Some Aspects of Poisson Transform on Homogeneous Trees

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July, 2020

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A thesis submitted
in partial fulfilment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

by

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to the

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July, 2020

Declaration

I hereby declare that the work contained in the thesis entitled **"Some Aspects of Poisson Transform on Homogeneous Trees"** has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under the guidance of **Dr. Pratyoosh Kumar**, Indian Institute of Technology Guwahati, for the award of **Doctor of Philosophy** and that this work has not been submitted elsewhere for a degree.

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Certificate

It is certified that the work contained in the thesis titled **"Some Aspects of Poisson Transform on Homogeneous Trees"** by **Sumit Kumar Rano (156123004)**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of **Doctor of Philosophy** has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

Guwahati
July 2020

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Dedicated

To

My Family

Acknowledgement

It is with a sense of gratitude, I acknowledge the efforts of entire hosts of well wishers who have in some way or the other contributed in their own special ways to the success and completion of this dissertation.

To my supervisor Dr. Pratyoosh Kumar, I owe my greatest thanks. His passion towards learning and humble nature has always created a productive environment around me. Through the countless discussions we had, I have always been fascinated by his way of looking at things and his mathematical maturity. I am deeply indebted and sincerely grateful for his patient guidance, encouragement and advice he has provided throughout my time as his student. I have been extremely lucky to have a supervisor who cared so much about my work and who responded to my questions and queries so promptly. I am sure that his teachings will stay with me and create a positive impact in times to come.

I wish to express my warm and sincere thanks to Prof. M. G. P. Prasad, Head of the Department and Prof. Swaroop Nandan Bora, Prof. N. Selvaraju, former Heads of the Department for providing excellent research facilities during my Ph.D programme. I further wish to express my deepest appreciation to the members of my dissertation committee: Dr. Anjan Kumar Chakrabarty, Dr. Jitendriya Swain and Dr. Arup Chattopadhyay for generously offering their time, support, guidance and goodwill throughout my tenure as a research scholar. I am also grateful to all my teachers at IIT Guwahati for their teachings and the various discussions I had with them.

To my dearest friends Sunil and Rakesh, I am very much indebted for their timely help and suggestion in preparing the figures of this dissertation. I equally thank my friends Somnath, Rupak, Shantiram, Abhijit, Shyam and Shamik who made my stay enjoyable. I owe a word of thanks to all the research scholars whom I could not mention personally but helped me directly or indirectly. I further take a moment to thank my school friends Shwetabh, Vishal, Manish and Nitish, and my IITM friends Ujjal and Sukanta for continuously entertaining me.

I extend my sincere thanks to all the staff members of Department of Mathematics for their cooperation and help during this period.

I am highly grateful to the Ministry of Human Resource Development, Government of India for the necessary financial supports. I sincerely acknowledge Indian Institute of Technology Guwahati for providing a very nice educational environment and all kinds of support.

At last but not the least, I would like to express my sincere gratitude towards my parents whose continuous support, blessings and encouragement has played a positive role throughout my life. I am also grateful to my brothers Subham, Amit and Parimal for standing beside me in every aspect. In this occasion I feel extremely honoured to dedicate this thesis to my family.

Sumit Kumar Rano

Abstract

In this thesis, we study certain properties of the eigenfunctions of the Laplacian and their application in harmonic analysis on homogeneous trees. The topics we study in the thesis are the following:

First we characterize all eigenfunctions of the Laplacian on homogeneous trees, which are the Poisson transform of L^p functions defined on the boundary. Using the duality argument, we also prove the restriction theorem for the Helgason-Fourier transforms on a homogeneous tree.

In 1980, J. Roe proved that if $\{f_k\}_{k\in\mathbb{Z}}$ is doubly infinite sequence of functions in $\mathbb{R}$ which is uniformly bounded and satisfies $df_k/dx = f_{k+1}$ for all $k \in \mathbb{Z}$ then $f_0(x) = a\sin(x+\theta)$ for some $a, \theta \in \mathbb{R}$. Later in 1993, Strichartz suitably extended the above result to $\mathbb{R}^n$. The second topic of this thesis mainly deals with some variant of their results on homogeneous trees.

At last we study the chaotic dynamics of semigroups generated by the Laplacian. Let f be a non-constant complex holomorphic function defined on a connected open set containing the L^p-spectrum of the Laplacian $\mathcal{L}$ on a homogeneous tree. In this work we give a necessary and sufficient condition for the semigroup $T(t) = e^{tf(\mathcal{L})}$ to be chaotic on L^p-spaces. Apart from this, we also study the dynamical behaviour of the semigroups generated by affine functions. It includes some of the important semigroups such as the heat semigroup and the Schrödinger semigroup.

Abbreviation and Notation

$\mathbb{N}$	The set of all natural numbers
$\mathbb{Z}^n$	$\{(k_1, k_2, \ldots, k_n) \mid k_i \in \mathbb{Z}\}$, $n \geq 1$ and $\mathbb{Z}$ the set of all integers
$\mathbb{Z}_+$	$\{k \in \mathbb{Z} \mid k \geq 0\}$
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$, $n \geq 1$ and $\mathbb{R}$ the set of all real numbers
$\mathbb{C}^n$	$\{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C}\}$, $n \geq 1$ and $\mathbb{C}$ the set of all complex numbers
$\operatorname{Re} z$	The real part of $z \in \mathbb{C}$
$\Im z$	The imaginary part of $z \in \mathbb{C}$
τ	$2\pi/\log q$
$\mathbb{T}$	$[-\tau/2, \tau/2)$
$f \asymp g$	There exists positive constants C_1 and C_2 such that $C_1 f \leq g \leq C_2 f$
p'	The conjugate exponent of p, page 15
δ_p	$\frac{1}{p} - \frac{1}{2}$, page 15
S_p	$\{z \in \mathbb{C} : \lvert\Im z\rvert \leq \lvert\delta_p\rvert\}$, page 15
S_p°	interior of S_p, page 15
∂S_p	boundary of S_p, page 15
S^{n-1}	The unit sphere in $\mathbb{R}^n$, page 4
$\mathcal{X}_E$	The characteristic function of the set E, page 20
$d_f(s)$	distribution function of f, page 15

$f^*(t)$	non increasing rearrangement of f, page 15
$L^{p,q}(M)$	Lorentz space, page 15
$\|f\|_{p,q}$	Lorentz space norm of f, page 15
$L^p(M)$	$\{f : M \to \mathbb{C} \mid f$ is measurable and $\int_M \|f\|^p dm < \infty\}$, page 15
$\sigma_p(T)$	L^p-spectrum of T, page 15
$P\sigma_p(T)$	L^p point spectrum of T, page 15
$\|T\|_{p\to p}$	The operator norm of T, page 15
X	A symmetric space of non-compact type, page 2
Δ	Laplace-Beltrami operator on symmetric spaces, page 2
$\Delta_{\mathbb{R}^n}$	Laplacian on $\mathbb{R}^n$, page 6
$\mathfrak{X}$	A homogeneous tree of degree $q+1$, page 1, 16
$S(x,n)$	page 16
$B(x,n)$	page 16
G	group of isometries on $\mathfrak{X}$, page 16
K	The rotation subgroup of G, page 16
$f * g$	convolution of f and g, page 17
$E(\mathfrak{X})^\#$	The subspace of all radial functions, in a function space $E(\mathfrak{X})$, page 17
Rf	The radialization of f defined on $\mathfrak{X}$, page 17
Ω	The boundary of $\mathfrak{X}$, page 19
$c(x,\omega)$	page 19
$E_j(x)$	page 19
$E(x)$	$E_{\|x\|}(x)$, page 19
ν	The probability measure on Ω, page 19
$h_\omega(x)$	The height of x with respect to ω, page 20
$p(x,\omega)$	The Poisson kernel on $\mathfrak{X}$, page 20
$\mathcal{L}$	Laplacian on $\mathfrak{X}$, page 20
δ_o	page 21
μ_1	page 21
$\mathbb{E}_z(\mathfrak{X})$	Eigenspace of $\mathcal{L}$, page 27
$\gamma(z)$	page 2, 26
$\mathcal{L}_\mathbb{Z}$	Laplacian on $\mathbb{Z}$, page 65

τ_g	page 21
$\mathcal{M}_n$	page 22
$\mathcal{K}_n(\Omega)$	page 22
$\mathcal{K}(\Omega)$	The space of all cylinderical functions, page 22
$\mathcal{E}_n F$	The conditional expectation of F defined on Ω, page 22
$\Delta_n(F)$	n-th difference of F, page 23
$\mathcal{K}'(\Omega)$	The space of all martingales on Ω, page 24
$\mathcal{P}_z F$	The Poisson transform on Ω, page 1, 25
π_z	The representation of G, page 25
ϕ_z	The elementary spherical function, page 27
$\mathbf{c}(\cdot)$	Harish-Chandra's c-function, page 27
$\hat{f}(z)$	The spherical transform of f defined on $\mathfrak{X}$, page 31
$\mathcal{S}_p(\mathfrak{X})$	The p-Schwartz spaces of rapidly decreasing functions, page 32
$\nu_m(\cdot)$	seminorm on $\mathcal{S}_p(\mathfrak{X})$, page 32
$\mathcal{S}(\mathbb{Z})$	The Schwartz on $\mathbb{Z}$,page 83
$\lambda'_n(\cdot)$	seminorm on $\mathcal{S}(\mathbb{Z})$, page 83
$\mathcal{H}(S_p)^\#$	page 33
$\mu_m(\cdot)$	seminorm on $\mathcal{H}(S_p)^\#$, page 33
$\mathcal{C}^\infty(\mathbb{T})$	page 83
$\mu'_l(\cdot)$	seminorm on $\mathcal{C}^\infty(\mathbb{T})$, page 83
$\widetilde{f}(z,\omega)$	The Helgason Fourier transform of a function f defined on $\mathfrak{X}$, page 34
I_z	The intertwining operator, page 53
b	$\frac{2\sqrt{q}}{q+1}$, page 68
$T(t)$	A semigroup, page 8
$\mathbb{B}_{per}$	Set of all periodic points in $\mathbb{B}$, page 8
X_0	page 89
X_∞	page 89
$\Phi_p(a)$	page 95

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CHAPTER 1

Introduction

A homogeneous tree $\mathfrak{X}$ of degree $q+1$ is a connected graph with no loops, in which every vertex is adjacent to $q+1$ other vertices. We shall henceforth assume $q \geq 2$, unless specified. We denote by $d(x,y)$ the natural distance between any two vertices x and y, which is the number of edges joining them. The canonical Laplacian $\mathcal{L}$ on $\mathfrak{X}$ is then defined by

$$\mathcal{L}f(x) = f(x) - \frac{1}{q+1} \sum_{y:d(x,y)=1} f(y).$$

Let Ω be the boundary of $\mathfrak{X}$. For $z \in \mathbb{C}$, the Poisson transform $\mathcal{P}_z F$ of a function $F \in L^2(\Omega)$ is defined by

$$\mathcal{P}_z F(x) = \int_{\Omega} p^{1/2+iz}(x,\omega) F(\omega) d\nu(\omega),$$

where $p(x,\omega)$ denotes the Poisson kernel on $\mathfrak{X}$. A detailed information regarding these topics has been gathered in Chapter 2, however for more exhaustive treatments we refer [12, 13, 14]. Here it is worth mentioning that we use a different parametrization of the Poisson transform from that of the above mentioned references. Our $\mathcal{P}_z F$ corresponds to their $\mathcal{P}_{1/2+iz} F$ and all other related quantities are accordingly reparametrized.

1.1 Eigenfunctions of the Laplacian

In [13] Figà-Talamanca and Picardello proved that if $F \in L^2(\Omega)$, then $\mathcal{P}_z F$ is an eigenfunction of $\mathcal{L}$ with the eigenvalue $\gamma(z)$ where

$$\gamma(z) = 1 - \frac{q^{1/2+iz} + q^{1/2-iz}}{q+1} \quad \text{for all } z \in \mathbb{C}.$$

Later Mantero and Zappa [29] gave a complete characterization of the eigenfunctions of the Laplacian as Poisson transforms of martingales, for almost every $z \in \mathbb{C}$. More precisely, for $z \in \mathbb{C}$, let $\mathbb{E}_z(\mathfrak{X})$ be the eigenspace of the Laplacian with eigenvalue $\gamma(z)$, $\mathcal{K}'(\Omega)$ be the space of all martingales defined on the boundary Ω, and also let $\tau = 2\pi/\log q$. Then the following result holds.

Theorem 1.1.1 ([29, Theorem A]). *Let $z \in \mathbb{C} \setminus \{(k\tau + i)/2 : k \in \mathbb{Z}\}$. Then the map $\mathcal{P}_z : \mathcal{K}'(\Omega) \to \mathbb{E}_z(\mathfrak{X})$ is a bijection.*

A detailed treatment of the above characterization is also given in [12, chapter 2].

The Poisson transform defined on a homogeneous tree is a natural analogue of the Poisson transform on hyperbolic spaces. In fact a homogeneous tree is often considered as a discrete analogue of a hyperbolic space. Many authors have shown the analogy between the harmonic analysis on homogeneous trees and that on hyperbolic spaces or more generally on non-compact symmetric spaces of rank one. For details we refer [5, 7, 12, 14, 39, 43]. Let X denote a non-compact symmetric space of rank one, K/M be its boundary and Δ be the Laplace-Beltrami operator on X. Theorem 1.1.1 can be considered as an analogue of the following celebrated result by Helgason [17].

Theorem 1.1.2. *If u is an eigenfunction of Δ on X then u is a real analytic function and is the Poisson transform of an analytic functional T defined on K/M.*

In [23] the authors extended the above result for symmetric spaces with higher rank. Further many authors continued to investigate the fine mapping properties of the Poisson transform on symmetric spaces by putting some additional size estimates on the eigenfunctions of Δ. See for example [20, 26, 28, 38, 41]. One such important result is the following characterization proved by Lohoué and Rychener:

Theorem 1.1.3 ([28, Proposition 1]). *Let* $1 < p < 2$, $\lambda = \alpha + i\gamma_{p'}\rho$, $\alpha \in \mathbb{R}$ *and* $\Delta u = -(\lambda^2 + \rho^2)u$. *Then* $u \in L^{p',\infty}(X)$ *if and only if* $u = \mathcal{P}_\lambda F$ *for some* $F \in L^{p'}(K/M)$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|\mathcal{P}_{\alpha+i\gamma_{p'}\rho}F\|_{p',\infty} \leq C\|F\|_{p'}. \tag{1.1.1}$$

In the above theorem, $\gamma_p = (\frac{2}{p} - 1)$ where ρ is the half sum of positive roots. In the same paper [28], the authors also obtained the Lorentz space version of Kunze-Stein phenomenon by using the estimate (1.1.1). In the context of homogeneous trees, Cowling etl. [4, 5] gave a beautiful generalization of this result. They proved the following analogous convolution relation on G, the group of isometries of $\mathfrak{X}$ (see also [32]):

$$L^{p,r}(G) * L^{p,s}(G) \subseteq L^{p,t}(G), \tag{1.1.2}$$

where $1 < p < 2$ and $r, s, t \in [1, \infty]$ satisfies the relation

$$\frac{1}{t} \leq \frac{1}{r} + \frac{1}{s}.$$

These convolution relations are based on Herz's principe de majoration. The main ingredient in the proof is the norm estimates of matrix coefficients of the representation π_z. For $z \in \mathbb{C}$, the representations π_z of G on $L^2(\Omega)$ is defined by the formula

$$\pi_z(g)\eta(\omega) = p^{1/2+iz}(g \cdot o, \omega)\eta(g^{-1}\omega) \quad \forall g \in G \text{ and } \forall \omega \in \Omega.$$

Since the Poisson transform defined above is a matrix coefficient of the representation π_z, that is $\mathcal{P}_z F(x) = \langle \pi_z(x)1, F \rangle$, it is known that an estimate of the form (1.1.1) is true for homogeneous trees. Now it is natural to ask whether a complete extension of Theorem 1.1.3 is possible on homogeneous trees. In this thesis we answer this question by proving a version of Theorem 1.1.3 in the context of homogeneous trees.

Coming back to the hyperbolic spaces, another important aspect is the generalization of Theorem 1.1.3 for $p = 2$. In [41] Strichartz initiated the study of the Poisson transform on hyperbolic spaces for the parameter $\lambda \in \mathbb{R} \setminus \{0\}$. Later Ionescu [20] established a bijection between $L^2(K/M)$ and a suitable subspace of eigenspaces of the Laplacian on non-compact symmetric spaces of rank one. A generalization of this result to arbitrary symmetric spaces was proved in [22]. In [3] the authors also obtained a similar result for $p \geq 2$ on hyperbolic spaces. In [25] the author proved an extension of Theorem 1.1.3 for the case $p = 2$.

Theorem 1.1.4 ([25, Theorem 1.1]). *If $\lambda \in \mathbb{R} \setminus \{0\}$ then there exists a constant $C_\lambda > 0$ such that*

$$\|\mathcal{P}_\lambda F\|_{2,\infty} \leq C_\lambda \|F\|_{L^2(K/M)}, \quad \textit{for all } F \in L^2(K/M). \tag{1.1.3}$$

If $\Delta u = -(\lambda^2 + \rho^2)u$ then $u \in L^{2,\infty}(X)$ if and only if $u = \mathcal{P}_\lambda F$ for some $F \in L^2(K/M)$.

Our inspiration to study this circle of ideas came from the following behaviour of the elementary spherical function ϕ_z, which is defined as $\mathcal{P}_z 1$. Using the explicit formula for ϕ_z given by Figà-Talamanca and Picardello in [13], we obtain

$$|\phi_z(x)| \asymp q^{-|x|/2}(1 + |x|) \text{ if } z \in (\tau/2)\mathbb{Z} \text{ and } |\phi_z(x)| \leq C_z q^{-|x|/2} \text{ if } z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}.$$

The above pointwise estimates clearly implies that $\phi_z \notin L^{2,\infty}(\mathfrak{X})$ if $z \in (\tau/2)\mathbb{Z}$ and $\phi_z \in L^{2,\infty}(\mathfrak{X})$ if $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$. In view of this and Theorem 1.1.4 we may ask the following questions in the context of homogeneous trees:

1. Does an estimate of the form (1.1.3) holds if $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$ and $F \in L^2(\Omega)$?

2. Are all weak L^2 eigenfunctions of $\mathcal{L}$ with eigenvalue $\gamma(z)$, $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$ necessarily of the form $\mathcal{P}_z F$ for some $F \in L^2(\Omega)$?

In this thesis we aim to address these questions and characterize all weak L^2 eigenfunctions of $\mathcal{L}$ on a homogeneous tree $\mathfrak{X}$.

1.2 Fourier Restriction Theorem

Our next aim is to prove a version of Fourier restriction theorem for the Helgason-Fourier transform on homogeneous trees. Historically the formulation of Fourier restriction theorem on $\mathbb{R}^n$ $(n \geq 2)$ emerged explicitly by the work of Stein. It says that S^{n-1} the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ satisfies a (p, r) restriction theorem if

$$\left(\int_{S^{n-1}} |\mathcal{F}(f)(\xi)|^r d\sigma(\xi) \right)^{1/r} \leq C_{p,r} \|f\|_{L^p(\mathbb{R}^n)}$$

holds for each $f \in L^1 \cap L^p$, where $\mathcal{F}(f)$ is the Euclidean Fourier transform of f. One of the celebrated result in this context is the Tomas-Stein restriction theorem. It says that the Fourier transform $\mathcal{F}(f)$ of a function $f \in L^p(\mathbb{R}^n)$ has a well defined restriction on S^{n-1} via the inequality

$$\left(\int_{S^{n-1}} |\mathcal{F}(f)(\xi)|^2 d\sigma(\xi)\right)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } 1 \leq p \leq \frac{2n+2}{n+3}.$$

It turns out that the above estimate can also be viewed as $(2, p')$ boundedness of the Poisson transform via the relation

$$\int_{S^{n-1}} F(\xi)\mathcal{F}f(\xi)d\sigma(\xi) = \int_{\mathbb{R}^n} P_1 F(x) f(x) dx,$$

where, for $\lambda > 0$ and $F \in C(S^{n-1})$, the Poisson transform $P_\lambda F$ is defined as

$$P_\lambda F(x) = \int_{S^{n-1}} e^{-2\pi i \lambda x \cdot \xi} F(\xi) d\sigma(\xi), \quad x \in \mathbb{R}^n.$$

More precisely the Tomas-Stein theorem can also be proved by using duality to the following boundedness of the Poisson transform and vice-versa.

$$\|P_1 F\|_{L^p(\mathbb{R}^n)} \leq C\|F\|_{L^2(\mathbb{R}^n)}, \quad \text{where } p > \frac{2n+2}{n-1}.$$

By using a similar duality relation between the Helgason-Fourier transform $\mathcal{H}$ and the Poisson transform, in [26] the authors proved the following restriction theorem on a rank one symmetric space of non-compact type.

Theorem 1.2.1 ([26, Theorem 1.1]). *Let f be a measurable function on X and $\alpha \in \mathbb{R}$. Then*

1. *for $f \in L^{p,1}(X)$, $1 \leq p \leq 2$ and $p \leq r \leq p'$,*

$$\left(\int_{K/M} |\mathcal{H}(f)(\alpha + i\gamma_r \rho, b)|^r db\right)^{1/r} \leq C_{p,r} \|f\|_{L^{p,1}(X)}.$$

2. *for $f \in L^{p,\infty}(X)$, $1 < p < 2$ and $p < r < p'$,*

$$\left(\int_{K/M} |\mathcal{H}(f)(\alpha + i\gamma_r \rho, b)|^r db\right)^{1/r} \leq C_{p,r} \|f\|_{L^{p,\infty}(X)}.$$

An extension of the above result for $p = 2$ was later proved in [25]. The result can be stated as follows:

Theorem 1.2.2 ([25, Theorem 4.1]). *If $\lambda \in \mathbb{R} \setminus \{0\}$ and $f \in L^{2,1}(X)$ then*

$$\left(\int_{K/M} |\mathcal{H}(f)(\lambda, b)|^2 db \right)^{1/2} \leq C_\lambda \|f\|_{L^{2,1}(X)}.$$

In this thesis we prove an analogue of Theorem 1.2.1 and 1.2.2 for the Helgason-Fourier transform on $\mathfrak{X}$.

1.3 A Theorem of Roe and Strichartz

In 1980, J. Roe [34] proved the following characterization of the sine functions in terms of the size of their derivatives and antiderivatives.

Theorem 1.3.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of real-valued functions of a real variable with*

$$\frac{d}{dx} f_k = f_{k+1} \quad \text{for all } k \in \mathbb{Z}.$$

If there exists a constant $M > 0$ such that

$$|f_k(x)| \leq M \quad \text{for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R},$$

then $f_0(x) = a \sin(x + \theta)$ for some $a, \theta \in \mathbb{R}$.

This striking characterization was later refined by R. Howard [18] where the author proved the validity of Roe's result by assuming a rather weaker condition on the size estimates of the sequence $\{f_k\}_{k\in\mathbb{Z}}$. Our starting point however is the following elegant observation made by Strichartz in his paper [42]. He viewed Roe's result as a representative theorem which characterizes all eigenfunctions of the operator d^2/dx^2 with eigenvalue -1. This observation helped him to get an n-dimensional generalization of Theorem 1.3.1. Consequently Strichartz [42] extended the above result to $\mathbb{R}^n$ by substituting d/dx with the Laplacian $\Delta_{\mathbb{R}^n}$ on $\mathbb{R}^n$. Strichartz's result can be stated as follows.

Theorem 1.3.2 ([42, Theorem 1.1]). *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of functions in $\mathbb{R}^n$ satisfying $\Delta_{\mathbb{R}^n} f_k = f_{k+1}$ for all $k \in \mathbb{Z}$ and $|f_k(x)| \leq M$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, where M is a real number. Then $\Delta_{\mathbb{R}^n} f_0 = -f_0$.*

In the same paper, Strichartz also proved that the above result holds for Heisenberg groups but fails for hyperbolic 3-spaces. A somewhat deeper understanding of the counterexample mentioned in [42] revealed that the negative result on hyperbolic 3-space can indeed be extended to homogeneous trees.

Consider the spherical function ϕ_z which is a radial eigenfunction of the Laplacian with eigenvalue $\gamma(z)$. Let us define the doubly infinite sequence $\{f_k\}_{k\in\mathbb{Z}}$ as follows:

$$f_k(x) = \gamma(z_1)^k \phi_{z_1}(x) + \gamma(z_2)^k \phi_{z_2}(x), \quad x \in \mathfrak{X},$$

where z_1, z_2 are two distinct points inside the strip $\{z \in \mathbb{C} : |\Im z| \leq 1/2\}$ for which $\gamma(z_1) \neq \gamma(z_2)$ and $|\gamma(z_1)| = |\gamma(z_2)| = 1$. In Chapter 5 it is proved that the infinite sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfy all the hypothesis of Theorem 1.3.2 but f_0 fails to be an eigenfunction of $\mathcal{L}$.

A careful analysis of the above counterexample reveals that the failure of Strichartz's result on $\mathfrak{X}$ is mainly due to the spectrum of $\mathcal{L}$. It was also observed in [27] that the failure of Strichartz's result is deeply rooted in the p-dependence of the L^p-spectrum of the Laplace-Beltrami operator Δ on the hyperbolic spaces. In [27] it was in fact proved that Theorem 1.3.2 indeed remains valid when uniform boundedness is replaced by uniform "almost L^p boundedness". Here it is worth mentioning that these size estimates arise naturally due to the behaviour of the Poisson transforms, which also acts as eigenfunctions of the Laplace-Beltrami operator with the eigenvalues lying on the boundary of the p-depending parabolic region. The version of Roe's theorem proved in the context of a Riemannian symmetric space of non-compact type is as follows (see [27, 33]).

Theorem 1.3.3 ([27, Theorem B]). *Let f be a measurable function on X and $p \in (1, 2)$. If $\|\Delta^k f\|_{p',\infty} \leq M(4\rho^2/pp')^k$ for $k = 0, 1, 2, \ldots$, for some $M > 0$, then $\Delta f = -(4\rho^2/pp')f$. In particular, $f = \mathcal{P}_{i\gamma_p\rho}F$ for some $F \in L^{p'}(K/M)$.*

An extension of the above result for the case $p = 2$ is the following.

Theorem 1.3.4 ([27, Theorem A]). *Let f be a measurable function on X and λ be a non-zero real number. If $\|\Delta^k f\|_{2,\infty} \leq M(\lambda^2 + \rho^2)^k$ for all $k \in \mathbb{Z}$, for some $M > 0$, then $\Delta f = -(\lambda^2 + \rho^2)f$. In particular, $f = \mathcal{P}_\lambda F$ for some $F \in L^2(K/M)$.*

In this thesis we extend Theorem 1.3.3 and 1.3.4, and hence prove a version of Roe's and Strichartz's result in the context of homogeneous trees. We also prove a version of Roe's result on a homogeneous tree of degree 1.

1.4 Dynamics of Semigroups Generated by Laplacian

The study of chaotic dynamical systems has always been a central theme as it brings out problems deeply rooted in several classical areas of mathematics, such as ergodic theory, analysis and geometry. As a result this topic attracts a wide variety of literature. However we shall restrict our attention only to those which suits our purpose. To begin with, we first introduce some basic ergodic theoretic terminologies which we shall require further. These definitions are an adaptaton of that introduced by Devaney [10] (see also [8, 9]).

Let $\mathbb{B}$ be a Banach space and $\mathcal{B}(\mathbb{B})$ be the space of all bounded linear operators from $\mathbb{B}$ into itself. A semigroup on $\mathbb{B}$ is a map $T : [0, \infty) \to \mathcal{B}(\mathbb{B})$ such that

1. $T(0) = I$, the identity map on $\mathbb{B}$,
2. $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$.

Furthermore we say that $T(t)$ is strongly continuous if $T(t)x \to x$ as $t \to 0$ for all $x \in \mathbb{B}$. An infinitesimal generator A of a strongly continuous semigroup $T(t)$ is then defined by

$$Ax = \lim_{t\to 0} \frac{(T(t) - I)x}{t},$$

whenever the above limit exists. In that case we also write $T(t) = e^{tA}$. A strongly continuous semigroup $T(t)$ is said to be hypercyclic if there exists $x \in \mathbb{B}$ such that $\{T(t)x : t \geq 0\}$ is dense in $\mathbb{B}$. In addition, a point $x \in \mathbb{B}$ is said to be periodic for $T(t)$ if there exists $t > 0$ such that $T(t)x = x$. The set of all periodic points will henceforth be denoted by $\mathbb{B}_{per}$. Finally the semigroup $T(t)$ is said to be chaotic if it is hypercyclic and its set of periodic points is dense in $\mathbb{B}$.

In a recent paper [21] Ji and Weber started the study of chaotic dynamics of the heat semigroup on a non-compact Riemannian symmetric space with real rank one. They studied the chaotic behaviour of certain shifts of the heat semigroup corresponding to the Laplace-Beltrami operator Δ endowed with Riemannian structure, namely

$$T(t) = e^{-t(\Delta - c)}, \quad t \geq 0, \; c \in \mathbb{R}$$

on the space of all radial functions on $L^p(X)$. In [31] Pramanik and Sarkar extended the above result and gave a complete characterization for the chaotic behaviour of the semigroup $T(t)$ on the whole of $L^p(X)$ and its related subspaces. A similar result concerning the chaotic behaviour of the heat semigroup and the Dunkl heat semigroup is also known for harmonic NA groups and Euclidean spaces respectively. For details see [2] and [36]. In [31] the authors proved the following result.

Theorem 1.4.1. *Let X be a Riemannian symmetric space of non-compact type, with $T(t)$ defined as above and $c_p = \frac{4|\rho|^2}{pp'}$. Then we have the following.*

1. *For $1 \leq p \leq 2$, $T(t)$ is non-chaotic on $L^p(X)$ for all $c \in \mathbb{R}$.*

2. *For $2 < p < \infty$, $T(t)$ is chaotic on $L^p(X)$ if and only if $c > c_p$.*

3. *For $p = \infty$, $T(t)$ is non-chaotic on $L^\infty(X)$ for all $c \in \mathbb{R}$.*

A detailed study revealed that the proof of the above theorem and the subsequent results of Ji and Weber [21] are largely influenced by the work of Desch, Schappacher and Webb. In [9] Desch etl. gave a sufficient condition for a semigroup to be chaotic in terms of spectral properties of its generator. Their result can be stated as follows:

Theorem 1.4.2 ([9, Theorem 3.1])**.** *Let $T(t)$ denote a strongly continuous semigroup on a separable Banach space $\mathbb{B}$ with generator A. Assume there is an open, connected subset U inside the point spectrum of A and a function $F : U \to \mathbb{B}$ such that:*

1. *$U \cap i\mathbb{R} \neq \emptyset$;*

2. *$F(\lambda) \in Ker\ (A - \lambda I)$ for all $\lambda \in U$;*

3. *For all f in the dual space $\mathbb{B}'$ of $\mathbb{B}$, the mapping $F_f : U \to \mathbb{C}$ defined by*

$$F_f(\lambda) = f \circ F(\lambda)$$

is analytic. Furthermore, if there is a $f \in \mathbb{B}'$ such that $F_f = 0$, then already $f = 0$ holds. Then $T(t)$ is chaotic.

The above result clearly indicates that the L^p-spectrum of Δ, which is a p-depending parabolic region, plays a vital role in examining chaoticity and determining the range of perturbation for which the heat semigroup is chaotic. In fact the p-dependence of the perturbation given in Theorem 1.4.1 relies heavily on the spectral properties of the generator Δ and demands that its point spectrum must contain infinitely many purely imaginary points. Interestingly the L^p-spectrum of the Laplacian $\mathcal{L}$ is also a p-depending conic region. In fact it is an elliptic region of all complex numbers w which satisfy

$$\left[\frac{1-\text{Re }(w)}{b\cosh(\delta_p \log q)}\right]^2 + \left[\frac{\Im(w)}{b\sinh(\delta_p \log q)}\right]^2 \leq 1, \text{ where } b = \frac{2\sqrt{q}}{q+1} \text{ and } \delta_p = \frac{1}{p} - \frac{1}{2}.$$

This observation paved the way for a detailed exploration of this problem in the context of homogeneous trees.

Having said that we also have certain differences due to the structural difference of these spaces. One such notable difference is that unlike Δ, the Laplacian $\mathcal{L}$ is a bounded operator on $L^p(\mathfrak{X})$. Therefore unlike the symmetric spaces, we consider a rather larger class of semigroups on homogeneous trees, of which the heat semigroups falls under a particular case. More precisely let $\sigma_p(\mathcal{L})$ denote the L^p-spectrum of $\mathcal{L}$ and f be a non-constant complex holomorphic function defined on a connected open set containing $\sigma_p(\mathcal{L})$. Then we study the chaotic behaviour of the semigroup $T(t) = e^{tf(\mathcal{L})}$. In addition we separately study the dynamical behaviour of the semigroups generated by affine functions, namely, $T(t) = \exp\{t(a\mathcal{L}+b)\}$, $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{R}$ and derive a sharp p-depending range for which the affine semigroup is hypercyclic.

Another motivation to study this class of semigroups came from the work of deLaubenfels and Emamirad. In [8] the authors studied the dynamical behaviour of a class of operators, which is generated by non-constant analytic functions of the shift operator L on weighted $L^p(\mathbb{N})$ spaces. To state their result more precisely we need the following notations. Let D be an open unit disc in the complex plane and $\beta = \{b_k\}$ be a complex-valued sequence such that

$$\sum_{k=1}^{\infty} b_k z^k \text{ converges for } |z| < R \quad \text{and} \quad \sum_{k=1}^{\infty} b_k R^k \text{ diverges.}$$

On the weighted space $L^p(\mathbb{N}, \beta)$, consider the semigroup $T(t)$ generated by a non-constant analytic function g defined in a neighbourhood of $\|L\|\overline{D}$, that is,

$$T(t) = e^{tg(L)} \quad \text{where } t \geq 0.$$

Then in [8] the authors proved the following.

Theorem 1.4.3. *For the semigroup $T(t)$ as defined above, the following are equivalent.*

1. *$T(t)$ is chaotic on $L^p(\mathbb{N}, \beta)$.*

2. *$T(t)$ has a non-trivial periodic point in $L^p(\mathbb{N}, \beta)$.*

3. *$g(R^{1/p}D) \cap i\mathbb{R} \neq \emptyset$.*

In this thesis we prove an extension of Theorem 1.4.1 and 1.4.3 for the above mentioned semigroups on homogeneous trees.

1.5 Outline of the Thesis

In Chapter 2 we introduce all such notations and conventions which we will follow throughout the thesis. By referring appropriately we also quote all necessary results on homogeneous trees which we shall employ later. Most of these results can be treated as discrete counterparts of that on hyperbolic spaces. However for the sake of simplicity we will present the proof of a few of them which we were unable to locate in the literature.

