsetminus \{9, 10, 13\}) = (1, 2, 3, 4, 5, 6, 7)$. *Clearly both of* $\mathrm{top}(C^+)$ *and* $\mathrm{bot}(C^-)$ *belong to* $I(d)$.

7.5 Gröbner basis for ideals of tangent cones

We now specify the **term order** $\rhd$ on monomials in the coordinate functions $\{X_{(r,c)} \mid (r,c) \in \mathfrak{OR}(\beta)\}$ with respect to which the initial ideal of the ideal $I^{\gamma}_{\alpha,\beta}$ of the tangent cone is to be taken.

Definition 7.5.1. *Let $>$ be the total order on $\mathfrak{OR}(\beta)$ satisfying the following condition:*

- $X_{(r,c)} > X_{(r',c')}$ *if either (a) $r > r'$ or (b) $r = r'$ and $c < c'$.*

Let $\rhd$ be the term order on monomials in $\mathfrak{OR}(\beta)$ given by the homogeneous lexicographic order with respect to $>$.

Example 7.5.2 below gives an illustration of the term order $\rhd$.

Example 7.5.2. *Let $d = 7$ and $\beta = (1, 3, 4, 7, 9, 10, 13)$. Now, all of $(14, 1), (12, 3), (11, 3), (11, 1)$ are elements in $\mathfrak{OR}(\beta)$, and according to the above term order we have, $X_{(14,1)} > X_{(12,3)} > X_{(11,1)} > X_{(11,3)}$. Let $X_{\mathfrak{S}_1} = X^3_{(14,1)}X_{(11,1)}X^2_{(11,3)}$, $X_{\mathfrak{S}_2} = X_{(14,1)}X_{(12,3)}X_{(11,3)}$, and $X_{\mathfrak{S}_3} = X_{(14,1)}X^2_{(11,1)}$. Clearly, $\mathfrak{S}_1$, $\mathfrak{S}_2$, and $\mathfrak{S}_3$ all are monomials in $\mathfrak{OR}(\beta)$. Now, the degree of the polynomial $X_{\mathfrak{S}_1}$ is greater than that of $X_{\mathfrak{S}_2}$ and $X_{\mathfrak{S}_3}$. So, $X_{\mathfrak{S}_1} \rhd X_{\mathfrak{S}_2}$ and $X_{\mathfrak{S}_1} \rhd X_{\mathfrak{S}_3}$. Though the degree of $X_{\mathfrak{S}_2}$ is equal to the degree of $X_{\mathfrak{S}_3}$, but $X_{(12,3)} > X_{(11,1)}$ and in $X_{\mathfrak{S}_2}$, the degree of $X_{(12,3)}$ is one and in $X_{\mathfrak{S}_3}$, the degree of $X_{(12,3)}$ is zero. So, according to the definition of homogeneous lexicographic order, we have $X_{\mathfrak{S}_2} \rhd X_{\mathfrak{S}_3}$. Hence, $X_{\mathfrak{S}_1} \rhd X_{\mathfrak{S}_2} \rhd X_{\mathfrak{S}_3}$.*

Now, recall that the ideal of the tangent cone to X^{γ}_{α} at e^{β} is the ideal $I^{\gamma}_{\alpha,\beta}$ given by (7.3.0.3). Let $\rhd$ be as in §7.5. For any element $f \in I^{\gamma}_{\alpha,\beta}$, let $in_{\rhd}f$ denote the initial term of f with respect to the term order $\rhd$. We define $in_{\rhd}I^{\gamma}_{\alpha,\beta}$ to be the ideal $\langle in_{\rhd}f \mid f \in I^{\gamma}_{\alpha,\beta}\rangle$ inside the polynomial ring $P := K[X_{(r,c)} \mid (r,c) \in \mathfrak{OR}(\beta)]$.

Definition 7.5.3. *An admissible pair $\mathfrak{w} = (t, u)$ (where $t \geq u$) is called a **good admissible pair** if it satisfies both of the following 2 properties:*

1. $\alpha \nleq u$ *or* $t \nleq \gamma$.
2. *Either $in_{\rhd}f_{\mathfrak{w},\beta}$ forms a positive upper extended β-chain C^+ such that $C^+_{(1)} - C^+_{(2)} \nleq \gamma$ or $in_{\rhd}f_{\mathfrak{w},\beta}$ forms a negative upper extended β-chain C^- such that $C^-_{(1)} - C^-_{(2)} \ngeq \alpha$.*

Notation: Let $\mathcal{G}^{\gamma}_{\alpha,\beta}$ denote the set $\{f_{\mathfrak{w},\beta} \mid \mathfrak{w}$ is good$\}$.

Example 7.5.4 below illustrates a good admissible pair.

Example 7.5.4. *Let $d = 4$, $\alpha = (1, 2, 3, 5)$, $\beta = (1, 2, 5, 6)$, and $\gamma = (2, 3, 5, 8)$. Let $\mathfrak{w} = (t, u)$ be an admissible pair, where $t = (3, 4, 7, 8)$ and $u = (1, 2, 5, 6)$. Clearly, $t \nleq \gamma$. Now, in § 7.3, we have already defined that $\theta = (t \cap [d]) \cup (u \cap [d]^c)$, and $f_{\mathfrak{w},\beta} = f_{\theta,\beta}$.*

Hence, in this example $\theta = (3,4,5,6)$ *and from the matrix which is given in (7.3.0.1), we have*

$$f_{\mathfrak{w},\beta} = \begin{vmatrix} x_{31} & x_{32} \\ x_{41} & x_{42} \end{vmatrix} \tag{7.5.0.1}$$

Observe that $in_{\rhd} f_{\mathfrak{w},\beta} = -x_{41}x_{32}$. *Clearly,* $in_{\rhd} f_{\mathfrak{w},\beta}$ *forms a positive upper extended* β*-chain* C^{+} *such that* $C^{+}_{(1)} - C^{+}_{(2)} = \{3,4\} \cup (\beta \smallsetminus \{1,2\}) = (3,4,5,6) \not\leq \gamma$. *Hence,* $\mathfrak{w} = (t,u)$ *is a good admissible pair.*

Definition 7.5.5. *If* S *is any nonempty subset of the polynomial ring* $P := K[X_{(r,c)} \mid (r,c) \in \mathfrak{O}\mathfrak{R}(\beta)]$ *such that* $S \neq \{0\}$. *We define* $in_{\rhd} S$ *to be the ideal* $\langle in_{\rhd}(s) \mid s \in S\rangle$.

The main result of this chapter is the following:

Theorem 7.5.6. *The set* $\mathcal{G}^{\gamma}_{\alpha,\beta}$ *is a Gröbner basis for the ideal* $I^{\gamma}_{\alpha,\beta}$.

7.5.1 Strategy of the proof

To explain the strategy of the proof of Theorem 7.5.6, we need the following definition.

Definition 7.5.7. *We call* $f = f_{\mathfrak{w}_1,\beta} \cdots f_{\mathfrak{w}_r,\beta} \in P = K[X_{(r,c)} \mid (r,c) \in \mathfrak{O}\mathfrak{R}(\beta)]$ *a* ***standard monomial*** *if*

$$\mathfrak{w}_1 \leq \cdots \leq \mathfrak{w}_r, \tag{7.5.1.1}$$

and for each $i \in \{1,\ldots,r\}$, *we have*

$$\textit{Either } \beta \geq \mathrm{top}(\mathfrak{w}_i) \textit{ or } \mathrm{top}(\mathfrak{w}_i) \geq \beta, \tag{7.5.1.2}$$

$$\textit{and either } \mathrm{bot}(\mathfrak{w}_i) \geq \beta \textit{ or } \beta \geq \mathrm{bot}(\mathfrak{w}_{\mathrm{i}}) \tag{7.5.1.3}$$

$$\textit{and } \mathfrak{w}_i \neq (\beta,\beta). \tag{7.5.1.4}$$

If in addition, for $\alpha, \gamma \in I(d)$, *we have*

$$\alpha \leq \mathrm{bot}(\mathfrak{w}_1) \textit{ and } \mathrm{top}(\mathfrak{w}_r) \leq \gamma, \tag{7.5.1.5}$$

then we say that f *is* ***standard on*** $Y^{\gamma}_{\alpha}(\beta)$.

Example 7.5.8 below gives an illustration of a standard monomial on $Y^{\gamma}_{\alpha}(\beta)$.

Example 7.5.8. *Let* $d = 4$, $\alpha = (1,2,3,5)$, $\beta = (1,2,5,6)$, *and* $\gamma = (3,4,7,8)$. *For this* β, *the* 8×4 *matrix is given by (7.3.0.1). Let* $\mathfrak{w}_1 = ((1,2,4,6),(1,2,3,5))$ *and* $\mathfrak{w}_2 = ((2,4,6,8),(2,3,5,8))$. *Clearly,* $\mathfrak{w}_1$ *and* $\mathfrak{w}_2$ *both are admissible pairs. Let* θ_1 *and* θ_2 *be the images of* $\mathfrak{w}_1$ *and* $\mathfrak{w}_2$ *respectively, under the correspondence given by* $\mathfrak{w} = (x,y) \mapsto \theta = (x \cap [d]) \cup (y \cap [d]^c)$ *as mentioned in [GR06, Proposition 3.4]. So, we have,* $\theta_1 = (1,2,4,5)$ *and* $\theta_2 = (2,4,5,8)$. *Now, using the* 8×4 *matrix of (7.3.0.1) we have,*

$$f_{\mathfrak{w}_1,\beta} = f_{\theta_1,\beta} = x_{35} \in P$$
$$\textit{and } f_{\mathfrak{w}_2,\beta} = f_{\theta_2,\beta} = -x_{41}x_{31} - x_{35}x_{81} \in P.$$

Hence, $f = f_{\mathfrak{w}_1,\beta} f_{\mathfrak{w}_2,\beta} \in P$*. Clearly,* $\mathfrak{w}_1 \leq \mathfrak{w}_2$*. Again,* $\beta \geq \mathrm{top}(\mathfrak{w}_1)$ *and* $\beta \geq \mathrm{bot}(\mathfrak{w}_1)$*. Also,* $\beta \leq \mathrm{top}(\mathfrak{w}_2)$ *and* $\beta \leq \mathrm{bot}(\mathfrak{w}_2)$*, and* $\mathfrak{w}_i \neq (\beta, \beta)$ *for all* $i \in \{1, 2\}$*. Again,* α*,* $\gamma \in I(d)$ *are such that* $\alpha \leq \mathrm{bot}(\mathfrak{w}_1)$ *and* $\gamma \geq \mathrm{top}(\mathfrak{w}_2)$*. Hence,* f *is standard on* $Y_\alpha^\gamma(\beta)$*.*

Definition 7.5.9. *Let* $f = f_{\mathfrak{w}_1,\beta} \cdots f_{\mathfrak{w}_r,\beta}$ *be a standard monomial on* $Y_\alpha^\gamma(\beta)$*. We define the degree of* f *to be the sum of the* β*-degrees of* $\mathfrak{w}_1, \ldots, \mathfrak{w}_r$*, where given any admissible pair* $\mathfrak{w} = (x, y)$*, the* β*-degree of* $\mathfrak{w}$ *is defined to be* $\frac{1}{2}(|x \setminus \beta| + |y \setminus \beta|)$*.*

We now briefly sketch the proof of Theorem 7.5.6 (the details can be found in §7.7). Clearly, $\mathcal{G}_{\alpha,\beta}^\gamma$ is contained in the ideal $I_{\alpha,\beta}^\gamma$. So, $in_\triangleright \mathcal{G}_{\alpha,\beta}^\gamma \subseteq in_\triangleright I_{\alpha,\beta}^\gamma$. Hence, to prove Theorem 7.5.6, we only need to show that in any degree, the number of monomials of $in_\triangleright \mathcal{G}_{\alpha,\beta}^\gamma$ is at least as great as the number of monomials of $in_\triangleright I_{\alpha,\beta}^\gamma$ (the other inequality being trivial). Equivalently, we need to prove that in any degree, the number of monomials of $P \setminus in_\triangleright \mathcal{G}_{\alpha,\beta}^\gamma$ is no greater than the number of monomials of $P \setminus in_\triangleright I_{\alpha,\beta}^\gamma$. Both the monomials of $P \setminus in_\triangleright I_{\alpha,\beta}^\gamma$ and the standard monomials on $Y_\alpha^\gamma(\beta)$ (the definition of a standard monomial on $Y_\alpha^\gamma(\beta)$ is given in Definition 7.5.7) form a basis for $P/I_{\alpha,\beta}^\gamma$, and thus agree in cardinality in any degree. Therefore it suffices to prove that, in any degree, the number of monomials of $P \setminus in_\triangleright \mathcal{G}_{\alpha,\beta}^\gamma$ is less than or equal to the number of standard monomials on $Y_\alpha^\gamma(\beta)$. In this chapter, we consider two sets, namely, the set of all "nonvanishing special multisets on $\bar{\beta} \times \beta$ (bounded by T_α, W_γ)", and the set of all "nonvanishing semistandard notched bitableaux on $(\bar{\beta} \times \beta)^*$ (bounded by T_α, W_γ)". The meaning attached to these two sets is given in §7.6 below. In §7.7 below, we will first show that there exists a degree doubling injection from the set of all monomials of $P \setminus in_\triangleright \mathcal{G}_{\alpha,\beta}^\gamma$ to the former set. Then we will show that, there exists a degree-halving injection from the later set (namely, the set of all "nonvanishing semistandard notched bitableaux on $(\bar{\beta} \times \beta)^*$ (bounded by T_α, W_γ)") to the set of all standard monomials on $Y_\alpha^\gamma(\beta)$. And then we will prove that the map $BRSK$ of Chapter 5 is a degree preserving bijection from the former set to the later. This will complete the proof.

Example 7.5.10 below gives an illustration of a Gröbner basis.

Example 7.5.10. *Let* $d = 4$*,* $\alpha = (1, 2, 3, 5)$*,* $\beta = (1, 2, 5, 6)$*, and* $\gamma = (2, 3, 5, 8)$*. Then from Example 7.5.4, we know that,* $\mathfrak{w} = (t, u)$ *is a good admissible pair, where* $t = (3, 4, 7, 8)$ *and* $u = (1, 2, 5, 6)$*. For the above* α, β, γ *if we consider all the admissible pairs which satisfying both the conditions of good admissible pair, then we will get the set* $\mathcal{G}_{\alpha,\beta}^\gamma$ *which is given by* $\{f_{\mathfrak{w},\beta} \mid \mathfrak{w} \in G\}$*, where* G *is the following set (of all good admissible pairs):*
$G = \{((1, 2, 3, 4), (1, 2, 3, 4)), ((1, 4, 6, 7), (1, 2, 5, 6)), ((2, 4, 6, 8), (1, 2, 5, 6)), ((3, 4, 7, 8), (1, 2, 5, 6)), ((1, 5, 6, 7), (1, 5, 6, 7)), ((2, 5, 6, 8), (1, 5, 6, 7)), ((3, 5, 7, 8), (1, 5, 6, 7)), ((4, 6, 7, 8), (1, 5, 6, 7)), ((2, 5, 6, 8), (2, 5, 6, 8)), ((3, 5, 7, 8), (2, 5, 6, 8)), ((4, 6, 7, 8), (2, 5, 6, 8)), ((5, 6, 7$

$,8),(5,6,7,8))\}$. *As in Example 7.5.4, we can easily find the initial term of the above good admissible pairs. In this case, we have*

$$in_{\triangleright}\mathcal{G}^{\gamma}_{\alpha,\beta} = \langle\{x_{71}x_{32}, x_{71}, x_{72}, x_{41}x_{32}, x_{41}, x_{42}, x_{45}x_{36}, x_{81}x_{72}, x_{81}x_{42}, x_{81}x_{32}, x_{81}, x_{71}x_{42}\}\rangle.$$

7.6 The two sets

As mentioned towards the end of the previous subsection, the two sets under consideration are "nonvanishing special multisets on $\bar{\beta}\times\beta$ (bounded by T_α, W_γ)" and "nonvanishing semistandard notched bitableaux on $(\bar{\beta}\times\beta)^*$ (bounded by T_α, W_γ)". We will now explain the meaning of these two sets.

Let α, β, γ be as before (Chapter 2). Let I_β be the set of all pairs (R,S) such that $R \subset \bar{\beta}$, $S \subset \beta$, and $|R|=|S|$. Let I^*_β be the subset of I_β consisting of all pairs (R,S) such that $R = S^*$. Clearly then, the map $(R,S) \mapsto R-S$ is a bijection from I^*_β to $I(d)$ (indeed, the inverse map is given by $\theta \mapsto (\theta\setminus\beta, \beta\setminus\theta)$). Let (R_α, S_α) and (R_γ, S_γ) be the preimages of α and γ respectively under the bijection from I^*_β to $I(d)$. Define T_α and W_γ to be any subsets of $\bar{\beta}\times\beta$ such that $(T_\alpha)_{(1)} = R_\alpha, (T_\alpha)_{(2)} = S_\alpha, (W_\gamma)_{(1)} = R_\gamma$, and $(W_\gamma)_{(2)} = S_\gamma$. Note that there always exist subsets T_α and W_γ of $\bar{\beta}\times\beta$ such that T_α is negative and W_γ is positive [this is because $\beta \leq \gamma$, apply Proposition 4.1.13 to γ (which is $\geq \beta$) to get a distinguished monomial corresponding to γ. This distinguished monomial will serve as a positive subset W_γ of $\bar{\beta}\times\beta$. Similarly, we can get a negative subset T_α of $\bar{\beta}\times\beta$ corresponding to α (which is $\leq \beta$)]. Hence, we can choose T_α and W_γ in such a way that the former is negative and the later is positive.

Example 7.6.1 below gives an illustration of the above paragraph.

