ARITHMETIC PROPERTIES OF CERTAIN PARTITION FUNCTIONS AND MODULAR FORMS

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Arithmetic Properties of Certain Partition Functions and Modular Forms

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This work is dedicated

to

My Parents

and

Sister

Certificate

This is to certify that the thesis entitled **Arithmetic Properties of Certain Partition Functions and Modular Form** submitted by **Mr. Chiranjit Ray** to the **Indian Institute of Technology Guwahati**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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"As we express our gratitude, we must never forget that the highest appreciation is not to utter words, but to live by them."

-John F. Kennedy

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Abstract

This thesis studies arithmetic properties of ℓ-regular overpartitions, Andrews' singular overpartitions, overpartitions into odd parts, cubic and overcubic partition pairs, and Andrews' integer partitions with even parts below odd parts. We use various dissections of Ramanujan's theta functions to find infinite families of arithmetic identities and Ramanujan-type congruences for ℓ-regular overpartitions and overpartitions into odd parts. We find certain congruences satisfied by $\overline{A}_\ell(n)$ for $\ell = 4, 8$ and 9, where $\overline{A}_\ell(n)$ denotes the number of ℓ-regular overpartitions of n. We find several infinite families of congruences including some Ramanujan-type congruences satisfied by $\overline{A}_{2\ell}(n)$ and $\overline{A}_{4\ell}(n)$ for any $\ell \geq 1$. We next prove several congruences for $\overline{p}_o(n)$ modulo 8 and 16, where $\overline{p}_o(n)$ denotes the number of overpartitions of n into odd parts. We also obtain the generating functions for $\overline{p}_o(16n+2)$, $\overline{p}_o(16n+6)$, and $\overline{p}_o(16n+10)$; and some new p-dissection formulas.

In a very recent paper, Andrews introduced the partition function $\mathcal{EO}(n)$ which counts the number of partitions of n where every even part is less that each odd part. He denoted by $\overline{\mathcal{EO}}(n)$, the number of partitions counted by $\mathcal{EO}(n)$ in which *only* the largest even part appears an odd number of times. We use arithmetic properties of modular forms and eta-quotients to study distribution of Andrews' singular overpartitions, cubic and overcubic partition pairs, and Andrews' integer partitions with even parts below odd parts. We use q-series manipulations and

Radu's algorithm on modular forms to derive certain congruences satisfied by cubic partition pairs, overcubic partition pairs and Andrews' integer partitions with even parts below odd parts. Along the way, we affirm two conjectures on Andrews' singular overpartitions and cubic partition pairs. We find two infinite families of congruences for $\overline{\mathcal{EO}}(n)$ using the theory of Hecke eigenforms. We also prove that there are infinitely many integers N in every arithmetic progression for which $\overline{\mathcal{EO}}(N)$ is even; and that there are infinitely many integers M in every arithmetic progression for which $\overline{\mathcal{EO}}(M)$ is odd so long as there is at least one.

Contents

Introduction

A partition of a positive integer n is any nonincreasing sequence of positive integers whose sum is n. The positive integers in the partition are called parts. The number of partitions of n is denoted by $p(n)$. For example, the partitions of 5 are $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$, so $p(5) = 7$. Conventionally, we set $p(0) = 1$.

In 1740, Naude raised a number of questions in his letter to Euler. In one of his questions, he asked in how many ways can an integer n be represented as a sum of integers? In response to this question, Euler [21] found many basic results on the partitions of numbers. His fundamental works in this subject using the generating functions and formal power series firmly established the additive number theory. For example, Euler proved that, for any positive integer n, the number of partitions of n using only odd parts equals the number of partitions of n into distinct parts. One of the most difficult challenges was to determine an explicit formula for $p(n)$. Hardy and Ramanujan [23], Rademacher [47], and Selberg [52] answered this question quite completely.

Partition theory has numerous connections to other areas of mathematics, including combinatorics, representation theory, and even mathematical physics. However, there are many questions about the basic properties of partitions which are not yet solved. Partitions reflect fundamental additive properties of the integers, so it is surprising to learn that $p(n)$ has divisibility properties as well. In 1919, Ramanujan announced that he had found three simple congruences satisfied by $p(n)$, namely,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5},\\ p(7n+5) &\equiv 0 \pmod{7},\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Ramanujan proved these congruences in a series of papers. He gave proofs of the first two congruences in [50] and derived the following q-series identities:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6} = 5 + 30q + 135q^2 + \ldots,$$
$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q^7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8} = 7 + 77q + \cdots.$$

Ramanujan also remarked that there are no simple congruences for $p(n)$ whose moduli involve primes other than $5, 7$ and 11. Ahlgren and Boylan [1] justifies Ramanujan's claim by proving that if ℓ is a prime and $0 \leq \beta < \ell$ is an integer for which

$$p(\ell n + \beta) \equiv 0 \pmod{\ell}$$

for every nonnegative integer n, then $(\ell, \beta) \in \{(5,4), (7,5), (11,6)\}$.

There are many generalizations of the Ramanujan's congruences for $p(n)$. In addition to the study of Ramanujan-type congruences, it is natural to consider the general distribution of the partition function modulo positive integers M. Ono revolutionized the subject by a seminal paper in 2000 [44]. He developed aspects of the p-adic theory of half-integral weight modular forms and used this to prove the existence of infinite families of partition congruences modulo every prime $\ell \geq 5$. There still remain many open questions on the distribution of $p(n)$. If M is a positive integer and $0 \leq r < M$, then define $\delta_r(M;X)$ by

$$\delta_r(M;X) := \frac{\#\{0 \leq n < X : p(n) \equiv r \pmod{M}\}}{X}.ama$$

The well known parity conjecture of Parkin and Shanks [46] predicts that the values of $p(n)$ are evenly distributed modulo 2, and little is known regarding this conjecture.

Conjecture (Parkin and Shanks)**.** If $r \in \{0,1\}$, then

$$\lim_{X \to +\infty} \delta_r(2;X) = \frac{1}{2}.$$

Different types of partition functions are introduced by many mathematicians and found that such partition functions too satisfy many interesting arithmetic properties. For example, ℓ-regular overpartitions, Andrews' singular overpartitions, overpartitions into odd parts, cubic and overcubic partition pairs, and Andrews' integer partitions with even parts below odd parts are some of the partition functions studied by many mathematicians. In this thesis, we study arithmetic properties of these partition functions. We find infinite families of arithmetic identities and Ramanujan-type congruences for these partition functions, and affirm certain conjectures along the way. We also find distribution of some of these partition functions modulo certain positive integers, namely, Andrews' singular overpartitions, cubic and overcubic partition pairs, and Andrews' integer partitions with even parts below odd parts. We use Ramanujan's theta functions identities and classical q-series methods to prove some of our results. We also use certain arithmetic properties of modular forms in our proofs. In particular, we use congruence properties of certain modular forms, basic properties of eta-quotients, and the theory of Hecke eigenforms in our proofs.

Organization of the Thesis

We present the entire work of this thesis in six chapters as described below.

- Chapter 1: Preliminaries
- Chapter 2: ℓ-Regular overpartitions
- Chapter 3: Overpartitions into odd parts
- Chapter 4: Andrews' singular overpartitions
- Chapter 5: Cubic and overcubic partition pairs
- Chapter 6: Andrews' integer partitions with even parts below odd parts

In Chapter 1 we introduce q-series, Ramanujan's theta functions and various dissections of Ramanujan's theta functions. We also recall some definitions and basic results on modular forms.

In Chapter 2 we prove some arithmetic properties of ℓ-regular overpartitions. We find certain congruences satisfied by $\overline{A}_\ell(n)$ for $\ell = 4, 8$ and 9, where $\overline{A}_\ell(n)$ denotes the number of ℓ-regular overpartitions of n. We next find several infinite families of congruences including some Ramanujan-type congruences satisfied by $\overline{A}_{2\ell}(n)$ and $\overline{A}_{4\ell}(n)$ for any $\ell \geq 1$. Our proofs use basic properties of q-series and various dissection formulas of Ramanujan's theta functions.

In Chapter 3 we study arithmetic properties of overpartitions into odd parts. We prove several congruences for $\overline{p}_o(n)$ modulo 8 and 16, where $\overline{p}_o(n)$ denotes the number of overpartitions of n into odd parts. We also obtain the generating functions for $\overline{p}_o(16n+2)$, $\overline{p}_o(16n+6)$, and $\overline{p}_o(16n+10)$. We use Ramanujan's theta function identities and some new p-dissections in our proofs.

In Chapter 4 we study the Andrews' singular overpartitions $\overline{C}_{3,1}(n)$. We affirm a conjecture of Naika and Gireesh by proving that

$$\overline{C}_{3,1}(12n+11) \equiv 0 \pmod{144}$$

for all $n \geq 0$. We next study the distribution of the partition function $\overline{C}_{3,1}(n)$. We establish congruences between certain eta-quotients and the partition function $\overline{C}_{3,1}(n)$, and then use a deep theorem of Serre to prove that $\overline{C}_{3,1}(n)$ is almost always divisible by 2^k and 3^k for any positive integer k.

In Chapter 5 we study various types of cubic partitions, namely, overcubic partition, cubic partition pairs, and overcubic partition pairs. Let $b(n)$ be the number of cubic partition pairs of n. We affirm a conjecture of Lin by proving that

$$b(49n + 37) \equiv 0 \pmod{49}$$

for all $n \geq 0$ using Radu's algorithm. We also prove two congruences modulo 256 satisfied by $\overline{b}(n)$, where $\overline{b}(n)$ denotes the number of overcubic partition pairs of n. We next prove some distribution results satisfied by overcubic partition and overcubic partition pairs using arithmetic of eta-quotients.

In Chapter 6 we study a partition function which appeared in a very recent paper of Andrews. He introduced the partition function $\mathcal{EO}(n)$ which counts the number of partitions of n where every even part is less that each odd part. He denoted by $\overline{\mathcal{EO}}(n)$, the number of partitions counted by $\mathcal{EO}(n)$ in which *only* the largest even part appears an odd number of times. Andrews proved that, for all $n \geq 0$

$$\overline{\mathcal{EO}}(10n + 8) \equiv 0 \pmod{5}.$$

In the same paper, he proposed to undertake a more extensive investigation of the properties of $\overline{\mathcal{EO}}(n)$. We show that the Andrews congruence is in fact true modulo 20 if $n \not\equiv 0 \pmod{5}$. We find two infinite families of congruences for $\overline{\mathcal{EO}}(n)$ using the theory of Hecke eigenforms. We also prove that there are infinitely many integers N in every arithmetic progression for which $\overline{\mathcal{EO}}(N)$ is even; and that there are infinitely many integers M in every arithmetic progression for which $\overline{\mathcal{EO}}(M)$ is odd so long as there is at least one. We further prove that $\overline{\mathcal{EO}}(n)$ is even for almost all n.

1

Preliminaries

1.1 q-Series and Ramanujan's theta functions

For complex numbers a and q, $(a;q)_\infty$ stands for the q-shifted factorial

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}), \quad |q|<1.$$

For $|ab|<1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{1.1}$$

In Ramanujan's notation, the Jacobi triple product identity [9, Entry 19, p. 36] takes the shape

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2}$$

The most important special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty = \frac{(q^2;q^2)^5_\infty}{(q;q)^2_\infty(q^4;q^4)^2_\infty}, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty}, \tag{1.4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.5}$$

We also have

$$\varphi(-q) = \frac{(q;q)^2_\infty}{(q^2;q^2)_\infty}, \tag{1.6}$$

$$\psi(-q) = \frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty}. \tag{1.7}$$

We now recall two definitions from [24, p. 225]. Let Π represent a pentagonal number (a number of the form $\frac{3n^2+n}{2}$) and Ω represent an octagonal number (a number of the form $3n^2+2n$). Let $\Pi(q) := \sum_{n=-\infty}^{\infty} q^{\frac{3n^2+n}{2}}$ and $\Omega(q) := \sum_{n=-\infty}^{\infty} q^{3n^2+2n}$. Then,

$$\Pi(q) = \frac{(q^2;q^2)_\infty(q^3;q^3)^2_\infty}{(q;q)_\infty(q^6;q^6)_\infty}, \tag{1.8}$$

$$\Omega(q) = \frac{(q^2;q^2)^2_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty(q^4;q^4)_\infty(q^6;q^6)_\infty}.$$

Also,

$$\Omega(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n}$$
$$= \prod_{n\geq 1}(1-q^{6n-5})(1-q^{6n-1})(1-q^{6n}) = \frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty (q^3;q^3)_\infty}. \qquad (1.9)$$

We state the following dissection formulas of Hirschhorn and Sellers [26].

Lemma 1.1. *We have*

$$\frac{1}{\varphi(-q)}$$
$$= \frac{1}{\varphi(-q^2)^2}\left(\varphi(q^4) + 2q\psi(q^8)\right) \qquad (1.10)$$
$$= \frac{\varphi(-q^9)}{\varphi(-q^3)^4}\left(\varphi(-q^9)^2 + 2q\varphi(-q^9)\Omega(-q^3) + 4q^2\Omega(-q^3)^2\right) \qquad (1.11)$$
$$= \frac{1}{\varphi(-q^4)^4}\left(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3\right). \qquad (1.12)$$

We state the following 4-dissection formula from [24, (1.9.4)] and [9, Entry 25, p. 40].

Lemma 1.2. *We have*

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \qquad (1.13)$$

That is,

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5 (q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^8)_\infty}. \qquad (1.14)$$

We next recall the following 3-dissection formula from [24, (26.1.2)] and [9, Corollary (i), p. 49].

Lemma 1.3. *We have*

$$\psi(q)=\Pi(q^3)+2q\psi(q^9). \tag{1.15}$$

That is,

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}=\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty}+q\frac{(q^{18};q^{18})_\infty^2}{(q^9;q^9)_\infty}.$$

The following is a consequence of dissection formulas of Ramanujan collected in [9, Entry 25, p. 40].

Lemma 1.4. *We have*

$$\frac{1}{(q;q)_\infty^4}=\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4}+4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}. \tag{1.16}$$

The following 2-dissection formula is due to Baruah and Ojah [8].

Lemma 1.5. *We have*

$$\begin{aligned}\frac{1}{(q;q)_\infty(q^3;q^3)_\infty}&=\frac{(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^4(q^{24};q^{24})_\infty^2}\\&\quad+q\frac{(q^4;q^4)_\infty^5(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4(q^6;q^6)_\infty^2(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty}.\end{aligned} \tag{1.17}$$

We also need the following lemma from [56, Lemma 3.5].

Lemma 1.6. *We have*

$$\frac{(q^9;q^9)_\infty}{(q;q)_\infty}=\frac{(q^{12};q^{12})_\infty^3(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty(q^{36};q^{36})_\infty}+q\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{36};q^{36})_\infty}{(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty}. \tag{1.18}$$

In the following lemma, we give p-dissections of $\varphi(q)$.

Lemma 1.7 (p. 49 [9]). *For any prime p,*

$$\varphi(q)=\varphi(q^{p^2})+\sum_{r=1}^{p-1} q^{r^2} f(q^{p(p-2r)},q^{p(p+2r)}). \tag{1.19}$$

In the following lemma, we give p-dissections of $\psi(q)$.

Lemma 1.8 (Theorem 2.1 [20]). *For any odd prime p,*

$$\psi(q)=\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(\frac{p^2+(2k+1)p}{2},\frac{p^2-(2k+1)p}{2}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{1.20}$$

Furthermore, $\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8} \pmod p$*, for* $0\le k\le\frac{p-3}{2}$.

The following lemma gives a p-dissection of $f(-q)$.

Lemma 1.9 (Theorem 2.2 [20]). *For any prime* $p\ge 5$,

$$\begin{aligned}(q;q)_\infty=f(-q)=&\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}})\\ &+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}),\end{aligned} \tag{1.21}$$

where

$$\frac{\pm p-1}{6}:=\begin{cases}\frac{p-1}{6} & \text{if } p\equiv 1 \pmod 6;\\ \frac{-p-1}{6} & \text{if } p\equiv -1 \pmod 6.\end{cases}$$

Furthermore, for $\frac{-(p-1)}{2}\le k\le\frac{p-1}{2}$ *and* $k\neq\frac{\pm p-1}{6}$,

$$\frac{3k^2+k}{2}\not\equiv\frac{p^2-1}{24} \pmod p.$$

The following lemma gives a p-dissection of $\psi(q^2)f(-q)^2$.

Lemma 1.10 (Lemma 2.4 [3]). *If $p \geq 5$ is a prime and*

$$\frac{\pm p-1}{3} := \begin{cases} \frac{p-1}{3} & \text{if } p \equiv 1 \pmod 3; \\ \frac{-p-1}{3} & \text{if } p \equiv -1 \pmod 3, \end{cases}$$

then

$$\psi(q^2)f(-q)^2 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{3}}}^{\frac{p-1}{2}} q^{3k^2+2k} \sum_{n=-\infty}^{\infty} (3pn+3k+1)q^{pn(3pn+6k+2)} \pm pq^{\frac{p^2-1}{3}} \psi(q^{2p^2}) f(-q^{p^2})^2. \tag{1.22}$$

Furthermore, if $k \neq \frac{\pm p-1}{3}$ and $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, then $3k^2 + 2k \not\equiv \frac{p^2-1}{3} \pmod p$.

The following lemma readily follows from [3, Lemma 2.3] by putting q^2 in place of q. The lemma gives a p-dissection of $f(-q^2)^3$.

Lemma 1.11. For any prime $p \geq 3$, we have

$$f(-q^2)^3 = \frac{1}{2} \sum_{\substack{k=0 \\ k \neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{k^2+k} \sum_{n=-\infty}^{\infty} (-1)^n (2pn+2k+1) q^{pn(pn+2k+1)} + (-1)^{\frac{p-1}{2}} pq^{\frac{p^2-1}{4}} f(-q^{2p^2})^3. \tag{1.23}$$

Furthermore, if $0 \leq k \leq p-1$ and $k \neq \frac{p-1}{2}$, then $k^2 + k \not\equiv \frac{p^2-1}{4} \pmod p$.

1.2 Modular forms

In this section, we recall some definitions and basic facts about modular forms. For more details, see for example [45, 33]. Let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ denote the full modular group of 2-by-2 matrices with determinant 1. Γ acts on points z in the upper half-plane $\mathcal{H}$ by linear fractional transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z := \frac{az+b}{cz+d}.$$

Let N be a positive integer. We now list some important subgroups of Γ below.

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \equiv 0 \pmod N \right\},$$
$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod N \right\},$$
$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1(N) : b \equiv 0 \pmod N \right\}.$$

Definition 1. *A subgroup Γ' of the full modular group Γ is called a congruence subgroup if $\Gamma(N) \subset \Gamma'$ for some N. The smallest N such that $\Gamma(N) \subset \Gamma'$ is called the level of Γ'.*

The index of $\Gamma_0(N)$ in Γ is

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}),$$

where p denotes a prime. The cusps of a subgroup $\Gamma' \leq \Gamma$ are the equivalence classes of $i\infty$ (also known as "the cusp at infinity") under the action of Γ'.

Definition 2. *Suppose that f is a meromorphic function on the upper half plane $\mathcal{H}$, and that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. For $k \in \mathbb{Z}$, the slash operator of weight k is defined by*

$$(f|_k\gamma)(z) := (ad-bc)^{\frac{k}{2}}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

A key property of this operator is that it gives a group action on the ring of meromorphic functions on $\mathcal{H}$, as described in the next proposition.

Proposition 1.12. *If* $\gamma, \gamma^{'} \in \mathrm{GL}_2^+(\mathbb{R})$*, then*

$$\left((f|_k\gamma)\,|_k\gamma^{'} \right)(z) = \left(f|_k(\gamma.\gamma^{'}) \right)(z)$$

Definition 3. *Let* $k \in \mathbb{Z}$ *and* $\Gamma^{'} \leq \Gamma$ *be a congruence subgroup of level* N*. A meromorphic function* f *on* $\mathcal{H}$ *is called a meromorphic modular form of integer weight* k *on* $\Gamma^{'}$ *if the following hold:*

1. *We have*
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$
for all $z \in \mathcal{H}$ *and all* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^{'}$.

2. *If* $\gamma_0 \in \Gamma$*, then* $(f|_k\gamma_0)(z)$ *has a Fourier expansion of the form*
$$(f|_k\gamma_0)(z) = \sum_{n \geq n_{\gamma_0}} c_{\gamma_0}(n) q_N^n,$$
where $q_N := e^{2\pi i z/N}$ *and* $c_{\gamma_0}(n_{\gamma_0}) \neq 0$.

Remark 1.2.1. *The condition (2) of the above definition means that* $f(z)$ *is meromorphic at the cusps of* $\Gamma^{'}$*. If* $n_{\gamma_0} \geq 0$ *for all* $\gamma_0 \in \Gamma$*, then we say that* $f(z)$ *is holomorphic at the cusps of* Γ'*.*

Definition 4. *Suppose that* $f(z)$ *is an integer weight meromorphic modular form on a congruence subgroup* $\Gamma^{'}$*. We say that* $f(z)$ *is a holomorphic modular (resp. cusp) form if* $f(z)$ *is holomorphic on* $\mathcal{H}$ *and is holomorphic (resp. vanishes) at the cusps of* $\Gamma^{'}$*. We say that* $f(z)$ *is a weakly holomorphic modular form if its poles (if there are any) are supported at the cusps of* $\Gamma^{'}$*.*

Holomorphic (resp. cusp) modular forms of weight k on a congruence subgroup Γ' naturally form $\mathbb{C}$-vector spaces. We denote the complex vector space of modular forms (resp. cusp forms) of weight k with respect to $\Gamma'(N)$ by $M_k\left(\Gamma'(N)\right)$ $\left(\text{resp. } S_k\left(\Gamma'(N)\right)\right)$.

Definition 5. *If χ is a Dirichlet character modulo N, then we say that a form $f(z) \in M_k(\Gamma_1(N))$ has Nebentypus character χ if*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all z in the upper half complex plane and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_k(\Gamma_0(N),\chi)$. If χ is the trivial character then we denote $M_k(\Gamma_0(N),\chi)$ by $M_k(\Gamma_0(N))$.