Chapter 3 is mainly devoted to the study of Poisson transforms on $\mathfrak{X}$. In this chapter we will prove two representative theorems which closely relates the size estimates of the Poisson transform with the problem of characterizing the eigenfunctions of the Laplacian.

In Chapter 4 we deal with certain consequences of the size estimates of the Poisson transform proved in Chapter 3. More precisely we prove the restriction theorems for the Helgason-Fourier transform on $\mathfrak{X}$. This is done by carefully formulating a duality relation between the Poisson transform on Ω and the Helgason-Fourier transform on $\mathfrak{X}$.

In Chapter 5 we study yet another form of characterization of eigenfunctions of the Laplacian on homogeneous trees. Here we prove a version of Roe's and Strichartz's result on $\mathfrak{X}$, in particular proving an analogue of Theorem 1.3.3 and Theorem 1.3.4 by analysing the behaviour of the Poisson transform and its relation with the geometry of the L^p-spectrum of the Laplacian.

In Chapter 6 we begin the study of chaotic dynamical systems on homogeneous trees. Our aim is to prove an analogue of Theorem 1.4.3 for semigroups generated by analytic function of the Laplacian on homogeneous trees. In view of Theorem 1.4.1 we will see how the elliptic region of the spectrum of $\mathcal{L}$ plays an important role in examining chaoticity. In addition we also study the chaotic behaviour of the affine semigroup separately and derive sharp p depending perturbations for which the affine semigroup is chaotic on $L^p(\mathfrak{X})$.

In Chapter 7 we briefly describe some open problems which are closely related to the topics discussed in this thesis.

CHAPTER 2

Preliminaries

This chapter is mainly devoted to introducing some general notations and preliminary facts regarding the homogeneous trees. Although there are a variety of expositions, we only gather those informations which we shall require further. Most of these preliminaries about the homogeneous trees are extracted from [5, 12, 13, 14]. For a detailed study, we advise all interested readers to go through the above mentioned references and the references therein.

However, it is worthwhile to mention that we use a different parametrization from that of [12, 14] while introducing the terms such as the Poisson transform, the Helgason-Fourier transform and many more which appears later in this chapter. In fact this change in the parametrization makes the analogy with symmetric spaces more transparent.

2.1 Generalities

The letters $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ will respectively denote the sets of all natural numbers, integers, rational numbers, real numbers and complex numbers. For $z \in \mathbb{C}$ we use the notation $\operatorname{Re} z$ and $\Im z$ for real and imaginary parts of z respectively. We will use the standard practice of using the letter C for constant, whose value may change from one line to another line. Initially the constants will be suffixed to show its dependency on the related parameters. Given two positive functions A and B defined on a set M, we say that $A \asymp B$ if there exist positive constants C_1 and C_2 such that $C_1 A(t) \leq B(t) \leq C_2 A(t)$ for all t in M.

For every Lebesgue exponent $p \in (1, \infty)$, we write p' to denote the conjugate exponent $p/(p-1)$. Further we define $p' = \infty$ when $p = 1$ and vice-versa. For $p \in (1, \infty)$ let

$$\delta_p = \frac{1}{p} - \frac{1}{2} \quad \text{and} \quad S_p = \{z \in \mathbb{C} : |\Im z| \leq |\delta_p|\}.$$

We assume $\delta_1 = -\delta_\infty = 1/2$ so that $S_1 = \{z \in \mathbb{C} : |\Im z| \leq 1/2\}$. It is important to note that $\delta_p = -\delta_{p'}$ and $S_p = S_{p'}$ for any $p \in [1, \infty]$. When $p = 2$, the infinite strip reduces to the real line. We shall henceforth write S_p° and ∂S_p to denote the usual interior and boundary of S_p respectively.

Let (M, m) be a σ-finite measure space and $f : M \to \mathbb{C}$ be a measurable function. The distribution function $d_f : (0, \infty) \to (0, \infty]$ and the nonincreasing rearrangement $f^* : (0, \infty) \to (0, \infty]$ are defined by the formulae

$$d_f(s) = m\big(\{x \in M : |f(x)| > s\}\big) \quad \text{and} \quad f^*(t) = \inf\{s : d_f(s) \leq t\}.$$

For $p \in [1, \infty)$ and $q \in [0, \infty]$, we define the Lorentz space $L^{p,q}(M)$ as follows:

$$L^{p,q}(M) = \{f : M \to \mathbb{C} : f \text{ measurable and } \|f\|_{p,q} < \infty\}.$$

where

$$\|f\|_{p,q} = \begin{cases} \left(\frac{q}{p} \int\limits_0^\infty [f^*(t) t^{1/p}]^q \frac{dt}{t}\right)^{1/q} & \text{when } q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t) = \sup\limits_{t>0} t d_f(t)^{1/p} & \text{when } q = \infty. \end{cases}$$

For $1 < p < \infty$ the space $L^{p,\infty}(M)$ is known as the weak L^p space and $L^{p,q} \subset L^{p,s}$ for all $1 \leq q \leq s \leq \infty$. For $1 \leq p \leq \infty$, $L^{p,p}(M)$ coincides with the usual Lebesgue space $L^p(M)$ with the norm $\|\cdot\|_p$ which is given by the formulae

$$\|f\|_p = \begin{cases} \left(\int\limits_M |f|^p dm\right)^{1/p} & \text{when } 1 \leq p < \infty, \\ \inf\{C \geq 0 : |f(x)| \leq C \text{ for almost all } x \in M\} & \text{when } p = \infty. \end{cases}$$

For any bounded linear operator T defined on $L^p(M)$, we shall write $\sigma_p(T)$, $P\sigma_p(T)$ to respectively denote the set of spectrum and point spectrum of T in $L^p(M)$. We further use the notation $\|T\|_{p \to p}$ to denote the operator norm of T.

2.2 Homogeneous Trees

We begin our literature survey by introducing some basic definitions related to the homogeneous trees and associating a group structure to it.

2.2.1 Basic Structure

Definition 2.2.1. *A homogeneous tree $\mathfrak{X}$ of degree $q+1$ is a connected graph with no loops, in which every vertex is adjacent to $q+1$ other vertices.*

When $q = 1$, a homogeneous tree can be identified to the set of all integers whose geometric and analytic properties are quite different from that of others. Therefore we shall assume $q \geq 2$ unless specified. Figure 2.1 represents a part of homogeneous tree of degree 3 and 4 respectively.

Figure 2.1: A part of homogeneous tree of degree 2 and 3 resp.

We identify $\mathfrak{X}$ with the set of all vertices where the natural distance $d(x, y)$ between any two vertices x and y, is defined as the number of edges between them. We endow $\mathfrak{X}$ with the counting measure and write $\#E$ to denote the measure of a finite set E in $\mathfrak{X}$. Let $S(x, n) = \{y \in \mathfrak{X} : d(x, y) = n\}$ and $B(x, n) = \{y \in \mathfrak{X} : d(x, y) \leq n\}$ respectively denote a sphere and a ball centered at x with radius n. Clearly

$$\#S(x, n) = (q+1)q^{n-1} \asymp q^n \quad \text{and} \quad \#S(x, n) \asymp \#B(x, n).$$

Let o be a fixed reference point in $\mathfrak{X}$. Let G be the group of isometries of the metric space $(\mathfrak{X}, d)$ and K be the stabilizer of o in G, that is $K = \{g \in G : g(o) = o\}$. Every element of K rotates the elements of $S(o,n)$ about o, into itself. Hence K is often referred to as the rotation subgroup of G. Under a suitable topology, G is a locally compact group and K is a compact subgroup of G (see [12, Chapter I] for details).

The group G acts transitively on $\mathfrak{X}$ via the map $g \to g \cdot o$ where $g \cdot o = g(o)$. This group action identifies $\mathfrak{X}$ with the coset space G/K, so that functions on $\mathfrak{X}$ corresponds to K-right-invariant functions on G and vice-versa. Using this identification, the convolution of two suitable functions f_1 and f_2 on $\mathfrak{X}$ is defined by the formula

$$f_1 * f_2(g \cdot o) = \int_G f_1(h \cdot o)\ f_2(h^{-1} g \cdot o)\ dh = \int_G f_1(gh \cdot o)\ f_2(h^{-1} \cdot o)\ dh \quad \text{for all } g \in G.$$

In the above expression, dh represents the left Haar measure on G. For any two functions f_1 and f_2 on $\mathfrak{X}$, we use the notation

$$\langle f_1, f_2 \rangle = \int_G f_1(g \cdot o) f_2(g \cdot o)\ dg.$$

It is easy to see that for measurable functions f_1, f_2, f_3 on $\mathfrak{X}$,

$$\langle f_1 * f_2, f_3 \rangle = \langle f_1, f_2 * f_3^{\#} \rangle \quad \text{where } f_3^{\#}(g \cdot o) = f_3(g^{-1} \cdot o) \text{ for all } g \in G.$$

For the sake of simplicity we shall henceforth write $|x|$ to denote $d(o,x)$. We now introduce the notion of a radial function on $\mathfrak{X}$.

Definition 2.2.2. *A complex-valued function f defined on $\mathfrak{X}$ is said to be radial if $f(x) = f(y)$ whenever $|x| = |y|$.*

Using the group action defined above, it follows that radial functions on $\mathfrak{X}$, that is, functions which only depend on $|x|$, corresponds to K-bi-invariant functions on G. Throughout this thesis, we shall use the notation $E(\mathfrak{X})^{\#}$ to denote the subspace of all radial functions on $\mathfrak{X}$, in a function space $E(\mathfrak{X})$.

Definition 2.2.3. *For a suitable function f on $\mathfrak{X}$, its radialization Rf is defined by the formula*

$$Rf(x) = \int_K f(k \cdot x)\ dk \quad \text{for all } x \in \mathfrak{X}.$$

In the above expression, dk represents the normalized Haar measure on K. Writing the above expression as an integral on $\mathfrak{X}$, it follows that

$$Rf(x) = \begin{cases} f(o) & \text{if } x = o, \\ \frac{q^{1-|x|}}{q+1} \sum_{y:|y|=|x|} f(y) & \text{if } x \neq o. \end{cases}$$

Before ending this section, we enlist some important properties of the radialization map which will be used further.

Proposition 2.2.4. *For a suitable function f, let Rf be as in Definition (2.2.3). Then we have the following.*

1. *Rf is always a radial function on $\mathfrak{X}$.*

2. *If f is radial then $Rf = f$.*

3. *$\langle Rf, g\rangle = \langle f, Rg\rangle$, for any two functions f and g for which the above integrals are well-defined.*

4. *For $p \in [1, \infty]$, $\|Rf\|_{L^p(\mathfrak{X})} \leq \|f\|_{L^p(\mathfrak{X})}$ for all $f \in L^p(\mathfrak{X})$.*

Proof. (1): To prove *(1)*, it is enough to show that Rf is also a K-left invariant function on G. To do so, we calculate

$$Rf(k_1 g \cdot o) = \int_K f(kk_1 g \cdot o)\, dk = \int_K f(k_2 g \cdot o)\, dk_2 = Rf(g \cdot o),$$

where $kk_1 = k_2$ so that $dk = dk_2$ (since K is unimodular). This completes the proof. Statements *(2)* and *(3)* easily follows from the definition above and hence we omit the proof.

(4): For a finitely supported function f on $\mathfrak{X}$,

$$\int_G |Rf(g \cdot o)|\, dg \leq \int_G \left(\int_K |f(kg \cdot o)|\, dk \right) dg = \int_K \left(\int_G |f(kg \cdot o)|\, dg \right) dk$$
$$= \|f\|_{L^1(\mathfrak{X})}.$$

Therefore for all $f \in L^1(\mathfrak{X})$, $\|Rf\|_{L^1(\mathfrak{X})} \leq \|f\|_{L^1(\mathfrak{X})}$. Interpolating (see [16, Theorem 1.3.4]) this with the trivial L^∞-boundedness, that is $\|Rf\|_{L^\infty(\mathfrak{X})} \leq \|f\|_{L^\infty(\mathfrak{X})}$, we finally conclude that for every $p \in [1, \infty]$,

$$\|Rf\|_{L^p(\mathfrak{X})} \leq \|f\|_{L^p(\mathfrak{X})} \quad \text{for all } f \in L^p(\mathfrak{X}).$$

This completes the proof. □

2.2.2 Boundary and Poisson Kernel

An infinite geodesic ray ω in $\mathfrak{X}$ is a one-sided sequence $\{\omega_n : n = 0, 1, 2 \ldots\}$ where the ω_n are in $\mathfrak{X}$. Two infinite geodesic rays $\omega = \{\omega_n : n = 0, 1, 2 \ldots\}$ and $\omega' = \{\omega'_n : n = 0, 1, 2 \ldots\}$ are said to be equivalent if there exist natural numbers n and m such that $\omega_i = \omega'_{i+m}$ for all i greater than n. This identification is an equivalence relation and partitions the set of all infinite geodesic rays into disjoint classes. In every such equivalence class there exists a unique geodesic ray which starts from o.

Definition 2.2.5. *The boundary of $\mathfrak{X}$ is the set of all infinite geodesic rays starting at o and will be denoted by Ω.*

The subgroup K acts transitively on Ω via the group action $(k, \omega) \to k \cdot \omega$. This transitive action of the rotation group gives rise to a unique K-invariant measure on Ω which we discuss below. We begin by defining the topology on Ω.

Given $\omega, \omega' \in \Omega$, we define $c(\omega, \omega')$ to be the confluence point of ω and ω', that is, the last common point between the infinite geodesics $\omega = \{\omega_n : n = 0, 1, 2 \ldots\}$ and $\omega' = \{\omega'_n : n = 0, 1, 2 \ldots\}$. Next we introduce the sets

$$E_n(\omega) = \{\omega' \in \Omega : |c(\omega, \omega')| \geq n\} \quad \text{for all } n \geq 0.$$

Similarly for $x \in \mathfrak{X}$ and $\omega \in \Omega$, we define $c(x, \omega) = x_l$ where x_l is the last point lying on ω in the geodesic path $\{o, x_1, \ldots, x\}$ joining o to x. For $x \in \mathfrak{X}$, let

$$E_j(x) = \{\omega \in \Omega : |c(x, \omega)| \geq j\} \quad \text{for all } j \geq 0.$$

Note that $E_0(x) = \Omega$ and $E_j(x) = \emptyset$ whenever j is greater than $|x|$. The sets $E_j(x)$ are open subsets which indeed form a basis of Ω. For the sake of simplicity we shall use the notation $E(x)$ to denote $E_{|x|}(x)$. Observe that the sets of the form $E(x)$ where x varies over $S(o,n)$ partition Ω into $(q+1)q^{n-1}$ disjoint open sets. Consequently there exists a unique K-invariant probability measure ν on Ω such that

$$\nu(E_j(x)) = \frac{q}{(q+1)q^j}, \quad \text{where } 1 \leq j \leq |x|.$$

Hence $(\Omega, \mathcal{M}, \nu)$ becomes a probability measure space with the σ-algebra $\mathcal{M}$ being generated by the set $\{E(x) : x \in \mathfrak{X}\}$.

To proceed further we need the notion of Poisson kernel. The measure ν on the boundary Ω is a G-quasi-invariant probability measure and the Poisson kernel $p(g \cdot o, \omega)$ is defined to be the Radon-Nikodym derivative $d\nu(g^{-1}\omega)/d\nu(\omega)$ which is explicitly given by the formula

$$p(x,\omega) = q^{h_\omega(x)} \quad \forall x \in \mathfrak{X} \quad \forall \omega \in \Omega,$$

where the height $h_\omega(x)$ is defined by

$$h_\omega(x) = 2|c(x,\omega)| - |x|.$$

For more equivalent formulae for $h_\omega(x)$ and its relevant details, we refer [5, 7]. From the explicit formula above, it turns out that the Poisson kernel is a non-negative function on $\mathfrak{X} \times \Omega$, which for any fixed x, takes only finitely many values as a function of ω. More precisely, for every fixed x, $|c(x,\omega)|$ can range from 0 to $|x|$ and consequently

$$-|x| \leq h_\omega(x) \leq |x| \quad \text{for all } \omega \in \Omega.$$

Recalling the definition of $E_j(x)$, we find that the Poisson kernel can also be written as

$$p(x,\omega) = \sum_{j=0}^{|x|} q^{2j-|x|} \mathcal{X}_{E_j(x) \setminus E_{j+1}(x)}(\omega) \quad \forall x \in \mathfrak{X} \quad \forall \omega \in \Omega. \tag{2.2.1}$$

2.2.3 Laplacian

Definition 2.2.6. *For a complex-valued function f defined on $\mathfrak{X}$, the nearest neighbour average Lf is defined by the formula*

$$Lf(x) = \frac{1}{q+1} \sum_{y:d(x,y)=1} f(y) \quad \text{for all } x \in \mathfrak{X}.$$

Corresponding to the operator L, the Laplacian or the Laplace operator $\mathcal{L}$ is then defined by

$$\mathcal{L}f(x) = (I-L)f(x) = f(x) - \frac{1}{q+1} \sum_{y:d(x,y)=1} f(y) \quad \text{for all } x \in \mathfrak{X}.$$

Many authors have also adopted the formula L to denote the Laplacian on $\mathfrak{X}$ (see for example [12, page 34]). But more appropriately, the name Laplacian is generally given to the operator $\mathcal{L}$. It is easy to see that $\mathcal{L}$ defines a bounded operator on $L^p(\mathfrak{X})$ for every $p \in [1, \infty]$.

For $g \in G$, we define

$$(\tau_g f)(x) = f(g \cdot x) \quad \text{for all } x \in \mathfrak{X}.$$

Operators which commute with τ_g for all $g \in G$ are often referred to as G-invariant operators.

Proposition 2.2.7. *The Laplacian $\mathcal{L}$ satisfies the following conditions:*

1. *$\mathcal{L}$ is a G-invariant operator, that is $\mathcal{L}(\tau_g f) = \tau_g(\mathcal{L} f)$ for every $g \in G$.*

2. *$\mathcal{L}$ commutes with the radialization operator R, that is $\mathcal{L}(Rf) = R(\mathcal{L} f)$.*

Proof. Instead of considering $\mathcal{L}$, it is enough to prove the above facts for the operator L. For $g \in G$ and $x \in \mathfrak{X}$,

$$L(\tau_g f)(x) = \frac{1}{q+1} \sum_{y:d(x,y)=1} f(g \cdot y) = \frac{1}{q+1} \sum_{g \cdot y:d(g \cdot x, g \cdot y)=1} f(g \cdot y) = \tau_g(Lf)(x).$$

This completes the proof of (*1*). Next we prove (*2*). If $x = o$ it immediately follows from the definitions that $L(Rf)(o) = R(Lf)(o)$. Hence we assume $x \neq o$. Let $|x| = n > 0$ and x_{n-1}, x_{n+1} be two vertices adjacent to x with $|x_{n\pm1}| = n \pm 1$. Then

$$\begin{aligned}
L(Rf)(x) &= \frac{1}{q+1} \sum_{y:d(x,y)=1} Rf(y) = \frac{1}{q+1} \left(q \; Rf(x_{n+1}) + Rf(x_{n-1})\right) \\
&= \frac{q}{q+1} \left(\frac{1}{(q+1)q^n} \sum_{w:|w|=n+1} f(w) \right) + \frac{1}{q+1} \left(\frac{1}{(q+1)q^{n-2}} \sum_{z:|z|=n-1} f(z) \right) \\
&= \frac{1}{(q+1)q^{n-1}} \left(\frac{1}{q+1} \sum_{w:|w|=n+1} f(w) + \frac{q}{q+1} \sum_{z:|z|=n-1} f(z) \right).
\end{aligned}$$

Note that every z in $S(o, n-1)$ is adjacent to q number of x_n's with $|x_n| = n$. Therefore the above expression takes the form

$$L(Rf)(x) = \frac{1}{(q+1)q^{n-1}} \left(\sum_{x_n:|x_n|=n} Lf(x_n) \right) = R(Lf)(x).$$

This completes the proof. ∎

The Laplacian $\mathcal{L}$ being a G-invariant and bounded operator on $L^p(\mathfrak{X})$, can be given by convolution on the right by a radial function on $\mathfrak{X}$, namely

$$\mathcal{L}f(x) = f * (\delta_o - \mu_1)(x) \quad \text{for all } x \in \mathfrak{X},$$

where δ_o denotes the Dirac measure at the reference point o and μ_1 denotes the uniformly distributed probability measure supported on $S(o,1)$. Consequently for any two functions f_1 and f_2,

$$\langle f_1 * (\delta_o - \mu_1), f_2 \rangle_{L^2(\mathfrak{X})} = \langle f_1, f_2 * (\delta_o - \mu_1) \rangle_{L^2(\mathfrak{X})},$$

which implies that $\mathcal{L}$ is self-adjoint on $L^2(\mathfrak{X})$.

2.3 Poisson Transform and Eigenfunction

We now begin to introduce the technical hearts of this thesis. However to proceed further we need the notion of conditional expectations and martingales on the boundary. Martingales are important in our context due of the fact that the first fundamental result related to the characterization of eigenfunctions of the Laplacian, which concerns us, is given in terms of the Poisson transform of martingales.

2.3.1 Conditional Expectations and Martingales

For each $n \geq 0$, let $\mathcal{M}_n$ denotes the sub-algebra of $\mathcal{M}$ generated by the sets of the form $E(x)$ where $|x| \leq n$. In addition, let $\mathcal{K}_n(\Omega)$ be a linear space of functions on Ω which are finite linear combinations of characteristic functions of the sets $E(x)$ where x varies over $B(o,n)$. It is easy to see that $(\mathcal{M}_n)_{n\geq 0}$ is an expanding sequence of sigma algebras on Ω and $\mathcal{K}_n(\Omega)$ is properly contained in $\mathcal{K}_{n+1}(\Omega)$ for each n. Now define the space

$$\mathcal{K}(\Omega) = \bigcup_{n\geq 0} \mathcal{K}_n(\Omega).$$

Since $\mathcal{K}_n(\Omega) \subset \mathcal{K}_{n+1}(\Omega)$ for each $n \geq 0$, we find that $F \in \mathcal{K}(\Omega)$ if and only if $F \in \mathcal{K}_n(\Omega)$ for some n. Functions belonging to this space are often referred to as the cylindrical functions (see for example [14, page 52]).

Definition 2.3.1. *The conditional expectation $\mathcal{E}_n(F)$ of an integrable function F on $(\Omega, \mathcal{M}, \nu)$ relative to the sub-algebra $\mathcal{M}_n$ is defined by*

$$\mathcal{E}_n(F)(\omega) = \frac{1}{\nu(E_n(\omega))} \int\limits_{E_n(\omega)} F(\omega')d\nu(\omega'). \tag{2.3.2}$$

We now summarize some important properties of $\mathcal{E}_n$ which will be frequently used in this thesis.

Proposition 2.3.2. *For $n \geq 0$, let $\mathcal{E}_n$ be as in Definition (2.3.1). Then for every $n, m \geq 0$ and $F, G \in L^1(\Omega)$, we have the following.*

1. *$\mathcal{E}_n$ is a projection of $L^1(\Omega)$ onto $\mathcal{K}_n(\Omega)$.*
2. *$\int\limits_{\Omega} \mathcal{E}_n(F)(\omega)G(\omega)d\nu(\omega) = \int\limits_{\Omega} \mathcal{E}_n(G)(\omega)F(\omega)d\nu(\omega)$.*
3. *$\mathcal{E}_m(\mathcal{E}_n(F)) = \mathcal{E}_k(F)$, where $k = \min\{m, n\}$.*

Proof. The proof of *(1)* easily follows from the definition above. For the proof of *(2)*, we refer [40, Page 90]. Since $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ for all n, we have *(3)*. □

For further details about the conditional expectations, we refer [40, Chapter IV].

Definition 2.3.3. *The nth difference $\Delta_n(F)$ of an integrable function F on $(\Omega, \mathcal{M}, \nu)$ is defined by the formula*

$$\Delta_n(F) = \mathcal{E}_n(F) - \mathcal{E}_{n-1}(F), \quad \text{where } \mathcal{E}_{-1} = 0.$$

Note that $\mathcal{E}_n(F) = \sum\limits_{j=0}^{n} \Delta_j(F)$ if $n \geq 1$ and $\mathcal{E}_0(F) = \Delta_0(F)$. Using Proposition 2.3.2 (*1*) and (*2*), it follows that

$$\langle \Delta_m(F), \Delta_n(F) \rangle_{L^2(\Omega)} = \begin{cases} 0 & \text{when } m \neq n, \\ \|\Delta_n(F)\|^2_{L^2(\Omega)} & \text{when } m = n, \end{cases} \tag{2.3.3}$$

where $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ denotes the standard inner product on $L^2(\Omega)$. By virtue of the above orthogonality of Δ_n we find that for any bounded sequence $\alpha = (\alpha_n)$ and F in $L^2(\Omega)$, the expression $\sum\limits_{n=1}^{\infty} \alpha_n \Delta_n(F)$ is well-defined. It is thus legitimate to define the operator

$$T_\alpha(F) = \sum_{n=1}^{\infty} \alpha_n \Delta_n(F), \qquad \text{where } F \in L^2(\Omega). \tag{2.3.4}$$

The L^2-boundedness of T_α is addressed in the following theorem. The proof of this theorem can be found in [40, page 94].

Theorem 2.3.4. *Suppose that* $\alpha = (\alpha_n)$ *is any sequence of numbers such that* $|\alpha_n| \leq 1$ *for all* $n \in \mathbb{N}$. *For* $F \in L^2(\Omega)$, *let* $T_\alpha(F)$ *be as in (2.3.4). Then there exists a constant* $C > 0$ *such that*

$$\|T_\alpha(F)\|_{L^2(\Omega)} \leq C\|F\|_{L^2(\Omega)} \quad \textit{for all } F \in L^2(\Omega).$$

We are now in a position to define a martingale on the boundary.

Definition 2.3.5. *A martingale on the boundary is a sequence* $\mathbf{F} = (F_n)_{n\geq 0}$ *of measurable functions on* Ω *satisfying the following conditions:*

1. $F_n \in \mathcal{K}_n(\Omega)$ *for each* $n \geq 0$.

2. *For every* $m, n \geq 0$, $\mathcal{E}_m(F_n) = F_k$ *where* $k = \min\{m, n\}$.

It follows from Definition 2.3.1 and the relevant properties of $\mathcal{E}_n$ (that is Proposition 2.3.2) that each function F in $L^p(\Omega)$ identifies to a martingale $\mathbf{F} = (\mathcal{E}_n(F))_{n\geq 0}$ via the conditional expectation. In fact for every $p \in [1, \infty]$, $\mathcal{E}_n(F)$ converges to F in L^p norm as n tends to infinity. Further using Theorem 2.3.6 (given below), it can also be shown that $\mathcal{E}_n(F)$ converges to the function F pointwise almost everywhere. The proof of this maximal theorem can be found in [40, page 91].

Theorem 2.3.6. *The maximal operator defined by the formula*

$$\mathcal{E}(F)(\omega) = \sup_{n\geq 0} |\mathcal{E}_n(F)(\omega)|, \tag{2.3.5}$$

is weak type $(1, 1)$ *and strong type* (p, p) *whenever* $p > 1$.

A natural question that arises in our mind is whether every martingale is a conditional expectation of some L^p-function defined on the boundary. The answer to this question is negative. In fact the following theorem clarifies this inquiry.

Theorem 2.3.7. *Let* $\mathbf{F} = (F_n)_{n\geq 0}$ *be a martingale such that*

$$\sup_{n\geq 0} \|F_n\|_{L^p(\Omega)} < \infty \quad \textit{for some } p > 1.$$

Then there exists a unique function F *in* $L^p(\Omega)$ *such that* $F_n = \mathcal{E}_n F$ *for every* $n \geq 0$.

2.3.2 Poisson Transform

Definition 2.3.8. *For $z \in \mathbb{C}$, the Poisson transform $\mathcal{P}_z F$ of a function $F \in \mathcal{K}(\Omega)$ is defined by*

$$\mathcal{P}_z F(x) = \int_\Omega p^{1/2+iz}(x,\omega)F(\omega)d\nu(\omega) \quad \text{for all } x \in \mathfrak{X}.$$

Alternatively, the Poisson transform can also be viewed as a matrix coefficient of a representation π_z. For $z \in \mathbb{C}$, we define the representations π_z of G on $\mathcal{K}(\Omega)$ by the formula

$$\pi_z(g)\eta(\omega) = p^{1/2+iz}(g \cdot o, \omega)\eta(g^{-1}\omega) \quad \forall g \in G \text{ and } \forall \omega \in \Omega.$$

The Poisson transform of $F \in \mathcal{K}(\Omega)$ is then given by the formula

$$\mathcal{P}_z F(x) = \langle \pi_z(x)1, F \rangle .$$

From the above definitions it is clear that $\mathcal{P}_{z+\tau} = \mathcal{P}_z$ where $\tau = 2\pi/\log q$. We shall henceforth write $\mathbb{T}$ to denote the torus $\mathbb{R}/\tau\mathbb{Z}$, which we usually identify with the interval $[-\tau/2, \tau/2)$. Clearly for $z = (k\tau + i)/2,\ k \in \mathbb{Z}$, the Poisson transform $\mathcal{P}_z$ is not injective on $\mathcal{K}(\Omega)$. For other values of z, we have the following result (see [14, page 53] for the proof).

Proposition 2.3.9. *The Poisson transform $\mathcal{P}_z$ is injective on $\mathcal{K}(\Omega)$ if and only if $z \notin (k\tau + i)/2,\ k \in \mathbb{Z}$.*

Our next aim is to extend the notion of Poisson transform to martingales. It is proved in [14, Proposition 2.1] that the dual space $\mathcal{K}'(\Omega)$ of $\mathcal{K}(\Omega)$ identifies to the space of all martingales. This means that every linear functional Φ defined on $\mathcal{K}(\Omega)$ corresponds to a unique martingale $\mathbf{F} = (F_n)$ and is given by

$$\Phi(\eta) = \lim_{n\to\infty} \int_\Omega F_n(\omega)\eta(\omega)d\nu(\omega) \quad \forall\, \eta \in \mathcal{K}(\Omega).$$

Since $\eta \in \mathcal{K}_n(\Omega)$ for some n, using Proposition 2.3.2 (*2*) it follows that the above limit is well-defined. In fact the corresponding sequence is eventually constant for each η.

Using the duality above, we now extend the definition of the Poisson transformation to a martingale $\mathbf{F} = (F_n)$ as

$$\mathcal{P}_z \mathbf{F}(x) = \lim_{n\to\infty} \int_\Omega p^{1/2+iz}(x,\omega)F_n(\omega)d\nu(\omega).$$

The limit exists because, for each x, the function $p^{1/2+iz}(x,\cdot) \in \mathcal{K}_{|x|}(\Omega)$ (see equation (2.2.1)). Using this fact together with Proposition 2.3.2 (*2*), it follows that for every $x \in B(o,N)$

$$\mathcal{P}_z\mathbf{F}(x) = \lim_{n\to\infty} \int_\Omega p^{1/2+iz}(x,\omega)F_n(\omega)d\nu(\omega) = \int_\Omega p^{1/2+iz}(x,\omega)F_N(\omega)d\nu(\omega). \tag{2.3.6}$$

2.3.3 Eigenfunctions of the Laplacian

We now begin our discussion on some basic eigenfunctions of $\mathcal{L}$. Observe that if y is a nearest neighbour of x, then $h_\omega(y) = h_\omega(x) \pm 1$. In fact as y varies among the $q+1$ nearest neighbours of x, the sign $+$ occurs once and the sign $-$ occurs q times. Since the expression of $p(x,\omega)$ involves powers of $h_\omega(x)$, it follows that for every fixed ω, the function $x \to p^{1/2+iz}(x,\omega)$ satisfies

$$\mathcal{L}p^{1/2+iz}(x,\omega) = \gamma(z)p^{1/2+iz}(x,\omega), \tag{2.3.7}$$

where γ is an analytic function on $\mathbb{C}$ defined by the formula

$$\gamma(z) = 1 - \frac{q^{1/2+iz}+q^{1/2-iz}}{q+1}. \tag{2.3.8}$$

Let $x \in \mathfrak{X}$ be such that $x = g\cdot o$ for some g in G. For a suitable function F defined on the boundary Ω, we find that

$$\begin{aligned} L(\mathcal{P}_zF)(g\cdot o) = \mathcal{P}_zF * \mu_1(g\cdot o) &= \frac{1}{q+1}\sum_{g_1\cdot o: d(o,g_1\cdot o)=1} \mathcal{P}_zF(gg_1\cdot o) \\ &= \frac{1}{q+1}\sum_{g_1\cdot o: d(o,g_1\cdot o)=1} \left(\int_\Omega p^{1/2+iz}(gg_1\cdot o,\omega)F(\omega)d\nu(\omega)\right). \end{aligned}$$

Now using the identity $p(gg_1\cdot o,\omega) = p(g_1\cdot o, g^{-1}\omega)\ p(g\cdot o,\omega)$ (see [14, page 36]) together with equation (2.3.7), we obtain

$$L(\mathcal{P}_zF)(x) = \frac{q^{1/2+iz}+q^{1/2-iz}}{q+1}\mathcal{P}_zF(x) \quad \text{for all } x \in \mathfrak{X}.$$

Consequently for every $z \in \mathbb{C}$ and $\mathbf{F} \in \mathcal{K}'(\Omega)$, $\mathcal{P}_z\mathbf{F}$ is in $\mathbb{E}_z(\mathfrak{X})$ where

$$\mathbb{E}_z(\mathfrak{X}) = \{u : \mathfrak{X} \to \mathbb{C} : \mathcal{L}u(x) = \gamma(z)u(x) \text{ for all } x \in \mathfrak{X}\}$$

is the eigenspace of the Laplace operator $\mathcal{L}$ with eigenvalue $\gamma(z)$. Also using Proposition 2.3.9, it is easy to see that if $z \in (k\tau + i)/2$, $k \in \mathbb{Z}$, the map $\mathcal{P}_z : \mathcal{K}'(\Omega) \to \mathbb{E}_z(\mathfrak{X})$ is not injective.

2.3.4 The Spherical Functions

We now draw our attention to the study of radial eigenfunctions of the Laplacian $\mathcal{L}$. To begin with, we first introduce the notion of the elementary spherical functions.