Example 7.6.1. *Let $d = 7$ and $\beta = (1,3,4,7,9,10,13)$. So, $\bar{\beta} = (2,5,6,8,11,12,14)$. Let $\alpha = (1,2,3,5,7,9,11)$ and $\gamma = (4,5,6,7,12,13,14)$. So, $\alpha \leq \beta \leq \gamma$. Now, I_β is the set of all pairs (R,S) such that $R \subset (2,5,6,8,11.12.14)$, $S \subset (1,3,4,7,9,10,13)$, and $|R|=|S|$. Let $R = (2,6,8,11)$ and $S = (4,7,9,13)$. Clearly, $|R|=|S|$ and $R = S^\star$. So, according to the definition of $I^\star_\beta$, $(R,S) \in I^\star_\beta$. Now, $R-S = (1,2,3,6,8,10,11)$ is in $I(d)$. Again, both of*

$$(R_\alpha, S_\alpha) = ((2,5,11),(4,10,13)) \text{ and } (R_\gamma, S_\gamma) = ((5,6,12,14),(1,3,9,10))$$

are in $I^\star_\beta$. Let $T_\alpha = \{(2,4),(5,10),(11,13)\}$ and $W_\gamma = \{(5,1),(6,3),(12,9),(14,10)\}$. Then clearly, T_α is a negative and W_γ is a positive subset of $\bar{\beta}\times\beta$.

7.6.1 The first set

A nonvanishing multiset on $\bar{\beta} \times \beta$, bounded by T_α, W_γ has the same meaning as in §3.1. Such a multiset $\mathfrak{S}$ is called a **nonvanishing special multiset on** $\bar{\beta} \times \beta$ **(bounded by** T_α, W_γ**)** if moreover, the following two properties are satisfied:

1. $\mathfrak{S} = \mathfrak{S}^{\#}$.

2. the multiplicity of any diagonal element in $\mathfrak{S}$ is even.

Example 7.6.2 below gives an illustration of the first set.

Example 7.6.2. *Let $d = 7$ and $\beta = (1, 3, 4, 7, 9, 10, 13)$. Let $\alpha = (1, 2, 3, 5, 7, 9, 11)$ and $\gamma = (4, 5, 6, 7, 12, 13, 14)$. Let $\mathfrak{S} = \{(2, 3), (12, 13), (5, 10), (5, 10)\}$. Clearly, $\mathfrak{S} = \mathfrak{S}^{\#}$ and the multiplicity of any diagonal element in $\mathfrak{S}$ is even. The only β-chains in $\mathfrak{S}$ are*

$$C_1 = \{(2, 3)\},\ C_2 = \{(12, 13)\},\ \text{and } C_3 = \{(5, 10)\}.$$

Let us take $T_\alpha = \{(2, 4), (5, 10), (11, 13)\}$ and $W_\gamma = \{(5, 1), (6, 3), (12, 9), (14, 10)\}$. Clearly, $(T_\alpha)_1 - (T_\alpha)_2 = \alpha$ and $(W_\gamma)_1 - (W_\gamma)_2 = \gamma$. We have to check that $T_\alpha \le C_i \le W_\gamma$ for all $i \in \{1, 2, 3\}$.
Now,

$$\{2\} - \{3\} = \{2\} \cup (\beta \setminus \{3\}) = (1, 2, 4, 7, 9, 10, 13).$$

So, $T_\alpha \le C_1 \le W_\gamma$. Similarly, one can check that $T_\alpha \le C_2 \le W_\gamma$ and $T_\alpha \le C_3 \le W_\gamma$. Hence, $\mathfrak{S}$ is a special multiset.

7.6.2 The second set

A nonvanishing semistandard notched bitableau on $\bar{\beta} \times \beta$ bounded by T_α, W_γ has the same meaning as in §3.3.3. Such a notched bitableau (P, Q) is said to be a **nonvanishing semistandard notched bitableau on** $(\bar{\beta} \times \beta)^*$ **(bounded by** T_α, W_γ**)** if moreover, the following 5 conditions are satisfied:

1. $P_i = Q_i^*$ for every row number i of (P, Q).

2. (P, Q) doesn't contain any empty rows.

3. The total number of rows in P (or Q) is either even, or it is odd but

$$P_1 - Q_1 \le \cdots \le P_n - Q_n \le \beta \le P_{n+1} - Q_{n+1} \le \cdots \le P_{n+p} - Q_{n+p},$$

where $n + p$ is the total number of rows in P (or Q), and (P_i, Q_i) (for $1 \le i \le n$) is the negative part of (P, Q), and (P_{n+i}, Q_{n+i}) (for $1 \le i \le p$) is the positive part of

(P, Q).
Notation: Let us denote by $\delta_1 \leq \cdots \leq \delta_{n+p+1}$ the sequence $P_1 - Q_1 \leq \cdots \leq P_n - Q_n \leq \beta \leq P_{n+1} - Q_{n+1} \leq \cdots \leq P_{n+p} - Q_{n+p}$, where $n + p$ is odd.

4. Either the total number of rows of P (or Q) is even, and the ϵ-degrees (where $\epsilon = (1, 2, \ldots, d) \in I(d)$) of $P_j - Q_j$ and $P_{j+1} - Q_{j+1}$ are equal for each j odd, or the total number of rows in P (or Q) is odd (say, $n + p$), and the ϵ-degrees of δ_j and δ_{j+1} are equal for each j odd, where the δ_j's are as mentioned in item (3) above.

5. The total number of boxes in P (or Q) is even.

Example 7.6.3 below gives an illustration of the second set.

Example 7.6.3. *Let $d = 7$ and $\beta = (1, 3, 4, 7, 9, 10, 13)$. Let*

$$(P, Q) = \left(\begin{array}{|c|c|} \hline 2 & 11 \\ \hline 5 & 12 \\ \hline 6 & 14 \\ \hline \end{array} \,,\; \begin{array}{|c|c|} \hline 4 & 13 \\ \hline 3 & 10 \\ \hline 1 & 9 \\ \hline \end{array} \right).$$

Clearly, (P, Q) is a notched bitableau on $\bar{\beta} \times \beta$. Let

$$\alpha = (1, 2, 3, 5, 7, 9, 11) \text{ and } \gamma = (4, 5, 6, 7, 12, 13, 14).$$

Let us take

$$T_\alpha = \{(2, 4), (5, 10), (11, 13)\}$$

$$\text{and } W_\gamma = \{(5, 1), (6, 3), (12, 9), (14, 10)\}.$$

Observe that

$$P_1 - Q_1 = (1, 2, 3, 7, 9, 10, 11),$$

$$P_2 - Q_2 = (1, 4, 5, 7, 9, 12, 13),$$

$$\text{and } P_3 - Q_3 = (3, 4, 6, 7, 10, 13, 14).$$

Since $P_1 \lessdot Q_1$, $P_2 \gtrdot Q_2$, and $P_3 \gtrdot Q_3$, we have (P, Q) is nonvanishing. Also, $P_1 - Q_1 \leq P_2 - Q_2 \leq P_3 - Q_3$. Hence, (P, Q) is semistandard. Again,

$$(T_\alpha)_{(1)} - (T_\alpha)_{(2)} = \alpha \leq P_1 - Q_1 \text{ and}$$

$$P_3 - Q_3 \leq \gamma = (W_\gamma)_{(1)} - (W_\gamma)_{(2)}.$$

So, (P, Q) is bounded by T_α and W_γ. Observe now that,

1. *$P_i = Q_i^*$ for all $i \in \{1, 2, 3\}$.*

2. *(P, Q) does not contain any empty rows.*

3. *The total number of rows in P (or Q) is 3, which is odd, but*

$$P_1 - Q_1 \leq \beta \leq P_2 - Q_2 \leq P_3 - Q_3.$$

4. *The ϵ-degrees of $P_1 - Q_1$ and β are the same (both are 3). Also, the ϵ-degrees of $P_2 - Q_2$ and $P_3 - Q_3$ are the same (both are 3).*

5. *The total number of boxes in P (or Q) is 6, which is even.*

Hence, (P, Q) is on $(\bar{\beta} \times \beta)^*$.

7.7 The proof

The main result (Theorem 7.5.6) of this chapter will be obtained as a consequence of Theorem 7.7.6, Theorem 7.7.9, and Theorem 7.7.13. For this subsection, we fix α, β, and γ in $I(d)$ such that $\alpha \leq \beta \leq \gamma$. Also, we will recall the definitions of $SM^{v,v}$ and special monomial from Chapter 5, which we will state first.

Definition 7.7.1. *An ordered sequence $(\mathfrak{w}_1, \ldots, \mathfrak{w}_t)$ of admissible pairs is called a* ***standard sequence of admissible pairs*** *if $\mathfrak{w}_i \geq \mathfrak{w}_{i+1}$ for $1 \leq i < t$. We often write $\mathfrak{w}_1 \geq \cdots \geq \mathfrak{w}_t$ to denote the standard sequence $(\mathfrak{w}_1, \ldots, \mathfrak{w}_t)$ of admissible pairs. Given any $v \in I(d)$, we say that a standard sequence $\mathfrak{w}_1 \geq \cdots \geq \mathfrak{w}_t$ of admissible pairs is v-**compatible** if for each $\mathfrak{w}_i$, either $v \geq \mathrm{top}(\mathfrak{w}_i)$ or $\mathrm{bot}(\mathfrak{w}_i) \geq v$, and $\mathfrak{w}_i \neq (v, v)$. A standard sequence $\mathfrak{w}_1 \geq \cdots \geq \mathfrak{w}_t$ of admissible pairs is called* ***anti-dominated*** *by v if $\mathrm{bot}(\mathfrak{w}_t) \geq v$. Let $SM^{v,v}$ denote the set of all v-compatible standard sequences of admissible pairs that are anti-dominated by v.*

Example 7.7.2 below gives an illustration of $SM^{v,v}$.

Example 7.7.2. *Let $d = 4$ and $v = (1, 2, 3, 5)$. Let $\mathfrak{w}_1 = ((2, 4, 6, 8), (2, 3, 5, 8))$ and $\mathfrak{w}_2 = ((1, 2, 4, 6), (1, 2, 3, 5))$. Then $\mathfrak{w}_1 \geq \mathfrak{w}_2$ is an element of $SM^{v,v}$.*

Definition 7.7.3. *A monomial $\mathfrak{S}$ of T^β (which is mentioned in §5.3.1) is* ***special*** *if*

1. *$\mathfrak{S} = \mathfrak{S}^\#$ and*

2. *the multiplicity of any diagonal element in $\mathfrak{S}$ is even.*

Example 7.7.4 below gives an illustration of the special monomial.

Example 7.7.4. *Let $d = 4$ and $\beta = (1, 2, 5, 6)$. Then the monomial*

$$\mathfrak{S} = \{(8, 1), (8, 1), (7, 1), (8, 2)\}$$

is a special monomial of T^β (the notation of T^β is mentioned in §5.3.1).

Now, we recall [GR06, Proposition 4.1], which has been used in the proof of the Theorem 7.7.6. [GR06, Proposition 4.1] is stated below as Proposition 7.7.5.

Proposition 7.7.5. *There is a bijection between $SM^{\beta,\beta}$ and T^{β} that respects domination and degree.*

Before we start the proof of the Theorem 7.7.6, let us recall the notation P and $in_{\triangleright}S$, which will be used in the proof of the Theorem 7.7.6.
Notation: If S is any nonempty subset of the polynomial ring $P := K[X_{(r,c)} \mid (r,c) \in \mathfrak{OR}(\beta)]$, such that $S \neq \{0\}$. We define $in_{\triangleright}S$ to be the ideal $\langle in_{\triangleright}(s) \mid s \in S\rangle$, where $in_{\triangleright}$ is as in Definition 7.5.1.

Theorem 7.7.6. *There exists a degree doubling injection from the set of all monomials of $P \setminus in_{\triangleright}\mathcal{G}^{\gamma}_{\alpha,\beta}$ to the set of all nonvanishing special multisets on $\bar{\beta} \times \beta$ (bounded by T_α, W_γ).*

Proof. Clearly,

$$in_{\triangleright}\mathcal{G}^{\gamma}_{\alpha,\beta} = \langle in_{\triangleright} f_{\mathfrak{w},\beta} : \mathfrak{w} \text{ is good}\rangle = \langle G^{+} \cup G^{-}\rangle,$$

where

$$G^{+} = \{x_{C^{+}} : C^{+} \text{ a positive upper extended } \beta\text{-chain such that } C^{+}_{(1)} - C^{+}_{(2)} \not\leq \gamma\},$$

$$\text{and } G^{-} = \{x_{C^{-}} : C^{-} \text{ a negative upper extended } \beta\text{-chain such that } \alpha \not\leq C^{-}_{(1)} - C^{-}_{(2)}\}.$$

Let

$$G'^{+} := \{x_{C^{+}} : C^{+} \text{ a positive upper extended } \beta\text{-chain such that } C^{+} \not\leq W_\gamma\},$$

$$\text{and } G'^{-} := \{x_{C^{-}} : C^{-} \text{ a negative upper extended } \beta\text{-chain such that } T_\alpha \not\leq C^{-}\}.$$

It is then easy to observe that $G^{+}= G'^{+}$ and $G^{-} = G'^{-}$. Therefore

$$in_{\triangleright}\mathcal{G}^{\gamma}_{\alpha,\beta} = \langle G'^{+} \cup G'^{-}\rangle.$$

The definition of a generating set for an ideal will now imply that x_U is a monomial in $in_{\triangleright}\mathcal{G}^{\gamma}_{\alpha,\beta}$ if and only if x_U is a multiple of some $x_{C^{+}}$ or some $x_{C^{-}}$, where C^{+} is a positive upper extended β-chain such that $C^{+} \not\leq W_\gamma$ and C^{-} is a negative upper extended β-chain such that $T_\alpha \not\leq C^{-}$. Therefore

$$x_U \text{ is a monomial in } P \setminus in_{\triangleright}\mathcal{G}^{\gamma}_{\alpha,\beta}$$

$\Leftrightarrow$ x_U is not divisible by any $x_{C^{+}}$ (where C^{+} is a positive upper extended β-chain such that $C^{+} \not\leq W_\gamma$) or by any $x_{C^{-}}$ (where C^{-} is a negative upper extended β-chain such that $T_\alpha \not\leq$

C^-)

$$\Rightarrow U \text{ contains no extended upper } \beta\text{-chains } C \text{ such that } T_\alpha \not\leq C^- \text{ or } C^+ \not\leq W_\gamma$$

Observe now that, as the bijection of Proposition 5.3.10 respects domination, and Corollary 5.2.20 holds true, so $C^+ \not\leq W_\gamma$ implies that $\text{top}(C^+) \not\leq \gamma$. A similar argument will show that $T_\alpha \not\leq C^-$ implies $\alpha \not\leq \text{bot}(C^-)$. So, we now have:

$$U \text{ contains no extended upper } \beta\text{-chains } C \text{ such that } T_\alpha \not\leq C^- \text{ or } C^+ \not\leq W_\gamma$$

$$\Rightarrow U \text{ contains no extended upper } \beta\text{-chains } C \text{ such that } \alpha \not\leq \text{bot}(C^-) \text{ or } \text{top}(C^+) \not\leq \gamma$$

$$\Leftrightarrow \alpha \leq \text{bot}(C^-) \text{ and } \text{top}(C^+) \leq \gamma \text{ for any extended upper } \beta\text{-chain } C \text{ in } U$$

$$\Leftrightarrow C \cup C^\# \text{ is bounded by } T_\alpha, W_\gamma \text{ for any extended upper } \beta\text{-chain } C \text{ in } U,$$

where the last $\Leftrightarrow$ follows because $\text{bot}(C^-)$ and $\text{top}(C^+)$ are the two elements of $I(d)$ (as mentioned in Definition 7.4.3) obtained by applying the map $BRSK$ to the monomial $C \cup C^\#$, and the map $BRSK$ preserves domination.

Observe now that, given any extended β-chain D in $U \cup U^\#$, we can naturally get hold of an extended upper β-chain C (in U) from it in the following way:

If $D = (r_1, c_1) > \cdots > (r_t, c_t)$ and $(r_{i_1}, c_{i_1}), \ldots, (r_{i_k}, c_{i_k})$ (where $i_1 < i_2 < \cdots < i_k$) are such that $r_{i_j} > c^*_{i_j}$ for all $1 \leq j \leq k$, then it is easy to check that the monomial formed by replacing all (r_{i_j}, c_{i_j}) $(1 \leq j \leq k)$ in D by $(c^*_{i_j}, r^*_{i_j})$ forms an extended upper β-chain in U. Call this extended upper β-chain in U as C.

Note that D is an extended β-chain in the monomial $C \cup C^\#$. So, if the monomial $C \cup C^\#$ is bounded by T_α, W_γ, then $T_\alpha \leq D \leq W_\gamma$.

Therefore

$$C \cup C^\# \text{ is bounded by } T_\alpha, W_\gamma \text{ for any extended upper } \beta\text{-chain } C \text{ in } U$$

$$\Rightarrow T_\alpha \leq D \leq W_\gamma \text{ for any extended } \beta\text{-chain } D \text{ in } U \cup U^\#$$

$$\Leftrightarrow U \cup U^\# \text{ is bounded by } T_\alpha, W_\gamma.$$

The map $U \mapsto U \cup U^\#$ from the set of all monomials of $P \setminus in_\rhd \mathcal{G}^\gamma_{\alpha,\beta}$ to the set of all nonvanishing special multisets on $\bar{\beta} \times \beta$ (bounded by T_α, W_γ) is the required degree-doubling injection. □

Example 7.7.7 below gives an illustration of Theorem 7.7.6.

Example 7.7.7. *Let $d = 4$ and α, β, γ be as given in Example 7.5.10. Here $T_\alpha = \{(3,6)\}$ and $W_\gamma = \{(3,1),(8,6)\}$. Take the monomial $U = \{(3,1),(3,2),(3,5)\}$ in $P \setminus in_\rhd \mathcal{G}^\gamma_{\alpha,\beta}$. It* *is now easy to verify that*

$$U \cup U^{\#} = \{(3,1),(3,2),(3,5),(8,6),(7,6),(4,6)\}$$

is a nonvanishing special multiset on $\bar{\beta} \times \beta$ *(bounded by* T_α*,* W_γ*).*

The following result (Proposition 7.7.8) follows easily from [BL03, Propositions 6 and 7] followed by a proof similar to the proof of [GR06, Proposition 3.9]:

Proposition 7.7.8. *The standard monomials on* $Y_\alpha^\gamma(\beta)$ *form a basis for* $K[Y_\alpha^\gamma(\beta)]$.