The spaces $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$ have the following decomposition (where the sums are over all Dirichlet characters χ modulo N):

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(\Gamma_0(N),\chi),$$

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(\Gamma_0(N),\chi).$$

The Hecke operators are natural linear transformations which act on spaces of modular forms. We recall the definition of the Hecke operators on spaces of integer weight modular forms.

Definition 6. *Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N),\chi)$. Then the action of Hecke operator T_m on $f(z)$ is defined by*

$$f(z)|T_m := \sum_{n=0}^{\infty}\left(\sum_{d|\gcd(n,m)} \chi(d)d^{k-1}a\left(\frac{nm}{d^2}\right)\right)q^n.$$

In particular, if $m = p$ is prime, we have

$$f(z)|T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p)p^{k-1}a\left(\frac{n}{p}\right)\right) q^n. \tag{1.24}$$

We note that $a(n) = 0$ unless n is a nonnegative integer.

Definition 7. *A modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ is called a Hecke eigenform if for every $m \geq 2$ there exists a complex number $\lambda(m)$ for which*

$$f(z)|T_m = \lambda(m)f(z). \tag{1.25}$$

We prove some distribution and parity results of certain partition functions in Chapters 4, 5, and 6. We will use the following result due to Serre.

Let $\mathcal{A}$ denote the subset of integer weight modular forms in $M_k(\Gamma_0(N), \chi)$ whose Fourier coefficients are in $\mathcal{O}_K$, the ring of algebraic integers in a number field K. Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$.

Theorem 1.13 (Theorem 2.65 [45]). *If $f(z) \in \mathcal{A}$ has Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n$$

then there is a constant $\alpha > 0$ such that

$$\#\{n \leq X : a(n) \not\equiv 0 \pmod{\mathfrak{m}}\} = \mathcal{O}\left(\frac{X}{(\log X)^{\alpha}}\right).$$

From Theorem 1.13, we have the following corollary.

Corollary 1.14. *Let m be a positive integer. If $f(z) \in M_k(\Gamma_0(N), \chi)$ has Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]],$$

then there is a constant $\alpha > 0$ *such that*

$$\#\{n \leq X : c(n) \not\equiv 0 \pmod{m}\} = \mathcal{O}\left(\frac{X}{(\log X)^{\alpha}}\right).$$

2

ℓ-Regular Overpartitions

2.1 Introduction

In [19], Corteel and Lovejoy introduced the notion of overpartitions. Since then, many interesting arithmetic properties of overpartitions are found by many mathematicians, for example, see Mahlburg [41], Hirschhorn and Sellers [26], and Kim [29, 31]. An overpartition of a nonnegative integer n is a partition of n in which the first occurrence of a part may be over-lined. For example, the eight overpartitions of 3 are $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1$. Let $\overline{p}(n)$ denote the number

[1]The contents of this chapter have been published in *The Ramanujan J.* (2018) and *Int. J. Number Theory* (2018).

of overpartitions of n. The generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}=\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}. \tag{2.1}$$

In [39], Lovejoy investigated the function $\overline{A}_\ell(n)$, which counts the number of overpartitions of n into parts not divisible by ℓ. In a very recent paper [53], Shen calls the overpartitions enumerated by the function $\overline{A}_\ell(n)$ as ℓ-regular overpartitions. The generating function for $\overline{A}_\ell(n)$ is

$$\sum_{n=0}^{\infty}\overline{A}_\ell(n)q^n=\frac{(-q;q)_\infty(q^\ell;q^\ell)_\infty}{(q;q)_\infty(-q^\ell;q^\ell)_\infty}=\frac{\varphi(-q^\ell)}{\varphi(-q)}. \tag{2.2}$$

For $\overline{A}_3(n)$ and $\overline{A}_4(n)$, Shen [53] finds some explicit results on the generating function dissections and derives some congruences modulo 3, 6 and 24.

2.2 Congruences for $\overline{A}_4(n)$

We prove the following new congruences for $\overline{A}_4(n)$ modulo 3, 32, and 96.

Theorem 2.1. *We have*

$$\overline{A}_4(8n+6)\equiv 0 \pmod{32}, \tag{2.3}$$
$$\overline{A}_4(24n+14)\equiv 0 \pmod{96}, \tag{2.4}$$
$$\overline{A}_4(24n+22)\equiv 0 \pmod{96}, \tag{2.5}$$
$$\overline{A}_4(72n+30)\equiv 0 \pmod{96}, \tag{2.6}$$
$$\overline{A}_4(72n+54)\equiv 0 \pmod{96}, \tag{2.7}$$
$$\overline{A}_4(24n+8)\equiv 0 \pmod{3}, \tag{2.8}$$
$$\overline{A}_4(24n+16)\equiv 0 \pmod{3}. \tag{2.9}$$

Proof. From (2.2), we have

$$\sum_{n=0}^{\infty}\overline{A}_4(n)q^n=\frac{\varphi(-q^4)}{\varphi(-q)}. \tag{2.10}$$

Putting (1.10) into (2.10) yields

$$\sum_{n=0}^{\infty}\overline{A}_4(n)q^n=\frac{\varphi(-q^4)}{\varphi(-q)}=\frac{\varphi(-q^4)}{\varphi(-q^2)^2}\left(\varphi(q^4)+2q\psi(q^8)\right).$$

Extracting the term containing q^{2n}, we have

$$\sum_{n=0}^{\infty}\overline{A}_4(2n)q^n=\frac{\varphi(q^2)\varphi(-q^2)}{\varphi(-q)^2}. \tag{2.11}$$

From [9, Entry 25, p. 40], we have

$$\varphi(q)\varphi(-q)=\varphi(-q^2)^2. \tag{2.12}$$

By (2.11) and (2.12), we obtain

$$\sum_{n=0}^{\infty}\overline{A}_4(2n)q^n=\frac{\varphi(-q^4)^2}{\varphi(-q)^2}.$$

Applying the 4-dissection formula (1.12) in above, we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{A}_4(2n)q^n\\
&=\frac{1}{\varphi(-q^4)^6}\left(\varphi(q^4)^3+2q\varphi(q^4)^2\psi(q^8)+4q^2\varphi(q^4)\psi(q^8)^2+8q^3\psi(q^8)^3\right)^2\\
&=\frac{1}{\varphi(-q^4)^6}\left(\varphi(q^4)^6+4q\varphi(q^4)^5\psi(q^8)+12q^2\varphi(q^4)^4\psi(q^8)^2+32q^3\varphi(q^4)^3\psi(q^8)^3\right.\\
&\qquad\left.+48q^4\varphi(q^4)^2\psi(q^8)^4+64q^5\varphi(q^4)\psi(q^8)^5+64q^6\psi(q^8)^6\right).
\end{aligned} \tag{2.13}$$

Extracting the terms containing q^{4n+i} for $i = 0, 1, 2, 3$, respectively, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_4(8n)q^n = \frac{1}{\varphi(-q)^6}\left(\varphi(q)^6 + 48q\varphi(q)^2\psi(q^2)^4\right), \tag{2.14}$$

$$\sum_{n=0}^{\infty} \overline{A}_4(8n+2)q^n = \frac{1}{\varphi(-q)^6}\left(4\varphi(q)^5\psi(q^2) + 64q\varphi(q)\psi(q^2)^5\right), \tag{2.15}$$

$$\sum_{n=0}^{\infty} \overline{A}_4(8n+4)q^n = \frac{1}{\varphi(-q)^6}\left(12\varphi(q)^4\psi(q^2)^2 + 64q\psi(q^2)^6\right), \tag{2.16}$$

$$\sum_{n=0}^{\infty} \overline{A}_4(8n+6)q^n = 32\frac{\varphi(q)^3\psi(q^2)^3}{\varphi(-q)^6}. \tag{2.17}$$

The congruence (2.3) now readily follows from (2.17).

From the binomial theorem, for any positive integers k, we have

$$(q^k;q^k)_\infty^3 \equiv (q^{3k};q^{3k})_\infty \pmod 3. \tag{2.18}$$

Therefore, (2.17) and (2.18) yield

$$\sum_{n=0}^{\infty} \overline{A}_4(8n+6)q^n \equiv 32\frac{\varphi(q^3)\psi(q^6)}{\varphi(-q^3)^2} \pmod{96}. \tag{2.19}$$

The congruences (2.4) and (2.5) now readily follow from (2.19).

We now prove the remaining four congruences of the theorem. From (2.19), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_4(24n+6)q^n \equiv 32\frac{\varphi(q)\psi(q^2)}{\varphi(-q)^2} \pmod{96}. \tag{2.20}$$

Also, from [9, Entry 25, p. 40], we have

$$\varphi(q)\psi(q^2) = \psi(q)^2. \tag{2.21}$$

Putting (2.21) into (2.20), we find that

$$\sum_{n=0}^{\infty} \overline{A}_4(24n+6)q^n \equiv 32\frac{\psi(q)^2}{\varphi(-q)^2} \pmod{96}. \tag{2.22}$$

Applying (1.4), (1.6) and (2.18), we deduce that

$$\begin{aligned} \frac{\psi(q)^2}{\varphi(-q)^2} &= \frac{\psi(q)^3}{\varphi(-q)^3} \times \frac{\varphi(-q)}{\psi(q)} \\ &= \frac{\psi(q)^3}{\varphi(-q)^3} \times \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty^3} \\ &\equiv \frac{\psi(q^3)}{\varphi(-q^3)} \times \frac{(q^3;q^3)_\infty}{(q^6;q^6)_\infty} \pmod{3}. \end{aligned} \tag{2.23}$$

Now, (2.22) and (2.23) imply

$$\sum_{n=0}^{\infty} \overline{A}_4(24n+6)q^n \equiv 32\frac{\psi(q^3)}{\varphi(-q^3)}\frac{(q^3;q^3)_\infty}{(q^6;q^6)_\infty} \pmod{96}. \tag{2.24}$$

Extracting the term containing q^{3n+1} and q^{3n+2} we get (2.6) and (2.7).

Again, using the binomial theorem we obtain form (2.14) that

$$\sum_{n=0}^{\infty} \overline{A}_4(8n)q^n \equiv \frac{\varphi(q^3)^2}{\varphi(-q^3)^2} \pmod{3}. \tag{2.25}$$

The congruences (2.8) and (2.9) now readily follow from (2.25). This completes the proof of Theorem 2.1. ∎

For an odd prime p, the Legendre symbol is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square modulo } p \text{ and } a \not\equiv 0 \pmod{p}; \\ -1 & \text{if } a \text{ is not a square modulo } p; \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

We also prove the following infinite family of congruences modulo 8 for $\overline{A}_4(n)$.

Theorem 2.2. *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for all nonnegative integers α and n, we have*

$$\sum_{n=0}^{\infty} \overline{A}_4(8p^{2\alpha}n + 4p^{2\alpha})q^n \equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 8. \tag{2.26}$$

Proof. From (2.16), we have

$$\sum_{n=0}^{\infty} \overline{A}_4(8n+4)q^n \equiv 4\frac{\varphi(q)^4\psi(q^2)^2}{\varphi(-q)^6} \pmod 8. \tag{2.27}$$

From the binomial theorem, for any positive integer k, we have

$$(q^k;q^k)_\infty^2 \equiv (q^{2k};q^{2k})_\infty \pmod 2. \tag{2.28}$$

Applying (1.3), (1.4), (1.6) and (2.28), we deduce that

$$\begin{aligned}\frac{\varphi(q)^4\psi(q^2)^2}{\varphi(-q)^6} &= \left(\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\right)^4\left(\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\right)^2\left(\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\right)^6\\ &\equiv (q;q)_\infty^{12} \pmod 2\\ &\equiv (q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 2. \end{aligned} \tag{2.29}$$

Hence, (2.27) and (2.29) yield

$$\sum_{n=0}^{\infty} \overline{A}_4(8n+4)q^n \equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 8, \tag{2.30}$$

which is (2.26) with $\alpha = 0$. We now use induction on α to complete the proof.

Observe that (2.26) can also be written as

$$\sum_{n=0}^{\infty} \overline{A}_4\left(8\left(p^{2\alpha}n + 12\frac{p^{2\alpha}-1}{24}\right)+4\right)q^n \equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 8. \tag{2.31}$$

We suppose that (2.31) holds for some $\alpha > 0$. Substituting (1.21) into (2.31), we have, modulo 8

$$\sum_{n=0}^{\infty} \overline{A}_4\left(8\left(p^{2\alpha}n + 12\frac{p^{2\alpha}-1}{24}\right)+4\right)q^n \tag{2.32}$$

$$\equiv 4\left[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{4\frac{3k^2+k}{2}} f(-q^{4\frac{3p^2+(6k+1)p}{2}}, -q^{4\frac{3p^2-(6k+1)p}{2}}) + (-1)^{\frac{\pm p-1}{6}} q^{4\frac{p^2-1}{24}} f(-q^{4p^2})\right]$$

$$\times\left[\sum_{\substack{m=-\frac{p-1}{2}\\ m\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{8\frac{3m^2+m}{2}} f(-q^{8\frac{3p^2+(6m+1)p}{2}}, -q^{8\frac{3p^2-(6m+1)p}{2}}) + (-1)^{\frac{\pm p-1}{6}} q^{8\frac{p^2-1}{24}} f(-q^{8p^2})\right].$$

For a prime $p \geq 5$ and $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$, consider

$$4\frac{3k^2+k}{2} + 8\frac{3m^2+m}{2} \equiv 12\frac{p^2-1}{24} \pmod{p} \tag{2.33}$$

which is equivalent to

$$4(6k+1)^2 + 8(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right) = -1$, we have $k = m = \frac{\pm p-1}{6}$ is the only solution of (2.33). Therefore, extracting the terms containing $q^{pn+12\frac{p^2-1}{24}}$ from both sides of (2.32), and then replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty} \overline{A}_4\left(8\left(p^{2\alpha+1}n + 12\,\frac{p^{2\alpha+2}-1}{24}\right)+4\right)q^n \equiv 4(q^{4p};q^{4p})_\infty(q^{8p};q^{8p})_\infty \pmod{8}. \tag{2.34}$$

Similarly, extracting the terms containing q^{pn} from both sides of (2.34), and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_4\left(8\left(p^{2(\alpha+1)}n + 12\,\frac{p^{2(\alpha+1)}-1}{24}\right)+4\right)q^n \equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 8.$$

Thus, (2.31) holds for $\alpha+1$ as well. This completes the proof of Theorem 2.2. ■

Corollary 2.3. *Let $p \geq 5$ be a prime, such that $\left(\frac{-2}{p}\right) = -1$. Then for all nonnegative integer n and $\alpha \geq 1$, we have*

$$\overline{A}_4(8p^{2\alpha}n + (8j+4p)p^{2\alpha-1}) \equiv 0 \pmod 8,$$

where $j = 1, 2, \ldots, p-1$.

Proof. From (2.34), it follows that

$$\overline{A}_4(8p^{2\alpha+1}(pn+j) + 4p^{2\alpha+2})) \equiv 0 \pmod 8,$$

where $j = 1, 2, \ldots, p-1$. This completes the proof for $\alpha \geq 1$. ■

2.3 Congruences for $\overline{A}_8(n)$

For $\overline{A}_8(n)$, we prove six congruences modulo 7 and three infinite families of congruences modulo 4, 8, and 16 in the following two theorems.

Theorem 2.4. *For $i = 1, 2, \ldots, 6$, we have*

$$\overline{A}_8(28n + 4i) \equiv 0 \pmod 7.$$

Proof. From (2.2), we have

$$\sum_{n=0}^{\infty} \overline{A}_8(n)q^n = \frac{\varphi(-q^8)}{\varphi(-q)}. \tag{2.35}$$

Putting (1.12) into (2.35), we arrive at

$$\sum_{n=0}^{\infty} \overline{A}_8(n)q^n = \frac{\varphi(-q^8)}{\varphi(-q^4)^4}\left(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3\right). \tag{2.36}$$

Extracting the terms containing q^{4n}, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_8(4n)q^n = \frac{\varphi(-q^2)}{\varphi(-q)^4}\varphi(q)^3. \tag{2.37}$$

Substituting (2.12) into (2.37), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_8(4n)q^n &= \frac{\varphi(-q^2)}{\varphi(-q)^4} \times \left(\frac{\varphi(-q^2)^2}{\varphi(-q)}\right)^3 \\ &= \frac{\varphi(-q^2)^7}{\varphi(-q)^7}.\end{aligned} \tag{2.38}$$

For any prime p, the binomial theorem gives

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}. \tag{2.39}$$

Thus, (2.38) and (2.39) yield

$$\sum_{n=0}^{\infty} \overline{A}_8(4n)q^n \equiv \frac{\varphi(-q^{14})}{\varphi(-q^7)} \pmod{7}. \tag{2.40}$$

The required congruences now readily follow from (2.40). This completes the proof of Theorem 2.4. ■

Theorem 2.5. *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for all nonnegative integers α and n, we have*

$$\sum_{n=0}^{\infty} \overline{A}_8(4p^{2\alpha}n + p^{2\alpha})q^n \equiv 2(q^2;q^2)_\infty(q^4;q^4)_\infty \pmod{4}, \tag{2.41}$$

$$\sum_{n=0}^{\infty} \overline{A}_8(4p^{2\alpha}n + 2p^{2\alpha})q^n \equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 8, \tag{2.42}$$

$$\sum_{n=0}^{\infty} \overline{A}_8(4p^{2\alpha}n + 3p^{2\alpha})q^n \equiv 8(q^2;q^2)_\infty(q^{16};q^{16})_\infty \pmod{16}. \tag{2.43}$$

Proof. We prove all the three congruences using induction on α. To prove (2.41), we extract the terms containing q^{4n+1} from (2.36) and we have

$$\sum_{n=0}^{\infty} \overline{A}_8(4n+1)q^n = 2\frac{\varphi(-q^2)}{\varphi(-q)^4}\varphi(q)^2\psi(q^2). \tag{2.44}$$

Now substitute (1.3), (1.4) and (1.6) into (2.44), and then using (2.28) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_8(4n+1)q^n &= 2\left(\frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty}\right)\left(\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\right)^4\left(\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\right)^2\left(\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\right)\\ &\equiv 2(q^2;q^2)_\infty^3 \pmod 4\\ &\equiv 2(q^2;q^2)_\infty(q^4;q^4)_\infty \pmod 4. \end{aligned} \tag{2.45}$$

Hence, (2.41) is true when $\alpha = 0$. We now observe that (2.41) can also be written as

$$\sum_{n=0}^{\infty} \overline{A}_8\left(4\left(p^{2\alpha}n + 6\frac{p^{2\alpha}-1}{24}\right)+1\right)q^n \equiv 2(q^2;q^2)_\infty(q^4;q^4)_\infty \pmod 4. \tag{2.46}$$

Suppose that (2.46) holds for some $\alpha > 0$. Substituting (1.21) into (2.46), and then proceeding by similar steps as shown in the proof of Theorem 2.2, we deduce that

$$\sum_{n=0}^{\infty} \overline{A}_8\left(4\left(p^{2\alpha+1}n + 6\frac{p^{2\alpha+2}-1}{24}\right)+1\right)q^n \equiv 2(q^{2p};q^{2p})_\infty(q^{4p};q^{4p})_\infty \pmod 4. \tag{2.47}$$

Extracting the terms containing q^{pn} from both sides of (2.47), and then replacing

q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_8\left(4\left(p^{2(\alpha+1)}n+6\frac{p^{2(\alpha+1)}-1}{24}\right)+1\right)q^n \equiv 2(q^2;q^2)_\infty(q^4;q^4)_\infty \pmod 4.$$

This completes the proof of (2.41).

To prove (2.42), we first extract the terms containing q^{4n+2} from (2.36) and we obtain

$$\sum_{n=0}^{\infty} \overline{A}_8(4n+2)q^n = 4\frac{\varphi(-q^2)}{\varphi(-q)^4}\varphi(q)\psi(q^2)^2. \tag{2.48}$$

Then substitute (1.3), (1.4) and (1.6) into (2.48); and then using (2.28) we get

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_8(4n+2)q^n &= 4\left(\frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty}\right)\left(\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\right)^4\left(\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\right)\left(\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\right)^2\\ &\equiv 4\left((q^2;q^2)_\infty^3\right)^2 \pmod 8\\ &\equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 8.\end{aligned} \tag{2.49}$$

Hence, (2.42) is true when $\alpha = 0$. We now observe that (2.42) can also be written as

$$\sum_{n=0}^{\infty} \overline{A}_8\left(4\left(p^{2\alpha}n+12\frac{p^{2\alpha}-1}{24}\right)+2\right)q^n \equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty. \tag{2.50}$$

Suppose that (2.50) holds for some $\alpha > 0$. Substituting (1.21) into (2.50), and then proceeding by similar steps as show in the proof of Theorem 2.2, we deduce that

$$\sum_{n=0}^{\infty} \overline{A}_8\left(4\left(p^{2\alpha+1}n+12\,\frac{p^{2\alpha+2}-1}{24}\right)+2\right)q^n \equiv 4(q^{4p};q^{4p})_\infty(q^{8p};q^{8p})_\infty \pmod 8. \tag{2.51}$$

Extracting the terms containing q^{pn} from both sides of (2.51), and then replacing

q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_8\left(4\left(p^{2(\alpha+1)}n + 12\,\frac{p^{2(\alpha+1)}-1}{24}\right)+2\right)q^n \equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 8.$$

This completes the proof of (2.42).