Definition 2.3.10. *For $z \in \mathbb{C}$, the elementary spherical function ϕ_z is the radial eigenfunction of the Laplacian normalized by $\phi_z(o) = 1$.*

Alternatively, the spherical function ϕ_z can also be represented in terms of the Poisson transform as follows:

$$\phi_z(x) = \mathcal{P}_z 1(x) = \int_\Omega p^{1/2+iz}(x,\omega)d\nu(\omega) \quad \forall x \in \mathfrak{X}.$$

Note that for every x in $\mathfrak{X}$, the map $z \to \phi_z(x)$ is an entire function. One of the most intriguing property of ϕ_z is that any radial eigenfunction of $\mathcal{L}$ with eigenvalue $\gamma(z)$ is a constant multiple of ϕ_z (see [12, Theorem 2.1],[13]). It is also known that the elementary spherical function ϕ_z has the following explicit formula (see for example [7]):

$$\phi_z(x) = \begin{cases} \left(\frac{q-1}{q+1}|x|+1\right) q^{-|x|/2} & \forall z \in \ \tau\mathbb{Z}, \\ \left(\frac{q-1}{q+1}|x|+1\right) q^{-|x|/2}(-1)^{|x|} & \forall z \in \tau/2+\tau\mathbb{Z}, \\ \mathbf{c}(z)q^{(iz-1/2)|x|} + \mathbf{c}(-z)q^{(-iz-1/2)|x|} & \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}, \end{cases} \tag{2.3.9}$$

where $\mathbf{c}(\cdot)$ is the Harish-Chandra's c-function given by

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1}\frac{q^{1/2+iz} - q^{-1/2-iz}}{q^{iz} - q^{-iz}} \quad \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}. \tag{2.3.10}$$

From the explicit formula above, it is easy to see that $\phi_z(x) = \phi_{-z}(x) = \phi_z(x^{-1})$ for every $z \in \mathbb{C}$ and $x \in \mathfrak{X}$. Also $|\phi_z(x)| \le 1$ for all $x \in \mathfrak{X}$ whenever $z \in S_1$. For other values of p, we have the following pointwise estimates of $\phi_z(x)$ (see [24]).

Lemma 2.3.11. *For every $x \in \mathfrak{X}$, $\phi_z(x)$ satisfies the following:*

1. *For $1 < p < 2$, $|\phi_z(x)| \asymp q^{-\frac{|x|}{p'}}$ if $\Im z = \delta_{p'}$.*

2. *For $p = 2$, $|\phi_z(x)| \asymp q^{-|x|/2}(1+|x|)$ if $z \in (\tau/2)\mathbb{Z}$ and $|\phi_z(x)| \le C_z q^{-|x|/2}$ if $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$.*

As a consequence of the above lemma, we have the following size estimates for the spherical function ϕ_z.

Lemma 2.3.12. *Let ϕ_z be the spherical function as defined above. Then the following holds.*

1. *For $1 < p < 2$, $1 \leq q < \infty$, $\phi_z \in L^{p',q}(\mathfrak{X})^{\#}$ if and only if $z \in S_p^{\circ}$.*

2. *For $1 \leq p < 2$, $\phi_z \in L^{p',\infty}(\mathfrak{X})^{\#}$ if and only if $z \in S_p$.*

3. *$\phi_z \notin L^{2,\infty}(\mathfrak{X})^{\#}$ if $z \in (\tau/2)\mathbb{Z}$ and $\phi_z \in L^{2,\infty}(\mathfrak{X})^{\#}$ if $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$.*

Proof. We omit the proofs as they easily follow from the definition of the Lorentz norms and the pointwise estimates of ϕ_z given in Lemma 2.3.11. □

2.3.5 Spectrum of the Laplacian

All necessary informations concerning the L^p-spectrum of the Laplacian are already known and can be retrieved from [12, 14]. However in an attempt to make this thesis self-contained, we shall recall some of its important properties. To begin with, we briefly study the map γ (defined in Subsection 2.3.3) which generates all eigenvalues of $\mathcal{L}$. Recalling equation (2.3.8), $\gamma : \mathbb{C} \to \mathbb{C}$ is defined by

$$\gamma(z) = 1 - \frac{q^{1/2+iz} + q^{1/2-iz}}{q+1} \quad \text{for all } z \in \mathbb{C}.$$

We now enlist some important properties of γ, most of which easily follows from the definition above.

Proposition 2.3.13. *For $z \in \mathbb{C}$, the map $z \to \gamma(z)$ satisfies the following conditions.*

1. *$z \to \gamma(z)$ is an entire function.*

2. *$\gamma(\cdot)$ is a surjective function.*

3. *$\gamma(z) = \gamma(-z) = \gamma(z+\tau)$ for all $z \in \mathbb{C}$.*

Substituting $z = x + iy$ in the above expression of $\gamma(z)$, it follows that the real and the imaginary parts of $\gamma(z)$ can be explicitly written as

$$\text{Re } \gamma(z) = 1 - \frac{q^{1/2+y} + q^{1/2-y}}{q+1} \cos(x \log q) \text{ and } \Im\gamma(z) = \frac{q^{1/2+y} - q^{1/2-y}}{q+1} \sin(x \log q). \tag{2.3.11}$$

In an attempt to get a more transparent idea concerning the geometry of the map γ, in the Proposition below, we enlist some important properties of the functions Re $\gamma(\cdot)$ and $\Im\gamma(\cdot)$. We omit the proofs as they easily follow using elementary calculus.

Proposition 2.3.14. *Let $z = x + iy$, $x, y \in \mathbb{R}$ and $\gamma(z)$ be as defined above. Then we have the following.*

1. *If $\cos(x \log q) \geq 0$, the map $y \to$ Re $\gamma(x+iy)$ is monotonically decreasing on $[0, 1/2]$.*

2. *If $\sin(x \log q) \geq 0$, the map $y \to \Im\gamma(x + iy)$ is monotonically increasing on $[0, 1/2]$.*

We are now ready to investigate the image of S_p under the map γ. Using Proposition 2.3.13, we first note that $\gamma(S_p) = \gamma(S_{p'})$ and $\gamma(S_p^\circ) = \gamma(S_{p'}^\circ)$ for every $p \in [1, 2]$. Since $\delta_p = 1/p - 1/2$ is a decreasing function on $[1, 2]$, using Proposition 2.3.14, it follows that for $1 \leq p \leq r \leq 2$,

$$\gamma(S_2) \subseteq \gamma(S_r) \subseteq \gamma(S_p) \subseteq \gamma(S_1).$$

More precisely, γ maps the lines $\{z \in \mathbb{C} : z = s + i\delta_p, s \in \mathbb{R}\}$ onto concentric ellipses in the complex plane, which degenerates into a line segment when $p = 2$. The following result about the L^p-point spectrum may be known to the experts. Nevertheless we try to sketch the proof as we were unable to locate it in the literature.

Proposition 2.3.15. *Regarding the L^p-point spectrum of $\mathcal{L}$, we have the following results.*

1. *For $1 \leq p \leq 2$, the point spectrum of $\mathcal{L}$ on $L^p(\mathfrak{X})$ is empty.*

2. *For $2 < p < \infty$, the point spectrum of $\mathcal{L}$ on $L^p(\mathfrak{X})$ is the set $\gamma(S_p^\circ)$.*

3. *The point spectrum of $\mathcal{L}$ on $L^\infty(\mathfrak{X})$ is the set $\gamma(S_1)$.*

Proof. We begin with the proof of *(2)* and *(3)*. Using Lemma 2.3.12 *(1)* and *(2)* it easily follows that for $2 < p < \infty$ the sets $\gamma(S_p^\circ)$ and $\gamma(S_1)$ are respectively contained inside the sets $P\sigma_p(\mathcal{L})$ and $P\sigma_\infty(\mathcal{L})$. So we prove the converse parts. Seeking a contradiction, let us assume that there exists a non-zero function u in $L^p(\mathfrak{X})$ (or $L^\infty(\mathfrak{X})$) such that $\mathcal{L}u = \gamma(z)u$ for some $z \notin S_p^\circ$ (or S_1). Further suppose that $u(g_o \cdot o) \neq 0$ for some $g_o \in G$. Then define the function

$$f(x) = \int_K u(g_o k \cdot x) dk.$$

Using Proposition 2.2.4 it is easy to see that $f \in L^p(\mathfrak{X})^\#$ (or $L^\infty(\mathfrak{X})^\#$). On the other hand, observe that $f(x) = R(\tau_{g_o}u)(x)$. Since $\mathcal{L}$ commutes with both these operators (see Proposition 2.2.7), it follows that f is a radial eigenfunction of the Laplacian with eigenvalue $\gamma(z)$. Hence $f(x) = u(g_o \cdot o)\phi_z(x)$ for every $x \in \mathfrak{X}$, which is impossible since $f \in L^p(\mathfrak{X})^\#$ (or $L^\infty(\mathfrak{X})^\#$) but $\phi_z \notin L^p(\mathfrak{X})^\#$ (or $L^\infty(\mathfrak{X})^\#$) for $z \notin S_p^\circ$ (or S_1) (see Lemma 2.3.12). This completes the proof of *(2)* and *(3)*.

Proceeding in a similar way as above, we find that in order to prove *(1)*, it is enough to show that for $1 \le p \le 2$, $\phi_z \notin L^p(\mathfrak{X})^\#$ for any $z \in \mathbb{C}$. Most of these facts easily follow from the pointwise estimates of ϕ_z given in Lemma 2.3.11. The only non-trivial part is to show that $\phi_z \notin L^2(\mathfrak{X})^\#$ for any $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, which we shall prove now. Let us on the contrary assume that $\phi_z \in L^2(\mathfrak{X})^\#$ for some $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$. Using the explicit formula 2.3.9 we find that

$$\|\phi_z\|_{L^2(\mathfrak{X})} \asymp \sum_{n=1}^{\infty} |\mathbf{c}(z)q^{izn} + \mathbf{c}(-z)q^{-izn}|^2.$$

Since $\|\phi_z\|_{L^2(\mathfrak{X})} < \infty$, it follows that $\frac{1}{N}\sum_{n=1}^{N} |\mathbf{c}(z)q^{izn} + \mathbf{c}(-z)q^{-izn}|^2$ tends to 0 as $N \to \infty$. On the other hand a simple computation shows that for each N,

$$\sum_{n=1}^{N} |\mathbf{c}(z)q^{izn} + \mathbf{c}(-z)q^{-izn}|^2 = 2N|\mathbf{c}(z)|^2 + \mathbf{c}(z)\overline{\mathbf{c}(-z)}q^{2iz}\frac{q^{2izN}-1}{q^{2iz}-1} + \overline{\mathbf{c}(z)}\mathbf{c}(-z)q^{-2iz}\frac{q^{-2izN}-1}{q^{-2iz}-1}.$$

Now using the fact that $|\mathbf{c}(z)| = |\mathbf{c}(-z)|$ for every $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, we obtain

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\mathbf{c}(z)q^{izn} + \mathbf{c}(-z)q^{-izn}|^2 = 2|\mathbf{c}(\mathbf{z})|^2 \neq 0 \quad \text{for any } z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}.$$

This is clearly not possible and hence we arrive at a contradiction. This completes the proof. □

We end this subsection by giving a complete description of the L^p-spectrum of $\mathcal{L}$. For details we refer [6, 14].

Proposition 2.3.16. *For every $p \in [1, \infty]$, the L^p-spectrum $\sigma_p(\mathcal{L})$ of $\mathcal{L}$ is the image of S_p under the map γ, which is precisely the set of all w in $\mathbb{C}$ which satisfies*

$$\left[\frac{1 - Re\ (w)}{b\cosh(\delta_p \log q)}\right]^2 + \left[\frac{\Im(w)}{b\sinh(\delta_p \log q)}\right]^2 \le 1, \ \ where\ b = \frac{2\sqrt{q}}{q+1}. \tag{2.3.12}$$

In particular, $\sigma_2(\mathcal{L})$ degenerates into the line segment $[1-b, 1+b]$.

Figure 2.2 illustrates the geometrical view of the L^p-spectrum of $\mathcal{L}$ when $2 < p < \infty$. In addition, the open elliptic region in this figure also represents the L^p point spectrum of $\mathcal{L}$ (i.e., $\gamma(S_p^\circ)$).

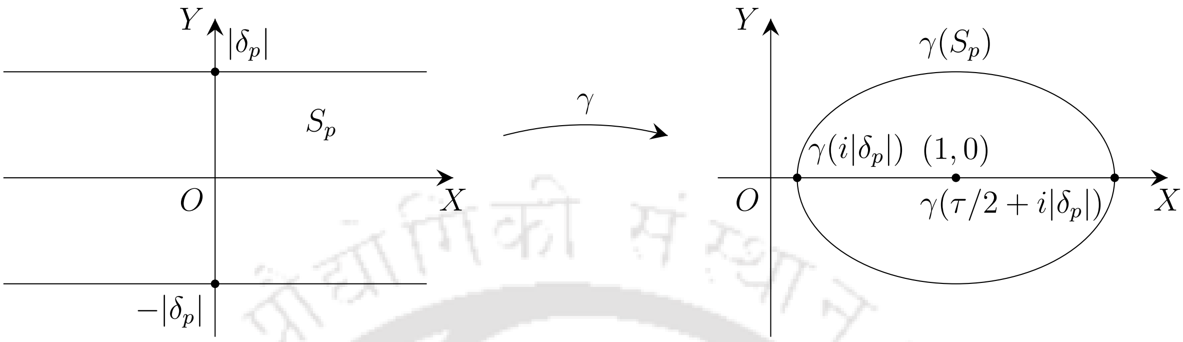

Figure 2.2: L^p-spectrum of $\mathcal{L}$.

2.4 Fourier Transform on Homogeneous Trees

We begin this section by introducing the notion of Fourier transform of radial functions on $\mathfrak{X}$.

2.4.1 Spherical Fourier Transform

Definition 2.4.1. *The spherical Fourier transform $\hat{f}$ of a finitely supported radial function f is defined by*

$$\hat{f}(z) = \sum_{x \in \mathfrak{X}} f(x)\phi_z(x) \quad \textit{where } z \in \mathbb{C}. \tag{2.4.13}$$

The symmetric property and the τ-periodicity of ϕ_z together implies that $\hat{f}$ is even and τ-periodic on $\mathbb{C}$. Using Proposition 2.2.7 and the relevant properties of the spherical function, it follows that

$$\phi_z(x)\phi_z(y) = \int_K \phi_z(xky)dk \quad \text{for all } x, y \in \mathfrak{X}. \tag{2.4.14}$$

This in turn implies that the spherical Fourier transform defines a multiplicative linear functional on the Banach algebra $L^1(\mathfrak{X})^\#$ (see [12, Lemma 4.2]), that is,

$$\widehat{(f_1 * f_2)}(z) = \hat{f}_1(z)\hat{f}_2(z) \quad \text{for all } z \in S_1.$$

For $1 \le p \le 2$, let $\mathcal{S}_p(\mathfrak{X})$ be the space of all functions f on $\mathfrak{X}$ for which

$$\nu_m(f) = \sup_{x\in\mathfrak{X}}(1+|x|)^m q^{|x|/p}|f(x)| < \infty \quad \text{for all } m \in \mathbb{N}. \tag{2.4.15}$$

It is known that $\mathcal{S}_p(\mathfrak{X})$ forms a Fréchet space with respect to these countable seminorms $\nu_m(\cdot)$ and they are also known as the p-Schwartz spaces of rapidly decreasing functions on $\mathfrak{X}$ (see for example [5]). We now aim to characterize the range of the spherical Fourier transform with $\mathcal{S}_p(\mathfrak{X})^\#$ as its domain. It was observed in [5] that the spherical Fourier transform of a finitely supported radial function f can also be written as

$$\hat{f}(z) = \sum_{n\in\mathbb{Z}} \mathcal{A}f(n) q^{inz}, \quad \text{where } \mathcal{A}f(n) = q^{n/2}\sum_{m\in\mathbb{N}} b(m,n)f(m) \tag{2.4.16}$$

denotes the Abel transformation of f and $b(m,n)$ is as in [5, Proposition 2.1]. In the same paper, Cowling, Meda and Setti proved that for every $p \in [1,2]$, the map $f \to \mathcal{A}f$ is a topological isomorphism from $\mathcal{S}_p(\mathfrak{X})^\#$ onto $q^{-\delta_p|\cdot|}S_{ev}(\mathbb{Z})$, where $S_{ev}(\mathbb{Z})$ is the space of all even functions on $\mathbb{Z}$ such that

$$\lambda_m(F) = \sup_{n\in\mathbb{Z}}(1+|n|)^m|F(n)| < \infty \quad \text{for all } m \in \mathbb{N}.$$

More precisely, the map $f \to \mathcal{A}f$ is a bijection from $\mathcal{S}_p(\mathfrak{X})^\#$ onto $q^{-\delta_p|\cdot|}S_{ev}(\mathbb{Z})$. Further for any natural number $m \ge 2$, there exists a constant $C(p,m) > 0$ such that for all $f \in \mathcal{S}_p(\mathfrak{X})^\#$,

$$C^{-1}\lambda_{(m-2)}(q^{\delta_p|\cdot|}\mathcal{A}f) \le \nu_m(f) \le C\lambda_m(q^{\delta_p|\cdot|}\mathcal{A}f). \tag{2.4.17}$$

Using the above result as a tool, we are now ready to prove the isomorphism theorem for the spherical Fourier transform on $\mathfrak{X}$. The proof of this theorem is mainly influenced by the technique given in [11]. For $1 \le p \le 2$, let $\mathcal{H}(S_p)^\#$ be the space of all even, τ-periodic functions g on S_p which are analytic on S_p°, continuous on ∂S_p and satisfies

$$\mu_m(g) = \sup_{z\in S_p}\left|\frac{d^m}{dz^m}g(z)\right| < \infty \quad \text{for all } m \in \mathbb{N}. \tag{2.4.18}$$

When $p = 2$, the strip reduces to the real line $\mathbb{R}$ and $\mathcal{H}(S_2)^\#$ is the space of all even, τ-periodic and infinitely differentiable functions g on $\mathbb{R}$ which satisfies

$$\mu_m(g) = \sup_{s\in\mathbb{R}}\left|\frac{d^m}{ds^m}g(s)\right| < \infty \quad \text{for all } m \in \mathbb{N}.$$

Theorem 2.4.2. *The map $f \to \hat{f}$ is a topological isomorphism from $\mathcal{S}_p(\mathfrak{X})^\#$ onto $\mathcal{H}(S_p)^\#$, for every $p \in [1, 2]$.*

Proof. The $\mathcal{S}_2(\mathfrak{X})^\#$ isomorphism theorem is already proved in [1, Theorem 3.3]. Hence we choose $p \in [1, 2)$. Let $f \in \mathcal{S}_p(\mathfrak{X})^\#$ and $z \in S_p$. Then it is clear that the infinite series (2.4.16) converges uniformly on S_p and consequently $\hat{f}$ is well-defined. The analyticity of $\hat{f}$ on S_p° follows directly from the analyticity of q^{inz} together with the fact that the infinite series (2.4.16) converges uniformly on any compact subset of S_p^o. In fact for every $m \in \mathbb{N}$,

$$\hat{f}^{(m)}(z) = \sum_{n\in\mathbb{Z}} (in \log q)^m \mathcal{A}f(n) q^{inz}, \quad \text{for all } z \in S_p^\circ.$$

The above expression along with equation (2.4.17) implies that for every semi-norm μ_m of $\mathcal{H}(S_p)^\#$, there exists a semi-norm $\nu_{(m+4)}$ of $\mathcal{S}_p(\mathfrak{X})^\#$ such that,

$$\mu_m(\hat{f}) \leq C\nu_{(m+4)}(f) \quad \text{for all } f \in \mathcal{S}_p(\mathfrak{X})^\#.$$

Conversely, assume $g \in \mathcal{H}(S_p)^\#$. Then for all r with $p < r \leq 2$, the function $g(\cdot + i\delta_r)$ is an infinitely differentiable function of period τ. Hence g has a Fourier series representation of the form $g(s) = \sum_{n\in\mathbb{Z}} F(n) q^{ins}$, where

$$F(n) = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} g(s) q^{-ins} ds$$

yields the n^{th} Fourier co-efficient of the function g. Our aim is to prove that $F \in q^{-\delta_p|\cdot|} S_{ev}(\mathbb{Z})$. Applying the Cauchy's integral theorem to g, it is easy to verify that for every $r \in (p, 2]$ and $n \in \mathbb{Z}$,

$$F(n) = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} g(s + i\delta_r) q^{-in(s+i\delta_r)} ds = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} g(s - i\delta_r) q^{-in(s-i\delta_r)} ds. \tag{2.4.19}$$

Infact the first equality in (2.4.19) can be proved using the closed rectangle

$$\Gamma(z) = \{z \in \mathbb{C} : \Im z = 0, -\tau/2 \leq \text{Re } z \leq \tau/2\} \cup \{z \in \mathbb{C} : \text{Re } z = \tau/2, 0 \leq \Im z \leq \delta_r\}$$
$$\cup \{z \in \mathbb{C} : \Im z = \delta_r, \tau/2 \leq \text{Re } z \leq -\tau/2\} \cup \{z \in \mathbb{C} : \text{Re } z = -\tau/2, \delta_r \leq \Im z \leq 0\}.$$

Using the identities (2.4.19) and noting that g is even, one can easily prove that $F(-n) = F(n)$ for all $n \in \mathbb{N}$, that F is even in $\mathbb{Z}$. Integrating by parts the second equation in

(2.4.19) (m times) and further using the Dominated convergence theorem and letting $r \to p$ we have,

$$\lambda_m(q^{\delta_p|\cdot|}F) \leq C\mu_m(g) \quad \text{for every } m \in \mathbb{N}. \tag{2.4.20}$$

Hence there exists an unique $f \in \mathcal{S}_p(\mathfrak{X})^{\#}$ such that $\mathcal{A}f = F$ and $g = \hat{f}$. Further using equations (2.4.17) and (2.4.20) we conclude that

$$\nu_m(f) \leq C\mu_m(\hat{f}) \quad \text{for every } m \in \mathbb{N}.$$

This completes the proof. □

2.4.2 Helgason-Fourier Transform

We will now introduce the notion of Fourier transform for functions which are not necessarily radial.

Definition 2.4.3. *The Helgason-Fourier transform $\widetilde{f}$ of a finitely supported function f on $\mathfrak{X}$ is a function on $\mathbb{C} \times \Omega$ defined by the formula*

$$\widetilde{f}(z, \omega) = \sum_{x \in \mathfrak{X}} f(x)p^{1/2+iz}(x, \omega).$$

It is clear that $\widetilde{f}(z, \omega) = \widetilde{f}(z + \tau, \omega)$ for every $z \in \mathbb{C}$. A simple computation shows that if f is radial, its Helgason-Fourier transform becomes independent of the variable ω and

$$\widetilde{f}(z, \omega) = \hat{f}(z) = \sum_{x \in \mathfrak{X}} f(x)\phi_z(x),$$

that is, the Helgason-Fourier transform reduces to the spherical Fourier transform. Also for every finitely supported function f_1 and finitely supported radial function f_2

$$\widetilde{(f_1 * f_2)}(z, \omega) = \widetilde{f}(z, \omega)\hat{f}_2(z). \tag{2.4.21}$$

We end this chapter by stating the Fourier inversion formula and the Plancherel theorem for the Helgason-Fourier transform on $\mathfrak{X}$. For details we refer [12, Chapter II] and [14, Chapter 3 and Chapter 5].

Theorem 2.4.4. *For every function $f \in \mathcal{S}_2(\mathfrak{X})$,*

$$f(x) = \int_{-\tau/2}^{\tau/2} \int_{\Omega} p^{1/2-is}(x, \omega)\widetilde{f}(s, \omega)d\nu(\omega)d\mu(s) \quad \textit{for all } x \in \mathfrak{X}.$$

In particular, if $f \in \mathcal{S}_2(\mathfrak{X})^\#$, then

$$f(x) = \int_{-\tau/2}^{\tau/2} \hat{f}(s)\phi_s(x)d\mu(s) \quad \text{for all } x \in \mathfrak{X}.$$

Theorem 2.4.5. *The Helgason-Fourier transform is an isometry from $L^2(\mathfrak{X})$ onto the space $L^2(\mathbb{T} \times \Omega, \mu \times \nu)$, that is, for $f \in L^2(\mathfrak{X})$,*

$$\|f\|^2_{L^2(\mathfrak{X})} = \int_{-\tau/2}^{\tau/2} \int_{\Omega} |\widetilde{f}(s,\omega)|^2 d\nu(\omega)d\mu(s).$$

In particular, if $f \in L^2(\mathfrak{X})^\#$, then

$$\|f\|^2_{L^2(\mathfrak{X})} = \int_{-\tau/2}^{\tau/2} |\hat{f}(s)|^2 d\mu(s).$$

In the above theorems μ denotes the Plancherel measure whose density with respect to the Lebesgue measure is given by $\tau^{-1}(q+1)^{-1}q/2|\mathbf{c}(s)|^{-2}$.

CHAPTER 3

Characterization of eigenfunctions of the Laplacian on homogeneous trees

3.1 Introduction

In this chapter we aim to prove certain results that characterizes all eigenfunctions of the Laplacian on homogeneous trees, which are the Poisson transform of $L^{p'}$ functions defined on the boundary. This is mostly done by carefully analyzing the size estimates of the Poisson transform of $L^{p'}$ functions whenever $1 < p \leq 2$. Our results mainly drew their stimulation from the works [25, 26, 27, 28] where the authors proved the similar results on symmetric spaces of non-compact type.

In 1983, Mantero and Zappa proved that if $z \neq (k\tau + i)/2$, $k \in \mathbb{Z}$, then every $\gamma(z)$-eigenfunction of $\mathcal{L}$ can be represented as the Poisson transform of martingales on the boundary (see Theroem 1.1.1). Taking a step forward, we now impose some size conditions on these eigenfunctions of the Laplacian in order to ensure that they are actually the Poisson transform of $L^{p'}$ functions on Ω. Here it is worth mentioning that these size estimates arise naturally due to the behaviour of the Poisson transform (of $L^{p'}$ functions defined on the boundary) which in turn is a generalization of the behaviour of the spherical functions, that is $\mathcal{P}_z 1$. In fact recalling Lemma 2.3.11, we find that

$$\text{for } 1 < p < 2,\ |\phi_z(x)| \asymp q^{-\frac{|x|}{p'}} \text{ if } \Im z = \delta_{p'} \text{ and } |\phi_z(x)| \leq C_z q^{-|x|/2} \text{ if } z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}.$$

To our surprise, the above point wise exponential decay also holds if we replace $\mathcal{P}_z 1$ by $\mathcal{P}_z F$ for any $F \in L^{p'}(\Omega)$, whenever $1 < p \leq 2$. This in particular leads to the fact that $\mathcal{P}_z F$ is a weak $L^{p'}$ eigenfunction of $\mathcal{L}$, which in turn signifies the use of $L^{p',\infty}$ as the size condition to characterize the eigenfunctions of $\mathcal{L}$ as the Poisson transform of $L^{p'}$ functions.

3.2 Some Important Results

In this section we prove certain complementary results which we shall require further. We begin with the following preparatory lemma whose proof follows directly from the definition of weak L^p spaces given in Chapter 2. Nevertheless for the sake of completeness, we provide the proof of this result.

Lemma 3.2.1. *If $1 < p \leq 2$, then there exists a positive constant C such that for every $u \in L^{p',\infty}(\mathfrak{X})$,*

$$\frac{1}{N} \sum_{x \in B(o,N)} |u(x)|^{p'} \leq C \|u\|^{p'}_{L^{p',\infty}} \quad \textit{for all } N \in \mathbb{N}. \tag{3.2.1}$$

Proof. Using the well known relation (see [16, Proposition 1.4.5])

$$\int_X |f|^{p'} d\mu = \int_0^\infty f^*(t)^{p'} dt,$$

we obtain

$$\sum_{x \in B(o,N)} |u(x)|^{p'} = \int_0^\infty (\chi_{B(o,N)}|u|)^*(t)^{p'} dt. \tag{3.2.2}$$

Since $u \in L^{p',\infty}(\mathfrak{X})$, it follows from the definition of the Lorentz norms that

$$u^*(t)^{p'} \leq \frac{1}{t} \|u\|^{p'}_{L^{p',\infty}}, \quad \text{for all } t > 0. \tag{3.2.3}$$

Further for every $x \in \mathfrak{X}$,

$$|u(x)| = \sum_{y \in \mathfrak{X}} |u(y)|\ \chi_{\{x\}}(y) \leq \|u\|_{L^{p',\infty}} \|\chi_{\{x\}}\|_{L^{p,1}} \leq C_p \|u\|_{L^{p',\infty}}.$$

Therefore

$$|\chi_{B(o,N)}(x)u(x)| \leq C_p \|u\|_{L^{p',\infty}} \chi_{B(o,N)}(x).$$

Since $(\chi_{B(o,N)})^* = \chi_{(0,\#B(o,N))}$ ([16, Page 45]), the above expression yields

$$(\chi_{B(o,N)}|u|)^*(t)^{p'} \leq C_p \|u\|^{p'}_{L^{p',\infty}} \chi_{(0,\#B(o,N))}(t). \tag{3.2.4}$$

Combining equations (3.2.2), (3.2.3) and (3.2.4) and using the fact that $\#B(o,N) \asymp q^N$, we finally obtain

$$\begin{aligned}
\sum_{x \in B(o,N)} |u(x)|^{p'} &\leq C_p \int_0^{\#B(o,N)} \min\{\|u\|^{p'}_{L^{p',\infty}}, \|u\|^{p'}_{L^{p',\infty}} \frac{1}{t}\} dt \\
&\leq C_p \|u\|^{p'}_{L^{p',\infty}} \int_0^{q^N} \min\{1, \frac{1}{t}\} dt \\
&= C_p \|u\|^{p'}_{L^{p',\infty}} \left(\int_0^1 dt + \int_1^{q^N} \frac{dt}{t} \right) \\
&= C_p \|u\|^{p'}_{L^{p',\infty}} (1 + \log q^N - \log 1) \\
&\leq C_p \|u\|^{p'}_{L^{p',\infty}} N.
\end{aligned}$$

This completes the proof. □

For proving the main result of this chapter, we also need the following result concerning the asymptotic behaviour of the Poisson transform. See [24, Proposition 1] for details.

Theorem 3.2.2. *Let $1 < p < 2$ and $z = \alpha + i\delta_{p'}$ for some $\alpha \in \mathbb{R}$. Then for every $F \in L^{p'}(\Omega)$,*

$$\int_K \left| \frac{\mathcal{P}_z F(k \cdot \omega_n^0)}{\phi_z(\omega_n^0)} \right|^{p'} dk \longrightarrow \int_K |F(k \cdot \omega_0)|^{p'} dk \quad \text{as } n \to \infty, \tag{3.2.5}$$

where $\omega_0 = \{o, \omega_1^0, \ldots, \omega_n^0, \ldots\}$ is a fixed element in Ω.

We now enlist some important properties of the Harish-Chandra's c-function which are useful in the sequel.

Proposition 3.2.3. *For $z \in \mathbb{R} \setminus (\tau/2)$, the Harish-Chandra's c-function satisfies the following properties.*

1. $\mathbf{c}(-z) = \overline{\mathbf{c}(z)}$.

2. $\mathbf{c}(z) + \overline{\mathbf{c}(z)} = 1$.

Proof. Recalling (2.3.10), the Harish-Chandra's c-function is explicitly given by

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1} \frac{q^{1/2+iz} - q^{-1/2-iz}}{q^{iz} - q^{-iz}}.$$

We omit the proof of *(1)* as it easily follows from the expression above. Further using *(1)*, the explicit formula for ϕ_z and the fact that $\phi_z(0) = 1$, we get *(2)*. □

3.3 Weak L^p Eigenfunctions

The purpose of this section is to prove our main results regarding the characterization of the eigenfunctions of the Laplacian, in terms of certain size estimates of the Poisson transform. We begin with the case $1 < p < 2$.

3.3.1 The Case $1 < p < 2$

Theorem 3.3.1. *Let $1 < p < 2$. Suppose u is a complex valued function defined on $\mathfrak{X}$ and that $z = \alpha + i\delta_{p'}$, $\alpha \in \mathbb{R}$. Then $u(x) = \mathcal{P}_z F(x)$ for some $F \in L^{p'}(\Omega)$ if and only if $u \in L^{p',\infty}(\mathfrak{X})$ and $\mathcal{L}u(x) = \gamma(z)u(x)$. Moreover there exist positive constants C_1 and C_2 such that for all $F \in L^{p'}(\Omega)$ we have*

$$C_1 \|F\|_{L^{p'}(\Omega)} \leq \|\mathcal{P}_z F\|_{L^{p',\infty}(\mathfrak{X})} \leq C_2 \|F\|_{L^{p'}(\Omega)}. \tag{3.3.6}$$

Proof. Fix $p \in (1, 2)$ and suppose that $z = \alpha + i\delta_{p'}$ for some $\alpha \in \mathbb{R}$. We first prove that for all $F \in L^{p'}(\Omega)$,

$$\|\mathcal{P}_z F\|_{L^{p',\infty}(\mathfrak{X})} \leq C_p \|F\|_{L^{p'}(\Omega)}. \tag{3.3.7}$$

The estimate (3.3.7) is already known (see for example [5]), however for the sake of completeness, we give the sketch of the proof. Let $x \in \mathfrak{X}$ and $\{o = x_0, x_1, x_2, \ldots, x_n = x\}$

be the geodesic connecting o to x. Using (2.2.1) we find that

$$\begin{aligned}\mathcal{P}_z F(x) &= \int_{\Omega} \left(\sum_{j=0}^{|x|} q^{(2j-|x|)(i\alpha+1/p)} \mathcal{X}_{E_j(x)\setminus E_{j+1}(x)}(\omega) \right) F(\omega) d\nu(\omega) \\ &= \sum_{j=0}^{|x|} q^{(2j-|x|)(i\alpha+1/p)} \int_{E_j(x)\setminus E_{j+1}(x)} F(\omega) d\nu(\omega).\end{aligned}$$

Taking modulus on both sides, we obtain

$$\begin{aligned}|\mathcal{P}_z F(x)| &\leq q^{-|x|/p} \sum_{j=0}^{|x|} q^{2j/p} \int_{E_j(x)\setminus E_{j+1}(x)} |F(\omega)| d\nu(\omega) \\ &\leq q^{-|x|/p} \sum_{j=0}^{|x|} q^{2j/p} \int_{E_j(x)} |F(\omega)| d\nu(\omega).\end{aligned}$$

Now using (2.3.2) and the fact that $\nu(E_j(x)) = q/(q+1)q^j$, we have

$$\begin{aligned}|\mathcal{P}_z F(x)| &\leq q^{-|x|/p} \left(\frac{q}{q+1} \sum_{j=1}^{|x|} q^{(2/p-1)j} \mathcal{E}_j(|F|)(\omega) + \mathcal{E}_0(|F|)(\omega) \right) \\ &\leq \mathcal{E}(|F|)(\omega) q^{-|x|/p} \sum_{j=0}^{|x|} q^{(2/p-1)j} \\ &= q^{-|x|/p'} \mathcal{E}(|F|)(\omega) \frac{q^{2/p-1} - q^{|x|(1/p'-1/p)}}{q^{2/p-1} - 1} \\ &\leq C_p q^{-|x|/p'} \mathcal{E}(|F|)(\omega) \quad \text{for all } x \in \mathfrak{X},\ \omega \in E_{|x|}(x).\end{aligned}$$

For $\lambda > 0$, define the set $E_\lambda = \{x \in \mathfrak{X} : |\mathcal{P}_z F(x)| > \lambda\}$. Then we have

$$\begin{aligned}E_\lambda &\subseteq \{x \in \mathfrak{X} : C_p q^{-|x|/p'} \mathcal{E}(|F|)(\omega) > \lambda\} \\ &= \left\{ x \in \mathfrak{X} : 1 \leq q^{|x|} < \frac{C_p^{p'} \mathcal{E}(|F|)^{p'}(\omega)}{\lambda^{p'}} \right\} \\ &= \left\{ x \in \mathfrak{X} : 0 \leq |x| < \frac{1}{\log q} \log \left(\frac{C_p^{p'} \mathcal{E}(|F|)^{p'}(\omega)}{\lambda^{p'}} \right) \right\}.\end{aligned}$$

Let $\beta = \frac{1}{\log q}\log(\frac{C_p^{p'}\mathcal{E}(|F|)^{p'}(\omega)}{\lambda^{p'}})$ and $[\beta]$ denote its greatest integer function. Then

$$\begin{aligned}
\#E_\lambda &\leq \#\{x \in \mathfrak{X} : 0 \leq |x| < \beta\} \\
&\leq \int_\Omega \sum_{j=0}^{[\beta]} q^j d\nu(\omega) = \int_\Omega \frac{q^{[\beta]+1}-1}{q-1} d\nu(\omega) \\
&\leq C \int_\Omega q^\beta d\nu(\omega) = C \int_\Omega \frac{C_p^{p'}\mathcal{E}(|F|)^{p'}(\omega)}{\lambda^{p'}} d\nu(\omega) \\
&= C_p \frac{\|\mathcal{E}(|F|)\|^{p'}_{L^{p'}(\Omega)}}{\lambda^{p'}}.
\end{aligned}$$

Therefore for $\lambda > 0$, we finally obtain

$$\lambda^{p'} d_{\mathcal{P}_z F}(\lambda) \leq C_p \|\mathcal{E}(|F|)\|^{p'}_{L^{p'}(\Omega)}.$$

Now the estimate (3.3.7) follows using Theorem 2.3.6. Further from the subsequent discussions in Section 2.3.3, it is clear that $\mathcal{L}(\mathcal{P}_z F) = \gamma(z)\mathcal{P}_z F$. This completes the proof of the first part.

