Some spaces of holomorphic functions and their applications

by

Arun Kumar Bhardwaj

Roll No.: 186123005

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

GUWAHATI-781039, INDIA

August 23, 2023

Some spaces of holomorphic functions and their applications

A dissertation submitted
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

by

Arun Kumar Bhardwaj

(Roll Number: 186123005)

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI-781039, INDIA
August 23, 2023

Declaration

I do hereby declare that this thesis entitled **"Some spaces of holomorphic functions and their applications"** is a presentation of my original research work done under the supervision of **Dr. Rajesh Kumar Srivastava**, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of doctor of philosophy. The results embodied in this thesis have not been submitted to any other university or institute for the award of a degree or diploma.

Arun Kumar Bhardwaj
Roll No: 186123005
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati-781039, India

Guwahati
August 2023

Certificate

This is certified that the work contained in the thesis entitled **"Some spaces of holomorphic functions and their applications"** by **Mr. Arun Kumar Bhardwaj** (Roll No. 186123005) has been carried out under my supervision. In my opinion, the thesis has reached the standard of fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other university or institute for the award of a degree or diploma.

Guwahati
August 2023

Dr. Rajesh Kumar Srivastava
Associate Professor
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati-781039, India

DEDICATION

This dissertation is dedicated to my parents

Smt. Renu Jha and Shri Binay Kumar Jha

Acknowledgement

First, I owe a special thanks to my thesis supervisor, Dr. Rajesh Kumar Srivastava. Thank you for helping me when I needed help and encouraging me when I needed encouragement. Thank you for always listening to my ideas and endlessly supporting my independent thought. Thank you for being exactly the supervisor I needed during my Ph.D.

Many thanks to Prof. Javad Mashreghi and Dr. Arup Chattopadhyay for numerous useful and fruitful discussions. I am grateful to Prof. Bhaba Sarma, Prof. S Nateson and Dr. Bikash Bhattacharjya for being my teachers at IIT Guwahati. Thanks to Prof. Sukanta Pati and Dr. Anjan K. Chakrabarty for serving on my doctoral committee and making several comments and remarks.

I am highly grateful to the Indian Institute of Technology Guwahati for giving me the opportunity to study on a beautiful campus and providing financial assistance and facilities to pursue my degree. I am also grateful to all the staff members of the Department of Mathematics for their official and technical help.

I am very thankful to my professors at Patna University and Patna Science College for introducing me to the beautiful subject of mathematics.

I want to convey my most profound appreciation to Dr. Deb Kumar Giri, Dr. Somnath Ghosh, and Dr. Shyam Swarup Mondal for their continuous encouragement and valuable suggestions.

I am grateful to my parents, Shri. Binay Kumar Jha and Smt. Renu Jha and sister Sakshi, who have always encouraged me, stood by my side, especially when I needed it most.

I apologize to my lovely little daughter Anaisha. I cut the time that must had been spent with her to concentrate on my research.

Finally, a huge thanks goes to my wife Nidhi. Thank you for encouraging me when it was difficult. Thank you for taking care of so many small things so that I could concentrate on my research. But most of all, thank you for being with me. Everything is so much better when you are around.

Arun Kumar Bhardwaj

Abstract

In this dissertation, we consider several problems in complex analysis.

In the first part, we study about an explicit formula for the Hilbert transform. The celebrated integral transforms such as Fourier transform, Laplace transform, and Hilbert transform have tremendous applications in various branches of science and engineering. However, unlike to Fourier or Laplace transform, very few functions have an explicit formula for their Hilbert transforms. In this dissertation, we obtain an explicit formula for the Hilbert transform of $\log |f|$, for the function f in Nevanlinna class having continuous extension to the real line. This family is the largest possible for which such a formula for the Hilbert transform of $\log |f|$, can be obtained. The formula is very general and implies several previously known results.

In the second part, we consider the multipliers between model spaces. The main interest of this chapter is to obtain a characterization of the algebra of multipliers in the non-Hilbert setting, that is, for the case $p \neq 2$. The main thrust of this chapter is that: for the case $p \neq 2$ the algebra of multipliers denies to obey some results obtained before for the case $p = 2$. In this chapter, we also proved that the algebra of multipliers does not allow a "perturbation" in inner functions without altering the algebra.

In the last part, we prove some results on the triviality of Toeplitz kernels with certain symbols, uniqueness set for model spaces, and determination of meromorphic inner functions with some data.

Abbreviation and Notation

$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$\mathbb{D}$	The open unit disk in $\mathbb{C}$
$\mathbb{T}$	The Unit circle in $\mathbb{C}$
$\text{Re } z$	The real part of $z \in \mathbb{C}$
$\Im z$	The imaginary part of $z \in \mathbb{C}$
$\mathbb{C}_+$	The upper half-plane, that is, $z \in \mathbb{C}$ such that $\Im z > 0$
$\bar{z}$	The complex conjugate of z
b	Blaschke product on $\mathbb{D}$ or $\mathbb{T}$
Θ	Inner functions on $\mathbb{D}$ or $\mathbb{T}$
s	Singular inner function on $\mathbb{D}$ or $\mathbb{T}$
B	Blaschke product on $\mathbb{C}_+$ or $\mathbb{R}$
I	Inner functions on $\mathbb{C}_+$ or $\mathbb{R}$
S	Singular inner function on $\mathbb{C}_+$ or $\mathbb{R}$
$\bar{X}$	$\{f : \bar{f} \in X\}$
$L^p(X)$	$\{f : X \to \mathbb{C} \mid f \text{ is measurable and } \int_X \|f\|^p dx < \infty\}$
$C(\mathbb{R})$	The set of all complex valued continuous functions on $\mathbb{R}$
$C^\omega(\mathbb{R})$	The set of all real analytic functions.
$\text{Hol}(X)$	The set of all holomorphic functions on $X \subset \mathbb{C}$

N	The Nevanlinna class on $\mathbb{D}$ or $\mathbb{T}$
N^+	The Smirnov class on $\mathbb{D}$ or $\mathbb{T}$
H^p	The Hardy space on $\mathbb{D}$ or $\mathbb{T}$
K_Θ^p	The modal space on $\mathbb{D}$ or $\mathbb{T}$ associated with the inner function Θ
$\ker_p[u]$	Toeplitz kernel on $\mathbb{D}$ or $\mathbb{T}$ associated with the symbol $u \in L^\infty(\mathbb{T})$
$\mathcal{N}$	The Nevanlinna class on $\mathbb{C}_+$ or $\mathbb{R}$
$\mathcal{N}^+$	The Smirnov class on $\mathbb{C}_+$ or $\mathbb{R}$
$\mathcal{H}^p$	The Hardy space on $\mathbb{C}_+$ or $\mathbb{R}$
$\mathcal{K}_I^p$	The modal space on $\mathbb{C}_+$ or $\mathbb{R}$ associated with the inner function Θ
$\mathrm{Ker}_p[U]$	Toeplitz kernel on $\mathbb{C}_+$ or $\mathbb{R}$ associated with the symbol $U \in L^\infty(\mathbb{R})$

Contents

CHAPTER 1

Introduction

We start with recalling that a collection of vectors from a Banach space $\mathfrak{B}$ is complete if the set of finite linear combinations of the vectors from the given collection is dense in the space $\mathfrak{B}$. By the Hahn-Banach theorem a system $\{f_k\}$ of vectors is incomplete in $\mathfrak{B}$ if and only if there exists a nontrivial functional $\Gamma \in \mathfrak{B}^*$ such that $\Gamma(f_k) = 0$ for all k. In many cases where the general form of functionals from $\mathfrak{B}^*$ is known, the problem of completeness can be reduced to uniqueness problems for Holomorphic functions.

Completeness problems appear in the many areas of analysis and its applications. For instance, a function on a subset of the real line may represent a wave and one may ask if it can be approximated, in a specified sense, by a linear combination of specially selected functions, often called harmonics. In modern terms, defining the criterion of approximation amounts to defining the norm in the space. Verifying whether any function in the space can be approximated by a finite linear combination of harmonics is equivalent to proving that harmonics are complete in the space. Problems of this kind gave name to a large and important part of mathematics, Harmonic analysis.

The role of harmonics in the above set-up can be played by a number of different sets of functions, such as trigonometric polynomials, monomials, complex exponentials, or special functions such as Bessel function, Jacobi or Chebyshev polynomials or Airy functions originating from Physics. In the 20th century, the most popular choice of Banach spaces are the function spaces such as L^p spaces, spaces of continuous functions or smooth functions with various norms.

As we know that any bounded linear functional on $L^2(-\pi,\pi)$, has an integral representation, the study of completeness of complex exponentials can easily be reduced to the study of uniqueness sets for some classes of entire functions as follows: let $\Lambda = \{\lambda_n\} \subset \mathbb{C}$ be a given sequence of pairwise distinct points, and let $\mathcal{E}_\Lambda = \{e^{i\lambda t} : \lambda \in \Lambda\}$ be the corresponding system of complex exponentials. If $\mathcal{E}_\Lambda$ is not complete in $L^2(-\pi,\pi)$, then there exists a non zero function, say, $f \in L^2(-\pi,\pi)$ such that

$$\int_{-\pi}^{\pi} f(t)e^{i\lambda t}dt = 0, \quad \forall\lambda \in \Lambda.$$

But by the Paley-Wiener theorem, we know that the function

$$F(z) = \int_{-\pi}^{\pi} f(t)e^{izt}dt$$

is an entire function that lives in the Paley-Wiener space of type π, that is, the class of entire functions of exponential type at most π and its restriction to the real line belongs to $L^2(\mathbb{R})$. Thus above discussion tells us that $\mathcal{E}_\Lambda$ is complete in $L^2(-\pi,\pi)$ if and only if Λ is a uniqueness set for Paley-Wiener space $\mathcal{PW}_\pi$ of type π. It is well known that $\mathcal{PW}_\pi$ can also be realized as $e^{-\pi iz}$ times the orthogonal complement of $e^{2\pi iz}H^2$, in H^2, where H^2 is the Hardy space in the upper half-plane, that is

$$\mathcal{PW}_\pi = e^{-\pi iz}\left(H^2 \ominus e^{2\pi iz}H^2\right).$$

The space $H^2 \ominus e^{2\pi iz}H^2$ is denoted by $K_{e^{2\pi iz}}$ and is known as the model space. Thus the completeness (incompleteness) of complex exponentials in $L^2(-\pi,\pi)$ is equivalent to studying uniqueness (non-uniqueness) sets for the model space $K_{e^{2\pi iz}}$. It is straightforward to observe that the function $e^{2\pi iz}$ is a bounded Holomorphic function in the upper half plane whose nontangential boundary value is unimodular almost everywhere on the real line. Any bounded Holomorphic function in the upper half plane having unimodular nontangential boundary is known as inner functions. One can define model space [17]

$$K_\Theta := H^2 \ominus \Theta H^2$$

corresponding to any inner function Θ. Model spaces play a vital role in modern function theory, complex analysis, Harmonic analysis, etc., see [27, 36–38]. As we saw above the study of uniqueness sets for the model space $K_{e^{2\pi iz}}$ is equivalent to the study of the completeness of complex exponentials on $L^2(-\pi,\pi)$, the study of uniqueness (non-uniqueness) sets for general model space is intriguing in its own right, in fact, it answers many questions in the inverse spectral problems of second order differential operators as can be seen in [29,31,32,43], and references therein. It is easy to see that a set $\Lambda \subset \mathbb{C}_+$ is a uniqueness set for the model space K_Θ if and only if the kernel of the Toeplitz operator $T_{B_\Lambda \bar{\Theta}}$ is trivial, where B_Λ is the Blaschke product associated with the sequence Λ. Thus the completeness problem of complex exponentials in $L^2(a,b)$ or the completeness of reproducing kernels in model spaces is directly connected with the injectivity of certain Toeplitz operators. The problem of injectivity for the Toeplitz operator is very handy in the study of the spectral theory of Toeplitz operators, see [6, 26, 37, 38]. As compared to the problem of invertibility of the Toeplitz operator, the problem of injectivity has received considerably less attention. In recent years, the approach developed by Makarov and Poltoratski, which uses the injectivity of Toeplitz operators with certain symbols and some other tools, has brought solutions to some of the long-standing problems of Complex and Harmonic analysis, see [29, 30, 35, 39–41]. One of them is a generalization of Beurling-Malliavin theory on the completeness radius.

In the 1960's Beurling and Malliavin [2, 3] obtained a solution to the long-standing problem of finding the completeness radius R_Λ, for a given sequence Λ, where

$$R_\Lambda = \sup\{a > 0 : \mathcal{E}_\Lambda \text{ is complete in } \; L^2(-a,a)\}.$$

This result is considered as one of the deepest results of the 20th century harmonic analysis. Since the solution of the completeness radius problems, the possibility of its generalization have been studied by many prominent analysts, but it resisted those attempts until 2010. In 2010 Makarov and Poltoratski [30] obtained an extension of this result in Toeplitz form. The main tools used in the generalization are The Hilbert transform, model spaces and Toeplitz kernels.

Inspired by these works the main objects of study in this thesis are model spaces, Toeplitz kernels and the Hilbert transform.

The thesis is organized as follows. In Chapter 2 we provide some notations, terminologies and definitions.

In Chapter 3 we consider The Hilbert transform. Unlike to celebrated integral transforms such as Fourier or Laplace transform, very few functions have an explicit formula for their Hilbert transforms. In this chapter we obtain an explicit formula for the Hilbert transform of $\log|f|$, for the function f in Nevanlinna class having continuous extension to the real line. This family is the largest possible for which such a formula for the Hilbert transform of $\log|f|$, can be obtained. The formula is very general. To highlight the generality of the result, we apply our formula to certain model spaces, certain Toeplitz kernels, and Cartwright-de Branges spaces and obtain several known results from [29,34,41] rather easily.

In Chapter 4 we study the multipliers between model spaces for the case $1 \leq p \leq \infty$. The case $p = 2$ was investigated in [15,16]. The main interest of this work is that unlike to the case $p = 2$ one cannot recover the entire class of multipliers in the upper half plane given the corresponding knowledge of the class of multipliers in the unit disk setting for $p > 2$ and vice-versa for the case $1 \leq p < 2$. We also obtain some uniqueness results on the class of multipliers.

In Chapter 5 we obtain some results on the uniqueness set for model spaces and some results on the mixed data problems.

In Chapter 6 we give a summary of our results and discuss some future directions for research.

CHAPTER 2

Preliminaries

Here we summarize some well-known facts that we will use in forthcoming chapters.

2.1 Function theory in the unit disk

References for this section are [11, 17, 18, 25, 37, 38].

2.1.1 Hardy Spaces

The most older members of the spaces of holomorphic function on the open unit disk $\mathbb{D}$ are the Hardy spaces $H^p(\mathbb{D})$. To each $f \in \mathrm{Hol}(\mathbb{D})$ we consider the following integral means

$$\mathfrak{M}_p(r,f) := ||f_r||_p = \left(\int_{-\pi}^{\pi} |f(re^{it})|^p \frac{dt}{2\pi}\right)^{\frac{1}{p}}, \quad p \in (0,\infty) \tag{2.1.1}$$

and,

$$\mathfrak{M}_\infty(r,f) := ||f||_\infty = \max_{|w|=r} |f(w)|.$$

In each case, the parameter r varies from 0 to 1. Now the maximum principle for holomorphic functions immediately implies that $\mathfrak{M}_\infty(r,f)$ is an increasing function of r. If we recall the very classical Hadamard three-circle theorem, we see that $\log \mathfrak{M}_\infty(r,f)$ is a convex function of $\log r$. In 1915, G. H. Hardy extended both the observations of Hadamard to all values of $p \in (0,\infty)$, that is, $\mathfrak{M}_p(r,f)$ is an increasing function of r, and $\log \mathfrak{M}_p(r,f)$ is a convex function of $\log r$. This result is considered as the starting point of the theory of Hardy spaces.

The Hardy space $H^p(\mathbb{D})$, $P \in (0,\infty]$, is defined as the collection of all functions f holomorphic on the open unit disk such that

$$||f||_{H^p} = \sup_{r\in(0,1)} \mathfrak{M}_p(r,f) < \infty. \tag{2.1.2}$$

We note that as a consequence of the previous discussions one can replace the supremum in (2.1.2) by the limit as $r \to 1^-$. The space $\big(H^p(\mathbb{D}), ||\cdot||_{H^p}\big)$ turns out to be a Banach space for $p \in [1,\infty]$. The triangle inequality fails for $p \in (0,1)$, but the map

$$d_p(f,g) := ||f-g||_{H^p}^p,$$

defined on $H^p(\mathbb{D}) \times H^p(\mathbb{D})$, is a translation invariant complete metric and gives the topological vector space $\big(H^p(\mathbb{D}), d_p\big)$ which is not locally convex.

By a generalized version of Fatou's theorem [25, Page 11] it is well-known that for every $f \in H^p(\mathbb{D})$, $p \in (0,\infty]$ the radial limits

$$f(e^{it}) = \lim_{r\to 1^-} f_r(e^{it}) := \lim_{r\to 1^-} f(re^{it}) \tag{2.1.3}$$

exists for almost all $e^{it} \in \mathbb{T}$, moreover the function $f(e^{it})$ belongs to $L^p(\mathbb{T})$ and

$$\sup_{r\in(0,1)} \left(\int_{-\pi}^{\pi} |f(re^{it})|^p \frac{dt}{2\pi}\right)^{\frac{1}{p}} = \left(\int_{-\pi}^{\pi} |f(e^{it})|^p \frac{dt}{2\pi}\right)^{\frac{1}{p}}. \tag{2.1.4}$$

The function $f(e^{it})$ is known as the boundary value function of the function $f(re^{it})$. Thus (2.1.4) says that

$$||f||_{H^p} = ||f||_p.$$

It is also well-known that for the case $p \in [1,\infty]$ one can recover the function $f(re^{it})$ from its boundary value function via the well-known Poisson integral formula

$$f(re^{it}) = \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)} f(e^{i\theta}) \frac{d\theta}{2\pi}. \tag{2.1.5}$$

The collection of all boundary value functions will be denoted by $H^p(\mathbb{T})$. The above discussion tells us that there is a one-to-one correspondence between $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ for $p \in [1,\infty]$. This true even for $p \in (0,1)$. And hence we do not make any distinction between $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$, and write H^p for both spaces and understand from the context which space we are considering.