Theorem 7.7.9. *There exists a degree-halving injection from the set of all nonvanishing semistandard notched bitableaux on* $(\bar{\beta}\times\beta)^*$ *(bounded by* T_α, W_γ*) to the set of all standard monomials on* $Y_\alpha^\gamma(\beta)$.

Proof. Given any nonvanishing semistandard notched bitableau (P,Q) on $(\bar{\beta}\times\beta)^*$, let $P_1,\ldots,P_r$ (resp. $Q_1,\ldots,Q_r$) denote the rows of P (resp. Q) from top to bottom. If r is even (say, $r=2s$), then let us denote the sequence $P_1 - Q_1 \leq \cdots \leq P_{2s} - Q_{2s}$ by

$$\mu_1 \leq \cdots \leq \mu_{2s}.$$

If r is odd (say, $r = 2s-1$), then let us denote the sequence $P_1 - Q_1 \leq \cdots \leq P_n - Q_n \leq \beta \leq P_{n+1} - Q_{n+1} \leq \cdots \leq P_{2s-1} - Q_{2s-1}$ by

$$\mu_1 \leq \cdots \leq \mu_{2s},$$

where (P_i, Q_i) (for $1 \leq i \leq n$) is the negative part of (P,Q), and (P_j, Q_j) (for $n+1 \leq j \leq 2s-1$) is the positive part of (P,Q). We can then form the monomial

$$f = f_{(\mu_2,\mu_1),\beta} f_{(\mu_4,\mu_3),\beta} \cdots f_{(\mu_{2s},\mu_{2s-1}),\beta},$$

which belongs to $K[X_{(r,c)} \mid (r,c) \in \mathfrak{OR}(\beta)]$.

The notched bitableau (P,Q) is nonvanishing which implies that, any μ_i $(1 \leq i \leq 2s)$ which is not equal to β is such that either $\mu_i < \beta$ or $\mu_i > \beta$. And this in turn implies that (7.5.1.2), (7.5.1.3), and (7.5.1.4) are satisfied by $f_{(\mu_2,\mu_1),\beta} \cdots f_{(\mu_{2s},\mu_{2s-1}),\beta}$. The notched bitableau (P,Q) is semistandard $\Rightarrow$ $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{2s-1} \leq \mu_{2s}$. Further, (P,Q) is a notched bitableau on $(\bar{\beta}\times\beta)^*$ which implies that the pairs $(\mu_2,\mu_1),\ldots,(\mu_{2s},\mu_{2s-1})$ are all admissible pairs. These two facts together imply that (7.5.1.1) is satisfied by $f_{(\mu_2,\mu_1),\beta} \cdots f_{(\mu_{2s},\mu_{2s-1}),\beta}$. If in addition, (P,Q) is bounded by T_α, W_γ, then it is implied that (7.5.1.5) is satisfied by $f_{(\mu_2,\mu_1),\beta} \cdots f_{(\mu_{2s},\mu_{2s-1}),\beta}$. It is now easy to verify that (P,Q) is a nonvanishing, semistandard, notched bitableau on $(\bar{\beta}\times\beta)^*$ (bounded by T_α, W_γ) $\Rightarrow$ $f_{(\mu_2,\mu_1),\beta} f_{(\mu_4,\mu_3),\beta} \cdots f_{(\mu_{2s},\mu_{2s-1}),\beta}$ is standard on $Y_\alpha^\gamma(\beta)$. Moreover, the degree of (P,Q)

equals the total number of boxes in P (or Q). The total number of boxes in P clearly equals $\Sigma_{i=1}^{r}|(P_i - Q_i) \setminus \beta|$, which in turn equals $\Sigma_{l=1}^{2s}|\mu_l \setminus \beta|$, which in turn equals twice the degree of $f_{(\mu_2,\mu_1),\beta} \cdots f_{(\mu_{2s},\mu_{2s-1}),\beta}$.

The map $(P,Q) \mapsto f_{(\mu_2,\mu_1),\beta} \cdots f_{(\mu_{2s},\mu_{2s-1}),\beta}$ is the required degree-halving injection. $\square$

Example 7.7.10 below illustrates Theorem 7.7.9.

Example 7.7.10. *Let $d = 4$ and α, β, γ be as given in Example 7.5.10. Here $T_\alpha = \{(3,6)\}$ and $W_\gamma = \{(3,1),(8,6)\}$. Let*

$$(P,Q) = \left(\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \hline 3 & 7 \\ \hline 3 & 8 \\ \hline \end{array} \; , \; \begin{array}{|c|c|} \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 2 & 6 \\ \hline 1 & 6 \\ \hline \end{array} \right).$$

Now, $(T_\alpha)_{(1)} - (T_\alpha)_{(2)} = (1,2,3,5)$, $(W_\gamma)_{(1)} - (W_\gamma)_{(2)} = (2,3,5,8)$, $P_1 - Q_1 = (1,2,3,5)$, $P_2 - Q_2 = (1,2,4,6)$, $P_3 - Q_3 = (1,3,5,7)$, and $P_4 - Q_4 = (2,3,5,8)$.
Since $P_i \lessdot Q_i$ for $i = 1,2$ and $P_i \gtrdot Q_i$ for $i = 3,4$, so (P,Q) is nonvanishing. Again, $P_1 - Q_1 \leq P_2 - Q_2 \leq P_3 - Q_3 \leq P_4 - Q_4$, so (P,Q) is semistandard. Also, $(T_\alpha)_{(1)} - (T_\alpha)_{(2)} \leq P_1 - Q_1$ and $P_4 - Q_4 \leq (W_\gamma)_{(1)} - (W_\gamma)_{(2)}$, so (P,Q) is bounded by T_α and W_γ.

Observe now that, (P,Q) is on $(\bar{\beta} \times \beta)^\star$ because:

1. *$P_i = Q_i^\star$ for every row number i.*

2. *(P,Q) does not contain any empty rows.*

3. *The total number of rows in P (or Q) is 4, which is even.*

4. *The ϵ-degrees of both $P_1 - Q_1$ and $P_2 - Q_2$ is 1. Also, the ϵ-degrees of both $P_3 - Q_3$ and $P_4 - Q_4$ is 2. That is, the ϵ-degrees of $P_i - Q_i$ and $P_{i+1} - Q_{i+1}$ are equal for every i odd.*

5. *The total number of box in P (or Q) is 6 that is even.*

So, (P,Q) is on $(\bar{\beta} \times \beta)^\star$. Hence, (P,Q) is a nonvanishing semistandard notched bitableau on $(\bar{\beta} \times \beta)^*$ (bounded by T_α, W_γ).
Here $\mu_1 = (1,2,3,5), \mu_2 = (1,2,4,6), \mu_3 = (1,3,5,7), \mu_4 = (2,3,5,8)$. Under the degree-halving injective map of Theorem 7.7.9, (P,Q) maps to $f_{(\mu_2,\mu_1),\beta} f_{(\mu_4,\mu_3),\beta}$, which is a standard monomial on $Y_\alpha^\gamma(\beta)$.

Before we start with the last theorem of this subsection, we will state [Kre08, Lemma 7.2] and we will recall Proposition 3.3.10, which are going to be used in the proof of the Theorem 7.7.13. [Kre08, Lemma 7.2] and Proposition 3.3.10 are stated below as Lemma 7.7.11 and Proposition 7.7.12 respectively.

Lemma 7.7.11. *If a nonvanishing multiset U on $\mathbb{N}^2$ is bounded by T, W (which is defined in §3.1), then $BRSK(U)$ is bounded by T, W.*

Proposition 7.7.12. *The map $BRSK$ is a degree-preserving bijection from the set of nonvanishing (resp. negative, positive) multisets on $\mathbb{N}^2$ to the set of nonvanishing (resp. negative, positive) semistandard notched bitableaux.*

Theorem 7.7.13. *The map $BRSK$ of Chapter 3 is a degree-preserving bijection from the set of all nonvanishing special multisets on $\bar{\beta} \times \beta$ (bounded by T_α, W_γ) to the set of all nonvanishing semistandard notched bitableaux on $(\bar{\beta} \times \beta)^*$ (bounded by T_α, W_γ).*

Proof. The fact that the map $BRSK$ of Chapter 3 is degree-preserving is obvious from Proposition 7.7.12 itself. For the rest of this proof, we will follow the notation and terminology of [GR06, §4.1] as well as the notation and terminology of Chapter 3.

There exists a natural injection from $SM^{\beta,\beta}$ to $\widetilde{SM^{\beta,\beta}}$ as given in [GR06, §4.1]. Let $\widetilde{A^{\beta,\beta}}$ denote the image of $SM^{\beta,\beta}$ in $\widetilde{SM^{\beta,\beta}}$ under this injection. Let $\mathfrak{E}$ denote the set of all special monomials in T^β (mentioned in §5.3.1). The map $\tilde{\pi}$ of Chapter 4 is a degree and domination preserving bijection between the sets $\mathfrak{E}$ and $\widetilde{A^{\beta,\beta}}$. The set $\mathfrak{E}$ is the same as the set of all positive special multisets on $\bar{\beta} \times \beta$, and the set $\widetilde{A^{\beta,\beta}}$ is in a natural bijection (induced by the bijective map from I^*_β to $I(d)$) with the set of all positive semistandard notched bitableaux on $(\bar{\beta} \times \beta)^*$. Also, we know from Corollary 5.1.3 that the map $BRSK$ of Chapter 3 and the map $\tilde{\pi}$ of Chapter 4 are the same on positive multisets on $\bar{\beta} \times \beta$. Moreover, it follows from Lemma 7.7.11 and the fact that inside the proof of [Kre08, Proposition 2.3], the inequality b is actually an equality) that a positive multiset U on $\bar{\beta} \times \beta$ is bounded by $\emptyset$, W_γ if and only if $BRSK(U)$ is bounded by $\emptyset$, W_γ. Therefore, we can now conclude that the map $BRSK$ of Chapter 3 is a degree-preserving bijection from the set of all positive special multisets on $\bar{\beta} \times \beta$ (bounded by $\emptyset$, W_γ) to the set of all positive semistandard notched bitableaux on $(\bar{\beta} \times \beta)^*$ (bounded by $\emptyset$, W_γ).

The proof for the negative part is similar. For the negative part, the multisets (as well as the notched bitableaux) will be bounded by T_α, $\emptyset$ instead. □

Example 7.7.14 below illustrates Theorem 7.7.13.

Example 7.7.14. *Let $d = 4$ and α, β, γ be as given in Example 7.5.10. Here $T_\alpha = \{(3,6)\}$ and $W_\gamma = \{(3,1),(8,6)\}$. Let*

$$\mathfrak{S} = \{(3,1),(3,2),(3,5),(8,6),(7,6),(4,6)\}.$$

Then $\mathfrak{S}$ is a nonvanishing special multiset on $\bar{\beta} \times \beta$ (bounded by T_α, W_γ).

$$BRSK(\mathfrak{S}) = (P, Q) = \left(\begin{array}{|c|c|} \hline 3 & \\ \hline 4 & \\ \hline 3 & 7 \\ \hline 3 & 8 \\ \hline \end{array} \, , \, \begin{array}{|c|c|} \hline 6 & \\ \hline 5 & \\ \hline 2 & 6 \\ \hline 1 & 6 \\ \hline \end{array} \right).$$

By Example 7.7.10, $BRSK(\mathfrak{S}) = (P, Q)$ is a nonvanishing semistandard notched bitableau on $(\bar{\beta} \times \beta)^*$ (bounded by T_α, W_γ).

The proof of Theorem 7.5.6 now follows easily from Theorems 7.7.6, 7.7.9, and 7.7.13.

CHAPTER 8

MULTIPLICITY AT ANY TORUS-FIXED POINT IN A RICHARDSON VARIETY IN THE SYMPLECTIC GRASSMANNIAN

8.1 Multiplicity

In this subsection everything is taken from [Eis95] except Remark 8.1.7.

Definition 8.1.1. *A **chain** of sub-modules of a module M is a sequence $\{M_i\}_{i=0}^{n}$ of sub-modules of M such that,*

$M = M_0 \supset M_1 \supset \ldots \supset M_n = (0)$ *(strict inclusions).*

*The **length** of the chain is n (the number of "links").*

Definition 8.1.2. *A **composition series** of M is a maximal chain, that is, one in which no extra sub-modules can be inserted.*

Definition 8.1.3. *Let A be a commutative ring with unity. By a chain of prime ideals of A, we mean a finite strictly increasing sequence*

$$\mathscr{P}_0 \subset \mathscr{P}_1 \ldots \subset \mathscr{P}_n;$$

the length of this chain is n.

*We define the **dimension** of A to be the supremum of all the lengths of all chains of prime ideals in A.*

It is an integer ≥ 0 , or ∞ (assuming that $A \neq 0$).

Definition 8.1.4. *A **graded ring** is a ring A together with a direct sum decomposition*

$A = A_0 \bigoplus A_1 \bigoplus \ldots$ *as abelian groups such that $A_i A_j \subset A_{i+j}$ $\forall i, j \geq 0$.*

Definition 8.1.5. *Let I be an ideal of a ring A. We define the* ***associated graded ring*** *of A with respect to I, written $gr_I A$ to be the graded ring*

$$gr_I A = A/I \bigoplus I/I^2 \bigoplus I^2/I^3 \bigoplus \dots$$

We will now define what is meant by the multiplicity of a maximal ideal on a local ring.

Definition 8.1.6. *Let R be a local ring with maximal ideal $\mathfrak{M}$. The* ***Hilbert-Samuel function*** *of R with respect to $\mathfrak{M}$ is defined as*

$$H_{\mathfrak{M},R}(n) = length\left(\frac{\mathfrak{M}^n}{\mathfrak{M}^{n+1}}\right),$$

where by length $(\frac{\mathfrak{M}^n}{\mathfrak{M}^{n+1}})$, we mean the length of a composition series of $(\frac{\mathfrak{M}^n}{\mathfrak{M}^{n+1}})$. There exists a polynomial $P_{\mathfrak{M},R}$ such that $P_{\mathfrak{M},R}(n) = H_{\mathfrak{M},R}(n)$ for n sufficiently large. The leading coefficient of $P_{\mathfrak{M},R}$ is of the form $\frac{e(\mathfrak{M},R)}{(dim R-1)!}$ ($dim R$ denotes the dimension of the local ring R) for some positive integer $e(\mathfrak{M}, R)$. This positive integer $e(\mathfrak{M}, R)$ is called the ***multiplicity of $\mathfrak{M}$ on*** *R.*

Remark 8.1.7. *Let α, β, and γ be the elements of $I(d)$, which were fixed at the end of the Chapter 2. In our case, R is the local ring $O_{X^\gamma_\alpha, e^\beta}$ of germs at e^β of function on the Richardson variety X^γ_α, and $\mathfrak{M}$ is its unique maximal ideal $\mathfrak{M}_{e^\beta}$.*

8.2 Some necessary definitions and notation

In this subsection we will recall some definitions and notation from [Kre08].
Let $(e, f),\ (g, h) \in \mathbb{N}^2$, both negative, then $(e, f) \prec (g, h)$ if $f < h$ and $e > g$, $(e, f) \trianglelefteq (g, h)$ if $f \leq h$ and $e \geq g$. Let $(c, d),\ (e, f) \in (\mathbb{N}^2)^-$, then define $(c, d) \wedge (e, f) = (\max(c, e), \min(d, f)) \in (\mathbb{N}^2)^-$.
If $T = \{(e_1, e_2), \dots, (e_m, e_{m+1})\}$ is a subset of $\mathbb{N}^2$, then T is said to be **completely disjointed** if $e_i \neq e_j$ when $i \neq j$. A **negative twisted chain** is a completely disjointed negative subset of $\mathbb{N}^2$ such that for any $u, v \in T$, either $u \prec v$, or $v \prec u$, or $u \wedge v \notin (\mathbb{N}^2)^-$. If T is a positive subset of $\mathbb{N}^2$, then T is called a **positive twisted chain** if $\iota(T)$ is a negative twisted chain. A **twisted chain** is a subset of $\mathbb{N}^2$ which is either a positive or a negative twisted chain.

Let $T = \{(e_1, f_1), \dots, (e_m, f_m)\}$ be a completely disjointed negative subset of $\mathbb{N}^2$ such that $f_1 < \dots < f_m$. For $\sigma \in S_m$, the permutation group on m elements, define $\sigma(T) = \{(e_{\sigma(1)}, f_1), \dots, (e_{\sigma(m)}, f_m)\}$. Let $\tau = \{\sigma(T) |\ \sigma \in S_m,\ \sigma(T) \text{ negative}\}$. Impose the following total order on τ : If both

$$R = \{(a_1, f_1), \dots, (a_m, f_m)\}$$

$$\text{and } S = \{(b_1, f_1), \dots, (b_m, f_m)\}$$

belong to τ, then $R \overset{\text{lex}}{<} S$ if, for the smallest i for which $a_i \neq b_i$, $a_i > b_i$. Since $\overset{\text{lex}}{<}$ is total order, τ has a unique minimal element, which Kreiman has denoted by $\widetilde{T}$ in [Kre08]. From [Kre08, Lemma 9.2], we know that, $\widetilde{T}$ **is a negative twisted chain**.

