DIVISIBILITY OF CERTAIN PARTITION FUNCTIONS AND MODULAR FORMS

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Divisibility of Certain Partition Functions and Modular Forms

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"We are what our thoughts have made us; so take care about what you think. Words are secondary. Thoughts live; they travel far."

-Swami Vivekananda

This work is dedicated

to

My Mother

and

My Thesis Supervisor

Prof. Rupam Barman

for

encouraging me to chase my dreams!

Certificate

This is to certify that the thesis entitled **"Divisibility of certain partition functions and modular forms"** submitted by **Mr. Ajit Singh** to the **Indian Institute of Technology Guwahati**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Date: May 13, 2022
Guwahati, India

Prof. Rupam Barman
Professor
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"Little drops of water make the mighty ocean"

-Julia A.F. Carney

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Abstract

This thesis studies arithmetic properties of certain partition functions, namely Andrews' singular overpartitions, mex-related partition functions, t-regular partitions and 3-regular color partitions. Firstly, we study the mex-related partition function which has been introduced by Andrews and Newman very recently. The minimal excludant, or "mex" function, on a set S of positive integers is the least positive integer not in S. Andrews and Newman extended the mex-function to integer partitions and found numerous surprising partition identities connected with these functions. We study two of the families of functions they introduced, namely $p_{t,t}(n)$ and $p_{2t,t}(n)$. We use various basic q-series manipulations to establish identities connecting the ordinary partition function $p(n)$ to $p_{t,t}(n)$ and $p_{2t,t}(n)$ for all $t \geq 1$. Using these identities, we prove that Ramanujan's famous congruences for $p(n)$ are also satisfied by $p_{t,t}(n)$ and $p_{2t,t}(n)$ for infinitely many values of t. Next, we prove that the generating function of $p_{t,t}(n)$ for $t = 2^\alpha, 3 \cdot 2^\alpha$ where $\alpha \geq 1$, is a modular form modulo 2. We then use a result of Ono and Taguchi on nilpotency of Hecke operators to find infinite families of congruences modulo 2 satisfied by $p_{2^\alpha,2^\alpha}(n)$ and $p_{3\cdot 2^\alpha,3\cdot 2^\alpha}(n)$ for all $\alpha \geq 1$. We further prove that $p_{t,t}(n) \equiv \overline{C}_{4t,t}(n) \pmod 2$ for all $n \geq 0$ and $t \geq 1$, where $\overline{C}_{4t,t}(n)$ is Andrews' singular overpartition function.

Secondly, we study certain arithmetic properties of Andrews singular overpartitions $\overline{C}_{k,i}(n)$. We use arithmetic properties of modular forms and eta-quotients to study divisibility of $\overline{C}_{3\ell,\ell}(n)$, $\overline{C}_{4\ell,\ell}(n)$ and $\overline{C}_{6\ell,\ell}(n)$ by arbitrary powers of 2 and 3. Using a result of Ono and Taguchi on nilpotency of Hecke operators, we find infinite families of congruences modulo arbitrary powers of 2 satisfied by $\overline{C}_{4\cdot 2^\alpha,2^\alpha}(n)$ and $\overline{C}_{4\cdot 3\cdot 2^\alpha,3\cdot 2^\alpha}(n)$ for all $\alpha \geq 0$.

Thirdly, we study arithmetic properties of certain t-regular partitions. In a recent paper, Keith and Zanello established infinite families of congruences and self-similarity results for $b_t(n)$ modulo 2 for certain values of t. Further, they proposed some conjectures on self-similarities of $b_t(n)$ modulo 2 for certain values of t. For example, they conjectured that, for a positive proportion of primes p, $b_3(n)$ satisfies

$$\sum_{n=0}^{\infty} b_3(2(pn+\alpha))q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{pn} \pmod{2},$$

where $\alpha \equiv -24^{-1} \pmod{p^2}$, $0 < \alpha < p^2$. We prove their conjectures on $b_3(n)$ and $b_{25}(n)$. We also prove a self-similarity result for $b_{21}(n)$ modulo 2. The proofs use a result of Serre which says that if f is an integer weight cusp form on the congruence subgroup $\Gamma_0(N)$ then for any positive integer M, a positive proportion of the primes $p \equiv -1 \pmod{MN}$ have the property that $f(z) \mid T_p \equiv 0 \pmod{M}$. We also establish infinite families of congruences modulo 2 for $b_3(n)$ and $b_{21}(n)$ using an approach developed by Radu. Using the theory of Hecke operators, we next prove a self-similarity result for $b_3(n)$ modulo 2 for the prime $p = 17$. We further prove that the series $\sum_{n=0}^{\infty} b_9(2n+1)q^n$ is lacunary modulo arbitrary powers of 2. We also prove that the series $\sum_{n=0}^{\infty} b_9(4n)q^n$ is lacunary modulo 2.

Finally, we study the partition function $p_{\{3,3\}}(n)$ which counts the number of 3-regular partitions in three colours. In a very recent paper, da Silva and Sellers studied certain arithmetic properties of $p_{\{3,3\}}(n)$. They further conjectured four Ramanujan-like congruences modulo 5 satisfied by $p_{\{3,3\}}(n)$. We confirm the conjectural congruences of da Silva and Sellers using the theory of modular forms.

Contents

Introduction

A partition of a positive integer n is a finite non-increasing sequence of positive integers $n_1 \geq n_2 \geq \cdots \geq n_m > 0$ such that

$$n = \sum_{i=1}^{m} n_i,$$

where n_i's are called parts. The number of partitions of n is denoted by $p(n)$ and by convention $p(0) := 1$. For example, partitions of 5 are

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

Therefore, $p(5) = 7$.

Leibniz [33] is credited with being the first to mention about partitions. In his letter to Bernoulli, he raised the question of finding the number of all the essentially different representations of a given positive integer n as the sum of positive integers, such as representing 3 as $3, 2+1$, or $1+1+1$. Despite the fact that Leibniz is credited with inventing what we now term as integer partitions, the Theory of Partitions as we know it now can be said to have excelled with the great Euler [20]. Euler found many basic results on the partitions of numbers. His groundbreaking work in this area, which included the use of generating functions and formal power series, solidified additive number theory. For example, Euler proved that, for any positive integer n, the number of partitions of n using only odd parts equals the number of partitions of n into distinct parts. Developing an explicit formula for $p(n)$ was one of the most difficult challenges. Hardy and Ramanujan [25], Rademacher [46], and Selberg [52] answered this question quite completely.

The seemingly easy question of representing positive numbers as a sum of positive integers turns out to have a wide range of applications, including combinatorics, representation theory, computer science, theoretical physics, statistics. The Theory of Partitions, which lies at the intersection of Number Theory, Analysis, and Combinatorics, is a rich field that has piqued the curiosity of many.

Partitions reflect the fundamental additive features of integers, hence it is startling to learn that $p(n)$ has divisibility properties as well. Euler gave a generating function for $p(n)$ using the q-series

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1}$$

where $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$ for any complex numbers a and q with $|q|<1$. Euler noted that the series representation of the infinite product $(q;q)_\infty$ is given by

$$\begin{aligned}(q;q)_\infty &= \sum_{n=-\infty}^{\infty} (-1)^k q^{k(3k+1)/2} \\ &= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots .\end{aligned} \tag{2}$$

This identity is known as Euler's pentagonal number theorem. Combining (1) and (2) we obtain the following recurrence relation:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots .$$

A British mathematician, Percy Alexander MacMahon in 1916, was the first person who computed $p(n)$ for n up to 200, using the Euler's recurrence relation. In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

He gave the proofs of the first two congruences in [51] and derived the following q-series identities:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6} = 5+30q+135q^2+\cdots,$$
$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q^7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8} = 7+77q+\cdots.$$

Later in a short one page note [50], he announced that he had also found a proof of the third congruence. In 1919, Ramanujan [51] offered a more general conjecture after studying a table of values of $p(n)$, $0 \leq n \leq 200$, made by MacMahon. Let $\delta = 5^a7^b11^c$ and let λ be an integer such that $24\lambda \equiv 1 \pmod{\delta}$. Then

$$p(n\delta+\lambda) \equiv 0 \pmod{\delta}. \tag{3}$$

Ramanujan gave the proof of (3) for arbitrary a and $b = c = 0$. He also began a proof of his conjecture for arbitrary b and $a = c = 0$, but he did not complete it. Later Gupta extended MacMahon's table up to $n = 300$. Upon examining Gupta's table in 1934, Chowla [13] found that $p(243) = 133978259344888 \not\equiv 0 \pmod{7^3}$, despite the fact that $24 \times 243 \equiv 1 \pmod{7^3}$. To correct Ramanujan's conjecture, for given $\delta = 5^a7^b11^c$, define $\delta' = 5^a7^{b'}11^c$, where $b' = b$, if $b = 0, 1, 2$ and $b' = \lfloor\frac{b+2}{2}\rfloor$, if

$b > 2$. Then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta'}. \tag{4}$$

In 1938, Watson [61] gave a proof of (4) for $a = c = 0$, with a more detailed version of Ramanujan's proof of (4) in the case $b = c = 0$. It was not until 1967 that Atkin [6] proved (4) for arbitrary c and $a = b = 0$.

In addition to the study of Ramanujan-type congruences, it is an interesting problem to study the distribution of the partition function modulo positive integers M. In [41], Ono revolutionized the subject by developing aspects of the p-adic theory of half-integral weight modular forms and using this he proved the existence of infinite families of partition congruences modulo every prime $\ell \geq 5$. There are still many unanswered questions on the distribution of the partition function. Let $F(q) := \sum_{n=0}^{\infty} a(n)q^n$ be a given integral power series and $0 \leq r < M$. We define

$$\delta_r(F, M; X) := \frac{\#\{0 \leq n \leq X : a(n) \equiv r \pmod{M}\}}{X}.$$

An integral power series F is called *lacunary modulo M* if

$$\lim_{X\to\infty} \delta_0(F, M; X) = 1,$$

that is, almost all of the coefficients of F are divisible by M. A conjecture of Parkin and Shanks [44] predicts that, for $r \in \{0, 1\}$,

$$\lim_{X\to\infty} \delta_r(P, 2; X) = \frac{1}{2},$$

where $P(q)$ is the generating function for $p(n)$. Little is known regarding this conjecture.

Many mathematicians have proposed various forms of partition functions and discovered that they too satisfy a number of intriguing arithmetic properties.

In this thesis, we study arithmetic properties of certain partition functions, namely, Andrews' singular overpartitions, Mex-related partition functions, t-regular partitions and 3-regular partitions in three colors. For these partition functions, we uncover infinite families of arithmetic identities and Ramanujan-type congruences, as well as we confirm some recent conjectures. We also find distribution of some families of these partition functions modulo certain positive integers. We use certain arithmetic properties of modular forms in our proofs. In particular, we use congruence properties of certain modular forms, basic properties of eta-quotients, and the theory of Hecke operators in our proofs. We also use classical q-series methods to prove some of our results.

Organization of the Thesis

We present the entire work of this thesis in eight chapters as described below.

- Chapter 1: Preliminaries
- Chapter 2: Mex-related partition functions and relations to certain partition functions
- Chapter 3: Congruences for mex-related partition functions
- Chapter 4: Divisibility of singular overpartitions $\overline{C}_{3\ell,\ell}(n)$
- Chapter 5: Divisibility of singular overpartitions $\overline{C}_{4\ell,\ell}(n)$ and $\overline{C}_{6\ell,\ell}(n)$
- Chapter 6: Congruences and self-similarity results on t-regular partitions
- Chapter 7: Divisibility of certain t-regular partitions by 2
- Chapter 8: Congruences for 3-regular partitions in three colors

In Chapter 1 we introduce Ramanujan's theta functions. We also recall some definitions and basic results on modular forms and eta-quotients.

In Chapter 2 we study a partition function which appeared in a very recent paper of Andrews and Newman. The minimal excludant, or "mex" function, on a set S of positive integers is the least positive integer not in S. They extended the mex-function to integer partitions and found numerous surprising partition identities connected with these functions. We study two of the families of functions Andrews and Newman introduced, namely $p_{t,t}(n)$ and $p_{2t,t}(n)$. We establish identities connecting the ordinary partition function $p(n)$ to $p_{t,t}(n)$ and $p_{2t,t}(n)$ for all $t \geq 1$. Using these identities, we prove that the Ramanujan's famous congruences for $p(n)$ are also satisfied by $p_{t,t}(n)$ and $p_{2t,t}(n)$ for infinitely many values of t. We further prove that $p_{t,t}(n) \equiv \overline{C}_{4t,t}(n) \pmod{2}$ for all $n \geq 0$ and $t \geq 1$, where $\overline{C}_{4t,t}(n)$ is the Andrews' singular overpartition function. We also give elementary proofs of certain congruences already proved by da Silva and Sellers. Our proofs use basic properties of q-series.

In Chapter 3 we study relationships between the generating function of $p_{t,t}(n)$ and certain modular forms. We then use a result of Ono and Taguchi on nilpotency of Hecke operators to find infinite families of congruences modulo 2 satisfied by $p_{2^\alpha,2^\alpha}(n)$ and $p_{3\cdot 2^\alpha,3\cdot 2^\alpha}(n)$ for all $\alpha \geq 1$.

In Chapter 4 we study the Andrews' singular overpartitions $\overline{C}_{3\ell,\ell}(n)$. We prove that $\overline{C}_{3\ell,\ell}(n)$ is almost always divisible by arbitrary powers of 2. We further prove that $\overline{C}_{3\ell,\ell}(n)$ is almost always divisible by arbitrary powers of 3 when $\ell = 2, 3, 4, 6, 12, 24$. We next prove that the eta-quotient which arises naturally as generating function for $\overline{C}_{3\cdot 3\cdot 2^\alpha,3\cdot 2^\alpha}(n)$ is not a modular form if $\alpha \geq 4$. Proofs of our density results rely on the modularity of certain eta-quotients which arise naturally as generating functions for the Andrews' singular overpartion functions. We also find infinite families of congruences modulo arbitrary powers of 2 satisfied by $\overline{C}_{6,2}(n)$.

In Chapter 5 we study the divisibility properties of the Andrews' singular overpartitions $\overline{C}_{4\ell,\ell}(n)$ and $\overline{C}_{6\ell,\ell}(n)$ by arbitrary powers of 2 and 3 for infinite families of k. We prove that, for prime $p \geq 3$ and integers $\alpha, \beta \geq 0$ satisfying $3 \cdot 2^\alpha \geq p^\beta$,

$\overline{C}_{4\cdot 2^\alpha p^\beta, 2^\alpha p^\beta}(n)$ is almost always divisible by arbitrary powers of 2. We also prove that $\overline{C}_{6\cdot 3^\alpha, 3^\alpha}(n)$ is almost always divisible by arbitrary powers of 3 for all $\alpha \geq 0$. Using a result of Ono and Taguchi on nilpotency of Hecke operators, we find infinite families of congruences modulo arbitrary powers of 2 satisfied by $\overline{C}_{4\cdot 2^\alpha, 2^\alpha}(n)$ and $\overline{C}_{4\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n)$ for all $\alpha \geq 0$.

In Chapter 6 we study certain t-regular partitions. In a recent paper, Keith and Zanello established infinite families of congruences and self-similarity results for $b_t(n)$ modulo 2 for certain values of t. Further, they proposed some conjectures on self-similarities of $b_t(n)$ modulo 2 for certain values of t. For example, they conjectured that, for a positive proportion of primes p, $b_3(n)$ satisfies

$$\sum_{n=0}^{\infty} b_3(2(pn+\alpha))q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{pn} \pmod{2},$$

where $\alpha \equiv -24^{-1} \pmod{p^2}$, $0 < \alpha < p^2$. In this chapter, we prove their conjectures on $b_3(n)$ and $b_{25}(n)$. We also prove a self-similarity result for $b_{21}(n)$ modulo 2. The proofs use a result of Serre which says that if f is an integer weight cusp form on the congruence subgroup $\Gamma_0(N)$ then for any positive integer M, a positive proportion of the primes $p \equiv -1 \pmod{MN}$ have the property that $f(z) \mid T_p \equiv 0 \pmod{M}$.

In Chapter 7 we establish infinite families of congruences modulo 2 for $b_3(n)$ and $b_{21}(n)$ using an approach developed by Radu. This approach reduces the number of coefficients that one must check as compared with the classical method which uses Sturm's bound alone. We next prove a particular case of self-similarity result of $b_3(n)$ modulo 2 for the prime $p = 17$ using the theory of Hecke operators. We further prove that the series $\sum_{n=0}^{\infty} b_9(2n+1)q^n$ is lacunary modulo arbitrary powers of 2. We also prove that the series $\sum_{n=0}^{\infty} b_9(4n)q^n$ is lacunary modulo 2.

In Chapter 8 we study the partition function $p_{\{3,3\}}(n)$ which counts the number of 3-regular partitions in three colours. In a very recent paper, da Silva and Sellers conjectured four Ramanujan-like congruences modulo 5 satisfied by $p_{\{3,3\}}(n)$. We confirm the conjectural congruences of da Silva and Sellers using the theory of modular forms.

1 Preliminaries

1.1 q-series and Ramanujan's theta functions

Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

The product representation of $f(a, b)$ arises from Jacobi's triple product identity [9, p.35, Entry 19] as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,$$

where $(c;q)_\infty = \prod_{k=0}^{\infty}(1-cq^k)$ for any complex numbers c and q with $|q|<1$. The following are the most important special cases of $f(a,b)$:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}; \tag{1.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}; \tag{1.2}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{1.3}$$

In partition theory, these special types of theta functions have a very important role. There are many remarkable q-series identities and p-dissection formulas involving the theta functions, which are used to study various types of partition functions.

We now recall Euler's pentagonal number theorem.

Theorem 1.1. [10, Corollary 1.3.5] *For* $|q|<1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty.$$

1.2 Spaces of modular forms

In this section, we review some definitions and fundamental facts about modular forms and eta-quotients. For more details, see for example [31, 42].

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc>0 \right\}$$

acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \dfrac{az+b}{cz+d}$. We identify ∞ with $\dfrac{1}{0}$ and define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \dfrac{r}{s} = \dfrac{ar+bs}{cr+ds}$, where $\dfrac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$, more presicely,

Definition 1.1. *Suppose $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, the slash operator $|_\ell$ is defined by*

$$(f|_\ell \gamma)(z) := (\det \gamma)^{\ell/2}(cz+d)^{-\ell} f(\gamma z).$$

One of the most important characteristics of this operator is that it gives a group action on the ring of meromorphic functions on $\mathbb{H}$, as stated in the following proposition.

Proposition 1.2. *For all $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{R})$, it follows that*

$$\left((f|_\ell \gamma_1)|_\ell \gamma_2\right)(z) = \left(f|_\ell(\gamma_1 \cdot \gamma_2)\right)(z).$$

The slash operator's shorthand is extremely useful for computing a function's order of vanishing at cusps, as the matrix $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ maps ∞ to the cusps $\frac{a}{c}$.

Let N be a fixed positive integer. Then the following matrix sets are subgroups of the group $\mathrm{GL}_2^+(\mathbb{R})$.

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N, \text{ and } b \equiv c \equiv 0 \pmod N \right\}.$$

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some N. The smallest N such that $\Gamma(N) \subseteq \Gamma$ is called the level of Γ. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

Definition 1.2. *Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ on Γ if the following hold:*

1. *We have*
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\ell} f(z)$$
for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.

2. *If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_{\ell}\gamma)(z)$ has a Fourier expansion of the form*
$$(f|_{\ell}\gamma)(z) = \sum_{n \geq 0} a_{\gamma}(n) q_N^n,$$
where $q_N := e^{2\pi i z/N}$.

In addition, if $a_{\gamma}(0) = 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then f is called a cusp form.

The set of all modular forms (resp. cusp forms) of weight ℓ on a congruence subgroup Γ naturally form $\mathbb{C}$-vector spaces. The complex vector space of modular forms (resp. cusp forms) of weight ℓ with respect to a congruence subgroup Γ is denoted by $M_{\ell}(\Gamma)$ (resp. $S_{\ell}(\Gamma)$).

Definition 1.3. *If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$* (resp. $S_\ell(\Gamma_1(N))$) *has Nebentypus character χ if*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms (resp. cusp forms) *is denoted by $M_\ell(\Gamma_0(N), \chi)$* (resp. $S_\ell(\Gamma_0(N), \chi)$). *If χ is the trivial character then we denote $M_\ell(\Gamma_0(N), \chi)$* (resp. $S_\ell(\Gamma_0(N), \chi)$) *by $M_\ell(\Gamma_0(N))$* (resp. $S_\ell(\Gamma_0(N))$).

The spaces $M_\ell(\Gamma_1(N))$ and $S_\ell(\Gamma_1(N))$ have the following decomposition (where the sums are over all Dirichlet characters χ modulo N):

$$M_\ell(\Gamma_1(N)) = \bigoplus_\chi M_\ell(\Gamma_0(N), \chi),$$
$$S_\ell(\Gamma_1(N)) = \bigoplus_\chi S_\ell(\Gamma_0(N), \chi).$$

The Hecke operators are natural linear transformations that act on spaces of modular forms. The Hecke operators on spaces of integer weight modular forms are defined as follows:

Definition 1.4. *Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then the action of Hecke operator T_m on $f(z)$ is defined by*

$$f(z)|T_m := \sum_{n=0}^{\infty} \left(\sum_{d|\gcd(n,m)} \chi(d) d^{\ell-1} a\left(\frac{nm}{d^2}\right) \right) q^n.$$

In particular, if $m = p$ is prime, we have

$$f(z)|T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p)p^{\ell-1} a\left(\frac{n}{p}\right) \right) q^n.$$

We adopt the convention that $a(n/p) = 0$ when $p \nmid n$.

Definition 1.5. *A modular form* $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$ *is called a Hecke eigenform if for every* $m \geq 2$ *there exists a complex number* $\lambda(m)$ *for which*

$$f(z)|T_m = \lambda(m)f(z).$$

We now recall a result of Serre [56] (also see [59, Proposition 4.2]) about the action of Hecke operator on cusp forms. For a number field K, let $\mathcal{O}_K$ denotes its ring of integers.

Proposition 1.3. [56, Exercise 6.4] *Suppose that*

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N), \chi)$$

has coefficients in $\mathcal{O}_K$*, and* M *is a positive integer. Furthermore, suppose that* $k > 1$*. Then a positive proportion of the primes* $p \equiv -1 \pmod{MN}$ *have the property that*

$$f(z) \mid T_p \equiv 0 \pmod{M\mathcal{O}_K}.$$

1.2.1 Modularity of eta-quotients

The Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer.

The following general result of Gordon, Hughes, and Newman [23, 39, 40] is very useful when working with eta-quotients. We will use these two results to verify modularity of certain eta-quotients appearing in the proofs of our main results. We state the theorems as mentioned in [42, p. 18].

Theorem 1.4. [42, Theorem 1.64] *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(\bullet) := \left(\frac{(-1)^\ell s}{\bullet}\right)$, where $s := \prod_{\delta|N} \delta^{r_\delta}$.

Suppose that f is an eta-quotient satisfying the conditions of Theorem 1.4 and that the associated weight ℓ is a positive integer. If $f(z)$ is holomorphic (resp. vanishes) at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$ (resp. $S_\ell(\Gamma_0(N), \chi)$). The following theorem (see, for example [11, 35, 38]) gives the necessary criterion for determining orders of an eta-quotient at cusps.

Theorem 1.5. [42, Theorem 1.65] *Let c, d and N be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If f is an eta-quotient satisfying the conditions of Theorem 1.4 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

We now recall the following proposition which gives a complete set of representatives for the cusps of $\Gamma_0(N)$ (see, for example [18, p. 99]).

Proposition 1.6. *Let*

$$C_0(N) := \left\{ \frac{c}{d} : d \mid N, \gcd(c, N) = 1 \right\},$$

where c runs through a complete residue system modulo $\gcd(d, \frac{N}{d})$. Then $C_0(N)$ is a complete set of representatives of the cusps on $\Gamma_0(N)$. Moreover, $C_0(N)$ is minimal.

1.2.2 Congruences for modular forms

In Chapters 4, 5, and 7, we establish several distribution and parity results for certain partiton functions. To prove such results, we require knowledge of the divisibility properties of the coefficients of integral weight modular forms. One of the most useful tools is Serre's work on divisibility of the coefficients of modular forms. Let $\mathcal{A}$ denote the subset of integer weight modular forms in $M_\ell(\Gamma_0(N), \chi)$ whose Fourier coefficients are in $\mathcal{O}_K$, the ring of algebraic integers in a number field K. Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$. Using p-adic Galois representations attached to certain modular forms by Deligne, Serre [53] proved the following remarkable theorem about the divisibility of Fourier coefficients of modular forms.