2.1.2 Canonical factorization

One of the pillars of the function space theory is the canonical factorization. In 1915, Blaschke showed that for given a sequence $\{w_k\} \subset \mathbb{D} \setminus \{0\}$, a necessary and sufficient condition for the infinite product

$$b(w) := cw^m \prod_k \frac{|w_k|}{w_k} \frac{w_k - w}{1 - \bar{w}_k w}, \tag{2.1.6}$$

where $|c| = 1$ and m is any non-negative integer, converges to a well-defined not identically-zero holomorphic function on the open unit disk $\mathbb{D}$ is that the series $\sum_k (1 - |w_k|)$ is convergent. Furthermore $|b| = 1$ almost everywhere on the circle $\mathbb{T}$. The product (2.1.6) is known as the Blaschke product. In order to state the canonical factorization theorem first we need to define inner and outer functions.

An inner function Θ is a bounded holomorphic function on $\mathbb{D}$ such that $|\Theta(e^{it})| = 1$ for almost all e^{it} on the circle $\mathbb{T}$. In particular, each Blaschke product is an inner function. There is another category of inner functions that are zero-free in $\mathbb{D}$. Such inner functions can always be represented as

$$s(w) := \exp\left(- \int_{\mathbb{T}} \frac{\zeta + w}{\zeta - w} d\mu(\zeta) \right), \quad w \in \mathbb{D}, \tag{2.1.7}$$

where μ is a finite positive singular Borel measure on $\mathbb{T}$. Conversely any finite positive singular Borel measure on $\mathbb{T}$ gives rise to a singular inner function via 2.1.7.

Given a non-negative function k on the unit circle $\mathbb{T}$ with $\log |k| \in L^1(\mathbb{T})$, the function

$$o(w) := \exp\left(\int_{\mathbb{T}} \frac{e^{it} + w}{e^{it} - w} \log k(e^{it}) \frac{dt}{2\pi} \right), \quad w \in \mathbb{D}, \tag{2.1.8}$$

is a holomorphic function in the open unit disk. Again by a general version of Fatou's theorem [25, Page 11] we see that

$$|o(e^{it})| = k(e^{it}),$$

for almost all $e^{it} \in \mathbb{T}$. Consequently, $O(w)$ belongs to the space $H^p(\mathbb{D})$ if and only if the function k belongs to the space $L^p(\mathbb{T})$. Any holomorphic function on the open unit disk of the form (2.1.8) is called an outer function. Now we are ready to state the celebrated canonical factorization theorem.

Theorem 2.1.1 (Smirnov). *Let $f \in H^p$, $p \in (0, \infty]$, then we have the following factorization*

$$f = bso, \tag{2.1.9}$$

where b is a Blaschke product cooked from the zeros (counted according to the multiplicity) of f in $\mathbb{D}$, s is a singular inner function and o is an outer function. The factorization (2.1.9) is unique upto a constant multiple of unit modulus.

A striking outcome of the canonical factorization theorem is that any inner function can be expressed as a product of a Blaschke product and a singular inner function.

Another interesting application of canonical factorization is the following estimate for $f \in H^p$

$$|f(w)| \leq 2^{1/p} \frac{||f||_p}{1 - |w|^{1/p}}, \quad w \in \mathbb{D}.$$

An immediate consequence of the above estimate is the fact that convergence in H^p implies uniform convergence on compact subsets of the unit disk.

One more consequence of canonical factorization is the fact that $\log |f| \in L^1(\mathbb{T})$, whenever $f \in H^p$. This immediately implies that functions in H^p cannot vanish on a subset of the circle $\mathbb{T}$ having a positive arc-length measure.

2.1.3 Nevanlinna class and Smirnov class

It is inherited from the structure of $L^p(\mathbb{T})$ that the Hardy spaces H^p forms a chain, that is, for $0 < p \leq q \leq \infty$ we have

$$H^q \subset H^p.$$

This tells us that as p gets smaller the size of the space gets larger. So it motivates to look for spaces containing $\cup_{p>0} H^p$. Nevanlinna constructed such a space by putting a milder growth condition on the functions as compared to the Hardy space. To define this class let

$$\mathfrak{L}(r, f) = \left(\int_{-\pi}^{\pi} \log^+ |f(re^{it})| \frac{dt}{2\pi} \right). \tag{2.1.10}$$

It is well-known that the integral means (2.1.10) as a function of r is increasing.

The Nevanlinna class $N(\mathbb{D})$ is defined as the collection of holomorphic functions on the open unit disk such that

$$\sup_{r\in(0,1)} \mathfrak{L}(r,f) = \lim_{r\to 1^-} \mathfrak{L}(r,f) < \infty. \tag{2.1.11}$$

There is an important characterization of function in the class $\mathcal{N}$ due to F. Nevanlinna and R. Nevanlinna as follows:

Theorem 2.1.2. $N(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) : f = g/h, \ \textit{where} \ \ g,h \in H^\infty\}$.

Due to this theorem functions in the class $N(\mathbb{D})$ are also known as the functions of bounded type. Again an immediate consequence of Theorem 2.1.2 is that the functions in $N(\mathbb{D})$ have boundary values which cannot vanish on a set of positive measure. Thus we have a one-to- one correspondence between functions in $N(\mathbb{D})$ and their boundary values, and hence we do not make any distinction between them. From now onward, N will be the common notation for both spaces, namely, the space $N(\mathbb{D})$ or the collection of boundary values of functions in $N(\mathbb{D})$. It will be clear from the context that the function we are considering belongs to which class.

If we apply the canonical factorization theorem (in view of Theorem 2.1.2) to a function $f \in N$, we see that

$$f = b\frac{s_1}{s_2}o, \tag{2.1.12}$$

where b is a Blaschke product cooked from the zeros (counted according to the multiplicity) of f in $\mathbb{D}$, s_1 and s_2 are singular inner functions and o is an outer function. Moreover, we can assume that s_1 and s_2 have no common divisors [17, Page 88]. Thus we see that inner-outer factorization breaks down in the case of Nevanlinna class as compared to H^p functions. Hence the Nevanlinna class is so big that it contains all the H^p spaces properly. More precisely we have

$$\cup_{p>0} H^p \subsetneq N.$$

The appearance of the singular inner function in the denominator in the factorization (2.1.12) has several topological disadvantages [10, 45]. To overcome such difficulties Smirnov considered the subclass

$$N^+ := \{f \in N : \quad f = bso\}.$$

The class N^+ is known as Smirnov class. Now it is straightforward that

$$H^p = N^+ \cap L^p(\mathbb{T}), \quad p \in (0, \infty].$$

One can easily check that if $f, g \in N$ and $|f| = |g|$ almost everywhere on $\mathbb{T}$ then $o_f = \lambda o_g$, where o_f and o_g are respectively the outer parts of f and g, and $|\lambda| = 1$.

2.1.4 Model spaces

Each inner function Θ in $\mathbb{D}$ produces a subspace of H^p as follows:

$$K_\Theta^p := H^p \cap \Theta \bar{z} \bar{H}^p.$$

In the above definition the intersection should be understood on the circle. The space K_Θ^p is known as the model spaces [17]. These subspaces play an important role in complex and harmonic analysis, operator theory, control theory and system analysis [36–38].

One can easily see that for each $\lambda \in \mathbb{D}$ and for each $1 \leq p \leq \infty$ the function

$$k_\lambda^\Theta(w) := \frac{1 - \bar{\Theta}(\lambda)\Theta(w)}{1 - \bar{\lambda}w}$$

belongs to the model space K_Θ^p. Given any function $f \in K_\Theta^p$ it is well-known that

$$f(\lambda) = \int_{-\pi}^{\pi} \overline{k_\lambda^\Theta(e^{it})} f(e^{it}) \frac{dt}{2\pi},$$

that is, the functions k_λ^Θ reproduces the functional value at the point λ of the open unit disk. Due to this reason the functions k_λ^Θ are known as the reproducing kernel.

By definition of K_Θ^p it is obvious that $f \in K_\Theta^p$ if and only if there exist $g \in H^p$ such that

$$f = \Theta \bar{w} \bar{g}$$

almost everywhere on the circle $\mathbb{T}$. Obviously such a g itself belong to K_Θ^p.

2.1.5 Operators on Hardy spaces

One of the most important operators on the Hardy spaces is the backward shift. The operator $S^* : H^p \longrightarrow H^p$ is defined as

$$S^* f(w) = \frac{f(w) - f(0)}{w} \tag{2.1.13}$$

is known as the backward shift on H^p. A crucial result of Beurling says that all the closed subspaces of H^p, $1 \leq p \leq \infty$ which are invariant with respect to the backward shift are precisely the model spaces.

Another important operator on the Hardy spaces is the Toeplitz operator. Given a function $u \in L^\infty(\mathbb{T})$, the Toeplitz operator T_u on H^p, $p \in [1, \infty]$, with the symbol u is defined as

$$(T_u f)(w) := \int_{-\pi}^{\pi} \frac{u(e^{it}) f(e^{it})}{e^{it} - w} \frac{dt}{2\pi}, \quad w \in \mathbb{D}.$$

It is a well-known fact that the operator T_u acts boundedly on H^p for $p \in (1, \infty)$ and fails to be bounded at both endpoints, that is, for $p = 1$ and $p = \infty$. However, we have

$$T_u(H^1) \subset \cap_{p \in (0,1)} H^p,$$

and

$$T_u(H^\infty) \subset \cap_{p \in (0,\infty)} H^p.$$

For a function $u \in L^\infty$, $\ker_p[u]$ will denote the kernel of the Toeplitz operator T_u in space H^p. One can easily check that

$$\ker_p[u] = H^p \cap u^{-1} \bar{w} \bar{H}^p, \; p \in [1, \infty].$$

Again the above intersection must be understood on $\mathbb{T}$. Thus for the inner function Θ we have $\ker_p[\bar{\Theta}] = K^p_\Theta$. It is easy to verify that, for any inner function I,

$$S^* I \in \ker_p[\bar{\Theta}] = K^p_\Theta \tag{2.1.14}$$

2.2 Function theory in the upper half-plane

In this section we recapitulate some definition and facts about the function theory in the upper half plan and draw connections with the function theory in the unit disk. The standard references for this section are [11, 18, 25, 33, 37].

Consider the linear fractional transformation

$$\omega(z) := \frac{z - i}{z + i}.$$

It is obvious that the map ω takes the open upper half-plane onto the open unit disk conformally and $\mathbb{R}$ onto $\mathbb{T} \setminus \{1\}$.

Clearly, the inverse map is again a linear fractional transformation and is given by

$$Z(w) := \omega^{-1}(w) = i\frac{1+w}{1-w}.$$

The map Z maps the the open unit disk onto the open upper half-plane and $\mathbb{T} \setminus \{1\}$ onto $\mathbb{R}$.

It is well-known that the map $\mathcal{U}_p : L^p(\mathbb{T}) \to L^p(\mathbb{R})$, defined as

$$(\mathcal{U}_p f)(t) = \left(\frac{1}{\sqrt{\pi}(t+i)}\right)^{\frac{2}{p}} \big(f \circ \omega\big)(t),$$

is an isometric isomorphism for all $p \in (0, \infty]$ [37, Page143].

2.2.1 Hardy spaces $\mathcal{H}^p(\mathbb{C}_+)$

Given $x + iy = z \in \mathbb{C}_+$ and for a function $F \in \mathrm{Hol}(\mathbb{C}_+)$ we set

$$\mathfrak{M}_p(y, f) := ||F_y||_p^p = \int_{\mathbb{R}} |F_y(x)|^p dx, \tag{2.2.1}$$

where $F_y(x) := F(z)$ and $p \in (0, \infty)$. For $p = \infty$ we let

$$\mathfrak{M}_\infty(y, F) = \sup_{x \in \mathbb{R}} |F_y(x)|. \tag{2.2.2}$$

The Hardy space $\mathcal{H}^p(\mathbb{C}_+)$ is defined as the collection of holomorphic functions F in the upper half-plane for which the integral means (2.2.1) are uniformly bounded.

For $p \in [1, \infty]$,

$$||F||_{H^p} := \sup_{y>0} ||F_y||_p$$

defines a norm on $\mathcal{H}^p(\mathbb{C}_+)$, and the space $\big(\mathcal{H}^p(\mathbb{C}_+), || \cdot ||_{\mathcal{H}^p}\big)$ turns out to be a Banach space and for $p \in (0, 1)$ the triangle inequality fails. However, the quantity

$$d_p(F, G) = ||F - G||_{\mathcal{H}^p}^p$$

defines a translation invariant metric on $H^p(\mathbb{C}_+)$.

As in the case of the unit disk the limit

$$\lim_{y \to 0} F_y(x) = F(x)$$

exists for almost all $x \in \mathbb{R}$, moreover the limit function $F(x)$ belongs to $L^p(\mathbb{R})$ and we have the following equality:

$$||F||_{\mathcal{H}^p} = ||F||_p.$$

The function $F(x)$ will be called the boundary value function of the function $F(z)$. It is also well-known (similar to the case of the unit disk) that for $p \in [1, \infty]$ one can recover the function $F(z)$ in the upper half-plane by its boundary value function $F(x)$ via Poisson integral formula as follows:

$$F(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\Im z}{|t - z|^2} F(t) dt.$$

The collection of all boundary value functions will be denoted by $\mathcal{H}^p(\mathbb{R})$. The above discussion tells us that there is a one-to-one correspondence between $\mathcal{H}^p(\mathbb{C}_+)$ and $\mathcal{H}^p(\mathbb{R})$ for $p \in [1, \infty]$. This is true even for $p \in (0, 1)$. And hence we do not make any distinction between $\mathcal{H}^p(\mathbb{C}_+)$ and $\mathcal{H}^p(\mathbb{R})$, and write $\mathcal{H}^p$ for both spaces and understand from the context what we are talking about.

The following theorem reveals the connection between $\mathcal{H}^p$ and H^p.

Theorem 2.2.1. *The map*

$$\mathcal{U}_p : H^p :\longrightarrow \mathcal{H}^p,$$

defined by

$$\mathcal{U}_p f(z) := \frac{1}{\pi^{1/p}(z + i)^{2/p}} f(\omega(z)), \qquad 0 < p < \infty, \tag{2.2.3}$$

and

$$\mathcal{U}_\infty f(z) = f(\omega(z)),$$

is an isometric isomorphism from H^p onto $\mathcal{H}^p$.

2.2.2 Canonical factorization

Let the sequence $\{z_k = x_k + iy_k\}$ in the upper half-plane be such that

$$\sum_{k=1}^{\infty} \frac{y_k}{x_k^2 + (y_k + 1)^2} < \infty,$$

and let the sequence $\{\alpha_k\}$ of real numbers be chosen so that

$$e^{i\alpha_k} \frac{i - z_k}{i - \bar{z}_k} \geq 0.$$

Then the infinite product

$$B(z) = \prod_{k=1}^{\infty} e^{i\alpha_k} \frac{z - z_k}{z - \bar{z}_k}$$

is a well-defined holomorphic function on $\mathbb{C} \setminus \mathrm{Cl}\{\bar{z}_k\}$, where $\mathrm{Cl}\{\bar{z}_k\}$ denotes the closure of the sequence $\{z_k\}$. The function B is called a Blaschke product. Note that if $|z_k| \to \infty$ then B turns out to be a meromorphic function in the complex plane with zeros at $\{z_k\}$ and poles at $\{\bar{z}_k\}$. Such a Blaschke product is said to be a *meromorphic Blaschke product.* Suppose that B is a meromorphic Blaschke product, then it is continuous $\mathbb{R}$ and hence $B(0)$ is a nonzero complex number. Defining

$$\hat{B}(z) := \frac{B(z)}{B(0)} = \prod_{k=1}^{\infty} B_k(z) = \prod_{k=1}^{\infty} \frac{1 - \frac{z}{z_k}}{1 - \frac{z}{\bar{z}_k}}, \tag{2.2.4}$$

then $\hat{B}$ itself is a meromorphic Blaschke product with the same zeros and poles as that of B and also satisfies $\hat{B}(0) = 1$. One can easily check that b is a Blaschke product in $\mathbb{D}$ if and only if $B = b \circ \omega$ is a Blaschke product in $\mathbb{C}_+$.

As in the case of the unit disk, a function $I \in \mathcal{H}^\infty$ is said to be inner if $|I(x)| = 1$ almost everywhere on the real line $\mathbb{R}$. It is not hard to check that $|B(x)| = 1$ almost everywhere on $\mathbb{R}$ for any Blaschke product B. In particular, Blaschke product are inner functions. It is straightforward to check that Θ is inner on $\mathbb{D}$ if and only if $I = \Theta \circ \omega$ is inner in $\mathbb{C}_+$. Another type of inner function are those having no zeros in $\mathbb{C}_+$. Such inner functions are known as singular inner functions. Again s is a singular inner function in $\mathbb{D}$ if and only if $S = s \circ \omega$ is singular in $\mathbb{C}_+$. Every singular inner function S on $\mathbb{C}_+$ has the representation

$$S(z) = e^{-iaz} \exp\left(i \int_{\mathbb{R}} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\mu(t) \right), \tag{2.2.5}$$

where $a \leq 0$, and μ is a nonnegative singular Borel measure on the real line without point mass satisfying

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty. \tag{2.2.6}$$

An outer function O is an holomorphic function in the upper half-plane which can be represented as

$$O(z) = \exp\left(\frac{1}{i\pi} \int_{\mathbb{R}} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] \log K(t) dt \right),$$

where $K(t) \geq 0$, for almost all $t \in \mathbb{R}$, and

$$\int_{\mathbb{R}} \frac{|\log K(t)|}{1+t^2} dt < \infty.$$

As before o is outer in $\mathbb{D}$ if and only if $O = o \circ \omega$ is outer in $\mathbb{C}_+$. Similar to the case of the unit disk, given any function $F \in \mathcal{H}^p$ we have

$$F = BSO,$$

where B is a Blaschke product cooked from the zeros (counted according to the multiplicity) of F in $\mathbb{C}_+$, S is a singular inner function and O is an outer function. The factorization is unique upto a constant multiple of unit modulus.

Again as in the case of the unit disk, an important consequence of canonical factorization is the fact that $F \in \mathcal{H}^p \Rightarrow \log|F| \in L^1(\mathbb{R})$, and this immediately implies that any $\mathcal{H}^p$ function cannot vanish on a set of positive Lebesgue measure unless vanishes identically.