In [Kre08], for R a negative subset of $\mathbb{N}^2$ and $x \in (\mathbb{N}^2)^-$, Kreiman has defined $\mathbf{depth}_R(x)$, which is maximum r such that there exists a chain $u_1 \prec \dots \prec u_r$ in R with $u_r \trianglelefteq x$, and for any two negative subsets R and S of $\mathbb{N}^2$, $R \trianglelefteq S$ (or $S \trianglerighteq R$), if $\text{depth}_R(x) \geq \text{depth}_S(x)$ for every negative $x \in \mathbb{N}^2$, which is equivalent to $\text{depth}_R(x) \geq \text{depth}_S(x)$ for every $x \in S$. If R and S are positive subsets of $\mathbb{N}^2$, then $S \trianglerighteq R$ if $\iota(S) \trianglelefteq \iota(R)$. If R is a negative subset of $\mathbb{N}^2$ and S is a positive subset, then $R \trianglelefteq S$. Also, from [Kre08, Lemma 9.4] we know that, if R and S are twisted chains then

$$R \trianglelefteq S \Longleftrightarrow R \leq S, \tag{8.2.0.1}$$

where the relation $\leq$ on multisets on $\mathbb{N}^2$ is defined in (3.1.0.5) of Chapter 3.

Let R and S be negative and positive twisted chains respectively. Then a multiset U on $\mathbb{N}^2$ is said to be **chain-bounded by** R**,** S if $R \trianglelefteq U^-$ and $U^+ \trianglelefteq S$, or equivalently, if for every chain C in U, $R \trianglelefteq C^-$ and $C^+ \trianglelefteq S$.

For the rest of this chapter let β be an arbitrary element of $I(d)$ and let $\bar{\beta} = \{1, \dots, 2d\} \setminus \beta$.

8.3 Some necessary definitions and lemmas

Definition 8.3.1. *A multiset U in $\bar{\beta} \times \beta$ is called a **star set** if*
(i) $(r, c) \in U$ and $r \neq c^\star \Rightarrow (c^\star, r^\star) \in U$ and
(ii) Multiplicity of every $(r, c) \in U$ is one.
*A star set U in $(\bar{\beta} \times \beta)^-$ is called a **negative star set**. Similarly, a star set U in $(\bar{\beta} \times \beta)^+$ is called a **positive star set**.*

Example 8.3.2. *Let $d = 7$, $\beta = (1, 3, 6, 8, 10, 11, 13)$. So, $\bar{\beta} = (2, 4, 5, 7, 9, 12, 14)$. Let $U = \{(2, 3), (5, 8), (9, 6), (12, 11), (12, 13), (7, 10), (4, 3)\}$. Since multiplicity of every element of U is one, and for every $(r, c) \in U$ with $r \neq c^\star$, $(c^\star, r^\star) \in U$. Hence, U is a star set.*

Again, $\{(2, 3), (5, 8), (7, 10), (12, 13)\}$ is a negative star set and $\{(4, 3), (9, 6),$ $(12, 11)\}$ is a positive star set.

Definition 8.3.3. *A multiset U in $\bar{\beta} \times \beta$ is called $\star\star$**-multiset** if*
(i) $(r, c) \in U \Rightarrow (c^\star, r^\star) \in U$ and
(ii) multiplicity of every $(r, c) \in U$ is one except the diagonal elements and for the diagonal elements, multiplicity of every element is two.

Example 8.3.4. *Let d and β are same as in Example 8.3.2. Let*

$$U = \{(2,3)(5,8), (9,6)^2, (7,10)(12,11)\}.$$

Since for every $(r,c) \in U$, $(c^\star, r^\star) \in U$. Also, multiplicity of every $(r,c) \in U$ is one except the diagonal element $(9,6)$, and the multiplicity of $(9,6)$ is two in U. Hence, U is a $\star\star$-multiset.

Let v be a fixed element of $I(d,N)$, where d and N are positive integer with $d < N$, and w be any element of $I(d,N)$.

We will now prove Lemma 8.3.7. But to prove Lemma 8.3.7, we need [KR03, Proposition 4.3] and [GR06, Proposition 5.7]. We first state these propositions below as Proposition 8.3.5 and Proposition 8.3.6 respectively.

Proposition 8.3.5. *There exists a bijection between elements w of $I(d,N)$ satisfying $w \geq v$ on the one hand and subsets $\mathfrak{S}$ of $\mathfrak{N}(v)$ satisfying both the conditions of distinguished subset (Definition 4.1.11). If $\mathfrak{S}_w$ be the subset of $\mathfrak{N}(v)$ corresponding to w under this bijection, then the v-degree of w equals the cardinality of $\mathfrak{S}_w$.*

Proposition 8.3.6. *The association $w \mapsto \mathfrak{S}_w$ respects $\#$, that is, $(\mathfrak{S}_w)^\# = \mathfrak{S}_{w^\#}$. In particular, $w^\# = w$ (for an element $w \in I(d)$, $w^\#$ is defined in §7.4) if and only if $(\mathfrak{S}_w)^\# = \mathfrak{S}_w$ (for an monomial $\mathfrak{S}$ in $\mathfrak{N}(v)$, $\mathfrak{S}^\#$ is mentioned in Definition 5.3.5).*

Lemma 8.3.7. *Let $\alpha, \beta, \gamma \in I(d)$ with $\alpha \leq \beta \leq \gamma$, then $\widetilde{T}_\alpha$ is a negative star set in $(\bar{\beta} \times \beta)$ and $\widetilde{W}_\gamma$ is a positive star set in $(\bar{\beta} \times \beta)$.*

Proof. It is enough to show that $\widetilde{W}_\gamma$ is a star set in $\bar{\beta} \times \beta$ (for $\widetilde{T}_\alpha$, the proof is similar). Now, our claim is, $\widetilde{W}_\gamma$ is the distinguished monomial corresponding to γ in the sense of Proposition 8.3.5.

Since $\gamma \in I(d)$, so $\gamma = \gamma^\#$. Hence, if we assume the claim, then the proof of the lemma follows from Proposition 8.3.6.

Proof of the claim: Since $\widetilde{W}_\gamma$ is a positive twisted chain, therefore $\iota(\widetilde{W}_\gamma)$ is a negative twisted chain.

Say, $\iota(\widetilde{W}_\gamma) = \{(c_1, r_1), \ldots, (c_m, r_m)\}$ with $r_1 < \ldots < r_m$. Hence, $\iota(\widetilde{W}_\gamma)$ is a completely disjointed negative subset of $\bar{\beta} \times \beta$ such that for any $(c_i, r_i), (c_j, r_j) \in \iota(\widetilde{W}_\gamma)$ with $i \neq j$, we have :

either (a) $(c_i, r_i) \prec (c_j, r_j)$, that is, $r_i < r_j$ and $c_i > c_j$

or (b) $(c_j, r_j) \prec (c_i, r_i)$, that is, $r_j < r_i$ and $c_j > c_i$

or (c) $(c_i, r_i) \wedge (c_j, r_j) \notin (\mathbb{N}^2)^-$.

Without loss of generality, let us assume $r_i < r_j$ (proof will be similar for $r_i > r_j$). Then either (a) or (c) holds. That is, either by (a), $c_i > c_j$ or by (c), $(c_j, r_i) \notin (\mathbb{N}^2)^-$.

Now, $(c_j, r_i) \notin (\mathbb{N}^2)^-$ means $c_j \not< r_i$, that is $c_j \geq r_i$. But here $\iota(\widetilde{W}_\gamma)$ is completely disjointed, so equality cannot happen. So, $c_j > r_i$. So, we have if $r_i < r_j$, then either $c_i > c_j$ or $r_i < c_j$. This is nothing but the condition for distinguished monomials. □

Example 8.3.8. *Let d and β be as in Example 8.3.2. Let $\alpha = (1, 2, 4, 5, 7, 9, 12)$ and $\gamma = (2, 4, 7, 9, 10, 12, 14)$. Clearly, $\alpha \leq \beta \leq \gamma$. Now, $\beta \setminus \gamma = \{1, 3, 6, 8, 11, 13\}$ and $\gamma \setminus \beta = \{2, 4, 7, 9, 12, 14\}$. So,*

$$\iota(\widetilde{W}_\gamma) = \{(1, 2), (3, 4), (6, 7), (8, 9)(11, 12), (13, 14)\}.$$

Since $\iota(\widetilde{W}_\gamma)$ is a negative twisted chain in $\bar{\beta} \times \beta$, so

$$\widetilde{W}_\gamma = \{(2, 1), (4, 3), (7, 6), (9, 8), (12, 11), (14, 13)\}$$

is a positive twisted chain. Again, $\widetilde{W}_\gamma$ satisfies both the conditions of distinguished subset. As $\gamma \in I(d)$, so $\widetilde{W}_\gamma = (\widetilde{W}_\gamma)^\#$. Hence, $\widetilde{W}_\gamma$ is a positive star set in $(\bar{\beta} \times \beta)$.

Lemma 8.3.9. *There exists a bijection from the set of all $\star\star$-multisets in $\bar{\beta} \times \beta$ to the set of all star sets in $\bar{\beta} \times \beta$.*

Proof. Consider the map ψ from the set of all $\star\star$-multisets in $\bar{\beta} \times \beta$ to the set of all star sets in $\bar{\beta} \times \beta$ given by $\psi(U) =$ the underlying set of U. Clearly, this map ψ is a bijection. □

Example 8.3.10. *Let β be same as in Example 8.3.8. Let*

$$U = \{(2, 3), (5, 8), (9, 6)^2, (12, 10), (7, 10)\}.$$

Then from Example 8.3.4, we know that, U is a $\star\star$-multiset. Take

$$\psi(U) = \{(2, 3), (5, 8), (9, 6), (7, 10), (12, 10)\}.$$

Then clearly, $\psi(U)$ is a star set. Again, let

$$U = \{(2, 3), (4, 3), (5, 8), (9, 6), (7, 10), (12, 11), (12, 13)\}.$$

So clearly, U is a star set. Take

$$\psi^{-1}(U) = \{(2, 3), (4, 3), (5, 8), (9, 6)^2, (7, 10), (12, 11), (12, 13)\}.$$

Clearly, $\psi^{-1}(U)$ is a $\star\star$-multiset.

8.4 The main theorem

In this subsection, we prove one of the main theorems of this chapter, namely Theorem 8.4.5. But before going to the proof of Theorem 8.4.5, we need to prove Theorem 8.4.2 and Theorem 8.4.3, and to prove Theorem 8.4.2, we need [Kre08, Lemma 8.5]. This lemma is stated below as Lemma 8.4.1.

Lemma 8.4.1. *Let $R = F[x_1, \dots, x_m]$ (F is an algebraically closed field) be a polynomial ring, let $I \subset R$ be a homogeneous ideal, and let $G = \{g_1, \dots, g_k\}$ be a Gröbner basis for I, such that in $_\triangleright(g_i)$ ($\triangleright$ is mentioned in Definition 7.5.1) is square-free, $i = 1, \dots, k$. Let M be the maximal degree of a square-free monomial in $R \setminus in_\triangleright(G)$. Then dim$(R/I) = M$, and deg$(R/I)$ is the number of square-free monomials of degree M in $R \setminus in_\triangleright(G)$.*

Theorem 8.4.2. *Degree $Y^\gamma_\alpha(\beta)$ ($= mult_{e_\beta} X^\gamma_\alpha$) is the number of square free monomials of maximal degree in $P \setminus in_\triangleright \mathcal{G}^\gamma_{\alpha,\beta}$, where $P := k[X_{(r,c)} \mid (r, c) \in \mathfrak{OR}(\beta)]$*

Proof. Since the initial term of $f_{\mathfrak{w},\beta}$ (where $\mathfrak{w}$ is good) is a positive or negative upper extended β chain, therefore it is square free. Then by Lemma 8.4.1, we have the proof. □

Theorem 8.4.3. *There exists a degree doubling bijection from the set of all monomials in $P \setminus in_\triangleright \mathcal{G}^\gamma_{\alpha,\beta}$ to the set of all nonvanishing special multisets on $\bar{\beta} \times \beta$ bounded by T_α, W_γ.*

Proof. Let $a :=$ cardinality of all degree m monomials on $P \setminus in_\triangleright \mathcal{G}^\gamma_{\alpha,\beta}$.
$b :=$ cardinality of all degree m monomials on $P \setminus in_\triangleright I$.
$c :=$ cardinality of all degree m standard monomials on $Y^\gamma_\beta(\beta)$.
$d :=$ cardinality of all degree $2m$ nonvanishing semistandard notched bitableaux on $(\bar{\beta} \times \beta)^\star$.
$e :=$ cardinality of all degree $2m$ nonvanishing special multisets on $\bar{\beta} \times \beta$ bounded by T_α, W_γ .
So, we have to prove $a = e$. Now, $\mathcal{G}^\gamma_{\alpha,\beta} \subseteq I$ (this follows from Theorem 7.5.6), which implies $in_\triangleright \mathcal{G}^\gamma_{\alpha,\beta} \subseteq in_\triangleright I$. So,

$$P \setminus in_\triangleright \mathcal{G}^\gamma_{\alpha,\beta} \supseteq P \setminus in_\triangleright I. \tag{8.4.0.1}$$

Again, both the monomials of $P \setminus in_\triangleright I$ and the standard monomials of $Y^\gamma_\alpha(\beta)$ form a basis for P/I, and thus agree in cardinality in any degree. Therefore, $b = c$. Again, from Theorem 7.7.13, we have $d = e$. Also, from Theorem 7.7.9, we have $d \leq c$. Now, we want to show $d \geq c$. Using (8.4.0.1) we have, $a \geq b$. Again, by Theorem 7.7.6, we have, $a \leq e$. Therefore, $d = e \geq a \geq b = c$, hence $d \geq c$. Therefore, $d = c$ and hence, $a \geq b = c = d = e$. We also have, $a \leq e$. Hence, $a = e$. □

Remark 8.4.4. *It follows from Theorem 8.4.3 that the number of square free monomials of maximal degree in $P \setminus in_\triangleright \mathcal{G}^\gamma_{\alpha,\beta}$ equals the number of $\star\star$-multisets in $\bar{\beta} \times \beta$ which are of* *maximal degree among those bounded by T_α, W_γ (because, the bijection of Theorem 8.4.3 above is given by the map $U \mapsto U \cup U^{\#}$).*

Theorem 8.4.5. *$Mult_{e_\beta} X_\alpha^\gamma$ is the number of star sets U in $\bar{\beta} \times \beta$, which are of maximal degree among those which are chain-bounded by $\widetilde{T}_\alpha$ and $\widetilde{W}_\gamma$.*

Proof. Recall from [Kre08] that, if U is a multiset on $\bar{\beta} \times \beta$, then the monomial X_U is square-free if and only if U is a subset of $\bar{\beta} \times \beta$, that is each of its elements has degree one. By Theorem 8.4.2, $Mult_{e_\beta} X_\alpha^\gamma$ is the number of square free monomials of maximal degree in $P \setminus in_{\triangleright} \mathcal{G}_{\alpha,\beta}^\gamma$. By Remark 8.4.4, this equals the number of $\star\star$-multisets in $\bar{\beta} \times \beta$, which are of maximal degree among those bounded by T_α, W_γ. Again, by Lemma 8.3.9, this equals the number of star sets in $\bar{\beta} \times \beta$, which are of maximal degree among those bounded by T_α, W_γ. However, a subset of $\bar{\beta} \times \beta$ is bounded by T_α, W_γ if and only if it is bounded by $\widetilde{T}_\alpha$ and $\widetilde{W}_\gamma$ if and only if it is chain bounded by $\widetilde{T}_\alpha$ and $\widetilde{W}_\gamma$, where the last equivalence is due to (8.2.0.1). □

8.5 Path families and multiplicities

In this subsection the write up is same as the write up of [Kre08].
For this subsection, let R and S be fixed positive and negative twisted chains contained in $\bar{\beta} \times \beta$ respectively. Let

$$\mathcal{M}_R = \max \{U \subset (\bar{\beta} \times \beta)^- \mid R \trianglelefteq U \text{and } U \text{ is a star set }\},$$
$$\mathcal{M}^S = \max \{V \subset (\bar{\beta} \times \beta)^+ \mid V \trianglelefteq S \text{ and } V \text{ is a star set }\},$$
$$\mathcal{M}_R^S = \max \{W \subset (\bar{\beta} \times \beta) \mid R \trianglelefteq W^- \text{ and } W^+ \trianglelefteq S \text{ and } W \text{ is a star set }\},$$

where in each case by 'max' we mean the star sets U, V, or W respectively of maximal degree. For example, $\mathcal{M}_R^S$ consists of the collection of all star sets W of $(\bar{\beta} \times \beta)$ which are of maximal degree among those which are chain bounded by R, S. When $R = \widetilde{T}_\alpha$ and $S = \widetilde{W}_\gamma$, $\mathcal{M}_R^S$ consists precisely of the star sets U of Theorem 8.4.5. In order to give a better formulation of Theorem 8.4.5, we study the combinatorics of $\mathcal{M}_R^S$. Clearly,

$$\mathcal{M}_R^S = \{U \dot{\cup} V \mid U \in \mathcal{M}_R, \ V \in \mathcal{M}^S\}, \qquad (8.5.0.1)$$

where $U \dot{\cup} V$ is defined in (3.1.0.1) of Chapter 3. To study $\mathcal{M}_R^S$, just like in [Kre08], we begin by considering $\mathcal{M}_R$, and thus restricting attention to negative star sets of $(\bar{\beta} \times \beta)$. Just like in [Kre08], a subset $P \subset (\bar{\beta} \times \beta)^-$ is **depth-one** if it contains no two-element chains and if P is depth-one, then it is a **negative-path** if the consecutive points are 'as close as possible' to each other, so that the points form a continuous path on $(\bar{\beta} \times \beta)^-$

which moves only down or to the right. For any $r = (e, f) \in (\bar{\beta} \times \beta)^-$, define $\lfloor r \rfloor$ and $\lceil r \rceil$ are just like in [Kre08], that is

$$\lfloor r \rfloor = (e, f'), \text{ where } f' = \min \{y \in \beta \mid (e, y) \in (\bar{\beta} \times \beta)^-\},$$

$$\lceil r \rceil = (e', f), \text{ where } e' = \max \{x \in \bar{\beta} \mid (x, f) \in (\bar{\beta} \times \beta)^-\}.$$

Now, we form the path P_r as follows:

1. it begins at $\lfloor r \rfloor$, and ends at $\lceil r \rceil$ and
2. if $r = (e_1, f_1)$, $r' = (e'_1, f'_1)$ and $r = (r')^{\#}$, then $P_r = P_{r'}{}^{\#}$. Also, if $r = (e, f)$ and $e = f^{\star}$, then P_r is a negative star set in $(\bar{\beta} \times \beta)$.