To prove the converse part of this theorem, we first show that for all $F \in L^{p'}(\Omega)$, there exists a constant C (independent of F and α) such that

$$\|F\|_{L^{p'}(\Omega)} \leq C \|\mathcal{P}_z F\|_{L^{p',\infty}(\mathfrak{X})}. \tag{3.3.8}$$

Let $\omega_0 = \{o = \omega_0^0, \omega_1^0, \ldots, \omega_n^0, \ldots\}$ be some fixed element in Ω. The estimates $|\varphi_z(x)| \asymp q^{-|x|/p'}$, $\#S(o,n) \asymp (q+1)q^{n-1}$ and (3.2.1) altogether implies that

$$\begin{aligned}
\sum_{n=0}^{N} \int_K \left|\frac{\mathcal{P}_z F(k\cdot\omega_n^0)}{\phi_z(\omega_n^0)}\right|^{p'} dk &\asymp \sum_{n=0}^{N} (q+1)q^{n-1} \int_K |\mathcal{P}_z F(k\cdot\omega_n^0)|^{p'} dk \\
&= \sum_{n=0}^{N} \sum_{x\in S(o,n)} |\mathcal{P}_z F(x)|^{p'} \\
&= \sum_{x \in B(o,N)} |\mathcal{P}_z F(x)|^{p'} \leq C \|\mathcal{P}_z F\|^{p'}_{L^{p',\infty}} N.
\end{aligned}$$

Therefore we finally have

$$\frac{1}{N}\sum_{n=0}^{N} \int_K \left|\frac{\mathcal{P}_z F(k\cdot\omega_n^0)}{\phi_z(\omega_n^0)}\right|^{p'} dk \leq C \|\mathcal{P}_z F\|^{p'}_{L^{p',\infty}} \quad \text{for all } N \in \mathbb{N}.$$

Now (3.3.8) follows from above inequality and Theorem 3.2.2.

Now we will show that if $u \in \mathbb{E}_z(\mathfrak{X}) \cap L^{p',\infty}(\mathfrak{X})$ then $u(x) = \mathcal{P}_z F(x)$ for some $F \in L^{p'}(\Omega)$. By Theorem 1.1.1 there exists a martingale $\mathbf{F} = (F_n)_{n\geq 0}$ such that

$$u(x) = \mathcal{P}_z\mathbf{F}(x) = \lim_{n\to\infty} \mathcal{P}_z F_n(x).$$

Since $F_n \in \mathcal{K}_n(\Omega) \subset L^{p'}(\Omega)$ for each n, therefore it follows from (3.3.8) that

$$\|F_n\|_{L^{p'}(\Omega)} \leq C\|\mathcal{P}_z F_n\|_{L^{p',\infty}(\mathfrak{X})}. \tag{3.3.9}$$

It was proved in [29] that for all n, the function $\mathcal{P}_z F_n$ is given by

$$\mathcal{P}_z F_n(x) = \varepsilon_n(u)(x) \quad \text{for all } x \in \mathfrak{X}, \tag{3.3.10}$$

where ε_n is defined as

$$\varepsilon_n u(x) = \frac{1}{\#\mathcal{S}(n,x)} \sum_{y\in\mathcal{S}(n,x)} u(y) \quad \text{for all } x \in \mathfrak{X},$$

and

$$\mathcal{S}(n,x) = \begin{cases} \{x\} & \text{if } |x| \leq n, \\ \{y \in \mathfrak{X} : |y| = |x|, x_n = y_n\} & \text{if } |x| > n. \end{cases}$$

Let us define a maximal function

$$\varepsilon^* u(x) = \sup_{n\geq 0} |\varepsilon_n u(x)| \quad \text{for all } x \in \mathfrak{X}.$$

Assume for a moment that the operator ε^* is a bounded map from $L^{p',\infty}(\mathfrak{X})$ into itself. Then from (3.3.9) and (3.3.10) we have

$$\|F_n\|_{L^{p'}(\Omega)} \leq C\|u\|_{L^{p',\infty}(\mathfrak{X})} \quad \text{for all } n \in \mathbb{N}.$$

Since the constant C above is independent of n, it follows from Theorem 2.3.7 that the martingale $\mathbf{F}$ is given by a $L^{p'}(\Omega)$ function, say F. Hence we have $u(x) = \mathcal{P}_z F(x)$.

Now we show that the operator ε^* is a bounded map from $L^{p',\infty}(\mathfrak{X})$ into itself. Note that $\varepsilon_n u(x) = u(x)$ if $|x| \leq n$. Now assume that $|x| > n$, that is, $|x| = n + k$ for some $k > 0$. Let $\{o, x_1, \ldots, x_n, \ldots, x_{n+k} = x\}$ be the geodesic connecting o to x. Observe that $\mathcal{S}(n,x) \subset B(x, 2k)$ and

$$\#\mathcal{S}(n,x) = q^k \asymp \#B(x,2k)^{1/2}.$$

From the above facts we find that for every x satisfying $|x| > n$,

$$|\varepsilon_n u(x)| \leq C \frac{1}{\#B(x,2k)^{1/2}} \sum_{y \in B(x,2k)} |u(y)| \leq C\mathcal{M}u(x),$$

where $\mathcal{M}$ is the operator defined by the formula

$$\mathcal{M}u(x) = \sup_{r \in \mathbb{N}} \frac{1}{\#B(x,r)^{1/2}} \sum_{y \in B(x,r)} |u(y)|.$$

Therefore we finally have

$$\varepsilon^* u(x) \leq C(\mathcal{M}u(x) + |u(x)|) \quad \text{for all } x \in \mathfrak{X}. \tag{3.3.11}$$

In [43] Veca proved that $\mathcal{M}$ is a bounded operator from $L^{2,1}(\mathfrak{X})$ to $L^{2,\infty}(\mathfrak{X})$. Hence from (3.3.11) we can say that ε^* is bounded from $L^{2,1}(\mathfrak{X})$ to $L^{2,\infty}(\mathfrak{X})$. Interpolating with the trivial L^∞-boundedness of ε^*, we conclude that

$$\|\varepsilon^* u\|_{L^{p',\infty}(\mathfrak{X})} \leq C\|u\|_{L^{p',\infty}(\mathfrak{X})} \quad \text{for all } u \in L^{p',\infty}(\mathfrak{X}).$$

This completes the proof. □

3.3.2 The Case $p = 2$

Our next theorem can be considered as an extension of the above result for $p = 2$. Here the standard trick of dominating $|\mathcal{P}_z F(x)|$ by $\mathcal{P}_{i\delta_{p'}}(|F|)(x)$ does not work. This is clearly evident from the fact that $\phi_0 \notin L^{2,\infty}(\mathfrak{X})$ (see Lemma 2.3.12). Hence we treat this case separately. Unlike the p case, here we compute $\mathcal{P}_z F$ directly and one has to get into the realms of the Harish-Chandra's c-function to get the size estimates of Poisson transforms.

Theorem 3.3.2. *Let u be a complex valued function defined on $\mathfrak{X}$ and $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$. Then $u(x) = \mathcal{P}_z F(x)$ for some $F \in L^2(\Omega)$ if and only if $u \in L^{2,\infty}(\mathfrak{X})$ and $\mathcal{L}u(x) = \gamma(z)u(x)$. Moreover there exist positive constants C_1 and C_2 such that for all $F \in L^2(\Omega)$*

$$C_1|\mathbf{c}(z)|\|F\|_{L^2(\Omega)} \leq \|\mathcal{P}_z F\|_{L^{2,\infty}(\mathfrak{X})} \leq C_2|\mathbf{c}(z)|\|F\|_{L^2(\Omega)}, \tag{3.3.12}$$

where $\mathbf{c}(\cdot)$ is the Harish-Chandra's c-function given by (2.3.10).

Proof. Since $\mathcal{P}_z F = \mathcal{P}_{z+\tau} F$, therefore it is enough to prove the above estimates for $z \in (-\tau/2, \tau/2)\setminus\{0\}$. To begin with, we first prove that if $F \in L^2(\Omega)$ and $z \in (-\tau/2, \tau/2)\setminus\{0\}$ then

$$\|\mathcal{P}_z F\|_{L^{2,\infty}(\mathfrak{X})} \leq C_2 |\mathbf{c}(z)| \|F\|_{L^2(\Omega)}. \tag{3.3.13}$$

For the simplicity in calculation let us assume $z = s/2$ where $0 < |s| < \tau$. Let $x \in \mathfrak{X}$ and $\{o = x_0, x_1, \dots, x_n = x\}$ be the geodesic connecting o to x. Then using (2.2.1), we find that

$$\begin{aligned}
\mathcal{P}_z F(x) &= \int\limits_{\Omega} p^{(1+is)/2}(x,\omega) F(\omega) d\nu(\omega) \\
&= q^{-|x|(1+is)/2} \sum_{j=0}^{|x|} q^{(1+is)j} \int\limits_{E_j(x)\setminus E_{j+1}(x)} F(\omega) d\nu(\omega) \\
&= q^{-|x|(1+is)/2} \Bigg[\mathcal{E}_0(F)(\omega) + \sum_{j=1}^{|x|} q^{(1+is)j} \int\limits_{E_j(x)} F(\omega) d\nu(\omega) \\
&\qquad - \sum_{j=0}^{|x|-1} q^{(1+is)j} \int\limits_{E_{j+1}(x)} F(\omega) d\nu(\omega) \Bigg].
\end{aligned}$$

Now expanding $\mathcal{P}_z F$ in terms of $\mathcal{E}_j(F)$ by using the fact that $\nu(E_j(x)) = q/(q+1)q^j$, we get

$$\begin{aligned}
\mathcal{P}_z F(x) &= q^{-|x|(1+is)/2} \left[\mathcal{E}_0(F)(\omega) + \frac{q}{q+1} \left(\sum_{j=1}^{|x|} q^{isj} \mathcal{E}_j(F)(\omega) - \sum_{j=0}^{|x|-1} q^{isj-1} \mathcal{E}_{j+1}(F)(\omega) \right) \right] \\
&= q^{-|x|(1+is)/2} \left[\mathcal{E}_0(F)(\omega) + \frac{q}{q+1} \left(\sum_{j=1}^{|x|} q^{isj} \mathcal{E}_j(F)(\omega) - \sum_{j=1}^{|x|} q^{is(j-1)-1} \mathcal{E}_j(F)(\omega) \right) \right] \\
&= q^{-|x|(1+is)/2} \left[\mathcal{E}_0(F)(\omega) + \frac{q}{q+1} (1 - q^{-1-is}) \sum_{j=1}^{|x|} q^{isj} \mathcal{E}_j(F)(\omega) \right].
\end{aligned}$$

Putting $\mathcal{E}_j(F) = \sum\limits_{m=0}^{j} \Delta_j(F)$ if $j \geq 1$ and $\mathcal{E}_0(F) = \Delta_0(F)$ in the above expression and then changing the order of summation, we obtain

$$\begin{aligned}
\mathcal{P}_z F(x) &= q^{-|x|(1+is)/2} \left[\Delta_0(F)(\omega) + \frac{q}{q+1} (1 - q^{-1-is}) \sum_{j=1}^{|x|} q^{isj} \sum_{m=0}^{j} \Delta_m(F)(\omega) \right] \\
&= q^{-|x|(1+is)/2} [\Delta_0(F)(\omega)
\end{aligned}$$

$$+ \frac{q}{q+1}(1-q^{-1-is})\left(\Delta_0(F)(\omega)\sum_{j=1}^{|x|} q^{isj} + \sum_{m=1}^{|x|}\Delta_m(F)(\omega)\sum_{j=m}^{|x|} q^{isj}\right)\Bigg]$$

$$= q^{-|x|(1+is)/2}\left[\left(1+\frac{1-q^{is|x|}}{1-q^{is}}\frac{q^{1+is}}{q+1}(1-q^{-1-is})\right)\Delta_0(F)(\omega)\right.$$

$$\left. + \frac{q}{q+1}(1-q^{-1-is})\sum_{m=1}^{|x|} q^{ism}\frac{1-q^{is(|x|-m+1)}}{1-q^{is}}\Delta_m(F)(\omega)\right]$$

$$= q^{-|x|(1+is)/2}\left[\left(1+\frac{(1-q^{-1-is})}{1-q^{is}}\frac{q^{1+is}}{q+1}\right)\mathcal{E}_0(F)(\omega)\right.$$

$$\left. - \frac{q(1-q^{-1-is})}{(q+1)(1-q^{is})}\left(q^{is(|x|+1)}\mathcal{E}_{|x|}(F)(\omega) - \sum_{m=1}^{|x|} q^{ism}\Delta_m(F)(\omega)\right)\right].$$

Recalling formula (2.3.10), a simple computation shows that

$$\mathbf{c}(s/2) = -\frac{q^{1+is}(1-q^{-1-is})}{(q+1)(1-q^{is})}.$$

Using the above fact together with Proposition 3.2.3, we obtain

$$\mathcal{P}_z F(x) = q^{-|x|(1/2+iz)}\left[\overline{\mathbf{c}(z)}\mathcal{E}_0(F)(\omega) + \mathbf{c}(z)q^{2iz|x|}\mathcal{E}_{|x|}(F)(\omega)\right.$$

$$\left. - \mathbf{c}(z)q^{-2iz}\sum_{m=1}^{|x|} q^{2izm}\Delta_m(F)(\omega)\right].$$

Taking modulus on both sides and using the fact that $|\mathbf{c}(z)| = |\mathbf{c}(-z)|$, we finally get

$$|\mathcal{P}_z F(x)| \leq q^{-|x|/2}|\mathbf{c}(z)|\left(2\mathcal{E}(|F|)(\omega) + \sup_{|x|\in\mathbb{N}}\left|\sum_{m=1}^{|x|} q^{2izm}\Delta_m(F)(\omega)\right|\right) \qquad \forall x\in\mathfrak{X}, \omega\in E(x).$$

Now let us define a function G by

$$G(\omega) = \sum_{m=1}^{\infty} q^{2izm}\Delta_m(F)(\omega) \quad \text{for all } \omega\in\Omega.$$

By Theorem 2.3.4, G is a well defined L^2 function and $\|G\|_{L^2(\Omega)} \leq C\|F\|_{L^2(\Omega)}$. Further using Propsition 2.3.2 *(3)*, we find that

$$\begin{aligned}\mathcal{E}(G)(\omega) &= \sup_{n\geq 0}|\mathcal{E}_n(G)(\omega)| \\ &= \sup_{n\geq 0}\left|\mathcal{E}_n\left(\sum_{m=1}^{\infty} q^{2izm}\Delta_m(F)\right)(\omega)\right| \\ &= \sup_{n\geq 0}\left|\sum_{m=1}^{n} q^{2izm}\Delta_m(F)(\omega)\right|.\end{aligned}$$

Therefore $|\mathcal{P}_z F(x)|$ is dominated by the sum of two factors namely, $q^{-|x|/2}\mathcal{E}(|F|)$ and $q^{-|x|/2}\mathcal{E}(G)$. Now using a similar technique as in the previous theorem, estimate (3.3.13) follows from Theorem 2.3.6.

Conversely we assume $u \in \mathbb{E}_z(\mathfrak{X})$ for some $z \in (-\tau/2, \tau/2)\setminus\{0\}$. Using Theorem 1.1.1 and formula (2.3.6) there exists a martingale $\mathbf{F} = (F_n)_{n\geq 0}$ such that

$$\begin{aligned} u(x) &= \mathcal{P}_z \mathbf{F}(x) \\ &= \mathcal{P}_z F_N(x) \text{ whenever } |x| \leq N. \end{aligned}$$

To complete the proof we need to show that $\mathbf{F}$ is in $L^2(\Omega)$. In view of Theorem 2.3.7, it is enough to show that

$$\sup_{n\geq 0} \|F_n\|_{L^2(\Omega)} < \infty.$$

To do so, we will use the given assumption that $u \in L^{2,\infty}(\mathfrak{X})$. From Lemma 3.2.1, for all $N \in \mathbb{N}$, we have

$$\begin{aligned} \frac{1}{N}\sum_{m=0}^{N} q^m \int_K |\mathcal{P}_z F_N(k\cdot\omega_m^0)|^2 dk &\asymp \frac{1}{N}\sum_{m=0}^{N}\sum_{x\in S(o,m)} |\mathcal{P}_z F_N(x)|^2 \\ &= \frac{1}{N}\sum_{x\in B(o,N)} |\mathcal{P}_z F_N(x)|^2 \\ &= \frac{1}{N}\sum_{x\in B(o,N)} |u(x)|^2 \leq C\|u\|^2_{L^{2,\infty}}, \end{aligned} \tag{3.3.14}$$

where $\omega_0 = \{o = \omega_0^0, \omega_1^0, \ldots, \omega_m^0, \ldots\}$ is some fixed element in Ω. Our next aim is to find an explicit formula for $\mathcal{P}_z F_N(k\cdot\omega_m^0)$. Recall that for all $N \in \mathbb{N}$,

$$F_N = \sum_{n=0}^{N} \Delta_n(F_N) \quad \text{with} \quad \Delta_n(\Delta_n(F_N)) = \Delta_n(F_N).$$

Now we need the following formula for $\mathcal{P}_z(\Delta_n(F_N))$ given in [29, Page 377].

$$\mathcal{P}_z(\Delta_n(F_N))(k\cdot\omega_m^0) = \begin{cases} 0 & \text{if } m < n, \\ B(n,m,z)\Delta_n(F_N)(k\cdot\omega_0) & \text{if } m \geq n, \end{cases}$$

where $B(n,m,z)$ is defined as follows.

Case I: Suppose $n=0$. Using [29, equation (2.5)], for all $m\geq 0$ we have

$$\begin{aligned}B(0,m,z) &= q^{-(1/2+iz)m}\left(1+\frac{q}{q+1}(1-q^{-1-2iz})\sum_{j=1}^{m}q^{2izj}\right)\\ &= q^{-(1/2+iz)m}\left(1+\frac{q^{1/2}}{q+1}(q^{1/2}-q^{-1/2-2iz})q^{2iz}\frac{1-q^{2izm}}{1-q^{2iz}}\right)\\ &= q^{-(1/2+iz)m}\left(1+\frac{q^{1/2}}{q+1}\frac{q^{1/2+iz}-q^{-1/2-iz}}{q^{iz}-q^{-iz}}(q^{2izm}-1)\right).\end{aligned}$$

Substituting the value of $\mathbf{c}(z)$ from (2.3.10), the above expression finally takes the form

$$B(0,m,z)=q^{-m/2}(q^{-izm}+\mathbf{c}(z)(q^{izm}-q^{-izm}))\quad\text{for all } m\geq 0.$$

Case II: In a similar way as above, for $n>0$ and $m\geq n$, we have

$$\begin{aligned}B(n,m,z) &= q^{-(1/2+iz)m}\frac{q}{q+1}(1-q^{-1-2iz})\sum_{j=n}^{m}q^{2izj}\\ &= q^{-(1/2+iz)m}\frac{q^{1/2}}{q+1}(q^{1/2}-q^{-1/2-2iz})q^{2izn}\frac{q^{2iz(m-n+1)}-1}{q^{2iz}-1}\\ &= q^{-m/2}\frac{q^{1/2}}{q+1}\frac{q^{1/2+iz}-q^{-1/2-iz}}{q^{iz}-q^{-iz}}q^{iz(n-1)}(q^{iz(m-n+1)}-q^{-iz(m-n+1)})\\ &= q^{-m/2}\mathbf{c}(z)q^{iz(n-1)}(q^{iz(m-n+1)}-q^{-iz(m-n+1)}).\end{aligned}$$

Hence for every m with $0\leq m\leq N$,

$$\mathcal{P}_zF_N(k\cdot\omega_m^0)=\sum_{n=0}^{m}\mathcal{P}_z(\Delta_n(F_N))(k\cdot\omega_m^0)=\sum_{n=0}^{m}B(n,m,z)\Delta_n(F_N)(k\cdot\omega_0).$$

Using the above expression of $\mathcal{P}_zF_N$ and (2.3.3), we compute

$$\begin{aligned}&\frac{1}{N}\sum_{m=0}^{N}q^m\int_K|\mathcal{P}_zF_N(k\cdot\omega_m^0)|^2dk=\frac{1}{N}\sum_{m=0}^{N}q^m\int_K\mathcal{P}_zF_N(k\cdot\omega_m^0)\overline{\mathcal{P}_zF_N(k\cdot\omega_m^0)}dk\\ =&\frac{1}{N}\sum_{m=0}^{N}q^m\int_K\left(\sum_{j=0}^{m}B(j,m,z)\Delta_j(F_N)(k\cdot\omega_0)\right)\left(\overline{\sum_{n=0}^{m}B(n,m,z)\Delta_n(F_N)(k\cdot\omega_0)}\right)dk\\ =&\frac{1}{N}\sum_{m=0}^{N}q^m\sum_{n=0}^{m}|B(n,m,z)|^2\|\Delta_n(F_N)\|^2_{L^2(\Omega)}\\ =&\frac{1}{N}\sum_{n=0}^{N}\|\Delta_n(F_N)\|^2_{L^2(\Omega)}\sum_{m=n}^{N}q^m|B(n,m,z)|^2.\end{aligned}$$

From (3.3.14), we have

$$\sum_{n=0}^{N}\|\Delta_n(F_N)\|^2_{L^2(\Omega)}\left(\frac{1}{N}\sum_{m=n}^{N}|B'(n,m,z)|^2\right)\leq C\|u\|^2_{L^{2,\infty}}\quad\text{for all } N\in\mathbb{N},\qquad(3.3.15)$$

where $B(n,m,z) = q^{-m/2}B'(n,m,z)$. More precisely,

$$B'(n,m,z) = \begin{cases} q^{-izm} + \mathbf{c}(z)(q^{izm} - q^{-izm}) & \text{if } n = 0, \\ \mathbf{c}(z)q^{iz(n-1)}(q^{iz(m-n+1)} - q^{-iz(m-n+1)}) & \text{if } n > 0. \end{cases} \tag{3.3.16}$$

Now we claim that for all $n \geq 0$,

$$\frac{1}{N}\sum_{m=n}^{N} |B'(n,m,z)|^2 \to 2|\mathbf{c}(z)|^2 \quad \text{as } N \to \infty. \tag{3.3.17}$$

Case 1: First we assume $n = 0$. Using the explicit formula (3.3.16), we find that

$$\begin{aligned}\sum_{m=0}^{N} |B'(0,m,z)|^2 &= 1 + \sum_{m=1}^{N}(q^{-izm} + \mathbf{c}(z)(q^{izm} - q^{-izm})) \cdot \overline{(q^{-izm} + \mathbf{c}(z)(q^{izm} - q^{-izm}))} \\ &= 1 + \sum_{m=1}^{N}(1 + |\mathbf{c}(z)|^2(2 - q^{2izm} - q^{-2izm}) + \mathbf{c}(z)(q^{2izm} - 1) \\ &\qquad + \overline{\mathbf{c}(z)}(q^{-2izm} - 1)) \\ &= (N+1) + |\mathbf{c}(z)|^2\left(2N - q^{2iz}\frac{1 - q^{2izN}}{1 - q^{2iz}} - q^{-2iz}\frac{1 - q^{-2izN}}{1 - q^{-2iz}}\right) \\ &\qquad + \mathbf{c}(z)\left(q^{2iz}\frac{1 - q^{2izN}}{1 - q^{2iz}} - N\right) + \overline{\mathbf{c}(z)}\left(q^{-2iz}\frac{1 - q^{-2izN}}{1 - q^{-2iz}} - N\right).\end{aligned}$$

Using the above expression and the fact that $\mathbf{c}(z) + \overline{\mathbf{c}(z)} = 1$, the convergence (3.3.17) follows.

Case 2: Similarly for $n > 0$, (3.3.17) follows from the following explicit formula.

$$\begin{aligned}\sum_{m=n}^{N} |B'(n,m,z)|^2 = |\mathbf{c}(z)|^2 \Bigg(2(N-n+1) &- \frac{q^{2iz}}{1 - q^{2iz}}(1 - q^{2iz(N-n+1)}) \\ &- \frac{q^{-2iz}}{1 - q^{-2iz}}(1 - q^{-2iz(N-n+1)})\Bigg).\end{aligned}$$

Hence for any fixed $k \in \mathbb{N}$ there exists a large positive integer (say) N_k greater than k such that

$$\frac{1}{N_k}\sum_{m=n}^{N_k} |B'(n,m,z)|^2 > |\mathbf{c}(z)|^2, \text{ for all } n \text{ with } 0 \leq n \leq k. \tag{3.3.18}$$

Since $k < N_k$, therefore by using (3.3.15), (3.3.18) and (2.3.3) we finally have

$$C|\mathbf{c}(z)|^2 \|F_k\|^2_{L^2(\Omega)} \leq \|u\|^2_{L^{2,\infty}(\mathfrak{X})}. \tag{3.3.19}$$

Since k is arbitrary, therefore $\mathbf{F}$ coincides with some $F \in L^2(\Omega)$ and $u(x) = \mathcal{P}_z F(x)$. Finally inequality (3.3.12) follows from (3.3.13), (3.3.19) and the fact that for all $F \in L^2(\Omega)$,

$$\|F_k - F\|_{L^2(\Omega)} \to 0 \quad \text{as } k \to \infty.$$

This completes the proof. □

Remark 3.3.3. *Before ending this chapter, we would like to highlight some remarkable differences in the statements of our main results. Note that Theorem 3.3.2 holds only for $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, however Theorem 3.3.1 is true for all parameter z that belongs to the line $\mathbb{R} + i\delta_{p'}$. This difference occurs because of the fact that the constants that appeared in norm estimate (3.3.6) is independent of the parameter $z \in \mathbb{R} + i\delta_{p'}$. On the other hand, the constants that appeared in the norm estimate (3.3.12) depends on the Harish-Chandra's c-function. It follows from (2.3.10) that $|\mathbf{c}(z)| \to \infty$ as z approaches to 0 or any $z_0 = \tau n/2$ for $n \in \mathbb{Z}$. This indicates that the quantity $\|\mathcal{P}_z F\|_{L^{2,\infty}(\mathfrak{X})}$ will go to infinity if z approaches to 0 or any number belonging to $(\tau/2)\mathbb{Z}$. Hence $\mathcal{P}_{z_0} F \notin L^{2,\infty}(\mathfrak{X})$ for any $z_0 \in (\tau/2)\mathbb{Z}$. An illustration of this observation can also be seen from the estimates of ϕ_z given in Lemma 2.3.12.*

CHAPTER 4

Restriction Theorem for the Helgason-Fourier transform on homogeneous trees

4.1 Introduction

In this chapter we prove the restriction theorems for the Helgason-Fourier transform on $\mathfrak{X}$. These results can also be considered as an application of the size estimates of the Poisson transform proved in Chapter 3. The key concept involved here is the formulation of a duality relation between the Poisson transform and the Helgason-Fourier transform. This idea is in turn adapted from [25, 26] where the authors proved the similar results on symmetric spaces and harmonic NA groups. In Section 4.2 we recall certain important facts related to the intertwining operators which plays a vital role in proving our main results. We then prove the restriction theorems in Section 4.3. Finally in Section 4.4, we discuss about the existence and analyticity of the Helgason-Fourier transform on $\mathfrak{X}$.

4.2 The Intertwining Operators

It is known that if $z \neq (k\tau + i)/2$, $k \in \mathbb{Z}$ then $\mathcal{P}_z$ is a bijective map from $\mathcal{K}'(\Omega)$ onto $\mathbb{E}_z(\mathfrak{X})$. Using this fact, Mantero and Zappa [29] proved the existence of an operator I_z on $\mathcal{K}'(\Omega)$

which intertwines the representations π_z and π_{-z}. More precisely, for $z \neq (k\tau + i)/2$, $k \in \mathbb{Z}$, the intertwining operator $I_z = \mathcal{P}_z^{-1}\mathcal{P}_{-z}$ is such that

$$\pi_{-z}(g)I_z = I_z\pi_z(g) \quad \text{for all } g \in G.$$

Furthermore they also proved that for some selective values of z, I_z turns out to be an integral operator, which in our context can be stated in the following manner. See [29, Proposition 4.4] for details.

Theorem 4.2.1. *For $1 < p < 2$ and $z = i\delta_{p'}$, I_z can be explicitly written as*

$$I_{i\delta_{p'}}F(\omega) = \mathcal{P}_{i\delta_{p'}}^{-1}\mathcal{P}_{i\delta_p}F(\omega) = \int_\Omega k_p(\omega,\omega')F(\omega')d\nu(\omega'), \quad F \in \mathcal{K}(\Omega), \tag{4.2.1}$$

where the kernel $k_p(\omega,\omega')$ is given by the formula

$$k_p(\omega,\omega') = \rho(p)q^{2|c(\omega,\omega')|/p'} \quad \textit{and} \quad \rho(p) = \frac{(q+1)(1-q^{1-2/p})}{q(1-q^{-2/p})}.$$

The main aim in this section is to prove certain important facts about the operator $I_{i\delta_{p'}}$ which will use further.

Theorem 4.2.2. *For $1 < p < 2$ and $z = i\delta_{p'}$, let $I_{i\delta_{p'}}$ be as defined in (4.2.1). Then we have the following.*

1. *$I_{i\delta_{p'}} = (F * K_p)(k_1 \cdot \omega_0)$ for all $F \in \mathcal{K}(\Omega)$ and $k_1 \in K$, where*

$$K_p(k_1 \cdot \omega_0) = k_p(\omega_0, k_1^{-1} \cdot \omega_0),$$

ω_0 is some fixed element in Ω, and '$*$' denotes the convolution (whenever it makes sense) of two functions defined on K.

2. *$I_{i\delta_{p'}}$ is a continuous linear operator from $L^p(\Omega)$ to $L^{p'}(\Omega)$. More precisely, there exists a constant $C_p > 0$ such that*

$$\|I_{i\delta_{p'}}F\|_{L^{p'}(\Omega)} \leq C\|F\|_{L^p(\Omega)} \quad \textit{for all } F \in L^p(\Omega). \tag{4.2.2}$$

Proof. *(1):* For $F \in \mathcal{K}(\Omega)$ and $k_1 \in K$, we have

$$\begin{aligned}
(F * K_p)(k_1 \cdot \omega_0) &= \int_K F(k_2 \cdot \omega_0)\ K_p(k_2^{-1}k_1 \cdot \omega_0)\ dk_2 \\
&= \int_K F(k_2 \cdot \omega_0)\ k_p(\omega_0, k_1^{-1}k_2 \cdot \omega_0)\ dk_2 \\
&= \int_K F(k_2 \cdot \omega_0)\ k_p(k_1 \cdot \omega_0, k_2 \cdot \omega_0)\ dk_2 \\
&= I_{i\delta_{p'}} F(k_1 \cdot \omega_0),
\end{aligned}$$

where the equality in the second and the third line is due to the fact that

$$|c(k_1 \cdot \omega_0, k_2 \cdot \omega_0)| = |k_1 \cdot c(\omega_0, k_1^{-1}k_2 \cdot \omega_0)| = |c(\omega_0, k_1^{-1}k_2 \cdot \omega_0)|. \tag{4.2.3}$$

(2): In view of [15, Proposition 1.10], it is enough to prove that $K_p \in L^{p'/2,\infty}(\Omega)$. For $\lambda > 0$, define the set $E_\lambda = \{\omega' \in \Omega : |K_p(\omega')| > \lambda\}$. Using (4.2.3), we obtain

$$\begin{aligned}
\nu(E_\lambda) &= \nu\left(\{k_1 \cdot \omega_0 \in \Omega : |k_p(\omega_0, k_1^{-1} \cdot \omega_0)| > \lambda\}\right) \\
&= \nu\left(\{\omega' \in \Omega : \rho(p) q^{2|c(\omega_0,\omega')|/p'} > \lambda\}\right) \\
&= \nu\left(\left\{\omega' \in \Omega : |c(\omega_0, \omega')| > \frac{1}{\log q} \log\left(\frac{\lambda}{\rho(p)}\right)^{p'/2}\right\}\right) \\
&\leq \frac{C_p}{\lambda^{p'/2}}.
\end{aligned}$$

The assertion now follows from the definition of Lorentz norm given in Chapter 2.

□

4.3 Fourier Restriction Theorems

In this section we prove a version of restriction theorems for the Helgason-Fourier transform on $\mathfrak{X}$. Recall that the Helgason-Fourier transform $\widetilde{f}$ of a finitely supported function f on $\mathfrak{X}$ is a function on $\mathbb{C} \times \Omega$ defined by the formula

$$\widetilde{f}(z, \omega) = \sum_{x \in \mathfrak{X}} f(x) p^{1/2+iz}(x, \omega).$$

For fixed $z \in \mathbb{C}$, we say that the boundary Ω of $\mathfrak{X}$ satisfies a (p, r) restriction theorem if

$$\left(\int_\Omega |\widetilde{f}(z, \omega)|^r d\nu(\omega)\right)^{1/r} \leq C_{p,r} \|f\|_{L^p(\mathfrak{X})} \quad \text{for all } f \in L^p(\mathfrak{X}).$$

Before the formulation of main result for general functions let us first discuss the special case of radial functions. If f is a radial function then it is immediate that

$$\widetilde{f}(z,\omega)=\hat{f}(z)=\sum_{x\in\mathfrak{X}} f(x)\phi_z(x).$$

Therefore the following theorem is an immediate consequence of Lemma 2.3.12.