2.2.3 Nevanlinna and Smirnov class

Bounded holomorphic functions are building blocks for the *Nevanlinna class*

$$\mathcal{N}(\mathbb{C}_+) := \left\{ \frac{G}{H} \in \mathrm{Hol}(\mathbb{C}_+) : G, H \in \mathcal{H}^\infty(\mathbb{C}_+),\ H \neq 0 \right\}.$$

The functions in $\mathcal{N}(\mathbb{C}_+)$ also have non-tangential boundary limits. This is a direct consequence of the previous discussion about bounded functions and a uniqueness theorem which ensures the non-tangential limits of the function in the denominator does not vanish on a set of positive Lebesgue measure. The collection of boundary values will be denoted by $\mathcal{N}(\mathbb{R})$. Thus, we have a one-to-one correspondence between $\mathcal{N}(\mathbb{C}_+)$ and $\mathcal{N}(\mathbb{R})$. From now onward we do not make any distinction between the classes $\mathcal{N}(\mathbb{C}_+)$ and $\mathcal{N}(\mathbb{R})$ and commonly denote both of them by $\mathcal{N}$. There is a vast literature on Nevanlinna class as it appears on many topics in complex and harmonic analysis. See [29, 30, 35, 39–41] and references therein. The following factorization [44, 6.13, p.128] of functions in $\mathcal{N}(\mathbb{C}_+)$ is a celebrated factorization theorem.

Theorem 2.2.2. 1. *Every nonzero function $F \in \mathcal{N}(\mathbb{C}_+)$ has the factorization*

$$F(z) = e^{-i\tau_F z} B_F \frac{S_+(z)}{S_-(z)} O_F, \tag{2.2.7}$$

where τ_F is a real number, B_F is a Blaschke product associated to the zeros of F, O_F is an outer function, and S_+ and S_- have the form

$$S_\pm(z) = \exp\left(i \int_{\mathbb{R}} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\mu_\pm(t) \right), \tag{2.2.8}$$

where μ_+ and μ_- are nonnegative singular and mutually singular Borel measures on $\mathbb{R}$ satisfying (2.2.6). This factorization is essentially unique: $e^{-i\tau_F z}$, S_+ and S_- are uniquely determined, and B and O are determined up to unimodular multiplicative constants.

2. *If F has holomorphic continuation across some open interval I on $\mathbb{R}$, then the measures $\mu_\pm$ in (i)satisfy $\mu_+|_I = \mu_-|_I = 0$.*

Given a function $F \in \mathcal{N}(\mathbb{C}_+)$, the real number τ_F that appeared in the factorization (2.2.7) is known as the *mean type* of F. The following result [44, 6.15, p.130] is a well-known formula to evaluate the mean type of F.

Theorem 2.2.3. *The mean type τ_F of any nonzero function $F \in \mathcal{N}(\mathbb{C}_+)$ is given by*

$$\tau_F = \limsup_{y\to\infty} \frac{\log|F(iy)|}{y}. \tag{2.2.9}$$

Again it is straightforward form the definition that a function $f \in N$ if and only if the function $F := f \circ \omega$ belong to $\mathcal{N}$. We consider the subclass $\mathcal{N}_* \subset \mathcal{N}$ as follows

$$\mathcal{N}_\star = \mathcal{N} \cap C^\omega(\mathbb{R}),$$

where $C^\omega(\mathbb{R})$ denotes the class of real analytic functions. It is easy to check that given $F \in \mathcal{N}_*$ we have

$$F = e^{-i\tau_F} BO, \tag{2.2.10}$$

where $\tau_F \in \mathbb{R}$, B is a meromorphic Blaschke product and $O \in C^\omega(\mathbb{R})$ is an outer function.

The Smirnov class $\mathcal{N}^+$ is defined as

$$\mathcal{N}^+ := \{F \in \mathcal{N} : F = e^{-i\tau_F} BS_{wp}O, \text{ such that } \tau_F \le 0\}.$$

Throughout this dissertation the singular inner function S_{wp} will have the form

$$S_{wp}(z) = \exp\left(i\int_{\mathbb{R}}\left[\frac{1}{t-z} - \frac{t}{1+t^2}\right]d\mu(t)\right),$$

where $a \le 0$, and μ is a nonnegative singular Borel measure on the real line without point mass satisfying (2.2.6.)

The subclass $\mathcal{N}^+_* \subset \mathcal{N}^+$ is defined as

$$\mathcal{N}^+_* = \mathcal{N}^+ \cap \mathcal{N}_*.$$

Again it is easy to check that if $F \in \mathcal{N}_*^+$ then

$$F = e^{-i\tau_F} BO,$$

where $\tau_F \leq 0$, B is a meromorphic Blaschke product and $O \in C^\omega(\mathbb{R})$ is an outer function.

2.2.4 Model spaces

Given an inner function I one can define the model space

$$\mathcal{K}_I = \mathcal{H}^2 \ominus I\mathcal{H}^2,$$

where I is an inner function are the most important subclasses of the Hardy space H^2 [17]. These subspaces play an important role in complex and harmonic analysis, operator theory, control theory and system analysis [36–38]. For a meromorphic inner function I, we can also define model spaces in the Smirnov class [29, 41] by

$$\mathcal{K}_I^+ = \mathcal{N}_*^+ \cap I\bar{\mathcal{N}}_*^+$$

and then in other Hardy spaces by

$$\mathcal{K}_I^p = \mathcal{K}_I^+ \cap L^p(\mathbb{R}).$$

2.2.5 Toeplitz operators

The Toeplitz operator T_U with the symbol $U \in L^\infty(\mathbb{R})$ is the mapping $T_U : \mathcal{H}^2 \longrightarrow \mathcal{H}^2$ defined by

$$T_U(f) = P_+ Uf,$$

where P_+ denotes the orthogonal projection of L^2 onto $\mathcal{H}^2$. We denote the kernel of the operator T_U by Ker$[U]$ and call it Toeplitz kernel associated with the symbol U. We can also define Toeplitz kernel associated with the unimodular symbol U in Smirnov class by

$$\text{Ker}^+[U] = \left\{ f \in \mathcal{N}^+ \cap L^1_{\text{loc}}(\mathbb{R}) : \bar{U}\bar{f} \in \mathcal{N}^+ \right\}, \tag{2.2.11}$$

and then in other Hardy spaces by

$$\text{Ker}_p[U] = \text{Ker}^+[U] \cap L^p(\mathbb{R}).$$

The condition of local summability in the definition of the kernel has a function theoretic reason: if $0 < p < 1$ and I is a meromorphic inner function, one needs the local summability to have $\text{Ker}_p[\bar{I}] = \mathcal{K}_I^p$, See [29, 41]. The following result [29, Lemma 3.12] or [41, Lemma 33] establishes a connection between Toeplitz kernel and $C^\omega(\mathbb{R})$.

Lemma 2.2.4. *Let $\gamma : \mathbb{R} \to \mathbb{R}$ and $\gamma \in C^\omega(\mathbb{R})$, then $\text{Ker}^+[e^{i\gamma}] \subset C^\omega(\mathbb{R})$.*

2.2.6 Cartwright-de Branges space

An entire function E is said to be a *de Branges function* if

$$|E^\sharp(z)| < |E(z)|, \qquad z \in \mathbb{C}_+,$$

where $E^\sharp(z) = \overline{E(\overline{z})}$. Roughly speaking, de Branges functions are generalizations of polynomials having no zeros in the upper half-plane. It is a well-known result that an inner function I is meromorphic if and only if there exists a de Branges function E such that $I = E^\sharp/E$. We define the *Cartwright-de Branges space* $\mathcal{B}^+(E)$ associated with a de Branges function E as

$$\mathcal{B}^+(E) := \left\{ F \in \text{Hol}(\mathbb{C}) : \frac{F}{E} \in \mathcal{N}^+(\mathbb{C}_+) \text{ and } \frac{F^\sharp}{E} \in \mathcal{N}^+(\mathbb{C}_+) \right\}. \tag{2.2.12}$$

In the special case $E(z) = e^{-2\pi i a z}$, $a \geq 0$, the Cartwright-de Branges space $\mathcal{B}^+(E)$ coincides with the more classical *Cartwright class* $\mathcal{C}_a$, the space of entire functions F of exponential type at most $2\pi a$ such that $\log^+|F|$ is Poisson summable on the real line. Cartwright class $\mathcal{C}$ is the collection of entire functions F of exponential type such that $\log^+|F|$ is Poisson summable on the real line, i.e.,

$$\mathcal{C} = \bigcup_{a \geq 0} \mathcal{C}_a.$$

The following result establishes the connection between the Cartwright-de Branges spaces and model spaces. Various versions of this proposition in available in the literature. E.g., see [1, Theorem 2.1], [20, Theorem 2.10], [29, Proposition 2.8] and [41, Proposition 7].

Proposition 2.2.5. *Let E be a de Branges function. Then*

$$\mathcal{B}^+(E) = E\mathcal{K}^+_{I_E},$$

where I_E is the meromorphic inner function associated to E, i.e., $I_E = E^\sharp/E$.

2.2.7 Schrodinger operator and Weyl inner function

Consider the Schrodinger equation

$$-\ddot{u} + qu = zu \tag{2.2.13}$$

on some interval (a, b) with the locally integrable potential q. Let us recall that the point a is called regular if (i) a is finite and (ii) $q \in L^1(a, c)$ for some $c \in (a, b)$. The point a is said to be singular if it is not regular.

We consider the Schrodinger equation (2.2.13) and assume that the point a is regular and fix a self-adjoint boundary condition at the point b then one can solve the equation (2.2.13) for any $z \in \mathbb{C}$. Suppose that for each z the corresponding solution be $u(x, z)$. It is well-known that the function $u(x, z)$ is entire with respect to z. Now the function

$$m(z) := \frac{\dot{u}(a, z)}{u(a, z)}$$

turns out to be a meromorphic function and is known as the Weyl m-function. Similarly if the point b is regular and we fix a self-adjoint boundary condition at the point a then the Weyl m-function turns out to be

$$m(z) := -\frac{\dot{u}(b, z)}{u(b, z)}.$$

Recall that a meromorphic function F in the $\mathbb{C}$ is said to be a Herglotz function if $F^\sharp(z) = F(z)$, and the map $F : \mathbb{C}_+ \to \mathbb{C}_+$ is holomorphic. One can easily check that each Herglotz function F gives rise to a meromorphic inner function

$$I = \frac{F - i}{F + i}.$$

Conversely, each meromorphic inner function I gives rise to a Herglotz function

$$F = i\frac{1 + I}{1 - I}.$$

It is well-known that the Weyl m-function as defined above is a Herglotz function. Thus each Schrodinger equation gives rise to a meromorphic inner function I_m as per the scheme discussed above.

CHAPTER 3

Hilbert transform: an explicit formula

The celebrated integral transforms such as Fourier transform, Laplace transform, and Hilbert transform have tremendous applications in various branches of science and engineering. However, unlike to Fourier or Laplace transform, very few functions have an explicit formula for their Hilbert transforms. In this chapter we obtain an explicit formula for the Hilbert transform of $\log|F|$, for the function F in Nevanlinna class having real analytic boundary value. This family is the largest possible for which such a formula for the Hilbert transform of $\log|F|$, can be obtained. The formula is very general and implies several previously known results. Results of this chapter are present in the contributed articles 1 and 2.

3.1 Introduction

Let F be an integrable function on the real line. In principle, we can evaluate the Hilbert transform of F with the help of complex analysis (boundary values of the conjugate function) or real analysis (a singular integral). More explicitly, in the complex analysis method one needs to perform the following steps to obtain the Hilbert transform: Extend the function F to a harmonic function in the upper half-plane $\mathbb{C}_+ := \{z = x + iy \in$

$\mathbb{C}|\ y > 0\}$. Then identify its harmonic conjugate. Finally, take the non-tangential limit of the harmonic conjugate to obtain the Hilbert transform of F. In real variable method one needs to use a well-known singular integral formula. We briefly discuss both methods below. Since in general both involve cumbersome limiting processes, they turn out to be highly complicated to evaluate the Hilbert transform using any one of these methods. Thus any explicit formula of the Hilbert transform is worthwhile and finds its place in applications.

The complex analysis technique: Let G be a real valued function on the real line with finite logarithmic integral, that is,

$$\int_{\mathbb{R}} \frac{|G(t)|}{1+t^2}\, dt < \infty. \tag{3.1.1}$$

Then the *Schwartz integral* of G is defined as

$$\mathcal{S}G(z) := \frac{1}{i\pi} \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) G(t)\, dt, \tag{3.1.2}$$

which is an analytic function in the upper half-plane. We note that

$$\mathrm{Re} \left[\frac{1}{i\pi} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \right] = \frac{1}{\pi} \frac{y}{(t-x)^2 + y^2},$$

and

$$\Im \left[\frac{1}{i\pi} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \right] = \frac{1}{\pi} \left[\frac{x-t}{(t-x)^2 + y^2} + \frac{t}{1+t^2} \right].$$

Let us define K and $\widetilde{K}$ in the upper half-plane by

$$K(z) := \mathrm{Re}\, \mathcal{S}G(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(t-x)^2 + y^2} G(t)\, dt,$$

and

$$\widetilde{K}(z) := \Im \mathcal{S}G(z) = \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{x-t}{(t-x)^2 + y^2} + \frac{t}{1+t^2} \right) G(t)\, dt.$$

Observe that K is the unique harmonic extension of G to the upper half-plane, and $\widetilde{K}$ is the unique harmonic conjugate of K normalized such that $\widetilde{K}(i) = 0$. It is well-known that the functions K and $\widetilde{K}$ have non-tangential limits on the real line almost everywhere. The non-tangential limit of K is G and the non-tangential limit of $\widetilde{K}$ is denoted by $\widetilde{G}$ which is known as the Hilbert transform of G.

The real analysis technique: If we do not want to move away from the real line, then in order to define the Hilbert transform we need to apply the theory of singular integrals. Suppose $k \in L^1(\mathbb{R})$. Then, for any $1 \le p \le \infty$, the operator $T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ given by

$$TG(x) = \int_{\mathbb{R}} k(x-t)G(t)\,dt \tag{3.1.3}$$

is well-defined and bounded on L^p. However, operators that usually appear in applications have non-integrable kernels. A celebrated example is the Hilbert transform. This point of view takes us to the fascinating theory of singular operators, which has a very rich history and leads to several applications in analysis, mathematical physics, image processing, signal processing, and many other disciplines of science and engineering, see [14, 22, 23, 29, 30, 41] and references therein. If we take $k(t) = \frac{1}{\pi t}$, then the integral

$$Hg(x) = \widetilde{g}(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{g(t)}{x-t}\,dt, \qquad x \in \mathbb{R}, \tag{3.1.4}$$

may fail to exist, for instance, as a Lebesgue integral. Nevertheless, the Cauchy principal value

$$HG(x) = \widetilde{G}(x) = \lim_{\epsilon \to 0} \frac{1}{\pi}\int_{|x-t|>\epsilon} \frac{G(t)}{x-t}\,dt, \qquad x \in \mathbb{R}, \tag{3.1.5}$$

exists almost everywhere and produces a nicely behaved operator H, known as the Hilbert transform of G. The operator H has many deep and intriguing properties, for instance, the classical Riesz theorem is one of them which asserts that the operator H is bounded on $L^p(\mathbb{R})$ for $p \in (1, \infty)$. If the function G is not in $L^p(\mathbb{R})$, but at least satisfies the milder condition (3.1.1) then, in this case, the Hilbert transform of G is defined by the modified formula

$$HG(x) = \widetilde{G}(x) = \lim_{\epsilon \to 0} \frac{1}{\pi}\int_{|x-t|>\epsilon} \left(\frac{1}{x-t} + \frac{t}{1+t^2}\right) G(t)\,dt.$$

In most cases, it is highly difficult to evaluate the Hilbert transform of functions directly by any one of the above discussed methods. In this chapter, we obtain an explicit formula for the Hilbert of $\log|F|$, where F is any function in the Nevanlinna class having real analytic boundary value. One motivation to carry out this work stems from the work [34], where an explicit formula for the Hilbert transform of $\log|F|$ is given. We extend the main result in [34] to a larger class. Another motivation is due to the recent work of Makarov and Poltoratski [29, 41], where they treat the uniqueness sets for model spaces. As a consequence of the explicit formula, we derive all the results of [34] and some results of [29, 41].

3.2 Argument of a meromorphic Blaschke product on $\mathbb{R}$

Let us recall the meromorphic Blaschke product (2.2.4)

$$\hat{B}(z) := \frac{B(z)}{B(0)} = \prod_{k=1}^{\infty} B_k(z) = \prod_{k=1}^{\infty} \frac{1-\frac{z}{z_k}}{1-\frac{z}{\bar{z}_k}},$$

where $\{z_k\} \subset \mathbb{C}_+$ is a Blaschke sequence with $|z_k| \to \infty$. Since $\hat{B}$ is analytic and unimodular on $\mathbb{R}$, there is a $\mathcal{C}^\infty$-function $\vartheta(t)$ such that $\hat{B}(t) = e^{i\vartheta(t)}$ on $\mathbb{R}$ [7]. To find a formula for ϑ note that

$$B_k(t) = \frac{1-\frac{t}{z_k}}{1-\frac{t}{\bar{z}_k}} = \frac{\left(1-\frac{\text{Re } z_k}{|z_k|^2}t\right) + i\left(\frac{\Im z_k}{|z_k|^2}t\right)}{\left(1-\frac{\text{Re } z_k}{|z_k|^2}t\right) - i\left(\frac{\Im z_k}{|z_k|^2}t\right)} = \frac{e^{i\vartheta_{z_k}(t)}}{e^{-i\vartheta_{z_k}(t)}} = e^{2i\vartheta_{z_k}(t)}.$$

The first candidate for ϑ_{z_k} is

$$\vartheta_{z_k}(t) = \arctan\left(\frac{\frac{\Im z_k}{|z_k|^2}t}{1-\frac{\text{Re } z_k}{|z_k|^2}t}\right), \qquad t \in \mathbb{R}. \tag{3.2.1}$$

As a matter of fact, if Re $z_k = 0$, then

$$\vartheta_{z_k}(t) = \arctan\left(\frac{t}{\Im z_k}\right), \qquad t \in \mathbb{R}, \tag{3.2.2}$$

is a well-defined $\mathcal{C}^\infty$ function on $\mathbb{R}$. But, if Re $z_k \neq 0$ then the candidate function (3.2.1) is not continuous at $t = \frac{|z_k|^2}{\text{Re } z_k}$. In this case, we need to consider the following branches of arctan [34]. If Re $z_k > 0$, let

$$\vartheta_{z_k}(t) = \begin{cases} \arctan\left(\frac{\frac{\Im z_k}{|z_k|^2}t}{1-\frac{\text{Re } z_k}{|z_k|^2}t}\right) + \pi, & t > \frac{|z_k|^2}{\text{Re } z_k}, \\ \frac{\pi}{2}, & t = \frac{|z_k|^2}{\text{Re } z_k}, \\ \arctan\left(\frac{\frac{\Im z_k}{|z_k|^2}t}{1-\frac{\text{Re } z_k}{|z_k|^2}t}\right), & t < \frac{|z_k|^2}{\text{Re } z_k}, \end{cases} \tag{3.2.3}$$

and, if Re $z_k < 0$, let

$$\vartheta_{z_k}(t) = \begin{cases} \arctan\left(\dfrac{\frac{\Im z_k}{|z_k|^2}t}{1 - \frac{\text{Re } z_k}{|z_k|^2}t}\right), & t > \frac{|z_k|^2}{\text{Re } z_k}, \\ -\frac{\pi}{2}, & t = \frac{|z_k|^2}{\text{Re } z_k}, \\ \arctan\left(\dfrac{\frac{\Im z_k}{|z_k|^2}t}{1 - \frac{\text{Re } z_k}{|z_k|^2}t}\right) - \pi, & t < \frac{|z_k|^2}{\text{Re } z_k}. \end{cases} \tag{3.2.4}$$