It is clear that if R is a twisted chain and if $r' \neq r$ then $P_{r'} \cap P_r = \emptyset$, where by $\emptyset$ we mean the empty intersection. Furthermore, $R \trianglelefteq \dot{\bigcup}_{r \in R} P_r$. Define $d_R = \sum_{r \in R} |P_r|$. Now, the following lemma is a straight forward consequence of the definitions.

Lemma 8.5.1. *Let Q be a depth-one negative star set in $\bar{\beta} \times \beta$ such that $P_r \trianglelefteq Q$. Then $|Q| \leq |P_r|$, with equality if and only if Q is a negative-path from $\lfloor r \rfloor$ to $\lceil r \rceil$.*

If $U \subset (\bar{\beta} \times \beta)^-$ is a star set, $R \trianglelefteq U$, and $r \in R$, then define

$$U_{R,r} := \{u \in U \mid r \trianglelefteq u, \ \text{depth}_U(u) = \text{depth}_R(r)\}.$$

It follows from the definition that $U_{R,r}$ is depth-one. Indeed, if u and u' are two elements of U which form a chain, then without loss of generality let $u \prec u'$. Thus $\text{depth}_U(u) < \text{depth}_U(u')$, and in particular, $\text{depth}_U(u) \neq \text{depth}_U(u')$. Thus u and u' cannot both lie in $U_{R,r}$.

Proof of Lemma 8.5.2 below is same as the proof of [Kre08, Lemma 10.2], so here we are omitting the proof.

Lemma 8.5.2. *Let $U \subset (\bar{\beta} \times \beta)^-$ be a star set. Then*

1. *if $R \trianglelefteq U$, then $U = \dot{\bigcup}_{r \in R} U_{R,r}$.*
2. *If $R \trianglelefteq U$, then $|U| \leq d_R$, with equality if and only if $U = \dot{\bigcup}_{r \in R} Q_r$, where Q_r is a negative-path from $\lfloor r \rfloor$ to $\lceil r \rceil$.*
3. *Let $U = \dot{\bigcup}_{r \in R} Q_r \subset (\bar{\beta} \times \beta)$, where Q_r is a negative-path from $\lfloor r \rfloor$ to $\lceil r \rceil$. Then $R \trianglelefteq U$.*

Condition 2, of Lemma 8.5.2 implies that any $U \in \mathcal{M}_R$ is a disjoint union $U = \dot{\bigcup}_{r \in R} Q_r$, where Q_r is a negative-path from $\lfloor r \rfloor$ to $\lceil r \rceil$. Condition 3, of Lemma 8.5.2 implies that any disjoint union $U = \dot{\bigcup}_{r \in R}$, where Q_r is a negative-path from $\lfloor r \rfloor$ to $\lceil r \rceil$, is an element of $\mathcal{M}_R$. Consequently we have,

Corollary 8.5.3. *$\mathcal{M}_R$ consists of the set of all possible disjoint unions $U = \dot{\bigcup}_{r \in R} Q_r$, where Q_r is a negative-path from $\lfloor r \rfloor$ to $\lceil r \rceil$.*

Similar analysis can be done for positive star sets on $\bar{\beta} \times \beta$. Here the notion of a positive-path is as follows:
if $P \in (\bar{\beta} \times \beta)^+$ is depth-one, then it is a *positive-path* if the consecutive points are 'as close as possible' to each other, so that the points form a continuous path on $(\bar{\beta} \times \beta)^+$ which moves only up or to the left.

Likewise, the notions of $\lfloor r \rfloor$ and $\lceil r \rceil$ for $s \in (\bar{\beta} \times \beta)^+$ can be defined as follows:
if $r = (e, f) \in (\bar{\beta} \times \beta)^+$, then we define

$$\lfloor r \rfloor = (e, f'), \text{ where } f' = \max \{y \in \beta \mid (e, y) \in (\bar{\beta} \times \beta)^+\} \text{ and}$$

$$\lceil r \rceil = (e', f), \text{ where } e' = \min \{x \in \bar{\beta} \mid (x, f) \in (\bar{\beta} \times \beta)^+\}.$$

Corollary 8.5.4. *$\mathcal{M}^S$ consists of the set of all possible disjoint unions $V = \dot{\bigcup}_{s \in S} Q_s$, where Q_s is a positive-path from $\lfloor s \rfloor$ to $\lceil s \rceil$.*

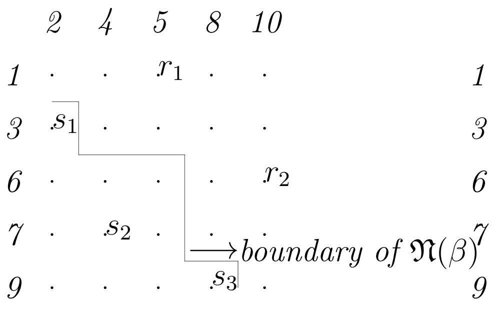
Corollary 8.5.3, Corollary 8.5.4, and (8.5.0.1) together imply the following corollary.

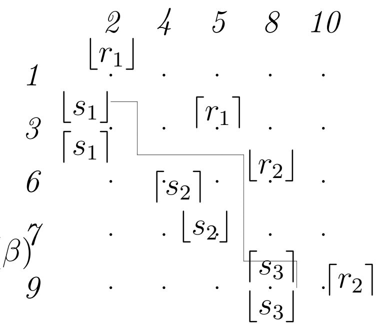
Corollary 8.5.5. *$\mathcal{M}_R^S$ consists of the set of all possible disjoint unions $W = \dot{\bigcup}_{r \in R \cup S} Q_r$, where Q_r is either a negative-path or a positive-path from $\lfloor r \rfloor$ to $\lceil r \rceil$ depending on whether r is negative or positive.*

The star sets U of Theorem 8.4.5 are precisely the elements of $\mathcal{M}_R^S$, when $R = \tilde{T}_\alpha$ and $S = \tilde{W}_\gamma$. Therefore, combining Theorem 8.4.5 and Corollary 8.5.5, we obtain the following theorem.

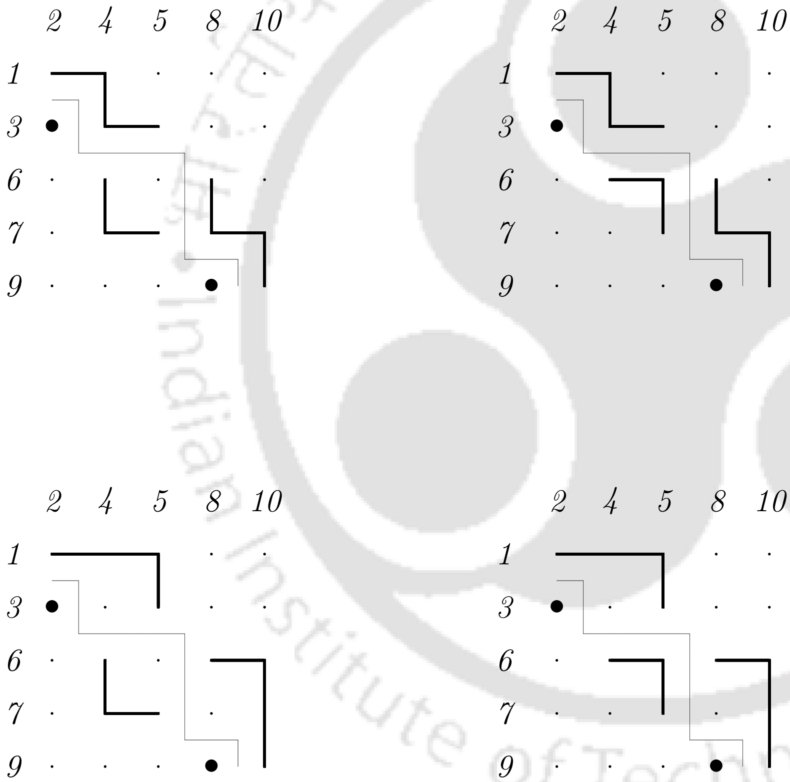
Theorem 8.5.6. *$Mult_{e_\beta} X_\alpha^\gamma$ is the number of disjoint unions $\dot{\bigcup}_{r \in \widetilde{T_\alpha} \cup \widetilde{W_\gamma}} P_r$, where P_r is either negative-path or a positive-path from $\lfloor r \rfloor$ to $\lceil r \rceil$, depending on whether r is negative or positive.*

Example 8.5.7. *Let $d = 5$, that is, $2d = 10$. Let $\alpha = (1, 2, 4, 6, 8), \beta = (2, 4, 5, 8, 10), \gamma = (3, 5, 7, 9, 10)$. Clearly, $\alpha, \beta, \gamma \in I(d)$, and $\alpha \leq \beta \leq \gamma$. Again, as $\alpha, \beta, \gamma \in I(d)$ so by Lemma 8.3.7, $\widetilde{T}_\alpha, \widetilde{W}_\gamma$ both are star sets. We want to compute $Mult_{e_\beta} X_\alpha^\gamma$. The following two diagrams show the negative and positive twisted chains $\widetilde{T}_\alpha = \{r_1, r_2\}$ and $\widetilde{W}_\gamma = \{s_1, s_2, s_3\}$ in $\bar{\beta} \times \beta$; and the set of $\lfloor r \rfloor$'s and $\lceil r \rceil$'s for all $r \in \widetilde{T}_\alpha \cup \widetilde{W}_\gamma$. Note that, $s_1 = \lfloor s_1 \rfloor = \lceil s_1 \rceil$ and $s_3 = \lfloor s_3 \rfloor = \lceil s_3 \rceil$.*





There are four non intersecting path families from $\lfloor r \rfloor$ *to* $\lceil r \rceil$, $r \in \widetilde{T}_\alpha \cup \widetilde{W}_\gamma$, *as shown below. Thus* $Mult_{e_\beta} X_\alpha^\gamma = 4$.



CHAPTER 9

MULTIPLICITY AT A TORUS-FIXED POINT IN A RICHARDSON VARIETY IN THE ORTHOGONAL GRASSMANNIAN

A Richardson variety is the intersection of a Schubert variety and an opposite Schubert variety. In [Upa08], Raghavan and Upadhyay have already found the multiplicity at any torus-fixed point in a Schubert variety in the orthogonal Grassmannian. In this chapter, we define and elaborate everything only for the opposite Schubert variety in the orthogonal Grassmannian and then merge the two results.

For the rest of this chapter, let d be a positive integer and $w' \leq v \leq w$ be elements of $\mathfrak{O}I(d)$. Also, recall the notation $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ from §4.1.

9.1 Vertical and horizontal projections of an element α in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$

Let $\alpha = (r, c)$ be an element in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$, then the element $p_v(\alpha) := (c^\star, c)$ is called the **vertical projection** of α and $p_h(\alpha) := (r, r^\star)$ is called the **horizontal projection** of α. The lines joining α to its projections are called the **legs** of α. For a monomial $\mathfrak{C}$ of $\mathfrak{R}(v)$, we define

$$\mathfrak{C}(\text{down}) = \mathfrak{C} \cap (\widetilde{\mathfrak{O}}\mathfrak{R}(v) \cup \mathfrak{d}(v)).$$

Example 9.1.1. *Let $d = 7$ and $v = (4, 6, 7, 10, 12, 13, 14)$. Let the monomial*

$$\mathfrak{S} = \{(9, 10)^2, (8, 14), (11, 6), (5, 7), (2, 12)^2\}.$$

Then

$$\mathfrak{S}\textit{(down)} = \{(9, 10)^2, (8, 14), (11, 6)\}.$$

9.2 A connected anti-v-chain and the subset $\tilde{\mathfrak{D}}mon_C$ in $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$

Let $C : \alpha_1 = (r_1, c_1) > \cdots > \alpha_t = (r_t, c_t)$ be an anti-v-chain (mentioned in Definition 5.2.10) in $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$. Two consecutive elements α_j and α_{j+1} of C are said to be **connected** if the following conditions are satisfied:

- $c_{j+1} \geq r_j^\star$.
- the point $(r_{j+1}, r_j^\star)$ belongs to $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$.

The above relation generates an equivalence relation on the elements of C. The equivalence classes of C corresponding to this equivalence relation are called the **connected components** of C.

Example 9.2.1 below illustrates the definition of connected components.

Example 9.2.1. *Let $d = 7$ and $v = (4, 6, 7, 10, 12, 13, 14)$.*

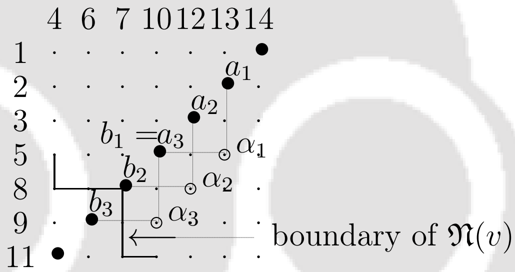


Figure 9.1: Illustration of the grid representing $\mathfrak{N}(v)$

The anti-v-chain $C : \alpha_1 > \alpha_2 > \alpha_3$ in Figure 9.1 has $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3\}$ as its connected components. Here a_i denotes the vertical projection of α_i and b_i denotes the horizontal projection of α_i.

Corresponding to a connected anti-v-chain C in $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$, we will now define a subset $\tilde{\mathfrak{D}}mon_C$ of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$.

Definition 9.2.2. *For a connected anti-v-chain $C : \alpha_1 = (r_1, c_1) > \cdots > \alpha_t = (r_t, c_t)$ in $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ it is clear that, if the horizontal projection $p_h(\alpha_j)$ does not belong to $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$, then $j = t$. Define*

$$\tilde{\mathfrak{D}}mon_C = \begin{cases} \{p_v(\alpha_1), \cdots, p_v(\alpha_t)\} & \text{if } t \text{ is even} \\ \{p_v(\alpha_1), \cdots, p_v(\alpha_t)\} \cup \{p_h(\alpha_t)\} & \text{if } t \text{ is odd and } p_h(\alpha_t) \in \mathfrak{R}(v) \setminus \mathfrak{N}(v) \\ \{p_v(\alpha_1, \cdots, p_v(\alpha_{t-1}))\} \cup \{\alpha_t, \alpha_t^\#\} & \text{if } t \text{ is odd and } p_h(\alpha_t) \notin \mathfrak{R}(v) \setminus \mathfrak{N}(v). \end{cases}$$

Remark 9.2.3. *Suppose C is not connected, let $C = C_1 \cup C_2 \cup \cdots$ be the connected components of C. Then we define*

$$\tilde{\mathfrak{O}}mon_C := \tilde{\mathfrak{O}}\mathfrak{mon}_{C_1} \cup \tilde{\mathfrak{O}}\mathfrak{mon}_{C_2} \cup \cdots$$

Example 9.2.4. *Let d and v be as in Example 9.2.1. Also, from Example 9.2.1, we know that, the anti-v-chain $C = \alpha_1 = (5,13) > \alpha_2 = (8,12) > \alpha_3 = (9,10)$ has two connected components $C_1 = \{\alpha_1, \alpha_2\}$ and $C_2 = \{\alpha_3\}$. Then*

$$\tilde{\mathfrak{O}}\mathfrak{mon}_{C_1} = \{(2,13),(3,12)\} \text{ and } \tilde{\mathfrak{O}}\mathfrak{mon}_{C_2} = \{(9,10),(5,6)\}.$$

So

$$\tilde{\mathfrak{O}}mon_C = \{(2,13),(3,12),(9,10),(5,6)\}.$$

Remark 9.2.5. *It is obvious from Definition 9.2.2 that $\tilde{\mathfrak{O}}mon_C$ is symmetric in the sense of Definition 5.3.5 and has evenly many elements on the diagonal. It is also distinguished in the sense of Definition 5.2.11.*

9.3 Definition of anti-$\mathfrak{O}$-domination

Definition 9.3.1. *For any two elements v and w' of $\mathfrak{O}I(d)$, we say w' anti-dominates v if $w' \leq v$.*

Definition 9.3.2. *An element w' of $\mathfrak{O}I(d)$ is said to anti-$\mathfrak{O}$-dominate an anti-v-chain C if w' anti-dominates the monomial $\tilde{\mathfrak{O}}mon_C$ in the sense of Definition 5.2.10.*

Definition 9.3.3. *An element w' of $\mathfrak{O}I(d)$ is said to anti-$\mathfrak{O}$-dominate a monomial C of $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ if it anti-$\mathfrak{O}$-dominates every anti-v-chain in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$.*

Definition 9.3.4. *We say that a monomial $\mathfrak{S}$ in $\mathfrak{O}\mathfrak{N}(v) \cup \widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ is $\mathfrak{O}$- dominated by w as well as anti-$\mathfrak{O}$-dominated by w' if $\mathfrak{S} \cap \mathfrak{O}\mathfrak{N}(v)$ is $\mathfrak{O}$-dominated by w and $\mathfrak{S} \cap \widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ is anti-$\mathfrak{O}$-dominated by w', where w', v, and w are defined at the beginning of this chapter.*

9.4 An element $\tilde{\mathfrak{O}}w'_C$ of $\mathfrak{O}I(d)$ attached to an anti-v-chain C

Recall the definition of $u^\#$ for an element u in $I(d, 2d)$ from notation (A) of §7.4. In this case, it is clear that $u = u^\#$ if and only if u is an element of $\mathfrak{O}I(d)$.

