Uniqueness pairs for the Fourier transform on the Euclidean spaces and certain Lie groups

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DECLARATION

I do hereby declare that this thesis entitled **"Uniqueness pairs for the Fourier transform on the Euclidean spaces and certain Lie groups"** is a presentation of my original research work done under the supervision of **Dr. Rajesh Kumar Srivastava**, Assistant Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of doctor of philosophy. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

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CERTIFICATE

This is certified that the work contained in the thesis entitled **"Uniqueness pairs for the Fourier transform on the Euclidean spaces and certain Lie groups"** by **Mr. Somnath Ghosh** (Roll No. 156123010) has been carried out under my supervision. In my opinion, the thesis has reached the standard fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

June, 2021

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Dedicated to

my *Parents*

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Somnath Ghosh

Abstract

In general, the uncertainty principle states that a non-zero function and its Fourier transform cannot both be sharply localized. And depending on different localization assumptions, various types of results related to the uncertainty principle for Fourier transform appeared. In this thesis, localization is described through the support of the function and its Fourier transform, and we consider two variants of the uncertainty principle, namely, the Heisenberg uniqueness pair and Benedicks-Amrein-Berthier theorem.

Let Γ be a disjoint union of finitely many smooth curves in the plane $\mathbb{R}^2$. Suppose $X(\Gamma)$ be the space of all finite complex-valued Borel measures μ in $\mathbb{R}^2$, which are supported on Γ and absolutely continuous with respect to the arc length measure of Γ. For $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of μ can be defined by

$$\hat{\mu}(\xi, \eta) = \int_\Gamma e^{-i(x\cdot\xi + y\cdot\eta)} d\mu(x, y).$$

For a set Λ in $\mathbb{R}^2$, the pair (Γ, Λ) is called a Heisenberg uniqueness pair (HUP) for $X(\Gamma)$ if the only $\mu \in X(\Gamma)$ that satisfies $\hat{\mu}|_\Lambda = 0$ is $\mu = 0$.

The concept of Heisenberg uniqueness pair was first introduced by Hedenmalm and Montes-Rodríguez, and in the article [25], they have shown that the pair (hyperbola, some discrete set) is a HUP by constructing a weak* dense subspace of $L^\infty(\mathbb{R})$, as a dual problem. Consequently, $\hat{\mu}$ solves the one-dimensional Klein-Gordon equation.

First, we work out Heisenberg uniqueness pair in the the Euclidean spaces and find out HUPs corresponding to measures supported on parabola type curve and paraboloid. Further, for measures supported on a sphere, prove a necessary and sufficient when the union of spheres does not form HUP. Then we move to the Heisenberg group, and deal with symplectic, modified Fourier transform and spectral projections. For measures

supported on a sphere in $\mathbb{C}^n$, we derive that sphere and non-harmonic cone become determining sets corresponding to the above transforms.

Heisenberg uniqueness pair has significant similarity with the Benedicks-Amrein-Berthier theorem. In [8], Benedicks consider functions supported on a set of finite measure and proved that if $f \in L^1(\mathbb{R}^n)$, then both the sets $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ and $\{\xi \in \mathbb{R}^n : \hat{f}(\xi) \neq 0\}$ cannot have finite Lebesgue measure, unless $f = 0$. Concurrently, in the article [2], Amrein-Berthier reached to the same conclusion via the Hilbert space theory. The aforesaid fundamental result got further attention in the setups of general Lie groups.

In [33], Narayanan and Ratnakumar proved that if $f \in L^1(\mathbb{H}^n)$ is supported on $B \times \mathbb{R}$, where B is a compact subset of $\mathbb{C}^n$, and $\hat{f}(\lambda)$ has finite rank for each λ, then $f = 0$. Thereafter, Vemuri [54] replaced the compactness condition on B by finite measure. In [13], authors consider B as a rectangle in $\mathbb{R}^{2n}$ while proving a similar result on step two nilpotent Lie groups and a version of this result, with a strong assumption on rank, derived therein for the Heisenberg motion group.

We work on an example of step two nilpotent Lie group, known as the quaternion Heisenberg group $\mathbb{Q} \times \mathbb{R}^3$, where $\mathbb{Q}$ is the set of all quaternions. In this setup, we prove that if an integrable quaternion-valued function f supported on $A \times \mathbb{R}^3$, where $A \subset \mathbb{Q}$ is of finite measure, and its Fourier transform $\hat{f}(a)$ is a finite rank operator for each $a \in \mathbb{R}^3 \setminus \{0\}$, then f is zero. Finally, we define strong annihilating pair for the Weyl transform and obtain such pairs.

Afterward, we consider connected, simply connected step two nilpotent Lie groups. The Lie algebra $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. We prove the following result for the groups with MW-condition. For $A \subset \mathfrak{b}$ with finite measure, if an integrable function is supported on $A \times \mathfrak{z}$ and each of its Fourier transform is a finite rank operator, then the function must be zero. We conclude by obtaining a quantitative estimate of this result.

Finally, we consider the Heisenberg motion group $G = \mathbb{H}^n \rtimes U(n)$, where $\mathbb{H}^n$ is the Heisenberg group and $U(n)$ is the unitary group on $\mathbb{C}^n$. Then, due to the fact that $(G, U(n))$ is a Gelfand pair [9], Fourier transform of an integrable $U(n)$-bi-invariant function will necessarily be of rank one, irrespective of the support of the function. Thus, an exact analogue of the Heisenberg group result is not possible for the Heisenberg motion group. However, we reach out to a variant of such result by considering finite rank condition on the Weyl transform, which is non-zero for finitely many Fourier-Wigner pieces, together with the finite support condition.

Contents

Chapter 1

Introduction

In general, the uncertainty principle states that a non-zero function and its Fourier transform cannot both be sharply localized. And depending on different localization assumptions, various types of results related to the uncertainty principle for Fourier transform appeared. In this thesis, localization is described through the support of the function and its Fourier transform, and we consider two variants of the uncertainty principle, namely, the Heisenberg uniqueness pair and Benedicks-Amrein-Berthier theorem. For more details and history, we would like to refer to [19, 24, 28, 53] and references therein.

1.1 Heisenberg uniqueness pairs

Let Γ be a disjoint union of finitely many smooth curves in the plane $\mathbb{R}^2$. Suppose $X(\Gamma)$ be the space of all finite complex-valued Borel measures μ in $\mathbb{R}^2$, which are supported on Γ and absolutely continuous with respect to the arc length measure of Γ. For $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of μ can be defined by

$$\hat{\mu}(\xi, \eta) = \int_\Gamma e^{-i(x\cdot\xi + y\cdot\eta)} d\mu(x, y).$$

For a set Λ in $\mathbb{R}^2$, the pair (Γ, Λ) is called a Heisenberg uniqueness pair (HUP) for $X(\Gamma)$ if the only $\mu \in X(\Gamma)$ that satisfies $\hat{\mu}|_\Lambda = 0$ is $\mu = 0$. In general, the problem of HUP is about the determining property of the finite Borel measures, which are supported on some lower dimensional entities and whose Fourier transform too vanishes on lower dimensional entities. The following invariance properties about HUP play an important role in reducing the problem to a simpler form.

(i) Let $u_o, v_o \in \mathbb{R}^2$. Then the pair (Γ, Λ) is a HUP if and only if the pair $(\Gamma + u_o, \Lambda + v_o)$ is a HUP.

(ii) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear transform whose adjoint is denoted by T^*. Then (Γ, Λ) is a HUP if and only if $(T^{-1}\Gamma, T^*\Lambda)$ is a HUP.

The concept of Heisenberg uniqueness pair was first introduced by Hedenmalm and Montes-Rodríguez, and in the article [25], they have shown that the pair (hyperbola, some discrete set) is a HUP by constructing a weak* dense subspace of $L^\infty(\mathbb{R})$, as a dual problem. Consequently, $\hat{\mu}$ solves the one-dimensional Klein-Gordon equation. Further, a complete characterization of HUP corresponding to any two parallel lines have been given in [25].

In [45], Sjölin shown that parabola becomes HUP with a certain system of lines. It has been extended to the case of paraboloid and hyperplanes by Vieli [56]. Some HUPs corresponding to exponential parabola is considered in [21]. While we look for such pairs, we observed that odd function plays an essential role. In this thesis, we consider some curves which can be thought of as generalized parabola, and the latter one is a candidate for HUP with certain line or union of lines. To get this, we define an operator on $L^1(\mathbb{R})$, which eventually generalized the concept of odd functions. Further, we prove that paraboloid with sphere and non-harmonic cone form HUPs.

Lev [31] and Sjölin [44] have independently shown that the unit circle S^1 forms HUP with certain circles and system of lines. Further corresponding to sphere, Vieli [55] has shown that individual sphere form HUP. Whereas Srivastava [49] has derived that the sphere forms HUP with non-harmonic cone. Furthermore, Lev [31] determined the conditions, which ensure that the pair (circle, union of circles) will not form HUP. We extend this result to higher dimension and corresponding to sphere, prove a necessary and sufficient conditions for union of spheres does not form HUP.

As an extension of [31, 44, 45], Jaming and Kellay [30] established that the hyperbola, polygon, ellipse, cross and graph of the function $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$, form HUPs with certain union of intersecting lines, through the dynamical system approach. Thereafter, Gröchenig and Jaming [23] have worked out some of the HUPs correspondings to some quadratic surfaces. Although, by applying basic Fourier analysis, we can make some remarks regarding cross section, which forms HUP with certain system of two lines.

This topic further proceeds through the dynamical system and ergodic theory approach. A list of articles regarding measures supported on union of parallel lines, and HUP through ergodic theory is [6, 7, 11, 20, 21, 26, 27, 30].

1.2 Benedicks-Amrein-Berthier type theorem

By the Paley-Wiener theorem, Fourier transform of a compactly supported function can be extended to an entire function with exponential decay. Thus for a non-zero compactly supported function, Fourier transform can vanish only on a very thin set. In [8], Benedicks consider functions supported on a set of finite measure and proved that if $f \in L^1(\mathbb{R}^n)$, then both the sets $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ and $\{\xi \in \mathbb{R}^n : \hat{f}(\xi) \neq 0\}$ cannot have finite Lebesgue measure, unless $f = 0$. Concurrently, in the article [2], Amrein-Berthier reached to the same conclusion via the Hilbert space theory. The aforesaid fundamental result got further attention in the setups of general Lie groups, see [38, 43].

Let G be a locally compact group and $\hat{m}$ denotes the Plancherel measure on the unitary dual $\hat{G}$. Then G is said to satisfy qualitative uncertainty principle (QUP) if for each $f \in L^2(G)$ with $m\{x \in G : f(x) \neq 0\} < m(G)$ and

$$\int_{\hat{G}} \operatorname{rank}\hat{f}(\lambda)\, d\hat{m}(\lambda) < \infty, \tag{1.2.1}$$

implies $f = 0$. Arnal and Ludwig [5] proved QUP for certain unimodular groups of type I. A brief survey of QUP is presented in [19]. In case of the Heisenberg group $\mathbb{H}^n$, condition (1.2.1) of QUP implies $\hat{f}$ should be supported on a set of finite Plancherel measure together with $\operatorname{rank}\hat{f}(\lambda)$ is finite for almost all λ.

In [33], Narayanan and Ratnakumar proved that if $f \in L^1(\mathbb{H}^n)$ is supported on $B \times \mathbb{R}$, where B is a compact subset of $\mathbb{C}^n$, and $\hat{f}(\lambda)$ has finite rank for each λ, then $f = 0$. Thereafter, Vemuri [54] replaced the compactness condition on B by finite measure. In [13], authors consider B as a rectangle in $\mathbb{R}^{2n}$ while proving a similar result on step two nilpotent Lie groups and a version of this result, with a strong assumption on rank, derived therein for the Heisenberg motion group.

Heisenberg uniqueness pair has a link with the Benedicks-Amrein-Berthier theorem through annihilating pair. Let $A \subseteq \mathbb{R}$ and $\Sigma \subseteq \hat{\mathbb{R}}$ be measurable sets. Then the pair (A, Σ) is called *weak annihilating pair* if $\operatorname{supp} f \subseteq A$ and $\operatorname{supp} \hat{f} \subseteq \Sigma$, implies $f = 0$. The pair (A, Σ) is called *strong annihilating pair* if there exists a positive number $C = C(A, \Sigma)$ such that

$$\|f\|_2^2 \leq C \left(\|f\|^2_{L^2(A^c)} + \|\hat{f}\|^2_{L^2(\Sigma^c)} \right) \tag{1.2.2}$$

for every $f \in L^2(\mathbb{R})$. It is obvious that every strong annihilating pair is a weak annihilating pair. In [8], Benedicks proved that (A, Σ) is a weak annihilating pair when A and Σ both have finite measure. In [2], Amrein-Berthier proved that (A, Σ) is a strong annihilating pair under the identical assumption as in [8]. In this thesis, we further take-up the concept of strong annihilating pair in terms of group Fourier transform, which is an operator-valued function, for step two nilpotent Lie groups and Heisenberg motion group.

This thesis is organized as follows:

In Chapter 2, we consider Heisenberg uniqueness pair. First, we focus on the Euclidean spaces and find out HUPs corresponding to measures supported on parabola type curve, paraboloid and sphere as described above. Then we move to the Heisenberg group, and deal with symplectic, modified Fourier transform and spectral projections. For measures supported on a sphere in $\mathbb{C}^n$, we derive that sphere and non-harmonic cone become determining sets corresponding to the above transforms.

The next part of thesis deals with the Benedicks-Amrein-Berthier type theorem. In Chapter 3, we work on an example of step two nilpotent Lie group, known as the quaternion Heisenberg group $\mathbb{Q} \times \mathbb{R}^3$, where $\mathbb{Q}$ is the set of all quaternions. In this setup, we prove that if an integrable quaternion-valued function f supported on $A \times \mathbb{R}^3$, where $A \subset \mathbb{Q}$ is of finite measure, and its Fourier transform $\hat{f}(a)$ is a finite rank operator for each $a \in \mathbb{R}^3 \setminus \{0\}$, then f is zero. Finally, we define strong annihilating pair for the Weyl transform and obtain such pairs.

In chapter 4, we consider connected, simply connected step two nilpotent Lie groups. Then the Lie algebra $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. We prove the following result for the groups with MW-condition. For $A \subset \mathfrak{b}$ with finite measure, if an integrable function is supported on $A \times \mathfrak{z}$ and each of its Fourier transform is a finite rank operator, then the function must be zero. This chapter concludes by obtaining a quantitative estimate of this result.

In Chapter 5, we consider the Heisenberg motion group $G = \mathbb{H}^n \rtimes U(n)$, where $\mathbb{H}^n$ is the Heisenberg group and $U(n)$ is the unitary group on $\mathbb{C}^n$. Then, due to the fact that $(G, U(n))$ is a Gelfand pair [9], Fourier transform of an integrable $U(n)$-bi-invariant function will necessarily be of rank one, irrespective of the support of the function. Thus, an exact analogue of the Heisenberg group result, as described in Chapter 4, is not possible for the Heisenberg motion group. However, we reach out to a variant of such result by considering finite rank condition on the Weyl transform, which is non-zero for finitely many Fourier-Wigner pieces, together with the finite support condition.

Chapter 2

Heisenberg uniqueness pairs

In this chapter, we discuss Heisenberg uniqueness pair on the Euclidean spaces and the Heisenberg group. First, we consider measures supported on some curves in the plane, ellipsoid, sphere, and find out some uniqueness sets as well as non-uniqueness sets. After that, we discuss uniqueness sets on the Heisenberg group corresponding to the symplectic Fourier transform, some modified Fourier transform and spectral projections.

2.1 Spherical and bi-graded spherical harmonics

Let $\mathbb{Z}_+$ denote the set of all non-negative integers. For $l \in \mathbb{Z}_+$, let P_l denote the space of all homogeneous polynomials of degree l in n variables. Let $H_l = \{P \in P_l : \Delta P = 0\}$, where Δ is the standard Laplacian on $\mathbb{R}^n$. The elements of H_l are called solid spherical harmonics of degree l. It is worth mentioning that H_l is invariant under the natural action of $SO(n)$. A spherical harmonic of degree l is $P|_{S^{n-1}}$, where $P \in H_l$. By homogeneity, H_l can be identified with its restriction to S^{n-1}. We write d_l for the dimension of H_l and let $\{Y_{lj} : 1 \leq j \leq d_l\}$ be an orthonormal basis of H_l. Let $d\sigma$ be the normalized surface measure on S^{n-1}, then any two spherical harmonics of different degrees are orthogonal with respect to the usual inner product on $L^2(S^{n-1}, d\sigma)$. Further, the set $\{Y_{lj} : 1 \leq j \leq d_l, l \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{n-1}, d\sigma)$. For each fixed $\xi \in S^{n-1}$, define a linear functional on H_l by $Y \mapsto Y(\xi)$. Then there exists a unique spherical harmonic, say $Z_\xi^{(l)} \in H_l$ such that

$$Y(\xi) = \int_{S^{n-1}} Z_\xi^{(l)}(\eta) Y(\eta) d\sigma(\eta). \tag{2.1.1}$$

The spherical harmonic $Z_\xi^{(l)}$ is called the zonal harmonic of degree l with pole at ξ. For details, see [50], chapter IV.

For $1 \leq p \leq \infty$, let $f \in L^p(S^{n-1}, d\sigma)$. For each $l \in \mathbb{Z}_+$, define the l-th spherical harmonic projection of f by

$$\Pi_l f(\xi) = \int_{S^{n-1}} Z_\xi^{(l)}(\eta) f(\eta) d\sigma(\eta) \tag{2.1.2}$$

for all $\xi \in S^{n-1}$. The function $\Pi_l f$ is a spherical harmonic of degree l. For $\delta \geq 0$ and $l \in \mathbb{Z}_+$, write $A_l^m(\delta) = \binom{m-l+\delta}{\delta}\binom{m+\delta}{\delta}^{-1}$. Then we have $\lim\limits_{m\to\infty} A_l^m(\delta) = 1$. In addition, for $\delta > (n-2)/2$, the Fourier-Laplace series $\sum\limits_{l=0}^{\infty} \Pi_l f$ is δ-Cesaro summable to f. That is,

$$f = \lim_{m\to\infty} \sum_{l=0}^{m} A_l^m(\delta) \Pi_l f, \tag{2.1.3}$$

where the limit in the right-hand side of (2.1.3) exists in $L^p\left(S^{n-1}\right)$. Further, the convergence in (2.1.3) can be extended to $L^p\left(rS^{n-1}\right)$ for $r > 0$. For more details, see [46].

Since any continuous function F on $[-1, 1]$ generates a continuous function on $S^{n-1} \times S^{n-1}$ via $g(\xi, \eta) = F(\xi \cdot \eta)$, by fixing one variable it can be thought of as a function on S^{n-1}. Further, the formula (2.1.1) can be extended for any continuous function F on $[-1, 1]$. That is, for any $\xi \in S^{n-1}$

$$\int_{S^{n-1}} F(\xi \cdot \eta) Y(\eta) d\sigma(\eta) = C_l Y(\xi), \tag{2.1.4}$$

where the constant C_l is given by

$$C_l = \alpha_l \int_{-1}^{1} F(t) G_l^{\frac{n-2}{2}}(t)(1-t^2)^{\frac{n-3}{2}}\ dt$$

and G_l^{β} stands for the Gegenbauer polynomial of degree l and order β. This is known as the Funk-Hecke formula, and using this, the following identity can be obtained, see [3], page no. 458-464. Let $Y \in H_l$, then for $r > 0$ and $\zeta \in S^{n-1}$,

$$\int_{S^{n-1}} e^{-ir\zeta\cdot\xi} Y(\xi) d\sigma(\xi) = (2\pi)^{n/2} i^l \frac{J_{l+(n-2)/2}(r)}{r^{(n-2)/2}} Y(\zeta), \tag{2.1.5}$$

where J_k denotes the Bessel function of the first kind of order k.

Let $P_{p,q}$ denote the space of all bi-graded homogeneous polynomials on $\mathbb{C}^n$ of the form

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^{\alpha} \bar{z}^{\beta}, \tag{2.1.6}$$

where $p, q \in \mathbb{Z}_+$. Denote $H_{p,q} = \{P \in P_{p,q} : \Delta P = 0\}$. The restriction of bi-graded homogeneous harmonic polynomial to the unit sphere S^{2n-1} is called bi-graded spherical harmonic and the restriction space can be identified with $H_{p,q}$.

We need the following basic facts about the bi-graded spherical harmonics, (see [16, 22, 53] for details). Let $K = U(n)$ be the unitary group and $M = U(n-1)$. Then, $S^{2n-1} \cong K/M$ under the map $kM \to k.e_n$, where $k \in U(n)$ and $e_n = (0, \dots, 1) \in \mathbb{C}^n$. Let $\hat{K}_M$ denote the set of all equivalence classes of irreducible unitary representations of K, which have a non-zero M-fixed vector.

