Dynamics of certain One-Parameter and Two-Parameter Families of Transcendental Functions

by

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A thesis submitted
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by

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Declaration

I hereby declare that the work contained in the thesis entitled **"Dynamics of certain One-Parameter and Two-Parameter Families of Transcendental Functions"** has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under the guidance of **Prof. M. Guru Prem Prasad**, Indian Institute of Technology Guwahati, for the award of **Doctor of Philosophy** and that this work has not been submitted elsewhere for a degree.

Guwahati
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Certificate

It is certified that the work contained in the thesis titled **"Dynamics of certain One-Parameter and Two-Parameter Families of Transcendental Functions"** by **Madhusudan Bera (136123003)**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of **Doctor of Philosophy** has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

Guwahati
May 2019

Prof. M. Guru Prem Prasad
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Dedicated

To

My Family

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ABSTRACT

Let $f : \mathbb{C} \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a nonconstant and nonlinear transcendental entire or meromorphic function. The function f^n (n-times composition of f) is called the n-th iterate of f. The Fatou set of the function f, denoted by $F(f)$, and defined by

$$F(f) = \left\{ z \in \widehat{\mathbb{C}} \ : \ \{f^n \ : \ n \in \mathbb{N}\} \text{ is defined and normal in some neighborhood of } z \right\}.$$

The complement of the Fatou set $F(f)$ of f in the extended complex plane $\widehat{\mathbb{C}}$, is called the Julia set of f and denoted by $J(f)$. In complex dynamics, the Fatou sets and the Julia sets of entire and meromorphic functions are mainly studied. By definition the Fatou set is open. A maximally connected subset of the Fatou set is known as a Fatou component. A Fatou component U is called p-periodic if p is the smallest natural number such that $f^p(U) \subseteq U$. Based on behaviour of function on the Fatou components, the p-periodic Fatou components are classified into five categories: (1) Basin of attraction, (2) Parabolic domain, (3) Siegel disk, (4) Herman ring and (5) Baker domain. For a Fatou component U, it is also possible $f^m(U) \cap f^n(U) = \emptyset$ for all $m \neq n$. Such a component is known as wandering domain.

In the present work, we have investigated the change of dynamics of transcendental entire and meromorphic functions in one-parameter and two-parameter families. **Chapter 1**, is the introductory chapter reviewing the basics and some useful results related to our work.

In **Chapter 2**, the dynamics of functions in two-parameter family $\mathcal{F}_b \equiv \{f_{\lambda,\mu}(z) = \lambda b^z - \mu b^{-z}$ for $z \in \mathbb{C} : \lambda,\ \mu > 0\}$ where $b > 1$, is investigated. The function $f_{\lambda,\mu}$ is skew symmetric about $x^*_{\lambda,\mu} = \ln(\mu/\lambda)/(2\ln b)$ and it has no singular values other than two critical values $\pm 2i\sqrt{\lambda\mu}$. On the real line $\mathbb{R}$, the existence and the nature of the real fixed points are described. The dynamical behaviour of $f_{\lambda,\mu}$ on the real line is exhibited. The dynamics of $f_{\lambda,\mu}$ on the extended complex, is studied by tracking forward orbits of the singular values of $f_{\lambda,\mu}$ whenever real attracting and rationally indifferent fixed points

exist. In the case of real attracting fixed point, it is proved that the basin of attraction corresponding to the real attracting fixed point is the Fatou set of $f_{\lambda,\mu}$. For the cases of the real rationally indifferent fixed points, it is shown that the Fatou set is the parabolic domain corresponding to the real rationally indifferent fixed point. Remaining cases, it is established that the Julia set $J(f_{\lambda,\mu})$ contains the real line $\mathbb{R}$ equivalently the Fatou set $F(f_{\lambda,\mu})$ is not connected if it exists.

Chapter 3 deals with the chaotic burst in the two-parameter family $\mathcal{G}_b \equiv \{g_{\lambda,\mu}(z) = \lambda b^z + \mu b^{-z}$ for $z \in \mathbb{C} : \lambda \geq \mu > 0\}$ where $b > 1$. The function $g_{\lambda,\mu}$ is symmetric about $x^0_{\lambda,\mu} = \ln(\mu/\lambda)/(2\ln b)$ and it has only two singular values $\pm 2\sqrt{\lambda\mu}$. The dynamics of $g_{\lambda,\mu}$ depends on $t_{\lambda,\mu} = \sqrt{1 + 4\lambda\mu(\ln b)^2} - \ln\left(\frac{1+\sqrt{1+4\lambda\mu(\ln b)^2}}{2\lambda\ln b}\right)$. In the family $\mathcal{G}_b$, the chaotic burst is observed when $t_{\lambda,\mu}$ becomes greater than 0. The Fatou set $F(g_{\lambda,\mu})$ for $t_{\lambda,\mu} < 0$, is the basin of attraction of a real attracting fixed point where as the Fatou set $F(g_{\lambda,\mu})$ for $t_{\lambda,\mu} = 0$, is the parabolic domain corresponding to a real rationally indifferent fixed point and the Fatou set $F(g_{\lambda,\mu})$ is empty for $t_{\lambda,\mu} > 0$. The Julia set $J(g_{\lambda,\mu})$ is nowhere dense, not locally connected and contains a vertical line for $t_{\lambda,\mu} \leq 0$.

In **Chapter 4**, we studied the dynamics of transcendental meromorphic functions from one-parameter families $\mathcal{F} \equiv \left\{f_\lambda(z) = \lambda\left(\cosh z + \frac{1}{\cosh z}\right) \text{ for } z \in \mathbb{C} : \lambda > 0\right\}$ and $\mathcal{G} \equiv \left\{g_\lambda(z) = \lambda\left(\cosh z - \frac{1}{\cosh z}\right) \text{ for } z \in \mathbb{C} : \lambda > 0\right\}$. The change in the dynamics of f_λ is observed when $t_\lambda := \lambda\left(\cosh x^*_\lambda + \frac{1}{\cosh x^*_\lambda}\right) - x^*_\lambda$ increases through 0 where x^*_λ is the real root of $\lambda\sinh^3 x - \cosh^2 x = 0$. For $t_\lambda < 0$, the Fatou set $F(f_\lambda)$ is the basin of attraction of a real attracting fixed point of f_λ, the Fatou set $F(f_\lambda)$ is the parabolic domain corresponding to a real rationally indifferent fixed point of f_λ for $t_\lambda = 0$ and the Fatou $F(f_\lambda)$ set is empty for $t_\lambda > 0$. Thus, bifurcation occurs at $t_\lambda = 0$. But for the function g_λ, the origin is always a superattracting fixed point for any value of the parameter λ. The Fatou set $F(g_\lambda)$ is the basin of attraction $A(0)$. The major differences between the dynamics of f_λ and g_λ, are pointed.

A one-parameter family $\mathcal{F} \equiv \{f_\lambda(z) = \lambda + \cosh z : \lambda \in \mathbb{R}\}$ of translated hyperbolic cosine functions is considered in **Chapter 5**. On the real line $\mathbb{R}$, the fixed points and the dynamics are studied. The function f_λ has no singular values other than two critical values $\lambda \pm 1$. The singular value $\lambda - 1$ always tends to ∞ under iteration of f_λ. In case of real attracting fixed points, the Fatou set $F(f_\lambda)$ is the basin of attraction of the real attracting fixed point of f_λ. In case of rationally indifferent fixed points, corresponding parabolic domain is the Fatou set $F(f_\lambda)$. The Fatou set $F(f_\lambda)$ is empty when both the singular values of f_λ tend to ∞ under iteration of f_λ. Remaining cases the dynamics is obtained by numerically computing the higher order real periodic points. It is observed that a period doubling bifurcation occurs in the family $\mathcal{F}$.

ABBREVIATION AND NOTATION

$\mathbb{N}$	The set of all natural numbers
$\mathbb{Z}$	The set of all integers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$\widehat{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$
$F(f)$	The Fatou set of the function f
$J(f)$	The Julia set of the function f
$I(f)$	The escaping set of f
$\mathcal{E}$	The set of all transcendental entire functions
$\mathcal{P}$	The set of all transcendental meromorphic functions having exactly one pole and that pole is an omitted value
$\mathcal{M}$	The set of all transcendental meromorphic functions having either at least two poles or exactly one pole and that pole is not an omitted value
$\text{sing}(f^{-1})$	The set of all critical values and finite asymptotic values of f and finite limit points of these values
$\mathcal{S}$	The set of all functions f such that $\text{sing}(f^{-1})$ is a finite set
$\mathcal{B}$	The set of all functions f such that $\text{sing}(f^{-1})$ is a bounded set

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CHAPTER 1
INTRODUCTION

1.1 Overview

A dynamical system is a system whose state evolves with time over a state space according to a fixed rule. Complex dynamics is the study of iteration of entire and meromorphic functions on the Riemann sphere. An elaborate study on iteration of polynomials and rational functions was done by two French mathematicians Pierre Fatou [39–41] and Gaston Julia [47, 48] during 1918 to 1920. At the same time, Ritt [65] also investigated the iteration of rational functions. After few years, some of the results are extended to the transcendental entire functions by Fatou [42] in 1926. But he did not consider transcendental meromorphic functions because of the presence of infinitely many essential singularities of the iterates. After long period of inactivity, there was renewed interest in this area in the last few decades. To describe complex dynamics, Mandelbrot used computer graphics successfully [56, 57] in 1980. His discovery of Mandelbrot set which is a fractal set inspired many researchers to study the field again. Sullivan [77, 78] introduced the use of quasi-conformal mapping into the subject. He proved the non-existence of wandering domains for rational functions using quasi-conformal mappings. For an introduction to dynamics of rational functions, one can see books by Beardon [12], Carleson and Gamelin [21], Steinmetz [76] and Milnor [58]. More about iteration theory of rational functions can be found in [19, 26, 32, 53, 55]. Many researchers tried to extend results of rational functions to the transcendental case. Baker proved some dynamical properties of entire functions that are quite different from those of rational maps [2]. Devaney [31], Bergweiler [17], Fagella [38], Rempe [43, 62, 67] and many other [5, 13, 36, 61] studied the iteration of transcendental entire functions. For iteration of transcendental meromorphic functions one can refer [12, 14, 19, 21, 22, 58, 76].

1.2 Fatou and Julia sets

For an entire or meromorphic function f, let $z_0 = f^0(z_0)$, $z_n = f(z_{n-1}) = f^n(z_0)$ for $n = 1,\ 2,\ 3, \cdots$. Then, the sequence $\{z_n = f^n(z_0)\}_{n=0}^{\infty}$ is called the sequence of iterates or the (forward) orbit of the point z_0. In the iteration theory, we investigate the behaviour of the orbits of various initial points. While studying the long term behaviour of different initial points, it is observed that for some initial points z_0, the orbits of all points z in some neighbourhood of z_0 exhibit similar behaviour, but certain other initial points z_0, the orbits of all points z in every neighbourhood of z_0 differ drastically. In other words, the forward orbits of some points remain stable under small perturbation which is not true for the other case. Fatou and Julia first studied these things. In honour of the contributions of these two mathematicians, the stable region is now known as the Fatou set and the chaotic region is known as the Julia set. In order to give the precise definitions of the Fatou set and the Julia set, we need to define the concept of normality. Montel [60] introduced the concept of normality in 1927. To know more about normality, one can refer to [74].

Definition 1.2.1. *A family $\mathcal{F}$ of analytic functions defined on a common domain $\Omega \subseteq \mathbb{C}$ is said to be normal in Ω if every sequence $\{f_n\} \subseteq \mathcal{F}$ contains either a subsequence which converges uniformly to a limit function $f \not\equiv \infty$ on each compact subset of Ω, or a subsequence which converges uniformly to ∞ on each compact subset of Ω.*

Let $z_1,\ z_2$ be any two points in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The

chordal distance between z_1 and z_2 is defined by

$$\chi(z_1, z_2) = \begin{cases} \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} & \text{if } z_1, z_2 \in \mathbb{C}, \\ \frac{1}{\sqrt{1+|z_1|^2}} & \text{if } z_2 = \infty. \end{cases}$$

A sequence of functions $\{f_n\}$ converges **spherically uniformly** to f on a set $E \subseteq \mathbb{C}$ if, for any $\epsilon > 0$, there is a number $n_0 \equiv n_0(\epsilon)$ such that $n \geq n_0$ implies that $\chi(f_n(z), f(z)) < \epsilon$ for all $z \in E$.

Note that if $\{f_n\}$ converges uniformly to f on E, then it also converges spherically uniformly to f on E. The converse is true if the limit function is bounded.

Definition 1.2.2. *A family $\mathcal{F}$ of meromorphic functions defined on a common domain $\Omega \subseteq \mathbb{C}$ is normal in Ω if every sequence $\{f_n\} \subseteq \mathcal{F}$ contains a subsequence which converges spherically uniformly on each compact subset of Ω.*

Some examples of normal family are given below.

Example 1.2.1. *The family $\mathcal{F} = \{f_n(z) = z^n \text{ for all } z \in \mathbb{C} : \ n \in \mathbb{N}\}$ is normal in $\Omega = \{z \in \mathbb{C} : |z| \neq 1\}$.*

Example 1.2.2. *The family $\mathcal{F} = \{f_c(z) = ce^z \text{ for all } z \in \mathbb{C} : c \in \mathbb{R}\}$ is normal in $\mathbb{C}$.*

Example 1.2.3. *Let $\mathcal{F} = \{f_n(z) = nz \text{ for all } z \in \mathbb{C} : \ n \in \mathbb{N}\}$. Then $f_n(0) \to 0$ as $n \to \infty$ and $f_n(z) \to \infty$ as $n \to \infty$ for $z \neq 0$. Thus, $\mathcal{F}$ cannot be normal in any domain that contains 0.*

From the definition, it is difficult to examine the normality of a family of functions. However the normality of a family of functions can be determined by the Fundamental Normality Test.

The following Fundamental Normality Test for the family of analytic functions, is given by Montel [59].

Theorem 1.2.1. *Let $\mathcal{F}$ be a family of analytic functions defined on a common domain Ω in $\mathbb{C}$. If $\mathcal{F}$ omits two distinct values a and b in $\mathbb{C}$ (i.e., $f(z) \neq a, b$ for all $z \in \Omega$ and for all $f \in \mathcal{F}$), then $\mathcal{F}$ is normal in Ω.*

For the family of meromorphic functions, the Fundamental Normality Test is given in the following theorem.

Theorem 1.2.2. *Let $\mathcal{F}$ be a family of meromorphic functions defined on a common domain $\Omega \subseteq \mathbb{C}$ such that $f(z) \neq a_j$ for all $j \in \{1, 2, 3\}$, for all $f \in \mathcal{F}$ and for all $z \in \Omega$, where $a_1, a_2, a_3 \in \mathbb{C}$ are distinct. Then $\mathcal{F}$ is normal in Ω.*

Throughout this thesis, we assume that f is nonconstant and nonlinear.

Definition 1.2.3. *[14] The **Fatou set** of a meromorphic function f, denoted by $F(f)$, is defined as*

$$F(f) = \{z \in \widehat{\mathbb{C}} : \{f^n : n \in \mathbb{N}\} \text{ is defined and normal in some neighbourhood of } z\}$$

where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane.

Definition 1.2.4. *The **Julia set** of f denoted by $J(f)$, is the complement of the Fatou set of f in the extended complex plane $\widehat{\mathbb{C}}$.*

Example 1.2.4. *Let $f(z) = z^2$ for $z \in \mathbb{C}$. Then $f^n(z) \to 0$ as $n \to \infty$ for $|z| < 1$ and $f^n(z) \to \infty$ as $n \to \infty$ for $|z| > 1$. Thus, the Fatou set $F(f) = \{z \in \mathbb{C} : |z| \neq 1\}$ and the Julia set $J(f) = \{z \in \mathbb{C} : |z| = 1\}$.*

For a transcendental function f, $f(\infty)$ is not defined, so ∞ always is in the Julia set of f. Similarly, the Julia set contains all such points whose forward orbit contains ∞. By definition, the Fatou set is open and the Julia set is closed. The Julia set has empty interior unless it is whole of $\widehat{\mathbb{C}}$. For example, the Julia set of the function e^z is $J(e^z) = \widehat{\mathbb{C}}$. Elementary properties of the Fatou set and the Julia set of $f(z)$ are given in the following propositions [14].

Proposition 1.2.1. *Let f be an entire function or a rational function. Then $F(f) = F(f^n)$ and $J(f) = J(f^n)$ for all $n \geq 2$.*

Proposition 1.2.1 is true if f is a transcendental meromorphic function having exactly one pole and that pole is an omitted value. For a meromorhic function f having at least two poles or exactly one pole which is not an omitted value, f^n is not meromorphic in $\mathbb{C}$ so that $F(f^n)$ and $J(f^n)$ are not defined. Proposition 1.2.1 holds true if it is taken more generally as f is meromorphic in $\mathbb{C}$ except for countably many points.

Definition 1.2.5. *Let f be a function. A set S is called forward invariant if $f(z) \in S$ for all $z \in S$ whenever $f(z)$ is defined. A set S is called backward invariant if $w \in S$ implies $z \in S$ for all z satisfying $f(z) = w$. A set S is called completely invariant if S is both forward and backward invariant.*

Proposition 1.2.2. *Let f be a meromorphic function. Then the Fatou set $F(f)$ and the Julia set $J(f)$ are completely invariant.*

Definition 1.2.6. *A set S is said to be perfect if S is nonempty, closed and does not contain isolated points.*

Proposition 1.2.3. *Let f be a meromorphic function. Then the Julia set $J(f)$ is perfect.*

For a meromorphic function f, let $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \text{ and } f^n(z) \neq \infty\}$. The points of the set $I(f)$ are known as escaping points of f. the set $I(f)$ is known as the **escaping set** of f. For a meromorphic function, any iterated preimage of infinity is not in the escaping set. Clearly, poles are not in the escaping set. The following theorem gives a characterization of the Julia set in terms of escaping points [34, 35].

Theorem 1.2.3. *If f is a transcendental entire or meromorphic function, then $J(f) = \partial I(f)$ and $I(f) \bigcap J(f) \neq \emptyset$, where $\partial I(f)$ is the boundary of the escaping set $I(f)$.*

The escaping sets of entire functions are studied in [43, 52, 62, 67, 75]. The escaping set of meromorphic functions are investigated in [16, 51, 63, 64, 79].

Definition 1.2.7. *The backward orbit of z_0, denoted by $O^-(z_0)$ is defined as $O^-(z_0) = \{z \in \mathbb{C} : f^n(z) = z_0 \text{ for some } n \in \mathbb{N}\}$. A point z_0 is said to be an exceptional value for a function f if the set $O^-(z_0)$ is finite. The point z_0 is called an omitted value of f if $O^-(z_0)$ is empty.*

A characterization of the Julia set in terms of nonexceptional value is given in the following theorem [7].

Theorem 1.2.4. *If $z_0 \in J(f)$ is not an exceptional value of f then $J(f) = \overline{O^-(z_0)}$, the closure of the backward orbit of z_0.*

Bergweiler [14] classified all transcendental meromorphic functions into three classes.

- $\mathcal{E} = \{f \ : \ f$ is transcendental entire$\}$.
- $\mathcal{P} = \{f \ : \ f$ is transcendental meromorphic, has exactly one pole, and this pole is an omitted value$\}$.
- $\mathcal{M} = \{f \ : \ f$ is transcendental meromorphic and has either at least two poles or exactly one pole which is not an omitted value$\}$.

The functions in the class $\mathcal{E}$ are called entire. The functions in $\mathcal{P}$ are known as analytic self maps of the punctured plane. In the class $\mathcal{P}$, each function has exactly one pole and that pole is an omitted value. If we choose 0 as the pole, then functions in $\mathcal{P}$ look like $\frac{e^{f(z)}}{z^n}$ where f is an entire function and $n \in \mathbb{N}$. According to Bergweiler [14], the functions f in $\mathcal{M}$ are called general meromorphic functions. There are major differences in the dynamics of functions belonging to the above three classes. For $f \in \mathcal{E}$, the iterates $f^n(z)$ are defined for all $z \in \mathbb{C}$ and the point at ∞ is an exceptional (omitted) value. In the class $\mathcal{P}$, if function is chosen of the form $\frac{e^{f(z)}}{z^n}$, then 0 and ∞ have finite orbits. The point at ∞ is not an exceptional value of f for $f \in \mathcal{M}$. Therefore, the backward orbit $O^-(\infty)$ of ∞ is an infinite set where f^n fails to be defined for some n. For general meromorphic functions, we have yet another characterization of the Julia set.

Theorem 1.2.5. *If $f \in \mathcal{M}$, then $J(f) = \overline{O^-(\infty)}$.*

1.3 Periodic points

In complex dynamics, there are points with finite forward orbits play an important role. Most oftenly, these points control the local dynamics.

Definition 1.3.1. *A point $z_0 \in \mathbb{C}$ is called a* ***periodic point*** *of period p for the function f if p is a natural number such that $f^p(z_0) = z_0$. The smallest positive integer p which satisfies $f^p(z_0) = z_0$, is called minimal or prime period of z_0. For a periodic point z_0 of minimal period p, the set $\{z_0,\ z_1 = f(z_0),\ z_2 = f^2(z_0),\ \ldots,\ z_{p-1} = f^{p-1}(z_0)\}$ is called a cycle of the periodic point z_0. The value $\lambda = (f^p)'(z_0)$ is called the* ***multiplier*** *or* ***eigenvalue*** *of the periodic point z_0 with minimal period p.*

The periodic points of period one are called fixed points. If for a point z_0, $f^{n_0}(z_0)$ is periodic for some $n_0 \in \mathbb{N}$, then z_0 is called a preperiodic point. The periodic points are classified into three categories according to the magnitude of their multipliers.

- The periodic point z_0 is called attracting if $|\lambda| < 1$. An attracting periodic point is called superattracting if $\lambda = 0$.

- The periodic point z_0 is called indifferent or neutral if $|\lambda| = 1$. In this case λ can be written as $\lambda = e^{2\pi i\alpha}$ for some real number α. If α is rational then z_0 is called a rationally indifferent periodic point and if α is irrational then z_0 is called an irrationally indifferent periodic point.

- The periodic point z_0 is called repelling if $|\lambda| > 1$.

Example 1.3.1 (Attracting periodic point). *Let $f(z) = z^2 - 1$ be the entire function. Then 0 is an attracting periodic point of $f(z)$ of minimal period 2.*

Example 1.3.2 (Rationally indifferent periodic point). *Let $f(z) = z + z^5$ be the entire function. Then 0 is a rationally indifferent fixed point of $f(z)$.*

Example 1.3.3 (Irrationally indifferent periodic point). *Let $f(z) = e^{i2\pi\alpha}z + z^3$ be the entire function where α is an irrational number. Then 0 is an irrationally indifferent fixed point of $f(z)$.*

Example 1.3.4 (Repelling periodic point). *Let $f(z) = z^2$ be the entire function. Then 1 is a repelling fixed point of $f(z)$.*

The attracting periodic points are in the Fatou set while the repelling and rationally indifferent periodic points are in the Julia set [14]. But for the irrationally indifferent periodic points both possibilities do occur.

Example 1.3.5 (Irrationally indifferent periodic point lies in the Fatou set). *Let $f(z) = e^{2\pi i\alpha}z + z^2$ be the entire function where α is the golden ratio. Then the origin is an irrationally indifferent fixed point and it lies in the Fatou set (see Example 1.5.4).*

Example 1.3.6 (Irrationally indifferent periodic point lies in the Julia set). *Let $f(z) = \alpha z + \cdots + z^d$, where $d \geq 2$ and $|\alpha| = 1$ but α is not a root of unity. If $|\alpha^n - 1| \leq (1/n)^{d^{n-1}}$ for infinitely many n, then the origin is an irrationally indifferent fixed point of f and it lies in the Julia set $J(f)$ (Theorem 6.7.1 of [12]).*

For any $n \in \mathbb{N}$, a rational function has periodic points of period n (not necessarily minimal period). But a transcendental entire function need not have a fixed point. For example, the function $f(z) = e^z + z$ has no fixed points. Fatou [42] proved that transcendental entire function has at least one periodic point of period 2. Fatou's result is generalized by Rosenbloom [66], which is stated below.

Theorem 1.3.1. *Let f be an transcendental entire function. Then f has infinitely many periodic points of period n for all $n \geq 2$.*

For transcendental meromorphic function Bergweiler [14] proved the following theorem.

Theorem 1.3.2. *If f is a transcendental meromorphic function and $n \geq 2$, then f has infinitely many periodic points of minimal period n.*

The Julia set of a transcendental meromorphic function can be characterized in terms of repelling periodic points as follows [7].

Theorem 1.3.3. *Let f be a meromorphic function. Then the Julia set $J(f)$ is the closure of the set of repelling periodic points of f.*

1.4 Singular values

Like periodic points, there are other points which play crucial role to study the dynamics of a function. The singular values are one such points.

Definition 1.4.1. *A point z_c is said to be a critical point of the meromorphic function f if $f'(z_c) = 0$. The value $f(z_c)$ at the critical point z_c, is called a critical value of f. A point a is called an asymptotic value of the function f if there exists a continuous curve $\gamma(t)$, $t > 0$ such that $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = a$. The curve γ is called an asymptotic path. A point s is called a* ***singular value*** *of a function f if s is either a critical value or a finite asymptotic value of f.*

Example 1.4.1. *Let $f(z) = e^z$ be an entire function. Let $\gamma(t) = -t$ for $t \in (0, \infty)$. Then $\lim_{t\to\infty} \gamma(t) = -\infty$ and $\lim_{t\to\infty} f(\gamma(t)) = 0$. So. 0 is an asymptotic value of f.*

Let $\operatorname{sing}(f^{-1})$ denote the set of all critical values and finite asymptotic values of f and finite limit points of these values [14]. The following two classes of functions were introduced depending on the presence of the singular values:

- $\mathcal{S} = \{f : f$ has only finitely many critical and asymptotic values$\}$;
- $\mathcal{B} = \{f : \operatorname{sing}(f^{-1})$ is bounded$\}$.

Clearly, $\mathcal{S} \subset \mathcal{B}$. According to Eremenko and Lyubich, the class $\mathcal{S}$ was chosen in honor of Speiser, who introduced this class in a different context. The class $\mathcal{B}$ was considered by Eremenko and Lyubich [37].

1.5 Components of Fatou set

A maximally connected open subset of the Fatou set is called a component of the Fatou set. Also it is called as a Fatou component. Since the Fatou set is completely invariant, any Fatou component U is mapped into a Fatou component but not necessarily in U.

Definition 1.5.1. *A component U of the Fatou set $F(f)$ of a function f is called p-periodic if p is the smallest natural number satisfying $f^p(U) \subseteq U$. Then the set $\{U, f(U), f^2(U), \ldots, f^{p-1}(U)\}$ is called a periodic cycle of Fatou components. If $p = 1$ then the component U is called invariant.*

Definition 1.5.2. *A component U of the Fatou set $F(f)$ of a function f is said to be preperiodic if there exists a natural number k such that $f^k(U)$ is periodic. A preperiodic Fatou component which is not periodic is called strictly preperiodic.*

Definition 1.5.3. *A component W of the Fatou set $F(f)$, is called a* **wandering domain** *if $f^m(W) \cap f^n(W) = \emptyset$ for all $m, n \in \mathbb{N}$ with $m \neq n$.*

Example 1.5.1 (Wandering domain). *Let $f(z) = z - 1 + e^{-z} + 2\pi i$ be an entire function. To prove f has a wandering domain, define $g(z) = z - 1 + e^{-z}$. Let $z_k = 2\pi k i$ where $k \in \mathbb{Z}$. Then z_k is a superattracting fixed point of the function g. Denote by U_k the immediate basin of attraction corresponding to z_k, that is, the component of $F(g)$ that contains z_k. It can be proved $J(g) = J(f)$ [3, 14]. Therefore U_k is also a component of the Fatou set $F(f)$. It is easy to see that $f(U_k) \subseteq U_{k+1}$ and hence U_k is wandering.*

Sullivan [78] proved that the Fatou set of a rational function has no wandering domains. But it exists for transcendental meromorphic function. For a study on wandering domain, one can refer to [3, 4, 11]. Wandering domains are related to the singular values. In [14], it is shown that some classes of meromorphic functions do not have wandering domains. The following theorem is one such result from [14].

Theorem 1.5.1. *The functions in $\mathcal{S}$ do not have wandering domains.*

The periodic Fatou components can be classified into five categories depending upon the behaviour of the sequence $\{f^n\}$ on the components, given in the following theorem.

Theorem 1.5.2. *Let f be a meromorphic function. Let U be a periodic component of the Fatou set of f of minimal period p. Then we have one of the following possibilities.*

- *U is a **basin of attraction**: In this case, the periodic component U contains an attracting periodic point z_0 of minimal period p and $\lim_{n\to\infty} f^{np}(z) = z_0$ for all $z \in U$. U is also known as immediate basin of attraction or immediate attracting basin of z_0. In this case $\{U,\ f(U),\ ...,\ f^{p-1}(U)\}$ is called an attracting periodic cycle of the component U. U is called superattracting domain if z_0 is a superattracting periodic point.*

- *U is a **parabolic domain**: In this case, the boundary ∂U of the component U contains a rationally indifferent periodic point z^* of minimal period p and $\lim_{n\to\infty} f^{np}(z) = z^*$ for all $z \in U$. Parabolic domain is also known as Leau domain. In this case $\{U,\ f(U),\ ...,\ f^{p-1}(U)\}$ is called a parabolic periodic cycle of the component U.*

- *U is a **Siegel disk**: In this case, there exists an analytic homeomorphism $\varphi : U \to D$ where $D = \{z : |z|< 1\}$, such that $\varphi(f^p(\varphi^{-1}(z))) = e^{i2\pi\alpha}z$ for some irrational number α.*

- *U is a **Herman ring**: In this case, there exists an analytic homeomorphism $\varphi : U \to A$ where A is the annulus $\{z : 1 < |z|< r\}, r > 1$, such that $\varphi(f^p(\varphi^{-1}(z))) = e^{i2\pi\alpha}z$ for some irrational number α. Herman rings are also known as Arnold rings.*

- *U is a **Baker domain**: In this case, there exists a point z^* on the boundary ∂U of U such that $\lim_{n\to\infty} f^{np}(z) = z^*$ for all $z \in U$ and $f^p(z^*)$ is not defined.*

Example 1.5.2 (Basin of attraction). *Let $f(z) = z^2$ be the entire function. The point $z = 0$ is an (super) attracting fixed point of f. Here $U = A(0) = \{z \in \widehat{\mathbb{C}} : |z|< 1\}$ is a basin of attraction of the Fatou set $F(f)$.*

Example 1.5.3 (Parabolic domain). *Let $f(z) = e^{z-1}$ be the entire function. The point $z = 1$ is a rationally indifferent fixed point of $f(z)$. Let $U = interior\{z \in \widehat{\mathbb{C}} : f^n(z) \to 1 \text{ as } n \to \infty\}$. Note that $f^n(x) \to \infty$ as $n \to \infty$ for $x > 1$. Thus the fixed point $z = 1$ lies on the boundary of U. Hence U is a parabolic domain of the Fatou set $F(f)$.*

Example 1.5.4 (Siegel disk). *Let $f(z) = e^{2\pi i\alpha}z + z^2$ be the entire function where α is the golden ratio. The point $z = 0$ is an irrationally indifferent fixed point of $f(z)$. Further, there exists an analytic homeomorphism $\varphi : U \to D$ where D is the unit disk and U is an open neighbourhood containing 0 such that $\varphi(f^p(\varphi^{-1}(z))) = e^{i2\pi\alpha}z$. Thus, U is a Siegel disk.*

Example 1.5.5 (Herman ring). *Consider the function $f(z) = \lambda z^2 \left(\frac{1+\bar{\alpha}z}{z+\alpha}\right)$ where $|\lambda|= 1$ and $0 < |\alpha|< 1$. It is proved in [12] that for suitable choices of λ and α, the function $f(z)$ has a Herman ring.*

Example 1.5.6 (Baker domain). *Let $f(z) = 1 + z + e^{-z}$ be the entire transcendental function. Let $H^+ = \{z \in \mathbb{C} : Re(z) > 0\}$ and $U = \{z \in \mathbb{C} : f^n(z) \to \infty$ as $n \to \infty\}$. Note that $Re(f(z)) = 1 + Re(z) + Re(e^{-z}) > Re(z)$ for all $z \in H^+$. So, $f^n(z) \to \infty$ as $n \to \infty$ for $z \in H^+$. Therefore all points in the half plane H^+ lie in stable domain U of points whose orbits tend to the essential singularity ∞. Hence U is a Baker domain of the Fatou set $F(f)$.*

If U is an attracting or parabolic domain then all the limit functions of $\{f^n(z)\}_{n>0}$ for $z \in U$ are constants which are nothing but the attracting or rationally indifferent periodic points. If U is a Siegel disc or Herman ring, then all the limit functions of $\{f^n(z)\}_{n>0}$ for $z \in U$ are nonconstants. If f is rational, Fatou [39–41] proved that if $\{f^n|_U\}_{n>0}$ has only constant limit functions then U is an attracting domain or a parabolic domain. Cremer proved if $\{f^n|_U\}_{n>0}$ has a nonconstant limit functions, then U is a Siegel disc or a Hermann ring [23]. These results are also true for transcendental meromorphic functions.

The attracting domains and Siegel disks contain periodic points. The parabolic domains contain periodic points on the boundary but the Baker domains and Hermann rings do not contain any periodic points.

Let U be a p-periodic Siegel disk or Herman ring. Since the function $f^p|_U$ is conformally conjugate to a rotation on a unit disk or an annulus, U is known as rotational domain. Thus the map f^p is one-one on U. Polynomials do not have Herman rings. In fact, the transcendental entire maps do not have Herman rings.

The existence of Baker domains are first proved by Baker. It is well known that for a transcendental entire function f, $z = \infty$ is the only point in $\widehat{\mathbb{C}}$ where $f(z)$ is not defined. Thus for an entire transcendental entire function, Baker domain is called as the domain at infinity. Note that if f is rational, then for all $z \in \widehat{\mathbb{C}}$ and $p \in \mathbb{N}$, $f^p(z)$ is defined. Thus, Baker domains do not exist for rational functions [14].

The relation between attracting domains, parabolic domains and rotational domains with the singular values [14] is given in the following theorem.

Theorem 1.5.3. *Let f be a meromorphic function and $C = \{U_0,\ U_1, ...,\ U_{p-1}\}$ be a periodic cycle of components of the Fatou set of f.*

1. *If C is a cycle of basins of attraction or parabolic domains, then $U_j \bigcap \mathrm{sing}(f^{-1}) \neq \emptyset$ for some $j \in \{0,\ 1, ...,\ p-1\}$.*
2. *If C is a cycle of Siegel disks or Herman rings, then $\partial U_j \subseteq \overline{O^+(\mathrm{sing}(f^{-1}))}$ for all $j \in \{0,\ 1, ...,\ p-1\}$.*

Eremenko and Lyubich [37] proved that the following class of functions do not have Baker domains.

