Local Geometry of Curve Graphs of Closed Surfaces

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BY

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CERTIFICATE

This is to certify that the thesis entitled **"Local Geometry of Curve Graphs of Closed Surfaces"** submitted by **Ms. Kuwari Mahanta** to the **Indian Institute of Technology Guwahati** for the award of the degree of **Doctor of Philosophy** is a record of the original bona fide research work carried out by her under my supervision. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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"Questions about the metric relations of Space in the immeasurably small are thus not idle ones."

-Bernhard Riemann

ABSTRACT

Let S_g denote a closed, orientable surface of genus $g \geq 2$. Let $\mathcal{C}(S_g)$ be the associated curve graph and d be the associated path metric. Let α and β be curves on S_g and $T_\beta(\alpha)$ be the Dehn twist of α about β.

If $d(\alpha, \beta) = 3$, we show that $d(\alpha, T_\beta(\alpha)) = 4$. This produces many tractable examples of distance 4 vertices in $\mathcal{C}(S_g)$. As an application we show that the minimum intersection number of any two curves at a distance 4 on S_g is at most $(2g-1)^2$.

Let $d(\alpha, \beta) = 4$. We fix the vertex α and apply T_β to it in an attempt to create pairs of curves at a distance 5 apart. We give a necessary and sufficient topological condition for $d(\alpha, T_\beta(\alpha))$ to be 4. We then characterise the pairs of α and β for which $5 \leq d(\alpha, T_\beta(\alpha)) \leq 6$. Lastly, we give an example of a pair of curves on S_2 which represent vertices at a distance 5 in $\mathcal{C}(S_2)$ with intersection number 144. This example gives that $i_{min}(2, 5) \leq 144$.

Our proofs majorly rely on cut and paste techniques.

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CONTENTS

CHAPTER 1

INTRODUCTION

A surface, S, is a real two-dimensional oriented differential manifold. It's natural to wonder how many distinct complex structures can S be equipped with. This problem is popularly known as the Riemann's moduli problem and the corresponding space of these structures is known as the moduli space of S. We briefly look into the moduli space of S when S is a closed surface. For the case when S has genus zero, its moduli space consists of a single point which is the Riemann sphere. When the genus of S is one, it has been proven that its moduli space can be identified by the complex plane. Consider $S = S_g$ has genus, g, greater than one. Riemann claimed that the moduli space of S_g is determined by $3g - 3$ complex parameters. After Riemann, the structure of the moduli space of S_g became an active field of interest over the next few decades.

Teichmüller gave a new approach to the moduli problem by defining a cover of the moduli space and studying its structure intrinsically. This space, known as the Teichmüller space corresponding to S_g, is denoted by $Teich(S_g)$ and is defined as follows : By a hyperbolic structure on S_g we will mean a diffeomorphism $f : S_g \longrightarrow T$, where T is a surface with a complete, finite-area hyperbolic metric and denote it by (T, f). We refer to (T, f) as a marked hyperbolic surface. We say that (T_1, f_1) and (T_2, f_2) are equivalent if there is an isometry $I : T_1 \longrightarrow T_2$ such that $I \circ f_1$ and f_2 are homotopic. The space of distinct marked hyperbolic surfaces of S_g is $Teich(S_g)$. The group of isotopy classes of orientation preserving homeomorphisms of S_g is known as the mapping class group of S_g and is denoted by $Mod(S_g)$. A classical fact is that the moduli space of S comes out to be $Teich(S_g)/Mod(S_g)$. An elaborate account of the historical development of the moduli space of surfaces and the corresponding Techmüller spaces can be found in [**1**].

In **[11]**, William J. Harvey associated a finite dimensional simplical complex corresponding to a surface, called the complex of curves, as a tool to study the corresponding Teichmüller space. By a curve on S_g we will mean an essential simple closed curve on it. Harvey considered the vertices of the complex of curves to be the isotopy classes of curves on S_g and any collection of $k+1$ mutually disjoint curves comprised to form a k-simplex. In **[13]**, Ivanov used the complex of curves to give a geometric proof to the famous theorem by Royden in **[23]** which states that the isometry group of the $Teich(S_g)$ is the corresponding extended mapping class group. The complex of curves becomes a natural geometric object on which $Mod(S_g)$ acts and thus becomes an intriguing space to study. Although the higher dimensional simplices find a number of applications (see, **[14]**), the combinatorial properties of the complex of curves is completely determined by its 1-skeleton. This 1-dimensional simplical complex is called the curve graph and is denoted by $\mathcal{C}(S_g)$. The curve graph is a connected graph (see, **[9]**) and thus, it can be equipped with a path-metric d. More precisely, the distance d between any two vertices of $\mathcal{C}(S_g)$ is the minimal number of edges in any edge path between them.

Masur and Minsky in their seminal work in **[18]** discovered that $\mathcal{C}(S_g)$ with the metric d is an infinite diameter δ-hyperbolic space. Later it was shown that the δ can be chosen to be independent of the surface S_g, see **[2]**, **[7]**, **[8]**, **[12]**, **[22]**. The coarse geometry of the curve complex plays a pivotal role in understanding the hyperbolic structure of 3-manifolds, the mapping class group of surfaces and Teichmuller theory. One can see **[21]** for many such applications.

In comparison the local geometry of $\mathcal{C}(S_g)$ remains relatively unexplored. For instance there is no characterisation of a 3-sphere around a vertex in $\mathcal{C}(S_g)$. A fundamental hindrance while studying $\mathcal{C}(S_g)$ is that there are infinitely many distinct vertices adjacent to any vertex in $\mathcal{C}(S_g)$. In **[19]**, the authors circumvented this local infinitude of $\mathcal{C}(S_g)$ by defining a set of geodesics called the tight geodesics in $\mathcal{C}(S_g)$. They prove that between any two vertices of $\mathcal{C}(S_g)$ there are only finitely many tight geodesics. Similar notions have been used in **[24]**, **[25]**, **[26]** and **[5]** to overcome this local pathology of $\mathcal{C}(S_g)$ and to compute distances between any two vertices. In **[4]**, the authors show the existence of infinite geodesic rays in $\mathcal{C}(S_g)$. The intersection number between the vertices of these geodesic rays is bounded above by a polynomial of the complexity of the surface and hence, is asymptotically low. Knowing the local geometry of the curve graph promises aid in determining exact distances between its vertices more efficiently than the existing methods. Further, this information can also be employed in calculating the translation length of pseudo-Anosov mapping classes and studying the action of $Mod(S_g)$ on $\mathcal{C}(S_g)$ more precisely.

Ivanov proved in **[13]** that the group of automorphisms of the complex of curves is $Mod(S_g)$. Since $Mod(S_g)$ is generated by Dehn twists about a finite collection of curves on S_g, we attempt a study of $\mathcal{C}(S_g)$ at a granular scale by looking at the impact of powers of Dehn twists on vertices of $\mathcal{C}(S_g)$ which are at shorter distances apart. Let α and γ be two curves on S_g and $p \in \mathbb{N}$.

REMARK 1. If $d(\alpha, \gamma) = 1$, then $d(\alpha, T_\gamma^p(\alpha)) = d(\alpha, \gamma) - 1$.

REMARK 2. If $d(\alpha, \gamma) = 2$, then $d(\alpha, T_\gamma^p(\alpha)) = d(\alpha, \gamma)$.

Remark 1 follows from the fact that if $d(\alpha, \gamma) = 1$, then $T_\gamma^p(\alpha) = \alpha$. Remark 2 can be arrived at as follows : if $d(\alpha, \gamma) = 2$, then there exists a curve c on S_g such that $d(\alpha, c) = 1$ and $d(c, \gamma) = 1$. Since $i(\alpha, T_\gamma^p(\alpha)) = p(i(\alpha, \gamma))^2$ therefore, $i(\alpha, T_\gamma^p(\alpha)) \neq 0$. Hence, $d(\alpha, T_\gamma^p(\alpha)) \geq 2$. Since α and γ are essential, simple closed curves in $S_g \setminus c$ we have, $T_\gamma^p(\alpha)$ is also an essential, simple closed curve in $S_g \setminus c$. It follows that α, c, $T_\gamma^p(\alpha)$ forms a geodesic in $\mathcal{C}(S_g)$.

In general, one can ask the following question:

QUESTION 1. If $d(\alpha, \gamma) > 2$, then what is the relation between $d(\alpha, \gamma)$ and $d(\gamma, T_\gamma^p(\alpha))$?

We note that if $d(\alpha, \gamma) = n$ then $d(T_\gamma^p(\alpha), \gamma) = n$. This follows from taking the image of the geodesic in $\mathcal{C}(S_g)$ between α and γ under the action of the isometry T_γ^p.

1.1. Overview of the thesis

In chapter 2 we define and state some properties of the mapping class group, curve graph, minimal intersection number and efficient geodesics.

Let α and γ be curves on S_g. In chapter 3, we state and prove the following theorem 5.

THEOREM 5. *Let α and γ be a filling pair of curves on S_g. Then α and $T_\gamma^p(\alpha)$ also fills S_g.*

In chapter 4, we apply theorem 5 to prove theorem 6 as stated below. This answers question 1 for $d(\alpha, \gamma) = 3$ and shows that $d(\alpha, T_\gamma^p(\alpha)) = d(\alpha, \gamma) + 1$.

THEOREM 6. *If α and γ be two curves on S_g with $d(\alpha, \gamma) = 3$, then $d(\alpha, T_\gamma(\alpha))$ $= 4$.*

A byproduct of theorem 6 is infinitely many examples of vertices at a distance 4 in $\mathcal{C}(S_g)$. These examples are the first examples of curves at a distance 4 apart on $\mathcal{C}(S_{g>3})$ which can be explicitly seen as a three dimensional rendering. As a demonstration of our method we construct a pair of distance 4 curves on S_4 (Figure 2) from a minimally intersecting pair of distance 3 curves (Figure 1) as described in [**3**]. In general, Aougab and Huang give a method to construct pairs of minimally

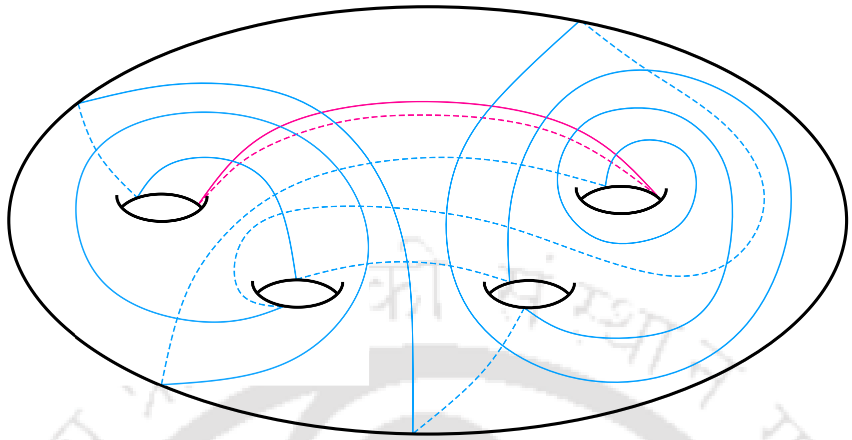


FIGURE 1. Minimally intersecting curves representing vertices at a distance 3 in $\mathcal{C}(S_4)$

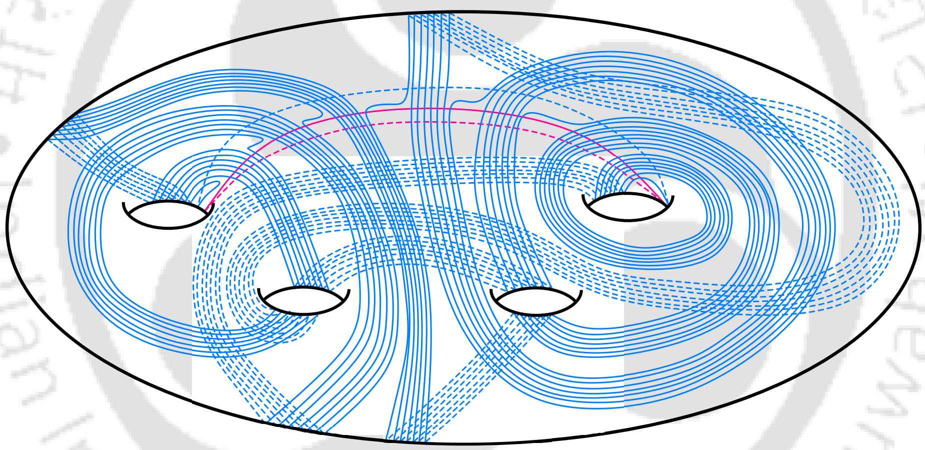


FIGURE 2. Curves on S_4 representing vertices at a distance 4 in $\mathcal{C}(S_4)$. The intersection number of these curves is 49.

intersecting pair of curves which are at a distance 3 in $\mathcal{C}(S_{g\geq 3})$. Using any such pair of curves on S_g we can explicitly construct a pair of curves which are at a distance 4 in $\mathcal{C}(S_g)$.

The minimal intersection number between any two curves on S_g which are at a distance n is denoted by $i_{min}(g,n)$. In [**4**], Aougab and Taylor proved that for $g \geq 3$, $i_{min}(g,3) = 2g-1$ and $i_{min}(2,3) = 4$. In [**4**], Aougab and Taylor proved that in general, $i_{min}(g,4) = O(g^2)$. In chapter 4, we apply our examples of curves which are at a distance 4 in $\mathcal{C}(S_g)$ to improve the known upper bound of $i_{min}(g_{\geq 4},4)$ to $(2g-1)^2$.

COROLLARY 1. *For a surface of genus $g \geq 3$, $i_{min}(g,4) \leq (2g-1)^2$.*

As a natural extension to theorem 6, we look into the analogous question 2. We were motivated to look into this process with the long term promise of creating examples of curves at a distance $n+1$ by using curves at a distance n apart.

QUESTION 2. For curves, α and γ, on S_g with $d(\alpha,\gamma)=4$, what are the possible values of $d(\alpha, T_\gamma^p(\alpha))$ for $p \in \mathbb{N}$?

In chapter 5 we define a family of curves known as the scaling curves. These curves are formed using arcs of $\gamma \setminus \alpha$ and we show that they fill along with α. The idea behind a scaling curve is the intuition that γ encodes the information of a few naturally occurring curves which are at a distance 3 from α and distance 1 from γ.

Let $a_0=\alpha$, a_1, a_2, a_3, $a_4=\gamma$ be a geodesic in $\mathcal{C}(S_g)$. The authors in **[4]** have shown that for some large enough constant $B \in \mathbb{N}$, $d(a_0, T^B_{T^B_{a_3}(a_0)}(a_0))=6$. The authors use arguments involving subsurface projection to show that

$$d(a_0, T^B_{T^B_{a_3}(a_0)}(a_0))=6$$

for a large enough constant $B \in \mathbb{N}$. In chapter 6, we employ the same arguments to show that for any general γ there exists a constant $K \in \mathbb{N}$ such that $d(\alpha, T^k_\gamma(\alpha))=6$, for every $k \geq K$, instead of the particular case when $\gamma = T^B_{a_3}(a_0)$. We then show that

$$4 \leq d(\alpha, T_\gamma(\alpha) \leq 6.$$

We use the scaling curves introduced in chapter 5 to give a necessary and sufficient condition for $d(a_0, T_{a_4}(a_0))=4$ in lemma 9.

Let N be an annular neighbourhood of a_4 and $B_m(T_{a_4}(a_0))$ be the sphere of radius m around $T_{a_4}(a_0)$. Let $\delta \in B_1(T_{a_4}(a_0))$ and $c \in B_2(T_{a_4}(a_0)) \cap B_1(\delta)$. Then, c is a standard single strand curve if $i(c,a_4)=1$ and if there exists an isotopic representative of c such that $(c \cap a_0) \subset N$. In section 6.3, we describe a placement of the components of $N \setminus (a_0 \cup a_4)$ which is equivalent to there being a curve on S_g which is mutually disjoint from a_0 and c. We call this arrangement of the components as the stacking property. We then apply lemma 9 to arrive at the following theorem which gives that $5 \leq d(a_0, T_{a_4}(a_0)) \leq 6$ for a judicious choice of a_0 and a_4.

THEOREM 9. *Let α and γ be curves on S_g such that $d(\alpha,\gamma)=4$ and the components of $S_g \setminus (\alpha \cup \gamma)$ doesn't contain any hexagons. Then, $d(\alpha, T_\gamma(\alpha)) \geq 5$ if and only if there doesn't exist any standard single strand curve $c \in B_2(T_\gamma(\alpha))$ having the stacking property.*

In chapter 7 we give a pair of curves on S_2 which are at a distance 5 apart on $\mathcal{C}(S_2)$ with intersection number 144. An immediate conclusion of this example is:

COROLLARY 4. $i_{min}(2,5) \leq 144$.

1.2. Prospects

A conclusion from the above results is that as we ascend distances from $d(a_0, a_3) = 3$ to $d(a_0, a_4) = 4$, the neatness of the result $d(a_0, T^p_{a_{n\leq 3}}(a_0)) = d(a_0, a_n) + C(n)$, where $C(n)$ is a constant function, doesn't carry over to the value of $d(a_0, T^p_{a_4}(a_0))$. Rather we have a pair of curves b_0, b_4 on S_2 and a constant $K \in \mathbb{N}$ such that $d(b_0, b_4) = 4$ and $d(b_0, T^k_{b_4}(b_0)) = 6$, $\forall k \geq K$ but $d(b_0, T_{b_4}(b_0)) = 5$. This thus prompts the following questions :

QUESTION 3. What are the values of $k \in \mathbb{N}$ such that $d(a_0, T^k_{a_4}(a_0)) = 6$?

Let v and w be curves on S_g such that $d(v, w) = n \geq 3$. Further suppose $i(v, w) = i_{min}(g, n)$. The components of $S_g \setminus (v \cup w)$ can be regarded as polygons whose edges are arcs from the set $(v \setminus w) \cup (w \setminus v)$. In **[6]**, the authors remark from their observations regarding $i_{min}(g, n)$ when $g = 2$ and $n = 3, 4$ that for lower distances the minimal intersection number is not only dependent on g and n but also on the combinatorics of the polygons in $S_g \setminus (v \cup w)$. We observe from our core proof idea of theorem 5 that the polygonal composition of $S_g \setminus (a_0 \cup T^k_{a_4}(a_0))$ differs from the polygonal composition of $S_g \setminus (a_0 \cup a_4)$ only in the number of rectangles. The number of rectangles depend on k and $i(a_0, a_4)$. This observation along with the finding that $d(b_0, T^k_{b_4}(b_0))$ can either be 5 or, 6 depending on the value of k helps us deduce that rectangles in $S_g \setminus (a_0 \cup a_4)$ have a significant role in determining distances in $\mathcal{C}(S_g)$.

QUESTION 4. Let a_0 and a_4 be curves on S_g such that $d(a_0, a_4) = 4$ and $i(a_0, a_4) = i_{min}(g, 4)$. Is $d(a_0, T_{a_4}(a_0)) = 5$?

We conjecture that question 4 has a positive answer. We further conjecture that for any general γ with $d(a_0, \gamma) = 4$ such that a_0 and γ have arbitrary intersection number, it need not be true that $d(a_0, T_\gamma(a_0)) = 5$. A possible counterexample to this might be $\gamma = T_{a_3}(a_0)$ where $d(a_0, a_3) = 3$ and $d(a_3, a_4) = 1$. We conjecture that $d(a_0, T_{T_{a_3}(a_0)}(a_0)) = 6$.

From our observations that $i_{min}(g, 4) \leq i_{min}(g, 3)^2$, it prompts us to ask the following question :

QUESTION 5. Is $i_{min}(g, 5) \leq i_{min}(g, 4)^2$?

In **[4]**, Aougab and Taylor showed that for large enough distances, $i_{min}(g, n)$ is independent of g. From **[17]**, we note that the geodesic triangle in $\mathcal{C}(S_g)$ with vertices a_0, a_3 and $T^k_{a_3}(a_0)$ is 0-hyperbolic for every $k \in \mathbb{N}$. The results in **[4]** give that the geodesic triangle in $\mathcal{C}(S_g)$ with vertices a_0, a_3 and $T^B_{T^B_{a_3}(a_0)}(a_0)$ is 0-hyperbolic for some constant $B \in \mathbb{N}$. As a consequence of the aforementioned pair of curves, b_0 and $T_{b_4}(b_0)$, we observe that the geodesic triangle formed with vertices b_0, b_4 and $T_{b_4}(b_0)$ is 1-hyperbolic. This leads to the conclusion that geodesic triangles formed with vertices a_0, a_n and $T^k_{a_n}(a_0)$ need not be 0-hyperbolic for all values of $k \in \mathbb{N}$. This leads to the following prospective question that was suggested by Joan Birman:

QUESTION 6. Let a_0 and a_n be a pair of curves on S_g such that $d(a_0, a_n) = n$. Let Δ be the geodesic triangle in $\mathcal{C}(S_g)$ with vertices a_0, a_n and $T_{a_n}(a_0)$. What is the minimum value of δ for which Δ is δ-hyperbolic?

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce definitions and theorems that will be used in this thesis. Since our work involves only closed surfaces, our definitions and theorems are tailored accordingly. For the analogous definitions involving surfaces of finite type, see **[9]**. This chapter doesn't contain any original work of the author.

2.1. Surfaces

A *surface* is a real 2-manifold. In this thesis we will consider only closed, oriented surfaces. By the classification of surfaces theorem, any closed, connected and oriented surface is homeomorphic to the connected sum of a 2-dimensional sphere with $g \geq 0$ tori. Here g is called the *genus* of the surface. We will denote a surface with genus g by S_g. The *Euler-characteristic* of S_g comes out to be $2 - 2g$ and it is a homeomorphic invariant of S_g. For the purpose of this thesis, we will consider that $g \geq 2$. We insist on this restriction on g as the geometry of the sphere and the torus is different and relatively well explored for our purpose.

2.2. Curves

A *curve* on S_g is an embedding of the unit circle into S_g. A curve on S_g is called *essential* if it is not null-homotopic. Throughout the thesis, by a curve on S_g, we will mean essential curves on S_g. For any curve γ on S_g, $[\gamma]$ is used to denote the isotopy class of γ on S_g. Let μ and λ be two curves on S_g in transverse position. The *geometric intersection number* of μ, λ is denoted as $i(\mu, \lambda)$ and is given as

$$i(\mu, \lambda) = min\{|a \cap b| : a \in [\mu], b \in [\lambda]\}.$$

We say that curves, μ and λ, are in *minimal position* if μ and λ intersect at $i(\mu, \lambda)$ points. A *bigon* is said to have been formed by μ and λ if an embedded disc in S_g is enclosed by the union of two arcs, one from μ and the other from λ. For any given pair of curves, the following bigon criteria gives an algorithm to figure out representatives that are in minimal positions.

FACT 1. *(The Bigon Criterion)* Two transverse curves on S_g are in minimal position if and only if they do not form any bigons.

PROOF. See [**9**, Proposition 1.7]. $\square$

FACT 2. *(Existence of minimal representatives)* Given $\mu_1, \ldots, \mu_k$ are curves in S_g which are pairwise in minimal position and nonisotopic. Then any curve μ_{k+1} on S_g has a representative that is in minimal position with μ_i for all $i \in \{1, \ldots, k\}$.

Thus, as is the common practice in this subject whenever we consider a collection of curves on S_g, we will consider isotopic representatives of these curves which are in minimal position with each other.

We say that μ and λ forms a filling pair of curves on S_g if the components of $S_g \setminus (\mu \cup \lambda)$ are topological discs. If μ and λ fill S_g then S_g can also be considered as a 2 dimensional CW-complex in the following manner : The 0 skeleton comprises of the distinct points in $\mu \cap \lambda$. The edge set comprises of the arcs of $\mu \setminus \lambda$ and $\lambda \setminus \mu$. The faces comprises of the discs in $S_g \setminus (\mu \cup \lambda)$.