Theorem 1.7. [42, Theorem 2.65] *If $f(z) \in \mathcal{A}$ has Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{O}_K[[q]],$$

then there is a constant $\alpha > 0$ such that

$$\#\left\{n \leq X : a(n) \not\equiv 0 \pmod{\mathfrak{m}}\right\} = \mathcal{O}\left(\frac{X}{(\log X)^\alpha}\right).$$

Theorem 1.7 yields

$$\lim_{X\to\infty} \frac{\#\{0 < n \leq X : a(n) \equiv 0 \pmod{\mathfrak{m}}\}}{X} = 1.$$

In a recent paper [15], Cotron et al. extended Serre's result to integral weight eta-quotients which are not modular forms, modulo arbitrary powers of primes under certain strong conditions. To be specific, we consider eta-quotients of the form

$$G(z) := \frac{\eta^{r_1}(\delta_1 z)\eta^{r_2}(\delta_2 z)\cdots\eta^{r_u}(\delta_u z)}{\eta^{s_1}(\gamma_1 z)\eta^{s_2}(\gamma_2 z)\cdots\eta^{s_t}(\gamma_t z)} = q^{\frac{E_G}{24}} \sum_{n=0}^{\infty} a(n)q^n, \tag{1.4}$$

where r_i, s_i, δ_i and γ_i are positive integers with $\delta_1, \delta_2, \ldots, \delta_u, \gamma_1, \gamma_2, \ldots, \gamma_t$ are distinct, $u, t \geq 0$, and

$$E_G = \sum_{i=1}^{u} \delta_i r_i - \sum_{i=1}^{t} \gamma_i s_i.$$

As defined in [15], $G(z)$ is called lacunary modulo M when the series $\sum_{n=0}^{\infty} a(n)q^n$ has that property. The main result of Cotron et al. [15, Theorem 1.1] is as follows:

Theorem 1.8. *Suppose $G(z)$ is an eta-quotient of the form (1.4) with integer weight. If p is a prime such that p^a divides $\gcd(\delta_1, \delta_2, \ldots, \delta_u)$ and*

$$p^a \geq \sqrt{\frac{\sum_{i=1}^{t} \gamma_i s_i}{\sum_{i=1}^{u} \frac{r_i}{\delta_i}}}, \tag{1.5}$$

then $G(z)$ is lacunary modulo p^j for any positive integer j. That is,

$$\lim_{X\to\infty} \frac{\#\{0 < n \leq X : a(n) \equiv 0 \pmod{p^j}\}}{X} = 1,$$

where the $a(n)$'s are given by (1.4).

Further, we recall the following classical result due to Landau [32].

Lemma 1.9. *Let $r(n)$ and $s(n)$ be quadratic polynomials. Then*

$$\left(\sum_{n\in\mathbb{Z}} q^{r(n)}\right)\left(\sum_{n\in\mathbb{Z}} q^{s(n)}\right)$$

is lacunary modulo 2.

We now state a result of Sturm [57] which gives a criterion to test whether two modular forms are congruent modulo a given prime.

Theorem 1.10. *Let p be a prime number, and $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=n_1}^{\infty} b(n)q^n$ be modular forms of weight k for $\Gamma_0(N)$ of characters χ and ψ, respectively, where $n_0, n_1 \geq 0$. If either $\chi = \psi$ and*

$$a(n) \equiv b(n) \pmod{p} \quad \textit{for all} \quad n \leq \frac{kN}{12} \prod_{d \text{ prime};\ d|N} \left(1 + \frac{1}{d}\right),$$

or $\chi \neq \psi$ and

$$a(n) \equiv b(n) \pmod{p} \quad \textit{for all} \quad n \leq \frac{kN^2}{12} \prod_{d \text{ prime};\ d|N} \left(1 - \frac{1}{d^2}\right),$$

then $f(z) \equiv g(z) \pmod{p}$ (i.e., $a(n) \equiv b(n) \pmod{p}$ for all n).

1.3 Hecke nilpotency

An ideal $\mathfrak{I}$ in a commutative ring A is locally nilpotent at a prime ideal $\mathfrak{p}$ if $\mathfrak{I}_{\mathfrak{p}}$, the localization of $\mathfrak{I}$ at $\mathfrak{p}$, is a nilpotent in $A_{\mathfrak{p}}$. Serre observed and Tate proved [54, 55, 58] that the action of Hecke algebras on spaces of modular forms of level 1 modulo 2 is locally nilpotent. This implies that if $f(z) \in M_k \cap \mathbb{Z}[[q]]$, there is a positive integer i with the property that

$$f(z)\,|T_{p_1}|\,T_{p_2}|\cdots|T_{p_i} \equiv 0 \pmod{2}$$

for every collection of odd primes $p_1, p_2, \ldots, p_i$. Ono and Taguchi [43] showed that this phenomenon generalizes to higher levels. We recall the following result which is implied by a much more general result of Ono and Taguchi [43, Theorem 1.3]. This result was also used by Aricheta (see for example [5, Theorem 4.5]).

Theorem 1.11. *Let n be a nonnegative integer and k be a positive integer. Let χ be a quadratic Dirichlet character of conductor $9 \cdot 2^n$. There is an integer $c \geq 0$ such that for every $f(z) \in M_k(\Gamma_0(9 \cdot 2^n), \chi) \cap \mathbb{Z}[[q]]$ and every $t \geq 1$*

$$f(z)\,|T_{p_1}|\,T_{p_2}|\cdots|T_{p_{c+t}} \equiv 0 \pmod{2^t}$$

whenever the primes $p_1, \ldots, p_{c+t}$ are coprime to 6.

Remark 1.3.1. *Theorem 1.3 of Ono and Taguchi is stated for the space of cusp forms; however, there is a remark right after the theorem which guarantees that we can use their result for any modular forms. Ono and Taguchi remarked that one merely needs to verify that the conclusion holds for the subspace of Eisenstein series. This is easily done using well-known formulas for the Fourier expansions of Eisenstein series which are given in terms of generalized divisor functions.*

2

Mex-Related Partition Functions and Relations to Certain Partition Functions

2.1 Introduction

In this chapter, we study the minimal excludant or mex function in integer partition. The minimal excludant function (mex-function) appears extensively in combinatorial game theory (see, for example [21]). For each set S of positive integers

[1] The contents of this chapter have been published in *J. Integer Sequences* (2021).

the mex-function is defined as follows:

$$\mathrm{mex}(S) = \min(\mathbb{Z}_{>0} \setminus S).$$

Andrews and Newman [4] recently generalized this function to integer partitions. Given a partition λ of n, they defined the mex-function $\mathrm{mex}_{A,a}(\lambda)$ to be the smallest positive integer congruent to a modulo A that is not a part of λ. They then defined $p_{A,a}(n)$ to be the number of partitions λ of n satisfying

$$\mathrm{mex}_{A,a}(\lambda) \equiv a \pmod{2A}.$$

For example, consider $n = 5$, $A = 2$, and $a = 2$. In the table below, we list the seven partitions λ of 5 and the corresponding values of $\mathrm{mex}_{2,2}(\lambda)$ for each λ:

Partition λ	$\mathrm{mex}_{2,2}(\lambda)$
5	2
$4+1$	2
$3+2$	4
$3+1+1$	2
$2+2+1$	4
$2+1+1+1$	4
$1+1+1+1+1$	2

We see that four of the partitions of 5 satisfy $\mathrm{mex}_{2,2}(\lambda) \equiv 2 \pmod 4$. Therefore, $p_{2,2}(5) = 4$. Andrews and Newman [4, Lemma 9] proved that the generating function for $p_{t,t}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{t,t}(n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{tn(n+1)/2} \tag{2.1}$$

and the generating function for $p_{2t,t}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{2t,t}(n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{tn^2}. \tag{2.2}$$

In order to state the results of Andrews and Newman on $p_{1,1}(n)$ and $p_{3,3}(n)$, we now recall two partition statistics, the rank and the crank.

Definition 2.1 (Dyson). *Suppose that Λ is a partition of n whose parts in nonincreasing order are the positive integers $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{k-1}$. Then $n = \lambda_0 + \lambda_1 + \cdots + \lambda_{k-1}$ and $l(\Lambda) = k$. The rank of Λ is defined by*

$$rank(\Lambda) := \lambda_0 - l(\Lambda).$$

Definition 2.2. *Suppose that Λ is a partition of n as in Definition 2.1. If $o(\Lambda)$ denotes the number of ones in Λ and $\mu(\Lambda)$ denotes the number of parts larger than $o(\Lambda)$, then the crank of Λ is defined by*

$$c(\Lambda) = \begin{cases} \lambda_0 & \text{if } o(\Lambda) = 0; \\ \mu(\Lambda) - o(\Lambda) & \text{if } o(\Lambda) > 0. \end{cases}$$

For more details on rank and crank, see for example [3, 19]. In [4], Andrews and Newman proved that $p_{1,1}(n)$ equals the number of partitions of n with non-negative crank and that $p_{3,3}(n)$ equals the number of partitions of n with rank ≥ -1. They also proved that $p_{2,1}(n)$ is equal to the number of partitions of n into even parts. They further proved that $p_{4,2}(n) - p_o(n)$ equals the number of partitions of n into parts congruent to $\pm 4, \pm 6, \pm 8, \pm 10$ modulo 32 and $p_{6,3}(n) - p_o(n)$ equals the number of partitions of n into parts congruent to $\pm 2, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8$ modulo 24. Here $p_o(n)$ denotes the number of partitions of n into odd parts. Very recently, da Silva and Sellers [16] provide complete parity characterizations of $p_{1,1}(n)$ and $p_{3,3}(n)$.

They prove that, for all $n \geq 1$,

$$p_{1,1}(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = k(3k \pm 1) \text{ for some } k; \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Similarly, they prove that, for all $n \geq 1$,

$$p_{3,3}(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } 3n+1 \text{ is a square;} \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

2.2 Mex-related partitions and relations to ordinary partition

In this section, we express $p_{t,t}(n)$ and $p_{2t,t}(n)$ in terms of the ordinary partition function $p(n)$ for all $t \geq 1$. Using our identities, we find that the partition functions $p_{t,t}(n)$ and $p_{2t,t}(n)$ satisfy the Ramanujan's famous congruences for $p(n)$ for infinitely many values of t. We adopt the convention that $p(n) = 0$ when n is a negative integer. In the following theorem, we express $p_{t,t}(n)$ and $p_{2t,t}(n)$ in terms of $p(n)$.

Theorem 2.1. *Let t be a positive integer. Then, for all non-negative integers n, we have*

$$p_{t,t}(n) = p(n) + \sum_{r=1}^{\infty} p(n - tr(2r+1)) - \sum_{s=1}^{\infty} p(n - ts(2s-1)) \tag{2.3}$$

and

$$p_{2t,t}(n) = p(n) + \sum_{r=1}^{\infty} p(n - 4tr^2) - \sum_{s=1}^{\infty} p(n - t(2s-1)^2). \tag{2.4}$$

Proof. We know that the generating function for the partition function $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

From (2.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{t,t}(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{tn(n+1)/2}\\ &= \left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(1+\sum_{r=1}^{\infty} q^{tr(2r+1)} - \sum_{s=1}^{\infty} q^{ts(2s-1)}\right)\\ &= \sum_{n=0}^{\infty}\left(p(n)+\sum_{r=1}^{\infty} p(n-tr(2r+1)) - \sum_{s=1}^{\infty} p(n-ts(2s-1))\right)q^n.\end{aligned}$$

Thus, for all non-negative integers n, we have

$$p_{t,t}(n) = p(n) + \sum_{r=1}^{\infty} p(n-tr(2r+1)) - \sum_{s=1}^{\infty} p(n-ts(2s-1)). \tag{2.5}$$

Again, from (2.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{2t,t}(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{tn^2}\\ &= \left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(1+\sum_{r=1}^{\infty} q^{4tr^2} - \sum_{s=1}^{\infty} q^{t(2s-1)^2}\right)\\ &= \sum_{n=0}^{\infty}\left(p(n)+\sum_{r=1}^{\infty} p(n-4tr^2) - \sum_{s=1}^{\infty} p(n-t(2s-1)^2)\right)q^n.\end{aligned}$$

Thus, for all non-negative integers n, we have

$$p_{2t,t}(n) = p(n) + \sum_{r=1}^{\infty} p(n-4tr^2) - \sum_{s=1}^{\infty} p(n-t(2s-1)^2).$$

This completes the proof of the theorem. ■

In the following theorem, we prove that $p_{t,t}(n)$ and $p_{2t,t}(n)$ satisfy Ramanujan-type congruences, and these congruences follow from those satisfied by the ordinary partition function $p(n)$.

Theorem 2.2. *Let $m, a \geq 1$ and b be integers. Suppose that $p(an+b) \equiv 0 \pmod{m}$ for all non-negative integers n. Then, for all $t \geq 1$, we have*

$$p_{at,at}(an+b) \equiv 0 \pmod{m}$$

and

$$p_{2at,at}(an+b) \equiv 0 \pmod{m}$$

for all non-negative integers n.

Proof. Let $n \geq 0$. From (2.3), we obtain

$$\begin{aligned} &p_{at,at}(an+b) \\ &= p(an+b) + \sum_{r=1}^{\infty} p(a(n - tr(2r+1)) + b) - \sum_{s=1}^{\infty} p(a(n - ts(2s-1)) + b). \end{aligned} \tag{2.6}$$

We note that the terms remaining in the sums in (2.5) satisfy that $n - tr(2r+1)$ and $n - ts(2s-1)$ are non-negative. Hence, the same is true in (2.6). Now, if $p(\ell a + b) \equiv 0 \pmod{m}$ for every non-negative integer ℓ, then (2.6) yields that $p_{at,at}(an+b) \equiv 0 \pmod{m}$. This completes the proof of the first congruence of the theorem.

Using (2.4) and proceeding along similar lines, we prove the second congruence. This completes the proof of the theorem. ■

As an application of Theorem 2.2, we find that $p_{t,t}(n)$ and $p_{2t,t}(n)$ satisfy the Ramanujan's famous congruences for certain infinite families of t. Much to Ramanujan's credit, the "Ramanujan's congruences" for $p(n)$ are given below. If $k \geq 1$, then for every non-negative integer n, we have

$$p\left(5^k n + \delta_{5,k}\right) \equiv 0 \pmod{5^k};$$
$$p\left(7^k n + \delta_{7,k}\right) \equiv 0 \pmod{7^{[k/2]+1}};$$
$$p\left(11^k n + \delta_{11,k}\right) \equiv 0 \pmod{11^k},$$

where $\delta_{p,k} := 1/24 \pmod{p^k}$ for $p = 5, 7, 11$. In the following, we prove that $p_{at,at}(n)$ and $p_{2at,at}(n)$ satisfy the Ramanujan's congruences when $a = 5^k, 7^k, 11^k$.

Corollary 2.2.1. *For all $k, t \geq 1$ and for every non-negative integer n, we have*

$$p_{5^k t,5^k t}\left(5^k n + \delta_{5,k}\right) \equiv 0 \pmod{5^k};$$
$$p_{7^k t,7^k t}\left(7^k n + \delta_{7,k}\right) \equiv 0 \pmod{7^{[k/2]+1}};$$
$$p_{11^k t,11^k t}\left(11^k n + \delta_{11,k}\right) \equiv 0 \pmod{11^k};$$
$$p_{2\cdot 5^k t,5^k t}\left(5^k n + \delta_{5,k}\right) \equiv 0 \pmod{5^k};$$
$$p_{2\cdot 7^k t,7^k t}\left(7^k n + \delta_{7,k}\right) \equiv 0 \pmod{7^{[k/2]+1}};$$
$$p_{2\cdot 11^k t,11^k t}\left(11^k n + \delta_{11,k}\right) \equiv 0 \pmod{11^k}.$$

Proof. Combining Ramanujan congruences for $p(n)$ and Theorem 2.2 we readily obtain that $p_{at,at}(n)$ and $p_{2at,at}(n)$ satisfy the Ramanujan-type congruences when $a = 5^k, 7^k, 11^k$. This completes the proof. ■

2.3 Mex-related partitions and singular overpartitions

In Chapter 4, we shall study Andrews' singular overpartition in details but for the sake of completeness of this section we define the so called Andrews' singular overpartition. Andrews' singular overpartition function $\overline{C}_{k,i}(n)$ counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. For $k \geq 3$ and $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by (see, for example [2, Theorem 1])

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n = \frac{(q^k;q^k)_\infty (-q^i;q^k)_\infty (-q^{k-i};q^k)_\infty}{(q;q)_\infty}. \tag{2.7}$$

In this section, we relate the mex-related partition functions to Andrews' singular overpartion functions. This helps us to find new congruences satisfied by the mex-related partition functions. We prove that $p_{t,t}(n)$ and $\overline{C}_{4t,t}(n)$ have the same parity for all $t \geq 1$. Using this, we find new congruences satisfied by $p_{t,t}(n)$. We also give elementary proofs of certain congruences proved by da Silva and Sellers [16]. In the following theorem, we prove that both $p_{t,t}(n)$ and $\overline{C}_{4t,t}(n)$ have the same parity.

Theorem 2.3. *Let t be a positive integer. Then, for all $n \geq 0$, we have*

$$p_{t,t}(n) \equiv \overline{C}_{4t,t}(n) \pmod{2}.$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} p_{t,t}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{tn(n+1)/2}. \tag{2.8}$$

Employing the Ramanujan's theta function (1.2)

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}$$

into (2.8), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} p_{t,t}(n) q^n &\equiv \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{tn(n+1)/2} \pmod{2} \\ &= \frac{1}{(q;q)_\infty} \frac{(q^{2t};q^{2t})_\infty^2}{(q^t;q^t)_\infty} \end{aligned}$$

$$\equiv \frac{1}{(q;q)_\infty}\frac{(q^t;q^t)_\infty^4}{(q^t;q^t)_\infty} \pmod 2$$
$$\equiv \frac{(q^t;q^t)_\infty^3}{(q;q)_\infty} \pmod 2. \tag{2.9}$$

From (2.7), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{C}_{4t,t}(n)q^n &= \frac{(q^{4t};q^{4t})_\infty(-q^t;q^{4t})_\infty(-q^{3t};q^{4t})_\infty}{(q;q)_\infty}\\ &= \frac{(q^{4t};q^{4t})_\infty(-q^t;q^{4t})_\infty(-q^{3t};q^{4t})_\infty(-q^{2t};q^{4t})_\infty(-q^{4t};q^{4t})_\infty}{(q;q)_\infty(-q^{2t};q^{4t})_\infty(-q^{4t};q^{4t})_\infty}\\ &= \frac{(q^{4t};q^{4t})_\infty(-q^t;q^t)_\infty}{(q;q)_\infty(-q^{2t};q^{2t})_\infty}\\ &= \frac{(q^{4t};q^{4t})_\infty(q^{2t};q^{2t})_\infty^2}{(q;q)_\infty(q^t;q^t)_\infty(q^{4t};q^{4t})_\infty}\\ &\equiv \frac{(q^t;q^t)_\infty^3}{(q;q)_\infty} \pmod 2.\end{aligned} \tag{2.10}$$

Thus, combining (2.9) and (2.10), we have

$$\sum_{n=0}^{\infty}p_{t,t}(n) \equiv \sum_{n=0}^{\infty}\overline{C}_{4t,t}(n) \pmod 2.$$

This completes the proof of the theorem. ■

Chen, Hirschhorn, and Sellers [12] proved that, for all $n \geq 1$,

$$\overline{C}_{4,1}(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = k(3k-1) \text{ for some } k;\\ 0 \pmod 2, & \text{otherwise.}\end{cases}$$

Due to Theorem 2.3, we have the same parity characterization for $p_{1,1}(n)$. Recently, da Silva and Sellers [16, Theorem 4] also found the same parity characterization for $p_{1,1}(n)$. To the best of our knowledge, the parity characterization for $\overline{C}_{12,3}(n)$ is not known till date. da Silva and Sellers [16, Theorem 7] found the parity characterization for $p_{3,3}(n)$. Combining [16, Theorem 7] and Theorem 2.3, we have the following parity characterization for $\overline{C}_{12,3}(n)$:

Corollary 2.3.1. *For all $n \geq 1$, we have*

$$\overline{C}_{12,3}(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } 3n+1 \text{ is a square;} \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

We next use known congruences for Andrews' singular overpartition functions $\overline{C}_{4t,t}(n)$ and combine them with Theorem 2.3 to deduce new congruences for $p_{t,t}(n)$ for different values of t. Our list of congruences obtained this way need not be exhaustive.

Theorem 2.4. *Let $p \geq 5$ be a prime and $p \not\equiv 1 \pmod{12}$. Then, for all $k, n \geq 0$ with $p \nmid n$, we have*

$$p_{1,1}\left(p^{2k+1}n + \frac{p^{2k+2}-1}{12}\right) \equiv 0 \pmod 2.$$

Proof. Taking $t = 1$ in Theorem 2.3, we have

$$p_{1,1}(n) \equiv \overline{C}_{4,1}(n) \pmod 2. \tag{2.11}$$

Thanks to Chen, Hirschhorn and Sellers ([12, Corollary 3.6]) we know that, for all $n \geq 0$,

$$\overline{C}_{4,1}\left(p^{2k+1}n + \frac{p^{2k+2}-1}{12}\right) \equiv 0 \pmod 4. \tag{2.12}$$

Combining (2.11) and (2.12), we deduce the required congruence. ■

Theorem 2.5. *If p is a prime such that $p \equiv 3 \pmod 4$ and $1 \leq j \leq p-1$, then for* *all non-negative integers* α *and* n*, we have*

$$p_{2,2}\left(p^{2\alpha+1}(pn+j)+\frac{5\left(p^{2(\alpha+1)}-1\right)}{24}\right)\equiv 0 \pmod 2.$$

Proof. Ahmed and Baruah [1, Theorem 1.7] proved that, if p is a prime such that $p\equiv 3 \pmod 4$ and $1\le j\le p-1$, then for all non-negative integers α and n,

$$\overline{C}_{8,2}\left(p^{2\alpha+1}(pn+j)+\frac{5\left(p^{2(\alpha+1)}-1\right)}{24}\right)\equiv 0 \pmod 2.$$

Now, Theorem 2.3 yields that the same congruence is also satisfied by $p_{2,2}(n)$. ■

In the following theorem, we find congruences satisfied by $p_{3,3}(n)$.

Theorem 2.6. *We have:*

1. *For all* $n\ge 0$,

$$p_{3,3}(16n+11)\equiv p_{3,3}(16n+15)\equiv 0 \pmod 2. \tag{2.13}$$

2. *If* n *cannot be represented as the sum of a pentagonal number and four times a pentagonal number, then*

$$p_{3,3}(16n+3)\equiv 0 \pmod 2. \tag{2.14}$$

3. *If* n *cannot be represented as the sum of two times a pentagonal number and three times a triangular number, then*

$$p_{3,3}(16n+7)\equiv 0 \pmod 2. \tag{2.15}$$

Proof. Taking $t = 3$ in Theorem 2.3, for all $n \geq 0$, we have

$$p_{3,3}(n) \equiv \overline{C}_{12,3}(n) \pmod{2}. \tag{2.16}$$

By Theorems 4.1, 4.2 and 4.3 of Li and Yao [34] we know that, for all $n \geq 0$,

$$\overline{C}_{12,3}(16n+11) \equiv \overline{C}_{12,3}(16n+15) \equiv 0 \pmod{8}; \tag{2.17}$$

$$\overline{C}_{12,3}(16n+3) \equiv 0 \pmod{8}; \tag{2.18}$$

and

$$\overline{C}_{12,3}(16n+7) \equiv 0 \pmod{8}. \tag{2.19}$$

Combining (2.16), (2.17), (2.18) and (2.19) we complete the proof of the theorem. ■

Corollary 2.3.2. *We have:*

1. *Let $p \geq 5$ be a prime with $p \equiv 3 \pmod{4}$. For $\alpha, n \geq 0$, if $p \nmid n$ then*

$$p_{3,3}\left(16p^{2\alpha+1}n + \frac{10p^{2\alpha+2}-1}{3}\right) \equiv 0 \pmod{2}.$$

2. *Let $p \geq 5$ be a prime with $\left(\frac{-2}{p}\right) = -1$. For $\alpha, n \geq 0$, if $p \nmid n$ then*

$$p_{3,3}\left(16p^{2\alpha+1}n + \frac{22p^{2\alpha+2}-1}{3}\right) \equiv 0 \pmod{2}.$$

Proof. By Corollary 4.2 and Corollary 4.3 of Li and Yao [34], for all n, we have

$$\overline{C}_{12,3}\left(16p^{2\alpha+1}n + \frac{10p^{2\alpha+2}-1}{3}\right) \equiv 0 \pmod{8} \tag{2.20}$$

and

$$\overline{C}_{12,3}\left(16p^{2\alpha+1}n+\frac{22p^{2\alpha+2}-1}{3}\right)\equiv 0 \pmod{8}. \tag{2.21}$$

Now, combining (2.16), (2.20) and (2.21) we complete the proof. ■

Pore and Fathima [45] found congruences for $\overline{C}_{20,5}(n)$. Combining their results and Theorem 2.3 for $t=5$, we obtain the following two theorems.