Theorem 4.3.1. *Let* $1\leq p<2$. *Then*

1. $\hat{f}(z)$ *exists if* $f\in L^{p,\infty}(\mathfrak{X})^{\#}$ *and* $z\in S_p^{\circ}$.
2. $\hat{f}(z)$ *exists if* $f\in L^{p,1}(\mathfrak{X})^{\#}$ *and* $z\in S_p$.
3. $\hat{f}(z)$ *exists if* $f\in L^{2,1}(\mathfrak{X})^{\#}$ *and* $z\in\mathbb{R}\setminus(\tau/2)\mathbb{Z}$.

4.3.1 Restriction Theorem for the Helgason-Fourier Transform

Now we prove our main results for the Helgason-Fourier transform on $\mathfrak{X}$, which can also be considered as an extension of Theorem 4.3.1. Our proofs are mainly based on the following duality relation: For $z\in\mathbb{C}$, a finitely supported function f on $\mathfrak{X}$ and $F\in\mathcal{K}(\Omega)$,

$$\begin{aligned}\int_{\Omega}\widetilde{f}(z,\omega)F(\omega)d\nu(\omega)&=\int_{\Omega}\left(\sum_{x\in\mathfrak{X}} f(x)p^{1/2+iz}(x,\omega)\right)F(\omega)d\nu(\omega)\\&=\sum_{x\in\mathfrak{X}} f(x)\left(\int_{\Omega}p^{1/2+iz}(x,\omega)F(\omega)d\nu(\omega)\right)\\&=\sum_{x\in\mathfrak{X}} f(x)\mathcal{P}_zF(x).\end{aligned}\tag{4.3.4}$$

Theorem 4.3.2. *Let f be a complex valued function defined on $\mathfrak{X}$ and $\alpha\in\mathbb{R}$.*

(a) Let $1\leq p<2$ *and* $p\leq r\leq p'$. *If* $f\in L^{p,1}(\mathfrak{X})$ *then*

$$\left(\int_{\Omega}|\widetilde{f}(\alpha+i\delta_{r'},\omega)|^r d\nu(\omega)\right)^{1/r}\leq C_{p,r}\|f\|_{L^{p,1}(\mathfrak{X})}\quad\text{and}\quad C_{1,r}=1.$$

(b) Let $1<p<2$ *and* $p<r<p'$. *If* $f\in L^{p,\infty}(\mathfrak{X})$ *then*

$$\left(\int_{\Omega}|\widetilde{f}(\alpha+i\delta_{r'},\omega)|^r d\nu(\omega)\right)^{1/r}\leq C_{p,r}\|f\|_{L^{p,\infty}(\mathfrak{X})}.$$

Proof. *(a):* First we assume $p = 1$ and $f \in L^1(\mathfrak{X})$. For $1 \leq r < \infty$ and $z = \alpha + i\delta_{r'}$, using Minkowski's inequality and the fact that

$$|\phi_z(x)| \leq 1 \quad \text{for all } z \in S_1 \text{ and } x \in \mathfrak{X},$$

we obtain

$$\begin{aligned}
\left(\int_\Omega |\widetilde{f}(z,\omega)|^r d\nu(\omega)\right)^{1/r} &= \left(\int_\Omega \left|\sum_{x\in\mathfrak{X}} f(x)p^{1/r+i\alpha}(x,\omega)\right|^r d\nu(\omega)\right)^{1/r} \\
&\leq \sum_{x\in\mathfrak{X}} \left(\int_\Omega |f(x)p^{1/r+i\alpha}(x,\omega)|^r d\nu(\omega)\right)^{1/r} \\
&= \sum_{x\in\mathfrak{X}} |f(x)| \left(\int_\Omega p(x,\omega)d\nu(\omega)\right)^{1/r} \\
&= \sum_{x\in\mathfrak{X}} |f(x)|\phi_{i\delta_\infty}(x)^{1/r} \leq \|f\|_{L^1(\mathfrak{X})}.
\end{aligned}$$

If $r = \infty$ and $z = \alpha + i\delta_1$ then

$$\widetilde{f}(z,\omega) = \sum_{x\in\mathfrak{X}} f(x)p^{i\alpha}(x,\omega).$$

Since $|p^{i\alpha}(x,\omega)| = 1$, we have

$$|\widetilde{f}(z,\omega)| \leq \sum_{x\in\mathfrak{X}} |f(x)| = \|f\|_{L^1(\mathfrak{X})}.$$

Consequently for $p = 1$ the result follows with $C_{1,r} = 1$.

Next we deal with the case $1 < p < 2$. Suppose that $r = p$ and $z = \alpha + i\delta_{p'}$, $\alpha \in \mathbb{R}$. In view of (4.3.4) and the estimate (3.3.7) we have

$$\begin{aligned}
\left|\int_\Omega \widetilde{f}(z,\omega)F(\omega)d\nu(\omega)\right| &= \left|\sum_{x\in\mathfrak{X}} f(x)\mathcal{P}_z F(x)\right| \\
&\leq \|f\|_{L^{p,1}(\mathfrak{X})}\|\mathcal{P}_z F\|_{L^{p',\infty}(\mathfrak{X})} \\
&\leq C_p\|f\|_{L^{p,1}(\mathfrak{X})}\|F\|_{L^{p'}(\Omega)}.
\end{aligned}$$

Since the above inequality holds for every $F \in \mathcal{K}(\Omega)$, therefore taking supremum over all $F \in \mathcal{K}(\Omega)$ with $\|F\|_{L^{p'}(\Omega)} = 1$, we finally get the relation

$$\left(\int_\Omega |\widetilde{f}(z,\omega)|^p d\nu(\omega)\right)^{1/p} \leq C_p\|f\|_{L^{p,1}(\mathfrak{X})}.$$

Now let us consider the case when $r = p'$ and $z = \alpha + i\delta_p$, $\alpha \in \mathbb{R}$. Using a similar argument as above, it is enough to show that

$$\|\mathcal{P}_{\alpha+i\delta_p} F\|_{L^{p',\infty}(\mathfrak{X})} \leq C_p \|F\|_{L^p(\Omega)} \quad \text{for all } F \in L^p(\Omega). \tag{4.3.5}$$

To prove (4.3.5), we first note that for all $x \in \mathfrak{X}$,

$$\begin{aligned} |\mathcal{P}_{\alpha+i\delta_p} F(x)| &= \left| \int_\Omega p^{1/p'+i\alpha}(x,\omega) F(\omega) d\nu(\omega) \right| \\ &\leq \int_\Omega p^{1/p'}(x,\omega) |F(\omega)| d\nu(\omega) = \mathcal{P}_{i\delta_p}(|F|)(x). \end{aligned}$$

In view of the above expression, it is enough to prove that

$$\|\mathcal{P}_{i\delta_p} F\|_{L^{p',\infty}(\mathfrak{X})} \leq C_p \|F\|_{L^p(\Omega)} \quad \text{for all } F \in L^p(\Omega).$$

Using Theorem 1.1.1 and formula (4.2.1), we find that

$$\begin{aligned} \mathcal{P}_{i\delta_p} F(x) &= \mathcal{P}_{i\delta_{p'}} \mathcal{P}^{-1}_{i\delta_{p'}} \mathcal{P}_{i\delta_p} F(x) \\ &= \mathcal{P}_{i\delta_{p'}}(I_{i\delta_{p'}} F)(x) \quad \text{for all } x \in \mathfrak{X}. \end{aligned}$$

Hence using the estimates (3.3.7) and (4.2.2), we finally obtain

$$\begin{aligned} \|\mathcal{P}_{i\delta_p} F\|_{L^{p',\infty}(\mathfrak{X})} &= \|\mathcal{P}_{i\delta_{p'}}(I_{i\delta_{p'}} F)\|_{L^{p',\infty}(\mathfrak{X})} \\ &\leq C_p \|I_{i\delta_{p'}} F\|_{L^{p'}(\mathfrak{X})} \\ &\leq C_p \|F\|_{L^p(\mathfrak{X})}. \end{aligned}$$

This completes part *(a)* except for the cases $p < r < p'$ which are included in the proof of part *(b)* keeping in mind that $L^{p,1}(\mathfrak{X}) \subset L^{p,\infty}(\mathfrak{X})$ and

$$\|f\|_{L^{p,\infty}(\mathfrak{X})} \leq \|f\|_{L^{p,1}(\mathfrak{X})} \quad \text{for all } f \in L^{p,1}(\mathfrak{X}).$$

(b): Let $1 < p < 2$ and $f \in L^{p,\infty}(\mathfrak{X})$. In view of the duality relation (4.3.4) it follows that to prove our assertion, it is enough to prove that for every $p < r < p'$,

$$\|\mathcal{P}_{\alpha+i\delta_{r'}} F\|_{L^{p',1}(\mathfrak{X})} \leq C_{p,r} \|F\|_{L^{r'}(\Omega)} \quad \text{for all } F \in L^{r'}(\Omega). \tag{4.3.6}$$

We subdivide this proof into the following steps.

Case 1: First we assume $p < r < 2$ and $z = \alpha + i\delta_{r'}$. Using Hölder's inequality and the fact

$$|\phi_z(x)| \leq 1 \quad \text{for all } z \in S_1 \text{ and } x \in \mathfrak{X},$$

we find that for all $x \in \mathfrak{X}$,

$$\begin{aligned}
|\mathcal{P}_{\alpha+i\delta_{r'}} F(x)| &\leq \int_\Omega |p^{1/r+i\alpha}(x,\omega)||F(\omega)| d\nu(\omega) \\
&= \int_\Omega p^{1/r}(x,\omega)|F(\omega)| d\nu(\omega) \\
&\leq \|F\|_{L^{r'}(\mathfrak{X})} \left(\int_\Omega p(x,\omega) d\nu(\omega) \right)^{1/r} \\
&= \|F\|_{L^{r'}(\mathfrak{X})} \phi_{i\delta_\infty}(x)^{1/r} \leq \|F\|_{L^{r'}(\mathfrak{X})} \quad \text{for all } F \in \mathcal{K}(\Omega).
\end{aligned}$$

Taking supremum over all $x \in \mathfrak{X}$, we obtain

$$\|\mathcal{P}_{\alpha+i\delta_{r'}} F\|_{L^\infty(\mathfrak{X})} \leq \|F\|_{L^{r'}(\Omega)} \quad \text{for all } F \in L^{r'}(\Omega). \tag{4.3.7}$$

Now we interpolate between the two norm inequalities (4.3.7) and (3.3.7) (see [16, Page 64, 1.4.2]) to obtain

$$\|\mathcal{P}_{\alpha+i\delta_{r'}} F\|_{L^{t,s}(\mathfrak{X})} \leq C_{t,s,r} \|F\|_{L^{r'}(\mathfrak{X})} \quad \text{for all } r' < t < \infty \text{ and } 0 < s \leq \infty. \tag{4.3.8}$$

Since $r' < p' < \infty$, the desired estimate (4.3.6) follows by putting $t = p'$ and $s = 1$ in (4.3.8).

Case 2: For $2 < r < p'$ and $z = \alpha + i\delta_{r'}$, by using a similar interpolation between the estimates (4.3.5) and (4.3.7), we get

$$\|\mathcal{P}_{\alpha+i\delta_{r'}} F\|_{L^{t,s}(\mathfrak{X})} \leq C_{t,s,r} \|F\|_{L^{r'}(\mathfrak{X})} \quad \text{for all } r < t < \infty \text{ and } 0 < s \leq \infty. \tag{4.3.9}$$

Now the estimate (4.3.6) follows by putting $t = p'$ and $s = 1$ in (4.3.9).

Case 3: Finally we consider $r = 2$ and $z = \alpha \in \mathbb{R}$. Choose two real numbers r_1, r_2 satisfying $p < r_1 < 2 < r_2 < p'$. Define a family of linear operators T_z on the strip

$$\{z \in \mathbb{C} : 1/r_2 \leq \text{Re } z \leq 1/r_1\}$$

by the formula

$$T_z F(x) = \int_\Omega p^z(x,\omega) F(\omega) d\nu(\omega). \tag{4.3.10}$$

Note that the family T_z satisfies all the hypothesis of [16, 1.3.4, Page 43]. In addition, using the estimates (4.3.8) and (4.3.9), it follows that for $j = 1, 2$, the operators

$$T_{1/r_j+i\alpha}F(x) = \int_{\Omega} p^{1/r_j+i\alpha}(x,\omega)F(\omega)d\nu(\omega) = \mathcal{P}_{\alpha+i\delta_{r'_j}}F(x) \quad \text{for all } x \in \mathfrak{X},$$

are bounded from $L^{r'_j}(\mathfrak{X})$ to $L^{p_1}(\mathfrak{X})$ for some p_1 satisfying, $\max\{r'_1, r_2\} < p_1 < p'$. Therefore using the complex interpolation [16, 1.3.4, Page 43], we find that

$$\|\mathcal{P}_\alpha F\|_{L^{p_1}(\mathfrak{X})} = \|T_{1/2+i\alpha}F\|_{L^{p_1}(\mathfrak{X})} \leq C_{p,2}\|F\|_{L^2(\mathfrak{X})}.$$

Interpolating this with the trivial L^∞ boundedness, that is

$$\|\mathcal{P}_\alpha F\|_{L^\infty(\mathfrak{X})} \leq \|F\|_{L^2(\Omega)}$$

and noting that $p_1 < p' < \infty$, the desired estimate (4.3.6) follows. This completes the proof. □

Now we consider the case $p = 2$. Since the behaviour of the Poisson transform differ when $p = 2$ (see Chapter 3 for details), we prove this case separately.

Theorem 4.3.3. *Let $f \in L^{2,1}(\mathfrak{X})$. If $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, then there exists a constant $C > 0$ such that*

$$\left(\int_{\Omega} |\widetilde{f}(z,\omega)|^2 d\nu(\omega) \right)^{1/2} \leq C|\mathbf{c}(z)|\|f\|_{L^{2,1}(\mathfrak{X})},$$

where $\mathbf{c}(\cdot)$ is the Harish-Chandra's c-function given by (2.3.10).

Proof. Using the duality relation (4.3.4), the above restriction theorem will follow from (3.3.13). □

4.4 Existence of the Helgason-Fourier Transform

The above 'restriction type' theorem reveals that if $f \in L^{p,\infty}(\mathfrak{X})$ (or $L^{p,1}(\mathfrak{X})$) then for each $z \in S_p^\circ$ (or $z \in S_p$), there exists a subset Ω_z of Ω of full Haar measure such that $\widetilde{f}(z,\omega)$ exists for all $\omega \in \Omega_z$. In this section we intend to improve this result by using some basic measure theoretic arguments.

Theorem 4.4.1. *Let f be a complex valued function defined on $\mathfrak{X}$.*

1. *Suppose $f \in L^{p,\infty}(\mathfrak{X})$ for $1 < p < 2$. Then there exists a subset Ω_p of Ω of full Haar measure such that $\widetilde{f}(z,\omega)$ exists for all $\omega \in \Omega_p$ and $z \in S_p^\circ$. Moreover for every $\omega \in \Omega_p$, the map $z \to \widetilde{f}(z,\omega)$ is analytic on S_p°.*

2. *Suppose $f \in L^{p,1}(\mathfrak{X})$ for $1 \le p < 2$. Then there exists a subset Ω_p of Ω of full Haar measure such that $\widetilde{f}(z,\omega)$ exists for all $\omega \in \Omega_p$ and $z \in S_p$. Moreover for every $\omega \in \Omega_p$, the map $z \to \widetilde{f}(z,\omega)$ is analytic on S_p° and continuous on ∂S_p.*

Proof. (1): Fix $1 < p < 2$ and suppose that $f \in L^{p,\infty}(\mathfrak{X})$. Consider a sequence $\{r_n\}$ of real numbers satisfying the following conditions.

(a). For all $n \in \mathbb{N}$,

$$p < r_n < 2 \quad \text{and hence} \quad 2 < r_n' < p'. \tag{4.4.11}$$

(b). The following convergences hold.

$$\frac{1}{r_n} \uparrow \frac{1}{p} \quad \text{and hence} \quad \frac{1}{r_n'} \downarrow \frac{1}{p'} \quad \text{as } n \to \infty. \tag{4.4.12}$$

Using (4.4.11) and Theorem 4.3.2 it follows that for each n, there exist subsets Ω_{r_n} and $\Omega_{r_n'}$ of Ω of full Haar measure such that $\widetilde{|f|}(i\delta_{r_n},\omega)$ and $\widetilde{|f|}(i\delta_{r_n'},\omega)$ exist for all $\omega \in \Omega_{r_n}$ and $\omega \in \Omega_{r_n'}$ respectively. Now define the set

$$\Omega_p = \bigcap_{k=r_n,r_n'} \Omega_k.$$

Then $\widetilde{|f|}(\pm i\delta_{r_n},\omega)$ exists for every $n \in \mathbb{N}$ and $\omega \in \Omega_p$. Furthermore

$$\begin{aligned}\nu(\Omega \setminus \Omega_p) &= \nu\left(\Omega \setminus \bigcap_{k=r_n,r_n'} \Omega_k\right) \\ &= \nu\left(\bigcup_{k=r_n,r_n'} \Omega \setminus \Omega_k\right) \\ &\le \sum_{k=r_n,r_n'} \nu(\Omega \setminus \Omega_k) = 0,\end{aligned}$$

that is, Ω_p is a subset of Ω with full Haar measure. We claim that $\widetilde{f}(z,\omega)$ exists for all $\omega \in \Omega_p$ and $z \in S_p^\circ$. To prove our claim, let us first assume that $z \in S_p^\circ$. Then $z = \alpha + i\delta_{r'}$

for some r satisfying $p < r < p'$. Using this fact and (4.4.12), we can find a natural number k such that

$$\frac{1}{p'} < \frac{1}{r'_k} < \frac{1}{r} < \frac{1}{r_k} < \frac{1}{p}. \tag{4.4.13}$$

Hence for each $\omega \in \Omega_p$,

$$\begin{aligned}
|\widetilde{f}(z,\omega)| &= \left|\sum_{x\in\mathfrak{X}} p^{1/2-\delta_{r'}+i\alpha}(x,\omega)f(x)\right| \\
&= \left|\sum_{x\in\mathfrak{X}} p^{1/r+i\alpha}(x,\omega)f(x)\right| \\
&\le \sum_{x\in\mathfrak{X}} p^{1/r}(x,\omega)|f(x)| \\
&= \sum_{x\in E_{\omega,1}} p^{1/r}(x,\omega)|f(x)| + \sum_{x\in E_{\omega,2}} p^{1/r}(x,\omega)|f(x)|,
\end{aligned}$$

where the sets $E_{\omega,1}$ and $E_{\omega,2}$ are defined as follows.

$$E_{\omega,1} = \{x \in \mathfrak{X} : p(x,\omega) < 1\} \quad \text{and} \quad E_{\omega,2} = \{x \in \mathfrak{X} : p(x,\omega) \ge 1\}.$$

Now using (4.4.13) in the last inequality, we finally get

$$\begin{aligned}
|\widetilde{f}(z,\omega)| &\le \sum_{x\in E_{\omega,1}} p^{1/r}(x,\omega)|f(x)| + \sum_{x\in E_{\omega,2}} p^{1/r}(x,\omega)|f(x)| \\
&\le \sum_{x\in E_{\omega,1}} p^{1/r'_k}(x,\omega)|f(x)| + \sum_{x\in E_{\omega,2}} p^{1/r_k}(x,\omega)|f(x)| \\
&\le \sum_{x\in\mathfrak{X}} p^{1/r'_k}(x,\omega)|f(x)| + \sum_{x\in\mathfrak{X}} p^{1/r_k}(x,\omega)|f(x)| \\
&= \widetilde{|f|}(i\delta_{r_k},\omega) + \widetilde{|f|}(i\delta_{r'_k},\omega) < \infty.
\end{aligned}$$

This completes the proof of the existence while the analyticity of the Helgason-Fourier transform follows from the standard use of Fubini's theorem, Morera's theorem and the fact that for each x and ω, the map $z \to p^{1/2+iz}(x,\omega)$ is analytic on S_p°.

(2): The proof of this part follows from a similar argument as above. □

CHAPTER 5

A Theorem of Roe and Strichartz on homogeneous trees

5.1 Introduction

In this chapter we will witness yet another way of characterizing the eigenfunctions of the Laplacian $\mathcal{L}$ on a homogeneous tree $\mathfrak{X}$. In particular our main aim here is to prove a version of Roe's and Strichartz's theorem on $\mathfrak{X}$. In 1980, J. Roe [34] proved that a function on the line with the property that all its derivatives and anti derivatives are uniformly bounded must be a linear combination of $\sin x$ and $\cos x$. This result was later extended to $\mathbb{R}^n$ (see Theorem 1.3.2) by Strichartz [42] where he also proved that the result is no longer valid on hyperbolic 3-spaces. We begin this chapter by extending this counterexample on homogeneous trees.

Counterexample: It is known from the work of Figà-Talamanca etl. [13, Page 288] and a subsequent refinement by Cowling etl. [6, Page 4273] that for all x in $\mathfrak{X}$,

$$|\phi_z(x)| \leq 1 \quad \text{if and only if} \quad z \in S_1.$$

From the subsequent discussions in Chapter 2, it also follows that $\mathcal{L}$ is a bounded linear operator from L^∞ to itself and the L^∞ point spectrum of $\mathcal{L}$ is the set $\gamma(S_1)$ which is an elliptic region comprising of all complex numbers w which satisfy

$$(1 - \text{Re }(w))^2 + \left(\frac{q+1}{q-1}\right)^2 \Im(w)^2 \leq 1.$$

Note that the above elliptic region intersects $\{w \in \mathbb{C} : |w| = 1\}$ in infinitely many points. Choose two distinct points z_1, z_2 in S_1 such that $\gamma(z_1) \neq \gamma(z_2)$ and $|\gamma(z_1)| = |\gamma(z_2)| = 1$. Now let us define the doubly infinite sequence $\{f_k\}_{k\in\mathbb{Z}}$ as follows:

$$f_k(x) = \gamma(z_1)^k \phi_{z_1}(x) + \gamma(z_2)^k \phi_{z_2}(x), \quad \text{where } x \in \mathfrak{X}.$$

Then we find that for all $k \in \mathbb{Z}$ and for all $x \in \mathfrak{X}$,

$$\begin{aligned}\mathcal{L}f_k(x) &= \gamma(z_1)^k \mathcal{L}\phi_{z_1}(x) + \gamma(z_2)^k \mathcal{L}\phi_{z_2}(x) \\ &= \gamma(z_1)^{k+1} \phi_{z_1}(x) + \gamma(z_2)^{k+1} \phi_{z_2}(x) = f_{k+1}(x).\end{aligned}$$

Furthermore for all $k \in \mathbb{Z}$ and for all $x \in \mathfrak{X}$,

$$|f_k(x)| \leq |\phi_{z_1}(x)| + |\phi_{z_2}(x)| \leq 2.$$

This shows that the infinite sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfy all the hypothesis of Theorem 1.3.2 but f_0 fails to be an eigenfunction of $\mathcal{L}$. □

As described in Chapter 1, Srichartz's result can also be viewed as a representative theorem which characterizes all eigenfunctions of the Laplacian $\Delta_{\mathbb{R}^n}$ with eigenvalue -1. In fact with the same line of arguments, Strichartz's result can also be stated in the following manner (see [27]).

Theorem 5.1.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of functions in $\mathbb{R}^n$ satisfying $\Delta_{\mathbb{R}^n} f_k = \lambda f_{k+1}$ for all $k \in \mathbb{Z}$, for some $\lambda > 0$ and $|f_k(x)| \leq M$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, where M is a real number. Then $\Delta_{\mathbb{R}^n} f_0 = -\lambda f_0$.*

Observe that the most elementary eigenfunction which satisfies the hypothesis of Theorem 5.1.1 is of the form $e^{i\sqrt{\lambda}\langle\omega,x\rangle}$ where $\omega \in S^{n-1}$. An obvious analogue of these functions on homogeneous trees are the complex powers of the Poisson kernel, that is, $p^{1/2+i\lambda}(x,\omega)$, which also act as elementary eigenfunctions of the Laplacian $\mathcal{L}$ (see Section 2.3.3, Chapter 2 for details). However the most concerning part is that these functions do not belong to any L^p, weak L^p or in general in any Lorentz space. They are not even bounded, unlike their Euclidean counterparts. Consequently any extension of Strichartz's theorem with certain size restrictions would lead to a prior exclusion of these functions. As a result, we try to adapt a different approach.

Apart from just being a bounded function, another way of looking at $e^{i\sqrt{\lambda}\langle\omega,x\rangle}$ is that it identifies to a tempered distribution on $\mathbb{R}^n$. This approach was in fact exploited by Strichartz in his article [42] where he proved Theorem 1.3.2 by extensively using Euclidean distribution theory as a tool. In view of this, our primary objective is to introduce an analogue of these distribution theories on homogeneous trees, which we shall do in Section 5.2. We then prove our version of Roe's result for distributions, namely Theorem 5.2.4, 5.2.5 and 5.2.6, all of which are capable of accommodating a large class of eigenfunctions of $\mathcal{L}$, some of which are not even well-behaved in terms of size estimates. A vivid description of such functions are also given in Section 5.2.2.

Finally in Section 5.3, we focus our attention to get an exact analogue of Roe's and Strichartz's theorem on homogeneous trees by imposing various uniform size estimates close to L^p, instead of uniform boundedness. An intuitive idea of these L^p-estimates are mainly obtained by carefully analysing the size estimates of the Poisson transforms which we have already proved in Chapter 3 (see Theorem 3.3.1 and 3.3.2 for details). Among other things, in this chapter we also prove the sharpness of our main results in Section 5.3.1.

After dealing with all viable generalizations of Roe's result on homogeneous trees of degree $q+1$ where $q \geq 2$, it becomes quite plausible to consider this problem on a homogeneous tree of degree 2 (i.e., when $q = 1$), which may be identified to $\mathbb{Z}$. The reason for handling this case separately can be ascribed to the difference between the analysis on $\mathbb{Z}$ which has a polynomial growth and that of others which has an exponential growth. However, the most intriguing difference in the context of this chapter lies in the spectrum of their respective Laplace operators. The Laplace operator on $\mathbb{Z}$ is defined as

$$\mathcal{L}_{\mathbb{Z}} f(m) = f(m) - \frac{f(m-1) + f(m+1)}{2} \quad \text{for all } m \in \mathbb{Z}.$$

Unlike the spectrum of $\mathcal{L}$ which is a p-depending elliptic region, the spectrum of the $\mathcal{L}_{\mathbb{Z}}$ is always a line segment, that is, $[0, 2]$. It must also be noted that the term $-2\mathcal{L}_{\mathbb{Z}}$ is in some way, a 'discrete' representation of the operator $-d^2/dx^2$ whose spectrum is also a line segment in $\mathbb{R}$. These observations motivated us to emphasize an analogy with Roe's result on $\mathbb{R}$. A detailed survey in this direction is carried out in Section 5.4.

5.2 Tempered distributions on Homogeneous Trees

We begin this section by recalling the definition of p-Schwartz spaces given in Section 2.4, Chapter 2. For $1 \leq p \leq 2$, $\mathcal{S}_p(\mathfrak{X})$ is the space of all such functions f on $\mathfrak{X}$ for which

$$\nu_m(f) = \sup_{x \in \mathfrak{X}} (1 + |x|)^m q^{|x|/p} |f(x)| < \infty \quad \text{for all } m \in \mathbb{N}. \tag{5.2.1}$$

Definition 5.2.1. *A linear functional $T : \mathcal{S}_p(\mathfrak{X}) \to \mathbb{C}$ is said to be an L^p-tempered distribution if $\langle T, f_n \rangle \to 0$ whenever $\nu_m(f_n) \to 0$ for all $m \in \mathbb{N}$.*

Recall the definition of the radialization operator R mentioned in Section 2.2, Chapter 2. In an attempt to retain the notion of a radial function, we say that a distribution T is radial if

$$\langle T, f \rangle = \langle T, Rf \rangle, \text{ for all } f \in \mathcal{S}_p(\mathfrak{X}).$$

In general the radial part $R(T)$ of an L^p-tempered distribution T is again an L^p-tempered distribution defined by the rule

$$\langle R(T), f \rangle = \langle T, Rf \rangle, \text{ for all } f \in \mathcal{S}_p(\mathfrak{X}).$$

In view of the above definition, it follows that when T is radial, $R(T) = T$. Furthermore we shall say that T has no radial part if $R(T) = 0$, in other words, if $\langle T, f \rangle = 0$ for all $f \in \mathcal{S}_p(\mathfrak{X})^\#$.

The left translation τ_x of T by an element $x \in G$ is defined as

$$\langle \tau_x T, f \rangle = T(\tau_{x^{-1}} f) = T * f^\#(x^{-1}), \text{ where } f \in \mathcal{S}_p(\mathfrak{X}) \text{ and } f^\#(x) = f(x^{-1}).$$

Having set down all basic definitions of a tempered distribution, we now focus our attention to define its Fourier transform. We begin by stating the following isomorphism theorem for the spherical Fourier transform, which can also be considered as one of the technical hearts of this chapter. For all necessary informations surrounding this theorem, we refer Section 2.4, Chapter 2.

Theorem 5.2.2. *The map $f \to \hat{f}$ is a topological isomorphism from $\mathcal{S}_p(\mathfrak{X})^\#$ onto $\mathcal{H}(S_p)^\#$, for every $p \in [1, 2]$. In particular, for every $m \in \mathbb{N}$,*

$$C_1 \mu_m(\hat{f}) \leq \nu_{m+4}(f) \leq C_2 \mu_{m+4}(\hat{f}), \quad \text{for all } f \in \mathcal{S}_p(\mathfrak{X})^\#. \tag{5.2.2}$$

In view of the above isomorphism theorem, we shall write $g^\vee$ to denote the inverse spherical Fourier transform of $g \in \mathcal{H}(S_p)^\#$. Furthermore for the sake of simplicity, we shall mostly use the notation ν to represent any arbitrarily fixed seminorm ν_m which is given by (5.2.1). Similarly we also use μ to denote a seminorm on $\mathcal{H}(S_p)^\#$ which is related to ν by the relation (5.2.2). Occasionally these seminorms will also be suffixed to show its dependency on the parameter m, as and when necessary.

Definition 5.2.3. *The spherical Fourier transform $\hat{T}$ of a radial L^p-tempered distribution T is a linear functional on $\mathcal{H}(S_p)^\#$ defined by the following rule:*

$$\langle \hat{T}, \psi \rangle = \langle T, f \rangle, \text{ where } \psi \in \mathcal{H}(S_p)^\# \text{ and } \hat{f} = \psi.$$

5.2.1 Roe's Theorem for Tempered Distributions

After gathering all necessary informations, we are now ready to prove our results for L^p-tempered distributions. Our approach is motivated by the proof given in [27], which in turn is influenced by Strichartz's approach. It is worthwhile to mention that in both these papers, the Fourier transform of a tempered distribution played an important role.

Theorem 5.2.4. *Let $\{T_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of L^2-tempered distributions on $\mathfrak{X}$ satisfying,*

1. *$\mathcal{L}T_k = z_0 T_{k+1}$ for some non-zero $z_0 \in \mathbb{C}$ and*

2. *$|\langle T_k, \phi \rangle| \leq M\nu(\phi)$ for all $\phi \in \mathcal{S}_2(\mathfrak{X})$, where ν is some fixed semi-norm of $\mathcal{S}_2(\mathfrak{X})$ and $M > 0$.*

Then we have the following results.

(a) If $|z_0| \in [1-b, 1+b]$, then $\mathcal{L}T_0 = |z_0| T_0$ and,

(b) if $|z_0| \notin [1-b, 1+b]$, then $T_k = 0$ for all $k \in \mathbb{Z}$,

where $b = \frac{2\sqrt{q}}{q+1}$.

Proof. Part (a): For a simplicity in understanding we have divided this proof into the following steps.

Step I: We first prove part *(a)* with an additional assumption that the distributions T_k are radial. Fix $z_0 \in \mathbb{C}$ such that $|z_0| \in (1-b, 1+b)$. The cases $|z_0| = 1 \pm b$ can be dealt with similarly. Since $|z_0| \in (1-b, 1+b)$, therefore $z_0 = \gamma(\alpha)e^{i\theta}$ for a unique $\alpha \in (0, \tau/2)$ where $\theta = \arg z_0$. It follows from hypothesis *(1)* of the theorem that $\mathcal{L}^k T_0 = e^{ik\theta}\gamma(\alpha)^k T_k$ for every $k \in \mathbb{Z}$. This implies

$$\widehat{T_0} = e^{ik\theta}\left(\frac{\gamma(\alpha)}{\gamma(\cdot)}\right)^k \widehat{T_k}. \tag{5.2.3}$$

Let $\phi \in \mathcal{H}(S_2)^{\#}$ be such that $\text{supp}(\phi) \subseteq [-\tau/2, -\alpha - r] \cup [\alpha + r, \tau/2]$ where $r > 0$. Observing the fact that $\gamma(\alpha)^k/\gamma(\cdot)^k\phi \in \mathcal{H}(S_2)^{\#}$ and using hypothesis *(2)* of the theorem, we have

$$\begin{aligned}|\langle \widehat{T_0}, \phi\rangle| = \left|\left\langle \widehat{T_k}, e^{ik\theta}\left(\frac{\gamma(\alpha)}{\gamma(\cdot)}\right)^k \phi\right\rangle\right| &= \left|\left\langle T_k, \left(\left(\frac{\gamma(\alpha)}{\gamma(\cdot)}\right)^k \phi\right)^{\vee}\right\rangle\right| \\ &\leq M\nu\left[\left(\left(\frac{\gamma(\alpha)}{\gamma(\cdot)}\right)^k \phi\right)^{\vee}\right].\end{aligned}$$

By the Isomorphism Theorem 5.2.2, there exists a fixed seminorm μ on $\mathcal{H}(S_2)^{\#}$ such that

$$\nu\left[\left(\left(\frac{\gamma(\alpha)}{\gamma(\cdot)}\right)^k \phi\right)^{\vee}\right] \leq C\mu\left[\left(\frac{\gamma(\alpha)}{\gamma(\cdot)}\right)^k \phi\right] = \sup_{\alpha+r\leq|s|\leq\tau/2}\left|\frac{d^m}{ds^m}\left(\left(\frac{\gamma(\alpha)}{\gamma(s)}\right)^k \phi\right)\right|,$$

which tends to zero as k tends to infinity. In fact by using a similar argument as above and letting $k \to -\infty$, we can show that $\langle \widehat{T_0}, \phi\rangle = 0$ for every $\phi \in \mathcal{H}(S_2)^{\#}$ with $\text{supp}(\phi) \subseteq [-\alpha + r, \alpha - r]$. Hence we proved that for any $r > 0$ and for every $\phi \in \mathcal{H}(S_2)^{\#}$ such that $\text{supp}(\phi) \subseteq [-\tau/2, -\alpha - r] \cup [-\alpha + r, \alpha - r] \cup [\alpha + r, \tau/2]$, $\langle \widehat{T_0}, \phi\rangle = 0$.