Theorem 1.5.4. *If $f \in \mathcal{E} \cap \mathcal{B}$, then f does not have Baker domains.*

In the next two theorems, the relations between the singular values and Baker domains are given [14].

Theorem 1.5.5. *Let f be a meromorphic function, and let $C = \{U_0,\ U_1, ...,\ U_{p-1}\}$ be a periodic cycle of Baker domains of f. Let z_j denote the limit of $\{f^{np}(z)\}_{n>0}$ for $z \in U_j$ and define $z_p = z_0$. Then $z_j \in \bigcup_{n=0}^{p-1} f^{-n}(\infty)$ for all $j \in \{0,\ 1, ...,\ p-1\}$, and $z_j = \infty$ for at least one $j \in \{0,\ 1, ...,\ p-1\}$. If $z_j = \infty$, then z_{j+1} is an asymptotic value of f.*

Theorem 1.5.6. *Let f be a meromorphic function and $\{U_0, U_1, \cdots, U_{p-1}\}$ be a periodic cycle of Baker domains of f. Then ∞ is in the derived set of*

$$\bigcup_{j=0}^{p-1} f^j(\mathrm{sing}(f^{-1})).$$

Remark 1.5.1. *If $f \in \mathcal{S}$, then f does not have Baker domains.*

1.6 Topology of Fatou components

The topology of the Fatou components are described in this section. The mapping properties of a function play a crucial role to determine the topology of the Fatou components.

Definition 1.6.1. *The connectivity of a domain* $\Omega \subseteq \widehat{\mathbb{C}}$ *is defined as the number of components of* $\widehat{\mathbb{C}} \setminus \Omega$. *The domains with connectivity one are called **simply connected** and the domains with connectivity more than one are called **multiply connected**. In particular, the domains having connectivity two are called **doubly connected** and the domains having connectivity* ∞ *are called **infinitely connected**.*

Siegel disk is simply connected since it is homeomorphic to the unit disc. Herman ring is homeomorphic to an annulus and hence is doubly connected. Baker [1] proved that multiply connected Fatou components of an entire function are bounded. Bergweiler [14] proved the following theorem.

Theorem 1.6.1. *If* $f \in \mathcal{E}$, *then any preperiodic Fatou component of* $F(f)$ *is simply connected. Hence the Fatou set of* f *does not contain Herman rings.*

The connectivity of an invariant Fatou component of a meromorphic function was given in the next two theorems [8].

Theorem 1.6.2. *Let* f *be a meromorphic function and* U *be an invariant Fatou component. Then the connectivity of* U *has one of the values* 1, 2 *or* ∞.

Theorem 1.6.3. *Let* f *be a meromorphic function and* U *be a completely invariant Fatou component. Then the connectivity of* U *is* 1 *or* ∞.

Bolsch [20] proved the connectivity of the periodic Fatou components of a meromorphic function.

Theorem 1.6.4. *Let* U *be a periodic Fatou component of a meromorphic function. Then the connectivity of* U *is* 1, 2 *or* ∞.

Baker *et al.* [6] constructed meromorphic functions which have wandering domains of any connectivity including infinity. They also proved in [8], for a meromorphic function, the connectivity of the preperiodic Fatou component can be any natural number.

The Fatou set of a transcendental entire function can contain at most one completely invariant domain, was proved by Baker. The Fatou set of a rational function contains at most two completely invariant Fatou components [12].

Definition 1.6.2. *[15] A transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ is called hyperbolic if $f \in \mathcal{B}$ and furthermore every element of* $\text{sing}(f^{-1})$ *belongs to the basin of some attracting periodic cycle of f.*

For hyperbolic entire functions, the following two theorems can be found in [15].

Theorem 1.6.5. *Let $f \in \mathcal{B}$ be an entire transcendental function and let f be hyperbolic. Then the following are equivalent:*

(1) every component of $F(f)$ is bounded;

(2) f has no asymptotic values and every component of $F(f)$ contains finitely many critical points.

Theorem 1.6.6. *Let $f \in \mathcal{B}$ be hyperbolic without finite asymptotic values and exactly two critical values. Then either*

(1) every connected component U of $F(f)$ is unbounded and ∂U is not locally connected at any finite point, or

(2) every connected component of $F(f)$ is a bounded quasidisk.

1.7 Bifurcation

In the one-parameter family of functions, it is mainly studied that how the dynamics of functions change as the parameter changes. Any sudden changes in the dynamics is normally described by the concept called bifurcation.

Definition 1.7.1. *Let f_λ be a parametrized family of functions where λ is a real parameter. Then there is a bifurcation at λ^* if there exists $\epsilon > 0$ such that whenever a and b satisfy $\lambda^* - \epsilon < a < \lambda^*$ and $\lambda^* < b < \lambda^* + \epsilon$, then the dynamics of f_a are different from the dynamics of f_b. In other words, the dynamics of the function changes when the parameter value crosses through the point λ^*.*

Some of the important types of bifurcations are described below [46].

Example 1.7.1 (Saddle-node bifurcation). *Consider the family of functions $E_\lambda(x) = \lambda e^x$ where $\lambda > 0$ is the parameter. If $\lambda < 1/e$, then $E_\lambda(x)$ has an attracting fixed point a_λ (say) and a repelling fixed point r_λ (say). If $\lambda < 1/e$, then $E_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in (-\infty, r_\lambda)$ and $E_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (r_\lambda, \infty)$. If $\lambda = 1/e$, then 1 is the only fixed point of $E_\lambda(x)$ which is rationally indifferent. Also if $\lambda = 1/e$, $E_\lambda^n(x) \to 1$ as $n \to \infty$ for $x \in (-\infty, 1]$ and $E_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (1, \infty)$. If $\lambda > 1/e$, $E_\lambda(x)$ has no fixed points and $E_\lambda^n(x) \to \infty$ for all $x \in \mathbb{R}$. First, as the parameter λ grows and approaches $1/e$, the two fixed points a_λ and r_λ gradually approach one another until, when $\lambda = 1/e$, they join to become one fixed point. Immediately thereafter, they disappear all together. This type of bifurcation is called a saddle-node bifurcation.*

Example 1.7.2 (Pitchfork bifurcation). *Consider the family of functions $A_k(x) = k\arctan(x)$ where $k > 0$ is the parameter. When $0 < k < 1$, the function has one attracting fixed point near 0 and all real number are in the stable set of 0. If $k = 1$, then 0 is a rationally indifferent fixed point and stable set of 0 is still $\mathbb{R}$. When $k > 1$, 0 is a repelling fixed point ($A_k'(0) = k > 1$) and two other fixed points have been formed, both of which are attracting. If the two fixed points are a and b and $a < 0 < b$, then stable set of a is $(-\infty.0)$ and the stable set of b is $(0, \infty)$. Here bifurcation occurs at $k = 1$ and this type of bifurcation is called pitchfork bifurcation. In general, pitchfork bifurcations occur when either an attracting periodic point splits into a repelling periodic point with an attracting periodic point of the same period as the original point on each side of it, or alternatively, a repelling periodic point splits into an attracting periodic point surrounded by two repelling periodic points of the same period.*

Example 1.7.3 (Transcritical bifurcation). *Consider the family of functions* $h_r(x) = rx(1-x)$ *where* $r > 0$ *is the parameter. When* $0 < r < 1$, $h_r(x)$ *has two fixed points,* 0 *which is attracting, and another one less than one and repelling. When* $r = 1$, *these two points merges to form one fixed point at* 0 *which attracts points which are greater than* 0 *and repels points which are less than* 0. *When* $1 < r < 3$, *we see that* 0 *becomes a repelling fixed point and there is an attracting fixed point which is greater than* 0. *Here bifurcation occurs at* $r = 1$. *This type of bifurcation is called transcritical bifurcation.*

Example 1.7.4 (Period doubling bifurcation). *Consider the family of functions* $h_r(x) = rx(1-x)$ *where* r *is the parameter. For* $r > 1$, *the nonzero fixed point is* $p_r = \frac{r-1}{r}$. *If* $1 < r < 3$, *by Example 1.7.3,* $h_r(x)$ *has a repelling fixed point at* 0 *and an attracting fixed at* p_r. *At* $r = 3$ *the largest fixed point is at* p_r *which is rationally indifferent. When* $r > 3$, *the largest fixed point* p_r *is repelling. As the parameter* r *increases through* 3, *the value of* $(h_r^2)'(p_r)$ *changes from being less than* 1 *to being greater than* 1. *Hence the fixed point* p_r *changes from attracting to repelling. In addition, the continuity of* h_r *requires that as this happen a period* 2 *attracting orbit must be added. This type of bifurcation is called a period doubling bifurcation since a periodic orbit of twice the period of the original periodic point is added.*

A period doubling bifurcation in a discrete dynamical system is a bifurcation in which a slight change in a parameter value in the governing equations of the system leads to the system switching to a new behaviour with twice the period of the original system.

1.8 Motivation of Present Work

Complex dynamics is one of the most popular area in dynamical system. The iterations of entire and meromorphic functions are studied in complex dynamical system. The main objects studied in the dynamics of a function are its Fatou and Julia sets. There are two basic approaches to study the dynamics of a function.

- studying the iterative behaviour of an individual function,

- studying changes in the iterative behaviour due to slight perturbations in the function.

The dynamics of one-parameter family of functions have been studied by many researchers in the end of 20th and the beginning of 21st century. Devaney vastly studied the one-parameter family $\{\lambda e^z : \lambda > 0\}$. Devaney [24] and Devaney and Durkin [31] proved that the Julia set $J(\lambda e^z)$ is a nowhere dense subset entirely contained in the right half plane for $0 < \lambda < (1/e)$ and the Julia set of λe^z is the whole of extended complex plane for $\lambda > 1/e$. So, when the parameter λ increases through $1/e$, the Julia set $J(\lambda e^z)$ changes from a nowhere dense subset of $\widehat{\mathbb{C}}$ to the whole of extended complex plane $\widehat{\mathbb{C}}$. This phenomenon is called explosion in the Julia set or chaotic burst in the dynamics of one-parameter family $\{\lambda e^z : \lambda > 0\}$. It is also shown that a saddle node bifurcation occurs in the one-parameter family $\{\lambda e^z : \lambda > 0\}$ at the parameter value $\lambda = 1/e$. More dynamics properties of λe^z can also be found in [9, 10, 18, 25, 27, 31, 33, 54, 75]. Like in the one-parameter family of exponential map, a bifurcation also occurs in many other one-parameter families of transcendental entire and meromorphic functions. This explosion is also seen in the family $\{i\lambda \cos z : \lambda > 0\}$, was observed by Devaney [24]. Kapoor and Prasad studied the dynamics of function $\lambda(e^z - 1)/z$ [49] and certain one-parameter family of entire functions whose singular values are bounded [50]. Sajid and Kapoor investigated the dynamics of $\lambda(\sinh z)/z^2$ [71] and meromorphic functions having rational Schwarzian derivatives [72, 73]. The dynamics of $\lambda \sinh(z)/z$ for $\lambda \in \mathbb{R} \setminus \{0\}$ is studied by Prasad [44] and it is shown that a bifurcation occurs in this family at $|\lambda| \approx 1.104$. Nayak and Prasad investigated the dynamics of $\lambda \tanh(e^z)$ [45]. Sajid attempted in [68] to study the dynamics of one-parameter family $\lambda(b^z - 1)/z$ which is a generalization of the work of Kapoor and Prasad [49]. Devaney [28–30] recently studied the dynamics of family of maps $F_\lambda(z) = z^n + \lambda(1/z^d)$ for $n,\ d \geq 2$. Recently there is an interest in studying the dynamics of two-parameter family of functions. Sajid [69] studied the fixed points and singular values of two-parameter families $\lambda(e^{az} - 1)/z$ and $\lambda z/(e^{az} - 1)$. Sajid also studied the singular values of two-parameter families $\lambda((b^z - 1)/z)^\mu$ and $\lambda(z/(b^z - 1))^\mu$ [70].

Those things inspire us to study the dynamics of two-parameter family of functions. We are mainly interested in studying the changes in the dynamics of functions in the two-parameter family $\{\lambda f + \mu g\ : f$ and g are fixed functions and λ and μ are parameters$\}$ as the parameters λ and μ change. From the dynamics of two-parameter family of functions, we can deduce easily the dynamics of certain function and certain one-parameter family of functions. For example, by the dynamics of the two-parameter family $\mathcal{G}_b \equiv \{g_{\lambda,\mu}(z) = \lambda b^z + \mu b^{-z}$ for $z \in \mathbb{C} : \lambda \geq \mu > 0\}$, the dynamics of $\cosh z$ can be obtained by choosing $\lambda = \mu = 1/2$ and $b = e$. The dynamics of the one-parameter family $\{\rho \cosh z\ :\ \rho > 0$ is the parameter$\}$ can be obtained from the dynamics of the two-parameter family $\mathcal{G}_b$ by choosing $\lambda = \mu = \rho/2$ and $b = e$. Also we understand the change in the dynamics when both the functions present with different weightage/scaling.

In this work, we investigate the dynamics of one-parameter families of both transcendental entire and transcendental meromorphic functions. We also study some two-parameter families of transcendental entire functions.

1.9 Organization of Present Work

The thesis investigates dynamics of certain transcendental entire and meromorphic functions. The current work is presented in five chapters. Basic theory and a brief literature survey of the complex dynamics are given in **Chapter 1**. The remaining chapters are organized as follows.

In **Chapter 2**, the two-parameter family of functions $\mathcal{F}_b \equiv \{f_{\lambda,\mu}(z) = \lambda b^z - \mu b^{-z}$ for $z \in \mathbb{C} : \lambda,\ \mu > 0\}$ where $b > 1$, is considered. In Section 2.1, the singular values, relation between the complex conjugate points and some other properties are found. The nature of the real fixed points are described in Section 2.2. In Section 2.3, we reported the dynamics on the real line $\mathbb{R}$. Finally, in Section 2.4, we investigate the forward orbits of the singular values and the dynamics for the cases when real attracting and real rationally indifferent fixed points exist. Also, some properties of the Fatou and Julia sets are proved.

In **Chapter 3**, we consider the two-parameter family of functions $\mathcal{G}_b \equiv \{g_{\lambda,\mu}(z) =$

$\lambda b^z + \mu b^{-z}$ for $z \in \mathbb{C} : \lambda \geq \mu > 0\}$ where $b > 1$. Some kind of symmetry of the function and its derivatives, the singular values and some other basis properties are proved in Section 3.1. Section 3.2 describes the fixed points and the dynamics on the real line $\mathbb{R}$. In this section, it is shown that a saddle-node bifurcation occurs in the real dynamics. In Section 3.3, the dynamics on the complex plane is explored. The Fatou set is the basin of attraction if $t_{\lambda,\mu} < 0$ and the Fatou set is the parabolic domain for $t_{\lambda,\mu} = 0$. But when $t_{\lambda,\mu} > 0$, the Fatou set is empty and the Julia set is the whole of the extended complex plane, and hence a bifurcation occurs at $t_{\lambda,\mu} = 0$. Also the pictures of the Fatou set are generated. In Section 3.4, the dynamics of one-parameter family of functions $\rho \cosh(z)$ where $\rho > 0$ is the parameter and the dynamics of two-parameter family of functions $m \cosh(az) + n \sinh(az)$ where m, n are parameters with $m \geq n > 0$, are deduced from the dynamics $g_{\lambda,\mu}$.

The dynamics of two one-parameter families $\mathcal{F} \equiv \{f_\lambda(z) = \lambda\left(\cosh z + \frac{1}{\cosh z}\right)$ for $z \in \mathbb{C} : \lambda > 0\}$ and $\mathcal{G} \equiv \{g_\lambda(z) = \lambda\left(\cosh z - \frac{1}{\cosh z}\right)$ for $z \in \mathbb{C} : \lambda > 0\}$ are investigated in **Chapter 4**. In Section 4.1, the dynamics of function f_λ in $\mathcal{F}$ is studied. Initially some basic properties are proved. Then the existence and nature of the real fixed points are explored. With the help of the real fixed points, the dynamics of f_λ on the real line is obtained. Then, with all these knowledge, the dynamics of f_λ is concluded on the complex plane. It is also observed that a bifurcation occurs in this family $\mathcal{F}$. The dynamics of functions g_λ in $\mathcal{G}$, are given in Section 4.2. At first, we prove $g_\lambda \in \mathcal{S}$. On the real line, the fixed points and the dynamics are exhibited. It is shown that the origin is always an superattracting fixed point of g_λ. Finally, we prove the Fatou set of g_λ is the basin of attraction corresponding to the origin.

In **Chapter 5**, we consider the one-parameter family $\mathcal{F} \equiv \{f_\lambda(z) = \lambda + \cosh z : \lambda \in \mathbb{R}\}$ of translated hyperbolic cosine functions. The real fixed points and the dynamics on the real line $\mathbb{R}$ are studied in Section 5.1. In this section, we also find f_λ has two singular values and both are real. It is proved that under iteration of f_λ, one singular value always tends to ∞ and other one is either bounded or tending to ∞ depending upon the parameter λ.

The dynamics of f_λ on the complex plane is described in Section 5.2. Beginning of this section, the dynamics is given in case of real attracting and rationally indifferent fixed points. We also proved that the Fatou of f_λ is empty when both the singular values tend to ∞ under the iteration of f_λ. Remaining cases we compute higher order real periodic points and their nature. By observing numerical computation, we conclude that period doubling bifurcation is seen in the dynamics of functions in the family $\mathcal{F}$.

CHAPTER 2

DYNAMICS OF TWO-PARAMETER FAMILY OF HYPERBOLIC SINE LIKE FUNCTIONS

The dynamics of one-parameter families of entire and meromorphic functions have been studied by many researchers in the end of 20-th and the beginning of the 21-st century. Recently, fixed points and singular values of functions in some two-parameter families, are studied by some researchers [69, 70]. In this chapter, for $b > 1$, the two-parameter family $\mathcal{F}_b \equiv \{f_{\lambda,\mu}(z) = \lambda b^z - \mu b^{-z}$ for $z \in \mathbb{C} : \lambda,\ \mu > 0\}$ is considered and the dynamics of $f_{\lambda,\mu} \in \mathcal{F}_b$ is studied. Since $f_{\lambda,\mu}(z)$ involved with multiple-valued functions, throughout this chapter, for the function $f_{\lambda,\mu} \in \mathcal{F}_b$, the principal value of $\log b$ is taken so that $f_{\lambda,\mu}$ becomes an entire function. If $\lambda = \mu = \rho/2$, $\rho > 0$ and $b = e$, then $f_{\lambda,\mu}(z) = \rho \sinh(z)$. So, the dynamics of one-parameter family $\rho \sinh(z)$ where $\rho > 0$ is the parameter, can be obtained from the dynamics of $f_{\lambda,\mu}(z)$. It is shown that $f_{\lambda,\mu} \in \mathcal{S}$. By tracking the singular values, the dynamics of $f_{\lambda,\mu}$ on the complex plane is obtained.

2.1 Basic Properties of Functions in $\mathcal{F}_b$

Some of the basic properties of the function $f_{\lambda,\mu} \in \mathcal{F}_b$ are investigated in this section.

On the real line $\mathbb{R}$, the function $f_{\lambda,\mu}(x) \to \pm\infty$ as $x \to \pm\infty$ and $f'_{\lambda,\mu}(x) = (\ln b)(\lambda b^x + \mu b^{-x}) > 0$ for $x \in \mathbb{R}$. This implies that $f_{\lambda,\mu}(x)$ has a unique zero $x^*_{\lambda,\mu} = \ln(\mu/\lambda)/(2 \ln b)$

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in $\mathbb{R}$. Therefore it follows that

$$f_{\lambda,\mu}(x)\begin{cases} <0 & \text{for } x < x^*_{\lambda,\mu}, \\ =0 & \text{for } x = x^*_{\lambda,\mu}, \\ >0 & \text{for } x > x^*_{\lambda,\mu}. \end{cases} \tag{2.1}$$

The following proposition shows some kind of symmetry of $f_{\lambda,\mu}$ and $f'_{\lambda,\mu}$ about the point $x^*_{\lambda,\mu}$.

Proposition 2.1.1. *Let $f_{\lambda,\mu} \in \mathcal{F}_b$. Then, $f_{\lambda,\mu}(x^*_{\lambda,\mu}+z) = -f_{\lambda,\mu}(x^*_{\lambda,\mu}-z)$ and $f'_{\lambda,\mu}(x^*_{\lambda,\mu}+z) = f'_{\lambda,\mu}(x^*_{\lambda,\mu}-z)$ for all $z \in \mathbb{C}$.*

Proof. Observe that $\lambda b^{x^*_{\lambda,\mu}} - \mu b^{-x^*_{\lambda,\mu}} = 0$ as $x^*_{\lambda,\mu}$ is a zero of $f_{\lambda,\mu}(x)$. Now for $z \in \mathbb{C}$, $f_{\lambda,\mu}(x^*_{\lambda,\mu}+z) + f_{\lambda,\mu}(x^*_{\lambda,\mu}-z) = \lambda b^{x^*_{\lambda,\mu}+z} - \mu b^{-x^*_{\lambda,\mu}-z} + (\lambda b^{x^*_{\lambda,\mu}-z} - \mu b^{-x^*_{\lambda,\mu}+z}) = (b^z + b^{-z})(\lambda b^{x^*_{\lambda,\mu}} - \mu b^{-x^*_{\lambda,\mu}}) = 0$. Thus, $f_{\lambda,\mu}(x^*_{\lambda,\mu}+z) = -f_{\lambda,\mu}(x^*_{\lambda,\mu}-z)$ for all $z \in \mathbb{C}$. Now differentiating both sides with respect to z, we get $f'_{\lambda,\mu}(x^*_{\lambda,\mu}+z) = f'_{\lambda,\mu}(x^*_{\lambda,\mu}-z)$ for all $z \in \mathbb{C}$. □

For $\lambda = 4$, $\mu = 1$ and $b = 2$, the graphs of $f_{\lambda,\mu}(x)$ and $f'_{\lambda,\mu}(x)$ are shown in Figure 2.1. In this figure $x^*_{\lambda,\mu} = -1$ is the point of symmetry.

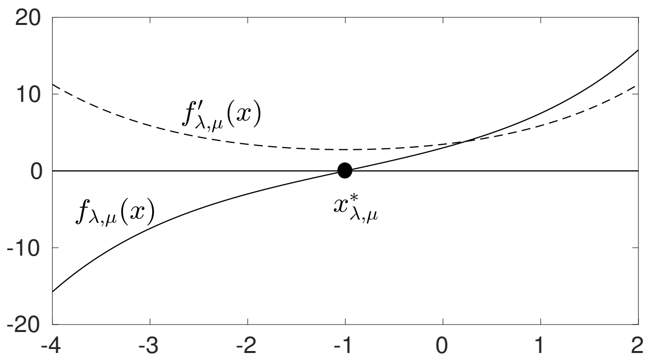


Figure 2.1: Skew symmetry of $f_{\lambda,\mu}(x)$ and symmetry of $f'_{\lambda,\mu}(x)$ about $x^*_{\lambda,\mu}$.

In the following proposition, it is shown that the function $f_{\lambda,\mu}$ has finite number of singular values and hence it is in the class $\mathcal{S}$.

Proposition 2.1.2. *Let $f_{\lambda,\mu} \in \mathcal{F}_b$. Then, $f_{\lambda,\mu}$ has only two critical values $\pm 2i\sqrt{\lambda\mu}$ and no finite asymptotic value.*

Proof. Observe that $f'_{\lambda,\mu}(z) = 0$ if and only if $\lambda b^z + \mu b^{-z} = 0$. That is, $b^z = \pm i\sqrt{(\mu/\lambda)}$. Thus, the critical values of $f_{\lambda,\mu}$ are $\pm 2i\sqrt{\lambda\mu}$. Next, we will show that $f_{\lambda,\mu}$ has no finite asymptotic value. If possible, let w^* be a finite asymptotic value of $f_{\lambda,\mu}$. Then, there exists a continuous curve $\gamma(t)$, $t > 0$ such that $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f_{\lambda,\mu}(\gamma(t)) = w^*$. Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ where $\gamma_1(t)$ and $\gamma_2(t)$ are real functions of t. Then, we can write $f_{\lambda,\mu}(\gamma(t))$ as

$$f_{\lambda,\mu}(\gamma(t)) = \cos(\gamma_2(t)\ln b)\left(\lambda b^{\gamma_1(t)} - \mu b^{-\gamma_1(t)}\right) + i\sin(\gamma_2(t)\ln b)\left(\lambda b^{\gamma_1(t)} + \mu b^{-\gamma_1(t)}\right).$$

It is easy to see that $|f_{\lambda,\mu}(\gamma(t))|^2 = \left(\lambda b^{\gamma_1(t)} + \mu b^{-\gamma_1(t)}\right)^2 - 4\lambda\mu\cos^2(\gamma_2(t)\ln b)$. Since w^* is an asymptotic value, $\lim_{t\to\infty}\left[\left(\lambda b^{\gamma_1(t)} + \mu b^{-\gamma_1(t)}\right)^2 - 4\lambda\mu\cos^2(\gamma_2(t)\ln b)\right] = |w^*|^2$. This implies $\gamma_1(t)$ is bounded and $|\gamma_2(t)| \to \infty$ as $t \to \infty$.

Assume that $\gamma_2(t) \to \infty$ as $t \to \infty$. On the curve $\gamma(t)$, choose two sequences $\{z_n\}$ and $\{z'_n\}$ such that $z_n = x_n + i(2n\pi + \frac{\pi}{2})/(\ln b)$ and $z'_n = x'_n + i((2n+1)\pi + \frac{\pi}{2})/(\ln b)$ for $n \in \mathbb{N}$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Note that both the sequences tend to ∞ along the curve $\gamma(t)$. Clearly, $f_{\lambda,\mu}(z_n) = (\lambda b^{x_n} + \mu b^{-x_n})i$. The asymptotic value w^* obtained as the limit of $f_{\lambda,\mu}(z_n)$ as $n \to \infty$, it must be of the form $w^* = pi$ where $p \geq 2\sqrt{\lambda\mu}$. Again in view of $f_{\lambda,\mu}(z'_n) = -(\lambda b^{x'_n} + \mu b^{-x'_n})i$, the asymptotic value w^* must be of the form $w^* = qi$ where $q \leq -2\sqrt{\lambda\mu}$ which is not possible. So, the function $f_{\lambda,\mu}$ has no finite asymptotic value.

If $\gamma_2(t) \to -\infty$ as $t \to \infty$, similarly it can be proved that $f_{\lambda,\mu}$ has no finite asymptotic value. This completes the proof. □

The iterative behaviour of the points which are complex conjugates to each other, is stated in the following proposition.

Proposition 2.1.3. *Let $f_{\lambda,\mu} \in \mathcal{F}_b$. Then, $f^n_{\lambda,\mu}(\bar{z}) = \overline{f^n_{\lambda,\mu}(z)}$ for all $z \in \mathbb{C}$ and for all $n \in \mathbb{N}$.*

Proof. We have proved this by induction principle on n. The Taylor series of $f_{\lambda,\mu}(z)$ is given by

$$f_{\lambda,\mu}(z) = \sum_{m=0}^{\infty} \frac{(\lambda-\mu)(\ln b)^{2m}}{(2m)!} z^{2m} + \sum_{m=0}^{\infty} \frac{(\lambda+\mu)(\ln b)^{2m+1}}{(2m+1)!} z^{2m+1} \text{ for all } z \in \mathbb{C}.$$

Since all the coefficients of the power series of $f_{\lambda,\mu}(z)$ are real, $f_{\lambda,\mu}(\bar{z}) = \overline{f_{\lambda,\mu}(z)}$ for all $z \in \mathbb{C}$. So, this is true for $n = 1$. Assume that it holds for $n = k$, for some $k \in \mathbb{N}$. That is, $f^k_{\lambda,\mu}(\bar{z}) = \overline{f^k_{\lambda,\mu}(z)}$. Then,

$$f^{k+1}_{\lambda,\mu}(\bar{z}) = f_{\lambda,\mu}\left(f^k_{\lambda,\mu}(\bar{z})\right) = f_{\lambda,\mu}\left(\overline{f^k_{\lambda,\mu}(z)}\right) = \overline{f_{\lambda,\mu}\left(f^k_{\lambda,\mu}(z)\right)} = \overline{f^{k+1}_{\lambda,\mu}(z)}.$$

This completes the proof. □

2.2 Real Fixed Points of Functions in $\mathcal{F}_b$

The existence of the real fixed points of $f_{\lambda,\mu}$ is investigated in this section and it is useful to study the dynamics of $f_{\lambda,\mu} \in \mathcal{F}_b$.

Let $h_{\lambda,\mu}(x) = f_{\lambda,\mu}(x) - x$ where $f_{\lambda,\mu}(x) = \lambda b^x - \mu b^{-x}$ for $x \in \mathbb{R}$ and λ, μ are real parameters with $\lambda > 0$, $\mu > 0$. Clearly, the zeros of $h_{\lambda,\mu}(x)$ are the fixed points of $f_{\lambda,\mu}(x)$. It is easy to see that $h_{\lambda,\mu}(x) \to \pm\infty$ as $x \to \pm\infty$. Observe that $h'_{\lambda,\mu}(x) = (\ln b)(\lambda b^x + \mu b^{-x}) - 1$ and $h'_{\lambda,\mu}(x) \to \infty$ as $|x| \to \infty$.

Further, $h''_{\lambda,\mu}(x) = (\ln b)^2(\lambda b^x - \mu b^{-x}) = (\ln b)^2 f_{\lambda,\mu}(x)$ implies that $h''_{\lambda,\mu}(x) \to \pm\infty$ as $x \to \pm\infty$. It is worth to note that the zeros of $h''_{\lambda,\mu}(x)$ and $f_{\lambda,\mu}(x)$ are same. So, $h''_{\lambda,\mu}(x)$ has unique zero at $x = x^*_{\lambda,\mu}$. Thus, we can write

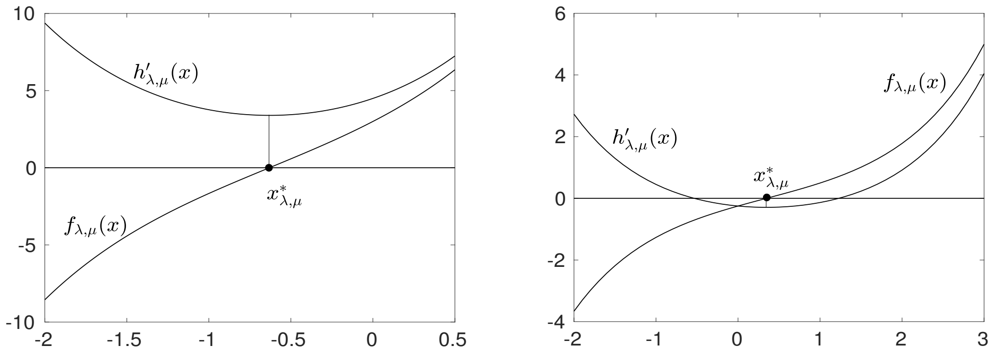
$$h''_{\lambda,\mu}(x) \begin{cases} < 0 & \text{for } x < x^*_{\lambda,\mu}, \\ = 0 & \text{for } x = x^*_{\lambda,\mu}, \\ > 0 & \text{for } x > x^*_{\lambda,\mu}. \end{cases} \tag{2.2}$$

In view of the fact that $h'''_{\lambda,\mu}(x^*_{\lambda,\mu}) = (\ln b)^3 \left(\lambda b^{x^*_{\lambda,\mu}} + \mu b^{-x^*_{\lambda,\mu}}\right) > 0$, it follows that $h'_{\lambda,\mu}(x)$ has global minimum at $x = x^*_{\lambda,\mu}$ (see Figure 2.2) and $h'_{\lambda,\mu}(x^*_{\lambda,\mu}) = 2\sqrt{\lambda\mu}\ln b - 1$.

The following proposition gives the local maxima and local minima of $f_{\lambda,\mu}(x) - x$ when the parameters λ and μ are such that $2\sqrt{\lambda\mu}\ln b < 1$.

Proposition 2.2.1. *Let $h_{\lambda,\mu}(x) = f_{\lambda,\mu}(x) - x$ for all $x \in \mathbb{R}$ where λ and μ are real parameters with $\lambda > 0$ and $\mu > 0$. If $2\sqrt{\lambda\mu}\ln b < 1$, then there exists $x'_{\lambda,\mu}$ and $x''_{\lambda,\mu}$ with $x'_{\lambda,\mu} < x^*_{\lambda,\mu} < x''_{\lambda,\mu}$ such that the following holds.*

(i) $h_{\lambda,\mu}(x)$ has a local maxima at $x'_{\lambda,\mu}$,



(a) For $\lambda = 4, \mu = 1$ and $b = 3$. (b) For $\lambda = 1/4, \mu = 1/2$ and $b = e$.

Figure 2.2: $h'_{\lambda,\mu}(x)$ has minima at $x = x^*_{\lambda,\mu}$.

(ii) $h_{\lambda,\mu}(x)$ has a local minima at $x''_{\lambda,\mu}$.

Proof. Since $2\sqrt{\lambda\mu}\ln b < 1$, we have $h'_{\lambda,\mu}(x^*_{\lambda,\mu}) < 0$. So, we conclude that $h'_{\lambda,\mu}(x)$ has only two zeros, as $h'_{\lambda,\mu}(x) \to \infty$ for $|x| \to \infty$ and $h''_{\lambda,\mu}(x) = 0$ has unique solution. Let $x'_{\lambda,\mu}$ and $x''_{\lambda,\mu}$ be the zeros of $h'_{\lambda,\mu}(x)$ with $x'_{\lambda,\mu} < x^*_{\lambda,\mu} < x''_{\lambda,\mu}$. Thus,

$$h'_{\lambda,\mu}(x) \begin{cases} > 0 & \text{for } x \in (-\infty, x'_{\lambda,\mu}) \cup (x''_{\lambda,\mu}, \infty), \\ = 0 & \text{for } x = x'_{\lambda,\mu},\ x''_{\lambda,\mu}, \\ < 0 & \text{for } x \in (x'_{\lambda,\mu}, x''_{\lambda,\mu}). \end{cases} \tag{2.3}$$

Now by (2.2), we get $h''_{\lambda,\mu}(x'_{\lambda,\mu}) < 0$ and $h''_{\lambda,\mu}(x''_{\lambda,\mu}) > 0$. Hence the function $h_{\lambda,\mu}(x)$ has a local maxima at $x'_{\lambda,\mu}$ and a local minima at $x''_{\lambda,\mu}$. □

The graphs of $h_{\lambda,\mu}(x)$ and $h'_{\lambda,\mu}(x)$ are shown for $\lambda = 1/4$, $\mu = 1/2$ and $b = e$ in Figure 2.3. Figure 2.3 shows that $h_{\lambda,\mu}(x)$ has local maxima and local minima at the points $x'_{\lambda,\mu}$ and $x''_{\lambda,\mu}$ respectively.