To prove (2.43), we extract the terms containing q^{4n+3} from (2.36) and we obtain

$$\sum_{n=0}^{\infty} \overline{A}_8(4n+3)q^n = 8\frac{\varphi(-q^2)}{\varphi(-q)^4}\psi(q^2)^3. \tag{2.52}$$

Now substitute (1.4) and (1.6) into (2.52), and then using (2.28) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_8(4n+3)q^n &= 8\left(\frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty}\right)\left(\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\right)^4\left(\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\right)^3\\ &\equiv 8(q^2;q^2)_\infty^9 \pmod{16}\\ &\equiv 8(q^2;q^2)_\infty(q^{16};q^{16})_\infty \pmod{16}.\end{aligned} \tag{2.53}$$

Hence, (2.43) is true when $\alpha = 0$. We now observe that (2.43) can also be written as

$$\sum_{n=0}^{\infty} \overline{A}_8\left(4\left(p^{2\alpha}n + 18\,\frac{p^{2\alpha}-1}{24}\right)+3\right)q^n \equiv 8(q^2;q^2)_\infty(q^{16};q^{16})_\infty \pmod{16}. \tag{2.54}$$

Suppose that (2.54) holds for some $\alpha > 0$. Substituting (1.21) in (2.54), and then proceeding by similar steps as show in the proof of Theorem 2.2, we deduce that

$$\sum_{n=0}^{\infty} \overline{A}_8\left(4\left(p^{2\alpha+1}n + 18\frac{p^{2\alpha+2}-1}{24}\right)+3\right)q^n \equiv 8(q^{2p};q^{2p})_\infty(q^{16p};q^{16p})_\infty \pmod{16}. \tag{2.55}$$

Extracting the terms containing q^{pn} from both sides of (2.55), and then replacing

q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_8 \left(4\left(p^{2(\alpha+1)}n + 18\frac{p^{2(\alpha+1)}-1}{24}\right)+3\right) q^n \equiv 8(q^2;q^2)_\infty (q^{16};q^{16})_\infty \pmod{16},$$

completing the proof of (2.43). This completes the proof of Theorem 2.5. ∎

Corollary 2.6. *Let $p \geq 5$ be a prime, such that $\left(\frac{-2}{p}\right) = -1$. Then for all nonnegative integer n and $\alpha \geq 1$, we have*

$$\overline{A}_8(4p^{2\alpha}n + (4j+p)p^{2\alpha-1}) \equiv 0 \pmod{4},$$

where $j = 1, 2, \ldots, p-1$.

Proof. From (2.47), it follows that

$$\overline{A}_8(4p^{2\alpha+1}(pn+j) + p^{2\alpha+2}) \equiv 0 \pmod{4},$$

where $j = 1, 2, \ldots, p-1$. This completes the proof for $\alpha \geq 1$. ∎

Corollary 2.7. *Let $p \geq 5$ be a prime, such that $\left(\frac{-2}{p}\right) = -1$. Then for all nonnegative integer n and $\alpha \geq 1$, we have*

$$\overline{A}_8(4p^{2\alpha}n + (4j+2p)p^{2\alpha-1}) \equiv 0 \pmod{8},$$

where $j = 1, 2, \ldots, p-1$.

Proof. From (2.51), it follows that

$$\overline{A}_8(4p^{2\alpha+1}(pn+j) + 2p^{2\alpha+2}) \equiv 0 \pmod{8},$$

where $j = 1, 2, \ldots, p-1$. This completes the proof for $\alpha \geq 1$. ∎

Corollary 2.8. *Let $p \geq 5$ be a prime, such that $\left(\frac{-2}{p}\right) = -1$. Then for all nonnegative integer n and $\alpha \geq 1$, we have*

$$\overline{A}_8(4p^{2\alpha}n + (4j+3p)p^{2\alpha-1}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$.

Proof. From (2.55), it follows that

$$\overline{A}_8(4p^{2\alpha+1}(pn+j) + 3p^{2\alpha+2}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$. This completes the proof for $\alpha \geq 1$. ■

2.4 Congruences for $\overline{A}_{2\ell}(n)$ and $\overline{A}_{4\ell}(n)$

In this section we prove infinite families of congruences for $\overline{A}_{2\ell}(n)$ and $\overline{A}_{4\ell}(n)$.

Theorem 2.9. *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for all positive integers n, k and α, we have*

$$\overline{A}_{2\ell}(2p^{2\alpha}n + (2j+p)p^{2\alpha-1}) \equiv 0 \pmod{4},$$

where $j = 1, 2, \ldots, p-1$.

To prove Theorem 2.9, we first prove the following result.

Theorem 2.10. *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for all integers $n, k \geq 1$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{A}_{2\ell}(2p^{2\alpha}n + p^{2\alpha})q^n \equiv 2(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod{4}. \tag{2.56}$$

Proof. From (2.2), we have

$$\sum_{n=0}^{\infty} \overline{A}_{2\ell}(n)q^n = \frac{\varphi(-q^{2\ell})}{\varphi(-q)}. \tag{2.57}$$

Putting (1.10) into (2.57) yields

$$\sum_{n=0}^{\infty} \overline{A}_{2\ell}(n)q^n = \frac{\varphi(-q^{2\ell})}{\varphi(-q^2)^2}\left(\varphi(q^4) + 2q\psi(q^8)\right).$$

Extracting the terms containing q^{2n+1}, and then using (1.4) and (1.6), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_{2\ell}(2n+1)q^n &= 2\frac{\varphi(-q^{\ell})\psi(q^4)}{\varphi(-q)^2} \\ &= 2\frac{(q^{\ell};q^{\ell})_\infty^2}{(q^{2\ell};q^{2\ell})_\infty}\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4}\frac{(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}.\end{aligned} \tag{2.58}$$

For any prime number p, the binomial theorem gives

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}. \tag{2.59}$$

By (2.58) and (2.59), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_{2\ell}(2n+1)q^n &\equiv 2(q^4;q^4)_\infty^3 \pmod{4} \\ &\equiv 2(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod{4}\end{aligned} \tag{2.60}$$

which is (2.56) with $\alpha = 0$. We now use induction on α to complete the proof.

Observe that (2.56) can also be written as

$$\sum_{n=0}^{\infty} \overline{A}_{2\ell}\left(2\left(p^{2\alpha}n + 12\frac{p^{2\alpha}-1}{24}\right)+1\right)q^n \equiv 2(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod{4}. \tag{2.61}$$

We suppose that (2.61) holds for some $\alpha > 0$. Substituting (1.21) in (2.61), and then proceeding by similar steps as show in the proof of Theorem 2.2, we deduce that

$$\sum_{n=0}^{\infty} \overline{A}_{2\ell}\left(2\left(p^{2\alpha+1}n + 12\frac{p^{2\alpha+2}-1}{24}\right)+1\right)q^n \equiv 2(q^{4p};q^{4p})_\infty(q^{8p};q^{8p})_\infty \pmod 4. \tag{2.62}$$

Similarly, extracting the terms containing q^{pn} from both sides of (2.62), and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{2\ell}\left(2\left(p^{2(\alpha+1)}n + 12\frac{p^{2(\alpha+1)}-1}{24}\right)+1\right)q^n \equiv 2(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod 4. \tag{2.63}$$

This completes the proof of the Theorem 2.10. ∎

Proof of Theorem 2.9. From (2.62), it follows that

$$\overline{A}_{2\ell}(2p^{2\alpha+1}(pn+j)+p^{2\alpha+2}) \equiv 0 \pmod 4,$$

where $j = 1, 2, \ldots, p-1$. This completes the proof of Theorem 2.9, for $\alpha \geq 1$. ∎

We obtain the following Ramanujan-type congruences for $\overline{A}_{2\ell}(n)$.

Corollary 2.11. *We have*

$$\overline{A}_{2\ell}(8n+2j+1) \equiv 0 \pmod 4,$$

where $j = 1, 2, 3$.

Proof. We readily obtain the congruences by extracting the terms containing q^{4n+j} from (2.60), where $j = 1, 2, 3$. ∎

We prove the following infinite families of congruences modulo 4, 8 and 16 for $\overline{A}_{4\ell}(n)$.

Theorem 2.12. *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for all positive integers n, k and α, we have*

$$\overline{A}_{4\ell}(4p^{2\alpha}n + (4j+p)p^{2\alpha-1}) \equiv 0 \pmod 4,$$
$$\overline{A}_{4\ell}(4p^{2\alpha}n + (4j+2p)p^{2\alpha-1}) \equiv 0 \pmod 8,$$
$$\overline{A}_{4\ell}(4p^{2\alpha}n + (4j+3p)p^{2\alpha-1}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$.

To prove Theorem 2.12, we first prove the following result.

Theorem 2.13. *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for all integers $n, k \geq 1$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4p^{2\alpha}n + p^{2\alpha})q^n \equiv 2(q^2;q^2)_\infty (q^4;q^4)_\infty \pmod 4, \tag{2.64}$$
$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4p^{2\alpha}n + 2p^{2\alpha})q^n \equiv 4(q^4;q^4)_\infty (q^8;q^8)_\infty \pmod 8, \tag{2.65}$$
$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4p^{2\alpha}n + 3p^{2\alpha})q^n \equiv 8(q^2;q^2)_\infty (q^{16};q^{16})_\infty \pmod{16}. \tag{2.66}$$

Proof. From (2.2), we have

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(n)q^n = \frac{\varphi(-q^{4\ell})}{\varphi(-q)}. \tag{2.67}$$

Putting (1.12) into (2.67), we arrive at

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(n)q^n = \frac{\varphi(-q^{4\ell})}{\varphi(-q^4)^4} \left(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3\right).$$

Extracting the terms containing q^{4n+i} for $i = 1, 2, 3$, respectively, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4n+1)q^n = 2\frac{\varphi(-q^\ell)}{\varphi(-q)^4}\varphi(q)^2\psi(q^2), \tag{2.68}$$

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4n+2)q^n = 4\frac{\varphi(-q^\ell)}{\varphi(-q)^4}\varphi(q)\psi(q^2)^2, \tag{2.69}$$

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4n+3)q^n = 8\frac{\varphi(-q^\ell)}{\varphi(-q)^4}\psi(q^2)^3. \tag{2.70}$$

We now substitute the values of $\varphi(q)$, $\psi(q)$ and $\varphi(-q)$ from (1.3), (1.4) and (1.6) into (2.68), (2.69) and (2.70), and then using the binomial theorem, we deduce that

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4n+1)q^n \equiv 2(q^2;q^2)_\infty(q^4;q^4)_\infty \pmod{4}, \tag{2.71}$$

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4n+2)q^n \equiv 4(q^4;q^4)_\infty(q^8;q^8)_\infty \pmod{8}, \tag{2.72}$$

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}(4n+3)q^n \equiv 8(q^2;q^2)_\infty(q^{16};q^{16})_\infty \pmod{16}. \tag{2.73}$$

Hence, (2.64), (2.65) and (2.66) are true when $\alpha = 0$. We now use induction on α to complete the proof.

For a prime $p \geq 5$ and $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$, we consider the congruences

$$2\frac{3k^2+k}{2} + 4\frac{3m^2+m}{2} \equiv 6\frac{p^2-1}{24} \pmod{p}, \tag{2.74}$$

$$4\frac{3k^2+k}{2} + 8\frac{3m^2+m}{2} \equiv 12\frac{p^2-1}{24} \pmod{p}, \tag{2.75}$$

$$2\frac{3k^2+k}{2} + 16\frac{3m^2+m}{2} \equiv 18\frac{p^2-1}{24} \pmod{p}. \tag{2.76}$$

We have the following equivalent congruences, respectively.

$$(6k+1)^2 + 2(6m+1)^2 \equiv 0 \pmod{p},$$

$$(6k+1)^2 + 2(6m+1)^2 \equiv 0 \pmod{p},$$

$$(6k+1)^2 + 8(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right) = -1$, therefore $k = m = \frac{\pm p-1}{6}$ is the only solution of (2.74), (2.75) and (2.76). By (1.21) and proceeding by similar steps as show in the proof of Theorem 2.2, we deduce the following congruences:

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}\left(4\left(p^{(2\alpha+1)}n + 6\frac{p^{2(\alpha+1)}-1}{24}\right)+1\right)q^n \equiv 2(q^{2p};q^{2p})_\infty(q^{4p};q^{4p})_\infty \pmod{4}; \tag{2.77}$$

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}\left(4\left(p^{(2\alpha+1)}n + 12\frac{p^{2(\alpha+1)}-1}{24}\right)+2\right)q^n \equiv 4(q^{4p};q^{4p})_\infty(q^{8p};q^{8p})_\infty \pmod{8}, \tag{2.78}$$

and, modulo 16, we have

$$\sum_{n=0}^{\infty} \overline{A}_{4\ell}\left(4\left(p^{(2\alpha+1)}n + 18\frac{p^{2(\alpha+1)}-1}{24}\right)+3\right)q^n \equiv 8(q^{2p};q^{2p})_\infty(q^{16p};q^{16p})_\infty. \tag{2.79}$$

We next extract the terms containing q^{pn} from both sides of the above congruences, and observe that (2.64), (2.65) and (2.66) are true when α is replaced by $\alpha+1$. This completes the proof of the result. ■

Proof of Theorem 2.12. By extracting the terms containing q^{pn+j} from both sides of (2.77), (2.78) and (2.79), we readily find that

$$\overline{A}_{4\ell}(4p^{2\alpha+1}(pn+j)+p^{2\alpha+2}) \equiv 0 \pmod{4},$$
$$\overline{A}_{4\ell}(4p^{2\alpha+1}(pn+j)+2p^{2\alpha+2}) \equiv 0 \pmod{8},$$
$$\overline{A}_{4\ell}(4p^{2\alpha+1}(pn+j)+3p^{2\alpha+2}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$. This completes the proof of Theorem 2.12 for $\alpha \geq 1$. ■

We obtain the following Ramanujan-type congruences for $\overline{A}_{4\ell}(n)$.

Corollary 2.14. *We have*

$$\overline{A}_{4\ell}(8n+7) \equiv 0 \pmod{16},$$
$$\overline{A}_{4\ell}(16n+4j+2) \equiv 0 \pmod{8},$$

where $j = 1, 2, 3$.

Proof. The first congruence follows from (2.73) by extracting the terms containing q^{2n+1}. We obtain the last congruences from (2.72) by extracting the terms containing q^{4n+j}, where $j = 1, 2, 3$. ■

Putting (1.12) into the generating function of $\overline{A}_\ell(n)$ we obtain the following remark.

Remark 2.4.1. *For any positive integer* $\ell \equiv 0 \pmod{4}$*, we have* $\overline{A}_\ell(4n+i) \equiv 0 \pmod{2^i}$*, for* $i = 1, 2, 3$.

2.5 Congruences for $\overline{A}_9(n)$

We prove the following congruences modulo 4 and 16 for $\overline{A}_9(n)$.

Theorem 2.15. *We have*

$$\overline{A}_9(6n+3) \equiv 0 \pmod{4}, \tag{2.80}$$
$$\overline{A}_9(36n+21) \equiv 0 \pmod{16}, \tag{2.81}$$
$$\overline{A}_9(36n+30) \equiv 0 \pmod{16}. \tag{2.82}$$

Proof. From (2.2), we have

$$\sum_{n=0}^{\infty} \overline{A}_9(n) q^n = \frac{\varphi(-q^9)}{\varphi(-q)}. \tag{2.83}$$

Putting (1.11) into (2.83), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_9(n)q^n = \frac{\varphi(-q^9)^2}{\varphi(-q^3)^4}\left(\varphi(-q^9)^2 + 2q\varphi(-q^9)\Omega(-q^3) + 4q^2\Omega(-q^3)^2\right).$$

After simplification, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_9(3n)q^n = \frac{\varphi(-q^3)^4}{\varphi(-q)^4}, \tag{2.84}$$

$$\sum_{n=0}^{\infty} \overline{A}_9(3n+1)q^n = 2\frac{\varphi(-q^3)^3\Omega(-q)}{\varphi(-q)^4}, \tag{2.85}$$

$$\sum_{n=0}^{\infty} \overline{A}_9(3n+2)q^n = 4\frac{\varphi(-q^3)^2\Omega(-q)^2}{\varphi(-q)^4}. \tag{2.86}$$

From the binomial theorem, we have

$$(q^k;q^k)_\infty^4 \equiv (q^{2k};q^{2k})_\infty^2 \pmod 4. \tag{2.87}$$

Hence, (2.84) and (2.87) yield

$$\sum_{n=0}^{\infty} \overline{A}_9(3n)q^n \equiv \frac{\varphi(-q^6)^2}{\varphi(-q^2)^2} \pmod 4. \tag{2.88}$$

Now, (2.80) follows from (2.88).

Again, putting (1.11) into (2.84), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{A}_9(3n)q^n \\
&= \frac{\varphi(-q^9)^4}{\varphi(-q^3)^{12}}\left(\varphi(-q^9)^2 + 2q\varphi(-q^9)\Omega(-q^3) + 4q^2\Omega(-q^3)^2\right)^4 \\
&= \frac{\varphi(-q^9)^4}{\varphi(-q^3)^{12}}\left(\varphi(-q^9)^8 + 8q\varphi(-q^9)^7\Omega(-q^3) + 40q^2\varphi(-q^9)^6\Omega(-q^3)^2\right. \\
&\quad +128q^3\varphi(-q^9)^5\Omega(-q^3)^3 + 304q^4\varphi(-q^9)^4\Omega(-q^3)^4 + 512q^5\varphi(-q^9)^3\Omega(-q^3)^5 \\
&\quad \left.+640q^6\varphi(-q^9)^2\Omega(-q^3)^6 + 512q^7\varphi(-q^9)\Omega(-q^3)^7 + 256q^8\Omega(-q^3)^8\right).
\end{aligned}$$

Extracting the term containing q^{3n+1}, we obtain

$$\sum_{n=0}^{\infty}\overline{A}_9(9n+3)q^n = \frac{\varphi(-q^3)^4}{\varphi(-q)^{12}}\left(8\varphi(-q^3)^7\Omega(-q)+304q\varphi(-q^3)^4\Omega(-q)^4\right.$$
$$\left.+512q^2\varphi(-q^3)\Omega(-q)^7\right). \tag{2.89}$$

Here we recall that $\Omega(-q) = \dfrac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}$. Hence, from (2.89) and (1.6) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{A}_9(9n+3)q^n\\
&\equiv 8\frac{\varphi(-q^3)^{11}}{\varphi(-q)^{12}}\Omega(-q) \pmod{16}\\
&\equiv 8\left(\frac{(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty}\right)^{11}\left(\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\right)^{12}\left(\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\right) \pmod{16}.
\end{aligned}$$

Then (2.28) and (1.17) yield

$$\begin{aligned}
\sum_{n=0}^{\infty}\overline{A}_9(9n+3)q^n &\equiv 8\frac{(q^6;q^6)_\infty^2}{(q;q)_\infty(q^3;q^3)_\infty} \pmod{16}\\
&\equiv 8\left(\frac{(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^2(q^{24};q^{24})_\infty^2}\right.\\
&\quad\left.+q\frac{(q^4;q^4)_\infty^5(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty}\right) \pmod{16}.
\end{aligned} \tag{2.90}$$

From (2.90), we find that

$$\sum_{n=0}^{\infty}\overline{A}_9(18n+3)q^n \equiv 8\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^5}{(q;q)_\infty^2(q^2;q^2)_\infty(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^2} \pmod{16}, \tag{2.91}$$

$$\sum_{n=0}^{\infty}\overline{A}_9(18n+12)q^n \equiv 8\frac{(q^2;q^2)_\infty^5(q^{12};q^{12})_\infty^2}{(q;q)_\infty^4(q^4;q^4)_\infty^2(q^6;q^6)_\infty} \pmod{16}. \tag{2.92}$$

Putting (1.17) into (2.91), we obtain, modulo 16

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{A}_9(18n+3)q^n \\
&\equiv 8\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty^2} \\
&\quad \times \left(\frac{(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^2(q^{24};q^{24})_\infty^2} + q\frac{(q^4;q^4)_\infty^5(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty}\right)^2,
\end{aligned}$$

which gives (2.81).

Again, using (1.16) in (2.92), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{A}_9(18n+12)q^n \qquad (2.93)\\
&\equiv 8\frac{(q^2;q^2)_\infty^5(q^{12};q^{12})_\infty^2}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty}\left(\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right) \pmod{16}.
\end{aligned}$$

Now, (2.82) follows from (2.93). This completes the proof of Theorem 2.15. ■

In the following theorem we prove a congruence relation modulo 3 for $\overline{A}_9(n)$.

Theorem 2.16. *For any positive integers k and n, we have*

$$\overline{A}_9(n) \equiv \overline{A}_9(3^k n) \pmod{3}. \qquad (2.94)$$

If n is odd, then for any positive integer k we have

$$\overline{A}_9(n) \equiv \overline{A}_9(3^k n) \pmod{6}. \qquad (2.95)$$

Proof. Using (1.6) on the generating function of $\overline{A}_9(3n)$ in (2.84) and then applying the binomial theorem, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_9(3n)q^n = \frac{\varphi(-q^3)^4}{\varphi(-q)^4}$$

$$
\begin{aligned}
&= \frac{(q^3;q^3)_\infty^8}{(q^6;q^6)_\infty^4}\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^8}\\
&= \frac{(q^3;q^3)_\infty^8}{(q^6;q^6)_\infty^4}\left(\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^6}\right)\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\\
&\equiv \frac{(q^3;q^3)_\infty^8}{(q^6;q^6)_\infty^4}\left(\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty^2}\right)\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \pmod 3\\
&= \frac{(q^3;q^3)_\infty^6}{(q^6;q^6)_\infty^3}\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}\\
&\equiv \frac{(q^9;q^9)_\infty^2}{(q^{18};q^{18})_\infty}\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \pmod 3\\
&= \frac{\varphi(-q^9)}{\varphi(-q)}.
\end{aligned}
\tag{2.96}
$$

Now, from (2.2) and (2.96), we obtain

$$\overline{A}_9(n) \equiv \overline{A}_9(3n) \pmod 3.$$

This proves the congruence (2.94).

We next prove the congruence (2.95). Putting (1.6) into (2.2), and then using (1.18), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\overline{A}_9(n)q^n\\
&= \frac{(q^2;q^2)_\infty(q^9;q^9)_\infty^2}{(q;q)_\infty^2(q^{18};q^{18})_\infty}\\
&= \frac{(q^2;q^2)_\infty}{(q^{18};q^{18})_\infty}\left(\frac{(q^{12};q^{12})_\infty^3(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty(q^{36};q^{36})_\infty}+q\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{36};q^{36})_\infty}{(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty}\right)^2.
\end{aligned}
$$

Extracting the terms containing q^{2n+1} we find that

$$\sum_{n=0}^{\infty}\overline{A}_9(2n+1)q^n = 2\frac{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2}{(q;q)_\infty^4}. \tag{2.97}$$

This proves that $\overline{A}_9(2n+1)$ is always even. Thus, if n is odd then (2.95) readily follows from (2.94). This completes the proof of the theorem. ■

Finally, we prove the following infinite families of congruences modulo 16 and 6 for $\overline{A}_9(n)$, respectively.