Since $B_k(0) = 1$, we normalized ϑ_{z_k} such that $\vartheta_{z_k}(0) = 0$. Therefore,

$$\hat{B}(t) = \prod_{k=1}^{\infty} b_k(t) = \prod_{k=1}^{\infty} e^{2i\vartheta_{z_k}(t)} = \exp\left(2i\sum_{k=1}^{\infty}\vartheta_{z_k}(t)\right).$$

Hence, the $\mathcal{C}^\infty$-argument is

$$\vartheta(t) = 2\sum_{k=1}^{\infty}\vartheta_{z_k}(t), \qquad t \in \mathbb{R}.$$

3.3 Representation of functions in the class $\mathcal{N}_*$ on $\mathbb{R}$

In this section we obtain an important representation of a function $F \in \mathcal{N}_*$ on the real line. By the definition of $\mathcal{N}_\star(\mathbb{C}_+)$, for each $z = x + iy \in \mathbb{C}_+$,

$$F(z) = e^{-i\tau_F z} B_F(z) O_F(z).$$

Since the Blaschke product B_F is meromorphic, then using (**??**) and the expression of O_f we obtain

$$\begin{aligned} F(z) &= B_F(0)e^{-i\tau_F z}\hat{B}_F(z)\exp\left(\frac{1}{i\pi}\int_{\mathbb{R}}\left[\frac{1}{t-z} - \frac{t}{1+t^2}\right]\log|F(t)|dt\right) \\ &= B_F(0)e^{-i\tau_F z}\hat{B}_F(z)\exp\left\{\frac{1}{\pi}\int_{\mathbb{R}}\frac{y}{(t-x)^2+y^2}\log|F(t)|dt\right. \\ &\qquad \left. + i\frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{x-t}{(t-x)^2+y^2} + \frac{t}{1+t^2}\right)\log|F(t)|dt\right\}. \end{aligned}$$

Now, we let z approaches non-tangentially to $t \in \mathbb{R} \setminus \{x_k\}$, where $\{x_k\}$ is the sequence of real zeros (counting multiplicity) of F. Then, by [33, Corollary 10.6],

$$\lim_{z\to t}\exp\left(\frac{1}{\pi}\int_{\mathbb{R}}\frac{y}{(t-x)^2+y^2}\log|F(t)|dt\right) = |F(t)|,$$

and, by [33, Theorems 14.4 and 14.7],

$$\lim_{z\to t} \exp\left\{ i\frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{x-t}{(t-x)^2+y^2} + \frac{t}{1+t^2} \right) \log|F(t)|dt \right\} = \exp\left(i\widetilde{\log|F|}(t) \right).$$

The function $B_F(z)$ behaves well at all points of $\mathbb{R}$. Hence, we conclude,

$$F(t) = B_F(0)e^{-i\tau_F t}\hat{B}_F(t)|F(t)|\exp\left(i\widetilde{\log|F|}(t) \right).$$

Thus above representation proves the following technical lemma.

Lemma 3.3.1. *Let F be a nonzero function in $\mathcal{N}_\star$. Let $\{x_k\}$ be the sequence of real zeros of F. Then, for every $t \in \mathbb{R} \setminus \{x_k\}$,*

$$\frac{F(t)}{|F(t)|} = B_F(0)e^{-i\tau_F t}\hat{B}_F(t)e^{i\widetilde{\log|F|}(t)}, \qquad (3.3.1)$$

where B_F is a Blaschke product associated to the upper half-plane zeros of F.

3.4 Hilbert Transform of $\log|F|$

Let $\Lambda \subset \mathbb{R}$ be a discrete set of real numbers whose elements are arranged in increasing order. Then the *counting function* N_Λ of Λ is defined by

$$N_\Lambda(t) = \sum_{\lambda\in\Lambda} N_\lambda(t), \qquad t \in \mathbb{R},$$

where, for $\lambda \geq 0$,

$$N_\lambda(t) = \begin{cases} 1, & t \geq \lambda, \\ 0, & t < \lambda, \end{cases}$$

and, for $\lambda < 0$,

$$N_\lambda(t) = \begin{cases} -1, & t < \lambda, \\ 0, & t \geq \lambda. \end{cases}$$

It is obvious that the counting function is constant between any two consecutive points and jumps down at left end points and jumps up at right end points. The following result [34, Lemma 5.1] is very crucial in obtaining the explicit formula for the Hilbert transform of $\log|F|$.

Lemma 3.4.1. *Let* $F \in C^{\omega}(\mathbb{R})$*, and* $\{x_k\}$ *be its sequence of zeros, counting multiplicity. Suppose that* $\log|F| \in L^1(\frac{dt}{1+t^2})$*. Then* $\widetilde{\log|F|} + N_\Lambda$ *is a continuous function on* $\mathbb{R}$*.*

Lemma 3.4.1 tells us that the counting function N_Λ is defined to take care of the real zeros of F. this lemma is implicitly used in the following lemma.

Lemma 3.4.2. *Let* F *be a nonzero function in* $\mathcal{N}_\star$*. Suppose that* $G \in \mathcal{N}_\star$ *is such that* $|F| = |G|$ *on the real line. Let* $\{x_k\}$ *and* $\{z_k\}$ *be the sequence of real and upper half-plane zeros of* F *respectively, and* $\{w_k\}$ *be the sequence of upper half plane zeros of* G*. Then, for each* $t \in \mathbb{R} \setminus \{x_k\}$*,*

$$\widetilde{\log|F|}(t) \equiv \theta + \left(\frac{\tau_F + \tau_G}{2}\right)t + \frac{1}{2}\arg\frac{F(t)}{\bar{G}(t)} - \sum_k \vartheta_{z_k}(t) - \sum_k \vartheta_{w_k}(t) \pmod{\pi}.$$

Proof. Applying Lemma (3.3.1) to G, we get

$$\frac{G(t)}{|F(t)|} = B_G(0)e^{-i\tau_G t}\hat{B}_G(t)e^{i\widetilde{\log|F|}(t)}, \tag{3.4.1}$$

for every $t \in \mathbb{R} \setminus \{x_k\}$. Comparing (3.3.1) and (3.4.1) we obtain

$$\frac{F(t)}{\bar{G}(t)} = B_F(0)B_G(0)e^{-i(\tau_F+\tau_G)t}\hat{B}_F(t)\hat{B}_G(t)e^{2i\widetilde{\log|F|}(t)}. \tag{3.4.2}$$

Since $|F| = |G|$, it is obvious that

$$\frac{F(t)}{\bar{G}(t)} = \exp\left(i\arg\frac{F(t)}{\bar{G}(t)}\right). \tag{3.4.3}$$

Also, as we have discussed in Section 3.2, we can write

$$\begin{cases} \hat{B}_F(t) & = \exp 2i\sum_k \vartheta_{z_k}, \\ \hat{B}_G(t) & = \exp 2i\sum_k \vartheta_{w_k}. \end{cases} \tag{3.4.4}$$

Combining (3.4.2), (3.4.3) and (3.4.4) gives

$$e^{2i\widetilde{\log|F|}(t)} = e^{i2\left(\theta + \left(\frac{\tau_F+\tau_G}{2}\right)t + \frac{1}{2}\arg\frac{F(t)}{\bar{G}(t)} - \sum_k \vartheta_{z_k} - \sum_k \vartheta_{w_k}\right)}, \tag{3.4.5}$$

where θ is a constant so chosen that $B_F(0)B_G(0) = e^{-2i\theta}$. Thus, for every $t \in \mathbb{R} \setminus \{x_k\}$, we have

$$\widetilde{\log|F|}(t) \equiv \theta + \left(\frac{\tau_F + \tau_G}{2}\right)t + \frac{1}{2}\arg\frac{F(t)}{\bar{G}(t)} - \sum_k \vartheta_{z_k} - \sum_k \vartheta_{w_k} \pmod{\pi}.$$

□

Recall the formulas for the argument $\vartheta_{z_k}(t)$, which was given by (3.2.2), (3.2.3) and (3.2.4), and the mean type τ by (2.2.9). The main theorem of this chapter is the following.

Theorem 3.4.3. *Let F be a nonzero function in $\mathcal{N}_\star$. Suppose that $G \in \mathcal{N}_\star$ is such that $|f| = |g|$ on the real line. Let $\{x_k\}$ and $\{z_k\}$ be the sequence of real and upper half-plane zeros of F respectively, and $\{w_k\}$ be the sequence of upper half plane zeros of G. Then, for each $t \in \mathbb{R} \setminus \{x_k\}$,*

$$\begin{aligned} \widetilde{\log|F|}(t) &= -\pi N_{\{x_k\}}(t) + \left(\frac{\tau_F + \tau_G}{2}\right)t + \frac{1}{2}\arg\left(\frac{F(t)}{\bar{G}(t)}\right) \\ &\quad - \sum_k \vartheta_{z_k}(t) - \sum_k \vartheta_{w_k}(t) + \theta. \end{aligned} \tag{3.4.6}$$

Proof. As we observed before, the counting function $N_{\{x_k\}}$ is a step function which jumps up at each x_k by an integer. Thus, for every $t \in \mathbb{R} \setminus \{x_k\}$,

$$\pi N_{\{x_k\}} \equiv \theta_1$$

mod π, where θ_1 is a constant. Hence, by Lemma 3.4.2, for every $t \in \mathbb{R} \setminus \{x_k\}$,

$$\begin{aligned} \widetilde{\log|F|}(t) + \pi N_{\{x_k\}}(t) &\equiv \widetilde{\log|F|}(t) + \theta_1 \\ &\equiv \theta_1 + \theta + \left(\frac{\tau_F + \tau_G}{2}\right)t \\ &\quad + \frac{1}{2}\arg\frac{F(t)}{\bar{G}(t)} - \sum_k \vartheta_{z_k} - \sum_k \vartheta_{w_k} \end{aligned}$$

mod π. However, the left hand side is a continuous function of t. Since the sums are locally uniformly convergent, the right hand side is also a continuous function of t. Hence, there is a constant θ_2 such that

$$\begin{aligned} \widetilde{\log|F|}(t) + \pi N_{\{x_k\}}(t) = \theta_2 + \theta_1 + \theta + \left(\frac{\tau_F + \tau_G}{2}\right)t \\ + \frac{1}{2}\arg\frac{F(t)}{\bar{G}(t)} - \sum_k \vartheta_{z_k} - \sum_k \vartheta_{w_k}. \end{aligned}$$

□

Theorem 3.4.3 is a very general result which implies several previously known results as a special case. We first prove this result and then collect some immediate corollaries of Theorem 3.4.3 in the following Section.

3.5 Some immediate consequences of Theorem 3.4.3

In this section, we have collected several interesting results which follow rather easily from Theorem 3.4.3. The first corollary gives us an explicit formula for $\widetilde{\log|F|}$ for functions in the Toeplitz kernels with certain symbols.

Corollary 3.5.1. *Let $\gamma : \mathbb{R} \to \mathbb{R}$ and $\gamma \in C^{\omega}(\mathbb{R})$. Suppose that F be any nonzero function in the Toeplitz kernel $\mathrm{Ker}^{+}[e^{i\gamma}]$. Let $\{x_k\}$ and $\{z_k\}$ be respectively the sequence of real and upper half-plane zeros of F. Suppose that $\{w_k\}$ be the sequence of zeros of $e^{-i\gamma}\bar{F}$ in the upper half-plane. Then for $t \in \mathbb{R} \setminus \{x_k\}$,*

$$\widetilde{\log|F|}(t) = -\pi N_{\{x_k\}}(t) + \left(\frac{\tau_F + \tau_{e^{-i\gamma}\bar{F}}}{2}\right)t - \frac{\gamma(t)}{2} - \sum_k \vartheta_{z_k}(t) - \sum_k \vartheta_{w_k}(t) + \theta,$$

where θ is a constant.

Proof. Take $G = e^{-i\gamma}\bar{F}$. Lemma 2.2.4 tells us that $F,\ G \in \mathcal{N}_{\star}$. Now apply Theorem 3.4.3. □

Corollary 3.5.2. *Let I be a meromorphic inner function, and let $F \in K_I^{+}$. Let $\{x_k\}$ and $\{z_k\}$ be respectively the sequence of real and upper half-plane zeros of F. Let $\{w_k\}$ and $\{\alpha_k\}$ be the sequence of zeros of $I\bar{F}$ and I in the upper half-plane. Then, for every $t \in \mathbb{R} \setminus \{x_k\}$,*

$$\widetilde{\log|F|}(t) = -\pi N_{\{x_k\}}(t) + \left(\frac{\tau_F + \tau_{I\bar{F}} - \tau_I}{2}\right)t + \sum_k \vartheta_{\alpha_k}(t) - \sum_k \vartheta_{z_k}(t) - \sum_k \vartheta_{w_k}(t) + \theta,$$

where θ is a constant.

Proof. Since I is meromorphic we have

$$I(t) = B_I(0)e^{-i\tau_I t}\hat{B}_I(t) = \exp i\Big(\theta_3 - \tau_I t + 2\sum_k \vartheta_{\alpha_k}(t)\Big),$$

where θ_3 is so chosen that $B_I(0) = e^{i\theta_3}$. Take $\gamma(t) = \tau_I t - 2\sum_k \vartheta_{\alpha_k}(t) - \theta_3$, then apply Corollary 3.5.1. □

Corollary 3.5.3. *Let E be a de Branges function having non real zeros only, and let $\mathcal{B}^{+}(E)$ be the associated Cartwright-de Branges space. Let $F \in \mathcal{B}^{+}(E)$ and let $\{z_k\}$ and*

$\{w_k\}$ be respectively the sequence of upper half-plane and lower half-plane zeros of F. Let $\{\alpha_n\}$ be the sequence of zeros of E in the lower half-plane and let $\{x_k\}$ be the sequence of real zeros of $\frac{F}{E}$. Then, for every $t \in \mathbb{R} \setminus \{x_k\}$,

$$\widetilde{\log\left|\frac{F}{E}\right|}(t) = -\pi N_{\{x_k\}}(t) + \Big(\frac{\tau_{F/E} + \tau_{F^\sharp/E} - \tau_{I_E}}{2}\Big)t + \sum_k \vartheta_{\bar{\alpha}_k}(t) - \sum_k \vartheta_{z_k}(t) - \sum_k \vartheta_{\bar{w}_k}(t) + \theta,$$

where, $I_E = E^\sharp/E$, and θ is a constant.

Proof. Take $F = \frac{F}{E}$. Then by Proposition 2.2.5, $F \in K^+_{I_E}$. Note that $\frac{F^\sharp}{E} = I_E\bar{F}$. Now apply Corollary 3.5.2. □

The next result, which was first proved in [34], here we obtain it via Theorem 4.3.7

Corollary 3.5.4. *Let $F \in \mathcal{C}$. Let $\{x_k\}, \{z_k\}$ and $\{w_k\}$ be respectively the sequence of real, upper and lower half-plane zeros of F. Then, for every $t \in \mathbb{R} \setminus \{x_k\}$,*

$$\widetilde{\log|F|}(t) = -\pi N_{\{x_k\}}(t) + \left(\frac{\tau_F + \tau_{F^\sharp}}{2}\right)t - \sum_k \vartheta_{z_k}(t) - \sum_k \vartheta_{\bar{w}_k}(t) + \theta,$$

where θ is a constant.

Proof. Since $F \in \mathcal{C}$, By definition, there exists $a \geq 0$ such that $F \in \mathcal{C}_a$. Thus, we have $E(z) = e^{-2\pi i a z}$. Then it is straightforward to see that

$$\tau_{F/E} = \tau_F - 2\pi a,$$

$$\tau_{F^\sharp/E} = \tau_{F^\sharp} - 2\pi a,$$

$$\tau_{I_E} = -4\pi a.$$

Substituting these values in the formula of Corollary 3.5.3 and observing that E has no zeros in the lower half-plane gives the result. □

The next four results were obtained by Makarov and Poltoratski [29, 41]. Here, we deduce them from Theorem 3.4.3. We say that Λ is an exact zero set of a function space (in our case, $\mathcal{B}^+(E)$), if there is a function F in the space whose zeros, counting multiplicities, are precisely the points of Λ.

Corollary 3.5.5. *Let $\gamma \in C^\omega(\mathbb{R})$ be real valued. Suppose that $N^+[e^{i\gamma}]$ is nontrivial. Then*

$$\gamma = -\alpha + \tilde{h},$$

where $\alpha \in C^\omega(\mathbb{R})$ is an increasing function and $h \in L^1(\frac{dt}{1+t^2})$.

Proof. Take a function $F \in N^+[e^{i\gamma}]$ which does not vanish on $\mathbb{R}$ and apply Corollary 3.5.1. □

Corollary 3.5.6. *Let $\Lambda \subset \mathbb{R}$ be an exact zero set of $\mathcal{B}^+(E)$, where the de Branges function E has only non real zeros. Then there is a constant $b \geq 0$ and a function*

$$h \in L^1\left(\frac{dt}{1+t^2}\right)$$

such that

$$2\pi N_\Lambda - \arg I_E = -bx + \widetilde{h}.$$

Proof. By definition, there exists $F \in \mathcal{B}^+(E)$ such that $\mathcal{Z}(F) = \Lambda$, where $\mathcal{Z}(F)$ denotes the zero set of F. Now take $b = -(\tau_+ + \tau_-)$ and $h = -2\log\left|\frac{F}{E}\right|$ and apply Theorem 3.4.3. □

Corollary 3.5.7. *Let I be a meromorphic inner function. Suppose that $\Lambda \subset \mathbb{R}$ is not a uniqueness set for K_I^p. Then there exists $h \in L^1(\frac{dt}{1+t^2})$ such that $e^{-h} \in L^{\frac{p}{2}}(\mathbb{R})$ and*

$$2\pi N_\Lambda - \arg I = -\phi + \widetilde{h},$$

where ϕ is the argument of a meromorphic inner function.