Throughout this chapter, we denote $M_{\lambda,\mu} = h_{\lambda,\mu}(x'_{\lambda,\mu})$ and $m_{\lambda,\mu} = h_{\lambda,\mu}(x''_{\lambda,\mu})$ whenever $2\sqrt{\lambda\mu}\ln b < 1$. Clearly, $M_{\lambda,\mu} > m_{\lambda,\mu}$.

Remark 2.2.1. *When $2\sqrt{\lambda\mu}\ln b < 1$,*

- *The value of $x'_{\lambda,\mu}$ is given by $x'_{\lambda,\mu} = \ln\left(\frac{1-\sqrt{1-4\lambda\mu(\ln b)^2}}{2\lambda\ln b}\right)/(\ln b)$.*
- *The value of $x''_{\lambda,\mu}$ is given by $x''_{\lambda,\mu} = \ln\left(\frac{1+\sqrt{1-4\lambda\mu(\ln b)^2}}{2\lambda\ln b}\right)/(\ln b)$.*

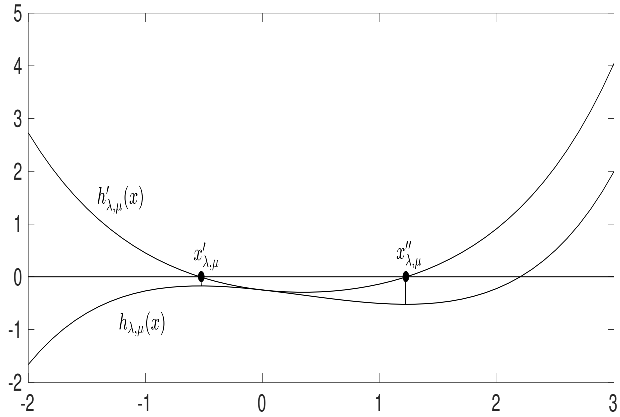


Figure 2.3: Local maxima and local minima of $h_{\lambda,\mu}(x)$ for $2\sqrt{\lambda\mu}\ln b < 1$.

- *Local maxima $M_{\lambda,\mu}$ is given by*

$$M_{\lambda,\mu} = \left(-\sqrt{1-4\lambda\mu(\ln b)^2} - \ln\left(\frac{1-\sqrt{1-4\lambda\mu(\ln b)^2}}{2\lambda\ln b}\right)\right)/(\ln b)\,.$$

- *Local minima $m_{\lambda,\mu}$ is given by*

$$m_{\lambda,\mu} = \left(\sqrt{1-4\lambda\mu(\ln b)^2} - \ln\left(\frac{1+\sqrt{1-4\lambda\mu(\ln b)^2}}{2\lambda\ln b}\right)\right)/(\ln b)\,.$$

The existence and nature of the real fixed points of the functions in the family $\mathcal{F}_b$ are investigated in the following theorem.

Theorem 2.2.1. *Let $f_{\lambda,\mu}(x) = \lambda b^x - \mu b^{-x}$ for $x \in \mathbb{R}$ where λ and μ are real parameters with $\lambda > 0$ and $\mu > 0$.*

(a) If $2\sqrt{\lambda\mu}\ln b > 1$, then $f_{\lambda,\mu}(x)$ has a unique repelling fixed point $r_{\lambda,\mu}$ (say).

(b) If $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$, then $f_{\lambda,\mu}(x)$ has a unique rationally indifferent fixed point 0.

(c) If $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda \neq \mu$, then $f_{\lambda,\mu}(x)$ has a unique repelling fixed point $r_{\lambda,\mu}$ (say).

(d) If $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} < 0$, then $f_{\lambda,\mu}(x)$ has a unique repelling fixed point $r_{\lambda,\mu}$ (say) with $r_{\lambda,\mu} > x''_{\lambda,\mu}$.

(e) *If* $2\sqrt{\lambda\mu}\ln b < 1$ *and* $m_{\lambda,\mu} > 0$, *then* $f_{\lambda,\mu}(x)$ *has a unique repelling fixed point* $r_{\lambda,\mu}$ *(say) with* $r_{\lambda,\mu} < x'_{\lambda,\mu}$.

(f) *If* $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ *and* $m_{\lambda,\mu} < 0$, *then* $f_{\lambda,\mu}(x)$ *has two repelling fixed points* $r'_{\lambda,\mu}$, $r''_{\lambda,\mu}$ *(say) and an attracting fixed point* $a_{\lambda,\mu}$ *(say) with* $r'_{\lambda,\mu} < x'_{\lambda,\mu} < a_{\lambda,\mu} < x''_{\lambda,\mu} < r''_{\lambda,\mu}$.

(g) *If* $2\sqrt{\lambda\mu}\ln b < 1$ *and* $M_{\lambda,\mu} = 0$, *then* $f_{\lambda,\mu}(x)$ *has a rationally indifferent fixed point* $x'_{\lambda,\mu}$ *and a repelling fixed point* $r_{\lambda,\mu}$ *(say) with* $r_{\lambda,\mu} > x''_{\lambda,\mu}$.

(h) *If* $2\sqrt{\lambda\mu}\ln b < 1$ *and* $m_{\lambda,\mu} = 0$, *then* $f_{\lambda,\mu}(x)$ *has a rationally indifferent fixed point* $x''_{\lambda,\mu}$ *and a repelling fixed point* $r_{\lambda,\mu}$ *(say) with* $r_{\lambda,\mu} < x'_{\lambda,\mu}$.

Proof. Set $h_{\lambda,\mu}(x) = f_{\lambda,\mu}(x) - x$ for $x \in \mathbb{R}$.

(a) **Case:** $2\sqrt{\lambda\mu}\ln b > 1$

If $2\sqrt{\lambda\mu}\ln b > 1$, then $h'_{\lambda,\mu}(x) \geq h'_{\lambda,\mu}(x^*_{\lambda,\mu}) > 0$ for all $x \in \mathbb{R}$. So, the function $h_{\lambda,\mu}(x)$ is strictly increasing. Therefore $h_{\lambda,\mu}(x)$ has a unique zero $r_{\lambda,\mu}$ (say). Hence $f_{\lambda,\mu}(x)$ has a unique fixed point $r_{\lambda,\mu}$. So,

$$h_{\lambda,\mu}(x) \begin{cases} < 0 & \text{for } x < r_{\lambda,\mu}, \\ = 0 & \text{for } x = r_{\lambda,\mu}, \\ > 0 & \text{for } x > r_{\lambda,\mu}. \end{cases} \tag{2.4}$$

Now $h'_{\lambda,\mu}(r_{\lambda,\mu}) > 0$ implies $f'_{\lambda,\mu}(r_{\lambda,\mu}) > 1$. Hence $r_{\lambda,\mu}$ is the repelling fixed point of $f_{\lambda,\mu}(x)$ for $2\sqrt{\lambda\mu}\ln b > 1$.

In Figure 2.4, $r_{\lambda,\mu}$ is the repelling fixed point of $f_{\lambda,\mu}(x)$ for $\lambda = 4$, $\mu = 1$ and $b = 3$. Here λ, μ and b satisfy $2\sqrt{\lambda\mu}\ln b > 1$.

(b) **Case:** $2\sqrt{\lambda\mu}\ln b = 1$ **and** $\lambda = \mu$

Clearly, $x^*_{\lambda,\mu} = 0$ as $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$. Then $h_{\lambda,\lambda}(0) = 0$ and $h'_{\lambda,\lambda}(0) = 0$ which implies $f_{\lambda,\lambda}(0) = 0$ and $f'_{\lambda,\lambda}(0) = 1$. Hence 0 is a rationally indifferent fixed point of $f_{\lambda,\lambda}(x)$.

Therefore,

$$h_{\lambda,\lambda}(x) \begin{cases} < 0 & \text{for } x < 0, \\ = 0 & \text{for } x = 0, \\ > 0 & \text{for } x > 0. \end{cases} \tag{2.5}$$

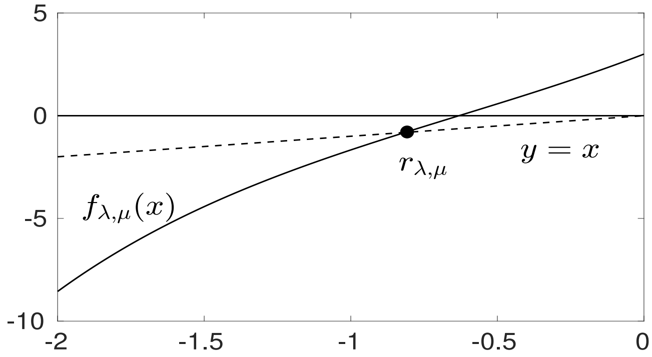


Figure 2.4: Repelling fixed point of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b > 1$.

In the following figure fixed point of $f_{\lambda,\mu}(x)$ is shown for $\lambda = \mu = 1/2$ and $b = e$. In Figure 2.5, the origin is the rationally indifferent fixed point of $f_{\lambda,\lambda}(x)$.

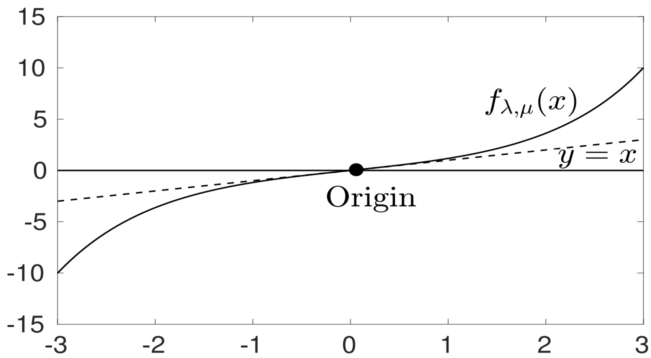


Figure 2.5: Rationally indifferent fixed point of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$.

(c) **Case:** $2\sqrt{\lambda\mu}\ln b = 1$ **and** $\lambda \neq \mu$

If $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda \neq \mu$, then $x^*_{\lambda,\mu} \neq 0$, $h_{\lambda,\mu}(x^*_{\lambda,\mu}) = -x^*_{\lambda,\mu} \neq 0$ and $h'_{\lambda,\mu}(x^*_{\lambda,\mu}) = 0$. So, $x^*_{\lambda,\mu}$ is not a fixed point of $f_{\lambda,\mu}(x)$. Therefore, there exists $r_{\lambda,\mu} \in \mathbb{R}$ such that

$$h_{\lambda,\mu}(x) \begin{cases} < 0 & \text{for } x < r_{\lambda,\mu}, \\ = 0 & \text{for } x = r_{\lambda,\mu}, \\ > 0 & \text{for } x > r_{\lambda,\mu}. \end{cases} \tag{2.6}$$

So, $r_{\lambda,\mu}$ is a fixed point of $f_{\lambda,\mu}(x)$. Now $h'_{\lambda,\mu}(r_{\lambda,\mu}) > h'_{\lambda,\mu}(x^*_{\lambda,\mu}) = 0$ implies $f'_{\lambda,\mu}(r_{\lambda,\mu}) > 1$. Hence $r_{\lambda,\mu}$ is a repelling fixed point of $f_{\lambda,\mu}(x)$ completing the proof of (c).

In Figure 2.6, the repelling fixed point of $f_{\lambda,\mu}(x)$ is shown for $\lambda = 1$, $\mu = 1/4$ and $b = e$. Here $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda \neq \mu$.

(d) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$ **and** $M_{\lambda,\mu} < 0$

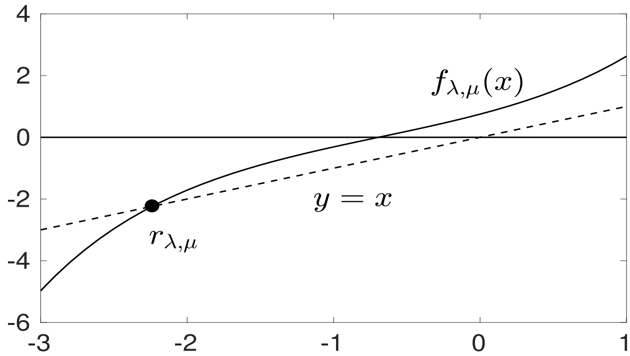


Figure 2.6: Repelling fixed point of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda \neq \mu$.

If $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} < 0$, by using (2.3), we get $h_{\lambda,\mu}(x) < 0$ for all $x \leq x''_{\lambda,\mu}$. Notice that $h_{\lambda,\mu}(x)$ is strictly increasing for $x > x''_{\lambda,\mu}$ and $h_{\lambda,\mu}(x) \to \infty$ as $x \to \infty$. So, $h_{\lambda,\mu}(x)$ has a unique zero at $r_{\lambda,\mu}$ (say) with $r_{\lambda,\mu} > x''_{\lambda,\mu}$. Therefore,

$$h_{\lambda,\mu}(x) \begin{cases} < 0 & \text{for } x < r_{\lambda,\mu}, \\ = 0 & \text{for } x = r_{\lambda,\mu}, \\ > 0 & \text{for } x > r_{\lambda,\mu}. \end{cases} \tag{2.7}$$

Thus, $r_{\lambda,\mu}$ is the only fixed point of $f_{\lambda,\mu}(x)$. By (2.3), $h'_{\lambda,\mu}(r_{\lambda,\mu}) > 0$ and consequently $f'_{\lambda,\mu}(r_{\lambda,\mu}) > 1$. Hence, $r_{\lambda,\mu}$ is the repelling fixed point of $f_{\lambda,\mu}(x)$.

In Figure 2.7, $r_{\lambda,\mu}$ is the repelling fixed point of $f_{\lambda,\mu}(x)$ for $\lambda = 1/4$, $\mu = 1/2$ and $b = e$. Here λ, μ and b satisfy $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} < 0$.

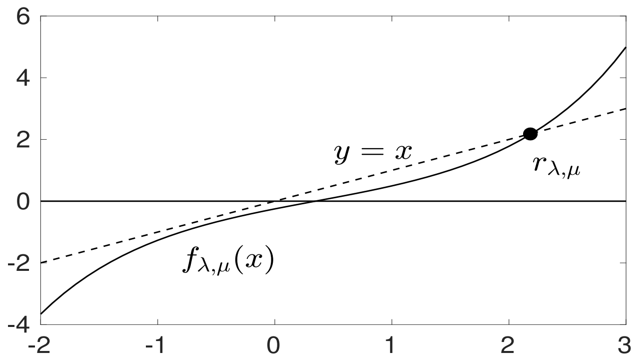


Figure 2.7: Repelling fixed point of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} < 0$.

(e) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$ **and** $m_{\lambda,\mu} > 0$

If $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} > 0$, then by using (2.3), $h_{\lambda,\mu}(x) > 0$ for all $x \geq x'_{\lambda,\mu}$. Since $h_{\lambda,\mu}(x)$ is strictly increasing for $x < x'_{\lambda,\mu}$ and $h_{\lambda,\mu}(x) \to -\infty$ as $x \to -\infty$, it follows that

$h_{\lambda,\mu}(x)$ has a unique zero $r_{\lambda,\mu}$ (say) with $r_{\lambda,\mu} < x'_{\lambda,\mu}$. Thus,

$$h_{\lambda,\mu}(x) \begin{cases} < 0 & \text{for } x < r_{\lambda,\mu}, \\ = 0 & \text{for } x = r_{\lambda,\mu}, \\ > 0 & \text{for } x > r_{\lambda,\mu}. \end{cases} \tag{2.8}$$

By (2.3), $h'_{\lambda,\mu}(r_{\lambda,\mu}) > 0$ and consequently $f'_{\lambda,\mu}(r_{\lambda,\mu}) > 1$. Therefore, $r_{\lambda,\mu}$ is the repelling fixed point of $f_{\lambda,\mu}(x)$.

For $\lambda = 1/2$, $\mu = 1/4$ and $b = e$, the repelling fixed point of $f_{\lambda,\mu}(x)$ is shown in Figure 2.8. Here λ, μ and b satisfy $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} > 0$.

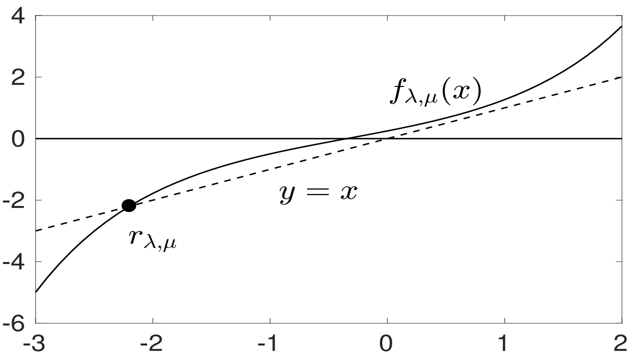


Figure 2.8: Repelling fixed point of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} > 0$.

(f) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$**,** $M_{\lambda,\mu} > 0$ **and** $m_{\lambda,\mu} < 0$

If $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$, then $h_{\lambda,\mu}(x)$ has only three zeroes $r'_{\lambda,\mu}$, $r''_{\lambda,\mu}$ and $a_{\lambda,\mu}$ (say) with $r'_{\lambda,\mu} < x'_{\lambda,\mu} < a_{\lambda,\mu} < x''_{\lambda,\mu} < r''_{\lambda,\mu}$. So,

$$h_{\lambda,\mu}(x) \begin{cases} < 0 & \text{for } x \in (-\infty, r'_{\lambda,\mu}) \cup (a_{\lambda,\mu}, r''_{\lambda,\mu}), \\ = 0 & \text{for } x = a_{\lambda,\mu},\ r'_{\lambda,\mu} \text{ and } r''_{\lambda,\mu}, \\ > 0 & \text{for } x \in (r'_{\lambda,\mu}, a_{\lambda,\mu}) \cup (r''_{\lambda,\mu}, \infty). \end{cases} \tag{2.9}$$

Now by (2.3), $h'_{\lambda,\mu}(r'_{\lambda,\mu}) > 0$ and $h'_{\lambda,\mu}(r''_{\lambda,\mu}) > 0$. This gives that $f'_{\lambda,\mu}(r'_{\lambda,\mu}) > 1$ and $f'_{\lambda,\mu}(r''_{\lambda,\mu}) > 1$. Thus, $r'_{\lambda,\mu}$ and $r''_{\lambda,\mu}$ are repelling fixed points of $f_{\lambda,\mu}(x)$. Again, by (2.3), $h'_{\lambda,\mu}(a_{\lambda,\mu}) < 0$ implies $0 < f'_{\lambda,\mu}(a_{\lambda,\mu}) < 1$. Hence $a_{\lambda,\mu}$ is the attracting fixed point of $f_{\lambda,\mu}(x)$, completing the proof of (f).

For $\lambda = 1/4$, $\mu = 1/8$ and $b = e$, the fixed points of $f_{\lambda,\mu}(x)$ are shown in Figure 2.9. Here λ, μ and b satisfy $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$. In this figure, $r'_{\lambda,\mu}$ and $r''_{\lambda,\mu}$ are repelling fixed points, and $a_{\lambda,\mu}$ is the attracting fixed point of $f_{\lambda,\mu}(x)$.

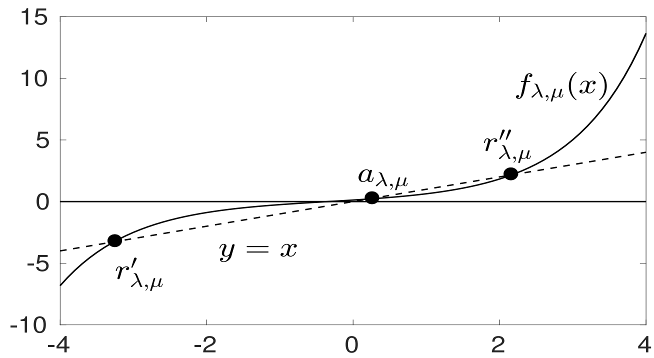


Figure 2.9: Three fixed points of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$.

(g) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$ **and** $M_{\lambda,\mu} = 0$

Note that $h_{\lambda,\mu}(x'_{\lambda,\mu}) = 0$ and $h'_{\lambda,\mu}(x'_{\lambda,\mu}) = 0$ as $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} = 0$. This implies $f_{\lambda,\mu}(x'_{\lambda,\mu}) = x'_{\lambda,\mu}$ and $f'_{\lambda,\mu}(x'_{\lambda,\mu}) = 1$. So, $x'_{\lambda,\mu}$ is a rationally indifferent fixed point of $f_{\lambda,\mu}(x)$. Observe that $h_{\lambda,\mu}(x) < 0$ for $x \in (-\infty, x''_{\lambda,\mu}] \setminus \{x'_{\lambda,\mu}\}$. Since $h_{\lambda,\mu}(x)$ is strictly increasing for $x > x''_{\lambda,\mu}$ and $h_{\lambda,\mu}(x) \to \infty$ as $x \to \infty$, it follows that $h_{\lambda,\mu}(x)$ has a unique zero $r_{\lambda,\mu}$ (say) in $(x''_{\lambda,\mu}, \infty)$. So,

$$h_{\lambda,\mu}(x) \begin{cases} < 0 & \text{for } x \in (-\infty, x'_{\lambda,\mu}) \cup (x'_{\lambda,\mu}, r_{\lambda,\mu}), \\ = 0 & \text{for } x = x'_{\lambda,\mu} \text{ and } r_{\lambda,\mu}, \\ > 0 & \text{for } x > r_{\lambda,\mu}. \end{cases} \tag{2.10}$$

Now by (2.3), $h'_{\lambda,\mu}(r_{\lambda,\mu}) > 0$ gives $f'_{\lambda,\mu}(r_{\lambda,\mu}) > 1$. Therefore, $r_{\lambda,\mu}$ is a repelling fixed point of $f_{\lambda,\mu}(x)$.

For $\lambda = 1/6$, $\mu = 2/5$ and $b = e$, the fixed points of $f_{\lambda,\mu}(x)$ are shown in Figure 2.10. Here λ, μ and b satisfy $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} = 0$. In Figure 2.10, $r_{\lambda,\mu}$ is the repelling fixed point and $x'_{\lambda,\mu}$ is the rationally indifferent fixed point of $f_{\lambda,\mu}(x)$.

(h) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$ **and** $m_{\lambda,\mu} = 0$

Clearly, $h_{\lambda,\mu}(x''_{\lambda,\mu}) = 0$ and $h'_{\lambda,\mu}(x''_{\lambda,\mu}) = 0$ as $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} = 0$. Thus, $f_{\lambda,\mu}(x''_{\lambda,\mu}) = x''_{\lambda,\mu}$ and $f'_{\lambda,\mu}(x''_{\lambda,\mu}) = 1$. Hence $x''_{\lambda,\mu}$ is a rationally indifferent fixed point of $f_{\lambda,\mu}(x)$. It is easy to see that $h_{\lambda,\mu}(x) > 0$ for $x \in [x'_{\lambda,\mu}, \infty) \setminus \{x''_{\lambda,\mu}\}$. Note that $h_{\lambda,\mu}(x)$ is strictly increasing for $x < x'_{\lambda,\mu}$ and $h_{\lambda,\mu}(x) \to -\infty$ as $x \to -\infty$. So, in $(-\infty, x'_{\lambda,\mu})$, the

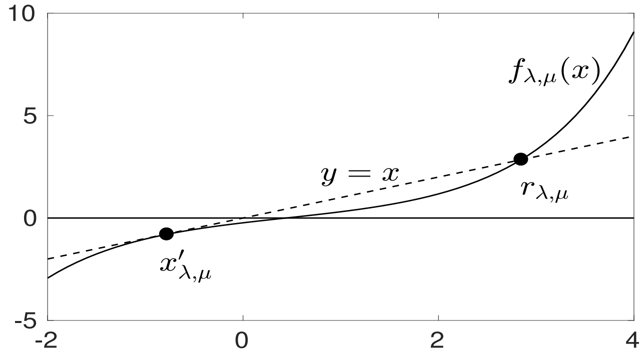


Figure 2.10: Two fixed points of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} = 0$.

function $h_{\lambda,\mu}(x)$ has a unique zero say $r_{\lambda,\mu}$. Thus,

$$h_{\lambda,\mu}(x) \begin{cases} < 0 & \text{for } x < r_{\lambda,\mu}, \\ = 0 & \text{for } x = r_{\lambda,\mu} \text{ and } x''_{\lambda,\mu}, \\ > 0 & \text{for } x \in (r_{\lambda,\mu}, x''_{\lambda,\mu}) \cup (x''_{\lambda,\mu}, \infty). \end{cases} \tag{2.11}$$

By (2.3), $h'_{\lambda,\mu}(r_{\lambda,\mu}) > 0$ and hence $f'_{\lambda,\mu}(r_{\lambda,\mu}) > 1$. Therefore, $r_{\lambda,\mu}$ is a repelling fixed point of $f_{\lambda,\mu}(x)$. This completes the proof.

For $\lambda = 2/5$, $\mu = 1/6$ and $b = e$, the fixed points of $f_{\lambda,\mu}(x)$ are shown in Figure 2.11. Here λ, μ and b satisfy $2\sqrt{\lambda\mu}\ln b < 1$, $m_{\lambda,\mu} = 0$. In Figure 2.11, $r_{\lambda,\mu}$ is the repelling fixed point and $x''_{\lambda,\mu}$ is the rationally indifferent fixed point of $f_{\lambda,\mu}(x)$. □

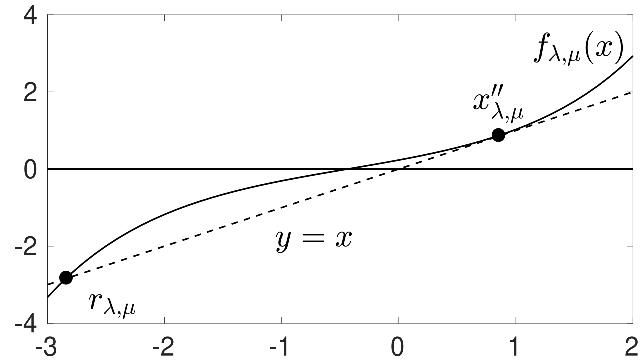


Figure 2.11: Two fixed points of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b < 1$, $m_{\lambda,\mu} = 0$.

2.3 Dynamics of $f_{\lambda,\mu}$ on the Real line $\mathbb{R}$

In Section 2.2, the real fixed points of $f_{\lambda,\mu}$ are determined. In this section, the dynamics of $f_{\lambda,\mu}$ on the real line $\mathbb{R}$, is studied with the help of the nature of the real fixed points.

The dynamics of $f_{\lambda,\mu} \in \mathcal{F}_b$ on the real line is established in the following theorem.

Theorem 2.3.1. *Let $f_{\lambda,\mu}(x) = \lambda b^x - \mu b^{-x}$ for $x \in \mathbb{R}$ where λ and μ are real parameters with $\lambda > 0$ and $\mu > 0$. Then the following holds.*

(a) If $2\sqrt{\lambda\mu}\ln b > 1$, then $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x > r_{\lambda,\mu}$ and $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for $x < r_{\lambda,\mu}$.

(b) If $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$, then $f^n_{\lambda,\lambda}(x) \to \infty$ as $n \to \infty$ for $x > 0$ and $f^n_{\lambda,\lambda}(x) \to -\infty$ as $n \to \infty$ for $x < 0$.

(c) If $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda \neq \mu$, then $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x > r_{\lambda,\mu}$ and $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for $x < r_{\lambda,\mu}$.

(d) If $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} < 0$, then $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x > r_{\lambda,\mu}$ and $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for $x < r_{\lambda,\mu}$.

(e) If $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} > 0$, then $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x > r_{\lambda,\mu}$ and $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for $x < r_{\lambda,\mu}$.

(f) If $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$, then $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x > r''_{\lambda,\mu}$, $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for $x < r'_{\lambda,\mu}$ and $f^n_{\lambda,\mu}(x) \to a_{\lambda,\mu}$ as $n \to \infty$ for $x \in \left(r'_{\lambda,\mu}, r''_{\lambda,\mu}\right)$.

(g) If $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} = 0$, then $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x > r_{\lambda,\mu}$, $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for $x < x'_{\lambda,\mu}$ and $f^n_{\lambda,\mu}(x) \to x'_{\lambda,\mu}$ as $n \to \infty$ for $x \in \left[x'_{\lambda,\mu}, r_{\lambda,\mu}\right)$.

(h) If $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} = 0$, then $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x > x''_{\lambda,\mu}$, $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for $x < r_{\lambda,\mu}$ and $f^n_{\lambda,\mu}(x) \to x''_{\lambda,\mu}$ as $n \to \infty$ for $x \in \left(r_{\lambda,\mu}, x''_{\lambda,\mu}\right]$.

Proof. (a) **Case:** $2\sqrt{\lambda\mu}\ln b > 1$

If $2\sqrt{\lambda\mu}\ln b > 1$, by Theorem 2.2.1(a), $f_{\lambda,\mu}(x)$ has a repelling fixed point $r_{\lambda,\mu}$. By using (2.4), $f_{\lambda,\mu}(x) > x$ for $x > r_{\lambda,\mu}$. Since $f_{\lambda,\mu}(x)$ is strictly increasing, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above for $x > r_{\lambda,\mu}$. Therefore, $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x > r_{\lambda,\mu}$.

Again by (2.4), $f_{\lambda,\mu}(x) < x$ for $x < r_{\lambda,\mu}$. As $f_{\lambda,\mu}(x)$ is strictly increasing, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically decreasing and not bounded below for $x < r_{\lambda,\mu}$. Thus, $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for all $x < r_{\lambda,\mu}$.

(b) **Case:** $2\sqrt{\lambda\mu}\ln b = 1$ **and** $\lambda = \mu$

If $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$, by Theorem 2.2.1(b), 0 is a rationally indifferent fixed point of $f_{\lambda,\lambda}(x)$. If $x > 0$, by (2.5), $f_{\lambda,\lambda}(x) > x$. Since $f_{\lambda,\lambda}(x) > x$ for all $x > 0$ and $f_{\lambda,\lambda}$ is strictly increasing, the sequence $\{f^n_{\lambda,\lambda}(x)\}$ is monotonically increasing and not bounded above for $x > 0$. Thus, $f^n_{\lambda,\lambda}(x) \to \infty$ as $n \to \infty$ for all $x > 0$.

Again by (2.5), $f_{\lambda,\lambda}(x) < x$ for $x < 0$. Since $f_{\lambda,\lambda}(x)$ is strictly increasing, the sequence $\{f^n_{\lambda,\lambda}(x)\}$ is monotonically decreasing and not bounded below for $x < 0$. Therefore, $f^n_{\lambda,\lambda}(x) \to -\infty$ as $n \to \infty$ for all $x < 0$.

(c) **Case:** $2\sqrt{\lambda\mu}\ln b = 1$ **and** $\lambda \neq \mu$

If $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda \neq \mu$, by Theorem 2.2.1(c), $f_{\lambda,\mu}(x)$ has a repelling fixed point $r_{\lambda,\mu}$. By (2.6), it follows that $f_{\lambda,\mu}(x) > x$ for $x > r_{\lambda,\mu}$. As $f_{\lambda,\mu}(x)$ is strictly increasing, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above for $x > r_{\lambda,\mu}$. So, $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x > r_{\lambda,\mu}$.

Again by (2.6), we get $f_{\lambda,\mu}(x) < x$ for $x < r_{\lambda,\mu}$. Therefore, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically decreasing and not bounded below for $x < r_{\lambda,\mu}$. Thus, $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for all $x < r_{\lambda,\mu}$.

(d) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$ **and** $M_{\lambda,\mu} < 0$

If $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} < 0$, by Theorem 2.2.1(d), $f_{\lambda,\mu}(x)$ has a repelling fixed point $r_{\lambda,\mu}$ with $r_{\lambda,\mu} > x''_{\lambda,\mu}$. By (2.7), $f_{\lambda,\mu}(x) > x$ for $x > r_{\lambda,\mu}$. Also $f'_{\lambda,\mu}(x) > 1$ for $x > r_{\lambda,\mu}$. Thus, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above for $x > r_{\lambda,\mu}$ and hence $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x > r_{\lambda,\mu}$.

Again by (2.7), $f_{\lambda,\mu}(x) < x$ for $x < r_{\lambda,\mu}$. Since $f_{\lambda,\mu}(x)$ is strictly increasing, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically decreasing and not bounded below for $x < r_{\lambda,\mu}$. This gives that $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for all $x < r_{\lambda,\mu}$.

(e) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$ **and** $m_{\lambda,\mu} > 0$

If $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} > 0$, by Theorem 2.2.1(e), $f_{\lambda,\mu}(x)$ has a repelling fixed point $r_{\lambda,\mu}$ with $r_{\lambda,\mu} < x'_{\lambda,\mu}$. If $x > r_{\lambda,\mu}$ by using (2.8), $f_{\lambda,\mu}(x) > x$. Since $f_{\lambda,\mu}(x)$ is increasing, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above for $x > r_{\lambda,\mu}$. Thus, $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x > r_{\lambda,\mu}$.

Again by (2.8), $f_{\lambda,\mu}(x) < x$ for $x < r_{\lambda,\mu}$. Also $f'_{\lambda,\mu}(x) > 1$ for $x < r_{\lambda,\mu}$. Therefore, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically decreasing and not bounded below for $x < r_{\lambda,\mu}$. Therefore $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for all $x < r_{\lambda,\mu}$.

(f) **Case:** $2\sqrt{\lambda\mu}\ln < 1$**,** $M_{\lambda,\mu} > 0$ **and** $m_{\lambda,\mu} < 0$

If $2\sqrt{\lambda\mu}\ln < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$, by Theorem 2.2.1(f), $f_{\lambda,\mu}(x)$ has two repelling fixed points $r'_{\lambda,\mu}$, $r''_{\lambda,\mu}$ and an attracting fixed point $a_{\lambda,\mu}$ with $r'_{\lambda,\mu} < x'_{\lambda,\mu} < a_{\lambda,\mu} < x''_{\lambda,\mu} < r''_{\lambda,\mu}$.

If $x > r''_{\lambda,\mu}$, by (2.9), $f_{\lambda,\mu}(x) > x$. Since $f'_{\lambda,\mu}(x) > 1$ for $x > r''_{\lambda,\mu}$, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above for $x > r''_{\lambda,\mu}$. So, $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x > r''_{\lambda,\mu}$.

Again by (2.9), $f_{\lambda,\mu}(x) < x$ for $x < r'_{\lambda,\mu}$. In view of the fact $f'_{\lambda,\mu}(x) > 1$ for $x < r'_{\lambda,\mu}$, it follows that the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically decreasing and not bounded below. Thus, $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for all $x < r'_{\lambda,\mu}$.

If $x \in (r'_{\lambda,\mu}, a_{\lambda,\mu})$, by (2.9), we have $f_{\lambda,\mu}(x) > x$. Thus, we have $|f_{\lambda,\mu}(x) - a_{\lambda,\mu}| < |x - a_{\lambda,\mu}|$ for $x \in (r'_{\lambda,\mu}, a_{\lambda,\mu})$. Also, if $x \in (a_{\lambda,\mu}, r''_{\lambda,\mu})$, by (2.9), $f_{\lambda,\mu}(x) < x$ gives $0 < f_{\lambda,\mu}(x) - a_{\lambda,\mu} < x - a_{\lambda,\mu}$. Thus, $|f_{\lambda,\mu}(x) - a_{\lambda,\mu}| < |x - a_{\lambda,\mu}|$ for $r'_{\lambda,\mu} < x < r''_{\lambda,\mu}$ and $x \neq a_{\lambda,\mu}$. Hence, $f^n_{\lambda,\mu}(x) \to a_{\lambda,\mu}$ as $n \to \infty$ for all $x \in (r'_{\lambda,\mu}, r''_{\lambda,\mu})$.