Theorem 2.17. *Let $p \geq 5$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for all nonnegative integers α and n, we have*

$$\sum_{n=0}^{\infty} \overline{A}_9(18p^{2\alpha}n + 3p^{2\alpha})q^n \equiv 8(q^4;q^4)_\infty \varphi(q^3) \pmod{16}. \tag{2.98}$$

Proof. We prove this theorem using induction on α. We first substitute (1.3) into (2.91), and then using (2.28) we obtain

$$\sum_{n=0}^{\infty} \overline{A}_9(18n+3)q^n \equiv 8(q^4;q^4)_\infty \varphi(q^3) \pmod{16}. \tag{2.99}$$

Hence, (2.98) is true when $\alpha = 0$. We now observe that (2.98) can also be written as

$$\sum_{n=0}^{\infty} \overline{A}_9\left(18\left(p^{2\alpha}n + 4\frac{p^{2\alpha}-1}{24}\right)+3\right)q^n \equiv 8(q^4;q^4)_\infty \varphi(q^3) \pmod{16}. \tag{2.100}$$

Suppose that (2.100) holds for some $\alpha > 0$. Substituting (1.19) and (1.21) into (2.100), and then proceeding by similar steps as show in the proof of Theorem 2.2, we deduce that

$$\sum_{n=0}^{\infty} \overline{A}_9\left(18\left(p^{2\alpha}\left(pn + 4\frac{p^2-1}{24}\right) + 4\frac{p^{2\alpha}-1}{24}\right)+3\right)q^n$$
$$\equiv 8(q^{4p};q^{4p})_\infty \varphi(q^{3p}) \pmod{16}. \tag{2.101}$$

Extracting the terms containing q^{pn} from both sides of (2.101), and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_9\left(18\left(p^{2(\alpha+1)}n + 4\frac{p^{2(\alpha+1)}-1}{24}\right)+3\right)q^n \equiv 8(q^4;q^4)_\infty \varphi(q^3) \pmod{16}.$$

This completes the proof of the Theorem 2.17. ■

Corollary 2.18. *Let $p \geq 5$ be a prime, such that $\left(\frac{-2}{p}\right) = -1$. Then for all nonnegative integer n and $\alpha \geq 1$, we have*

$$\overline{A}_9(18p^{2\alpha}n + (18j + 3p)p^{2\alpha-1}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$.

Proof. From (2.101), it follows that

$$\overline{A}_9(18p^{2\alpha+1}(pn + j) + 3p^{2\alpha+2}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$. This completes the proof for $\alpha \geq 1$. ■

Theorem 2.19. *Let p be a prime such that $\left(\frac{-3}{p}\right) = -1$. Then for all integers $n, k \geq 0$ and $\alpha \geq 1$, we have*

$$\overline{A}_9\left(3^k(2p^{2\alpha}n + (2j + p)p^{2\alpha-1})\right) \equiv 0 \pmod{6},$$

where $j = 1, 2, \ldots, p-1$.

We now prove a result before proving Theorem 2.19.

Theorem 2.20. *Let p be a prime such that $\left(\frac{-3}{p}\right) = -1$. Then for all nonnegative integers α, k and n, we have*

$$\sum_{n=0}^{\infty} \overline{A}_9\left(3^k(2p^{2\alpha}n + p^{2\alpha})\right) q^n \equiv 2\psi(q)\psi(q^3) \pmod{6}. \tag{2.102}$$

Proof. We note that due to Theorem 2.16 it is enough to prove the congruence (2.102) for $k = 0$. Putting (1.4) into (2.97), and then using the fact that $\overline{A}_9(2n+1)$

is even, we have

$$\sum_{n=0}^{\infty} \overline{A}_9(2n+1)q^n \equiv 2\psi(q)\psi(q^3) \pmod 6. \tag{2.103}$$

Hence, (2.102) is true when $\alpha = 0$ and $k = 0$. Suppose that for some $\alpha > 0$, we have

$$\sum_{n=0}^{\infty} \overline{A}_9\left(2\left(p^{2\alpha}n + 4\frac{(p^{2\alpha}-1)}{8}\right)+1\right) q^n \equiv 2\psi(q)\psi(q^3) \pmod 6. \tag{2.104}$$

Substituting (1.20) into (2.104), we have, modulo 6

$$\sum_{n=0}^{\infty} \overline{A}_9\left(2\left(p^{2\alpha}n + 4\frac{(p^{2\alpha}-1)}{8}\right)+1\right) q^n \tag{2.105}$$
$$\equiv 2\left[\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(\frac{p^2+(2k+1)p}{2}, \frac{p^2-(2k+1)p}{2}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right]$$
$$\times \left[\sum_{m=0}^{\frac{p-3}{2}} q^{3\frac{m^2+m}{2}} f\left(3\frac{p^2+(2m+1)p}{2}, 3\frac{p^2-(2m+1)p}{2}\right) + q^{3\frac{p^2-1}{8}}\psi(q^{3p^2})\right].$$

For any odd prime p and $0 \le k, m \le \frac{p-3}{2}$, we consider the congruence

$$\frac{k^2+k}{2} + 3\frac{m^2+m}{2} \equiv 4\frac{p^2-1}{8} \pmod p, \tag{2.106}$$

which is equivalent to

$$(2k+1)^2 + 3(2m+1)^2 \equiv 0 \pmod p.$$

Since $\left(\frac{-3}{p}\right) = -1$, therefore $k = m = \frac{p-1}{2}$ is the only solution of (2.106). Now, extracting the terms containing $q^{pn+4\frac{(p^2-1)}{8}}$ from both sides of (2.105), and then replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty} \overline{A}_9 \left(2 \left(p^{2\alpha} \left(pn + 4\frac{(p^2-1)}{8} \right) + 4\frac{(p^{2\alpha}-1)}{8} \right) + 1 \right) q^n \equiv 2\psi(q)\psi(q^3) \pmod{6}. \tag{2.107}$$

Again, extracting the terms containing q^{pn} from both sides of (2.107) and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_9 \left(2 \left(p^{2\alpha+2} n + 4\frac{(p^{2\alpha+2}-1)}{8} \right) + 1 \right) q^n \equiv 2\psi(q)\psi(q^3) \pmod{6}. \tag{2.108}$$

Thus, if $k = 0$ then (2.102) is true when α is replaced by $\alpha + 1$. This completes the proof of Theorem 2.20. ■

Proof of Theorem 2.19. From (2.107), it follows that

$$\overline{A}_9(2p^{2\alpha+1}(pn + j) + p^{2\alpha+2}) \equiv 0 \pmod{6}, \tag{2.109}$$

where $j = 1, 2, \ldots, p-1$. Hence, the result is true for $k = 0$. We now apply Theorem 2.16 to complete the proof of the Theorem 2.19. ■

3 Overpartitions Into Odd Parts

3.1 Introduction

In this chapter, we study overpartitions in which only odd parts are used. This function has arisen in a number of recent papers, but in contexts which are very different from overpartitions. For example, see Ardonne, Kedem and Stone [6], Bessenrodt [11], and Santos and Sills [51]. We denote by $\overline{p}_o(n)$ the number of overpartitions of n into odd parts. Hirschhorn and Sellers [27] obtain many interesting arithmetic properties of $\overline{p}_o(n)$. They observe that the generating function for $\overline{p}_o(n)$

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is given by

$$\sum_{n=0}^{\infty}\overline{p}_o(n)q^n=\prod_{n=1}^{\infty}\frac{1+q^{2n-1}}{1-q^{2n-1}}=\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2(q^4;q^4)_\infty}. \tag{3.1}$$

They establish a number of arithmetic results including several Ramanujan-type congruences satisfied by $\overline{p}_o(n)$, and some easily-stated characterizations of $\overline{p}_o(n)$ modulo small powers of 2. For example, the following two Ramanujan-type congruences can readily be seen from one of their main theorems.

$$\overline{p}_o(8n+5)\equiv 0 \pmod 8, \tag{3.2}$$

$$\overline{p}_o(8n+7)\equiv 0 \pmod{16}. \tag{3.3}$$

They also prove that, for $n\geq 1$, $\overline{p}_o(n)$ is divisible by 4 if and only if n is neither a square nor twice a square. In [16, Theorem 1], Chen proves that

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+14)q^n=112\frac{(q^2;q^2)_\infty^{27}}{(q;q)_\infty^{25}(q^4;q^4)_\infty^2}+256q\frac{(q^2;q^2)_\infty^3(q^4;q^4)_\infty^{14}}{(q;q)_\infty^{17}}, \tag{3.4}$$

from which it readily follows that $\overline{p}_o(16n+14)\equiv 0 \pmod{16}$. Using elementary theory of modular forms, he further proves infinitely many congruences for $\overline{p}_o(n)$ modulo 32 and 64. Let $t\geq 0$ be an integer and $p_1,p_2\equiv 1 \pmod 8$ be primes. Chen [16, Theorem 2] proves that

$$\overline{p}_o(p_1^{2t+1}(16n+14))\equiv 0 \pmod{32}, \tag{3.5}$$

$$\overline{p}_o(p_1^{4t+3}(16n+14))\equiv 0 \pmod{64}, \tag{3.6}$$

$$\overline{p}_o(p_1p_2(16n+14))\equiv 0 \pmod{64}. \tag{3.7}$$

The first two congruences are valid for all nonnegative integers n satisfying $8n\not\equiv -7 \pmod{p_1}$. The last congruence is valid for all nonnegative integers n satisfying $8n\not\equiv -7 \pmod{p_1}$ and $8n\not\equiv -7 \pmod{p_2}$.

3.2 Some generating functions for $\overline{p}_o(n)$

We prove the following identities for $\overline{p}_o(n)$ similar to (3.4) for other values of n. Along the way, we also obtain (3.4).

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} \overline{p}_o(4n)q^n = \frac{(q^2;q^2)_\infty^5 (q^4;q^4)_\infty^3}{(q;q)_\infty^6 (q^8;q^8)_\infty^2}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} \overline{p}_o(4n+1)q^n = 2\frac{(q^4;q^4)_\infty^7}{(q;q)_\infty^4 (q^2;q^2)_\infty (q^8;q^8)_\infty^2}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} \overline{p}_o(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^7 (q^8;q^8)_\infty^2}{(q;q)_\infty^6 (q^4;q^4)_\infty^3}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} \overline{p}_o(4n+3)q^n = 4\frac{(q^2;q^2)_\infty (q^4;q^4)_\infty (q^8;q^8)_\infty^2}{(q;q)_\infty^4}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+2)q^n = 2\frac{(q^2;q^2)_\infty^{45}}{(q;q)_\infty^{31} (q^4;q^4)_\infty^{14}} + 224q\frac{(q^2;q^2)_\infty^{21} (q^4;q^4)_\infty^2}{(q;q)_\infty^{23}}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+6)q^n = 12\frac{(q^2;q^2)_\infty^{39}}{(q;q)_\infty^{29} (q^4;q^4)_\infty^{10}} + 320q\frac{(q^2;q^2)_\infty^{15} (q^4;q^4)_\infty^6}{(q;q)_\infty^{21}}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+10)q^n = 40\frac{(q^2;q^2)_\infty^{23}}{(q;q)_\infty^{27} (q^4;q^4)_\infty^{6}} + 384q\frac{(q^2;q^2)_\infty^{9} (q^4;q^4)_\infty^{10}}{(q;q)_\infty^{19}}. \tag{3.14}$$

Proof. From (3.1) and (1.13), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{p}_o(n)q^n &= \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2 (q^4;q^4)_\infty} \\
&= \frac{(q^2;q^2)_\infty^3}{(q^4;q^4)_\infty}\left(\frac{1}{(q;q)_\infty^2}\right) \\
&= \frac{(q^2;q^2)_\infty^3}{(q^4;q^4)_\infty}\left(\frac{q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5}\varphi(q)\right) \\
&= \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2}\varphi(q) \\
&= \frac{\varphi(q)}{\varphi(-q^2)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\varphi(q)\varphi(q^2)}{\varphi(q^2)\varphi(-q^2)} \\
&= \frac{\varphi(q)\varphi(q^2)}{\varphi(-q^4)^2} \\
&= \frac{\left(\varphi(q^4)+2q\psi(q^8)\right)\left(\varphi(q^8)+2q^2\psi(q^{16})\right)}{\varphi(-q^4)^2}.
\end{aligned}$$

Now extracting the terms containing q^{4n+i} for $i = 0, 1, 2, 3$, respectively and using (1.3), (1.4) and (1.6), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{p}_o(4n)q^n &= \frac{\varphi(q)\varphi(q^2)}{\varphi(-q)^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4} \\
&= \frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty^3}{(q;q)_\infty^6(q^8;q^8)_\infty^2}, \\
\sum_{n=0}^{\infty} \overline{p}_o(4n+1)q^n &= 2\frac{\psi(q^2)\varphi(q^2)}{\varphi(-q)^2} = 2\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4} \\
&= 2\frac{(q^4;q^4)_\infty^7}{(q;q)_\infty^4(q^2;q^2)_\infty(q^8;q^8)_\infty^2}, \\
\sum_{n=0}^{\infty} \overline{p}_o(4n+2)q^n &= 2\frac{\varphi(q)\psi(q^4)}{\varphi(-q)^2} = 2\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\frac{(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4} \\
&= 2\frac{(q^2;q^2)_\infty^7(q^8;q^8)_\infty^2}{(q;q)_\infty^6(q^4;q^4)_\infty^3}, \\
\sum_{n=0}^{\infty} \overline{p}_o(4n+3)q^n &= 4\frac{\psi(q^2)\psi(q^4)}{\varphi(-q)^2} = 4\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\frac{(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4} \\
&= 4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^4}.
\end{aligned}$$

This completes the proof of (3.8), (3.9), (3.10) and (3.11). We note that the identity (3.11) is also found by Hirschhorn and Sellers [27, Theorem 2.12].

Using (1.13) and (1.12) in (3.10), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_o(4n+2)q^n$$

$$
\begin{aligned}
&= 2\psi(q^4)\varphi(q)\frac{1}{\varphi(-q)^2}\\
&= 2\psi(q^4)\left(\varphi(q^4)+2q\psi(q^8)\right)\\
&\quad\times\frac{1}{\varphi(-q^4)^8}\left(\varphi(q^4)^3+2q\varphi(q^4)^2\psi(q^8)+4q^2\varphi(q^4)\psi(q^8)^2+8q^3\psi(q^8)^3\right)^2\\
&= 2\frac{\psi(q^4)}{\varphi(-q^4)^8}\left(\varphi(q^4)+2q\psi(q^8)\right)\left(\varphi(q^4)^6+4q\varphi(q^4)^5\psi(q^8)+12q^2\varphi(q^4)^4\psi(q^8)^2\right.\\
&\quad\left.+32q^3\varphi(q^4)^3\psi(q^8)^3+48q^4\varphi(q^4)^2\psi(q^8)^4+64q^5\varphi(q^4)\psi(q^8)^5+64q^6\psi(q^8)^6\right)\\
&= 2\frac{\psi(q^4)}{\varphi(-q^4)^8}\left(\varphi(q^4)^7+6q\varphi(q^4)^6\psi(q^8)+20q^2\varphi(q^4)^5\psi(q^8)^2+56q^3\varphi(q^4)^4\psi(q^8)^3\right.\\
&\quad\left.+112q^4\varphi(q^4)^3\psi(q^8)^4+160q^5\varphi(q^4)^2\psi(q^8)^5+192q^6\varphi(q^4)\psi(q^8)^6+128q^7\psi(q^8)^7\right).
\end{aligned}
\tag{3.15}
$$

Extracting the terms containing q^{4n+i} for $i=0,1,2$, respectively, we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty}\overline{p}_o(16n+2)q^n &= 2\frac{\varphi(q)^7\psi(q)}{\varphi(-q)^8}+224q\frac{\varphi(q)^3\psi(q)\psi(q^2)^4}{\varphi(-q)^8}\\
&= 2\frac{(q^2;q^2)_\infty^{45}}{(q;q)_\infty^{31}(q^4;q^4)_\infty^{14}}+224q\frac{(q^2;q^2)_\infty^{21}(q^4;q^4)_\infty^{2}}{(q;q)_\infty^{23}},\\
\sum_{n=0}^{\infty}\overline{p}_o(16n+6)q^n &= 12\frac{\varphi(q)^6\psi(q)\psi(q^2)}{\varphi(-q)^8}+320q\frac{\varphi(q)^2\psi(q)\psi(q^2)^5}{\varphi(-q)^8}\\
&= 12\frac{(q^2;q^2)_\infty^{39}}{(q;q)_\infty^{29}(q^4;q^4)_\infty^{10}}+320q\frac{(q^2;q^2)_\infty^{15}(q^4;q^4)_\infty^{6}}{(q;q)_\infty^{21}},\\
\sum_{n=0}^{\infty}\overline{p}_o(16n+10)q^n &= 40\frac{\varphi(q)^5\psi(q)\psi(q^2)^2}{\varphi(-q)^8}+384q\frac{\varphi(q)\psi(q)\psi(q^2)^6}{\varphi(-q)^8}\\
&= 40\frac{(q^2;q^2)_\infty^{33}}{(q;q)_\infty^{27}(q^4;q^4)_\infty^{6}}+384q\frac{(q^2;q^2)_\infty^{9}(q^4;q^4)_\infty^{10}}{(q;q)_\infty^{19}}.
\end{aligned}
$$

This completes the proof of (3.12), (3.13) and (3.14), respectively. ■

Remark 3.2.1. *If we extract the coefficients of q^{4n+3} from (3.15), we readily obtain (3.4).*

3.3 Congruences for $\overline{p}_o(n)$

We prove the following congruences for $\overline{p}_o(n)$ modulo 8 and 16.

Theorem 3.2. *We have*

$$\sum_{n=0}^{\infty}\overline{p}_o(8n+3)q^n \equiv 4\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty} \pmod{16}, \tag{3.16}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+6)q^n \equiv 12\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty} \pmod{16}, \tag{3.17}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+10)q^n \equiv 8\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^3}{(q;q)_\infty} \pmod{16}, \tag{3.18}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+4)q^n \equiv 6\frac{(q;q)_\infty(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2} \pmod{8}, \tag{3.19}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+12)q^n \equiv 4\frac{(q;q)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} \pmod{8}, \tag{3.20}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(24n+1)q^n \equiv 2\frac{(q^2;q^2)_\infty}{(q;q)_\infty} \pmod{8}, \tag{3.21}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(24n+17)q^n \equiv 4\frac{(q;q)_\infty(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty} \pmod{8}, \tag{3.22}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(72n+9)q^n \equiv 6(q;q)_\infty(q^2;q^2)_\infty \pmod{8}. \tag{3.23}$$

Proof. From the binomial theorem, we have

$$(q;q)_\infty^4 \equiv (q^2;q^2)_\infty^2 \pmod{4}.$$

Now, applying the above congruences in (3.11), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_o(4n+3)q^n &= 4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^4}\\ &= 4\left(\frac{(q^2;q^2)_\infty}{(q;q)_\infty^4}\right)\left((q^4;q^4)_\infty(q^8;q^8)_\infty^2\right)\end{aligned}$$

$$\equiv 4\left(\frac{(q^2;q^2)_\infty}{(q^2;q^2)_\infty^2}\right)\left((q^4;q^4)_\infty(q^4;q^4)_\infty^4\right) \pmod{16}$$
$$= 4\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty} \pmod{16}.$$

Extracting the terms containing q^{2n}, we readily deduce (3.16).

Using the binomial theorem, from (3.13) and (3.14), we obtain, respectively,

$$\sum_{n=0}^{\infty}\overline{p}_o(16n+6)q^n \equiv 12\frac{(q^2;q^2)_\infty^{39}}{(q;q)_\infty^{29}(q^4;q^4)_\infty^{10}}$$
$$= 12\left(\frac{(q^2;q^2)_\infty^{14}}{(q;q)_\infty^{28}}\right)\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty}\left(\frac{(q^2;q^2)_\infty^{20}}{(q^4;q^4)_\infty^{10}}\right)$$
$$\equiv 12\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty} \pmod{16},$$
$$\sum_{n=0}^{\infty}\overline{p}_o(16n+10)q^n \equiv 8\frac{(q^2;q^2)_\infty^{33}}{(q;q)_\infty^{27}(q^4;q^4)_\infty^{6}}$$
$$= 8\left(\frac{(q^2;q^2)_\infty^{13}}{(q;q)_\infty^{26}}\right)\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^3}{(q;q)_\infty}\left(\frac{(q^2;q^2)_\infty^{18}}{(q^4;q^4)_\infty^{9}}\right)$$
$$\equiv 8\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^3}{(q;q)_\infty} \pmod{16}.$$

This completes the proof of (3.17) and (3.18).