2.3. Mapping Class Group

The mapping class group of S_g is the group of isotopy classes of orientation-preserving homeomorphisms of S_g. We denote this group by $Mod(S_g)$. Figure 1 and 2 give examples of finite order mapping classes. Both these examples can be generalised to obtain elements of $Mod(S_g)$ by considering the analogous rigid motions of S_g in $\mathbb{R}^3$.

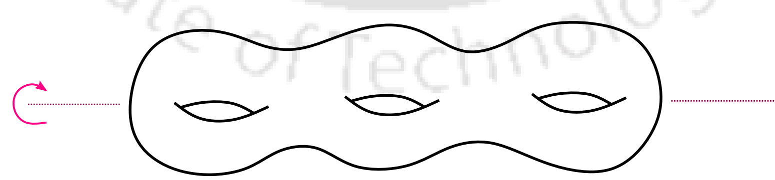
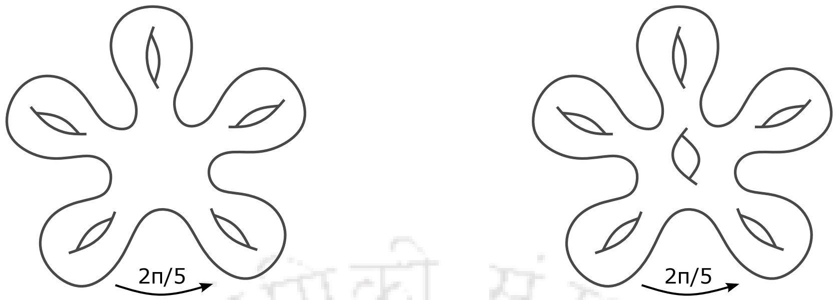


FIGURE 1. Rotating S_3 about the central axis by π gives an order 2 mapping class in $Mod(S_3)$ known as the hyperelliptic involution.

We now look at a class of infinite order mapping classes which were introduced by Max Dehn.



(A) (B)

FIGURE 2. Rotating S_5 and S_6 about the centre by $\frac{2\pi}{5}$ gives order 5 mapping classes in $Mod(S_5)$ and $Mod(S_6)$ respectively.

2.4. Dehn Twists

Consider the annulus, $A = S^1 \times [0,1]$ and define $T : A \longrightarrow A$ as $(\theta, r) \mapsto (\theta + 2\pi r, r)$. The action of T is called as "right twist" and by replacing $(\theta + 2\pi r)$ by $(\theta - 2\pi r)$, a "left twist" is obtained.

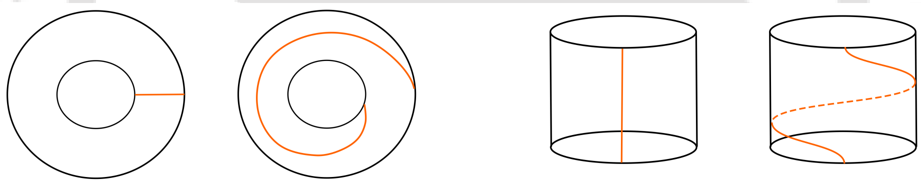


FIGURE 3. Annular and cylindrical view of the action of T.

Let α be a curve in S_g. Let N be an annular neighbourhood of α and $\phi : A \longrightarrow N$ be an orientation preserving homeomorphism. Then, the *Dehn twist about* α, $T_\alpha : S_g \longrightarrow S_g$ is defined as

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & x \in N \\ x & x \notin N \end{cases}$$

The action of T_α on S can be interpreted as "T acting on N" and keeping $S_g \setminus N$ fixed. The mapping class, T_α, is well-defined upto isotopy for the isotopy class of α.

Let λ, μ be curves on S_g and $p \in \mathbb{N}$. If $i(\mu, \lambda) = 0$, we have that $T_\mu^p(\lambda) = \lambda$. If $i(\mu, \lambda) \neq 0$, can obtain a picture of $T_\mu^p(\lambda)$ by performing a surgery of curves described as follows. Suppose $k = i(\mu, \lambda)$. Draw pk distinct and parallel copies of μ say, $\mu_1, \ldots, \mu_{pk}$ on S_g which are in minimal position with λ. For $i \in \{1, \ldots, pk\}$, at each point of $\mu_i \cap \lambda$, perform the surgery of curves as in figure 5. Performing this surgery gives a representative of $T_\mu^p(\lambda)$ on S_g. Details of this surgery can be found in [**9**, page 70].

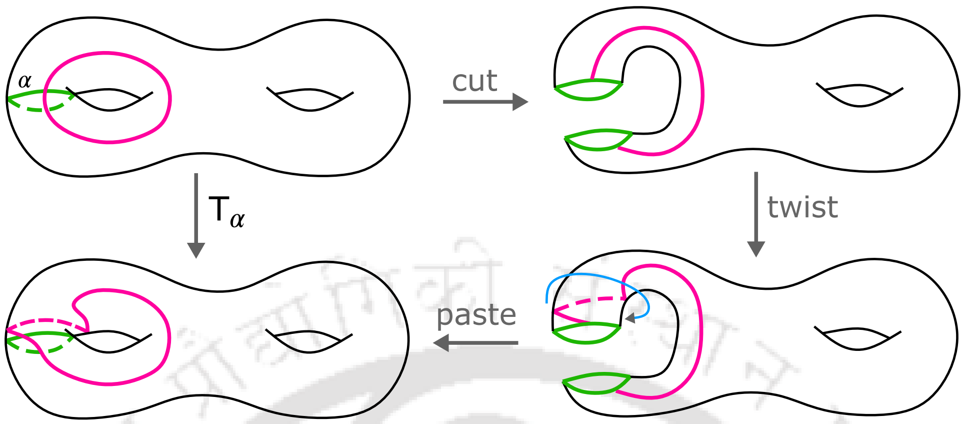


FIGURE 4. Action of T_α on the pink curve.

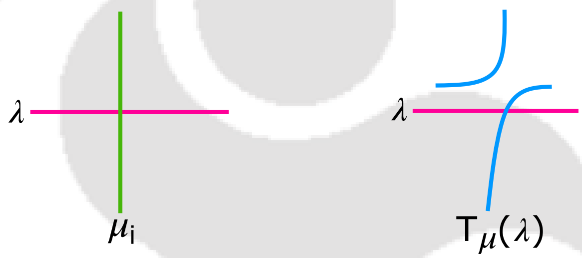


FIGURE 5. Surgery of curves performed to obtain $T_\mu(\lambda)$.

The following are a few facts about Dehn twists that are necessary for the work in this thesis.

FACT 3. Let μ and λ be curves on S_g and $p \in \mathbb{Z}$. Then

$$i(T_\mu^p(\lambda), \lambda) = |p|(i(\mu, \lambda))^2.$$

PROOF. See [**9**, Proposition 3.2] □

FACT 4. Let μ be a curve on S_g. Then T_μ is an infinite order mapping class.

PROOF. Corollary of Fact 3. □

THEOREM 1. *(Dehn-Lickorish theorem) $Mod(S_g)$ is generated by finitely many Dehn twists about non-separating curves.*

PROOF. See [**9**, Theorem 4.1]. □

2.5. Curve graph

The *curve graph* of S_g, denoted by $\mathcal{C}(S_g)$ is a 1 dimensional simplical complex defined as follows : the 0-skeleton is in one-to-one correspondence with isotopy classes of essential simple closed curves on S_g. Two vertices span an edge in $\mathcal{C}(S_g)$ if and only if these vertices have mutually disjoint representatives. By an excusable abuse of notation, for any curve γ on S_g we will use γ to denote the curve as well its isotopy class whenever the context is clear.

FACT 5. $\mathcal{C}(S_g)$ is a connected graph.

PROOF. See [**9**, Theorem 4.3]. □

By virtue of fact 5, we can make $\mathcal{C}(S_g)$ into a path-metric space. We define the distance, d, between any two vertices in $\mathcal{C}(S_g)$ as the minimum of lengths of all the paths between these two vertices.

FACT 6. $(\mathcal{C}(S_g), d)$ is a δ-hyperbolic space.

PROOF. See [**19**]. □

FACT 7. Let μ and λ be curves on S_g. Then, $d(\mu, \lambda) \geq 3$ if and only if μ and λ fill S_g.

PROOF. If μ and λ fill then $i(\mu, \lambda) \neq 0$ and also, there doesn't exist any curve c on S_g that is disjoint from both μ and λ. Thus, $d(\mu, \lambda)$ can't be 1 or 2. Hence, $d(\mu, \lambda) \geq 3$.

Suppose μ and λ don't fill S_g. If $i(\mu, \lambda) = 0$, then $d(\mu, \lambda) = 1$. If $i(\mu, \lambda) \neq 0$, then by the classification of surfaces theorem there exists a non-disc, non-annular component of $S_g \setminus (\mu \cup \lambda)$. Considering a curve in this particular component gives a distance 2 path between μ and λ. Thus, $d(\mu, \lambda) < 3$. □

2.6. Minimal intersection number

For a given distance $n \in \mathbb{N}$ and genus g, $i_{min}(g, n)$ is the quantity defined as

$$i_{min}(g, n) = \min\{i(\alpha, \beta) : d(\alpha, \beta) = n\}.$$

Since any two curves on $\mathcal{C}(S_g)$ which are at a distance 1 apart are disjoint, we have that $i_{min}(g, 1) = 0$. By the classification of surfaces theorem, we can always find curves which intersect once and don't fill S_g. Thus, such curves are at a distance 2 apart on $\mathcal{C}(S_g)$ and hence, $i_{min}(g, 2) = 1$. By Euler characteristic considerations the theoretical minimum for $i_{min}(g, 3)$ is $2g - 1$. In [**4**], the Aougab and Taylor proved that for $g \geq 3$, $i_{min}(g, 3) = 2g - 1$ and $i_{min}(2, 3) = 4$. For showing $i_{min}(g, 3) = 2g - 1$, they used the list of minimally intersecting filling pairs of curves given by Aougab and Huang in [**3**]. In [**10**], the authors using the MICC software showed that $i_{min}(2, 4) = 12$ by listing all minimally intersecting pairs of curves at distance 4. In [**20**], the author provides a pair of distance 4 curves on S_3 with intersection number 21. Thus, $i_{min}(3, 4) \leq 21$. In [**4**], Aougab and Taylor proved that $i_{min}(g, 4) = O(g^2)$

by answering a more general question by Dan Margalit that $i_{min}(g,n) = O(g^{n-2})$. With this information, the following questions still remains open :

Question 7. What is $i_{min}(g,4)$ for $g \geq 3$?

In general,

Question 8. What is $i_{min}(g,n)$ for $n \geq 5$?

2.7. Subsurface projection

We briefly define subsurface projections and state the bounded geodesic theorem which were introduced in detail by Masur and Minsky in [**19**]. Let Y be an isotopy class of an incompressible, non-peripheral, connected proper open subsurface of S_g which is not an annulus. An arc in Y is a homotopy class of properly embedded paths in Y which cannot be deformed to a point. We define the arc complex of Y, $\mathcal{A}(Y)$ as : the set of vertices comprises of arcs and curves in Y and any two vertex share an edge if they are disjoint. Let $\mathcal{A}_0(Y)$ and $\mathcal{C}_0(S_g)$ denote the vertex set of $\mathcal{A}(Y)$ and $\mathcal{C}(S_g)$, respectively. Corresponding to a set X, we use $\mathcal{P}(X)$ to denote the set of finite subsets of X. We define the following two functions :

- $\psi_Y : \mathcal{A}_0(Y) \longrightarrow \mathcal{P}(\mathcal{A}_0(Y))$ such that
  - if α is a curve on Y, $\psi(\alpha) = \{\alpha\}$
  - if α is an arc on Y, $\psi(\alpha)$ are the boundary curves of a neighbourhood of $\alpha \cup \partial(Y)$
- $\pi'_Y : \mathcal{C}_0(S_g) \longrightarrow \mathcal{P}(\mathcal{A}_0(Y))$ such that
  - if $\alpha \cap Y = \phi$, $\pi'_Y(\alpha) = \phi$
  - $\pi'_Y(\alpha)$ is otherwise the set of all the essential arcs in $Y \cap \alpha$.

We define the subsurface projection π_Y by $\pi_Y : \mathcal{C}(S_g) \longrightarrow \mathcal{P}(\mathcal{A}_0(Y))$, $\alpha \mapsto \psi_Y(\pi'_Y(\alpha))$. Let d_Y be a metric on $\mathcal{C}(S_g)$ such that $d_Y(v,w) = dist(\pi_Y(v), \pi_Y(w))$.

Suppose Y is an annular subsurface in S_g whose core curve, γ, is non-trivial. Let $\hat{Y}$ be the natural compactification of the annular cover of S_g such that Y lifts to this cover homeomorphically. Such a compactification is obtained by equipping S_g with a choice of hyperbolic metric. We define the curve graph corresponding to Y, $\mathcal{C}(Y)$, as follows : the set of vertices, $\mathcal{C}_0(Y)$, comprises of paths with end points on the boundary component of $\hat{Y}$, modulo end points fixing homotopies. Any two vertices share an edge if they have disjoint interiors. The subsurface projection π_Y from $\mathcal{C}_0(S_g)$ to $\mathcal{P}(\mathcal{C}_0(Y))$: If $\beta \cap \gamma = \phi$, $\pi_Y(\beta) = \phi$. Otherwise, $\pi_Y(\beta)$ consists of the lifts of arcs of $\beta \cap Y$ in $\hat{Y}$ with well-defined end points on the distinct components of $\partial(\hat{Y})$. We define d_α analogous to d_X where X is a non-annular subsurface.

The *bounded geodesic theorem* was discovered and proved by Masur and Minsky in [**19**]. The version of this theorem stated below is given by Webb in [**27**].

THEOREM 2 (Bounded geodesic theorem). *There is an $M \geq 0$ so that for S_g and any geodesic g in $\mathcal{C}(S_g)$, if each vertex of g meets the subsurface Y, then $diam(\pi_Y(g)) \leq M$.*

2.8. Efficient geodesics in $\mathcal{C}(S_g)$

A fundamental hindrance while studying $\mathcal{C}(S_g)$ is that there are infinitely many distinct vertices adjacent to any vertex in $\mathcal{C}(S_g)$. This pathological property of $\mathcal{C}(S_g)$ is commonly referred to as its local-infinitude. In [**19**], the authors circumvented this local infinitude of $\mathcal{C}(S_g)$ by defining a set of geodesics called the tight geodesics in $\mathcal{C}(S_g)$. They proved that between any two vertices of $\mathcal{C}(S_g)$ there are only finitely many tight geodesics. Similar notions to consider special classes of geodesics have been developed in [**24**], [**25**], [**26**] and [**5**] to overcome the local infinitude of $\mathcal{C}(S_g)$ and compute distances between any two vertices. The algorithm in [**5**] is by far the most effective in calculating small distances in $\mathcal{C}(S_g)$.

Consider a geodesic, $\nu_0, \ldots, \nu_N$ of length $N \geq 3$ in $\mathcal{C}(S_g)$. An arc, ω in S is a *reference arc* for the triple ν_0, ν_1, ν_N if ω and ν_1 are in minimal position and the interior of ω is disjoint from $\nu_0 \cup \nu_N$. Such arcs were considered by Leasure in [**15**]. The authors of [**5**] define the following concept of efficient geodesics in $\mathcal{C}(S_g)$ and prove that there exists finitely many initially efficient geodesic between any two vertices of $\mathcal{C}(S)$. The oriented geodesic $\nu_0, \ldots, \nu_N$ is said to be *initially efficient* if $i(\nu_1, \omega) \leq N - 1$ for all choices of reference arc, ω. Finally, the geodesic $\nu_0, \ldots, \nu_N$ is *efficient* if the oriented geodesic $\nu_k, \ldots, \nu_N$ is initially efficient for each $0 \leq k \leq N - 3$ and the oriented geodesic ν_N, ν_{N-1}, ν_{N-2}, ν_{N-3} is also initially efficient. The following theorem says that between any two vertices in $\mathcal{C}(S_g)$ there are finitely many efficient geodesics.

THEOREM 3. *If v and w are vertices of $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$, then there exists an efficient geodesic from v to w. Further, there is an explicitly computable list of at most n^{6g-6} vertices v_1 that can appear as the first vertex on an initially efficient geodesic*

$$v = v_0, v_1, \ldots, v_n = w.$$

In particular, there are finitely many efficient geodesics from v to w.

PROOF. See [**5**, Theorem 1.1]. □

The following theorem from [**10**] gives a criterion for detecting vertices in $\mathcal{C}(S)$ at distance at-least 4. This criteria is based on the results proved in [**5**] which involves the efficient geodesics.

THEOREM 4. *For the filling pair, κ, ω, let $\Gamma \subset \mathcal{C}^0(S)$ be the collection of all vertices such that the following hold :*

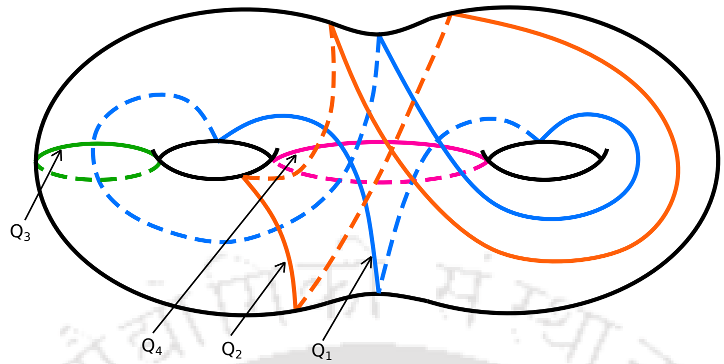


FIGURE 6. Q_1, Q_2, Q_3, Q_4 is an example of an initially efficient geodesic in $\mathcal{C}(S_2)$.

(1) *for* $\overline{\gamma} \in \Gamma$, $d(\kappa, \overline{\gamma}) = 1$; *and*
(2) *for* $\overline{\gamma} \in \Gamma$; *for each segment,* $b \subset \omega \setminus \kappa$, $i(\overline{\gamma}, b) \leq 1$.

Then $d(\kappa, \omega) \geq 4$ *if and only if* $d(\overline{\gamma}, \omega) \geq 3$ *for all* $\overline{\gamma} \in \Gamma$.

PROOF. See [**10**, Theorem 1.3]. $\square$

CHAPTER 3

SETUP

Let α and β be a filling pair of curves on S_g. The goal of this chapter is to pick nice enough representatives for $T_\beta(\alpha)$ w.r.t. the representatives of α, β and their annular neighbourhoods. In section 3.1 we pick suitable representatives for α, β and their corresponding annular neighbourhoods. In section 3.2, we describe the arcs of $T_\beta(\alpha)$ in the components common to the chosen annular neighbourhoods of α and β. Finally, in section 3.3, we show that α and $T_\beta^p(\alpha)$ fill S_g, for $p \in \mathbb{N}$.

This chapter comprises of results from [**17**, Section 2] and [**17**, Step 1, Section 3].

For any ordered index in this thesis, we follow cyclical ordering. For instance, if $i \in \{1, 2, \ldots, k\}$, $i = k + 1$ will indicate $i = 1$.

3.1. Amenable to Dehn twist in special position

Let λ and μ be two simple closed curves on S_g, R_λ and R_μ be closed regular neighbourhoods of λ and μ respectively. We choose R_λ and R_μ to be nice enough as described in the following definition and follow the algorithm in [**9**] to obtain $T_\lambda(\mu)$. However, our representative of the isotopy class of $T_\lambda(\mu)$ is chosen such that $T_\lambda(\mu)$ is linear in the components of $R_\lambda \cap R_\mu$.

Definition 1. Let λ and μ be two simple closed curves on S_g and let R_λ and R_μ be closed regular neighbourhoods of λ and μ respectively. We say that the 4-tuple $(\lambda, \mu, R_\lambda, R_\mu)$ is *amenable to Dehn twist in special position* if the following hold:

(1) λ and μ intersect transversely and minimally on S_g,
(2) λ and μ fill S_g,

(3) the number of components of $R_\lambda \cap R_\mu$ is equal to the intersection number of λ and μ and each of these components is a disc.
(4) μ and λ are in minimal position with the components of $\partial(R_\lambda)$ and $\partial(R_\mu)$, respectively.

Let λ and μ be two minimally and transversely intersecting simple closed curves which fill S_g. In the following, we show that there exist closed regular neighborhoods R_λ and R_μ of λ and μ respectively such that the 4-tuple $(\lambda, \mu, R_\lambda, R_\mu)$ is amenable to Dehn twist in special position. Consider a closed regular neighborhood, R_λ, of λ. The two components of ∂R_λ are disjoint simple closed curves each of which is isotopic to λ on S_g. Since μ intersects λ transversely, we can assume that μ intersects the closed annulus R_λ in essential arcs which are not boundary reducible. So, the number of these arcs will be precisely as many as the intersection number of λ and μ. We can take small closed regular neighborhoods of these arcs of $\mu \cap R_\lambda$ in R_λ such that each such neighborhood is a *rectangular* disk, the length of which runs parallel to the arcs of μ and the two breadth lines of which lie on the boundary curves of R_λ with each breadth line lying on a different component of ∂R_λ. The number of these disks is precisely the intersection number of λ and μ. Let r_1, r_2, $\dots$, r_k denote these discs. For $i \in \{1, \dots k\}$, we call any component of $r_i \cap \partial R_\lambda$ as the breadth line of r_i. Now we extend r_i's into $S_g \setminus R_\lambda$ to form R_μ. To do this, we take a disks-with-handles presentation, Σ, of $S_g \setminus R_\lambda$. Σ is homeomorphic to $S_{g-1,2}$, via a homeomorphism ϕ, where each of the two boundary components of Σ is an image of each of the boundary component ∂R_λ via ϕ. Let $\mathcal{A}$ be a maximal collection of properly embedded essential arcs that are pairwise non-parallel in Σ. The image under ϕ of the closure of each arc of μ contained in the complement of the annulus R_λ is an essential arc in Σ. These arcs cut Σ into disks because λ and μ fill S_g. These arcs of $\phi(\mu)$ can be assumed to be a disjoint collection of arcs, each of which is parallel to exactly one of the arcs in $\mathcal{A}$. We now take small closed regular neighborhoods of these k arcs in Σ, call them s_1, s_2 $\dots$, s_k, such that these s_i's are mutually disjoint. Now, ϕ^{-1} of these s_i's glue to r_j's in some order along the breadth lines of r_j's by suitably adjusting the breadth of r_j's to give a regular neighbourhood of μ, R_μ. This completes the construction of R_μ as required. Note that r_1, r_2, $\dots$, r_k are the disks of intersection of R_λ and R_μ by construction and their number is equal to the intersection number of λ and μ.

3.2. Discs of transformation

Consider a 4-tuple $(\lambda, \mu, R_\lambda, R_\mu)$ which is amenable to Dehn twist in special position. Let $i(\lambda, \mu) = k$ and $K := \{1, 2, ..., k\}$. We construct a curve in the isotopy class of $T_\lambda(\mu)$ which we call $T_\lambda(\mu)$ *in special position w.r.t. the* 4*-tuple*

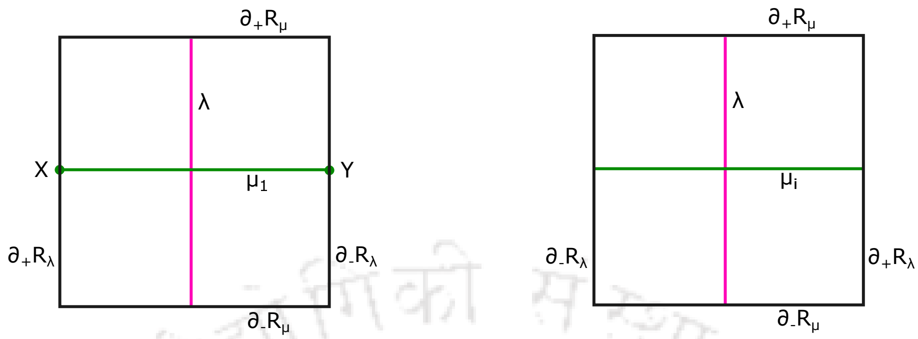


FIGURE 1. The figure to the left depicts A_1 and the figure to the right depicts a possible A_i.