(g) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$ *and* $M_{\lambda,\mu} = 0$

If $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} = 0$, by Theorem 2.2.1(g), $f_{\lambda,\mu}(x)$ has a rationally indifferent fixed point $x'_{\lambda,\mu}$ and a repelling fixed point $r_{\lambda,\mu}$ with $r_{\lambda,\mu} > x''_{\lambda,\mu}$.

If $x > r_{\lambda,\mu}$, by (2.10), $f_{\lambda,\mu}(x) > x$. Observe that $f'_{\lambda,\mu}(x) > 1$ for $x > r_{\lambda,\mu}$. So, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above for $x > r_{\lambda,\mu}$. Thus, $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x > r_{\lambda,\mu}$.

Again by (2.10), $f_{\lambda,\mu}(x) < x$ for $x < x'_{\lambda,\mu}$. In view of the fact $f'_{\lambda,\mu}(x) > 1$ for $x < x'_{\lambda,\mu}$, it follows that the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically decreasing and not bounded below. Thus, $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for all $x < x'_{\lambda,\mu}$.

If $x \in (x'_{\lambda,\mu}, r_{\lambda,\mu})$, by (2.10), $f_{\lambda,\mu}(x) < x$ gives $f_{\lambda,\mu}(x) - x'_{\lambda,\mu} < x - x'_{\lambda,\mu}$. Thus, we have $|f_{\lambda,\mu}(x) - x'_{\lambda,\mu}| < |x - x'_{\lambda,\mu}|$. Hence $f^n_{\lambda,\mu}(x) \to x'_{\lambda,\mu}$ as $n \to \infty$ for all $x \in [x'_{\lambda,\mu}, r_{\lambda,\mu})$.

(h) **Case:** $2\sqrt{\lambda\mu}\ln b < 1$ *and* $m_{\lambda,\mu} = 0$

If $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} = 0$, by Theorem 2.2.1(h), $f_{\lambda,\mu}(x)$ has a rationally indifferent fixed point $x''_{\lambda,\mu}$ and a repelling fixed point $r_{\lambda,\mu}$ with $r_{\lambda,\mu} < x'_{\lambda,\mu}$.

If $x > x''_{\lambda,\mu}$, by (2.11), $f_{\lambda,\mu}(x) > x$. Since $f'_{\lambda,\mu}(x) > 1$ for $x > x''_{\lambda,\mu}$, the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above for $x > x''_{\lambda,\mu}$. So, $f^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x > x''_{\lambda,\mu}$.

If $x < r_{\lambda,\mu}$, by (2.11), $f_{\lambda,\mu}(x) < x$. Since $f'_{\lambda,\mu}(x) > 1$ for $x < r_{\lambda,\mu}$, it follows that the sequence $\{f^n_{\lambda,\mu}(x)\}$ is monotonically decreasing and not bounded below. Thus, $f^n_{\lambda,\mu}(x) \to -\infty$ as $n \to \infty$ for all $x < r_{\lambda,\mu}$.

If $x \in (r_{\lambda,\mu}, x''_{\lambda,\mu})$, by (2.11), $f_{\lambda,\mu}(x) > x$ implies $f_{\lambda,\mu}(x) - x''_{\lambda,\mu} > x - x''_{\lambda,\mu}$. Thus, we have $|f_{\lambda,\mu}(x) - x''_{\lambda,\mu}| < |x - x''_{\lambda,\mu}|$. Hence $f^n_{\lambda,\mu}(x) \to x''_{\lambda,\mu}$ as $n \to \infty$ for $x \in (r_{\lambda,\mu}, x''_{\lambda,\mu}]$. This completes the proof. □

2.4 Complex Dynamics of Functions in $\mathcal{F}_b$

The complex dynamics of $f_{\lambda,\mu} \in \mathcal{F}_b$ for certain cases is investigated in this section.

2.4.1 Dynamics of $f_{\lambda,\mu}$ in case of Real Attracting Fixed Point

The function $f_{\lambda,\mu}$ has a real attracting fixed point if $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$. Also if $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$, by Theorem 2.2.1(f), $f_{\lambda,\mu}$ has two real repelling fixed points $r'_{\lambda,\mu}$ and $r''_{\lambda,\mu}$ with $r'_{\lambda,\mu} < a_{\lambda,\mu} < r''_{\lambda,\mu}$ where $a_{\lambda,\mu}$ is the real attracting fixed point. Then, the basin of attraction $A(a_{\lambda,\mu})$ of the real attracting fixed point $a_{\lambda,\mu}$ is defined as

$$A(a_{\lambda,\mu}) := \{z \in \mathbb{C} : f^n_{\lambda,\mu}(z) \to a_{\lambda,\mu} \text{ as } n \to \infty\}.$$

Remark 2.4.1. *If* $2\sqrt{\lambda\mu}\ln b < 1$*,* $M_{\lambda,\mu} > 0$ *and* $m_{\lambda,\mu} < 0$*,*

- *By Theorem 2.3.1(f), $f^n_{\lambda,\mu}(x) \to a_{\lambda,\mu}$ as $n \to \infty$ for $x \in (r'_{\lambda,\mu}, r''_{\lambda,\mu})$. Thus, the basin of attraction $A(a_{\lambda,\mu})$ contains the interval $(r'_{\lambda,\mu}, r''_{\lambda,\mu})$ in the real line $\mathbb{R}$.*

- *Observe that $f_{\lambda,\mu}\left(a_{\lambda,\mu} + i\frac{2n\pi}{\ln b}\right) = a_{\lambda,\mu}$ for $n \in \mathbb{Z}$. Hence the basin of attraction $A(a_{\lambda,\mu})$ is unbounded.*

In the following proposition, the fate of forward orbits of all the singular values of $f_{\lambda,\mu}$ is described for $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$.

Proposition 2.4.1. *Let $A(a_{\lambda,\mu})$ be the basin of attraction of the real attracting fixed point $a_{\lambda,\mu}$ of $f_{\lambda,\mu}$ for $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$. Then, the basin of attraction $A(a_{\lambda,\mu})$ contains all the singular values and their forward orbits.*

Proof. By Proposition 2.1.2, the function $f_{\lambda,\mu}$ has only two singular values $\pm 2i\sqrt{\lambda\mu}$. By Theorem 1.5.3, the basin of attraction $A(a_{\lambda,\mu})$ contains at least one singular value. Let w be a singular value that is contained in the basin of attraction $A(a_{\lambda,\mu})$. Then, the singular values of $f_{\lambda,\mu}$ are w and $\bar{w}$. Now $f^n_{\lambda,\mu}(w) \to a_{\lambda,\mu}$ as $n \to \infty$, since $w \in A(a_{\lambda,\mu})$. This implies $\overline{f^n_{\lambda,\mu}(w)} \to \overline{a_{\lambda,\mu}} = a_{\lambda,\mu}$ as $n \to \infty$. By Proposition 2.1.3, $\overline{f^n_{\lambda,\mu}(w)} = f^n_{\lambda,\mu}(\bar{w})$ for all $n \in \mathbb{N}$. Therefore, it follows that $f^n_{\lambda,\mu}(\bar{w}) \to a_{\lambda,\mu}$ as $n \to \infty$. Thus, $\bar{w} \in A(a_{\lambda,\mu})$. This completes the proof. □

The following theorem shows that the Fatou set of $f_{\lambda,\mu}$ is the basin of attraction $A(a_{\lambda,\mu})$ for $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$.

Theorem 2.4.1. *Let $A(a_{\lambda,\mu})$ be the basin of attraction of the real attracting fixed point $a_{\lambda,\mu}$ of $f_{\lambda,\mu}$ for $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$. Then, the Fatou set of $f_{\lambda,\mu}$ is equal to the basin of attraction $A(a_{\lambda,\mu})$.*

Proof. The Fatou set of $f_{\lambda,\mu}$ has no basin of attraction other than $A(a_{\lambda,\mu})$. If possible, let $A(z_{\lambda,\mu})$ be a basin of attraction of the attracting periodic point $z_{\lambda,\mu} \neq a_{\lambda,\mu}$. Clearly, $A(z_{\lambda,\mu}) \cap A(a_{\lambda,\mu})$ is an empty set. Then, by Theorem 1.5.3, $A(z_{\lambda,\mu})$ contains at least one singular value. This contradicts Proposition 2.4.1 that $A(a_{\lambda,\mu})$ contains all the singular values and their forward orbits.

The Fatou set of $f_{\lambda,\mu}$ cannot contain a parabolic domain. If the Fatou set of $f_{\lambda,\mu}$ contains a parabolic domain P, then P must contain a singular value (see Theorem 1.5.3), which contradicts Proposition 2.4.1.

Again the Fatou set of $f_{\lambda,\mu}$ cannot contain Siegel disks. If U is a Siegel disk, then by Theorem 1.5.3, the boundary of U is contained in the closure of the forward orbits of the singular values of f_{λ}. But by Proposition 2.4.1, all the singular values and their forward orbits are contained in $A(a_{\lambda,\mu})$, giving a contradiction.

By Proposition 2.1.2, $f_{\lambda,\mu}$ has only two singular values. So the function $f_{\lambda,\mu}$ is in the class $\mathcal{S}$. By Theorems 1.5.1 and 1.5.4, the Fatou set of $f_{\lambda,\mu}$ does not contain wandering domains and Baker domains.

So, all possible Fatou components other than $A(a_{\lambda,\mu})$ is excluded. Thus, the Fatou set of $f_{\lambda,\mu}$ is $A(a_{\lambda,\mu})$. □

The basin of attraction of $f_{\lambda,\mu}$ is shown in Figure 2.12 when $\lambda = 1/4$, $\mu = 1/4$ and $b = e$. In this figure, the points for which the forward orbits are attracted by the attracting fixed point are colored in black color.

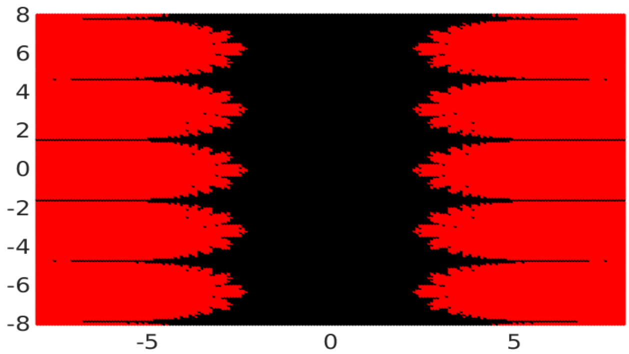


Figure 2.12: Basin of attraction in the Fatou set of $f_{\lambda,\mu}$ when $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$.

Remark 2.4.2. *When* $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ *and* $m_{\lambda,\mu} < 0$, *the following may be noted.*

- *If* $z \in A(a_{\lambda,\mu})$, *then* $f^n_{\lambda,\mu}(z) \to a_{\lambda,\mu}$ *by definition of the basin of attraction. This gives* $\overline{f^n_{\lambda,\mu}(z)} \to \overline{a_{\lambda,\mu}} = a_{\lambda,\mu}$. *Now by using Proposition 2.1.3,* $f^n_{\lambda,\mu}(\bar{z}) \to a_{\lambda,\mu}$ *and* *consequently* $\bar{z} \in A(a_{\lambda,\mu})$*. Thus,* $z \in A(a_{\lambda,\mu})$ *implies* $\bar{z} \in A(a_{\lambda,\mu})$*. So, the Fatou set* $A(a_{\lambda,\mu})$ *is symmetric with respect to the real line* $\mathbb{R}$*.*

- *Since* $f_{\lambda,\mu}$ *is transcendental entire, the Fatou set* $A(a_{\lambda,\mu})$ *has at most one completely invariant component [14].*

- *Since* $f_{\lambda,\mu} \in \mathcal{B}$ *is transcendental entire, by Proposition 3 of [37], all the components of* $A(a_{\lambda,\mu})$ *are simply connected.*

2.4.2 Dynamics of $f_{\lambda,\mu}$ in case of Real Rationally Indifferent Fixed Points

When $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$, the parabolic domain $P \equiv P(0)$ corresponding to the real rationally indifferent fixed point 0 is defined as

$$P \equiv P(0) := \{z \in \mathbb{C} : f^n_{\lambda,\mu}(z) \to 0 \text{ as } n \to \infty\}.$$

Similarly, if $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} = 0$, then the parabolic domain $P \equiv P(x'_{\lambda,\mu})$ corresponding to the real rationally indifferent fixed point $x'_{\lambda,\mu}$ is defined as

$$P \equiv P(x'_{\lambda,\mu}) := \{z \in \mathbb{C} : f^n_{\lambda,\mu}(z) \to x'_{\lambda,\mu} \text{ as } n \to \infty\}.$$

Similarly, if $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} = 0$, then the parabolic domain $P \equiv P(x''_{\lambda,\mu})$ corresponding to the real rationally indifferent fixed point $x''_{\lambda,\mu}$ is defined as

$$P \equiv P(x''_{\lambda,\mu}) := \{z \in \mathbb{C} : f^n_{\lambda,\mu}(z) \to x''_{\lambda,\mu} \text{ as } n \to \infty\}.$$

Remark 2.4.3.

- *If* $2\sqrt{\lambda\mu}\ln b = 1$ *and* $\lambda = \mu$*, by Theorem 2.3.1(b), the origin lies on the boundary of the Parabolic domain* $P(0)$*.*

- *If* $2\sqrt{\lambda\mu}\ln b < 1$ *and* $M_{\lambda,\mu} = 0$*, by Theorem 2.3.1(g), the Parabolic domain* $P(x'_{\lambda,\mu})$ *contains the interval* $[x'_{\lambda,\mu}, r_{\lambda,\mu})$ *on the real axis.*

- *If* $2\sqrt{\lambda\mu}\ln b < 1$ *and* $m_{\lambda,\mu} = 0$*, by Theorem 2.3.1(h), the Parabolic domain* $P(x''_{\lambda,\mu})$ *contains the interval* $(r_{\lambda,\mu}, x''_{\lambda,\mu}]$ *on the real axis.*

The dynamics of $f_{\lambda,\mu} \in \mathcal{F}_b$ for the above cases is similar to the dynamics of $f_{\lambda,\mu}(z)$ for $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$ except that instead of the basin of attraction as its Fatou set for $2\sqrt{\lambda\mu}\ln b < 1$, $M_{\lambda,\mu} > 0$ and $m_{\lambda,\mu} < 0$, a parabolic domain corresponding to the real rationally indifferent fixed point, is its Fatou set.

Theorem 2.4.2. *Let P be the parabolic domain corresponding to the real rationally indifferent fixed point of $f_{\lambda,\mu}$ for any one of these cases (i) $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$, (ii) $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} = 0$ and (iii) $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} = 0$. Then, the parabolic domain P contains all the singular values and their forward orbits and P is equal to the Fatou set of $f_{\lambda,\mu}$.*

Remark 2.4.4. *When the parameters λ and μ satisfy any one of the conditions (i) $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$, (ii) $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} = 0$ and (iii) $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} = 0$,*

- *By Proposition 2.1.3, the Fatou set P is symmetric with respect to the real axis.*
- *The Fatou set P has at most one completely invariant component as $f_{\lambda,\mu}$ is transcendental entire.*
- *Since $f_{\lambda,\mu} \in \mathcal{B}$ is transcendental entire, all the components of P are simply connected.*

In this chapter, we have described the dynamics of $f_{\lambda,\mu}$ for the cases (b) $2\sqrt{\lambda\mu}\ln b = 1$ and $\lambda = \mu$, (g) $2\sqrt{\lambda\mu}\ln b < 1$ and $M_{\lambda,\mu} = 0$ and (h) $2\sqrt{\lambda\mu}\ln b < 1$ and $m_{\lambda,\mu} = 0$. In the remaining cases, the Julia set contains the real line $\mathbb{R}$, equivalently the Fatou set is disconnected if it is non empty.

CHAPTER 3

DYNAMICS OF TWO-PARAMETER FAMILY OF HYPERBOLIC COSINE LIKE FUNCTIONS

The dynamics of transcendental entire functions in two-parameter family

$$\mathcal{G}_b \equiv \{g_{\lambda,\mu}(z) = \lambda b^z + \mu b^{-z} \text{ for } z \in \mathbb{C} : \lambda \geq \mu > 0\}$$

is studied in this chapter where $b > 1$. The dynamics of one-parameter family of functions $E_\lambda(z) = \lambda e^z$ for $\lambda > 0$, is mainly studied by Devaney. The Julia set $J(E_\lambda)$ changes from a nowhere dense subset of $\widehat{\mathbb{C}}$ to the whole of extended complex plane when the parameter λ increases through the value $1/e$. We prove that a similar kind of bifurcation occurs in the dynamics of functions in $\mathcal{G}_b$ when the parameters λ and μ are such that $t_{\lambda,\mu} = \sqrt{1 + 4\lambda\mu(\ln b)^2} - \ln\left(\frac{1+\sqrt{1+4\lambda\mu(\ln b)^2}}{2\lambda \ln b}\right)$ increases through the value 0. The function $g_{\lambda,\mu}$ has two critical values and no finite asymptotic values. Clearly, $g_{\lambda,\mu}$ is in $\mathcal{S}$, the class of entire or meromorphic functions having finitely many critical and asymptotic values. The change in the dynamics of two-parameter family $\mathcal{G}_b$ is investigated in this chapter. Also certain properties of the Fatou set of $g_{\lambda,\mu}$ are proved. If $\lambda = \mu = 1/2$ and $b = e$, then $g_{\lambda,\mu}(z) = \cosh(z)$. So, the dynamics of $\cosh(z)$ can be obtained from the dynamics of $g_{\lambda,\mu}(z)$. The dynamics of one-parameter family of functions $\rho\cosh(z)$ where $\rho(> 0)$ is the parameter, can be deduced from the dynamics of $g_{\lambda,\mu}(z)$ by choosing $\lambda = \mu = \rho/2$ and $b = e$.

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3.1 Basic Properties of the Functions in $\mathcal{G}_b$

In this section, some basic properties of the functions in $\mathcal{G}_b$ are established which are useful to study the dynamics. The function $g_{\lambda,\mu}(z) = \lambda b^z + \mu b^{-z}$ is periodic of minimal period $(2\pi i/\ln b)$. Since $g_{\lambda,\mu}(z)$ involved with a multiple-valued function, throughout this chapter, for the function $g_{\lambda,\mu} \in \mathcal{G}_b$, the principal branch of log is taken so that $g_{\lambda,\mu}$ becomes an entire function.

On the real line $\mathbb{R}$, the function $g_{\lambda,\mu}(x)$ is positive and $g_{\lambda,\mu}(x) \to \infty$ as $|x| \to \infty$. Observe that $g'_{\lambda,\mu}(x) = (\ln b)(\lambda b^x - \mu b^{-x})$ for $x \in \mathbb{R}$. Clearly, $g'_{\lambda,\mu}(x) \to \pm\infty$ as $x \to \pm\infty$. Since $g''_{\lambda,\mu}(x) = (\ln b)^2(\lambda b^x + \mu b^{-x}) > 0$, it follows that $g'_{\lambda,\mu}(x)$ has a unique zero at $x^0_{\lambda,\mu} = \ln(\mu/\lambda)/(2\ln b)$. Thus, $g_{\lambda,\mu}(x)$ has a unique minimum at $x = x^0_{\lambda,\mu}$ and the minimum value is $g_{\lambda,\mu}(x^0_{\lambda,\mu}) = 2\sqrt{\lambda\mu}$. Thus $g_{\lambda,\mu}$ maps real line $\mathbb{R}$ onto $[2\sqrt{\lambda\mu}, \infty)$.

In the following proposition, it is shown that the functions $g_{\lambda,\mu}$ and $g'_{\lambda,\mu}$ have some sort of symmetry about $x^0_{\lambda,\mu}$.

Proposition 3.1.1. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$. Then $g_{\lambda,\mu}(x^0_{\lambda,\mu}+z) = g_{\lambda,\mu}(x^0_{\lambda,\mu}-z)$ and $g'_{\lambda,\mu}(x^0_{\lambda,\mu}+z) = -g'_{\lambda,\mu}(x^0_{\lambda,\mu}-z)$ for all $z \in \mathbb{C}$.*

Proof. Since $x^0_{\lambda,\mu}$ is a zero of $g'_{\lambda,\mu}$, we have $\lambda b^{x^0_{\lambda,\mu}} - \mu b^{-x^0_{\lambda,\mu}} = 0$. Now for all $z \in \mathbb{C}$, $g_{\lambda,\mu}(x^0_{\lambda,\mu}+z) - g_{\lambda,\mu}(x^0_{\lambda,\mu}-z) = \lambda b^{x^0_{\lambda,\mu}+z} + \mu b^{-x^0_{\lambda,\mu}-z} - (\lambda b^{x^0_{\lambda,\mu}-z} + \mu b^{-x^0_{\lambda,\mu}+z}) = (b^z - b^{-z})(\lambda b^{x^0_{\lambda,\mu}} - \mu b^{-x^0_{\lambda,\mu}}) = 0$. Thus, $g_{\lambda,\mu}(x^0_{\lambda,\mu}+z) = g_{\lambda,\mu}(x^0_{\lambda,\mu}-z)$ for all $z \in \mathbb{C}$. Differentiating both sides with respect to z, it yields $g'_{\lambda,\mu}(x^0_{\lambda,\mu}+z) = -g'_{\lambda,\mu}(x^0_{\lambda,\mu}-z)$ for all $z \in \mathbb{C}$. This completes the proof. □

For $\lambda = 4$, $\mu = 1$ and $b = 2$, the graphs of $g_{\lambda,\mu}(x)$ and $g'_{\lambda,\mu}(x)$ are shown in Figure 3.1. In this figure $x^0_{\lambda,\mu} = -1$ is the point of symmetry.

The singular values of $g_{\lambda,\mu}$ are given in the next proposition. The function $g_{\lambda,\mu}$ has two critical values and no finite asymptotic value.

Proposition 3.1.2. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$. Then $g_{\lambda,\mu}$ has only two critical values $\pm 2\sqrt{\lambda\mu}$ and no finite asymptotic value.*

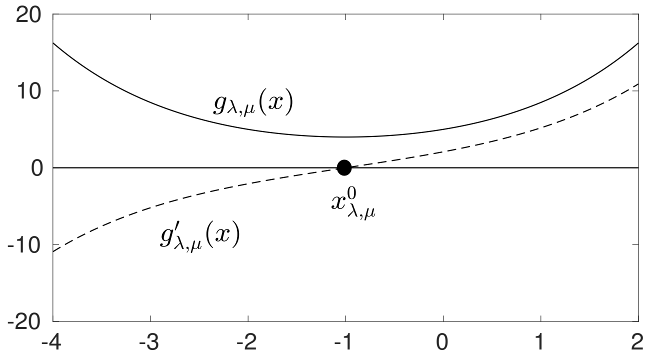

Figure 3.1: Symmetry of $g_{\lambda,\mu}(x)$ and skew symmetry of $g'_{\lambda,\mu}(x)$ about $x^0_{\lambda,\mu}$.

Proof. Observe that $g'_{\lambda,\mu}(z) = 0$ if and only if $\lambda b^z - \mu b^{-z} = 0$. That is, $b^z = \pm\sqrt{(\mu/\lambda)}$. Hence the critical values of $g_{\lambda,\mu}(z)$ are $\pm 2\sqrt{\lambda\mu}$. Now, we claim that $g_{\lambda,\mu}$ has no finite asymptotic value. If possible, let w^* be a finite asymptotic value of $g_{\lambda,\mu}$. Then, there exists a continuous curve $\gamma(t)$, $t > 0$ such that $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} g_{\lambda,\mu}(\gamma(t)) = w^*$. Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ where $\gamma_1(t)$ and $\gamma_2(t)$ are real functions of t. We can write

$$g_{\lambda,\mu}(\gamma(t)) = \cos(\gamma_2(t)\ln b)\left(\lambda b^{\gamma_1(t)} + \mu b^{-\gamma_1(t)}\right) + i\sin(\gamma_2(t)\ln b)\left(\lambda b^{\gamma_1(t)} - \mu b^{-\gamma_1(t)}\right).$$

Clearly, it follows that $|g_{\lambda,\mu}(\gamma(t))|^2 = \left(\lambda b^{\gamma_1(t)} + \mu b^{-\gamma_1(t)}\right)^2 - 4\lambda\mu\sin^2(\gamma_2(t)\ln b)$. Now $\lim_{t\to\infty}\left[\left(\lambda b^{\gamma_1(t)} + \mu b^{-\gamma_1(t)}\right)^2 - 4\lambda\mu\sin^2(\gamma_2(t)\ln b)\right] = |w^*|^2$, since w^* is an asymptotic value. This implies that $\gamma_1(t)$ is bounded and $|\gamma_2(t)| \to \infty$ as $t \to \infty$.

Assume that $\gamma_2(t) \to \infty$ as $t \to \infty$. On the curve $\gamma(t)$, we are choosing two sequences $\{z_n\}$ and $\{z'_n\}$ such that $z_n = x_n + i(2n\pi)/(\ln b)$ and $z'_n = x'_n + i((2n+1)\pi)/(\ln b)$ for $n \in \mathbb{N}$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Note that both the sequences tend to ∞ along the curve $\gamma(t)$. Clearly, $g_{\lambda,\mu}(z_n) = (\lambda b^{x_n} + \mu b^{-x_n}) \geq 2\sqrt{\lambda\mu}$ and $g_{\lambda,\mu}(z'_n) = -(\lambda b^{x'_n} + \mu b^{-x'_n}) \leq -2\sqrt{\lambda\mu}$. So, w^* to be an asymptotic value of $g_{\lambda,\mu}$, $\lim_{t\to\infty} g_{\lambda,\mu}(\gamma(t))$ must exist which is not possible for the above choices of curves $\gamma(t)$.

Again, if $\gamma_2(t) \to -\infty$ as $t \to \infty$, it can be proved that $\lim_{t\to\infty} g_{\lambda,\mu}(\gamma(t))$ does not exist. Thus, $g_{\lambda,\mu}(z)$ has no finite asymptotic value. □

In view of Proposition 3.1.2, the function $g_{\lambda,\mu}$ is in the class $\mathcal{S}$.

The iterative behaviour between the complex conjugate points is shown in the following proposition which is useful to determine the symmetry of the Fatou set about the real line $\mathbb{R}$.

Proposition 3.1.3. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$. Then for all $n \in \mathbb{N}$,*

$$g^n_{\lambda,\mu}(\bar{z}) = \overline{g^n_{\lambda,\mu}(z)} \tag{3.1}$$

holds for all $z \in \mathbb{C}$.

Proof. To prove this proposition, we apply induction principle on n. The Taylor series of $g_{\lambda,\mu}(z)$ can be expressed as

$$g_{\lambda,\mu}(z) = \sum_{m=0}^{\infty} \frac{(\lambda+\mu)(\ln b)^{2m}}{(2m)!} z^{2m} + \sum_{m=0}^{\infty} \frac{(\lambda-\mu)(\ln b)^{2m+1}}{(2m+1)!} z^{2m+1} \text{ for all } z \in \mathbb{C}.$$

Notice that $g_{\lambda,\mu}(\bar{z}) = \overline{g_{\lambda,\mu}(z)}$ for all $z \in \mathbb{C}$, as all the coefficients of the power series are real. Thus, (3.1) is true for $n = 1$. Assume that (3.1) is true for $n = k$, for some $k \in \mathbb{N}$. That is, $g^k_{\lambda,\mu}(\bar{z}) = \overline{g^k_{\lambda,\mu}(z)}$. Then

$$g^{k+1}_{\lambda,\mu}(\bar{z}) = g_{\lambda,\mu}\left(g^k_{\lambda,\mu}(\bar{z})\right) = g_{\lambda,\mu}\left(\overline{g^k_{\lambda,\mu}(z)}\right) = \overline{g_{\lambda,\mu}\left(g^k_{\lambda,\mu}(z)\right)} = \overline{g^{k+1}_{\lambda,\mu}(z)}.$$

Therefore, (3.1) is true for $n = k + 1$. This completes the proof. □

Proposition 3.1.4. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$. Then, $g_{\lambda,\mu}(x) \geq g_{\lambda,\mu}(-x)$ for all $x \geq 0$.*

Proof. Observe that $g_{\lambda,\mu}(x) - g_{\lambda,\mu}(-x) = (\lambda - \mu)(b^x - b^{-x})$ for all $x \in \mathbb{R}$. Since $\lambda \geq \mu$, it follows that $g_{\lambda,\mu}(x) \geq g_{\lambda,\mu}(-x)$ for all $x \geq 0$. □

The following theorem is a consequence of Proposition 3.1.2 and Theorems 1.5.1 and 1.5.4.

Theorem 3.1.1. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$. Then, the Fatou set of $g_{\lambda,\mu}$ does not contain wandering domains and Baker domains.*

3.2 Real Dynamics of the Functions in $\mathcal{G}_b$

In this section, the dynamics of $g_{\lambda,\mu} \in \mathcal{G}_b$ on the real line $\mathbb{R}$ is obtained by investigating the nature of the real fixed points of $g_{\lambda,\mu}$. We find that a saddle-node bifurcation occurs in the dynamics of $g_{\lambda,\mu}(x)$ whenever λ and μ satisfy $\sqrt{1+4\lambda\mu(\ln b)^2} = \ln\left(\frac{1+\sqrt{1+4\lambda\mu(\ln b)^2}}{2\lambda \ln b}\right)$.

Define $h_{\lambda,\mu}(x) = g_{\lambda,\mu}(x) - x$ where $g_{\lambda,\mu}(x) = \lambda b^x + \mu b^{-x}$ for $x \in \mathbb{R}$ and λ, μ are real parameters with $\lambda \geq \mu > 0$. Clearly the zeros of $h_{\lambda,\mu}(x)$ are the fixed points of $g_{\lambda,\mu}(x)$. Note that $h_{\lambda,\mu}(x) \to \infty$ as $|x| \to \infty$. Observe that $h'_{\lambda,\mu}(x) = (\ln b)(\lambda b^x - \mu b^{-x}) - 1$ and $h'_{\lambda,\mu}(x) \to \pm\infty$ as $x \to \pm\infty$.

In view of $h''_{\lambda,\mu}(x) = (\ln b)^2(\lambda b^x + \mu b^{-x}) > 0$, it follows that $h'_{\lambda,\mu}(x)$ has a unique zero at

$$x^*_{\lambda,\mu} = \left(\frac{1}{\ln b}\right) \ln\left(\frac{1+\sqrt{1+4\lambda\mu(\ln b)^2}}{2\lambda \ln b}\right).$$

So, the function $h_{\lambda,\mu}(x)$ has a global minimum at $x^*_{\lambda,\mu}$ and the minimum value is

$$h_{\lambda,\mu}(x^*_{\lambda,\mu}) = \left(\frac{1}{\ln b}\right)\left[\sqrt{1+4\lambda\mu(\ln b)^2} - \ln\left(\frac{1+\sqrt{1+4\lambda\mu(\ln b)^2}}{2\lambda \ln b}\right)\right].$$

Throughout this chapter, let $t_{\lambda,\mu}$ denote

$$t_{\lambda,\mu} := \sqrt{1+4\lambda\mu(\ln b)^2} - \ln\left(\frac{1+\sqrt{1+4\lambda\mu(\ln b)^2}}{2\lambda \ln b}\right).$$

Note that,

$$h'_{\lambda,\mu}(x)\begin{cases} < 0 & \text{for } x < x^*_{\lambda,\mu}, \\ = 0 & \text{for } x = x^*_{\lambda,\mu}, \\ > 0 & \text{for } x > x^*_{\lambda\mu}. \end{cases} \tag{3.2}$$

Observe that $g'_{\lambda,\mu}(x^*_{\lambda,\mu}) = 1$. Also there exists a unique $x = x^{**}_{\lambda,\mu}$ such that $g'_{\lambda,\mu}(x^{**}_{\lambda,\mu}) = -1$, since $g'_{\lambda,\mu}(x)$ is strictly increasing and $g'_{\lambda,\mu}(x) \to \pm\infty$ as $x \to \pm\infty$. This $x^{**}_{\lambda,\mu}$ is given by

$$x^{**}_{\lambda,\mu} = \left(\frac{1}{\ln b}\right) \ln\left(\frac{-1+\sqrt{1+4\lambda\mu(\ln b)^2}}{2\lambda \ln b}\right).$$

In the following theorem, the existence and nature of the real fixed points of $g_{\lambda,\mu}$ are established.

Theorem 3.2.1. *Let $g_{\lambda,\mu}(x) = \lambda b^x + \mu b^{-x}$ for $x \in \mathbb{R}$ where λ and μ are parameters with $\lambda \geq \mu > 0$.*

(a) *If* $t_{\lambda,\mu} < 0$, *then* $g_{\lambda,\mu}(x)$ *has a repelling fixed point* $r_{\lambda,\mu}$ *(say) and an attracting fixed point* $a_{\lambda,\mu}$ *(say) with* $a_{\lambda,\mu} < x^*_{\lambda,\mu} < r_{\lambda,\mu}$.

(b) *If* $t_{\lambda,\mu} = 0$, *then* $g_{\lambda,\mu}(x)$ *has a unique rationally indifferent fixed point* $x^*_{\lambda,\mu}$.

(c) *If* $t_{\lambda,\mu} > 0$, *then* $g_{\lambda,\mu}(x)$ *has no fixed points.*

Proof. Set $h_{\lambda,\mu}(x) = g_{\lambda,\mu}(x) - x$ for $x \in \mathbb{R}$ and $\lambda \geq \mu > 0$.