Theorem 2.7. *For all $\alpha, n \geq 0$, we have*

$$p_{5,5}\left(2\cdot 5^{2\alpha+1}n+\frac{31\cdot 5^{2\alpha}-7}{12}\right)\equiv 0 \pmod{2}; \tag{2.22}$$

$$p_{5,5}\left(2\cdot 5^{2\alpha+1}n+\frac{79\cdot 5^{2\alpha}-7}{12}\right)\equiv 0 \pmod{2}; \tag{2.23}$$

$$p_{5,5}\left(2\cdot 5^{2\alpha+2}n+\frac{83\cdot 5^{2\alpha+1}-7}{12}\right)\equiv 0 \pmod{2}; \tag{2.24}$$

$$p_{5,5}\left(2\cdot 5^{2\alpha+2}n+\frac{107\cdot 5^{2\alpha+1}-7}{12}\right)\equiv 0 \pmod{2}. \tag{2.25}$$

Proof. Combining [45, Theorem 1.6] and Theorem 2.3 for $t=5$, we obtain the desired congruences satisfied by $p_{5,5}(n)$. ■

Theorem 2.8. *Let $p \geq 5$ be a prime such that $\left(\frac{-10}{p}\right)=-1$ and $j=1,2,\ldots,p-1$. Then, for all $\alpha, n \geq 0$,*

$$p_{5,5}\left(2p^{2\alpha+1}(pn+j)+7\times\frac{p^{2\alpha+2}-1}{12}\right)\equiv 0 \pmod{2}.$$

Proof. Combining [45, Theorem 1.7] and Theorem 2.3 for $t=5$, we obtain the desired congruences satisfied by $p_{5,5}(n)$. ■

Theorem 2.9. *For all $\alpha, n \geq 0$, we have*

$$p_{7,7}\left(2\cdot 7^{2\alpha+1}n+\frac{(11+12r)\cdot 49^{\alpha}-5}{6}\right)\equiv 0 \pmod{2}, \quad r\in\{3,4,6\} \tag{2.26}$$

and

$$p_{7,7}\left(2\cdot 49^{\alpha+1}n+\frac{(12s+5)\cdot 7^{2\alpha+1}-5}{6}\right)\equiv 0 \pmod 2, \quad s\in\{2,4,5\} \tag{2.27}$$

Proof. In the proof of Theorem 5.1, Li and Yao [34] found two congruences satisfied by $\overline{C}_{28,7}(n)$, for example see [34, (5.22) & (5.23)]. Combining those two congruences and Theorem 2.3 for $t=7$, we complete the proof of the theorem. ∎

Remark 2.3.1. *da Silva and Sellers* [16, Theorem 11] *found relations between* $p_{t,t}(n)$ *and the* t*-core partition functions. They used certain congruences for* t*-core partition functions* (obtained by Radu and Sellers [49] using modular forms) *to find several congruences satisfied by* $p_{t,t}(n)$ *when* $t=5,7,11,13,17,19,23$*. They also proposed to find a fully elementary proof of their congruences listed in their Theorem 11. Putting* $\alpha=0$ *in* (2.22) *and* (2.23)*, we obtain the congruences for* $p_{5,5}(n)$ *listed in* [16, Theorem 11]. *Again, putting* $\alpha=0$ *in* (2.26)*, we obtain the congruences for* $p_{7,7}(n)$ *listed in* [16, Theorem 11].

Theorem 2.10. *Let* $p\geq 5$ *be a prime with* $\left(\frac{-21}{p}\right)=-1$*. We have:*

1. *For* $n,\alpha,\beta\geq 0$,

$$p_{7,7}\left(2\cdot 7^{2\alpha+1}p^{2\beta}n+\frac{(11+12r)\cdot 49^{\alpha}p^{2\beta}-5}{6}\right)\equiv 0 \pmod 2$$

and

$$p_{7,7}\left(2\cdot 49^{\alpha+1}p^{2\beta}n+\frac{(5+12s)\cdot 7^{2\alpha+1}p^{2\beta}-5}{6}\right)\equiv 0 \pmod 2,$$

where $r\in\{3,4,6\}$ *and* $s\in\{2,4,5\}$.

2. *For* $n,\alpha,\beta\geq 0$*, if* $p\nmid n$ *then*

$$p_{7,7}\left(2\cdot 49^{\alpha}p^{2\beta+1}n+\frac{11\cdot 49^{\alpha}p^{2\beta+2}-5}{6}\right)\equiv 0 \pmod 2.$$

Proof. Li and Yao [34, Theorem 5.1 and 5.2] proved that, for all $n \geq 0$,

$$\overline{C}_{28,7}\left(2\cdot 7^{2\alpha+1}p^{2\beta}n+\frac{(11+12r)\cdot 49^{\alpha}p^{2\beta}-5}{6}\right)\equiv 0 \pmod 4; \tag{2.28}$$

$$\overline{C}_{28,7}\left(2\cdot 49^{\alpha+1}p^{2\beta}n+\frac{(5+12s)\cdot 7^{2\alpha+1}p^{2\beta}-5}{6}\right)\equiv 0 \pmod 4 \tag{2.29}$$

and

$$\overline{C}_{28,7}\left(2\cdot 49^{\alpha}p^{2\beta+1}n+\frac{11\cdot 49^{\alpha}p^{2\beta+2}-5}{6}\right)\equiv 0 \pmod 4. \tag{2.30}$$

Combining (2.28), (2.29), (2.30), and Theorem 2.3 with $t = 7$, we find the desired congruences satisfied by $p_{7,7}(n)$. ■

3

Congruences for Mex-Related Partition Functions

3.1 Introduction

In Chapter 2, we found several congruences satisfied by $p_{t,t}(n)$ and $p_{2t,t}(n)$ by using their relations to ordinary partition function and Andrews' singular overpartition function. In this chapter, we use modular forms techniques to establish infinite families of congruences satisfied by $p_{t,t}(n)$.

[1]The contents of this chapter have been published in *Res. number theory* (2021).

3.2 Hecke nilpotency and congruences for $p_{t,t}(n)$

In this section, we use a result of Ono and Taguchi [43] to find infinite families of congruences satisfied by $p_{2^\alpha,2^\alpha}(n)$ and $p_{3\cdot 2^\alpha,3\cdot 2^\alpha}(n)$ for all $\alpha \geq 1$. We find that the eta-quotients associated to the generating functions of $p_{2^\alpha,2^\alpha}(n)$ and $p_{3\cdot 2^\alpha,3\cdot 2^\alpha}(n)$ are modular forms modulo 2 whose levels land in Ono and Taguchi's list. In the following theorems, we prove two infinite families of congruences satisfied by $p_{t,t}(n)$.

Theorem 3.1. *Let α be a positive integer. Then there is an integer $c_1 \geq 0$ such that for every $d_1 \geq 1$ and distinct primes $p_1, \ldots, p_{c_1+d_1}$ coprime to 6, we have*

$$p_{2^\alpha,2^\alpha}\left(\frac{p_1 \cdots p_{c_1+d_1} \cdot n + 1 - 3 \cdot 2^\alpha}{24}\right) \equiv 0 \pmod{2}$$

whenever n is coprime to $p_1, \ldots, p_{c_1+d_1}$.

Theorem 3.2. *Let α be a positive integer. Then there is an integer $c_2 \geq 0$ such that for every $d_2 \geq 1$ and distinct primes $p_1, \ldots, p_{c_2+d_2}$ coprime to 6, we have*

$$p_{3\cdot 2^\alpha,3\cdot 2^\alpha}\left(\frac{p_1 \cdots p_{c_2+d_2} \cdot n + 1 - 9 \cdot 2^\alpha}{24}\right) \equiv 0 \pmod{2}$$

whenever n is coprime to $p_1, \ldots, p_{c_2+d_2}$.

In order to prove Theorem 3.1 and Theorem 3.2, we require to prove two lemmas. Let α be a positive integer. From (2.1), we have

$$\sum_{n=0}^{\infty} p_{2^\alpha,2^\alpha}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{2^\alpha \cdot n(n+1)/2}. \tag{3.1}$$

For a prime p and positive integer j, it is easy to find that

$$(1-q)^{p^j} \equiv (1-q^p)^{p^{j-1}} \pmod{p^j}. \tag{3.2}$$

Employing the Ramanujan's theta function (1.2)

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}$$

into (3.1) and then using (3.2), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{2^\alpha,2^\alpha}(n)q^n &\equiv \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{2^\alpha n(n+1)/2} \pmod 2\\
&= \frac{1}{(q;q)_\infty}\frac{\left(q^{2^{\alpha+1}};q^{2^{\alpha+1}}\right)_\infty^2}{(q^{2^\alpha};q^{2^\alpha})_\infty}\\
&\equiv \frac{1}{(q;q)_\infty}\frac{(q^{2^\alpha};q^{2^\alpha})_\infty^4}{(q^{2^\alpha};q^{2^\alpha})_\infty} \pmod 2\\
&= \frac{(q^{2^\alpha};q^{2^\alpha})_\infty^3}{(q;q)_\infty}\\
&= q^{\frac{1-3\cdot 2^\alpha}{24}}\frac{\eta^3(2^\alpha z)}{\eta(z)}.
\end{aligned} \tag{3.3}$$

Let

$$G_\alpha(z) := \prod_{n=1}^{\infty}\frac{(1-q^{(3\cdot 2^{\alpha+3})n})^2}{(1-q^{(3\cdot 2^{\alpha+4})n})} = \frac{\eta^2(3\cdot 2^{\alpha+3}z)}{\eta(3\cdot 2^{\alpha+4}z)}.$$

Then (3.2) yields

$$G_\alpha^{2^\alpha}(z) = \frac{\eta^{2^{\alpha+1}}(3\cdot 2^{\alpha+3}z)}{\eta^{2^\alpha}(3\cdot 2^{\alpha+4}z)} \equiv 1 \pmod{2^{\alpha+1}}. \tag{3.4}$$

Define $H_\alpha(z)$ by

$$H_\alpha(z) := \left(\frac{\eta^3(3\cdot 2^{\alpha+3}z)}{\eta(24z)}\right)G_\alpha^{2^\alpha}(z) = \frac{\eta^{3+2^{\alpha+1}}(3\cdot 2^{\alpha+3}z)}{\eta(24z)\eta^{2^\alpha}(3\cdot 2^{\alpha+4}z)}.$$

Due to (3.4), we have

$$\begin{aligned} H_\alpha(z) &\equiv \frac{\eta^3(3\cdot 2^{\alpha+3}z)}{\eta(24z)} \pmod{2^{\alpha+1}} \\ &\equiv q^{3\cdot 2^\alpha-1}\left(\frac{(q^{3\cdot 2^{\alpha+3}};q^{3\cdot 2^{\alpha+3}})^3_\infty}{(q^{24};q^{24})_\infty}\right) \pmod{2^{\alpha+1}}. \end{aligned} \tag{3.5}$$

Combining (3.3) and (3.5), we obtain

$$H_\alpha(z) \equiv \sum_{n=0}^{\infty} p_{2^\alpha,2^\alpha}(n)q^{24n+3\cdot 2^\alpha-1} \pmod 2. \tag{3.6}$$

Lemma 3.3. *Let α be a positive integer. Then $H_\alpha(z)\in M_{2^{\alpha-1}+1}(\Gamma_0(N),\chi_1)$, where $N=9\cdot 2^{\alpha+6}$ and the quadratic character χ_1 is given by $\chi_1(\bullet)=(\frac{-2^{(\alpha+2)(2^\alpha+3)}3^{2^\alpha+2}}{\bullet})$.*

Proof. First we calculate the level of the eta-quotient $H_\alpha(z)$ by using Theorem 1.4. The level of $H_\alpha(z)$ is equal to $3\cdot 2^{\alpha+4}\cdot m$, where m is the smallest positive integer such that

$$3\cdot 2^{\alpha+4}\cdot m\left[\frac{3+2^{\alpha+1}}{3\cdot 2^{\alpha+3}}-\frac{1}{24}-\frac{2^\alpha}{3\cdot 2^{\alpha+4}}\right]\equiv 0 \pmod{24}.$$

Equivalently,

$$m[6+2^\alpha]\equiv 0 \pmod{24}.$$

Therefore, $m=12$ and the level of $H_\alpha(z)$ is $9\cdot 2^{\alpha+6}$. By Proposition 1.6, the cusps of $\Gamma_0(9\cdot 2^{\alpha+6})$ are represented by fractions $\frac{c}{d}$ where $d\mid 9\cdot 2^{\alpha+6}$ and $\gcd(c,d)=1$. By Theorem 1.5, we find that $H_\alpha(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$(2^{\alpha+1}+3)\frac{\gcd(d,3\cdot 2^{\alpha+3})^2}{3\cdot 2^{\alpha+3}}-\frac{\gcd(d,24)^2}{24}-2^\alpha\frac{\gcd(d,3\cdot 2^{\alpha+4})^2}{3\cdot 2^{\alpha+4}}\geq 0.$$

Equivalently, if and only if

$$L := (2^{\alpha+2}+6)\frac{\gcd(d,3\cdot 2^{\alpha+3})^2}{\gcd(d,3\cdot 2^{\alpha+4})^2} - 2^{\alpha+1}\frac{\gcd(d,24)^2}{\gcd(d,3\cdot 2^{\alpha+4})^2} - 2^{\alpha} \geq 0.$$

In Table 3.1, we find all the possible values of L. Since $L \geq 0$ for all $d \mid 9\cdot 2^{\alpha+6}$ and

$d \mid 9\cdot 2^{\alpha+6}$	$\frac{\gcd(d,3\cdot 2^{\alpha+3})^2}{\gcd(d,3\cdot 2^{\alpha+4})^2}$	$\frac{\gcd(d,24)^2}{\gcd(d,3\cdot 2^{\alpha+4})^2}$	L
$1,2,3,4,6,8,9,12,18,24,36,72$	1	1	$6+2^{\alpha}$
$2^r,\ 3\cdot 2^r,\ 9\cdot 2^r : 4\leq r\leq \alpha+3$	1	$1/2^{2r-6}$	$6+3\cdot 2^{\alpha}-2^{7+\alpha-2r}$
$2^s, 3\cdot 2^s, 9\cdot 2^s : \alpha+4\leq s\leq \alpha+6$	$1/4$	$1/2^{2\alpha+2}$	$2/3-1/2^{\alpha+1}$

Table 3.1: Possible values of L

$\alpha \geq 1$, therefore $H_\alpha(z)$ is holomorphic at every cusp $\frac{c}{d}$. Using Theorem 1.4, we find that the weight of $H_\alpha(z)$ is $\ell = 2^{\alpha-1}+1$. Also, the associated character for $H_\alpha(z)$ is given by $\chi_1(\bullet) = \left(\frac{-2^{(\alpha+2)(2^\alpha+3)}3^{2^\alpha+2}}{\bullet}\right)$. This completes the proof of the lemma. ■

We next prove our second lemma. Let α be a positive integer. From (2.1), we have

$$\sum_{n=0}^{\infty} p_{3\cdot 2^\alpha,3\cdot 2^\alpha}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3\cdot 2^\alpha\cdot n(n+1)/2}. \tag{3.7}$$

Employing the Ramanujan's theta function (1.2)

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}$$

into (3.7) and then using (3.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{3\cdot 2^\alpha,3\cdot 2^\alpha}(n)q^n &\equiv \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{3\cdot 2^\alpha n(n+1)/2} \pmod 2\\ &= \frac{1}{(q;q)_\infty}\frac{\left(q^{3\cdot 2^{\alpha+1}};q^{3\cdot 2^{\alpha+1}}\right)_\infty^2}{\left(q^{3\cdot 2^\alpha};q^{3\cdot 2^\alpha}\right)_\infty}\\ &\equiv \frac{1}{(q;q)_\infty}\frac{\left(q^{3\cdot 2^\alpha};q^{3\cdot 2^\alpha}\right)_\infty^4}{\left(q^{3\cdot 2^\alpha};q^{3\cdot 2^\alpha}\right)_\infty} \pmod 2\end{aligned}$$

$$\begin{aligned}&\equiv \frac{(q^{3\cdot 2^\alpha};q^{3\cdot 2^\alpha})_\infty^3}{(q;q)_\infty} \pmod 2\\ &\equiv q^{\frac{1-9\cdot 2^\alpha}{24}}\frac{\eta^3(3\cdot 2^\alpha z)}{\eta(z)} \pmod 2. \end{aligned} \tag{3.8}$$

Let

$$R_\alpha(z) := \prod_{n=1}^{\infty}\frac{(1-q^{(9\cdot 2^{\alpha+3})n})^2}{(1-q^{(9\cdot 2^{\alpha+4})n})} = \frac{\eta^2(9\cdot 2^{\alpha+3}z)}{\eta(9\cdot 2^{\alpha+4}z)}.$$

Then (3.2) yields

$$R_\alpha^{2^{\alpha+1}}(z) = \frac{\eta^{2^{\alpha+2}}(9\cdot 2^{\alpha+3}z)}{\eta^{2^{\alpha+1}}(9\cdot 2^{\alpha+4}z)} \equiv 1 \pmod{2^{\alpha+2}}. \tag{3.9}$$

Define $S_\alpha(z)$ by

$$S_\alpha(z) := \left(\frac{\eta^3(9\cdot 2^{\alpha+3}z)}{\eta(24z)}\right)R_\alpha^{2^{\alpha+1}}(z) = \frac{\eta^{3+2^{\alpha+2}}(9\cdot 2^{\alpha+3}z)}{\eta(24z)\eta^{2^{\alpha+1}}(9\cdot 2^{\alpha+4}z)}.$$

Due to (3.9), we have

$$\begin{aligned}S_\alpha(z) &\equiv \frac{\eta^3(9\cdot 2^{\alpha+3}z)}{\eta(24z)} \pmod{2^{\alpha+2}}\\ &\equiv q^{9\cdot 2^\alpha-1}\left(\frac{(q^{9\cdot 2^{\alpha+3}};q^{9\cdot 2^{\alpha+3}})_\infty^3}{(q^{24};q^{24})_\infty}\right) \pmod{2^{\alpha+2}}.\end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10), we obtain

$$S_\alpha(z) \equiv \sum_{n=0}^{\infty} p_{3\cdot 2^\alpha,3\cdot 2^\alpha}(n)q^{24n+9\cdot 2^\alpha-1} \pmod 2. \tag{3.11}$$

Lemma 3.4. *Let α be a positive integer. Then $S_\alpha(z)\in M_{2^{\alpha+1}}(\Gamma_0(N),\chi_2)$, where $N = 9{\cdot}2^{\alpha+6}$ and the quadratic charecter χ_2 is given by $\chi_2(\bullet) = \left(\frac{-2^{(\alpha+2)(2^{\alpha+1}+3)}3^{2^{\alpha+1}+2}}{\bullet}\right)$.*

Proof. The level of $S_\alpha(z)$ is equal to $9\cdot 2^{\alpha+4}\cdot m$, where m is the smallest positive integer such that

$$9 \cdot 2^{\alpha+4} \cdot m \left[\frac{3+2^{\alpha+2}}{9 \cdot 2^{\alpha+3}} - \frac{1}{24} - \frac{2^{\alpha+1}}{9 \cdot 2^{\alpha+4}}\right] \equiv 0 \pmod{24}.$$

Equivalently,

$$6 \cdot m \equiv 0 \pmod{24}.$$

Therefore, $m = 4$ and the level of $S_\alpha(z)$ is $9 \cdot 2^{\alpha+6}$. By Proposition 1.6, the cusps of $\Gamma_0(9 \cdot 2^{\alpha+6})$ are represented by fractions $\frac{c}{d}$ where $d \mid 9 \cdot 2^{\alpha+6}$ and $\gcd(c,d) = 1$. By Theorem 1.5, we find that $S_\alpha(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$(2^{\alpha+2}+3)\frac{\gcd(d, 9 \cdot 2^{\alpha+3})^2}{9 \cdot 2^{\alpha+3}} - \frac{\gcd(d,24)^2}{24} - 2^{\alpha+1}\frac{\gcd(d, 9 \cdot 2^{\alpha+4})^2}{9 \cdot 2^{\alpha+4}} \geq 0.$$

Equivalently, if and only if

$$K := (2^{\alpha+3}+6)\frac{\gcd(d, 9 \cdot 2^{\alpha+3})^2}{\gcd(d, 9 \cdot 2^{\alpha+4})^2} - 3 \cdot 2^{\alpha+1}\frac{\gcd(d,24)^2}{\gcd(d, 9 \cdot 2^{\alpha+4})^2} - 2^{\alpha+1} \geq 0.$$

As shown in the proof of Lemma 3.3, we verify that $K \geq 0$ for all $d \mid 9 \cdot 2^{\alpha+6}$ and $\alpha \geq 1$. Hence, $S_\alpha(z)$ is holomorphic at every cusp $\frac{c}{d}$. Now using Theorem 1.4, we find that the weight of $S_\alpha(z)$ is $\ell = 2^\alpha + 1$, and the associated character for $S_\alpha(z)$ is given by $\chi_2(\bullet) = (\frac{-2^{(\alpha+2)(2^{\alpha+1}+3)}3^{2^{\alpha+1}+2}}{\bullet})$. This completes the proof of the lemma. ■

3.2.1 Proof of Theorem 3.1 and Theorem 3.2

Now, we prove Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1. From (3.6), we have

$$H_\alpha(z) \equiv \sum_{n=0}^{\infty} p_{2^\alpha,2^\alpha}(n) q^{24n+3\cdot 2^\alpha - 1} \pmod{2}.$$

This yields

$$H_\alpha(z) := \sum_{n=0}^{\infty} A_\alpha(n)q^n \equiv \sum_{n=0}^{\infty} p_{2^\alpha,2^\alpha}\left(\frac{n+1-3\cdot 2^\alpha}{24}\right) q^n \pmod 2. \tag{3.12}$$

Note that $H_\alpha(z) \in M_{2^{\alpha-1}+1}\left(\Gamma_0(9\cdot 2^{\alpha+6}), \chi_1\right)$. Using Theorem 1.11 we find that there is an integer $c_1 \geq 0$ such that for any $d_1 \geq 1$,

$$H_\alpha(z)\,|T_{p_1}|\,T_{p_2}|\cdots|T_{p_{c_1+d_1}} \equiv 0 \pmod 2$$

whenever the primes $p_1, \ldots, p_{c_1+d_1}$ are coprime to 6. It follows from the definition of Hecke operators that if $p_1, \ldots, p_{c_1+d_1}$ are distinct primes and if n is coprime to $p_1 \cdots p_{c_1+d_1}$ then

$$A_\alpha\left(p_1 \cdots p_{c_1+d_1} \cdot n\right) \equiv 0 \pmod 2. \tag{3.13}$$

Combining (3.12) and (3.13) we complete the proof of the theorem. ■

Proof of Theorem 3.2. From (3.11), we have

$$S_\alpha(z) \equiv \sum_{n=0}^{\infty} p_{3\cdot 2^\alpha,3\cdot 2^\alpha}(n) q^{24n+9\cdot 2^\alpha-1} \pmod 2.$$

This yields

$$S_\alpha(z) := \sum_{n=0}^{\infty} B_\alpha(n)q^n \equiv \sum_{n=0}^{\infty} p_{3\cdot 2^\alpha,3\cdot 2^\alpha}\left(\frac{n+1-9\cdot 2^\alpha}{24}\right) q^n \pmod 2. \tag{3.14}$$

We now proceed similarly as shown in the proof of Theorem 3.1. Applying Theorem 1.11 to $S_\alpha(z)$ we find that there is an integer $c_2 \geq 0$ such that for any $d_2 \geq 1$ and distinct primes $p_1, \ldots, p_{c_2+d_2}$ coprime to 6,

$$B_\alpha\left(p_1 \cdots p_{c_2+d_2} \cdot n\right) \equiv 0 \pmod 2 \tag{3.15}$$

whenever n is coprime to $p_1, \ldots, p_{c_2+d_2}$. Combining (3.14) and (3.15) we complete the proof of the theorem. ■

4

Divisibility of Singular Overpartitions $\overline{C}_{3\ell,\ell}(n)$

4.1 Introduction

Beginning with the paper [14], Corteel and Lovejoy introduced and developed the theory of overpartitions. An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. Thus the eight overpartitions of 3 are $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$. Since the overlined parts form a partition into distinct parts and the

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non-overlined parts form an ordinary partition, we have the generating function for overpartitions [14, p.1],

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

Mahlburg [37] showed $\overline{p}(n) \equiv 0 \pmod{64}$ for a set of integers of arithmetic density 1 and he conjectured that for all positive integers k, $\overline{p}(n) \equiv 0 \pmod{2^k}$ for almost all integers n. Kim [30] proved this conjecture when $k = 7$. Xiong [63] with the help of some ternary quadratic forms, proved this conjecture when $k = 8$. Recently, Xue and Yao [64] proved that the set of positive integers $S = \{n : \overline{p}(n) \equiv 0 \pmod{2^{11}}\}$ has a positive density.

In order to give overpartition analogues of Rogers-Ramanujan type theorems for the ordinary partition function with restricted successive ranks, Andrews [2] defined the so-called singular overpartitions. Andrews' singular overpartition function $\overline{C}_{k,i}(n)$ counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. For example, $\overline{C}_{3,1}(4) = 10$ with the relevant partitions being

$$4, \overline{4}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1.$$

For $k \geq 3$ and $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by (see, for example [2, Theorem 1])

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}. \tag{4.1}$$

Andrews proved the following Ramanujan-type congruences satisfied by $\overline{C}_{3,1}(n)$: For $n \geq 0$,

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}.$$

Numerous other congruences for Andrews' singular overpartitions are obtained by many authors, see for example [1, 7, 12, 36].

In [12], Chen, Hirschhorn and Sellers studied the parity of $\overline{C}_{k,i}(n)$. They established a complete parity characterizations of $\overline{C}_{3,1}(n)$ and $\overline{C}_{6,2}(n)$. More precisely, they proved that, for all $n \geq 1$, $\overline{C}_{3,1}(n) \equiv 0 \pmod 2$. Similarly, they proved that, for all $n \geq 1$,

$$\overline{C}_{6,2}(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n \text{ is a pentagonal number;} \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Recently, Aricheta [5] studied the parity of $\overline{C}_{3\ell,\ell}(n)$. He proved that

$$\overline{C}_{3\ell,\ell}(n) \equiv b_\ell(n) \pmod 2, \tag{4.2}$$

where $b_\ell(n)$ denotes the number of partitions of n into parts none of which are multiples of ℓ. In [24], Gordon and Ono proved the following density result about $b_\ell(n)$.