Proof. Pick any function $F \in K_I^p$ whose sequence of real zeros is exactly Λ and then apply Corollary 3.5.2. □

Corollary 3.5.8. *Let $\Lambda \subset \mathbb{R}$ be an exact zero set of $\mathcal{C}$. Then there exist an constant $c \geq 0$ and a function*

$$h \in L^1\left(\frac{dt}{1+t^2}\right)$$

such that

$$N_\Lambda(t) = ct + \widetilde{h}(t).$$

Proof. Pick $F \in \mathcal{C}$ such that $\mathcal{Z}(f) = \Lambda$. Put $h = -2\log|F|$ and $c = \frac{h_+ + h_-}{2\pi}$. Then apply Corollary 3.5.4 and [44, Theorem 6.18]]. □

The following result is due to Cartwright and Levinson and can be found in [34, 41]. It follows from Corollary 3.5.8 and the fact that $\frac{\widetilde{\log|F|}(t)}{t} \to 0$ as $|t| \to \infty$. See also [24, 29].

Corollary 3.5.9. *Let $F \in \mathcal{C}$. Suppose F has only real zeros $\{x_n\}$. Then*

$$\lim_{|t|\to\infty} \frac{N_{\{x_n\}}(t)}{t} = \frac{h_+ + h_-}{2\pi}.$$

Recall that $\mathrm{Lip}_\alpha(\mathbb{R})$, $\alpha \in (0,1]$, denotes the class of functions f on the real line such that

$$\sup_{t\neq s} \frac{|F(t) - F(s)|}{|t-s|^\alpha} < \infty.$$

Corollary 3.5.10. *Let $F \in \mathcal{N}_\star^+ \cap L^p(\mathbb{R}), 1 \leq p < \infty$, be such that the following conditions are satisfied*

(i) *F has no real zeros.*

(ii) *$\hat{B}_F$ has bounded derivative.*

(iii) $\widetilde{\log|F|} \in \mathrm{Lip}_1(\mathbb{R})$.

Then $f' \in H^p$.

Proof. Take $G = F$. Apply Theorem 3.4.3 and Corollary of [12, Theorem 11] □

CHAPTER 4

Multipliers between model spaces

In this chapter, we discuss the problems on the multipliers between model spaces under non Hilbert setting that is for the case $p \neq 2$. The multipliers between two different model spaces of H^2 were first studied by E. Fricain, A. Hartmann and W. Ross in [15]. In this chapter, we establish the corresponding results for multipliers between two model spaces in the non-Hilbert setting. Our main observation is that, contrary to the Hilbert space setting, in the non-Hilbert setting one cannot travel back and forth between model spaces of the open unit disc and model spaces of the upper half-plane to transfer the multiplier results of one setting to another. Inspired by the work of Crofoot [9] on the family of surjective multipliers, we completely characterize equalities such as $\mathcal{M}^p(\Theta_1, \Theta_3) = \mathcal{M}^p(\Theta_2, \Theta_3)$ and $\mathcal{M}^p(\Theta_1, \Theta_2) = \mathcal{M}^p(\Theta_1, \Theta_3)$. The results of this chapter are contained in the contributed article 3.

4.1 Introduction

Given a pair of inner functions Θ_1 and Θ_2, our goal is to investigate the algebra of multipliers

$$\mathcal{M}^p(\Theta_1, \Theta_2) := \{\vartheta \in \mathrm{Hol}(\mathbb{D}) : \ \vartheta K^p_{\Theta_1} \subseteq K^p_{\Theta_2}\}.$$

The class $\mathcal{M}^2(\Theta_1, \Theta_2)$ was investigated in the pioneering work of E. Fricain, A. Hartmann and W. Ross in [15]. This work was the inspiration for the current line of research. They were also investigated by C. Camara, J. Partington [8] and E. Fricain, R. Rupam [16].

The main motivation for us to carry out this work was that the extension to cases $p \neq 2$ were not so straightforward. As a matter of fact, according to Lemmas 6.1 and 6.2 of [15], on can deduce the multipliers of the function spaces in the upper half-plane with the knowledge of multipliers of the function spaces in the open unit disc and vice-versa. We observed that for $p \neq 2$ this technique is no longer applicable. This crucial fact is crystalized in a uniqueness-type result in Proposition 4.3.3.

Another problem that we study in this article is motivated by the work of Crofoot [9], where the class of onto multipliers was investigated. He showed that the space of onto multipliers, if not void, is basically one dimensional. More explicitly, Corollary 13 of [9] says that if Θ_1 and Θ_2 are nonconstant inner functions on the unit disc, and if there exists a multiplier ϑ from K_{Θ_1} onto K_{Θ_2}, then

$$\mathcal{M}_{onto}(\Theta_1, \Theta_2) := \{\varphi \in Hol(\mathbb{D}) : \ \varphi K_{\Theta_1} = K_{\Theta_2}\} = (\mathbb{C} \setminus \{0\})\vartheta. \qquad (4.1.1)$$

Suppose now that Θ_1, Θ_2 and Θ_3 are nonconstant inner functions and that

$$\mathcal{M}_{\text{onto}}(\Theta_1, \Theta_3) = \mathcal{M}_{\text{onto}}(\Theta_2, \Theta_3) \neq \emptyset.$$

Then, by (4.1.1), it is immediate that $\Theta_1 = C\Theta_2$, for some unimodular constant C. At the same token, if

$$\mathcal{M}_{\text{onto}}(\Theta_1, \Theta_2) = \mathcal{M}_{\text{onto}}(\Theta_1, \Theta_3) \neq \emptyset,$$

then, once more by (4.1.1), we have $\Theta_2 = C\Theta_3$. Considering the above observation, a natural question to ask is as follows. If we have

$$\mathcal{M}^p(\Theta_1, \Theta_3) = \mathcal{M}^p(\Theta_2, \Theta_3) \neq \{0\},$$

can we conclude that $\Theta_1 = C\Theta_2$, for some unimodular constant C? Similarly, does $\mathcal{M}^p(\Theta_1, \Theta_2) = \mathcal{M}^p(\Theta_1, \Theta_3) \neq \{0\}$ imply that $\Theta_2 = C\Theta_3$? We answer affirmatively to both questions.

4.2 Multipliers of spaces on the open unit disc

In this section, we give a characterization of $\mathcal{M}^p(\Theta_1,\Theta_2)$, $1 \le p \le \infty$, via the Toeplitz kernel and Carleson measure for model spaces. Recall that a measure μ on the unit circle $\mathbb{T}$ is a *Carleson measure* for the model space K^p_Θ if $K^p_\Theta \subseteq L^p(\mu)$. The novel work for $\mathcal{M}^2(\Theta_1,\Theta_2)$ is done by E. Fricain, A. Hartmann, and W. Ross [15].

As a simple starting observation, note that

$$\mathcal{M}^p(\Theta_1,\Theta_2) \subset H^p. \tag{4.2.1}$$

In fact, let $\vartheta \in \mathcal{M}^p(\Theta_1,\Theta_2)$. Since the Cauchy kernel

$$k_0 = 1 - \bar{\Theta}_1(0)\Theta_1 \tag{4.2.2}$$

belong to $K^p_{\Theta_1} \bigcap (H^\infty)^{-1}$, we must have $\vartheta k_0 \in K^p_{\Theta_2}$. But, since $1/k_0 \in H^\infty$, we conclude that $\vartheta \in H^p$.

Theorem 4.2.1. *Let Θ_1 and Θ_2 be inner functions and $\vartheta \in H^p$. Then the following are equivalent.*

(i) $\vartheta \in \mathcal{M}^p(\Theta_1,\Theta_2)$.

(ii) $\vartheta\, S^*\Theta_1 \in K^p_{\Theta_2}$ *and* $K^p_{\Theta_1} \subseteq L^p(|\vartheta|^p dm)$.

(iii) $\vartheta \in \ker_p[\bar{z}\bar{\Theta}_2\Theta_1]$ *and* $K^p_{\Theta_1} \subseteq L^p(|\vartheta|^p dm)$.

Proof. *(i)* $\Longrightarrow$ *(ii)*: Since $S^*\Theta_1 \in K^p_{\Theta_1}$, we must have $\vartheta\, S^*\Theta_1 \in K^p_{\Theta_2}$. That $K^p_{\Theta_1} \subseteq L^p(|\vartheta|^p dm)$ follows directly from the definitions of multipliers and of the norm in model spaces.

(ii) $\Longrightarrow$ *(iii)*: Since $\vartheta S^*\Theta_1 \in K^p_{\Theta_2}$, by the definition $K^p_{\Theta_2}$, we have

$$\vartheta\, S^*\Theta_1 = \bar{z}\bar{g}\,\Theta_2, \text{ for some } g \in H^p. \tag{4.2.3}$$

But, according to (2.1.13) and (4.2.2), we see that

$$S^*\Theta_1 = \bar{z}\Theta_1\bar{k}_0. \tag{4.2.4}$$

Thus, combining (4.2.3) and (4.2.4), we obtain

$$\bar{z}\bar{\Theta}_2\Theta_1\vartheta = \bar{z}\bar{g}/\bar{k}_0.$$

Since $k_0 \in (H^\infty)^{-1}$, it follows that $g/k_0 \in H^p$, and thus the above identity implies that $\vartheta \in \ker_p[\bar{z}\bar{\Theta}_2\Theta_1]$.

$(iii) \implies (i)$: Suppose that $f \in K^p_{\Theta_1}$. By the definition of $K^p_{\Theta_1}$, we have

$$\bar{\Theta}_1 f = \bar{z}\bar{h}, \text{ for some } \; h \in H^p. \tag{4.2.5}$$

Hence,

$$\bar{\Theta}_2\vartheta f = \bar{z}\bar{\Theta}_2\Theta_1\vartheta\bar{h}. \tag{4.2.6}$$

The first assumption in (iii) implies that $\bar{z}\bar{\Theta}_2\Theta_1\vartheta \in \bar{z}\overline{H^p} \subseteq \bar{z}\overline{N^+}$, and the second assumption implies that $\vartheta f \in L^p(\mathbb{T})$. Therefore, by (4.2.6), $\bar{\Theta}_2\vartheta f \in \bar{z}\overline{N^+} \cap L^p(\mathbb{T})$. □

Theorem 4.2.1 immediately implies the following result for bounded multipliers.

Corollary 4.2.2. *Let Θ_1 and Θ_2 be inner functions, and let $\vartheta \in H^p$. Then the following are equivalent.*

(i) $\vartheta \in \mathcal{M}^p(\Theta_1, \Theta_2) \cap H^\infty$.

(ii) $\vartheta S^*\Theta_1 \in K^p_{\Theta_2} \cap H^\infty$.

(iii) $\vartheta \in \ker_p[\bar{z}\bar{\Theta}_2\Theta_1] \cap H^\infty$.

Proof. The implications $(i) \implies (ii)$ and $(iii) \implies (i)$ are immediate. For the implication $(ii) \implies (iii)$, observe that the assumption (ii) implies $\vartheta \in H^\infty$. This follows from (4.2.4) and that $k_0 \in (H^\infty)^{-1}$. □

Corollary 4.2.2 reveals that

$$\ker_p[\bar{z}\bar{\Theta}_2\Theta_1] \cap H^\infty = \mathcal{M}^p(\Theta_1, \Theta_2) \cap H^\infty \subseteq \mathcal{M}^p(\Theta_1, \Theta_2) \subseteq \ker_p[\bar{z}\bar{\Theta}_2\Theta_1].$$

However, in general, the inclusion $\mathcal{M}^p(\Theta_1, \Theta_2) \subseteq \ker_p[\bar{z}\bar{\Theta}_2\Theta_1]$ is strict. See [15, Example 3.6].

Corollary 4.2.3. *Let* Θ_1 *and* Θ_2 *be inner functions such that* $\Theta_1 | \Theta_2$*, i.e.,*

$$\Theta_2 = \Theta_1 \Theta_3$$

for some inner function Θ_3*, and let* $\vartheta \in H^p$*. Then the following are equivalent.*

(i) $\vartheta \in \mathcal{M}^p(\Theta_1, \Theta_2)$.

(ii) $\vartheta \in K^p_{z\Theta_3}$ *and* $K^p_{\Theta_2} \subseteq L^p(|\vartheta|^p dm)$.

Moreover,

$$\mathcal{M}^p(\Theta_1, \Theta_2) \cap H^\infty = K^p_{z\Theta_3} \cap H^\infty.$$

4.3 Multipliers of spaces of the upper half-plane

In this section we investigate the multiplier results in the upper half-plane. The common strategy is to use a conformal mapping between the open unit disc and the upper half plane and then transfer the results from one space to another. In the same spirit, the following Lemmas 6.1 and 6.2 were proved in [15]. For further reference, we mention them below.

Lemma 4.3.1 (Lemma 6.1 of [15])**.** *Let* $\varphi \in L^\infty(\mathbb{T})$*. Then*

$$\mathcal{U}_2 \text{ker}_2[\varphi] = \text{Ker}_2[\mathcal{U}_\infty \varphi].$$

Lemma 4.3.2 (Lemma 6.2 of [15])**.** *Let* Θ_1 *and* Θ_2 *be inner functions on the unit disc. Suppose that* $I_1 = \mathcal{U}_\infty \Theta_1$ *and* $I_2 = \mathcal{U}_\infty \Theta_2$*. Then*

$$\mathcal{U}_\infty \mathcal{M}^2(\Theta_1, \Theta_2) = \mathcal{M}^2(I_1, I_2).$$

As the first step, we show that these results are valid just for the case $p = 2$. Therefore, the knowledge of multipliers for the model spaces of the open unit disc does not help in getting the corresponding results for the class of multipliers of model spaces in the upper half plane in $p \neq 2$. To see that Lemma 4.3.1 is specific for the case of $p = 2$, we prove the following uniqueness-type result.

Proposition 4.3.3. *Let* Θ *be inner in the open unit disc. Then*

$$\mathcal{U}_p \text{ker}_2[\Theta] = \text{Ker}_p[\mathcal{U}_\infty \Theta] \quad \Longleftrightarrow \quad p = 2.$$

Proof. Lemma 4.3.1 tells us $\mathcal{U}_2\mathrm{ker}_2[\Theta] = \mathrm{Ker}_2[\mathcal{U}_\infty\Theta]$. For the inverse implication assume that

$$\mathcal{U}_p\mathrm{ker}_p[\Theta] = \mathrm{Ker}_p[\mathcal{U}_\infty\Theta] \tag{4.3.1}$$

holds for some p. According to (2.1.14),

$$S^*\Theta \in \mathcal{U}_p\mathrm{ker}_p[\Theta], \qquad 1 < p < \infty.$$

Hence, by the assumption (4.3.1), $\mathcal{U}_pS^*\Theta \in \mathrm{Ker}_p[I]$, where $I = \mathcal{U}_\infty\Theta = \Theta \circ \omega$. On the one hand, by (2.2.3),

$$\begin{aligned}
(\mathcal{U}_pS^*\Theta)(z) &= \frac{1}{\pi^{1/p}(z+i)^{2/p}}\, S^*\Theta(\omega(z)) \\
&= \frac{1}{\pi^{1/p}(z+i)^{2/p}} \frac{\Theta(\omega(z)) - \Theta(0)}{\omega(z)} \\
&= \frac{1}{\pi^{1/p}(z+i)^{2/p}} \frac{I(z) - I(i)}{\omega(z)}.
\end{aligned}$$

Hence, according to (2.2.11), $\mathcal{U}_pS^*\Theta \in \mathrm{Ker}_p[\bar{I}]$ if and only if

$$\frac{1}{\pi^{1/p}(x+i)^{2/p}} \frac{I(x) - I(i)}{\omega(x)} = I(x)\overline{G(x)}, \tag{4.3.2}$$

for some $G \in \mathcal{H}^p$. Note that in the last equation, we consider the boundary functions on $\mathbb{R}$ and thus the identity should hold for almost all $x \in \mathbb{R}$. A simplification of (4.3.2) gives

$$\frac{1}{(x+i)^{2/p}} = \pi^{1/p}\omega(x)\overline{\left(\frac{G(x)}{1-\bar{I}(i)I(x)}\right)}.$$

Since $1 - \bar{I}(i)I \in (\mathcal{H}^\infty)^{-1}$, by (2.2.11), the above representation implies

$$\frac{1}{(x+i)^{2/p}} \in \mathrm{Ker}_p[\bar{\omega}] = \mathcal{K}^p_\omega,$$

which happens if and only if $p = 2$. For the only if part, recall that

$$\mathcal{K}^p_\omega = \mathrm{Span}\left\{\frac{1}{x+i}\right\}$$

is a one-dimensional subspace of H^p. □

In order to observe the failure of the Lemma 4.3.2 for $p \in (1,2) \cup (2,\infty]$ we produce a counter example. An insight to look for a counter example came through the following key elusive example.

Example 1. Let $\Theta_1(w) = w$, $w \in \mathbb{D}$, and Θ_2 be any inner functions on the unit disc. Clearly $K^p_{\Theta_1} = \mathbb{C}$. Then let

$$I_1(z) = \mathcal{U}_\infty \Theta_1(z) = \frac{z-i}{z+i}, \qquad z \in \mathbb{C}_+,$$

and $I_2 = \mathcal{U}_\infty \Theta_2$. It is immediate that

$$\mathcal{M}^p(\Theta_1, \Theta_2) = K^p_{\Theta_2}$$

but

$$\mathcal{M}^p(I_1, I_2) = (z+i)\mathcal{K}^p_{I_2},$$

since as before $\mathcal{K}^p_{I_1} = \text{Span}\,\{1/(z+i)\}$, $1 < p \le \infty$.

An easy observation: If possible, suppose that Lemma 4.3.2 does hold for all values $1 < p \le \infty$. Applying this assumption to the case of Example 1 we get,

$$\mathcal{U}_\infty K^p_{\Theta_2} = (z+i)\mathcal{K}^p_{I_2}, \tag{4.3.3}$$

for all inner functions Θ_2 on the unit disk. Equation 4.3.3 immediately tells that the map $\mathcal{U}_2 : K^p_{\Theta_2} \to \mathcal{K}^p_{I_2}$ is surjective for every inner function Θ_2 on the unit disk. We extract an inner function Θ_2 (See, Theorem 4.3.9 and 4.3.12) on the unit disk such that the above conclusion fails. Such a Θ_2 together with the inner function $\Theta_1(w) = w$ will serve as our counter example.

Above observation led us to following propositions.

Proposition 4.3.4. *For $p > 2$ there exist a pair of inner functions Θ_1 and Θ_2 on the unit disc. Such that*

$$\mathcal{U}_\infty \mathcal{M}^p(\Theta_1, \Theta_2) \neq \mathcal{M}^p(I_1, I_2),$$

where $I_1 = \mathcal{U}_\infty \Theta_1$ and $I_2 = \mathcal{U}_\infty \Theta_2$.