(a) **Case:** $t_{\lambda,\mu} < 0$

Note that $h_{\lambda,\mu}(x^*_{\lambda,\mu}) < 0$, since $t_{\lambda,\mu} < 0$. This implies that $h_{\lambda,\mu}(x)$ has only two zeros $a_{\lambda,\mu}$ and $r_{\lambda,\mu}$(say) with $a_{\lambda,\mu} < x^*_{\lambda,\mu} < r_{\lambda,\mu}$. Thus,

$$h_{\lambda,\mu}(x) \begin{cases} > 0 & \text{for } x \in (-\infty, a_{\lambda,\mu}) \cup (r_{\lambda,\mu}, \infty), \\ = 0 & \text{for } x = a_{\lambda,\mu} \text{ or } r_{\lambda,\mu}, \\ < 0 & \text{for } x \in (a_{\lambda,\mu}, r_{\lambda,\mu}). \end{cases} \tag{3.3}$$

So, $a_{\lambda,\mu}$ and $r_{\lambda,\mu}$ are the fixed points of $g_{\lambda,\mu}(x)$. Now by (3.2), $h'_{\lambda,\mu}(r_{\lambda,\mu}) > 0$ which gives $g'_{\lambda,\mu}(r_{\lambda,\mu}) > 1$. Thus, $r_{\lambda,\mu}$ is a repelling fixed point of $g_{\lambda,\mu}(x)$. Now $h'_{\lambda,\mu}(x)$ is strictly increasing and $a_{\lambda,\mu} < x^*_{\lambda,\mu}$ implies $h'_{\lambda,\mu}(a_{\lambda,\mu}) < h'_{\lambda,\mu}(x^*_{\lambda,\mu}) = 0$. Thus, $g'_{\lambda,\mu}(a_{\lambda,\mu}) < 1$. Observe that $a_{\lambda,\mu} > x^0_{\lambda,\mu}$. Now, $g'_{\lambda,\mu}(a_{\lambda,\mu}) > g'_{\lambda,\mu}(x^0_{\lambda,\mu}) = 0$, as $g'_{\lambda,\mu}(x)$ is strictly increasing. Thus, $0 < g'_{\lambda,\mu}(a_{\lambda,\mu}) < 1$ for $t_{\lambda,\mu} < 0$. Hence $a_{\lambda,\mu}$ is an attracting fixed point of $g_{\lambda,\mu}(x)$. This completes the proof of (a).

In the following figure, the fixed points of $g_{\lambda,\mu}(x)$ are shown for $\lambda = 1/4$, $\mu = 1/8$ and $b = e$. In Figure 3.2, $a_{\lambda,\mu}$ is the attracting fixed point and $r_{\lambda,\mu}$ is the repelling fixed point.

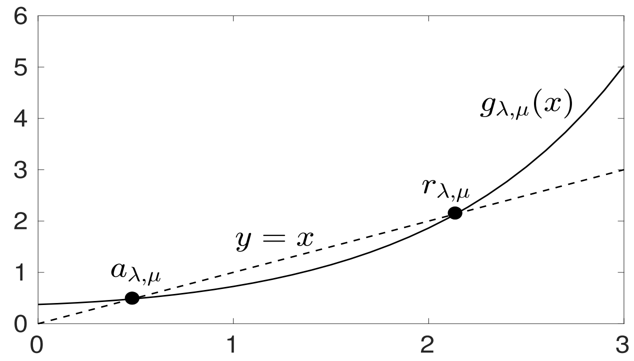

Figure 3.2: Two fixed points of $g_{\lambda,\mu}(x)$ when $t_{\lambda,\mu} < 0$.

(b) **Case:** $t_{\lambda,\mu} = 0$

If $t_{\lambda,\mu} = 0$, then $h_{\lambda,\mu}(x^*_{\lambda,\mu}) = 0$ and $h'_{\lambda,\mu}(x^*_{\lambda,\mu}) = 0$. This implies $g_{\lambda,\mu}(x^*_{\lambda,\mu}) = x^*_{\lambda,\mu}$ and $g'_{\lambda,\mu}(x^*_{\lambda,\mu}) = 1$. So, $x^*_{\lambda,\mu}$ is a rationally indifferent fixed point of $g_{\lambda,\mu}(x)$. Now $h_{\lambda,\mu}(x) > h_{\lambda,\mu}(x^*_{\lambda,\mu}) = 0$ for $x \neq x^*_{\lambda,\mu}$. Thus, $h_{\lambda,\mu}(x)$ has no zero other than $x^*_{\lambda,\mu}$. Hence $g_{\lambda,\mu}(x)$ has only one fixed point at $x = x^*_{\lambda,\mu}$.

In Figure 3.3, $x^*_{\lambda,\mu}$ is the rationally indifferent fixed point of $g_{\lambda,\mu}(x)$ for $\lambda = \mu = 1/3.018$ and $b = e$.

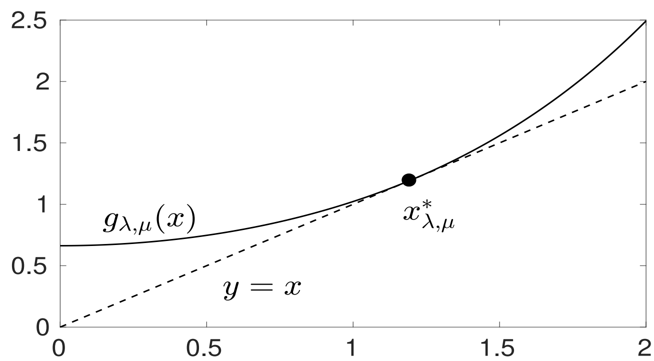

Figure 3.3: Rationally indifferent fixed point of $g_{\lambda,\mu}(x)$ when $t_{\lambda,\mu} = 0$.

(c) **Case:** $t_{\lambda,\mu} > 0$

If $t_{\lambda,\mu} > 0$, then $h_{\lambda,\mu}(x) \geq h_{\lambda,\mu}(x^*_{\lambda,\mu}) > 0$ for all $x \in \mathbb{R}$. Thus $h_{\lambda,\mu}(x)$ has no zeros and consequently $g_{\lambda,\mu}(x)$ has no fixed points. This completes the proof of the theorem.

In Figure 3.4, it is shown that the function $g_{\lambda,\mu}(x)$ has no fixed points for $\lambda = 1$, $\mu = 1/4$ and $b = e$. □

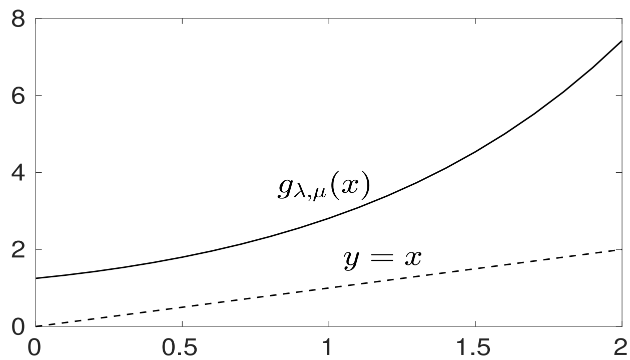

Figure 3.4: No fixed points of $g_{\lambda,\mu}(x)$ when $t_{\lambda,\mu} > 0$.

The dynamics of $g_{\lambda,\mu} \in \mathcal{G}_b$ on the real line $\mathbb{R}$, is investigated and described in the following theorem:

Theorem 3.2.2. *Let $g_{\lambda,\mu}(x) = \lambda b^x + \mu b^{-x}$ for $x \in \mathbb{R}$ where λ and μ are real parameters with $\lambda \geq \mu > 0$.*

(a) If $t_{\lambda,\mu} < 0$, then $g^n_{\lambda,\mu}(x) \to a_{\lambda,\mu}$ as $n \to \infty$ for $x \in (r'_{\lambda,\mu}, r_{\lambda,\mu})$ and $g^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, r'_{\lambda,\mu}) \cup (r_{\lambda,\mu}, \infty)$ where $r'_{\lambda,\mu} = g^{-1}_{\lambda,\mu}(r_{\lambda,\mu}) \in \mathbb{R}$ and $r'_{\lambda,\mu} < r_{\lambda,\mu}$.

(b) If $t_{\lambda,\mu} = 0$, then $g^n_{\lambda,\mu}(x) \to x^*_{\lambda,\mu}$ as $n \to \infty$ for $x \in \left[x^{**}_{\lambda,\mu}, x^*_{\lambda,\mu}\right]$ and $g^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, x^{**}_{\lambda,\mu}) \cup (x^*_{\lambda,\mu}, \infty)$.

(c) If $t_{\lambda,\mu} > 0$, then $g^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x \in \mathbb{R}$.

Proof. (a) **Case:** $t_{\lambda,\mu} < 0$

If $t_{\lambda,\mu} < 0$, by Theorem 3.2.1(a), $g_{\lambda,\mu}(x)$ has an attracting fixed point $a_{\lambda,\mu}$ and a repelling fixed point $r_{\lambda,\mu}$ with $2\sqrt{\lambda\mu} < a_{\lambda,\mu} < x^*_{\lambda,\mu} < r_{\lambda,\mu}$. By (3.3), for $a_{\lambda,\mu} < x < r_{\lambda,\mu}$,

$$g_{\lambda,\mu}(x) - a_{\lambda,\mu} < x - a_{\lambda,\mu}. \tag{3.4}$$

If $x^0_{\lambda,\mu} \leq x < a_{\lambda,\mu}$, then by mean value theorem, $|g_{\lambda,\mu}(x) - a_{\lambda,\mu}| = |g'_{\lambda,\mu}(c)||x - a_{\lambda,\mu}|$, where $x^0_{\lambda,\mu} \leq x < c < a_{\lambda,\mu}$. Since $g'_{\lambda,\mu}(x^0_{\lambda,\mu}) = 0$, $g'_{\lambda,\mu}(a_{\lambda,\mu}) < 1$ and $g'_{\lambda,\mu}(x)$ is strictly increasing, it follows that $|g'_{\lambda,\mu}(c)| < 1$. Consequently, $|g_{\lambda,\mu}(x) - a_{\lambda,\mu}| < |x - a_{\lambda,\mu}|$ for $x^0_{\lambda,\mu} \leq x < a_{\lambda,\mu}$. This inequality together with (3.4) gives that $|g_{\lambda,\mu}(x) - a_{\lambda,\mu}| < |x - a_{\lambda,\mu}|$ for $x^0_{\lambda,\mu} \leq x < r_{\lambda,\mu}$ and $x \neq a_{\lambda,\mu}$. Thus, if $x^0_{\lambda,\mu} \leq x < r_{\lambda,\mu}$, then $g^n_{\lambda,\mu}(x) \to a_{\lambda,\mu}$ as $n \to \infty$. Now in view of $g_{\lambda,\mu}\left((r'_{\lambda,\mu}, x^0_{\lambda,\mu})\right) = \left(2\sqrt{\lambda\mu}, r_{\lambda,\mu}\right) \subseteq (x^0_{\lambda,\mu}, r_{\lambda,\mu})$, it follows that $g^n_{\lambda,\mu}(x) \to a_{\lambda,\mu}$ as $n \to \infty$ for $x \in (r'_{\lambda,\mu}, r_{\lambda,\mu})$.

If $x > r_{\lambda,\mu}$, observe that $g_{\lambda,\mu}(x) > x$ and $g'_{\lambda,\mu}(x) > 1$. Therefore, the sequence $\{g^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above, and hence $g^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x > r_{\lambda,\mu}$. Again, in view of $g_{\lambda,\mu}\left((-\infty, r'_{\lambda,\mu})\right) = (r_{\lambda,\mu}, \infty)$, $g^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x < r'_{\lambda,\mu}$. This completes the proof of (a).

(b) **Case:** $t_{\lambda,\mu} = 0$

By Theorem 3.2.1(b), $g_{\lambda,\mu}(x)$ has a unique rationally indifferent fixed point $x^*_{\lambda,\mu}$ for $t_{\lambda,\mu} = 0$. If $x^{**}_{\lambda,\mu} \leq x < x^*_{\lambda,\mu}$, then by mean value theorem, $|g_{\lambda,\mu}(x) - x^*_{\lambda,\mu}| = |g'_{\lambda,\mu}(c)||x - x^*_{\lambda,\mu}|$, where $x^{**}_{\lambda,\mu} \leq x < c < x^*_{\lambda,\mu}$. Since $g'_{\lambda,\mu}(x^{**}_{\lambda,\mu}) = -1$, $g'_{\lambda,\mu}(x^*_{\lambda,\mu}) = 1$ and $g'_{\lambda,\mu}(x)$ is strictly increasing, it follows that $|g'_{\lambda,\mu}(c)| < 1$. Thus, $|g_{\lambda,\mu}(x) - x^*_{\lambda,\mu}| < |x - x^*_{\lambda,\mu}|$ for $x^{**}_{\lambda,\mu} \leq x < x^*_{\lambda,\mu}$. Hence $g^n_{\lambda,\mu}(x) \to x^*_{\lambda,\mu}$ as $n \to \infty$ for $x \in [x^{**}_{\lambda,\mu}, x^*_{\lambda,\mu}]$.

Observe that $g_{\lambda,\mu}(x) > x$ for $x > x^*_{\lambda,\mu}$. Since $g_{\lambda,\mu}(x) > x$ and $g_{\lambda,\mu}(x)$ is strictly increasing for $x > x^*_{\lambda,\mu}$, the sequence $\{g^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above for $x > x^*_{\lambda,\mu}$. Thus, $g^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x \in (x^*_{\lambda,\mu}, \infty)$.

Notice that $g_{\lambda,\mu}\left((-\infty, x^{**}_{\lambda,\mu})\right) = (x^*_{\lambda,\mu}, \infty)$. So, $g^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, x^{**}_{\lambda,\mu})$.

(c) **Case:** $t_{\lambda,\mu} > 0$

If $t_{\lambda,\mu} > 0$, by Theorem 3.2.1(c), $g_{\lambda,\mu}(x)$ has no fixed points. Since $g_{\lambda,\mu}(x) \geq 2\sqrt{\lambda\mu}$, $g_{\lambda,\mu}(x) > x$ for all $x \in \mathbb{R}$ and $g_{\lambda,\mu}(x)$ is strictly increasing for $x > x^0_{\lambda,\mu}$, the sequence $\{g^n_{\lambda,\mu}(x)\}$ is monotonically increasing and not bounded above. Therefore, it follows that $g^n_{\lambda,\mu}(x) \to \infty$ as $n \to \infty$ for all $x \in \mathbb{R}$. This completes the proof. □

Remark 3.2.1. *Theorem 3.2.2 shows that a bifurcation occurs in the dynamics of* $g_{\lambda,\mu}(x)$, $x \in \mathbb{R}$, *in the family* $\mathcal{G}_b$. *If* $t_{\lambda,\mu} < 0$ *it follows from Theorem 3.2.2(a), that under iteration of* $g_{\lambda,\mu}$ *the orbits of all the points in* $[r'_{\lambda,\mu}, r_{\lambda,\mu}]$ *remain bounded and the orbits of all the points in* $\mathbb{R} \setminus [r'_{\lambda,\mu}, r_{\lambda,\mu}]$ *become unbounded. Similarly, if* $t_{\lambda,\mu} = 0$ *by Theorem 3.2.2(b), orbits of all the points in* $[x^{**}_{\lambda,\mu}, x^*_{\lambda,\mu}]$ *are bounded under iteration of* $g_{\lambda,\mu}$ *and orbits of points in* $\mathbb{R} \setminus [x^{**}_{\lambda,\mu}, x^*_{\lambda,\mu}]$ *are unbounded under iteration of* $g_{\lambda,\mu}$. *But when* $t_{\lambda,\mu} > 0$, *by Theorem 3.2.2(c), there are no real points whose orbits remain bounded under iteration of* $g_{\lambda,\mu}$. *Thus, a saddle-node bifurcation occurs in the dynamics of* $g_{\lambda,\mu}(x)$, $x \in \mathbb{R}$, *when the parameters* λ *and* μ *satisfy* $t_{\lambda,\mu} = 0$.

3.3 Complex Dynamics of $g_{\lambda,\mu}$

The dynamics of $g_{\lambda,\mu}$ on the complex plane is studied in this section. It is observed that the Fatou set of $g_{\lambda,\mu}$ is non empty whenever $t_{\lambda,\mu} \leq 0$.

3.3.1 Dynamics of $g_{\lambda,\mu}$ for $t_{\lambda,\mu} < 0$

If $t_{\lambda,\mu} < 0$, by Theorem 3.2.1(a), $g_{\lambda,\mu}(z)$ has a real attracting fixed point $a_{\lambda,\mu}$ and a real repelling fixed point $r_{\lambda,\mu}$ with $2\sqrt{\lambda\mu} < a_{\lambda,\mu} < x^*_{\lambda,\mu} < r_{\lambda,\mu}$. The basin of attraction $A(a_{\lambda,\mu})$ corresponding to the attracting fixed point $a_{\lambda,\mu}$ is defined as

$$A(a_{\lambda,\mu}) := \{z \in \mathbb{C} : g^n_{\lambda,\mu}(z) \to a_{\lambda,\mu} \text{ as } n \to \infty\}.$$

Remark 3.3.1. *When $t_{\lambda,\mu} < 0$,*

- *By Theorem 3.2.2(a), $g^n_{\lambda,\mu}(x) \to a_{\lambda,\mu}$ as $n \to \infty$ for $x \in (r'_{\lambda,\mu}, r_{\lambda,\mu})$. Hence the basin of attraction $A(a_{\lambda,\mu})$ contains the interval $(r'_{\lambda,\mu}, r_{\lambda,\mu})$ on the real line $\mathbb{R}$.*
- *Observe that $g_{\lambda,\mu}(x^0_{\lambda,\mu} + iy) = 2\sqrt{\lambda\mu}\cos(y \ln b) \in (r'_{\lambda,\mu}, r_{\lambda,\mu})$ for $y \in \mathbb{R}$. Thus, the basin of attraction $A(a_{\lambda,\mu})$ contains the vertical line $\{z = x^0_{\lambda,\mu} + iy : y \in \mathbb{R}\}$. Hence the basin of attraction $A(a_{\lambda,\mu})$ is unbounded.*

In the following proposition, the fate of forward orbits of all the singular values of $g_{\lambda,\mu}(z)$ is described for $t_{\lambda,\mu} < 0$.

Proposition 3.3.1. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$ and $A(a_{\lambda,\mu})$ be the basin of attraction of the real attracting fixed point $a_{\lambda,\mu}$ of $g_{\lambda,\mu}(z)$ for $t_{\lambda,\mu} < 0$. Then, $A(a_{\lambda,\mu})$ contains all the singular values and their forward orbits.*

Proof. By Proposition 3.1.2, the function $g_{\lambda,\mu}$ has only two singular values $\pm 2\sqrt{\lambda\mu}$. If $t_{\lambda,\mu} < 0$, notice that the critical value $2\sqrt{\lambda\mu} \in (r'_{\lambda,\mu},\ r_{\lambda,\mu})$. By using Theorem 3.2.2(a), the critical value $2\sqrt{\lambda\mu}$ is contained in $A(a_{\lambda,\mu})$. Observe that $r'_{\lambda,\mu} < x^0_{\lambda,\mu} \leq 0$. Now by Proposition 3.1.4, it follows that $g_{\lambda,\mu}(-2\sqrt{\lambda\mu}) \leq g_{\lambda,\mu}(2\sqrt{\lambda\mu})$. Clearly, $r'_{\lambda,\mu} < g_{\lambda,\mu}(-2\sqrt{\lambda\mu}) \leq g_{\lambda,\mu}(2\sqrt{\lambda\mu}) < r_{\lambda,\mu}$. Thus, $-2\sqrt{\lambda\mu} \in A(a_{\lambda,\mu})$ and hence $A(a_{\lambda,\mu})$ contains all the singular values and their forward orbits. □

For $t_{\lambda,\mu} < 0$, the Fatou set of $g_{\lambda,\mu}(z)$ is described in the following theorem.

Theorem 3.3.1. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$ and $A(a_{\lambda,\mu})$ be the basin of attraction of the real attracting fixed point $a_{\lambda,\mu}$ of $g_{\lambda,\mu}$ for $t_{\lambda,\mu} < 0$. Then, the Fatou set of $g_{\lambda,\mu}$ equals the basin of attraction $A(a_{\lambda,\mu})$.*

Proof. By Theorem 3.1.1, the Fatou set of $g_{\lambda,\mu}$ does not contain Baker domains and wandering domains.

The Fatou set of $g_{\lambda,\mu}$ has no basin of attraction other than $A(a_{\lambda,\mu})$. Let $A(z_{\lambda,\mu})$ be a basin of attraction of the attracting periodic point $z_{\lambda,\mu} \neq a_{\lambda,\mu}$. Obviously, $A(z_{\lambda,\mu}) \cap A(a_{\lambda,\mu})$ is an empty set. By Theorem 1.5.3, basin of attraction $A(z_{\lambda,\mu})$ contains at least one singular value. This contradicts the fact that all the singular values and their forwards orbits are in $A(a_{\lambda,\mu})$, since $A(z_{\lambda,\mu}) \cap A(a_{\lambda,\mu})$ is an empty set.

The Fatou set of $g_{\lambda,\mu}$ cannot contain a parabolic domain. If the Fatou set of $g_{\lambda,\mu}$ contains a parabolic domain U, the invariant domain U must contain at least one singular value (Theorem 1.5.3), which contradicts Proposition 3.3.1.

Again the Fatou set of $g_{\lambda,\mu}$ cannot contain Siegel disks. If U is a Siegel disk, then by Theorem 1.5.3, the boundary of U is contained in the closure of the forward orbits of the singular values of $g_{\lambda,\mu}$. But $A(a_{\lambda,\mu})$ contains all the singular values and their forward orbits, giving a contradiction.

Thus, $A(a_{\lambda,\mu})$ is the only possible Fatou component. Therefore, the Fatou set of $g_{\lambda,\mu}$ is the basin of attraction $A(a_{\lambda,\mu})$. □

Figure 3.5 gives an idea on the basin of attraction of $g_{\lambda,\mu}$ when $\lambda = 1/4$, $\mu = 1/8$ and $b = e$. In this figure, the points for which the forward orbits are attracted by the attracting cycle are colored in black color.

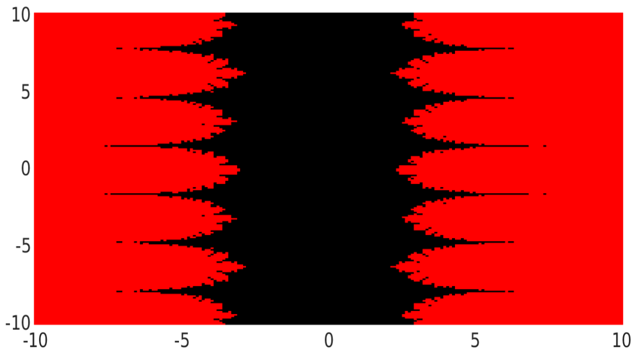

Figure 3.5: Basin of attraction in the Fatou set of $g_{\lambda,\mu}$ when $t_{\lambda,\mu} < 0$.

Remark 3.3.2. *When* $t_{\lambda,\mu} < 0$,

- *The Fatou set of $g_{\lambda,\mu}$ contains the interval $(r'_{\lambda,\mu}, r_{\lambda,\mu})$ on the real line $\mathbb{R}$.*

- *By Remark 3.3.1, the Fatou set of $g_{\lambda,\mu}$ contains the vertical line $\{z = x^0_{\lambda,\mu} + iy : y \in \mathbb{R}\}$. Hence the Julia set $J(g_{\lambda,\mu})$ is not connected.*

- *If $z \in A(a_{\lambda,\mu})$, then $g^n_{\lambda,\mu}(z) \to a_{\lambda,\mu}$ as $n \to \infty$. This gives $\overline{g^n_{\lambda,\mu}(z)} \to \overline{a_{\lambda,\mu}} = a_{\lambda,\mu}$. Now by using Proposition 3.1.3, $g^n_{\lambda,\mu}(\bar{z}) \to a_{\lambda,\mu}$ and consequently $\bar{z} \in A(a_{\lambda,\mu})$. Thus if $z \in A(a_{\lambda,\mu})$, then $\bar{z} \in A(a_{\lambda,\mu})$. So, the Fatou set $A(a_{\lambda,\mu})$ is symmetric with respect to the real axis.*

- *In view of Proposition 3.1.1, the Fatou set $A(a_{\lambda,\mu})$ is symmetric about the point $x^0_{\lambda,\mu}$.*

Theorem 3.3.2. *Let U be a connected component of the Fatou set of $g_{\lambda,\mu}$ for $t_{\lambda,\mu} < 0$. Then, U is unbounded and ∂U is not locally connected at any finite point.*

Proof. Observe that the immediate basin of attraction contains the interval $(r'_{\lambda,\mu}, r_{\lambda,\mu})$ on the real axis. In view of $g_{\lambda,\mu}$ maps the vertical line $\{z = x^0_{\lambda,\mu} + iy : y \in \mathbb{R}\}$ into $[-2\sqrt{\lambda\mu}, 2\sqrt{\lambda\mu}]$ which is a subset of $(r'_{\lambda,\mu}, r_{\lambda,\mu})$, the immediate basin of attraction contains the vertical line $\{z = x^0_{\lambda,\mu} + iy : y \in \mathbb{R}\}$. So the Fatou set of $g_{\lambda,\mu}$ has an unbounded connected component. It is easy to see that $g_{\lambda,\mu}$ is hyperbolic for $t_{\lambda,\mu} < 0$. Then by Theorem 1.6.6, every connected component U of $F(g_{\lambda,\mu})$ is unbounded and ∂U is not locally connected at any finite point. □

Remark 3.3.3. *By Theorem 3.3.2, $F(g_{\lambda,\mu})$ has an unbounded Fatou component for $t_{\lambda,\mu} < 0$. By Lemma 2.4 of [15], the Julia set $J(g_{\lambda,\mu})$ is not locally connected.*

3.3.2 Dynamics of $g_{\lambda,\mu}$ for $t_{\lambda,\mu} = 0$

When $t_{\lambda,\mu} = 0$, the parabolic domain $P \equiv P(x^*_{\lambda,\mu})$ corresponding to the real rationally indifferent fixed point $x^*_{\lambda,\mu}$ is given by

$$P \equiv P(x^*_{\lambda,\mu}) := \{z \in \mathbb{C} : g^n_{\lambda,\mu}(z) \to x^*_{\lambda,\mu} \text{ as } n \to \infty\}.$$

The dynamics of $g_{\lambda,\mu} \in \mathcal{G}_b$ for $t_{\lambda,\mu} = 0$ is similar to that of the dynamics of $g_{\lambda,\mu}$ for $t_{\lambda,\mu} < 0$ except that instead of basin of attraction, a parabolic domain is its Fatou set and it is proved in the following theorem.

Theorem 3.3.3. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$ and P be the parabolic domain of the real rationally indifferent fixed point $x^*_{\lambda,\mu}$ of $g_{\lambda,\mu}(z)$ for $t_{\lambda,\mu} = 0$. Then P contains all the singular values and their forward orbits and the Fatou set of $g_{\lambda,\mu}$ is equal to the parabolic domain P.*

Figure 3.6 gives an idea on the parabolic domain of $g_{\lambda,\mu}$ when $\lambda = \mu = 1/3.018$ and $b = e$. In Figure 3.6, the points for which the forward orbits are attracted by the parabolic cycle are colored in black color.

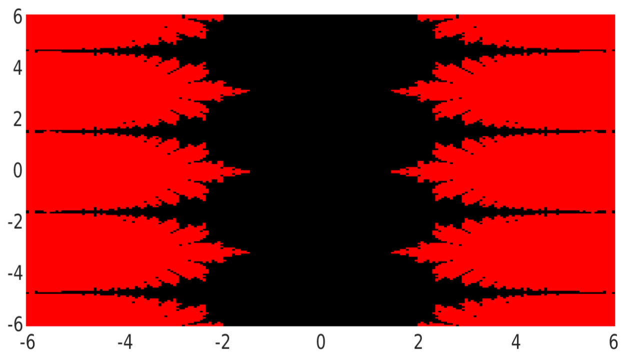

Figure 3.6: Parabolic domain in the Fatou set of $g_{\lambda,\mu}$ when $t_{\lambda,\mu} = 0$.

Remark 3.3.4. *When $t_{\lambda,\mu} = 0$,*

- *By Theorem 3.2.2(b), $g^n_{\lambda,\mu}(x) \to x^*_{\lambda,\mu}$ as $n \to \infty$ for $x \in [x^{**}_{\lambda,\mu}, x^*_{\lambda,\mu}]$. Hence the parabolic domain $P(x^*_{\lambda,\mu})$ contains the interval $[x^{**}_{\lambda,\mu}, x^*_{\lambda,\mu}]$ on the real axis $\mathbb{R}$.*

- *Observe that $g_{\lambda,\mu}(x^0_{\lambda,\mu} + iy) = 2\sqrt{\lambda\mu}\cos(y \ln b) \in (x^{**}_{\lambda,\mu}, x^*_{\lambda,\mu})$ for $y \in \mathbb{R}$. Thus, the parabolic domain P contains the vertical line $\{z = x^0_{\lambda,\mu} + iy : y \in \mathbb{R}\}$. Hence the Fatou set P of $g_{\lambda,\mu}$ is unbounded and the Julia set is not connected.*

- *In view of Propositions 3.1.1 and 3.1.3, the Fatou set P is symmetric about the point $x^0_{\lambda,\mu}$ as well as the real axis $\mathbb{R}$.*

- *The Fatou set $F(g_{\lambda,\mu})$ has an unbounded component, since the parabolic domain P contains the vertical line $\{z = x^0_{\lambda,\mu} + iy : y \in \mathbb{R}\}$. By Lemma 2.4 of [15], the Julia set $J(g_{\lambda,\mu})$ is not locally connected.*

3.3.3 Dynamics of $g_{\lambda,\mu}$ for $t_{\lambda,\mu} > 0$

The dynamics of $g_{\lambda,\mu}(z)$ for $z \in \mathbb{C}$ and $t_{\lambda,\mu} > 0$ is discussed in this subsection. In this case the Julia set of $g_{\lambda,\mu}(z)$ explodes and becomes equal to the whole of the extended complex plane $\widehat{\mathbb{C}}$.

Theorem 3.3.4. *Let $g_{\lambda,\mu} \in \mathcal{G}_b$ and $t_{\lambda,\mu} > 0$. Then, the Julia set $J(g_{\lambda,\mu}) = \widehat{\mathbb{C}}$.*

Proof. If $t_{\lambda,\mu} > 0$, by Theorem 3.2.2(c), $g^n_{\lambda,\mu}(x) \to \infty$ for all $x \in \mathbb{R}$ as $n \to \infty$. Thus, the forward orbits of all the singular values of $g_{\lambda,\mu}(z)$, tend to infinity under iteration if $t_{\lambda,\mu} > 0$. It remains to show that the Fatou set of $g_{\lambda,\mu}$ is empty if the orbits of all the singular values of $g_{\lambda,\mu}(z)$ tend to ∞ under iteration.

In view of Theorem 3.1.1, the Fatou set of $g_{\lambda,\mu}$ cannot contain Baker domains and wandering domains. So the possible choices of Fatou components are (i) basin of attraction, (ii) parabolic domain or (iii) Siegel disk.

The Fatou set of $g_{\lambda,\mu}(z)$ can not contain attracting or parabolic domain. If the Fatou set of $g_{\lambda,\mu}(z)$ contains either a basin of attraction or a parabolic domain U, then by Theorem 1.5.3, U must contain at least one singular value w. Consequently the forward orbit of the singular value w will tend to the attracting cycle or to the parabolic cycle. This is not possible, since all the singular values of $g_{\lambda,\mu}(z)$ tend to ∞ under iteration and $\infty \in J(g_{\lambda,\mu})$.

Finally, the Fatou set of $g_{\lambda,\mu}(z)$ can not contain a Siegel disk. If possible, U is a Siegel disk of the Fatou set of $g_{\lambda,\mu}(z)$. By Theorem 1.5.3, its boundary ∂U is contained in the forward orbits of the singular values of $g_{\lambda,\mu}(z)$. Since as in the case of critically finite entire functions, the forward orbits of the singular values of $g_{\lambda,\mu}(z)$ tend to ∞ under iteration of $g_{\lambda,\mu}$, it is impossible for the Fatou set of $g_{\lambda,\mu}(z)$ to contain a Siegel disk. This completes the proof. □

Theorems 3.3.1 and 3.3.3 and Remarks 3.3.2 and 3.3.4 show that the Fatou set of $g_{\lambda,\mu}$ is non-empty and unbounded for $t_{\lambda,\mu} \leq 0$. Therefore, the Julia set of $g_{\lambda,\mu}$ for $t_{\lambda,\mu} \leq 0$ is a nowhere dense subset of the extended complex plane. Theorem 3.3.4 shows that the Julia set of $g_{\lambda,\mu}$ for $t_{\lambda,\mu} > 0$ equals to the extended complex plane. Thus, the chaotic burst in the dynamics of $g_{\lambda,\mu}$ occurs when parameters λ and μ are such that $t_{\lambda,\mu}$ increases through the value 0. It is worth to note that if $\lambda > 1/(e \ln b)$ then $t_{\lambda,\mu} > 0$ holds for all $\mu > 0$ and hence in this case the Julia set of $g_{\lambda,\mu}$ equals to the extended complex plane.

3.4 Applications

From the results obtained in previous sections of this chapter, we can deduce the dynamics of one-parameter family of functions $\rho \cosh(z)$ where $\rho(> 0)$ is the parameter and the dynamics of the two-parameter family of functions $m \cosh(az) + n \sinh(az)$ where m, n are parameters with $m \geq n > 0$ and a is a positive constant.

(i) Dynamics of the one-parameter family of functions $\rho \cosh(z)$ where $\rho(> 0)$ is the parameter

From the dynamics of the functions $g_{\lambda,\mu}$ in $\mathcal{G}_b$, we get the dynamics of one-parameter $\rho \cosh(z)$ where $\rho(> 0)$ is the parameter, by taking $b = e$ and parameters λ, μ as $\lambda = \mu = \rho/2$. For these choices of λ, μ and b, we get $t_{\lambda,\mu}$ as $t_{\lambda,\mu} = \sqrt{1+\rho^2} - \ln\left(\frac{1+\sqrt{1+\rho^2}}{\rho}\right) = \sqrt{1+\rho^2} - \sinh^{-1}(1/\rho)$. Let ρ^* be the root of $\sqrt{1+\rho^2} - \sinh^{-1}(1/\rho) = 0$.

If $\rho < \rho^*$, by using Theorem 3.2.1(a), the function $\rho \cosh(z)$ has a real attracting fixed point a_ρ (say) and a real repelling fixed point r_ρ (say). For $\rho < \rho^*$, by Theorem 3.3.1, the Fatou set $F(\rho \cosh(z))$ is the basin of attraction corresponding to the real attracting fixed point a_ρ.

For $\rho = 1/3$, the basin of attraction of the Fatou set $F(\rho \cosh(z))$ is shown in Figure 3.7. In this figure, the points for which the forward orbits are attracted by the real attracting fixed point are colored in black color.

By Theorem 3.2.1(b), the function $\rho^* \cosh(z)$ has a unique real rationally indifferent fixed point $\ln\left(\frac{1+\sqrt{1+\rho^{*2}}}{\rho^*}\right)$. By Theorem 3.3.3, the Fatou set $F(\rho^* \cosh(z))$ is the parabolic

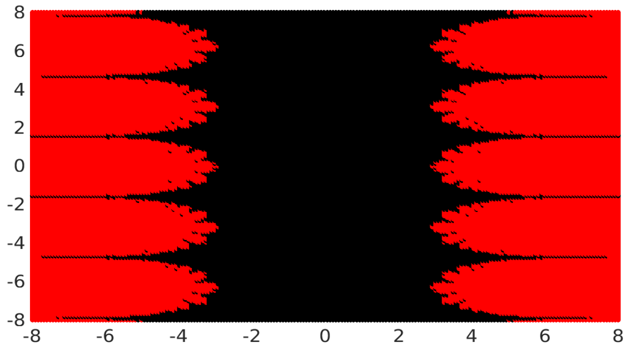

Figure 3.7: Basin of attraction in the Fatou set of $\rho\cosh(z)$ when $\rho < \rho^*$.

domain corresponding to the real rationally indifferent fixed point. For $\rho = \rho^*$, the parabolic domain is shown in Figure 3.6. In this figure, the black color points are attracted by the rationally indifferent fixed point $\ln\left(\frac{1+\sqrt{1+\rho^{*2}}}{\rho^*}\right)$.