From (3.8) and (1.13), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_o(4n)q^n = \frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty^3}{(q;q)_\infty^6(q^8;q^8)_\infty^2}$$
$$= \frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty^3}{(q^8;q^8)_\infty^2}\left(\frac{1}{(q;q)_\infty^2}\right)^3$$
$$= \frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty^3}{(q^8;q^8)_\infty^2}\left(\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^5}\varphi(q)\right)^3$$
$$= \frac{(q^4;q^4)_\infty^9}{(q^2;q^2)_\infty^{10}(q^8;q^8)_\infty^2}\varphi(q)^3$$
$$= \frac{(q^4;q^4)_\infty^9}{(q^2;q^2)_\infty^{10}(q^8;q^8)_\infty^2}\left(\varphi(q^4)+2q\psi(q^8)\right)^3$$
$$= \frac{(q^4;q^4)_\infty^9}{(q^2;q^2)_\infty^{10}(q^8;q^8)_\infty^2}$$

$$\times\left(\varphi(q^4)^3+6q\varphi(q^4)^2\psi(q^8)+12q^2\varphi(q^4)\psi(q^8)^2+8q^3\psi(q^8)^3\right).$$

Extracting the terms containing q^{2n+1}, and then using the binomial theorem, we obtain, modulo 8,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_o(8n+4)q^n&=\frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^{10}(q^4;q^4)_\infty^2}\left(6\varphi(q^2)^2\psi(q^4)+8q\psi(q^4)^3\right)\\&=\frac{(q^2;q^2)_\infty^9}{(q;q)_\infty^{10}(q^4;q^4)_\infty^2}\left(6\frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^4}\frac{(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}+8q\frac{(q^8;q^8)_\infty^6}{(q^2;q^2)_\infty^3}\right)\\&\equiv 6\frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty^7}{(q;q)_\infty^{10}(q^8;q^8)_\infty^2}\\&=6\left(\frac{(q^4;q^4)_\infty^2}{(q;q)_\infty^8}\right)\frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty}{(q;q)_\infty^2}\left(\frac{(q^4;q^4)_\infty^4}{(q^8;q^8)_\infty^2}\right)\\&\equiv 6\frac{(q^2;q^2)_\infty^5(q^4;q^4)_\infty}{(q;q)_\infty^2}.\end{aligned}\tag{3.24}$$

Applying (1.14) in (3.24) yields

$$\sum_{n=0}^{\infty}\overline{p}_o(8n+4)q^n\equiv 6(q^4;q^4)_\infty\left(\frac{(q^8;q^8)_\infty^5}{(q^{16};q^{16})_\infty^2}+2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}\right)\pmod 8.\tag{3.25}$$

Now, extracting the terms containing q^{4n} and q^{4n+1}, respectively, we have

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+4)q^n\equiv 6\frac{(q;q)_\infty(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2}=6(q;q)_\infty\varphi(-q^2)^2f(-q^2)\pmod 8,$$
$$\sum_{n=0}^{\infty}\overline{p}_o(32n+12)q^n\equiv 4\frac{(q;q)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}=4(q;q)_\infty\psi(q^2)f(-q)^2\pmod 8.$$

This completes the proof of (3.19) and (3.20).

Applying the congruences $(q;q)_\infty^4\equiv(q^2;q^2)_\infty^2 \pmod 4$ and $(q^8;q^8)_\infty^2\equiv(q^4;q^4)_\infty^4$

(mod 4) in (3.9) yields

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_o(4n+1)q^n &= 2\frac{(q^4;q^4)_\infty^7}{(q;q)_\infty^4(q^2;q^2)_\infty(q^8;q^8)_\infty^2}\\ &= 2\left(\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4}\right)\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^3}\left(\frac{(q^4;q^4)_\infty^4}{(q^8;q^8)_\infty^2}\right)\\ &\equiv 2\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^3} \pmod 8.\end{aligned}$$

Extracting the terms containing q^{2n}, we obtain, modulo 8,

$$\sum_{n=0}^{\infty}\overline{p}_o(8n+1)q^n \equiv 2\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^3} = 2\frac{\psi(q)}{\varphi(-q)}. \tag{3.26}$$

Now, (1.15) and (1.11) yield

$$\begin{aligned}\frac{\psi(q)}{\varphi(-q)} &= \frac{\varphi(-q^9)}{\varphi(-q^3)^4}\left(\Pi(q^3)+2q\psi(q^9)\right)\left(\varphi(-q^9)^2+2q\varphi(-q^9)\Omega(-q^3)+4q^2\Omega(-q^3)^2\right)\\ &= \frac{\varphi(-q^9)^3}{\varphi(-q^3)^4}\Pi(q^3)+3q\frac{\varphi(-q^9)^3}{\varphi(-q^3)^4}\psi(q^9)+6q^2\frac{\varphi(-q^9)^2}{\varphi(-q^3)^4}\Omega(-q^3)\psi(q^9)\\ &\quad+4q^3\frac{\phi(-q^9)}{\phi(-q^3)^4}\Omega(-q^3)^2\psi(q^9).\end{aligned} \tag{3.27}$$

Let us recall that $\Pi(q)=\dfrac{(q^2;q^2)_\infty(q^3;q^3)_\infty^2}{(q;q)_\infty(q^6;q^6)_\infty}$ and $\Omega(-q)=\dfrac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}$. From (3.26) and (3.27), and then extracting the terms containing q^{3n+i} for $i=0,1,2$, respectively, we find, modulo 8,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_o(24n+1)q^n &\equiv 2\frac{\varphi(-q^3)^3}{\varphi(-q)^4}\Pi(q) = 2\frac{(q^2;q^2)_\infty^5(q^3;q^3)_\infty^8}{(q;q)_\infty^9(q^6;q^6)_\infty^4}\\ &= 2\left(\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^8}\right)\frac{(q^2;q^2)_\infty}{(q;q)_\infty}\left(\frac{(q^3;q^3)_\infty^8}{(q^6;q^6)_\infty^4}\right)\\ &\equiv 2\frac{(q^2;q^2)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty}\overline{p}_o(24n+9)q^n &= 6\frac{\varphi(-q^3)^3}{\varphi(-q)^4}\psi(q^3) = 6\frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^5}{(q;q)_\infty^8(q^6;q^6)_\infty}\end{aligned}$$

$$\equiv 6\left(\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^8}\right)(q^3;q^3)_\infty(q^6;q^6)_\infty\left(\frac{(q^3;q^3)_\infty^4}{(q^6;q^6)_\infty^2}\right)$$

$$\equiv 6(q^3;q^3)_\infty(q^6;q^6)_\infty, \tag{3.28}$$

$$\sum_{n=0}^{\infty}\overline{p}_o(24n+17)q^n = 12\frac{\varphi(-q^3)^2}{\varphi(-q)^4}\Omega(-q)\psi(q^3)$$

$$\equiv 4\left(\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^8}\right)\frac{(q;q)_\infty(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty}$$

$$\equiv 4\frac{(q;q)_\infty(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty}.$$

This completes the proof of (3.21) and (3.22). Extracting the terms containing q^{3n} from (3.28) and using (1.5), we readily obtain (3.23). This complete the proof of Theorem 3.2. ∎

If we extract the terms containing q^{3n+1} and q^{3n+2} from (3.28), the following two Ramanujan-type congruences can readily be obtained.

Corollary 3.3. *For any $n \geq 0$,*

$$\overline{p}_o(72n+33) \equiv 0 \pmod 8, \quad \textit{and}$$

$$\overline{p}_o(72n+57) \equiv 0 \pmod 8.$$

Theorem 3.4. For nonnegative integers n and α we have

$$\overline{p}_o(2^\alpha(32n+20)) \equiv 0 \pmod 8, \tag{3.29}$$

$$\overline{p}_o(2^\alpha(32n+28)) \equiv 0 \pmod 8. \tag{3.30}$$

Proof. Extracting the terms containing q^{4n+2} and q^{4n+3} from (3.25), we have

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+20)q^n \equiv 0 \pmod 8,$$

$$\sum_{n=0}^{\infty}\overline{p}_o(32n+28)q^n \equiv 0 \pmod 8.$$

From [27, Corollary 2.10], we have that $\overline{p}_o(2n) \equiv 0 \pmod 8$ if $\overline{p}_o(n) \equiv 0 \pmod 8$. This proves (3.29) and (3.30) for any $\alpha \geq 0$. ■

3.4 Two new dissection formulas

In the following two lemmas, we deduce new p-dissections of $\dfrac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2}$ and $\Omega(-q)$, respectively.

Lemma 3.5. For a prime $p \geq 5$,

$$\frac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2} = \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{3k^2+k} \sum_{n=-\infty}^{\infty} (6pn+6k+1) q^{pn(3pn+6k+1)} \pm p q^{\frac{p^2-1}{12}} \frac{(q^{2p^2};q^{2p^2})_\infty^5}{(q^{4p^2};q^{4p^2})_\infty^2}. \tag{3.31}$$

In addition, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $k \neq \frac{\pm p-1}{6}$, then $3k^2+k \not\equiv \frac{p^2-1}{12} \pmod p$.

Proof. Due to Hirschhorn [24, (10.7.3)] and Berndt [10, (1.3.60)], we have

$$\begin{aligned}
\frac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2} &= \sum_{n=-\infty}^{\infty} (6n+1) q^{3n^2+n} \\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} \sum_{n=-\infty}^{\infty} [6(pn+k)+1] q^{3(pn+k)^2+(pn+k)} \\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} q^{3k^2+k} \sum_{n=-\infty}^{\infty} (6pn+6k+1) q^{pn(3pn+6k+1)} \\
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{3k^2+k} \sum_{n=-\infty}^{\infty} (6pn+6k+1) q^{pn(3pn+6k+1)} \\
&\quad \pm q^{\frac{p^2-1}{12}} \sum_{n=-\infty}^{\infty} p(6n+1) q^{p^2(3n^2+n)}
\end{aligned}$$

$$= \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} q^{3k^2+k} \sum_{n=-\infty}^{\infty} (6pn+6k+1)q^{pn(3pn+6k+1)} \pm pq^{\frac{p^2-1}{12}} \frac{(q^{2p^2};q^{2p^2})_\infty^5}{(q^{4p^2};q^{4p^2})_\infty^2}.$$

We observe that for $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, if $3k^2+k \equiv \frac{p^2-1}{12} \pmod p$, then we have $(6k+1)^2 \equiv 0 \pmod p$, which yields $k = \frac{\pm p-1}{6}$. ■

Lemma 3.6. If $p \geq 5$ is a prime and

$$\frac{\pm p+1}{3} := \begin{cases} \frac{-p+1}{3} & \text{if } p \equiv 1 \pmod 3; \\ \frac{p+1}{3} & \text{if } p \equiv -1 \pmod 3, \end{cases}$$

then

$$\Omega(-q) = \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq \frac{\pm p+1}{3}}}^{\frac{p-1}{2}} (-1)^k q^{3k^2-2k} \sum_{n=-\infty}^{\infty} (-1)^n q^{pn(3pn+6k-2)} + (-1)^{\frac{\pm p+1}{3}} q^{\frac{p^2-1}{3}} \Omega(-q^{p^2}). \tag{3.32}$$

Furthermore, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$ and $k \neq \frac{\pm p+1}{3}$, then $3k^2-2k \not\equiv \frac{p^2-1}{3} \pmod p$.

Proof. From (1.9), we have

$$\begin{aligned}\Omega(-q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} \\ &= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} \sum_{n=-\infty}^{\infty} (-1)^{pn+k} q^{3(pn+k)^2-2(pn+k)} \\ &= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{3k^2-2k} \sum_{n=-\infty}^{\infty} (-1)^n q^{pn(3pn+6k-2)} \\ &= \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq \frac{\pm p+1}{3}}}^{\frac{p-1}{2}} (-1)^k q^{3k^2-2k} \sum_{n=-\infty}^{\infty} (-1)^n q^{pn(3pn+6k-2)}\end{aligned}$$

$$+(-1)^{\frac{\pm p+1}{3}} q^{\frac{p^2-1}{3}} \sum_{n=-\infty}^{\infty} (-1)^n q^{p^2(3n^2-2n)}$$

$$= \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p+1}{3}}}^{\frac{p-1}{2}} (-1)^k q^{3k^2-2k} \sum_{n=-\infty}^{\infty} (-1)^n q^{pn(3pn+6k-2)}$$

$$+(-1)^{\frac{\pm p+1}{3}} q^{\frac{p^2-1}{3}} \Omega(-q^{p^2}).$$

Note that, if $3k^2 - 2k \equiv \frac{p^2-1}{3} \pmod{p}$, then $k = \frac{\pm p+1}{3}$. ■

3.5 Infinite families of congruences for $\overline{p}_o(n)$

We next prove certain infinite families of congruences for $\overline{p}_o(n)$ modulo 8 and 16 as stated in the following theorems.

Theorem 3.7. *Let $p \geq 3$ be a prime, $n \geq 0$ and $\alpha \geq 1$. If $\left(\frac{-2}{p}\right) = -1$, then we have*

$$\overline{p}_o\left(8p^{2\alpha}n + (3p+8j)p^{2\alpha-1}\right) \equiv 0 \pmod{16}, \tag{3.33}$$

$$\overline{p}_o\left(16p^{2\alpha}n + (6p+16j)p^{2\alpha-1}\right) \equiv 0 \pmod{16}, \tag{3.34}$$

$$\overline{p}_o\left(72p^{2\alpha}n + (9p+72j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}, \tag{3.35}$$

$$\overline{p}_o\left(24p^{2\alpha}n + (17p+24j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}, \tag{3.36}$$

$$\overline{p}_o\left(32p^{2\alpha}n + (4p+32j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}, \tag{3.37}$$

$$\overline{p}_o\left(32p^{2\alpha}n + (12p+32j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}. \tag{3.38}$$

where $j = 1, 2, \ldots, p-1$.

Before we prove Theorem 3.7, we first prove the following results.

Theorem 3.8. *Let $p \geq 3$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then, for all nonnegative integers n and α, we have*

$$\sum_{n=0}^{\infty} \overline{p}_o(8p^{2\alpha}n + 3p^{2\alpha})q^n \equiv 4f(-q^2)^3\psi(q) \pmod{16}. \tag{3.39}$$

Proof. From (3.16), we have that (3.39) is true for $\alpha = 0$. We now use induction on α to complete the proof. Observe that (3.39) can also be written as

$$\sum_{n=0}^{\infty} \overline{p}_o\left(8\left(p^{2\alpha}n + 3\frac{p^{2\alpha}-1}{8}\right) + 3\right) q^n \equiv 4f(-q^2)^3\psi(q) \pmod{16}. \tag{3.40}$$

We suppose that (3.40) holds for some $\alpha > 0$. Substituting (1.20) and (1.23) in (3.40), we have, modulo 16

$$\begin{aligned} &\sum_{n=0}^{\infty} \overline{p}_o\left(8\left(p^{2\alpha}n + 3\frac{p^{2\alpha}-1}{8}\right) + 3\right) q^n \\ &\equiv 4\left[\frac{1}{2}\sum_{\substack{k=0 \\ k\neq\frac{p-1}{2}}}^{p-1} (-1)^k q^{k^2+k} \sum_{n=-\infty}^{\infty} (-1)^n(2pn+2k+1)q^{pn(pn+2k+1)} \right. \\ &\qquad \left. + (-1)^{\frac{p-1}{2}} p q^{\frac{p^2-1}{4}} f(-q^{2p^2})^3 \right] \\ &\qquad \times \left[\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(\frac{p^2+(2m+1)p}{2}, \frac{p^2-(2m+1)p}{2}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right]. \end{aligned} \tag{3.41}$$

For a prime $p \geq 3$ and $0 \leq k \leq p-1$, $0 \leq m \leq \frac{p-1}{2}$, we consider

$$(k^2+k) + \frac{m^2+m}{2} \equiv 3\frac{p^2-1}{8} \pmod{p} \tag{3.42}$$

which is equivalent to

$$2(2k+1)^2 + (2m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right) = -1$, we have $k = m = \frac{p-1}{2}$ is the only solution of (3.42). Therefore, extracting the terms containing $q^{pn+3\frac{(p^2-1)}{8}}$ from both sides of (3.41), and then replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_o\left(8\left(p^{2\alpha+1}n + 3\frac{p^{2\alpha+2}-1}{8}\right)+3\right)q^n \equiv 4f(-q^{2p})^3\psi(q^p) \pmod{16}. \tag{3.43}$$

Similarly, extracting the terms containing q^{pn} from both sides of (3.43), and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_o\left(8\left(p^{2(\alpha+1)}n + 3\frac{p^{2(\alpha+1)}-1}{8}\right)+3\right)q^n \equiv 4f(-q^2)^3\psi(q) \pmod{16}, \tag{3.44}$$

proving the result for $\alpha+1$. This completes the proof of the theorem. ∎

Theorem 3.9. *Let $p \geq 3$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then, for all nonnegative integers n and α, we have*

$$\sum_{n=0}^{\infty} \overline{p}_o(72p^{2\alpha}n + 9p^{2\alpha})q^n \equiv 6f(-q)f(-q^2) \pmod{8}. \tag{3.45}$$

Proof. Clearly, (3.45) is true when $\alpha = 0$ due to (3.23). We now use induction on α to complete the proof.

For a prime $p \geq 5$ and $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$, we consider the congruence

$$\frac{3k^2+k}{2} + 2\frac{3m^2+m}{2} \equiv 3\frac{p^2-1}{24} \pmod{p}, \tag{3.46}$$

which is equivalent to

$$(6k+1)^2 + 2(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right) = -1$, therefore $k = m = \frac{\pm p-1}{6}$ is the only solution of (3.46). By Lemma 1.21 and proceeding similarly as shown in the proof of Theorem 3.8, we deduce the following congruence

$$\sum_{n=0}^{\infty} \overline{p}_o\left(72\left(p^{2\alpha+1}n+9\frac{p^{2\alpha+2}-1}{72}\right)+9\right)q^n \equiv 6f(-q^p)f(-q^{2p}) \pmod 8. \tag{3.47}$$

We next extract the terms containing q^{pn} from both sides of the above congruence, and observe that (3.45) is true when α is replaced by $\alpha+1$. This completes the proof of the result. ∎

Theorem 3.10. *Let $p \geq 3$ be a prime such that $\left(\frac{-2}{p}\right) = -1$. Then, for all nonnegative integers n and α, we have*

$$\sum_{n=0}^{\infty} \overline{p}_o\left(24\left(p^{2\alpha}n+17\frac{p^{2\alpha}-1}{24}\right)+17\right)q^n \equiv 4f(-q^3)^3\Omega(-q) \pmod 8. \tag{3.48}$$

Proof. From (3.22) we can see that (3.48) is true when $\alpha=0$. Suppose that (3.48) holds for some $\alpha > 0$. Substituting (1.23) and (3.32) into (3.48), we have, modulo 8

$$\sum_{n=0}^{\infty} \overline{p}_o\left(24\left(p^{2\alpha}n+17\frac{p^{2\alpha}-1}{24}\right)+17\right)q^n \tag{3.49}$$

$$= \left[\frac{1}{2}\sum_{\substack{m=0\\ m\neq\frac{p-1}{2}}}^{p-1}(-1)^m q^{3\frac{m^2+m}{2}}\sum_{n=-\infty}^{\infty}(-1)^n(2pn+2m+1)q^{\frac{3pn(pn+2m+1)}{2}} \right.$$
$$\left. +(-1)^{\frac{p-1}{2}}pq^{3\frac{p^2-1}{8}}f(-q^{3p^2})^3\right]$$
$$\times\left[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p+1}{3}}}^{\frac{p-1}{2}}(-1)^k q^{3k^2-2k}\sum_{n=-\infty}^{\infty}(-1)^n q^{pn(3pn+6k-2)}+(-1)^{\frac{\pm p+1}{3}}q^{\frac{p^2-1}{3}}\Omega(-q^{p^2})\right].$$

For a prime $p \geq 5$, $0 \leq m \leq p-1$ and $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, we consider

$$\frac{3m(m+1)}{2}+3k^2-2k \equiv 17\frac{p^2-1}{24} \pmod p, \tag{3.50}$$

which is equivalent to $(6m+3)^2+2(6k-2)^2 \equiv 0 \pmod{p}$. Since $\left(\frac{-2}{p}\right)=-1$, we have $m=\frac{p-1}{2}$ and $k=\frac{\pm p+1}{3}$ is the only solution of (3.50). Therefore, extracting the terms containing $q^{pn+17\frac{p^2-1}{24}}$ from both sides of (3.49), and then replacing q^p by q, we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_o\left(24\left(p^{2\alpha+1}n+17\frac{p^{2\alpha+2}-1}{24}\right)+17\right)q^n \equiv 4f(-q^{3p})^3\Omega(-q^p) \pmod{8}. \tag{3.51}$$

Similarly, extracting the terms containing q^{pn} from both sides of (3.51), and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_o\left(24\left(p^{2(\alpha+1)}n+17\frac{p^{2(\alpha+1)}-1}{24}\right)+17\right)q^n \equiv 4f(-q^{3})^3\Omega(-q) \pmod{8}. \tag{3.52}$$

This completes the proof of the result. ■

Proof of Theorem 3.7. From (3.43), it follows that

$$\overline{p}_o(8p^{2\alpha+1}(pn+j)+3p^{2\alpha+2}) \equiv 0 \pmod{16}, \tag{3.53}$$

where $j=1,2,\ldots,p-1$. This completes the proof of (3.33) for $\alpha \geq 1$.

Proof of (3.34) proceeds along similar lines to the proof of (3.33). Therefore, we omit the details for reasons of brevity.

Extracting the terms containing q^{pn+j} from (3.47), where $j=1,2,\ldots,p-1$, it follows that

$$\overline{p}_o\left(72p^{2\alpha}n+(9p+72j)p^{2\alpha-1}\right) \equiv 0 \pmod{8}.$$

This completes the proof of (3.35) for $\alpha \geq 1$.

From (3.51), it follows that

$$\overline{p}_o(24p^{2\alpha+1}(pn+j)+17p^{2\alpha+2}) \equiv 0 \pmod 8,$$

where $j = 1, 2, \dots, p-1$. This completes the proof of (3.36) for $\alpha \geq 1$.