We now show that

$$(\mathcal{L} - \gamma(\alpha))^{N+1} T_0 = 0 \quad \text{for some } N \in \mathbb{Z}_+. \tag{5.2.4}$$

In view of the fact that the spherical transform is an isomorphism from $\mathcal{S}_2(\mathfrak{X})^{\#}$ onto $\mathcal{H}(S_2)^{\#}$, it is enough to prove that

$$(\gamma(\alpha) - \gamma(s))^{N+1}\widehat{T_0} = 0 \quad \text{for some } N \in \mathbb{Z}_+. \tag{5.2.5}$$

Let g be an infinitely differentiable even function on $\mathbb{R}$ such that $g \equiv 1$ on $[-1/2, 1/2]$ and $\mathrm{supp}(g) \subseteq (-1, 1)$. Define

$$\psi_\epsilon(s) = \begin{cases} g((s-\alpha)/\epsilon) & s \in [0, \tau/2], \\ g((-s-\alpha)/\epsilon) & s \in [-\tau/2, 0]. \end{cases}$$

Here ϵ is a suitably chosen positive number for which $\psi_\epsilon \in \mathcal{H}(S_2)^\#$ with $\mathrm{supp}(\psi_\epsilon) \subseteq (-\alpha-\epsilon, -\alpha+\epsilon) \cup (\alpha-\epsilon, \alpha+\epsilon)$.

Note that if $\phi \in \mathcal{H}(S_2)^\#$, then $(\gamma(\alpha) - \gamma(\cdot))^{N+1}\phi(1-\psi_\epsilon) \in \mathcal{H}(S_2)^\#$ with its support inside $[-\tau/2, -\alpha-\epsilon/2] \cup [-\alpha+\epsilon/2, \alpha-\epsilon/2] \cup [\alpha+\epsilon/2, \tau/2]$. Observing this fact and using the result proved earlier, we obtain

$$\begin{aligned} |\langle(\gamma(\alpha)-\gamma(\cdot))^{N+1}\widehat{T_0}, \phi\rangle| &= |\langle\widehat{T_0}, (\gamma(\alpha)-\gamma(\cdot))^{N+1}\phi\rangle| \\ &\leq |\langle\widehat{T_0}, (\gamma(\alpha)-\gamma(\cdot))^{N+1}\phi(1-\psi_\epsilon)\rangle| \\ &\qquad + |\langle\widehat{T_0}, (\gamma(\alpha)-\gamma(\cdot))^{N+1}\phi\psi_\epsilon\rangle| \\ &= |\langle\widehat{T_0}, (\gamma(\alpha)-\gamma(\cdot))^{N+1}\phi\psi_\epsilon\rangle| \\ &\leq M\nu\left[\left((\gamma(\alpha)-\gamma(\cdot))^{N+1}\phi\psi_\epsilon\right)^\vee\right]. \end{aligned}$$

Once again using the Isomorphism Theorem 5.2.2, there exists a fixed seminorm μ on $\mathcal{H}(S_2)^\#$ such that

$$\begin{aligned} \nu\left[\left((\gamma(\alpha)-\gamma(\cdot))^{N+1}\phi\psi_\epsilon\right)^\vee\right] &\leq \mu\left[(\gamma(\alpha)-\gamma(\cdot))^{N+1}\phi\psi_\epsilon\right] \\ &= \sup_{s\in[-\tau/2,\tau/2]}\left|\frac{d^m}{ds^m}\left((\gamma(\alpha)-\gamma(s))^{N+1}\phi(s)\psi_\epsilon(s)\right)\right| \\ &= \sup_{\alpha-\epsilon\leq|s|\leq\alpha+\epsilon}\left|\frac{d^m}{ds^m}\left((\gamma(\alpha)-\gamma(s))^{N+1}\phi(s)\psi_\epsilon(s)\right)\right| \\ &\leq \sum_{i=0}^{m}\binom{m}{i}\sup_{\alpha-\epsilon\leq|s|\leq\alpha+\epsilon}\left|\frac{d^i}{ds^i}\left((\gamma(\alpha)-\gamma(s))^{N+1}\right)\right| \\ &\qquad \times \sup_{\alpha-\epsilon\leq|s|\leq\alpha+\epsilon}\left|\frac{d^{m-i}}{ds^{m-i}}\left(\phi(s)\psi_\epsilon(s)\right)\right|. \end{aligned} \tag{5.2.6}$$

Choose N large enough, for example $N = 10m+1$. Then for every $s \in (\alpha-\epsilon, \alpha+\epsilon)$ we have the following estimates:

(i) $\left|\frac{d^i}{ds^i}\left((\gamma(\alpha)-\gamma(s))^{10m+2}\right)\right| \leq B_m|\gamma(\alpha)-\gamma(s)|^{10m+2-i}$ where $0 \leq i \leq m$ and

(ii) $\left|\frac{d^{m-i}}{ds^{m-i}}\left(\phi(s)\psi_\epsilon(s)\right)\right| \leq C_{m,\phi}/\epsilon^{m-i}$.

The above estimates together with (5.2.6) implies that

$$\begin{aligned}\nu\left[\left((\gamma(\alpha)-\gamma(\cdot))^{N+1}\phi\psi_\epsilon\right)^\vee\right] &\leq M_{m,\phi}\sum_{i=0}^{m}\sup_{\alpha-\epsilon\leq|s|\leq\alpha+\epsilon}|\gamma(\alpha)-\gamma(s)|^{10m+2-i}\frac{1}{\epsilon^{m-i}}\\ &\leq D\epsilon^{9m+2}\to 0 \text{ as } \epsilon\to 0.\end{aligned}$$

This proves (5.2.5).

Finally we shall prove that

$$(\mathcal{L}-\gamma(\alpha))T_0 = 0 \text{ that is, } N \text{ must be 0 in (5.2.4).}$$

Equation (5.2.4) together with hypothesis *(1)* of the theorem implies that

$$\text{span}\{T_0, T_1, \ldots\} = \text{span}\{T_0, \mathcal{L}T_0, \ldots, \mathcal{L}^N T_0\} = \text{span}\{T_0, T_0, \ldots, T_N\}.$$

Now let us assume that $(\mathcal{L}-\gamma(\alpha))T_0 \neq 0$ and let k_0 be the largest positive integer such that $(\mathcal{L}-\gamma(\alpha))^{k_0}T_0 \neq 0$. Then $1 \leq k_0 \leq N$. Define $T = (\mathcal{L}-\gamma(\alpha))^{k_0-1}T_0 \in \text{span}\{T_0, T_0, \ldots, T_N\}$. Then we can write $T = c_0T_0 + c_1T_1 + \cdots + c_NT_N$ and observe that

(i) $(\mathcal{L}-\gamma(\alpha))T = (\mathcal{L}-\gamma(\alpha))^{k_0}T_0 \neq 0$,

(ii) $(\mathcal{L}-\gamma(\alpha))^2T = (\mathcal{L}-\gamma(\alpha))^{k_0+1}T_0 = 0$.

This implies that for any positive integer k,

$$\begin{aligned}\mathcal{L}^kT &= (\mathcal{L}-\gamma(\alpha)+\gamma(\alpha))^kT\\ &= k\gamma(\alpha)^{k-1}(\mathcal{L}-\gamma(\alpha))T + \gamma(\alpha)^kT.\end{aligned}$$

Thus for any $\psi \in \mathcal{S}_2(\mathfrak{X})^\#$, we have

$$|\langle(\mathcal{L}-\gamma(\alpha))T,\psi\rangle| \leq \frac{1}{k}|\gamma(\alpha)|^{1-k}|\langle\mathcal{L}^kT,\psi\rangle| + \frac{1}{k}|\gamma(\alpha)||\langle T,\psi\rangle|. \tag{5.2.7}$$

Using hypothesis *(2)* of the theorem and the expression of T, we now evaluate $|\langle\mathcal{L}^kT,\psi\rangle|$ to get

$$\begin{aligned}|\langle\mathcal{L}^kT,\psi\rangle| &= |\langle\mathcal{L}^k(c_0T_0 + c_1T_1 + \cdots + c_NT_N),\psi\rangle|\\ &= |\gamma(\alpha)|^k|\langle c_0T_k + c_1T_{1+k} + \cdots + c_NT_{N+k},\psi\rangle|\\ &\leq M\nu(\psi)|\gamma(\alpha)|^k(|c_0| + |c_1| + \ldots + |c_N|).\end{aligned}$$

Thus from equation (5.2.7), we have

$$|\langle(\mathcal{L}-\gamma(\alpha))T,\psi\rangle| \leq M\frac{|\gamma(\alpha)|}{k}(|c_0|+|c_1|+\ldots+|c_N|)\nu(\psi)+\frac{|\gamma(\alpha)|}{k}|\langle T,\psi\rangle|$$
$$\to 0 \text{ as } k\to\infty.$$

Therefore $(\mathcal{L}-\gamma(\alpha))T=(\mathcal{L}-\gamma(\alpha))^{k_0}T_0=0$ which is a contradiction to our assumption on k_0. Hence $N=0$ in (5.2.4) so that $(\mathcal{L}-\gamma(\alpha))T_0=0$. This completes the proof of *(a)* for radial distributions $\{T_k\}$.

Step II: Now we shall prove the result for general case. To avoid triviality, we further assume that T_k is nonzero for some and hence for all $k\in\mathbb{Z}$.

Observe that for any L^2-tempered distribution T, if $R(\tau_x T)=0$ for every $x\in\mathfrak{X}$ then $T=0$. Indeed the above assumption on T implies that

$$\langle\tau_x T,\delta_0\rangle = T*\delta_0(x^{-1})=0$$

for all $x\in\mathfrak{X}$, where δ_0 denotes the Dirac-Delta function at o. Since $T*\delta_0=T$ in the sense of distribution thus $T=0$. This shows that for every $k\in\mathbb{Z}$, there exists an $x\in\mathfrak{X}$ such that the distribution $\tau_x T_k$ has a non-zero radial part.

Now we claim that if $R(\tau_x T_0)\neq 0$ for some $x\in\mathfrak{X}$, then $R(\tau_x T_k)\neq 0$ for every $k\in\mathbb{Z}$. To prove this it is enough to show that if $R(\tau_x T_0)\neq 0$ for some $x\in\mathfrak{X}$, then $R(\tau_x T_{-1})\neq 0$ and $R(\tau_x T_1)\neq 0$. If $R(\tau_x T_{-1})=0$ then $\mathcal{L}R(\tau_x T_{-1})=0$. Since $\mathcal{L}$ commutes with translation and radialization (see Proposition 2.2.7) and $\mathcal{L}T_{-1}=z_0T_0$ for $z_0\neq 0$ thus $R(\tau_x T_0)=0$. On the other hand if $R(\tau_x T_1)=0$ then

$$\langle\tau_x T_1,\phi\rangle=\langle\tau_x\mathcal{L}T_0,\phi\rangle=\langle\tau_x T_0,\mathcal{L}\phi\rangle=0$$

for every $\phi\in\mathcal{S}_2(\mathfrak{X})^\#$. Since $\gamma(s)^{-1}\hat{\psi}(s)\in\mathcal{H}(S_2)^\#$ for every $\psi\in\mathcal{S}_2(\mathfrak{X})^\#$, therefore ψ can be written as $\psi=\mathcal{L}\phi$ for some $\phi\in\mathcal{S}_2(\mathfrak{X})^\#$. Hence $\tau_x T_0=0$ for all $x\in\mathfrak{X}$. This proves our claim.

Finally we shall show that for every $x\in\mathfrak{X}$, the sequence $\{R(\tau_x T_k)\}$ of radial distributions satisfies the hypothesis of this theorem. Once again using the fact that $\mathcal{L}$ commutes with translations and radialization (see Proposition 2.2.7) and $\mathcal{L}T_k=zT_{k+1}$, we have $\mathcal{L}(R(\tau_x T_k))=zR(\tau_x T_{k+1})$ for all $k\in\mathbb{Z}$. It is also easy to see that every $\phi\in\mathcal{S}_2(\mathfrak{X})^\#$,

$$|\langle R\tau_x T_k,\phi\rangle|=|\langle\tau_x T_k,\phi\rangle|=|\langle T_k,\tau_{x^{-1}}\phi\rangle|. \tag{5.2.8}$$

Thus the sequence $\{R(\tau_x T_k)\}$ satisfies the second hypothesis if for any seminorm ν_m of $\mathcal{S}_2(\mathfrak{X})$ and for every $\phi \in \mathcal{S}_2(\mathfrak{X})^{\#}$, $\nu_m(\tau_x\phi) \leq M_x\nu_m(\phi)$. Using the fact that for all $x, y \in \mathfrak{X}$, $|xy| \leq |x| + |y|$ and $1 + |xy| \leq (1 + |x|)(1 + |y|)$, we have for any seminorm ν_m of $\mathcal{S}_2(\mathfrak{X})$ and $\phi \in \mathcal{S}_2(\mathfrak{X})^{\#}$,

$$\begin{aligned}\nu_m(\tau_x\phi) &= \sup_{y\in\mathfrak{X}}(1 + |y|)^m q^{|y|/2}|\phi(x^{-1}y)| \\ &= \sup_{y\in\mathfrak{X}}(1 + |xy|)^m q^{|xy|/2}|\phi(y)| \\ &\leq (1 + |x|)^m q^{|x|/2} \sup_{y\in\mathfrak{X}}(1 + |y|)^m q^{|y|/2}|\phi(y)| \\ &= M_x\nu_m(\phi),\end{aligned}$$

where M_x is a constant which only depends on x. Using equation (5.2.8) together with the fact that $|\langle T_k, \tau_{x^{-1}}\phi\rangle| \leq M\nu(\tau_{x^{-1}}\phi)$, we finally have $|\langle R\tau_x T_k, \phi\rangle| \leq C_x\nu(\phi)$ for every $\phi \in \mathcal{S}_2(\mathfrak{X})^{\#}$.

Since the result is already proved for radial L^2-tempered distributions, we have

$$\mathcal{L}R(\tau_x T_0) = |z_0|R(\tau_x T_0) \quad \text{for all } x \in \mathfrak{X}.$$

Therefore $R(\tau_x(\mathcal{L}T_0 - |z_0|T_0)) = 0$ for all $x \in \mathfrak{X}$. From above observation we have $\mathcal{L}T_0 = |z_0|T_0$. This complete the proof of part *(a)*.

Part (b): We shall prove this part of the theorem only for radial distributions. The proof for the general case follows in a similar way as in part *(a)*. Now assuming that T_k are radial, we have for any $\phi \in \mathcal{S}_2(\mathfrak{X})^{\#}$,

$$|\langle \widehat{T_0}, \phi\rangle| = \left|\left\langle \widehat{T_k}, \left(\frac{z_0}{\gamma(\cdot)}\right)^k \phi \right\rangle\right| \leq M\nu\left[\left(\left(\frac{z_0}{\gamma(\cdot)}\right)^k \phi\right)^{\vee}\right] \leq M\mu\left[\left(\frac{z_0}{\gamma(\cdot)}\right)^k \phi\right].$$

If $|z_0| < \gamma(s)$ (resp. $|z_0| > \gamma(s)$) for $s \in [-\tau/2, \tau/2]$, then letting $k \to \infty$ (resp. $k \to -\infty$) in the above equation we conclude that $\langle T_0, \phi\rangle = 0$ for all $\phi \in \mathcal{S}_2(\mathfrak{X})^{\#}$. This completes the proof. □

Next we consider the case when $1 < p < 2$. The main difference from the previous theorem and the classical Euclidean case is that the L^p-tempered distributions act on holomorphic functions. Therefore the main technique of the Theorem 5.2.4, namely, the use of function whose Fourier transform are supported outside of an interval will not work.

Theorem 5.2.5. *For $1 < p < 2$, let $\{T_k\}_{k\in\mathbb{Z}^+}$ be an infinite sequence of L^p-tempered distributions on $\mathfrak{X}$ satisfying,*

1. $\mathcal{L}T_k = \lambda T_{k+1}$ *for some non-zero* $\lambda \in \mathbb{C}$ *and*

2. $|\langle T_k, \phi\rangle| \leq M\nu(\phi)$ *for all* $\phi \in \mathcal{S}_p(\mathfrak{X})$*, where* ν *is some fixed semi-norm of* $\mathcal{S}_p(\mathfrak{X})$ *and* $M > 0$.

Then we have the following results.

(a) If $|\lambda| = \gamma(i\delta_{p'})$*, then* $\mathcal{L}T_0 = |\lambda|T_0$.

(b) If $|\lambda| < \gamma(i\delta_{p'})$*, then* $T_k = 0$ *for all* $k \in \mathbb{Z}^+$ *and,*

(c) there are solutions which are not eigen-distributions whenever $\gamma(i\delta_{p'}) < |\lambda| < \gamma(\tau/2 + i\delta_{p'})$.

Proof. Part (a): We prove this result for radial distributions, while the general case follows in a similar way as in Theorem 5.2.4. For $p \in (1, 2)$ let $z_0 = i\delta_{p'}$. For a fixed $N \in \mathbb{Z}_+$ we claim that

$$(\gamma(z_0) - \gamma(z))^{N+1}\widehat{T_0} = 0.$$

As observed earlier, for any $\phi \in \mathcal{H}(S_p)^\#$ we have,

$$|\langle(\gamma(z_0) - \gamma(\cdot))^{N+1}\widehat{T_0}, \phi\rangle| \leq M\mu\left[\left(\left(\frac{\gamma(z_0)}{\gamma(\cdot)}\right)^k (\gamma(z_0) - \gamma(\cdot))^{N+1}\phi\right)\right].$$

Since $\gamma(z)$ and $\phi(z)$ are $\tau-$periodic, even functions on S_p, so the seminorm μ on $\mathcal{H}(S_p)^\#$ is given by

$$\begin{aligned}
\mu\left[\left(\left(\frac{\gamma(z_0)}{\gamma(\cdot)}\right)^k (\gamma(z_0) - \gamma(z))^{N+1}\phi\right)\right] &= \sup_{z\in S_p}\left|\frac{d^m}{dz^m}\left(\left(\frac{\gamma(z_0)}{\gamma(z)}\right)^k (\gamma(z_0) - \gamma(z))^{N+1}\phi(z)\right)\right| \\
&= \sup_{z\in S_p^+}\left|\frac{d^m}{dz^m}\left(\left(\frac{\gamma(z_0)}{\gamma(z)}\right)^k (\gamma(z_0) - \gamma(z))^{N+1}\phi(z)\right)\right| \\
&= \sup_{z\in S_p^+} F_k(z) \text{ (say)},
\end{aligned}$$

where $S_p^+ = \{z \in S_p : \ |\text{Re } z| \leq \tau/2 \text{ and } \Im z \geq 0\}$. Observe that in order to prove our claim it is enough to show that $\sup\limits_{z\in S_p^+} F_k \to 0$ as $k \to \infty$.

To proceed further, we need the following estimates. For every $z \in S_p^+$,

(i) $\left|\frac{\gamma(z_0)}{\gamma(z)}\right| \leq 1$,

(ii) $\left|\frac{d^i}{dz^i}\left(\left(\frac{\gamma(z_0)}{\gamma(z)}\right)^k\right)\right| \leq B_m \left|\frac{\gamma(z_0)}{\gamma(z)}\right|^k k(k+1)(k+2)\ldots(k+i-1)$, where $0 \leq i \leq m$.

In view of these observations we find that for all $z \in S_p^+$,

$$\begin{aligned} F_k(z) &\leq \sum_{i=0}^{m} \binom{m}{i} \left|\frac{d^i}{dz^i}\left(\left(\frac{\gamma(z_0)}{\gamma(z)}\right)^k\right)\right| \left|\frac{d^{m-i}}{dz^{m-i}}\left((\gamma(z_0)-\gamma(z))^{N+1}\phi(z)\right)\right| \\ &\leq \sum_{i=0}^{m} \binom{m}{i} B_m \left|\frac{\gamma(z_0)}{\gamma(z)}\right|^k k(k+1)(k+2)\ldots(k+i-1) \qquad (5.2.9) \\ &\qquad \times \left|\frac{d^{m-i}}{dz^{m-i}}\left((\gamma(z_0)-\gamma(z))^{N+1}\phi(z)\right)\right|. \end{aligned}$$

From the above inequality we also have

$$F_k(z) \leq A_m k^m \left|\frac{\gamma(z_0)}{\gamma(z)}\right|^k, \qquad (5.2.10)$$

where

$$A_m = \max_{0\leq i\leq m}\left[\sup_{z\in S_p^+}\left|\frac{d^i}{dz^i}\left((\gamma(z_0)-\gamma(z))^{N+1}\phi(z)\right)\right|\right].$$

In order to prove that $\sup_{z\in S_p^+} F_k \to 0$ as $k\to\infty$, it is enough to show that

(a) $\sup_{z\in V_k^c} F_k \to 0$ and,

(b) $\sup_{z\in V_k} F_k \to 0$ as $k \to \infty$.

where for k large enough,

$$V_k = \{z \in S_p^+ : |\text{Re } z| < (k^{1/4}\log q)^{-1} \text{ and } \delta_p - \log(1+1/k^{1/6})(\log q)^{-1} < \Im z \leq \delta_p\}.$$

Case I: First we deal with (a). In view of equation (5.2.10), it is sufficient to show that for every $z \in V_k^c$ there exists a constant $c > 0$ such that

$$\left|\frac{\gamma(z_0)}{\gamma(z)}\right|^k \leq \left(1+\frac{c}{\sqrt{k}}\right)^{-k}. \qquad (5.2.11)$$

Subcase I′: Let $z \in V_k^c$ be such that $0 \leq \Im z \leq \delta_p - \frac{\log(1+1/k^{1/6})}{\log q}$. Then using Proposition (2.3.14) and the fact that $|\gamma(z)| \geq |\text{Re }\gamma(z)|$ we obtain

$$|\gamma(z)| \geq \left|\gamma\left(i\left(\delta_p - \frac{\log(1+1/k^{1/6})}{\log q}\right)\right)\right|.$$

Using the explicit formula (2.3.11) we find that

$$|\gamma(z)| \geq \left(1 - \frac{q^{1/p}(1+k^{-1/6})^{-1} + q^{1/p'}(1+k^{-1/6})}{q+1}\right).$$

Therefore

$$\begin{aligned}|\gamma(z)| - |\gamma(z_0)| &\geq \left(\frac{q^{1/p} + q^{1/p'}}{q+1}\right) - \left(\frac{q^{1/p}(1+k^{-1/6})^{-1} + q^{1/p'}(1+k^{-1/6})}{q+1}\right)\\ &= \frac{1}{q+1}\frac{q^{1/p'}}{k^{1/6}(1+k^{1/6})}\left[k^{1/6}(q^{2/p-1}-1)-1\right]. \qquad (5.2.12)\end{aligned}$$

Since $q^{2/p-1} - 1 > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$k^{1/6}(q^{2/p-1} - 1) \geq 2 \quad \text{for every } k \geq k_0.$$

This together with (5.2.12) gives the desired inequality (5.2.11).

Subcase I″: On the other hand if $z \in V_k^c$ is such that $(k^{1/4}\log q)^{-1} \leq |\text{Re } z| \leq \tau/2$, then using Proposition (2.3.14) and the explicit formula (2.3.11) we obtain

$$\begin{aligned}|\gamma(z)| \geq |\text{Re } \gamma(z)| &= 1 - \frac{q^{1/2+\Im z} + q^{1/2-\Im z}}{q+1}\cos((\text{Re } z)(\log q))\\ &\geq 1 - \frac{q^{1/2+\delta_p} + q^{1/2-\delta_p}}{q+1}\cos((\text{Re } z)(\log q))\\ &\geq \left(1 - \frac{q^{1/p'} + q^{1/p}}{q+1}\cos(k^{-1/4})\right).\end{aligned}$$

Therefore

$$\begin{aligned}|\gamma(z)| - |\gamma(z_0)| &\geq \left(\frac{q^{1/p'} + q^{1/p}}{q+1}(1 - \cos(k^{-1/4}))\right)\\ &= \left(\frac{q^{1/p'} + q^{1/p}}{q+1}\right) 2\sin^2\left(\frac{1}{2k^{1/4}}\right)\\ &\geq \frac{c}{k^{1/2}}.\end{aligned}$$

Thus for every $z \in V_k^c$, inequality (5.2.11) holds. Consequently $\sup\limits_{z \in V_k^c} F_k \to 0$ as $k \to \infty$.

Case II: Now let us assume that $z \in V_k$. Then $z = a + i\delta_r$, where $|a| < (k^{1/4}\log q)^{-1}$ and $\delta_p - \log(1 + 1/k^{1/6})(\log q)^{-1} < \delta_r \leq \delta_p$. Hence we have

$$\begin{aligned}|\gamma(z) - \gamma(z_0)|^2 &= \left(\frac{q^{1/p'} + q^{1/p}}{q+1} - \frac{q^{1/r'} + q^{1/r}}{q+1}\cos(a\log q)\right)^2 + \left(\frac{q^{1/r} - q^{1/r'}}{q+1}\right)^2 \sin^2(a\log q)\\ &\leq \left(\frac{q^{1/p} + q^{1/p'}}{q+1} - \frac{q^{1/p}(1+k^{-1/6})^{-1} + q^{1/p'}(1+k^{-1/6})}{q+1}\cos(k^{-1/4})\right)^2\\ &\quad + \left(\frac{q^{1/p} - q^{1/p'}}{q+1}\right)^2 \sin^2(k^{-1/4}).\end{aligned}$$

It follows from the inequality $\sqrt{|x|^2+|y|^2} \leq |x|+|y|$ that

$$\begin{aligned}|\gamma(z)-\gamma(z_0)| &\leq \left(\frac{q^{1/p}+q^{1/p'}}{q+1} - \frac{q^{1/p}(1+k^{-1/6})^{-1}+q^{1/p'}(1+k^{-1/6})}{q+1}\cos(k^{-1/4})\right) + \frac{c_1}{k^{1/4}} \\ &= \left(\frac{q^{1/p}+q^{1/p'}}{q+1} - \frac{q^{1/p}(1+k^{-1/6})^{-1}+q^{1/p'}(1+k^{-1/6})}{q+1}\right) \\ &\qquad + \frac{q^{1/p}(1+k^{-1/6})^{-1}+q^{1/p'}(1+k^{-1/6})}{q+1}(1-\cos(k^{-1/4})) \\ &\qquad + \frac{c_1}{k^{1/4}} \\ &\leq \frac{c_3}{k^{1/6}} + \frac{c_2}{k^{1/2}} + \frac{c_1}{k^{1/4}} \leq \frac{c}{k^{1/6}},\end{aligned}$$

where the constants c_1, c_2, c_3 (and hence c) are independent of k. If we take $N = 7m+1$ then each term in equation (5.2.9) atleast contains the factor $(\gamma(z_0)-\gamma(z))^{N+1-m} = (\gamma(z_0)-\gamma(z))^{6m+2}$. Thus from above estimates and equation (5.2.9), we finally have $\sup_{z\in V_k} F_k \leq \frac{C}{k^{1/3}}$, where C is independent of k and $\sup_{z\in S_p} F_k \to 0$ as $k\to\infty$. This completes the proof of part *(a)* for radial and eventually for general distributions.

Part (b): We omit the proof of this part as it is similar to that of Part *(b)* of Theorem 5.2.4.

Part (c): Fix $1<p<2$ and assume that $\gamma(i\delta_{p'}) < |\lambda| < \gamma(\tau/2+i\delta_{p'})$. Then the $L^{p'}$ point spectrum of $\mathcal{L}$, that is, $\gamma(S_p^\circ)$ intersects $\{w\in\mathbb{C} : |w| = |\lambda|\}$ at infinitely many points. Therefore we can choose $p<r<s<2$ and two real numbers α, β such that

$$\gamma(\alpha+i\delta_{r'})e^{-i\theta_1} = \gamma(\beta+i\delta_{s'})e^{-i\theta_2} = \lambda \quad \text{for some } \theta_1,\theta_2\in(0,2\pi).$$

For all $k\in\mathbb{Z}_+$, let us define $T_k = e^{ik\theta_1}\phi_{\alpha+i\delta_{r'}} + e^{ik\theta_2}\phi_{\beta+i\delta_{s'}}$. Then it follows that

$$\begin{aligned}\mathcal{L}T_k &= e^{ik\theta_1}\gamma(\alpha+i\delta_{r'})\phi_{\alpha+i\delta_{r'}} + e^{ik\theta_2}\gamma(\beta+i\delta_{s'})\phi_{\beta+i\delta_{s'}} \\ &= \lambda(e^{i(k+1)\theta_1}\phi_{\alpha+i\delta_{r'}} + e^{i(k+1)\theta_2}\phi_{\beta+i\delta_{s'}}) = \lambda T_{k+1}.\end{aligned}$$

Furthermore using Lemma 2.3.11, we find that for every $\psi\in\mathcal{S}_p(\mathfrak{X})$,

$$\begin{aligned}|\langle T_k,\psi\rangle| &\leq \sum_{x\in\mathfrak{X}} |\psi(x)|\left(|\phi_{\alpha+i\delta_{r'}}(x)| + |\phi_{\beta+i\delta_{s'}}(x)|\right) \\ &\leq \sum_{x\in\mathfrak{X}} |\psi(x)|\left(q^{-|x|/r'} + q^{-|x|/r'}\right) \leq 2\sum_{x\in\mathfrak{X}} |\psi(x)|q^{-|x|/p'} \\ &\leq \left(2\sum_{x\in\mathfrak{X}} \frac{q^{-|x|}}{(1+|x|)^3}\right)\nu_3(\psi).\end{aligned}$$

Therefore the distributions T_k satisfy all the hypothesis of Theorem 5.2.5 but T_0 fails to be an eigen-distribution of $\mathcal{L}$. This completes the proof. □

Now we state the another important theorem whose proof is just a repetation of the arguments of Theorem 5.2.5.

Theorem 5.2.6. *For $1 < p < 2$, let $\{T_{-k}\}_{k\in\mathbb{Z}^+}$ be an infinite sequence of L^p-tempered distributions on $\mathfrak{X}$ satisfying,*

1. *$\mathcal{L}T_{-k} = \lambda T_{-k+1}$ for some non-zero $\lambda \in \mathbb{C}$ and*

2. *$|\langle T_{-k}, \phi\rangle| \leq M\nu(\phi)$ for all $\phi \in \mathcal{S}_p(\mathfrak{X})$, where ν is some fixed semi-norm of $\mathcal{S}_p(\mathfrak{X})$ and $M > 0$.*

Then we have the following results.

(a) *If $|\lambda| = \gamma(\tau/2 + i\delta_{p'})$, then $\mathcal{L}T_0 = |\lambda|T_0$.*

(b) *If $|\lambda| > \gamma(\tau/2 + i\delta_{p'})$, then $T_{-k} = 0$ for all $k \in \mathbb{Z}^+$ and,*

(c) *there are solutions which are not eigen-distributions whenever $\gamma(i\delta_{p'}) < |\lambda| < \gamma(\tau/2 + i\delta_{p'})$.*

5.2.2 Functions which are Tempered Distributions

The theory of distributions are often regarded as an important tool which furnishes a mathematical framework for many operations to functions which are not really well-behaved. In this section we aim to identify a few of these functions which qualifies to form an L^p-tempered distribution.

Lemma 5.2.7. *Let $1 < p \leq 2$ and $f \in L^{p',\infty}(\mathfrak{X})$. Then there exists a semi-norm ν on $\mathcal{S}_p(\mathfrak{X})$ such that for all $\phi \in \mathcal{S}_p(\mathfrak{X})$, $|\langle f, \phi\rangle| \leq C\|f\|_{p',\infty}\nu(\phi)$.*

Proof. Fix $p \in (1, 2]$. Using [32, Lemma 1], it follows that the radial function $h(x) =$

$q^{-|x|/p}(1+|x|)^{-m}$ is in $L^{p,1}(\mathfrak{X})$ if and only if $m > 1$. Therefore for every $\phi \in \mathcal{S}_p(\mathfrak{X})$,

$$\begin{aligned} |\langle f, \phi \rangle| &\le \sum_{x \in \mathfrak{X}} |f(x)||\phi(x)| \\ &= \sum_{x \in \mathfrak{X}} q^{|x|/p}(1+|x|)^m |\phi(x)||f(x)|h(x) \\ &\le \nu(\phi) \sum_{x \in \mathfrak{X}} |f(x)|h(x) \le \nu(\phi)\|h\|_{p,1}\|f\|_{p',\infty}. \end{aligned} \tag{5.2.13}$$

This gives us the desired result with $C = \|h\|_{p,1}$. □

Lemma 5.2.8. *Let $1 < p < 2$. If $z \in S_p$, then for each $\omega \in \Omega$, the function $x \to p^{1/2+iz}(x, \omega)$ identifies to an L^p-tempered distribution.*

Proof. We first recall that for $z \in S_p$, $\phi_z \in L^{p',\infty}(\mathfrak{X})$ (see Lemma 2.3.12). Furthermore from the previous lemma, $h(x) = q^{-|x|/p}(1+|x|)^{-m} \in L^{p,1}(\mathfrak{X})$ for all $m > 1$. Hence using Proposition 2.2.4 *(3)*, we find that for $\psi \in \mathcal{S}_p(\mathfrak{X})$ and $z \in S_p$,

$$\begin{aligned} |\langle \psi, p^{1/2+iz}(\cdot,\omega)\rangle| &= \left| \sum_{x \in \mathfrak{X}} \psi(x) p^{1/2+iz}(x,\omega) \right| \\ &= \left| \sum_{x \in \mathfrak{X}} q^{|x|/p}(1+|x|)^m \psi(x) p^{1/2+iz}(x,\omega) h(x) \right| \\ &\le \nu(\psi) \left| \sum_{x \in \mathfrak{X}} p^{1/2+iz}(x,\omega)h(x) \right| = \nu(\psi)\left|\langle h, p^{1/2+iz}(\cdot,\omega)\rangle\right| \\ &= \nu(\psi)\left|\langle Rh, p^{1/2+iz}(\cdot,\omega)\rangle\right| = \nu(\psi)\left|\langle h, Rp^{1/2+iz}(\cdot,\omega)\rangle\right| \\ &= \nu(\psi)\left|\langle h, \phi_z\rangle\right| \le \nu(\psi)\|h\|_{L^{p,1}(\mathfrak{X})}\|\phi_z\|_{L^{p',\infty}(\mathfrak{X})}. \end{aligned}$$

This completes the proof. □

5.3 Roe's Results for Almost L^p-Functions

As a corollary of the results proved earlier, we are now ready to prove an extension of Roe's and Strichartz's results on homogeneous trees by imposing certain size estimates on the eigenfunctions of $\mathcal{L}$.