Theorem 4.1. [24, Theorem 1] *Let $k = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization of a positive integer k and let $b_k(n)$ denote the number of partitions of n into parts none of which are multiples of k. If $p_i^{a_i} \geq \sqrt{k}$, then for every positive integer j*

$$\lim_{N\to\infty} \frac{\#\left\{0 < n \leq N : b_k(n) \equiv 0 \pmod{p_i^j}\right\}}{N} = 1.$$

In other words the set of those positive integers n for which $b_k(n) \equiv 0 \pmod{p_i^j}$ has arithmetic density one. In fact there exists a positive constant α depending on p_i, j, and k such that there are at most $\mathcal{O}\left(\frac{N}{\log^\alpha N}\right)$ many integers $n \leq N$ for which $b_k(n)$ is not divisible by p_i^j.

Combining Theorem 4.1 and the congruence (4.2), Aricheta found a density result about $\overline{C}_{3\ell,\ell}(n)$ modulo 2 for an infinite family of ℓ. More precisely, represent any positive integer ℓ as $\ell = 2^\alpha m$ where the integer $\alpha \geq 0$ and m is positive odd. Assume further that $2^\alpha \geq m$. Then Aricheta proved that the set $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3\ell,\ell}(n) \equiv 0 \pmod 2\}$ has arithmetic density 1.

It is worth noting that the generating function of $\overline{C}_{3\ell,\ell}(n)$ does not satisfy the conditions of Theorem 1.8 of Cotron et al. Therefore, it is an interesting problem to study the distribution of $\overline{C}_{3\ell,\ell}(n)$ modulo arbitrary powers of primes. In this chapter, we study divisibility of $\overline{C}_{3\ell,\ell}(n)$ modulo arbitrary powers of 2 and 3.

4.2 Distribution of $\overline{C}_{3\ell,\ell}(n)$ modulo arbitrary powers of 2

In this section, we study the arithmetic densities of $\overline{C}_{3\ell,\ell}(n)$ modulo arbitrary powers of 2 when $\ell = p \cdot 2^\alpha$, where p is a prime. Let k be a fixed positive integer. In a recent paper, Barman and Ray [8] prove that $\overline{C}_{3\cdot 2^\alpha,2^\alpha}(n)$ is almost always divisible by 2^k and 3^k for $\alpha = 0$, that is, the sets $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3,1}(n) \equiv 0 \pmod{2^k}\}$ and $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3,1}(n) \equiv 0 \pmod{3^k}\}$ have arithmetic density 1. Here we prove that, for all $\alpha \geq 1$ satisfying $2^\alpha \geq p$, the set $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3p\cdot 2^\alpha,p\cdot 2^\alpha}(n) \equiv 0 \pmod{2^k}\}$ has arithmetic density 1. To be specific, we prove the following result.

Theorem 4.2. *Let k be a fixed positive integer. Then for all $\alpha \geq 1$ and all prime p satisfying $2^\alpha \geq p$, $\overline{C}_{3p\cdot 2^\alpha,p\cdot 2^\alpha}(n)$ is almost always divisible by 2^k, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3p\cdot 2^\alpha,p\cdot 2^\alpha}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

Note that the function $\overline{C}_{6,2}(n)$ is not included in Theorem 4.2. In the following theorem, we prove that $\overline{C}_{6,2}(n)$ is divisible by arbitrary powers of 2 for almost all n.

Theorem 4.3. *Let k be a fixed positive integer. Then, $\overline{C}_{6,2}(n)$ is almost always* *divisible by* 2^k*, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \le X : \overline{C}_{6,2}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

Remark 4.2.1. *Let* k *be a fixed positive integer. If we take* $p = 2$ *in Theorem 4.2, then for all* $\alpha \geq 2$*, we have that* $\overline{C}_{3\cdot 2^\alpha, 2^\alpha}(n)$ *is almost always divisible by* 2^k*. Also, in Theorem 4.3, we proved that* $\overline{C}_{3\cdot 2,2}(n)$ *is almost always divisible by* 2^k*. In [8], the same has been proved for* $\overline{C}_{3,1}(n)$*. Hence,* $\overline{C}_{3\cdot 2^\alpha, 2^\alpha}(n)$ *is almost always divisible by* 2^k *for all* $\alpha \geq 0$*.*

Remark 4.2.2. *Let* $\ell = p\cdot 2^\alpha$*. To find the density of* $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3\ell,\ell}(n) \equiv 0 \pmod{2}\}$*, Aricheta used the condition* $2^\alpha \geq p$ *to apply Theorem 4.1. Interestingly, we also need the condition* $2^\alpha \geq p$ *in Theorem 4.2. However, we need this condition to prove the modularity of certain eta-quotients appearing in the proof of Theorem 4.2.*

4.2.1 Proof of Theorem 4.2 and Theorem 4.3

Here α is a positive integer and p is a prime. By (4.1), the generating function for $\overline{C}_{3p\cdot 2^\alpha, p\cdot 2^\alpha}(n)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{3p\cdot 2^\alpha, p\cdot 2^\alpha}(n) q^n &= \frac{(q^{3p\cdot 2^\alpha}; q^{3p\cdot 2^\alpha})_\infty (-q^{p\cdot 2^\alpha}; q^{3p\cdot 2^\alpha})_\infty (-q^{p\cdot 2^{\alpha+1}}; q^{3p\cdot 2^\alpha})_\infty}{(q;q)_\infty} \\ &= \frac{(q^{2p\cdot 2^\alpha}; q^{2p\cdot 2^\alpha})_\infty (q^{3p\cdot 2^\alpha}; q^{3p\cdot 2^\alpha})_\infty^2}{(q;q)_\infty (q^{p\cdot 2^\alpha}; q^{p\cdot 2^\alpha})_\infty (q^{6p\cdot 2^\alpha}; q^{6p\cdot 2^\alpha})_\infty}. \end{aligned} \tag{4.3}$$

We note that $\eta(3p\cdot 2^{\alpha+3} z) = q^{p\cdot 2^\alpha} \prod_{n=1}^{\infty} (1 - q^{(3p\cdot 2^{\alpha+3})n})$ is a power series of q. Let

$$A_{\alpha,p}(z) := \prod_{n=1}^{\infty} \frac{(1 - q^{(3p\cdot 2^{\alpha+3})n})^2}{(1 - q^{(3p\cdot 2^{\alpha+4})n})} = \frac{\eta^2(3p\cdot 2^{\alpha+3} z)}{\eta(3p\cdot 2^{\alpha+4} z)}.$$

Then using binomial theorem we have

$$A_{\alpha,p}^{2^k}(z) = \frac{\eta^{2^{k+1}}(3p\cdot 2^{\alpha+3}z)}{\eta^{2^k}(3p\cdot 2^{\alpha+4}z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $B_{\alpha,p,k}(z)$ by

$$B_{\alpha,p,k}(z) := \left(\frac{\eta(3p\cdot 2^{\alpha+4}z)\eta(9p\cdot 2^{\alpha+3}z)^2}{\eta(24z)\eta(3p\cdot 2^{\alpha+3}z)\eta(9p\cdot 2^{\alpha+4}z)}\right) A_{\alpha,p}^{2^k}(z). \tag{4.4}$$

Modulo 2^{k+1}, we have

$$\begin{aligned} B_{\alpha,p,k}(z) &\equiv \frac{\eta(3p\cdot 2^{\alpha+4}z)\eta(9p\cdot 2^{\alpha+3}z)^2}{\eta(24z)\eta(3p\cdot 2^{\alpha+3}z)\eta(9p\cdot 2^{\alpha+4}z)} \\ &= q^{p\cdot 2^{\alpha}-1}\left(\frac{(q^{3p\cdot 2^{\alpha+4}};q^{3p\cdot 2^{\alpha+4}})_\infty (q^{9p\cdot 2^{\alpha+3}};q^{9p\cdot 2^{\alpha+3}})_\infty^2}{(q^{24};q^{24})_\infty (q^{3p\cdot 2^{\alpha+3}};q^{3p\cdot 2^{\alpha+3}})_\infty (q^{9p\cdot 2^{\alpha+4}};q^{9p\cdot 2^{\alpha+4}})_\infty}\right). \end{aligned} \tag{4.5}$$

Combining (4.3) and (4.5), we obtain

$$B_{\alpha,p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{3p\cdot 2^{\alpha},p\cdot 2^{\alpha}}(n) q^{24n+p\cdot 2^{\alpha}-1} \pmod{2^{k+1}}. \tag{4.6}$$

In the following lemma, we prove that $B_{\alpha,p,k}(z)$ is a modular form for certain values of α, p and k.

Lemma 4.4. *Let p be a prime and α be a positive integer satisfying $2^\alpha \geq p$. Then $B_{\alpha,p,k}(z) \in M_{2^{k-1}}(\Gamma_0(9p\cdot 2^{\alpha+5}), \chi)$ for all $k \geq 2\alpha$, where the character χ is given by $\chi(\bullet) = (\frac{2^{\alpha+2^k\cdot(\alpha+2)}3^{2^k+1}p^{2^k+1}}{\bullet})$.*

Proof. From (4.4) we have

$$\begin{aligned} B_{\alpha,p,k}(z) &= \left(\frac{\eta(3p\cdot 2^{\alpha+4}z)\eta(9p\cdot 2^{\alpha+3}z)^2}{\eta(24z)\eta(3p\cdot 2^{\alpha+3}z)\eta(9p\cdot 2^{\alpha+4}z)}\right) A_{\alpha,p}^{2^k}(z) \\ &= \frac{\eta(3p\cdot 2^{\alpha+3}z)^{2^{k+1}-1}\eta(9p\cdot 2^{\alpha+3}z)^2}{\eta(3p\cdot 2^{\alpha+4}z)^{2^k-1}\eta(24z)\eta(9p\cdot 2^{\alpha+4}z)}. \end{aligned}$$

We now calculate the level of the eta-quotient $B_{\alpha,p,k}(z)$ by using Theorem 1.4. Thus, the level of $B_{\alpha,p,k}(z)$ is equal to $9p\cdot 2^{\alpha+4}\cdot m$, where m is the smallest positive integer such that

$$9p\cdot 2^{\alpha+4}m\left[\frac{2^{k+1}-1}{3p\cdot 2^{\alpha+3}}+\frac{2}{9p\cdot 2^{\alpha+3}}-\frac{2^k-1}{3p\cdot 2^{\alpha+4}}-\frac{1}{24}-\frac{1}{9p\cdot 2^{\alpha+4}}\right]\equiv 0 \pmod{24}.$$

Equivalently,

$$m\left[9\cdot 2^k-3p\cdot 2^{\alpha+1}\right]\equiv 0 \pmod{24}.$$

Therefore, if $k\geq 2$, then $m=2$ and the level of the eta-quotient is $9p\cdot 2^{\alpha+5}$. By Proposition 1.6, the cusps of $\Gamma_0(9p\cdot 2^{\alpha+5})$ are represented by fractions $\frac{c}{d}$ where $d\mid 9p\cdot 2^{\alpha+5}$ and $\gcd(c,d)=1$. By Theorem 1.5, we find that $B_{\alpha,p,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\begin{aligned}&\frac{\gcd(d,3p\cdot 2^{\alpha+3})^2}{3p\cdot 2^{\alpha+3}}(2^{k+1}-1)+\frac{\gcd(d,3p\cdot 2^{\alpha+4})^2}{3p\cdot 2^{\alpha+4}}(1-2^k)\\&+2\frac{\gcd(d,9p\cdot 2^{\alpha+3})^2}{9p\cdot 2^{\alpha+3}}-\frac{\gcd(d,24)^2}{24}-\frac{\gcd(d,9p\cdot 2^{\alpha+4})^2}{9p\cdot 2^{\alpha+4}}\geq 0.\end{aligned}$$

Equivalently, if and only if

$$L:=6G_1^2\cdot(2^{k+1}-1)+3G_2^2\cdot(1-2^k)+4G_3^2-3p\cdot 2^{\alpha+1}G_4^2-1\geq 0,$$

where

$$G_1=\frac{\gcd(d,3p\cdot 2^{\alpha+3})}{\gcd(d,9p\cdot 2^{\alpha+4})},\ G_2=\frac{\gcd(d,3p\cdot 2^{\alpha+4})}{\gcd(d,9p\cdot 2^{\alpha+4})},$$
$$G_3=\frac{\gcd(d,9p\cdot 2^{\alpha+3})}{\gcd(d,9p\cdot 2^{\alpha+4})},\ G_4=\frac{\gcd(d,24)}{\gcd(d,9p\cdot 2^{\alpha+4})},$$

respectively.

In Table 4.1, we find all the possible values of L. Using the given condition $2^\alpha\geq p$, we now find that $L\geq 0$ for all $d\mid 9p\cdot 2^{\alpha+5}$ and for all $k\geq 2\alpha$. Hence,

$d \mid 9p\cdot 2^{\alpha+5}$	G_1	G_2	G_3	G_4	L
$1,2,3,4,6,8,12,24$	1	1	1	1	$9\cdot 2^k - 3p\cdot 2^{\alpha+1}$
$p,2p,3p,4p,6p,8p,12p,24p$	1	1	1	$1/p$	$9\cdot 2^k - 3\cdot 2^{\alpha+1}/p$
$9,18,36,72$	$1/3$	$1/3$	1	$1/3$	$2^k+8/3-2^{\alpha+1}p/3$
$9p,18p,36p,72p$	$1/3$	$1/3$	1	$1/3p$	$2^k+8/3-2^{\alpha+1}/3p$
$2^{\alpha+4},2^{\alpha+5},3\cdot 2^{\alpha+4},3\cdot 2^{\alpha+5}$	$1/2$	1	$1/2$	$2^{-1-\alpha}$	$1.5-3p\cdot 2^{-1-\alpha}$
$p\cdot 2^{\alpha+4},p\cdot 2^{\alpha+5},3p\cdot 2^{\alpha+4},3p\cdot 2^{\alpha+5}$	$1/2$	1	$1/2$	$2^{-1-\alpha}/p$	$1.5-3\cdot 2^{-1-\alpha}/p$
$2^r,3\cdot 2^r : 4\le r\le \alpha+3$	1	1	1	2^{3-r}	$9\cdot 2^k-3p\cdot 2^{7+\alpha-2r}$
$p\cdot 2^r,3p\cdot 2^r : 4\le r\le \alpha+3$	1	1	1	$2^{3-r}/p$	$9\cdot 2^k-3\cdot 2^{7+\alpha-2r}/p$
$9\cdot 2^r : 4\le r\le \alpha+3$	$1/3$	$1/3$	1	$2^{3-r}/3$	$2^k+8/3-p\cdot 2^{7+\alpha-2r}/3$
$9p\cdot 2^r : 4\le r\le \alpha+3$	$1/3$	$1/3$	1	$2^{3-r}/3p$	$2^k+8/3-2^{7+\alpha-2r}/3p$
$9\cdot 2^{\alpha+4},9\cdot 2^{\alpha+5}$	$1/6$	$1/3$	$1/2$	$2^{-1-\alpha}/3$	$1/6-p\cdot 2^{-1-\alpha}/3$
$9p\cdot 2^{\alpha+4},9p\cdot 2^{\alpha+5}$	$1/6$	$1/3$	$1/2$	$2^{-1-\alpha}/3p$	$1/6-2^{-1-\alpha}/3p$

Table 4.1: Possible values of L

$B_{\alpha,p,k}(z)$ is holomorphic at every cusp $\frac{c}{d}$. Now, from Theorem 1.4 we find that the weight of $B_{\alpha,p,k}(z)$ is $\ell = 2^{k-1}$. Also, the associated character for $B_{\alpha,p,k}(z)$ is given by $\chi(\bullet) = (\frac{2^{\alpha+2^k\cdot(\alpha+2)}3^{2^k+1}p^{2^k+1}}{\bullet})$. Finally, Theorem 1.4 yields that $B_{\alpha,p,k}(z) \in M_{2^{k-1}}(\Gamma_0(9p\cdot 2^{\alpha+5}),\chi)$ for all $k\ge 2\alpha$. This completes the proof of the lemma. ∎

Proof of Theorem 4.2. For a fixed $\alpha \ge 1$, it is enough to prove Theorem 4.2 for all $k\ge 2\alpha$. From Lemma 4.4, we have $B_{\alpha,p,k}(z)\in M_{2^{k-1}}(\Gamma_0(9p\cdot 2^{\alpha+5}),\chi)$ for all $k\ge 2\alpha$ under the condition that $2^\alpha \ge p$. Also, the Fourier coefficients of $B_{\alpha,p,k}(z)$ are all integers. Hence by Theorem 1.7, the Fourier coefficients of $B_{\alpha,p,k}(z)$ are almost always divisible by $m=2^k$. Due to (4.6), $\overline{C}_{3p\cdot 2^\alpha,p\cdot 2^\alpha}(n)$ is also almost always divisible by 2^k. This completes the proof of the theorem. ∎

We now prove Theorem 4.3. Using (4.1), we find that the generating function of $\overline{C}_{6,2}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{C}_{6,2}(n)q^n = \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty^2}{(q;q)_\infty(q^2;q^2)_\infty(q^{12};q^{12})_\infty}. \tag{4.7}$$

Given a prime p, let

$$A_p(z) := \prod_{n=1}^{\infty} \frac{(1-q^{48n})^p}{(1-q^{48pn})} = \frac{\eta^p(48z)}{\eta(48pz)}.$$

Then using binomial theorem we have

$$A_p^{p^k}(z) = \frac{\eta^{p^{k+1}}(48z)}{\eta^{p^k}(48pz)} \equiv 1 \pmod{p^{k+1}}.$$

Define $B_{p,k}(z)$ by

$$B_{p,k}(z) := \left(\frac{\eta(96z)\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)}\right) A_p^{p^k}(z). \tag{4.8}$$

Modulo p^{k+1}, we have

$$B_{p,k}(z) \equiv \frac{\eta(96z)\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)} = q\left(\frac{(q^{96};q^{96})_\infty (q^{144};q^{144})_\infty^2}{(q^{24};q^{24})_\infty (q^{48};q^{48})_\infty (q^{288};q^{288})_\infty}\right). \tag{4.9}$$

Combining (4.7) and (4.9), we obtain

$$B_{p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{6,2}(n) q^{24n+1} \pmod{p^{k+1}}. \tag{4.10}$$

Proof of Theorem 4.3. We put $p=2$ in (4.8) to obtain

$$B_{2,k}(z) = \left(\frac{\eta(96z)\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)}\right) A_2^{2^k}(z) = \frac{\eta(48z)^{2^{k+1}-1}\,\eta(144z)^2}{\eta(96z)^{2^k-1}\,\eta(24z)\eta(288z)}.$$

Now, $B_{2,k}$ is an eta-quotient with $N = 576$. By Proposition 1.6, the cusps of $\Gamma_0(576)$ are represented by fractions $\frac{c}{d}$ where $d \mid 576$ and $\gcd(c,d) = 1$. By Theorem 1.5, we find that $B_{2,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,48)^2}{48}\left(2^{k+1}-1\right) + \frac{\gcd(d,96)^2}{96}\left(1-2^k\right)$$

$$+\frac{\gcd(d,144)^2}{72}-\frac{\gcd(d,24)^2}{24}-\frac{\gcd(d,288)^2}{288}\geq 0.$$

Equivalently, if and only if

$$K:=6\frac{\gcd(d,48)^2}{\gcd(d,288)^2}\left(2^{k+1}-1\right)+3\frac{\gcd(d,96)^2}{\gcd(d,288)^2}\left(1-2^k\right)$$
$$+4\frac{\gcd(d,144)^2}{\gcd(d,288)^2}-12\frac{\gcd(d,24)^2}{\gcd(d,288)^2}-1\geq 0.$$

In Table 4.2, we find all the possible values of K. Since $K\geq 0$ for all $d\mid 576$, we have

$d\mid 576$	$\frac{\gcd(d,48)^2}{\gcd(d,288)^2}$	$\frac{\gcd(d,96)^2}{\gcd(d,288)^2}$	$\frac{\gcd(d,144)^2}{\gcd(d,288)^2}$	$\frac{\gcd(d,24)^2}{\gcd(d,288)^2}$	K
$1,2,3,4,6,8,12,24$	1	1	1	1	$9\cdot 2^k-12$
$16,48$	1	1	1	0.2500	$9\cdot 2^k-3$
$32,64,96,192$	0.2500	1	0.2500	0.0625	0.7500
$9,18,36,72$	0.1111	0.1111	1	0.1111	$2^k+1.33$
144	0.1111	0.1111	1	0.0278	$2^k+2.33$
$288,576$	0.0278	0.1111	0.2500	0.0069	0.0833

Table 4.2: Possible values of K

that $B_{2,k}(z)$ is holomorphic at every cusp $\frac{c}{d}$. Using Theorem 1.4, we find that the weight of $B_{2,k}(z)$ is $\ell=2^{k-1}$. Also, the associated character for $B_{2,k}(z)$ is given by $\chi_1(\bullet)=\left(\frac{2^{3\cdot 2^k+1}3^{2^k+1}}{\bullet}\right)$. Finally, Theorem 1.4 yields that $B_{2,k}(z)\in M_{2^{k-1}}\left(\Gamma_0(576),\chi_1\right)$ for $k\geq 2$. Also, the Fourier coefficients of $B_{2,k}(z)$ are all integers. Hence by Theorem 1.7, the Fourier coefficients of $B_{2,k}(z)$ are almost always divisible by $m=2^k$. Now, using (4.10) we complete the proof of the theorem. ■

4.3 Distribution of $\overline{C}_{3\ell,\ell}(n)$ modulo arbitrary powers of 3

In this section, we study the arithmetic densities of $\overline{C}_{3\ell,\ell}(n)$ modulo arbitrary powers of 3. In the following theorem we find the arithmetic density of the set

$\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n) \equiv 0 \pmod{3^k}\}$ when $\alpha = 0, 1, 2, 3$.

Theorem 4.5. *Let k be a fixed positive integer. Then for each α, $0 \leq \alpha \leq 3$, $\overline{C}_{3\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n)$ is almost always divisible by 3^k, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n) \equiv 0 \pmod{3^k}\right\}}{X} = 1.$$

We further prove that the partition functions $\overline{C}_{6,2}(n)$ and $\overline{C}_{12,4}(n)$ are divisible by 3^k for almost all n.

Theorem 4.6. *Let k be a fixed positive integer. Then for $\ell = 2, 4$, $\overline{C}_{3\ell,\ell}(n)$ is almost always divisible by 3^k, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3\ell,\ell}(n) \equiv 0 \pmod{3^k}\right\}}{X} = 1.$$

It should be noted that only a few results analogous to Theorems 4.2, 4.3, 4.5 and Theorem 4.6 are known for the overpartition function. The generating function of the overpartition function is a modular form of half-integral weight. The coefficients for such functions are poorly understood, and conjectures such as Mahlburg conjecture are considered to be difficult with present techniques. However, our proofs rely on the fact that, for certain values of α and p, the generating functions of $\overline{C}_{3p\cdot 2^\alpha, p\cdot 2^\alpha}(n)$ are congruent to modular forms of integral weights, and this allows us to use the Serre's Theorem 1.7 regarding divisibility of the coefficients modulo powers of 2 and 3 of such functions.

4.3.1 Proof of Theorem 4.5 and Theorem 4.6

By (4.1), the generating function for $\overline{C}_{3\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{3\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n) q^n = \frac{(q^{9\cdot 2^\alpha}; q^{9\cdot 2^\alpha})_\infty (-q^{3\cdot 2^\alpha}; q^{9\cdot 2^\alpha})_\infty (-q^{3\cdot 2^{\alpha+1}}; q^{9\cdot 2^\alpha})_\infty}{(q;q)_\infty}$$

$$= \frac{(q^{3\cdot 2^{\alpha+1}}; q^{3\cdot 2^{\alpha+1}})_\infty (q^{3^2\cdot 2^{\alpha}}; q^{3^2\cdot 2^{\alpha}})^2_\infty}{(q;q)_\infty (q^{3\cdot 2^{\alpha}}; q^{3\cdot 2^{\alpha}})_\infty (q^{3^2\cdot 2^{\alpha+1}}; q^{3^2\cdot 2^{\alpha+1}})_\infty}. \tag{4.11}$$

For $\alpha \geq 0$, let

$$A_\alpha(z) := \prod_{n=1}^{\infty} \frac{(1-q^{(3^2\cdot 2^{\alpha+3})n})^3}{(1-q^{(3^3\cdot 2^{\alpha+3})n})} = \frac{\eta^3(3^2\cdot 2^{\alpha+3}z)}{\eta(3^3\cdot 2^{\alpha+3}z)}.$$

Then using binomial theorem we have

$$A_\alpha^{3^k}(z) = \frac{\eta^{3^{k+1}}(3^2\cdot 2^{\alpha+3}z)}{\eta^{3^k}(3^3\cdot 2^{\alpha+3}z)} \equiv 1 \pmod{3^{k+1}}.$$

Define $D_{\alpha,k}(z)$ by

$$D_{\alpha,k}(z) := \left(\frac{\eta(3^2\cdot 2^{\alpha+4}z)\eta(3^3\cdot 2^{\alpha+3}z)^2}{\eta(24z)\eta(3^2\cdot 2^{\alpha+3}z)\eta(3^3\cdot 2^{\alpha+4}z)}\right) A_\alpha^{3^k}(z). \tag{4.12}$$

Modulo 3^{k+1}, we have

$$\begin{aligned} D_{\alpha,k}(z) &\equiv \frac{\eta(3^2\cdot 2^{\alpha+4}z)\eta(3^3\cdot 2^{\alpha+3}z)^2}{\eta(24z)\eta(3^2\cdot 2^{\alpha+3}z)\eta(3^3\cdot 2^{\alpha+4}z)} \\ &= q^{3\cdot 2^{\alpha}-1}\left(\frac{(q^{3^2\cdot 2^{\alpha+4}}; q^{3^2\cdot 2^{\alpha+4}})_\infty (q^{3^3\cdot 2^{\alpha+3}}; q^{3^3\cdot 2^{\alpha+3}})^2_\infty}{(q^{24};q^{24})_\infty (q^{3^2\cdot 2^{\alpha+3}}; q^{3^2\cdot 2^{\alpha+3}})_\infty (q^{3^3\cdot 2^{\alpha+4}}; q^{3^3\cdot 2^{\alpha+4}})_\infty}\right). \end{aligned} \tag{4.13}$$

Combining (4.11) and (4.13), we obtain

$$D_{\alpha,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{3\cdot 3\cdot 2^{\alpha},3\cdot 2^{\alpha}}(n) q^{24n+3\cdot 2^{\alpha}-1} \pmod{3^{k+1}}. \tag{4.14}$$

We now study the modularity of the eta-quotient $D_{\alpha,k}(z)$ in the following lemma.