For a $\tau \in \hat{K}_M$, which is realized on V_τ, let $\{e_1, \dots, e_{d(\tau)}\}$ be an orthonormal basis of V_τ with e_1 as the M-fixed vector. Let $t_{ij}^{\tau}(k) = \langle e_i, \tau(k) e_j \rangle$, $k \in K$. By the Peter-Weyl theorem, the set $\{\sqrt{d(\tau)} t_{j1}^{\tau} : 1 \leq j \leq d(\tau), \tau \in \hat{K}_M\}$ forms an orthonormal basis for $L^2(K/M)$, (see [53], page 14). Define $Y_j^{\tau}(\omega) = \sqrt{d(\tau)} t_{j1}^{\tau}(k)$, where $\omega = k.e_n \in S^{2n-1}$,

$k \in K$. Then $\{Y_j^\tau : 1 \leq j \leq d(\tau), \tau \in \hat{K}_M,\}$ becomes an orthonormal basis for $L^2(S^{2n-1})$.

Since $H_{p,q}$ is K-invariant, let $\pi_{p,q}$ be the corresponding representation of K on $H_{p,q}$. Then $\hat{K}_M$ can be identified with $\{\pi_{p,q} : p, q \in \mathbb{Z}_+\}$. See [40], p.253, for more details. Thus, a bi-graded spherical harmonic on S^{2n-1} can be defined by $Y_j^{p,q}(\omega) = \sqrt{d(p,q)}t_{j1}^{p,q}(k)$, and hence $\{Y_j^{p,q} : 1 \leq j \leq d(p,q)$ and $p, q \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{2n-1})$. For $f \in L^2(S^{2n-1})$, the expression

$$\Pi_{p,q}(f)(\omega) = \sum_{j=1}^{d(p,q)} a_j^{p,q} Y_j^{p,q}(\omega), \tag{2.1.7}$$

where $a_j^{p,q} = \langle f, Y_j^{p,q} \rangle$, is called $(p,q)^{th}$ spherical harmonic projection of f.

Next, we shall figure out the relation between spherical and bi-graded spherical harmonics on S^{2n-1}. For instance, let the space H_l consists of spherical harmonics of degree l on S^{2n-1}, where $l \in \mathbb{Z}_+$. Then H_l is $SO(2n)$-invariant. Hence, H_l is $U(n)$-invariant as well, and under this action of $U(n)$, the space H_l breaks up into an orthogonal direct sum of $H_{p,q}$'s where $p + q = l$.

Note that we adopted the same notation H_l for the spherical harmonics of degree l, on S^{n-1} as well as on S^{2n-1}, since the dimension of the space will be clear from the context.

Lemma 2.1.1. [40] *Let $\omega \in S^{2n-1}$ and $Y \in H_l$. Then there exist $Y_{p,q} \in H_{p,q}$ with $p + q = l$, such that*

$$Y(\omega) = \sum_{p+q=l} Y_{p,q}(\omega). \tag{2.1.8}$$

Definition 2.1.2. *A set $\mathcal{C} \subset \mathbb{C}^n$ $(n \geq 2)$ that satisfies the scaling condition $\lambda\mathcal{C} \subseteq \mathcal{C}$ for all $\lambda \in \mathbb{C}$, is called a complex cone, whereas a set $\mathcal{K}$ in $\mathbb{R}^d$ $(d \geq 2)$ which satisfies $\kappa\mathcal{K} \subseteq \mathcal{K}$ for all $\kappa \in \mathbb{R}$ is called a real cone.*

We say that a cone is *non-harmonic* if it is not contained in the zero set of any homogeneous harmonic polynomial. An example of a non-harmonic complex cone was produced by Srivastava (see [48]). The zero set of the polynomial $H(z) = az_1\bar{z}_2 + |z|^2$, where $a \neq 0$ and $z \in \mathbb{C}^n$, is a complex cone which is not contained in the zero set of any bi-graded homogeneous harmonic polynomial.

An example of a non-harmonic real cone was given by Armitage, (see [4]). Let $0 < a < 1$. Then $K_a = \left\{x \in \mathbb{R}^d : \ |x_1|^2 = a^2|x|^2\right\}$ is a non-harmonic cone if and only if $D^m G_k^{\frac{d-2}{2}}(a) \neq 0$, for all $0 \leq m \leq k-2$, where D^m stands for the mth derivative on $\mathbb{R}$ and G_l^{β} is the Gegenbauer polynomial of degree l and of order β.

In view of Lemma 2.1.1, it is easy to prove the following result, which is required to prove our result.

Lemma 2.1.3. *Let $Y \in H_l$ be given as in* (2.1.8). *Suppose $\mathcal{C}$ be a complex cone, and denote $\widetilde{\mathcal{C}} = \left\{\frac{z}{|z|} : \ z \in \mathcal{C}, \ z \neq 0\right\}$. Then $Y = 0$ on $\widetilde{\mathcal{C}}$ if and only if $Y_{p,q} = 0$ on $\widetilde{\mathcal{C}}$ for all $p, q \in \mathbb{Z}_+$ such that $p + q = l$.*

Proof. Let $\omega \in \widetilde{\mathcal{C}}$ and $Y(\omega) = 0$. Since the cone $\mathcal{C}$ is closed under complex scaling, by replacing ω with $e^{i\theta}\omega$ in (2.1.8), we get

$$\sum_{p+q=l} e^{i(p-q)\theta} Y_{p,q}(\omega) = 0.$$

Thus, the proof of the required lemma will be followed by the fact that $\{e^{is\theta} : \ s \in \mathbb{Z}\}$ is an orthogonal set in $L^2(S^1)$. $\square$

We would like to mention that the proof of our main result is carried out by restricting the cone to the unit sphere and decomposing the integral on the sphere into averages over geodesic spheres. This is possible because the cone is closed under scaling.

For $\omega \in S^{2n-1}$ and $t \in (-1,1)$, the set $S^t_\omega = \{\nu \in S^{2n-1} : \omega \cdot \nu = t\}$ is a geodesic sphere on S^{2n-1} with pole at ω. Let f be an integrable function on S^{2n-1}. In view of Fubini's Theorem, we can define the geodesic spherical means of the function f by

$$\tilde{f}(\omega, t) = \int_{S^t_\omega} f d\sigma_{2n-2},$$

where σ_{2n-2} is the normalized surface measure on the geodesic sphere S^t_ω.

The following lemma percolates the geodesic mean vanishing conditions of $f \in L^1(S^{2n-1})$ to a vanishing condition of each spherical harmonic projection of f. For the class of continuous functions $C(S^{2n-1})$, this lemma was proved in [1]. As a consequence of the Cesaro summation formula (2.1.3), in [49], the result has been extended for functions in $L^1(S^{2n-1})$.

Lemma 2.1.4. [49] *Let $f \in L^1(S^{2n-1})$. Then $\tilde{f}(\omega, t) = 0$ for all $t \in (-1,1)$ if and only if $\Pi_l f(\omega) = 0$ for all $l \in \mathbb{Z}_+$.*

Notice that as a corollary to Lemma 2.1.4, it can be shown that if $\tilde{f}(\omega, t) = 0$ for all $(\omega, t) \in \widetilde{\mathcal{C}} \times (-1,1)$, then $f = 0$ on S^{2n-1} as long as $\mathcal{C}$ is non-harmonic.

2.2 HUP on the Euclidean spaces

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $\gamma(t) - t^2 \geq 0$ and γ' is a strictly increasing function that increases to ∞ as $t \to \infty$ and decreases to $-\infty$ as $t \to -\infty$. We shall consider HUP for measures supported on the set $\Gamma = \{(t, \gamma(t)) : t \in \mathbb{R}\}$. Let $\mu \in X(\Gamma)$, then there exists $g \in L^1(\mathbb{R}, \sqrt{1 + \gamma'(t)^2})$ such that $d\mu = g(t)dt$. If we write $f(t) = \sqrt{1 + \gamma'(t)^2} g(t)$, then the finiteness of μ implies that $f \in L^1(\mathbb{R})$ and Fourier transform is defined by

$$\hat{\mu}(x,y)=\int_{\mathbb{R}} e^{-i\pi(xt+y\gamma(t))}f(t)dt. \tag{2.2.1}$$

First, we discuss the following map. For $c\in\mathbb{R}\setminus\{0\}$, define $h_c:\mathbb{R}\to\mathbb{R}$ by

$$h_c(t)=\left(t+\frac{1}{2c}\right)^2+\gamma(t)-t^2. \tag{2.2.2}$$

Since $\gamma(t)-t^2\geq 0$, the range of h_c is $[a^2,\infty)$ for some $a\geq 0$. The second condition that γ' is strictly increasing and $\gamma'(t)\to\pm\infty$ as $t\to\pm\infty$, implies that there exists $\alpha\in\mathbb{R}$ such that $\gamma'(\alpha)=0$. Therefore $h_c(\alpha)=a^2$, and h_c is strictly increasing in $[\alpha,\infty)$ and strictly decreasing in $(-\infty,\alpha]$. We denote $\tilde{h}_c:=h_c|_{[\alpha,\infty)}$ and $\hbar_c:=h_c|_{(-\infty,\alpha]}$. Note that both the functions $\tilde{h}_c$ and $\hbar_c$ are invertible having range $[a^2,\infty)$. Now, consider the reflection type map $\rho_c:[\alpha,\infty)\to(-\infty,\alpha]$ such that $\hbar_c\circ\rho_c=\tilde{h}_c$. Thus, ρ_c is diffeomorphism and $\rho_c(\alpha)=\alpha$. To describe the dynamic of the function h_c, we need to define an operator $T_c:L^1(\mathbb{R})\to L^1([a,\infty))$ by

$$T_c[f](u)=\left[f\circ\tilde{h}_c^{-1}(u^2)-f\circ\rho_c\circ\tilde{h}_c^{-1}(u^2)\rho_c'(\tilde{h}_c^{-1}(u^2))\right]\frac{2u}{\tilde{h}_c'(\tilde{h}_c^{-1}(u^2))}. \tag{2.2.3}$$

Let $\mathcal{N}(T_c)$ denotes the null space of T_c. Then the following result holds.

Theorem 2.2.1. *Let $\Gamma=\{(t,\gamma(t)):\ t\in\mathbb{R}\}$, where γ is defined as above.*

(a). Let Λ_x be a straight line parallel to the X-axis, then (Γ,Λ_x) is a Heisenberg uniqueness pair. Further, for $c\neq 0$, let Λ_c be a straight line with slope c. Then (Γ,Λ_c) is a Heisenberg uniqueness pair if and only if $\mathcal{N}(T_c)=\{0\}$.

(b). Let $L_j,\ j=1,2$ be two parallel lines which are not parallel to either of the axes and $E_j\subset L_j$. If each E_j has positive one dimensional Lebesgue measure, then $(\Gamma,E_1\cup E_2)$ is a Heisenberg uniqueness pair.

Remark 2.2.2. (a). *Let Λ be a straight line not parallel to the Y-axis. Assume that $\mathcal{N}(T_c) \neq \{0\}$ for each $c \neq 0$. Then, in view of Theorem* 2.2.1(a), (Γ, Λ) *is a Heisenberg uniqueness pair if and only if Λ is parallel to the x-axis.*

(b). *We would like to mention that, when $h_c(t) = t^2$ or more generally $h_c(t) = \eta(t)$, where $\eta : \mathbb{R} \to \mathbb{R}_+$ be a differentiable even function such that $\eta|_{[0,\infty)}$ is invertible, then $\rho_c(t) = -t$. In this particular case, $\mathcal{N}(T_c)$ is the set of all odd integrable functions.*

(c). *In the sense of dynamical system and ergodic theory, the operator T_c defined in* (2.2.3), *can be thought of as a* transfer-type operator. *For instance, Perron-Frobenius operator is used to study HUP* (see [11, 25]). *For further details on the connection between dynamical system and HUP, we refer* [26,27,30]. *The null space of the operator T_c will be crucial for our main result about HUP.*

In order to prove Theorem 2.2.1, we need the following results. First, we state a result which can be found in Havin and Jöricke [24], p. 36.

Lemma 2.2.3. [24] *Let $\varphi \in L^1[0,\infty)$. If $\int\limits_{\mathbb{R}} \log|\hat{\varphi}| \dfrac{dx}{1+x^2} = -\infty$, then $\varphi = 0$.*

Next, we state the following form of the Radon-Nikodym derivative Theorem.

Proposition 2.2.4. *Let ν be a σ-finite signed measure which is absolutely continuous with respect to a σ-finite measure μ on the measure space $(X, \mathcal{M})$. If $g \in L^1(\nu)$, then $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and $\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$.*

As a consequence of Lemma 2.2.3 and Proposition 2.2.4, we prove the following result. Let $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}$.

Lemma 2.2.5. *Let $f \in L^1(\mathbb{R})$. Suppose $E \subset \mathbb{R}$ and $|E| > 0$. Then*

$$\int_{\mathbb{R}} e^{-i\pi c x h_c(t)} f(t) dt = 0 \tag{2.2.4}$$

for all $x \in E$ if and only if $f \in \mathcal{N}(T_c)$.

Proof. The left-hand side of (2.2.4) can be written as

$$\begin{aligned} I &= \int_{\alpha}^{\infty} e^{-i\pi cx\tilde{h}_c(t)} f(t)dt + \int_{-\infty}^{\alpha} e^{-i\pi cx\tilde{h}_c(t)} f(t)dt \\ &= \int_{\alpha}^{\infty} e^{-i\pi cx\tilde{h}_c(t)} f(t)dt - \int_{\alpha}^{\infty} e^{-i\pi cx\tilde{h}_c(s)} f(\rho_c(s)) \cdot \rho_c'(s)ds, \end{aligned}$$

by using the change of variables $s = \rho_c^{-1}(t)$. Further applying the change of variables $\tilde{h}_c(t) = u^2$ we get

$$\begin{aligned} I &= \int_{a}^{\infty} e^{-i\pi cxu^2} \left[f \circ \tilde{h}_c^{-1}(u^2) - f \circ \rho_c \circ \tilde{h}_c^{-1}(u^2) \cdot \rho_c'(\tilde{h}_c^{-1}(u^2)) \right] \frac{2u}{\tilde{h}_c'(\tilde{h}_c^{-1}(u^2))} du \\ &= \int_{a}^{\infty} e^{-i\pi cxu^2} T_c[f](u)du. \end{aligned}$$

In view of Proposition 2.2.4, the function $T_c[f] \in L^1([a,\infty))$, and by the change of variables $u^2 = v$, we have

$$I = \int_{a^2}^{\infty} e^{-i\pi cxv} T_c[f](\sqrt{v}) \frac{dv}{2\sqrt{v}}. \tag{2.2.5}$$

Let $\varphi(v) = T_c[f](\sqrt{v})/2\sqrt{v}\ \chi_{(a^2,\infty)}(v)$. Then $\varphi \in L^1(\mathbb{R})$ and from (2.2.5) we obtain $I = \hat{\varphi}(cx) = 0$ for all $x \in E$. That is, $\hat{\varphi}$ vanishes on the set cE of positive measure. Thus, by Lemma 2.2.3 we conclude that $\varphi = 0$ a.e. Hence, it follows that $T_c[f] = 0$ a.e. on $[a,\infty)$. Conversely, if $T_c[f] = 0$, then (2.2.4) trivially holds. □

From Remark 2.2.2(b) and Lemma 2.2.5, we have the following result.

Corollary 2.2.6. *Let $f \in L^1(\mathbb{R})$ and $\eta : \mathbb{R} \to \mathbb{R}_+$ be a differentiable even function such that $\eta|_{[0,\infty)}$ is invertible. Suppose $E \subset \mathbb{R}$ and $|E| > 0$. Then*

$$\int_{\mathbb{R}} e^{-i\pi x\eta(t)} f(t)dt = 0 \tag{2.2.6}$$

for all $x \in E$ if and only if f is an odd function.

Proposition 2.2.7. *Let $\mu \in X(\Gamma)$ and $f \in L^1(\mathbb{R})$, as appeared in (2.2.1). If $E \subset \mathbb{R}$ with $|E| > 0$, then for $c, d \in \mathbb{R} \setminus \{0\}$ following holds.*

(a) $\widehat{\mu}(x, cx) = 0$ for all $x \in E$ if and only if $f \in \mathcal{N}(T_c)$.

(b) $\widehat{\mu}(x, cx+d) = 0$ for all $x \in E$ if and only if $\chi \in \mathcal{N}(T_c)$, where $\chi(t) = e^{-i\pi d\gamma(t)} f(t)$.

Proof. (a). From (2.2.1) we can express

$$\hat{\mu}(x, cx) = \int_{\mathbb{R}} e^{-i\pi x(t+c\gamma(t))} f(t)dt = e^{i\pi x/4c} \int_{\mathbb{R}} e^{-i\pi cxh_c(t)} f(t)dt.$$

By Lemma 2.2.5, $T_c[f] = 0$ if and only if $\widehat{\mu}(x, cx) = 0$ for all $x \in E$.

(b). By a simple computation, we get

$$\hat{\mu}(x, cx+d) = \int_{\mathbb{R}} e^{-i\pi x(t+c\gamma(t))} \chi(t)dt = e^{i\pi x/4c} \int_{\mathbb{R}} e^{-i\pi cxh_c(t)} \chi(t)dt.$$

As similar to the above case, $T_c[\chi] = 0$ if and only if $\widehat{\mu}(x, cx+d) = 0$ for all $x \in E$. □

Proof of Theorem 2.2.1. (a). In view of the invariance property, we can assume that Λ_x is the x-axis. Recall from (2.2.1) that $\hat{\mu}$ satisfies

$$\hat{\mu}(x, y) = \int_{\mathbb{R}} e^{-i\pi(xt+y\gamma(t))} f(t)dt.$$

Hence $\hat{\mu}|_{\Lambda_x} = 0$ implies that $\hat{f}(x) = 0$ for all $x \in \mathbb{R}$. Thus, we conclude that $\mu = 0$.

On the other hand, assume that Λ_c be of the form $y = cx + d$, where $c \neq 0$ and $d \in \mathbb{R}$. Then by Proposition 2.2.7, it follows that $\hat{\mu} = 0$ on Λ_c if and only if $\chi \in \mathcal{N}(T_c)$. Since, $\chi(t) = e^{-i\pi d\gamma(t)} f(t) = 0$ if and only if $f(t) = 0$, it completes the proof.

(b). Let $L_1 = \{(x, cx) : \ x \in \mathbb{R}\}$ and $L_2 = \{(x, cx + d) : \ x \in \mathbb{R}\}$, where $c, d \neq 0$. Since $\hat{\mu}|_{L_j} = 0; \ j = 1, 2$, by Proposition 2.2.7 it follows that $T_c[f] = 0$ and $T_c[\chi] = 0$. That is,

$$f \circ \tilde{h}_c^{-1}(u^2) = f \circ \rho_c \circ \tilde{h}_c^{-1}(u^2)\, \rho_c'(\tilde{h}_c^{-1}(u^2)),$$

$$e^{i\pi d\gamma\left(\tilde{h}_c^{-1}(u^2)\right)} f \circ \tilde{h}_c^{-1}(u^2) = e^{i\pi d\gamma\left(\rho_c \circ \tilde{h}_c^{-1}(u^2)\right)} f \circ \rho_c \circ \tilde{h}_c^{-1}(u^2)\, \rho_c'(\tilde{h}_c^{-1}(u^2)).$$

Thus, we have

$$\left[e^{i\pi d\left\{\gamma\left(\tilde{h}_c^{-1}(u^2)\right) - \gamma\left(\rho_c \circ \tilde{h}_c^{-1}(u^2)\right)\right\}} - 1\right] f \circ \tilde{h}_c^{-1}(u^2) = 0. \tag{2.2.7}$$

Now, we prove that $e^{i\pi d\{\gamma(t) - \gamma(\rho_c(t))\}} \neq 1$ almost everywhere, where $t = \tilde{h}_c^{-1}(u^2)$. Since $h_c(t) = h_c(\rho_c(t))$, then from (2.2.2) we get

$$\frac{t}{c} + \gamma(t) = \frac{\rho_c(t)}{c} + \gamma(\rho_c(t)). \tag{2.2.8}$$

If $\gamma(t) - \gamma(\rho_c(t)) = \frac{k}{d}$, for some $k \in \mathbb{Z}$, then by (2.2.8) we have $\rho_c(t) = t + \frac{c}{d}k$. Since γ' is a strictly increasing function, for each $k \neq 0$, $\gamma(t) - \gamma(t + \frac{c}{d}k)$ is strictly monotone in t. Therefore for each $k \neq 0$, $\gamma(t) - \gamma(t + \frac{c}{d}k) = \frac{k}{d}$ is possible for at most one point and for $k = 0$, $\rho_c(t) = t$ holds only at $t = \alpha$.

Thus from (2.2.7), $f \circ \tilde{h}_c^{-1}(u^2) = 0$ a.a. $u \geq a$, that is, $f = 0$ a.e. on $[\alpha, \infty)$. As $T_c[f] = 0$ and $\rho_c' \neq 0$ a.e., it follows that $f = 0$ a.e. Thus, the pair $(\Gamma, L_1 \cup L_2)$ is a Heisenberg uniqueness pair. □

Remark 2.2.8. (a). *If we consider* $h_c(t) = \gamma(t)$, *then we can define an operator* T^γ, *similar to* T_c *as in* (2.2.3). *Hence, if* $\mathcal{N}(T^\gamma) \neq \{0\}$ *and* Λ *be the* Y*-axis, then* (Γ, Λ) *is not a HUP. Proof of this fact follows from* (2.2.1) *and Lemma 2.2.5.*

(b). *Let* $\gamma(t) = \eta(t)$, *where* $\eta : \mathbb{R} \to \mathbb{R}_+$ *be a differentiable even function such that* $\eta|_{[0,\infty)}$ *is invertible. If* Λ *be the* Y*-axis, by Corollary 2.2.6,* (Γ, Λ) *is not a HUP.*

2.2.1 Some remarks about cross

Let $\Gamma = (\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})$ is a cross. If $\mu \in X(\Gamma)$, then there exists $f, g \in L^1(\mathbb{R})$ such that

$$d\mu(x,y) = f(x)dxd\delta_0(y) + g(y)dyd\delta_0(x), \tag{2.2.9}$$

where $d\delta_u$ denotes the unit point mass at the point u. Hence $\hat{\mu}(\xi,\eta) = \hat{f}(\xi) + \hat{g}(\eta)$ for $(\xi,\eta)\in\mathbb{R}^2$.