$(\lambda, \mu, R_\lambda, R_\mu)$. Start at any one of the components of $R_\lambda \cap R_\mu$ and label it as A_1. Since μ intersects λ transversely, the arc μ_1 of μ contained in A_1 which has its endpoints X and Y on boundary arcs of R_λ is such that X and Y lie on distinct boundary components of ∂R_λ. We call the component of ∂R_λ containing X to be $\partial_+ R_\lambda$ and the other component containing Y to be $\partial_- R_\lambda$. Equip A_1 with the Euclidean metric such that it is a square in the $xy-$ plane. Two opposite sides of A_1 are formed from the arcs of ∂R_λ and the two remaining sides are formed from arcs of ∂R_μ. The x-axis lies along μ_1 and the value of the x-coordinate increases from X to Y. Orient μ_1 from X to Y. This induces an orientation on μ. Next we pick k distinct points $\{q_1, q_2, ..., q_k\}$ in the interior of μ_1 such that the x-coordinate of q_i is greater than the x coordinate of q_j whenever $i > j$ and $i, j \in K$. For each $i \in K$, let λ_i be a curve in R_λ which is isotopic to λ and passes through q_i. Further for each $i, j \in K, i \neq j$ let λ_i and λ_j be disjoint.

Orient λ_1 such that the y-coordinate on λ_1 increases when following this orientation in the disk A_1. Starting with A_1, label the subsequent disk components, $R_\lambda \cap R_\mu$, as $A_2, A_3, ..., A_k$, in the orientation of λ_1. For each $i \in K$, A_i contains a unique arc of μ which we label as μ_i. μ_i gets an induced orientation from μ. For each $i \in K$, equip A_i with Euclidean metric and assume it to be a square in the xy-plane where μ_i lies along the x-axis with the x coordinate increasing along the orientation of μ_i. Assume A_i to be positioned such that μ_i is the line segment joining the mid-points of the left and right sides of the square. In this orientation, call the component of ∂R_μ which appears above μ_i as $\partial_+ R_\mu$ and the the component of ∂R_μ below μ_i as $\partial_- R_\mu$. However, note that the side of A_i which is formed from the arcs of $\partial_+ R_\lambda$ could either be to the right or to the left of this square. Accordingly, the side of A_i which is formed from the arcs of $\partial_- R_\lambda$ could either be to the left or to the right of this square. For $i, j \in K$, by an isotopy inside A_i, we can assume that all the arcs of λ_j in A_i are straight lines.

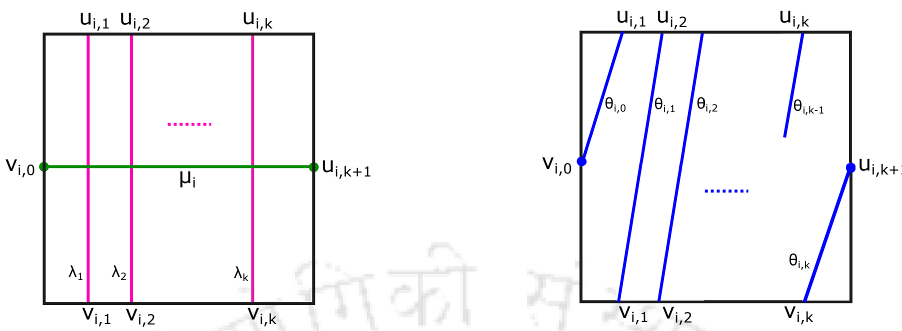


FIGURE 2. Disk of transformation before (figure on the left) and after (figure on the right) the Dehn twist

For each $i, j \in K$, let $u_{i,j} := A_i \cap \lambda_j \cap \partial_+ R_\mu$ and $v_{i,j} := A_i \cap \lambda_j \cap \partial_- R_\mu$. Also for each $i \in K$ let the left end point of μ_i in the square A_i be $v_{i,0}$ and the right end point of μ_i in the square A_i be $u_{i,k+1}$. Construct the Dehn twist of μ about λ as follows: For each $j \in K \cup \{0\}$ draw line segments, $\theta_{i,j}$, connecting $v_{i,j}$ to $u_{i,j+1}$. $T_\lambda(\mu)$ is the curve

$$((\mu \cup (\cup_{i \in K} \lambda_i)) \cap (S_g \setminus (\cup_{i \in K} A_i)) \cup (\cup_{i,j \in K} \theta_{i,j}).$$

The schematic, Figure 2, shows A_i before and after this transformation. In the complement of A_i's the transformation described above does not disturb the curves λ_i's and μ. In the previous chapter, an algorithm to obtain the Dehn twist, $T_\lambda(\mu)$ has been described such that the curves in the discs of transformation are as in Figure 3. The line segments in Figure 2 are isotopic to the corresponding curves in 3 which shows that the above transformation indeed results in $T_\lambda(\mu)$. When $T_\lambda(\mu)$ is constructed as above and as shown in Figure 2, we say that $T_\lambda(\mu)$ *is in special position w.r.t.* λ *and* μ. We call the k copies of λ, λ_i, $i \in K$, and μ to be the *scaffolding for* $T_\lambda(\mu)$ and denote it by $(\{\lambda_i\}_{i \in K}, \mu)$. We call the Euclidean disks A_i, $i \in K$, along with the line segments $\theta_{i,j}$'s for $j \in K$ to be the *disks of transformation for* $T_\lambda(\mu)$. The points $u_{i,j}$'s, $v_{i,j}$'s, $u_{i,k+1}$ and $v_{i,0}$ for $i, j \in K$ shall hold their meaning as defined in the context of the disks of transformations. So, using these phrases, when $T_\lambda(\mu)$ is in special position w.r.t. λ and μ, the scaffolding of $T_\lambda(\mu)$ remains unchanged outside its disks of transformation. Inside the disks of transformation for $T_\lambda(\mu)$, the schematic in Figure 2 describes the changes to its scaffolding.

3.3. Filling pairs of curves using Dehn twists

THEOREM 5. *Let α and γ be a filling pair of curves on S_g. Then, α and $T_\gamma^p(\alpha)$ also fills S_g.*

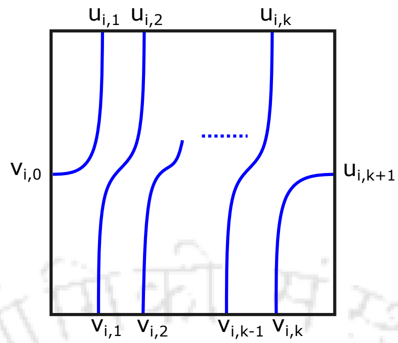


FIGURE 3. Surgery of the curves to obtain $T_\lambda(\mu)$

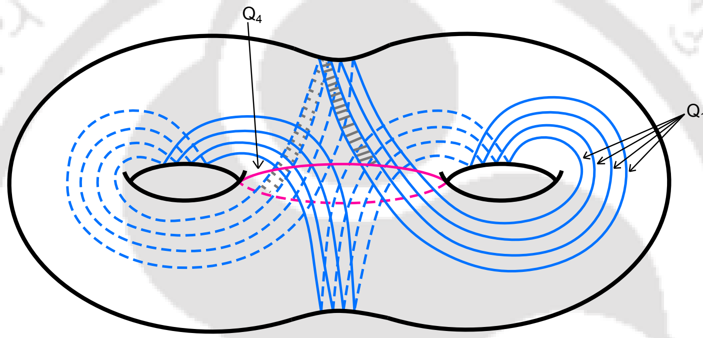


FIGURE 4. The scaffolding for $T_{Q_1}(Q_4)$, where Q_1 and Q_4 are from example in figure 6 and the shaded region is a rectangle of the scaffolding

PROOF. Let $i(\gamma, \alpha) = k$, $K := \{1, 2, ..., k\}$, $K_{-1} := \{1, 2, ..., k-1\}$ and $K_{2-2g} := \{1, 2, ..., k+2-2g\}$. We prove the theorem for $p = 1$. For $p > 1$, the proof remains as it is with just the arguments for k copies of γ replaced by pk copies of γ. The terminologies in the previous section can be adjusted accordingly to account for the pk copies of γ instead of k copies of γ. This is because the following proof relies on the idea of the surgery of curves, α and copies of γ, to obtain $T_\gamma(\alpha)$. And the surgery to obtain $T^p_\gamma(\alpha)$ is similar to this surgery.

Since α and γ fill S_g, there is a 4-tuple $(\alpha, \gamma, R_\alpha, R_\gamma)$ which is amenable to Dehn twist in special position. Let $T_\gamma(\alpha)$ be in special position w.r.t to α and γ. We denote the disks of transformation of $T_\gamma(\alpha)$ by A_i for $i \in K$. By an isotopy we assume the curve α to be disjoint from $T_\gamma(\alpha) \setminus A_i$ for $i \in K$ and in each A_i we further assume the arc $\alpha_i := \alpha \cap A_i$ to be a straight line segment below the segment connecting $v_{i,0}$ and $u_{i,k+1}$ (below μ_i in Figure 2).

Let $(\{g_i\}_{i\in K}, \alpha)$ be the scaffolding for $T_\gamma(\alpha)$. For $j \in K_{-1}$, one of the components of $S_g \setminus (g_j \cup g_{j+1})$ is an annulus, G_j. The core curve of the annuli G_j is isotopic in S_g to λ. Any component of $G_j \setminus \alpha$ is a 4-gon which we call as *a rectangle*

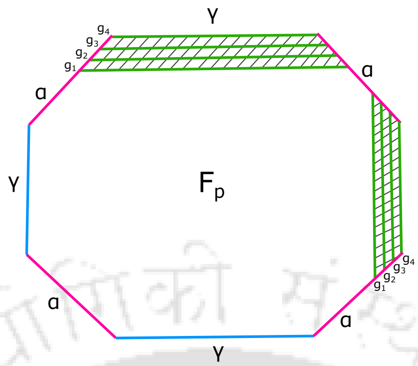


FIGURE 5. The shaded portion represent the portions of B along two edges corresponding to γ in F_p. The complement of the shaded portion in F_p is F'_p.

of the scaffolding for $T_\gamma(\alpha)$. Figure 4 shows an example of such a rectangle of the scaffolding. The disks A_i, $i \in K$, further divide each rectangle of the scaffolding into three components. There is a unique $i \in K$ such that A_i and A_{i+1} intersect a given rectangle of the scaffolding. Denote a rectangle of the scaffolding formed out of G_j with its arcs of α lying in A_i and A_{i+1} by $B_{i,j}$. Denote the sub-rectangles $B_{i,j} \cap A_i$, by $C'_{i,j}$ and $B_{i,j} \cap A_{i+1}$, by $C''_{i+1,j}$. Also let $B'_{i,j} := B_{i,j} \setminus (C'_{i,j} \cup C''_{i+1,j})$. Let

$$B = \cup_{i=1}^{k} \cup_{j=1}^{k-1} B_{i,j}.$$

$S_g \setminus (\alpha \cup \gamma)$ has $k + 2 - 2g$ disk components by Euler characteristic considerations. If F_p is a disk component of $S_g \setminus (\alpha \cup \gamma)$, for some $p \in K_{2-2g}$, then $F'_p := F_p \setminus B$ is a single disk as B intersects any F_p only in disks which contain a boundary arc of F_p, namely arcs of γ. Figure 5 is a schematic of possible portions of B in F_p. The components of $S_g \setminus (\alpha \cup g_1 \cup \cdots \cup g_k)$ comprise of $k(k-1)$ rectangles of the scaffolding for $T_\gamma(\alpha)$, namely $B_{i,j}$ where $i \in K$, $j \in K_{-1}$, and $k + 2 - 2g$ even sided polygonal discs, namely F'_p, where $p \in K_{2-2g}$. Let F''_p denote $F'_p \setminus R_\alpha$ for $p \in K_{2-2g}$.

For each $j \in K$ let $w_{i,j} := \theta_{i,j} \cap \alpha_i$. For each $i \in K$ and $j \in K_{-1}$, let $D''_{i,j}$ be the parallelogram with vertices $v_{i,j}, v_{i,j+1}, w_{i,j}$ and $w_{i,j+1}$ and $D'_{i,j+1}$ be the parallelogram with vertices $w_{i,j}, w_{i,j+1}, u_{i,j+1}, u_{i,j+2}$. In each disk A_i, for $i \in K$, there is a pentagon, $P_{i,1}$, which is above α_i and bounded by the lines $\theta_{i,0}$, ∂R_γ, α_i, $\theta_{i,1}$ and the line segment of $\partial_+ R_\alpha$ between $u_{i,1}$ and $u_{i,2}$. Likewise, in each disk A_i, for $i \in K$, there is a triangle, $T_{i,k+1}$, which is bounded by the lines α_i, $\theta_{i,k}$ and ∂R_γ. Figure 6 shows a schematic before and after the transformation to the disk A_i; the figure to the left shows the rectangles $C'_{i,1}$ and $C''_{i,k}$ and the figure on the right shows $P_{i,1}$ and $T_{i,k+1}$.

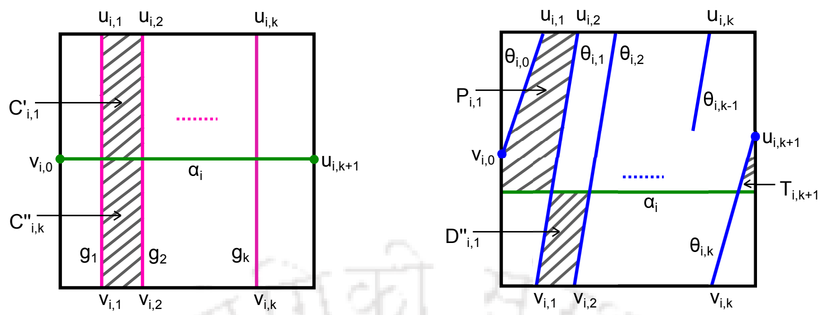


FIGURE 6. The disk of transformation for $T_\gamma(\alpha)$: the figure on the left shows the portion of the scaffolding for $T_\gamma(\alpha)$; the figure on the right shows the pentagon $P_{i,1}$, the triangle $T_{i,k+1}$ and the parallelograms formed due to α_i and $T_\gamma(\alpha)$

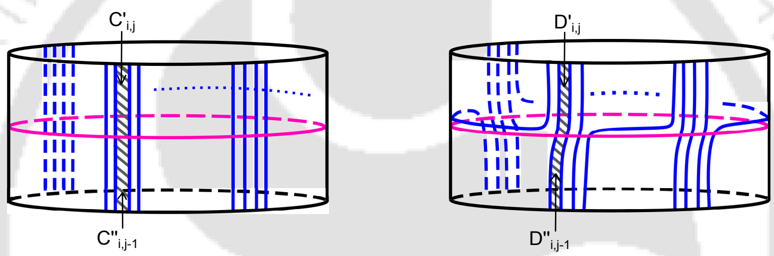


FIGURE 7. A schematic of R_α (figure on the left) and after (figure on the right) the Dehn twist

Figure 7 shows a schematic of R_α before and after the transformation to the scaffolding of $T_\gamma(\alpha)$. The shaded region in the figure on the left shows $C'_{i,j}$ and $C''_{i,j-1}$ for some indices i, j. The shaded region in the figure on the right shows $D'_{i,j}$ and $D''_{i,j-1}$ for some indices i, j.

For $i \in K$, note that all the disks A_i, occur in some sequence in the annulus R_α when moving along α. So, a disk A_i is connected to some disk A_j on the left and to some other disk A_p on the right by a single arc of $\alpha \setminus R_\gamma$, for some distinct indices $i, j, p \in K$. The schematic for two disks A_i and A_j, for some $i, j \in K$, which are connected via a single arc of $\alpha \setminus R_\gamma$ and an arc of $T_\gamma(\alpha) \setminus R_\gamma$ is as shown in the Figure 8. Note that this schematic is generic since for every $j \in K$, there is a distinct $i \in K$ such that A_j occurs to the left of A_i, in the sense mentioned above.

Figure 8 is a schematic of a portion of Figure 7 in which the following are the possibilities of how the edges corresponding to $\partial(R_\gamma)$ of the adjacent discs of transformation match up, namely ∂_+R_γ and ∂_+R_γ face each other, ∂_+R_γ and ∂_-R_γ face each other or ∂_-R_γ and ∂_-R_γ face each other. In this schematic, we see that the pentagon $P_{i,1}$ of the disk A_i is connected to the triangle $T_{j,k+1}$ of A_j via an arc of $\alpha \setminus R_\gamma$, $\omega_{i,j}$, and an arc of $T_\gamma(\alpha)$, $\eta_{i,j}$. The disk, $R_{i,j}$ outside R_γ bounded

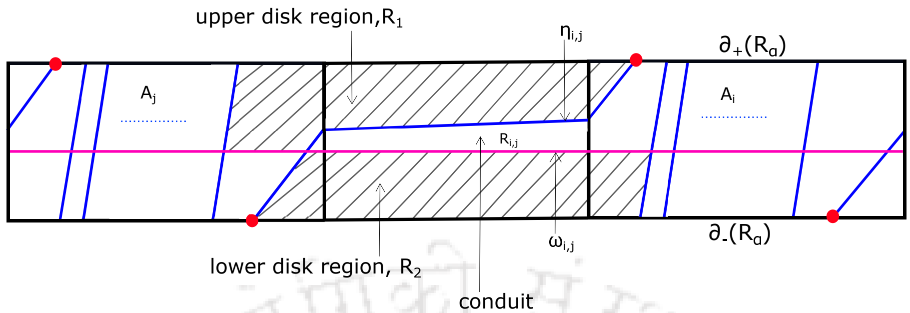


FIGURE 8. Two adjacent disks of transformation in R_α

by $\omega_{i,j}$, $\eta_{i,j}$ and two arcs of ∂R_γ, will be called a *conduit*. Equip the conduit with the Euclidean metric and assume that $R_{i,j}$ is a rectangle with two opposite sides $\omega_{i,j}$ and $\eta_{i,j}$. Now $P_{i,1} \cup R_{i,j} \cup T_{j,k+1}$ is a 4-gon bounded by four arcs *viz.* (i) $\theta_{i,0} \cup \eta_{i,j} \cup \theta_{j,k}$, (ii) $\alpha_j \cup \omega_{i,j} \cup \alpha_i$, (iii) $\theta_{i,1}$ and (iv) the arc of $\partial_+ R_\alpha$ between $u_{i,1}$ and $u_{i,2}$. This protracted 4-gon will be denoted by $D'_{i,1}$.

Let $S' = S_g \setminus R_\alpha$. The components of $S_g \setminus (\alpha \cup T_\gamma(\alpha))$ are the components of $S' \setminus T_\gamma(\alpha)$ and the components of $R_\alpha \setminus (\alpha \cup T_\gamma(\alpha))$ glued at the boundary of R_α. Since the changes to the scaffolding of $T_\gamma(\alpha)$ is restricted to R_α, the components of $S' \setminus T_\gamma(\alpha)$ are precisely the disc components of $S' \setminus (g_1 \cup \cdots \cup g_k)$.

The components of $S' \setminus (g_1 \cup \cdots \cup g_k)$ are $B'_{i,j}$, $i \in \{1, 2 \dots k\}$, $j \in K_{-1}$, along with disks F''_p, $p \in K_{2-2g}$, as explained above. The components of $R_\alpha \setminus (\alpha \cup T_\gamma(\alpha))$ will be examined using the schematic Figure 8 of a portion of R_α. There are four kinds of regions in R_α. The upper disk regions, like R_1 in the schematic Figure 8, the lower disk regions, like R_2 in the schematic Figure 8, and the disks $D'_{i,j}$, $D''_{i,j}$, $i \in K$, $j \in K_{-1}$. Figure 8 shows how the upper and lower disk regions are glued to disks F''_p for $p \in K_{2-2g}$. For each $p \in K_{2-2g}$, after gluing the lower disk regions and the upper disk regions to the respective disks F''_p, we get disks which we denote by F'''_p. We know that F'''_p is a disk because the upper and the lower disk regions are disjoint, except for the points $w_{i,j}$ on the boundary and share a single arc of ∂R_α with a unique F''_p. For each $p \in K_{2-2g}$, we call F'''_p to be the *modified disk* corresponding to the initial disk F_p.

For each $i \in K$ and $j \in K_{-1}$, the line segment of $\partial_+ R_\alpha$ between $u_{i,j}$ $u_{i,j+1}$ is the common boundary of $C'_{i,j}$ and $D'_{i,j}$. Likewise, for each such i, j, the line segment of $\partial_- R_\alpha$ between $v_{i,j}$ $v_{i,j+1}$ is the common boundary of $C''_{i,j}$ and $D''_{i,j}$. So, for such i, j, when considering the components of $S_g \setminus (\alpha \cup g_1 \cup \cdots \cup g_k)$ the rectangular core $B'_{i,j}$ is connected to $C'_{i,j}$ along the boundary segment $u_{i,j}$ $u_{i,j+1}$ and to $C''_{i+1,j}$ along the boundary segment $v_{i+1,j}$ $v_{i+1,j+1}$, whereas when considering the components of $S_g \setminus (\alpha \cup T_\gamma(\alpha))$, the rectangular core $B'_{i,j}$ is connected to $D'_{i,j}$ along the boundary segment $u_{i,j}$ $u_{i,j+1}$ and $D''_{i+1,j}$ along the boundary segment $v_{i+1,j}$

$v_{i+1,j+1}$. So the rectangles of the scaffolding for $T_\gamma(\alpha)$, $B_{i,j}$, which are components of $S_g \setminus (\alpha \cup g_1 \cup \cdots \cup g_k)$, after the transformation in the disks of transformation for $T_\gamma(\alpha)$ result in disks $E_{i,j} := B'_{i,j} \cup D'_{i,j} \cup D''_{i+1,j}$ which now are components of $S_g \setminus (\alpha \cup T_\gamma(\alpha))$. For each $p \in K_{2-2g}$, F'''_p is a disk as seen earlier. The components of $S_g \setminus (\alpha \cup T_\gamma(\alpha))$ are precisely the disks F'''_p and $E_{i,j}$ where $p \in K_{2-2g}$, $i \in K$ and $j \in K_{-1}$. This proves that the components of $S_g \setminus (\alpha \cup T_\gamma(\alpha))$ are all disks and hence proving $d(\alpha, T_\gamma(\alpha)) \geq 3$.

Thus, we have that α and $T_\gamma(\alpha)$ fills S_g.

□

CHAPTER 4

DISTANCE 4 CURVES ON $\mathcal{C}(S_g)$

In this chapter we produce examples of pairs of curves which are at a distance 4 apart on $\mathcal{C}(S_g)$ using Dehn twists and a pair of curves which are at a distance 3 apart. Let α and β be a pair of curves on S_g such that $d(\alpha, \beta) = 3$. In section 4.1, we label a few components of an annular neighbourhood of β cut along $T_\beta(\alpha) \cup \alpha$. In section 4.2, we prove that $d(\alpha, T^p_\beta(\alpha)) = 4$, for $p \in \mathbb{N}$. Using this result, in subsection 4.2.1, we show that $i_{min}(g, 4) \leq (2g-1)^2$. In subsection 4.2.2, we prove that the geodesic we consider between α and $T_\beta(\alpha)$ is an initially efficient geodesic.