Lemma 4.7. *Let $\alpha \geq 0$ be an integer. For a fixed $k \geq 1$, let χ_α denote the character $\left(-\frac{2^{\alpha+3^k\cdot(2\alpha+6)}3^{3^{k+1}+2}}{\bullet}\right)$. We have*

1. $D_{0,k} \in M_{3^k}(\Gamma_0(1728), \chi_0)$ *for all* $k \geq 1$.

2. $D_{1,k} \in M_{3^k}(\Gamma_0(1728), \chi_1)$ *for all* $k \geq 1$.

3. $D_{2,k} \in M_{3^k}\left(\Gamma_0(1728), \chi_2\right)$ *for all* $k \geq 2$.

4. $D_{3,k} \in M_{3^k}\left(\Gamma_0(3456), \chi_3\right)$ *for all* $k \geq 2$.

Furthermore, $D_{\alpha,k}(z)$ *is not a modular form if* $\alpha \geq 4$.

Proof. As before, using Theorem 1.4 we find that the levels of the eta-quotients $D_{0,k}(z)$ and $D_{1,k}(z)$ are equal to 1728 for all $k \geq 1$. Also, if $\alpha \geq 2$ then the level of $D_{\alpha,k}(z)$ is equal to $3^3 \cdot 2^{\alpha+4}$ for all $k \geq 1$. Again, Theorem 1.4 yields that the weight of the eta-quotient $D_{\alpha,k}(z)$ is 3^k and the associated character is $\chi_\alpha(\bullet) = (\frac{-2^{\alpha+3^k\cdot(2\alpha+6)}3^{3^{k+1}+2}}{\bullet})$.

We now first prove that $D_{\alpha,k}(z)$ is not a modular form if $\alpha \geq 4$. If $\alpha \geq 2$ then Theorem 1.5 yields that $D_{\alpha,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\begin{aligned}Q := 6\frac{\gcd(d, 3^2\cdot 2^{\alpha+3})^2}{\gcd(d, 3^3\cdot 2^{\alpha+4})^2}(3^{k+1}-1) + 2\frac{\gcd(d, 3^3\cdot 2^{\alpha+3})^2}{\gcd(d, 3^3\cdot 2^{\alpha+4})^2}(2-3^k)\\ + 3\frac{\gcd(d, 3^2\cdot 2^{\alpha+4})^2}{\gcd(d, 3^3\cdot 2^{\alpha+4})^2} - 3^2\cdot 2^{\alpha+1}\frac{\gcd(d, 24)^2}{\gcd(d, 3^3\cdot 2^{\alpha+4})^2} - 1 \geq 0.\end{aligned}$$

If we take $d = 27$ then we find that

$$Q = \frac{6}{9}(3^{k+1}-1) + 2(2-3^k) + \frac{3}{9} - \frac{2^{\alpha+1}}{9} - 1 = \frac{8}{3} - \frac{2^{\alpha+1}}{9} < 0$$

for all $\alpha \geq 4$. Hence, $D_{\alpha,k}(z)$ is not a modular form if $\alpha \geq 4$.

We next consider the remaining four values of α, namely $\alpha = 0, 1, 2, 3$. We prove that for each of these values of α, $D_{\alpha,k}(z)$ is a modular form. Putting $\alpha = 1$ in (4.12) we have

$$D_{1,k}(z) = \left(\frac{\eta(288z)\eta(432z)^2}{\eta(24z)\eta(144z)\eta(864z)}\right)A_1^{3^k}(z) = \frac{\eta(144z)^{3^{k+1}-1}\eta(288z)}{\eta(432z)^{3^k-2}\eta(24z)\eta(864z)}.$$

Now, $D_{1,k}$ is an eta-quotient with $N = 1728$. As before, by Proposition 1.6, the cusps of $\Gamma_0(1728)$ are represented by fractions $\frac{c}{d}$ where $d \mid 1728$ and $\gcd(c, d) = 1$.

By Theorem 1.5, we find that $D_{1,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,144)^2}{144}(3^{k+1}-1)+\frac{\gcd(d,432)^2}{432}(2-3^k) + \frac{\gcd(d,288)^2}{288}-\frac{\gcd(d,24)^2}{24}-\frac{\gcd(d,864)^2}{864}\geq 0.$$

Equivalently, if and only if

$$P := 6\frac{\gcd(d,144)^2}{\gcd(d,864)^2}(3^{k+1}-1)+2\frac{\gcd(d,432)^2}{\gcd(d,864)^2}(2-3^k) + 3\frac{\gcd(d,288)^2}{\gcd(d,864)^2}-36\frac{\gcd(d,24)^2}{\gcd(d,864)^2}-1\geq 0.$$

In Table 4.3, we find all the possible values of P. Since $P \geq 0$ for all $d \mid 1728$ and

$d \mid 1728$	$\frac{\gcd(d,144)^2}{\gcd(d,864)^2}$	$\frac{\gcd(d,432)^2}{\gcd(d,864)^2}$	$\frac{\gcd(d,288)^2}{\gcd(d,864)^2}$	$\frac{\gcd(d,24)^2}{\gcd(d,864)^2}$	P
$1,2,3,4,6,8,12,24$	1	1	1	1	$16\cdot 3^k-36$
$27,54,108,216$	0.1111	1	0.1111	0.0123	2.2222
9, 18, 36, 72	1	1	1	0.1111	$16\cdot 3^k-4$
32, 64, 96, 192	0.2500	0.2500	1	0.0625	$4\cdot 3^k-0.7500$
16, 48	1	1	1	0.2500	$16\cdot 3^k-9$
288, 576	0.2500	0.2500	1	0.0069	$4\cdot 3^k+1.2500$
144	1	1	1	0.0278	$16\cdot 3^k-1$
432	0.1111	1	0.1111	0.0031	2.5556
864, 1728	0.0278	0.2500	0.1111	0.0007	0.1389

Table 4.3: Possible values of P

for all $k \geq 1$, we have that $D_{1,k}(z)$ is holomorphic at every cusp $\frac{c}{d}$. Hence, Theorem 1.4 yields that $D_{1,k}(z) \in M_{3^k}(\Gamma_0(1728),\chi_1)$ for all $k\geq 1$. This completes the proof of the lemma when $\alpha = 1$.

The proof goes along similar lines when $\alpha = 0,2,3$, and so we omit the details for reasons of brevity. This completes the proof of the lemma. ■

Proof of Theorem 4.5. Throughout the proof we assume that $\alpha \leq 3$. Without loss of generality we assume that $k \geq 2$. From Lemma 4.7, we know that $D_{\alpha,k}(z)$ is a modular form. Also, the Fourier coefficients of $D_{\alpha,k}(z)$ are all integers. Hence by

Theorem 1.7, the Fourier coefficients of $D_{\alpha,k}(z)$ are almost always divisible by 3^k. Now using (4.14) we find that $\overline{C}_{3\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n)$ is almost always divisible by 3^k. This completes the proof of the theorem. ■

We now proof Theorem 4.6. By (4.1), the generating function of $\overline{C}_{12,4}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}(n) q^n = \frac{(q^{12};q^{12})_\infty(-q^4;q^{12})_\infty(-q^8;q^{12})_\infty}{(q;q)_\infty} = \frac{(q^8;q^8)_\infty(q^{12};q^{12})_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty(q^{24};q^{24})_\infty}. \tag{4.15}$$

Given a prime p, let

$$E_p(z) := \prod_{n=1}^{\infty} \frac{(1-q^{32n})^p}{(1-q^{32pn})} = \frac{\eta^p(32z)}{\eta(32pz)}.$$

Then using binomial theorem we have

$$E_p^{p^k}(z) = \frac{\eta^{p^{k+1}}(32z)}{\eta^{p^k}(32pz)} \equiv 1 \pmod{p^{k+1}}.$$

Define $R_{p,k}(z)$ by

$$R_{p,k}(z) := \left(\frac{\eta(64z)\eta(96z)^2}{\eta(8z)\eta(32z)\eta(192z)}\right) E_p^{p^k}(z). \tag{4.16}$$

Modulo p^{k+1}, we have

$$R_{p,k}(z) \equiv \frac{\eta(64z)\eta(96z)^2}{\eta(8z)\eta(32z)\eta(192z)} = q\left(\frac{(q^{64};q^{64})_\infty(q^{96};q^{96})_\infty^2}{(q^8;q^8)_\infty(q^{32};q^{32})_\infty(q^{192};q^{192})_\infty}\right). \tag{4.17}$$

Combining (4.15) and (4.17), we obtain

$$R_{p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{12,4}(n) q^{8n+1} \pmod{p^{k+1}}. \tag{4.18}$$

Proof of Theorem 4.6. We put $p = 3$ in (4.16) to obtain

$$R_{3,k}(z) = \left(\frac{\eta(64z)\eta(96z)^2}{\eta(8z)\eta(32z)\eta(192z)}\right) E_3^{3^k}(z) = \frac{\eta(32z)^{3^{k+1}-1}\,\eta(64z)}{\eta(96z)^{3^k-2}\,\eta(8z)\eta(192z)}. \quad (4.19)$$

Now, $R_{3,k}$ is an eta-quotient with $N = 192$. We know by Proposition 1.6, that the cusps of $\Gamma_0(192)$ are represented by fractions $\frac{c}{d}$ where $d \mid 192$ and $\gcd(c,d) = 1$. Hence, by Theorem 1.5, we find that $R_{3,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,32)^2}{32}\left(3^{k+1}-1\right) + \frac{\gcd(d,64)^2}{64} - \frac{\gcd(d,8)^2}{8} - \frac{\gcd(d,96)^2}{96}\left(3^k-2\right) - \frac{\gcd(d,192)^2}{192} \geq 0.$$

Equivalently, if and only if

$$S := 6\frac{\gcd(d,32)^2}{\gcd(d,192)^2}\left(3^{k+1}-1\right) + 3\frac{\gcd(d,64)^2}{\gcd(d,192)^2} - 24\frac{\gcd(d,8)^2}{\gcd(d,192)^2} - 2\frac{\gcd(d,96)^2}{\gcd(d,192)^2}\left(3^k-2\right) - 1 \geq 0.$$

In Table 4.4, we find all the possible values of S. Since $S \geq 0$ for all $d \mid 192$

$d \mid 192$	$\frac{\gcd(d,96)^2}{\gcd(d,192)^2}$	$\frac{\gcd(d,64)^2}{\gcd(d,192)^2}$	$\frac{\gcd(d,8)^2}{\gcd(d,192)^2}$	$\frac{\gcd(d,32)^2}{\gcd(d,192)^2}$	S
$1,2,4,8$	1	1	1	1	$16\cdot 3^k - 24$
$3,6,12,24$	1	0.1111	0.1111	0.1111	0
16	1	1	0.2500	1	$16\cdot 3^k - 6$
32	1	1	0.0625	1	$16\cdot 3^k - 1.5$
48	1	0.1111	0.0278	0.1111	2
64	0.2500	1	0.0156	0.2500	$4.3^k + 1.12$
96	1	0.1111	0.0069	0.1111	2.5
192	0.2500	0.1111	0.0017	0.0278	0.1250

Table 4.4: Possible values of S

and $k \geq 1$, we have that $R_{3,k}(z)$ is holomorphic at every cusp $\frac{c}{d}$. Using Theorem

1.4, we find that the weight of $R_{3,k}(z)$ is $\ell = 3^k$. Also, the associated character for $R_{3,k}(z)$ is given by $\chi_3(\bullet) = (\frac{-2^{10\cdot 3^k+2}3^{3^{-k}+1}}{\bullet})$. Finally, Theorem 1.4 yields that $R_{3,k}(z) \in M_{3^k}(\Gamma_0(192), \chi_3)$ for $k \geq 1$, and hence using Theorem 1.7, we find that the Fourier coefficients of $R_{3,k}(z)$ are almost always divisible by 3^k. This proves that $\overline{C}_{12,4}(n)$ is divisible by 3^k for almost all n due to (4.18).

We next put $p = 3$ in (4.8) to obtain

$$B_{3,k}(z) = \left(\frac{\eta(96z)\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)}\right) A_3^{3^k}(z) = \frac{\eta(48z)^{3^{k+1}-1}\,\eta(96z)}{\eta(144z)^{3^k-2}\,\eta(24z)\eta(288z)}.$$

As before, by Proposition 1.6, the cusps of $\Gamma_0(576)$ are represented by fractions $\frac{c}{d}$ where $d \mid 576$ and $\gcd(c, d) = 1$. By Theorem 1.5, $B_{3,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\begin{aligned}&\frac{\gcd(d,48)^2}{48}\left(3^{k+1}-1\right) + \frac{\gcd(d,144)^2}{144}\left(2-3^k\right)\\&+\frac{\gcd(d,96)^2}{96} - \frac{\gcd(d,24)^2}{24} - \frac{\gcd(d,288)^2}{288} \geq 0.\end{aligned}$$

Equivalently, if and only if

$$\begin{aligned}Q := {}&6\frac{\gcd(d,48)^2}{\gcd(d,288)^2}\left(3^{k+1}-1\right) + 2\frac{\gcd(d,144)^2}{\gcd(d,288)^2}\left(2-3^k\right)\\&+3\frac{\gcd(d,96)^2}{\gcd(d,288)^2} - 12\frac{\gcd(d,24)^2}{\gcd(d,288)^2} - 1 \geq 0.\end{aligned}$$

Using the Table 4.2, we find that $Q \geq 0$ for all $d \mid 576$. As before, by Theorem 1.4, we obtain that $B_{3,k}(z) \in M_{3^k}(\Gamma_0(576), \chi_2)$, where the character χ_2 is given by $\chi_2(\bullet) = (\frac{-2^{8\cdot 3^k+1}3^{3^k+1}}{\bullet})$. Using the same reasoning and (4.10), we find that $\overline{C}_{6,2}(n)$ is divisible by 3^k for almost all $n \geq 0$. This completes the proof of the theorem. ■

Remark 4.3.1. *Let k be a fixed positive integer. In this chapter, we have found the arithmetic density of the set $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3\ell,\ell}(n) \equiv 0 \pmod{2^k}\}$ for an infinite family of ℓ. But the arithmetic density of the set $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3\ell,\ell}(n) \equiv 0 \pmod{3^k}\}$*

is known only for a few values of ℓ till date. Here, we have found the density of $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3\ell,\ell}(n) \equiv 0 \pmod{3^k}\}$ when $\ell = 2, 3, 4, 6, 12, 24$, and the same is already known for $\ell = 1$, see for example [8]. Since $D_{\alpha,k}$ is not a modular form if $\alpha \geq 4$, therefore, using the method used in this chapter, it won't be possible to find the density of the set $\{n \in \mathbb{Z}_{\geq 0} : \overline{C}_{3\ell,\ell}(n) \equiv 0 \pmod{3^k}\}$ when $\ell = 3 \cdot 2^\alpha$ and $\alpha \geq 4$. It would be interesting to study this problem for an infinite family of ℓ.

4.4 Infinite family of congruences for $\overline{C}_{6,2}(n)$

In this section, using Theorem 1.11 we prove the following congruence for $\overline{C}_{6,2}(n)$ modulo arbitrary powers of 2.

Theorem 4.8. *Let n be a non-negative integer. Then there is an integer $s \geq 0$ such that for every $t \geq 1$ and distinct primes $p_1, \ldots, p_{s+t}$ coprime to 6, we have*

$$\overline{C}_{6,2}\left(\frac{p_1 \cdots p_{s+t} \cdot n - 1}{24}\right) \equiv 0 \pmod{2^t}$$

whenever n is coprime to $p_1, \ldots, p_{s+t}$.

Proof. Taking $p = 2$ in (4.10), we have

$$B_{2,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{6,2}(n) q^{24n+1} \pmod{2^{k+1}}.$$

This yields

$$B_{2,k}(z) := \sum_{n=0}^{\infty} A(n) q^n \equiv \sum_{n=0}^{\infty} \overline{C}_{6,2}\left(\frac{n-1}{24}\right) q^n \pmod{2^{k+1}}. \tag{4.20}$$

Note that $B_{2,k}(z) \in M_{2^{k-1}}(\Gamma_0(N), \chi_1)$, where the level $N = 576 = 9 \cdot 2^6$. Using Theorem 1.11 we find that there is an integer $s \geq 0$ such that for any $t \geq 1$,

$$B_{2,k}(z) \,|T_{p_1}|\, T_{p_2}| \cdots |T_{p_{s+t}} \equiv 0 \pmod{2^t}$$

whenever $p_1, \ldots, p_{s+t}$ are coprime to 6. It follows from the definition of Hecke operators that if $p_1, \ldots, p_{s+t}$ are distinct primes and if n is coprime to $p_1 \cdots p_{s+t}$ then

$$A\left(p_1 \cdots p_{s+t} \cdot n\right) \equiv 0 \pmod{2^t}. \tag{4.21}$$

Combining (4.20) and (4.21), we complete the proof of the theorem. ■

5

Divisibility of Singular Overpartitions $\overline{C}_{4\ell,\ell}(n)$ and $\overline{C}_{6\ell,\ell}(n)$

5.1 Introduction

In this chapter, we study the divisibility properties of Andrews' singular overpartitions $\overline{C}_{4\ell,\ell}(n)$ and $\overline{C}_{6\ell,\ell}(n)$ by arbitrary powers of 2 and 3 for infinitely many values of ℓ. We prove that, for prime $p \geq 3$ and integers $\alpha, \beta \geq 0$ satisfying $3 \cdot 2^{\alpha} \geq p^{\beta}$, $\overline{C}_{4\cdot 2^{\alpha}p^{\beta},2^{\alpha}p^{\beta}}(n)$ is almost always divisible by arbitrary powers of 2. We also prove that $\overline{C}_{6\cdot 3^{\alpha},3^{\alpha}}(n)$ is almost always divisible by arbitrary powers of 3 for all $\alpha \geq 0$.

[1]The contents of this chapter have been published in *J. Number Theory* (2021).

We further employ Theorem 1.11 to find infinite families of congruences modulo arbitrary powers of 2 satisfied by $\overline{C}_{4\cdot 2^\alpha, 2^\alpha}(n)$ and $\overline{C}_{4\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n)$ for all $\alpha \geq 0$. We also note that the generating functions of $\overline{C}_{4\ell,\ell}(n)$ and $\overline{C}_{6\ell,\ell}(n)$ do not satisfy the conditions of Theorem 1.8 of Cotron et al. Therefore, it is an interesting problem to study the distribution of $\overline{C}_{3\ell,\ell}(n)$ modulo arbitrary powers of primes.

5.2 Distribution of $\overline{C}_{4\ell,\ell}(n)$ modulo arbitrary powers of 2

In this section, we study the distribution of $\overline{C}_{4\ell,\ell}(n)$ modulo arbitrary powers of 2 for an infinitely many values of ℓ. In [12], Chen, Hirschhorn, and Sellers studied the parity of $\overline{C}_{4,1}(n)$. They proved that, for all $n \geq 1$,

$$\overline{C}_{4,1}(n) \equiv \begin{cases} 1 \pmod{2}, & \text{if } n = k(3k-1) \text{ for some } k; \\ 0 \pmod{2}, & \text{otherwise.} \end{cases}$$

Recently, Aricheta [5] also explored the parity of $\overline{C}_{4\ell,\ell}(n)$, when $\ell = 2, 3$. He proved that if $(d, \ell) = (3,3), (6,3), (12,3), (24,2), (24,3)$, then there is an integer $c \geq 0$ such that for every $t \geq 0$ and distinct primes $p_1, \ldots, p_{c+t}$ coprime to 6, one has

$$\overline{C}_{4\ell,\ell}\left(\frac{24p_1\cdots p_{c+t}n + d - 3kd}{24d}\right) \equiv 0 \pmod{2}$$

whenever n is coprime to $p_1, \ldots, p_{c+t}$.

In the following theorem, for positive integer $\ell = 2^\alpha m$, where $\alpha \geq 0$ and m is positive odd satisfying $2^\alpha \geq m$, we prove that $\overline{C}_{4\cdot 2^\alpha m, 2^\alpha m}(n)$ is almost always divisible by arbitrary powers of 2.

Theorem 5.1. *Let k be a fixed positive integer. Then, for a positive integer $\ell = 2^\alpha m$,* *where $\alpha \geq 0$ and m is positive odd satisfying $2^\alpha \geq m$, the set*

$$\{n \in \mathbb{N} : \overline{C}_{4\cdot 2^\alpha m, 2^\alpha m}(n) \equiv 0 \pmod{2^k}\}$$

has arithmetic density 1.

5.2.1 Proof of Theorem 5.1

Here $\ell = 2^\alpha m$, where $\alpha \geq 0$ and $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is positive odd such that $\alpha_i \geq 0$ and the distinct primes $p_i \geq 3$. Employing (4.1), we find that the generating function of $\overline{C}_{4\cdot 2^\alpha m, 2^\alpha m}(n)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{4\cdot 2^\alpha m, 2^\alpha m}(n) q^n &= \frac{(q^{2^{\alpha+2}m}; q^{2^{\alpha+2}m})_\infty (-q^{2^\alpha m}; q^{2^{\alpha+2}m})_\infty (-q^{3\cdot 2^\alpha m}; q^{2^{\alpha+2}m})_\infty}{(q;q)_\infty} \\ &= \frac{(q^{2^{\alpha+1}m}; q^{2^{\alpha+1}m})_\infty^2}{(q;q)_\infty (q^{2^\alpha m}; q^{2^\alpha m})_\infty}. \end{aligned} \tag{5.1}$$

Let

$$E_\alpha(z) := \prod_{n=1}^{\infty} \frac{(1 - q^{(3\cdot 2^{\alpha+3}m)n})^2}{(1 - q^{(3\cdot 2^{\alpha+4}m)n})} = \frac{\eta^2(3\cdot 2^{\alpha+3}mz)}{\eta(3\cdot 2^{\alpha+4}mz)}.$$

Then using the binomial theorem we have

$$E_\alpha^{2^k}(z) = \frac{\eta^{2^{k+1}}(3\cdot 2^{\alpha+3}mz)}{\eta^{2^k}(3\cdot 2^{\alpha+4}mz)} \equiv 1 \pmod{2^{k+1}}.$$

Define $F_{\alpha,k}(z)$ by

$$F_{\alpha,k}(z) := \left(\frac{\eta^2(3\cdot 2^{\alpha+4}mz)}{\eta(24z)\eta(3\cdot 2^{\alpha+3}mz)}\right) E_\alpha^{2^k}(z) = \frac{\eta^{2^{k+1}-1}(3\cdot 2^{\alpha+3}mz)}{\eta(24z)\eta^{2^k-2}(3\cdot 2^{\alpha+4}mz)}.$$

Modulo 2^{k+1}, we have

$$F_{\alpha,k}(z) \equiv \frac{\eta^2(3\cdot 2^{\alpha+4}mz)}{\eta(24z)\eta(3\cdot 2^{\alpha+3}mz)}$$
$$= q^{3\cdot 2^{\alpha}m-1}\left(\frac{(q^{3\cdot 2^{\alpha+4}m};q^{3\cdot 2^{\alpha+4}m})_\infty^2}{(q^{24};q^{24})_\infty(q^{3\cdot 2^{\alpha+3}m};q^{3\cdot 2^{\alpha+3}m})_\infty}\right). \tag{5.2}$$

Combining (5.1) and (5.2), we obtain

$$F_{\alpha,k}(z) \equiv \sum_{n=0}^{\infty}\overline{C}_{4\cdot 2^{\alpha}m,2^{\alpha}m}(n)q^{24n+3\cdot 2^{\alpha}m-1} \pmod{2^{k+1}}. \tag{5.3}$$

In the following lemma, we prove that $F_{\alpha,k}(z)$ is a modular form for certain values of α, m and k.

Lemma 5.2. *Let $\ell = 2^{\alpha}m$ be a positive integer, where $\alpha \geq 0$ and $m = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ such that $\alpha_i \geq 0$ and the distinct primes $p_i \geq 3$, is positive odd satisfying $2^{\alpha} \geq m$. We have:*

(1) If $p_i \geq 5$, then $F_{\alpha,k}(z) \in M_{2^{k-1}}(\Gamma_0(N),\chi_1)$ for all $k \geq 2\alpha$, where $N = 9\cdot 2^{\alpha+6}m$ and χ_1 is given by $\chi_1(\bullet) = (\frac{2^{(\alpha+2)(2^k+1)}3^{2^k}m^{(2^k+1)}}{\bullet})$.