If $\rho > \rho^*$, by Theorem 3.3.4, the Julia set $J(\rho\cosh(z))$ is the whole of the extended complex plane $\widehat{\mathbb{C}}$. Thus, a bifurcation occurs in the dynamics of one-parameter family of functions $\rho\cosh(z)$ at the parameter value $\rho = \rho^*$ and chaotic burst occurs if the parameter ρ crosses the value ρ^*.

(ii) Dynamics of the two-parameter family of functions $m\cosh(az) + n\sinh(az)$ where m, n are parameters with $m \geq n > 0$ and a is a positive constant

Notice that $g_{\lambda,\mu}(z) = \lambda b^z + \mu b^{-z} = (\lambda+\mu)\cosh(z\ln b) + (\lambda-\mu)\sinh(z\ln b)$ for all $z \in \mathbb{C}$. The dynamics of $m\cosh(az)+n\sinh(az)$ where m, n are positive parameters with $m \geq n$ and a is a positive constant can be deduced by our work for choices of parameters $\lambda = (m+n)/2$, $\mu = (m-n)/2$ and constant $b = e^a$. In this case

$$t_{\lambda,\mu} = \sqrt{1+(m^2-n^2)a^2} - \ln\left(\frac{1+\sqrt{1+(m^2-n^2)a^2}}{(m+n)a}\right).$$

By Theorem 3.2.1(a), the function $m\cosh(az) + n\sinh(az)$ has a real attracting fixed point $a_{m,n}$ (say) and a real repelling fixed point $r_{m,n}$ (say) if $\sqrt{1+(m^2-n^2)a^2} < \ln\left(\frac{1+\sqrt{1+(m^2-n^2)a^2}}{(m+n)a}\right)$. By Theorem 3.3.1, the Fatou set is the basin of attraction corresponding to the attracting fixed point $a_{m,n}$.

If $\sqrt{1+(m^2-n^2)a^2} = \ln\left(\frac{1+\sqrt{1+(m^2-n^2)a^2}}{(m+n)a}\right)$, by Theorem 3.2.1(b), the function

$m\cosh(az)+n\sinh(az)$ has real rationally indifferent fixed point $\ln\left(\frac{1+\sqrt{1+(m^2-n^2)a^2}}{(m+n)a}\right)/a$. By Theorem 3.3.3, the parabolic domain corresponding to the real rationally indifferent fixed point is the Fatou set.

If $\sqrt{1+(m^2-n^2)a^2} > \ln\left(\frac{1+\sqrt{1+(m^2-n^2)a^2}}{(m+n)a}\right)$, by Theorem 3.3.4, the Julia set of $m\cosh(az)+n\sinh(az)$ is the whole of the extended complex plane $\widehat{\mathbb{C}}$, leading to the chaotic burst. Thus, a bifurcation and chaotic burst occurs in the dynamics of two-parameter family of functions $m\cosh(az)+n\sinh(az)$ when the parameters m and n satisfy $\sqrt{1+(m^2-n^2)a^2} = \ln\left(\frac{1+\sqrt{1+(m^2-n^2)a^2}}{(m+n)a}\right)$.

CHAPTER 4

DYNAMICS OF TWO FAMILIES OF MEROMORPHIC FUNCTIONS INVOLVING HYPERBOLIC COSINE FUNCTION

The dynamics of one-parameter families $\mathcal{F}$ and $\mathcal{G}$ of transcendental meromorphic functions are studied in the present chapter where $\mathcal{F}$ and $\mathcal{G}$ are given by

$$\mathcal{F} \equiv \left\{ f_\lambda(z) = \lambda \left(\cosh z + \frac{1}{\cosh z} \right) \text{ for } z \in \mathbb{C} : \lambda > 0 \right\},$$

$$\mathcal{G} \equiv \left\{ g_\lambda(z) = \lambda \left(\cosh z - \frac{1}{\cosh z} \right) \text{ for } z \in \mathbb{C} : \lambda > 0 \right\}.$$

Here both the functions $f_\lambda(z)$ and $g_\lambda(z)$ are combinations of the functions $\cosh z$ and $1/\cosh z$. Both the functions f_λ and g_λ are in the class $\mathcal{S}$. The dynamics of $f_\lambda(z)$ and $g_\lambda(z)$ on the extended complex plane are investigated. The functions f_λ and g_λ exhibit a totally different dynamical behaviour. It is shown that a bifurcation and chaotic burst occur at a certain parameter value of λ for the functions f_λ in the family $\mathcal{F}$. The origin is always an attracting fixed point of g_λ and the basin of attraction corresponding to the origin, is the Fatou set of g_λ. Therefore, there is no bifurcation in the dynamics of g_λ. Also, some comparison between the dynamics of f_λ and g_λ are described.

4.1 Dynamics of functions in $\mathcal{F}$

In this section, the dynamics of $f_\lambda \in \mathcal{F}$ is studied. Some basic properties of the function f_λ is obtained. The dynamics of f_λ on the real line $\mathbb{R}$, is described with the help of the nature of the real fixed points. The dynamics of f_λ on the complex plane is established by tracking the forward orbits of the singular values of f_λ.

4.1.1 Basic Properties of Functions in $\mathcal{F}$

The function $f_\lambda(z)$ is an even function and it is periodic of minimal period $2\pi i$. On the real line $\mathbb{R}$, the function $f_\lambda(x)$ is positive and $f_\lambda(x) \to \infty$ as $|x| \to \infty$. It is easy to see that $f_\lambda(x)$ attains its minimum value 2λ at $x = 0$.

The singular values of f_λ are given in the following proposition.

Proposition 4.1.1. *Let $f_\lambda \in \mathcal{F}$. Then, f_λ has only two critical values $\pm 2\lambda$ and no finite asymptotic value.*

Proof. Note that $f'_\lambda(z) = \lambda \frac{\sinh^3 z}{\cosh^2 z}$ for all $z \in \mathbb{C}$. So, $f'_\lambda(z) = 0$ if and only if $z = n\pi i$ where $n \in \mathbb{Z}$. Therefore, the critical values of f_λ are $\pm 2\lambda$. Now, we show that f_λ has no finite asymptotic value. If possible, let w^* be a finite asymptotic value of f_λ. Then, there exists a continuous curve $\gamma(t)$, $t > 0$ such that $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f_\lambda(\gamma(t)) = w^*$. Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ where $\gamma_1(t)$ and $\gamma_2(t)$ are real functions of t. Then, we can write

$$\begin{aligned} f_\lambda(\gamma(t)) &= \lambda \cosh(\gamma_1(t)) \cos(\gamma_2(t)) \left(\frac{\cosh^2(\gamma_1(t)) + \cos^2(\gamma_2(t))}{\cosh^2(\gamma_1(t)) - \sin^2(\gamma_2(t))} \right) \\ &\quad + i\lambda \sinh(\gamma_1(t)) \sin(\gamma_2(t)) \left(\frac{\sinh^2(\gamma_1(t)) - \sin^2(\gamma_2(t))}{\cosh^2(\gamma_1(t)) - \sin^2(\gamma_2(t))} \right). \end{aligned}$$

Now $\lim_{t\to\infty} f_\lambda(\gamma(t)) = w^*$ implies $\lim_{t\to\infty} |f_\lambda(\gamma(t))|^2 = |w^*|^2$. Thus,

$$\begin{aligned} |w^*|^2 &= \lim_{t\to\infty} \Bigg[\cosh^2(\gamma_1(t)) - \sin^2(\gamma_2(t)) + \frac{1}{\cosh^2(\gamma_1(t)) - \sin^2(\gamma_2(t))} \\ &+ 2\frac{\cosh^2(\gamma_1(t))\cos^2(\gamma_2(t)) - \sinh^2(\gamma_1(t))\sin^2(\gamma_2(t))}{\cosh^2(\gamma_1(t))\cos^2(\gamma_2(t)) + \sinh^2(\gamma_1(t))\sin^2(\gamma_2(t))} \Bigg]. \end{aligned}$$

This implies that $\gamma_1(t)$ is bounded and $|\gamma_2(t)| \to \infty$ as $t \to \infty$.

Let us assume that $\gamma_2(t) \to \infty$ as $t \to \infty$. On the curve $\gamma(t)$, we are choosing two sequences $\{z_n\}$ and $\{z'_n\}$ such that $z_n = x_n + 2n\pi i$ and $z'_n = x'_n + (2n+1)\pi i$ for $n \in \mathbb{N}$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Note that both the sequences tend to ∞ along the curve $\gamma(t)$. It is easy to see that $g_\lambda(z_n) = \lambda \cosh(x_n) \frac{\cosh^2(x_n)+1}{\cosh^2(x_n)} \geq \lambda$ and $g_\lambda(z'_n) = -\lambda \cosh(x'_n) \frac{\cosh^2(x'_n)+1}{\cosh^2(x'_n)} \leq -\lambda$. So, w^* to be an asymptotic value, $\lim\limits_{t\to\infty} f_\lambda(\gamma(t))$ must exist which is not possible for the above choices of $\gamma(t)$. So, f_λ has no finite asymptotic value.

Again if $\gamma_2(t) \to -\infty$ as $t \to \infty$, by similar arguments it can be proved that f_λ has no finite asymptotic value. This completes the proof. □

By Proposition 4.1.1, $f_\lambda \in \mathcal{S}$. Now by using Theorem 1.5.1 and Remark 1.5.1, the following two results can be concluded.

Theorem 4.1.1. *Let $f_\lambda \in \mathcal{F}$. Then, the Fatou set of f_λ does not contain wandering domains.*

Theorem 4.1.2. *Let $f_\lambda \in \mathcal{F}$. Then, the Fatou set of f_λ has no Baker domains.*

4.1.2 Real Dynamics of Functions from $\mathcal{F}$

In this subsection, the dynamics of the functions $f_\lambda \in \mathcal{F}$ is investigated on the real line $\mathbb{R}$ with the help of the real fixed points.

Define $h_\lambda(x) = f_\lambda(x) - x$ where $f_\lambda(x) = \lambda\left(\cosh x + \frac{1}{\cosh x}\right)$ for $x \in \mathbb{R}$ and $\lambda(>0)$ be a real parameter. So, the zeros of $h_\lambda(x)$ are the fixed points of $f_\lambda(x)$. Clearly, $h_\lambda(x) \to \infty$ as $|x| \to \infty$. Observe that $h'_\lambda(x) = \lambda \frac{\sinh^3 x}{\cosh^2 x} - 1$ for all $x \in \mathbb{R}$. It is easy to see that $h'_\lambda(x) \leq -1$ for $x \leq 0$ and $h'_\lambda(x) \to \infty$ as $x \to \infty$. Since $h''_\lambda(x) = \lambda \frac{\sinh^2 x}{\cosh^3 x}\left(3 + \sinh^2 x\right) > 0$ for $x > 0$, it follows that $h'_\lambda(x)$ has a unique zero x^*_λ (say). This x^*_λ is given by

$$\sinh(x^*_\lambda) = \frac{1}{3\lambda} + \frac{1}{(2\lambda)^{\frac{1}{3}}}\left(\left[1 + \frac{1}{27\lambda^2} + \sqrt{1 + \frac{4}{27\lambda^2}}\right]^{\frac{1}{3}} + \left[1 + \frac{1}{27\lambda^2} - \sqrt{1 + \frac{4}{27\lambda^2}}\right]^{\frac{1}{3}}\right). \tag{4.1}$$

Note that $x^*_\lambda > 0$ and

$$h'_\lambda(x) \begin{cases} < 0 & \text{for } x < x^*_\lambda, \\ = 0 & \text{for } x = x^*_\lambda, \\ > 0 & \text{for } x > x^*_\lambda. \end{cases} \tag{4.2}$$

Thus the function $h_\lambda(x)$ has a unique minimum at x^*_λ and the minimum value is $h_\lambda(x^*_\lambda) = \lambda\left(\cosh x^*_\lambda + \frac{1}{\cosh x^*_\lambda}\right) - x^*_\lambda$. Throughout this chapter, we denote

$$t_\lambda := \lambda\left(\cosh x^*_\lambda + \frac{1}{\cosh x^*_\lambda}\right) - x^*_\lambda$$

where x^*_λ is given by (4.1).

The existence and nature of the real fixed points of the function f_λ are discussed in the following theorem.

Theorem 4.1.3. *Let $f_\lambda(x) = \lambda\left(\cosh x + \frac{1}{\cosh x}\right)$ for $x \in \mathbb{R}$ where $\lambda(>0)$ is a real parameter.*

(a) If $t_\lambda < 0$, then $f_\lambda(x)$ has a repelling fixed point r_λ (say) and an attracting fixed point a_λ (say) with $0 < a_\lambda < x^*_\lambda < r_\lambda$.

(b) If $t_\lambda = 0$, then $f_\lambda(x)$ has a unique rationally indifferent fixed point x^*_λ.

(c) If $t_\lambda > 0$, then $f_\lambda(x)$ has no fixed points.

Proof. Set $h_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$.

(a) **Case:** $t_\lambda < 0$

Notice that the minimum value $h_\lambda(x^*_\lambda) < 0$ if $t_\lambda < 0$. Therefore, it follows that $h_\lambda(x)$ has exactly two zeros. Let a_λ and r_λ be the zeros of $h_\lambda(x)$ with $0 < a_\lambda < x^*_\lambda < r_\lambda$. So,

$$h_\lambda(x)\begin{cases} >0 & \text{for } x \in (-\infty, a_\lambda) \cup (r_\lambda, \infty), \\ =0 & \text{for } x = a_\lambda \text{ or } r_\lambda, \\ <0 & \text{for } x \in (a_\lambda, r_\lambda). \end{cases} \tag{4.3}$$

Thus, a_λ and r_λ are the fixed points of $f_\lambda(x)$. Now by (4.2), $h'_\lambda(r_\lambda) > 0$ and consequently $f'_\lambda(r_\lambda) > 1$. So, r_λ is the repelling fixed point of $f_\lambda(x)$. Since $h'_\lambda(x)$ is increasing and $0 < a_\lambda < x^*_\lambda$, it follows that $-1 = h'_\lambda(0) < h'_\lambda(a_\lambda) < h'_\lambda(x^*_\lambda) = 0$. This gives, $0 < f'_\lambda(a_\lambda) < 1$. Hence a_λ is the attracting fixed point of $f_\lambda(x)$. This completes the proof of (a).

For $\lambda = 1/2$, the fixed points of $f_\lambda(x)$ are shown in Figure 4.1. In Figure 4.1, a_λ is the attracting fixed point and r_λ is the repelling fixed point.

(b) **Case:** $t_\lambda = 0$

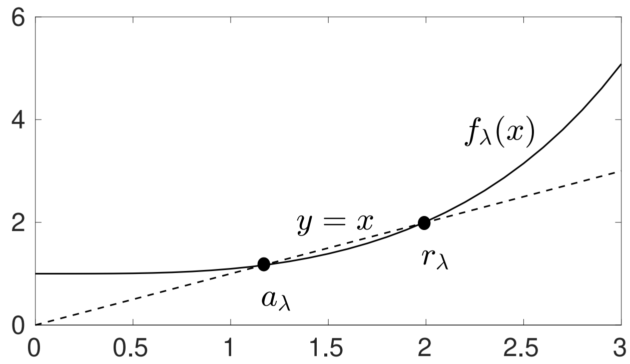

Figure 4.1: Two fixed points of f_λ for $t_\lambda < 0$.

If $t_\lambda = 0$, then $h_\lambda(x^*_\lambda) = 0$ and $h'_\lambda(x^*_\lambda) = 0$ consequently $f_\lambda(x^*_\lambda) = x^*_\lambda$ and $f'_\lambda(x^*_\lambda) = 1$. Thus, x^*_λ is a rationally indifferent fixed point of $f_\lambda(x)$. Now for $x \neq x^*_\lambda$, $h_\lambda(x) > h_\lambda(x^*_\lambda) = 0$. So, $h_\lambda(x)$ has no zero other than x^*_λ. Hence $f_\lambda(x)$ has only one fixed point x^*_λ which is rationally indifferent.

In Figure 4.2, x^*_λ is the rationally indifferent fixed point of $f_\lambda(x)$ for $\lambda = 0.54$.

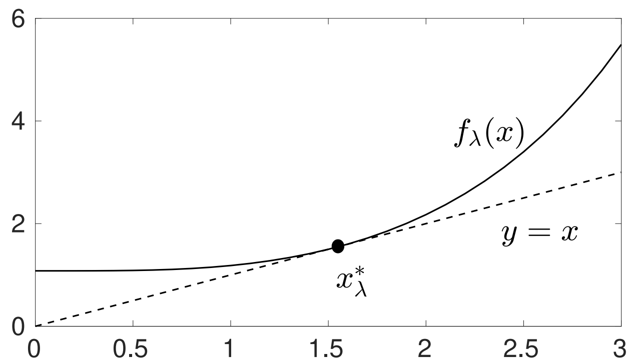

Figure 4.2: One rationally indifferent fixed point of f_λ for $t_\lambda = 0$.

(c) **Case:** $t_\lambda > 0$

If $t_\lambda > 0$, then $h_\lambda(x) \geq h_\lambda(x^*_\lambda) > 0$. Thus, $f_\lambda(x)$ has no fixed points for $t_\lambda > 0$. This completes the proof of the theorem.

Figure 4.3 shows that $f_\lambda(x)$ has no fixed points for $\lambda = 1$. $\square$

In the following theorem, the dynamics of $f_\lambda \in \mathcal{F}$ on the real line $\mathbb{R}$ is established.

Theorem 4.1.4. *Let* $f_\lambda(x) = \lambda\left(\cosh x + \frac{1}{\cosh x}\right)$ *for* $x \in \mathbb{R}$ *where* $\lambda(> 0)$ *is a real parameter.*

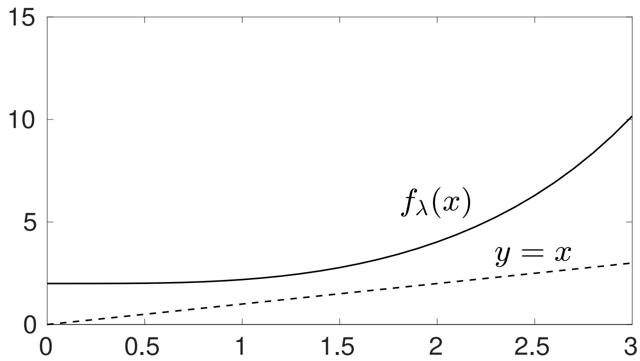

Figure 4.3: No fixed points of f_λ for $t_\lambda > 0$.

(a) If $t_\lambda < 0$, then $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $|x| < r_\lambda$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $|x| > r_\lambda$.

(b) If $t_\lambda = 0$, then $f_\lambda^n(x) \to x_\lambda^*$ as $n \to \infty$ for $|x| \leq x_\lambda^*$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $|x| > x_\lambda^*$.

(c) If $t_\lambda > 0$, then $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x \in \mathbb{R}$.

Proof. (a) **Case:** $t_\lambda < 0$

If $t_\lambda < 0$, by Theorem 4.1.3(a), $f_\lambda(x)$ has an attracting fixed point a_λ and a repelling fixed point r_λ with $0 < a_\lambda < x_\lambda^* < r_\lambda$. If $a_\lambda < x < r_\lambda$, by (4.3), we have

$$f_\lambda(x) - a_\lambda < x - a_\lambda. \tag{4.4}$$

If $0 \leq x < a_\lambda$, by mean value theorem, $|f_\lambda(x) - a_\lambda| = |f_\lambda'(c)||x - a_\lambda|$, where $0 \leq x < c < a_\lambda$. Since $f_\lambda'(x)$ is strictly increasing for $x > 0$, $f_\lambda'(0) = 0$ and $f_\lambda'(a_\lambda) < 1$, it follows that $|f_\lambda'(c)| < 1$. So, $|f_\lambda(x) - a_\lambda| < |x - a_\lambda|$ for $0 \leq x < a_\lambda$. This inequality together with (4.4), gives that $|f_\lambda(x) - a_\lambda| < |x - a_\lambda|$ for $0 \leq x < r_\lambda$ and $x \neq a_\lambda$. Thus, $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $0 \leq x < r_\lambda$. Since $f_\lambda(-x) = f_\lambda(x)$, it follows that $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $-r_\lambda < x \leq 0$. Notice that $f_\lambda(x) > x$ and $f_\lambda'(x) > 1$ if $x > r_\lambda$. Therefore, the sequence $\{f_\lambda^n(x)\}$ is monotonically increasing and not bounded above for $x > r_\lambda$. Hence $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x > r_\lambda$. Again in view of $f_\lambda(-x) = f_\lambda(x)$, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x < -r_\lambda$. This completes the proof of (a).

(b) **Case:** $t_\lambda = 0$

If $t_\lambda = 0$, by Theorem 4.1.3(b), $f_\lambda(x)$ has a unique rationally indifferent fixed point x_λ^*. If $0 \leq x < x_\lambda^*$, by mean value theorem, $|f_\lambda(x) - x_\lambda^*| = |f_\lambda'(c)||x - x_\lambda^*|$, where $0 \leq x < c < x_\lambda^*$. Since $f_\lambda'(0) = 0$, $f_\lambda'(x_\lambda^*) = 1$ and $f_\lambda'(x)$ is strictly increasing for $x > 0$, it follows that $|f_\lambda'(c)| < 1$. So, $|f_\lambda(x) - x_\lambda^*| < |x - x_\lambda^*|$ for $0 \leq x < x_\lambda^*$. Therefore, $f_\lambda^n(x) \to x_\lambda^*$ as $n \to \infty$ for $0 \leq x \leq x_\lambda^*$. If $x > x_\lambda^*$, then $f_\lambda(x) > x$ and $f_\lambda'(x) > 1$. Thus, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x > x_\lambda^*$. Since $f_\lambda(x)$ is an even function, it follows that $f_\lambda^n(x) \to x_\lambda^*$ as $n \to \infty$ for $-x_\lambda^* \leq x \leq 0$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x < -x_\lambda^*$.

(c) **Case:** $t_\lambda > 0$

If $t_\lambda > 0$, by Theorem 4.1.3(c), $f_\lambda(x)$ has no fixed points. Observe that $f_\lambda(x) \geq 2\lambda$ for all $x \in \mathbb{R}$ and $f_\lambda(x) - x > 0$ for all $x \in \mathbb{R}$. Since $f_\lambda(x)$ is strictly increasing for $x > 0$, the sequence $\{f_\lambda^n(x)\}$ is monotonically increasing and not bounded above. Thus, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x \in \mathbb{R}$. This completes the proof. □

Remark 4.1.1. *Theorem 4.1.4 shows that a bifurcation occurs in the dynamics of* $f_\lambda(x)$, $x \in \mathbb{R}$, *in the family* $\mathcal{F}$. *If* $t_\lambda < 0$ *(*t_λ*=0), by Theorem 4.1.4, it follows that under iteration of* f_λ, *the orbits of all the points in* $[-r_\lambda, r_\lambda]$ *(in* $[-x_\lambda^*, x_\lambda^*]$*), remain bounded and orbits of all the points in* $\mathbb{R} \setminus [-r_\lambda, r_\lambda]$ *(in* $\mathbb{R} \setminus [-x_\lambda^*, x_\lambda^*]$*), become unbounded. But when* $t_\lambda > 0$, *by Theorem 4.1.4, there are no real points whose orbits are bounded under iteration of* f_λ. *Thus, a saddle-node bifurcation occurs in the real dynamics of* f_λ *when* $t_\lambda = 0$.

4.1.3 Chaotic Burst in the Dynamics of f_λ

In the present subsection, the chaotic burst in the dynamics of functions in the family $\mathcal{F}$, is discussed by describing the dynamics of $f_\lambda \in \mathcal{F}$ in the complex plane $\mathbb{C}$.

Dynamics of f_λ for $t_\lambda \leq 0$

If $t_\lambda < 0$, by Theorem 4.1.3(a), f_λ has a real attracting fixed point a_λ and a real repelling fixed point r_λ. The basin of attraction $A(a_\lambda)$ of the real attracting fixed point a_λ is defined as

$$A(a_\lambda) := \{z \in \mathbb{C} : f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}.$$

Remark 4.1.2. *When $t_\lambda < 0$,*

- *By Theorem 4.1.4(a), $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in (-r_\lambda, r_\lambda)$. Therefore, the basin of attraction $A(a_\lambda)$ contains the interval $(-r_\lambda, r_\lambda)$ on the real axis.*
- *Observe that $f_\lambda(2n\pi i) = 2\lambda \in (-r_\lambda, r_\lambda)$ for all $n \in \mathbb{Z}$. Thus, the basin of attraction $A(a_\lambda)$ contains the set $\{2n\pi i : n \in \mathbb{Z}\}$. Hence the basin of attraction $A(a_\lambda)$ is unbounded.*

The following proposition describes the fate of the forward orbits of all the singular values of $f_\lambda(z)$ for $t_\lambda < 0$.

Proposition 4.1.2. *Let $A(a_\lambda)$ be the basin of attraction of the real attracting fixed point a_λ of $f_\lambda(z)$ for $t_\lambda < 0$. Then, $A(a_\lambda)$ contains all the singular values and their forward orbits.*

Proof. Observe that $0 < 2\lambda < a_\lambda < r_\lambda$ for $t_\lambda < 0$. By Theorem 4.1.4(a), $f_\lambda^n(\pm 2\lambda) \to a_\lambda$ as $n \to \infty$. Thus, the basin of attraction $A(a_\lambda)$ contains all the singular values and their forward orbits. □

In the following theorem the Fatou set of f_λ is described for $t_\lambda < 0$.

Theorem 4.1.5. *Let $A(a_\lambda)$ be the basin of attraction of the real attracting fixed point a_λ of $f_\lambda(z)$ for $t_\lambda < 0$. Then, the Fatou set of f_λ is equal to the basin of attraction $A(a_\lambda)$.*

Proof. We claim that $A(a_\lambda)$ is the only basin of attraction of the Fatou set of f_λ. If possible, let $A(z_\lambda)$ be a basin of attraction of the attracting periodic point $z_\lambda \neq a_\lambda$. Clearly, $A(a_\lambda) \cap A(z_\lambda)$ is an empty set. Then by Theorem 1.5.3, basin of attraction $A(z_\lambda)$ contains at least one singular value. But by Proposition 4.1.2, all the singular values are in $A(a_\lambda)$. Thus, basin of attraction of the Fatou set of f_λ is $A(a_\lambda)$ only.

The Fatou set of f_λ cannot contain a parabolic domain. If U is a parabolic domain, contained in the Fatou set of f_λ, then by Theorem 1.5.3, U must contain at least one singular value which contradicts that all the singular values and their forward orbits are in $A(a_\lambda)$.

The Fatou set of f_λ cannot contain Siegel disks and Herman rings. If U is a Siegel disk or a Herman ring, then the boundary of U is contained in the closure of the forward orbits of the singular values of f_λ (Theorem 1.5.3) but all the singular values and their forward orbits are in $A(a_\lambda)$.

By Theorems 4.1.1 and 4.1.2, the Fatou set of f_λ does not contain wandering domains and Baker domains.

Thus, $A(a_\lambda)$ is the only possible stable domain. Therefore, the Fatou set of f_λ is the basin of attraction $A(a_\lambda)$. □

The following figure gives an idea on the basin of attraction of f_λ when $\lambda = 1/2$. In Figure 4.4, the points for which the forward orbits are attracted by the attracting cycle are in black color.

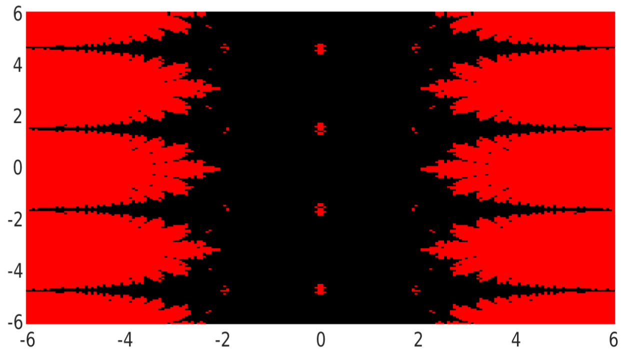

Figure 4.4: Basin of attraction in the Fatou set of f_λ for $t_\lambda < 0$.

If $t_\lambda = 0$, by Theorem 4.1.3(b), f_λ has a real rationally indifferent fixed point x^*_λ. Then, the parabolic domain $P \equiv P(x^*_\lambda)$ corresponding to the rationally indifferent fixed point x^*_λ is given by

$$P \equiv P(x^*_\lambda) := \{z \in \mathbb{C} : f^n_\lambda(z) \to x^*_\lambda \text{ as } n \to \infty\}.$$

Remark 4.1.3. *When* $t_\lambda = 0$,

- *By Theorem 4.1.4(b),* $f^n_\lambda(x) \to x^*_\lambda$ *as* $n \to \infty$ *for* $x \in [-x^*_\lambda, x^*_\lambda]$. *Hence* P *contains the interval* $[-x^*_\lambda, x^*_\lambda]$ *on the real axis.*

- *Note that $f_\lambda(2n\pi i) = 2\lambda \in (-x^*_\lambda, x^*_\lambda)$ for all $n \in \mathbb{Z}$. Thus, the parabolic domain P contains the set $\{2n\pi i : n \in \mathbb{Z}\}$. Hence the parabolic domain P is unbounded.*

The dynamics of $f_\lambda \in \mathcal{F}$ for $t_\lambda = 0$ is similar to that of the dynamics of f_λ for $t_\lambda < 0$, except that instead of a basin of attraction, a parabolic domain is its Fatou set.

Theorem 4.1.6. *Let P be the parabolic domain of f_λ corresponding to the real rationally indifferent fixed point x^*_λ for $t_\lambda = 0$. Then P contains all the singular values and their forward orbits and P is equal to the Fatou set of f_λ.*

Figure 4.5 gives an idea on the parabolic domain of f_λ when $\lambda = 0.54$. In this figure, the points for which the forward orbits are attracted by the parabolic cycle are in black color.

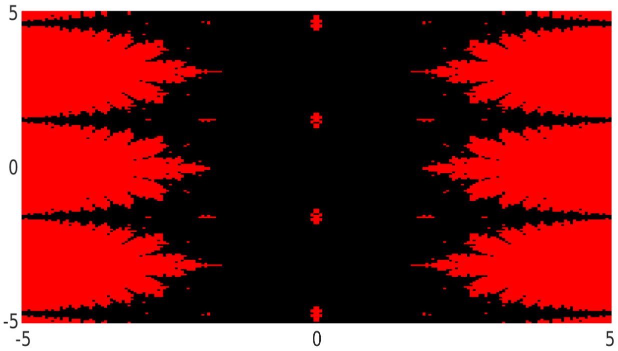

Figure 4.5: Parabolic Domain in the Fatou set of f_λ for $t_\lambda = 0$.

Dynamics of f_λ for $t_\lambda > 0$

The dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > 0$ is studied here. It is shown that the Julia set of f_λ is equal to the extended complex plane.

Theorem 4.1.7. *Let $f_\lambda \in \mathcal{F}$ and $t_\lambda > 0$. Then, the Julia set $J(f_\lambda) = \widehat{\mathbb{C}}$.*

Proof. If $t_\lambda > 0$, by Theorem 4.1.4(c), the forward orbits of all the singular values of f_λ tend to infinity under iteration of f_λ. It remains to show that the Fatou set of f_λ is empty if the forward orbits of all the singular values of f_λ tend to infinity under iteration.

In view of Theorems 4.1.1 and 4.1.2, the Fatou set of f_λ cannot contain wandering domains and Baker domains. So the possible choices of the Fatou components are (i) basin of attraction, (ii) parabolic domain, (iii) Siegel disk or (iv) Herman ring.

The Fatou set of f_λ has no basin of attraction and parabolic domain. If the Fatou set of f_λ contains either a basin of attraction or a parabolic domain U then by Theorem 1.5.3, U contains at least one singular value w. Then, the forward orbit of the singular value w will tend to the attracting cycle or to the parabolic cycle. But all the singular values of f_λ tend to ∞ under iteration of f_λ. So, the Fatou set of f_λ cannot contain basin of attraction or parabolic domains.

The Fatou set of f_λ cannot contain Siegel disks or Herman rings. If U is a Siegel disk or a Herman ring, then the boundary of U is contained in the closure of the forward orbits of the singular values of $f_\lambda(z)$ but this is not true, since closure of the forward orbits of the singular values of $f_\lambda(z)$ is a countable subset of $\mathbb{R}$ having all of its points are isolated. So, the Fatou set of f_λ is empty and hence the Julia set of f_λ is $\widehat{\mathbb{C}}$. □

Remark 4.1.4. *Theorems 4.1.5 and 4.1.6 and Remarks 4.1.2 and 4.1.3 show that the Fatou set of f_λ is non-empty and unbounded for $t_\lambda \leq 0$. So, the Julia set of f_λ is nowhere dense subset of the extended complex plane for $t_\lambda \leq 0$. Theorem 4.1.7 shows that the Julia set of f_λ is equal to the extended complex plane for $t_\lambda > 0$. Thus, the chaotic burst in the dynamics of f_λ occurs when t_λ increases through the value 0.*

4.2 Dynamics of functions in $\mathcal{G}$

In this section, the dynamics of the function $g_\lambda \in \mathcal{G}$ is investigated. At first the dynamics on the real line $\mathbb{R}$ is explored and then the dynamics on the complex plane is studied.

4.2.1 Basic Properties of Functions in $\mathcal{G}$

Observe that $g_\lambda(z)$ is an even function and it is periodic of minimal period $2\pi i$.

In the following proposition, it is shown that $g_\lambda(z)$ has finite number of singular values i.e., $g_\lambda \in \mathcal{S}$.

Proposition 4.2.1. *Let $g_\lambda \in \mathcal{G}$. Then, g_λ has only three singular values 0, $\pm 2\lambda i$.*

Proof. Observe that $g'_\lambda(z) = \lambda\left(1 + \frac{1}{\cosh^2 z}\right)\sinh z$ for $z \in \mathbb{C}$. Note that $g'_\lambda(z) = 0$ if and only if $\sinh z = 0$ or $\cosh z = \pm i$. Thus, the critical values of g_λ are 0, $\pm 2\lambda i$.