Unless otherwise defined, we will adhere to the notations used in the previous chapters.

4.1. Terminology

Let α and γ be a filling pair of curves on S_g. Thus, $d(\alpha, \gamma) \geq 3$. Let $i(\gamma, \alpha) = k$, $K := \{1, 2, ..., k\}$, $K_{-1} := \{1, 2, ..., k-1\}$ and $K_{2-2g} := \{1, 2, ..., k+2-2g\}$. Since α and γ fill S_g, there is a 4-tuple $(\alpha, \gamma, R_\alpha, R_\gamma)$ which is amenable to Dehn twist in special position. Let $T_\gamma(\alpha)$ be in special position w.r.t to α and γ. We denote the disks of transformation of $T_\gamma(\alpha)$ by A_i for $i \in K$.

The components of $R_\gamma \setminus T_\gamma(\alpha)$ are disks and their boundary consists of two arc segments of $T_\gamma(\alpha)$ and one each of $\partial_+ R_\gamma$ and $\partial_- R_\gamma$. We call these disks as *rectangular tracks*. The word tracks derives its motivation from how these tracks appear in R_γ. Figure 1 shows R_γ and rectangular tracks inside R_γ.

Since $i(\alpha, \gamma) = k$, there are k components of $\alpha \cap R_\gamma$. Every component of $\alpha \setminus T_\gamma(\alpha)$ is either contained in R_γ or, has a sub-arc which is contained in R_γ. For any $i \in K$, α_i intersects the rectangular tracks.

Let $i_0 \in K$. In the schematic Figure 6, A_{i_0} has exactly $k+1$ arcs of $T_\gamma(\alpha)$. Call $\theta_{i_0,0}$ to be the *leftmost arc of* A_{i_0} and $\theta_{i_0,k}$ to be the *rightmost arc of* A_{i_0}.

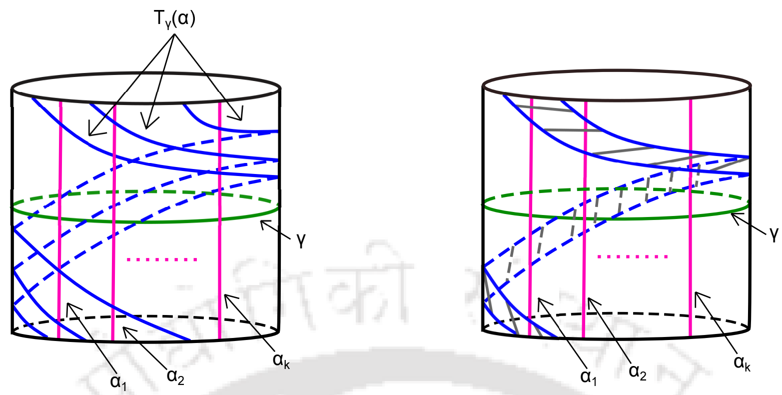


FIGURE 1. The rectangular tracks shown inside the annulus R_γ

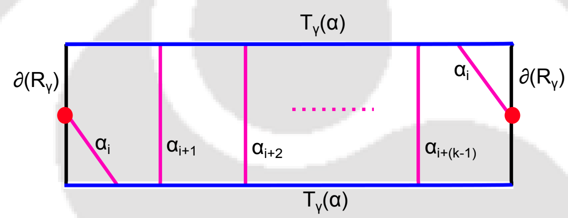


FIGURE 2. A rectangular track T_i along with arcs of α_i in it

Let us consider one component of $T_\gamma(\alpha) \cap R_\gamma$, call it ρ_{i_0}, which intersects A_{i_0} in its leftmost arc. This ρ_{i_0} intersects A_{i_0} precisely in the arcs $\theta_{i_0,0}$ and $\theta_{i_0,k}$ and it intersects A_j for every $j \in K \setminus \{i_0\}$ in the arcs $\theta_{j,m}$ where $m = (j - i_0)(mod\ k)$. We constructed $T_\gamma(\alpha)$ in special position w.r.t. α and γ with the motivation that ρ_{i_0} will intersect A_{i_0} and A_j in exactly these arcs.

From this discussion it is clear that ρ_{i_0} intersects each α_j, for $j \in K$, exactly once. It is also clear that, for $j \in K$, the points of $\rho_{i_0} \cap \alpha_j$ lie on ρ_{i_0} in the order $\alpha_{i_0+1}, ..., \alpha_k, \alpha_1, ..., \alpha_{i_0-1}, \alpha_{i_0}$ when ρ_{i_0} is traversed from $\partial_+ R_\gamma$ to $\partial_- R_\gamma$. We now consider two arc components, ρ_{i_0} and ρ_{i_0+1}, of $T_\gamma(\alpha) \cap R_\gamma$ and the rectangular track, T_{i_0}, which is enclosed by these two components in R_γ. We equip this rectangular track T_{i_0} with the Euclidean metric so that the boundary arcs ρ_{i_0}, ρ_{i_0+1}, and the arcs of $T_{i_0} \cap \partial R_\gamma$ are all straight lines and so that T_{i_0} is a rectangle. We refer to $T_{i_0} \cap \partial_+ R_\gamma$ as the left end of the rectangle and $T_{i_0} \cap \partial_- R_\gamma$ as the right end of this rectangular track. We can draw the arcs of α_j, for $j \in K$, as straight line segments in the rectangular tracks T_{i_0}. Figure 2 shows a schematic of T_i where $i \in K$.

From this schematic, at both the left and right end of this rectangular track T_i, a_i is a common boundary to a triangle and a pentagon. We call α_i as the *starting arc* of this rectangular track T_i.

Figure 3 shows the two possible schematics of A_i as pictured in R_γ.

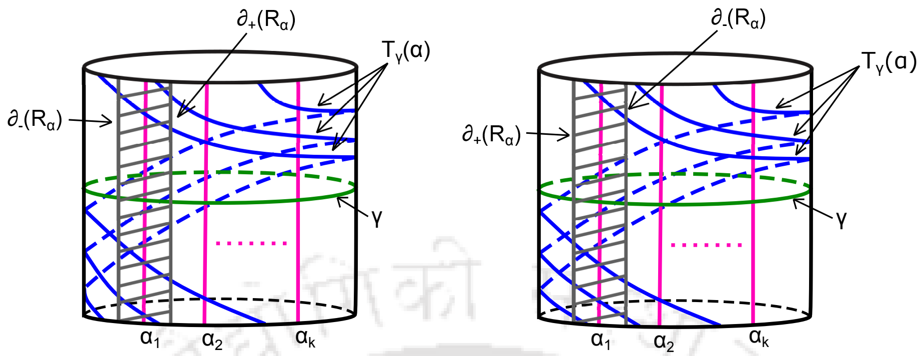


FIGURE 3. A_i shown inside R_γ in the two possible ways : the figure on the left shows α_i oriented from top to bottom; the figure on the right shows α_i oriented from bottom to top

For either of the two possible cases observed in Figure 3, a portion of one of the two pentagons of T_i appears in the A_i which is between α_i and $\partial_+ R_\alpha$, where α_i is the starting arc of this track. We call this pentagon *the upper pentagon of the rectangular track* T_i, owing to the viewpoint that $\partial_+ R_\alpha$ is the upper boundary of R_α. A portion of the other pentagon of T_i appears in A_i which is between α_i and $\partial_- R_\alpha$. We call this pentagon *the lower pentagon of the rectangular track*. Likewise, we define the *upper triangle* and the *lower triangle* of a rectangular track T_i.

4.2. Distance 4 curves

THEOREM 6. *If α and γ be two curves on S_g with $d(\alpha, \gamma) = 3$, then $d(\alpha, T_\gamma(\alpha)) =$ 4.*

PROOF. Let ν_0, ν_1, ν_2, ν_3 be a geodesic from the vertex ν_0 corresponding to α to the vertex ν_3 corresponding to γ in $\mathcal{C}(S_g)$. Let $T_\gamma(\nu_0)$ be the vertex in $\mathcal{C}(S_g)$ corresponding to $T_\gamma(\alpha)$. The existence of the path $T_\gamma(\nu_0)$, $T_\gamma(\nu_1)$, $T_\gamma(\nu_2) = \nu_2$, ν_1, ν_0 gives that $d(T_\gamma(\alpha), \alpha) \leq 4$

Let $\overline{\gamma} \in \Gamma$ as in the statement of the Theorem 4. We prove that $d(\overline{\gamma}, \alpha) \geq 3$ by showing that $\overline{\gamma}$ and α fill S_g. By Theorem 4, this will imply that $d(T_\gamma(\alpha), \alpha) \geq 4$. We prove the theorem in 2 steps : in step 1 we perform an isotopy of $\overline{\gamma}$ such that the arcs of $\overline{\gamma} \setminus \alpha$ in R_γ replicate the arcs of $\gamma \setminus \alpha$. In step 2 we prove that any $\overline{\gamma} \in \Gamma$ fills S_g with α and thus $d(\overline{\gamma}, \alpha) \geq 3$.

Step 1 : To prove that α and $\overline{\gamma}$ fill S_g it suffices to show that there exists a non-empty, finite subset of arcs of $\overline{\gamma} \setminus \alpha$, Υ, such that $(S_g \setminus \alpha) \setminus \Upsilon$ are discs. We show the existence of Υ by carefully choosing an isotopic copy of $\overline{\gamma}$. We obtain this isotopic copy of $\overline{\gamma}$ by first performing an isotopy, I_1, a finite number of times such that the end points of each of the arcs of $\overline{\gamma}$ in R_γ are essential and not boundary reducible.

We then define a second isotopy, I_2, which when performed a finite number of times will ensure that every intersection point of $\overline{\gamma}$ and α lie in R_γ. We then define an isotopy, I_3, called the normalization move, which proves that there is an arc of $\overline{\gamma}$ in R_γ, $\overline{\gamma}_0$, such that the arcs $\overline{\gamma}_0 \setminus \alpha$ act like the arcs of $\gamma \setminus \alpha$ in $S_g \setminus \alpha$.

It can be observed that $i(\overline{\gamma}, \alpha) \neq 0$ because if $\overline{\gamma}$ is disjoint from both α and $T_\gamma(\alpha)$ then we would get a path of length 2, namely α, $\overline{\gamma}$, $T_\gamma(\alpha)$. Using the triangular inequality and the fact that $Mod(S_g)$ acts on $\mathcal{C}(S_g)$ by isometries, we have that $d(\overline{\gamma}, \gamma) \geq d(T_\gamma(\alpha), \gamma) - d(\overline{\gamma}, T_\gamma(\alpha)) = d(T_\gamma(\alpha), T_\gamma(\gamma)) - d(\overline{\gamma}, T_\gamma(\alpha)) = 3 - 1 = 2$. Thus, we also conclude that $i(\overline{\gamma}, \gamma) \neq 0$. Since $\overline{\gamma}$ intersects γ, it intersects R_γ. It cannot be completely contained in R_γ because every simple closed curve contained in an annulus bounds a disk or is isotopic to the core curve of the annulus. Since neither of these is true, it follows that that $\overline{\gamma}$ intersects R_γ in arcs. Since $i(\overline{\gamma}, T_\gamma(\alpha)) = 0$, each component of $\overline{\gamma} \cap R_\gamma$ has to be completely contained in one of the rectangular tracks described by $T_\gamma(\alpha)$. Such a component arc of $\overline{\gamma}$ could either be boundary reducible or essential in R_γ.

We consider an isotopy I_1 of $\overline{\gamma}$, as follows: In the case that a component arc of $\overline{\gamma}$ in R_γ is boundary reducible in R_γ, we can perform the boundary reduction of $\overline{\gamma}$ preserving its minimal intersection position with α and $T_\gamma(\alpha)$. This is possible because an arc of $\overline{\gamma}$ which is boundary reducible in R_γ and is contained in the disk T_i will bound a bigon with one boundary arc of R_γ in T_i. Also, since $\overline{\gamma}$ was already in minimal intersection position with α, it does not bound bigons with the arcs α_j inside T_i. Call the isotopy of $\overline{\gamma}$ which reduces all the boundary-reducible arcs of $\overline{\gamma} \cap R_\gamma$ as I_1. After the isotopy I_1, we can assume that all the arcs of $\overline{\gamma}$ in R_γ are essential. We know that there is at-least one component of $\overline{\gamma} \cap R_\gamma$ which is an essential arc of R_γ as $\overline{\gamma}$ cannot be disjoint from R_γ. By the hypothesis that $i(\overline{\gamma}, b) \leq 1$ for $b \subset \alpha \setminus T_\gamma(\alpha)$ each rectangular track can contain at-most one component of $\overline{\gamma} \cap R_\gamma$.

Next, we describe an isotopy I_2 of $\overline{\gamma}$ such that all the points of $\overline{\gamma} \cap \alpha$ will lie inside R_γ and so that no new boundary reducible arc components of $\overline{\gamma} \cap R_\gamma$ are introduced and $\overline{\gamma}$'s minimal intersection position with α and $T_\gamma(\alpha)$ is retained. To this end, suppose that a point of $\overline{\gamma} \cap \alpha$ lies outside R_γ.

Following the construction of the disk $D'_{i,1}$ described above using Figure 8, we see that the upper pentagon of the rectangular track T_i is connected to the upper triangle of the rectangular track T_j via a conduit $R_{i,j}$ where $i, j \in K$ are such that A_j is to the left of A_i in R_α as in schematic 8.

If a point of $\overline{\gamma} \cap \alpha$, x_0, lies outside R_γ, then it has to lie on $\omega_{i,j}$ for some i and j such that $i, j \in K, i \neq j$. We now refer to the dotted line in Figure 4. Since the intersection of $\overline{\gamma}$ and α is transverse, an arc of $\overline{\gamma}$, call it δ lies on the two sides of the conduit $R_{i,j}$, one inside and one outside $R_{i,j}$. The endpoint P of the arc δ

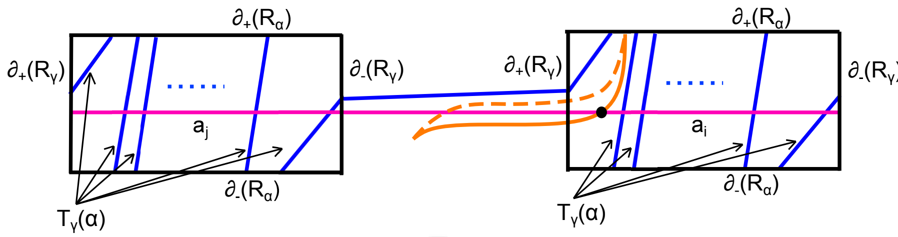


FIGURE 4. The isotopy I_2 moving points of $\overline{\gamma} \cap \alpha$ into R_γ

inside $R_{i,j}$ is also the endpoint of some other arc of $\overline{\gamma}$ as $\overline{\gamma}$ is a closed curve. If P connects to an arc of $\overline{\gamma}$ lying in the upper triangular region of the track T_j, then an essential arc δ_1 of $\overline{\gamma} \cap R_\gamma$ lies in T_j with its endpoint Q on ∂R_γ in the upper triangle of T_j so that δ, the arc PQ and δ_1 together form a bigon with α contradicting the minimal intersection position of $\overline{\gamma}$ with α. So, P connects to an arc of $\overline{\gamma}$ in the upper pentagon in the track T_i as is the dotted line in Figure 4. Consider an isotopy I_2 which slides the point x_0 onto α_i. The image of the arc component of $\overline{\gamma} \cap R_\gamma$ which is in T_i, under I_2 has its endpoint in the lower triangle of T_i and the image of x_0 lies in R_γ. A schematic for this isotopy I_2 is shown in Figure 4.

After finitely many such isotopies, we can now assume that all the points of $\overline{\gamma} \cap \alpha$ lie inside R_γ. Now consider an isotopy I_3 of $\overline{\gamma}$ as follows: If any of the components of $\overline{\gamma} \cap R_\gamma$ has its endpoint on the boundary of the upper triangle of T_j, for some $j \in K$, then by the above discussion, $\overline{\gamma}$ cannot intersect $\omega_{i,j}$ or $\eta_{i,j}$, for some $i \in K$ such that the arcs of T_i and T_j forms the opposite sides of a conduit $R_{i,j}$. So $\overline{\gamma} \cap R_{i,j}$ is an arc MN which has its endpoints $M \in T_j$ and $N \in T_i$ on ∂R_γ. Further, since $\overline{\gamma}$ is a closed curve, $\overline{\gamma} \cap T_i$ is an arc with its endpoint as N such that N necessarily lies in the upper pentagon of T_i. Conversely, if any of the components of $\overline{\gamma} \cap R_\gamma$ has its endpoint, z_0, on the boundary of the upper pentagon of T_i, then it should be connected to an arc, g, of $\overline{\gamma}$ in the conduit $R_{i,j}$. Note that the endpoints, z_0, z_0' of g are on ∂R_γ. There exists an arc component of $\overline{\gamma} \cap R_\gamma$ lying in T_j such that z_0' is on the boundary of the upper triangle of T_j, as the dotted line in Figure 5 shows. If any such arc g of $\overline{\gamma}$ exists, consider an isotopy, I_3, of g such that the image, $I_3(g)$, lies outside $R_{i,j}$. A schematic of this is Figure 5.

The component of $\overline{\gamma} \cap R_\gamma$ in T_j now has an endpoint on the boundary of the lower pentagon of T_j and the component of $\overline{\gamma} \cap R_\gamma$ in T_i has an endpoint on the boundary of the lower triangle of T_i. Also the image of $\overline{\gamma} \cap \alpha$ under I_3 moves a point of $\overline{\gamma} \cap \alpha$ from the boundary of the upper traingle of T_j to the boundary of the lower pentagon of T_i. We call I_3 to be a *normalization* move on $\overline{\gamma}$. After finitely many normalization moves performed on $\overline{\gamma}$, wherever applicable, we can assume that every component of $\overline{\gamma} \cap R_\gamma$ is contained in a rectangular track T_i for some

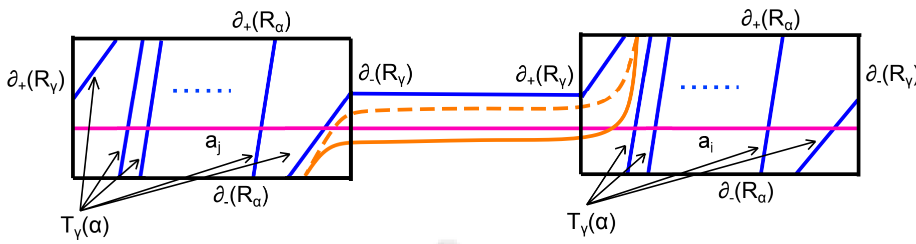


FIGURE 5. A schematic showing the normalization move, the isotopy I_3

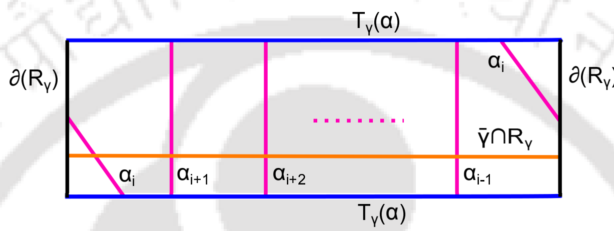


FIGURE 6. The portion of $\overline{\gamma}$ in rectified position inside T_i

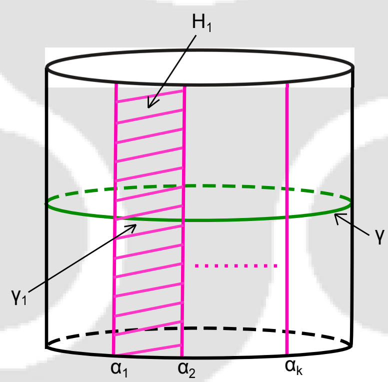


FIGURE 7. Schematic showing H_1 and γ_1 in R_γ

$i \in K$ such that the endpoints of that component lie on the boundary of the lower triangle and the lower pentagon of T_i. So a schematic of every component of $\overline{\gamma} \cap R_\gamma$ inside T_i is as in Figure 6.

After these isotopies I_1, I_2, I_3 of $\overline{\gamma}$, we say that $\overline{\gamma}$ is in a *rectified position*. We now prove that $\overline{\gamma}$ in rectified position and α fill S_g. From now on we assume that $\overline{\gamma}$ is in a rectified position.

Step 2 : For $i \in K$, let H_i be the rectangular component of $R_\gamma \setminus (\cup_{i\in K}\alpha_i)$ containing the arcs a_i and a_{i+1} on its boundary. Each of these H_i contains a unique segment, γ_i, of the core curve γ. The schematic 7 shows H_1 and γ_1 for instance.

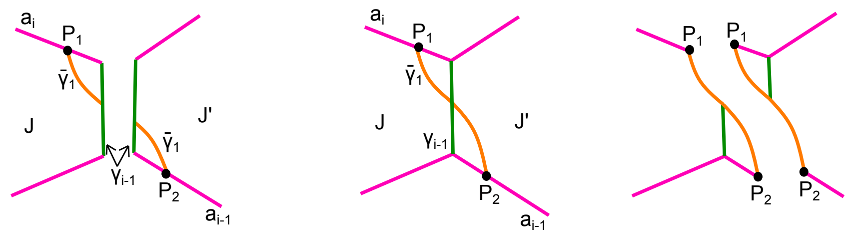


FIGURE 8. The figure on the left shows disks J and J' formed by cutting along γ_{i-1}; the figure on the right shows the new disks formed when $J \cup J'$ are cut along $\overline{\gamma}_1$

We say that an arc, g of $\overline{\gamma}$ *covers* γ_i if $g \subset H_i$ has its end points on α_i and α_{i+1} and g is isotopic in H_i to γ_i through arcs whose end points stay on α_i, α_{i+1}. Since γ and α form a filling pair, the set of essential arcs, $\{\gamma_1, \dots, \gamma_k\}$ fill $S_g \setminus \alpha$. It follows that $\overline{\gamma}$ fills S_g along with α if segments of $\overline{\gamma} \setminus \alpha$ cover γ_i for all i with $i \in K$.

Since $\overline{\gamma}$ is in rectified position, each component of $\overline{\gamma} \cap R_\gamma$ already covers all γ_i except one as in Figure 6. More precisely, if a component of $\overline{\gamma} \cap R_\gamma$ is in a rectangular track T_i, then $\overline{\gamma}$ covers every γ_j where j is such that $1 \le j \le k$ and $j \neq i-1$. So, if $\overline{\gamma} \cap R_\gamma$ has two distinct components, then each component has to lie in T_i for distinct i and hence $\overline{\gamma}$ covers γ_j for $j \in \{1, 2, \dots k\}$. We conclude that $\overline{\gamma}$ and α fill S_g in this case. Now it remains to show that if there is a single component of $\overline{\gamma} \cap R_\gamma$, which is an essential arc of R_γ and is contained in some rectangular track T_i, then $\overline{\gamma}$ and α fill S_g. As in the previous case, $\overline{\gamma}$ covers every γ_j where j is such that $1 \le j \le k$ and $j \neq i-1$. The components of $S_g \setminus (\alpha \cup_{1 \le j \le k, j \neq i-1} \gamma_j)$ will be disks except possibly one which could be a cylinder. This can be seen as follows. Since α and γ fill S_g, the components of $S_g \setminus \alpha \cup_{1 \le j \le k} \gamma_j$ are disks. Each segment of $\gamma_j \setminus \alpha$ for $j \in \{1, 2, ..., k\}$ contributes to two distinct edges of a component J_0 or two separate components J, J' of $S_g \setminus \alpha \cup_{1 \le j \le k} \gamma_j$.