Lemma 9.4.1. *An element w' $(\leq v)$ of $I(d, 2d)$ belongs to $\mathfrak{O}I(d)$ if and only if the distinguished subset (stated in Definition 5.2.11) $\mathfrak{S}_{w'}$ of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ corresponding to it as described in Remark 5.2.12 is symmetric (stated in Definition 5.3.5) and has evenly many diagonal elements.*

Proof. The proof of the above lemma is analogous to the proof of [Upa08, Proposition 5.2.1]. Hence we omit the proof. □

Notation: We denote by $\tilde{\mathfrak{O}}w'_C$, the element of $\mathfrak{O}I(d)$ associated to $\tilde{\mathfrak{O}}mon_C$ (using Remark 9.2.5 and Lemma 9.4.1).

Remark 9.4.2. *It follows from Definition 9.3.2 that an element w' of $\mathfrak{O}I(d)$ anti-$\mathfrak{O}$-dominates an anti-v-chain C in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ if and only if w' anti-dominates in the sense of Definition 5.2.10 the monomial $\tilde{\mathfrak{O}}mon_C$, that is, if and only if $w' \leq \tilde{\mathfrak{O}}w'_C$.*

9.5 The main theorem of this chapter

The main theorem of this chapter is the following:

Theorem 9.5.1. *Let V be a vector space of dimension $2d$ with symmetric non-degenerate bilinear form (over an algebraically closed field of characteristic not equal to 2). Let $X^w_{w'}$ be a Richardson variety in the orthogonal Grassmannian $\mathfrak{M}_d(V)$, and e^v the torus fixed point of $X^w_{w'}$ corresponding to v. Then for any non-negative integer m, the Hilbert function of $R^w_{w'}(v)$ equals the cardinality of the set $S^w_{w'}(v)(m)$, where*

- *$R^w_{w'}(v)$ denotes the co-ordinate ring of the tangent cone to $X^w_{w'}$ at the point e^v.*
- *$S^w_{w'}(v)$ denotes the set of w-dominated, w'-anti-dominated monomials in $\mathfrak{O}\mathfrak{N}(v) \cup \widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$.*
- *$S^w_{w'}(v)(m)$ denotes the set of w-dominated, w'-anti-dominated monomials in $\mathfrak{O}\mathfrak{N}(v) \cup \widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ of degree m.*

Corollary 9.5.2. *The multiplicity of $\mathfrak{M}_{e^v}$ on $O_{X^w_{w'}(v)}$ equals the number of monomials in $\mathfrak{O}\mathfrak{N}(v) \cup \widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ of maximal cardinality that are square-free, as well as $\mathfrak{O}$-dominated by w, and anti-$\mathfrak{O}$-dominated by w', where $\mathfrak{M}_{e^v}$ is the unique maximal ideal of the local ring $O_{X^w_{w'}(v)}$.*

9.6 Reduction of the proof to combinatorics

In this section, we will see how some well known results of [LMS79] and [Upa08] allow us to reduce the proof of Theorem 9.5.1 to the solution of a combinatorial problem.

9.6.1 Standard monomial theory

Let $\mathfrak{M}_d(V) \subseteq G_d(V) \hookrightarrow \mathbb{P}(\wedge^d V)$ be the Plücker embedding (where $G_d(V)$ denotes the Grassmannian of all d-dimensional subspaces of V). The pull-back to $\mathfrak{M}_d(V)$ of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\wedge^d V)$ is the square of the ample generator of the Picard group of $\mathfrak{M}_d(V)$. Let L denote the ample generator. There exists a section q_θ of the line bundle L on $\mathfrak{M}_d(V)$ such that $q_\theta^2 = p_\theta$, where p_θ denotes the corresponding plücker coordinate. These q_θ are called **Pfaffians**.

Definition 9.6.1. *A **standard monomial** is a totally ordered sequence $\theta_1 \geq \cdots \geq \theta_t$ (with repetitions allowed) of elements of $\mathfrak{O}I(d)$. Such a standard monomial is said to be*

- *w**-dominated** for $w \in \mathfrak{O}I(d)$ if $w \geq \theta_1$.*
- *w'**-anti-dominated** for $w' \in \mathfrak{O}I(d)$ if $\theta_t \geq w'$.*
- *v**-compatible** if for each k, $1 \leq k \leq t$, either $\theta_k > v$ or $v > \theta_k$ and none of θ_k is equal to v.*

Notation: Given $v, w,$ and $w' \in \mathfrak{O}I(d)$, we denote by $SM_{w'}^w(v)$ the set of all w**-dominated, w'-anti-dominated, and v-compatible standard monomials**.

To a standard monomial $\theta_1 \geq \cdots \geq \theta_t$ in $\mathfrak{O}I(d)$ we associate the product $q_{\theta_1} \cdots q_{\theta_t}$, where q_θ are the sections defined above of the line bundle L. Such a product is also called a standard monomial and it is said to be dominated by w and anti-dominated by w' for $w, w' \in \mathfrak{O}I(d)$ if the underlying monomial in $\mathfrak{O}I(d)$ is dominated by w and anti-dominated by w'. From Brion, Lakshmibai [BL03], standard monomial theory for $\mathfrak{M}_d(V)$ says:

Theorem 9.6.2. *Standard monomials $q_{\theta_1} \cdots q_{\theta_r}$ of degree r form a basis for the space of forms of degree r in the homogeneous coordinate ring of $\mathfrak{M}_d(V)$ in the embedding defined by the ample generator L of the Picard group. More generally, for $w, w' \in \mathfrak{O}I(d)$, the w-dominated and w'- anti-dominated standard monomials of degree r form a basis for the space of forms of degree r in the homogeneous coordinate ring of the Richardson variety $X_{w'}^w$ of $\mathfrak{M}_d(V)$.*

Let $\mathbb{A}^v$ be the affine patch of $\mathbb{P}(\wedge^d V)$ given by $q_v \neq 0$, and set $Y_{w'}^w(v) := X_{w'}^w \cap \mathbb{A}^v$. The point e^v is the origin of the affine space $\mathbb{A}^v$. The functions $f_u := q_u/q_v$, $u \in \mathfrak{O}I(d)$, provide a set of co-ordinate functions on $\mathbb{A}^v$. The co-ordinate ring $K[Y_{w'}^w(v)]$ of $Y_{w'}^w(v)$ is a quotient of the polynomial ring $K[f_u | u \in \mathfrak{O}I(d)]$, K being the underlying field. Now, we want to find a basis for $K[Y_{w'}^w(v)]$ as a K-vector space.

Proposition 9.6.3. *As $\theta_1 \geq \cdots \geq \theta_t$ varies over $SM_{w'}^w(v)$, that is, over all w-dominated, w'-anti-dominated, and v-compatible standard monomials, the elements $f_{\theta_1} \cdots f_{\theta_t}$ form a K-vector space basis of $K[Y_{w'}^w(v)]$.*

Proof. The proof is similar to the proof of Proposition 3.2.1 of [Upa08]. □

For $\theta \in \mathfrak{O}I(d)$, define the v-degree of θ to be the cardinality of the set $\theta \setminus v$ (as a subset of $\{1, 2, \cdots, 2d\}$). The proposition above tells us that the graded piece of degree m of $K[Y_{w'}^w(v)]$ is generated as a K-vector space by elements of $SM_{w'}^w(v)$ of degree m, where the degree of a standard monomial $f_{\theta_1} \cdots f_{\theta_t}$ is defined to be the sum of the v-degrees of $\theta_1, \cdots, \theta_t$. To prove Theorem 9.5.1 it therefore suffices to show that the set $SM_{w'}^w(v)(m)$ of w-dominated w'-anti-dominated v-compatible standard monomials of degree m is in bijection with $S_{w'}^w(v)(m)$, which is stated in Theorem 9.6.4 below.

Theorem 9.6.4. *The set $SM_{w'}^w(v)(m)$ of standard monomials in $\mathfrak{O}I(d)$ of degree m that are v-compatible, dominated by w, and anti-dominated by w' is in bijection with the set $S_{w'}^w(v)(m)$ of monomials in $\mathfrak{O}\mathfrak{N}(v) \cup \widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ of degree m that are $\mathfrak{O}$-dominated by w as well as anti-$\mathfrak{O}$-dominated by w'.*

9.7 The proof

In this subsection, we will show that $SM_{w'}^w(v)(m)$ and $S_{w'}^w(v)(m)$ are naturally bijective. As we saw in the previous subsection, this serves to complete the proof of our main Theorem 9.5.1. In [Upa08], there was a theorem (Theorem 3.2.2), which is similar to Theorem 9.6.4. Theorem 3.2.2 of [Upa08] was proved by means of two maps $\mathfrak{O}\pi$ and $\mathfrak{O}\phi$ described for the set $\mathfrak{O}\mathfrak{N}(v)$. We also had two propositions, namely Proposition 4.1.1 and Proposition 4.1.2 in [Upa08] which stated some good properties for the set $\mathfrak{O}\mathfrak{N}(v)$. Now, we will define the maps $\mathfrak{O}\pi$ and $\mathfrak{O}\phi$ for monomials in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$.

Let $mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ denote the **set of all monomials in** $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$. The map $\mathfrak{O}\pi$ is a function from $mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ to $\mathfrak{O}I(d) \times mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$. For any monomial $\mathfrak{S}$ in $mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$, set

$$\mathfrak{O}\pi(\mathfrak{S}) = (w', \mathfrak{S}').$$

The map $\mathfrak{O}\pi$ enjoys the following good properties:

Proposition 9.7.1. *1. $w' \leq v$.*

2. v-degree (w') + degree$(\mathfrak{S}')$ = degree$(\mathfrak{S})$.

3. w' anti-$\mathfrak{O}$-dominates $\mathfrak{S}'$.

4. w' is the largest element of $\mathfrak{O}I(d)$ that anti-$\mathfrak{O}$-dominates $\mathfrak{S}$.

Proof. The proof of this proposition is analogous to the proof of of Proposition 4.1.1 of [Upa08]. Hence we omit the proof. □

Notation: Let $mon_{resw'}\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ denote the **set of all monomials in** $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ **which are anti-$\mathfrak{O}$-dominated by** w'.
Now, $\mathfrak{O}\phi$ is a mapping from $\{w'\} \times mon_{resw'}\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ to $mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$. It takes a pair $(w', \mathfrak{T})$ as input (where $\mathfrak{T}$ is a monomial in $mon_{resw'}\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$) and produces an element $\mathfrak{T}'$ in $mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ as output. That is

$$\mathfrak{O}\phi(w', \mathfrak{T}) = \mathfrak{T}'.$$

Proposition 9.7.2. *The maps $\mathfrak{O}\pi$ and $\mathfrak{O}\phi$ for monomials in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ are inverses to each other.*

Proof. The proof of this proposition is analogous to the proof of of Proposition 4.1.2 of [Upa08]. Hence we omit the proof. □

Now we will prove Theorem 9.6.4. For that, first we will show that there exists a bijection between the sets $SM^v_{w'}(v)(m)$ (where $SM^v_{w'}(v)(m)$ denotes the set of all standard monomials in $\mathfrak{O}I(d)$ of degree m that are v-compatible, dominated by v, and anti-dominated by w') and $U_{w'}(m)$, where $U_{w'}$ denotes the set of all monomials in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ which are anti-$\mathfrak{O}$-dominated by w' and $U_{w'}(m)$ denotes such monomials of degree m.

Let S, T, U denote respectively the set of all monomials in $\mathfrak{O}\mathfrak{N}(v) \cup \widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$, $\mathfrak{O}\mathfrak{N}(v)$, $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ (please note that for the sake of convenience, we are using the notation U for $mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$). We define the anti-domination map from U to $\mathfrak{O}I(d)$ by sending a monomial in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ to the greatest element that anti-$\mathfrak{O}$-dominates it. Define the anti-domination map from $SM^v(v)$ to $\mathfrak{O}I(d)$ (where $SM^v(v)$ denotes the set of all v-compatible standard monomials which are dominated by v) by sending $\theta_1 \leq \cdots \leq \theta_t$ to θ_1.

Again, repeated application of $\mathfrak{O}\pi$ (for monomials in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$) gives a map from U to $SM^v(v)$ that commutes with anti-domination and preserves degree. Repeated application of $\mathfrak{O}\phi$ (for monomials in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$) gives a map from $SM^v(v)$ to U. These two maps are inverses to each other (Proposition 9.7.2) and so we have a bijection between $SM^v(v)$ and U. In fact, since anti-domination and degree are respected (Proposition 9.7.1), we get a bijection

$$SM^v_{w'}(v)(m) \overset{bij}{\to} U_{w'}(m).$$

Now, from [Upa08] we know there exists a bijection between the sets $SM^w_v(v)(m)$ and $T^w(m)$, where T^w denotes the set of all monomials in $\mathfrak{O}\mathfrak{N}(v)$ which are $\mathfrak{O}$-dominated by w and $T^w(m)$ denotes such monomials of degree m. This bijection is analogous to the bijection between $SM^v_{w'}(v)(m)$ and $U_{w'}(m)$, the only differences being that anti-$\mathfrak{O}$-domination is replaced here by $\mathfrak{O}$-domination, $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ is replaced by $\mathfrak{O}\mathfrak{N}(v)$, and the word “greatest element” is replaced by the word “least element”. Putting these bijections together, we get the desired result:

$$SM^{w}_{w'}(v)(m) = \bigcup_{k=0}^{m} SM^{w}_{v}(k) \times SM^{v}_{w'}(m-k)$$

$$= T^{w}(k) \times U_{w'}(m-k)$$

$$= S^{w}_{w'}(m).$$

9.8 Description of the map $\mathfrak{O}\pi$ for monomials in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$

9.8.1 The type of an element α in an anti-v-chain C

For a connected anti-v-chain $C : \alpha_1 > \cdots > \alpha_t$ in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$, define the type in C of an element α_j, $1 \leq j \leq t$, of C to be:

1. V if $j \neq t$, or $j = t$ and t is even,
2. H if $j = t$, t is odd, and $p_h(\alpha_t) \in \mathfrak{R}(v) \setminus \mathfrak{N}(v)$,
3. S if $j = t$, t is odd, and $p_h(\alpha_t) \notin \mathfrak{R}(v) \setminus \mathfrak{N}(v)$.

The type of an element in an anti-v-chain that is not necessarily connected is defined to be its type in its connected component.

9.8.2 $\mathfrak{O}$-depth of an element in a monomial in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$

The **$\mathfrak{O}$-depth** of an element α in an anti-v-chain C in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ is the depth in $\tilde{\mathfrak{O}}mon_C$ of $p_v(\alpha)$ in case α is of type V or H, and of α (equivalently $\alpha^{\#}$) in case α is of type S. It is denoted by $\mathfrak{O}$-depth$_C(\alpha)$. The $\mathfrak{O}$-depth of an element α in a monomial $\mathfrak{S}$ of $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ is the maximum over all anti-v-chains C in $\mathfrak{S}$ containing α, of the $\mathfrak{O}$-depth of α in C. It is denoted by $\mathfrak{O}$-depth$_{\mathfrak{S}}(\alpha)$.

Finally, the **$\mathfrak{O}$-depth of a monomial** $\mathfrak{S}$ in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ is the maximum of the $\mathfrak{O}$-depth in $\mathfrak{S}$ of all the elements in $\mathfrak{S}$.

9.8.3 Description of the map $\mathfrak{O}\pi$

Recall that $\mathfrak{O}\pi$ is a mapping from $mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ to $\mathfrak{O}I(d) \times mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ such that for a monomial $\mathfrak{S}$ in $mon\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$

$$\mathfrak{O}\pi(\mathfrak{S}) = (w', \mathfrak{S}'),$$

where w' and $\mathfrak{S}'$ are satisfying the conditions of Proposition 9.7.1.

If $\mathfrak{S}$ is empty, no output will be produced (by definition). Suppose now, $\mathfrak{S}$ is non-empty. We first partition $\mathfrak{S}$ into non-empty subsets according to the $\mathfrak{D}$-depth of its elements. Let $\mathfrak{S}_k^{pr}$ be the elements of $\mathfrak{S}$ of $\mathfrak{D}$-depth k. Let α_k denote the last element of $\mathfrak{S}_k^{pr}$ if its elements are arranged of their non-increasing order of their row and column indices. Let j be an odd integer. We set

$$\mathfrak{S}_{j,j+1}^{pr} := \mathfrak{S}_j^{pr} \cup \mathfrak{S}_{j+1}^{pr}.$$

we say that $\mathfrak{S}$ is **truly orthogonal** at j if $p_h(\alpha_j) \in \mathfrak{R}(v) \setminus \mathfrak{N}(v)$. Define:

$$\mathfrak{S}_{j,j+1} := \begin{cases} (\mathfrak{S}_{j,j+1}^{pr} \setminus \{\alpha_j\}) \cup (\mathfrak{S}_{j,j+1}^{pr} \setminus \{\alpha_j\})^{\#} \cup \{p_v(\alpha_j), p_h(\alpha_j)\} \text{ if } \alpha \text{ is truly orthogonal at } j, \\ (\mathfrak{S}_{j,j+1}^{pr}) \cup (\mathfrak{S}_{j,j+1}^{pr})^{\#} \text{ otherwise.} \end{cases}$$

It is clear that no two elements of $\mathfrak{S}_j^{pr}$ are comparable, so $\mathfrak{D}$-depth of any element of $\mathfrak{S}_{j,j+1}^{pr}$ is atmost 2. It is also clear that depth of any element of $\mathfrak{S}_{j,j+1}$ is also atmost 2. Let $\mathfrak{S}_j$ (resp. $\mathfrak{S}_{j+1}$) be the subset (as a multiset) of elements of depth 1 (resp. 2) of $\mathfrak{S}_{j,j+1}$.