Now we claim that either g_λ has no finite asymptotic value or only possible finite asymptotic value of g_λ is 0. Let w be a finite asymptotic value of $g_\lambda(z)$. Then, there exists a continuous curve $\gamma(t)$, $t > 0$ such that $\lim_{t\to\infty}\gamma(t) = \infty$ and $\lim_{t\to\infty} g_\lambda(\gamma(t)) = w$. Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ where $\gamma_1(t)$ and $\gamma_2(t)$ are real functions of t. Then, $g_\lambda(\gamma(t))$ can be written as

$$\begin{aligned} g_\lambda(\gamma(t)) = \lambda\cosh(\gamma_1(t))\cos(\gamma_2(t))\left(\frac{\sinh^2(\gamma_1(t)) - \sin^2(\gamma_2(t))}{\cosh^2(\gamma_1(t)) - \sin^2(\gamma_2(t))}\right) \\ +i\lambda\sinh(\gamma_1(t))\sin(\gamma_2(t))\left(\frac{\cosh^2(\gamma_1(t)) + \cos^2(\gamma_2(t))}{\cosh^2(\gamma_1(t)) - \sin^2(\gamma_2(t))}\right). \end{aligned}$$

Now $\lim_{t\to\infty}|g_\lambda(\gamma(t))| = |w|$, since w is asymptotic value. This gives

$$\lim_{t\to\infty}\frac{\cosh(\gamma_1(t)) - \frac{\cos^2(\gamma_2(t))}{\cosh(\gamma_1(t))}}{\sqrt{1 - \frac{\sin^2(\gamma_2(t))}{\cosh^2(\gamma_1(t))}}} = |w|.$$

So, w to be an asymptotic value $\gamma_1(t)$ should be bounded as $t \to \infty$ and $\gamma_2(t)$ is unbounded as $t \to \infty$. Without loss of generality, let us assume that $\gamma_2(t) \to \infty$ as $t \to \infty$. On the curve $\gamma(t)$, we are choosing two sequences $\{z_n\}$ and $\{z'_n\}$ such that $z_n = x_n + 2n\pi i$ and $z'_n = x'_n + (2n+1)\pi i$ for $n \in \mathbb{N}$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Clearly, along the curve $\gamma(t)$, both the sequences tend to ∞. Note that $g_\lambda(z_n) = \lambda\frac{\sinh^2(x_n)}{\cosh(x_n)} \geq 0$ and $g_\lambda(z'_n) = -\lambda\frac{\sinh^2(x'_n)}{\cosh(x'_n)} \leq 0$. So, either asymptotic value w does not exist or if such w exists then w must be zero. Thus, the singular values of $g_\lambda(z)$ are 0, $\pm 2\lambda i$. □

4.2.2 Real Dynamics of Functions from $\mathcal{G}$

On the real line $\mathbb{R}$, the function $g_\lambda(x) = \lambda\left(\cosh x - \frac{1}{\cosh x}\right)$ is positive for $x \neq 0$ and $g_\lambda(0) = 0$. Thus, $g_\lambda(x)$ has a unique minimum value 0 at $x = 0$. It is easy to see that $g_\lambda(x) \to \infty$ as $|x| \to \infty$.

The existence and nature of the real fixed points of $g_\lambda(x)$ are investigated in the following theorem:

Theorem 4.2.1. *Let $g_\lambda \in \mathcal{G}$. Then, $g_\lambda(x)$ has a superattracting fixed point 0 and a repelling fixed point r_λ (say).*

Proof. Define $h_\lambda(x) = g_\lambda(x) - x$ for $x \in \mathbb{R}$ where λ is a positive real parameter. Clearly, $h_\lambda(x) \to \infty$ as $|x| \to \infty$. Observe that $h'_\lambda(x) = \lambda\left(1 + \frac{1}{\cosh^2 x}\right)\sinh x - 1$ for $x \in \mathbb{R}$. So, $h'_\lambda(x) \to \pm\infty$ as $x \pm \infty$. Since $h'_\lambda(x) \leq -1$ for $x \leq 0$ and $h''_\lambda(x) = \lambda\left(\frac{\sinh^2 x}{\cosh x} + \frac{2}{\cosh^3 x}\right) > 0$ for all $x \in \mathbb{R}$, it follows that $h'_\lambda(x)$ has a unique zero x^*_λ (say) in $(0, \infty)$. Hence $h_\lambda(x)$ has global minimum at $x = x^*_\lambda$. Now $h_\lambda(x^*_\lambda) < h_\lambda(0) = 0$ implies $h_\lambda(x)$ has two zeros 0 and r_λ (say) with $0 < x^*_\lambda < r_\lambda$. So,

$$h_\lambda(x) \begin{cases} > 0 & \text{for } x \in (-\infty, 0) \cup (r_\lambda, \infty), \\ = 0 & \text{for } x = 0 \text{ or } r_\lambda, \\ < 0 & \text{for } x \in (0, r_\lambda). \end{cases} \tag{4.5}$$

Thus, the fixed points of $g_\lambda(x)$ are 0 and r_λ. Now $g'_\lambda(0) = 0$ implies 0 is a superattracting fixed point of $g_\lambda(x)$. Since $h'_\lambda(x)$ is strictly increasing, it follows that $h'_\lambda(r_\lambda) > h'_\lambda(x^*_\lambda) = 0$. So, $g'_\lambda(r_\lambda) > 1$ and consequently r_λ is a repelling fixed point of $g_\lambda(x)$. This completes the proof.

For $\lambda = 1$, the fixed points of $g_\lambda(x)$ are shown in Figure 4.6. In Figure 4.6, the origin is the superattracting fixed point and r_λ is the repelling fixed point. □

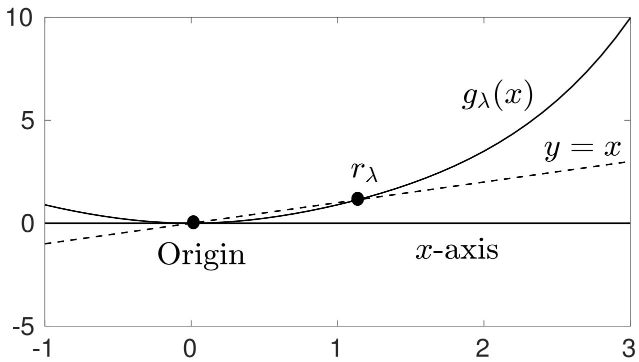

Figure 4.6: Two fixed points of $g_\lambda(x)$.

The dynamics of $g_\lambda \in \mathcal{G}$ on the real line $\mathbb{R}$ is investigated in the following theorem:

Theorem 4.2.2. *Let $g_\lambda(x) = \lambda\left(\cosh x - \frac{1}{\cosh x}\right)$ for $x \in \mathbb{R}$, where $\lambda (> 0)$ is a real parameter. Then $g^n_\lambda(x) \to 0$ as $n \to \infty$ for $|x| < r_\lambda$ and $g^n_\lambda(x) \to \infty$ as $n \to \infty$ for $|x| > r_\lambda$.*

Proof. By Theorem 4.2.1, $g_\lambda(x)$ has a superattracting fixed point 0 and a repelling fixed point r_λ with $0 < x^*_\lambda < r_\lambda$. By (4.5), for $0 < x < r_\lambda$, we have $0 \leq g_\lambda(x) < x$. Thus, $g^n_\lambda(x) \to 0$ as $n \to \infty$ for $0 \leq x < r_\lambda$. Since $g_\lambda(-x) = g_\lambda(x)$, it follows that $g^n_\lambda(x) \to 0$ as $n \to \infty$ for $-r_\lambda < x \leq 0$.

If $x > r_\lambda$, then $g_\lambda(x) > x$ and $g'_\lambda(x) > 1$. Therefore, the sequence $\{g^n_\lambda(x)\}$ is monotonically increasing and not bounded above and hence $g^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x > r_\lambda$. Since $g_\lambda(x)$ is an even function, it follows that $g^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x < -r_\lambda$. This completes the proof. □

4.2.3 Complex Dynamics of g_λ

In this subsection, the dynamics of $g_\lambda \in \mathcal{G}$ on the complex plane is investigated.

By Theorem 4.2.1, $g_\lambda(z)$ has a superattracting fixed point 0 and a real repelling fixed point r_λ. The basin of attraction $A(0)$ of the attracting fixed point 0 is defined as

$$A(0) := \{z \in \mathbb{C} : \ g^n_\lambda(z) \to 0 \text{ as } n \to \infty\}.$$

Remark 4.2.1. *By Theorem 4.2.2, the basin of attraction $A(0)$ contains the interval $(-r_\lambda, r_\lambda)$ on the real line $\mathbb{R}$.*

The following proposition describes the fate of the forward orbits of the singular values of $f_\lambda(z)$.

Proposition 4.2.2. *Let $g_\lambda \in \mathcal{G}$. Then, the basin of attraction $A(0)$ either contains all the singular values and their forward orbits or contains no singular values other than* 0.

Proof. By Proposition 4.2.1, the singular values of $g_\lambda(z)$ are 0, $\pm 2\lambda i$. The basin of attraction $A(0)$ always contains the singular value 0, since $g_\lambda(0) = 0$. Observe that $g_\lambda(\pm 2\lambda i) = -\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda} \in \mathbb{R}$. If $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| < r_\lambda$, then by Theorem 4.2.2, $g^n_\lambda(-\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}) \to 0$ as $n \to \infty$. Thus, $A(0)$ contains all the singular values and their forward orbits for $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| < r_\lambda$.

If $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| = r_\lambda$, then $g^n_\lambda(\pm 2\lambda i) = r_\lambda$ for all $n \in \mathbb{N}$ with $n \geq 2$. In this case 0 is the only singular value contained in $A(0)$.

Again, if $|\lambda\frac{\sin^2 2\lambda}{\cos 2\lambda}|> r_\lambda$, by Theorem 4.2.2, $g_\lambda^n(-\lambda\frac{\sin^2 2\lambda}{\cos 2\lambda}) \to \infty$ as $n \to \infty$ and consequently $g_\lambda^n(\pm 2\lambda i) \to \infty$ as $n \to \infty$. Hence $A(0)$ contains only the singular value 0 if $|\lambda\frac{\sin^2 2\lambda}{\cos 2\lambda}|> r_\lambda$. This completes the proof. □

Remark 4.2.2.

- *Observe that $g_\lambda(\pm 2\lambda i) = 0$ for $\lambda = \frac{n\pi}{2}$ where $n \in \mathbb{N}$. So, the basin of attraction $A(0)$ contains all the singular values and their forward orbits when $\lambda = \frac{n\pi}{2}$ and $n \in \mathbb{N}$.*
- *Note that $\lim\limits_{\lambda\to\frac{n\pi}{2}} g_\lambda(\pm 2\lambda i) = -\lim\limits_{\lambda\to\frac{n\pi}{2}} \lambda\frac{\sin^2 2\lambda}{\cos 2\lambda} = 0$ for all $n \in \mathbb{N}$. So, if λ is sufficiently closed to $\frac{n\pi}{2}$, $A(0)$ contains all the singular values and their forward orbits.*

The following theorem describes the Fatou set of g_λ:

Theorem 4.2.3. *Let $g_\lambda \in \mathcal{G}$. Then, the Fatou set of g_λ is equal to the basin of attraction $A(0)$.*

Proof. The Fatou set of g_λ does not contain Baker domains and wandering domains, since g_λ has three singular values. So, only possible choices of the Fatou components of $F(g_\lambda)$ are basin of attraction, parabolic domain, Siegel disk and Herman ring. Now we will show that $F(g_\lambda)$ has no Fatou components other than $A(0)$. We prove this by dividing into two cases: (i) $A(0)$ contains all the singular values and their forward orbits and (ii) only singular value contained in $A(0)$ is 0.

Case(i): $A(0)$ contains all the singular values

The Fatou set of $g_\lambda(z)$ has no basin of attraction other than $A(0)$. If possible, let $A(z_\lambda)$ be a basin of attraction of the attracting periodic point $z_\lambda \neq 0$. Clearly, $A(z_\lambda) \cap A(0) = \emptyset$. Then by Theorem 1.5.3, $A(z_\lambda)$ contains at least one singular value which contradicts the fact that $A(0)$ contains all the singular values and their forward orbits.

The Fatou set of $g_\lambda(z)$ cannot contain a parabolic domain. If the Fatou set of $g_\lambda(z)$ contains a parabolic domain P, then by Theorem 1.5.3, P must contain a singular value but all the singular values and their forward orbits are in $A(0)$, giving a contradiction.

Now, the Fatou set of $g_\lambda(z)$ cannot contain a Siegel disk or a Herman ring. If the Fatou set of $g_\lambda(z)$ contains a Siegel disk or a Herman ring U, then by Theorem 1.5.3, the boundary of U is contained in the closure of the forward orbits of the singular values of $g_\lambda(z)$. But this is not true, since $A(0)$ contains all the singular values and their forward orbits. So, all possible Fatou components other than $A(0)$ is excluded. Thus, the Fatou set of $g_\lambda(z)$ is $A(0)$.

Case(ii): $A(0)$ contains only singular value 0

Note that only singular value contained in $A(0)$ is 0 whenever $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| \geq r_\lambda$.

Observe that if $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| = r_\lambda$, forward orbits of the singular values of $g_\lambda(z)$ is a subset of $\{0, \pm 2\lambda i, \pm r_\lambda\}$. Note that $\pm 2\lambda i \in J(g_\lambda)$, since $g_\lambda^2(\pm 2\lambda i) = r_\lambda \in J(g_\lambda)$ for $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| = r_\lambda$.

If $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| > r_\lambda$, define $S = \{g_\lambda^n(\pm 2\lambda i) : n \in \mathbb{N}\}$. Clearly S is a countable set of $\mathbb{R}$ and all points of S are isolated points in $\mathbb{R}$. So closure of the forward orbits of the singular values of $g_\lambda(z)$ are $\{0, \pm 2\lambda i\} \cup S$ for $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| > r_\lambda$. Again $\pm 2\lambda i \in J(g_\lambda)$, since $g_\lambda^n(\pm 2\lambda i) \to \infty$ as $n \to \infty$ for $|\lambda \frac{\sin^2 2\lambda}{\cos 2\lambda}| > r_\lambda$ and $\infty \in J(g_\lambda)$.

So, in this case $J(g_\lambda)$ contains the singular values $\pm 2\lambda i$.

The Fatou set of g_λ cannot contain a parabolic domain or a basin of attraction other than $A(0)$. If U is a parabolic domain or a basin of attraction with $U \neq A(0)$, then by Theorem 1.5.3, U must contains at least one singular value. Now $0 \in A(0)$ implies at least one of $\pm 2\lambda i$ is in U. But this is not possible, since $\pm 2\lambda i \in J(g_\lambda)$.

Again the Fatou set of g_λ cannot contain a Siegel disk or a Herman ring. If U is a Siegel disk or a Herman ring, then the boundary of U is contained in the closure of the forward orbits of the singular values of g_λ (by Theorem 1.5.3). But closure of the forward orbits of the singular values of g_λ is either a finite set or a countable set having all points are isolated points, giving a contradiction. In this case also, all possible Fatou components other than $A(0)$ is excluded. So, the basin of attraction $A(0)$ equals to the Fatou set of g_λ. □

Figure 4.7 gives an idea on the basin of attraction of g_λ when $\lambda = 1$. In this figure, the points for which the forward orbits are attracted by the superattracting fixed point are in black color.

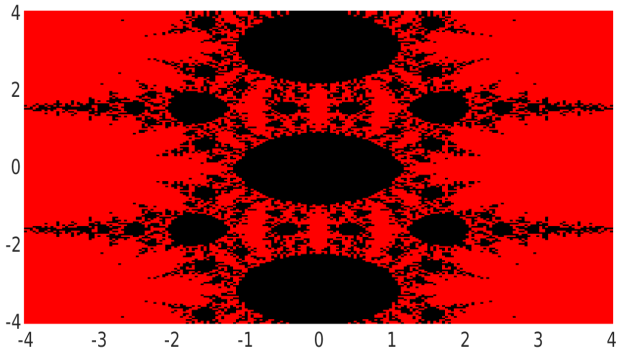

Figure 4.7: Basin of attraction in the Fatou set of g_λ.

Theorem 4.2.3 shows that the Fatou set of g_λ is non empty for any value of the parameter λ. Also the Fatou set of g_λ is unbounded, since $g_\lambda(z)$ is a periodic function.

4.3 Comparison of the dynamics of f_λ and g_λ

In this subsection, the important dynamical properties of $f_\lambda(z) = \lambda(\cosh z + \frac{1}{\cosh z})$ and $g_\lambda(z) = \lambda(\cosh z - \frac{1}{\cosh z})$ are compared and presented in a tabular form.

Table 4.1: Comparison of the dynamics of f_λ and g_λ

Dynamics of $f_\lambda(z) = \lambda(\cosh z + \frac{1}{\cosh z})$, $\lambda > 0$	**Dynamics of** $g_\lambda(z) = \lambda(\cosh z - \frac{1}{\cosh z})$, $\lambda > 0$
f_λ is critically finite	g_λ is critically finite
Singular values of f_λ are $\pm 2\lambda$	Singular values of g_λ are $0, \pm 2\lambda i$
$f_\lambda(z)$ is an even function	$g_\lambda(z)$ is an even function
$f_\lambda(z)$ is periodic function of period $2\pi i$	$g_\lambda(z)$ is periodic function of period $2\pi i$
$f_\lambda(x)$ has no real fixed point for $t_\lambda > 0$ $f_\lambda(x)$ has real rationally fixed point x_λ^* for $t_\lambda = 0$ $f_\lambda(x)$ has real attracting fixed points a_λ and real repelling fixed point r_λ for $t_\lambda < 0$	$g_\lambda(x)$ has super-attracting fixed point 0 and real repelling fixed point r_λ for any $\lambda > 0$
Bifurcation in the dynamics of f_λ occurs when $t_\lambda = 0$	No bifurcation in the dynamics of g_λ
The Julia set of f_λ is $\widehat{\mathbb{C}}$ for $t_\lambda > 0$	The Julia set of g_λ can never be $\widehat{\mathbb{C}}$

CHAPTER 5

PERIOD DOUBLING BIFURCATION IN THE DYNAMICS OF ONE-PARAMETER FAMILY OF TRANSLATED HYPERBOLIC COSINE FUNCTIONS

A one-parameter family $\mathcal{F} \equiv \{f_\lambda(z) = \lambda + \cosh z : \lambda \in \mathbb{R}\}$ of translated hyperbolic cosine functions is considered in this chapter. The function $f_\lambda(z)$ is symmetric and periodic of minimal period $2\pi i$. Also, the function f_λ has two singular values $\lambda \pm 1$ and the singular value $\lambda - 1$ always tends to infinity under iteration of f_λ for any $\lambda \in \mathbb{R}$. The existence and nature of the real fixed points of $f_\lambda(x)$ are studied. The dynamics of f_λ on the complex plane is investigated for some cases theoretically. Remaining cases we have numerically computed the higher order periodic points and their nature for different values of the parameter λ and then we conclude the dynamics. It is observed that a period doubling bifurcation occurs in the dynamics of functions in the family $\mathcal{F}$. It may be noted that the dynamics of $\cosh z$ is very simple where as the dynamics of $\lambda + \cosh z$ changes dramatically.

5.1 Real Dynamics of Functions in $\mathcal{F}$

The dynamics of $f_\lambda \in \mathcal{F}$ on the real line $\mathbb{R}$ is investigated and obtained results are discussed in this section.

On the real line $\mathbb{R}$, the function $f_\lambda(x)$ has a unique minimum at 0 and the minimum value of $f_\lambda(x)$ is $\lambda + 1$. Observe that $f_\lambda(x) \to \infty$ as $|x| \to \infty$. Thus, f_λ maps real line $\mathbb{R}$

into $[\lambda + 1, \infty)$.

Define $g_\lambda(x) = f_\lambda(x) - x$ where $f_\lambda(x) = \lambda + \cosh x$ for $x \in \mathbb{R}$ and λ is a real parameter. So, the zeros of $g_\lambda(x)$ are the fixed points of $f_\lambda(x)$. Note that $g_\lambda(x) \to \infty$ as $|x| \to \infty$. It is easy to see that $g_\lambda(x)$ has a unique minimum at $\tilde{x} = \sinh^{-1}(1)$ and the minimum value of $g_\lambda(x)$ is $g_\lambda(\tilde{x}) = \lambda + \cosh(\tilde{x}) - \tilde{x} = \lambda + \sqrt{2} - \tilde{x}$.

Throughout this chapter, we denote $\tilde{x} = \sinh^{-1}(1)$, $\tilde{\lambda} = \tilde{x} - \sqrt{2}$ and $\lambda_0^* = -\tilde{x} - \sqrt{2}$.

Remark 5.1.1. *The values of $\tilde{x}$, $\tilde{\lambda}$ and λ_0^* are computed numerically and given by $\tilde{x} \approx 0.881373587$, $\tilde{\lambda} \approx -0.532839975$ and $\lambda_0^* \approx -2.2955871$.*

The existence and nature of the real fixed points of f_λ are proved in the following theorem.

Theorem 5.1.1. *Let $f_\lambda(x) = \lambda + \cosh x$ for $x \in \mathbb{R}$ where λ is a real parameter.*

(a) If $\lambda > \tilde{\lambda}$, then $f_\lambda(x)$ has no fixed points.

(b) If $\lambda = \tilde{\lambda}$, then $f_\lambda(x)$ has a unique rationally indifferent fixed point $\tilde{x}$.

(c) If $\lambda_0^* < \lambda < \tilde{\lambda}$, then $f_\lambda(x)$ has an attracting fixed point a_λ(say) and a repelling fixed point r_λ(say) with $a_\lambda < \tilde{x} < r_\lambda$.

(d) If $\lambda = \lambda_0^*$, then $f_\lambda(x)$ has a rationally indifferent fixed point $-\tilde{x}$ and a repelling fixed point r_λ(say) with $r_\lambda > \tilde{x}$.

(e) If $\lambda < \lambda_0^*$, then $f_\lambda(x)$ has two repelling fixed points r_λ and r'_λ(say) with $r_\lambda > \tilde{x}$ and $r'_\lambda < -\tilde{x}$.

Proof. Set $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$.

(a) **Case:** $\lambda > \tilde{\lambda}$

If $\lambda > \tilde{\lambda}$, then $g_\lambda(x) \geq g_\lambda(\tilde{x}) = \lambda + \sqrt{2} - \tilde{x} = \lambda - \tilde{\lambda} > 0$ for all $x \in \mathbb{R}$. So, $f_\lambda(x)$ has no fixed points.

In Figure 5.1, $f_\lambda(x)$ has no fixed points for $\lambda = -2/5$.

(b) **Case:** $\lambda = \tilde{\lambda}$

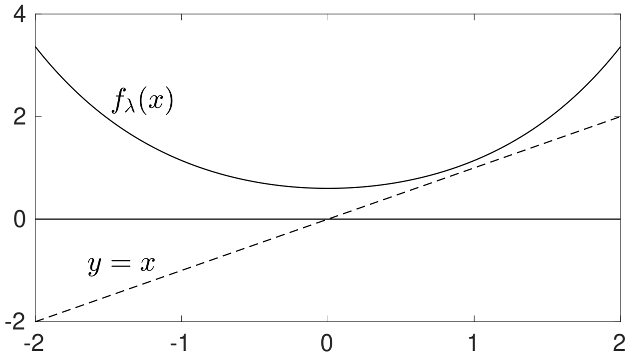

Figure 5.1: No fixed points of $f_\lambda(x)$ for $\lambda > \tilde{\lambda}$.

Note that $g_{\tilde{\lambda}}(\tilde{x}) = \tilde{\lambda} + \sqrt{2} - \tilde{x} = 0$ and $g'_{\tilde{\lambda}}(\tilde{x}) = 0$. Thus, $f_{\tilde{\lambda}}(\tilde{x}) = \tilde{x}$ and $f'_{\tilde{\lambda}}(\tilde{x}) = 1$. So, $\tilde{x}$ is a rationally indifferent fixed point of $f_{\tilde{\lambda}}(x)$. Now $g_{\tilde{\lambda}}(x) > g_{\tilde{\lambda}}(\tilde{x}) = 0$ for $x \neq \tilde{x}$. Thus, $g_{\tilde{\lambda}}(x)$ has no zero other than $\tilde{x}$ and consequently $\tilde{x}$ is the only fixed point of $f_{\tilde{\lambda}}(x)$.

In Figure 5.2, $\tilde{x}$ is the rationally indifferent fixed point of $f_\lambda(x)$ for $\lambda = \tilde{\lambda}$.

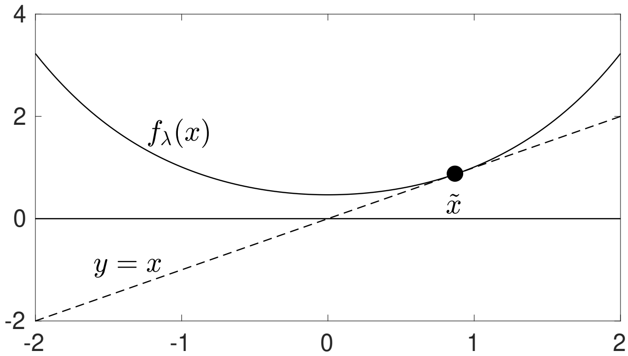

Figure 5.2: One rationally indifferent fixed point of $f_\lambda(x)$ when $\lambda = \tilde{\lambda}$.

(c) **Case:** $\lambda_0^* < \lambda < \tilde{\lambda}$

Observe that $g_\lambda(\tilde{x}) = \lambda + \sqrt{2} - \tilde{x} = \lambda - \tilde{\lambda} < 0$. This implies $g_\lambda(x)$ has two zeros. Since $g_\lambda(-\tilde{x}) = \lambda + \sqrt{2} + \tilde{x} = \lambda - \lambda_0^* > 0$, it follows that $g_\lambda(x)$ has zeros a_λ and r_λ(say) with $-\tilde{x} < a_\lambda < \tilde{x} < r_\lambda$. So, a_λ and r_λ are the fixed points of $f_\lambda(x)$. Note that $-1 < f'_\lambda(a_\lambda) < 1$ and $f'_\lambda(r_\lambda) > 1$. Thus, a_λ is an attracting fixed point and r_λ is a repelling fixed point of $f_\lambda(x)$.

The fixed points of $f_\lambda(x)$ are shown in Figure 5.3 for $\lambda = -4/5$. In Figure 5.3, a_λ is the attracting fixed point and r_λ is the repelling fixed point of $f_\lambda(x)$.

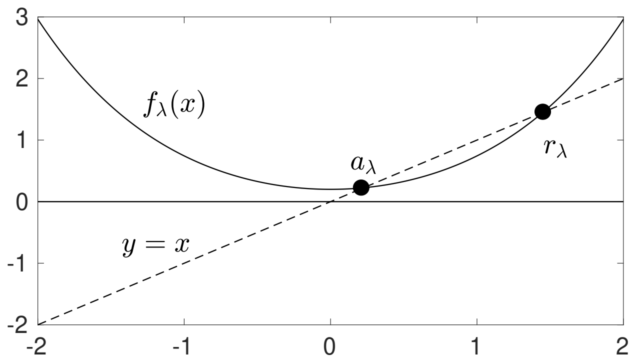

Figure 5.3: Two fixed points of $f_\lambda(x)$ for $\lambda_0^* < \lambda < \tilde{\lambda}$.

(d) **Case:** $\lambda = \lambda_0^*$

Observe that $g_{\lambda_0^*}(\tilde{x}) = \lambda_0^* + \sqrt{2} - \tilde{x} = -2\tilde{x} < 0$. So, $g_{\lambda_0^*}(x)$ has two zeros. Now $g_{\lambda_0^*}(-\tilde{x}) = \lambda_0^* + \sqrt{2} + \tilde{x} = 0$ implies $f_{\lambda_0^*}(-\tilde{x}) = -\tilde{x}$. Since $f'_{\lambda_0^*}(-\tilde{x}) = -1$, $-\tilde{x}$ is a rationally indifferent fixed point of $f_{\lambda_0^*}(x)$. It is easy to see that $g_{\lambda_0^*}(x) \to \infty$ as $x \to \infty$ and $g'_{\lambda_0^*}(x) > 0$ for $x > \tilde{x}$. Thus, $g_{\lambda_0^*}(x)$ has a zero r_λ(say) with $r_\lambda > \tilde{x}$. Since $f'_{\lambda_0^*}(r_\lambda) > 1$, r_λ is a repelling fixed point of $f_{\lambda_0^*}(x)$.

For $\lambda = \lambda_0^*$, the fixed points of $f_\lambda(x)$ are shown in Figure 5.4. In Figure 5.4, $-\tilde{x}$ is the rationally indifferent fixed point and r_λ is the repelling fixed point of $f_\lambda(x)$.

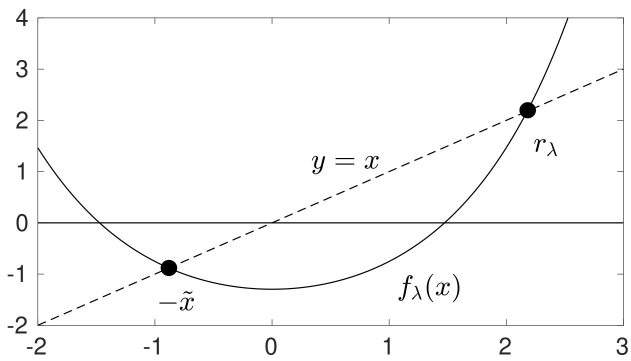

Figure 5.4: Two fixed points of $f_\lambda(x)$ when $\lambda = \lambda_0^*$.

(e) **Case:** $\lambda < \lambda_0^*$

If $\lambda < \lambda_0^*$, then $g_\lambda(\tilde{x}) = \lambda + \sqrt{2} - \tilde{x} = \lambda - \tilde{\lambda} < 0$ and $g_\lambda(-\tilde{x}) = \lambda + \sqrt{2} + \tilde{x} = \lambda - \lambda_0^* < 0$. So, $g_\lambda(x)$ has two zeros r_λ and r'_λ(say) with $r_\lambda > \tilde{x}$ and $r'_\lambda < -\tilde{x}$. It is easy to see that $f'_\lambda(r_\lambda) > 1$ and $f'_\lambda(r'_\lambda) < -1$. Hence r_λ and r'_λ are the repelling fixed points of $f_\lambda(x)$. This completes the proof.

For $\lambda = -3$, the fixed points of $f_\lambda(x)$ are shown in Figure 5.5. In Figure 5.5, r_λ and r'_λ are the repelling fixed points of $f_\lambda(x)$. $\square$

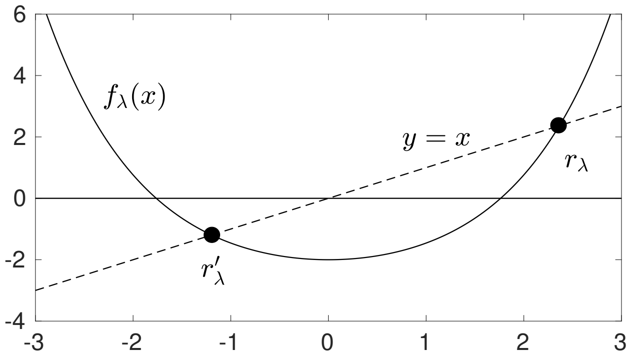

Figure 5.5: Two repelling fixed points of $f_\lambda(x)$ when $\lambda < \lambda_0^*$.

Remark 5.1.2. *If $\lambda < \tilde{\lambda}$, by Theorem 5.1.1(c), (d) and (e), $f_\lambda(x)$ has a repelling fixed point r_λ with $r_\lambda > \tilde{x}$. Clearly, $\cosh(r_\lambda) - r_\lambda = -\lambda$ as r_λ is a fixed point of $f_\lambda(x)$. Observe that $\cosh(x) - x$ is strictly increasing for $x > \tilde{x}$. Therefore, it follows that if λ decreases, then r_λ increases (if λ increases, then r_λ decreases).*

The dynamics of $f_\lambda(x)$ on the real line $\mathbb{R}$ is discussed in the following theorem.

Theorem 5.1.2. *Let $f_\lambda(x) = \lambda + \cosh x$ for $x \in \mathbb{R}$ where λ is a real parameter.*

(a) If $\lambda > \tilde{\lambda}$, then $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x \in \mathbb{R}$.

(b) If $\lambda = \tilde{\lambda}$, then $f_\lambda^n(x) \to \tilde{x}$ as $n \to \infty$ for $|x| \leq \tilde{x}$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $|x| > \tilde{x}$.

(c) If $\lambda < \tilde{\lambda}$, then $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $|x| > r_\lambda$.

Proof. (a) **Case:** $\lambda > \tilde{\lambda}$

If $\lambda > \tilde{\lambda}$, by Theorem 5.1.1(a), $f_\lambda(x)$ has no fixed points. Clearly, $f_\lambda(x) > x$ for all $x \in \mathbb{R}$. Observe that $f_\lambda(x) > 0$ for all $x \in \mathbb{R}$ as $f_\lambda(x)$ has minimum value $\lambda + 1 > 0$. Also, $f_\lambda(x)$ is strictly increasing for $x \geq 0$. Therefore, the sequence $\{f_\lambda^n(x)\}$ is monotonically increasing and not bounded above. Hence $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x \in \mathbb{R}$.

(b) **Case:** $\lambda = \tilde{\lambda}$

If $\lambda = \tilde{\lambda}$, by Theorem 5.1.1(b), $f_\lambda(x)$ has a unique rationally indifferent fixed point $\tilde{x}$. If $-\tilde{x} \leq x < \tilde{x}$, by mean value theorem, $|f_{\tilde{\lambda}}(x) - \tilde{x}| = |f'_{\tilde{\lambda}}(c)||x - \tilde{x}|$ where $-\tilde{x} \leq x < c < \tilde{x}$. Since $|f'_{\tilde{\lambda}}(c)| < 1$, it follows that $|f_{\tilde{\lambda}}(x) - \tilde{x}| < |x - \tilde{x}|$ for $-\tilde{x} \leq x < \tilde{x}$. Thus, $f^n_{\tilde{\lambda}}(x) \to \tilde{x}$ as $n \to \infty$ for $x \in [-\tilde{x}, \tilde{x}]$.

Notice that $f_{\tilde{\lambda}}(x) > x$ and $f'_{\tilde{\lambda}}(x) > 1$ for $x > \tilde{x}$. So, the sequence $\{f^n_{\tilde{\lambda}}(x)\}$ is monotonically increasing and not bounded above for $x > \tilde{x}$. Thus, $f^n_{\tilde{\lambda}}(x) \to \infty$ as $n \to \infty$ for $x \in (\tilde{x}, \infty)$.

Since $f_{\tilde{\lambda}}(x)$ is an even function, it follows that $f^n_{\tilde{\lambda}}(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, -\tilde{x})$.

(c) **Case:** $\lambda < \tilde{\lambda}$

If $\lambda < \tilde{\lambda}$, by Theorem 5.1.1(c), (d) and (e), $f_\lambda(x)$ has always a repelling fixed point r_λ with $r_\lambda > \tilde{x}$. If $x > r_\lambda$, observe that $f_\lambda(x) > x$ and $f'_\lambda(x) > 1$. Thus, the sequence $\{f^n_\lambda(x)\}$ is monotonically increasing and not bounded above for $x > r_\lambda$. Therefore, $f^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x > r_\lambda$. Since $f_\lambda(-x) = f_\lambda(x)$, it follows that $f^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x < -r_\lambda$. □

In the following proposition it is shown that f_λ has only real singular values.