Let $P_1 := \overline{\gamma} \cap \alpha_i$ and $P_2 := \overline{\gamma} \cap \alpha_{i-1}$ be points in T_i which appear on the unique component of $\overline{\gamma} \cap R_\gamma$. Let $[P_1, P_2]$ represent the arc of $\overline{\gamma}$ in R_γ with endpoints P_1 and P_2 and $\overline{\gamma}_1 := \overline{\gamma} \setminus [P_1, P_2]$. $\overline{\gamma}_1$ is contained in all the components of $S_g \setminus \alpha \cup_{1 \le j \le k, j \neq i-1} \gamma_j$ which contain the arcs α_{i-1} and α_i on their boundary. We know that there is at-least one such component because γ_{i-1} is also such an arc which joins α_{i-1} to α_i. If γ_{i-1} is the boundary of J, J', then it would have been an arc which connected α_{i-1} on one disk to α_i on another disk. Note that both α_i and α_{i-1} are also boundary arcs of both J and J'. So, we would find P_1 on the disk containing α_i and P_2 on the disk containing α_{i-1}. When we join J and J' along γ_{i-1} we get a disk where $\overline{\gamma}_1$ is an arc from P_1 to P_2 intersecting γ_{i-1}. Cutting along $\overline{\gamma}_1$ still yields two different disks. The schematic, Figure 8 shows this situation.

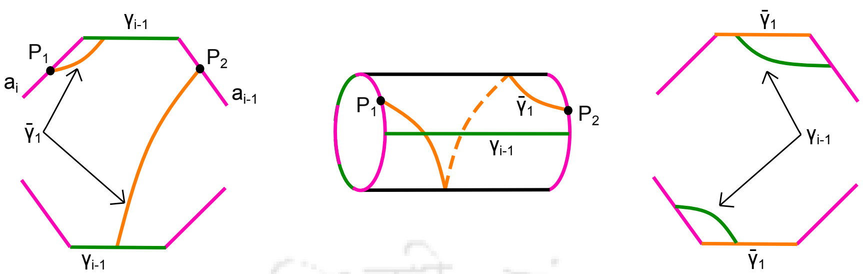


FIGURE 9. The disk J_0 glued to itself along γ_{i-1} and cut along $\overline{\gamma}_1$

If γ_{i-1} were on the boundary of J_0 representing two edges of J_0 then it would have been an arc which connected α_{i-1} to α_i. When we glue J_0 to itself along γ_{i-1}, we get a cylinder, A, where α_i and α_{i-1} will be arcs on different boundary components of A. So we would find P_1 and P_2 on distinct boundaries of A and hence $\overline{\gamma}_1$ would be an essential arc on A. So cutting A along this arc $\overline{\gamma}_1$ would yield a disk as shown in the schematic, Figure 9.

In any case, we get disks by cutting $S_g \setminus \alpha$ along the arcs of $\overline{\gamma} \setminus \alpha$.

Thus, we have finished our application of the distance ≥ 4 test and we have that $d(T_\gamma(\alpha), \alpha) \geq 4$. This along with the existence of the length 4 path between α and $T_\gamma(\alpha)$ proves the theorem. □

THEOREM 7. *If α and γ is a pair of curves on S_g with $d(\alpha, \gamma) = 3$ then for $p \geq 2$, $d(\alpha, T^p_\gamma(\alpha)) = 4$.*

PROOF. Let $\nu_0 = \alpha$, ν_1, ν_2, $\nu_3 = \gamma$ be a geodesic in $\mathcal{C}(S_g)$. For $p \geq 2$, the existence of the path $T^p_\gamma(\nu_0)$, $T^p_\gamma(\nu_1)$, $T^p_\gamma(\nu_2) = \nu_2$, ν_1, ν_0 gives that $d(\alpha, T^p_\gamma(\alpha)) \leq 4$.

Let $k = i(\alpha, \gamma)$. $T^p_\gamma(\alpha)$ is obtained by performing a surgery on pk copies of γ and α similar to the surgery (Figure 3) performed on k copies of γ and α to obtain $T_\gamma(\alpha)$.

Since only the k copies of γ in the surgery of $T_\gamma(\alpha)$ is changed to pk copies of γ to obtain $T^p_\gamma(\alpha)$ and as k is arbitrary throughout the definitions and proofs in the previous part, we can prove that $d(\alpha, T^p_\gamma(\alpha)) \geq 4$ in exactly the same way as the proof of Theorem 6. □

4.2.1. Upper bound for $i_{min}(g, 4)$. As an application of Theorem 6 we are able to obtain an upper bound for the minimum intersection number for a pair of curves at a distance 4 in $\mathcal{C}(S_g)$.

COROLLARY 1. *For a surface of genus $g \geq 3$, $i_{min}(g, 4) \leq (2g-1)^2$.*

PROOF. Aougab and Huang **[3]** proved that $i_{min}(g, 3) = 2g-1$ for $g \geq 3$. Now, on S_g, for $g \geq 3$, suppose that α and β are two such minimally intersecting curves with $d(\alpha, \beta) = 3$. Then $i(\alpha, T_\beta(\alpha)) = (2g-1)^2$ and by Theorem 6, $d(\alpha, T_\beta(\alpha)) = 4$. So $i_{min}(g, 4) \leq (2g-1)^2$. □

4.2.2. An initially efficient geodesic.

THEOREM 8. *If $\alpha = \nu_0, \nu_1, \nu_2, \nu_3 = \gamma$ is an initially efficient geodesic in $\mathcal{C}(S_g)$ then so is $T_\gamma(\alpha), T_\gamma(\nu_1), \nu_2, \nu_1, \alpha$.*

PROOF. For $p \in K_{2-2g}$, let F''_p be the components of $S_g \setminus \{\alpha, R_\gamma\}$ as in the proof of Theorem 6. Since the geodesic $\alpha, \nu_1, \nu_2, \gamma$ is an initially efficient one, each segment of ν_1 intersects every reference arc in E_i at most twice. In particular, arcs of $\partial(R_\gamma)$ that form the edges of E_i intersect ν_1 at most twice. It follows from here that there are at the most two segments of ν_1 in each rectangular track T_i as defined in. A schematic of this is shown in figure 10. Further, since the interior of a reference arc is disjoint from $\alpha \cup T_\gamma(\alpha)$, it is sufficient to check for the initial efficiency of the geodesic, $T_\gamma(\alpha), T_\gamma(\nu_1), \nu_2, \nu_1, \alpha$ in the modified disks F'''_p, abbreviated F, corresponding to F_p , abbreviated E.

Since E and F are homeomorphic to a $2g$-gon. Without loss of generality assume E and F to be a regular Euclidean regular polygon with $2g$ sides. Starting at any segment of α in E, we label the edge as α_1. Label the edges of E in a clockwise direction, starting at α_1 as γ_1, α_2, γ_2, $\ldots$, γ_g. Let $S' = S_g \setminus R_\gamma$. Since the components of $S' \setminus \{\alpha, \gamma\}$ and $S' \setminus \{\alpha, T_\gamma(\alpha)\}$ are the same, it follows that for every edge, a_{j_0} in F corresponding to α, there exists a unique $i_0 \in \{1, \ldots, g\}$ such that $a_{j_0} \subset \alpha_{i_0}$. Index the edges, a_{j_0} of F such that $j_0 = i_0$. Label the edge of $T_\gamma(\alpha)$ in F between a_i and a_{i+1} as t_i. Let ω be a reference arc in F with end points on t_p and t_q for some $p, q \in \{1, \ldots, g\}$. Suppose to the contrary that $\omega \cap T_\gamma(\nu_1) \geq 3$. Then there exists three segments, z_1, z_2, z_3 of $T_\gamma(\nu_1)$ in F such that $z_j \cap \omega \neq \phi$. For $j \in \{1, 2, 3\}$, let the end points of z_j lie on a_{j_1} and a_{j_2}. From our previous discussion on Dehn twist and figure 11, there exists arcs of ν_1 in E with end points on γ_{j_1} and γ_{j_2} for all $j \in \{1, 2, 3\}$. Consider a line segment, ω' in E from an interior point of a_p to an interior point of a_q. Then ω' is a reference arc for the triple, α, ν_1, γ and $\omega' \cap \nu_1 \geq 3$. This contradicts that α, ν_1, ν_2, γ is an initially efficient geodesic. Hence, $\omega \cap T_\gamma(\nu_1) \leq 2$ for any choice of reference arc, ω for the triple $T_\gamma(\alpha)$, $T_\gamma(\nu_1)$, α.

Since $T_\gamma(\alpha), T_\gamma(\nu_1), \nu_2, \nu_1, \alpha$ is already a geodesic we have that $d(T_\gamma(\nu_1), \alpha) = 3$. This gives that $T_\gamma(\nu_1)$ is an initially efficient geodesic of distance 4 from $T_\gamma(\alpha)$ to α.

□

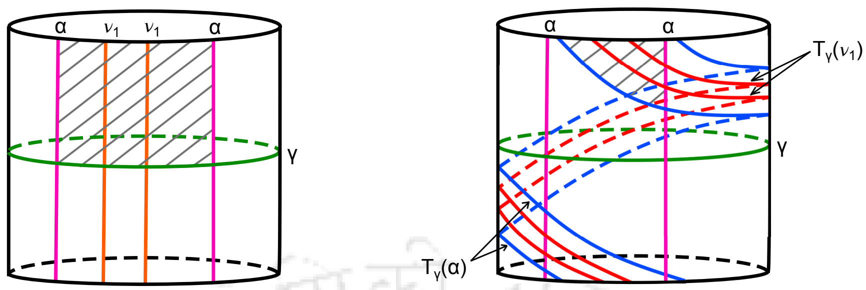


FIGURE 10. There can be at-most two distinct segments of $T_\gamma(\nu_1)$ in any rectangular component of $S_g \setminus (\alpha \cup T_\gamma(\alpha))$ in R_γ

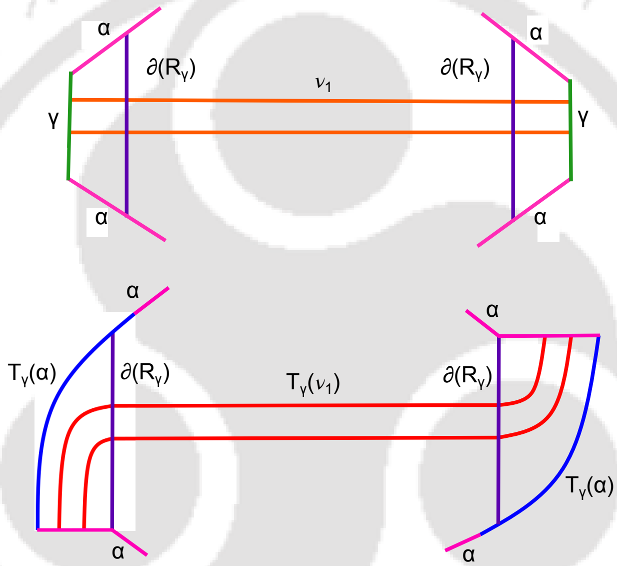


FIGURE 11. Initial efficiency of $T_\gamma(a_1)$ follows from the initial efficiency of a_1

CHAPTER 5

SCALING CURVES

Let γ be an arbitrary curve on S_g. In this chapter, sections 5.1 and 5.5 characterises certain sets of arcs that fill $S_g \setminus \gamma$. Let α and β be curves on S_g with $d(\alpha, \beta) = 4$. In section 5.2, we describe particular components of the annular neighbourhood of β cut along α. We observe some of the properties of these components in section 5.4. In section 5.3, we construct curves on S_g, which we call the scaling curves, from arcs of $\beta \setminus \alpha$ and prove that these curves are at a distance at least 3 from α.

The objective of this chapter is to introduce a few terminologies and certain properties of scaling curves that will aid us in analysing the values of $d(\alpha, T_\alpha(\beta))$ in the following chapter. The work in this chapter is part of the preprint [**16**].

5.1. Filling system of arcs

Let α and β be a filling pair of curves on S_g. Let $\beta' \in \beta \setminus \alpha$. Let D be the polygonal disc obtained by gluing the two components of $S_g \setminus (\alpha \cup \beta)$ along β'. Let b_1 be an arc in D such that b_1 and β' have their end points on the same arcs of $\alpha \cap D$. We say that b_1 *covers* β' if b_1 is isotopic to β' by an isotopy of arcs in D having end points on the same arcs of $\alpha \cap D$ as β' and b_1.

Let $\mathcal{A}$ be a non-empty set of essential arcs on $S_g \setminus \alpha$ such that the end points of every arc in $\mathcal{A}$ lies on the boundary. We call $\mathcal{A}$ a *filling system* of arcs of $S_g \setminus \alpha$ if the components of $(S_g \setminus \alpha) \setminus \mathcal{A}$ are discs.

LEMMA 1. *Let α and β be a pair of filling curves on S_g. Let $i(\alpha, \beta) = n$ and the components of $\beta \setminus \alpha$ be $\{\beta_i : 1 \leq i \leq n\}$. Let Γ be a non-empty set of essential arcs on $S_g \setminus \alpha$. If for every $i \in \{1, \ldots, n\}$, there exists $g_i \in \Gamma$ such that g_i covers β_i, then Γ is a filling system of arcs of $S_g \setminus \alpha$.*

PROOF. Consider the components of $(S_g \setminus \alpha) \setminus \cup\{g_i\}_{1\le i\le n}$. These components coincide with the components of $(S_g \setminus \alpha) \setminus \beta$ and hence, are discs. Since $\cup_{1\le i\le n}\{g_i\} \subset \Gamma$, the components of $(S_g \setminus \alpha) \setminus \Gamma$ are also discs. □

For any curve α on S_g, we denote the annular neighbourhood of α as R_α. Let a_0, a_1, a_2, a_3, a_4 be a geodesic in $\mathcal{C}(S_g)$. Let $(a_0, a_4, R_{a_0}, R_{a_4})$ be amenable to Dehn twist in special position. The following lemma 2 states that for the purpose of cutting $S_g \setminus a_0$ into discs, not every arc of $a_4 \setminus a_0$ is necessary. We can forgo any one of the arcs of $a_4 \setminus a_0$.

LEMMA 2. *Let b be a component of $a_4 \setminus a_0$. Then $(a_4 \setminus a_0) \setminus b$ is a filling system of arcs of $S_g \setminus a_0$.*

PROOF. Each component of $a_4 \setminus a_0$ is common to two components of $S_g \setminus (a_0 \cup a_4)$. Let the components of $S_g \setminus (a_0 \cup a_4)$ that share the edge corresponding to b be D_1 and D_2.

We first show that $D_1 \neq D_2$. On the contrary, if $D_1 = D_2$ let p be the central curve of the annulus obtained by gluing D_1 along b. Being in minimal position with a_0 and a_4, p forms an essential curve on S_g. Since $i(p, a_0) = 0$ and $i(p, a_4) = 1$, we get a path of distance 3 between a_0 and a_4 via p, which is not possible.

Call the disc obtained by gluing D_1 and D_2 along b as D. Components of $S_g \setminus ((a_4 \setminus a_0) \setminus b)$ comprise of the components of $(S_g \setminus (a_0 \cup a_4)) \setminus (D_1 \cup D_2)$ and D. Since each component is a disc, it follows that $(a_4 \setminus a_0) \setminus b$ forms a filling system of $S_g \setminus a_0$. □

5.2. Buckets

Let α be a curve on S_g which intersects μ and λ minimally. Any arc in $\alpha \cap R_\lambda$ with end points on distinct components of $\partial(R_\lambda)$ is called a *strand of* α in R_λ. If α is such that $i(\alpha, \lambda) = 1$ and $\alpha \cap \mu \subset R_\lambda$ then α is called a *standard single strand curve*.

Given an ordered set of points on μ, we now give a shorthand notation to represent the arcs of μ between these points. Let μ be with a preferred orientation and $x_1, \ldots, x_{m\ge 3}$ be distinct points on μ. Considering μ as the embedding $\mu : [0,1] \longrightarrow S_g$ with $\mu(0) = \mu(1)$, we say that $x_1, \ldots, x_m$ are *along the orientation of* μ if $\mu^{-1}(x_i) < \mu^{-1}(x_{i+1})$ for $i \in \{1, \ldots, m-1\}$. We use $\mu_{[x_i, x_{i+1}]}$ to denote the undirected arc of a with end points x_i, x_{i+1} and which has no other x_j's on it. Since $\mu_{[x_i,x_{i+1}]}$ is undirected, we set $\mu_{[x_i,x_{i+1}]} = \mu_{[x_{i+1},x_i]}$. For $i \in \{1, \ldots, m\}$, let b_i be curves or essential arcs on S_g such that $b_i \cap \mu = x_i$. When the context is clear, we will interchangeably use $\mu_{[x_i,x_{i+1}]}$ and $\mu_{[b_i,b_{i+1}]}$.

Select some orientation for a_0 and a_4. Let $a_0 \cap a_4 = \{w_i : i \in K\}$ be ordered along the orientation of a_4. For $i \in K$, let a_0^i be the arc of $a_0 \cap R_{a_4}$ containing

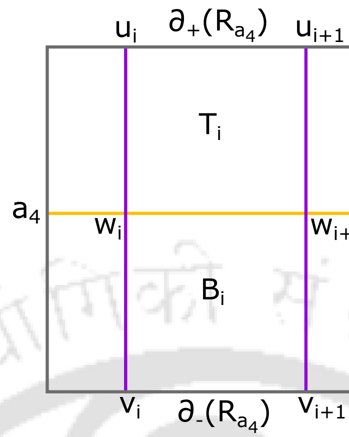


FIGURE 1. Top bucket T_i and bottom bucket B_i

w_i. Let the two component curves of $\partial(R_{a_4})$ be $\partial_+(R_{a_4})$ and $\partial_-(R_{a_4})$ such that a_0^1 with the induced orientation from a_0 goes from $\partial_+(R_{a_4})$ to $\partial_-(R_{a_4})$. There is a natural orientation of $\partial_+(R_{a_4})$ and $\partial_-(R_{a_4})$ induced by the orientation of a_4. For $i \in K$, let $u_i = a_0^i \cap \partial_+(R_{a_4})$ and $v_i = a_0^i \cap \partial_-(R_{a_4})$. We call the rectangle in R_{a_4} with boundaries $a_{4_{[w_i,w_{i+1}]}}$, $\partial_+(R_{a_4})_{[u_i,u_{i+1}]}$, $a_{0_{[u_i,w_i]}}$ and $a_{0_{[u_{i+1},w_{i+1}]}}$ as a *top bucket* and denote it by T_i. Similarly, we call the rectangle in R_{a_4} with boundaries $a_{4_{[w_i,w_{i+1}]}}$, $\partial_-(R_{a_4})_{[v_i,v_{i+1}]}$, $a_{0_{[w_i,v_i]}}$ and $a_{0_{[w_{i+1},v_{i+1}]}}$ as a *bottom bucket* and denote it by B_i. Figure 1 gives a schematic of a top and a bottom bucket. We note that each top and bottom bucket is contained in a unique component of $S_g \setminus (a_0 \cup a_4)$. Let H be a top (or, bottom) bucket and let O be the component of $S_g \setminus (a_0 \cup a_4)$ containing H. We then call H to be a *top (or, bottom) bucket in* O.

5.3. Scaling curves

Let D be a component of $S_g \setminus (a_0 \cup a_4)$. Let T_p, T_q be top buckets in R_{a_4} for some $p, q \in K$, $p < q$ such that $(T_p \cup T_q) \subset D$. Let γ'' be an arc in $\cup_{i=p+1}^{q-1} T_i$ parallel to a_4 with end points on $a_0^{p+1} \cap T_{p+1}$ and $a_0^q \cap T_{q-1}$. Let γ' be an arc in the interior of D with end points $(\gamma'' \cap a_0) \cap D$. Let γ be the curve obtained by concatenation of the arcs γ' and γ''. A schematic of γ is shown in figure 2. We call γ a *scaling curve from* T_p *to* T_q. Since R_{a_4} is a cylinder, we can similarly define a *scaling curve from* T_q *to* T_p as follows. Let γ_1'' be an arc in $\cup_{i=q+1}^{p-1} T_i$ parallel to a_4 with end points on $a_0^{q+1} \cap T_{q+1}$ and $a_0^p \cap T_{p-1}$. Let γ_1' be an arc in the interior of D with end points $(\gamma_1'' \cap a_0) \cap D$. Then the curve obtained by concatenation of the arcs γ' and γ'' is a scaling curve from T_q to T_p. By replacing top buckets with their bottom buckets counterpart, we can define *scaling curve from* B_p *to* B_q and B_q *to* B_p.

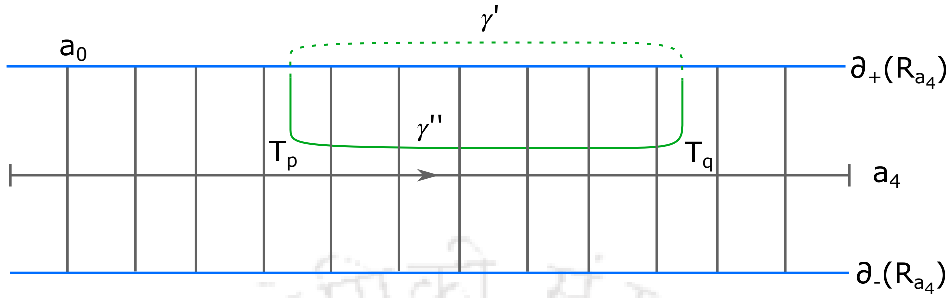


FIGURE 2. A schematic of the scaling curve γ from T_p to T_q. The dashed arc is a schematic of γ'.

LEMMA 3. *Scaling curves are not null-homotopic.*

PROOF. We prove the lemma when γ is a scaling curve from a top bucket T_p to T_q, $p < q$ and $(T_p \cup T_q) \subset D$ for some component D of $S_g \setminus (a_0 \cup a_4)$. A similar proof follows if γ is a scaling curve from a bottom bucket B_p to B_q, $p < q$ by replacing T_p, T_q with B_p, B_q, respectively, in the proof below. Similar proofs work for scaling curves from T_q to T_p and B_q to B_p. We show that γ is not null-homotopic by considering a minimal representative of γ along with a_0 and showing that this representative has non-zero intersections with a_0. We obtain this minimal representative of γ and a_0 by removing bigons in iterations.

Suppose if possible that γ and a_0 are not in minimal position. Since there exists an isotopic copy of γ such that γ'' overlaps with $a_{4_{[w_{p+1},w_{q-1}]}}$, if γ and a_0 are not in minimal position then a bigon is formed by γ' and a subarc of a_0. This subarc of a_0 is a component of $a_0 \setminus a_4$ because otherwise, if there is a point of $a_0 \cap a_4$ on the boundary of this bigon then as $\gamma \cap a_4 = \phi$ we get a bigon between a_0 and a_4 which contradicts the minimality of a_0, a_4. The closed component of $a_0 \setminus a_4$ that contains this subarc also contains the arcs $T_p \cap a_{0_{[u_{p+1},w_{p+1}]}}$ and $T_q \cap a_{0_{[u_{q-1},w_{q-1}]}}$. Thus, $(T_{p+1} \cup T_{q-1}) \subset D_1$ for some component D_1 of $S_g \setminus (a_0 \cup a_4)$. We remove this bigon between γ and a_0 to obtain an isotopic copy of γ. This isotopic copy of γ is in turn a scaling curve from T_{p+1} to T_{q-1}. By abuse of notation, we denote this isotopic copy as γ.

If we have that γ is not in minimal position with a_0, then by similar arguments as in the previous paragraph, $T_{p+1} \cap a_{0_{[u_{p+2},w_{p+2}]}}$ and $T_{q-1} \cap a_{0_{[u_{q-2},w_{q-2}]}}$ are contained in the same closed component of $a_0 \setminus a_4$. Thus, we have that D_1 is a rectangle. As previously, we remove this bigon between γ and a_0 and consider denote the new isotopic copy which is also a scaling curve from T_{p+2} to T_{q-2} by γ. Further, we have that $(T_{p+2} \cup T_{q-2}) \subset D_2$ for some component D_2 of $S_g \setminus (a_0 \cup a_4)$.