Now, for every positive integer k, we apply the map π of [KR03] as defined in §5.2.5 to $\mathfrak{S}_k$ to obtain a pair $(w'(k), \mathfrak{S}'_k)$, where $w'(k)$ is an element of $I(d, 2d)$ and $\mathfrak{S}'_k$ is a monomial in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. Let $\mathfrak{S}_{w'(k)}$ be the distinguished monomial in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ associated to $w'(k)$. Now, we want to define the image $(w', \mathfrak{S}')$ of $\mathfrak{S}$ under $\mathfrak{D}\pi$. We let w' be the element of $I(d, 2d)$ associated to the distinguished subset $\cup_k \mathfrak{S}_{w'(k)}$ of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. Since $\cup_k \mathfrak{S}_{w'(k)}$ is symmetric and has evenly many diagonal elements, it follows from Lemma 9.4.1 that w' is in fact an element of $\mathfrak{D}I(d)$. And we take

$$\mathfrak{S}' := \bigcup_k \mathfrak{S}'_k \cap \widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v)).$$

Define

$$\pi(\mathfrak{S}_{j,j+1}) := (w'_{j,j+1}, \mathfrak{S}'_{j,j+1}) \text{ and } \mathfrak{S}' := \bigcup_{j \text{ odd}} \mathfrak{S}'_{j,j+1} \cap \widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v)).$$

9.9 Description of the map $\mathfrak{D}\phi$ for monomials in $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$

In this section, we mainly describe the map $\mathfrak{D}\phi$ for monomials in $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$. But before going to the description of $\mathfrak{D}\phi$ we need to state some lemmas and a corollary which are given below in §9.9.1. The proofs of these lemmas and corollary are similar to the proofs given in [KR03] of Lemma 4.5, Lemma 4.6, and Corollary 4.7 respectively. The lemmas and the corollary are needed to describe the map $\mathfrak{O}\phi$.

9.9.1 Some important lemmas

Lemma 9.9.1. *Let $\alpha_1 = (r_1, c_1) > \cdots > \alpha_t = (r_t, c_t)$ be an anti-v-chain. Let w' be an element of $I(d, 2d)$ with $w' \leq v$ and $\mathfrak{S}_{w'}$ the distinguished subset of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ associated to w' as defined in Remark 5.2.12. Then w' anti-dominates $\alpha_1 > \cdots > \alpha_t$ if and only if there exists an anti-v-chain $\beta_1 = (R_1, C_1) > \cdots > \beta_t = (R_t, C_t)$ in $\mathfrak{S}_{w'}$ such that $R_j \leq r_j$ and $C_j \geq c_j$ for $1 \leq j \leq t$.*

Lemma 9.9.2. *Let $\beta = (r, c)$ be an element of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ and w' be an element of $I(d, 2d)$ such that $w' \leq s_\beta v$. Then there exists an element (R, C) in the distinguished monomial $\mathfrak{S}_{w'}$ associated to w' such that $R \leq r$ and $C \geq c$.*

Corollary 9.9.3. *let w' be an element of $I(d, 2d)$ such that $w' \leq v$ and $\mathfrak{S}_{w'}$ the corresponding distinguished subset of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. For a positive integer j, let $\mathfrak{S}^j_{w'}$ denote the subset of $\mathfrak{S}_{w'}$ of those elements that are j-deep, and w'^j the corresponding element of $I(d, 2d)$. Let $\alpha_1 = (r_1, c_1) > \cdots > \alpha_t = (r_t, c_t)$ be an anti-v-chain.*

1. *If w'^k anti-dominates $\alpha_1 > \cdots > \alpha_t$, then w'^{k+1} anti-dominates $\alpha_2 > \cdots > \alpha_t$ and w'^{k+2} anti-dominates $\alpha_3 > \cdots > \alpha_t$ and so on.*
2. *If for positive integers $m > k$, there exists (R, C) in $\mathfrak{S}^m_{w'}$ such that $R \leq r_1$ and $C \geq c_1$, and w'^{k+1} does not anti-dominate $\alpha_1 > \cdots > \alpha_t$, then w'^{k+2} does not anti-dominate $\alpha_2 > \cdots > \alpha_t$ and w'^{k+3} does not anti-dominate $\alpha_3 > \cdots > \alpha_t$ and so on until, finally, w'^{m+1} does not anti-dominate $\alpha_{m-k+1} > \cdots > \alpha_t$.*

9.9.2 Description of $\mathfrak{O}\phi$

Before describing the map $\mathfrak{O}\phi$, first we need to define a map ϕ in a similar way as in [KR03].

Consider a monomial $\mathfrak{T}$ in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ which is anti-dominated by w'. Let $\mathfrak{S}_{w'}$ be the distinguished monomial corresponding to w' as in the sense of Remark 5.2.12. Since w' belongs to $\mathfrak{O}I(d)$, so $\mathfrak{S}_{w'}$ is distinguished as well as symmetric and has evenly many elements on the diagonal (by Lemma 9.4.1). The map ϕ is described in the following way: Let k be the maximum length of an anti-v-chain in $\mathfrak{S}_{w'}$. For a positive integer j, $1 \leq j \leq k$, let $\mathfrak{S}^j_{w'}$ be the subset of elements of $\mathfrak{S}_{w'}$ that are j-deep. Let w'^j be the element associated to $\mathfrak{S}^j_{w'}$. Since $w' \leq v$, so we have, $w' = w'^1 \leq w'^2 \leq \cdots \leq w'^k \leq v$. Clearly, $\mathfrak{S}_{w'} = \mathfrak{S}^1_{w'} \supseteq \cdots \supseteq \mathfrak{S}^k_{w'}$.

For a positive integer j, $1 \leq j \leq k$, let $\mathfrak{T}^j_{w'}$ be the subset of $\mathfrak{T}$ of elements β such that β is the head of an anti-v-chain in $\mathfrak{T}$ which is anti-dominated by w'^j but not by w'^{j+1} (we set $w'^{k+1} = v$), and every anti-v-chain in $\mathfrak{T}$ with head β is anti-dominated by w'^j. Then $\mathfrak{T}^j_{w'}$ forms a partition of $\mathfrak{T}$ (some of the $\mathfrak{T}^j_{w'}$ could be empty). Two distinct elements belonging to the same $\mathfrak{T}^j_{w'}$ are not comparable: Let β, β' both be elements of $\mathfrak{T}^j_{w'}$ such that $\beta > \beta'$. Since β' belongs to $\mathfrak{T}^j_{w'}$ so there exists an anti-v-chain $\beta' > \cdots$ in $\mathfrak{T}$ that is anti-dominated by w'^j but not w'^{j+1}, so by Lemma 9.9.3 the anti-v-chain $\beta > \beta' > \cdots$ is not anti-dominated by w'^j, a contradiction because β belongs to $\mathfrak{T}^j_{w'}$.

We now further partition each $\mathfrak{T}^j_{w'}$ into subsets called **pieces** as follows:
Let $\mathfrak{S}_{w',j}$ denote the set of all elements of $\mathfrak{S}_{w'}$ that are j-deep but not $(j+1)$-deep. Clearly, no two distinct elements of $\mathfrak{S}_{w',j}$ are comparable.

Lemma 9.9.4. *For an element (r, c) of $\mathfrak{T}^j_{w'}$, there exists a unique element (R, C) of $\mathfrak{S}_{w',j}$ such that $R \leq r$ and $c \leq C$.*

Proof. The proof of this lemma is analogous to the proof of [KR03, Lemma 4.17]. Hence we omit the proof. □

We will index the pieces of $\mathfrak{T}^j_{w'}$ by elements of $\mathfrak{S}_{w',j}$. Let $\beta = (R, C)$ be an element of $\mathfrak{S}_{w',j}$. Then the corresponding piece p_β of $\mathfrak{T}^j_{w'}$ consists of all those (r, c) in $\mathfrak{T}^j_{w'}$ with $R \leq r$ and $c \leq C$. Of course, some of these piece could be empty. Let us arrange the elements of p_β in non-increasing order of the row entries; among those with equal row entries the arrangement is by non-increasing order of column entries. Since no two distinct elements of $\mathfrak{T}^j_{w'}$ are comparable, the column entries are also in non-increasing order. Suppose the arrangement is

$$(r_1, c_1), (r_2, c_2), \cdots, (r_p, c_p).$$

Note that $C \geq c_1$ and $r_p \geq R$. Let $p^\star_\beta$ denote the monomial

$$\{(r_1, C), (r_2, c_1), \cdots, (r_p, c_{p-1}), (R, c_p)\}.$$

Clearly, the elements of $p^\star_\beta$ also belong to $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$. Set

$$(\mathfrak{T}^j_{w'})^\star := \cup_{\beta \in \mathfrak{S}_{w',j}} p^\star_\beta,$$

$$\phi(w', \mathfrak{T}) := \cup_j (\mathfrak{T}^j_{w'})^\star.$$

This finishes the description of ϕ.

We now describe the map $\mathfrak{D}\phi$ for monomials in $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$. Let w', v, and $\mathfrak{S}_{w'}$ be as in above. Let $\mathfrak{T}$ be a monomial in $\widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ which is anti-$\mathfrak{D}$-dominated by w'. For an odd integer j, let $\mathfrak{S}^j_{w'}$ (respectively $\mathfrak{S}_{w',j,j+1}$) denote the subset $\mathfrak{S}_{w'}$ consisting of those elements that are j-deep (respectively that are j-deep but not $(j+2)$-deep, or equivalently of depth j and $(j+1)$) in $\mathfrak{S}_{w'}$ in the sense of §5.2.5. Since $\mathfrak{S}_{w'}$ is distinguished, symmetric, and has evenly many elements on the diagonal, so $\mathfrak{S}^j_{w'}$ and $\mathfrak{S}_{w',j,j+1}$ too have these properties. In fact, the number of diagonal elements of $\mathfrak{S}_{w',j,j+1}$ is either 0 or 2 (in the latter case, the elements have to be distinct, since $\mathfrak{S}_{w'}$ is distinguished and so is multiplicity free). Let us denote by w'^j and $w'_{j,j+1}$ the elements of $\mathfrak{O}I(d)$ corresponding to $\mathfrak{S}^j_{w'}$ and $\mathfrak{S}_{w',j,j+1}$ in the sense of Remark 5.2.12.

Let $\mathfrak{T}_{w',j,j+1}$ denote the subset of $\mathfrak{T}$ consisting of those elements α such that

- every anti-v-chain in $\mathfrak{T}$ with head α that is anti-$\mathfrak{O}$-dominated by w'^j,
- there exists an anti-v-chain in $\mathfrak{T}$ with head α that is not anti-$\mathfrak{O}$-dominated by w'^{j+2}.

It is evident that the subsets $\mathfrak{T}_{w',j,j+1}$ are disjoint (as j varies over the odd integers) and that their union is all of $\mathfrak{T}$ ($w' = w'^1$ anti-$\mathfrak{O}$-dominates all anti-v-chains in $\mathfrak{T}$). In other words, $\mathfrak{T}_{w',j,j+1}$ forms a partition of $\mathfrak{T}$.

Lemma 9.9.5. *1. The length of an anti-v-chain in $\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1}$ is at most 2.*

2. $w'_{j,j+1}$ anti-$\mathfrak{O}$-dominates $\mathfrak{T}_{w',j,j+1}$.

Proof. The proof of this lemma is analogous to the proof of [Upa08, Lemma 8.1.1]. Hence we omit the proof. □

Corollary 9.9.6. *$w_{j,j+1}$ anti-dominates $\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1}$ in the sense of Definition 9.3.3.*

Proof. The proof of this corollary is analogous to the proof of [Upa08, Corollary 8.1.2]. Hence we omit the proof. □

We now apply the map ϕ to the pair $(w_{j,j+1}, \mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1})$ in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ to obtain a monomial $(\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1})^{\star}$. According to the description of ϕ, it is clear that

$$(\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1})^{\star} = \cup_{\beta\in\mathfrak{S}_{w',j,j+1}} p^{\star}_{\beta}.$$

Suppose that $(\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1})^{\star}$ contains the pair $(a, a^*), (b, b^*)$ of diagonal elements with $a^* > b^*$. We call the pair $(b, a^*), (a, b^*)$ the "twists", and set $\delta_j := (b, a^*)$ (if $(\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1})^{\star}$ has diagonal elements: δ_j exists; then we say w' is diagonal at j).

With this notation, define the new monomial:

$$(\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1})^{*}_{w'} := \begin{cases} (\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1})^{*} \text{ if } w' \text{ is not diagonal at } j \\ \{(\mathfrak{T}_{w',j,j+1} \cup \mathfrak{T}^{\#}_{w',j,j+1})^{*} \setminus \mathfrak{d}(v)\} \cup \{\delta_j, \delta^{\#}_j\} \text{ if } w' \text{ is diagonal at } j, \end{cases}$$

where $\mathfrak{d}(v)$ denotes the diagonal.

This new monomial is symmetric and contains no diagonal elements. Its intersection with $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ is denoted by $\mathfrak{T}^*_{w',j,j+1}$. The union of $\mathfrak{T}^*_{w',j,j+1}$ over all odd integers j is defined by $\mathfrak{T}^*_{w'}$, the result of $\mathfrak{O}\phi$ applied to $(w', \mathfrak{T})$. This finishes the description of the map $\mathfrak{O}\phi$.

Example 9.9.7 below elaborate that $\mathfrak{O}\pi \circ \mathfrak{O}\phi = \text{identity}$.

Example 9.9.7. *Let $v = (4.6, 7, 10, 12, 13, 14)$ and $w' = (1, 2, 3, 4, 5, 8, 9)$. Clearly, v and w' belongs to $\mathfrak{O}I(d)$. Now, $w' \setminus v = \{1, 2, 3, 5, 8, 9\}$ and $v \setminus w' = \{6, 7, 10, 12, 13, 14\}$. So,*

$$\mathfrak{S}_{w'} = \{(9, 10), (8, 12), (5, 6), (3, 7), (2, 13), (1, 14)\}$$

$$\textit{Now } \mathfrak{S}_{w'} = \mathfrak{S}^1_{w'} = \{(9, 10), (8, 12), (5, 6), (3, 7), (2, 13), (1, 14)\},$$
$$\mathfrak{S}^2_{w'} = \{(5, 6), (3, 7), (2, 13), (9, 10), (8, 12)\},$$
$$\mathfrak{S}^3_{w'} = \{(5, 6), (3, 7), (9, 10), (8, 12)\},$$
$$\mathfrak{S}^4_{w'} = \{(5, 6), (9, 10)\}.$$

Then

$$w' = w'^1 = (1, 2, 3, 4, 5, 8, 9),\ \ w'^2 = (2, 3, 4, 5, 8, 9, 14),\ \ w'^3 = (3, 4, 5, 8, 913, 14),$$

$$\textit{and } w'^4 = (4, 5, 7, 9, 12, 13, 14),$$

where $\mathfrak{S}^j_{w'}$ denotes the elements of $\mathfrak{S}_{w'}$ of depth-j and w'^j denotes the associated element in $\mathfrak{O}I(d)$. Clearly, $w'^1 \le w'^2 \le w'^3 \le w'^4$. Let

$$\mathfrak{T} = \{(3, 14), (5, 13), (8, 13), (11, 12)\}.$$

Clearly, $\mathfrak{T}$ is a monomial in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$.

In Figure 9.2, the open circles denote the points of $\mathfrak{S}_{w'}$, the dark circles denote the points of $\mathfrak{T}$, and d_i denotes the depth of the corresponding elements of $\mathfrak{S}_{w'}$ and $\mathfrak{O}$-depths of the elements of $\mathfrak{T}$. Clearly, $\mathfrak{T}$ is anti-$\mathfrak{O}$-dominated by w'. Now,

$$\mathfrak{T}_{w',1,2} = \{(3, 4), (5, 13), (8, 13)\},\ \ \mathfrak{T}_{w',3,4} = \{(11, 12)\},\ \ \textit{and}$$

$$\mathfrak{S}_{w',1,2} = \{(1, 14), (2, 13)\},\ \ \mathfrak{S}_{w',3,4} = \{(9, 10), (8, 12), (5, 6), (3, 7)\}.$$

Again,

$$p_{(2,13)} = \{(8, 13), (5, 13), (2, 10), (2, 7)\},$$
$$p_{(1,14)} = \{(3, 14), (1, 12)\},$$

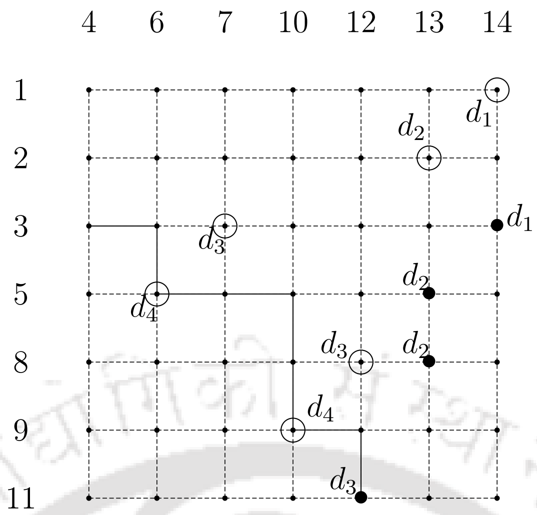


Figure 9.2: The monomials $\mathfrak{S}_{w'}$ and $\mathfrak{T}$.