We now substitute the p-dissection identities, namely (1.21), (1.22) and (3.31) into (3.19) and (3.20). For $p \geq 3$, $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$ and $\left(\frac{-2}{p}\right) = -1$, the congruences

$$\frac{3k^2+k}{2} + 3m^2 + m \equiv 3\frac{p^2-1}{24} \pmod p,$$
$$\frac{3k^2+k}{2} + 3m^2 + 2m \equiv 9\frac{p^2-1}{24} \pmod p$$

have the only solutions $k = m = \frac{\pm p-1}{6}$, and $k = \frac{\pm p-1}{6}$, $m = \frac{\pm p-1}{3}$, respectively. Proceeding similarly as shown in the proof of (3.36), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_o\left(32\left(p^{2\alpha+1}n + 3\frac{p^{2\alpha+2}-1}{24}\right)+4\right)q^n \equiv 6\frac{(q^p;q^p)_\infty (q^{2p};q^{2p})_\infty^5}{(q^{4p};q^{4p})_\infty^2} \pmod 8, \tag{3.54}$$

$$\sum_{n=0}^{\infty} \overline{p}_o\left(32\left(p^{2\alpha+1}n + 9\frac{p^{2\alpha+2}-1}{24}\right)+12\right)q^n \equiv 4\frac{(q^p;q^p)_\infty^3 (q^{4p};q^{4p})_\infty^2}{(q^{2p};q^{2p})_\infty} \pmod 8. \tag{3.55}$$

Now, from (3.54) and (3.55), it follows that

$$\overline{p}_o(32p^{2\alpha+1}(pn+j)+4p^{2\alpha+2}) \equiv 0 \pmod 8,$$
$$\overline{p}_o(32p^{2\alpha+1}(pn+j)+12p^{2\alpha+2}) \equiv 0 \pmod 8,$$

where $j = 1, 2, \dots, p-1$. This completes the proof of (3.37) and (3.38) for $\alpha \geq 1$. ■

Theorem 3.11. *Let $p \equiv 3 \pmod 4$ be a prime , $n \geq 0$ and $\alpha \geq 1$. If $\left(\frac{-2}{p}\right) = -1$,* *then we have*

$$\overline{p}_o\left(16p^{2\alpha}n+(10p+16j)p^{2\alpha-1}\right)\equiv 0 \pmod{16}, \tag{3.56}$$

where $j=1,2,\ldots,p-1$.

Before we prove Theorem 3.11, we first prove the following result.

Theorem 3.12. *Let* $p\geq 3$ *be a prime such that* $p\equiv 3 \pmod 4$. *Then, for all nonnegative integers* n *and* α, *we have*

$$\sum_{n=0}^{\infty}\overline{p}_o\left(16\left(p^{2\alpha}n+5\frac{p^{2\alpha}-1}{8}\right)+10\right)q^n\equiv 8f(-q^4)^3\psi(q) \pmod{16}. \tag{3.57}$$

Proof. We use induction on α to proof the theorem. Clearly, (3.57) is true when $\alpha=0$ due to (3.18). Suppose that (3.57) holds for some $\alpha>0$. For a prime $p\geq 5$ and $0\leq k\leq p-1$, $0\leq m\leq \frac{p-1}{2}$, the equation

$$2(k^2+k)+\frac{m^2+m}{2}\equiv 5\frac{p^2-1}{8} \pmod p,$$

which is equivalent to $4(2k+1)^2+(2m+1)^2\equiv 0 \pmod p$, has the only solution $k=m=\frac{p-1}{2}$ as $p\equiv 3 \pmod 4$. Applying (1.20) and (1.23) in (3.57), and then proceeding similarly as shown in the proof of Theorem 3.8, we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_o\left(16\left(p^{2\alpha+1}n+5\frac{p^{2\alpha+2}-1}{8}\right)+10\right)q^n\equiv 4f(-q^{4p})^3\psi(q^p) \pmod{16}. \tag{3.58}$$

Extracting the terms containing q^{pn} from both sides of (3.58), we find that

$$\sum_{n=0}^{\infty}\overline{p}_o\left(16\left(p^{2(\alpha+1)}n+5\frac{p^{2(\alpha+1)}-1}{8}\right)+10\right)q^n\equiv 4f(-q^4)^3\psi(q) \pmod{16},$$

completing the proof of (3.57). ∎

Proof of Theorem 3.11. From (3.58), it follows that

$$\overline{p}_o(16p^{2\alpha+1}(pn+j)+10p^{2\alpha+2}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$. This completes the proof of the Theorem 3.11. ■

4

Andrews' Singular Overpartitions

4.1 Introduction

In [4], Andrews defined the partition function $\overline{C}_{k,i}(n)$, called singular overpartition, which counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. For example, $\overline{C}_{3,1}(4) = 10$ with the relevant partitions being $4, \overline{4}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1$. For $k \geq 3$ and $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$, the generating function

[1]The contents of this chapter have been published in *The Ramanujan J.* (2018) and some parts are under review.

for $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}. \tag{4.1}$$

Andrews proved the following Ramanujan-type congruences

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod 3$$

using some q-series identities. Chen, Hirschhorn and Sellers [17] later showed that Andrews' congruences modulo 3 are two examples of an infinite family of congruences modulo 3 which hold for the function $\overline{C}_{3,1}(n)$. Recently, Ahmed and Baruah [2] found congruences for $\overline{C}_{3,1}(n)$ modulo 4, 18 and 36, infinite families of congruences modulo 2 and 4 for $\overline{C}_{8,2}(n)$, congruences modulo 2 and 3 for $\overline{C}_{12,2}(n)$ and $\overline{C}_{12,4}(n)$; and congruences modulo 2 for $\overline{C}_{28,8}(n)$ and $\overline{C}_{48,16}(n)$.

In a very recent work, Naika and Gireesh [40] prove that two congruences for $\overline{C}_{3,1}(n)$ modulo 36 proved by Ahmed and Baruah hold for modulo 72. They also prove congruences modulo 6, 12, 16, 18, and 24 for $\overline{C}_{3,1}(n)$ and infinite families of congruences modulo 12, 18, 48, and 72 for $\overline{C}_{3,1}(n)$. They further conjecture that

$$\overline{C}_{3,1}(12n+11) \equiv 0 \pmod{144}$$

for all $n \geq 0$. In this chapter we prove the conjecture.

4.2 Proof of Naika and Gireesh's conjecture

Theorem 4.1. *For each nonnegative integer n,*

$$\overline{C}_{3,1}(12n+11) \equiv 0 \pmod{144}.$$

Proof. From [40, (3.19)], we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(4n+3)q^n = 6\frac{(q^2;q^2)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^6}. \tag{4.2}$$

Also, (1.11) can be written as

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} = \frac{(q^6;q^6)_\infty^4 (q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^8 (q^{18};q^{18})_\infty^3} + 2q\frac{(q^6;q^6)_\infty^3 (q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^7} + 4q^2\frac{(q^6;q^6)_\infty^2 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^6}. \tag{4.3}$$

Substituting (4.3) into (4.2), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{C}_{3,1}(4n+3)q^n \\
&= 6(q^6;q^6)_\infty^3 \left(\frac{(q^6;q^6)_\infty^4 (q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^8 (q^{18};q^{18})_\infty^3} + 2q\frac{(q^6;q^6)_\infty^3 (q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^7} + 4q^2\frac{(q^6;q^6)_\infty^2 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^6} \right)^3 \\
&= 6(q^6;q^6)_\infty^3 \left(\frac{(q^6;q^6)_\infty^{12} (q^9;q^9)_\infty^{18}}{(q^3;q^3)_\infty^{24} (q^{18};q^{18})_\infty^9} + 6q\frac{(q^6;q^6)_\infty^{11} (q^9;q^9)_\infty^{15}}{(q^3;q^3)_\infty^{23} (q^{18};q^{18})_\infty^6} + 24q^2\frac{(q^6;q^6)_\infty^{10} (q^9;q^9)_\infty^{12}}{(q^3;q^3)_\infty^{22} (q^{18};q^{18})_\infty^3} \right. \\
&\quad +56q^3\frac{(q^6;q^6)_\infty^9 (q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{21}} + 96q^4\frac{(q^6;q^6)_\infty^8 (q^9;q^9)_\infty^6 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^{20}} \\
&\quad \left. +96q^5\frac{(q^6;q^6)_\infty^7 (q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^6}{(q^3;q^3)_\infty^{19}} + 64q^6\frac{(q^6;q^6)_\infty^6 (q^{18};q^{18})_\infty^9}{(q^3;q^3)_\infty^{18}} \right).
\end{aligned} \tag{4.4}$$

Extracting the term containing q^{3n+2} from (4.4), it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{C}_{3,1}(12n+11)q^n &= 144\frac{(q^2;q^2)_\infty^{13} (q^3;q^3)_\infty^{12}}{(q;q)_\infty^{22} (q^6;q^6)_\infty^3} + 576q\frac{(q^2;q^2)_\infty^{10} (q^3;q^3)_\infty^3 (q^6;q^6)_\infty^6}{(q;q)_\infty^{19}} \\
&\equiv 0 \pmod{144}.
\end{aligned}$$

This completes the proof of Theorem 4.1. ■

Extracting the term containing q^{3n+1} from (4.4), we have the following corollary.

Corollary 4.2. *For each nonnegative integer n,*

$$\overline{C}_{3,1}(12n+7) \equiv 0 \pmod{36}.$$

4.3 Distribution of $\overline{C}_{3,1}(n)$

In [17], Chen, Hirschhorn and Sellers studied the parity of $\overline{C}_{k,i}(n)$. They showed that $\overline{C}_{3,1}(n)$ is always even and that $\overline{C}_{6,2}(n)$ is always even unless n is a pentagonal number. In a very recent paper, Aricheta [7] studied the parity of $\overline{C}_{3k,k}(n)$. To be specific, let $k = 2^a m$ be a positive integer where the integer $a \geq 0$ and m is positive odd. Assume further that $2^a \geq m$. Then Aricheta proved that $\overline{C}_{3k,k}(n)$ is almost always even, that is

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3k,k}(n) \equiv 0 \pmod{2}\right\}}{X} = 1.$$

Let k be a fixed positive integer. In this chapter, we study divisibility of $\overline{C}_{3,1}(n)$ by 2^k and 3^k. Similar studies are done for many other partition functions. For example, Gordon and Ono [22] proved that the number of partitions of n into distinct parts is divisible by 2^k for almost all n. Bringmann and Lovejoy [12] showed that this is also true for the number of overpartition pairs, and Lin [35] did the same for the number of overpartition pairs into odd parts. Here we prove that the partition function $\overline{C}_{3,1}(n)$ has this property as well.

Theorem 4.3. *Let k be a positive integer. Then $\overline{C}_{3,1}(n)$ is almost always divisible by 2^k, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3,1}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1. \tag{4.5}$$

We further prove that the partition function $\overline{C}_{3,1}(n)$ is also divisible by 3^k for almost all n.

Theorem 4.4. *Let k be a positive integer. Then $\overline{C}_{3,1}(n)$ is almost always divisible by 3^k, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3,1}(n) \equiv 0 \pmod{3^k}\right\}}{X} = 1. \tag{4.6}$$

Chen, Hirschhorn and Sellers [17] showed that $\overline{C}_{3,1}(n)$ is even for all $n \geq 1$. Hence, we have the following corollary.

Corollary 4.5. *Let k be a positive integer. Then $\overline{C}_{3,1}(n)$ is almost always divisible by $2 \cdot 3^k$.*

4.4 Modularity of eta-quotient

Recall that Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i z}$ and z is in the upper half complex plane. A function $f(z)$ is called an eta-quotient if

$$f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$$

for some positive integer N and r_δ is an integer. Recall that, for a positive integer k, the complex vector space of modular forms of weight k with respect to $\Gamma_1(N)$ is denoted by $M_k(\Gamma_1(N))$.

We now recall two theorems from [45, p. 18] which will be used to prove our result.

Theorem 4.6 (Theorem 1.64 and Theorem 1.65 [45])**.** *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an* *eta-quotient such that* $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ *satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. *Here the character* χ *is defined by* $\chi(d) := \left(\frac{(-1)^k \prod_{\delta|N} \delta^{r_\delta}}{d}\right)$. *In addition, if* $c, d,$ *and* N *are positive integers with* $d|N$ *and* $\gcd(c,d) = 1$, *then the order of vanishing of* $f(z)$ *at the cusp* $\frac{c}{d}$ *is* $\frac{N}{24}\sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d}) d\delta}$.

Suppose that $f(z)$ is an eta-quotient satisfying the conditions of Theorem 4.6. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi)$.

4.5 Proof of Theorem 4.3 and Theorem 4.4

Proof of Theorem 4.3 . The generating function of $\overline{C}_{3,1}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(n) q^n = \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty^2 (q^6;q^6)_\infty}. \tag{4.7}$$

We note that $\eta(24z) = q\prod_{n=1}^{\infty}(1-q^{24n})$ is a power series of q. Given a prime p, let

$$A_p(z) = \prod_{n=1}^{\infty} \frac{(1-q^{24n})^p}{(1-q^{24pn})} = \frac{\eta^p(24z)}{\eta(24pz)}.$$

Then using binomial theorem we have

$$A_p^{p^k}(z) = \frac{\eta^{p^{k+1}}(24z)}{\eta^{p^k}(24pz)} \equiv 1 \pmod{p^{k+1}}. \tag{4.8}$$

Define $B_{p,k}(z)$ by

$$B_{p,k}(z) = \left(\frac{\eta(48z)\eta(72z)^2}{\eta(24z)^2\eta(144z)}\right) A_p^{p^k}(z). \tag{4.9}$$

Modulo p^{k+1}, we have

$$B_{p,k}(z) \equiv \frac{\eta(48z)\eta(72z)^2}{\eta(24z)^2\eta(144z)} = \frac{(q^{48};q^{48})_\infty (q^{72};q^{72})_\infty^2}{(q^{24};q^{24})_\infty^2 (q^{144};q^{144})_\infty}. \tag{4.10}$$

Combining (4.7) and (4.10), we obtain

$$B_{p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{3,1}(n) q^{24n} \pmod{p^{k+1}}. \tag{4.11}$$

We put $p = 2$ in (4.9) to obtain

$$B_{2,k}(z) = \left(\frac{\eta(48z)\eta(72z)^2}{\eta(24z)^2\eta(144z)}\right) A_2^{2^k}(z) = \frac{\eta(24z)^{2^{k+1}-2}\ \eta(72z)^2}{\eta(48z)^{2^k-1}\ \eta(144z)}.$$

The cusps of $\Gamma_0(576)$ are represented by fractions $\frac{c}{d}$ where $d \mid 576$ and $\gcd(c,d) = 1$. By Theorem 4.6, it is easily seen that $B_{2,k}(z)$ is a form of weight 2^{k-1} on $\Gamma_0(576)$. Therefore, $B_{2,k}(z)$ is a modular form if and only if $B_{2,k}(z)$ is holomorphic at the cusp $\frac{c}{d}$. We know that $B_{2,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,24)^2}{24}\left(2^{k+1}-2\right) + \frac{\gcd(d,48)^2}{48}\left(1-2^k\right) + \frac{\gcd(d,72)^2}{36} - \frac{\gcd(d,144)^2}{144} \geq 0.$$

Equivalently, if and only if

$$D := 6\frac{\gcd(d,24)^2}{\gcd(d,144)^2}\left(2^{k+1}-2\right) + 3\frac{\gcd(d,48)^2}{\gcd(d,144)^2}\left(1-2^k\right) + 4\frac{\gcd(d,72)^2}{\gcd(d,144)^2} - 1 \geq 0.$$

In the following table, we find all the possible values of D by taking $d = 24, 48, 72,$ and 144 only. We note that all other values of d will give the following four values of D listed in the table.

d	$\frac{\gcd(d,24)^2}{\gcd(d,144)^2}$	$\frac{\gcd(d,48)^2}{\gcd(d,144)^2}$	$\frac{\gcd(d,72)^2}{\gcd(d,144)^2}$	D
24	1	1	1	$9 \cdot 2^k - 6$
48	1/4	1	1/4	0
72	1/9	1/9	1	$2^k + 2$
144	1/36	1/9	1/4	0

Since $D \geq 0$ for all $d \mid 576$, hence we have $B_{2,k}(z) \in M_{2^{k-1}}\left(\Gamma_0(576), (\frac{(-1)^{2^{k-1}} 2^{2^{k+1}} 3^{2^k+1}}{\bullet})\right)$.

Let m be a positive integer. Suppose that $f(z) \in M_k(\Gamma_0(N), \chi)$ has Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c(n) q^n \in \mathbb{Z}[[q]].$$

Then by Corollary 1.14, there exists a constant $\alpha > 0$ such that

$$\#\left\{0 < n \leq X : c(n) \not\equiv 0 \pmod{m}\right\} = \mathcal{O}\left(\frac{X}{(\log X)^{\alpha}}\right).$$

This implies that

$$\lim_{X \to \infty} \frac{\#\left\{0 < n \leq X : c(n) \not\equiv 0 \pmod{m}\right\}}{X} = 0.$$

Hence

$$\lim_{X \to \infty} \frac{\#\left\{0 < n \leq X : c(n) \equiv 0 \pmod{m}\right\}}{X} = 1. \tag{4.12}$$

Since $B_{2,k}(z) \in M_{2^{k-1}}\left(\Gamma_0(576), (\frac{3}{\bullet})\right)$, the Fourier coefficients of $B_{2,k}(z)$ also satisfy (4.12). In particular, the Fourier coefficients of $B_{2,k}(z)$ are almost always divisible by $m = 2^k$. Now, using (4.11) we complete the proof of the theorem. ■

Proof of Theorem 4.4. We put $p = 3$ in (4.9) to obtain

$$B_{3,k}(z) = \left(\frac{\eta(48z)\eta(72z)^2}{\eta(24z)^2\eta(144z)}\right) A_3^{3^k}(z) = \frac{\eta(24z)^{3^{k+1}-2}\ \eta(48z)}{\eta(72z)^{3^k-2}\ \eta(144z)}.$$

The cusps of $\Gamma_0(576)$ are represented by fractions $\frac{c}{d}$ where $d \mid 576$ and $\gcd(c,d) = 1$. By Theorem 4.6, it is easily seen that $B_{3,k}(z)$ is a form of weight 3^k on $\Gamma_0(576)$. Therefore, $B_{3,k}(z) \in M_{3^k}\left(\Gamma_0(576), \left(\frac{-1}{\bullet}\right)\right)$ if and only if $B_{3,k}(z)$ is holomorphic at the cusp $\frac{c}{d}$. We know that $B_{3,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,24)^2}{24}\left(3^{k+1}-2\right) + \frac{\gcd(d,48)^2}{48} + \frac{\gcd(d,72)^2}{72}\left(2-3^k\right) - \frac{\gcd(d,144)^2}{144} \geq 0.$$

Equivalently, if and only if

$$L := 6\frac{\gcd(d,24)^2}{\gcd(d,144)^2}\left(3^{k+1}-2\right) + 3\frac{\gcd(d,48)^2}{\gcd(d,144)^2} + 2\frac{\gcd(d,72)^2}{\gcd(d,144)^2}\left(2-3^k\right) - 1 \geq 0.$$

In the following table, we find all the possible values of L by taking $d = 24, 48, 72,$ and 144 only.

d	$\frac{\gcd(d,24)^2}{\gcd(d,144)^2}$	$\frac{\gcd(d,48)^2}{\gcd(d,144)^2}$	$\frac{\gcd(d,72)^2}{\gcd(d,144)^2}$	L
24	1	1	1	$16 \cdot 3^k - 6$
48	1/4	1	1/4	$4 \cdot 3^k$
72	1/9	1/9	1	2
144	1/36	1/9	1/4	0

Since $L \geq 0$ for all $d \mid 576$, hence we have $B_{3,k}(z) \in M_{3^k}\left(\Gamma_0(576), \left(\frac{-2^{2\cdot 3^{k+1}}3^{3^k+1}}{\bullet}\right)\right)$. Using the same reasoning as given in the proof of Theorem 4.3 and (4.11), we find that $\overline{C}_{3,1}(n)$ is divisible by 3^k for almost all $n \geq 0$. This completes the proof of the theorem. ■

5

Cubic and Overcubic Partition Pairs

5.1 Introduction

In a series of papers [13, 14, 15], Chan studied the cubic partition function $a(n)$ which counts the number of partitions of n where the even parts can appear in two colors. For example, there are four cubic partitions of 3, namely $3, 2_1+1, 2_2+1$ and $1+1+1$, where the subscripts 1 and 2 denote the colors. The generating function of $a(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty}.$$

[1]The contents of this chapter have been accepted in *The Ramanujan J.* (2019).

The function $a(n)$ satisfies many Ramanujan-type congruences, for example

$$a(3n+2) \equiv 0 \pmod 3$$

for all $n \geq 0$. Inspired by Chan's work, Zhao and Zhong [57] studied the cubic partition pair function $b(n)$ defined by

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2},$$

and proved that

$$\begin{aligned} b(5n+4) &\equiv 0 \pmod 5, \\ b(7n+i) &\equiv 0 \pmod 7, \ i \in \{2,3,4,6\}, \\ b(9n+7) &\equiv 0 \pmod 9. \end{aligned}$$

Recently, Lin [37] studied the arithmetic properties of $b(n)$ modulo 27, and conjectured the following four congruences:

$$b(49n+37) \equiv 0 \pmod{49}, \tag{5.1}$$

$$b(81n+61) \equiv 0 \pmod{243}, \tag{5.2}$$

$$\sum_{n\geq 0} b(81n+7)q^n \equiv 9\frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q^6;q^6)_\infty} \pmod{81}, \tag{5.3}$$

$$\sum_{n\geq 0} b(81n+34)q^n \equiv 36\frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^3;q^3)_\infty} \pmod{81}. \tag{5.4}$$

In two recent papers, Lin, Wang, and Xia [38] and Chern [18] independently proved (5.2), (5.3) and (5.4), and moreover showed that the congruence in (5.2) holds modulo 729. In [25], Hirschhorn also proved (5.2) using properties of Ramanujan's theta functions.