Proposition 4.3.4 shows that in general for $p > 2$ one cannot recover the class of multipliers in the upper half plane with the corresponding knowledge in the unit disc.

Proposition 4.3.5. *For $1 < p < 2$ there exist a pair of inner functions Θ_1 and Θ_2 on the unit disc. Such that*

$$\mathcal{M}^p(\Theta_1, \Theta_2) \neq \mathcal{U}_\infty^{-1} \mathcal{M}^p(I_1, I_2),$$

where $I_1 = \mathcal{U}_\infty \Theta_1$ and $I_2 = \mathcal{U}_\infty \Theta_2$.

Proposition 4.3.5 shows that in general for $1 < p < 2$ one cannot recover the class of multipliers in the unit disc with the corresponding knowledge in the upper half plane.

As observed above our prime goal reduces to look for some mapping properties of the map $\mathcal{U}_2$ on model spaces. We note that for an inner function Θ on the unit disc and $I = \mathcal{U}_\infty \Theta$, the map

$$\mathcal{U}_2 : K^p_\Theta \to \mathcal{K}^p_I$$

is well defined for all $p \geq 2$. Indeed, it is straightforward that $f \in K^p_\Theta$ implies that $I\overline{\mathcal{U}_2 f} = \mathcal{U}_2 g$ for some $g \in H^p$ and it is also not hard to see that $||\mathcal{U}_2 f||_p \lesssim ||f||_p$, whenever $p \geq 2$. It is obvious that the map is a bijection for $p = 2$, and at least injective for $p > 2$. We will see in Theorem 4.3.9 that this map is not surjective in general, which is a key result of this paper. A consequence of the surjectivity of the map $\mathcal{U}_2$ on some model spaces is that there exists a "wild" multiplier which multiplies the underlying model space into the corresponding Hardy space. The precise statement is given in the form of following lemma.

Lemma 4.3.6. *Let Θ be an inner function on the unit disc and $p > 2$. If the map $\mathcal{U}_2 : K^p_\Theta \to \mathcal{K}^p_I$ is surjective, then $(z+i)^{1-\frac{2}{p}} \mathcal{K}^p_I \subseteq \mathcal{H}^p$.*

Proof. Given $F \in \mathcal{K}^p_I$, put $g(w) = \mathcal{U}_2^{-1} F(w) = \frac{2\sqrt{\pi} i}{1-w} F\left(i \frac{1+w}{1-w}\right)$, $w \in \mathbb{D}$. Then $g \in K^p_\Theta$, and hence $\mathcal{U}_p g \in \mathcal{H}^p$. □

We note that the function $(z+i)^{1-\frac{2}{p}}$ does not belong to $\mathcal{H}^\infty$ whenever $p \in (2, \infty]$. Thus the surjectivity of the map $\mathcal{U}_2$ must fail if we can trap an inner function I in the upper half plane such that $\mathcal{K}^p_I$ contains a function F for which $(z+i)^{1-\frac{2}{p}} F$ goes outside $\mathcal{H}^p$. But in general it is extremely difficult to identify general functions in a model space. However there always exists a function F in $\mathcal{H}^p$ such that $(z+i)^{1-\frac{2}{p}} F$ goes outside $\mathcal{H}^p$, but we do not immediately know that whether or not this function belongs to some model space in the upper half plane. However it is straightforward that any non trivial function in $\mathcal{H}^p$ can be put in some Toeplitz kernel, indeed if $F \in \mathcal{H}^p$ then $F \in \mathrm{Ker}_p[\bar{F}/F]$. Here Dyakonov's [13] famous result on a characterization of Toeplitz kernels comes into the picture, which relates Toeplitz kernels and model spaces. Given any function in the Hardy space, Dyakonov's result gives us a way to manufacture a new function which has the same asymptotic behavior as that of the original function, in fact both the functions have the same integrability properties and the nice thing about the new function is that there are model spaces ready to embrace it (see, Lemma 4.3.8). To state Dyakonov's result the elements of the Cartesian product $\mathcal{B} \times \mathcal{B} \times (\mathcal{H}^\infty)^{-1}$, where $\mathcal{B}$ denotes the class of Blaschke products in the upper half plane, will be called triples. Dyakonov used a remarkable factorization theorem for $L^\infty(\mathbb{R})$ functions due to Bourgain [5] to obtain the following characterization of Toeplitz kernels.

Theorem 4.3.7 (Dyakonov [13]). *Let $\varphi \in L^\infty(\mathbb{R})$, $\varphi \neq 0$. Then there exists a triple (B_1, B_2, O) such that, for all $1 \le p \le \infty$,*

$$\mathrm{Ker}_p[\varphi] = \bar{B}_2 O \cdot (\mathcal{K}^p_{B_1} \cap B_2 \mathcal{H}^p). \tag{4.3.4}$$

Let us define a subset $\mathcal{K}^p$ of $\mathcal{H}^p$ as

$$\mathcal{K}^p := \bigcup_{B \in \mathcal{B}} \mathcal{K}^p_B. \tag{4.3.5}$$

As a consequence of Theorem 4.3.7 we obtain the following *pointwise multiplier result*,which tells that every function in $\mathcal{H}^p$ can be realized as a function in some weighted model space. The precise statement is the following.

Lemma 4.3.8. *Let F be an element of $\mathcal{H}^p$, $1 \le p \le \infty$. Then there exists a function $G \in \mathcal{H}^\infty$ such that $|G| \asymp 1$ almost everywhere on $\mathbb{R}$ and $GF \in \mathcal{K}^p$.*

Proof. Set $\varphi = \frac{\bar{F}}{F}$, then $F \in \mathrm{Ker}_p[\varphi]$. Theorem 4.3.7 gives the existence of a triple (B_1, B_2, O) such that (4.3.4) holds. Take $G = B_2 O^{-1}$, then G has required properties. □

Now we are ready to prove our prime goal of non surjevtivity of the map $\mathcal{U}_2$.

Theorem 4.3.9. *For $p > 2$ there exists an inner function Θ on the unit disc such that the map $\mathcal{U}_2 : K^p_\Theta \to \mathcal{K}^p_I$, where $I = \mathcal{U}_\infty \Theta$,is not surjective.*

Proof. Chose a function $F \in \mathcal{H}^p$ such that $(x+i)^{1-\frac{2}{p}} F(x)$ does not belong to $L^p(\mathbb{R})$. Applying Lemma 4.3.8 to F, we obtain a function G such that $|G| \asymp 1$ almost everywhere on $\mathbb{R}$ and $GF \in \mathcal{K}^p_B$ for some Blaschke product B in the upper half plane. Put

$$\Theta(w) = B\left(i\frac{1+w}{1-w}\right) = \mathcal{U}_\infty^{-1} B(w), \ w \in \mathbb{D}.$$

Obviously Θ is an inner function in the unit disc.

Claim. The map $\mathcal{U}_2 : K^p_\Theta \to \mathcal{K}^p_B$, is not surjective.

Proof of the claim. If the map is surjective, Lemma 5.3.4 implies that $(z+i)^{1-\frac{2}{p}}\mathcal{K}^p_B \subseteq \mathcal{H}^p$, consequently $(z+i)^{1-\frac{2}{p}}GF \in \mathcal{H}^p$. But this is not possible as $(x+i)^{1-\frac{2}{p}}F(x)$ does not belong to $L^p(\mathbb{R})$ and $|G| \asymp 1$ almost everywhere on $\mathbb{R}$. This completes the proof of the claim and hence the theorem. □

Now we are ready to prove Proposition 4.3.4.

Proof of Proposition 4.3.4. Suppose $p > 2$ and $\Theta_1(w) = w$, $w \in \mathbb{D}$. Now choose an inner function Θ_2 on the unit disc such that the map

$$\mathcal{U}_2 : K^p_{\Theta_2} \to \mathcal{K}^p_{I_2},$$

where $I_2 = \mathcal{U}_\infty \Theta_2$, is not surjective. The existence of such an inner function Θ_2 is guaranteed by Theorem 4.3.9. As discussed in Example 1 we have

$$\mathcal{M}^p(\Theta_1, \Theta_2) = K^p_{\Theta_2}$$

and

$$\mathcal{M}^p(I_1, I_2) = (z+i)\mathcal{K}^p_{I_2}.$$

Now obviously we have $\mathcal{U}_\infty \mathcal{M}^p(\Theta_1, \Theta_2) \neq \mathcal{M}^p(I_1, I_2)$. Indeed,

$$\mathcal{U}_\infty \mathcal{M}^p(\Theta_1, \Theta_2) = \mathcal{M}^p(I_1, I_2)$$

implies that

$$\mathcal{U}_2 K^p_{\Theta_2} = \mathcal{K}^p_{I_2}.$$

Which is certainly not true by the choice of the inner function Θ_2. This completes the proof.

Remark. It is not hard to see that the map $\mathcal{U}_2$ has dense range. As Theorem 4.3.9 suggests that the map $\mathcal{U}_2$ is not surjective in general, and hence by open mapping theorem the map $\mathcal{U}_2^{-1}$ does not have bounded inverse for the case $p > 2$.

In order to prove Proposition 4.3.5 one needs to use the unit disc version of the Theorem 4.3.7 to obtain analogues of Lemma 4.3.8, Lemma 5.3.4 and Theorem 4.3.9.

We note that for an inner function Θ on the unit disc and $I = \mathcal{U}_\infty\Theta$, the map

$$\mathcal{U}_2^{-1} : \mathcal{K}_I^p \to K_\Theta^p$$

is well defined for all $1 \leq p \leq 2$. It is obvious that the map is a bijection for $p = 2$, and at least injective for $1 \leq p < 2$. Let us define a subset K^p of H^p as

$$K^p := \bigcup_{b \in \mathcal{I}} K_b^p \tag{4.3.6}$$

where $\mathcal{I}$ denotes the class os Blaschke products on the unit disc. Now we have the following pointwise multiplier result on the unit disc.

Lemma 4.3.10. *Let f be an element of H^p, $1 \leq p \leq \infty$. Then there exists a function $g \in H^\infty$ such that $|g| \asymp 1$ almost everywhere on $\mathbb{T}$ and $gf \in K^p$.*

Proof. Proof is similar to the proof of Lemma 4.3.8. □

The following result is an analogue of Lemma 5.3.4 in the unit disc setting.

Lemma 4.3.11. *Let Θ be an inner function on the unit disc and $1 \leq p < 2$. If the map $\mathcal{U}_2^{-1} : \mathcal{K}_I^p \to K_\Theta^p$ is surjective, then $(1-w)^{1-\frac{2}{p}} \mathcal{K}_I^p \subseteq H^p$.*

Proof. Proof is similar to the proof of Lemma 5.3.4. □

The following result is the analogue of Theorem 4.3.9 in the unit disc setting.

Theorem 4.3.12. *For $1 \leq p < 2$ there exists an inner function Θ on the unit disc such that the map $\mathcal{U}_2^{-1} : \mathcal{K}_I^p \to K_\Theta^p$, where $I = \mathcal{U}_\infty\Theta$,is not surjective.*

Proof. Proof is similar to the proof of Theorem 4.3.9. □

Proof of the Proposition 4.3.5. Follows from Example 1 and Theorem 4.3.12.

Remark. It is not hard to see that the map $\mathcal{U}_2^{-1}$ has dense range. As Theorem 4.3.12 suggests that the map $\mathcal{U}_2^{-1}$ is not surjective in general, and hence by open mapping theorem the map $\mathcal{U}_2$ does not have bounded inverse for the case $1 \leq p < 2$.

Unlike to the case of unit disc, multipliers in the upper half-plane need not belong to $\mathcal{H}^p$ however it always belongs to the Smirnov class $\mathcal{N}^+$. Regarding this we have the following Proposition. From now onward we have $1 \leq p \leq \infty$.

Proposition 4.3.13. *Let I_1 and I_2 be inner functions and $1 \leq p \leq \infty$. Let $\Phi \in \mathcal{M}^p(I_1, I_2)$, then $\Phi \in \mathcal{N}^+$.*

Proof. Observe that the function

$$G_o = 1 - \bar{I}_1(i) I_1 \tag{4.3.7}$$

falls in $(\mathcal{H}^\infty)^{-1}$. It is straightforward that the function

$$F_o = \frac{G_o}{z + i} \tag{4.3.8}$$

belongs to $\mathcal{K}^p_{I_1}$. Thus $\Phi F_o \in \mathcal{K}^p_{I_2}$, and hence $\Phi \in \mathcal{N}^+$. □

Following results corresponding to the case $p = 2$ were obtained in [15]. However the proofs were obtained using Lemmas 4.3.1 and 4.3.2. Since these Lemmas are no longer true for $p \neq 2$ so we need to produce an independent argument.

Theorem 4.3.14. *Let I_1 and I_2 be inner functions and $\Phi \in \mathcal{N}^+$, then for $1 \leq p \leq \infty$ the following are equivalent:*

(i) $\Phi \in \mathcal{M}^p(I_1, I_2)$.

(ii) $\Phi \frac{I_1 - I_1(i)}{z - i} \in \mathcal{K}^p_{I_2}$ *and* $\mathcal{K}^p_{I_1} \subseteq L^p(|\Phi|^p dx)$.

(iii) $\Phi \in (z + i)\mathrm{Ker}_p[\bar{\omega}\bar{I}_2 I_1]$ *and* $\mathcal{K}^p_{I_1} \subseteq L^p(|\Phi|^p dx)$.

Proof. The implication (i)⇒(ii) follows directly from the definition. To see the implication (ii)⇒(iii) we argue as follows. Since $\Phi \frac{I - I(i)}{z - i} \in \mathcal{K}^p_{I_2}$ by definition we have

$$\bar{I}_2 \Phi \frac{I_1 - I_1(i)}{x - i} \in \mathcal{K}^p_{I_2} = \bar{G}, \text{ for some } G \in \mathcal{H}^p. \tag{4.3.9}$$

Combining (4.3.8) and (4.3.9) we obtain

$$\bar{\omega}\bar{I}_2 I_1 \frac{\Phi}{x + i} = \overline{\left(\frac{G}{F_o}\right)}$$

Since $F_o \in (\mathcal{H}^\infty)^{-1}$, it follows that $G/F_o \in \mathcal{H}^p$, and hence by definition $\Phi/(x + i) \in \mathrm{Ker}_p[\bar{\omega}\bar{I}_2 I_1]$. To complete the proof it remains to prove the implication (iii)⇒(i). Suppose that $F \in \mathcal{K}^p_{\Theta_1}$. By definition we have

$$\bar{I}_1 F = \bar{H}, \qquad H \in \mathcal{H}^p. \tag{4.3.10}$$

A simple manipulation of (4.3.10) gives us

$$\bar{I}_2\Phi F = \bar{I}_2 I_1 \Phi \bar{H}. \tag{4.3.11}$$

Now first condition in (iii) implies that $\bar{I}_2 I_1\Phi \in \bar{\mathcal{H}}^p \subseteq \overline{\mathcal{N}^+}$, and second condition implies that $\Phi F \in L^p(\mathbb{R})$. Hence (4.3.11) yields $\bar{I}_2\Phi F \in \overline{\mathcal{N}^+} \cap L^p(\mathbb{R})$. □

Now we come to the situation when the embedding condition of the model space in some weighted L^p is comparatively easier to handle. The following result can be obtained as a consequence of Volberg-Treil embedding theorem [46].

Theorem 4.3.15. *Let I be an inner function in the upper half-plane such that $|I'(x)| \asymp 1$, $x \in \mathbb{R}$. Given a positive Borel measure σ on $\mathbb{R}$, the following are equivalent:*

(i) *for $1 < p < \infty$, $\mathcal{K}_I^p \subseteq L^p(d\sigma)$.*

(ii) *the set $\{\sigma([x, x+1]) : x \in \mathbb{R}\}$ is bounded.*

For $p = 2$, Baranov proved this result independently [1, Theorem 5.1].

Using this embedding result one can give the following characterization of multipliers.

Corollary 4.3.16. *Let I_1 and I_2 be inner functions and $|I_1'| \asymp 1$ on $\mathbb{R}$. Suppose that $\Phi \in N^+$, then for $1 \leq p < \infty$. the following are equivalent:*

(i) $\Phi \in \mathcal{M}^p(I_1, I_2)$.

(ii) $\Phi\frac{I_1 - I_1(i)}{z-i} \in \mathcal{K}_{I_2}^p$ *and* $\sup_{x\in\mathbb{R}} \int_x^{x+1} |\Phi(x)|^p dx < \infty$.

(iii) $\Phi \in (z+i)\mathrm{Ker}_p[\bar{\omega}\bar{I}_2 I_1]$ *and* $\sup_{x\in\mathbb{R}} \int_x^{x+1} |\Phi(x)|^p dx < \infty$.

When the inner functions in the above characterizations turn out to be meromorphic the situation becomes easier to apply. The following result was obtained for $p = 2$ in [16]. For all $1 < p < \infty$ proof is more or less similar, so we omit the proof.

Theorem 4.3.17. *Let I_1 and I_2 be meromorphic inner functions such that $|I_1'| \asymp 1$ on $\mathbb{R}$. Set $\lambda := \arg(I_1) - \arg(bI_2)$ on $\mathbb{R}$. Assume that either $\lambda \notin \widetilde{L^1(\frac{dx}{1+x^2})}$ or if $\lambda = \tilde{h}$ for some $h \in L^1(\frac{dx}{1+x^2})$, then $e^{-h} \notin L^{\frac{p}{2}}(\frac{dx}{1+x^2})$. Then the following are equivalent.*

(i) $\dim \mathrm{Ker}_p[\bar{\omega}\bar{I}_2 I_1] \geq 2$.

(ii) $\mathrm{Ker}_p[\bar{\omega}\bar{I}_2 I_1] \neq \{0\}$.

(iii) $\mathcal{M}^p(I_1, I_2) \neq \{0\}$.

4.4 Uniqueness results on the class of multipliers

In this section, we prove some uniqueness results regarding the class of multipliers. More explicitly, we prove that if I_1 is fixed and I_2 and I_3 are distinct inner functions, that is, their ratio is not a unimodular constant, then $\mathcal{M}^p(I_1, I_2)$ and $\mathcal{M}^p(I_1, I_3)$ are distinct. All the results of this section, hold in the upper half-plane setting as well as in the open unit disc setting. However, we state the results in the upper half-plane setting. Throughout this section, we assume $1 \leq p \leq \infty$.