(2) If $p_1 = 3$, then $F_{\alpha,k}(z) \in M_{2^{k-1}}(\Gamma_0(N),\chi_2)$ for all $k \geq 2\alpha$, where $N = 2^{\alpha+6}3^{\alpha_1+1}p_2^{\alpha_2}\cdots p_r^{\alpha_r} = 2^{\alpha+6}3^{\alpha_1+1}m'$ and χ_2 is given by

$$\chi_2(\bullet) = (\frac{2^{(\alpha+2)(2^k+1)}3^{2^k(\alpha_1+1)+\alpha_1}m'^{2^k+1}}{\bullet}).$$

Proof. We first prove (1). Here, $p_i \geq 5$. Using Theorem 1.4, we find that the level of the eta-quotient $F_{\alpha,k}(z)$ is equal to $3M\cdot 2^{\alpha+4}m$, where m is the smallest positive integer such that

$$3M\cdot 2^{\alpha+4}m\left[\frac{2^{k+1}-1}{3\cdot 2^{\alpha+3}m} - \frac{1}{24} - \frac{2^k-2}{3\cdot 2^{\alpha+4}m}\right] \equiv 0 \pmod{24}.$$

Equivalently,

$$M\left[3\cdot 2^k - 2^{\alpha+1}\cdot m\right] \equiv 0 \pmod{24}.$$

Therefore, M is equal to 12 and the level of $F_{\alpha,k}(z)$ is $9\cdot 2^{\alpha+6}m$. By Proposition 1.6, the cusps of $\Gamma_0(9\cdot 2^{\alpha+6}m)$ are represented by fractions $\frac{c}{d}$ where $d \mid 9\cdot 2^{\alpha+6}m$ and $\gcd(c,d)=1$. By Theorem 1.5, we find that $F_{\alpha,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$(2^{k+1}-1)\frac{\gcd(d,3\cdot 2^{\alpha+3}m)^2}{3\cdot 2^{\alpha+3}m} - \frac{\gcd(d,24)^2}{24} - (2^k-2)\frac{\gcd(d,3\cdot 2^{\alpha+4}m)^2}{3\cdot 2^{\alpha+4}m} \geq 0.$$

Equivalently, if and only if

$$L := (2^{k+2}-2)G_1 - 2^{\alpha+1}mG_2 - (2^k-2) \geq 0,$$

where $G_1 = \dfrac{\gcd(d,3\cdot 2^{\alpha+3}m)^2}{\gcd(d,3\cdot 2^{\alpha+4}m)^2}$ and $G_2 = \dfrac{\gcd(d,24)^2}{\gcd(d,3\cdot 2^{\alpha+4}m)^2}$.

We now consider the following two cases according to the divisors of $9\cdot 2^{\alpha+6}m$ and find the values of G_1 and G_2. Let d be a divisor of $9\cdot 2^{\alpha+6}m$.

Case (i). For $d = 2^{r_1}3^{r_2}t$, where $0 \leq r_1 \leq \alpha+3, 0 \leq r_2 \leq 2$ and $t|m$, we find that $G_1 = 1$ and $1/t^2 2^{2\alpha} \leq G_2 \leq 1$. Hence,

$$L \geq 2^{k+2} - 2 - 2^{\alpha+1}m - 2^k + 2 \geq 2^{k+1} - 2^{\alpha+1}m.$$

Since $k \geq 2\alpha$ and $2^\alpha \geq m$, we have $L \geq 0$.

Case (ii). For $d = 2^{r_1}3^{r_2}t$, where $\alpha+4 \leq r_1 \leq \alpha+6, 0 \leq r_2 \leq 2$ and $t|m$, we find that $G_1 = 1/4$ and $1/t^2 2^{2\alpha+2} \leq G_2 \leq 2^{2\alpha+2}$. Hence,

$$L \geq \frac{3}{2} - \frac{m}{2^{\alpha+1}} \geq 0.$$

Therefore, $F_{\alpha,k}(z)$ is holomorphic at every cusp $\frac{c}{d}$. Using Theorem 1.4, we find that the weight of $F_{\alpha,k}(z)$ is equal to 2^{k-1}. Also, the associated character for $F_{\alpha,k}(z)$ is given by $\chi_1(\bullet) = (\frac{2^{(\alpha+2)(2^k+1)}3^{2^k}m^{(2^k+1)}}{\bullet})$. This completes the proof of (1).

Again, if $p_1 = 3$, then using Theorem 1.4 we find that the level of the eta-quotient $F_{\alpha,k}(z)$ is equal to $2^{\alpha+6}3^{\alpha_1+1}m'$, where $m' := p_2^{\alpha_2}\cdots p_r^{\alpha_r}$. Note that this level is different from what we would have obtained by putting $p_1 = 3$ in part (1). The rest of the proof goes along similar lines as shown in the proof of part (1) of the lemma. This completes the proof of the lemma. ■

Proof of Theorem 5.1. For a fixed $\alpha \geq 0$, it is enough to prove Theorem 5.1 for all $k \geq 2\alpha$. Now, if $2^\alpha \geq m$, then $F_{\alpha,k}(z) \in M_{2^{k-1}}(\Gamma_0(N), \chi)$ for all $k \geq 2\alpha$, where $N = 9 \cdot 2^{\alpha+6}m$ and $\chi = \chi_1$ when $p_i \geq 5$; and $N = 2^{\alpha+6}3^{\alpha_1+1}m'$ and $\chi = \chi_2$ when $p_1 = 3$. Hence by Theorem 1.7, the Fourier coefficients of $F_{\alpha,k}(z)$ are almost always divisible by $m = 2^k$. Due to (5.3), the same holds for $\overline{C}_{4\cdot 2^\alpha m, 2^\alpha m}(n)$. This completes the proof of the theorem. ■

5.3 Distribution of $\overline{C}_{6\ell,\ell}(n)$ modulo arbitrary powers of 3

We next study the divisibility properties of $\overline{C}_{6\ell,\ell}(n)$ by arbitrary powers of 3 for an infinite family of ℓ, namely $\ell = 3^\alpha m$ where $\alpha \geq 0$ and m is positive integer such that $3 \nmid m$. In [12], Chen, Hirschhorn and Sellers studied the divisibility of $\overline{C}_{6,1}(n)$ and $\overline{C}_{6,2}(n)$ by 3. They proved that, if n cannot be represented as the sum of a pentagonal number and twice a triangular number, or if n cannot be represented as the sum of a triangular number and four times a pentagonal number, then $\overline{C}_{6,1}(n) \equiv 0 \pmod 3$. They also proved that, if n cannot be represented as the sum of a pentagonal number and a square, or if n cannot be written as the sum of a pentagonal number and twice a square, then $\overline{C}_{6,2}(n) \equiv 0 \pmod 3$. In the following theorem we prove that $\overline{C}_{6\cdot 3^\alpha m, 3^\alpha m}(n)$ is almost always divisible by arbitrary powers of 3 for all $\alpha \geq 0$.

Theorem 5.3. *Let k be a fixed positive integer. Then, for all $\ell = 3^\alpha m$, where $\alpha \geq 0$ and m is positive integer such that $3 \nmid m$, the set*

$$\{n \in \mathbb{N} : \overline{C}_{6\cdot 3^\alpha m, 3^\alpha m}(n) \equiv 0 \pmod{3^k}\}$$

has arithmetic density 1.

5.3.1 Proof of Theorem 5.3

We first find the generating function of $\overline{C}_{6\ell,\ell}(n)$. We use the notation $f_\ell := (q^\ell; q^\ell) = \prod_{j=1}^{\infty}(1 - q^{j\ell})$ throughout this section.

Lemma 5.4. *Let ℓ be a positive integer. Then*

$$\sum_{n=0}^{\infty} \overline{C}_{6\ell,\ell}(n) q^n = \frac{f_{2\ell}^2 f_{3\ell} f_{12\ell}}{f_1 f_\ell f_{4\ell} f_{6\ell}}.$$

Proof. From (4.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{C}_{6\ell,\ell}(n) q^n &= \frac{(q^{6\ell}; q^{6\ell})_\infty (-q^\ell; q^{6\ell})_\infty (-q^{5\ell}; q^{6\ell})_\infty}{(q; q)_\infty} \\
&= \frac{(q^{6\ell}; q^{6\ell})_\infty (-q^\ell; q^{6\ell})_\infty (-q^{3\ell}; q^{6\ell})_\infty (-q^{5\ell}; q^{6\ell})_\infty}{(q; q)_\infty (-q^{3\ell}; q^{6\ell})_\infty} \\
&= \frac{(q^{6\ell}; q^{6\ell})_\infty (-q^\ell; q^{2\ell})_\infty}{(q; q)_\infty (-q^{3\ell}; q^{6\ell})_\infty} \\
&= \frac{(q^{6\ell}; q^{6\ell})_\infty (q^{3\ell}; q^{6\ell})_\infty (q^{2\ell}; q^{4\ell})_\infty}{(q; q)_\infty (q^{6\ell}; q^{12\ell})_\infty (q^\ell; q^{2\ell})_\infty} \\
&= \frac{(q^{2\ell}; q^{2\ell})_\infty^2 (q^{3\ell}; q^{3\ell})_\infty (q^{12\ell}; q^{12\ell})_\infty}{(q; q)_\infty (q^\ell; q^\ell)_\infty (q^{4\ell}; q^{4\ell})_\infty (q^{6\ell}; q^{6\ell})_\infty} \\
&= \frac{f_{2\ell}^2 f_{3\ell} f_{12\ell}}{f_1 f_\ell f_{4\ell} f_{6\ell}}.
\end{aligned}$$

This completes the proof of the lemma. ■

Proof of Theorem 5.3. Let k be a fixed positive integer. We have $\ell = 3^\alpha m$, where $\alpha \geq 0$ and m is positive integer such that $3 \nmid m$. Let

$$G_\alpha(z) := \prod_{n=1}^{\infty} \frac{(1-q^{(2^5\cdot 3^{\alpha+1}m)n})^3}{(1-q^{(2^5\cdot 3^{\alpha+2}m)n})} = \frac{\eta^3(2^5\cdot 3^{\alpha+1}mz)}{\eta(2^5\cdot 3^{\alpha+2}mz)}.$$

Then using the binomial theorem we have

$$G_\alpha^{3^k}(z) = \frac{\eta^{3^{k+1}}(2^5\cdot 3^{\alpha+1}mz)}{\eta^{3^k}(2^5\cdot 3^{\alpha+2}mz)} \equiv 1 \pmod{3^{k+1}}.$$

Define $H_{\alpha,k}(z)$ by

$$\begin{aligned} H_{\alpha,k}(z) :&= \left(\frac{\eta^2(2^4\cdot 3^{\alpha+1}mz)\eta(2^3\cdot 3^{\alpha+2}mz)\eta(2^5\cdot 3^{\alpha+2}mz)}{\eta(24z)\eta(2^3\cdot 3^{\alpha+1}mz)\eta(2^4\cdot 3^{\alpha+2}mz)\eta(2^5\cdot 3^{\alpha+1}mz)}\right) G_\alpha^{3^k}(z) \\ &= \frac{\eta^2(2^4\cdot 3^{\alpha+1}mz)\eta(2^3\cdot 3^{\alpha+2}mz)\eta^{3^{k+1}-1}(2^5\cdot 3^{\alpha+1}mz)}{\eta(24z)\eta(2^3\cdot 3^{\alpha+1}mz)\eta(2^4\cdot 3^{\alpha+2}mz)\eta^{3^k-1}(2^5\cdot 3^{\alpha+2}mz)}. \end{aligned}$$

Modulo 3^{k+1}, we have

$$\begin{aligned} H_{\alpha,k}(z) &\equiv \frac{\eta^2(2^4\cdot 3^{\alpha+1}mz)\eta(2^3\cdot 3^{\alpha+2}mz)\eta(2^5\cdot 3^{\alpha+2}mz)}{\eta(24z)\eta(2^3\cdot 3^{\alpha+1}mz)\eta(2^4\cdot 3^{\alpha+2}mz)\eta(2^5\cdot 3^{\alpha+1}mz)} \\ &= q^{8\cdot 3^\alpha m-1} \frac{f^2_{2^4\cdot 3^{\alpha+1}m} f_{2^3\cdot 3^{\alpha+2}m} f_{2^5\cdot 3^{\alpha+2}m}}{f_{24} f_{2^3\cdot 3^{\alpha+1}m} f_{2^4\cdot 3^{\alpha+2}m} f_{2^5\cdot 3^{\alpha+1}m}}. \end{aligned} \tag{5.4}$$

Employing Lemma 5.4 with $k = 3^\alpha m$, and then combining with (5.4), we obtain

$$H_{\alpha,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{6\cdot 3^\alpha m, 3^\alpha m}(n) q^{24n+8\cdot 3^\alpha m-1} \pmod{3^{k+1}}. \tag{5.5}$$

We next prove that $H_{\alpha,k}(z)$ is a modular form for all $k \geq 2\alpha+1$ and $3^\alpha \geq 4m$. First we calculate the level of the eta-quotient $H_{\alpha,k}(z)$ by using Theorem 1.4. The level of $H_{\alpha,k}(z)$ is equal to $2^5 3^{\alpha+2} mM$, where M is the smallest positive integer such that modulo 24,

$$2^5 3^{\alpha+2} mM \left[\frac{2}{2^4 3^{\alpha+1} m} + \frac{1}{2^3 3^{\alpha+2} m} + \frac{3^{j+1}-1}{2^5 3^{\alpha+1} m} - \frac{1}{24} - \frac{1}{2^3 3^{\alpha+1} m} - \frac{1}{2^4 3^{\alpha+2} m} - \frac{3^j - 1}{2^5 3^{\alpha+2} m}\right] \equiv 0.$$

Equivalently,

$$M\left[8\cdot 3^k - 4\cdot 3^{\alpha+1}\right] \equiv 0 \pmod{24}.$$

Therefore, $M = 2$ and the level of $H_{\alpha,k}(z)$ is $2^6 3^{\alpha+2} m$. We know by Proposition 1.6, that the cusps of $\Gamma_0(2^6 3^{\alpha+2} m)$ are represented by fractions $\frac{c}{d}$, where $d \mid 2^6 3^{\alpha+2} m$ and $\gcd(c,d) = 1$. By Theorem 1.5, we find that $H_{\alpha,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$(3^{k+1}-1)\frac{\gcd(d, 2^5 3^{\alpha+1} m)^2}{2^5 3^{\alpha+1} m} + 2\frac{\gcd(d, 2^4 3^{\alpha+1} m)^2}{2^4 3^{\alpha+1} m} + \frac{\gcd(d, 2^3 3^{\alpha+2} m)^2}{2^3 3^{\alpha+2} m} - \frac{\gcd(d,24)^2}{24} - \frac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{2^3 3^{\alpha+1} m} - \frac{\gcd(d, 2^4 3^{\alpha+2} m)^2}{2^4 3^{\alpha+2} m} - (3^k-1)\frac{\gcd(d, 2^5 3^{\alpha+2} m)^2}{2^5 3^{\alpha+2} m} \geq 0.$$

Equivalently, if and only if

$$L := (3^{k+2}-3)G_1 + 12G_2 + 4G_3 - 4\cdot 3^{\alpha+1} m G_4 - 12G_5 - 2G_6 - (3^k - 1) \geq 0,$$

where

$$G_1 = \frac{\gcd(d, 2^5 3^{\alpha+1} m)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}, G_2 = \frac{\gcd(d, 2^4 3^{\alpha+1} m)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2},$$
$$G_3 = \frac{\gcd(d, 2^3 3^{\alpha+2} m)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}, G_4 = \frac{\gcd(d, 24)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2},$$
$$G_5 = \frac{\gcd(d, 2^3 3^{\alpha+1} m)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2}, G_6 = \frac{\gcd(d, 2^4 3^{\alpha+2} m)^2}{\gcd(d, 2^5 3^{\alpha+2} m)^2},$$

respectively.

We now consider the following three cases according to the divisors of $2^6 3^{\alpha+2}m$ and find the values of G_i for $i = 1, 2, \ldots, 6$. Let d be a divisor of $N = 2^6 3^{\alpha+2}m$.

Case (i). For $d = 2^{r_1}3^{r_2}t$, where $0 \leq r_1 \leq 6, 0 \leq r_2 \leq \alpha+1$ and $t|m$, we find that $G_1 = 1$, $1/4 \leq G_2 = G_6 \leq 1$, $1/16 \leq G_3 = G_5 \leq 1$ and $1/2^4 \cdot 3^{2\alpha}t^2 \leq G_4 \leq 1$. Hence,

$$L \geq 3^{k+2} - 3 + 3 + \frac{4}{16} - 4 \cdot 3^{\alpha+1}m - 12 - 2 - 3^k + 1 = 8 \cdot 3^k - 4 \cdot 3^{\alpha+1}m - \frac{51}{4}.$$

Since $k \geq 2\alpha + 1$ and $3^\alpha \geq 4m$, we have $L \geq 0$.

Case (ii). For $d = 2^{r_1}3^{r_2}t$, where $0 \leq r_1 \leq 3, r_2 = \alpha+2$ and $t|m$, we find that $G_1 = G_2 = G_5 = 1/9$, $G_3 = G_6 = 1$ and $1/3^{2\alpha+2}t^2 \leq G_4 \leq 1/3^{2\alpha+2}$. Hence,

$$L \geq 3^k - \frac{1}{3} + \frac{12}{9} + 4 - \frac{4 \cdot 3^{\alpha+1}m}{3^{2\alpha+2}} - \frac{12}{9} - 2 - 3^k + 1 = \frac{8}{3} - \frac{4m}{3^{\alpha+1}} \geq 0.$$

Case (iii). For $d = 2^{r_1}3^{r_2}t$, where $4 \leq r_1 \leq 6, r_2 = \alpha+2$ and $t|m$, we find that $G_1 = 1/9$, $G_2 = 1/2^2 \cdot 3^2$, $1/16 \leq G_3 \leq 1/4$, $1/2^4 \cdot 3^2 \leq G_5 \leq 1/2^2 \cdot 3^2$, $G_6 = 1/4$ and $1/2^4 \cdot 3^{2\alpha+2}t^2 \leq G_4 \leq 1/2^2 \cdot 3^{2\alpha+2}t^2$. Hence,

$$L \geq 3^k - \frac{1}{3} + \frac{12}{36} + \frac{4}{16} - \frac{4 \cdot 3^{\alpha+1}m}{4 \cdot 3^{2\alpha+2}t^2} - \frac{12}{36} - \frac{2}{4} - 3^k + 1 = \frac{5}{12} - \frac{m}{3^{\alpha+1}t^2} \geq 0.$$

Hence, $H_{\alpha,k}(z)$ is holomorphic at every cusp $\frac{c}{d}$ for all $k \geq 2\alpha+1$ and $3^\alpha \geq 4m$. Using Theorem 1.4, we find that the weight of $H_{\alpha,k}(z)$ is equal to 3^k. Also, the associated character for $H_{\alpha,k}(z)$ is given by $\chi_5(\bullet) = \left(\frac{-2^{(1+10\cdot 3^k)}3^{(\alpha+1)+(2\alpha+1)3^k}m^{2\cdot 3^k+1}}{\bullet}\right)$. This proves that $H_{\alpha,k}(z) \in M_{3^k}\left(\Gamma_0(2^6 3^{\alpha+2}m), \chi_5\right)$ for all $k \geq 2\alpha+1$ and $3^\alpha \geq 4m$.

For a fixed $\alpha \geq 0$, it is enough to prove Theorem 5.3 for all $k \geq 2\alpha+1$. Hence by Theorem 1.7, the Fourier coefficients of $H_{\alpha,k}(z)$ are almost always divisible by $m = 3^k$. Due to (5.5), the same holds for $\overline{C}_{6\cdot 3^\alpha, 3^\alpha}(n)$. This completes the proof of the theorem. ∎

5.4 Congruences for $\overline{C}_{4\cdot 2^\alpha,2^\alpha}(n)$ and $\overline{C}_{4\cdot 3\cdot 2^\alpha,3\cdot 2^\alpha}(n)$

We find that the eta-quotients associated to the generating functions of $\overline{C}_{4\cdot 2^\alpha,2^\alpha}(n)$ and $\overline{C}_{4\cdot 3\cdot 2^\alpha,3\cdot 2^\alpha}(n)$ are modular forms whose levels land in Ono and Taguchi's list. More precisely, we have following theorems.

Theorem 5.5. *Let α be a non-negative integer. Then there is an integer $c \geq 0$ such that for every $d \geq 1$ and distinct primes $p_1, \ldots, p_{c+d}$ coprime to 6, we have*

$$\overline{C}_{4\cdot 2^\alpha,2^\alpha}\left(\frac{p_1 \cdots p_{c+d} \cdot n + 1 - 3\cdot 2^\alpha}{24}\right) \equiv 0 \pmod{2^d}$$

whenever n is coprime to $p_1, \ldots, p_{c+d}$.

Theorem 5.6. *Let $\alpha \geq 2$ be a integer. Then there is an integer $s \geq 0$ such that for every $t \geq 1$ and distinct primes $p_1, \ldots, p_{s+t}$ coprime to 6, we have*

$$\overline{C}_{4\cdot 3\cdot 2^\alpha,3\cdot 2^\alpha}\left(\frac{p_1 \cdots p_{s+t} \cdot n + 1 - 9\cdot 2^\alpha}{24}\right) \equiv 0 \pmod{2^t}$$

whenever n is coprime to $p_1, \ldots, p_{s+t}$.

5.4.1 Proof of Theorem 5.5 and Theorem 5.6

Proof of Theorem 5.5. Let k be a fixed positive integer. Taking $m = 1$ in (5.3), for all $\alpha \geq 0$, we have

$$R_\alpha(z) := F_{\alpha,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{4\cdot 2^\alpha,2^\alpha}(n) q^{24n+3\cdot 2^\alpha - 1} \pmod{2^{k+1}}.$$

This yields

$$R_\alpha(z) := \sum_{n=0}^{\infty} A_\alpha(n) q^n \equiv \sum_{n=0}^{\infty} \overline{C}_{4\cdot 2^\alpha,2^\alpha}\left(\frac{n+1-3\cdot 2^\alpha}{24}\right) q^n \pmod{2^{k+1}}. \tag{5.6}$$

Note that $R_\alpha(z) \in M_{2^{k-1}}\left(\Gamma_0(9\cdot 2^{\alpha+6}), \chi_3\right)$ for all $k \geq 2\alpha$, where χ_3 is the associated character (which is χ_1 evaluated at $m = 1$). Using Theorem 1.11, we find that there is an integer $c \geq 0$ such that for any $d \geq 1$,

$$R_\alpha(z) \,|T_{p_1}|\, T_{p_2}| \cdots |T_{p_{c+d}} \equiv 0 \pmod{2^d}$$

whenever $p_1, \ldots, p_{c+d}$ are coprime to 6. It follows from the definition of Hecke operators that if $p_1, \ldots, p_{c+d}$ are distinct primes and if n is coprime to $p_1 \cdots p_{c+d}$ then

$$A_\alpha\left(p_1 \cdots p_{c+d} \cdot n\right) \equiv 0 \pmod{2^d}. \tag{5.7}$$

Combining (5.6) and (5.7), we complete the proof of the theorem. ■

Proof of Theorem 5.6. Let k be a fixed positive integer. Taking $m = 3$ in (5.3), for all $\alpha \geq 0$, we have

$$S_\alpha(z) := F_{\alpha,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{4\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}(n) q^{24n+9\cdot 2^\alpha - 1} \pmod{2^{k+1}}.$$

This yields

$$S_\alpha(z) := \sum_{n=0}^{\infty} B_\alpha(n) q^n \equiv \sum_{n=0}^{\infty} \overline{C}_{4\cdot 3\cdot 2^\alpha, 3\cdot 2^\alpha}\left(\frac{n+1-9\cdot 2^\alpha}{24}\right) q^n \pmod{2^{k+1}}. \tag{5.8}$$

Note that $S_\alpha(z) \in M_{2^{k-1}}\left(\Gamma_0(9\cdot 2^{\alpha+6}), \chi_4\right)$ for all $k \geq 2\alpha$ and $\alpha \geq 2$, where χ_4 is the associated character. We now proceed along similar lines as shown in the proof of Theorem 5.5. Applying Theorem 1.11 to $S_\alpha(z)$ we find that there is an integer $s \geq 0$ such that for any $t \geq 1$ and distinct primes $p_1, \ldots, p_{s+t}$ coprime to 6

$$B_\alpha\left(p_1 \cdots p_{s+t} \cdot n\right) \equiv 0 \pmod{2^t}. \tag{5.9}$$

Combining (5.8) and (5.9), we complete the proof of the theorem. ■

6

Congruences and Self-Similarity Results on t-Regular Partitions

6.1 Introduction

A t-regular partition of a positive integer n is a partition of n such that none of its part is divisible by t. Let $b_t(n)$ denote the number of t-regular partitions of n. The generating function of $b_t(n)$ is given by

$$\sum_{n=0}^{\infty} b_t(n)q^n = \frac{f_t}{f_1}, \tag{6.1}$$

[1]The contents of this chapter are under review.

where $f_k := (q^k;q^k)_\infty = \prod_{j=1}^{\infty}(1-q^{jk})$ and k is a positive integer.

In a very recent paper [29], Keith and Zanello studied t-regular partition for certain values of t. They proved various congruences for $b_t(n)$ modulo 2 for certain values of $t \le 28$, and posed several open questions.