Continuing in a similar iterative manner as in the above paragraphs, if γ and a_0 are not in minimal intersection position, then we claim that there is a positive integer l with $l < \lceil \frac{q-p}{2} \rceil - 1$ such that

(1) $T_{p+l} \cup T_{q-l}$ is contained in the one component D_l of $S_g \setminus (a_0 \cup a_4)$
(2) $T_{p+l} \cap a_4$ and $T_{q-l} \cap a_4$ are separated by a single edge corresponding to a_0 in D_l

The fact that there exists l with $l \leq \lceil \frac{q-p}{2} \rceil - 1$ is immediate as there are at most $\lceil \frac{q-p}{2} \rceil$ pairs of buckets of the form T_{p+i} and T_{q-i} between T_p and T_q in R_{a_4}. We first show that if we assume $l = \lceil \frac{q-p}{2} \rceil - 1$ along with hypothesis (1) and (2) then we arrive at the following contradictions. If $q-p$ is odd then we have that T_{p+l} and T_{q-l} are adjacent top buckets. But if T_{p+l} and T_{q-l} are adjacent top buckets then a_0 has a self intersection, which is absurd. If $q-p$ is even then $p+l+2 = q-l$. But then the $a_0 \cap D_l$ arc containing the end points w_{p+l+1} and w_{q-l} encloses a disc with $a_{4_{[w_{p+l+1}, w_{q-l}]}}$, thus giving a bigon between a_0 and a_4. This contradicts that a_0 and a_4 are in minimal position.

We thus have that the scaling curve from T_{p+l} to T_{q-l} intersects a_0 minimally and is isotopic to the given γ.

□

As in the proof of lemma 3, whenever we consider a scaling curve we will work with an isotopic copy of it which is in minimal position with a_0, a_4 and $T_{a_4}(a_0)$.

Corollary 2. *Scaling curves fill with a_0.*

Proof. From the construction of a scaling curve, γ, $\gamma \cap a_4 = \phi$ and hence $d(\gamma, a_0) \geq 3$. Thus, a_0 and γ fill S_g. □

Remark 3. If a_0 and a_4 intersect $i_{min}(g,4)$ number of times then any scaling curve are at a distance 3 from a_0.

5.4. Properties of buckets

The following lemmas explain a few observations regarding the buckets in R_{a_4} and the components of $S_g \setminus (a_0 \cup a_4)$ that contain them.

From corollary 2 and the fact $i_{min}(g,3) \geq 4$ ([**3**]), we have the following corollary regarding the placement in R_{a_4} of the top buckets which are subset of the same disc of $S_g \setminus (a_0 \cup a_4)$. A similar version of corollary 3 holds true for bottom buckets.

Corollary 3. *Let D be a component of $S_g \setminus (a_0 \cup a_4)$ and T_p, T_q be distinct top buckets in R_{a_4} for some $p, q \in K$, $p < q$ such that $(T_p \cup T_q) \subset D$. Then $|q-p| \geq 4$.*

Lemma 4. *For any $i \in K$, T_i and B_i can't be subsets of the same component of $S_g \setminus (a_0 \cup a_4)$.*

PROOF. This follows directly from the proof of lemma 2 by considering $b = a_{4_{[i,i+1]}}$. □

Choice of δ : Considering $T_{a_4}(a_0)$ to be in special position w.r.t. a_0 and a_4, there exists a representative of a_0 such that the possible schematics of the strands of $T_{a_4}(a_0)$ in R_{a_4} are as in figure 3. The details to the choice of such a representative of a_0 can be found in the proof of 5. Any path between a_0 and $T_{a_4}(a_0)$ is of the form $T_{a_4}(a_0)$, δ, c, Θ, a_0 where $\delta \in B_1(T_{a_4}(a_0))$, $c \in B_1(\delta)$ and Θ is a non-trivial path. We now give an algorithm to select a representative of δ such that for each strand of δ in R_{a_4} there exists $i \in K$ such that the end points of the strand lies on $\partial_+(R_{a_4})_{[u_i,u_{i+1}]}$ and $\partial_-(R_{a_4})_{[v_i,v_{i+1}]}$. Applying T_{a_4} to a geodesic between a_0 and a_4, we get that $d(a_4, T_{a_4}(a_0)) = 4$. Since $\delta \in B_1(T_{a_4}(a_0))$, we have that $d(\delta, a_4) \geq 3$. From **[3]**, we have that $i_{min}(g, 3) \geq 4$. Thus $i(\delta, a_4) \geq 4$, i.e. there are at least 4 strands of δ in R_{a_4}. Consider a representative of δ which is in minimal position with a_0, a_4, $T_{a_4}(a_0)$, $\partial_+(R_{a_4})$ and $\partial_-(R_{a_4})$. We can choose a representative of δ such that $\delta \cap a_0 \subset R_{a_4}$ by performing the isotopy I_2 described in the step 1 of theorem 6. An intuitive picture of this isotopy is to finger push the points in $\delta \cap a_0$ which don't lie in R_{a_4}, along a_0, into R_{a_4}. This "finger pushing" doesn't disturb the minimal position of δ and $T_{a_4}(a_0)$. The strands of δ in R_{a_4} attained after performing the above isotopies can be one of the four possible schematics as in figure 3. If a strand of δ in R_{a_4}, say δ', is as in figure 3a or 3b, we can perform an isotopy of δ such that the isotopic image of δ' is as in figure 3c or 3d and the isotopy doesn't disturb the other strands of δ. This isotopy of δ is defined as I_3 in the step 1 of theorem 6. The isotopic copy thus obtained is said to be "δ in a rectified position".

Let δ_1 be the point on $\partial_+(R_{a_4})$ such that $\delta_1 = \delta \cap \partial_+(R_{a_4})_{[u_1,u_2]}$ and that one of the arcs $\partial_+(R_{a_4}) \setminus \{u_1 \cup \delta_1\}$ doesn't contain any points of $\delta \cap \partial_+(R_{a_4})$. If $m = i(a_4, \delta)$, let $\delta \cap \partial_+(R_{a_4}) = \{\delta_i : 1 \leq i \leq m\}$ such that the $\delta_1, \delta_2, \ldots, \delta_m$ are along the orientation of $\partial_+(R_{a_4})$. Let the strand of δ containing the point δ_i be δ^i.

Let δ^r, δ^s be any two distinct strands of δ in R_{a_4} such that δ^r and δ^s start in distinct top buckets, say T_r and T_s, and that there exists a component of $\partial_+(R_{a_4}) \setminus (\delta^r \cup \delta^s)$ that doesn't contain any points of $\partial_+(R_{a_4}) \cap \delta$ other than $\partial_+(R_{a_4}) \cap \delta^r$ and $\partial_+(R_{a_4}) \cap \delta^s$. We call the rectangular component of $R_{a_4} \setminus (\delta^r \cup \delta^s)$ which doesn't contain any other strand of δ as a *δ-track in R_{a_4}* and denote it by $\delta^{r,s}$. The boundary of $\delta^{r,s}$ comprises of the arcs δ^r, δ^s, $\partial_+(R_{a_4})_{[\delta^r,\delta^s]}$ and $\partial_-(R_{a_4})_{[\delta^r,\delta^s]}$. Further, assuming $r < s$, we call the set $\bigcup_{i=r}^{s}(T_i \cup B_i)$ as *inside of $\delta^{r,s}$*.

LEMMA 5. *Let O be a $2n$-gon disc of $S_g \setminus (a_0 \cup a_4)$ with $n \geq 4$. For any delta-track in R_{a_4}, $\delta^{r,s}$, there exists at-least one top bucket or, bottom bucket in O that is not in the inside of $\delta^{r,s}$.*

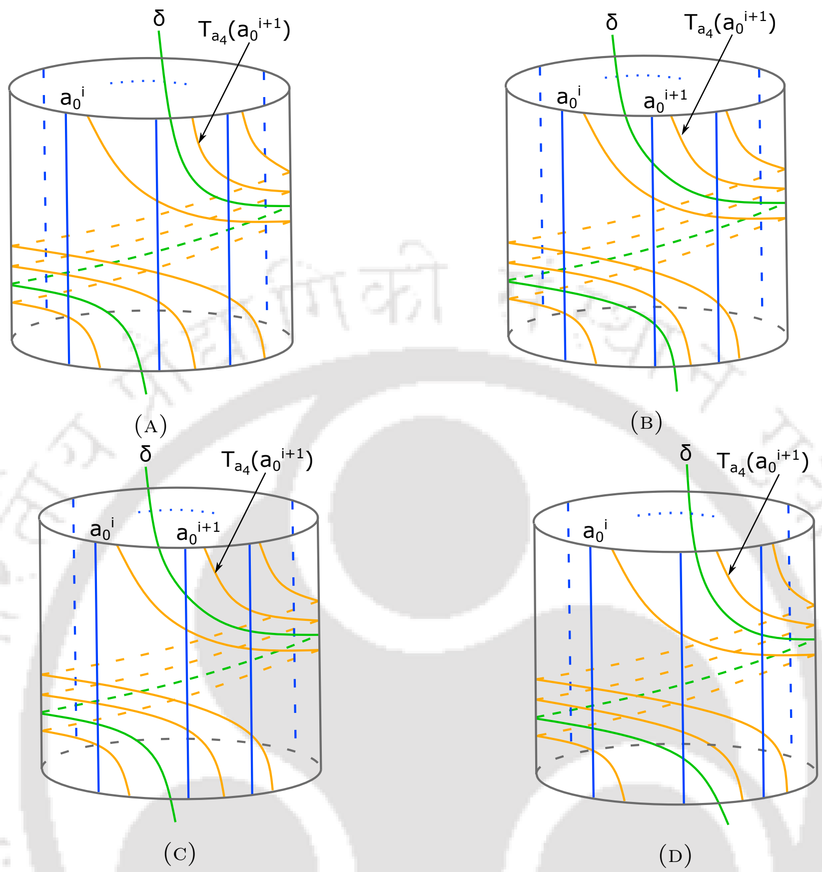
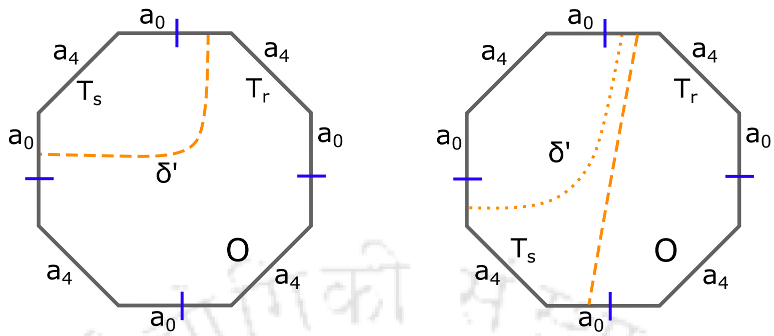


FIGURE 3. The possible starting and ending points of strands of δ in R_{a_4}

PROOF. Let us suppose on the contrary that there exists a δ-track $\delta^{r,s}$ such that all the top and bottom buckets in O are inside $\delta^{r,s}$. Without loss of generality, assume the top and bottom buckets containing the end points of δ^r are T_r and B_r, respectively. Similarly, for δ^s the top and bottom buckets are T_s and B_s, respectively. If every bucket in O are of the form T_i or, B_i for $r < i < s$ then, we get a scaling curve γ such that δ, γ, a_4 is a path. This contradicts $d(\delta, a_4) \geq 4$.

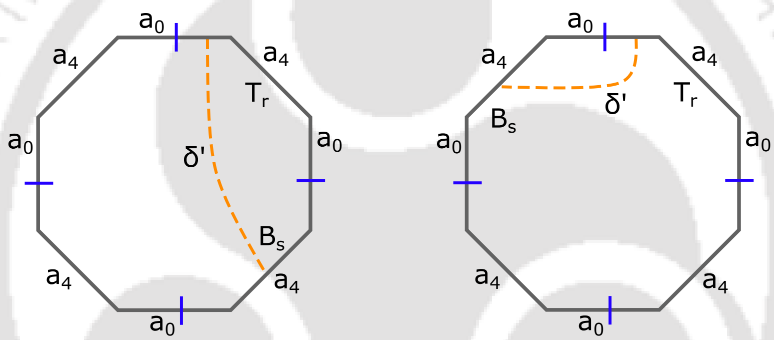
Also, since every bucket of O is inside $\delta^{r,s}$, any arc of $\delta \cap O$ are either an arc that covers $a_{4_{[i,i+1]}}$ for $r < i < s$ or, an arc, say δ', with end points $\delta^r \cap a_4$ and $\delta^s \cap a_4$. It follows along with lemma 4 that either T^r, T^s or, T^r, B^s or, B^r, T^s or, B^r, B^s are buckets of O. We show that all these possibilities, if they exist, lead to a contradiction. The following is a combinatorial proof and we give it for the case O is an octagon. As n increases, the proof remains intact with only the possibility of certain cases being redundant.

If T^r, T^s are buckets in O, figure 4 shows the distinct possible δ'. If T^r, B^s are buckets in O, figure 5 shows the distinct possible δ'. If B^r, B^s are buckets in



(A) (B)

FIGURE 4. Both the dotted line and the dashed line are possibilities for δ' if T_r and T_s are as in the schematic.



(A) (B)

FIGURE 5

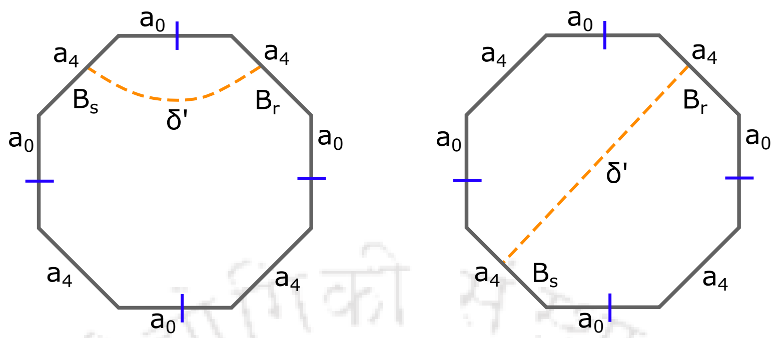
O, figure 6 shows the distinct possible δ'. In the possible cases of figure 4, 5 and 6a, by the pigeon hole principle, either component of $O \setminus \delta'$ contains either two top or, bottom buckets. Thus, if these cases occur, we can construct a scaling curve γ such that δ, γ, a_4 is a path, which is absurd. For figure 6b, if both the components of $O \setminus \delta'$ contains a top bucket, because S_g is an orientable surface, we will be able to find a non-trivial curve γ with properties as in the above cases. Here, $\gamma \cap O$ lies in the component of O containing the vertices w_{r+1} and w_s.

If B^r, T^s are buckets in O, the arguments are similar to the case of T^r and B^s being buckets of O.

□

5.5. Almost filling arcs

Suppose we have a filling system of arcs of $S_g \setminus a_0$ and there is another set of arcs on $S_g \setminus a_0$ that covers all the arcs in the former filling system of arcs except



(A) (B)

Figure 6

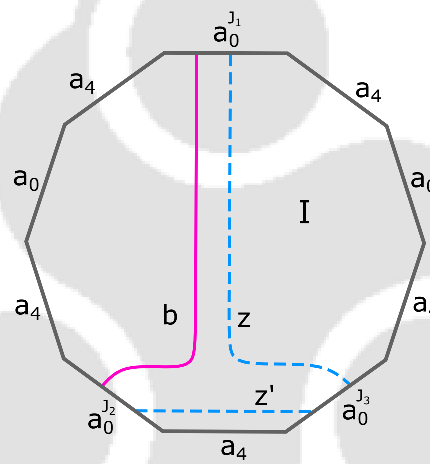


Figure 7. A schematic of $\mathcal{I}$ with (z, z') almost covering b.

for one. In the following we explore a sufficient condition on the latter system of arcs which ensures that it forms a filling system of arcs. To prove this condition we take the aid of the fact that a_0 and a_4 fill S_g.

Let $\mathcal{I}$ be some non-rectangular component of $S_g \setminus (a_0 \cup a_4)$ and $a_0^{J_1}$, $a_0^{J_2}$ be two distinct edges of $\mathcal{I}$, where $J_1, J_2 \in K$. Consider an arc, b, in $\mathcal{I}$ with end points on $a_0^{J_1}$ and $a_0^{J_2}$. Let I be one of the two components of $\mathcal{I} \setminus J$ such that I contains an edge a_0^J for some $J \in K$. By our assumption, there exists an edge, $a_0^{J_3}$, in I such that $a_0^{J_2} \cap I$ and $a_0^{J_3}$ are adjacent in the polygon I. Consider arcs, z and z', in I such that the end points of z are $a_0^{J_1} \cap I$, $a_0^{J_3}$ and the endpoints of z' are $a_0^{J_2}$, $a_0^{J_3}$. Clearly, z' covers the a_4 edge in I adjacent to $a_0^{J_2} \cap I$ and $a_0^{J_3}$. We call such a pair of arcs (z, z') to *almost cover* b. Figure 7 gives a schematic of (z, z').

Lemma 6. *Let a_0 and a_4 be curves on S_g with $d(a_0, a_4) = 4$. Let κ be another curve on S_g such that a_0 and κ fill S_g. Let Γ be a set of essential arcs on $S_g \setminus a_0$.*

If Γ consists of arcs that covers all but one arc of $\kappa \setminus a_0$ and almost covers the remaining arc of $\kappa \setminus a_0$, then Γ forms a filling system of arcs of $S_g \setminus a_0$.

Proof. Let $g, g' \in \Gamma$ and x be the component of $\kappa \setminus a_0$ such that (g, g') almost cover x. Let $\{g_j\}_{j\in J} \subset \Gamma$ be the arcs that cover the components of $(\kappa \setminus a_0) \setminus x$. Let $\mathcal{I}$ be the component of $S_g \setminus (a_0 \cup a_4)$ that contains x.

Starting with the components of $(S_g \setminus a_0) \setminus a_4$, we can obtain the components of $(S_g \setminus a_0) \setminus \kappa$ by gluing the components of $(S_g \setminus a_0) \setminus a_4$ along the components of $a_4 \setminus a_0$ and cutting along the components of $\kappa \setminus a_0$. In the components $(S_g \setminus (a_0 \cup a_4)) \setminus \mathcal{I}$, the action of cutting along the components of $\kappa \setminus a_0$ coincides with the action of cutting along $\{g_j\}_{j\in J}$.

Let I and I' be the two components of $\mathcal{I} \setminus x$ such that $(g \cup g') \subset I$. In $\mathcal{I}$, the action of gluing along x is such that it separates the a_4-edges of $\mathcal{I}$ into two sets i.e. $(a_4 \setminus a_0) \cap I$ and $(a_4 \setminus a_0) \cap I'$. Let G_1, G_2 and G_3 be the components of $\mathcal{I} \setminus (g \cup g')$. From the schematic in figure 7, we can see that the components of $\mathcal{I} \setminus (g \cup g')$ can be named such that $I' \subset G_1$, $G_2 \subset I$ and $G_3 \subset I$. Since, the components of $(S_g \setminus a_0) \setminus (\kappa \setminus x)$ and the components of $(S_g \setminus a_0) \setminus \cup_{j\in J} g_j$ are the same, we have that the components of $(S_g \setminus (a_0 \setminus \cup_{j\in J} g_j) \setminus (g \cup g')$ will be discs if the action of cutting along $g \cup g'$ doesn't put an arc from $(a_4 \setminus a_0) \cap I$ and another from $(a_4 \setminus a_0) \cap I'$ in the same G_i. Such a phenomenon never occurs by our definition of almost filling.

□

CHAPTER 6

CRITERIA FOR $d(a_0, t_{a_4}(a_0)) \geq 5$

We have that a_0, a_1, a_2, $a_3 = T_{a_4}(a_3)$, $T_{a_4}(a_2)$, $T_{a_4}(a_1)$, $T_{a_4}(a_0)$ is a path of length 6 in $\mathcal{C}(S_g)$. Existence of a path of length 6 between $T_{a_4}(a_0)$, a_0 and theorem 5 gives that

$$4 \leq d(a_0, T_{a_4}(a_0)) \leq 6.$$

Thus, geodesics between a_0 and $T_{a_4}(a_0)$ in $\mathcal{C}(S_g)$ can be of the form $z_0 = T_{a_4}(a_0), z_1 = \delta, z_2 = c, \ldots, z_N = a_0$ for some $N \in \{4,5,6\}$ and $c \in B_1(\delta)$. We have that $d(a_0, T_{a_4}(a_0)) \geq 5$ if and only if $d(a_0, c) \geq 3$ for all possible $c \in B_2(T_{a_4}(a_0)) \cap B_1(\delta)$. In this chapter we identify the characteristics of δ and c which results in $d(c, a_0) \geq 3$.

The notations and representatives for a_0, a_4, $T_{a_4}(a_0)$, $\delta \in B_1(T_{a_4}(a_0))$, R_{a_0} and R_{a_4} in this chapter are the same as the ones made in the previous chapter.

Let $k \in \mathbb{N}$. The path a_0, a_1, a_2, $a_3 = T^k_{a_4}(a_3)$, $T^k_{a_4}(a_2)$, $T^k_{a_4}(a_1)$, $T^k_{a_4}(a_0)$ in $\mathcal{C}(S_g)$ gives that $d(a_0, T^k_{a_4}(a_0)) \leq 6$. In [**4**], the authors showed that $d(a_0, T^B_{T^B_{a_3}(a_0)}(a_0)) \geq 6$ for some large enough constant B. We replicate their arguments to show that $d(a_0, T^k_{a_4}(a_0)) \geq 6$, $\forall k \geq K$ for some large enough constant K. We will be using the notations as introduced in section 2.7. For any curve γ, we have that $d_\gamma(\alpha, T^N_\gamma(\alpha)) \geq N - 2$. Choose a large enough constant K such that

$$d_{a_4}(a_0, T^K_{a_4}(a_0)) \geq K - 2 > M.$$

By theorem 2, any geodesic, g, between a_0 and $T^K_{a_4}(a_0)$ has to pass through the one neighbourhood of a_4. Suppose for g, the node p lies in the one neighbourhood of

a_4. Then,

$$\begin{aligned} d(a_0, T_{a_4}^K(a_0)) &= d(a_0, p) + d(p, T_{a_4}^K(a_0)) \\ &\geq (d(a_0, a_4) - 1) + (d(T_{a_4}^K(a_4), T_{a_4}^K(a_0)) - 1) \\ &= 6. \end{aligned}$$

The above arguments follows for all $k \geq K$ and thus $d(a_0, T_{a_4}^k(a_0)) = 6$, $\forall k \geq K$.

