Abstract

The goal of this thesis is to study module of derivations of certain rings of invariants and of certain hypersurface rings. The thesis is divided into two parts.

In the first part we study module of derivarions of ring of invariants under the linear action of cyclic subgroups of $GL(n, \mathbf{k})$. The motivation to study this comes from a result by R. V. Gurjar and V. Wagh. They have proved [2] that for a finite cyclic subgroup G of $GL(2, \mathbb{C})$ the module of derivations of the ring of invariants is minimally generated by 4 elements.

A generalization of this result is discussed in the thesis. The main theorem of the first part of this thesis is as follows:

Theorem 1. *[1] Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and m be a natural number ≥ 2. Let $G \leq GL(m, \mathbf{k})$ be a finite cyclic diagonal subgroup which does not contain non-trivial pseudo-reflections. Let $R = \mathbf{k}[X_1, \ldots, X_m]^G$ be the ring of invariants under the linear action of G on $\mathbf{k}[X_1, \ldots, X_m]$. Then*

$$\mu(\mathrm{Der}\, R) \leq m(1 + n^{m-2})$$

This gives a bound on $\mu(\mathrm{Der}\, R)$. In case of $m = 2$ equality holds. This bound however, is not a strict one. We believe that this result can be improved further.

The method used in the proof of this theorem gives an algorithmic way to compute an explicit generating set of the the module of derivation.

In the second part of the thesis we consider $\mathbf{k}$ to be an algebraically closed field of characteristic zero and $f \in \mathbf{k}[X_1, \ldots, X_n]$. For the ring $S = \frac{\mathbf{k}[X_1,\ldots,X_n]}{\langle f+1 \rangle}$ we study the module of derivations of the ring S. We showed that finding $\mathrm{Der}\, S$ is equivalent to find $\mathrm{syz}_S \left(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \right)$. Then we prove the following theorem:

Theorem 2. *Let $f \in \mathbf{k}[X_1, \ldots, X_n]$ be such that*

$$f \in \left\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \right\rangle .$$

We consider $S = \mathbf{k}[X_1, \ldots, X_n]/\langle f + 1 \rangle$. Then $\mathrm{Der}\, S$ is generated by regular derivations Δ_{ij}, for $1 \leq i < j \leq n$, where

$$\Delta_{ij} = \frac{\partial f}{\partial X_j}\frac{\partial}{\partial X_i} - \frac{\partial f}{\partial X_i}\frac{\partial}{\partial X_j}.$$

In this case it can be proved that $\mathrm{Der}\, S$ is stably free projective module of rank $n-1$. Thus we can further conclude, using a result by Suslin [3], that $\mathrm{Der}\, S$ is free of rank $n - 1$. The generating set for $\mathrm{Der}\, S$ in Theorem 2 comprises of $\binom{n}{2}$ elements. Hence the size of this generating set can be further reduced.

We have found a minimal generating set for $\mathrm{Der}\, S$ when $n = 3$ and f is quasi-homogeneous. To prove this we first showed that, it suffices to compute minimal generating set for $\mathrm{Der}\, S$ when f is homogeneous. Hence we prove the following theorem.

Theorem 3. *For* $S = \mathbf{k}[X_1, X_2, X_3]/\langle f+1 \rangle$ *where* f *is a homogeneous polynomial of degree* m *in* $\mathbf{k}[X_1, X_2, X_3]$ *such that* ${X_3}^m$ *does not appear in* f *(without loss of generality), a minimal generating set for* $\operatorname{Der} S$ *can be given by*

$$\left\{-X_2\Delta_{23} - X_1\Delta_{13}, PX_3\Delta_{23} - \frac{Q}{X_2}X_3\Delta_{13} + \Delta_{12}\right\}$$

where P *and* Q *are such that* $f = PX_1 + Q$ *and* Q *belongs to* $\mathbf{k}[X_2, X_3]$*. (We note that* $X_2|Q$ *as* ${X_3}^m$ *does not appear in* Q*, hence* $\frac{Q}{X_2}$ *makes sense).*

References

[1] Arindam Dey and Vinay Wagh. On the module of derivations of certain rings of invariants. *J. Ramanujan Math. Soc.*, 33(2):149–158, 2018.

[2] R. V. Gurjar and Vinay Wagh. On the number of generators of the module of derivations and multiplicity of certain rings. *J. Algebra*, 319(5):2030–2049, 2008.

[3] A. A. Suslin. Stably free modules. *Mat. Sb. (N.S.)*, 102(144)(4):537–550, 632, 1977.