Proposition 2.2.9. *Suppose* $\Gamma = (\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})$ *is a cross and let* $\Lambda\subset\mathbb{R}^2$ *be any of the following sets:*

(a) $\Lambda = (\mathbb{R}\times\{y_0\})\cup(\{x_0\}\times\mathbb{R})$ *for some* $x_0, y_0\in\mathbb{R}$.

(b) $\Lambda = L_1\cup L_2$, *where* L_1, L_2 *are two straight lines which are parallel to each other but not parallel to any axis.*

(c) $\Lambda = L_1\cup L_2$, *where* L_1, L_2 *are two straight lines perpendicular to each other and the angle between* L_1 *and* x*-axis is not* $\frac{\pi}{4}$ *or* $-\frac{\pi}{4}$.

Then (Γ,Λ) *is a HUP.*

Proof. (a) Follows by the Riemann-Lebesgue lemma.

(b) Without loss of generality, assume that $\Lambda = \{(x,y)\in\mathbb{R}^2 : y = mx; m\neq 0\}\cup\{(x,y)\in\mathbb{R}^2 : y = mx+c; m,c\neq 0\}$. By hypothesis $\hat{\mu}|_\Lambda = 0$ implies $\hat{f}(\xi)+\hat{g}(m\xi) = 0$ and $\hat{f}(\xi)+\hat{h}(m\xi) = 0$ for all $\xi\in\mathbb{R}$, where $h(y) = g(y)e^{-i\pi cy}$. Thus we have $\hat{g}(m\xi) = \hat{h}(m\xi)$, that is, $(1-e^{-i\pi cy})g(y) = 0$. Since for a non-zero constant c, $e^{-i\pi cy} = 1$ is possible on a set of measure zero, we get $g = 0$ a.e. Therefore, $f = 0$ a.e. and hence $\mu = 0$.

(c) Let $\Lambda = \{(x,y) \in \mathbb{R}^2 : y = mx; m \neq 0,1\} \cup \{(x,y) \in \mathbb{R}^2 : y = -\frac{1}{m}x; m \neq 0,1\}$. By hypothesis $\hat{\mu}|_\Lambda = 0$ implies $\hat{f}(\xi) + \hat{g}(m\xi) = 0$ and $\hat{f}(\xi) + \hat{g}(-\frac{1}{m}\xi) = 0$ for all $\xi \in \mathbb{R}$. Therefore $\hat{g}(\xi) = \hat{g}(-\frac{1}{m^2}\xi) = \cdots = \hat{g}((-1)^k \frac{1}{m^{2k}}\xi)$ for all $\xi \in \mathbb{R}$. Since $\hat{g}$ is a continuous function so it is constant function. Hence by Riemann-Lebesgue lemma $\hat{g}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Thus $g = 0$ a.e. and $f = 0$ a.e. Hence $\mu = 0$.

□

Remark 2.2.10. *Suppose* $\Lambda = \{(x,y) \in \mathbb{R}^2 : y = mx; m \neq 0\} \cup \{(x,y) \in \mathbb{R}^2 : y = -mx; m \neq 0\}$. *Choosing* $f = \chi_{[-1,1]}$ *and* $g = -m.\chi_{[-\frac{1}{m},\frac{1}{m}]}$, *we can conclude that* (Γ, Λ) *is not a HUP.*

Suppose $\Gamma = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ is a cross. Consider a subset $\mathbf{C} \subset X(\Gamma)$ such that for any $\mu \in \mathbf{C}$ there exists two function $f, g \in L^1(\mathbb{R})$, atleast one is odd, satisfying (2.2.9). Then the pair (Γ, Λ) is called a $\mathbf{C}$-HUP if any measure $\mu \in \mathbf{C}$ satisfying $\hat{\mu}|_\Lambda = 0$ implies $\mu = 0$ identically. Then we have the following remark.

Remark 2.2.11. *Suppose* $\Lambda = \{(x,y) \in \mathbb{R}^2 : y = mx; m \neq 0\} \cup \{(x,y) \in \mathbb{R}^2 : y = -mx; m \neq 0\}$. *Then* (Γ, Λ) *is a* $\mathbf{C}$*-HUP.*

2.2.2 Paraboloid and sphere

Let Γ be the paraboloid $x_{n+1} = x_1^2 + \cdots + x_n^2$ in $\mathbb{R}^{n+1}$ and $\mu \in X(\Gamma)$. Then there exists $g \in L^1(\mathbb{R}^n)$ such that $d\mu = g(u)\sqrt{1 + 4||u||^2}du$. We write $f(u) = g(u)\sqrt{1 + 4||u||^2}$, then the Fourier transform of μ can be expressed as

$$\hat{\mu}(x_1, \ldots, x_{n+1}) = \int_{\mathbb{R}^n} e^{-i(x, x_{n+1})\cdot(u, ||u||^2)} f(u) du \qquad (2.2.10)$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x_{n+1} \in \mathbb{R}$.

Proposition 2.2.12. *Suppose Γ be a paraboloid in $\mathbb{R}^{n+1}$. Let E be a set of positive measure in $\mathbb{R}$ and S^{n-1} be the unit sphere in $\mathbb{R}^n$, then the following holds.*

(a). If $\Lambda = S^{n-1} \times E$, then (Γ, Λ) is a Heisenberg uniqueness pair.

(b). If $\Lambda = \mathcal{K} \times E$, where $\mathcal{K}$ be a cone in $\mathbb{R}^n$, then (Γ, Λ) is a Heisenberg uniqueness pair if and only if $\mathcal{K}$ is **non-harmonic**.

In order to prove Proposition 2.2.12, we need the following result, which follows from Lemma 2.2.3.

Lemma 2.2.13. [44] *Let $\varphi \in L^1((0,\infty))$ and E be a measurable subset of $\mathbb{R}$ with $|E| > 0$. If*

$$\int_0^\infty e^{-ixt^2}\varphi(t)dt = 0$$

for all $x \in E$, then $\varphi = 0$.

Proof of proposition 2.2.12. (a). In view of (2.2.10), $\hat{\mu}$ vanishes on Λ implies

$$\int_{\mathbb{R}^n} e^{-i(x,x_{n+1})\cdot(u,||u||^2)} f(u)du = 0,$$

for all $(x, x_{n+1}) \in S^{n-1} \times E$. Next, by converting the above integral into the polar coordinates, we get

$$\int_0^\infty e^{-ix_{n+1}r^2} \int_{S_r^{n-1}} e^{-ix\cdot\eta} f(\eta)d\sigma_r(\eta)dr = 0.$$

It follows from Lemma 2.2.13 that for each $r > 0$,

$$\int_{S_r^{n-1}} e^{-ix\cdot\eta} f(\eta)d\sigma_r(\eta) = 0 \tag{2.2.11}$$

for all $x \in S^{n-1}$. If we write $f_r(\xi) = f(r\xi)$ for $\xi \in S^{n-1}$, then (2.2.11) reduce to

$$\int_{S^{n-1}} e^{-iy\cdot\xi} f_r(\xi) d\sigma(\xi) = 0$$

for all $y \in S_r^{n-1}$ and $r > 0$. In view of ([55], Proposition 1.2), we have $f_r(\xi) = 0$ for *a.e.* $\xi \in S^{n-1}$ if and only if $J_{d+(n-2)/2}(r) \neq 0$ for all $d \in \mathbb{Z}_+$.

Thus, we infer that $f(\eta) = 0$ for *a.e.* $\eta \in S_r^{n-1}$ if and only if $J_{d+(n-2)/2}(r) \neq 0$ for all $d \in \mathbb{Z}_+$. Since the set $\{r > 0 : J_{d+(n-2)/2}(r) = 0, \text{ for some } d \in \mathbb{Z}_+\}$ is countable, we conclude that $f = 0$ *a.e.*, and hence $\mu = 0$.

(b). Let $\mathcal{K}$ be a non-harmonic cone. In view of (2.2.11), we have

$$\int_{S_r^{n-1}} e^{-ix\cdot\eta} f(\eta) d\sigma_r(\eta) = 0$$

for all $x \in \mathcal{K}$ and $r > 0$. Since $\mathcal{K}$ is non-harmonic, by ([49], Theorem 3.1),

$$\int_{S^{n-1}} e^{-iy\cdot\xi} f_r(\xi) d\sigma(\xi) = 0$$

for all $y \in \mathcal{K}$ and $r > 0$, where $f_r(\xi) = f(r\xi)$ for all $\xi \in S^{n-1}$, implies $f_r = 0$ a.e for each $r > 0$. Hence $f = 0$ *a.e.* that is, (Γ, Λ) is a Heisenberg uniqueness pair.

Conversely, assume that $\mathcal{K}$ is contained in the zero set of a homogeneous harmonic polynomial P of degree l. Let $Y = P|_{S^{n-1}} \in H_l$. Define a function f on $\mathbb{R}^n$ by $f(y) = e^{-r^2} Y(\xi)$, where $y = r\xi, r > 0$ and $\xi \in S^{n-1}$. Then $f \in L^1(\mathbb{R}^n)$. Thus, we can construct a finite complex Borel measure μ in $\mathbb{R}^n$ by $d\mu = f(u)du$. Hence from the identity

(2.1.5), for each $r > 0$ and $x \in \mathcal{K}$, we have

$$\int_{S_r^{n-1}} e^{-ix\cdot\eta} f(\eta) d\sigma_r(\eta) = \int_{S^{n-1}} e^{-irx\cdot\xi} e^{-r^2} Y(\xi) d\sigma(\xi)$$
$$= (2\pi)^{n/2} e^{-r^2} i^l \frac{J_{l+(n-2)/2}(rs)}{(rs)^{(n-2)/2}} Y(\zeta),$$

where $x = s\,\zeta$ for some $s > 0$ and $\zeta \in S^{n-1}$. This shows that $\hat{\mu}|_\Lambda = 0$ but μ is non-zero measure supported on Γ. □

Remark 2.2.14. *If we consider* $\Lambda = S_r^{n-1} \times E$ *for some* $r > 0$, *then Proposition* 2.2.12(a) *remains true.*

In [55], Vieli proved that (S^{n-1}, S_r^{n-1}) is a HUP if and only if $J_{d+(n-2)/2}(r) \neq 0$ for all $d \in \mathbb{Z}_+$. It is natural to ask if Λ is a union of spheres with their radii lay in the zero sets of Bessel functions, then whether Λ will be a determining set for the sphere. The following result extends the result due to Lev [31] to higher dimensions, which gives examples of non-uniqueness sets for the sphere.

Theorem 2.2.15. *Let* S^{n-1} *be the unit sphere in* $\mathbb{R}^n$ *and* Λ *be a union of two or more spheres in* $\mathbb{R}^n$. *Then* (S^{n-1}, Λ) *is not a Heisenberg uniqueness pair if and only if these spheres in* Λ *are concentric and their radii lay in the zero set of the same Bessel function* $J_{d+(n-2)/2}$ *where* $d \in \mathbb{Z}_+$.

Proof. Let Λ be a union of two or more spheres with center at $a \in \mathbb{R}^n$. Further, assume that their radius lies in the zero set of $J_{d+(n-2)/2}$, for some $d \in \mathbb{Z}_+$. Consider $f(\eta) = e^{ia.\eta} Y(\eta)$, where $Y \in H_d$ and write $d\mu = f d\sigma$. For $x \in \Lambda$, there exist $r > 0$ and $\xi \in S^{n-1}$ such that $x = a + r\xi$.

Hence the expression

$$\begin{aligned}\hat{\mu}(x) &= \int_{S^{n-1}} e^{-i(a+r\xi).\eta} f(\eta) d\sigma(\eta) \\ &= (2\pi)^{n/2} i^d r^{-(n-2)/2} J_{d+(n-2)/2}(r) Y_d(\xi)\end{aligned}$$

shows that (S^{n-1}, Λ) is not a Heisenberg uniqueness pair.

Conversely, let Λ be a union of two spheres such that (S^{n-1}, Λ) is not a HUP. Due to invariance properties of HUP, we can assume that $\Lambda = \Lambda_r \cup \Lambda_\rho$, where Λ_r be the sphere of radius r centered at origin and Λ_ρ be the sphere of radius ρ centered at $(a_1, 0, \ldots, 0)$ such that $J_{l_1+(n-2)/2}(r) = 0$ and $J_{l_2+(n-2)/2}(\rho) = 0$ for some $l_1, l_2 \in \mathbb{Z}_+$. Since the zero sets of the above two Bessel functions can intersect at most at the origin ([57], p.484), $J_{m+(n-2)/2}(r) = 0$ only if $m = l_1$ and a similar conclusion holds for ρ. Now, $\hat{\mu} = 0$ on Λ_r implies

$$(2\pi)^{n/2} i^l r^{-(n-2)/2} J_{l+(n-2)/2}(r) \|\Pi_l f\|_2^2 = 0$$

for all $l \in \mathbb{Z}_+$ (see [55]). It follows that $\Pi_l f = 0$ for all $l \in \mathbb{Z}_+$ except $l = l_1$. Thus from (2.1.3) we have $f(\eta) = \Pi_{l_1} f(\eta)$. Further, $\hat{\mu}$ vanishes on Λ_ρ gives $g(\eta) = \Pi_{l_2} g(\eta)$, where $g(\eta) = e^{-ia_1\eta_1} f(\eta)$, that is, $f(\eta) = e^{ia_1\eta_1} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta)$, where η_1 be the first coordinate of η. Therefore

$$\Pi_{l_1} f(\eta) = e^{ia_1\eta_1} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta) \tag{2.2.12}$$

for all $\eta \in S^{n-1}$.

Next, we show that $a_1 = 0$ and $l_1 = l_2$. If $a_1 = 0$, then by the orthogonality of spherical harmonics we get $l_1 = l_2$. Observe from (2.2.12) that $e^{ia_1\eta_1} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta)$ is a spherical harmonic of degree l_1. Hence for all $\alpha > 0$, we have

$$\begin{aligned}\alpha^{l_1} e^{ia_1\eta_1} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta) &= e^{i\alpha a_1\eta_1} \Pi_{l_2} e^{-i\alpha a_1\eta_1} f(\alpha\eta) \\ &= e^{i\alpha a_1\eta_1} \alpha^{l_2} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta).\end{aligned} \tag{2.2.13}$$

We claim that $\Pi_{l_2} e^{-ia_1\eta_1} f(\eta) \neq 0$ for some η such that $\eta_1 \neq 0$. In contrary, if $\Pi_{l_2} e^{-ia_1\eta_1} f(\eta) = 0$ for all η such that $\eta_1 \neq 0$, then

$$\Pi_{l_2} e^{-ia_1\eta_1} f(\eta) = 0$$

for almost all η, which implies $g = 0$ and hence $f = 0$, which is not possible.

Thus by the above claim and (2.2.13) we get

$$\alpha^{l_1} e^{ia_1\eta_1} = e^{i\alpha a_1\eta_1} \alpha^{l_2}$$

for all $\alpha > 0$, which is possible only if $l_1 = l_2$ and $a_1 = 0$. This completes the proof while Λ is the union of two spheres.

If Λ is a union of more than two spheres, then by applying the above argument for each pair of spheres in Λ, we can reach to the conclusion. $\square$

2.3 Preliminaries on the Heisenberg group

In this section, we describe some basic facts about Fourier transform on the Heisenberg group, Weyl transform, special Hermite functions and expansion of functions on $\mathbb{C}^n$ accordingly.

The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is a step two nilpotent Lie group having center $\mathbb{R}$ that equipped with the group law

$$(z,t)\cdot(w,s) = \left(z+w, t+s+\frac{1}{2}\mathrm{Im}(z\cdot\bar{w})\right).$$

By the Stone-von Neumann theorem, the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ can be parametrized by $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. That is, each of $\lambda \in \mathbb{R}^*$ defines a Schrödinger representation π_λ of $\mathbb{H}^n$ by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi+y),$$

where $z = x+iy$ and $\varphi \in L^2(\mathbb{R}^n)$. Hence, the group Fourier transform of $f \in L^1(\mathbb{H}^n)$ defined by

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)\,dz\,dt,$$

is a bounded operator. When $f \in L^2(\mathbb{H}^n)$, $\hat{f}(\lambda)$ is a Hilbert-Schmidt operator. An important technique in many problems on $\mathbb{H}^n$ is to take partial Fourier transform in the t-variable to reduce matters to $\mathbb{C}^n$. Let

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t)e^{i\lambda t}\,dt$$

be the inverse Fourier transform of f in the t-variable.

The group convolution of the functions $f, g \in L^1(\mathbb{H}^n)$ is given by

$$f * g(z,t) = \int_{H^n} f((z,t)(-w,-s))g(w,s)\, dwds. \tag{2.3.1}$$

A simple calculation shows that

$$\begin{aligned}(f*g)^\lambda(z) &= \int_{-\infty}^{\infty} f*g(z,t)e^{i\lambda t}dt \\ &= \int_{\mathbb{C}^n} f^\lambda(z-w)g^\lambda(w)e^{\frac{i\lambda}{2}\operatorname{Im}(z.\bar{w})}\, dw \\ &=: f^\lambda \times_\lambda g^\lambda(z).\end{aligned}$$

Thus, the group convolution $f * g$ on the Heisenberg group can be studied using the λ-twisted convolution $f^\lambda \times_\lambda g^\lambda$ on $\mathbb{C}^n$. For $\lambda \neq 0$, by a scaling argument, it is enough to study the twisted convolution for the case $\lambda = 1$.

Now, we recall the Weyl transform, which is an important constituent of the group Fourier transform on the Heisenberg group. Denote by $\pi_\lambda(z) = \pi_\lambda(z,0)$. Then $\pi_\lambda(z,t) = e^{i\lambda t}\pi_\lambda(z)$. For a suitable function g on $\mathbb{C}^n$, the Weyl transform of g can be expressed as

$$W_\lambda(g) = \int_{\mathbb{C}^n} g(w)\pi_\lambda(w)dw.$$

This, in turn, implies $\hat{f}(\lambda) = W_\lambda(f^\lambda)$. It is easy to see that $W_\lambda(g)$ is a bounded operator whenever $g \in L^1(\mathbb{C}^n)$. On the other hand, if $g \in L^2(\mathbb{C}^n)$, then $W_\lambda(g)$ is a Hilbert-Schmidt operator and satisfies the Plancherel formula

$$|\lambda|^{\frac{n}{2}}\|W_\lambda(g)\|_{HS} = (2\pi)^{\frac{n}{2}}\|g\|_2.$$

Next, we describe the special Hermite expansion for function on $\mathbb{C}^n$. Let

$$T = \frac{\partial}{\partial t},\ X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t} \text{ and } Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}$$

be the left-invariant vector fields on $\mathbb{H}^n$. Then $\{T, X_j, Y_j : j = 1, \ldots, n\}$ forms a basis for the Lie algebra $\mathfrak{h}^n$ and the representation π_λ induces a representation π^*_λ of $\mathfrak{h}^n$ on the space of C^∞ vectors in $L^2(\mathbb{R}^n)$ via

$$\pi^*_\lambda(X)f = \frac{d}{dt}\bigg|_{t=0} \pi_\lambda(\exp tX)f.$$

It is easy to see that $\pi^*_\lambda(X_j) = i\lambda x_j$ and $\pi^*_\lambda(Y_j) = \frac{\partial}{\partial x_j}$. Hence for the sub-Laplacian $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$, it follows that $\pi^*_\lambda(\mathcal{L}) = -\Delta_x + \lambda^2|x|^2 =: H_\lambda$, the scaled Hermite operators. Let $\phi^\lambda_\alpha(x) = |\lambda|^{\frac{n}{4}}\phi_\alpha(\sqrt{|\lambda|}x);\ \alpha \in \mathbb{Z}^n_+$, where ϕ_α are the Hermite functions on $\mathbb{R}^n$. Then ϕ^λ_α is an eigenfunction of H_λ with eigenvalue $(2|\alpha| + n)|\lambda|$. Hence the entry functions $E^\lambda_{\alpha\beta}$'s of the representation π_λ are eigenfunctions of the sub-Laplacian $\mathcal{L}$ satisfying

$$\mathcal{L}E^\lambda_{\alpha\beta} = (2|\alpha| + n)|\lambda|E^\lambda_{\alpha\beta},$$

where $E^\lambda_{\alpha\beta}(z,t) = \left\langle \pi_\lambda(z,t)\phi^\lambda_\alpha, \phi^\lambda_\beta \right\rangle$. Since $E^\lambda_{\alpha\beta}(z,t) = e^{i\lambda t}\left\langle \pi_\lambda(z)\phi^\lambda_\alpha, \phi^\lambda_\beta \right\rangle$, the eigenfunctions $E^\lambda_{\alpha\beta}$'s are not in $L^2(\mathbb{H}^n)$. However, for a fixed t, they are in $L^2(\mathbb{C}^n)$. Now, define an operator L_λ by $\mathcal{L}\left(e^{i\lambda t}f(z)\right) = e^{i\lambda t}L_\lambda f(z)$. Then the special Hermite functions

$$\phi^\lambda_{\alpha\beta}(z) = (2\pi)^{-\frac{n}{2}}|\lambda|^{\frac{n}{2}}\left\langle \pi_\lambda(z)\phi^\lambda_\alpha, \phi^\lambda_\beta \right\rangle$$

are eigenfunctions of L_λ with eigenvalue $2|\alpha| + n$. Now, we summarize by mentioning that the special Hermite functions $\phi^\lambda_{\alpha\beta}$'s form an orthonormal basis for $L^2(\mathbb{C}^n)$ [53].