6.1. Representative of c

Since $c \in B_1(\delta)$, $d(\delta, a_4) \geq 3$ implies that $c \cap a_4 \neq \phi$. Thus, there is at least one strand of c in R_{a_4}. We now perform an isotopy of c so that c is in a favourable position with respect to a_0, a_4, $\partial(R_{a_4})$ on S_g. The idea behind this isotopy is to get a representative of c such that in R_{a_4}, the strands of c resemble the "spiral pattern" of the strands of $T_{a_4}(a_0)$ and δ. Consider an isotopic copy of c such that c is in minimal position with $\partial_+(R_{a_4})$, $\partial_-(R_{a_4})$, a_0, a_4 and δ. Consider any strand, c', of c in R_{a_4}. Let $c_+ = c' \cap \partial_+(R_{a_4})$ and $c_- = c' \cap \partial_-(R_{a_4})$. Let $i_0 \in K = \{1, \ldots, k\}$ such that c_+ lies on the boundary of the top bucket T_{i_0}. There exists $0 \leq l \leq k-1$ such that c_- lies on the boundary of the bottom bucket B_{i_0+l}. Note that since in any top bucket T_i, there is an arc of delta with end points on a_0^i and a_0^{i+1}, if $l = 0$ then $c' \cap a_0^{i_0+1} \neq \phi$. There exists $i_1, i_2 \in K$ such that c_+ lies in $\partial_+(R_{a_4})_{[\delta^{i_1}, \delta^{i_2}]}$. Since $c' \cap (\delta^{i_1} \cup \delta^{i_2}) = \phi$, we have $i_1 \leq i_0 + l \leq i_2$. Consider the arcs c_1, c_2 and c_3 in the annulus R_{a_4} as follows :

- c_1 starts at c_+ passing through T_{i_0}, T_{i_0+1} $\ldots$, T_{i_1-1} and ends in some interior point on $a_0^{i_1} \cap T_{i_1-1}$
- c_2 is an arc parallel to a_4 which starts at $c_1 \cap a_0^{i_1}$, passes through T_{i_1}, T_{i_1+1} $\ldots$, T_{i_0+l-1} and ends in some interior point on $a_0^{i_0+l} \cap T_{i_0+l}$
- c_3 is an arc in the rectangle $T_{i_0+l} \cup B_{i_0+l}$ with end points $c_2 \cap a_0^{i_0+l}$ and c_-

Let c'' be the arc obtained by concatenating c_1, c_2 and c_3. Note that c'' intersects a_4 only once. Since both c' and c'' are arcs in the rectangle with edges δ^{i_1}, δ^{i_2}, $\partial_+(R_{a_4})_{[\delta^{i_1}, \delta^{i_2}]}$ and $\partial_-(R_{a_4})_{[\delta^{i_1}, \delta^{i_2}]}$ such that both c' and c'' have end points c_+ and c_-, there is a end point fixing isotopy, $\mathcal{I}$, of arcs in the rectangle from c' to c''. The isotopy $\mathcal{I}$ can be extended to an isotopy of c to $(c \setminus c') \cup c''$ such that the action on $c \setminus c'$ remains identity. By abuse of notation, we denote $\mathcal{I}(c')$ i.e. c'' by c'. Since the strand of c, c', is arbitrary and $\mathcal{I}$ is identity on $c \setminus c'$, we can apply $\mathcal{I}$ to every strand of c to obtain a representative of c which remains in minimal position with a_0, a_4, $T_{a_4}(a_0)$ and δ. We will always consider such a representative of c.

Suppose $i(c, a_4) = k_0$. Let $c_1, \ldots, c_{k_0}$ be the strands of c. Let the end points of c_i be in the top bucket T_{d_i} and the bottom bucket $B_{d_i - l_i}$. We call the set $C_i = \bigcup_{j=d_i-l_i}^{d}(T_j \cup B_j)$ as the inside of c_i. We define $\bigcap_{j=1}^{k_0}(C_j)$ as the *inside of c*.

6.2. Values of $d(c, a_0)$

We now look into the possible values for $d(c, a_0)$ by considering the following two cases depending on the number of strands of c :

Case i : there is a single strand of c in R_{a_4}

Case ii : there are multiple strands of c in R_{a_4}

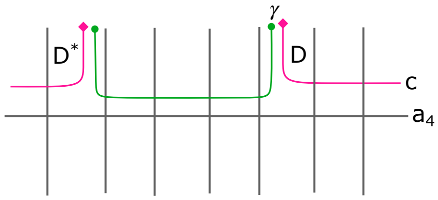
Case i : Suppose there exists a single strand, c', of c in R_{a_4}. Let $c' \cap \partial_+(R_{a_4})$ lie on $\partial_+(R_{a_4})_{[\delta^p, \delta^q]}$ where $1 \leq p < q \leq m$. We first consider the case when $\delta^p \cap \partial_+(R_{a_4})$ and $\delta^q \cap \partial_+(R_{a_4})$ lie in the same top bucket then $c' \cap \partial_+(R_{a_4})$ and $c' \cap \partial_-(R_{a_4})$ lie in T_e and B_e, respectively, for some $e \in K = \{1, \ldots, k\}$. By lemma 2 we have that c' and a_0 fills.

We now consider the case that $\delta^p \cap \partial_+(R_{a_4})$ and $\delta^q \cap \partial_+(R_{a_4})$ lie in distinct top buckets. Without loss of generality in the arguments below, we can assume that $\delta^p \cap \partial_+(R_{a_4})$ and $\delta^q \cap \partial_+(R_{a_4})$ lie in T_p and T_q, respectively. Let the end points of c' be in the top bucket $D = T_d$ and the bottom bucket $T = B_{d-l}$ where $r \leq d - l \leq d \leq s$. The set $\bigcup_{i=d-l}^{d}(T_i \cup B_i)$ forms the inside of c. For a c with a single strand c', if $c \cap a_0 = c' \cap a_0$, we recall from section **??** that such a c is said to be a standard single strand curve. We first show that any c with $i(c, a_4) = 1$ is a standard single strand curve.

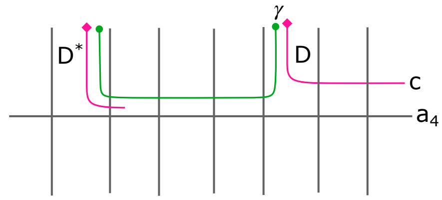
We recall that if c is such that $i(c, a_4) = 1$ and $c \cap a_0 \subset R_{a_4}$ then c is a *standard single strand curve*. If $\mathcal{T}$ is the disc which contains T, then there exists another top or, bottom bucket T^* which contains an endpoint of the arc of $c \cap \mathcal{T}$ containing the subarc $c' \cap \mathcal{T}$. If c is a standard single strand curve then $T^* = D$. If $T^* \neq D$, let $\mathcal{D}$ be the component of $S_g \setminus (a_0 \cup a_4)$ containing D. Let D^* be the bucket in $\mathcal{D}$ where the other end of the arc in $(c \setminus a_0) \cap \mathcal{D}$ containing $c' \cap D$ lies.

Lemma 7. *Let $T^* \neq D$. If either T^* or, D^* are inside c then there exists a representative of c that is a standard single strand curve.*

Proof. Without loss of generality, suppose that D^* is inside c. We show that D^* can't be a top bucket. Similar arguments ensure that whenever T^* is inside c, it can't be a bottom bucket. Assume on the contrary that D^* is a top bucket. Then the arc of c in D^* is either as in figure 1a or, 1b. In either case, consider γ as shown in figure 1. We observe that in figure 1, the arc $(\gamma \setminus a_0) \cap \mathcal{D}$ from D^* to D is parallel to the arc $(c \setminus a_0) \cap \mathcal{D}$ from D^* to D. By corollary 2, γ is essential. Since, γ is inside c it implies that γ is inside $\delta^{p,q}$. Thus, $\gamma \cap \delta = \phi$. Further, by the construction of γ, we have that δ, γ, a_4 is a path. But this contradicts $d(\delta, a_4) \geq 3$.



(A)



(B)

FIGURE 1

We give the isotopy for the case when T^* is inside c. Let c_D and c_{T^*} be any two points on the interior of the arcs $c \cap D$ and $c \cap T^*$, respectively. Let $\tilde{c}$ be the component of $c \setminus \{c_D \cup c_{T^*}\}$ that contains the point $c \cap a_4$. Let ζ_1 be the closed arc $c \setminus \tilde{c}$. Recall that $D = T_{i_0}$. If $T^* = T_{i_0 - s}$, let ζ_2 be the arc passing through T_{i_0-s}, T_{i_0-s+1}, $\ldots$, T_{i_0} parallel to a_4 and having end points c_D and c_{T^*}. Let ζ be the curve on S_g formed from the concatenation of ζ_1 and ζ_2.

We now show that ζ has to be a trivial curve on S_g. By construction, we have $\zeta \cap a_4 = \phi$. Since ζ_1 is an arc of c, $\zeta_1 \cap \delta = \phi$. Since T^* is inside c and hence, inside $\delta^{p,q}$, it follows that $\zeta_2 \cap \delta = \phi$. Thus, $\zeta \cap \delta = \phi$. If ζ is non-trivial it contradicts the fact that $d(\delta, a_4) \geq 3$.

Since ζ is trivial, we have that ζ_1 is isotopic to ζ_2 by an isotopy, say L'. We perform an isotopy, L, of c such that $L(\tilde{c}) = \tilde{c}$ and $L(c \setminus \tilde{c}) = L'(\zeta_1) = \zeta_2$. Thus $L(c)$ is a standard single strand curve on S_g whose strand has its end points in T and T^*.

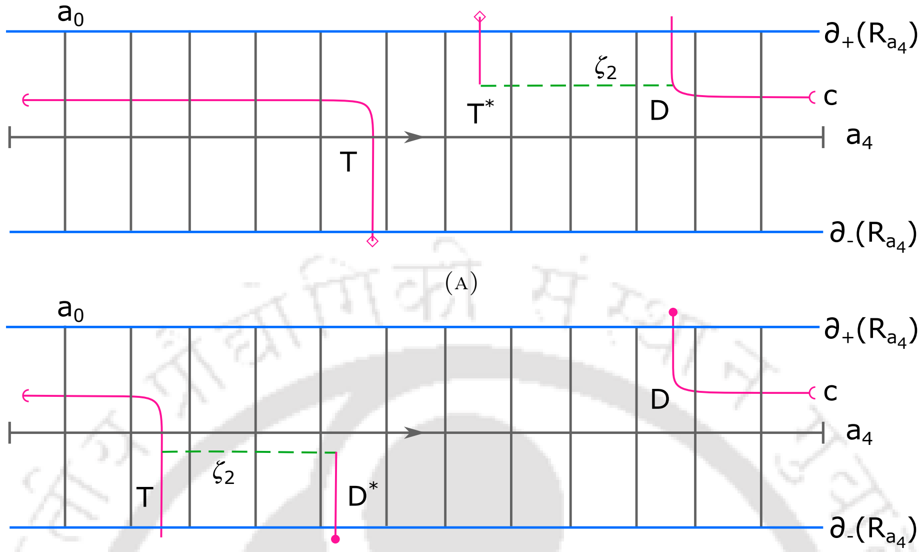
A similar proof follows for the case if D^* is inside c and T^* is outside c by reversing the roles of T^* with D^* and D with T.

$\square$

LEMMA 8. *If c is a standard single strand curve then, there exists a representative of c such that the end points of its strand lies in a top and a bottom bucket of a non-rectangular component of $S_g \setminus (a_0 \cup a_4)$.*

PROOF. Let us suppose $\mathcal{T}$ is a rectangle. From the notations defined in the previous section, it follows that the four vertices of $\mathcal{T}$ are w_d, w_{d+1}, w_{d-l} and w_{d-l+1}. Since c has a single strand, we have that $i(c, a_4) = 1$. As c and a_0 are in minimal position, the two parallel edges corresponding to the a_0-arcs in $\mathcal{T}$ are as follows : one edge is between w_{d-l} and w_d and the other edge is between w_{d-l+1} and w_{d+1}. A schematic of the bigon formed between c snd a_0 if the a_0-edges in $\mathcal{T}$ are otherwise is shown in figure 3. We note that, since $\delta \cap c = \phi$, we have that $c \cap \partial_+(R_{a_4})_{[d,d+1]} = \phi$. This means that $D = T_d$ and T_q don't coincide.

We first show that there exists j such that $d \leq j \leq q$ and T_j is a top bucket of some non-rectangular component of $S_g \setminus (a_0 \cup a_4)$. On the contrary assume that T_j



(A)

(B)

FIGURE 2

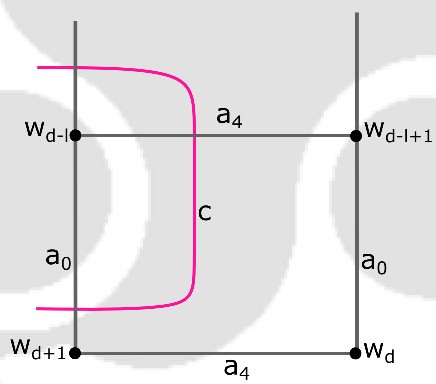


FIGURE 3. The possible bigon formed if the a_4 edges in $\mathcal{T}$ aren't parallel.

for every $d \leq j \leq q$ is a top bucket of some rectangular disc. Let T_q be contained in the rectangular disc $\mathcal{R}$ and let the component of $\delta^q \cap \mathcal{R}$ with end points on a_0 and $\partial_+(R_{a_4})$ be δ^q_*. Then the a_4-edges in $\mathcal{R}$ are $a_{4_{[w_q, w_{q+1}]}}$ and $a_{4_{[w_{q-l}, w_{q-l+1}]}}$. The union of the rectangular components of $S_g \setminus (a_0 \cup a_4)$ containing T_j for every $d \leq j \leq q$ is again a rectangle, say R. In particular, $\mathcal{T}$ and $\mathcal{R}$ are contained in R. Since the a_4 edges of $\mathcal{T}$ are in oriented parallelly, it gives that the a_4 edges of R are also oriented parallelly. In particular, the a_4 edges of $\mathcal{R}$ are oriented parallelly. As a result $B_{q-l} \subset \mathcal{R}$. Thus the component of $\delta \cap \mathcal{R}$ that contains δ^q_* has an end point on $a_{4_{[w_{q-l}, w_{q-l+1}]}}$, say x. A schematic of the above description is as in figure

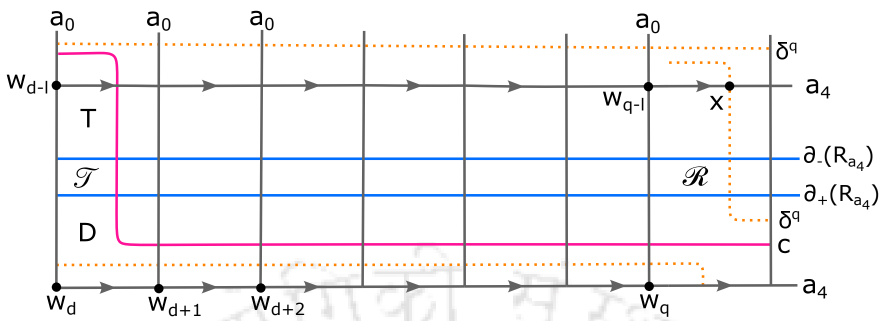


FIGURE 4. A schematic of c

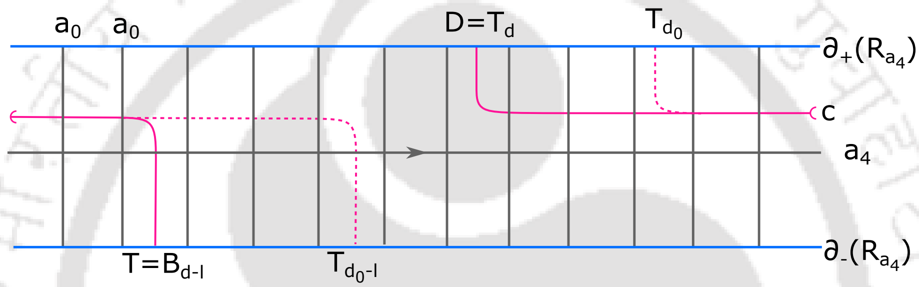


FIGURE 5. The dotted line is a schematic of $L(c)$

4. Since $\delta^q \setminus a_0$ covers every arc of $a_4 \setminus a_0$ except $a_{4_{[w_q, w_{q+1}]}}$, there is an arc of δ^q parallel to $a_{4_{[w_{d-l}, w_{q-l+1}]}}$ in R_{a_4}. Since there are no points of $\delta \cap a_0$ on a_0^{q-l} between $\delta^q \cap a_0^{q-l}$ and w_{q-l}, there is no possibility of x joining to any arc of $\delta \setminus a_0$. Thus, we have that there is a T_j for some $d \leq j \leq q$ such that T_j is contained in some non-rectangular component of $S_g \setminus (a_0 \cup a_4)$.

Consider the non-rectangular disc H such that there is a top bucket T_{d_0} of H where $d < d_0 \leq q$ and T_j for $d \leq j < d_0$ are top buckets of rectangular discs. Since T_j are rectangles for $d \leq j < d_0$, we have that B_{d-l+i} for $0 \leq i < d_0 - d$ are bottom buckets of rectangles. Further, for $0 \leq i < d_0 - d$, T_{d+i} and B_{d-l+i} are buckets of the same rectangular disc. Consider an isotopy, L, of c that moves the point $c \cap a_4$ along the increasing direction of a_4 from $a_{4_{[w_{d-l}, w_{d-l+1}]}}$ to $a_{4_{[w_{d_0-l}, w_{d_0-l+1}]}}$ such that $L(c)$ is in minimal position with a_0 and a_4. A schematic of L is shown in figure 5. Since T_{d_0} is in the same δ-track as D and T, $L(c)$ remains to be in minimal position with δ. □

As a consequence of lemma 8, we can now assume that the strand of c has its end points in a top and bottom buckets, A_t, A_b of a non-rectangular disc, say H. If H is a $2n$-gon with $n \geq 4$, then by lemma 5 there exists either a top or, bottom bucket A of H outside the delta track $\delta^{p,q}$. We will assume that A is a top bucket.

When A is a bottom bucket, similar conclusions can be made about c and a_0 by interchanging the roles of A_t and A_b in the following arguments.

Let γ be a scaling curve from A_t to A. Let γ' be the arc in $\gamma \cap H$ which contains $(\gamma \cap A) \cup (\gamma \cap A_t)$. By corollary 2, we have that γ and a_0 fill. The construction of γ gives that every arc in $(\gamma \setminus a_0) \setminus \gamma'$ is covered by some arc in $c \setminus a_0$. Let Λ_1 and Λ_2 be the two polygonal components of $S_g \setminus (a_0 \cup \gamma)$ containing the two edges corresponding to γ'. Let Λ be the component formed by gluing Λ_1 and Λ_2 along γ'. If Λ_1 and Λ_2 are distinct, then Λ is a disc. It then follows that the components of $(S_g \setminus a_0) \setminus c$ correspond to the components of $\{(S_g \setminus a_0) \setminus (\gamma \setminus \gamma')\} \cup \Lambda$. Thus, c and a_0 fill whenever $\Lambda_1 \neq \Lambda_2$. If $\Lambda_1 = \Lambda_2$, then Λ is an annulus. Note that by construction of γ there exists arcs of $a_4 \setminus a_0$ that covers every arc of $(\gamma \setminus a_0) \setminus \gamma'$. Consider a representative of a_4 such that for every arc of $(\gamma \setminus a_0) \setminus \gamma'$, the respective arc of $a_4 \setminus a_0$ which covers it also overlaps it. Let $\mathcal{P}$ be the central curve of the annulus Λ. We have that $\mathcal{P}$ will be an essential curve on S_g. If not, a_0 ceases to be connected. Let the two boundary components of Λ be $\partial_+(\Lambda)$ and $\partial_-(\Lambda)$.

Let Y be a component of $S_g \setminus (a_0 \cup a_4)$ such that one of the arcs in $(a_4 \setminus a_0) \cap Y$, say y, has one of its end points on $\partial_+(\Lambda)$ and another on $\partial_-(\Lambda)$. We have that $(a_0 \cap a_4) \cap Y \subset \partial(\Lambda)$. Since $\gamma \cap a_4 = \phi$, the arcs in $(a_4 \setminus a_0) \cap Y$ are either in the interior of Λ or, in $\partial(\Lambda)$. Thus, Y is a polygon in the annulus Λ such that its edge y is in the interior of Λ. It follows that there is another arc in $(a_4 \setminus a_0) \cap Y$ with its end points on $\partial_+(\Lambda)$ and $\partial_-(\Lambda)$.

Since $\mathcal{P}$ is a curve on S_g with $\mathcal{P} \cap a_0 = \phi$ and a_0 and a_4 fill S_g, there must exist an arc y_1 in $a_4 \setminus a_0$ such that $y_1 \cap \mathcal{P} \neq \phi$. As $\mathcal{P}$ is the core curve of Λ, the end points of y_1 must lie on $\partial_+(\Lambda)$ and $\partial_-(\Lambda)$. Let Y_1 be one of the discs of $S_g \setminus (a_0 \cup a_4)$ containing an edge corresponding to y_1. By the argument in the previous paragraph, there exists another arc, y_2 in $a_4 \setminus a_0$ with end points on $\partial_+(\Lambda)$ and $\partial_-(\Lambda)$. Let Y_2, if exists, be the other disc which contains the other edge corresponding to y_2. We apply this process inductively to obtain all the discs Y_1, Y_2, $\ldots$, Y_l with edges y_1, y_2, $\ldots$, y_l having end points on distinct components of $\partial(\Lambda)$. If any y_i is not inside c we have that an arc of $c \setminus a_0$ covers this particular y_i. Thus, c and a_0 fill S_g. If all $y_i's$ lie inside c, then $T_{a_4}(a_0)$, δ, c, $\mathcal{P}$, a_0 is a geodesic of distance 4. If such a geodesic of length 4 exists with $l = 4$, a schematic of R_{a_4} and $\mathcal{P} \cap R_{a_4}$ is as in figure 6 upto renaming of the components Y_i for $1 \leq i \leq 4$.

If $i(c, a_4) = 1$ but c is not a standard single strand curve, by lemma 7 neither T^* nor D^* are inside c. Then T^* can be either a top or, bottom bucket. If T^* is a bottom bucket, consider the scaling curve γ as in 7a and let the arc in $\gamma \setminus a_0$ between T and T^* be γ'. It can be seen from 7a that by virtue of the choice of γ, $c \setminus a_0$ contains a subset of arcs that cover all the arcs in $(\gamma \setminus a_0) \setminus \gamma'$. Since T^* doesn't lie inside c, there exists a pair of arcs in $c \setminus a_0$ that almost covers γ'.

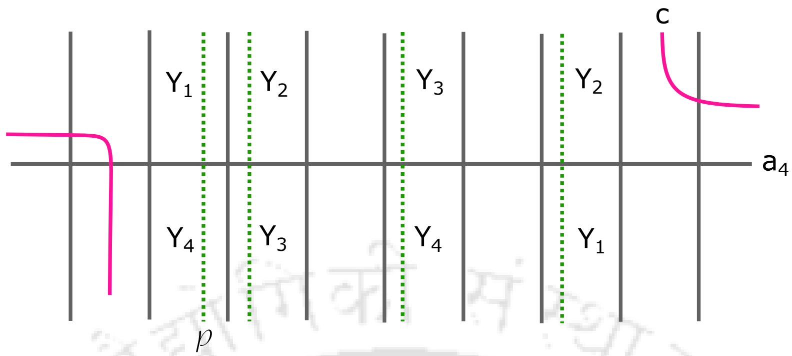
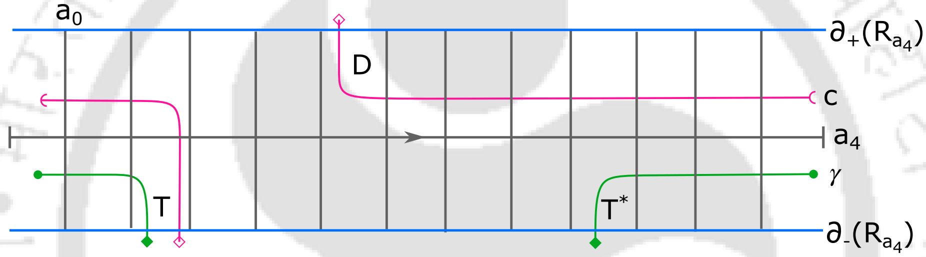
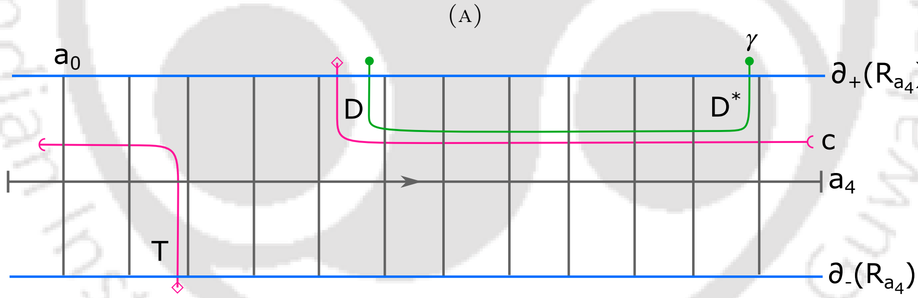


FIGURE 6. A schematic of $\mathcal{P}$ (dotted lines) if $Y_1, \ldots, Y_4$ occur as above in R_{a_4}.