Lemma 4.4.1. *For a pair of inner functions I_1 and I_2, if $\mathcal{M}^p(I_1, I_2) \neq \{0\}$, then $\mathcal{M}^p(I_1, I_2)$ contains an outer function.*

Proof. For inner functions I_1 and I_2, suppose that $\mathcal{M}^p(I_1, I_2)$ be nontrivial and let $\Phi \in \mathcal{M}^p(I_1, I_2)$. Applying Proposition 4.3.13, it follows that $\Phi \in \mathcal{N}^+$ and hence $\Phi = IO$, where I is inner and O is outer. For each $F \in \mathcal{K}^p_{I_1}$, by assumption $\Phi F \in \mathcal{K}^p_{I_2}$. Thus we have

$$\bar{I}_2 \Phi F = \bar{I}_2 IOF \in \overline{\mathcal{H}}^p.$$

This in turn implies that $OF \in \mathcal{K}^p_{I_2}$, and hence $O \in \mathcal{M}^p(I_1, I_2)$. □

A similar argument as in the case of multipliers gives the well-known result about model spaces, i.e., for each non-constant inner function I, the space $\mathcal{K}^p_I$ contains an outer function. We exploit these results to give a characterization of multipliers $\Phi \in \mathcal{M}^p(I_1, I_2)$ as a solution to the Riemann-Hilbert (RH) problem

$$\Phi I_1 \overline{I_2} = \Phi_-, \text{ for some } \Phi_- \in \overline{\mathcal{N}}^+.$$

The RH approach to study the class of multipliers sheds more light on the question of when two classes of multipliers coincide.

Proposition 4.4.2. *Let I_1 and I_2 be non-constant inner functions and let $\Phi \in \mathcal{N}^+$. Then the following are equivalent.*

(i) $\Phi \in \mathcal{M}^p(I_1, I_2)$.

(ii) $\overline{\Phi}\, \overline{I_1} I_2 \in \mathcal{M}^p(I_1, I_2)$.

Proof. (*i*) $\Longrightarrow$ (*ii*): Let $\Phi \in \mathcal{M}^p(I_1, I_2)$. If $\Phi = 0$, the equivalence is trivial. Hence, assume that $\Phi \neq 0$. By definition, we have $F \in \mathcal{K}^p_{I_1}$ if and only if $I_1\bar{F} \in \mathcal{K}^p_{I_1}$. Thus we have

$$\Phi I_1\bar{F} \in \mathcal{K}^p_{I_2}, \text{ for all } F \in \mathcal{K}^p_{I_1}.$$

Hence, based on the same characterization but applied for $\mathcal{K}^p_{I_2}$, we also have

$$I_2\bar{\Phi}\bar{I}_1 F \in \mathcal{K}^p_{I_2}, \text{ for all } F \in \mathcal{K}^p_{I_1}. \tag{4.4.1}$$

Therefore, in order to complete the proof, we only need to show that $I_2\bar{\Phi}\bar{I}_1 \in \mathcal{N}^+$. But this is straightforward, since (4.4.1) is true for all $F \in \mathcal{K}^p_{I_1}$ and, in particular, by taking F to be any outer function in $\mathcal{K}^p_{I_1}$, the conclusion follows.

The above argument is almost reversible and gives the implication (*ii*) $\Longrightarrow$ (*i*). □

According to Proposition 4.4.2, the map

$$\begin{aligned} T: \ \mathcal{M}^p(I_1, I_2) &\longrightarrow \mathcal{M}^p(I_1, I_2) \\ \Phi &\longmapsto \overline{\Phi}\,\overline{I_1} I_2 \end{aligned}$$

is well-defined. It is easy to see that this map is involutive, i.e., $T^2 = \mathbf{I}$. Hence, the point spectrum of T fulfills $\sigma_p(T) \subseteq \{-1, 1\}$.

Theorem 4.4.3. *Let I_1, I_2 and I_3 be inner functions. Assume that there is an outer function $\Phi \in \mathcal{N}^+$ such that*

$$\overline{\Phi}\,\overline{I_1} I_3 \in \mathcal{M}^p(I_2, I_3).$$

Then I_2 divides I_1.

Proof. We use the following characterization for model space inclusion: I_2 divides I_1 if and only if $K^p_{I_2} \subset K^p_{I_1}$. E.g., see [36, Corollary 8]. Let $F \in \mathcal{K}^p_{I_2}$. Hence, by the hypothesis, we have

$$\overline{\Phi}\,\overline{I_1} I_3 F \in \mathcal{K}^p_{I_3}.$$

Applying the conjugation for $K^p_{I_3}$, we see that $\Phi I_1\bar{F} \in \mathcal{K}^p_{I_3}$. Since Φ is outer it follows that

$$I_1\bar{F} \in \mathcal{N}^+.$$

Finally, as $F \in H^p(\mathbb{R}) \subset L^p(\mathbb{R})$, the Smirnov maximum principle implies $I_1\bar{F} \in H^p(\mathbb{R})$. Therefore, $F \in \mathcal{K}^p_{I_1}$. The result now follows from the above-mentioned characterization. □

Corollary 4.4.4. *Let $I_1,\ I_2$ and I_3 be inner functions. Let $\Phi,\ \Psi \in \mathcal{N}^+$ be outer functions such that*

$$\{I_3\overline{\Phi}\,\overline{I_1},\ I_3\overline{\Psi}\,\overline{I_2}\} \subseteq \mathcal{M}^p(I_1, I_3) \cap \mathcal{M}^p(I_2, I_3).$$

Then $I_2 = CI_1$, for some unimodular constant C.

By Corollary 4.4.4, it is immediate that

$$\mathcal{M}^p(I_1, I_3) = \mathcal{M}^p(I_2, I_3) \neq \{0\} \Longleftrightarrow I_2 = CI_1,$$

for some unimodular constant C. Using a similar argument, we have

$$\mathcal{M}^p(I_1, I_2) = \mathcal{M}^p(I_1, I_3) \neq \{0\} \Longleftrightarrow I_2 = CI_3.$$

CHAPTER 5

Uniqueness results and mixed data problem

In this article, we discuss on the problems of uniqueness sets for model spaces and mixed data problems. The Results of this chapter are contained in the contributed article 4.

5.1 Introduction

The injectivity problem of the Toeplitz operator has a connection with the following inverse spectral problem. Let us consider the Schrodinger equation

$$-u'' + qu = \lambda u$$

on some interval (a, b) and assume that the potential q is locally integrable and the point a is a regular point, i.e., a is finite and q is summable near a. Let us fix the following boundary conditions:

$$\cos(\alpha)u(a) + \sin(\alpha)u'(a) = 0$$
$$\cos(\beta)u(b) + \sin(\beta)u'(b) = 0.$$

Suppose that a separated sequence $\Lambda \subset \mathbb{R}$ be the spectral data of the above operator. Then a natural question is to ask is that, can we uniquely reconstruct the operator with the help of the spectral data Λ? In [4], Borg proved that, in most cases, two spectra are enough to recover the potential. In [28] further, Levinson relaxed the condition of Borg's paper. Hochstadt and Liebermann in [21] established that if half of the potential is known, then one spectrum is enough to reconstruct the potential on the other half. Simon and Gesztesy in [19] and [26] proved sufficient conditions for partial potential and one spectrum to recover the potential uniquely. In [29], the authors established that recovering the operator uniquely is equivalent to recovering the meromorphic inner function (MIF) with some information about potential and spectrum. For more on spectral theory see [29] and references therein.

The above discussion tells us that the problem of injectivity of Toeplitz operators, determining uniqueness sets for model spaces, and determination of a meromorphic inner function with data about potential and its spectrum is of great importance. In this chapter we explore some of these aspects.

5.2 Uniqueness sets for model spaces

Let I be a meromorphic inner function. A set $\Lambda \subset \mathbb{C}_+$ is said to be a uniqueness set for $\mathcal{K}_I$ if any function in $\mathcal{K}_I$ vanishes on Λ must vanish identically. We prove the following theorems on the uniqueness set for model spaces in $\mathcal{H}^2$.

Theorem 5.2.1. *Let I be a meromorphic inner function and $\Lambda \subset \mathbb{C}_+$. Set*

$$K = \{f \in \mathcal{K}_I : f|_\Lambda = 0\}.$$

Then Λ is a uniqueness set for $\mathcal{K}_I$ if and only if $\dim K = 0$ *or* 1.

Proof. If $\dim K = 0$, then we have nothing to prove. So, we assume that $\dim K = 1$. Let $f \in \mathcal{K}_I$ and $f|_\Lambda = 0$. This implies that

$$\bar{I}f = \bar{H}, H \in \mathcal{H}^2$$

and

$$f = B_\Lambda g, g \in \mathcal{H}^2.$$

Define G and F as follows

$$G = \omega B_\Lambda H,$$

and

$$F = B_\Lambda H.$$

Obviously $G|_\Lambda = F|_\Lambda = 0$. A simple calculation shows that

$$\bar{\omega}\bar{I}G = \bar{g}.$$

and

$$\bar{\omega}\bar{I}F = \bar{b}\bar{g}.$$

Consequently, $G, F \in K$. By the hypothesis we have,

$$F = \alpha G, \text{ for some } \alpha \in \mathbb{C}.$$

This implies

$$B_\Lambda H(1 - \alpha b) = 0,$$

which yields

$$H \equiv 0, \text{ and hence } f \equiv 0.$$

Conversely, let Λ be a uniqueness set for $\mathcal{K}_I$. If possible, suppose that $\dim K \geq 2$. Let $F_1 \in K$, then we have $F_1 = B_\Lambda G_1$, for some $G_1 \in \mathcal{H}^2$ with $G_1(\lambda) \neq 0$ for any $\lambda \in \Lambda$. Next, since $F_1 \in K_{bI}$ we have

$$\bar{\omega}\bar{I}F_1 = \bar{H}, \text{ for some } H \in \mathcal{H}^2.$$

Using the fact that $F_1 = B_\Lambda G_1$ it is immediate that

$$\bar{\omega}\bar{I}G_1 = \bar{B}_\Lambda \bar{H}. \tag{5.2.1}$$

This shows that $G_1 \in K_{bI}$.

Case I. If $G_1(i) = 0$, then $\bar{\omega}G_1 \in \mathcal{H}^2$. Hence by (5.2.1) $B_\Lambda \bar{\omega} G_1 \in \mathcal{K}_I$, but this contradicts the fact that Λ is a uniqueness set for $\mathcal{K}_I$.

Case II. Let $G_1(i) \neq 0$. Since $\dim K \geq 2$, there exists a function $F_2 \in K$ such that F_1 and F_2 are linearly independent in K. Now, we have $F_2 = B_\Lambda G_2$, for some $G_2 \in \mathcal{H}^2$ with $G_2(\lambda) \neq 0$ for any $\lambda \in \Lambda$. Assume that $G_2(i) \neq 0$, for if $G_2(i) = 0$ then as in Case I, we reach a contradiction. Since F_1 and F_2 are linearly independent, it follows that G_1 and G_2 are linearly independent. Now, we construct a function G as follows:

$$G = G_2(i)G_1 - G_1(i)G_2.$$

Since $G_1, G_2 \in \mathcal{K}_I$, it follows that $G \in \mathcal{K}_I$, and $G(i) = 0$. Again as in Case I we reach a contradiction. □

Remark. Theorem 5.2.1 holds good for $\mathcal{K}_I^p$ for all $p > 0$.

Theorem 5.2.2. *Let I be a meromorphic inner function and $\Lambda \subset \mathbb{C}_+$. Set*

$$K = \{f \in \mathcal{K}_I : f|_\Lambda = 0\}.$$

Then $\Lambda \setminus \{\lambda_o\}$ is not a uniqueness set for $\mathcal{K}_I$ for any $\lambda_o \in \Lambda$ if and only if $K \neq \{0\}$.

Proof. Suppose that $\Lambda_o = \Lambda \setminus \{\lambda_o\}$ is not a uniqueness for $\mathcal{K}_I$. Set

$$K_o := \{f \in \mathcal{K}_I : f|_{\Lambda_o} = 0\}.$$

By Theorem 5.2.1, we have $\dim K_o \geq 2$. This implies that there exists a non zero function $g \in K_o$ such that $g(\lambda_o) = 0$ and hence $g \in K$.

Conversely, assume that $K \neq \{0\}$. Since $K \subset K_o$, this implies that $K_o \neq 0$. Let

$$\omega_{\lambda_o}(z) = c_o \frac{z - \lambda_o}{z - \bar{\lambda}_o}, \text{ where } |c_o| = 1$$

be such that

$$B_\Lambda = \omega_{\lambda_o} B, \text{ for some Blaschke product } B \text{ with } B|_{\Lambda_o} = 0.$$

Let $f \in K$, then we have

$$f = B_\Lambda g, \text{ for some } g \in \mathcal{H}^2 \text{ with } g(\lambda) \neq 0, \text{ for any } \lambda \in \Lambda.$$

Obviously, $f \in K_o$. Next, we define f_1 as follows:

$$f_1 = Bg.$$

Then $f_1|_{\Lambda_o} = 0$. Next, since $f \in \mathcal{K}_I$, we have

$$\bar{\omega}\bar{I}B_\Lambda g = \bar{h}, \text{ for some } h \in \mathcal{H}^2.$$

This implies

$$\bar{\omega}\bar{I}\omega_{\lambda_o}Bg = \bar{h}.$$

Hence, we get

$$\bar{\omega}\bar{I}Bg = \bar{\omega}_{\lambda_o}\bar{h}.$$

It follows that $f_1 \in \mathcal{K}_I$ and hence $f_1 \in K_o$. Since f and f_1 are linearly independent, we conclude that $\dim K_o \geq 2$. Thus, the result follows from Theorem 5.2.1. □

Remarks. (a) By the same reasoning as in Theorem 5.2.1, one can show that Λ is a uniqueness set for $\mathcal{K}_I$ if and only if $\dim K \in \{0, 1, \ldots, n\}$, where

$$K = \{f \in \mathcal{K}_{\omega^n I} : f|_\Lambda = 0\}.$$

(b) Theorem 5.2.2 tells us that if $\Lambda \setminus \{\lambda_o\}$ is a uniqueness set for $\mathcal{K}_I$ for some $\lambda_o \in \Lambda$ then we must have $K = 0$. In contrast of Theorem 5.2.1, it discards the possibility of being $\dim K = 1$. So removing a point from Λ can be interpreted as eliminating a one-dimensional subspace of K.

Let consider the Schrodinger equation

$$-\ddot{u} + qu = zu \tag{5.2.2}$$

on some interval (a, b) with the locally integrable potential q. assume that the point a is regular and fix a self-adjoint boundary condition at the point b. Let $\Lambda \subset C$, then as we discussed in the Subsection 2.2.7 for each $\lambda \in \Lambda$ there is a non-trivial solution $u_{(.,}\lambda)$.

Let us recall that the system $U_\Lambda := \{u(.,\lambda) : \lambda \in \Lambda\}$ is complete if closer of the span of U_Λ coincides with $L^2(a,b)$, and is minimal if for each $\lambda_o \in \Lambda,\ u(.,\lambda_o)$ does not belong to the closer of the span of $U_{\Lambda \setminus \{\lambda_o\}}$. Let I_m be the meromorphic inner function associated with the Schrodinger equation as discussed in the Subsection 2.2.7. The following result of Makarov and Poltoratski, see [29, Theorem 4.3] or [41, Theorem 48] can be deduced from the Theorems 5.2.1, 5.2.2 and the Lemma 5.2.4.

Corollary 5.2.3. *Let $\Lambda = \Lambda_+ \cup \Lambda_-$, $\Lambda_\pm \subset \mathbb{C}_+$, and let B be Blaschke product corresponding to the sequence $\Lambda_+ \cup \bar{\Lambda}_-$. Then the family $\{u(.,\lambda) : \lambda \in \Lambda\}$*

1. *is complete in $L^2(a,b)$ if and only if $\mathrm{Ker}_2[\bar{I}_m B] = \{0\}$,*

2. *is minimal if and only if $\mathrm{Ker}_2[\bar{b}\bar{I}_m B] \neq \{0\}$, and*

3. *is complete and minimal if and only if $\dim \mathrm{Ker}_2[\bar{b}\bar{I}_m B] = 1$.*

Lemma 5.2.4. *Let I be a meromorphic inner function and $\Lambda \in \mathbb{C}_+$ be any Blaschke sequence. Set*

$$K := \{F \in \mathcal{K}_I :\ F|_\Lambda = 0\}.$$

Then the operator $T : \mathrm{Ker}_2[\bar{I}B_\lambda] :\to K$ defined as

$$TF = I\bar{F}$$

is unitary.

Proof. Let $F \in \mathrm{Ker}_2[\bar{I}B_\Lambda]$, by definition we have

$$I\bar{B}_\Lambda \bar{F} = G, \text{ for some } G \in \mathcal{H}^2,$$

and this yields that the function $I\bar{F}$ belongs to $\mathcal{H}^2$ and vanishes on Λ consequently we have $I\bar{F} \in K$. Also note that if $F_1,\ F_2 \in \mathrm{Ker}_2[\bar{I}B_\Lambda]$ with $F_1 = F_2$, then we must have $TF_1 = TF_2$. Thus the map T is well defined. Obviously the map T is an isometry as $|TF| = |F|$ on the real line $\mathbb{R}$. So in order to prove that T is unitary we only require to prove that T is surjective. Suppose that $F \in K$, then by definition F vanishes on Λ and hence we have $F = B_\Lambda K$ for some $K \in \mathcal{H}^2$. Now as $B_\Lambda K \in \mathcal{K}_I$ we have $I\bar{B}_\Lambda \bar{K} \in \mathcal{H}^2$. Put $H = I\bar{B}_\Lambda \bar{K} = I\bar{F}$, then $H \in \mathrm{Ker}_2[\bar{I}B_\Lambda]$ and $TH = F$. □

5.3 Mixed data problem

Let us recall that given a meromorphic inner function I we define the point spectrum of I as

$$\sigma(I) := \begin{cases} \{x \in \mathbb{R} : I(x) = 1\} & \text{if } 1 - I \notin L^2(\mathbb{R}) \\ \{x \in \mathbb{R} : I(x) = 1\} \cup \{\infty\} & \text{if } 1 - I \in L^2(\mathbb{R}). \end{cases}$$

Let Φ and Ψ be meromorphic inner functions and $I = \Psi\Phi$. Suppose that $\sigma(I)$ denotes the point spectrum of I. We say that the data $[\Psi, \sigma(I)]$ determine I if any inner function $\tilde{I}$ satisfies the following conditions:

- Ψ divides $\tilde{I}$, and
- $\sigma(I) = \sigma(\tilde{I})$,

then we must have $I = \tilde{I}$. Alternatively, we can say that Ψ and $\sigma(I)$ determine Φ.

In [29], the authors proved the following three results related to the mixed data problem.