Hence $g \in L^2(\mathbb{C}^n)$ can be expressed as

$$g = \sum_{\alpha,\beta} \langle g, \phi^\lambda_{\alpha\beta} \rangle \, \phi^\lambda_{\alpha\beta}. \tag{2.3.2}$$

By employing a correlation of the special Hermite functions with the Laguerre function, expression (2.3.2) can be further simplified. To do so, we need to recall the definition of Laguerre functions. Given $\upsilon \in \mathbb{C}$, the Laguerre polynomial of degree $k \in \mathbb{Z}_+$ is defined by

$$L^\upsilon_k(x) = \sum_{j=0}^{k} \binom{\upsilon + k}{k - j} \frac{(-x)^j}{j!}.$$

Now, the Laguerre function on $\mathbb{C}^n$ of order $n-1$ and degree k can be defined by $\varphi^{n-1}_k(z) = L^{n-1}_k(\frac{|z|^2}{2})e^{-\frac{|z|^2}{4}}$. Denote $\varphi^{n-1}_{k,\lambda}(z) = \varphi^{n-1}_k(\sqrt{|\lambda|}z)$, where $\lambda \in \mathbb{R}^*$. Then the special Hermite functions $\phi^\lambda_{\alpha\alpha}$ will satisfy the relation

$$\sum_{|\alpha|=k} \phi^\lambda_{\alpha,\alpha}(z) = (2\pi)^{-\frac{n}{2}} |\lambda|^{\frac{n}{2}} \varphi^{n-1}_{k,\lambda}(z). \tag{2.3.3}$$

Thus, $g \in L^2(\mathbb{C}^n)$ can be expressed as

$$g(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^{\infty} g \times_\lambda \varphi^{n-1}_{k,\lambda}(z),$$

whenever $\lambda \in \mathbb{R}^*$, (see [53]). In particular, for $\lambda = 1$, we have

$$g(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} g \times \varphi^{n-1}_k(z), \tag{2.3.4}$$

which is the special Hermite expansion for g. Hence g can be completely determined by its spectral projections $g \times \varphi^{n-1}_k$.

The following weighted functional relations can be obtained by considering the Hecke-Bochner identity for the spectral projection of compactly supported functions. For more details, see [53], p. 98.

Lemma 2.3.1. [53] *Let $P \in H_{p,q}$ and $d\nu_r = Pd\sigma_r$, where σ_r is the surface measure on the sphere S_r. Then, for $z \in \mathbb{C}^n$,*

$$\varphi_k^{n-1} \times \nu_r(z) = (2\pi)^{-n} \frac{\Gamma(k-q+1)}{\Gamma(k+n+p)} r^{2(p+q)} \varphi_{k-q}^{n+p+q-1}(r) P(z) \varphi_{k-q}^{n+p+q-1}(z),$$

if $k \geq q$ and 0 otherwise.

2.4 Uniqueness pairs on the Heisenberg group

In this section, we find some uniqueness pairs corresponding to symplectic Fourier transform, modified Fourier transform and spectral projection.

2.4.1 Symplectic Fourier transform

We prove that the unit sphere S^{2n-1} together with a non-harmonic complex cone forms a Heisenberg uniqueness pair for the symplectic Fourier transform.

Let $X(S^{2n-1})$ be the space of all finite Borel measures μ in $\mathbb{C}^n$ which are supported on S^{2n-1} and absolutely continuous with respect to the surface measure of S^{2n-1}. Then by the Radon-Nikodym Theorem, there exists $f \in L^1(S^{2n-1})$ such that $d\mu = fd\sigma$. Define symplectic Fourier transform (SFT) of a measure $\mu \in X(S^{2n-1})$ by

$$\mathcal{F}_S\mu(z) = \int_{S^{2n-1}} e^{-\frac{i}{2}\text{Im}(z\cdot\bar{\zeta})} f(\zeta) d\sigma(\zeta),$$

where $z = x + iy \in \mathbb{C}^n$ and $\zeta = \xi + i\eta \in \mathbb{C}^n$. Hence $\mathcal{F}_S\mu$ is a bounded uniformly continuous function on $\mathbb{C}^n$. In other words, $\mathcal{F}_S\mu$ can be expressed as

$$\mathcal{F}_S\mu(x,y) = \int_{S^{2n-1}} e^{-\frac{i}{2}(-x\cdot\eta+y\cdot\xi)} f(\xi,\eta)d\sigma(\xi,\eta). \tag{2.4.1}$$

We are going to prove the following result.

Theorem 2.4.1. *Let $\mathcal{C}$ be a complex cone in $\mathbb{C}^n$. If $\mu \in X(S^{2n-1})$ satisfies $\mathcal{F}_S\mu(z) = 0$ for all $z \in \mathcal{C}$, then $\mu = 0$ if and only if $\mathcal{C}$ is non-harmonic.*

Proof. Let $(x,y) = r\omega$, where $\omega = (\omega_1,\ldots,\omega_n,\omega_1',\ldots,\omega_n') \in S^{2n-1}$. Denote $\tilde{\omega} = (\omega_1',\ldots,\omega_n',-\omega_1,\ldots,-\omega_n)$. Then from (2.4.1), it follows that

$$\int_{S^{2n-1}} e^{-\frac{i}{2}r\tilde{\omega}\cdot(\xi,\eta)} f(\xi,\eta)d\sigma(\xi,\eta) = 0, \tag{2.4.2}$$

whenever $r\omega \in \mathcal{C}$. Since $\mathcal{C}$ is closed under complex scaling, $r\omega \in \mathcal{C}$ implies $r\tilde{\omega} \in \mathcal{C}$. By decomposing the integral in (2.4.2) over geodesic spheres at pole ω, we obtain

$$\int_{-1}^{1}\left(\int_{S_\omega^t} e^{-\frac{i}{2}r\omega\cdot\nu} f(\nu)d\sigma_{2n-2}(\nu)\right)dt = 0,$$

where $S_\omega^t = \{\nu \in S^{2n-1} : \omega\cdot\nu = t\}$. That is,

$$\int_{-1}^{1} e^{-\frac{i}{2}rt}\tilde{f}(\omega,t)dt = 0, \tag{2.4.3}$$

for all $r > 0$, where $\tilde{f}(\omega,t) = \int_{S_\omega^t} f d\sigma_{2n-2}$. Hence $\tilde{f}(\omega,t) = 0$, for all $t \in (-1,1)$. Thus by Lemma 2.1.4, it follows that $\Pi_l(f)(\omega) = 0$ for all $l \in \mathbb{Z}_+$. Further by Lemma 2.1.3, we get $\Pi_{p,q}f(\omega) = 0$ for all $p,q \in \mathbb{Z}_+$. Thus, by the given condition $w \notin Y_{pq}^{-1}(0)$ for any $p,q \in \mathbb{Z}_+$, it follows that $f = 0$. That is, $\mu = 0$.

Conversely, assume that $\mathcal{C}$ is contained in the zero set of a bi-graded homogeneous harmonic polynomial $P \in H_{p,q}$ and denote $Y = P|_{S^{2n-1}}$. For $\zeta \in S^{2n-1}$, let $d\mu(\zeta) = Y(\zeta)d\sigma(\zeta)$. Then μ is a finite complex Borel measure supported on S^{2n-1}. By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we find that $H_{p,q} \subseteq H_{p+q}$ and hence $Y \in H_{p+q}$. Therefore, using the identity (2.1.5), we have

$$\mathcal{F}_S\mu(z) = \int_{S^{2n-1}} e^{-ir\tilde{\omega}\cdot\zeta} Y(\zeta)d\sigma(\zeta) = (2\pi)^n i^{p+q} \frac{J_{p+q+n-1}(r)}{r^{n-1}} Y(\tilde{\omega}), \tag{2.4.4}$$

where $z = r\omega \in \mathcal{C}$. Recall that $r\omega \in \mathcal{C}$ implies $r\tilde{\omega} \in \mathcal{C}$. Thus, from (2.4.4), we can conclude that $\mathcal{F}_S\mu|_{\mathcal{C}} = 0$. □

Remark 2.4.2. (*a*) *Further, we observed that Theorem 2.4.1 holds for a* non-harmonic real cone. *Let $\mathcal{K}$ be a non-harmonic real cone. Write $\tilde{\omega} = \sigma_o\omega$, where σ_o is the symplectic matrix, which in fact, belongs to $U(n) \subset SO(2n)$. Suppose $\mu \in X(S^{2n-1})$ satisfies $\mathcal{F}_S\mu|_{\mathcal{K}} = 0$. Then $\Pi_l f(\sigma_o\omega) = 0$ for all $l \in \mathbb{Z}_+$. Since $\sigma_o^{-1} \cdot \Pi_l f$ would also be a spherical harmonic, we infer that $(S^{2n-1}, \mathcal{K})$ is a HUP for the SFT.*

(*b*) *Suppose Γ be a smooth sub-manifold in $\mathbb{R}^{2n}$ and Λ be a subset of $\mathbb{R}^{2n}$. Let $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be defined by $T(x, y) = (\frac{y}{2}, -\frac{x}{2})$ for $x, y \in \mathbb{R}^n$. It is easy to see that $\mathcal{F}_S\mu(x, y) = \hat{\mu}(\frac{y}{2}, -\frac{x}{2})$, where $\hat{\mu}$ is the Euclidean Fourier transform (EFT) of μ. Thus, (Γ, Λ) is a HUP for the SFT if and only if $(\Gamma, T\Lambda)$ is a HUP for the EFT.*

For instance, by the Euclidean result ([55], Proposition 1.2), *(S^{2n-1}, S_r^{2n-1}) is a HUP for the SFT as long as $\frac{r}{2} \notin J^{-1}_{(n+k-1)}(0)$ for any $k \in \mathbb{Z}_+$.*

(*c*) *Consider the map $\tilde{T} := (T^{-1})^*$ on $\mathbb{R}^{2n}$. It is known for the EFT that (Γ, Λ) is a HUP if and only if $(\tilde{T}^{-1}\Gamma, \tilde{T}^*\Lambda)$ is a HUP. Hence, from Remark (b), we have (Γ, Λ) is a HUP for the SFT if and only if $(T^*\Gamma, \Lambda)$ is a HUP for the EFT. Thus, in view of the Euclidean result (* [49], Theorem 3.1), *Remark (a) holds.*

2.4.2 Modified Fourier transform

In this subsection, we prove that a finite measure supported on the cylinder $S_r^{2n-1} \times \mathbb{R}$ can be determined by a non-harmonic cone as well as the boundary of a bounded domain in $\mathbb{C}^n$.

We know that the modified Fourier transform of $f \in L^1(\mathbb{H}^n)$ is defined by (see [52])

$$\mathcal{F}_M f(z,\lambda) = \pi_\lambda(-z) W_\lambda(f^\lambda)\pi_\lambda(z),$$

where $W_\lambda(f^\lambda)$ is the Weyl transform of f^λ and $(z,\lambda) \in \mathbb{C}^n \times \mathbb{R}^*$, see [52]. This, in turn, can be expressed as

$$\begin{aligned}\mathcal{F}_M f(z,\lambda) &= \int_{\mathbb{C}^n} \pi_\lambda(-z)\pi_\lambda(w) f^\lambda(w)\pi_\lambda(z)dw, \\ &= \int_{\mathbb{C}^n} e^{-i\lambda \mathrm{Im}(z\cdot\bar{w})} f^\lambda(w)\pi_\lambda(w)dw.\end{aligned}$$

Consider the measure $\mu \in X(S_r^{2n-1} \times \mathbb{R})$. Then there exists $f \in L^1(S_r^{2n-1} \times \mathbb{R})$ such that $d\mu(\zeta,t) = f(\zeta,t)d\sigma_r(\zeta)dt$. For $(z,\lambda) \in \mathbb{C}^n \times \mathbb{R}^*$, define the modified Fourier transform of μ by

$$\mathcal{F}_M \mu(z,\lambda) = \int_{S_r^{2n-1}} e^{-i\lambda \mathrm{Im}(z\cdot\bar{\zeta})} f^\lambda(\zeta)\pi_\lambda(\zeta)d\sigma_r(\zeta).$$

For $\lambda \in \mathbb{R}^*$, consider the one dimensional subspace $U_\lambda = \operatorname{span}\{\phi_\alpha^\lambda\}$ of $L^2(\mathbb{R}^n)$, where ϕ_α^λ is the scaled Hermite function for some $\alpha \in \mathbb{Z}_+^n$. Here for different λ, different α can be chosen. Denote $\tilde{\Lambda} = \mathcal{C} \times \mathbb{R}^*$, where $\mathcal{C}$ could be a real (or complex) cone. We have proved the following result.

Proposition 2.4.3. *Let $\mu \in X(S_r^{2n-1} \times \mathbb{R})$ and range of $\mathcal{F}_M\mu(z,\lambda)$ is a subspace of $U_\lambda^\perp$ for all $(z,\lambda) \in \tilde{\Lambda}$. If $\mathcal{C}$ is non-harmonic, then $\mu = 0$.*

Proof. Since, for each $\lambda \in \mathbb{R}^*$, range of $\mathcal{F}_M\mu(z,\lambda)$ is a subspace of $U_\lambda^\perp$, it follows that

$$\langle \mathcal{F}_M\mu(z,\lambda)\varphi, \phi_\alpha^\lambda \rangle = 0 \tag{2.4.5}$$

for all $\varphi \in L^2(\mathbb{R}^n)$. We know from [52] that $\langle \pi_\lambda(w)\phi_o^\lambda, \phi_\alpha^\lambda \rangle = c_\alpha |\lambda|^{\alpha/2} w^\alpha e^{\frac{-|\lambda||w|^2}{4}}$, for all $w \in \mathbb{C}^n$. If we choose $\varphi = \phi_o^\lambda$, then from (2.4.5), we have

$$\int_{S_r^{2n-1}} e^{-i\lambda \mathrm{Im}(z\cdot\bar{\zeta})} f^\lambda(\zeta) c_\alpha \zeta^\alpha e^{\frac{-|\lambda||\zeta|^2}{4}} d\sigma_r(\zeta) = 0 \tag{2.4.6}$$

for all $z \in \mathcal{C}$. This, in fact, reduces to the case of SFT on $\mathbb{C}^n$. That is, for each $\lambda \in \mathbb{R}^*$, we get $\mathcal{F}_S(g_r^\lambda)(2r\lambda z) = 0$ for each $z \in \mathcal{C}$, where $g_r^\lambda(\nu) = f^\lambda(r\nu)\nu^\alpha$ for $\nu \in S^{2n-1}$. Since $(2r\lambda)\mathcal{C} \subseteq \mathcal{C}$, in view of Theorem 2.4.1 and Remark 2.4.2 (a), we infer that $f^\lambda = 0$ if and only if $\mathcal{C}$ is non-harmonic. Thus, $f = 0$. □

Theorem 2.4.4. *Let $\partial\Omega$ be the boundary of the bounded domain Ω in $\mathbb{C}^n$. Suppose $\mu \in X(S_r^{2n-1} \times \mathbb{R})$ satisfies $\mathcal{F}_M\mu(z,\lambda) = 0$ for all $(z,\lambda) \in \partial\Omega \times \mathbb{R}^*$. Then $\mu = 0$.*

Proof. Since $\mathcal{F}_M\mu$ can be extended holomorphically to a function $F(.,\lambda)$ on $\mathbb{C}^{2n}$, it follows that $F(.,\lambda)|_{\mathbb{R}^{2n}} = \mathcal{F}_M\mu$ is a real analytic function. Consider

$$\mathcal{F}_M\mu(z,\lambda) = \int_{S_r^{2n-1}} e^{-i\lambda \mathrm{Im}(z\cdot\bar{\zeta})} f^\lambda(\zeta)\pi_\lambda(\zeta) d\sigma_r(\zeta).$$

Then

$$\frac{\partial}{\partial z_j}\mathcal{F}_M\mu(z,\lambda) = \frac{\lambda}{2}\int_{S_r^{2n-1}} \bar{\zeta}_j e^{-i\lambda \mathrm{Im}(z\cdot\bar{\zeta})} f^\lambda(\zeta)\pi_\lambda(\zeta) d\sigma_r(\zeta)$$

and

$$\frac{\partial^2}{\partial \bar{z}_j \partial z_j}\mathcal{F}_M\mu(z,\lambda) = -\frac{\lambda^2}{4}\int_{S_r^{2n-1}} \bar{\zeta}_j \zeta_j e^{-i\lambda \mathrm{Im}(z\cdot\bar{\zeta})} f^\lambda(\zeta)\pi_\lambda(\zeta) d\sigma_r(\zeta).$$

This, in turn, implies

$$\triangle_z \mathcal{F}_M \mu(z,\lambda) + (r\lambda)^2 \mathcal{F}_M \mu(z,\lambda) = 0.$$

Now, if $\varphi, \psi \in L^2(\mathbb{R}^n)$, then we have

$$\triangle_z \langle \mathcal{F}_M \mu(z,\lambda)\varphi, \psi\rangle + (r\lambda)^2 \langle \mathcal{F}_M \mu(z,\lambda)\varphi, \psi\rangle = 0.$$

Let $g(z,\lambda) = \langle \mathcal{F}_M \mu(z,\lambda)\varphi, \psi\rangle$. Then g will be a real analytic function satisfying

$$\triangle_z g(z,\lambda) + (r\lambda)^2 g(z,\lambda) = 0.$$

Hence $g(.,\lambda);\ \lambda \in \mathbb{R}^*$ are eigenfunctions of the Dirichlet boundary value problem in Ω. By the discreteness of eigenvalues of the Dirichlet problem in the bounded domain, it follows that $g(.,\lambda) = 0$ for all most all $\lambda \in \mathbb{R}^*$. Since $g(.,\lambda)$ is continuous in λ, we infer that $g(z,\lambda) = 0$ for all $(z,\lambda) \in \mathbb{C}^n \times \mathbb{R}^*$. Thus, $\mu = 0$. □

Remark 2.4.5. *Let $\mu \in X(\Gamma \times \mathbb{R})$ be such that range of $\mathcal{F}_M \mu(z,\lambda)$ is a subspace of $U_\lambda^\perp$, whenever $(z,\lambda) \in \Lambda \times \mathbb{R}^*$. If $(\Gamma, sT\Lambda)$ is a HUP for the EFT, for almost all $s \in \mathbb{R}$, then Remark 2.4.2* (b) *and Equation* (2.4.6) *allow to conclude that $\mu = 0$.*

For instance, consider $\Gamma = S^{2n-1}$ and $\Lambda = S_r^{2n-1}$. Since the set $\{J_{n+k-1}^{-1}(0) : k \in \mathbb{Z}_+\}$ has measure zero, in view of the Euclidean result ([55], Proposition 1.2*), we conclude that $\mu = 0$.*

Remark 2.4.6. *Let u be a solution of the Helmholtz equation $\triangle u + c^2 u = 0$ on $\mathbb{R}^{2n}$ and $\Sigma \subset \mathbb{R}^{2n}$ be such that $u = 0$ on Σ implies $u = 0$ a.e. Then the statement of Theorem 2.4.4 will remain true if we replace $\partial\Omega$ by Σ. Regular Jordan curves and two intersecting curves separated by an angle which is an irrational multiple of π, are examples of such Σ. For details, see* [17].

2.4.3 Spectral projections

In this subsection, we derive that the sphere determines the spectral projections of finite measures on $\mathbb{C}^n$, which are supported on S^{2n-1}. Further, we deduce that non-harmonic complex cone as well as NA-set can determine the spectral projections of the above class of measures.

For $\mu \in X(S_r^{2n-1})$, we define the spectral projection of μ by

$$\varphi_k^{n-1} \times \mu(z) = \int_{S_r^{2n-1}} \varphi_k^{n-1}(z-\omega) e^{\frac{i}{2}\mathrm{Im}(z\cdot\bar{\omega})} d\mu(\omega).$$

Then the following result holds.

Theorem 2.4.7. *Let $\mu \in X(S_{r_1}^{2n-1})$ be such that $\varphi_k^{n-1} \times \mu(z) = 0$ for all $z \in S_{r_2}^{2n-1}$ and $k \in \mathbb{Z}_+$. Then $\mu = 0$.*

In order to prove Theorem 2.4.7, we need the following results about the irreducibility of the Laguerre polynomials.

Theorem 2.4.8. [18] *Let v be a rational number, which is not a negative integer. Then for all but finitely many $k \in \mathbb{Z}_+$, the Laguerre polynomial L_k^v is irreducible over the field of rationals.*

Notice that the disjointness of the zero set of Laguerre functions can be understood by the following description of the zero set of Laguerre polynomials. This follows from Theorem 2.4.8 and has been worked out in [47].