Theorem 5.3.1. *Let f be a complex valued function defined on $\mathfrak{X}$ and $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$. If there exists an $M > 0$ such that $\|\mathcal{L}^k f\|_{L^{2,\infty}(\mathfrak{X})} \le M|\gamma(z)|^k$ for all $k \in \mathbb{Z}$ then $\mathcal{L}f \equiv \gamma(z)f$. In particular, there exists $F \in L^2(\Omega)$ such that $f \equiv \mathcal{P}_z F$.*

Proof. Fix $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$ so that $\gamma(z) \in [1-b, 1+b]$. Let us assume that

$$T_k = \gamma(z)^{-k}\mathcal{L}^k f \quad \text{for all } k \in \mathbb{Z}.$$

From the hypothesis of this theorem, we find that $\gamma(z)^{-k}\mathcal{L}^k f \in L^{2,\infty}(\mathfrak{X})$. Therefore by Lemma 5.2.7, T_k is an L^2-tempered distribution and

$$|\langle T_k, \phi\rangle| \leq C\|T_k\|_{2,\infty}\nu(\phi) \leq M\nu(\phi) \quad \text{for all } \phi \in \mathcal{S}_2(\mathfrak{X}).$$

We also note that

$$\mathcal{L}T_k = \gamma(z)^{-k}\mathcal{L}^{k+1}f = \gamma(z)\gamma(z)^{-(k+1)}\mathcal{L}^{k+1}f = \gamma(z)T_{k+1}.$$

Hence the sequence $\{T_k\}_{k\in\mathbb{Z}}$ satisfies all the hypothesis of Theorem 5.2.4 and consequently we have $\mathcal{L}T_0 = \gamma(z)T_0$, that is, $\mathcal{L}f = \gamma(z)f$. Furthermore using Theorem 3.3.2 we conclude that $f \equiv \mathcal{P}_z F$ for some $F \in L^2(\Omega)$. This completes the proof. □

Theorem 5.3.2. *Let f be a complex valued function defined on $\mathfrak{X}$ and $1 < p < 2$.*

1. *Suppose that $z = n\tau + i\delta_{p'}$ for some $n \in \mathbb{Z}$. If there exists an $M > 0$ such that $\|\mathcal{L}^k f\|_{L^{p',\infty}(\mathfrak{X})} \leq M|\gamma(z)|^k$ for all $k \in \mathbb{Z}_+$ then $\mathcal{L}f \equiv \gamma(z)f$.*

2. *Suppose that $z = (2n+1)\tau/2 + i\delta_{p'}$ for some $n \in \mathbb{Z}$. If there exists an $M > 0$ such that $\|\mathcal{L}^{-k} f\|_{L^{p',\infty}(\mathfrak{X})} \leq M|\gamma(z)|^{-k}$ for all $k \in \mathbb{Z}_+$ then $\mathcal{L}f \equiv \gamma(z)f$.*

In either of these cases, there exists $F \in L^{p'}(\Omega)$ such that $f \equiv \mathcal{P}_z F$.

Proof. The proof of this theorem follows a similar procedure as the previous one. However for the sake of completeness we shall sketch the proof of part *(1)*.
Part (1): Fix $z = n\tau + i\delta_{p'}$ for some $n \in \mathbb{Z}$. Since $\gamma(\cdot)$ is a τ-periodic function therefore $\gamma(z) = \gamma(i\delta_{p'})$. Assume that $T_k = \gamma(i\delta_{p'})^{-k}\mathcal{L}^k f$ where $k \in \mathbb{Z}_+$. Using the hypothesis of this theorem and Lemma 5.2.7, It is easy to show that each T_k is an L^p-tempered distribution which satisfies all the hypothesis of Theorem 5.2.5. Hence $\mathcal{L}f = \gamma(i\delta_{p'})f$. Furthermore using Theorem 3.3.1, we conclude that $f \equiv \mathcal{P}_z F$ for some $F \in L^{p'}(\Omega)$. This completes the proof. □

Remark 5.3.3. *In Comparison with the classical Euclidean case, that is Theorem 1.3.2, we would like to point out certain important facts regarding our main results.*

1. *Since $\gamma(z) \in \mathbb{R}$ whenever $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, Theorem 5.3.1 can be thought of as a suitable extension of Strichartz's result on homogeneous tree. In particular if we define $f_k = \gamma(z)^{-k}\mathcal{L}^k f$, then the statement of Theorem 5.3.1 resembles Theorem 1.3.2 with the only difference that the L^∞ boundedness is being replaced by weak L^2 boundedness.*

2. *Unlike the Euclidean counterpart, our results on homogeneous trees are more precise in the sense that it not only characterizes the eigenfunctions of $\mathcal{L}$ but also represents the eigenfunctions as Poisson transform of functions on the boundary.*

5.3.1 Sharpness of the Results

In this section we will try to establish the sharpness our main results by answering the following natural questions. In fact from the relevant discussions, we will also get a transparent idea regarding the formulation of Theorem 5.3.1 and 5.3.2.

Question 5.3.4. *Is Theorem 5.3.1 valid for $z \in (\tau/2)\mathbb{Z}$?*

Proof. Seeking a contradiction, let us assume that there exists a non-zero function u in $L^{2,\infty}(\mathfrak{X})$ such that $\mathcal{L}u = \gamma(z)u$ for some $z \in (\tau/2)\mathbb{Z}$. Further suppose that $u(x_o) \neq 0$ for some $x_o \in \mathfrak{X}$. Then define the function

$$f(x) = \int_K u(x_o k x) dk.$$

Using Proposition 2.2.4 *(4)*, it is easy to see that $f \in L^{2,\infty}(\mathfrak{X})^\#$. On the other hand, observe that $f(x) = R(\tau_{x_o} u)(x)$. Since $\mathcal{L}$ commutes with both these operators (see Proposition 2.2.7), it follows that f is a radial eigenfunction of the Laplacian with eigenvalue $\gamma(z)$. Hence $f(x) = u(x_o)\phi_z(x)$ for every $x \in \mathfrak{X}$, which is impossible since $\phi_z \notin L^{2,\infty}(\mathfrak{X})$ for $z \in (\tau/2)\mathbb{Z}$ (see Lemma 2.3.12). This observation shows that Theorem 5.3.1 is no longer valid for any $z \in (\tau/2)\mathbb{Z}$. □

Remark 5.3.5. *However if we replace the $L^{2,\infty}$ estimate by $\|\phi_0^{-1}\mathcal{L}^k f\|_{L^\infty(\mathfrak{X})} \leq M|\gamma(z)|^k$ for all $k \in \mathbb{Z}$, then Theorem 5.3.1 holds true.*

Question 5.3.6. *Is it possible to replace the $L^{2,\infty}$ (resp. the $L^{p',\infty}$) estimates in Theorem 5.3.1 (resp. in Theorem 5.3.2) by $L^{2,r}$, $r < \infty$ (resp. $L^{s',r}$ with $s > p$ or $L^{p',r}$ with $r < \infty$) ?*

Proof. In view of the radialization technique used in the proof of the previous question, it is enough to prove the following.

(a) If $z = \alpha \pm i\delta_{p'}$ then $\phi_z \notin L^{s',r}(\mathfrak{X})$ for any $s > p$ or $s = p$ and $r < \infty$.

(b) If $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$ then $\phi_z \notin L^{2,r}(\mathfrak{X})$ for any $r < \infty$.

(a) Let $1 < p < 2$ and $z = \alpha \pm i\delta_{p'}$. Using Lemma 2.3.11, we find that $|\phi_z(x)| \asymp q^{-\frac{|x|}{p'}}$. Using this estimate, we now evaluate the distribution function $d_{\phi_z}(\cdot)$ of ϕ_z as

$$\begin{aligned} d_{\phi_z}(t) &= \#\{x \in \mathfrak{X} : |\phi_z(x)| > t\} \asymp \#\{x \in \mathfrak{X} : q^{-\frac{|x|}{p'}} > t\} \\ &\asymp \#\{x \in \mathfrak{X} : q^{|x|} < t^{-p'}\} \asymp \frac{1}{t^{p'}}. \end{aligned} \tag{5.3.14}$$

Therefore if $s > p$, $t^{s'} d_{\phi_z}(t) \asymp 1/t^{p'-s'}$ which tends to infinity as t tends to zero. Hence $\phi_z \notin L^{s',\infty}(\mathfrak{X})$ for any $s > p$.

On the other hand if $s = p$, then using [16, Proposition 1.4.9] and equation (5.3.14), it follows that for $r < \infty$,

$$\|\phi_z\|_{p',r} \asymp \left(\int_0^\infty \frac{dt}{t} \right)^{1/r},$$

which is not convergent by the Monotone Convergence Theorem. Therefore $\phi_z \notin L^{p',r}(\mathfrak{X})$ for any $r < \infty$.

(b): The proof uses a similar technique as above.

Hence we finally conclude that the use of $L^{2,r}$ estimate, $r < \infty$ (resp. $L^{s',r}$ estimate with $s > p$ or $L^{p',r}$ with $r < \infty$) in Theorem 5.3.1 (resp. in Theorem 5.3.2) can never yield a non-zero eigenfunction of $\mathcal{L}$. □

Question 5.3.7. *Do the conclusions of Theorem 5.3.2, Part (1) (resp. Part (2)) hold for $z = \alpha \pm i\delta_{p'}$ where $\alpha \in \mathbb{R} \setminus (n\tau)\mathbb{Z}$ (resp. $\alpha \in \mathbb{R} \setminus ((2n+1)\tau/2)\mathbb{Z}$)?*

Proof. Let $1 < p < 2$. On the contrary, let us assume that $z = \alpha + i\delta_{p'}$ for some $\alpha \in \mathbb{R} \setminus (n\tau)\mathbb{Z}$. Then there are two possibilities.

Case 1: Suppose that α is not of the form $((2n+1)\tau/2)\mathbb{Z}$. Then the set $\{w \in \mathbb{C} : |w| = |\gamma(z)|\}$ intersects the $L^{p'}$-spectrum of $\mathcal{L}$ at infinitely many points. Therefore we can choose r, s satisfying $p < r < s < 2$ and two real numbers β, η such that $|\gamma(\beta + i\delta_{r'})| = |\gamma(\eta + i\delta_{s'})| = |\gamma(z)|$. This further implies that there exists θ and ψ in $(0, 2\pi)$ such that

$$\gamma(\beta + i\delta_{r'})e^{-i\theta} = \gamma(\eta + i\delta_{s'})e^{-i\psi} = \gamma(z). \tag{5.3.15}$$

Now let us define

$$f(x) = \phi_{\beta+i\delta_{r'}}(x) + \phi_{\eta+i\delta_{s'}}(x) \quad \text{for all } x \in \mathfrak{X}.$$

Using (5.3.15), it follows that

$$\begin{aligned}
|\mathcal{L}^k f(x)| &= \left|\gamma(\beta+i\delta_{r'})^k \phi_{\beta+i\delta_{r'}}(x) + \gamma(\eta+i\delta_{s'})^k \phi_{\eta+i\delta_{s'}}(x)\right| \\
&= |\gamma(z)|^k \left(\left|e^{ik\theta}\phi_{\beta+i\delta_{r'}}(x) + e^{ik\psi}\phi_{\eta+i\delta_{s'}}(x)\right|\right) \\
&\leq |\gamma(z)|^k \left(|\phi_{\beta+i\delta_{r'}}(x)| + |\phi_{\eta+i\delta_{s'}}(x)|\right).
\end{aligned}$$

Using Lemma 2.3.12, we find that $\|\mathcal{L}^k f\|_{p',\infty} \leq M|\gamma(z)|^k$ where $M = \|\phi_{\beta+i\delta_{r'}}\|_{p',\infty} + \|\phi_{\eta+i\delta_{s'}}\|_{p',\infty}$. Hence f satisfies all the hypothesis of Theorem 5.3.2 but f is not an eigenfunction of $\mathcal{L}$.

Case 2: Now let us assume that $z = \alpha + i\delta_{p'}$ for some $\alpha \in ((2n+1)\tau/2)\mathbb{Z}$. If we choose $f(x) = \phi_{i\delta_{p'}}(x)$, then it is easy to see that

$$|\mathcal{L}^k f(x)| = |\gamma(i\delta_{p'})|^k |\phi_{i\delta_{p'}}(x)| \leq |\gamma(z)|^k |\phi_{i\delta_{p'}}(x)|.$$

Hence f satisfies all the hypothesis of Theorem 5.3.2 but $\mathcal{L}f \neq \gamma(z)f$.

The cases $z = \alpha - i\delta_{p'}$, $\alpha \in \mathbb{R} \setminus (n\tau)\mathbb{Z}$ and $z = \alpha \pm i\delta_{p'}$, $\alpha \in \mathbb{R} \setminus ((2n+1)\tau/2)\mathbb{Z}$ are analogous. □

Question 5.3.8. *Unlike Theorem 5.3.2, is it necessary to consider all integral powers of $\mathcal{L}$ in Theorem 5.3.1?*

Proof. Yes, it is necessary to consider all the integral powers of $\mathcal{L}$. Otherwise a counterexample can be constructed in the following manner: For $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, choose $s_1, s_2 \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$ such that $\gamma(s_i) \leq \gamma(z)$ for each $i = 1, 2$. Now define $f = \phi_{s_1} + \phi_{s_2}$. Then for every $k \in \mathbb{Z}_+$,

$$\begin{aligned}
|\mathcal{L}^k f| &\leq |\gamma(s_1)|^k |\phi_{s_1}| + |\gamma(s_2)|^k |\phi_{s_2}| \\
&= |\gamma(z)|^k \left(\left|\frac{\gamma(s_1)}{\gamma(z)}\right|^k |\phi_{s_1}| + \left|\frac{\gamma(s_2)}{\gamma(z)}\right|^k |\phi_{s_2}| \right) \\
&\leq |\gamma(z)|^k \left(|\phi_{s_1}| + |\phi_{s_2}|\right).
\end{aligned}$$

Therefore $\|\mathcal{L}^k f\|_{2,\infty} \leq M|\gamma(z)|^k$ where $M = \|\phi_{s_1}\|_{2,\infty} + \|\phi_{s_2}\|_{2,\infty}$, which is finite by Lemma 2.3.12. Hence f satisfies all the hypothesis of Theorem 5.3.1 but $\mathcal{L}f \neq \gamma(z)f$. □

5.4 Roe's Theorem on $\mathbb{Z}$ ($q = 1$ case)

The Fourier transform $\widehat{f}$ of a finitely supported function f defined on $\mathbb{Z}$, is a function on $\mathbb{T}$ given by

$$\widehat{f}(s) = \sum_{m\in\mathbb{Z}} f(m)e^{ims}, \quad \text{where } f(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(s)e^{-ims}ds$$

represents it's inverse Fourier transform. Let us define the Schwartz space $\mathcal{S}(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} : \lambda'_n(f) < \infty$ for all $n \in \mathbb{N}\}$, equipped with the countable family of semi-norms

$$\lambda'_n(f) = \sup_{m\in\mathbb{Z}} (1 + |m|)^n |f(m)|.$$

It is easy to see that the map $f \to \widehat{f}$ is a topological isomorphism from $\mathcal{S}(\mathbb{Z})$ onto $\mathcal{C}^\infty(\mathbb{T})$ where $\mathcal{C}^\infty(\mathbb{T}) = \{g : \mathbb{R} \to \mathbb{C} : g$ is infintely differentiable on $\mathbb{R}$, $g(x + \pi) = g(x)$ and $\mu'_l(g) < \infty$ for all $x \in \mathbb{R}, l \in \mathbb{N}$ respectively$\}$ where

$$\mu'_l(g) = \sup_{s\in[-\pi,\pi]} \left| \frac{d^l g}{dx^l}(s) \right|$$

defines a countable family of semi-norms. More precisely for all $f \in \mathcal{S}(\mathbb{Z})$,

$$C_1\lambda'_{n+2}(f) \le \mu'_n(\hat{f}) \le C_2\lambda'_{n+2}(f) \quad \text{for all } n \in \mathbb{N}.$$

Analogous to the Euclidean case, a distribution T is a continuous linear functional on $\mathcal{S}(\mathbb{Z})$ whose Fourier transform is defined as

$$\langle \widehat{T}, \phi\rangle = \langle T, (\phi^\vee)^\#\rangle, \text{ where } \widehat{\phi^\vee} = \phi \text{ and } (\phi^\vee)^\# (m) = \phi^\vee(-m).$$

Now we state our result on $\mathbb{Z}$, which can be proved by a similar argument developed by Roe in [34] (see also [19] for details). However for the sake of completeness, we provide an outline of the proof.

Theorem 5.4.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a doubly infinte sequence of functions on $\mathbb{Z}$ which satisfies*

$$\Delta_{\mathbb{Z}} f_k = (1 - \cos\alpha) f_{k+1} \quad \textit{for } \alpha \in (0, \pi], \tag{5.4.16}$$

and there exists constants $M_k \ge 0$, $\beta \in (0, 1]$ and a non-negative integer n such that

$$|f_k(m)| \le M_k(1 + |m|)^{n+\beta} \textit{ for all } k, m \in \mathbb{Z}. \tag{5.4.17}$$

If

$$\liminf_{k\to\infty} \frac{M_k}{(1+\epsilon)^k} = 0 \quad \text{for all } \epsilon > 0 \tag{5.4.18}$$

and

$$\liminf_{k\to\infty} \frac{M_{-k}}{(1+\epsilon)^k} = 0 \quad \text{for all } \epsilon > 0, \tag{5.4.19}$$

then

$$f_0(m) = p(m)e^{im\alpha} + q(m)e^{-im\alpha}$$

where p, q are polynomials of degree at most n.

Proof. The proof uses the same techniques that were used to prove Theorem 5.2.4. Let $\phi \in \mathcal{C}^\infty(\mathbb{T})$ be such that $\text{supp}(\phi) \subseteq [-\pi, -\alpha - r] \cup [\alpha + r, \pi]$ where $r > 0$. We shall show that $\langle \widehat{f_0}, \phi \rangle = 0$. Using the conditions (5.4.16) and (5.4.17) we have

$$\begin{aligned} |\langle \widehat{f_0}, \phi \rangle| = \left| \left\langle \widehat{f_k}, \left(\frac{1-\cos\alpha}{1-\cos s} \right)^k \phi \right\rangle \right| &= \left| \sum_{m\in\mathbb{Z}} f_k(m) \left(\left(\frac{1-\cos\alpha}{1-\cos s} \right)^k \phi \right)^\vee (-m) \right| \\ &\leq M_k \sum_{m\in\mathbb{Z}} (1+|m|)^{n+\beta} \left| \left(\left(\frac{1-\cos\alpha}{1-\cos s} \right)^k \phi \right)^\vee (-m) \right|. \end{aligned} \tag{5.4.20}$$

For every $k \in \mathbb{Z}$, let us define

$$\psi_k(m) = \left(\left(\frac{1-\cos\alpha}{1-\cos s} \right)^k \phi \right)^\vee (-m), \quad \text{where } m \in \mathbb{Z}.$$

We now try to find a suitable pointwise estimate of ψ_k. Integrating by parts $(l+2)$ times, for $m \neq 0$ we have

$$\begin{aligned} |\psi_k(m)| &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \frac{d^{l+2}}{ds^{l+2}} \left(\left(\frac{1-\cos\alpha}{1-\cos s} \right)^k \phi(s) \right) \frac{e^{ims}}{(-im)^{l+2}} ds \right| \\ &\leq \frac{1}{2\pi |m|^{l+2}} \int_{-\pi}^{\pi} \left| \frac{d^{l+2}}{ds^{l+2}} \left(\left(\frac{1-\cos\alpha}{1-\cos s} \right)^k \phi(s) \right) \right| ds \\ &\leq \frac{C_{l,\phi} k^{l+2}}{|m|^{l+2}} \left(\frac{1-\cos\alpha}{1-\cos(\alpha+r)} \right)^k. \end{aligned}$$

Further when $m = 0$,

$$|\psi_k(m)| \leq C_\phi \left(\frac{1-\cos\alpha}{1-\cos(\alpha+r)} \right)^k.$$

Putting the above estimates in (5.4.20) and then using the hypothesis (5.4.18), we find that $\langle \widehat{f_0}, \phi \rangle = 0$. In a similar way as above, using (5.4.19) and letting $k \to -\infty$, one can prove that $\langle \widehat{f_0}, \phi \rangle = 0$ whenever $\sup(\phi) \subseteq [-\alpha + r, \alpha - r]$. We now use the structured theorem to conclude that

$$\widehat{f_0} = \sum_{N=0}^{n_1} (c_N \delta_{\alpha}^{(N)} + d_N \delta_{-\alpha}^{(N)}) \quad \text{for some } n_1 \in \mathbb{N},$$

where $\delta_{\pm\alpha}^{(N)}$ represent the N-th derivative of the Dirac delta distributions supported at the points $\pm\alpha$. Taking the inverse Fourier transform and using equation (5.4.17), we conclude that $n_1 \leq n$ and hence $f_0(m) = p(m)e^{im\alpha} + q(m)e^{-im\alpha}$, where p, q are polynomials of degree at most n. This completes the proof. □

CHAPTER 6

Dynamics of semigroups generated by analytic functions of the Laplacian on homogeneous trees

6.1 Introduction

Let $\sigma_p(\mathcal{L})$ denote the L^p-spectrum of the Laplacian $\mathcal{L}$. Also assume that f is a non-constant complex holomorphic function defined on a connected open set containing $\sigma_p(\mathcal{L})$. Then by the usual Riesz functional calculus (see [35, Page 261]), it follows that the semigroup

$$T(t) = e^{tf(\mathcal{L})} \quad \text{where } t \geq 0, \tag{6.1.1}$$

is a bounded linear operator on the Lebesgue space $L^p(\mathfrak{X})$ for every $p \in [1, \infty]$. In the first part of this chapter, we study the chaotic behaviour of the semigroup $T(t)$ in terms of its set of all periodic points. After stating all necessary results in Section 6.2, we prove our main results in Section 6.3. The main results of this part are Theorem 6.3.1 and 6.3.2. Theorem 6.3.1 can also be considered as an analogue of Theorem 1.4.3 proved for weighted shift operators. Furthermore Theorem 6.3.2 is a generalization of [31, Theorem 1.3] proved for the heat semigroups on symmetric spaces. For details concerning the dynamical system terminology we refer Section 1.4, Chapter 1.

In the second part we separately study the dynamical behaviour of the semigroups generated by affine functions, namely,

$$T(t) = \exp\{t(a\mathcal{L} + b)\} \quad \text{where } a \in \mathbb{C} \setminus \{0\},\ b \in \mathbb{R}. \tag{6.1.2}$$

Apart from studying chaoticity in terms of periodic points, here we also derive a sharp p-depending range for which the semigroup (6.1.2) is hypercyclic. This range relies heavily on the p-depending elliptic spectrum of $\mathcal{L}$ and the sharp operator norm estimates of $e^{\xi\mathcal{L}}$, $\xi \in \mathbb{C}$. A detailed survey of such operators started with the work of Cowling and his collaborators (see [6]). They considered the heat semigroup $\{\mathcal{H}_t\}_{t\geq 0}$ generated by the Laplacian $\mathcal{L}$, that is,

$$\mathcal{H}_t = e^{-t\mathcal{L}} = \sum_{n=0}^{\infty} \frac{(-t\mathcal{L})^n}{n!}.$$

In [6] Cowling, Meda and Setti studied the behaviour of $\mathcal{H}_t$ and its $L^p - L^r$ operator norm. Further Setti considered the analogous problem for the complex-time heat operator $\mathcal{H}_\xi$, and derived precise estimates for the $L^p - L^r$ operator norms of $\mathcal{H}_\xi$, for ξ belonging to the half-plane Re $\xi \geq 0$ in his paper [37]. Since our main result largely depends on the sharp L^p norm estimates of the operator $e^{\xi\mathcal{L}}$ for $\xi \in \mathbb{C}$, subsequently we have extended Setti's result for all $\xi \in \mathbb{C}$. See Lemma 6.4.1 for details.

As a consequence of the results obtained so far, at last we separately study the dynamical behaviour of some classical semigroups such as the the heat semigroup and the Schrödinger semigroup. Works in this direction are contained in Section 6.5.

6.2 Basic Results

Before going into further details, here we quote several important facts involving the concepts introduced above. In fact these observations will be frequently used in the remaining part of this chapter. We begin with the following lemma whose proof follows immediately from the definition of hypercyclicity given in Section 1.4, Chapter 1.

Lemma 6.2.1. *A Banach space $\mathbb{B}$ which admits a hypercyclic semigroup, must be separable.*

Proof. Suppose that the semigroup $T(t)$ is hypercyclic on $\mathbb{B}$, and that there exists $x \in \mathbb{B}$ such that $\{T(t)x : t \geq 0\}$ is dense in $\mathbb{B}$. We claim that the countable set $\{T(q)x : q \in \mathbb{Q}\}$

is also dense in $\mathbb{B}$. To prove our assertion it is enough to prove that every element of the form $T(t)x$ where $t \in \mathbb{R} \setminus \mathbb{Q}$, belong to the closure of $\{T(q)x : q \in \mathbb{Q}\}$ in $\mathbb{B}$.

Using the density of rational numbers in $\mathbb{R}$ and the strong continuity of $T(t)$ it follows that for every $t \in \mathbb{R} \setminus \mathbb{Q}$ there exists a sequence $\{q_n\}$ of rational numbers such that $q_n \downarrow t$ and $T(q_n - t)x \to x$ as $n \to \infty$. Also for all $n \in \mathbb{N}$, we have

$$0 \leq \|T(q_n)x - T(t)x\| \leq \|T(t)\|\|T(q_n - t)x - x\|.$$

The assertion now follows by passing the limits to both the sides of the above inequality. This completes the proof. □

As mentioned earlier, chaoticity of a semigroup is mainly triggered due to the abundance of purely imaginary point spectrum of its generators, which in our case is of the form $f(\mathcal{L})$. So we need the following spectral mapping theorem. For details we refer [35, Theorem 10.28, Theorem 10.33].

Theorem 6.2.2. *Suppose T is a bounded linear operator on $L^p(\mathfrak{X})$ and g is a non-constant complex holomorphic function defined on a connected open set containing $\sigma_p(T)$. Then we have the following.*

(a) $\sigma_p(g(T)) = g(\sigma_p(T))$.

(b) $P\sigma_p(g(T)) = g(P\sigma_p(T))$.

For $1 \leq p < \infty$ define the sets

$$X_0 = \{x \in L^p(\mathfrak{X}) : T(t)x \to 0 \text{ as } t \to \infty\}, \tag{6.2.3}$$

$$X_\infty = \{x \in L^p(\mathfrak{X}) : \forall \epsilon > 0 \; \exists \; w \in L^p(\mathfrak{X}), t_0 > 0 \text{ such that } \|w\| < \epsilon, \|T(t_0)w - x\| < \epsilon\}. \tag{6.2.4}$$

The following sufficient condition for hypercyclicity which was proved by Desch, Schappacher and Webb is also useful in the sequel. See [9, Theorem 2.3] for details.

Proposition 6.2.3. *Let $T(t)$, $t \geq 0$ be a strongly continuous semigroup on $L^p(\mathfrak{X})$ for $1 \leq p < \infty$. If both the sets X_0 and X_∞ are dense in $L^p(\mathfrak{X})$, then $T(t)$ is hypercyclic.*

6.3 Chaotic Dynamics and Periodic Points

Using Lemma 6.2.1 it is obvious that the semigroup (6.1.1) cannot be hypercyclic on $L^\infty(\mathfrak{X})$ and hence not chaotic on $L^\infty(\mathfrak{X})$. Therefore we exclude the case when $p = \infty$. For other values of p, we have the following results.

Theorem 6.3.1. *Let $2 < p < \infty$ and $T(t) = e^{tf(\mathcal{L})}$ be a semigroup on $L^p(\mathfrak{X})$ as defined in (6.1.1). Then the following statements are equivalent.*

(1) $T(t)$ is chaotic on $L^p(\mathfrak{X})$.

(2) $T(t)$ has a non-trivial periodic point, that is $L^p(\mathfrak{X})_{per} \neq \{0\}$.

(3) The set of periodic points of $T(t)$ is dense in $L^p(\mathfrak{X})$, that is $\overline{L^p(\mathfrak{X})}_{per} = L^p(\mathfrak{X})$.

Proof. Fix $p \in (2, \infty)$. It follows from the definition that *(1)* implies *(2)* and *(3)*. To complete the proof we only need to show that $(2) \implies (3)$ and $(3) \implies (1)$.
$(2) \implies (3)$: For a clear understanding, we have divided this proof into the following steps.
Step 1: In this step, we prove that $P\sigma_p(f(\mathcal{L})) \cap i\mathbb{R}$ is an infinite set. Condition *(2)* implies that there exists a nonzero function $h \in L^p(\mathfrak{X})$ such that $T(t_o)h = h$ for some $t_o > 0$, that is $1 \in P\sigma_p(e^{t_o f(\mathcal{L})})$. Using Proposition 2.3.15 *(2)* and Theorem 6.2.2 *(b)* we also have $P\sigma_p(e^{t_o f(\mathcal{L})}) = e^{t_o f(\gamma(S_p^\circ))}$. Therefore there exists $z_0 \in S_p^\circ$ such that $f(\gamma(z_0)) = 2n\pi i/t_o$ for some $n \in \mathbb{Z}$.

Let $\Gamma : S_p^\circ \to \mathbb{C}$ be defined by

$$\Gamma(z) = (f \circ \gamma)(z) = f(\gamma(z)). \tag{6.3.5}$$

It follows from the assumption on f that Γ is a non-constant holomorphic function on S_p°. Since $\Gamma(z_0) = 2n\pi i/t_o$ for some $z_0 \in S_p^\circ$ and $\Gamma(S_p^\circ) = P\sigma_p(f(\mathcal{L}))$ therefore by the open mapping theorem it follows that $P\sigma_p(f(\mathcal{L}))$ must contains some open ball centered at $2n\pi i/t_o$. Hence $P\sigma_p(f(\mathcal{L})) \cap i\mathbb{R}$ is an infinite set. In particular the set

$$V_1 = \{z \in S_p^\circ : \Gamma(z) \in i\mathbb{Q}\}$$

is an infinite set which contains a cluster point in S_p°.

Step 2: Let $z \in V_1$. Since $V_1 \subseteq S_p^\circ$, therefore $z = \alpha + i\delta_{r'}$ for some $r \in (p', p)$ and $\alpha \in \mathbb{R}$. Hence the set

$$\mathcal{V}_1 = \bigcup_{z \in V_1} \{\mathcal{P}_z F : F \in L^{r'}(\Omega) \text{ whenever } \Im z = \delta_{r'} \text{ with } p' < r < p\}$$

is well defined. It follows from inequality (4.3.6) that $\mathcal{V}_1 \subseteq L^p(\mathfrak{X})$. We now show that $\text{span}(\mathcal{V}_1) \subseteq L^p(\mathfrak{X})_{per}$. Since $f(\mathcal{L})\mathcal{P}_z F = \Gamma(z)\mathcal{P}_z F$ for every $z \in S_p^\circ$, thus $T(t)\mathcal{P}_z F = e^{t\Gamma(z)}\mathcal{P}_z F$. Now if $g \in \text{span}(\mathcal{V}_1)$ then

$$g = \sum_{j=1}^{k} \beta_j \mathcal{P}_{z_j} F_j, \text{ where } z_j \in V_1 \text{ and } \beta_j \in \mathbb{C}, 1 \leq j \leq k.$$

Now for every $j \in \{1, 2, \ldots, n\}$, $z_j \in V_1$ and hence, $\Gamma(z_j) = ip_j/q_j$ such that $p_j, q_j \in \mathbb{Z}$ with $q_j \neq 0$. If we choose $s = 2\pi q_1 \cdots q_n$, then $T(s)g = g$. Hence $\text{span}(\mathcal{V}_1) \subseteq L^p(\mathfrak{X})_{per}$.

Step 3: To prove that $\overline{L^p(\mathfrak{X})_{per}} = L^p(\mathfrak{X})$, it is enough to prove that $\text{span}(\mathcal{V}_1)$ is dense in $L^p(\mathfrak{X})$. Let $f \in L^{p'}(\mathfrak{X})$ annihilates $\mathcal{V}_1$, that is

$$\sum_{x \in \mathfrak{X}} f(x)\mathcal{P}_z F(x) = 0 \quad \text{for all } \mathcal{P}_z F \in \mathcal{V}_1.$$

Using the duality relation (4.3.4) we have,

$$\int_\Omega \widetilde{f}(z, \omega) F(\omega) d\nu(\omega) = \sum_{x \in \mathfrak{X}} f(x)\mathcal{P}_z F(x) = 0 \quad \text{for all } \mathcal{P}_z F \in \mathcal{V}_1.$$

Fix $z \in V_1$ and suppose that $z = \alpha + i\delta_{r'}$ for some $r \in (p', p)$ and $\alpha \in \mathbb{R}$. Then for every $F \in L^{r'}(\mathfrak{X})$, we have

$$\int_\Omega \widetilde{f}(\alpha + i\delta_{r'}, \omega) F(\omega) d\nu(\omega) = 0.$$

Since $F \in L^{r'}(\mathfrak{X})$ is arbitrary, therefore from Theorem 4.3.2 and the above equation, we have $\widetilde{f}(\alpha + i\delta_{r'}, \omega) = 0$ for almost every $\omega \in \Omega$. Thus for every $z \in V_1$, $\widetilde{f}(z, \omega) = 0$ for almost every $\omega \in \Omega$. By Theorem 4.4.1, for almost every ω the function $z \to \widetilde{f}(z, \omega)$ is analytic on S_p°. So we conclude that for almost every ω, the set of zeros of $\widetilde{f}$ has a cluster point in S_p°, and hence $\widetilde{f}(z, \omega) = 0$ for every $z \in S_p^\circ$ and for almost every ω. Since $f \in L^{p'}(\mathfrak{X}) \subseteq L^2(\mathfrak{X})$ (as $\mathfrak{X}$ is a discrete space) whenever $2 < p < \infty$, therefore by Plancherel Theorem 2.4.5 we conclude that $f \equiv 0$. This proves that $\text{span}(\mathcal{V}_1)$ is dense in $L^p(\mathfrak{X})$.