Proposition 5.1.1. *Let $f_\lambda \in \mathcal{F}$. Then $f_\lambda(z)$ has two critical values $\lambda \pm 1$ and no finite asymptotic value.*

Proof. Observe that $f'_\lambda(z) = 0$ if and only if $\sinh z = 0$. So, the critical values of $f_\lambda(z)$ are $\lambda \pm 1$. In Chapter 3, Proposition 3.1.2 gives that the function $\cosh z$ has no finite asymptotic value when $\lambda = \mu = 1/2$ and $b = e$. This gives the function $f_\lambda(z) = \lambda + \cosh z$ has no finite asymptotic value. This completes the proof. □

By Proposition 5.1.1, the function f_λ is in the class $\mathcal{S}$.

The following two propositions describe the forward orbits of the singular values of the function f_λ. The forward orbits of $\lambda - 1$ and $\lambda + 1$ are presented in Proposition 5.1.2 and Proposition 5.1.3 respectively.

Proposition 5.1.2. *Let $f_\lambda \in \mathcal{F}$. Then, $f_\lambda^n(\lambda - 1) \to \infty$ as $n \to \infty$.*

Proof. **Case (i):** $\lambda > \tilde{\lambda}$

If $\lambda > \tilde{\lambda}$, by Theorem 5.1.2(a), $f_\lambda^n(\lambda - 1) \to \infty$ as $n \to \infty$.

Case (ii): $\lambda = \tilde{\lambda}$

If $\lambda = \tilde{\lambda}$, it is easy to see that $1-\lambda = 1+\sqrt{2}-\tilde{x} > \tilde{x}$. By Theorem 5.1.2(b), $f_\lambda^n(1-\lambda) \to \infty$ as $n \to \infty$. Now $f_\lambda^n(\lambda - 1) \to \infty$ as $n \to \infty$, since $f_\lambda(x) = f_\lambda(-x)$.

Case (iii): $\lambda < \tilde{\lambda}$

If $\lambda < \tilde{\lambda}$, $f_\lambda(x)$ has a repelling fixed point r_λ with $r_\lambda > \tilde{x}$. Note that $1 - \lambda - r_\lambda = 1 + \cosh(r_\lambda) - 2r_\lambda$. On the real line $\mathbb{R}$, the function $1 + \cosh x - 2x$ has a unique minimum value which is a positive real number. So, $1 - \lambda > r_\lambda$ and by Theorem 5.1.2(c), $f_\lambda^n(1 - \lambda) \to \infty$ as $n \to \infty$. Since $f_\lambda(x)$ is an even function, it follows that $f_\lambda^n(\lambda - 1) \to \infty$ as $n \to \infty$. This completes the proof. □

Proposition 5.1.3. *If $\lambda \in [\lambda^*, \tilde{\lambda}]$, then $\{f_\lambda^n(\lambda + 1)\}$ is a bounded sequence and if $\lambda \in \mathbb{R} \setminus [\lambda^*, \tilde{\lambda}]$, then $f_\lambda^n(\lambda+1) \to \infty$ as $n \to \infty$ where $\lambda^* = x^* - \cosh(x^*)$ and x^* is the positive real root of the equation $1 + 2x - \cosh x = 0$.*

Proof. **Case (i):** $\lambda > \tilde{\lambda}$

If $\lambda > \tilde{\lambda}$, by Theorem 5.1.2(a), $f_\lambda^n(\lambda + 1) \to \infty$ as $n \to \infty$.

Case (ii): $\lambda = \tilde{\lambda}$

If $\lambda = \tilde{\lambda}$, then $0 < \lambda + 1 = \tilde{x} - \sqrt{2} + 1 < \tilde{x}$. By Theorem 5.1.2(b), $f_\lambda^n(\lambda + 1) \to \tilde{x}$ as $n \to \infty$.

Case (iii): $\lambda < \tilde{\lambda}$

If $\lambda < \tilde{\lambda}$, by Theorem 5.1.1, $f_\lambda(x)$ has a repelling fixed point r_λ with $r_\lambda > \tilde{x}$. Notice that $f_\lambda(x)$ is an even function and $f_\lambda(x)$ is strictly increasing for $x > 0$. So, $f_\lambda([-r_\lambda, r_\lambda]) = [\lambda+1, r_\lambda]$. Thus, the sequence $\{f_\lambda^n(\lambda+1)\}$ is bounded if $\lambda + 1 \geq -r_\lambda$, and $f_\lambda^n(\lambda+1) \to \infty$ as $n \to \infty$ if $\lambda + 1 < -r_\lambda$. It is easy to see that $\lambda + 1 + r_\lambda = 1 + 2r_\lambda - \cosh(r_\lambda)$. For $x > 0$, we define $h(x) = 1 + 2x - \cosh x$. It can be verified

$$h(x) \begin{cases} > 0 & \text{for } x \in (0, x^*), \\ = 0 & \text{for } x = x^*, \\ < 0 & \text{for } x > x^*. \end{cases} \tag{5.1}$$

So, it follows that

$$\lambda + 1 \begin{cases} \geq -r_\lambda & \text{for } r_\lambda \leq x^*, \\ < -r_\lambda & \text{for } r_\lambda > x^*. \end{cases} \tag{5.2}$$

If $\lambda \geq \lambda^*$, by Remark 5.1.2, $r_\lambda \leq x^*$. Now by using (5.2), $\lambda+1 \geq -r_\lambda$ if $\lambda \geq \lambda^*$. Similarly, it can be shown that $\lambda + 1 < -r_\lambda$ if $\lambda < \lambda^*$.

Thus, in this case, $\{f_\lambda^n(\lambda+1)\}$ is a bounded sequence if $\lambda \geq \lambda^*$ and $f_\lambda^n(\lambda+1) \to \infty$ as $n \to \infty$ if $\lambda < \lambda^*$. This completes the proof. □

Remark 5.1.3. *The values of x^* and λ^* are numerically computed and found to be $x^* \approx 2.4664795$ and $\lambda^* \approx -3.46641163$.*

5.2 Chaotic Burst in the Dynamics of f_λ

The chaotic burst in the dynamics of functions in the family $\mathcal{F}$ is established by describing the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ in the present section.

The following theorem is a consequence of Proposition 5.1.1 and Theorems 1.5.1, 1.5.4.

Theorem 5.2.1. *Let $f_\lambda \in \mathcal{F}$. Then the Fatou set of f_λ does not contain Baker domains and wandering domains.*

5.2.1 Dynamics of $f_\lambda(z)$ for $\lambda \in [\lambda_0^*, \tilde{\lambda}]$

If $\lambda \in (\lambda_0^*, \tilde{\lambda})$, by Theorem 5.1.1(c), $f_\lambda(z)$ has a real attracting fixed point a_λ with $-\tilde{x} < a_\lambda < \tilde{x}$. The basin of attraction $A(a_\lambda)$ corresponding to the real attracting fixed point a_λ is defined as

$$A(a_\lambda) := \{z \in \mathbb{C} : f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}.$$

For $\lambda \in (\lambda_0^*, \tilde{\lambda})$, it is proved in the following theorem that the Fatou set of f_λ is the basin of attraction $A(a_\lambda)$.

Theorem 5.2.2. *Let $f_\lambda \in \mathcal{F}$ and $\lambda \in (\lambda_0^*, \tilde{\lambda})$. Then, the Fatou set $F(f_\lambda) = A(a_\lambda)$.*

Proof. By Theorem 1.5.3, the basin of attraction $A(a_\lambda)$ contains at least one singular value. Now $\lambda + 1 \in A(a_\lambda)$, since by Proposition 5.1.2, $f_\lambda^n(\lambda - 1) \to \infty$ as $n \to \infty$ and $\infty \in J(f_\lambda)$.

The Fatou set of f_λ has no basin of attraction other than $A(a_\lambda)$. If possible, let $A(z_\lambda)$ be a basin of attraction of the attracting periodic point $z_\lambda \neq a_\lambda$. Clearly, $\lambda + 1 \notin A(z_\lambda)$ as $A(a_\lambda) \cap A(z_\lambda) = \emptyset$. Now, $f_\lambda^n(\lambda - 1) \to \infty$ as $n \to \infty$ gives $\lambda - 1 \notin A(z_\lambda)$. Thus, $A(z_\lambda)$ does not contain a singular value which is not possible.

If U is a parabolic domain contained in the Fatou set $F(f_\lambda)$, then U contains at least one singular value (Theorem 1.5.3). Note that $\lambda + 1 \notin U$ as $U \cap A(a_\lambda) = \emptyset$. There exists a rationally indifferent periodic point $x'_\lambda \in \mathbb{C}$ of minimal period p such that $U = \{z \in \mathbb{C} : f_\lambda^{np}(z) \to x'_\lambda \text{ as } n \to \infty\}$. Clearly, $\lambda - 1 \notin U$. So, U does not contain any singular value which is not true. Thus, the Fatou set of $f_\lambda(z)$ has no parabolic domain.

The Fatou set of f_λ cannot contain Siegel disks. If $F(f_\lambda)$ contains a Siegel disk U, then by Theorem 1.5.3, the boundary ∂U of U is contained in the closure of the forward orbits of the singular values of f_λ. Since $\lambda + 1 \in A(a_\lambda)$, ∂U cannot contain $\lambda + 1$ and its forward orbits. Since $f_\lambda(z)$ is critically finite and $f_\lambda^n(\lambda - 1) \to \infty$ as $n \to \infty$, it is impossible for $F(f_\lambda)$ to have a Siegel disk.

By Theorem 5.2.1, the Fatou set of f_λ does not contain Baker domains and wandering domains.

Thus, the basin of attraction $A(a_\lambda)$ is the Fatou set of f_λ. □

Figure 5.6 gives an idea on the basin of attraction of f_λ when $\lambda = -2$. In Figure 5.6, the points for which the forward orbits are attracted by the attracting fixed point are in black color.

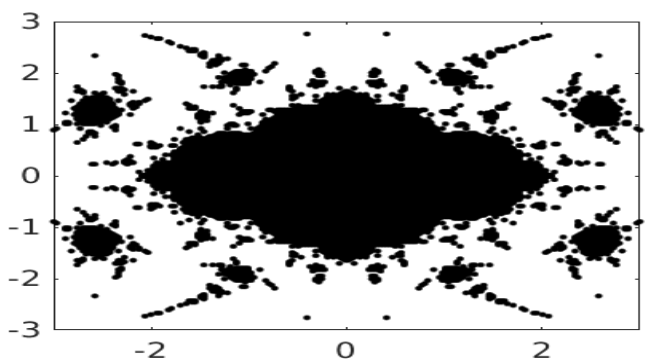

Figure 5.6: Basin of attraction in the Fatou set of f_λ for $\lambda \in (\lambda_0^*, \tilde{\lambda})$.

When $\lambda = \tilde{\lambda}$, the parabolic domain $P \equiv P(\tilde{x})$ corresponding to the real rationally indifferent fixed point $\tilde{x}$ is defined as

$$P \equiv P(\tilde{x}) := \{z \in \mathbb{C} : \; f_{\tilde{\lambda}}^n(z) \to \tilde{x} \text{ as } n \to \infty\}.$$

Similarly, if $\lambda = \lambda_0^*$ the parabolic domain $P \equiv P(-\tilde{x})$ corresponding to the real rationally indifferent fixed point $-\tilde{x}$ is defined as

$$P \equiv P(-\tilde{x}) := \{z \in \mathbb{C} : \; f_{\lambda_0^*}^n(z) \to -\tilde{x} \text{ as } n \to \infty\}.$$

For $\lambda = \tilde{\lambda}$ and $\lambda = \lambda_0^*$, the Fatou set $F(f_\lambda)$ is described in the following theorem and its proof is similar to the proof of Theorem 5.2.2.

Theorem 5.2.3. *Let $f_\lambda \in \mathcal{F}$ where $\lambda = \tilde{\lambda}$ or $\lambda = \lambda_0^*$. Then, the Fatou set of $f_\lambda(z)$ is equal to P.*

For $\lambda = \lambda_0^*$, the parabolic domain of f_λ is given in Figure 5.7. In Figure 5.7, the points for which the forward orbits are attracted by the rationally indifferent fixed point $-\tilde{x}$ are in black color.

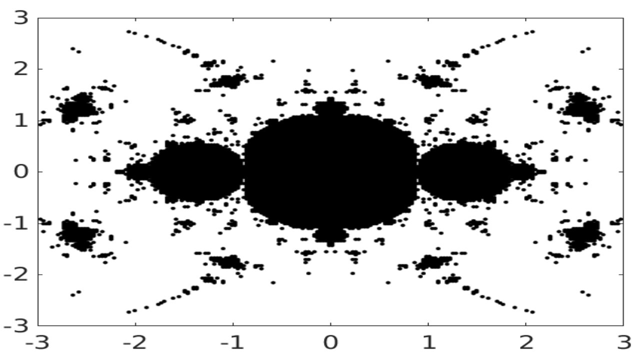

Figure 5.7: Parabolic domain in the Fatou set of f_λ for $\lambda = \lambda_0^*$.

5.2.2 Dynamics of $f_\lambda(z)$ for $\lambda \in [\lambda^*, \lambda_0^*)$

If $\lambda \in [\lambda^*, \lambda_0^*)$, by Theorem 5.1.1, f_λ has two real repelling fixed points r_λ and r'_λ with $r_\lambda > \tilde{x}$ and $r'_\lambda < -\tilde{x}$. Also, by Proposition 5.1.3, $\{f_\lambda^n(\lambda+1)\}$ is a bounded sequence in $\mathbb{R}$. So, there is a possibility of higher order real attracting or rationally indifferent periodic points of $f_\lambda(x)$. For different values of the parameter λ, the existence and the nature of the higher order periodic points of $f_\lambda(x)$ are computationally analyzed and given in Table 5.1.

Table 5.1: Nature of the periodic points of $f_\lambda(x)$ for different values of λ

Nature of the periodic points of $f_\lambda(x)$			
λ	**Minimal period one**	**Minimal period two**	**Minimal period four**
$\lambda_0^* = -2.29$	One rationally indifferent and one repelling		
-2.35	Two repelling	one attracting	
-2.40	Two repelling	one attracting	
-2.45	Two repelling	one attracting	
-2.50	Two repelling	one attracting	
-2.55	Two repelling	one attracting	
-2.60	Two repelling	one attracting	
-2.65	Two repelling	one attracting	
-2.70	Two repelling	one attracting	
-2.75	Two repelling	one attracting	
-2.80	Two repelling	one attracting	
-2.85	Two repelling	one attracting	
$\lambda_1^* = -2.885$	Two repelling	one rationally indifferent	
-2.90	Two repelling	one repelling	one attracting
-2.95	Two repelling	one repelling	one attracting
-2.97	Two repelling	one repelling	one attracting
$\lambda_2^* = -3.001$	Two repelling	one repelling	one rationally indifferent
-3.05	Two repelling	one repelling	one repelling
-3.10	Two repelling	one repelling	one repelling

Denote $\lambda_1^* = -2.885$. When $\lambda = \lambda_1^*$, from Table 5.1, $f_\lambda(x)$ has two repelling fixed points and a cycle consisting of one rationally indifferent periodic point of minimal period 2. Again, from Table 5.1, observe that $f_\lambda(x)$ has two repelling fixed points and a cycle consisting of an attracting periodic point of minimal period two for $\lambda = -2.35, -2.40, -2.45, -2.50, -2.55, -2.60, -2.65, -2.70, -2.75, -2.80, -2.85$. For $\lambda = -2.5$, the periodic points of minimal period one and two are shown in Figure 5.8. In Figure 5.8, the blue color points are repelling fixed points and the maroon color points are attracting periodic points of minimal period 2. In Figure 5.8, there are two maroon color points but one of them is the image of the other under f_λ. All of these λ lie in $(\lambda_1^*, \lambda_0^*)$. It can be concluded that

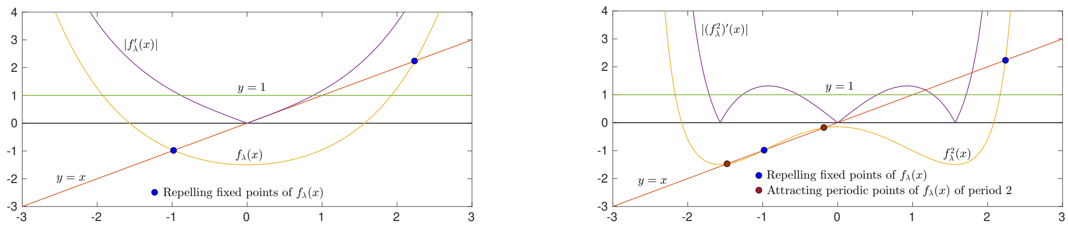

Figure 5.8: Periodic points of $f_\lambda(x)$ for $\lambda = -2.5$.

if $\lambda \in (\lambda_1^*, \lambda_0^*)$, $f_\lambda(x)$ has two repelling fixed points and one attracting periodic point of minimal period 2. Let $\{a_{\lambda,1}^0, a_{\lambda,1}^1 = f_\lambda(a_{\lambda,1}^0)\}$ be the cycle of the attracting periodic point of minimal period 2 for $\lambda \in (\lambda_1^*, \lambda_0^*)$. Then the corresponding basin of attraction is $A(a_\lambda; 1) = U_{\lambda,0} \cup U_{\lambda,1}$ where $U_{\lambda,0}$, $U_{\lambda,1}$ are given by

$$U_{\lambda,i} = \{z \in \mathbb{C} : \ f_\lambda^{2n}(z) \to a_{\lambda,1}^i \text{ as } n \to \infty\} \text{ for } i = 0,\ 1.$$

In this case, it can be proved that the Fatou set of $f_\lambda(z)$ is $A(a_\lambda; 1)$.

The basin of attraction of $F(f_\lambda)$ for $\lambda = -2.5$ is shown in Figure 5.9. In this figure, the basin of attraction consists of black and blue regions. Here, it is an attracting cycle of length 2. One of the cycles is the black region and another one is the blue region.

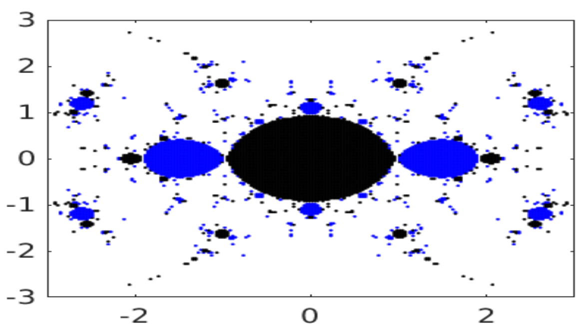

Figure 5.9: Basin of attraction in the Fatou set of f_λ for $\lambda \in (\lambda_1^*, \lambda_0^*)$.

For $\lambda = \lambda_1^*$, the fixed points and periodic points of minimal period 2 of $f_\lambda(x)$, are shown in Figure 5.10. In Figure 5.10, the blue color points are repelling fixed points and the maroon color points are rationally indifferent periodic points of minimal period 2 of $f_{\lambda_1^*}(x)$. Let $\{x_{\lambda_1^*}^0, x_{\lambda_1^*}^1 = f_{\lambda_1^*}(x_{\lambda_1^*}^0)\}$ be the cycle of the rationally indifferent periodic point for $\lambda = \lambda_1^*$. Then the corresponding parabolic domain is $P(x_{\lambda_1^*}) = P_{\lambda_1^*,0} \cup P_{\lambda_1^*,1}$ where

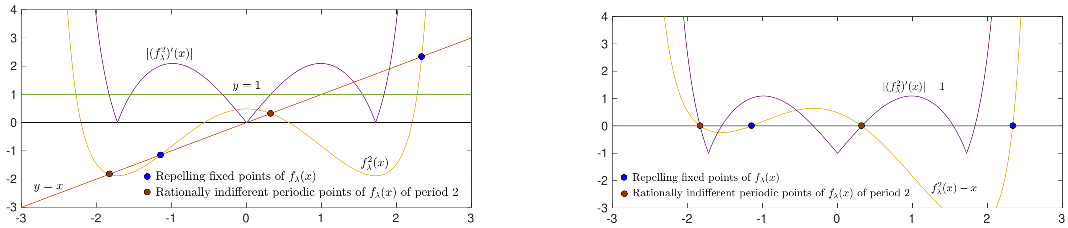

Figure 5.10: Periodic points of $f_\lambda(x)$ for $\lambda = -2.885$.

$P_{\lambda_1^*,0}$ and $P_{\lambda_1^*,1}$ are given by

$$P_{\lambda_1^*,i} = \{z \in \mathbb{C} : \; f_{\lambda_1^*}^{2n}(z) \to x_{\lambda_1^*}^i \text{ as } n \to \infty\} \text{ for } i = 0, \; 1.$$

For $\lambda = \lambda_1^*$, it can be proved that the Fatou set of $f_{\lambda_1^*}(z)$ is $P(x_{\lambda_1^*})$.

The parabolic domain of $F(f_\lambda)$ for $\lambda = \lambda_1^*$ is given in Figure 5.11. In this figure, the parabolic domain consists of black and blue regions. Here, it is a parabolic cycle of length 2. One of the cycles is the black region and another one is the blue region.

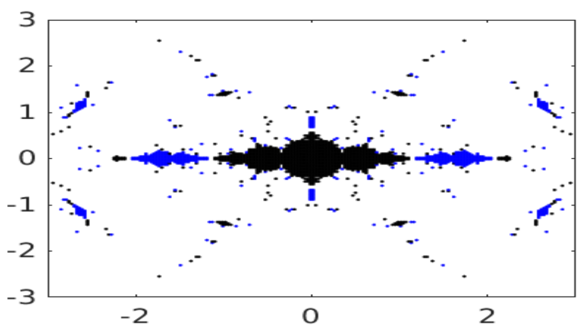

Figure 5.11: Parabolic domain in the Fatou set of f_λ for $\lambda = \lambda_1^*$.

Again from Table 5.1, notice that for $\lambda = -2.90, \; -2.95, \; -2.97$, the function $f_\lambda(x)$ has no attracting or rationally indifferent periodic points of minimal period one and two, but it has an attracting periodic point of minimal period 4. For $\lambda = -2.95$, the periodic points of $f_\lambda(x)$ are shown in Figure 5.12. In Figure 5.12, the blue color points are repelling fixed points, the maroon color points are repelling periodic points of minimal period 2 and the black color points are attracting periodic points of minimal period 4. There are two maroon color points but one of them is the image of the other. Although there are four black color points but they are in a periodic cycle. All of these λ lie in $(\lambda_2^*, \lambda_1^*)$.

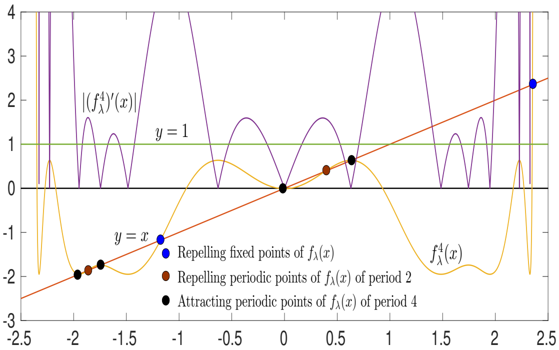

Figure 5.12: Periodic points of $f_\lambda(x)$ for $\lambda = -2.95$.

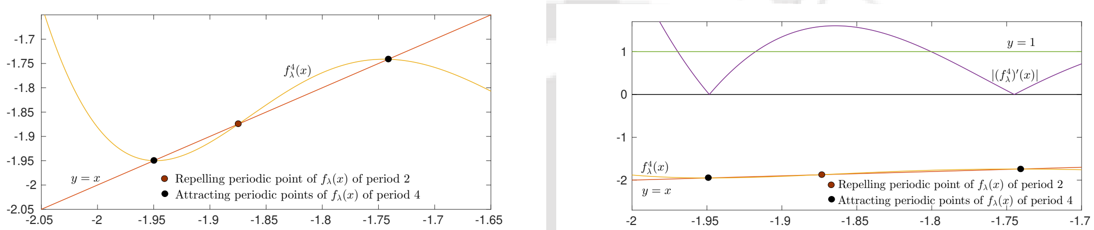

Figure 5.13: Zoom view of Figure 5.12

It can be concluded that the function $f_\lambda(x)$ has two repelling fixed points, one repelling periodic point of minimal period 2 and one attracting periodic point of minimal period 4 for $\lambda \in (\lambda_2^*, \lambda_1^*)$. Let $\{a_{\lambda,2}^0,\ a_{\lambda,2}^1 = f_\lambda(a_{\lambda,2}^0),\ a_{\lambda,2}^2 = f_\lambda^2(a_{\lambda,2}^0),\ a_{\lambda,2}^3 = f_\lambda^3(a_{\lambda,2}^0)\}$ be the cycle of attracting periodic point for $\lambda \in (\lambda_2^*, \lambda_1^*)$. Then, the corresponding basin of attraction is $A(a_\lambda; 2) = \bigcup_{i=0}^{3} U_{\lambda,i}$ where $U_{\lambda,i}$ are given by

$$U_{\lambda,i} = \{z \in \mathbb{C} :\ f_\lambda^{4n}(z) \to a_{\lambda,2}^i \text{ as } n \to \infty\} \text{ for } i = 0,\ 1,\ 2,\ 3.$$

In this case, it can be proved $A(a_\lambda; 2)$ is equal to the Fatou set of $f_\lambda(z)$.

The basin of attraction of $F(f_\lambda)$ for $\lambda = -2.95$ is given in Figure 5.14. In this figure, the basin of attraction consists of black, blue, green and red regions. Here, it is an attracting cycle of length 4. Each of the regions black, blue, green and red is one of the components of the attracting cycle.

For $\lambda = \lambda_2^*$, the periodic points of $f_\lambda(x)$ are shown in Figure 5.15. In Figure 5.15, the blue color points are repelling fixed points, the maroon color points are in the cycle of repelling periodic point of minimal period 2 and the black color points are in the cycle of

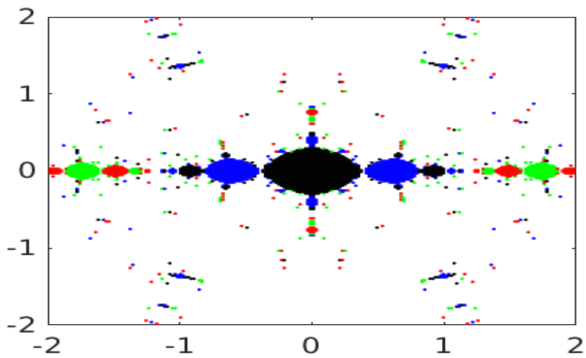

Figure 5.14: Attracting domain in the Fatou set of f_λ for $\lambda = -2.95$.

the rationally indifferent periodic point of minimal period 4.

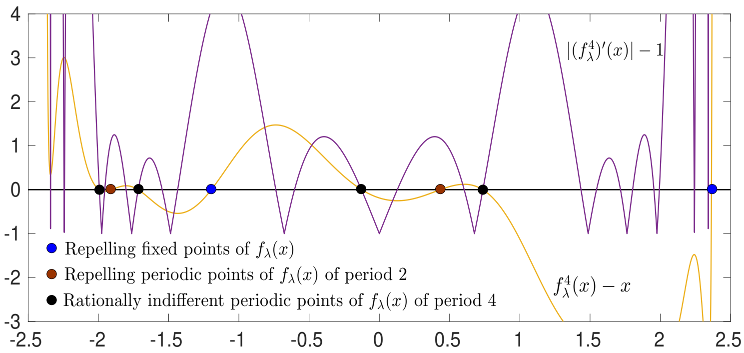

Figure 5.15: Periodic points of $f_\lambda(x)$ for $\lambda = \lambda_2^*$.

Similarly, for $\lambda = \lambda_2^*$, it can be defined $\{x^0_{\lambda_2^*},\ x^1_{\lambda_2^*} = f_{\lambda_2^*}(x^0_{\lambda_2^*})\ x^2_{\lambda_2^*} = f^2_{\lambda_2^*}(x^0_{\lambda_2^*})\ x^3_{\lambda_2^*} = f^3_{\lambda_2^*}(x^0_{\lambda_2^*})\}$ is the cycle of rationally indifferent periodic point and $P(x_{\lambda_2^*}) = \bigcup_{i=0}^{3} P_{\lambda_2^*,i}$ is the corresponding parabolic domain where

$$P_{x_{\lambda_2^*,i}} = \{z \in \mathbb{C} :\ f^{4n}_{\lambda_2^*}(z) \to x^i_{\lambda_2^*} \text{ as } n \to \infty\} \text{ for } i = 0,\ 1,\ 2,\ 3.$$

In this case, $P(x_{\lambda_2^*})$ is the Fatou set of $f_\lambda(z)$.

Figure 5.16 is the picture of the parabolic domain of $F(f_\lambda)$ for $\lambda = \lambda_2^*$. In Figure 5.16, the parabolic domain consists of black, blue, green and red regions. Here, it is a parabolic cycle of length 4. Each of the regions black, blue, green and red is one of the components of the parabolic cycle.

From Table 5.1, for $\lambda = -3.05,\ -3.10,\ f_\lambda(x)$ has no attracting or parabolic periodic points of minimal period one, two and four.

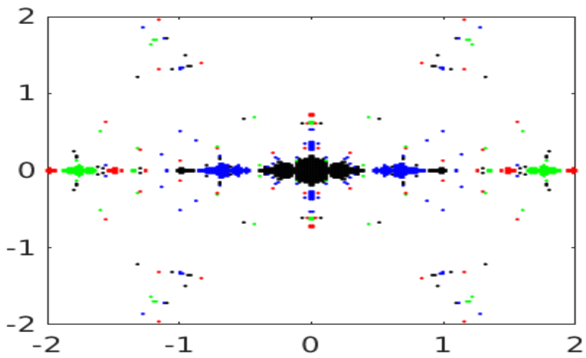

Figure 5.16: Parabolic domain in the Fatou set of f_λ for $\lambda = \lambda_2^*$.

From the above discussion, the following observation is listed in Table 5.2.

Table 5.2: Nature of the periodic cycle of f_λ

Nature of the periodic cycle of f_λ	
$\lambda \in (\lambda_1^*, \lambda_0^*)$	Attracting cycle of minimal period 2
$\lambda = \lambda_1^*$	Parabolic cycle of minimal period 2
$\lambda \in (\lambda_2^*, \lambda_1^*)$	Attracting cycle of minimal period 2^2
$\lambda = \lambda_2^*$	Parabolic cycle of minimal period 2^2

We conjecture that the following things hold when $\lambda \in [\lambda^*, \lambda_0^*)$.

Conjecture 5.2.1. *There exists a sequence $\{\lambda_k^*\}_{k=1}^{\infty}$ in $[\lambda^*, \lambda_0^*)$ such that the following holds.*

- *$\{\lambda_k^*\}$ is a strictly decreasing sequence converging to λ^*.*
- *If $\lambda \in (\lambda_k^*, \lambda_{k-1}^*)$, $f_\lambda(x)$ has an attracting periodic point of minimal period 2^k.*
- *If $\lambda = \lambda_k^*$, $f_\lambda(x)$ has a rationally indifferent periodic point of minimal period 2^k.*

If $\lambda \in (\lambda_k^*, \lambda_{k-1}^*)$, by Conjecture 5.2.1, let $\{a_{\lambda,k}^0,\ a_{\lambda,k}^1 = f_\lambda(a_{\lambda,k}^0), \ldots, a_{\lambda,k}^{2^k-1} = f_\lambda^{2^k-1}(a_{\lambda,k}^0)\}$ be the cycle of attracting periodic point of minimal period 2^k. Then the corresponding basin of attraction is $A(a_\lambda; k) = \bigcup_{i=0}^{2^k-1} U_{\lambda,i}$ where $U_{\lambda,i}$ are given by

$$U_{\lambda,i} = \{z \in \mathbb{C} :\ f_\lambda^{2^k n}(z) \to a_{\lambda,k}^i \text{ as } n \to \infty\} \text{ for } i = 0,\ 1, \ldots, 2^k - 1.$$

If $\lambda \in (\lambda_k^*, \lambda_{k-1}^*)$, it can be proved that $F(f_\lambda) = A(a_\lambda; k)$.

If $\lambda = \lambda_k^*$, by Conjecture 5.2.1, let $\{x_{\lambda_k^*}^0,\ x_{\lambda_k^*}^1 = f_{\lambda_k^*}(x_{\lambda_k^*}^0), \ldots, x_{\lambda_k^*}^{2^k-1} = f_{\lambda_k^*}^{2^k-1}(x_{\lambda_k^*}^0)\}$ be the cycle of rationally indifferent periodic point of minimal period 2^k. Then the corresponding parabolic domain is $P(x_{\lambda_k^*}) = \bigcup_{i=0}^{2^k-1} P_{\lambda_k^*,i}$ where $P_{\lambda_k^*,i}$ are given by

$$P_{\lambda_k^*,i} = \{z \in \mathbb{C} :\ f_{\lambda_k^*}^{2^k n}(z) \to x_{\lambda_k^*}^i \text{ as } n \to \infty\} \text{ for } i = 0,\ 1, \ldots, 2^k - 1.$$

If $\lambda = \lambda_k^*$, it can be proved that $F(f_\lambda) = P(x_{\lambda_k^*})$.

5.2.3 Dynamics of $f_\lambda(z)$ for $\lambda \in \mathbb{R} \setminus [\lambda^*, \tilde{\lambda}]$

The Dynamics of $f_\lambda(z)$ for $\lambda \in \mathbb{R} \setminus [\lambda^*, \tilde{\lambda}]$ is proved in the following theorem.

Theorem 5.2.4. *Let $f_\lambda \in \mathcal{F}$ and $\lambda \in \mathbb{R} \setminus [\lambda^*, \tilde{\lambda}]$. Then, the Julia set of f_λ is equal to the extended complex plane $\widehat{\mathbb{C}}$.*

Proof. f_λ has only two singular values $\lambda \pm 1$. If $\lambda \in \mathbb{R} \setminus [\lambda^*, \tilde{\lambda}]$, by Propositions 5.1.2 and 5.1.3, $f_\lambda^n(\lambda \pm 1) \to \infty$ as $n \to \infty$. Hence the forward orbit of all the singular values of f_λ tend to infinity under iteration of f_λ. Therefore, the Julia set $J(f_\lambda) = \widehat{\mathbb{C}}$. □

The Fatou set of f_λ is unbounded for $[\lambda^*, \tilde{\lambda}]$, since f_λ is a periodic function. From the Theorems 5.2.2, 5.2.3 and 5.2.4 and Conjecture 5.2.1, it is clear that a period doubling bifurcation occurs in the family $\mathcal{F}$.

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- M. Bera and M. G. P. Prasad. Fixed points and dynamics of two-parameter family of hyperbolic cosine like functions. *J. Math. Anal. Appl.*, 469(2), 1070-1079, 2019.
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- M. Bera and M. G. P. Prasad. Period doubling bifurcation in the dynamics of one-parameter family of translated hyperbolic cosine functions (communicated).