Proposition 2.4.9. [47] *Let $k \in \mathbb{Z}_+$. Then for all but finitely many k, the Laguerre polynomials L_k^{n-1} have distinct zeroes over the reals.*

Proof of Theorem 2.4.7. Since $\mu \in X(S_{r_1}^{2n-1})$, there exists $f \in L^1(S_{r_1}^{2n-1})$ such that $d\mu = f d\sigma_{r_1}$. Thus,

$$\varphi_k^{n-1} \times \mu(z) = \int_{S_{r_1}^{2n-1}} \varphi_k^{n-1}(z-\omega) e^{\frac{i}{2}\mathrm{Im}(z\cdot\bar{\omega})} f(\omega) d\sigma_{r_1}(\omega) = 0, \tag{2.4.7}$$

for all $k \in \mathbb{Z}_+$ and $z \in S_{r_2}^{2n-1}$. As $f \in L^1(S_{r_1}^{2n-1})$, f will satisfy

$$f = \lim_{m\to\infty} \sum_{l=0}^{m} A_l^m(\delta) \Pi_l f,$$

where $A_l^m(\delta) = \binom{m-l+\delta}{\delta}\binom{m+\delta}{\delta}^{-1}$ and $\delta > n-1$. Further, from Lemma 2.1.1, it follows that

$$f = \lim_{m\to\infty} \sum_{l=0}^{m} \sum_{p+q=l} A_{p+q}^m(\delta) \Pi_{p,q} f.$$

Now from condition (2.4.7), it follows that

$$\begin{aligned}\left|\sum_{l=0}^{m}\sum_{p+q=l} A_{p+q}^m(\delta)\varphi_k^{n-1} \times \Pi_{p,q} f(z)\right| &= \left|\sum_{l=0}^{m}\sum_{p+q=l} A_{p+q}^m(\delta)\varphi_k^{n-1} \times \Pi_{p,q} f(z) - \varphi_k^{n-1} \times f(z)\right| \\ &\leq M_k \int_{S_{r_1}^{2n-1}} \left|\sum_{l=0}^{m}\sum_{p+q=l} A_{p+q}^m(\delta)\Pi_{p,q} f(\omega) - f(\omega)\right| d\sigma_{r_1}(\omega),\end{aligned}$$

where $M_k = \sup_{(\omega,z)\in S_{r_1}^{2n-1}\times S_{r_2}^{2n-1}} |\varphi_k^{n-1}(z-\omega)|$. Hence in view of (2.1.3), we deduce that

$$\lim_{m\to\infty} \sum_{l=0}^{m}\sum_{p+q=l} A_{p+q}^m(\delta)\varphi_k^{n-1} \times \Pi_{p,q} f(z) = 0 \tag{2.4.8}$$

converges uniformly in $S_{r_2}^{2n-1}$, whenever $k \in \mathbb{Z}_+$. When $k \geq q$, Lemma 2.3.1 gives

$$\int_{S^{2n-1}} \varphi_k^{n-1}(z - r_1\eta) e^{\frac{i}{2} r_1 \mathrm{Im}(z\cdot\bar{\eta})} Y_{p,q}(\eta) d\eta = B_n^{k,\gamma} {r_1}^{p+q} \varphi_{k-q}^{\gamma-1}(r_1) \varphi_{k-q}^{\gamma-1}(z) P_{p,q}(z), \tag{2.4.9}$$

where $B_n^{k,\gamma} = (2\pi)^{-n} \dfrac{\Gamma(k-q+1)}{\Gamma(k+n+p)}$ and $\gamma = n+p+q$. Let $z = r_2\xi$, where $\xi \in S^{2n-1}$.

Then from (2.4.8) and (2.4.9) we infer that

$$\lim_{m\to\infty}\sum_{l=0}^{m}\sum_{p+q=l}A^m_{p+q}(\delta)B^{k,\gamma}_n(r_1r_2)^{p+q}\varphi^{\gamma-1}_{k-q}(r_1)\varphi^{\gamma-1}_{k-q}(r_2)\Pi_{p,q}f(\xi)=0. \tag{2.4.10}$$

Since (2.4.8) converges uniformly on $S^{2n-1}_{r_2}$, it follows that (2.4.10) converges in $L^2(S^{2n-1})$. Recall that the bi-graded spherical harmonic projections $\Pi_{p,q}f$ are orthogonal among themselves, and $A^m_{p+q}(\delta)B^{k,\gamma}_n\neq 0$ holds true for every choice of $p,q\in\mathbb{Z}_+$. From (2.4.10) we obtain that

$$\varphi^{\gamma-1}_{k-q}(r_1)\varphi^{\gamma-1}_{k-q}(r_2)\,\|\Pi_{p,q}f\|_2=0 \tag{2.4.11}$$

for $k\geq q$. Hence by invoking Proposition 2.4.9 for each $p,q\in\mathbb{Z}_+$, there exists $k_o\geq q$ such that $r_i\notin\left(\varphi^{n+p+q-1}_{k_o-q}\right)^{-1}(0)$ for $i=1,2$. Hence, from (2.4.11) we conclude that $\Pi_{p,q}f=0$ for all $p,q\in\mathbb{Z}_+$. Thus, $f=0$. □

Remark 2.4.10. *A set, which is a determining set for any real analytic function, is called an NA - set. For instance, the spiral is an NA - set in the plane* (see [37]). *Since the spectral projection $\varphi^{n-1}_k\times\mu$ can be extended holomorphically to $\mathbb{C}^{2n}$, the function $\varphi^{n-1}_k\times\mu$ must be real analytic on $\mathbb{C}^n$.*

Let Λ be an NA-set for real analytic functions on $\mathbb{C}^n$. If $\mu\in X(S^{2n-1}_r)$ satisfies $\varphi^{n-1}_k\times\mu|_\Lambda=0$ for all $k\in\mathbb{Z}_+$, then $\varphi^{n-1}_k\times\mu(z)=0$ for all $z\in\mathbb{C}^n$. Thus, in view of Theorem 2.4.7, we have $f=0$.

Next, we shall prove that spectral projections of a measure $\mu\in X(S^{2n-1}_r)$ can be determined by a non-harmonic complex cone.

Theorem 2.4.11. *Let $\mathcal{C}$ be a non-harmonic complex cone in $\mathbb{C}^n$. If $\mu\in X(S^{2n-1}_r)$ satisfies $\varphi^{n-1}_k\times\mu|_{\mathcal{C}}=0$ for all $k\in\mathbb{Z}_+$, then $\mu=0$.*

Proof. From (2.4.10), it follows that

$$\lim_{m\to\infty}\sum_{l=0}^{m}\sum_{p+q=l}A_{p+q}^{m}(\delta)B_n^{k,\gamma}(rs)^{p+q}\varphi_{k-q}^{\gamma-1}(r)\varphi_{k-q}^{\gamma-1}(s)\Pi_{p,q}f(\xi)=0,$$

whenever $s\xi\in\mathcal{C}$ and $k\in\mathbb{Z}_+$. Since $\mathcal{C}$ is closed under complex scaling, replacing ξ by $e^{i\theta}\xi$, we obtain

$$\lim_{m\to\infty}\sum_{l=0}^{m}\sum_{p+q=l}A_{p+q}^{m}(\delta)B_n^{k,\gamma}(rs)^{p+q}\varphi_{k-q}^{\gamma-1}(r)\varphi_{k-q}^{\gamma-1}(s)\Pi_{p,q}f(\xi)e^{i(p-q)\theta}=0.$$

Now, by induction on k, we show that each of the projection $\Pi_{p,q}f$ restricted to $\mathcal{C}$ must be zero. For $k=0$, the choice for q is only 0. Since $\{e^{ip\theta}:p\in\mathbb{Z}_+\}$ is an orthonormal set, we infer that $\Pi_{p,0}(f)(\xi)=0$. Similarly, for $k=1$, the choices for q are 0 and 1 only. The case $q=0$ is already settled. Now for $q=1$ the set $\{e^{i(p-1)\theta}:p\in\mathbb{Z}_+\}$ is an orthonormal set. Hence $\Pi_{p,1}(f)(\xi)=0$. This, in turn, implies that each of the projection $\Pi_{p,q}(f)$ vanishes on $\mathcal{C}$. Thus, $f=0$. □

2.5 Related problems for future work

Problem 1. Let $\Gamma=(\mathbb{R}\times\{0,1,\cdots,p\})\cup(\{0,1,\cdots,p\}\times\mathbb{R})$ be a system of cross. Characterization of the determining set Λ for Γ.

Problem 2. Let $\mu\in X(S^{2n-1}\times\mathbb{R})$. We define the group Fourier transform of μ by

$$\hat{\mu}(\lambda)=\int_\Gamma f(w,t)\pi_\lambda(w,t)dwdt.$$

Can we characterize the measures $\mu\in X(S^{2n-1}\times\mathbb{R})$ if $\hat{\mu}(\lambda)$ is a finite rank operator for all $\lambda\in\mathbb{R}^*$.

Chapter 3

Quaternion Heisenberg group

In this chapter, we prove an analogue of Benedicks-Amrein-Berthier theorem for the quaternion Heisenberg group. As the multiplication of two quaternions is not commutative, the generalization of the above result in the quaternion Heisenberg group is notable. The proof of our result follows in a similar spirit as described by Amrein-Berthier in [2].

3.1 Preliminaries and auxiliary results

In this section, we first describe basic preliminaries related to quaternion numbers and then discuss Fourier analysis on the quaternion Heisenberg group. Details can be found in [12, 14]. Further, we prove the representation is square integrability modulo the center.

The quaternion Heisenberg group is a step two nilpotent Lie group with centre $\mathbb{R}^3$. Let $\mathbb{Q}$ be the set of all quaternions. For $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{Q}$, the conjugate of q is defined by $\bar{q} = q_0 - iq_1 - jq_2 - kq_3$. Further $\mathrm{Re}(q)$, the real part of q, is the real number q_0, and the imaginary part $\mathrm{Im}(q) = (q_1, q_2, q_3) \in \mathbb{R}^3$. The product of two quaternions $q = q_0 + iq_1 + jq_2 + kq_3$ and $\tilde{q} = \tilde{q}_0 + i\tilde{q}_1 + j\tilde{q}_2 + k\tilde{q}_3$ is defined by $q\tilde{q} = (q_0\tilde{q}_0 - q_1\tilde{q}_1 - q_2\tilde{q}_2 -$

$q_3\tilde{q}_3)+i(q_0\tilde{q}_1+q_1\tilde{q}_0+q_2\tilde{q}_3-q_3\tilde{q}_2)+j(q_0\tilde{q}_2-q_1\tilde{q}_3+q_2\tilde{q}_0+q_3\tilde{q}_1)+k(q_0\tilde{q}_3+q_1\tilde{q}_2-q_2\tilde{q}_1+q_3\tilde{q}_0)$.

The inner product in $\mathbb{Q}$ is defined by $\langle q,\tilde{q}\rangle = \mathrm{Re}(\bar{q}\tilde{q})$. This leads to $|q|^2 = \langle q,q\rangle = \sum_{l=0}^{3} q_l^2$, and we get the relations $\overline{q\tilde{q}} = \bar{\tilde{q}}\bar{q}$ and $|q\tilde{q}| = |q||\tilde{q}|$. The set $\mathcal{Q} = \mathbb{Q}\times\mathbb{R}^3 = \{(q,t) : q\in\mathbb{Q}, t\in\mathbb{R}^3\}$ becomes a non-commutative group when equipped with the group law

$$(q,t)(\tilde{q},\tilde{t}) = (q+\tilde{q}, t+\tilde{t}-2\ \mathrm{Im}(\bar{\tilde{q}}q)).$$

The Lebesgue measure $dqdt$ on $\mathbb{Q}\times\mathbb{R}^3$ is the Haar measure on $\mathcal{Q}$. For $1\le p\le\infty$, $L^p(\mathcal{Q})$ denotes the usual L^p space of all quaternion-valued functions on $\mathbb{Q}\times\mathbb{R}^3$.

Let $a\in \mathrm{Im}\ \mathbb{Q}\smallsetminus\{0\}$. Then $J_a : q\mapsto q\cdot\frac{a}{|a|}$ defines a complex structure on $\mathbb{Q}$. Let $\mathcal{F}_a$ be the Fock space of all holomorphic functions F with quaternion values on $(\mathbb{Q}, J_a)$ such that

$$\|F\|_2^2 = \int_{\mathbb{Q}} |F(q)|^2 e^{-2|a||q|^2} dq < \infty.$$

An irreducible unitary representation π_a of $\mathcal{Q}$ realized on $\mathcal{F}_a$ is given by

$$\pi_a(q,t)F(\tilde{q}) = e^{i\langle a,t\rangle - |a|\left(|q|^2+2\langle\tilde{q},q\rangle-2i\langle\tilde{q}.\frac{a}{|a|},q\rangle\right)}F(\tilde{q}+q),$$

where $F\in\mathcal{F}_a$. Up to unitary equivalence, π_a's are all the infinite dimensional irreducible unitary representations of $\mathcal{Q}$. For $f\in L^1(\mathcal{Q})$, the group Fourier transform can be expressed as

$$\hat{f}(a) = \int_{\mathcal{Q}} f(q,t)\pi_a(q,t)dqdt.$$

For $f\in L^2(\mathcal{Q})$, the following Plancherel formula holds.

$$\|f\|_2^2 = \frac{1}{2\pi^5}\int_{\mathrm{Im}\ \mathbb{Q}\setminus\{0\}} \|\hat{f}(a)\|_{\mathrm{HS}}^2 |a|^2 da. \tag{3.1.1}$$

Let

$$f^a(q) = \int_{\mathbb{R}^3} f(q,t)e^{i\langle a,t\rangle}dt$$

be the inverse Fourier transform of f in the t variable.

If we denote $\pi_a(q) = \pi_a(q,0)$, then the Weyl transform of $g \in L^1(\mathbb{Q})$ can be defined by

$$W_a(g) = \int_{\mathbb{Q}} g(q)\pi_a(q)dq.$$

Hence it follows that $\hat{f}(a) = W_a(f^a)$. Further, $W_a(g)$ is a bounded operator if $g \in L^1(\mathbb{Q})$ and a Hilbert-Schmidt operator when $g \in L^2(\mathbb{Q})$. In addition, when $g \in L^2(\mathbb{Q})$, the Plancherel formula for the Weyl transform W_a is given by

$$\|W_a(g)\|_{\text{HS}}^2 = \frac{\pi^2}{4|a|^2}\|g\|^2. \tag{3.1.2}$$

Finally, the inversion formula for the Weyl transform is given by

$$g(q) = \frac{4|a|^2}{\pi^2}\text{tr}\left(W_a(g)\pi_a^*(q)\right). \tag{3.1.3}$$

In order to prove that the representation π_a is square integrable modulo the center, we need to recall the following Schur's orthogonality relation, see [32].

Proposition 3.1.1. [32] *Suppose G be a connected, simply connected nilpotent Lie group with centre Z. Let π be an irreducible unitary representation of G realized on a complex Hilbert space H and $\pi|_Z = \chi I_H$, where χ is a character of Z. Then $\langle \pi(x)h, h\rangle \in L^2(G/Z)$ for some non-zero $h \in H$ if and only if*

$$\int_{G/Z} \langle \pi(x)h_1, k_1\rangle \overline{\langle \pi(x)h_2, k_2\rangle} d\nu_{G/Z} = c_\pi \langle h_1, h_2\rangle \overline{\langle k_1, k_2\rangle}$$

for all $h_l, k_l \in H, l = 1, 2,$ where c_π is a constant depending only on π.

Lemma 3.1.2. *Let $\varphi, \psi \in \mathcal{F}_a$. Then,*

$$\int_{\mathbb{Q}} |\langle \pi_a(q)\varphi, \psi\rangle|^2 dq \leq 4c_a \|\varphi\|_2^2 \|\psi\|_2^2 \tag{3.1.4}$$

for some constant c_a.

Proof. For $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{Q}$, write $z_1 = q_0 + iq_1$ and $z_2 = q_3 + iq_4$. Then $\varphi(q) = \frac{4|a|^2}{\pi} z_1 z_2 \in \mathcal{F}_a$ and $\|\varphi\|_2 = 1$. Further,

$$\begin{aligned}\int_{\mathbb{Q}} |\langle \pi_a(q)\varphi, \varphi\rangle|^2 dq &= \int_{\mathbb{Q}} \Big| \int_{\mathbb{Q}} \pi_a(q)\varphi(\tilde{q})\overline{\varphi(\tilde{q})} e^{-2|a||\tilde{q}|^2} d\tilde{q}\Big|^2 dq \\ &= \int_{\mathbb{Q}} \Big| \int_{\mathbb{Q}} e^{-|a|(|q|^2 + 2\langle \tilde{q}, q\rangle - 2i\langle \tilde{q}\frac{a}{|a|}, q\rangle)} \varphi(\tilde{q}+q)\overline{\varphi(\tilde{q})} e^{-2|a||\tilde{q}|^2} d\tilde{q}\Big|^2 dq.\end{aligned}$$

By Minkowski's integral inequality, it follows that

$$\begin{aligned}\left(\int_{\mathbb{Q}} |\langle \pi_a(q)\varphi, \varphi\rangle|^2 dq\right)^{\frac{1}{2}} &\leq \int_{\mathbb{Q}} \left(\int_{\mathbb{Q}} |\varphi(\tilde{q}+q)|^2 e^{-2|a||\tilde{q}+q|^2} |\varphi(\tilde{q})|^2 e^{-2|a||\tilde{q}|^2} dq\right)^{\frac{1}{2}} d\tilde{q} \\ &= \|\varphi\|_2 \int_{\mathbb{Q}} |\varphi(\tilde{q})| e^{-|a||\tilde{q}|^2} d\tilde{q} \\ &= \frac{4|a|^2}{\pi} \int_{\mathbb{C}^2} |z_1||z_2| e^{-|a|(|z_1|^2 + |z_2|^2)} dz_1 dz_2 < \infty.\end{aligned}$$

Hence by Proposition 3.1.1, for complex valued functions $\varphi_l, \psi_l \in \mathcal{F}_a$, where $l = 1, 2$, we get

$$\int_{\mathbb{Q}} \langle \pi_a(q)\varphi_1, \psi_1\rangle \overline{\langle \pi_a(q)\varphi_2, \psi_2\rangle} dq = c_a \langle \varphi_1, \varphi_2\rangle \overline{\langle \psi_1, \psi_2\rangle}. \tag{3.1.5}$$

For arbitrary $\varphi, \psi \in \mathcal{F}_a$, we can write $\varphi = \varphi_1 + \varphi_2 j$ and $\psi = \psi_1 + \psi_2 j$, where φ_l, ψ_l are complex valued functions. Then by (3.1.5),

$$\int_{\mathbb{Q}} |\langle \pi_a(q)\varphi, \psi\rangle|^2 dq \leq 4c_a \sum_{l_1, l_2 = 1}^{2} \|\varphi_{l_1}\|_2^2 \|\psi_{l_2}\|_2^2 = 4c_a \|\varphi\|_2^2 \|\psi\|_2^2,$$

where the last equality true as $|\varphi(\tilde{q})|^2 = |\varphi_1(\tilde{q})|^2 + |\varphi_2(\tilde{q})|^2$. □

3.2 Benedicks-Amrein-Berthier type theorem

Let $g \in L^2(\mathbb{Q})$ and $W_a(g)$ be a finite rank operator. Then there exists an orthonormal basis $\{e_1, e_2, \ldots\}$ of $\mathcal{F}_a$ such that $\mathcal{R}(W_a(g)) = \mathcal{B}_N$, where $\mathcal{B}_N = \text{span}\{e_1, \ldots, e_N\}$. Define an orthogonal projection P_N of $\mathcal{F}_a$ onto $\mathcal{B}_N$. Let A be a measurable subset of $\mathbb{Q}$. Define a pair of orthogonal projections E_A and F_N of $L^2(\mathbb{Q})$ by

$$E_A g = \chi_A g \qquad \text{and} \qquad W_a(F_N g) = P_N W_a(g), \tag{3.2.1}$$

where χ_A denotes the characteristic function of A. Then $\mathcal{R}(E_A) = \{g \in L^2(\mathbb{Q}) : g = \chi_A g\}$ and $\mathcal{R}(F_N) = \{g \in L^2(\mathbb{Q}) : \mathcal{R}(W_a(g)) \subseteq \mathcal{B}_N\}$.

Next, we prove that $E_A F_N$ is a Hilbert-Schmidt operator. Throughout this section, we shall assume that A is a measurable subset of $\mathbb{Q}$ with finite measure.