6.2 Proof of a conjecture of Keith and Zanello on $b_3(n)$

One of the congruences Keith and Zanello proved for $b_3(n)$ is the following:

$$\sum_{n=0}^{\infty} b_3(26n+14)q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{13n} \pmod 2. \tag{6.2}$$

More generally, they conjectured that:

Conjecture 6.1. [29, Conjecture 6] *For any prime $p>3$, let $\alpha \equiv -24^{-1} \pmod{p^2}$, $0<\alpha<p^2$. It holds for a positive proportion of primes p that*

$$\sum_{n=0}^{\infty} b_3(2(pn+\alpha))q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{pn} \pmod 2. \tag{6.3}$$

The congruence (6.2) is a specific case of (6.3) corresponding to $p=13$. In the following theorem, we confirm that Conjecture 6.1 is true.

Theorem 6.2. *Conjecture 6.1 is true.*

Proof. We first recall the following even-odd dissection of the 3-regular partitions [29, (6)]:

$$\sum_{n=0}^{\infty} b_3(n)q^n = \frac{f_3}{f_1} \equiv \frac{f_1^8}{f_3^2} + q\frac{f_3^{10}}{f_1^4} \pmod 2.$$

Extracting the terms with even powers of q, we obtain

$$\sum_{n=0}^{\infty} b_3(2n)q^n \equiv \frac{f_1^4}{f_3} \pmod 2. \tag{6.4}$$

Let

$$A(z) := \prod_{n=1}^{\infty} \frac{(1-q^{24n})^2}{(1-q^{48n})} = \frac{\eta^2(24z)}{\eta(48z)}.$$

Then using the binomial theorem we have

$$A(z) = \frac{\eta^2(24z)}{\eta(48z)} \equiv 1 \pmod 2.$$

Define $B(z)$ by

$$B(z) := \left(\frac{\eta^4(24z)}{\eta(72z)}\right) A(z) = \frac{\eta^6(24z)}{\eta(72z)\eta(48z)}.$$

Modulo 2, we have

$$B(z) \equiv \frac{\eta^4(24z)}{\eta(72z)} = q\frac{(q^{24};q^{24})_\infty^4}{(q^{72};q^{72})_\infty}. \tag{6.5}$$

Combining (6.4) and (6.5), we obtain

$$B(z) \equiv \sum_{n=0}^{\infty} b_3(2n)q^{24n+1} \pmod 2. \tag{6.6}$$

Now, $B(z)$ is an eta-quotient with $N = 3456$. We next prove that $B(z)$ is a cusp form. We know by Proposition 1.6, that the cusps of $\Gamma_0(3456)$ are represented by fractions $\frac{c}{d}$, where $d \mid 3456$ and $\gcd(c,d) = 1$. By Theorem 1.5, we find that $B(z)$ vanishes at a cusp $\frac{c}{d}$ if and only if

$$L := 12\frac{\gcd(d,24)^2}{\gcd(d,48)^2} - \frac{2}{3}\frac{\gcd(d,72)^2}{\gcd(d,48)^2} - 1 > 0.$$

We now consider the following four cases according to the divisors of 3456 and find the values of $G_1 := \frac{\gcd(d,24)^2}{\gcd(d,48)^2}$ and $G_2 := \frac{\gcd(d,72)^2}{\gcd(d,48)^2}$. Let d be a divisor of $N = 3456$.
Case (i). For $d = 2^{r_1}3^{r_2}$, where $0 \le r_1 \le 3$ and $0 \le r_2 \le 1$, we find that $G_1 = G_2 = 1$. Hence, $L > 0$.
Case (ii). For $d = 2^{r_1}3^{r_2}$, where $0 \le r_1 \le 3$ and $2 \le r_2 \le 3$, we find that $G_1 = 1$ and $G_2 = 9$. Hence, $L > 0$.
Case (iii). For $d = 2^{r_1}3^{r_2}$, where $4 \le r_1 \le 7$ and $0 \le r_2 \le 1$, we find that $G_1 = G_2 = 1/4$. Hence, $L > 0$.
Case (iv). For $d = 2^{r_1}3^{r_2}$, where $4 \le r_1 \le 7$ and $2 \le r_2 \le 3$, we find that $G_1 = 1/4$ and $G_2 = 9/4$. Hence, $L > 0$.

Thus, $B(z)$ vanishes at every cusp $\frac{c}{d}$. Using Theorem 1.4, we find that the weight of $B(z)$ is equal to 2. Also, the associated character for $B(z)$ is given by $\chi_1(\bullet) = (\frac{2^{11}3^3}{\bullet})$. This proves that $B(z) \in S_2(\Gamma_0(3456), \chi_1)$. Also, the Fourier coefficients of $B(z)$ are all integers. Hence by Proposition 1.3, a positive proportion of the primes $p \equiv -1 \pmod{6912}$ have the property that

$$B(z) \mid T_p \equiv 0 \pmod{2}. \tag{6.7}$$

Let $B(z) := \sum_{n=1}^{\infty} a(n)q^n$. Then, (6.6) yields

$$\sum_{n=1}^{\infty} b_3\left(\frac{2(n-1)}{24}\right) q^n \equiv \sum_{n=1}^{\infty} a(n)q^n \pmod{2}. \tag{6.8}$$

Now, from (6.7) we obtain

$$B(z) \mid T_p = \sum_{n=1}^{\infty}(a(pn) + p\chi_1(p)a(n/p))q^n \equiv 0 \pmod{2}$$

which yields

$$\sum_{n=1}^{\infty} a(pn)q^n \equiv \sum_{n=1}^{\infty} a(n/p)q^n \pmod 2. \tag{6.9}$$

Combining (6.8) and (6.9), we find that

$$\begin{aligned}\sum_{n=1}^{\infty} b_3\left(\frac{2(pn-1)}{24}\right) q^n &\equiv \sum_{n=1}^{\infty} b_3\left(\frac{2(n/p-1)}{24}\right) q^n \pmod 2\\ &\equiv \sum_{n=1}^{\infty} b_3\left(\frac{2(n-1)}{24}\right) q^{pn} \pmod 2\\ &\equiv \sum_{n=0}^{\infty} b_3\left(\frac{2n}{24}\right) q^{pn+p} \pmod 2.\end{aligned}$$

Multiplying both sides by q^{-p} we obtain

$$\sum_{n=p}^{\infty} b_3\left(\frac{2(pn-1)}{24}\right) q^{n-p} \equiv \sum_{n=0}^{\infty} b_3\left(\frac{2n}{24}\right) q^{pn} \pmod 2$$

which yields

$$\sum_{n=0}^{\infty} b_3\left(\frac{2(pn+p^2-1)}{24}\right) q^n \equiv \sum_{n=0}^{\infty} b_3\left(\frac{2n}{24}\right) q^{pn} \pmod 2.$$

Let $\alpha = \frac{p^2-1}{24}$. Since $p \equiv -1 \pmod{6912}$, so α is a positive integer, and $\alpha \equiv -24^{-1} \pmod{p^2}$, $0 < \alpha < p^2$. Replacing n by $24n$ and then substituting q^{24} by q we get

$$\sum_{n=0}^{\infty} b_3(2(pn+\alpha))q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{pn} \pmod 2.$$

This completes the proof of the theorem. ■

6.3 Proof of a conjecture of Keith and Zanello on $b_{25}(n)$

Keith and Zanello also studied 2-divisibility of $b_{25}(n)$ and proved several congruences for primes $p \equiv 11, 13, 17, 19 \pmod{20}$ and $p \equiv 31, 39 \pmod{40}$. To be specific, if $p \equiv 11, 13, 17, 19 \pmod{20}$ is prime, then they proved that

$$b_{25}(8(p^2n + kp - 3 \cdot 4^{-1}) + 5) \equiv 0 \pmod{2}$$

for all $1 \leq k < p$, where $3 \cdot 4^{-1}$ is taken modulo p^2. Further, they conjectured the following:

Conjecture 6.3. [29, Conjecture 28] *For a positive proportion of primes p, it holds that*

$$\sum_{n=0}^{\infty} b_{25}(2pn + \alpha)q^n \equiv q^{\beta} \sum_{n=0}^{\infty} b_{25}(2n+1)q^{pn} \pmod{2},$$

for some α and β depending on p.

Our second theorem confirms that Conjecture 6.3 is true.

Theorem 6.4. *Conjecture 6.3 is true.*

Proof. Putting $t = 25$ in (6.1), we have

$$\sum_{n=0}^{\infty} b_{25}(n)q^n = \frac{f_{25}}{f_1}. \tag{6.10}$$

We use identity [27, (4)], namely

$$f_1 f_5 \equiv f_1^6 + q f_5^6 \pmod{2}.$$

Dividing both sides by f_1^2 we obtain

$$\frac{f_5}{f_1} \equiv f_1^4 + q\frac{f_5^6}{f_1^2} \pmod{2}. \tag{6.11}$$

Therefore, by (6.10) and (6.11), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_{25}(n)q^n &= \frac{f_{25}}{f_1} = \frac{f_{25}}{f_5}\frac{f_5}{f_1}\\ &\equiv f_1^4 f_5^4 + q^6 f_5^4 \frac{f_{25}^6}{f_1^2} + q\frac{f_5^{10}}{f_1^2} + q^5 f_1^4 \frac{f_{25}^6}{f_5^2} \pmod{2}.\end{aligned}$$

Extracting the terms involving q^{2n+1}, and then dividing by q and replacing q^2 by q, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} b_{25}(2n+1)q^n &\equiv \frac{f_5^5}{f_1} + q^2\frac{f_1^2 f_{25}^3}{f_5} \pmod{2}\\ &\equiv f_1^4 f_5^4 + q\frac{f_5^{10}}{f_1^2} + q^2 f_1^2 f_5^4 f_{25}^2 + q^7\frac{f_1^2 f_{25}^8}{f_5^2} \pmod{2}.\end{aligned}$$

Extracting the terms involving q^{2n}, we obtain

$$\sum_{n=0}^{\infty} b_{25}(4n+1)q^{2n} \equiv f_2^2 f_{10}^2 + q^2 f_2 f_{10}^2 f_{50} \pmod{2}. \tag{6.12}$$

Define $F(z)$ by

$$F(z) := \eta^2(2z)\eta^2(10z) + \eta(2z)\eta^2(10z)\eta(50z). \tag{6.13}$$

Combining (6.12) and (6.13), we obtain

$$F(z) \equiv \sum_{n=0}^{\infty} b_{25}(4n+1)q^{2n+1} \pmod{2}. \tag{6.14}$$

Now using Theorems 1.4 and 1.5, we find that $\eta^2(2z)\eta^2(10z) \in S_2(\Gamma_0(100), \chi_3)$ and

$\eta(2z)\eta^2(10z)\eta(50z) \in S_2(\Gamma_0(100), \chi_3)$ for some Nebentypus character χ_3 and hence $F(z) \in S_2(\Gamma_0(100), \chi_3)$. Also, the Fourier coefficients of $F(z)$ are all integers. Hence by Proposition 1.3, a positive proportion of the primes $p \equiv -1 \pmod{200}$ have the property that

$$F(z) \mid T_p \equiv 0 \pmod{2}. \tag{6.15}$$

Let $F(z) := \sum_{n=1}^{\infty} d(n)q^n$. Then, (6.14) yields

$$\sum_{n=1}^{\infty} b_{25}(2(n-1)+1)q^n \equiv \sum_{n=1}^{\infty} d(n)q^n \pmod{2}. \tag{6.16}$$

Now, from (6.15) we obtain

$$F(z) \mid T_p = \sum_{n=1}^{\infty} (d(pn) + p\chi_3(p)d(n/p))q^n \equiv 0 \pmod{2}$$

which yields

$$\sum_{n=1}^{\infty} d(pn)q^n \equiv \sum_{n=1}^{\infty} d(n/p)q^n \pmod{2}. \tag{6.17}$$

Combining (6.16) and (6.17), we find that

$$\begin{aligned}\sum_{n=1}^{\infty} b_{25}(2pn-1)q^n &\equiv \sum_{n=1}^{\infty} b_{25}(2(n/p-1)+1)q^n \pmod{2}\\ &\equiv \sum_{n=1}^{\infty} b_{25}(2(n-1)+1)q^{pn} \pmod{2}\\ &\equiv \sum_{n=0}^{\infty} b_{25}(2n+1)q^{pn+p} \pmod{2}.\end{aligned}$$

Replacing n by $n+1$ on the left side and then dividing both sides by q we obtain

$$\sum_{n=0}^{\infty} b_{25}(2pn+\alpha)q^n \equiv q^{\beta} \sum_{n=0}^{\infty} b_{25}(2n+1)q^{pn} \pmod 2,$$

where $\alpha = 2p-1$ and $\beta = p-1$. This completes the proof of the theorem. ■

6.4 A self-similarity result for $b_{21}(n)$

In this section, we prove a self-similarity result for $b_{21}(n)$ modulo 2. More precisely, we prove the following theorem:

Theorem 6.5. *For any prime $p > 3$, let $\gamma \equiv -6^{-1} \pmod{p^2}$, $0 < \gamma < p^2$. It holds for a positive proportion of primes p that*

$$\sum_{n=0}^{\infty} b_{21}(pn+11\gamma+1)q^n \equiv \sum_{n=0}^{\infty} b_{21}(n+1)q^{pn} \pmod 2.$$

Proof. We begin with the identity [29, Section 7], namely

$$\sum_{n=0}^{\infty} b_{21}(4n+1)q^n \equiv \frac{f_3^4}{f_1} \pmod 2. \tag{6.18}$$

Let

$$G(z) := \prod_{n=1}^{\infty} \frac{(1-q^{24n})^2}{(1-q^{48n})} = \frac{\eta^2(24z)}{\eta(48z)}.$$

Then using the binomial theorem we have

$$G(z) = \frac{\eta^2(24z)}{\eta(48z)} \equiv 1 \pmod 2.$$

Define $H(z)$ by

$$H(z) := \left(\frac{\eta^4(72z)}{\eta(24z)}\right) G(z) = \frac{\eta^4(72z)\eta(24z)}{\eta(48z)}.$$

Modulo 2, we have

$$H(z) \equiv \frac{\eta^4(72z)}{\eta(24z)} = q^{11}\frac{(q^{72};q^{72})_\infty^4}{(q^{24};q^{24})_\infty}. \tag{6.19}$$

Combining (6.18) and (6.19), we obtain

$$H(z) \equiv \sum_{n=0}^{\infty} b_{21}(4n+1)q^{24n+11} \pmod 2. \tag{6.20}$$

Now using Theorems 1.4 and 1.5, we find that $H(z) \in S_2(\Gamma_0(3456), \chi_2)$ for some Nebentypus character χ_2. Also, the Fourier coefficients of $H(z)$ are all integers. Hence by Proposition 1.3, a positive proportion of the primes $p \equiv -1 \pmod{6912}$ have the property that

$$H(z) \mid T_p \equiv 0 \pmod 2. \tag{6.21}$$

Let $H(z) := \sum_{n=1}^{\infty} c(n)q^n$. Then, (6.20) yields

$$\sum_{n=1}^{\infty} b_{21}\left(\frac{4(n-11)}{24}+1\right) q^n \equiv \sum_{n=1}^{\infty} c(n)q^n \pmod 2. \tag{6.22}$$

Now, from (6.21) we obtain

$$H(z) \mid T_p = \sum_{n=1}^{\infty} (c(pn) + p\chi_2(p)c(n/p))q^n \equiv 0 \pmod 2$$

which yields

$$\sum_{n=1}^{\infty} c(pn)q^n \equiv \sum_{n=1}^{\infty} c(n/p)q^n \pmod 2. \tag{6.23}$$

Combining (6.22) and (6.23), we find that

$$\begin{aligned}\sum_{n=1}^{\infty} b_{21}\left(\frac{4(pn-11)}{24}+1\right)q^n &\equiv \sum_{n=1}^{\infty} b_{21}\left(\frac{4(n/p-11)}{24}+1\right)q^n \pmod 2\\ &\equiv \sum_{n=1}^{\infty} b_{21}\left(\frac{4(n-11)}{24}+1\right)q^{pn} \pmod 2\\ &\equiv \sum_{n=0}^{\infty} b_{21}\left(\frac{4n}{24}+1\right)q^{pn+11p} \pmod 2.\end{aligned}$$

Multiplying both sides by q^{-11p} we obtain

$$\sum_{n=11p}^{\infty} b_{21}\left(\frac{pn-11}{6}+1\right)q^{n-11p} \equiv \sum_{n=0}^{\infty} b_{21}\left(\frac{n}{6}+1\right)q^{pn} \pmod 2$$

which yields

$$\sum_{n=0}^{\infty} b_{21}\left(\frac{pn+11(p^2-1)}{6}+1\right)q^n \equiv \sum_{n=0}^{\infty} b_{21}\left(\frac{n}{6}+1\right)q^{pn} \pmod 2.$$

Let $\gamma = \frac{p^2-1}{6}$. Since $p \equiv -1 \pmod{6912}$, so γ is a positive integer, and $\gamma \equiv -6^{-1} \pmod{p^2}$, $0 < \gamma < p^2$. Replacing n by $6n$ and then substituting q^6 by q we get

$$\sum_{n=0}^{\infty} b_{21}(pn+11\gamma+1)q^n \equiv \sum_{n=0}^{\infty} b_{21}(n+1)q^{pn} \pmod 2.$$

This completes the proof of the theorem. ■

7

Divisibility of Certain t-Regular Partitions by 2

7.1 Introduction

In this chapter, we establish infinite families of congruences modulo 2 for $b_3(n)$ and $b_{21}(n)$. We next prove that the series $\sum_{n=0}^{\infty} b_9(2n+1)q^n$ is lacunary modulo arbitrary powers of 2. We also prove that the series $\sum_{n=0}^{\infty} b_9(4n)q^n$ is lacunary modulo 2.

[1] Contents of this chapter has been published in Ramanujan J. (2022).

7.2 A specific case of self-similarity of $b_3(n)$

In [29], Keith and Zanello proved various congruences for $b_3(n)$ and proposed a conjecture regarding the self-similarity of $b_3(n)$. In Chapter 6, we proved their conjecture in our Theorem 6.2. In the same paper, they proved one specific case of their conjecture corresponding to $p = 13$. In the following theorem, we prove another specific case of Conjecture 6.1 corresponding to $p = 17$. We note that primes 13 and 17 do not fall in the family of primes for which we proved Theorem 6.2 because they are not congruent to -1 modulo 6912.

Theorem 7.1. *It holds that*

$$\sum_{n=0}^{\infty} b_3(34n+24)q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{17n} \pmod 2,$$

and therefore

$$b_3(2\cdot 17^2 n + 58) \equiv 0 \pmod 2,$$

and by iteration,

$$b_3\left(2\cdot 17^{2k} n + 17^{2k-2}\cdot 58 + 24\left(\frac{17^{2k-2}-1}{288}\right)\right) \equiv 0 \pmod 2$$

for all $k \geq 1$.

Proof. We first recall the following even-odd disection of the 3-regular partitions [29, (6)]:

$$\sum_{n=0}^{\infty} b_3(n)q^n = \frac{f_3}{f_1} \equiv \frac{f_1^8}{f_3^2} + q\frac{f_3^{10}}{f_1^4} \pmod 2. \tag{7.1}$$

Thus, extracting the terms with even powers of q, we obtain

$$\sum_{n=0}^{\infty} b_3(2n)q^n \equiv \frac{f_1^4}{f_3} \pmod 2. \tag{7.2}$$

Let

$$G_{3,1}(z) := \frac{\eta^4(z)\eta^2(51z)\eta(17z)}{\eta(3z)}$$

and

$$G_{3,2}(z) := \frac{\eta^4(17z)\eta^2(3z)\eta(z)}{\eta(51z)}.$$

By Theorems 1.4 and 1.5, we find that $G_{3,1}(z)$ and $G_{3,2}(z)$ are modular forms of weight 3 and level 51. Also, the Nebentypus character χ_0 is given by $\chi_0(\bullet) = (\frac{-3\cdot 17^3}{\bullet})$. By (7.2), the Fourier expansions of our forms satisfy

$$G_{3,1}(z) = \left(\sum_{n=0}^{\infty} b_3(2n)q^{n+5}\right) f_{51}^2 f_{17}$$

and

$$G_{3,2}(z) = \left(\sum_{n=0}^{\infty} b_3(2n)q^{17n+1}\right) f_3^2 f_1.$$

We then calculate that

$$G_{3,1}(z)|T_{17} \equiv \left(\sum_{n=0}^{\infty} b_3(34n+24)q^{n+1}\right) f_3^2 f_1 \pmod 2.$$

Since the Hecke operator is an endomorphism on $M_3(\Gamma_0(51), \chi_0)$, we have that $G_{3,1}(z)|T_{17} \in M_3(\Gamma_0(51), \chi_0)$. By Theorem 1.10, the Sturm bound for this space of forms is 18. We wish to verify the congruence

$$q\left(\sum_{n=0}^{\infty} b_3(34n+24)q^n\right) f_3^2 f_1 \equiv q\frac{f_{17}^4}{f_{51}} f_3^2 f_1 \pmod 2.$$

The coefficient of q^{18} on the left side involves the value $b_3(636)$; thus, f_3/f_1 must be expanded at least that far, and the product on the right side must be constructed up to the q^{18} terms. Finally, expansion with a calculation package such as *Sage* confirms that all coefficients up to the desired bound are congruent modulo 2, and the first part of the theorem is established.

Since only powers for which $17|n$ can be nonzero on the right side of the statement, we obtain:

$$b_3(34(17n+1)+24) = b_3(2\cdot 17^2 n + 58) \equiv 0 \pmod 2.$$

Finally, recursively applying the relation

$$b_3(2n) \equiv b_3(34\cdot 17n + 24) \pmod 2,$$

we obtain

$$\begin{aligned} b_3(2\cdot 17^2 n + 58) &\equiv b_3(2\cdot 17^2(17^2 n + 29) + 24) \pmod 2 \\ &= b_3(2\cdot 17^4 n + 17^2\cdot 58 + 24) \\ &\equiv b_3(2\cdot 17^6 n + 17^4\cdot 58 + 17^2\cdot 24 + 24) \pmod 2 \\ &\equiv \dots \\ &\equiv b_3\left(2\cdot 17^{2k} n + 17^{2k-2}\cdot 58 + 24\left(\frac{17^{2k-2}-1}{288}\right)\right) \equiv 0 \pmod 2, \end{aligned}$$

where the last line is given by a finite geometric summation. This completes the proof of the theorem. ■

7.3 Congruences for $b_3(n)$ modulo 2

If we assume that Theorem 6.2 is true for $p = 29$, then using the congruence $\sum_{n=0}^{\infty} b_3(2(29n+35))q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{29n} \pmod 2$, one can deduce infinite families of congruences of the form

$$b_3(2(29^2\cdot n + 29k + 35)) \equiv 0 \pmod 2, \tag{7.3}$$

where $1 \leq k \leq 28$. We do not know whether Theorem 6.2 is true or not for $p = 29$. However, in the following theorem, we prove the congruence (7.3) without assuming (6.3) for $p = 29$.

Theorem 7.2. *Let* $\alpha \in \{6, 64, 93, 122, 151, 180, 209, 238, 267, 296, 325, 354, 383, 412, 441, 470, 499, 528, 557, 586, 615, 644, 673, 702, 731, 760, 789, 818\}$. *Then for all* $n \geq 0$, *we have*

$$b_3(2(29^2 \cdot n + \alpha)) \equiv 0 \pmod 2.$$

We prove Theorem 7.2 using the approach developed in [47, 48]. Throughout this section, Γ denotes the full modular group $\mathrm{SL}_2(\mathbb{Z})$. We recall that the index of $\Gamma_0(N)$ in Γ is

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}),$$

where p denotes a prime.

For a positive integer M, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of M. If $r \in R(M)$ and $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ are the positive divisors of M, we write $r = (r_{\delta_1}, \ldots, r_{\delta_k})$. Define $c_r(n)$ by

$$\sum_{n=0}^{\infty} c_r(n) q^n := \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{n\delta})^{r_\delta}. \tag{7.4}$$

The approach to proving congruences for $c_r(n)$ developed by Radu [47, 48] reduces the number of coefficients that one must check as compared with the classical method which uses Sturm's bound alone.

Let m be a positive integer. For any integer s, let $[s]_m$ denote the residue class of s in $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $\mathbb{S}_m \subseteq \mathbb{Z}_m$ be the set of all squares in $\mathbb{Z}_m^*$. For $t \in \{0, 1, \ldots, m-1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subseteq \{0, 1, \ldots, m-1\}$ by

$$P_{m,r}(t) := \left\{ t' : \exists [s]_{24m} \in \mathbb{S}_{24m} \text{ such that } t' \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m} \right\}.$$

Definition 7.1. *Suppose m, M and N are positive integers, $r = (r_\delta) \in R(M)$ and $t \in \{0, 1, \ldots, m-1\}$. Let $k = k(m) := \gcd(m^2-1, 24)$ and write*

$$\prod_{\delta|M} \delta^{|r_\delta|} = 2^s \cdot j,$$

where s and j are nonnegative integers with j odd. The set Δ^* consists of all tuples $(m, M, N, (r_\delta), t)$ satisfying these conditions and all of the following.