(A)



(B)

FIGURE 7

This pair of arcs comprises of the following two arcs of $c \setminus a_0$: firstly the arc that contains the subarc $T^* \cap c$ and secondly, the arc that lies in the top bucket which has a common component of $a_4 \setminus a_0$ with T^*. This second arc of $c \setminus a_0$ covers the edge corresponding to $a_4 \cap T^*$. Thus by lemma 6, $c \setminus a_0$ forms a filling system of arcs on $S_g \setminus a_0$. If D^* is a top bucket, we construct γ as in figure 7b and a similar argument as above gives that c and a_0 fill S_g. If T^* is a top bucket and D^* is a bottom bucket then construct γ as in figure 8. Similar arguments as above along with lemma 6 concludes that c and a_0 fill S_g.

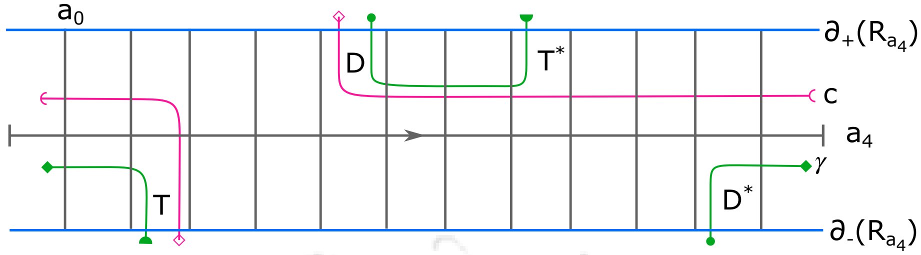


FIGURE 8

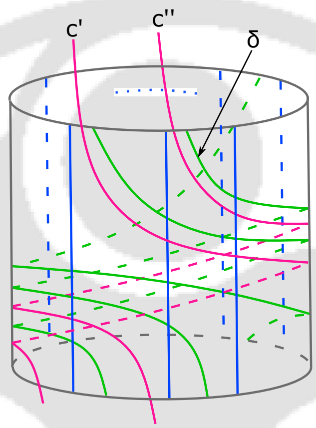


FIGURE 9. A schematic of c' and c'' in R_{a_4} when they do not cover every arc of $a_4 \setminus a_0$.

Case ii : Suppose there exists at least two distinct strands, c' and c'', of c in R_{a_4}. Assume that $c' \cap \partial_+(R_{a_4})$ lie on $\partial_+(R_{a_4})_{[\delta^p, \delta^q]}$ and $c'' \cap \partial_+(R_{a_4})$ lie on $\partial_+(R_{a_4})_{[\delta^r, \delta^s]}$ where $1 \leq r < s \leq m$. If either $\delta^p \cap \partial_+(R_{a_4})$ and $\delta^q \cap \partial_+(R_{a_4})$ lie in the same top bucket or, $\delta^r \cap \partial_+(R_{a_4})$ and $\delta^s \cap \partial_+(R_{a_4})$ lie in the same top bucket then by lemma 2, c and a_0 fills S_g. The argument is similar to that of case i when the end points of the strand lie in T_e and B_e for some $e \in K = \{1, \ldots, k\}$. We can thus assume that $p \leq r$.

Suppose $p < r$. If $q < r$, then for every $j \in K$, there exists an arc of $(c' \cup c'') \setminus a_0$ that covers a_4^j. Thus, $c \setminus a_0$ forms a filling system of arcs in $S_g \setminus a_0$. If $q = r$, then figure 9 gives the only instance when there exists a $J \in K$ such that a_4^J isn't covered by an arc of $(c' \cup c'') \setminus a_0$. It follows from lemma 2 that $c \setminus a_0$ forms a filling system of arcs in $S_g \setminus a_0$.

Suppose $p = r$. We rename c' to be the strand of c such that one of the components of $\partial_+(R_{a_4}) \setminus \{\delta^p, c'\}$ doesn't contain any points of $\partial_+(R_{a_4}) \cap c$ in its interior. Let z be the component of $c \setminus R_{a_4}$ containing $c' \cap \partial_-(R_{a_4})$. Rename c'' to be the strand of c such that $z \cap c'' \neq \phi$. We claim that $z \cap c''$ lies on $\partial_+(R_{a_4})$.

Let z' be the component of $\partial_-(R_{a_4}) \setminus z$ such that no points of $\partial_-(R_{a_4}) \cap c$ lies in its interior. On the contrary, if $z \cap c''$ lies on $\partial_-(R_{a_4})$ then we get that z is isotopic to z'. This follows because the curve obtained by concatenating z and z' is a curve disjoint from δ and a_4. Since $d(\delta, a_4) \geq 3$, this curve has to be non-essential. Therefore, we must have that $z \cap c''$ lies on $\partial_+(R_{a_4})$. Let $\tilde{z}$ be the component of $c \setminus R_{a_4}$ containing $c' \cap \partial_+(R_{a_4})$. A similar argument ensures that the end point of $\tilde{z}$ lies in $\partial_-(R_{a_4})$. Thus, following the naming convention of D, D^*, T and T^* for the strand c' as in case i, we can consider a scaling curve as in figure 8. Similar arguments as in case i gives that c and a_0 fills S_g.

6.3. Conclusion

If $d(a_0, T_{a_4}(a_0)) = 4$, then there exists $\delta \in B_1(T_{a_4}(a_0))$ and corresponding to δ there exists $c \in B_1(\delta) \cap B_2(T_{a_4}(a_0))$ such that a_0, p, c, δ, $T_{a_4}(a_0)$ is a geodesic. We consider the representatives of δ and c to be the ones as described in section 5.4 and section 6.1, respectively. Then a schematic of a possible p in R_{a_4} is as in figure 6. We now describe an equivalent condition for the existence of p in the form of buckets. Given such a curve p, consider the collection $\mathcal{Y}_p$ of all top and bottom buckets in R_{a_4} containing p. Since $p \cap a_0 = \phi$, if $T_i \in \mathcal{Y}_p$ for some $i \in \{1, \dots, k\}$ then $B_i \in \mathcal{Y}_p$. Conversely, we define a collection of pairs of top and bottom bucket $\{(T_i, B_i)\}_{i \in I}$ for some $I \subset K = \{1, \dots, k\}$ where for every T_i there exists unique $j \in I$ and $j \neq i$ such that $T_i \cup T_j \subset Y$ (or, $T_i \cup B_j \subset Y$) for some component Y of $S_g \setminus (a_0 \cup a_4)$ as a *stack of buckets*. We note that given a stack of buckets we can always construct a curve disjoint from a_0. The pattern in figure 6 can be described as the inside of c containing a stack of buckets. For any given $c \in B_2(T_{a_4}(a_0))$ if the inside of c contains a stack of buckets, we say c *has the stacking property*. Thus, we conclude that :

Lemma 9. *$d(a_0, T_{a_4}(a_0)) = 4$ if and only if there exists $c \in B_2(T_{a_4}(a_0))$ such that c has the stacking property.*

From our analysis of curves $c \in B_2(T_{a_4}(a_0)) \cap B_1(\delta)$ in section 6.2, we have that if c is not a standard single strand curve then c and a_0 always fill. Thus we have the following theorem :

Theorem 9. *Let a_0 and a_4 be curves on S_g such that $d(a_0, a_4) = 4$ and the components of $S_g \setminus (a_0 \cup a_4)$ doesn't contain any hexagons. Then, $d(a_0, T_{a_4}(a_0)) \geq 5$ if and only if there doesn't exist any standard single strand curve $c \in B_2(T_{a_4}(a_0))$ having the stacking property.*

An advantage of theorem 9 is that it reduces the number of possible vertices through which a path of length 4 between a_0 and $T_{a_4}(a_0)$ if it exists can pass.

CHAPTER 7

A PAIR OF DISTANCE 5 CURVES ON $\mathcal{C}(S_2)$

The work in this chapter is part of the preprint [**16**].

Let a_0 and a_4 be curves on S_2 as in figure 1. These curves are at a distance 4 in $\mathcal{C}(S_2)$ and are taken from [**5**]. In this section we show that $d(a_0, T_{a_4}(a_0)) = 5$ by giving a geodesic between them. Let $b_0 = a_0$, b_1, b_2, b_3 be curves on S_2 as in figure 2 and b_4 be as in figure 4. The juxtaposition of the curves in figure 2 and 4 shows that b_0, b_1, b_2, b_3, b_4 form a path of length 4 in $\mathcal{C}(S_2)$.

Since a_0 and a_4 fill S_2, we can give a schematic of S_2 by giving the components of $S_2 \setminus (a_0 \cup a_4)$ as polygons whose vertices are the points of $a_0 \cap a_4$ marked as in figure 1 and edges correspond to arcs of $a_0 \setminus a_4$ or, $a_4 \setminus a_0$. Figure 6a - 10 represent all polygons but the rectangle with vertices 10, 9, 4, 5 of $S \setminus (a_0 \cup a_4)$. We give a juxtaposition of the curves b_4 and $T_{a_4}(a_0)$ in minimal position on S_2 by giving their arcs on the polygonal discs of $S \setminus (a_0 \cup a_4)$. Since the representatives of b_4 and $T_{a_4}(a_0)$ that we pick don't have any arcs in the rectangle of $S_2 \setminus (a_0 \cup a_4)$ with vertices 10, 9, 4, 5, we exclude this rectangles from the figures. In figure 6a - 10, the straight lines correspond to the arcs of $T_{a_4}(a_0)$ and the dotted ones correspond to b_4. Since there is no intersection between these arcs, we conclude that b_4 and $T_{a_4}(a_0)$ are at a distance 1 in S_2.

We now show that $d(a_0, T_{a_4}(a_0)) > 4$ by using lemma 9. Consider the curves γ_1, γ_2, γ_3 and γ_4 as in figure 3 which are at a distance 1 from a_4. If for any $i_0 \in \{1, 2, 3, 4\}$, $T_{a_4}^{-1}(\delta) \cap \gamma_{i_0} = \phi$ then a_0, $T_{a_4}^{-1}(\delta)$, γ_{i_0}, a_4 will form a path of length 3, which is absurd. Thus, $d(T_{a_4}^{-1}(\delta), \gamma_i) \geq 2$ for $i = 1, 2, 3, 4$. Now, since $d(T_{a_4}^{-1}(\delta), a_0) = 1$ and $T_{a_4}^{-1}(\delta) \cap a_4 \neq \phi$, the arcs in the non-empty set $T_{a_4}^{-1}(\delta) \cap R_{a_4}$ are parallel to the arcs in $a_0 \cap R_{a_4}$. Since, $T_{a_4}^{-1}(\delta) \cap \gamma_i \neq \phi$ for every $i = 1, 2, 3, 4$, we refer to figure 3 and observe that for any two possible consecutive arcs of $T_{a_4}^{-1}(\delta) \cap R_{a_4}$ there are no stack of buckets between them. We note that we can circumvent

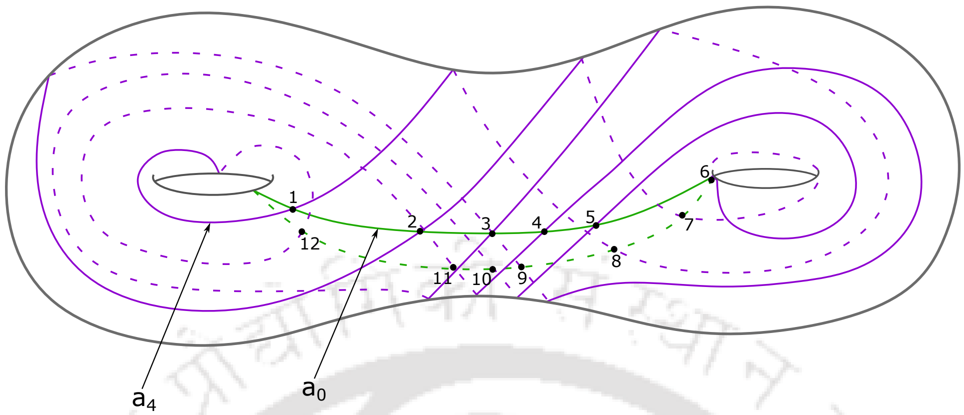


FIGURE 1

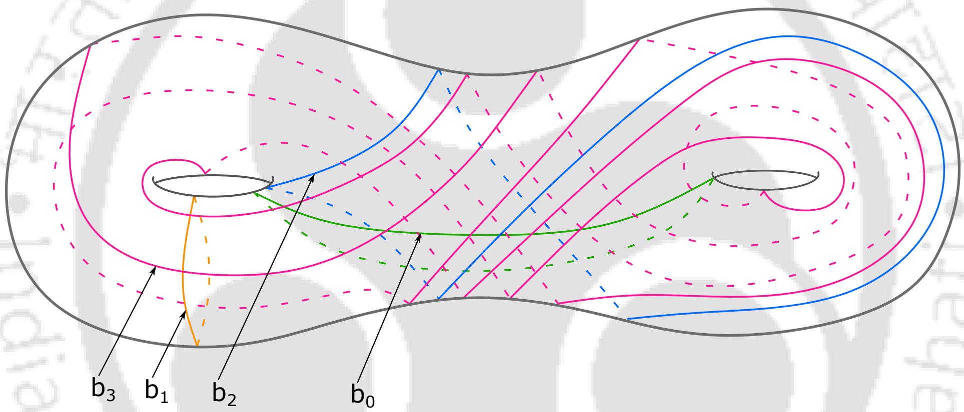


FIGURE 2

verifying the above for the set of all possible consecutive arcs of $T_{a_4}^{-1}(\delta) \cap R_{a_4}$ by looking at only the consecutive arcs of $T_{a_4}^{-1}(\delta) \cap R_{a_4}$ that has the maximum number of top buckets between them. Since the inside of a c is contained in some δ-track and the strands of δ that constitute the boundary of a δ-track are T_{a_4}-image of some arc of $T_{a_4}^{-1}(\delta)$ therefore, no c has the stacking property. Thus, $d(a_0, T_{a_4}(a_0)) > 4$.

From the above discussion, we conclude that the path in $\mathcal{C}(S_2)$ comprising of vertices $b_0 = a_0$, b_1, b_2, b_3, b_4, $b_5 = T_{a_4}(a_0)$ is a geodesic of length 5 in $\mathcal{C}(S_2)$. As an application of this example we give an upper bound on $i_{min}(2, 5)$ as follows :

COROLLARY 4. *$i_{min}(2, 5) \leq 144$.*

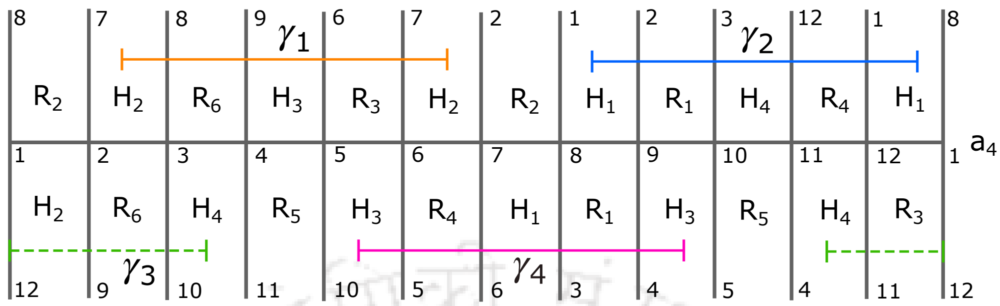


FIGURE 3. Regular neighbourhood of a_4 with $a_4 \cap a_0$ marked as in figure 1. The vertical arcs represent a_0.

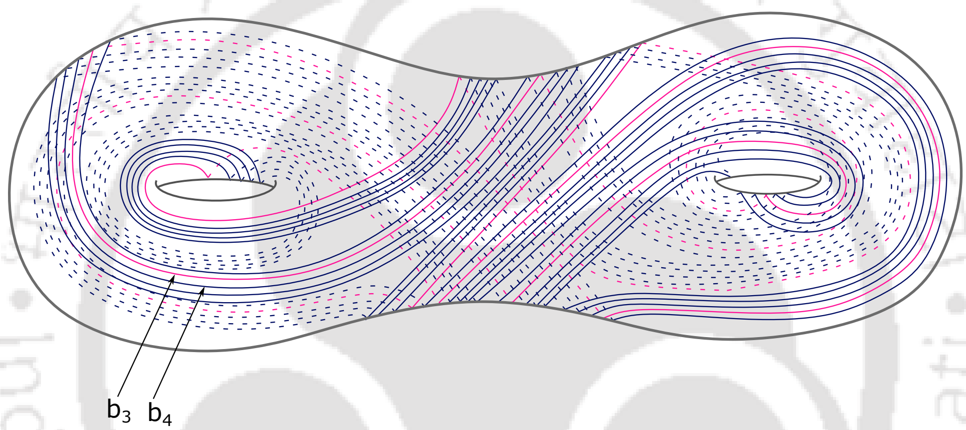


FIGURE 4

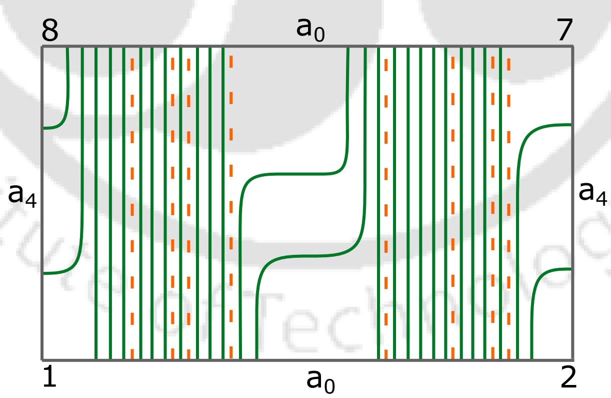
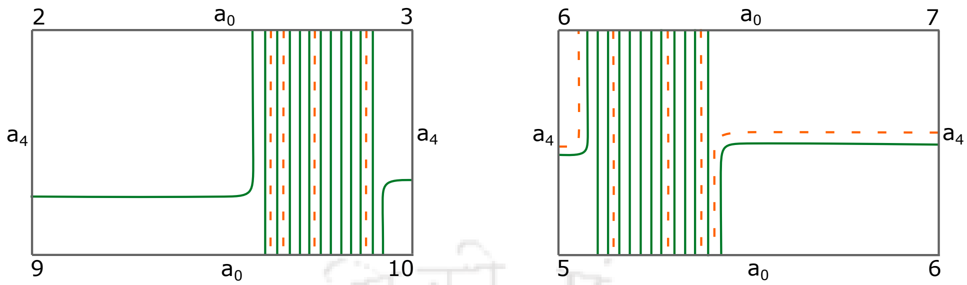
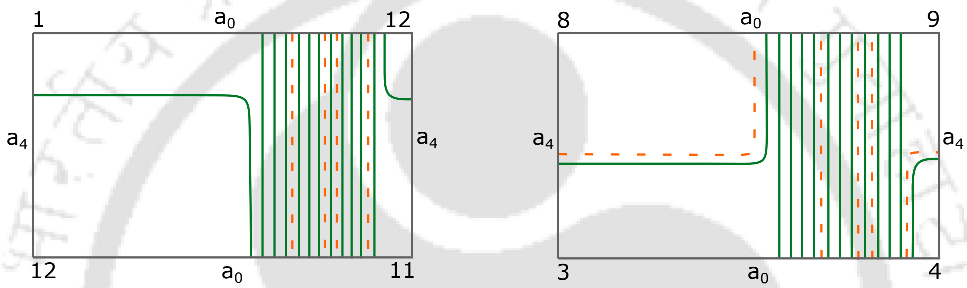


FIGURE 5. R_2



(A) R_1 (B) R_3

FIGURE 6



(A) R_4 (B) R_6

FIGURE 7

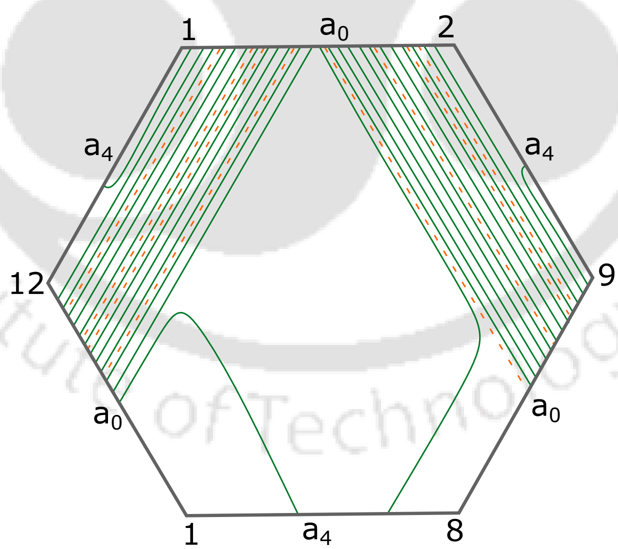


FIGURE 8. H_1

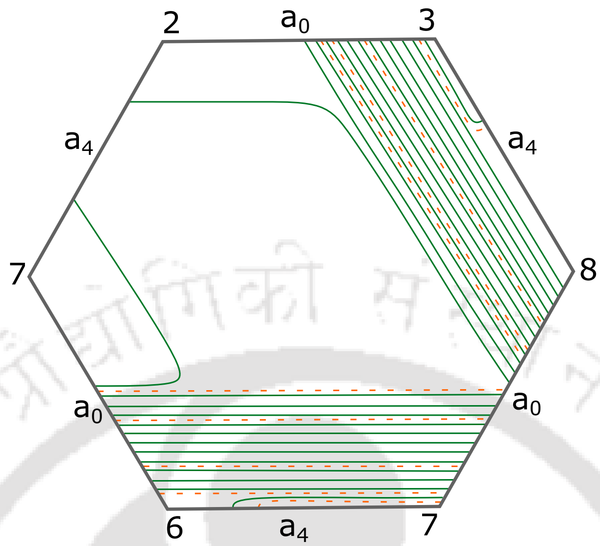
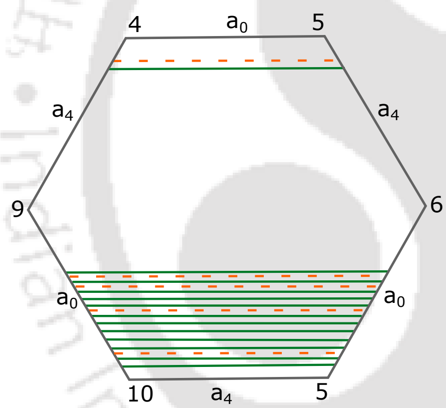
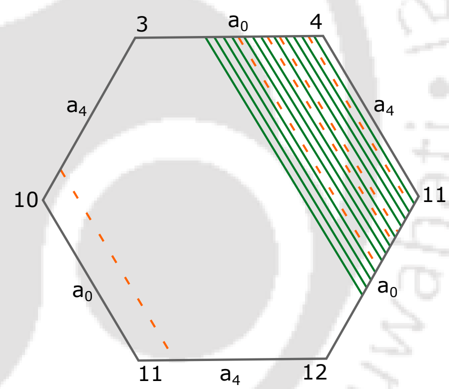


FIGURE 9. H_2



(A) H_3



(B) H_4

FIGURE 10

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