Lemma 3.2.1. *The operator $E_A F_N$ is an integral operator on $L^2(\mathbb{Q})$.*

Proof. For $g \in L^2(\mathbb{Q})$, we have $W_a(F_N g) = P_N W_a(g)$. Denote $\tilde{c}_a = \frac{4|a|^2}{\pi^2}$. Then by inversion formula for the Weyl transform

$$\begin{aligned}(F_N g)(q) &= \tilde{c}_a \ \text{tr}\,(W_a(F_N g)\pi_a^*(q)) = \tilde{c}_a \ \text{tr}\,(P_N W_a(g)\pi_a(-q)) \\ &= \tilde{c}_a \int_{\mathbb{Q}} g(\tilde{q}) \ \text{tr}\,(P_N \pi_a(\tilde{q})\pi_a(-q))\, d\tilde{q}.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}(E_A F_N g)(q) = \chi_A(q)(F_N g)(q) &= \tilde{c}_a \chi_A(q) \int_{\mathbb{Q}} g(\tilde{q}) \ \text{tr}\,(P_N \pi_a(\tilde{q})\pi_a(-q))\, d\tilde{q} \\ &= \int_{\mathbb{Q}} g(\tilde{q}) K(q, \tilde{q}) d\tilde{q},\end{aligned}$$

where $K(q, \tilde{q}) = \tilde{c}_a \ \chi_A(q) \ \text{tr}\,(P_N \pi_a(\tilde{q})\pi_a(-q))$. Thus, we infer that $E_A F_N$ is an integral operator with kernel K. □

Lemma 3.2.2. *E_AF_N is a Hilbert-Schmidt operator and $\|E_AF_N\|_{HS}^2 \leq c'm(A)N^2$, for some constant c', independent of the choice of A and N.*

Proof. From Lemma 3.2.1, it follows that

$$\begin{aligned}\|E_AF_N\|_{HS}^2 &= \tilde{c}_a^2 \int_{\mathbb{Q}}\int_{\mathbb{Q}} |K(q,\tilde{q})|^2 d\tilde{q}dq \\ &= \tilde{c}_a^2 \int_{\mathbb{Q}} |\chi_A(q)|^2 \left(\int_{\mathbb{Q}} |\mathrm{tr}\,(P_N\pi_a(\tilde{q})\pi_a(-q))|^2 d\tilde{q}\right) dq \\ &= \tilde{c}_a^2 \int_{\mathbb{Q}} \chi_A(q) \left(\int_{\mathbb{Q}} \Big|\sum_{l=1}^{N} \langle \pi_a(\tilde{q})\pi(-q)e_l, e_l\rangle\Big|^2 d\tilde{q}\right) dq.\end{aligned}$$

Since $\pi_a(\tilde{q})\pi_a(q) = e^{2i\langle \tilde{q}a,q\rangle}\pi_a(\tilde{q}+q))$, we get

$$\begin{aligned}\|E_AF_N\|_{HS}^2 &= \tilde{c}_a^2 \int_{\mathbb{Q}} \chi_A(q) \left(\int_{\mathbb{Q}} \Big|\sum_{l=1}^{N} e^{-2i\langle \tilde{q}a,q\rangle} \langle \pi_a(\tilde{q}-q)e_l, e_l\rangle\Big|^2 d\tilde{q}\right) dq \\ &\leq \tilde{c}_a^2 \int_{\mathbb{Q}} \chi_A(q) \left(N\int_{\mathbb{Q}} \sum_{l=1}^{N} \Big|\langle \pi_a(\tilde{q})e_l, e_l\rangle\Big|^2 d\tilde{q}\right) dq.\end{aligned}$$

Hence, from the square integrable property (3.1.4) of the representation, it follows that

$$\|E_AF_N\|_{HS}^2 \leq 4c_a\tilde{c}_a^2 m(A)N^2 = c'm(A)N^2 < \infty.$$

□

We need the following result that describes an interesting property of Lebesgue measurable sets in $\mathbb{R}^n$. Since, as a set $\mathbb{Q}$ can be realized as $\mathbb{R}^4$, this will also hold for any measurable set in $\mathbb{Q}$. For a measurable set $B \subseteq \mathbb{R}^n$ and $y \in \mathbb{R}^n$, denote $yB = \{x \in \mathbb{R}^n : x - y \in B\}$.

Lemma 3.2.3. [2] *Let B be a measurable set in $\mathbb{R}^n$ with $0 < m(B) < \infty$. If B_0 is a measurable subset of B with $m(B_0) > 0$, then for each $\epsilon > 0$ there exists $y \in \mathbb{R}^n$ such that*

$$m(B) < m(B \cup yB_0) < m(B) + \epsilon.$$

We also need the following basic fact about the orthogonal projection, which helps to decide the disjointness of the projections E_A and F_N while $m(A) < \infty$.

For given orthogonal projections E and F of a Hilbert space $\mathcal{H}$, let $E \cap F$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(E) \cap \mathcal{R}(F)$. Then

$$\|E \cap F\|_{HS}^2 = \dim \mathcal{R}(E \cap F) \leq \|EF\|_{HS}^2. \tag{3.2.2}$$

Let S be a closed subspace of $\mathcal{F}_a$. Define F_S by $W_a(F_S g) = P_S W_a(g)$, where P_S is the orthogonal projection of $\mathcal{F}_a$ onto S and $g \in L^2(\mathbb{Q})$. In particular, if $S = \mathcal{B}_N$, then $F_S = F_N$. Denote A^c for the complement of A and $F_S^\perp = I - F_S$.

Proposition 3.2.4. *Let $A \subset \mathbb{Q}$ with finite measure, and S be a closed subspace of $\mathcal{F}_a$. Then, either $E_A \cap F_S = 0$ or for each $\epsilon' > 0$ there exists $\tilde{A} \supset A$ with $m(\tilde{A} \setminus A) < \epsilon'$ such that* $\dim \mathcal{R}(E_{\tilde{A}} \cap F_S) = \infty$.

Proof. If $E_A \cap F_S \neq 0$, then there exists a non-zero function $g_0 \in \mathcal{R}(E_A \cap F_S)$. Let $A_0 = \{x \in A : g_0(x) \neq 0\}$ and $\tilde{A}_1 = A$. By Lemma 3.2.3, for $\epsilon = \frac{\epsilon'}{2^{l+1}}$, $B_0 = A_0$ and $B = \tilde{A}_l$, there exists $\tilde{q}_l \in \mathbb{Q}$ such that

$$m(\tilde{A}_l) < m(\tilde{A}_l \cup \tilde{q}_l A_0) < m(\tilde{A}_l) + \frac{\epsilon'}{2^{l+1}}, \tag{3.2.3}$$

where $l \in \mathbb{N}$. Put $\tilde{A}_{l+1} = \tilde{A}_l \cup \tilde{q}_l A_0$ and $\tilde{A} = \bigcup_{l=1}^{\infty} \tilde{A}_l$. Then $\tilde{A}_l$ is a non-decreasing sequence, and hence from (3.2.3) it follows that $m(\tilde{A} \setminus A) < \epsilon'$. For $l \in \mathbb{N}$, define

$g_l(q) = g_0(q-\tilde{q}_l)e^{2i\langle qa,\tilde{q}_l\rangle}$. We show that $g_l \in \mathcal{R}(E_{\tilde{A}} \cap F_S)$ for each l and they are linearly independent. Let $\mathcal{B}_S$ be an orthonormal basis of S. Then, we can extend $\mathcal{B}_S$ to an orthonormal basis $\mathcal{B}$ of $\mathcal{F}_a$. For $\varphi \in \mathcal{F}_a$ and $e_\beta \in \mathcal{B} \setminus \mathcal{B}_S$, we have

$$\begin{aligned}\langle W_a(g_l)\varphi, e_\beta\rangle &= \int_{\mathbb{Q}} g_l(q)\langle \pi_a(q)\varphi, e_\beta\rangle dq \\ &= \int_{\mathbb{Q}} g_0(q-\tilde{q}_l)e^{2i\langle qa,\tilde{q}_l\rangle}\langle \pi_a(q)\varphi, e_\beta\rangle dq.\end{aligned}$$

Since $\langle (q+\tilde{q}_l)a, \tilde{q}_l\rangle = \langle qa, \tilde{q}_l\rangle$, by the change of variables

$$\langle W_a(g_l)\varphi, e_\beta\rangle = \int_{\mathbb{Q}} g_0(q)e^{2i\langle qa,\tilde{q}_l\rangle}\langle \pi_a(q+\tilde{q}_l)\varphi, e_\beta\rangle dq.$$

We know that $\pi_a(q)\pi_a(\tilde{q}_l) = e^{2i\langle qa,\tilde{q}_l\rangle}\pi_a(q+\tilde{q}_l)$. Therefore,

$$\begin{aligned}\langle W_a(g_l)\varphi, e_j\rangle &= \int_{\mathbb{Q}} g_0(q)\langle \pi_a(q)\pi_a(\tilde{q}_l)\varphi, e_\beta\rangle dq \\ &= \int_{\mathbb{Q}} g_0(q)\langle \pi_a(q)\psi, e_\beta\rangle dq \\ &= \langle W_a(g_0)\psi, e_\beta\rangle = 0. \end{aligned} \tag{3.2.4}$$

Hence $\mathcal{R}(W_a(g_l)) \subseteq S$ and thus $g_l \in \mathcal{R}(E_{\tilde{A}} \cap F_S)$ for each $l \in \mathbb{N}$. Next, we shall show that g_l's are linearly independent. Let $A_l = A_{l-1} \cup \tilde{q}_l A_0$. Then $\tilde{A}_{l+1} = \tilde{A}_l \cup A_l$. Thus, we have $m(A_l \setminus A_{l-1}) \geq m(\tilde{A}_{l+1} \setminus \tilde{A}_l) > 0$. Let $s \in \mathbb{N}$. Since, $A_s = A_0 \cup \tilde{q}_1 A_0 \cup \cdots \cup \tilde{q}_s A_0$ and $g_l = 0$ on $(\tilde{q}_l A_0)^c$, we have $E_{A_s} g_l = g_l$ for $l = 0, 1, \ldots, s$. Furthermore, $E_{A_s \setminus A_{s-1}} g_l = 0$ for $l = 0, \ldots, s-1$ and $E_{A_s \setminus A_{s-1}} g_s \neq 0$. Therefore, it shows that g_s is not a linear combination of $g_0, \ldots, g_{s-1}$. Since s is arbitrary, $\{g_l : l \in \mathbb{N}\}$ is a linearly independent set in $\mathcal{R}(E_{\tilde{A}} \cap F_S)$. □

Proposition 3.2.5. *Let A be a measurable subset of $\mathbb{Q}$ having finite measure. Then the projection $E_A \cap F_N = 0$.*

Proof. In view of (3.2.2) and Lemma 3.2.2, we obtain the relations

$$\dim \mathcal{R}(E_{\tilde{A}} \cap F_N) \leq c' m(\tilde{A}) N^2 < \infty.$$

Therefore, as a corollary of Proposition 3.2.4, we get $E_A \cap F_N = 0$. □

The following theorem is our main result of this section, which is an analogue of Benedicks-Amrein-Berthier theorem on the quaternion Heisenberg group.

Theorem 3.2.6. *Let $A \subset \mathbb{Q}$ be a set of finite measure. Suppose $f \in L^1(\mathcal{Q})$ and $\{(q,t) \in \mathcal{Q} : f(q,t) \neq 0\} \subseteq A \times \mathbb{R}^3$. If $\hat{f}(a)$ is a finite rank operator for each $a \in Im\, \mathbb{Q} \setminus \{0\}$, then $f = 0$.*

Proof of Theorem 3.2.6 follows from the following result for the Weyl transform on $\mathbb{Q}$.

Proposition 3.2.7. *Let $g \in L^1(\mathbb{Q})$ and $\{q \in \mathbb{Q} : g(q) \neq 0\} \subseteq A$, where $m(A)$ is finite. If there exists $a \in Im\, \mathbb{Q} \setminus \{0\}$ such that $W_a(g)$ has finite rank, then $g = 0$.*

Since $W_a(g)$ is a finite rank operator, by the Plancherel theorem for the Weyl transform on $\mathbb{Q}$, it follows that $g \in L^2(\mathbb{Q})$. Hence, it is enough to prove Proposition 3.2.7 for $g \in L^2(\mathbb{Q})$. The prove of Proposition 3.2.7 follows from Proposition 3.2.5.

Remark 3.2.8. *If $0 < m(A) < \infty$, then $\dim \mathcal{R}(E_A) = \infty$. In view of Proposition 3.2.5 and the fact that $E_A = (E_A \cap F_N) + (E_A \cap F_N^{\perp}) = (E_A \cap F_N^{\perp})$, it follows that $\dim \mathcal{R}(E_A \cap F_N^{\perp}) = \infty$. Since $m(A^c) = \infty$, there exists a measurable set $B \subset A^c$ satisfying $0 < m(B) < \infty$. Hence $\mathcal{R}(E_{A^c} \cap F_N^{\perp}) \supseteq \mathcal{R}(E_B \cap F_N^{\perp})$. This implies $\dim \mathcal{R}(E_{A^c} \cap F_N^{\perp}) = \infty$. Similarly, it can be shown that $\dim \mathcal{R}(E_{A^c} \cap F_N) = \infty$.*

3.2.1 Strong annihilating pair

In this section, we discuss some strong annihilating pair for the Weyl transform. Since Fourier transform on the quaternion Heisenberg group is an operator valued function, we could not expect a similar conclusion as (1.2.2). However, we can adequately describe a strong annihilating pair.

Definition 3.2.9. *Let A be a measurable subset of $\mathbb{Q}$, and S be a closed subspace of $\mathcal{F}_a$. We say that the pair (A, S) is a* strong annihilating pair *for the Weyl transform W_a if there exists a positive number $C = C(A, S)$ such that for every $g \in L^2(\mathbb{Q})$,*

$$\|g\|_2^2 \leq C \left(\|g\|^2_{L^2(A^c)} + \|P_S^{\perp} W_a(g)\|^2_{HS} \right),$$

where P_S is the projection of $\mathcal{F}_a$ onto S.

We prove that if A has finite measure and dimension of S is finite, then (A, S) is a strong annihilating pair. For this, we need the following basic result.

Lemma 3.2.10. [24] *Let P and Q be two orthogonal projections on a complex Hilbert space H. Then $\|PQ\| < 1$ if and only if there exists a positive number C such that for each $x \in H$*

$$\|x\|^2 \leq C \left(\|P^{\perp} x\|^2 + \|Q^{\perp} x\|^2 \right).$$

Consider the projections E_A and F_N as defined by (3.2.1). By Lemma 3.2.2 and Proposition 3.2.5, $E_A F_N$ is a compact operator and $E_A \cap F_N = 0$. Therefore, we must have $\|E_A F_N\| < 1$. Since $\mathcal{R}(F_N^{\perp}) = \{g \in L^2(\mathbb{Q}) : \mathcal{R}(W_a(g)) \subseteq \mathcal{B}_N^{\perp}\}$, it follows that $W_a F_N^{\perp} = P_N^{\perp} W_a$. Thus, by Lemma 3.2.10, (A, S) is a strong annihilating pair, whenever $m(A)$ and $\dim S$ are finite. In this connection, it is worth to mention that one can also prove (A, S) is a strong annihilating pair by using the strategy available in [10].

Chapter 4

Step two Nilpotent Lie groups

In this chapter, we prove a version of Benedicks-Amrein-Berthier theorem for the step two nilpotent Lie groups in terms of rank of the Fourier transform.

4.1 Preliminaries on step two Nilpotent Lie groups

Let G be a connected, simply connected Lie group with real step two nilpotent Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. We can choose an inner product on $\mathfrak{g}$ to make the above decomposition orthogonal. Let $\{X_1, \ldots, X_m\}$ and $\{T_1, \ldots, T_k\}$ be orthonormal basis of $\mathfrak{b}$ and $\mathfrak{z}$ respectively. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp : \mathfrak{g} \to G$ is surjective. Hence G can be identified with $\mathfrak{b} \oplus \mathfrak{z}$. Thus, corresponding to $X + T \in \mathfrak{b} \oplus \mathfrak{z}$, $\exp(X + T) \in G$ and denote it by (X, T). Since $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{z}$ and $[\mathfrak{b}, [\mathfrak{b}, \mathfrak{b}]] = 0$, by the Baker-Campbell-Hausdorff formula, group law on G can be expressed as

$$(X, T)(X', T') = (X + X', T + T' + \frac{1}{2}[X, X'])$$

for $X, X' \in \mathfrak{b}$ and $T, T' \in \mathfrak{z}$. Let $\mathfrak{z}^*$ be the real dual of $\mathfrak{z}$. For each $\lambda \in \mathfrak{z}^*$, define the bilinear form B_λ on $\mathfrak{b}$ by $B_\lambda(X, Y) = \lambda([X, Y])$. Let m_λ be the orthogonal complement of $r_\lambda = \{X : B_\lambda(X,Y) = 0, \forall Y \in \mathfrak{b}\}$ in $\mathfrak{b}$. Then $\Lambda = \{\lambda \in \mathfrak{z}^* : \dim m_\lambda \text{ is maximum}\}$ is a Zariski open subset of $\mathfrak{z}^*$. For $(X,T) = \sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{k} t_i T_i$ we can identify (X,T) with $(x,t) \in \mathbb{R}^m \times \mathbb{R}^k$ and define norm on G by $\|(X,T)\|^2 = x_1^2 + \cdots + x_m^2 + t_1^2 + \cdots + t_k^2$. Hence, the Lebesgue measure on $\mathbb{R}^{m+k}$ can be taken as the Haar measure on G. The step two nilpotent Lie groups can be studied in two different cases. For more details, please refer to [15, 35, 36, 39].

4.1.1 Step two nilpotent Lie groups without MW-condition

In this case $r_\lambda \neq \{0\}$ for each $\lambda \in \Lambda$ and $B_\lambda|_{m_\lambda}$ is non-degenerate, hence $\dim m_\lambda$ even. Let $\{X_1(\lambda), \ldots, X_n(\lambda), Y_1(\lambda), \ldots, Y_n(\lambda), Z_1(\lambda), \ldots, Z_r(\lambda)\}$ be an orthonormal basis of $\mathfrak{b}$ and $d_j(\lambda) > 0$ be satisfying

1. $r_\lambda = \text{span}\{Z_1(\lambda), \ldots, Z_r(\lambda)\}$,
2. $\lambda([X_i(\lambda), Y_j(\lambda)]) = \delta_{ij} d_i(\lambda)$, for $1 \leq i, j \leq n$ and
   $\lambda([X_i(\lambda), X_j(\lambda)]) = 0, \lambda([Y_i(\lambda), Y_j(\lambda)]) = 0$, for $1 \leq i, j \leq n$.

The basis $\mathsf{B} = \{X_1(\lambda), \ldots, X_n(\lambda), Y_1(\lambda), \ldots, Y_n(\lambda), Z_1(\lambda), \ldots, Z_r(\lambda), T_1, \ldots, T_k\}$ of $\mathfrak{g}$ is called almost symplectic basis. Consider the subspaces $\xi_\lambda = \text{span}_{\mathbb{R}}\{X_1(\lambda), \ldots, X_n(\lambda)\}$ and $\eta_\lambda = \text{span}_{\mathbb{R}}\{Y_1(\lambda), \ldots, Y_n(\lambda)\}$. Then we have the decomposition $\mathfrak{g} = \xi_\lambda \oplus \eta_\lambda \oplus r_\lambda \oplus \mathfrak{z}$. Thus any element $\exp(X+Y+Z+T)$ of G can be written as (X,Y,Z,T), where

$$(X,Y,Z,T) = \sum_{i=1}^{n} x_i(\lambda) X_i(\lambda) + \sum_{i=1}^{n} y_i(\lambda) y_i(\lambda) + \sum_{i=1}^{r} z_i(\lambda) Z_i(\lambda) + \sum_{i=1}^{k} t_i T_i$$

and denote it by $(x,y,z,t) \in \mathbb{R}^{2n+r+k}$.

Let $\{X_1^*(\lambda), \ldots, X_n^*(\lambda), Y_1^*(\lambda), \ldots, Y_n^*(\lambda), Z_1^*(\lambda), \ldots, Z_r^*(\lambda), T_1^*, \ldots, T_k^*\}$ be the dual basis of B. Therefore, any element $\lambda \in \Lambda$ and $\mu \in r_\lambda^*$ can be expressed as $(\lambda, \mu) =$

$\sum_{i=1}^{k} \lambda_i T_i^* + \sum_{i=1}^{r} \mu_i(\lambda) Z_i^*(\lambda)$. In view of the almost symplectic basis B, for $\lambda \in \Lambda$ and $\mu \in r_\lambda^*$ the irreducible unitary representation $\pi_{\lambda,\mu}$ of G can be realized on $L^2(\eta_\lambda)$ and expressed as

$$(\pi_{\lambda,\mu}(x,y,z,t)\varphi)(\xi) = e^{i\sum_{j=1}^{k}\lambda_j t_j + i\sum_{j=1}^{r}\mu_j z_j + i\sum_{j=1}^{n} d_j(\lambda)(x_j\xi_j + \frac{1}{2}x_j y_j)}\varphi(\xi + y),$$

where $\varphi \in L^2(\eta_\lambda)$. Define the Fourier transform of $f \in L^1(G)$ by

$$\hat{f}(\lambda,\mu) = \int_{\mathfrak{z}}\int_{r_\lambda}\int_{\eta_\lambda}\int_{\xi_\lambda} f(x,y,z,t)\pi_{\lambda,\mu}(x,y,z,t)\,dxdydzdt.$$

Consider inverse Fourier transform of f in t and (z,t) variables as follows.

$$f^\lambda(x,y,z) = \int_{\mathfrak{z}} e^{i\sum_{j=1}^{k}\lambda_j t_j} f(x,y,z,t)\,dt,$$
$$f^{\lambda,\mu}(x,y) = \int_{r_\lambda}\int_{\mathfrak{z}} e^{i\sum_{j=1}^{k}\lambda_j t_j + i\sum_{j=1}^{r}\mu_j z_j} f(x,y,z,t)\,dtdz.$$

Let $\mathrm{Pf}(\lambda) = \prod_{j=1}^{n} d_j(\lambda)$ be the Pfaffian of λ. If $f \in L^1 \cap L^2(G)$, then $\hat{f}(\lambda,\mu)$ is a Hilbert-Schmidt operator on $L^2(\eta_\lambda)$ and satisfies (see [39])

$$\mathrm{Pf}(\lambda) \parallel \hat{f}(\lambda,\mu) \parallel_{HS}^2 = (2\pi)^n \int_{\eta_\lambda}\int_{\xi_\lambda} |f^{\lambda,\mu}(x,y)|^2\,dxdy. \tag{4.1.1}$$

For $f \in L^2(G)$, we have the Plancherel formula (see [35])

$$\int_\Lambda\int_{r_\lambda^*} \mathrm{Pf}(\lambda) \parallel \hat{f}(\lambda,\mu) \parallel_{HS}^2 \,d\mu d\lambda = (2\pi)^\gamma \int_{\mathfrak{z}}\int_{r_\lambda}\int_{\eta_\lambda}\int_{\xi_\lambda} |f(x,y,z,t)|^2\,dxdydzdt, \tag{4.1.2}$$

where $\gamma = n + r + k$.