1. *Each prime divisor of m is also a divisor of N.*

2. *$\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$.*

3. *$kN \sum_{\delta|M} r_\delta mN/\delta \equiv 0 \pmod{24}$.*

4. *$kN \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$.*

5. *$\frac{24m}{\gcd\left(-24kt - k\sum_{\delta|M} \delta r_\delta, 24m\right)}$ divides N.*

6. *If $2|m$, then either $4|kN$ and $8|sN$ or $2|s$ and $8|(1-j)N$.*

Let m, M, N be positive integers. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, $r \in R(M)$ and $r' \in R(N)$, set

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,1,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta a + \delta k \lambda c, mc)}{\delta m}$$

and

$$p^*_{r'}(\gamma) := \frac{1}{24} \sum_{\delta|N} r'_\delta \frac{\gcd^2(\delta, c)}{\delta}.$$

Lemma 7.3. [47, Lemma 4.5] *Let u be a positive integer, $(m, M, N, r = (r_\delta), t) \in \Delta^*$ and $r' = (r'_\delta) \in R(N)$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ be a complete set of representatives of the double cosets of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p^*_{r'}(\gamma_i) \geq 0$ for all $1 \leq i \leq n$. Let $t_{min} = \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24}\left\{\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta r'_\delta\right\} - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{min}}{m}.$$

If the congruence $c_r(mn+t') \equiv 0 \pmod{u}$ holds for all $t' \in P_{m,r}(t)$ and $0 \leq n \leq \lfloor \nu \rfloor$, then it holds for all $t' \in P_{m,r}(t)$ and $n \geq 0$.

To apply Lemma 7.3, we utilize the following result, which gives us a complete set of representatives of the double coset in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$.

Lemma 7.4. [60, Lemma 4.3] *If N or $\frac{1}{2}N$ is a square-free integer, then*

$$\bigcup_{\delta|N} \Gamma_0(N)\begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}\Gamma_\infty = \Gamma.$$

We are now ready to prove Theorem 7.2.

Proof of Theorem 7.2. From (7.1), we have

$$\sum_{n=0}^{\infty} b_3(n)q^n = \frac{f_3}{f_1} \equiv \frac{f_1^8}{f_3^2} + q\frac{f_3^{10}}{f_1^4} \pmod{2}.$$

Extracting the terms with even powers of q, we obtain

$$\sum_{n=0}^{\infty} b_3(2n)q^n \equiv \frac{f_1^4}{f_3} \pmod{2}.$$

Let $(m, M, N, r, t) = (841, 3, 87, (4, -1), 64)$. It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{6, 64, 151, 180, 209, 238, 296, 412, 499, 615, 673, 702, 731, 760\}$. By

Lemma 7.4, we know that $\left\{\begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} : \delta | 87\right\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let $\gamma_\delta = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}$. Let $r' = (0,0,0,0) \in R(87)$ and we use *Sage* to verify that $p_{m,r}(\gamma_\delta) + p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta | N$. We compute that the upper bound in Lemma 7.3 is $\lfloor \nu \rfloor = 14$. Using *Sage* we verify that $b_3(1682n + 2t') \equiv 0 \pmod 2$ for all $t' \in P_{m,r}(t)$ and for $n \leq 14$. By Lemma 7.3 we conclude that $b_3(1682n + 2t') \equiv 0 \pmod 2$ for all $t' \in P_{m,r}(t)$ and for all $n \geq 0$. To prove the remaining congruences, we take $(m, M, N, r, t) = (841, 3, 87, (4, -1), 93)$. It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{93, 122, 267, 325, 354, 383, 441, 470, 528, 557, 586, 644, 789, 818\}$. Following similar steps as shown before, we find that $b_3(1682n + 2t') \equiv 0 \pmod 2$ for all $t' \in P_{m,r}(t)$ and for all $n \geq 0$. This completes the proof of the theorem. ■

7.4 Congruences for $b_{21}(n)$ modulo 2

Keith and Zanello [29] also studied 2-divisibility of $b_{21}(n)$ and proved several congruences for primes $p \equiv 13, 17, 19, 23 \pmod{24}$. To be specific, if $p \equiv 13, 17, 19, 23 \pmod{24}$ is prime, then

$$b_{21}(4(p^2n + kp - 11 \cdot 24^{-1}) + 1) \equiv 0 \pmod 2$$

for all $1 \leq k < p$, where 24^{-1} is taken modulo p^2. For example, if $p = 13$, then one obtains

$$b_{21}(4 \cdot 13^2 n + 52k + 309) \equiv 0 \pmod 2$$

for all $k = 1, 2, \ldots, 12$. In the following theorem we prove similar type of congruences for the prime $p = 29$.

Theorem 7.5. *Let* $\beta \in \{8, 37, 66, 95, 124, 153, 182, 211, 240, 269, 298, 327, 356, 414, 443, 472, 501, 530, 559, 588, 617, 646, 675, 704, 733, 762, 791, 820\}$. *Then for all* $n \geq 0$, *we have*

$$b_{21}(4(29^2 \cdot n + \beta) + 1) \equiv 0 \pmod{2}.$$

To prove Theorem 7.5, again we use the approach developed by Radu in [47, 48].

Proof of Theorem 7.5. We begin our proof by recalling the following even-odd dissection formula of the 21-regular partitions [29, (9)]:

$$\sum_{n=0}^{\infty} b_{21}(n)q^n = \frac{f_{21}}{f_1} \equiv f_1^8 f_3^4 + q^3 f_1^8 f_{21}^4 + q^6 \frac{f_1^8 f_{21}^8}{f_3^4} + q \frac{f_3^{16}}{f_1^4} + q^4 \frac{f_3^{12} f_{21}^4}{f_1^4} + q^7 \frac{f_3^8 f_{21}^8}{f_1^4} \pmod{2}.$$

Extracting the terms with odd powers of q, we obtain

$$\sum_{n=0}^{\infty} b_{21}(2n+1)q^n \equiv q f_1^4 f_{21}^2 + \frac{f_3^8}{f_1^2} + q^3 \frac{f_3^4 f_{21}^4}{f_1^2} \pmod{2}.$$

Finally, extracting the terms with even powers of q, we obtain

$$\sum_{n=0}^{\infty} b_{21}(4n+1)q^n \equiv \frac{f_3^4}{f_1} \pmod{2}.$$

Let $(m, M, N, r, t) = (841, 3, 87, (-1, 4), 414)$. We verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{8, 124, 182, 211, 240, 269, 356, 414, 501, 530, 559, 588, 646, 762\}$. By Lemma 7.4, we know that $\left\{ \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} : \delta | 87 \right\}$ forms a complete set of double coset representatives of $\Gamma_0(N) \backslash \Gamma / \Gamma_\infty$. Let $\gamma_\delta = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}$. Let $r' = (0, 0, 0, 0) \in R(87)$ and we use *Sage* to verify that $p_{m,r}(\gamma_\delta) + p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta | N$. We compute that the upper bound in Lemma 7.3 is $\lfloor \nu \rfloor = 14$. Using *Sage* we verify that $b_{21}(4(841n + t') + 1) \equiv 0 \pmod{2}$ for all $t' \in P_{m,r}(t)$ and for $n \leq 14$. By

Lemma 7.3 we conclude that $b_{21}(4(841n+t')+1) \equiv 0 \pmod 2$ for all $t' \in P_{m,r}(t)$ and for all $n \geq 0$. To prove the remaining congruences, we take $(m, M, N, r, t) = (841, 3, 87, (-1, 4), 443)$. It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{37, 66, 95, 153, 298, 327, 443, 472, 617, 675, 704, 733, 791, 820\}$. Following similar steps as shown before, we find that $b_{21}(4(841n+t')+1) \equiv 0 \pmod 2$ for all $t' \in P_{m,r}(t)$ and for all $n \geq 0$. This completes the proof of the theorem. ∎

7.5 Distribution of $b_9(n)$

In this section, we study distribution of $b_9(2n+1)$ and $b_9(4n)$. In order to prove the main results of this section, we first prove the following lemma.

Lemma 7.6. *We have*

$$\sum_{n=0}^{\infty} b_9(2n+1)q^n = \frac{f_2^2 f_3 f_{18}}{f_1^3 f_6}; \tag{7.5}$$

$$\sum_{n=0}^{\infty} b_9(4n)q^n \equiv \frac{f_3^7}{f_1 f_9^2} \pmod 2. \tag{7.6}$$

Proof. Letting $\ell = 9$ in (6.1), we have

$$\sum_{n=0}^{\infty} b_9(n)q^n = \frac{f_9}{f_1}. \tag{7.7}$$

From Lemma 3.5 in [62], we have

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}. \tag{7.8}$$

Extracting the terms with odd powers of q and then using (7.8), we obtain

$$\sum_{n=0}^{\infty} b_9(2n+1)q^n = \frac{f_2^2 f_3 f_{18}}{f_1^3 f_6}.$$

From (7.8), extracting the terms with even powers of q and then using (7.7), we obtain

$$\sum_{n=0}^{\infty} b_9(2n)q^n = \frac{f_6^3 f_9}{f_1^2 f_3 f_{18}} \equiv \frac{f_3^5}{f_1^2 f_9} \pmod{2}. \tag{7.9}$$

From [28, (2.5)], we have

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}. \tag{7.10}$$

Magnifying equation (7.10) by $q \to q^3$ and combining with (7.9), we obtain

$$\sum_{n=0}^{\infty} b_9(2n)q^n \equiv \frac{f_3^2}{f_1^2}\left(\frac{f_{12}^3}{f_{36}} + q^3\frac{f_6^2 f_{36}^3}{f_{12} f_{18}^2}\right) \pmod{2}.$$

Extracting the terms with even powers of q, we obtain

$$\sum_{n=0}^{\infty} b_9(4n)q^n \equiv \frac{f_3^7}{f_1 f_9^2} \pmod{2}.$$

This completes the proof of the lemma. ■

7.5.1 Distribution of $b_9(2n+1)$ modulo arbitrary powers of 2

Keith and Zanello [29] studied lacunarity of the functions $b_3(2n)$, $b_{21}(4n)$, $b_{21}(4n+1)$ and $b_{25}(8n+3)$ modulo 2 using the technique developed by Landau [32]. We note that the generating function of $b_9(2n+1)$ does not satisfy the conditions of

Theorem 1.8, and hence the approach of Cotron et al. [15] can not be used to study the lacunarity of $b_9(4n+1)$. Also, we can not apply Theorem 1.9 of Landau as we are studying the lacunarity modulo arbitrary powers of 2. In the following theorem, we prove that $b_9(2n+1)$ is almost always divisible by arbitrary powers of 2 by using Serre's density result.

Theorem 7.7. *The series $\sum_{n=0}^{\infty} b_9(2n+1)q^n$ is lacunary modulo 2^k for any positive integer k.*

Proof. Let

$$A(z) := \prod_{n=1}^{\infty} \frac{(1-q^{54n})^2}{(1-q^{108n})} = \frac{\eta^2(54z)}{\eta(108z)}.$$

Then using the binomial theorem we have

$$A^{2^k}(z) = \frac{\eta^{2^{k+1}}(54z)}{\eta^{2^k}(108z)} \equiv 1 \pmod{2^{k+1}}.$$

Define $B_k(z)$ by

$$B_k(z) := \left(\frac{\eta^2(6z)\eta(9z)\eta(54z)}{\eta^3(3z)\eta(18z)}\right) A^{2^k}(z) = \frac{\eta^2(6z)\eta(9z)\eta^{2^{k+1}+1}(54z)}{\eta^3(3z)\eta(18z)\eta^{2^k}(108z)}.$$

Modulo 2^{k+1}, we have

$$B_k(z) \equiv \frac{\eta^2(6z)\eta(9z)\eta(54z)}{\eta^3(3z)\eta(18z)} = q^2\left(\frac{(q^6;q^6)_\infty^2(q^9;q^9)_\infty(q^{54};q^{54})_\infty}{(q^3;q^3)_\infty^3(q^{18};q^{18})_\infty}\right). \tag{7.11}$$

Combining (7.5) and (7.11), we obtain

$$B_k(z) \equiv \sum_{n=0}^{\infty} b_9(2n+1)q^{3n+2} \pmod{2^{k+1}}. \tag{7.12}$$

Now, $B_k(z)$ is an eta-quotient with $N = 324$. We next prove that $B_k(z)$ is a modular form for all $k \geq 6$. We know by Proposition 1.6, that the cusps of $\Gamma_0(324)$

are represented by fractions $\frac{c}{d}$, where $d \mid 324$ and $\gcd(c,d)=1$. By Theorem 1.5, we find that $B_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\left(2^{k+1}+1\right)\frac{\gcd(d,54)^2}{54}+2\frac{\gcd(d,6)^2}{6}+\frac{\gcd(d,9)^2}{9}-3\frac{\gcd(d,3)^2}{3}-\frac{\gcd(d,18)^2}{18} - 2^k\frac{\gcd(d,108)^2}{108}\geq 0.$$

Equivalently, if and only if

$$L := (2^{k+2}+2)G_1+36G_2+12G_3-108G_4-6G_5-2^k\geq 0,$$

where

$$G_1=\frac{\gcd(d,54)^2}{\gcd(d,108)^2},\quad G_2=\frac{\gcd(d,6)^2}{\gcd(d,108)^2},$$
$$G_3=\frac{\gcd(d,9)^2}{\gcd(d,108)^2},\quad G_4=\frac{\gcd(d,3)^2}{\gcd(d,108)^2},$$
$$G_5=\frac{\gcd(d,18)^2}{\gcd(d,108)^2},$$

respectively.

We now consider the following four cases according to the divisors of 324 and find the values of G_i for $i=1,2,\ldots,5$. Let d be a divisor of $N=324$.

Case (i). For $d|324$ and $d\notin\{4,12,36,108,324\}$, we find that $G_1=1$, $1/81\leq G_2\leq 1, 1/36\leq G_3\leq 1$, $1/324\leq G_4\leq 1$ and $1/9\leq G_5\leq 1$. Hence,

$$L\geq 2^{k+2}+2+36/81+12/36-108-6-2^k=3\cdot 2^k-112-7/9.$$

Since $k\geq 6$, we have $L\geq 0$.

Case (ii). For $d=4,12$, we find that $G_1=G_2=G_5=1/4$ and $G_3=G_4=1/16$. Hence, $L=2$.

Case (iii). For $d=36$, we find that $G_1=G_5=1/4$, $G_2=1/36$, $G_3=1/16$ and

$G_4 = 1/144$. Hence, the value of L is equal to 0.

Case (iv). For $d = 108, 324$, we find that $G_1 = 1/4$, $G_2 = 1/324$, $G_3 = 1/144$, $G_4 = 1/1296$ and $G_5 = 1/36$. Hence, we have value of L equal to $4/9$.

Hence, $B_k(z)$ is holomorphic at every cusp $\frac{c}{d}$ for all $k \geq 6$. Using Theorem 1.4, we find that the weight of $B_k(z)$ is equal to 2^{k-1}. Also, the associated character for $B_k(z)$ is given by $\chi_1(\bullet) = (\frac{4 \cdot 3^{3\cdot 2^k+2}}{\bullet})$. This proves that $B_k(z) \in M_{2^{k-1}}(\Gamma_0(324), \chi_1)$ for all $k \geq 6$. Also, the Fourier coefficients of $B_k(z)$ are all integers. Hence by Theorem 1.7, the Fourier coefficients of $B_k(z)$ are almost always divisible by $m = 2^k$, for any positive integer k. Due to (7.12), the same holds for $b_9(2n+1)$. This completes the proof of the theorem. ■

7.5.2 $b_9(4n)$ is almost always even

Keith and Zanello [29] derived several congruences for the partition function $b_9(n)$ modulo 2 using the theory of Hecke operators. In the following theorem we prove that $b_9(4n)$ is almost always divisible by 2.

Theorem 7.8. *The series* $\sum_{n=0}^{\infty} b_9(4n)q^n$ *is lacunary modulo* 2.

Proof. We first recall the following identity [29, (7)]:

$$f_1^3 \equiv f_3 + q f_9^3 \pmod 2.$$

We rewrite the above identity as

$$\frac{f_3}{f_1} \equiv f_1^2 + q\frac{f_9^3}{f_1} \pmod 2. \tag{7.13}$$

Combining (7.6) and (7.13), we obtain

$$\sum_{n=0}^{\infty} b_9(4n)q^n \equiv \frac{f_3^6 f_1^2}{f_9^2} + q\frac{f_3^6 f_9^2}{f_1} \pmod 2. \tag{7.14}$$

We note that the second term of (7.14) is lacunary modulo 2, by Theorem 1.8. For the first term of (7.14), we again recall the following identity [29, p. 12]

$$\frac{f_1^3}{f_3} \equiv 1 + \sum_{n\in\mathbb{Z}} q^{(3n-1)^2} \pmod 2,$$

which is quadratic. Hence, the same holds true for $(f_3^3/f_9)^2$, by substituting q with q^6. More precisely, we obtain

$$\left(\frac{f_3^3}{f_9}\right)^2 \equiv 1 + \sum_{n\in\mathbb{Z}} q^{6(3n-1)^2} \pmod 2. \tag{7.15}$$

Now, squaring the Euler's Pentagonal Number formula (Theorem 1.1), we have

$$f_1^2 \equiv \sum_{n\in\mathbb{Z}} q^{n(3n-1)} \pmod 2. \tag{7.16}$$

Finally combining (7.15) and (7.16), and then applying Lemma 1.9 we conclude that the first term of (7.14) is also lacunary modulo 2. This completes the proof of the theorem. ■

Remark 7.5.1. *We note that the generating function of $b_4(4n)$ does not satisfy the conditions of Theorem 1.8. Hence, Theorem 7.8 can not be proved by using the approach of Cotron et al. [15]. However, we can prove Theorem 7.8 using the Serre's density result as shown in the proof of Theorem 7.7. For this, we rewrite (7.6) in terms of η-quotients and obtain*

$$\sum_{n=0}^{\infty} b_9(4n)q^{12n+1} \equiv \frac{\eta^7(36z)}{\eta(12z)\eta^2(108z)} \pmod 2. \tag{7.17}$$

Let $F(z) = \frac{\eta^7(36z)}{\eta(12z)\eta^2(108z)}$. As shown in the proof of Theorem 7.7, one can prove that $F(z) \in M_2(\Gamma_0(1296), (\frac{2^8 3^7}{\bullet}))$. By Theorem 1.7, the Fourier coefficients of $F(z)$ are almost always divisible by $m = 2$. Due to (7.17), the same holds for $b_9(4n)$.

8

Congruences for 3-Regular Partitions in Three Colors

8.1 Introduction

In 2018, Hirschhorn [26] studied the number of partitions of n in three colors, $p_3(n)$, given by

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{f_1^3},$$

[1]Contents of this chapter has been published in Bull. Aust. Math. Soc. (2021).

where $f_k := (q^k; q^k)_\infty = \prod_{j=1}^{\infty}(1 - q^{jk})$ and k is a positive integer. He deduced a number of congruences for $p_3(n)$ modulo high powers of 3. He proved that for all $\alpha \geq 0$ and $n \geq 0$,

$$p_3(3^{2\alpha+1}n + \frac{5 \times 3^{2\alpha+1} + 1}{8}) \equiv 0 \pmod{3^{3\alpha+2}}$$

and

$$p_3(3^{2\alpha+2}n + \frac{7 \times 3^{2\alpha+2} + 1}{8}) \equiv 0 \pmod{3^{3\alpha+4}}$$

and that for all $\alpha \geq 1$ and $n \geq 0$,

$$p_3(3^{2\alpha}n + \frac{13 \times 3^{2\alpha-1} + 1}{8}) \equiv 0 \pmod{3^{3\alpha}}$$

and

$$p_3(3^{2\alpha+1}n + \frac{23 \times 3^{2\alpha} + 1}{8}) \equiv 0 \pmod{3^{3\alpha+4}}.$$

Let $p_{\{3,3\}}(n)$ denote the number of 3-regular partitions in three colours, whose generating function is given by

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(n)q^n = \frac{f_3^3}{f_1^3}. \tag{8.1}$$

In 2019, Gireesh and Mahadeva Naika [22] studied the function $p_{\{3,3\}}(n)$, and deduced some congruences modulo powers of 3 for $p_{\{3,3\}}(n)$. In a very recent paper [17], using elementary generating function manipulations and classical techniques, da Silva and Sellers significantly extended the list of proven arithmetic properties satisfied by $p_{\{3,3\}}(n)$. They obtained parity characterisation for $p_{\{3,3\}}(2n)$. They provided a complete characterisation for $p_{\{3,3\}}(n)$ modulo 3. For example, they proved that, for all $n \geq 0$

$$p_{\{3,3\}}(3n+1) \equiv p_{\{3,3\}}(3n+2) \equiv 0 \pmod 3,$$

and

$$p_{\{3,3\}}(3n) \equiv \begin{cases} (-1)^{k+\ell} \pmod 3, & \text{if } n = k(3k-1)/2 + \ell(3\ell-1)/2; \\ 0 \pmod 3, & \text{otherwise.} \end{cases}$$

They also found some congruences for $p_{\{3,3\}}(n)$ modulo 4 and 9. They further conjectured four Ramanujan-like congruences modulo 5 satisfied by $p_{\{3,3\}}(n)$.

Conjecture 8.1. [17, Conjecture 5.1] *For all $n \geq 0$,*

$$p_{\{3,3\}}(15n+6) \equiv 0 \pmod 5, \tag{8.2}$$

$$p_{\{3,3\}}(25n+6) \equiv 0 \pmod 5, \tag{8.3}$$

$$p_{\{3,3\}}(25n+16) \equiv 0 \pmod 5, \tag{8.4}$$

$$p_{\{3,3\}}(25n+21) \equiv 0 \pmod 5. \tag{8.5}$$

8.2 Proof of Conjecture 8.1

In this section we confirm that Conjecture 8.1 is true using the theory of modular forms.

Theorem 8.2. *Conjecture 8.1 is true.*

Proof. From (8.1), we have

$$\sum_{n=0}^{\infty} p_{\{3,3\}}(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty^3}.$$

We choose $(m, M, N, r, t) = (15, 3, 15, (-3, 3), 6)$. We verify that $(m, M, N, r, t) \in$

Δ^* and $P_{m,r}(t) = \{6\}$. By Lemma 7.4, we know that $\left\{ \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} : \delta | 15 \right\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let $\gamma_\delta = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}$. Let $r' = (30, 0, 0, 0) \in R(15)$ and we use *Sage* to verify that $p_{m,r}(\gamma_\delta) + p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta|N$. Using Lemma 7.3 we have

$$\nu := \frac{1}{24}\left\{\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta r'_\delta\right\} - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{min}}{m}$$
$$= \frac{1}{24}\{(0+30)\,24 - 30\} - \frac{1}{24\cdot 15}(-3+9) - \frac{6}{15} = \frac{85}{3}.$$

Therefore $\lfloor \nu \rfloor$ is equal to 28. Using *Sage* we verify that $p_{\{3,3\}}(15n+6) \equiv 0 \pmod 5$ for all $0 \leq n \leq 28$. By Lemma 7.3 we conclude that $p_{\{3,3\}}(15n+6) \equiv 0 \pmod 5$ for all $n \geq 0$. This completes the proof of (8.2). To prove (8.3), we take $(m, M, N, r, t) = (25, 3, 15, (-3, 3), 6)$. It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{6\}$. We compute that the upper bound in Lemma 7.3 is $\lfloor \nu \rfloor = 47$. Following similar steps as shown before, we find that $p_{\{3,3\}}(25n+6) \equiv 0 \pmod 5$ for all $n \geq 0$.

We now prove (8.4) and (8.5). We take $(m, M, N, r, t) = (25, 3, 15, (-3, 3), 16)$. It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{16, 21\}$. Here we also check that $t_{min} = 16$. By Lemma 7.4, we know that $\left\{ \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} : \delta | 15 \right\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let $\gamma_\delta = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}$. Let $r' = (50, 0, 0, 0) \in R(15)$ and we use *Sage* to verify that $p_{m,r}(\gamma_\delta) + p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta|N$. We compute that the upper bound in Lemma 7.3 is $\lfloor \nu \rfloor = 47$. Using *Sage* we verify that $p_{\{3,3\}}(25n + t') \equiv 0 \pmod 5$ for all $t' \in P_{m,r}(t)$ and for $0 \leq n \leq 47$. By Lemma 7.3 we conclude that $p_{\{3,3\}}(25n+t') \equiv 0 \pmod 5$ for all $t' \in P_{m,r}(t)$ and for all $n \geq 0$. This completes the proof of the theorem. ∎

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Publications

Publications from Thesis work

1. A. Singh and R. Barman, *Divisibility of certain singular overpartitions by powers of* 2 *and* 3, Bull. Aust. Math. Soc. 104 (2021), Article no. 2, 238–248.

2. A. Singh and R. Barman, *Certain eta-quotients and arithmetic density of Andrews' singular overpartitions*, J. Number Theory 229 (2021), 487–498.

3. R. Barman and A. Singh, *On mex-related partition functions of Andrews and Newman*, Res. Number Theory 7 (2021), Article no. 53, 29 pages.

4. A. Singh and R. Barman, *New density results and congruences for Andrews' singular overpartitions*, J. Number Theory 229 (2021), 328–341.

5. R. Barman and A. Singh, *Mex-related partition functions of Andrews and Newman*, Journal of Integer Sequences, 24 (2021), Article 21.6.3.

6. A. Singh and R. Barman, *Proof of some conjectural congruences of da Silva and Sellers*, Bull. Aust. Math. Soc., doi:10.1017/S000497272100099X.

7. A. Singh and R. Barman, *Divisibility of certain ℓ-regular partitions by 2*, Ramanujan J., doi:10.1007/s11139-022-00580-6.

8. A. Singh and R. Barman, *Proofs of some conjectures of Keith and Zanello on t-regular partition*, arXiv:2201.07046v1 [math.NT].