An overpartition analog of $a(n)$ was studied by Kim [30], who defined an overcubic partition to be a partition in which odd parts appear in two colors, one of which can occur at most once, and the even parts appear in four colors, two of which can occur at most once each. For example, $a(4) = 26$ with the relevant partitions being $4_1, 4_2, 4_3, 4_4, 3_1 + 1_1, 3_1 + 1_2, 3_2 + 1_1, 3_2 + 1_2, 2_1 + 2_1, 2_1 + 2_2, 2_1 + 2_3, 2_1 + 2_4, 2_2 + 2_2, 2_2 + 2_3, 2_2 + 2_4, 2_3 + 2_4, 2_1 + 1_1 + 1_1, 2_1 + 1_1 + 1_2, 2_2 + 1_1 + 1_1, 2_2 + 1_1 + 1_2, 2_3 + 1_1 + 1_1, 2_3 + 1_1 + 1_2, 2_4 + 1_1 + 1_1, 2_4 + 1_1 + 1_2, 1_1 + 1_1 + 1_1 + 1_1$ and $1_1 + 1_1 + 1_1 + 1_2$. The generating function of the overcubic partition function $\overline{a}(n)$ satisfies the identity

$$\overline{A}(q) := \sum_{n=0}^{\infty} \overline{a}(n)q^n = \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}, \tag{5.5}$$

and thus we have

$$\overline{B}(q) := \sum_{n=0}^{\infty} \overline{b}(n)q^n = \frac{(-q;q)_\infty^2(-q^2;q^2)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty^2}, \tag{5.6}$$

where $\overline{b}(n)$ denotes the number of overcubic partition pairs of n. Using arithmetic properties of quadratic forms and modular forms, Kim [32] derived the congruences $\overline{b}(8n + 7) \equiv 0 \pmod{64}$ and $\overline{b}(9n + 3) \equiv 0 \pmod{3}$. In [36], Lin proved two Ramanujan-type congruences and several infinite families of congruences modulo 3 satisfied by $\overline{b}(n)$, and also obtained some congruences for $\overline{b}(n)$ modulo 5.

5.2 Proof of a conjecture of Lin

In this section we prove the congruence given in (5.1).

Theorem 5.1. *For all $n \geq 0$ we have*

$$b(49n + 37) \equiv 0 \pmod{49}.$$

We prove Theorem 5.1 using the approach developed in [48, 49]. To this end, we first recall some definitions and results from [48, 49]. For a positive integer M, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of M. If $r \in R(M)$ and $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ are the positive divisors of M, we write $r = (r_{\delta_1}, \ldots, r_{\delta_k})$. Define $c_r(n)$ by

$$\sum_{n=0}^{\infty} c_r(n) q^n := \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{n\delta})^{r_\delta}. \tag{5.7}$$

The approach to proving congruences for $c_r(n)$ developed by Radu [48, 49] reduces the number of cases that one must check as compared with the classical method which uses Sturm's bound alone.

Let m be a positive integer. For any integer s, let $[s]_m$ denote the residue class of s in $\mathbb{Z}_m$, and $\mathbb{S}_m$ the set of squares in $\mathbb{Z}_m^*$. For $t \in \{0, 1, \ldots, m-1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subseteq \{0, 1, \ldots, m-1\}$ by

$$P_{m,r}(t) := \left\{ t' : \exists [s]_{24m} \in \mathbb{S}_{24m} \text{ such that } t' \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m} \right\}.$$

Definition 8. *Suppose m, M and N are positive integers, $r = (r_\delta) \in R(M)$ and $t \in \{0, 1, \ldots, m-1\}$. Let $k = k(m) := \gcd(m^2 - 1, 24)$ and write*

$$\prod_{\delta|M} \delta^{|r_\delta|} = 2^s \cdot j,$$

where s and j are nonnegative integers with j odd. The set Δ^* consists of all tuples $(m, M, N, (r_\delta), t)$ satisfying these conditions and all of the following.

1. *Each prime divisor of m is also a divisor of N.*

2. *$\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$.*

3. *$kN \sum_{\delta|M} r_\delta mN/\delta \equiv 0 \pmod{24}$.*

4. $kN\sum_{\delta|M} r_\delta \equiv 0 \pmod 8$.

5. $\dfrac{24m}{\gcd\left(-24kt - k\sum_{\delta|M}\delta r_\delta, 24m\right)}$ *divides* N.

6. *If* $2|m$, *then either* $4|kN$ *and* $8|sN$ *or* $2|s$ *and* $8|(1-j)N$.

Let m, M, N be positive integers. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, $r \in R(M)$ and $r' \in R(N)$, set

$$p_{m,r}(\gamma) := \min_{\lambda\in\{0,1,\ldots,m-1\}} \frac{1}{24}\sum_{\delta|M} r_\delta \frac{\gcd^2(\delta a + \delta k\lambda c, mc)}{\delta m}$$

and

$$p^*_{r'}(\gamma) := \frac{1}{24}\sum_{\delta|N} r'_\delta \frac{\gcd^2(\delta, c)}{\delta}.$$

Let us define a congruence subgroup Γ_∞ of level N as

$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

Lemma 5.2 (Lemma 4.5 [48]). *Let* u *be a positive integer,* $(m, M, N, r = (r_\delta), t) \in \Delta^*$ *and* $r' = (r'_\delta) \in R(N)$. *Let* $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ *be a complete set of representatives of the double cosets of* $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. *Assume that* $p_{m,r}(\gamma_i) + p^*_{r'}(\gamma_i) \geq 0$ *for all* $1 \leq i \leq n$. *Let* $t_{min} = \min_{t'\in P_{m,r}(t)} t'$ *and*

$$\nu := \frac{1}{24}\left\{\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N}\delta r'_\delta\right\} - \frac{1}{24m}\sum_{\delta|M}\delta r_\delta - \frac{t_{min}}{m}.$$

If the congruence $c_r(mn+t') \equiv 0 \pmod u$ *holds for all* $t' \in P_{m,r}(t)$ *and* $0 \leq n \leq \lfloor \nu \rfloor$, *then it holds for all* $t' \in P_{m,r}(t)$ *and* $n \geq 0$.

To apply Lemma 5.2 we utilize the following result, which gives a complete set of representatives of the double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$.

Lemma 5.3 (Lemma 4.3 [55]). *If* N *or* $\frac{1}{2}N$ *is a square-free integer, then*

$$\bigcup_{\delta|N} \Gamma_0(N) \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \Gamma_\infty = \Gamma.$$

We are now ready to give a proof of Theorem 5.1

Proof of Theorem 5.1. By the binomial theorem we have

$$\begin{aligned}\sum_{n=0}^{\infty} b(n)q^n &= \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2} \\ &\equiv \frac{(q;q)_\infty^{49}}{(q;q)_\infty^2 (q^2;q^2)_\infty^2 (q^7;q^7)_\infty^7} \\ &\equiv \frac{(q;q)_\infty^{47}}{(q^2;q^2)_\infty^2 (q^7;q^7)_\infty^7} \pmod{49}.\end{aligned}$$

Let us consider $(m, M, N, r, t) = (49, 14, 14, (47, -2, -7, 0), 37)$.
It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{37\}$. By Lemma 5.3, we know that $\left\{ \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} : \delta|14 \right\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let $\gamma_\delta = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}$ and $r' = (12, 0, 0, 0) \in R(14)$. We used *Sage* to verify that $p_{m,r}(\gamma_\delta) + p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta|N$. We compute that the upper bound in Lemma 5.2 is $\lfloor \nu \rfloor = 48$, and using *Mathematica* we verify that $b(49n + 37) \equiv 0 \pmod{49}$ for $n \leq 48$. By Lemma 5.2 we conclude that $b(49n + 37) \equiv 0 \pmod{49}$ for any $n \geq 0$. ∎

5.3 Ramanujan-type congruences for overcubic partition pairs

Theorem 5.4. *Let $t \in \{42, 66\}$. Then for all $n \geq 0$ we have*

$$\bar{b}(72n+t) \equiv 0 \pmod{256}.$$

Proof. We first recall the following 2-dissection formula from (1.16)

$$\frac{1}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}.$$

Combining (5.6) with the above 2-dissection formula, and using the fact that $(-q;q)_\infty = \dfrac{(q^2;q^2)_\infty}{(q;q)_\infty}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{b}(n)q^n &= \frac{(q^4;q^4)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty^2}\\ &= \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2}\left(\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}\right).\end{aligned}$$

Extracting the terms with even powers of q and then using (1.16), we obtain, modulo 256

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{b}(2n)q^n &= \frac{(q^2;q^2)_\infty^{16}}{(q;q)_\infty^{16}(q^4;q^4)_\infty^4}\\ &= \frac{(q^4;q^4)_\infty^{52}}{(q^2;q^2)_\infty^{40}(q^8;q^8)_\infty^{16}} + 16q\frac{(q^4;q^4)_\infty^{40}}{(q^2;q^2)_\infty^{36}(q^8;q^8)_\infty^{8}} + 96q^2\frac{(q^4;q^4)_\infty^{28}}{(q^2;q^2)_\infty^{32}}\\ &\quad + 256q^3\frac{(q^4;q^4)_\infty^{16}(q^8;q^8)_\infty^{8}}{(q^2;q^2)_\infty^{28}} + 256q^4\frac{(q^4;q^4)_\infty^{4}(q^8;q^8)_\infty^{16}}{(q^2;q^2)_\infty^{24}}\\ &\equiv \frac{(q^4;q^4)_\infty^{52}}{(q^2;q^2)_\infty^{40}(q^8;q^8)_\infty^{16}} + 16q\frac{(q^4;q^4)_\infty^{40}}{(q^2;q^2)_\infty^{36}(q^8;q^8)_\infty^{8}} + 96q^2\frac{(q^4;q^4)_\infty^{28}}{(q^2;q^2)_\infty^{32}}.\end{aligned}$$

Extracting the terms with odd powers of q, we obtain, modulo 256

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(4n+2)q^n &\equiv 16\frac{(q^2;q^2)_\infty^{40}}{(q;q)_\infty^{36}(q^4;q^4)_\infty^{8}}\\ &= 16\frac{(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{4}(q^4;q^4)_\infty^{8}}\left(\frac{(q^2;q^2)_\infty^{16}}{(q;q)_\infty^{32}}\right)\\ &\equiv 16\frac{(q^2;q^2)_\infty^{24}}{(q;q)_\infty^{4}(q^4;q^4)_\infty^{8}}\\ &\equiv 16\frac{(q^2;q^2)_\infty^{24}}{(q^4;q^4)_\infty^{8}}\left(\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^{4}}+4q\frac{(q^4;q^4)_\infty^{2}(q^8;q^8)_\infty^{4}}{(q^2;q^2)_\infty^{10}}\right).\end{aligned}$$

Finally, extracting the terms with even powers of q, we obtain

$$\sum_{n=0}^{\infty}\overline{b}(8n+2)q^n \equiv 16\frac{(q;q)_\infty^{10}(q^2;q^2)_\infty^{6}}{(q^4;q^4)_\infty^{4}} \pmod{256}.$$

Let $(m, M, N, r, t) = (9, 8, 12, (10, 6, -4, 0), 5)$.

It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{5\}$. From Lemma 5.3 we know that $\left\{\begin{bmatrix}1 & 0\\ \delta & 1\end{bmatrix} : \delta|12\right\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let $r' = (0, 0, 0, 0, 0, 0) \in R(12)$. We have used *Sage* to verify that $p_{m,r}(\gamma_\delta)+p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta|N$, where $\gamma_\delta = \begin{bmatrix}1 & 0\\ \delta & 1\end{bmatrix}$. We compute that the upper bound in Lemma 5.2 is $\lfloor\nu\rfloor = 11$. Using *Mathematica* we verify that $\overline{b}(72n+42) \equiv 0 \pmod{256}$ for $n \leq 11$, and conclude from Lemma 5.2 that $\overline{b}(72n+42) \equiv 0 \pmod{256}$ for all $n \geq 0$. A similar argument using $(m, M, N, r, t) = (9, 8, 12, (10, 6, -4, 0), 8)$ yields the other claimed congruence. ■

5.4 Distribution of $\overline{a}(n)$ and $\overline{b}(n)$

Theorem 5.5. *Let k be a positive integer. Then $\overline{a}(n)$ and $\overline{b}(n)$ are divisible by 2^k for almost all $n \geq 0$. That is,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{a}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1,$$
$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{b}(n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

Proof. Let

$$B_k(z) = \frac{\eta(48z)^{2^k-2}}{\eta(24z)^4\eta(96z)^{2^{k-1}-2}}.$$

Using Theorem 4.6, we find that $B_k(z) \in M_{2^{k-2}-2}(\Gamma_0(384))$ for $k \geq 4$.

By (5.6) we have

$$\mathcal{B}(z) := \overline{B}(q^{24}) = \frac{\eta(96z)^2}{\eta(24z)^4\eta(48z)^2}.$$

Defining

$$F_k(z) := \frac{\eta(48z)^{2^k}}{\eta(96z)^{2^{k-1}}},$$

we have that

$$B_k(z) = \mathcal{B}(z)F_k(z).$$

It is easy to show that $F_k(z) \equiv 1 \pmod{2^k}$, which implies

$$B_k(z) \equiv \mathcal{B}(z) \pmod{2^k}.$$

Since $B_k(z) \in M_{2^{k-2}-2}(\Gamma_0(384))$ for $k \geq 4$, applying Corollary 1.14 and proceeding similarly as shown in the proof of Theorem 4.3 we find that the Fourier coefficients of $B_k(z)$ are almost always divisible by 2^k. Hence the same is true for the Fourier coefficients of $\mathcal{B}(z)$. This proves that the same holds for the partition function $\overline{b}(n)$.

We complete the proof of the theorem by proving the same for the partition function $\overline{a}(n)$. To do this, we let

$$A_k(z) = \frac{\eta(48z)^{2^k-1}}{\eta(24z)^2\eta(96z)^{2^{k-1}-1}}.$$

Using Theorem 4.6 we find that $A_k(z) \in M_{2^{k-2}-1}(\Gamma_0(768), \chi)$ for $k > 2$, where χ is defined by $\chi(\bullet) = \left(\frac{-2}{\bullet}\right)$.

By (5.5) we have

$$\mathcal{A}(z) := \overline{A}(q^{24}) = \frac{\eta(96z)}{\eta(24z)^2\eta(48z)},$$

and hence

$$A_k(z) = \mathcal{A}(z)F_k(z) \equiv \mathcal{A}(z) \pmod{2^k}.$$

Using the same reasoning we now find that $\overline{a}(n)$ is divisible by 2^k for almost all $n \geq 0$. ■

6

Andrews' Integer Partitions With Even Parts Below Odd Parts

6.1 Introduction

In a recent paper, Andrews [5] studied the partition function $\mathcal{EO}(n)$ which counts the number of partitions of n where every even part is less than each odd part. He denoted by $\overline{\mathcal{EO}}(n)$, the number of partitions counted by $\mathcal{EO}(n)$ in which *only* the largest even part appears an odd number of times. For example, $\mathcal{EO}(8) = 12$ with the relevant partitions being $8, 6+2, 7+1, 4+4, 4+2+2, 5+3, 5+1+1+1, 2+2+2+2, 3+$

[1]The contents of this chapter are under review.

3+2, 3+3+1+1, 3+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1; and $\overline{\mathcal{EO}}(8) = 5$, with the relevant partitions being 8, 4+2+2, 3+3+2, 3+3+1+1, 1+1+1+1+1+1+1+1.

Andrews proved that the partition function $\overline{\mathcal{EO}}(n)$ has the following generating function [5, Eqn. (3.2)]:

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^4)^2_\infty} = \frac{(q^4;q^4)^3_\infty}{(q^2;q^2)^2_\infty}. \tag{6.1}$$

In the same paper, he proposed to undertake a more extensive investigation of the properties of $\overline{\mathcal{EO}}(n)$. The objective of this chapter is to study divisibility properties of $\overline{\mathcal{EO}}(n)$.

6.2 Infinite families of congruences for $\overline{\mathcal{EO}}(n)$

We use the theory of Hecke eigenforms to prove two infinite families of congruences for $\overline{\mathcal{EO}}(n)$ modulo 2 and 8.

Theorem 6.1. *Let k, n be nonnegative integers. For each i with $1 \leq i \leq k+1$, if $p_i \geq 5$ is prime such that $p_i \equiv 2 \pmod 3$, then for any integer $j \not\equiv 0 \pmod{p_{k+1}}$*

$$\overline{\mathcal{EO}}\left(p_1^2 \dots p_{k+1}^2 n + \frac{p_1^2 \dots p_k^2 p_{k+1}(3j + p_{k+1}) - 1}{3}\right) \equiv 0 \pmod 2.$$

Proof. We have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^n = \frac{(q^4;q^4)^3_\infty}{(q^2;q^2)^2_\infty} \equiv (q;q)^8_\infty \pmod 2.$$

This gives

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^{3n+1} \equiv \eta^8(3z) \pmod 2.$$

Let $\eta^8(3z) = \sum_{n=1}^{\infty} a(n)q^n$. Then $a(n) = 0$ if $n \not\equiv 1 \pmod 3$ and for all $n \geq 0$,

$$\overline{\mathcal{EO}}(n) \equiv a(3n+1) \pmod 2. \tag{6.2}$$

By Theorem 4.6, we have $\eta^8(3z) \in S_4(\Gamma_0(9))$. Since $\eta^8(3z)$ is a Hecke eigenform (see, for example [42]), (1.24) and (1.25) yield

$$\eta^8(3z)|T_p = \sum_{n=1}^{\infty} \left(a(pn) + p^3 a\left(\frac{n}{p}\right)\right) q^n = \lambda(p) \sum_{n=1}^{\infty} a(n)q^n,$$

which implies

$$a(pn) + p^3 a\left(\frac{n}{p}\right) = \lambda(p)a(n). \tag{6.3}$$

Putting $n = 1$ and noting that $a(1) = 1$, we readily obtain $a(p) = \lambda(p)$. Since $a(p) = 0$ for all $p \not\equiv 1 \pmod 3$, we have $\lambda(p) = 0$. From (6.3), we obtain

$$a(pn) + p^3 a\left(\frac{n}{p}\right) = 0. \tag{6.4}$$

From (6.4), we derive that for all $n \geq 0$ and $p \nmid r$,

$$a(p^2n + pr) = 0 \tag{6.5}$$

and

$$a(p^2n) = -p^3 a(n) \equiv a(n) \pmod 2. \tag{6.6}$$

Substituting n by $3n - pr + 1$ in (6.5) and together with (6.2), we find that

$$\overline{\mathcal{EO}}\left(p^2n + \frac{p^2-1}{3} + pr\frac{1-p^2}{3}\right) \equiv 0 \pmod 2. \tag{6.7}$$

Substituting n by $3n+1$ in (6.6) and using (6.2), we obtain

$$\overline{\mathcal{EO}}\left(p^2n+\frac{p^2-1}{3}\right)\equiv \overline{\mathcal{EO}}(n) \pmod 2. \tag{6.8}$$

Since $p\geq 5$ is prime, so $3\mid(1-p^2)$ and $\gcd\left(\frac{1-p^2}{3},p\right)=1$. Hence when r runs over a residue system excluding the multiple of p, so does $\frac{1-p^2}{3}r$. Thus (6.7) can be rewritten as

$$\overline{\mathcal{EO}}\left(p^2n+\frac{p^2-1}{3}+pj\right)\equiv 0 \pmod 2, \tag{6.9}$$

where $p\nmid j$.

Now, $p_i\geq 5$ are primes such that $p_i\not\equiv 1 \pmod 3$. Since

$$p_1^2\dots p_k^2n+\frac{p_1^2\dots p_k^2-1}{3}=p_1^2\left(p_2^2\dots p_k^2n+\frac{p_2^2\dots p_k^2-1}{3}\right)+\frac{p_1^2-1}{3},$$

using (6.8) repeatedly we obtain that

$$\overline{\mathcal{EO}}\left(p_1^2\dots p_k^2n+\frac{p_1^2\dots p_k^2-1}{3}\right)\equiv \overline{\mathcal{EO}}(n) \pmod 2. \tag{6.10}$$

Let $j\not\equiv 0 \pmod{p_{k+1}}$. Then (6.9) and (6.10) yield

$$\overline{\mathcal{EO}}\left(p_1^2\dots p_{k+1}^2n+\frac{p_1^2\dots p_k^2p_{k+1}(3j+p_{k+1})-1}{3}\right)\equiv 0 \pmod 2.$$

This completes the proof of the theorem. ■

Remark 6.2.1. *Let $p\geq 5$ be a prime such that $p\equiv 2 \pmod 3$. By taking all the primes $p_1,p_2,\dots,p_{k+1}$ to be equal to the same prime p in Theorem 6.1, we obtain the following infinite family of congruences for $\overline{\mathcal{EO}}(n)$:*

$$\overline{\mathcal{EO}}\left(p^{2(k+1)}n+p^{2k+1}j+\frac{p^{2(k+1)}-1}{3}\right)\equiv 0 \pmod 2,$$

where $j \not\equiv 0 \pmod{p}$. *In particular, for all* $n \geq 0$ *and* $j \not\equiv 0 \pmod{5}$, *we have*

$$\overline{\mathcal{EO}}\,(25n + 5j + 8) \equiv 0 \pmod{2}.$$

In the following Theorem, we prove an infinite family of congruences for $\overline{\mathcal{EO}}(n)$ modulo 8. In the proof, we use the fact that the eta-quotient $\eta^5(96z)/\eta(24z)$ is an eigenform for the Hecke operators T_p, where $p \equiv 1 \pmod{24}$. This has been observed to be true by Scott Ahlgren. We now present below the proof given by Ahlgren which was communicated to us through an email. Let $F_1 = \eta^5(24z)/\eta(96z)$, $F_7 = \eta^3(24z)\eta(96z)$, $F_{13} = \eta(24z)\eta^3(96z)$, and $F_{19} = \eta^5(96z)/\eta(24z)$. Then F_j is supported on exponents congruent to $j \pmod{24}$. The Hecke operators T_p for $p \equiv 5, 11, 17, 23 \pmod{24}$ annihilate each of these forms. The Hecke operators T_p for $p \equiv 1, 5, 13, 19 \pmod{24}$ map F_j to a multiple of $F_{j'}$, where $j' \equiv pj \pmod{24}$. It turns out that a linear combination of the forms F_j is an eigenform of all of the Hecke operators. In [34, p. 209], equation (13.84) expresses the linear combination as an eigenform. Since the F_j are supported on distinct classes of coefficients, it follows that F_j are eigenforms of all the Hecke operators.