4.1.2 Step two nilpotent Lie groups with MW-condition

A step two nilpotent Lie group is called MW group if there exists a $\lambda \in \mathfrak{z}^*$ such that B_λ is non-degenerate. Therefore, $r_\lambda = \{0\}$ for each $\lambda \in \Lambda$ and the irreducible unitary representations can be parametrized by Λ. Hence for $\lambda \in \Lambda$, the irreducible unitary representation π_λ of G can be realized on $L^2(\eta_\lambda)$ by

$$(\pi_\lambda(x,y,t)\varphi)(\xi) = e^{i\sum_{j=1}^{k}\lambda_j t_j + i\sum_{j=1}^{n} d_j(\lambda)(x_j\xi_j + \frac{1}{2}x_j y_j)}\varphi(\xi + y),$$

where $\varphi \in L^2(\eta_\lambda)$. Define the Fourier transform of $f \in L^1(G)$ by

$$\hat{f}(\lambda) = \int_{\mathfrak{z}}\int_{\eta_\lambda}\int_{\xi_\lambda} f(x,y,t)\pi_\lambda(x,y,t)dxdydt.$$

If $f \in L^1 \cap L^2(G)$, then $\hat{f}(\lambda,\mu)$ is a Hilbert Schmidt operator on $L^2(\eta_\lambda)$ and satisfies (see [39])

$$\mathrm{Pf}(\lambda) \parallel \hat{f}(\lambda) \parallel_{HS}^2 = (2\pi)^n \int_{\eta_\lambda}\int_{\xi_\lambda} |f^\lambda(x,y)|^2 dxdy, \tag{4.1.3}$$

where f^λ is the inverse Fourier transform of f in t variable. For $f \in L^2(G)$ we have the Plancherel formula (see [35])

$$\int_\Lambda \mathrm{Pf}(\lambda) \parallel \hat{f}(\lambda) \parallel_{HS}^2 d\lambda = (2\pi)^{n+k} \int_{\mathfrak{z}}\int_{\eta_\lambda}\int_{\xi_\lambda} |f(x,y,t)|^2 dxdydt. \tag{4.1.4}$$

4.2 Benedicks-Amrein-Berthier type theorem

In this section, we consider rank analogue of Benedicks-Amrein-Berthier theorem for the step two nilpotent Lie groups with MW-condition. Consequently, we obtain strong annihilating pair for the Weyl transform.

Theorem 4.2.1. *(MW group) Let* $f \in L^1(G)$ *and* $\{(x,y,t) : f(x,y,t) \neq 0\} \subseteq A \times \mathfrak{z}$, *where* $A \subset \mathfrak{b}$ *with finite measure. If* $\hat{f}(\lambda)$ *is a finite rank operator for each* $\lambda \in \Lambda$, *then* $f = 0$.

The fact that representation π_λ for the groups with MW-condition is square integrable modulo the center, plays an important role in proving the above result. But this is not true for the representation $\pi_{\lambda,\mu}$ of the groups without MW-condition. If we recall the decomposition $\mathfrak{g} = \xi_\lambda \oplus \eta_\lambda \oplus r_\lambda \oplus \mathfrak{z}$, then $\pi_{\lambda,\mu}(x,y,0,0)$ becomes square integrable. In this case, we need to consider functions supported on $A_\lambda \times r_\lambda \times \mathfrak{z}$, where $A_\lambda \subset \xi_\lambda \oplus \eta_\lambda$ has finite measure. Since the above decomposition is not global, depends on each λ, such a support condition will not make sense. Thus, we prove the result only for the groups with MW-condition.

In order to prove Theorem 4.2.1, we need to define Weyl type transform, which is the most non-commutative constituent of the group Fourier transform. Consider a $\lambda \in \Lambda$ and denote by $\pi_\lambda(x,y) = \pi_\lambda(x,y,0)$. Then define Weyl transform of $g \in L^1 \cap L^2(\mathfrak{b})$ by

$$W_\lambda(g) = \int_{\mathfrak{b}} g(x,y)\pi_\lambda(x,y)dxdy.$$

This, in turn, implies $\hat{f}(\lambda) = W_\lambda(f^\lambda)$ and satisfies the Plancherel formula (4.1.3). The inversion formula for W_λ can be expressed as

$$g(x,y) = (2\pi)^{-n}\mathrm{Pf}(\lambda)\mathrm{tr}\left(\pi_\lambda^*(x,y)W_\lambda(g)\right). \tag{4.2.1}$$

The Fourier-Wigner transform of $\varphi, \psi \in L^2(\eta_\lambda)$ is defined by the formula

$$V(\varphi,\psi)(x,y) = (2\pi)^{-n/2}\mathrm{Pf}(\lambda)^{1/2}\langle \pi_\lambda(x,y)\varphi, \psi\rangle.$$

A simple calculation, as in the case of the Heisenberg group (see [52]), leads to the following orthogonality relation.

Lemma 4.2.2. *Let* $\varphi_l, \psi_l \in L^2(\eta_\lambda)$ *for* $l = 1, 2$. *Then*

$$\int_{\mathfrak{b}} V(\varphi_1, \psi_1)(x, y)\overline{V(\varphi_2, \psi_2)}(x, y)\, dxdy = \langle \varphi_1, \varphi_2 \rangle \overline{\langle \psi_1, \psi_2 \rangle}.$$

Since $f \in L^1(G)$ implies $f^\lambda \in L^1(\mathfrak{b})$ and $\hat{f}(\lambda) = W_\lambda(f^\lambda)$, to prove Theorem 4.2.1, it is enough to consider the following result for the Weyl transform.

Proposition 4.2.3. *Let* $\lambda \in \Lambda$ *and* $A \subset \mathfrak{b}$ *be a set of finite measure. Suppose* $g \in L^1(\mathfrak{b})$ *and* $\{(x, y) \in \mathfrak{b} : g(x, y) \neq 0\} \subseteq A$. *If* $W_\lambda(g)$ *has finite rank, then* $g = 0$.

Let $g \in L^2(\mathfrak{b})$ and $W_\lambda(g)$ is of finite rank. Then, there exists an orthonormal basis $\{\varphi_1, \varphi_2, \ldots\}$ of $L^2(\eta_\lambda)$ such that $\mathcal{R}(W_\lambda(g)) = \mathcal{B}_N$, where $\mathcal{B}_N = \text{span}\{\varphi_1, \ldots, \varphi_N\}$ and $\mathcal{R}$ stands for the range. Consider the orthogonal projection P_N of $L^2(\eta_\lambda)$ onto $\mathcal{B}_N$. Let A be a measurable subset of $\mathfrak{b}$. Define a pair of orthogonal projections E_A and F_N of $L^2(\mathfrak{b})$ by

$$E_A g = \chi_A g \qquad \text{and} \qquad W_\lambda(F_N g) = P_N W_\lambda(g), \tag{4.2.2}$$

where χ_A denotes the characteristic function of A. Then

$$\mathcal{R}(E_A) = \{g \in L^2(\mathfrak{b}) : g = \chi_A g\} \text{ and } \mathcal{R}(F_N) = \{g \in L^2(\mathfrak{b}) : \mathcal{R}(W_\lambda(g)) \subseteq \mathcal{B}_N\}.$$

To prove Proposition 4.2.3, it is enough to show that $E_A \cap F_N = 0$. Further, proof of $E_A \cap F_N = 0$ will proceed through by describing the dimension of the space $\mathcal{R}(E_A \cap F_N)$. Since $\mathfrak{b}$ can be understood as $\mathbb{R}^{2n}$, it is adequate to think E_A, F_N are projections of $L^2(\mathbb{R}^{2n})$. First, we prove that $E_A F_N$ is a Hilbert-Schmidt operator that satisfies $\|E_A F_N\|_{HS}^2 = (2\pi)^{-n}\text{Pf}(\lambda)m(A)N$. Throughout this section, A will be considered as a set of finite measure and $c_\lambda = (2\pi)^{-n}\text{Pf}(\lambda)$.

Lemma 4.2.4. *The operator E_AF_N is an integral operator on $L^2(\mathbb{R}^{2n})$, with kernel $K(x,y,u,v) = c_\lambda \chi_A(x,y) tr\left(P_N \pi_\lambda(u,v)\pi_\lambda(-x,-y)\right)$.*

Proof. For $g \in L^2(\mathbb{R}^{2n})$, we have $W_\lambda(F_N g) = P_N W_\lambda(g)$. By the inversion formula for the Weyl transform (4.2.1), we have

$$\begin{aligned}(F_N g)(x,y) &= c_\lambda \ \mathrm{tr}\left(\pi^*_\lambda(x,y) W_\lambda(F_N g)\right) = c_\lambda \ \mathrm{tr}\left(\pi_\lambda(-x,-y) P_N W_\lambda(g)\right)\\ &= c_\lambda \ \mathrm{tr}\left(P_N W_\lambda(g)\pi_\lambda(-x,-y)\right)\\ &= c_\lambda \int_{\mathbb{R}^{2n}} g(u,v) \ \mathrm{tr}\left(P_N \pi_\lambda(u,v)\pi_\lambda(-x,-y)\right) dudv.\end{aligned}$$

Hence it follows that

$$\begin{aligned}(E_A F_N g)(x,y) &= \chi_A(x,y)(F_N g)(x,y)\\ &= c_\lambda \chi_A(x,y) \int_{\mathbb{R}^{2n}} g(u,v) \ \mathrm{tr}\left(P_N \pi_\lambda(u,v)\pi_\lambda(-x,-y)\right) dudv\\ &= \int_{\mathbb{R}^{2n}} g(u,v) K(x,y,u,v) dudv,\end{aligned}$$

where $K(x,y,u,v) = c_\lambda \chi_A(x,y) \ \mathrm{tr}\left(P_N \pi_\lambda(u,v)\pi_\lambda(-x,-y)\right)$. We infer that E_AF_N is an integral operator with kernel K. □

Lemma 4.2.5. *E_AF_N is Hilbert-Schmidt and $\|E_AF_N\|^2_{HS} = (2\pi)^{-n} \ Pf(\lambda) m(A) N$.*

Proof. From Lemma 4.2.4, we know that E_AF_N is an integral operator with kernel $K(x,y,u,v)$. Therefore,

$$\begin{aligned}\|E_AF_N\|^2_{HS} &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} |K(x,y,u,v)|^2 dudvdxdy\\ &= c^2_\lambda \int_{\mathbb{R}^{2n}} |\chi_A(x,y)|^2 \left(\int_{\mathbb{R}^{2n}} \left|\mathrm{tr}\left(P_N \pi_\lambda(u,v)\pi_\lambda(-x,-y)\right)\right|^2 dudv \right) dxdy\\ &= c^2_\lambda \int_{\mathbb{R}^{2n}} \chi_A(x,y) \int_{\mathbb{R}^{2n}} \left| \sum_{\alpha=1}^{N} \langle \pi_\lambda(u,v)\pi_\lambda(-x,-y)\varphi_\alpha, \varphi_\alpha\rangle \right|^2 dudvdxdy.\end{aligned}$$

Since $\pi_\lambda(u,v)\pi_\lambda(-x,-y) = e^{-i\sum_{j=1}^{n}\frac{d_j(\lambda)}{2}(x_jv_j - y_ju_j)}\pi_\lambda(u-x,v-y)$, the second integral of the above equation is equal to

$$\int_{\mathbb{R}^{2n}} \left| e^{-i\sum_{j=1}^{n}\frac{d_j(\lambda)}{2}(x_jv_j - y_ju_j)} \sum_{\alpha=1}^{N} \langle \pi_\lambda(u-x,v-y)\varphi_\alpha, \varphi_\alpha\rangle \right|^2 dudv$$
$$= \int_{\mathbb{R}^{2n}} \left| \sum_{\alpha=1}^{N} \langle \pi_\lambda(u,v)\varphi_\alpha, \varphi_\alpha\rangle \right|^2 dudv.$$

Hence from Lemma 4.2.2, orthogonality of the Fourier-Wigner transform, it follows that

$$\|E_AF_N\|_{HS}^2 = (2\pi)^{-n}\mathrm{Pf}(\lambda)m(A)N.$$

□

Let S be a closed subspace of $L^2(\mathbb{R}^n)$. Define the orthogonal projection F_S on $L^2(\mathbb{R}^{2n})$ by $W_\lambda(F_Sg) = P_SW_\lambda(g)$, where P_S is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto S and $g \in L^2(\mathbb{R}^{2n})$. In particular, if $S = \mathcal{B}_N$, then $F_S = F_N$. For $A \subseteq \mathbb{R}^{2n}$ and $w = (u,v) \in \mathbb{R}^{2n}$, denote $wA = \{(x,y) \in \mathbb{R}^{2n} : (x-u, y-v) \in A\}$.

Proposition 4.2.6. *Let $A \subset \mathbb{R}^{2n}$ with finite measure and S be a closed subspace of $L^2(\mathbb{R}^n)$. Then either $E_A \cap F_S = 0$ or for every $\epsilon' > 0$, there exists $\tilde{A} \supset A$ with $m(\tilde{A} \setminus A) < \epsilon'$ such that $\mathcal{R}(E_{\tilde{A}} \cap F_S)$ is of infinite dimension.*

Proof. If $E_A \cap F_S \neq 0$, then there exists a non-zero function $g_0 \in \mathcal{R}(E_A \cap F_S)$. Let $A_0 = \{(x,y) \in A : g_0(x,y) \neq 0\}$ and $\tilde{A}_1 = A$. By Lemma 3.2.3, for $\epsilon = \frac{\epsilon'}{2^{l+1}}$, $B_0 = A_0$ and $B = \tilde{A}_l$, there exists $w_l = (u^{(l)}, v^{(l)}) \in \mathbb{R}^{2n}$ such that

$$m(\tilde{A}_l) < m(\tilde{A}_l \cup w_lA_0) < m(\tilde{A}_l) + \frac{\epsilon'}{2^{l+1}}. \tag{4.2.3}$$

Put $\tilde{A}_{l+1} = \tilde{A}_l \cup w_lA_0$ and $\tilde{A} = \bigcup_{l=1}^{\infty} \tilde{A}_l$. Then $\tilde{A}_l$ is a non-decreasing sequence, and hence from (4.2.3) it follows that $m(\tilde{A} \smallsetminus A) < \epsilon'$. For $l \in \mathbb{N}$, consider the function

$$g_l(x,y) = e^{\frac{i}{2}\sum_{j=1}^n d_j(\lambda)(y_j u_j^{(l)} - x_j v_j^{(l)})} g_0(x - u^{(l)}, y - v^{(l)}),$$

where $(u^{(l)}, v^{(l)}) = (u_1^{(l)}, \ldots, u_n^{(l)}, v_1^{(l)}, \ldots, v_n^{(l)})$. We show that $g_l \in \mathcal{R}(E_{\tilde{A}} \cap F_S)$ for each $l \in \mathbb{N}$, and they are linearly independent. Let $\mathcal{B}_S$ be an orthonormal basis of S. Then we can extend $\mathcal{B}_S$ to an orthonormal basis $\mathcal{B}$ of $L^2(\mathbb{R}^n)$. For $\varphi \in L^2(\mathbb{R}^n)$ and $\psi \in \mathcal{B} \smallsetminus \mathcal{B}_S$, we have

$$\begin{aligned}
\langle W_\lambda(g_l)\varphi, \psi\rangle &= \int_{\mathbb{R}^{2n}} g_l(x,y)\langle \pi_\lambda(x,y)\varphi, \psi\rangle dxdy \\
&= \int_{\mathbb{R}^{2n}} e^{\frac{i}{2}\sum_{j=1}^n d_j(\lambda)(y_j u_j^{(l)} - x_j v_j^{(l)})} g_0(x - u^{(l)}, y - v^{(l)})\langle \pi_\lambda(x,y)\varphi, \psi\rangle dxdy \\
&= \int_{\mathbb{R}^{2n}} e^{\frac{i}{2}\sum_{j=1}^n d_j(\lambda)(y_j u_j^{(l)} - x_j v_j^{(l)})} g_0(x,y)\langle \pi_\lambda(x + u^{(l)}, y + v^{(l)})\varphi, \psi\rangle dxdy.
\end{aligned}$$

Since $\pi_\lambda(x,y)\pi_\lambda(u,v) = e^{\frac{i}{2}\sum_{j=1}^n d_j(\lambda)(y_j u_j^{(l)} - x_j v_j^{(l)})}\pi_\lambda(x + u^{(l)}, y + v^{(l)})$, we get

$$\begin{aligned}
\langle W_\lambda(g_l)\varphi, \psi\rangle &= \int_{\mathbb{R}^{2n}} g_0(x,y)\langle \pi_\lambda(x,y)\pi_\lambda(u^{(l)}, v^{(l)})\varphi, \psi\rangle dxdy \\
&= \int_{\mathbb{R}^{2n}} g_0(x,y)\langle \pi_\lambda(x,y)\tilde{\varphi}, \psi\rangle dxdy \\
&= \langle W_\lambda(g_0)\tilde{\varphi}, \psi\rangle = 0.
\end{aligned}$$

Hence $\mathcal{R}(W_\lambda(g_l)) \subseteq \mathcal{B}_S$. Let $A_l = A_{l-1} \cup w_l A_0$. Then $\tilde{A}_{l+1} = \tilde{A}_l \cup A_l$. Thus, $m(A_l \smallsetminus A_{l-1}) \geq m(\tilde{A}_{l+1} \smallsetminus \tilde{A}_l) > 0$. Let $s \in \mathbb{N}$. Since, $A_s = A_0 \cup w_1 A_0 \cup \cdots \cup w_s A_0$ and $g_l) = 0$ on $(w_l A_0)^c$, we have $E_{A_s} g_l = g_l$ for $l = 0, 1, \ldots, s$. Furthermore, $E_{A_s \smallsetminus A_{s-1}} g_l = 0$ for $l = 0, \ldots, s-1$ and $E_{A_s \smallsetminus A_{s-1}} g_s \neq 0$. Therefore, it shows that g_s is not a linear combination of $g_0, \ldots, g_{s-1}$. Since s is arbitrary, $\{g_l : l \in \mathbb{N}\}$ is a linearly independent set in $\mathcal{R}(E_{\tilde{A}} \cap F_S)$. □

Proposition 4.2.7. *Let A be a measurable subset of $\mathbb{C}^n$ having finite Lebesgue measure. Then the projection $E_A \cap F_N = 0$.*

Proof. In view of (3.2.2) and Lemma 4.2.5, we obtain

$$\dim \mathcal{R}(E_{\tilde{A}} \cap F_N) \leq (2\pi)^{-n}\mathrm{Pf}(\lambda) m(\tilde{A}) N < \infty.$$

Therefore, as a corollary of Proposition 4.2.6, we get $E_A \cap F_N = 0$. □

By assuming $W_\lambda(g)$ is a finite rank operator, by the Plancheral theorem for the Weyl transform, it is enough to prove Proposition 4.2.3 for $g \in L^2(\xi_\lambda \oplus \eta_\lambda)$. Further, Proposition 4.2.3 follows from Proposition 4.2.7.

4.2.1 Strong annihilating pair

Now, as a quantitative version of Proposition 4.2.3, we shall define and find out strong annihilating pair for the Weyl transform.

Definition 4.2.8. *Let A be a measurable subset of $\mathbb{R}^{2n}$, and S be a closed subspace of $L^2(\mathbb{R}^n)$. We say that the pair (A, S) is a* strong annihilating pair *for the Weyl transform W_λ if there exists a positive number $C = C(A, S)$ such that for every $g \in L^2(\mathbb{R}^{2n})$*

$$\|g\|_2^2 \leq C \left(\|g\|_{L^2(A^c)} + \|P_S^\perp W_\lambda(g)\|_{HS}^2 \right), \tag{4.2.4}$$

where P_S is the projection of $L^2(\mathbb{R}^n)$ onto S.

Consider the projections E_A and F_N as defined by (4.2.2). By Lemma 4.2.5 and Proposition 4.2.7, $E_A F_N$ is a compact operator and $E_A \cap F_N = 0$. Hence $\|E_A F_N\| < 1$. Since $\mathcal{R}(F_N)^\perp = \{g \in L^2(\mathbb{R}^{2n}) : \mathcal{R}(W_\lambda(g)) \subseteq \mathcal{B}_N^\perp\}$, it follows that $W_\lambda F_N^\perp = P_N^\perp W_\lambda$. Thus by Lemma 3.2.10, (A, S) is a strong annihilating pair, whenever $m(A) < \infty$ and $\dim S < \infty$.