(3) $\implies$ *(1)* : Since the density of the periodic points is already assumed, so to prove our assertion it is enough to show that $T(t)$ is hypercyclic. In view of Proposition 6.2.3, we only need to show that the sets X_0 and X_∞ (as defined in Proposition 6.2.3) are dense in $L^p(\mathfrak{X})$. We define the sets

$$V_2 = \{z \in S_p^\circ : \text{Re }(\Gamma(z)) < 0\} \text{ and } V_3 = \{z \in S_p^\circ : \text{Re }(\Gamma(z)) > 0\}.$$

By repeating the arguments of *Step 1* in the previous proof, one may prove that V_2 and V_3 are non-empty open sets and both contain a cluster point in S_p°. Corresponding to the sets V_i, we define (for $i = 2, 3$)

$$\mathcal{V}_i = \bigcup_{z \in V_i} \{\mathcal{P}_z F : F \in L^{r'}(\Omega) \text{ whenever } \Im z = \delta_{r'} \text{ with } p' < r < p\}.$$

Adopting a similar approach as in *Step 2* and *Step 3*, we may easily show that both $\text{span}(\mathcal{V}_2)$ and $\text{span}(\mathcal{V}_3)$ are dense in $L^p(\mathfrak{X})$. The proof will be complete once we show that $\text{span}(\mathcal{V}_2) \subseteq X_0$ and $\text{span}(\mathcal{V}_3) \subseteq X_\infty$. For every $z \in V_2$,

$$\lim_{t\to\infty} \|T(t)\mathcal{P}_z F\|_{L^p(\mathfrak{X})} = \lim_{t\to\infty} e^{t\text{Re }(\Gamma(z))} \|\mathcal{P}_z F\|_{L^p(\mathfrak{X})} = 0.$$

This shows that $\mathcal{V}_2$ and hence $\text{span}(\mathcal{V}_2)$ is a subset of X_0 (since $\mathcal{V}_2$ is a subspace of X_0).

Next we prove that $\text{span}(\mathcal{V}_3) \subseteq X_\infty$. Let $g \in \text{span}(\mathcal{V}_3)$ be of the form

$$g = \sum_{j=1}^{k} \alpha_j \mathcal{P}_{z_j} F_j, \text{ where } z_j \in V_3 \text{ and } \alpha_j \in \mathbb{C}, 1 \le j \le k.$$

If we choose $g_t = \sum_{j=1}^{k} e^{-t\Gamma(z_j)} \alpha_j \mathcal{P}_{z_j} F_j$, then $T(t)g_t = g$ for all $t \ge 0$. Since Re $(\Gamma(z_j)) > 0$ for each j, the limit $\|g_t\|_{L^p(\mathfrak{X})} \to 0$ as $t \to \infty$. Hence it follows from definition (6.2.4) that $\text{span}(\mathcal{V}_3) \subseteq X_\infty$. This completes the proof. □

Theorem 6.3.2. *Let $1 \le p \le 2$ and $T(t)$ be as in Theorem 6.3.1. Then we have the following.*

(1) $T(t)$ has no non-trivial periodic point in $L^p(\mathfrak{X})$.

(2) $T(t)$ is not hypercyclic on $L^p(\mathfrak{X})$.

In particular $T(t)$ is not chaotic on $L^p(\mathfrak{X})$.

Proof. Part (1): Let $p \in [1,2]$. We know from Proposition 2.3.15 *(1)* that $P\sigma_p(\mathcal{L}) = \emptyset$. Hence by Theorem 6.2.2 *(b)* it follows that $P\sigma_p(T(t)) = \emptyset$ for all $t > 0$. This show that only the zero function is a periodic point, that is, $T(t)$ has no non-trivial periodic point in $L^p(\mathfrak{X})$ for any $p \in [1,2]$.

Part (2): Now we will show that $T(t)$ is not hypercyclic on $L^p(\mathfrak{X})$ for any $p \in [1,2]$. Let us first assume that $p = 2$. We proof this assertion by contradiction. If possible, assume that there exists a non-zero $h \in L^2(\mathfrak{X})$ such that the set $\{T(t)h : t \geq 0\}$ is dense in $L^2(\mathfrak{X})$. Then for $g = 2h$, there exists a sequence of non-negative real numbers $\{t_n\}$ such that $T(t_n)h \to g$ in $L^2(\mathfrak{X})$ as $n \to \infty$.

Case I: If the sequence $\{t_n\}$ is bounded then there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and some number $t_0 \geq 0$ such that $t_{n_k} \to t_0$ as $k \to \infty$. Using the strong continuity of $T(t)$, it follows that $T(t_{n_k})h \to T(t_0)h$ as $k \to \infty$. By the uniqueness of limits we find that $T(t_0)h = g = 2h$, which is impossible as $P\sigma_2(T(t_0)) = \emptyset$.

Case II: If the sequence $\{t_n\}$ is unbounded, then there always exists a strictly increasing subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \to \infty$ as $k \to \infty$. With the abuse of notations we assume that $\{t_n\}$ is strictly increasing to infinity. Since the sequence $\{T(t_n)h\}$ converges to $g = 2h$ in $L^2(\mathfrak{X})$, using the Plancherel Theorem 2.4.5, we have

$$4\|h\|^2_{L^2(\mathfrak{X})} = \lim_{n\to\infty} \|T(t_n)h\|^2_{L^2(\mathfrak{X})} = \lim_{n\to\infty} \int_{-\tau/2}^{\tau/2} \int_{\Omega} \exp\{2t_n \text{Re } f(\gamma(s))\} |\widetilde{h}(s,\omega)|^2 d\nu(\omega) d\mu(s). \tag{6.3.6}$$

Now let us define the sets

$$S_1 = \{s \in [-\tau/2, \tau/2) : \exp\{2\text{Re } f(\gamma(s))\} \leq 1\} \quad \text{and}$$
$$S_2 = \{s \in [-\tau/2, \tau/2) : \exp\{2\text{Re } f(\gamma(s))\} > 1\}.$$

Then (6.3.6) can be further decomposed as

$$4\|h\|^2_{L^2(\mathfrak{X})} = \lim_{n\to\infty} \left(\int_{S_1} \int_{\Omega} \exp\{2t_n \text{Re } f(\gamma(s))\} |\widetilde{h}(s,\omega)|^2 d\nu(\omega) d\mu(s) + \int_{S_2} \int_{\Omega} \exp\{2t_n \text{Re } f(\gamma(s))\} |\widetilde{h}(s,\omega)|^2 d\nu(\omega) d\mu(s) \right). \tag{6.3.7}$$

If the Plancherel measure of S_2 is zero then $4\|h\|^2_{L^2(\mathfrak{X})} \leq \|h\|^2_{L^2(\mathfrak{X})}$. This further implies that $\|h\|_{L^2(\mathfrak{X})} = 0$, which is a contradiction to our assumption that $h \neq 0$. On the other hand if S_2 has a positive Plancherel measure, then

$$4\|h\|^2_{L^2(\mathfrak{X})} \geq \lim_{n\to\infty} \int\limits_{S_2}\int\limits_{\Omega} \exp\{2t_n \mathrm{Re}\ f(\gamma(s))\}|\widetilde{h}(s,\omega)|^2 d\nu(\omega)d\mu(s). \tag{6.3.8}$$

Note that the above integral converges due of the fact that the integral over S_1 in (6.3.7) can further be decomposed into the integral over the sets

$$S_1' = \{s \in [-\tau/2, \tau/2) : \exp\{2\mathrm{Re}\ f(\gamma(s))\} < 1\} \quad \text{and}$$
$$S_1'' = \{s \in [-\tau/2, \tau/2) : \exp\{2\mathrm{Re}\ f(\gamma(s))\} = 1\},$$

both of which converges by the Dominated Convergence Theorem. However by applying the Monotone Convergence Theorem, it follows that the integral in (6.3.8) tends to infinity as n tends to infinity. This again leads to a contradiction. Hence we conclude that for any $h \in L^2(\mathfrak{X})$ and $h \neq 0$, the set $\{T(t)h : t \geq 0\}$ can never approximate $2h$. This proves our assertion for $p = 2$.

Now we assume $p \in [1, 2)$. Since $\mathfrak{X}$ is a discrete space, hence $L^p(\mathfrak{X}) \subseteq L^2(\mathfrak{X})$ whenever $1 \leq p < 2$ and $\|h\|_{L^2(\mathfrak{X})} \leq \|h\|_{L^p(\mathfrak{X})}$ for every $h \in L^p(\mathfrak{X})$. This implies that for any $h \in L^p(\mathfrak{X})$ and $h \neq 0$, the set $\{T(t)h : t \geq 0\}$ can never approximate $2h$. This completes the proof. □

6.4 Chaotic Dynamics of Affine Semigroups

We begin this section by recalling some important facts related to the operator $e^{\xi\mathcal{L}}$ for $\xi \in \mathbb{C}$. For a detailed study we refer [37]. The operator $e^{\xi\mathcal{L}}$ is a G-invariant, bounded operator on $L^p(\mathfrak{X})$. Let

$$h_\xi(x) = e^{-\xi} \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \mu_1^{(*k)},$$

where μ_1 is the probability measure at a distance 1 from the reference point o and $\mu_1^{(*k)}$ denotes the k-th convolution power of μ_1 and $\mu_1^{(*0)} = \delta_o$. Then for $p \geq 1$,

$$e^{\xi\mathcal{L}} f = f * h_{-\xi}, \quad \text{for all } f \in L^p(\mathfrak{X}).$$

Note that we use a different parametrization, our $\gamma(z)$ corresponds to $1-\gamma(z)$ in [37]. Now we prove the following lemma, which gives a sharp norm estimate of the operator $e^{\xi\mathcal{L}}$ for all $\xi\in\mathbb{C}$. For $2<p<\infty$ let us define

$$\Phi_p(a)=(1-\gamma(i\delta_p))\cdot((\text{Re }a)^2+\tanh^2(\delta_p\log q)(\Im a)^2)^{1/2}. \tag{6.4.9}$$

Lemma 6.4.1. *Let $e^{\xi\mathcal{L}}$ be the operator as defined above. Then for $p>2$, the following hold.*

$$\exp\{Re\ \xi+\Phi_p(\xi)\}\leq\|e^{\xi\mathcal{L}}\|_{p\to p}\leq C\exp\{Re\ \xi+\Phi_p(\xi)\}. \tag{6.4.10}$$

Proof. For Re $\xi\leq 0$, this result is already known (see [37, Theorem 1]). Now we prove the result for Re $\xi>0$. It was proved in [37, Corollary 4] that for all non-zero ξ,

$$\|h_{-\xi}\|_{L^{p,1}(\mathfrak{X})}\leq C\frac{\exp\{\gamma(0)\text{Re }\xi\}}{|\xi|}\left(\sum_{d=0}^{\infty}dq^{d\delta_p}|h^{\mathbb{Z}}_{-\xi(1-\gamma(0))}(d)|\right), \tag{6.4.11}$$

where $h^{\mathbb{Z}}_{-\xi(1-\gamma(0))}(d)$ denotes the heat kernel associated to the heat operator on $\mathbb{Z}$. It was also proved in [37, Page 748] that

$$h^{\mathbb{Z}}_{-\xi(1-\gamma(0))}(d)=e^{\xi(1-\gamma(0))}I_d(-\xi(1-\gamma(0))),$$

where $I_d(\xi)$ denotes the modified Bessel function of order d. Since Re $\{-\xi(1-\gamma(0))\}<0$, so the arguments given in [37] cannot be applied here. However by using the formula (8.01) from [30, Page 379], we have

$$h^{\mathbb{Z}}_{-\xi(1-\gamma(0))}(d)=e^{i\pi d}e^{2\xi(1-\gamma(0))}h^{\mathbb{Z}}_{\xi(1-\gamma(0))}(d)\quad\forall d\in\mathbb{N}\cup\{0\}.$$

Now by substituting the pointwise estimates of $h^{\mathbb{Z}}_{\xi(1-\gamma(0))}(d)$ from [37, Lemma 6] in the above equation and then using (6.4.11), the result follows by just imitating the calculations given in that paper.

Now we will prove the lower bound of $\|e^{\xi\mathcal{L}}\|_{p\to p}$. For $p>2$,

$$\|e^{\xi\mathcal{L}}\|_{p\to p}\geq\frac{\|e^{\xi\mathcal{L}}\phi_z\|_{L^p(\mathfrak{X})}}{\|\phi_z\|_{L^p(\mathfrak{X})}}=\exp\{\text{Re }(\xi\gamma(z))\}\ \text{ for all } z\in S_p^\circ.$$

By taking the supremum over all $z\in S_p^\circ$, we find that

$$\sup_{z\in S_p^\circ}\{\exp\{\text{Re }(\xi\gamma(z))\}\}=\exp\{\text{Re }\xi+\Phi_p(\xi)\}.$$

The above supremum is also explicitly calculated in the proof of Lemma 6.4.4. This gives the desired lower bound. □

Using Theorem 6.3.2, it follows that the semigroup (6.1.2) cannot be hypercyclic on $L^p(\mathfrak{X})$ whenever $p \in [1,2]$. However for $2 < p < \infty$, the following lemma gives us a necessary condition in terms of the parameters a and b, for which the semigroup (6.1.2) is hypercyclic. The proof of this lemma depends immensely on the sharp norm estimates proved above.

Lemma 6.4.2. *Suppose that $T(t) = e^{t(a\mathcal{L}+b)}$ where $t \geq 0$, a is a non-zero complex number and b is real. Then for $2 < p < \infty$, $T(t)$ is not hypercyclic on $L^p(\mathfrak{X})$ whenever $b \leq -Re\ a - \Phi_p(a)$ or $b \geq -Re\ a + \Phi_p(a)$.*

Proof. Fix $p \in (2, \infty)$ and let a be a non-zero complex number. To prove that the semigroup $T(t)$ is not hypercyclic for the given range of b, it is enough to show that the set $\{T(t)h : t \geq 0\}$ is not dense in $L^p(\mathfrak{X})$ for any $h \in L^p(\mathfrak{X})$. If $b \leq -\text{Re } a - \Phi_p(a)$ then it follows from the Lemma 6.4.1 that for every $h \in L^p(\mathfrak{X})$,

$$\|T(t)h\|_{L^p(\mathfrak{X})} = e^{bt}\|e^{at\mathcal{L}}h\|_{L^p(\mathfrak{X})} \leq C\exp\{t(b + \text{Re } a + \Phi_p(a))\}\|h\|_{L^p(\mathfrak{X})} \leq C\|h\|_{L^p(\mathfrak{X})}.$$

This show that the set $\{T(t)h : t \geq 0\}$ is bounded and hence it cannot be dense.

Now we consider the case when $b \geq -\text{Re } a + \Phi_p(a)$. We prove this assertion by contradiction. Suppose that there exists a non-zero h in $L^p(\mathfrak{X})$ such that $\{T(t)h : t \geq 0\}$ is dense in $L^p(\mathfrak{X})$. Since for all $t \geq 0$, $e^{at\mathcal{L}}e^{-at\mathcal{L}}h = e^{-at\mathcal{L}}e^{at\mathcal{L}}h = h$, by using the norm estimate (6.4.10) we have

$$\|h\|_{L^p(\mathfrak{X})} = \|e^{-at\mathcal{L}}e^{at\mathcal{L}}h\|_{L^p(\mathfrak{X})} \leq C\exp\{t(-\text{Re } a + \Phi_p(a))\}\|e^{at\mathcal{L}}h\|_{L^p(\mathfrak{X})}.$$

Hence we find that

$$\|e^{at\mathcal{L}}h\|_{L^p(\mathfrak{X})} \geq C\exp\{t(\text{Re } a - \Phi_p(a))\}\|h\|_{L^p(\mathfrak{X})}.$$

From the above inequality we obtain

$$\|T(t)h\|_{L^p(\mathfrak{X})} = e^{bt}\|e^{at\mathcal{L}}h\|_{L^p(\mathfrak{X})} \geq C\exp\{t(b + \text{Re } a - \Phi_p(a))\}\|h\|_{L^p(\mathfrak{X})} \geq C\|h\|_{L^p(\mathfrak{X})}.$$

Thus we conclude that the function '0' does not belong to the closure of $\{T(t)h : t \geq 0\}$ in $L^p(\mathfrak{X})$ and we finally arrive at a contradiction. This shows that $\{T(t)h : t \geq 0\}$ cannot be dense in $L^p(\mathfrak{X})$ for any $h \in L^p(\mathfrak{X})$. This completes the proof. ∎

Theorem 6.4.3. *Let $2 < p < \infty$ and $T(t)$ be a semigroup on $L^p(\mathfrak{X})$ as in (6.1.2). Then the following are equivalent.*

(1) $T(t)$ is chaotic on $L^p(\mathfrak{X})$.

(2) $T(t)$ has a non-trivial periodic point, that is $L^p(\mathfrak{X})_{per} \neq \emptyset$.

(3) a and b satisfy $-Re\ a - \Phi_p(a) < b < -Re\ a + \Phi_p(a)$, where $\Phi_p(a)$ is given by (6.4.9).

(4) $T(t)$ is hypercyclic.

Proof. The equivalence of the conditions *(1)* and *(2)* is already proved in Theorem 6.3.1. Also *(4)* implies *(3)* is a consequence of the Lemma 6.4.2 and *(1)* implies *(4)* is obvious. Now to complete the proof we only need to show the equivalence of *(3)* and *(2)*, which will follow from the the following lemma. □

Lemma 6.4.4. *Let $T(t)$ be defined as in Theorem 6.4.3. Then for $2 < p < \infty$, $T(t)$ has a non-trivial periodic point on $L^p(\mathfrak{X})$ if and only if $-Re\ a - \Phi_p(a) < b < -Re\ a + \Phi_p(a)$.*

Proof. Assume that $T(t)$ has a non-trivial periodic point. As in the proof of Theorem 6.3.1 there exists $z_0 \in S_p^\circ$ such that $\exp\{(a\gamma(z_0) + b)t_0)\} = 1$ for some $t_0 > 0$. By applying the open mapping theorem on the analytic function $\exp\{(a\gamma(\cdot) + b)t_0\}$, it follows that the set $\exp\{(a\gamma(S_p^\circ) + b)t_0\}$ contains an open ball centered at 1. This implies that

$$\max_{z\in S_p} |e^{t_0(a\gamma(z)+b)}| > 1 \quad \text{and} \quad \min_{z\in S_p} |e^{t_0(a\gamma(z)+b)}| < 1. \tag{6.4.12}$$

Suppose $a = x + iy$ and that $z = s + i\delta_p$. Let $h(s) = \text{Re}\ (a\gamma(s + i\delta_p) + b)$ be a complex-valued function defined for all $s \in [-\tau/2, \tau/2]$. Using the explicit formula (2.3.11) we find that

$$\text{Re}\ \gamma(s + i\delta_p) = 1 - \frac{q^{1/p} + q^{1/p'}}{q+1}\cos(s\log q) \ \text{ and } \ \Im\gamma(s + i\delta_p) = \frac{q^{1/p} - q^{1/p'}}{q+1}\sin(s\log q).$$

Hence we finally obtain

$$h(s) = x - \frac{q^{1/p} + q^{1/p'}}{q+1}x\cos(s\log q) - \frac{q^{1/p} - q^{1/p'}}{q+1}y\sin(s\log q) + b.$$

A straightforward computation yield that the maximum and the minimum values of h on the interval $[-\tau/2, \tau/2]$ are $x + \Phi_p(a) + b$ and $x - \Phi_p(a) + b$ respectively. Since γ

is a τ-periodic function, therefore by applying the Maximum Modulus principle on the function $e^{t(a\gamma(\cdot)+b)}$, we obtain

$$\max_{z\in S_p}|e^{t(a\gamma(z)+b)}| = \exp\{(\text{Re } a + \Phi_p(a) + b)t\} \quad \text{and}$$
$$\min_{z\in S_p}|e^{t(a\gamma(z)+b)}| = \exp\{(\text{Re } a - \Phi_p(a) + b)t\}.$$

Hence it follows from the above estimates and (6.4.12) that $-\text{Re } a - \Phi_p(a) < b < -\text{Re } a + \Phi_p(a)$.

Now we prove the converse. It is follows from the above discussion that if $-\text{Re } a - \Phi_p(a) < b < -\text{Re } a + \Phi_p(a)$, then for any fix $t > 0$,

$$\max_{z\in S_p}|e^{t(a\gamma(z)+b)}| > 1 \quad \text{and} \quad \min_{z\in S_p}|e^{t(a\gamma(z)+b)}| < 1.$$

Recall that the maximum and the minimum values of $|e^{t(a\gamma(z)+b)}|$ are attained on the boundary ∂S_p. Let $s_1, s_2 \in [-\tau/2, \tau/2]$ be such that

$$|e^{t(a\gamma(s_1+i\delta_p)+b)}| = \exp\{(\text{Re } a + \Phi_p(a) + b)t\} \quad \text{and}$$
$$|e^{t(a\gamma(s_1+i\delta_p)+b)}| = \exp\{(\text{Re } a - \Phi_p(a) + b)t\}.$$

Let $\{s_j + i\delta_{r_n}\}_n$ be a sequence in S_p° which converges to the point $s_j + i\delta_p$ for $j = 1, 2$. Using the continuity of $|e^{t(a\gamma(z)+b)}|$ on S_p, it follows that $|e^{t(a\gamma(s_1+i\delta_{r_m})+b)}| > 1$ and $|e^{t(a\gamma(s_2+i\delta_{r_m})+b)}| < 1$ for some natural number m. Consequently there exists $s_3 \in (s_1, s_2)$ such that $|e^{t(a\gamma(s_3+i\delta_{r_m})+b)}| = 1$. The assertion now follows by using Theroem 6.2.2 *(b)*. This completes the proof. □

Remark 6.4.5. *Note that if we assume b to be a complex number, then there won't be any significant change in the proof of the Theorem 6.4.3 because $|e^{it\Im b}| = 1$ for all $t \geq 0$. However there will be a minor modification in the statement of Theorem 6.4.3 (3) where b will be replaced by Re b.*

6.5 Some Classical Semigroups

There are some well-known examples of semigroups which are generated by the affine functions. As a consequence of Theorem 6.4.3 we investigate the dynamics of these semigroups which yeilds some interesting results.

6.5.1 The Heat Semigroup

It is already mentioned in the introduction that the chaotic dynamics of the heat semigroup generated by certain shifts of the Laplace-Beltrami operator, are extensively studied on symmetric spaces [21, 31] and harmonic NA-groups [36]. Our objective here is to formulate these results for the heat semigroup on homogeneous trees by using Theorem 6.4.3 as a tool. The heat semigroup on $\mathfrak{X}$ generated by certain shifts of $\mathcal{L}$ is defined by the formula

$$T(t) = e^{-t(\mathcal{L}-b)} \quad \text{where } t \geq 0,\ b \in \mathbb{R}.$$

Using Theorem 6.3.2, it is clear that for any $b \in \mathbb{R}$, $T(t)$ is neither hypercyclic nor does it have any non-trivial periodic point on $L^p(\mathfrak{X})$ whenever $1 \leq p \leq 2$. However putting $a = -1$, the following result is an immediate consequence of Theorem 6.4.3.

Theorem 6.5.1. *Suppose that $T(t) = e^{-t(\mathcal{L}-b)}$ where $t \geq 0$. Then for $2 < p < \infty$, the following are equivalent.*

(1) $T(t)$ is chaotic on $L^p(\mathfrak{X})$.

(2) $T(t)$ has a non-trivial periodic point.

(3) b satisfies the relation $\gamma(i\delta_p) < b < \gamma(\tau/2 + i\delta_p)$.

(4) $T(t)$ is hypercyclic.

Proof. Substituting $a = -1$ in Theorem 6.4.3, we find that the above statements are equivalent if and only if b satisfies

$$1 - \Phi_p(-1) < b < 1 + \Phi_p(-1). \tag{6.5.13}$$

Now explicitly calculating the value of $\Phi_p(-1)$ using formula (6.4.9), we obtain

$$\Phi_p(-1) = (1 - \gamma(i\delta_p)) \cdot ((-1)^2)^{1/2} = (1 - \gamma(i\delta_p)).$$

Putting the above value of $\Phi_p(-1)$ in (6.5.13) and noting that $2 - \gamma(i\delta_p) = \gamma(\tau/2 + i\delta_p)$, we finally get the relation

$$\gamma(i\delta_p) < b < \gamma(\tau/2 + i\delta_p).$$

□

The geometrical interpretation of the above result can also be seen from the following figure. In this figure, the open elliptic region in the left hand side represents the L^p point spectrum of $\mathcal{L}$. It is easy to see that the L^p point spectrum of $\mathcal{L} - b$ cuts the imaginary axis at infinitely many points if and only if $\gamma(i\delta_p) < b < \gamma(\tau/2 + i\delta_p)$.

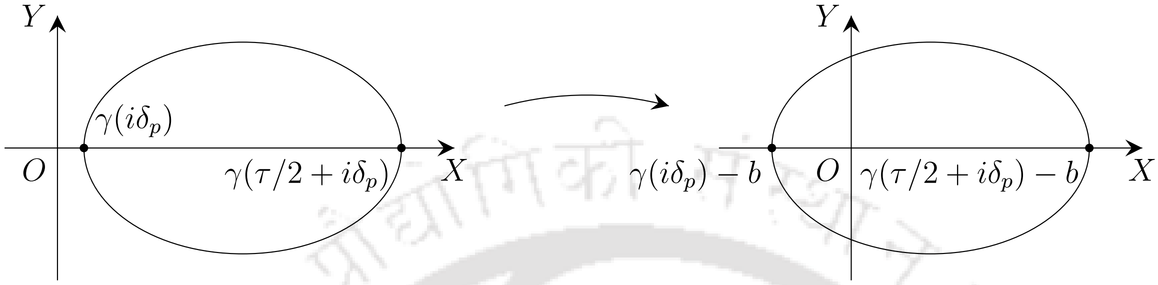

Figure 6.1

6.5.2 The Schrödinger Semigroup

Now we consider the Schrödinger semigroup generated by the perturbation of $i\mathcal{L}$. Once again using Theorem 6.4.3 with $a = i$, we have the following.

Theorem 6.5.2. *Suppose that $T(t) = e^{t(i\mathcal{L}+b)}$ where $t \geq 0$. Then for $2 < p < \infty$, the following are equivalent.*

(1) $T(t)$ is chaotic on $L^p(\mathfrak{X})$.

(2) $T(t)$ has a non-trivial periodic point.

(3) b satisfies the relation $\Im\gamma(\tau/4 + i\delta_p) < b < \Im\gamma(-\tau/4 + i\delta_p)$.

(4) $T(t)$ is hypercyclic.

Proof. Following a similar procedure as in Theorem 6.5.1, it is enough to prove that b satisfies the relation

$$\Im\gamma(\tau/4 + i\delta_p) < b < \Im\gamma(-\tau/4 + i\delta_p). \tag{6.5.14}$$

Putting $a = i$ in (6.4.9), we find that

$$\Phi_p(i) = (1 - \gamma(i\delta_p)) \cdot |\tanh(\delta_p \log q)| = \frac{q^{1/p} + q^{1/p'}}{q+1} \cdot \frac{q^{-\delta_p} - q^{\delta_p}}{q^{-\delta_p} + q^{\delta_p}} = \frac{q^{1/p'} - q^{1/p}}{q+1}.$$

Putting the above value of $\Phi_p(i)$ in $-\Phi_p(i) < b < \Phi_p(i)$ and noting that

$$\Im\gamma(\pm\tau/4 + i\delta_p) = \pm\frac{q^{1/p} - q^{1/p'}}{q+1},$$

we finally get the relation (6.5.14). This completes the proof. □

In a similar way as above, Theorem 6.5.2 can also be described geometrically using the following figure.

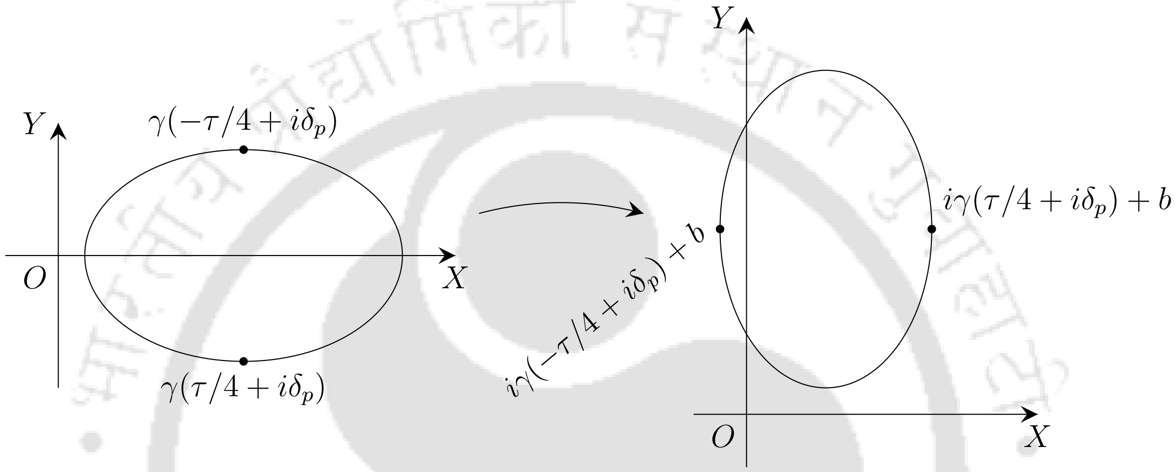

Figure 6.2

CHAPTER 7

Concluding Remarks

In this chapter we would like to point out certain interesting problems that evolved while carrying out a detailed survey on the topics discussed so far.

Problem 1: Let f be a finitely supported function on $\mathfrak{X}$ and $z \in \mathbb{R}$. Then for every $g \in G$, we have

$$f * \phi_z(g) = \int_G f(h)\phi_z(h^{-1}g)dh.$$

Using the formula for $\phi_z(h^{-1}g)$ from [12, pp. 55], we obtain

$$\begin{aligned} f * \phi_z(g) &= \int_G f(h) \left(\int_\Omega p^{1/2-iz}(h,\omega)p^{1/2+iz}(g,\omega)d\nu(\omega) \right) dh \\ &= \int_\Omega \left(\int_G f(h)p^{1/2-iz}(h,\omega)dh \right) p^{1/2+iz}(g,\omega)d\nu(\omega) \\ &= \int_\Omega \widetilde{f}(-z,\omega)p^{1/2+iz}(g,\omega)d\nu(\omega) \\ &= \mathcal{P}_z(\widetilde{f}(-z,\cdot))(g). \end{aligned}$$

Therefore using Theorem 3.3.2 and 4.3.3, it follows that for every $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, the operator given by convolution on the right by ϕ_z is restricted weak type $(2,2)$ and

$$\|f * \phi_z\|_{L^{2,\infty}(\mathfrak{X})} \leq C|\mathbf{c}(z)|^2 \|f\|_{L^{2,1}(\mathfrak{X})} \quad \text{for all } f \in L^{2,1}(\mathfrak{X}).$$

Next we observe that for $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, the following pointwise estimate of ϕ_z holds (see Lemma 2.3.11):

$$|\phi_z(x)| \leq C_z \phi(x) \quad \text{where } \phi(x) = q^{-|x|/2}.$$

In view of the above restricted weak type $(2,2)$ inequality, it is thus natural to investigate whether the same holds for an operator given by convolution on the right by a larger kernel, that is $\phi(x)$. By extending an example given in [25, pp. 5595], we now prove that this is not the case. Consider a non-negative radial function f on $\mathfrak{X}$ defined by

$$f(x) = (1 + |x|)^{-3/2} q^{-|x|/2}.$$

From the definition of the Lorentz norms, it follows that $f \in L^{2,1}(\mathfrak{X})$. We now show that $f * \phi \notin L^{2,\infty}(\mathfrak{X})$. Using the functional equation (2.4.14), the pointwise estimates of ϕ_0 from Lemma 2.3.11 and the fact that $|xy| \leq |x| + |y|$, we obtain

$$\begin{aligned}
f * \phi(x) &= \int_G f(y)\phi(y^{-1}x)dy \\
&= \int_G f(y) \left(\int_K \phi(y^{-1}kx) \frac{(1 + |y^{-1}kx|)}{(1 + |y^{-1}kx|)} dk \right) dy \\
&\geq \int_G \frac{f(y)}{(1 + |y| + |x|)} \left(\int_K \phi_0(y^{-1}kx) dk \right) dy \\
&= \int_G \frac{f(y)}{(1 + |y| + |x|)} \phi_0(y)\phi_0(x) dy \\
&\asymp \sum_{n \in \mathbb{N}} \frac{(1 + |x|)}{(1 + n + |x|)} (1 + n)^{-1/2} q^{-|x|/2} \\
&\geq \frac{(1 + |x|)}{(1 + 2|x|)} q^{-|x|/2} \sum_{n=0}^{|x|} (1 + n)^{-1/2} \\
&\geq C|x|^{1/2} q^{-|x|/2}.
\end{aligned}$$

Once again using the definition of the Lorentz norms, it follows that $f * \phi \notin L^{2,\infty}(\mathfrak{X})$.

The above observations signify the following facts regarding the restricted weak type $(2,2)$ boundedness of an operator given by convolution on the right by a kernel.

1. The end point version of the celebrated Kunze-Stein phenomenon (see [43]), that is,

$$L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G)$$

is not sharp in the sense that there exist functions ϕ_z, $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, which is in $L^{2,\infty}(G)$ and not in $L^{2,1}(G)$, but convolves with $L^{2,1}(G)$ boundedly to give $L^{2,\infty}$ functions on G.

2. The presence of the oscillatory factor, that is, Re $(\mathbf{c}(z)q^{iz})$ in ϕ_z is mainly responsible for the restricted weak type $(2, 2)$ inequality. In fact for $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$, ϕ_z has the following explicit formula:

$$\begin{aligned}\phi_z(x) &= \mathbf{c}(z)q^{(iz-1/2)|x|} + \mathbf{c}(-z)q^{(-iz-1/2)|x|}\\ &= q^{-|x|/2}\text{Re }(\mathbf{c}(z)q^{iz}).\end{aligned}$$

In view of these facts, it is natural to ask the following question in the context of homogeneous trees: For which function $\psi : \mathbb{Z}_+ \to \mathbb{C}$, does the function $\phi(x) = q^{-|x|/2}\psi(|x|)$, $x \in \mathfrak{X}$, defines a restricted weak type $(2, 2)$ operator ?

Problem 2: The restriction theorem (that is Theorem 4.3.2) which we have proved for the Helgason-Fourier transform on $\mathfrak{X}$ is a natural analogue of those on harmonic NA groups (see [26, Theorem 1.1]). However just like on NA groups, it would be interesting to know whether the estimates in Theorem 4.3.2 are sharp.

Problem 3: Observe that in Theorem 6.4.3 we have mainly derived an equivalence relation between the hypercyclicity and chaoticity of the affine semigroups on $L^p(\mathfrak{X})$, by using the sharp operator norm of $e^{\xi\mathcal{L}}$ as our tool. In spirit of this it would be interesting to know whether a similar equivalence holds for semigroups generated by the operator $a\mathcal{L}^2 + b\mathcal{L} + c$, or more generally for semigroups generated by the analytic functions of the Laplacian on $\mathfrak{X}$. The problems seem to be interesting due to the fact that the sharp operator norms of these semigroups are not yet known and one has to adopt a completely different approach to establish this equivalence (provided such an equivalence holds).

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List of Articles from Thesis Work

List of Accepted/Published Articles

- P. Kumar and S. K. Rano. A characterization of weak L^p -eigenfunctions of the Laplacian on homogeneous trees. *Ann. Mat. Pura Appl. (4) (Published Online)*. doi: 10.1007/s10231-020-01011-3.

List of Communicated Articles

- S. K. Rano. A theorem of Roe and Strichartz on homogeneous trees.
- P. Kumar and S. K. Rano. Dynamics of semigroups generated by analytic functions of the Laplacian on homogeneous trees.