Theorem 6.2. *Let* k, n *be nonnegative integers. For each* i *with* $1 \leq i \leq k+1$, *if* $p_i \equiv 1 \pmod{24}$ *is prime such that* $\overline{\mathcal{EO}}\left(\frac{19p_i - 1}{3}\right) \equiv 0 \pmod{8}$, *then for any integer* $j \not\equiv 0 \pmod{p_{k+1}}$

$$\overline{\mathcal{EO}}\left(8p_1^2 \dots p_{k+1}^2 n + \frac{p_1^2 \dots p_k^2 p_{k+1}(24j + 19p_{k+1}) - 1}{3}\right) \equiv 0 \pmod{8}.$$

Proof. We first recall the following 2-dissection formula from [9, Entry 25, p. 40]:

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5 (q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5 (q^8;q^8)_\infty}. \tag{6.11}$$

From (6.1), we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)q^n=\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}. \tag{6.12}$$

Combining (6.11) and (6.12), and then extracting the terms with odd powers of q, we deduce that

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(4n+2)q^n=2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^2(q^4;q^4)_\infty}. \tag{6.13}$$

We again combine (6.11) and (6.13), and then extract the terms with odd powers of q to obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^n=4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^3}.$$

Since $(q;q)_\infty^2\equiv(q^2;q^2)_\infty \pmod 2$, we have

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^n\equiv 4\frac{(q^4;q^4)_\infty^5}{(q;q)_\infty} \pmod 8.$$

This gives

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^{24n+19}\equiv 4\frac{\eta(96z)^5}{\eta(24z)} \pmod 8.$$

Let $\frac{\eta(96z)^5}{\eta(24z)}=\sum_{n=1}^{\infty}a(n)q^n$. It is clear that $a(n)=0$ if $n\not\equiv 19 \pmod{24}$. Also, for all $n\geq 0$,

$$\overline{\mathcal{EO}}(8n+6)\equiv 4a(24n+19) \pmod 8.$$

By Theorem 4.6, we have $\frac{\eta(96z)^5}{\eta(24z)}\in S_2\left(\Gamma_0(2304),\left(\frac{2}{\bullet}\right)\right)$. Since $\frac{\eta(96z)^5}{\eta(24z)}$ is a Hecke eigenform for the Hecke operator T_p, where $p \equiv 1 \pmod{24}$, (1.24) and (1.25) yield

$$a(pn) + p\left(\frac{2}{p}\right) a\left(\frac{n}{p}\right) = \lambda(p)a(n). \tag{6.14}$$

Putting $n = 19$ in (6.14) and noting that $p \not\equiv 19 \pmod{24}$, we obtain $a(19p) = \lambda(p)a(19)$. Also, $a(19) = 1$, and hence $a(19p) = \lambda(p)$. Thus (6.14) gives

$$a(pn) + p\left(\frac{2}{p}\right) a\left(\frac{n}{p}\right) = a(19p)a(n). \tag{6.15}$$

From (6.15), we obtain that for all $n \geq 0$ and $p \nmid r$,

$$a(p^2n) + a\,(n) \equiv a(19p)a(pn) \pmod{2} \tag{6.16}$$

and

$$a(p^2n + pr) = a(19p)a(pn + r). \tag{6.17}$$

Let $A(n) = a(24n+19)$. Let p be a prime such that $p \equiv 1 \pmod{24}$. Now, replacing n by $24n - pr + 19$ in (6.17), we obtain

$$A\left(p^2n + 19\frac{p^2-1}{24} + pr\frac{1-p^2}{24}\right) = A\left(19\frac{p-1}{24}\right) A\left(pn + 19\frac{p-1}{24} + r\frac{1-p^2}{24}\right). \tag{6.18}$$

We note that $\gcd\left(\frac{1-p^2}{24}, p\right) = 1$. Hence when r runs over a residue system excluding the multiple of p, so does $\frac{1-p^2}{24}r$. Thus, (6.18) can be rewritten as

$$A\left(p^2n + 19\frac{p^2-1}{24} + pj\right) = A\left(19\frac{p-1}{24}\right) A\left(pn + 19\frac{p-1}{24} + j\right), \tag{6.19}$$

where $p \nmid j$. Similarly, replacing n by $24n + 19$ in (6.16), we have, modulo 2

$$A\left(p^2 n + 19\frac{p^2-1}{24}\right) + A(n) \equiv A\left(19\frac{p-1}{24}\right) A\left(pn + 19\frac{p-1}{24}\right). \tag{6.20}$$

Let p be such that $\overline{\mathcal{EO}}\left(\frac{19p-1}{3}\right) \equiv 0 \pmod 8$. Then, using the relation $\overline{\mathcal{EO}}(8n+6) \equiv 4A(n) \pmod 8$, we have $A\left(19\frac{p-1}{24}\right) \equiv 0 \pmod 2$. Hence, (6.19) and (6.20) imply

$$A\left(p^2 n + 19\frac{p^2-1}{24} + pj\right) \equiv 0 \pmod 2 \tag{6.21}$$

and

$$A\left(p^2 n + 19\frac{p^2-1}{24}\right) \equiv A(n) \pmod 2. \tag{6.22}$$

From our hypothesis, we have $p_i \geq 5$ are primes such that $p_i \equiv 1 \pmod{24}$ and $A\left(19\frac{p_i-1}{24}\right) \equiv 0 \pmod 2$. Now, using (6.22) we deduce that

$$A\left(p_1^2 \dots p_k^2 n + 19\frac{p_1^2 \dots p_k^2 - 1}{24}\right) \equiv A(n) \pmod 2.$$

Replacing n by $p_{k+1}^2 n + 19\frac{p_{k+1}^2-1}{24} + p_{k+1}j$, and then using (6.21) we obtain

$$A\left(p_1^2 \dots p_k^2 p_{k+1}^2 n + 19\frac{p_1^2 \dots p_k^2 p_{k+1}^2 - 1}{24} + p_1^2 \dots p_k^2 p_{k+1} j\right) \equiv 0 \pmod 2.$$

We complete the proof by using the fact that $\overline{\mathcal{EO}}(8n+6) \equiv 4A(n) \pmod 8$. ∎

Remark 6.2.2. *Let p be a prime such that $p \equiv 1 \pmod{24}$ and $\overline{\mathcal{EO}}\left(\frac{19p-1}{3}\right) \equiv 0 \pmod 8$. By taking all the primes $p_1, p_2, \dots, p_{k+1}$ to be equal to the same prime p in Theorem 6.2, we obtain the following infinite family of congruences for $\overline{\mathcal{EO}}(n)$:*

$$\overline{\mathcal{EO}}\left(8p^{2(k+1)}n + 8p^{2k+1}j + \frac{19p^{2(k+1)}-1}{3}\right) \equiv 0 \pmod 8,$$

where $j \not\equiv 0 \pmod{p}$*. In particular, if we choose* $p = 1009$*, then* $1009 \equiv 1 \pmod{24}$ *and* $\frac{19\times 1009-1}{3} = 6390$*. Using Mathematica we verify that* $\overline{\mathcal{EO}}(6390) \equiv 0 \pmod{8}$*. Thus, for all* $n \geq 0$ *and* $j \not\equiv 0 \pmod{1009}$*, we have*

$$\overline{\mathcal{EO}}(8144648n + 8072j + 6447846) \equiv 0 \pmod{8}.$$

6.3 Ramanujan-type congruences for $\overline{\mathcal{EO}}(n)$

In [5], Andrews proved that, for all $n \geq 0$

$$\overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod{5}. \tag{6.23}$$

In this section, we prove that the congruence (6.23) is also true modulo 4 if $n \not\equiv 0 \pmod{5}$. To be specific, we prove the following result.

Theorem 6.3. *Let* $t \in \{1, 2, 3, 4\}$*. Then for all* $n \geq 0$ *we have*

$$\overline{\mathcal{EO}}(10(5n+t)+8) \equiv 0 \pmod{20}.$$

Proof. Due to (6.23) we need to prove our congruences modulo 4 only. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^n = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} &= \frac{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^4} \\ &\equiv \frac{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^3}{(q^4;q^4)_\infty^2} \pmod{4} \\ &= (q^2;q^2)_\infty^2 (q^4;q^4)_\infty \pmod{4}.\end{aligned}$$

Let us consider that $(m, M, N, r, t) = (50, 8, 10, (0, 2, 1, 0), 18)$.
It is easy to verify that $(m, M, N, r, t) \in \Delta^*$ and $P_{m,r}(t) = \{18, 28, 38, 48\}$. From Lemma 5.3 we know that $\left\{ \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} : \delta | 10 \right\}$ forms a complete set of double coset

representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Let $r' = (0,0,0,0,0,0,0) \in R(10)$. We have used *Sage* to verify that $p_{m,r}(\gamma_\delta) + p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta \mid N$, where $\gamma_\delta = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}$. We compute that the upper bound in Lemma 5.2 is $\lfloor \nu \rfloor = 1$. Using *Mathematica* we verify that $\overline{\mathcal{EO}}(50n+t') \equiv 0 \pmod 4$ for $n \leq 1$ and $t' \in P_{m,r}(t)$. Thus, by Lemma 5.2, we conclude that $\overline{\mathcal{EO}}(50n+t') \equiv 0 \pmod 4$ for any $n \geq 0$, where $t' \in \{18, 28, 38, 48\}$. This completes the proof of the theorem. ∎

Remark 6.3.1. *We note that Theorem 6.3 is not true if $t = 0$. For example, $\overline{\mathcal{EO}}(8)$ is not divisible by 4.*

6.4 Parity of $\overline{\mathcal{EO}}(2n)$

For a nonnegative integer n, recall that $p(n)$ denotes the number of partitions of n. In [43], Ono proved that there are infinitely many integers N in every arithmetic progression for which $p(N)$ is even; and that there are infinitely many integers M in every arithmetic progression for which $p(M)$ is odd so long as there is at least one. Ono's result gave an affirmative answer to a well-known conjecture on parity of $p(n)$ in arithmetic progression. In this section, we prove the same for the partition function $\overline{\mathcal{EO}}(n)$. We note that $\overline{\mathcal{EO}}(2n+1) = 0$ for all $n \geq 0$. In the following theorem, we prove the parity of $\overline{\mathcal{EO}}(2n)$ in any arithmetic progression.

Theorem 6.4. *For any arithmetic progression $r \pmod t$, there are infinitely many integers $N \equiv r \pmod t$ for which $\overline{\mathcal{EO}}(2N)$ is even. Also, for any arithmetic progression $r \pmod t$, there are infinitely many integers $M \equiv r \pmod t$ for which $\overline{\mathcal{EO}}(2M)$ is odd, provided there is one such M. Furthermore, if there does exist an $M \equiv r \pmod t$ for which $\overline{\mathcal{EO}}(2M)$ is odd, then the smallest such M is less than*

$$\frac{2^{9+j}3^7t^6}{d^2}\prod_{p \mid 6t}\left(1 - \frac{1}{p^2}\right) - 2^j,$$

where $d = \gcd(12r-1, t)$ *and* $2^j > \dfrac{t}{12}$.

We prove Theorem 6.4 by using the approach developed in [43]. Recently, Jameson and Wieczorek [28] have done a similar study for the generalized Frobenius partitions. We now recall two results from [28]. Also see [43]. Let $M_k^!(\Gamma_0(N_0), \chi)$ denote the space of weakly holomorphic modular forms.

Theorem 6.5 (Theorem 5 [28]). *Let N_0, α, β, t be integers with N_0, α, t positive, and let*

$$\sum_{n=0}^{\infty} c(n) q^{\alpha n+\beta} \in M_k^!\left(\Gamma_0(N_0), \chi\right),$$

where $c(n)$ are algebraic integers in some number field. For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $N \equiv r \pmod{t}$ for which $c(N)$ is even.

Theorem 6.6 (Theorem 6 [28]). *Let N_0, α, β, t be integers with N_0, α positive, and $t > 1$, and let*

$$\sum_{n=0}^{\infty} c(n) q^{\alpha n+\beta} \in M_k^!\left(\Gamma_0(N_0), \chi\right),$$

where $c(n)$ are algebraic integers in some number field. For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $M \equiv r \pmod{t}$ for which $c(M)$ is odd, provided there is one such M.

Furthermore, if there does exist an $M \equiv r \pmod{t}$ for which $c(M)$ is odd, then the smallest such M is less than $C_{r,t}$ for

$$C_{r,t} := \frac{2^j \cdot 12 + k}{12\alpha} \left[\frac{N\alpha^2 t^2}{d}\right]^2 \prod_{p \mid N\alpha t} \left(1 - \frac{1}{p^2}\right) - 2^j,$$

where $N := \operatorname{lcm}(\alpha t, N_0)$, $d := \gcd(\alpha r + \beta, t)$, and j is a sufficiently large integer.

Proof of Theorem 6.4. We have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}.$$

We rewrite the above identity in terms of η-quotients, and then use the binomial theorem to obtain

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^{6n+1} = \frac{\eta(12z)^3}{\eta(6z)^2} \equiv \frac{\eta(12z)^4}{\eta(6z)^4} \pmod 2.$$

By Theorem 4.6, we have

$$\frac{\eta(12z)^4}{\eta(6z)^4} \in M_0^!\left(\Gamma_0(72)\right).$$

Let $f_t(z) := \frac{\eta(12z)^4}{\eta(6z)^4}\Delta^{2^j}(6tz)$, where $\Delta(z) := \eta^{24}(z)$. The cusps of $\Gamma_0(72t)$ are represented by fractions $\frac{c}{d}$ where $d \mid 72t$ and $\gcd(c,d) = 1$. Now, $f_t(z)$ vanishes at the cusp $\frac{c}{d}$ if and only if

$$4\frac{\gcd(d,12)^2}{12} - 4\frac{\gcd(d,6)^2}{6} + 24 \cdot 2^j \frac{\gcd(d,6t)^2}{6t} > 0.$$

We have

$$4\frac{\gcd(d,12)^2}{12} - 4\frac{\gcd(d,6)^2}{6} + 24 \cdot 2^j \frac{\gcd(d,6t)^2}{6t} \geq 2^j\frac{6}{t} - \frac{1}{2}.$$

Hence, if j is an integer such that $2^j > \frac{t}{12}$, then $f_t(z) \in S_{12\cdot 2^j}\left(\Gamma_0(72t)\right)$. Finally, our desired result follows immediately by applying Theorems 6.5 and 6.6 to $\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^{6n+1}$. ■

6.5 $\overline{\mathcal{EO}}(n)$ is almost always even

From (6.1) it is clear that, for any nonnegative integer n, $\overline{\mathcal{EO}}(2n+1)$ is always even. In the following theorem we prove that $\overline{\mathcal{EO}}(2n)$ is almost always even.

Theorem 6.7. *Let $n \geq 0$. Then $\overline{\mathcal{EO}}(8n+6)$ is almost always divisible by 8, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{\mathcal{EO}}(8n+6) \equiv 0 \pmod 8\right\}}{X} = 1.$$

Proof. We first recall the following 2-dissection formula from [9, Entry 25, p. 40]:

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}. \tag{6.24}$$

From (6.1), we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(2n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2}. \tag{6.25}$$

Combining (6.24) and (6.25), and then extracting the terms with odd powers of q, we deduce that

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^2(q^4;q^4)_\infty}. \tag{6.26}$$

We again combine (6.24) and (6.26), and then extract the terms with odd powers of q to obtain

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8n+6)q^n = 4\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^3}.$$

Since $(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2$, we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(8n+6)q^n \equiv 4\frac{(q^4;q^4)_\infty^5}{(q;q)_\infty} \pmod 8.$$

We rewrite the above equation in terms of η-quotients and obtain

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(8n+6)q^{24n+19}\equiv 4\frac{\eta^5(96z)}{\eta(24z)} \pmod 8. \tag{6.27}$$

Let $A(z)=\dfrac{\eta^2(24z)}{\eta(48z)}$. Then, $A^2(z)\equiv 1 \pmod 4$. Also, let $B(z)=\dfrac{\eta^5(96z)\eta^3(24z)}{\eta^2(48z)}$. Then we have

$$B(z)=\frac{\eta^5(96z)}{\eta(24z)}A^2(z)\equiv\frac{\eta^5(96z)}{\eta(24z)} \pmod 4. \tag{6.28}$$

The cusps of $\Gamma_0(2304)$ are represented by fractions $\frac{c}{d}$ where $d\mid 2304$ and $\gcd(c,d)=1$. By Theorem 4.6, $B(z)$ is holomorphic at the cusp $\frac{c}{d}$ if and only if

$$5\frac{\gcd(d,96)^2}{96}+3\frac{\gcd(d,24)^2}{24}-2\frac{\gcd(d,48)^2}{48}\geq 0.$$

Now,

$$\begin{aligned}&5\frac{\gcd(d,96)^2}{96}+3\frac{\gcd(d,24)^2}{24}-2\frac{\gcd(d,48)^2}{48}\\&=\frac{\gcd(d,48)^2}{24}\left(\frac{5}{4}\frac{\gcd(d,96)^2}{\gcd(d,48)^2}+3\frac{\gcd(d,24)^2}{\gcd(d,48)^2}-1\right)\\&>0.\end{aligned}$$

Hence, by Theorem 4.6, $B(z)\in S_3(\Gamma_0(2304),\left(\frac{-4}{\bullet}\right))$.

Since $B(z)\in S_3(\Gamma_0(2304),\left(\frac{-4}{\bullet}\right))$, applying Corollary 1.14 and proceeding similarly as shown in the proof of Theorem 4.3 we find that the Fourier coefficients of $B(z)$ are almost always divisible by m. Hence, using (6.28) and (6.27) we complete the proof of the theorem. ■

6.6 Distribution of $\mathcal{EO}_u(n)$

Recently, Uncu [54] has treated a different subset of the partitions enumerated by $\mathcal{EO}(n)$. Also see [5, p. 435]. We denote by $\mathcal{EO}_u(n)$ the partition function defined by Uncu, and the generating function is given by

$$\sum_{n=0}^{\infty} \mathcal{EO}_u(n)q^n = \frac{1}{(q^2;q^4)_\infty^2}. \tag{6.29}$$

In the following theorem, we prove that $\mathcal{EO}_u(n)$ is almost always divisible by 2^k.

Theorem 6.8. *Let k be a positive integer. Then $\mathcal{EO}_u(2n)$ is almost always divisible by 2^k, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{n \le X : \mathcal{EO}_u(2n) \equiv 0 \pmod{2^k}\right\}}{X} = 1.$$

Proof. The generating function of $\mathcal{EO}_u(2n)$ is given by

$$\sum_{n=0}^{\infty} \mathcal{EO}_u(2n)q^n = \frac{1}{(q;q^2)_\infty^2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}. \tag{6.30}$$

We note that $\eta(24z) = q\prod_{n=1}^{\infty}(1-q^{24n})$ is a power series of q. As in the proof of Theorem 6.7, let

$$A(z) = \prod_{n=1}^{\infty} \frac{(1-q^{24n})^2}{(1-q^{48n})} = \frac{\eta^2(24z)}{\eta(48z)}.$$

Then using binomial theorem we have

$$A^{2^k}(z) = \frac{\eta^{2^{k+1}}(24z)}{\eta^{2^k}(48z)} \equiv 1 \pmod{2^{k+1}}. \tag{6.31}$$

Define $B_k(z)$ by

$$B_k(z) = \left(\frac{\eta(48z)}{\eta(24z)}\right)^2 A^{2^k}(z). \tag{6.32}$$

Modulo 2^{k+1}, we have

$$B_k(z) = \frac{\eta^2(48z)}{\eta^2(24z)} A^{2^k}(z) \equiv \frac{\eta^2(48z)}{\eta^2(24z)} = q^2 \frac{(q^{48};q^{48})_\infty^2}{(q^{24};q^{24})_\infty^2}. \tag{6.33}$$

Combining (6.30) and (6.33), we obtain

$$B_k(z) \equiv \sum_{n=0}^{\infty} \mathcal{EO}_u(2n) q^{24n+2} \pmod{2^{k+1}}. \tag{6.34}$$

The cusps of $\Gamma_0(576)$ are represented by fractions $\frac{c}{d}$ where $d \mid 576$ and $\gcd(c,d) = 1$. By Theorem 4.6, it is easily seen that $B_k(z)$ is a form of weight 2^{k-1} on $\Gamma_0(576)$. Therefore, $B_k(z) \in M_{2^{k-1}}(\Gamma_0(576))$ if and only if $B_k(z)$ is holomorphic at the cusp $\frac{c}{d}$. We know that $B_k(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\frac{\gcd(d,24)^2}{24}\left(2^{k+1}-2\right) + \frac{\gcd(d,48)^2}{24}\left(1-2^{k-1}\right) \geq 0.$$

Now,

$$\begin{aligned}
&\gcd(d,24)^2\left(2^{k+1}-2\right) + \gcd(d,48)^2\left(1-2^{k-1}\right) \\
&= \gcd(d,48)^2\left(\frac{\gcd(d,24)^2}{\gcd(d,48)^2}(2^{k+1}-2) + (1-2^{k-1})\right) \\
&\geq \frac{1}{4}(2^{k+1}-2) + (1-2^{k-1}) > 0.
\end{aligned}$$

Hence, $B_k(z) \in M_{2^{k-1}}(\Gamma_0(576))$. Now, using Corollary 1.14 and proceeding similarly as shown in the proof of Theorem 4.3, we arrive at the desired result due to (6.34).

■

Index

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Publications

<u>Publications from Thesis work</u>

1. R. Barman and C. Ray. Congruences for ℓ-regular overpartitions and Andrews' singular overpartitions. *The Ramanujan J.*, 45 (2018), no. 2, 497–515.

2. C. Ray and R. Barman. Infinite families of congruences for k-regular overpartitions. *Int. J. Number Theory*, 14 (2018), no. 1, 19–29.

3. C. Ray and R. Barman. New congruences for overpartitions into odd parts. *Integers*, 18 (2018), A50, 1–20.

4. C. Ray and R. Barman. Arithmetic properties of cubic and overcubic partition pairs. *The Ramanujan J.*, (DOI: 10.1007/s11139-019-00136-1).

5. C. Ray and R. Barman. Arithmetic properties of Andrews' integer partitions with even parts below odd parts. (under review at Journal of Number Theory).

6. C. Ray and R. Barman. Divisibility of Andrews' singular overpartitions by powers of 2 and 3. (under review at Research in Number Theory).