Proposition 5.3.1. *If* $\mathrm{Ker}_\infty[\bar{\Phi}\Psi] \neq \{0\}$*, then the data* $[\Psi, \sigma(I)]$ *does not determine* I*.*

Proposition 5.3.2. *If* $\mathrm{Ker}_p^\Pi[\bar{\Phi}\Psi] = \{0\}$ *for some* $p < 1$*, then the data* $[\Psi, \sigma(I)]$ *determine* I*.*

Proposition 5.3.3. *If the data* $[\Psi, \sigma(I)]$ *does not determine* I*, then there exists meromorphic inner functions* I_1 *and* J *such that* $\{x \in \mathbb{R} : I_1(x) = 1\} = \{x \in \mathbb{R} : J(x) = 1\}$ *and*

$$\mathrm{Ker}_\infty[\bar{I}_1 J \bar{\Phi}\Psi] \neq \{0\}.$$

In this direction we obtain some result. In order to prove the theorem on the mixed data problem we need the following technical lemma.

Lemma 5.3.4. *Let* $\Phi, \Psi, I,$ *and* J *be meromorphic inner functions. Let* $f \in \mathrm{Ker}_\infty[\bar{I} J \bar{\Phi}\Psi]$*. Assume that the following conditions are satisfied.*

(i) $\sigma(I) \subset \sigma(J)$,

(ii) $\Psi f = \Phi g$, *for some* $g \in \mathcal{H}^\infty$ *with* $g(a) = 0$, *for some* $a \in \mathbb{C}_+$.

Then $f \equiv 0$.

Proof. Since $f \in N^\infty[\bar{I}J\bar{\Phi}\Psi]$, by definition we have

$$\bar{I}J\bar{\Phi}\Psi f = \bar{H}, \text{ for some } H \in \mathcal{H}^\infty.$$

Using condition (ii), we get

$$\bar{I}Jg = \bar{H}. \tag{5.3.1}$$

Since $g(a) = 0$, this implies that there exists $h \in \mathcal{H}^\infty$ such that $(z - a)h = g$. By (5.3.1) we get,

$$\bar{I}J(x - a)h = \bar{H}.$$

This implies

$$\bar{I}Jh = \overline{\left(\frac{H}{x - \bar{a}}\right)}. \tag{5.3.2}$$

Next, we note that, since the function $\dfrac{1}{z - \bar{a}}$ belongs to $\mathcal{H}^2$, for any $a \in \mathbb{C}_+$, and $H \in \mathcal{H}^\infty$, we get $\dfrac{H}{z - \bar{a}} \in \mathcal{H}^2$. By (5.3.2), it follows that $h \in \text{Ker}_2[\bar{I}J]$. Since $\sigma(I) \subset \sigma(J)$ it follows from Theorem 3.20, [29] that $h \equiv 0$. Thus we have $f \equiv 0$. □

Now we prove the following Theorem on the mixed data problem.

Theorem 5.3.5. *Let* Ψ, *and* Φ *be two meromorphic inner functions and put* $I = \Psi\Phi$. *Suppose that following conditions are satisfied*

(i) $Z(\Phi) \subsetneq Z(\Psi)$, *where* $Z(f)$ *denotes the zero set of a typical function* f,

(ii) Φ *divides* Ψ, *and*

(iii) $\infty \notin \sigma(I)$.

Then the data $[\Psi, \sigma(I)]$ *determine* I.

Proof. If possible suppose that the data $[\Psi, \sigma(I)]$ does not determine I. Then there exist meromorphic inner functions I_1 and J such that

$$\{x \in \mathbb{R} : I(x) = 1\} = \{x \in \mathbb{R} : J(x) = 1\}$$

and

$$\text{Ker}_\infty[\bar{I}_1 J \bar{\Phi} \Psi] \neq \{0\}.$$

For $f \in \text{Ker}_\infty[\bar{I}_1 J \bar{\Phi} \Psi]$, we note that the following conditions are satisfied.

- $\dfrac{\Psi f}{\Phi} \in \mathcal{H}^\infty$, by condition (ii)
- $\dfrac{\Psi f}{\Phi}(a) = 0$, for some $a \in \mathbb{C}_+$, by condition (i), and
- $\sigma(I_1) \subseteq \sigma(J)$, by condition (iii).

Thus by Lemma 5.3.4, we must have $f \equiv 0$. Hence, we reach a contradiction and this completes the proof.

□

Remark. In Theorem 5.3.5, one can weaken the condition $Z(\Phi) \subsetneq Z(\Psi)$. By the proof of Theorem 5.3.5, it is obvious that one only needs to assume that Ψ has a zero of multiplicity higher than that of Φ.

Now, we construct a space for which only a single point in the upper half-plane turns out to be a uniqueness set.

Corollary 5.3.6. *Let I and J be two meromorphic inner functions such that $\sigma(I) \subset \sigma(J)$. Then for any meromorphic inner function Ψ in the upper half-plane $\mathbb{C}_+$ one and exactly one of the following is true.*

(i) Any singleton set in $\mathbb{C}_+$ is a uniqueness set for $\text{Ker}_\infty[\bar{I} J \Psi]$.

(ii) $\text{Ker}_\infty[\bar{I} J \Psi] = \{0\}$.

Proof. Let Ψ be any meromorphic inner function in the upper half-plane. Then, either $\Psi = Be^{iaz}$, where B is some non-trivial Blaschke product and $a \geq 0$, or $\Psi = Ce^{iaz}$, where C is a unimodular constant.

Case I. Let $\Psi = Be^{iaz}$. Consider $\Phi = 1$, then one can easily see that any function $f \in \mathrm{Ker}_\infty[\bar{I}JI^a]$ satisfies the condition $\Psi f \in \Phi\mathcal{H}^\infty$. Thus if f vanishes at any point in the upper half-plane, we must have $f \equiv 0$.

Case II. Let $\Psi = Ce^{iaz}$. Consider $\Phi = 1$, then for any $f \in \mathrm{Ker}_\infty[\bar{I}J\Psi]$, we automatically have $\dfrac{\Psi f}{\Phi} \in \mathcal{H}^\infty$ and vanishes at a point in the upper half-plane. □

CHAPTER 6

Summary and future direction

6.1 Summary

In this dissertation we have discussed three problems arising in the study of complex analysis and used in the completeness problems in function spaces and in the spectral theory.

In the third chapter we discussed about the explicit formula for Hilbert transform. We are able to capture the largest class which can be considered in this context. The idea was to generate two equations involving Hilbert transform of the same function and then solve them. The solution was obtained using the non-tangential limits and comparing the arguments of the Known equations. Then to highlight generality of our formula we have applied it to many situations and derived some known results rather easily.

In the fourth chapter we studied the multipliers between model spaces. The main interest of this chapter was to obtain a characterization of the algebra of multipliers in the non-Hilbert setting, that is, for the case $p \neq 2$. The main thrust of this chapter was that: for the case $p \neq 2$ the algebra of multipliers denies to obey some results obtained before for the case $p = 2$. The man tool that we used to prove the above claim was a remarkable characterization of Toeplitz kernels obtained by Dyakonov. In this chapter we also proved that the algebra of multipliers does not allow a "perturbation" in inner functions without altering the algebra.

In the fifth chapter we studied about uniqueness sets for model spaces and mixed data problem. The result on uniqueness sets was studied via the study of dimension of some auxiliary space and the results on the mixed data was obtained via injectivity of some Toeplitz operators.

6.2 Future direction

6.2.1 Hilbert transform

Since the Hilbert transform appears in many applications it seems to worthwhile look for some explicit formula for the other class of functions as well. In Chapter 2 we studied about an explicit formula for the functions $\log|F|$ for $F \in \mathcal{N}_*$. The main property of this class that we have used in the proof was that, such functions F belongs to the class $C^\omega(\mathbb{R})$ and this property allowed us to solve the two equations involving the Hilbert transform of $|\log|F|$. The approach that we have adopted will break-dawn if $F \notin C^\omega(\mathbb{R})$.

Problem 1. If $F \in \mathcal{N} \setminus C^\omega(\mathbb{R})$ can we adopt some other method to obtain some explicit formula for the Hilbert transform of $\log|f|$?

6.2.2 Multipliers between model spaces

In Chapter 4 we studied about the multipliers between the model spaces. We proved that, in general for the case $p > 2$ the map $\mathcal{U}_2$ fails to be surjective from the model space $\mathcal{K}^p_\Theta$ into the model space K^p_Θ. This was proved by giving the existence of an inner function for which such a phenomena occur. We believe that if $\mathit{\Theta}$ is any inner function other than a finite Blaschke, then such a phenomena always occur. So it seems to be interesting to investigate the case of non finite Blaschke product.

Problem 2. Let $\mathit{\Theta}$ be any inner function in the unit disk which is not a finite Blaschke product and let $\Theta = \mathit{\Theta} \circ \omega$ be the corresponding inner function in the upper half-plane. If $p > 2$ does it always imply that the map

$$\mathcal{U}_2 : \mathcal{K}^p_\Theta :\rightarrow K^p_\Theta$$

is not surjective?

As we have discussed in Chapter 4 that Crofoot [9] investigated onto multipliers between model spaces $\mathcal{K}^p_{\Theta_1}$ and $\mathcal{K}^p_{\Theta_2}$. In [9] he also investigated isometric multipliers. Recall that a multiplier ϑ is said to be isometric if $||\vartheta F||_p = ||F||_p$. Crofoot prove that a multiplier ϑ is isometric from $\mathcal{K}^p_{\Theta_1}$ onto $\mathcal{K}^p_{\Theta_2}$ if and only if $\Theta_2 = \tau \circ \Theta_1$ for some disk automorphism τ. It seems to be interesting to investigate isometric multipliers in the setting of into multipliers.

Problem 3. Let Θ_1 and Θ_2 be inner functions on the unit disk $\mathbb{D}$. Under what condition does there exist an into isometric multipliers between the model spaces $\mathcal{K}^p_{\Theta_1}$ onto $\mathcal{K}^p_{\Theta_2}$?

6.2.3 Uniqueness Results on Model Spaces

In Chapter 5 we investigated about the uniqueness sets for model spaces K^p_Θ and we obtain the criterion of uniqueness in terms of the dimension of some auxiliary spaces. It seems to a be very deep and challenging problem to obtain a criterion for uniqueness sets in terms terms of the inner function Θ itself. Any such criterion can be understood as a generalization of Beurling-Malliavin theory on the radius of completeness. Some results of this kind for a class of meromorphic inner function whose argument on the real line $\mathbb{R}$ grows polynomially are contained the work of Makarov and Poltoratski [30] or [41, Chapter 8].

Problem 4 Let Θ be an inner function and suppose that $\Lambda \subset \mathbb{C}_+$. Obtain a criterion in terms of Θ so that Λ turns out to be a uniqueness set for the model space K^p_Θ.

Bibliography

[1] A. Baranov, *Differentiation in the branges spaces and embedding theorems*, J. Math. Sciences volume 101 (2000), 2881–2913.

[2] A. Beurling, P. Malliavin, *On Fourier transforms of measures with com-pact support,* Acta Math. 107 (1962), 291-309.

[3] A. Beurling, P. Malliavin, *On the closure of characters and the zeros of entire functions,* Acta Math. 118 (1967), 79-93

[4] G. Borg, *Uniqueness theorems in the spectral theory of* $y'' + (\lambda - q(x))y = 0$, Proc. 11-th Scandinavian Congress of Mathematicians, Johan Grundt Tanums Forlag, Oslo, 1952, 276-287.

[5] J. Bourgain, *A problem of Douglas and Rudin on factorization,* Pacific J.Math. 121 (1986), 47-50.

[6] A. Böttcher, Silbermann B., *Analysis of Toeplitz operators.* Academie-Verlag and Springer-Verlag, Berlin, 1990.

[7] J. Conway, *Functions of one complex variable,* Second Edition, Springer-Verlag, 1978.

[8] M. Cristina Cmara, Jonathan R. Partington, *Multipliers and equivalences between Toeplitz kernels,* J. Math. Anal. Appl. 465 (2018), no. 1, 557–570.

[9] R. Bruce Crofoot, *Multipliers between invariant subspaces of the backward shift*, Pacific J. Math. 166 (1994), no. 2, 225–246.

[10] C. S. Davis, *Iterated limits in $N^*(U^n)$*, Trans. Amer. Math. Soc. 178 (1973), 139–146.

[11] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London 1970.

[12] K. M. D'yakonov, *Entire functions of exponential type and model subspaces in H^p*, (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 190 (1991), Issled. po Linen. Oper. i Teor. Funktsi. 19, 81–100, 186; translation in J. Math. Sci. 71 (1994), no. 1, 2222–2233

[13] K. M. Dyakonov, *Kernels of Toeplitz operators via Bourgain's factorization theorem*, J. Funct. Anal. 170 (2000), no. 1, 93–106

[14] E. Dynkin, *Methods of the theory of singular integrals: Hilbert transform and Caldern-Zygmund theory*, Commutative harmonic analysis I, Encyclopaedia Math. Sci., 15, Springer, Berlin, 1991, 167–259.

[15] E. Fricain, A. Hartmann, W. Ross, *Multipliers between model spaces,* Studia Math. 240 (2018), no. 2, 177–191.

[16] E. Fricain, R. Rupam, *A note on multipliers between model spaces,* New York J. Math. 24 (2018), 147156.

[17] S. Garcia, J. Mashreghi, W. Ross, *Introduction to Model Spaces and their Operators*, Cambridge Studies in Advanced Mathematics 148, Cambridge University Press, 2016.

[18] J.B. Garnett, *Bounded Analytic Functions*, revised first edition, Springer, New York, 2007.

[19] F. Gesztesy, and B. Simon, *Spherical means in annular regions,* Trans. Amer. Math. Soc. 189(1999), no. 2, 85-92.

[20] V. Havin, J. Mashreghi, *Admissible Majorants for Model Subspaces of* H^2*, Part I: Slow Winding of the Generating Inner Function*, Canad. J. Math. Vol. 55 (2003), no. 6, 1231–1263.

[21] H. Hochstadt, B. Lieberman, *An inverse Sturm-Liouville problem with mixed given data,* SIAM J. Appl. Math. 34 (1978), no. 4, 676-680.

[22] Frederick W. King, *Hilbert transforms, Vol 1*, Encyclopedia of Mathematics and its Applications, 124, Cambridge University Press, Cambridge, 2009.

[23] Frederick W. King, *Hilbert transforms, Vol 2*, Encyclopedia of Mathematics and its Applications, 125, Cambridge University Press, Cambridge, 2009.

[24] P. Koosis, *The Logarithmic Integral I*, Cambridge Studies in Advanced Mathematics 12, Cambridge University Press, 1988.

[25] P. Koosis, *Introduction to Hp spaces. Second edition. With two appendices by V. P. Havin* , Cambridge Tracts in Mathematics, 115. Cambridge University Press, Cambridge, 1998

[26] F. Gesztesy, B. Simon, *Inverse Spectral Analysis with partial information on the Potential, II. the case of discrete spectrum,* Trans. Amer. Math. Soc. 352 (2000), no. 6, 2765-2787.

[27] P. D. Lax, R. S. Phillips, *Scattering theory,* Academic Press, New York (1967)

[28] N. Levinson, *The Inverse Sturm Liouville problem* Mat. Tidsskr. B (1949), 25-30.

[29] N. Makarov, A. Poltoratski, *Meromorphic inner functions, Toeplitz kernels and the uncertainty principle,* Perspectives in Analysis, Springer, Berlin, 2005, pp. 185-252

[30] N. Makarov, A. Poltoratski, *Beurling-Malliavin theory for Toeplitz kernels,* Invent. Math. 180 (2010), no. 3, 443-480.

[31] N. Makarov, A. Poltoratski, *Two-spectra theorem with uncertainty,* J. Spectr. Theory 9 (2019), no. 4, 12491285.

[32] N. Makarov, A. Poltoratski, *Etudes for the inverse spectral problem,* J. Lond. Math. Soc. (2) 108 (2023), no. 3, 916-977

[33] J. Mashreghi, *Representation Theorems in Hardy Spaces*, London Mathematical Society Student Texts Series 74, Cambridge University Press, 2009.

[34] J. Mashreghi, *Hilbert transform of* $\log|f|$, Proc. Amer. Math. Soc. 130 (2002), no. 3, 683–688.

[35] M. Mitkovski, A. Poltoratski, *Polya sequences, Toeplitz kernels and gap theorems,* Adv. Math. 224 (2010), 1057-1070.

[36] N. K. Nikolskii, *Treatise on the shift operator* Springer-Verlag, Berlin-New York (1986).

[37] N. K. Nikolskii, *Operators, functions, and systems: an easy reading, Vol. I,* AMS, Providence, RI (2002).

[38] N. K. Nikolskii, *Operators, functions, and systems: an easy reading, Vol. II,* AMS, Providence, RI (2002).

[39] A. Poltoratski, *A problem on completeness of exponentials,* Ann. of Math. (2) 178 (2013), no. 3, 983-1016.

[40] A. Poltoratski, *Spectral gaps for sets and measures,* Acta Math. 208 (2012), no. 1, 151-209.

[41] A. Poltoratski, *Toeplitz Approach to Problems of the Uncertainty Principle,* CBMS, vol. 121, American Mathematical Soc., 2015.

[42] A. Poltoratski, *Pointwise convergence of the non-linear Fourier transform,* Ann. of Math. (2), to appear.

[43] C. Remling, *Spectral theory of canonical systems,* De Gruyter Studies in Mathematics, 70. De Gruyter, Berlin, 2018.

[44] M. Rosenblum, J. Rovnyak, *Topics in Hardy classes and univalent functions,* Birkhuser Verlag, Basel, 1994.

[45] J. H. Shapiro, A. L. Shields, *Unusual topological properties of the Nevanlinna class,* Amer. J. Math. 97 (1975), no. 4, 915–936.

[46] A. L. Volberg, S. R. Treil, *Embedding theorems for invariant subspaces of the inverse shift operator*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 149 (1986), 38-51; translation in J. Soviet Math. 42 (1988), no. 2, 15621572

Contributed articles

Based on the work in this thesis, the following research articles are published/communicated/to be communicated.

1. A. K. Bhardwaj, A. Chattopadhyay, J. Mashreghi, R. K. Srivastava, *Hilbert Transform in the Cartwright-de Branges space*, Operator Theory: Advances and Applications. Springer Nature. 14 pages, to appear.

2. A. K. Bhardwaj, J. Mashreghi, R. K. Srivastava, *Hilbert Transform, Nevanlinna Class and Toeplitz kernels.*
   Communicated.

3. A. K. Bhardwaj, A. Chattopadhyay, J. Mashreghi, R. K. Srivastava, *Multipliers between Model Spaces Revisited.*
   To be Communicated.

4. A. K. Bhardwaj, A. Chattopadhyay, R. K. Srivastava, *Some Uniqueness Results on Model Spaces.*
   To be Communicated.