$$p_{(5,6)} = \emptyset,$$

$$p_{(3,7)} = \{(3,4)\},$$

$$p_{(9,10)} = \emptyset,$$

$$p_{(8,12)} = \{(11,12)\},$$

where $\emptyset$ denotes the empty set. So,

$$p^*_{(2,13)} = \{(8,13),(5,13),(2,13),(2,10),(2,7)\},$$

$$p^*_{(1,14)} = \{(3,14),(1,14),(1,12)\},$$

$$p^*_{(5,6)} = \{(5,6)\},$$

$$p^*_{(3,7)} = \{(3,7),(3,4)\},$$

$$p^*_{(9,10)} = \{(9,10)\},$$

$$p^*_{(8,12)} = \{(11,12),(8,12)\}.$$

Therefore,

$$(\mathfrak{T}_{w',1,2} \cup \mathfrak{T}^{\#}_{w',1,2})^* = \bigcup_{\beta \in \mathfrak{S}_{w',1,2}} p^*_\beta$$

$$= \{(8,13),(5,13),(2,13),(2,10),(2,7),(3,14),(1,14),(1,12)\},$$

$$\text{and } (\mathfrak{T}_{w',3,4} \cup \mathfrak{T}^{\#}_{w',3,4})^* = \bigcup_{\beta \in \mathfrak{S}_{w',3,4}} p^*_\beta$$

$$= \{(11,12),(9,10),(8,12),(5,6),(3,7),(3,4)\}.$$

Now, $(\mathfrak{T}_{w',1,2} \cup \mathfrak{T}^{\#}_{w',1,2})^{*}$ *contains the pair* $(1,14)$ *and* $(2,13)$ *of diagonal elements with* $14 > 13$*, so* $\delta_1 = \{(2,14)\}$*. So,*

$$(\mathfrak{T}_{w',1,2} \cup \mathfrak{T}^{\#}_{w',1,2})^{\star}_{w'} = (\{(8,13),(5,13),(3,14),(2,13),(2,10),(2,7),(1,14),(1,12)\}$$

$$\setminus\{(1,14),(2,13)\}) \cup \{(2,14),(1,13)\}$$

$$= \{(8,13),(5,13),(3,14),(2,14),(1,13)\},$$

$$\textit{and } (\mathfrak{T}_{w',3,4} \cup \mathfrak{T}^{\#}_{w',3,4})^{*}_{w'} = \{(11,12),(9,10),(8,12),(5,6),(3,7),(3,4)\}.$$

$$\textit{Therefore } \mathfrak{T}^{*}_{w',1,2} = (\mathfrak{T}_{w',1,2} \cup \mathfrak{T}^{\#}_{w',1,2})^{*}_{w'} \cap \widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$$

$$= \{(3,14),(2,14),(8,13),(5,13)\}$$

$$\textit{and } \mathfrak{T}^{*}_{w',3,4} = (\mathfrak{T}_{w',3,4} \cup \mathfrak{T}^{\#}_{w',3,4})^{*}_{w'} \cap \widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$$

$$= \{(11,12),(8,12),(9,10)\}.$$

Hence,

$$\mathfrak{T}^{*}_{w'} = \{(3,14),(2,14),(8,13),(5,13),(11,12),(8,12),(9,10)\}.$$

Now, we want to apply the map $\mathfrak{D}\pi$ *on* $\mathfrak{T}^{*}_{w'}$ *and see that* $\mathfrak{D}\pi \circ \mathfrak{D}\phi =$ *identity. Let us denote*

$$\mathfrak{T}^{*}_{w'} = \{(3,14),(2,14),(8,13),(5,13),(11,12),(8,12),(9,10)\} \textit{ by } \mathfrak{S}.$$

The monomial $\mathfrak{S}$ *is shown in Figure 9.3. The dark circles indicate the elements that occur in* $\mathfrak{S}$ *with non-zero multiplicity and the stars indicate the diagonal elements. For every element we are considering multiplicity one. The* $\mathfrak{D}$*-depth of* $\mathfrak{S}$ *is four. Figure 9.4 shows the monomials* $\mathfrak{S}^{pr}_{1,2}$ *and* $\mathfrak{S}^{pr}_{3,4}$*. Solid dots and open circles indicate elements of these monomials respectively. Since the horizontal projection of* α_1 *(here* $\alpha_1 = (2,14)$*)* $\in \mathfrak{R}(v) \setminus \mathfrak{N}(v)$ *but the horizontal projection of* α_2 *(here* $\alpha_2 = (8,12)$*)* $\notin \mathfrak{R}(v) \setminus \mathfrak{N}(v)$*, so the monomial* $\mathfrak{S}$ *is truly orthogonal at 1 but not at 3. Figure 9.5 shows the monomials* $\mathfrak{S}_{1,2}$ *and* $\mathfrak{S}_{3,4}$*, and also their decomposition in blocks. Figure 9.6 shows the monomials* $\mathfrak{S}'_{1,2}$ *and* $\mathfrak{S}'_{3,4}$*.*

From Figure 9.5 we can say,

$$\mathfrak{S}_{w'} = \{(1,14),(2,13),(3,7),(8,12),(5,6),(9,10)\}.$$

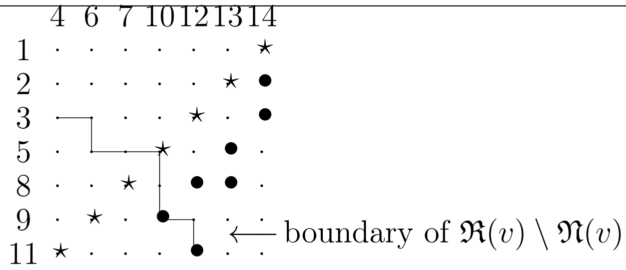


Figure 9.3: The monomial $\mathfrak{S}$

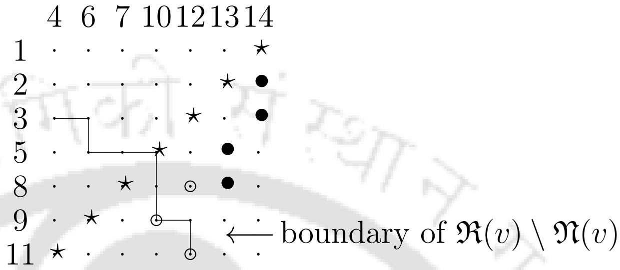


Figure 9.4: The monomials $\mathfrak{S}^{pr}_{1,2}$ and $\mathfrak{S}^{pr}_{3,4}$

Hence,

$$w' = (1, 2, 3, 4, 5, 8, 9).$$

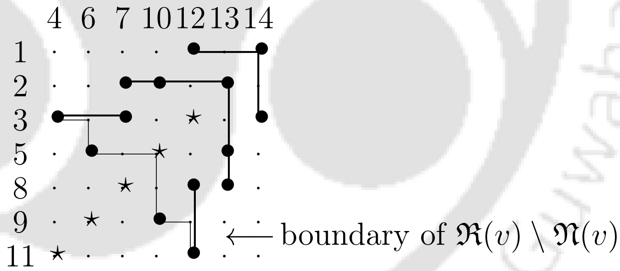


Figure 9.5: The grid representing the monomials $\mathfrak{S}_{1,2}$ and $\mathfrak{S}_{3,4}$

Now, we know

$$\mathfrak{S}' = \bigcup_{j \text{ odd}} \mathfrak{S}'_{j,j+1} \cap \widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v)).$$

So in this case,

$$\mathfrak{S}' = \{(3, 14), (5, 13), (8, 13), (11, 12)\},$$

which is clearly equal to $\mathfrak{T}$. *Hence,* $\mathfrak{O}\pi(\mathfrak{T}^*_{w'}) = \mathfrak{O}\pi(\mathfrak{S}) = (w', \mathfrak{T})$. *That is* $\mathfrak{O}\pi \circ \mathfrak{O}\phi =$ *identity.*

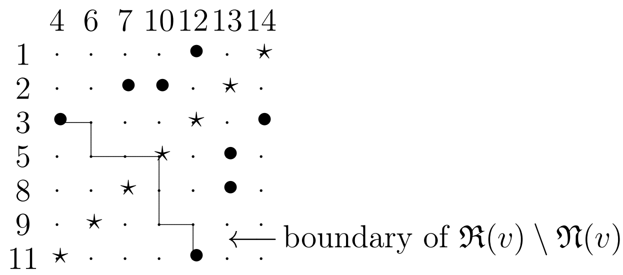


Figure 9.6: The monomials $\mathfrak{S}'_{1,2}$ and $\mathfrak{S}'_{3,4}$

9.10 Multiplicity counts using certain lattice paths

Using Corollary 9.5.2 below we show that the multiplicity of the Richardson variety $X^{w}_{w'}$ in $\mathfrak{M}_d(V)$ at the point e^v can be interpreted as the cardinality of a certain set of non-intersecting lattice paths.

9.10.1 Description and illustration

We will illustrate the above on the sets $\mathfrak{O}\mathfrak{N}(v)$ and $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$. On the set $\mathfrak{O}\mathfrak{N}(v)$ the illustration is already given in [Upa08]. So, we will illustrate the above on the set $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$.

The points $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ can be represented in a natural way, as the lattice points of a grid. The column indices of the points of the grid are the entries of v and the row indices are the entries of $\{1, \cdots, 2d\} \setminus v$. Let $\mathfrak{S}_{w'}$ denote the distinguished monomial in $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$ associated to w'. From any point β of $\mathfrak{S}_{w'}$(down) let us draw a vertical line downwards from β and let β**(start)** denote the bottom most point of $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ on this line. In case β is not on the diagonal draw also a horizontal line leftwards from β and let β**(finish)** denote the left most point of $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ on this line. In case β is on the diagonal, then β(finish) is not a fixed point but varies subject to the following constraints:

- β(finish) is one step away from the diagonal.
- The column index of β(finish) is less than or equal than that of β.
- If $\text{depth}_{\mathfrak{S}_{w'}}\beta$ is odd, then the horizontal projection of β(finish) is the same as the vertical projection of γ(finish), where γ is the diagonal element of $\mathfrak{S}_{w'}$ of depth 1 more than that of β.

A lattice path in $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ between a pair of points β(start) and β(finish) is a sequence $\alpha_1, \cdots, \alpha_q$ of elements of $\widetilde{\mathfrak{O}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ such that $\alpha_1 = \beta$(start) and $\alpha_q = \beta$(finish), and for $1 \leq j \leq q-1$, α_{j+1} is either the next point on the right of α_j or below α_j in the grid.

Consider the set $\mathbf{Paths}_{w'}^{w}$ of all tuples $(\Lambda_\beta)_{\beta\in\mathfrak{S}_{w'}(\text{up})\cup\mathfrak{S}_{w'}(\text{down})}$ (recall the notation $\mathfrak{N}(v)$, $\mathfrak{O}\mathfrak{N}(v)$, and $\mathfrak{d}(v)$ from §4.1, then for a monomial $\mathfrak{S}$ of $\mathfrak{N}(v)$, $\mathfrak{S}(\text{up}) = \mathfrak{S}\cap(\mathfrak{O}\mathfrak{N}(v)\cup \mathfrak{d}(v)))$ of paths, where

- Λ_β is a lattice path between $\beta(\text{start})$ and $\beta(\text{finish})$ (if β is on the diagonal, then $\beta(\text{finish})$ is allowed to vary in the manner described above).
- Λ_β and Λ_γ do not intersect for $\beta \neq \gamma$.

The number of such p-tuples, where $p := |\mathfrak{S}_{w'}(\text{up})\cup\mathfrak{S}_{w'}(\text{down})|$ is the multiplicity of $X_{w'}^{w}$ at e^v.

Example 9.10.1. *Let* $v = (4,6,7,10,12,13,14)$, $w = (5,6,7,11,12,13,14)$ *and* $w' = (1,2,3,4,5,8,9)$. *Clearly,* $w' \leq v \leq w$ *and* $v, w, w' \in \mathfrak{O}I(d)$. *Now,* $v\backslash w' = \{6,7,10,12,13,14\}$, $v \setminus w = \{4,10\}$, $w \setminus v = (5,11)$, *and* $w' \setminus v = \{1,2,3,5,8,9\}$. *So,*

$$\mathfrak{S}_{w'} = \{(5,4),(11,10)\}.$$

And

$$\mathfrak{S}_{w'}(up) = \{(5,4)\}.$$

$$\mathfrak{S}'_{w'} = \{(9,10),(8,12),(5,6),(3,7),(2,13),(1,14)\}.$$

And

$$\mathfrak{S}'_{w'}(down) = \{(1,14),(2,13),(8,12),(9,10)\}.$$

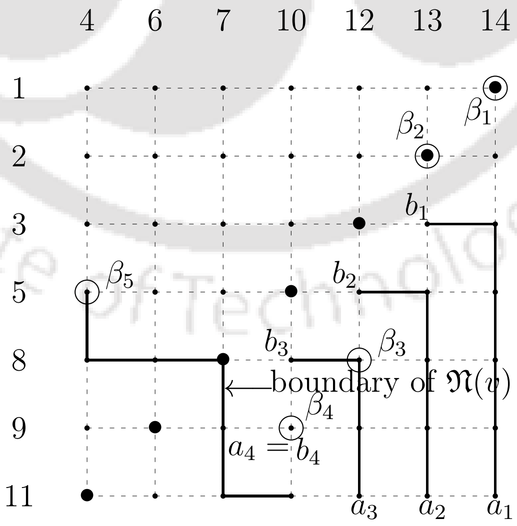


Figure 9.7: An element of $\text{Paths}_{w'}^{w}$.

In Figures 9.7 and 9.8, the zigzag line which starts from $(5,4)$ *and ends at* $(11,10)$ *denotes the boundary of* $\mathfrak{N}(v)$. *The solid dots indicate the points on the diagonal. The*

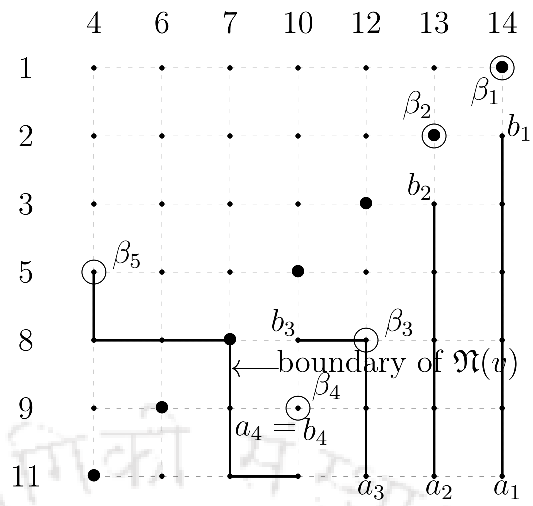


Figure 9.8: Another element of $\text{Paths}_{w'}^{w}$.

circles denote the points β_i, $1 \leq i \leq 5$, where β_i, $1 \leq i \leq 4$ denote the points in $\mathfrak{S}_{w'}$(down) and β_5 denotes the point in $\mathfrak{S}_w$(up). Here a_i denote the points β_i(start) and b_i denote the points β_i(finish) for $1 \leq i \leq 4$. If we draw the boundary of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$, then we can see that the point $\beta_4 = (9, 10)$ lies the boundary of $\mathfrak{R}(v) \setminus \mathfrak{N}(v)$, so it is the only possible path originating from β_4. Since $\mathfrak{S}_w$(up) is a singleton point which lies on the boundary of $\mathfrak{N}(v)$, the point $\beta_5 = (5, 4)$ itself is the only possible path in $\mathfrak{N}(v)$ originating from β_5. So, in this case, multiplicity is 2.

9.10.2 Justification for the interpretation

We now justify the interpretation of the multiplicity of Corollary 9.5.2. Corollary 9.5.2 says that the multiplicity is the number of monomials in $\mathfrak{D}\mathfrak{N}(v) \cup \widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ of maximal cardinality that are square-free, $\mathfrak{D}$-dominated by w, and anti-$\mathfrak{D}$-dominated by w'. We now establish a bijection between the set $S_{w'}^{w}$ of such monomials and the set $\text{Paths}_{w'}^{w}$ of non-intersecting lattice paths.

Each element Λ of $\text{Paths}_{w'}^{w}$ can be thought, as a monomial in $\mathfrak{D}\mathfrak{N}(v) \cup \widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$. Let us denote the corresponding monomial by Λ. It is clear that the monomial Λ is square-free and that all such monomials Λ have the same cardinality (in particular, that if $\Lambda_1 \subseteq \Lambda_2$ for two such monomials then $\Lambda_1 = \Lambda_2$). So, to establish the bijection it therefore suffices to prove the following.

Lemma 9.10.2. *1. w is the element of $\mathfrak{D}I(d)$ obtained on application of $\mathfrak{D}\pi$ to the monomial $\Lambda \cap \mathfrak{D}\mathfrak{N}(v)$ and w' is the element of $\mathfrak{D}I(d)$ obtained on application of $\mathfrak{D}\pi$ to the monomial $\Lambda \cap \widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$.*

2. Given any monomial $\mathfrak{T}$ of elements of $\mathfrak{D}\mathfrak{N}(v) \cup \widetilde{\mathfrak{D}}(\mathfrak{R}(v) \setminus \mathfrak{N}(v))$ that is square-free as well as $\mathfrak{D}$-dominated by w, anti-$\mathfrak{D}$-dominated by w', there exists Λ such that $\mathfrak{T} \subseteq \Lambda$.

Proof. The proof of the above lemma is analogous to the proof of [Upa08, Proposition 11.2.1]. So, here we omit the proof. □

CHAPTER 10

FUTURE PLANS

10.1 Future Questions

Problem 10.1.1. *To compute multiplicity at any point on a Schubert variety in G/P, where $G = Sp(2n)$ and P is any parabolic in G.*

Problem 10.1.2. *In [AIJK20], Ikeda and his co-authors have computed the multiplicity of any Schubert variety in the symplectic flag variety. I plan to extend this work to the orthogonal flag variety.*

Problem 10.1.3. *In the paper [BC12], the authors discuss three natural generalizations of Richardson varieties, which they call projection varieties, intersection varieties, and rank varieties.*
I am planning to compute the following:

1. *The singular loci of projective varieties of Type B and Type C Grassmannian.*
2. *The multiplicity of intersection varieties in a non-minuscule partial flag variety.*
3. *The multiplicity of rank varieties in any partial flag variety.*

PUBLICATIONS

1. Ray, P., Upadhyay, S. (2022). *Schubert varieties in the Grassmannian and the symplectic Grassmannian via a bounded RSK correspondence* (Published) (Indian Journal of Pure and Applied Mathematics, DOI: 10.1007/s13226-022-00334-6).
2. Upadhyay, S., Ray, P. : *Initial ideals of tangent cones to Richardson varieties in the Symplectic Grassmannian*, arXiv:1905.01660 (2019) (Submitted).
3. Ray, P., Upadhyay, S. : *Multiplicity at any torus-fixed point in a Richardson variety in the symplectic Grassmannian*, arXiv:2002.07074 (2020) (Submitted).

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INDEX