4.3 Related problems for future work

Problem 1 : Let G be a step two nilpotent Lie group with MW condition and $\mathcal{E}$ be a set of finite measure in G. It is natural to ask whether there exists a non-zero function $f \in L^1(G)$ such that $\{(x,y,t) \in G : f(x,y,t) \neq 0\} \subseteq \mathcal{E}$ and $\hat{f}(\lambda)$ has finite rank for every choice of $\lambda \in \Lambda$. If the projection of $\mathcal{E}$ into $\mathfrak{b}$ has finite measure, then by Theorem 4.2.1 we get $f = 0$. However, the other case and analogue result for the non-MW group is still open.

Problem 2 : In an interesting result on the real line, Nazarov [34] proved that the constant in (1.2.2) is of the form $C_1 e^{C_1 m(A) m(\Sigma)}$. Later, Jaming [29] described an improved version of the above constant in the higher dimensional Euclidean spaces. Thus, it is an interesting question to find out the constant in (4.2.4), in terms of $m(A)$ and the dimension of S.

Chapter 5

Heisenberg motion group

In this chapter, we prove a variant of Benedicks-Amrein-Berthier theorem for the Heisenberg motion group G. It is known that the set of all integrable functions on G, which are $U(n)$-bi-invariant, form a commutative convolution algebra. Hence, the Fourier transform of an $U(n)$-bi-invariant integrable function has rank one, irrespective of support of the function. Thus, an exact analogue of the Heisenberg group result (Theorem 4.2.1) is not true for the Heisenberg motion group.

However, we prove that if an integrable function is supported on a finite measure set, and its Weyl transform is non-zero only for finitely many Fourier-Wigner pieces and have finite rank, then the function is zero. Consequently, we obtain that if each Fourier-Wigner piece of a non-trivial function is supported on a set of finite measure, then all of its Fourier transform can not have finite rank. A quantitative interpretation of this result is described through strong annihilating pair for the Weyl transform.

5.1 Preliminaries and auxiliary results

In this section, we describe necessary notions and auxiliary results about the Heisenberg motion group and its irreducible representations. For more details, see [42]. The significance of the Fourier-Wigner transform is alluded to in the introduction. Later, we emphasize the efficacy of the Fourier-Wigner decomposition compared to the usual decomposition due to the Peter-Weyl theorem. We derive the inversion formula for the group Fourier transform.

Let $\mathbb{H}^n$ be the Heisenberg group and for each $\lambda \in \mathbb{R}^*$, π_λ's are the Schrödinger representation of $\mathbb{H}^n$, as mentioned in Section 2.3.

Having chosen sublaplacian $\mathcal{L}$ of the Heisenberg group $\mathbb{H}^n$ and its geometry, there is a larger group of isometries that commute with $\mathcal{L}$, known as Heisenberg motion group. The Heisenberg motion group G is the semidirect product of $\mathbb{H}^n$ with the unitary group $K = U(n)$. Since K defines a group of automorphisms on $\mathbb{H}^n$, via $k \cdot (z,t) = (kz, t)$, the group law on G can be expressed as

$$(z,t,k_1) \cdot (w,s,k_2) = \left(z + k_1 w, t + s - \frac{1}{2}\mathrm{Im}(k_1 w \cdot \bar{z}), k_1 k_2 \right).$$

Since a right K-invariant function on G can be thought of as a function on $\mathbb{H}^n$, the Haar measure on G is given by $dg = dzdtdk$, where $dzdt$ and dk are the normalized Haar measure on $\mathbb{H}^n$ and K, respectively.

For $k \in K$, define another set of representations of the Heisenberg group $\mathbb{H}^n$ by $\pi_{\lambda,k}(z,t) = \pi_\lambda(kz,t)$. Since $\pi_{\lambda,k}$ agrees with π_λ on the center of $\mathbb{H}^n$, it follows by the Stone-Von Neumann theorem for the Schrödinger representation that $\pi_{\lambda,k}$ is equivalent to π_λ. Hence, there exists an intertwining operator $\mu_\lambda(k)$ satisfying

$$\pi_\lambda(kz,t) = \mu_\lambda(k)\pi_\lambda(z,t)\mu_\lambda(k)^*.$$

By an appropriate selection of μ_λ, it becomes a unitary representation of K on $L^2(\mathbb{R}^n)$, called metaplectic representation. For details, we refer to [9]. Let $(\sigma, \mathcal{H}_\sigma)$ be an irreducible unitary representation of K and $\mathcal{H}_\sigma = \text{span}\{e_j^\sigma : 1 \leq j \leq d_\sigma\}$. For $k \in K$, the matrix coefficients of the representation $\sigma \in \hat{K}$ are given by $\varphi_{ij}^\sigma(k) = \langle \sigma(k)e_j^\sigma, e_i^\sigma \rangle$, where $i, j = 1, \ldots, d_\sigma$.

Let $\phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}}\phi_\alpha(\sqrt{|\lambda|}x);\ \alpha \in \mathbb{Z}_+^n$, where ϕ_α's are the Hermite functions on $\mathbb{R}^n$. Since for each $\lambda \in \mathbb{R}^*$, the set $\{\phi_\alpha^\lambda : \alpha \in \mathbb{Z}_+^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$, letting $P_m^\lambda = \text{span}\{\phi_\alpha^\lambda :\ |\alpha| = m\}$, μ_λ becomes an irreducible unitary representation of K on P_m^λ. Hence, the action of μ_λ can be realized on P_m^λ by

$$\mu_\lambda(k)\phi_\gamma^\lambda = \sum_{|\alpha|=|\gamma|} \eta_{\gamma\alpha}^\lambda(k)\phi_\alpha^\lambda, \tag{5.1.1}$$

where $\eta_{\gamma\alpha}^\lambda$'s are the matrix coefficients of $\mu_\lambda(k)$. Define a bilinear form $\phi_\alpha^\lambda \otimes e_j^\sigma$ on $L^2(\mathbb{R}^n) \times \mathcal{H}_\sigma$ by $\phi_\alpha^\lambda \otimes e_j^\sigma = \phi_\alpha^\lambda e_j^\sigma$. Then $\{\phi_\alpha^\lambda \otimes e_j^\sigma : \alpha \in \mathbb{Z}_+^n, 1 \leq j \leq d_\sigma\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$. Denote $\mathcal{H}_\sigma^2 = L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$.

Define a representation ρ_σ^λ of G on the space $\mathcal{H}_\sigma^2$ by

$$\rho_\sigma^\lambda(z, t, k) = \pi_\lambda(z, t)\mu_\lambda(k) \otimes \sigma(k).$$

Then ρ_σ^λ are all possible irreducible unitary representations of G that participate in the Plancherel formula [42]. Thus, in view of the above discussion, we shall denote the partial dual of the group G by $G' \cong \mathbb{R}^* \times \hat{K}$. For $(\lambda, \sigma) \in G'$, the Fourier transform of $f \in L^1(G)$, defined by

$$\hat{f}(\lambda, \sigma) = \int_K \int_{\mathbb{R}} \int_{\mathbb{C}^n} f(z, t, k)\rho_\sigma^\lambda(z, t, k)dzdtdk,$$

is a bounded linear operator on $\mathcal{H}_\sigma^2$. As the Plancherel formula [42]

$$\int_K \int_{\mathbb{H}^n} |f(z, t, k)|^2 dzdtdk = (2\pi)^{-n} \sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda, \sigma)\|_{HS}^2 |\lambda|^n d\lambda$$

holds for $f \in L^2(G)$, it follows that $\hat{f}(\lambda,\sigma)$ is a Hilbert-Schmidt operator on $\mathcal{H}^2_\sigma$.

In order to prove our main result on the Heisenberg motion group G, it is enough to consider a similar proposition for the Weyl transform on $G^\times = \mathbb{C}^n \times K$. For that, we require to set some preliminaries about the Weyl transform on $G^\times$.

Let f^λ be the inverse Fourier transform of the function f in the t variable defined by

$$f^\lambda(z,k) = \int_{\mathbb{R}} f(z,t,k)e^{i\lambda t}dt. \tag{5.1.2}$$

Then

$$\hat{f}(\lambda,\sigma) = \int_K \int_{\mathbb{C}^n} f^\lambda(z,k)\rho^\lambda_\sigma(z,k)dzdk,$$

where $\rho^\lambda_\sigma(z,k) = \rho^\lambda_\sigma(z,0,k)$. For $(\lambda,\sigma) \in G'$, define the Weyl transform W^λ_σ on $L^1(G^\times)$ by

$$W^\lambda_\sigma(g) = \int_K \int_{\mathbb{C}^n} g(z,k)\rho^\lambda_\sigma(z,k)dzdk.$$

Then $\hat{f}(\lambda,\sigma) = W^\lambda_\sigma(f^\lambda)$, and hence $W^\lambda_\sigma(g)$ is a bounded operator if $g \in L^1(G^\times)$. On the other hand, if $g \in L^2(G^\times)$ then $W^\lambda_\sigma(g)$ becomes a Hilbert-Schmidt operator satisfying the Plancherel formula

$$\int_K \int_{\mathbb{C}^n} |g(z,k)|^2 dzdk = (2\pi)^{-n}|\lambda|^n \sum_{\sigma\in\hat{K}} d_\sigma \left|\left|W^\lambda_\sigma(g)\right|\right|^2_{HS}. \tag{5.1.3}$$

Fourier-Wigner representation: Define the Fourier-Wigner transform V^η_ζ of the functions $\zeta,\eta \in \mathcal{H}^2_\sigma$ by

$$V^\eta_\zeta(z,k) = (2\pi)^{-\frac{n}{2}}|\lambda|^{\frac{n}{2}} \left\langle \rho^\lambda_\sigma(z,k)\zeta,\eta \right\rangle,$$

where $(z,k) \in G^\times$. The following orthogonality relation is derived in [13]. A version of this result also appeared in [42].

Lemma 5.1.1. *For $\zeta_l, \eta_l \in \mathcal{H}^2_\sigma,\ l = 1, 2,$ the corresponding Fourier-Wigner transform satisfies*

$$\int_K \int_{\mathbb{C}^n} V^{\eta_1}_{\zeta_1}(z, k)\overline{V^{\eta_2}_{\zeta_2}(z, k)}dzdk = \langle\zeta_1, \zeta_2\rangle\,\overline{\langle\eta_1, \eta_2\rangle}.$$

In particular, $V^\eta_\zeta \in L^2(G^\times)$. Let $V^\lambda_\sigma = \overline{\text{span}}\{V^\eta_\zeta : \zeta, \eta \in \mathcal{H}^2_\sigma\}$ and set $\Psi^\sigma_{\alpha,j} = \phi^\lambda_\alpha \otimes e^\sigma_j$. Since $B^\lambda_\sigma = \{\Psi^\sigma_{\alpha,j}\ : \alpha \in \mathbb{Z}^n_+, 1 \leq j \leq d_\sigma\}$ forms an orthonormal basis for $\mathcal{H}^2_\sigma$, by Lemma 5.1.1, we infer that

$$V_{B^\lambda_\sigma} = \left\{V^{\Psi^\sigma_{\alpha_2,j_2}}_{\Psi^\sigma_{\alpha_1,j_1}} :\ \Psi^\sigma_{\alpha_1,j_1}, \Psi^\sigma_{\alpha_2,j_2} \in B^\lambda_\sigma\right\}$$

is an orthonormal basis for V^λ_σ. The next result, which is followed as a corollary of the Peter-Weyl theorem [51], will be the desired decomposition of $L^2(G^\times)$.

Proposition 5.1.2. *The set $\bigcup_{\sigma\in\hat{K}} V_{B^\lambda_\sigma}$ is an orthonormal basis for $L^2(G^\times)$.*

Since $V_{B^\lambda_\sigma}$ is an orthonormal basis for V^λ_σ, by Proposition 5.1.2, we infer that $L^2(G^\times) = \bigoplus_{\sigma\in\hat{K}} V^\lambda_\sigma$. We shall call this as the ***Fourier-Wigner decomposition*** and V^λ_σ as ***Fourier-Wigner representation*** of $G^\times$.

Remark 5.1.3. *Though the decomposition in Proposition 5.1.2 is being followed by the Peter-Weyl theorem, it is quite finer than the usual Peter-Weyl decomposition of function on K, due to the presence of the metaplectic representation. And as an effect, even if $g \in L^2(G^\times)$ is K-bi-invariant on $G^\times$, it need not fall into the trivial Fourier-Wigner representation. This fact can be explained more explicitly via the following example.*

Consider the one dimensional Heisenberg motion group $\mathbb{H}^1 \rtimes U(1)$. Realize $U(1) \cong S^1$. Let $(z, t, e^{i\theta}) \in \mathbb{H}^1 \rtimes U(1)$. Then for each $(\lambda, \alpha) \in \mathbb{R}^* \times \mathbb{Z}$, the unitary irreducible representations $(\rho^\lambda_\alpha, L^2(\mathbb{R}))$ of $\mathbb{H}^1 \rtimes U(1)$ can be defined by

$$\rho^\lambda_\alpha(z, t, e^{i\theta}) = e^{-i\alpha\theta}\pi_\lambda(z, t)\mu_\lambda(e^{i\theta}).$$

In fact, the action of ρ_α^λ on Hermite function ϕ_β^λ, where $\beta \in \mathbb{Z}_+$, will be given by

$$\rho_\alpha^\lambda(z,t,e^{i\theta})\phi_\beta^\lambda = e^{-i(\alpha-\beta)\theta}\pi_\lambda(z,t)\phi_\beta^\lambda. \tag{5.1.4}$$

For more details, see [41]. By Proposition 5.1.2, we have $L^2(\mathbb{C}\times S^1) = \oplus_{\alpha\in\mathbb{Z}} V_\alpha^\lambda$, where

$$V_{\alpha'}^\lambda = \{\varphi \in L^2(\mathbb{C}\times S^1) : W_\alpha^\lambda(\varphi) = 0 \text{ for all } \alpha \neq \alpha'\}.$$

From (5.1.4), it follows that

$$\langle \rho_\alpha^\lambda(z,e^{i\theta})\phi_\beta^\lambda, \phi_\gamma^\lambda\rangle = e^{-i(\alpha-\beta)\theta}\langle \pi_\lambda(z)\phi_\beta^\lambda, \phi_\gamma^\lambda\rangle = e^{-i(\alpha-\beta)\theta}\Phi_{\beta,\gamma}^\lambda(z),$$

where $\Phi_{\beta,\gamma}^\lambda$ is the special Hermite function. Denote $\tilde{\Phi}_{\beta,\gamma}^{\alpha,\lambda}(z,e^{i\theta}) = e^{-i(\alpha-\beta)\theta}\Phi_{\beta,\gamma}^\lambda(z)$. Then $\{\tilde{\Phi}_{\beta,\gamma}^{\alpha,\lambda} : \beta,\gamma \in \mathbb{Z}_+\}$ will be an orthonormal basis for V_α^λ. In particular, corresponding to the trivial representation, $\tilde{\Phi}_{\beta,\gamma}^{0,\lambda}(z,e^{i\theta}) = e^{i\beta\theta}\Phi_{\beta,\gamma}^\lambda(z)$; $\beta,\gamma \in \mathbb{Z}_+$, forms an orthonormal basis for V_0^λ. Thus, the presence of $e^{i\beta\theta}$ in the basis concludes that an arbitrary $h \in L^2(\mathbb{C})$ need not be contained in V_0^λ. To be explicit, consider a function $h \in L^2(\mathbb{C})$ supported on a set of finite measure. Further, define a function g on $\mathbb{C}\times S^1$ by $g(z,e^{i\theta}) = h(z)$. Then, the Weyl transform of g acts on the Hermite functions ϕ_β^λ by

$$\begin{aligned} W_\alpha^\lambda(g)\phi_\beta^\lambda &= \int_{\mathbb{C}\times[0,2\pi]} g(z,e^{i\theta})\rho_\alpha^\lambda(z,e^{i\theta})\phi_\beta^\lambda\, dz d\theta \\ &= W_\lambda(h)\phi_\beta^\lambda \cdot \int_0^{2\pi} e^{-i(\alpha-\beta)\theta} d\theta. \end{aligned}$$

Hence, for each $\alpha \in \mathbb{Z}$, $W_\alpha^\lambda(g)$ can have rank at most one. Further, $W_\beta^\lambda(g)\phi_\beta^\lambda = W_\lambda(h)\phi_\beta^\lambda$ for $\beta \in \mathbb{Z}_+$. Since h is a non-zero function supported on a finite measure set, $W_\lambda(h)$ cannot be a finite rank operator, see [33, 54]. Therefore, there exist infinitely many β such that $W_\beta^\lambda(g)\Phi_\beta^\lambda \neq 0$. Hence, g is not contained in any V_α^λ. In fact, g fails to be a member of any finite union of V_α^λ's.

The above discussion brings forward the following question. Does there exist a non-trivial function $g \in L^1(G^\times)$ which is supported on a set of finite measure, and whose Weyl transform $W_\sigma^\lambda(g)$ has finite rank only for finitely many σ along with $W_\sigma^\lambda(g) = 0$ for other σ? The non-existence of such a function is guaranteed by Proposition 5.2.4.

Next, we prove the inversion formula for the Weyl transform W_σ^λ which is a key ingredient while proving our main result. For this, we need the fact that

$$\rho_\sigma^\lambda(z, k_1)\rho_\sigma^\lambda(w, k_2) = e^{-\frac{i\lambda}{2}\mathrm{Im}(k_1 w.\bar{z})}\rho_\sigma^\lambda(z + k_1 w, k_1 k_2), \tag{5.1.5}$$

where $(z, k_1), (w, k_2) \in G^\times$.

Theorem 5.1.4. (***Inversion formula***) *Let $g \in L^1 \cap L^2(G^\times)$. Then*

$$g(z, k) = (2\pi)^{-n}|\lambda|^n \sum_{\sigma \in \hat{K}} d_\sigma \ tr\left(W_\sigma^\lambda(g)(\rho_\sigma^\lambda)^*(z, k)\right), \tag{5.1.6}$$

where the series converges in $L^2(G^\times)$.

Proof. For $(z, k_1) \in G^\times$, we have

$$\begin{aligned} W_\sigma^\lambda(g)(\rho_\sigma^\lambda)^*(z, k_1) &= \int_{G^\times} g(w, k_2)\rho_\sigma^\lambda(w, k_2)\rho_\sigma^\lambda(-k_1^{-1}z, k_1^{-1})dwdk_2 \\ &= \int_{G^\times} g(w, k_2)e^{\frac{i\lambda}{2}\mathrm{Im}(k_2 k_1^{-1} z.\bar{w})}\rho_\sigma^\lambda(w - k_2 k_1^{-1} z, k_2 k_1^{-1})dwdk_2, \end{aligned}$$

where last equality follows from (5.1.5). Hence tr $\left(W_\sigma^\lambda(g)(\rho_\sigma^\lambda)^*(z, k_1)\right)$ is equal to

$$\sum_{\substack{\gamma \in \mathbb{N}^n \\ 1 \le j \le d_\sigma}} \int_{G^\times} g(w, k_2)e^{\frac{i\lambda}{2}\mathrm{Im}(k_2 k_1^{-1} z.\bar{w})} \left\langle \rho_\sigma^\lambda(w - k_2 k_1^{-1} z, k_2 k_1^{-1})(\phi_\gamma^\lambda \otimes e_j^\sigma), \phi_\gamma^\lambda \otimes e_j^\sigma \right\rangle dwds.$$

By (5.1.1), the above expression takes the form

$$\sum_{\substack{\gamma\in\mathbb{N}^n\\ 1\leq j\leq d_\sigma}}\sum_{|\alpha|=|\gamma|}\int_K \eta^\lambda_{\gamma\alpha}(k_2k_1^{-1})\int_{\mathbb{C}^n} g(w,k_2)e^{\frac{i\lambda}{2}\mathrm{Im}(k_2k_1^{-1}z.\bar{w})}\phi^\lambda_{\alpha\gamma}(w-k_2k_1^{-1}z)\varphi^\sigma_{jj}(k_2k_1^{-1})dwdk_2,$$

where $\phi^\lambda_{\alpha\gamma}(x)=\langle\pi_\lambda(x)\phi^\lambda_\alpha,\phi^\lambda_\gamma\rangle$. Then, by the Peter-Weyl theorem (inversion) for the compact groups (see [51]), we derive that

$$\begin{aligned}\sum_{\sigma\in\hat{K}} d_\sigma\ \mathrm{tr}(W^\lambda_\sigma(g)(\rho^\lambda_\sigma)^*(z,k_1)) &= \sum_{\gamma\in\mathbb{N}^n}\sum_{|\alpha|=|\gamma|}\eta^\lambda_{\gamma\alpha}(\mathsf{e})\int_{\mathbb{C}^n} g(w,k_1)e^{\frac{i\lambda}{2}\mathrm{Im}(z.\bar{w})}\phi^\lambda_{\alpha\gamma}(w-z)dw\\ &= \sum_{\gamma\in\mathbb{N}^n}\int_{\mathbb{C}^n} g(w,k_1)e^{\frac{i\lambda}{2}\mathrm{Im}(z.\bar{w})}\phi^\lambda_{\gamma\gamma}(w-z)dw,\end{aligned}$$

where e is the identity element in K. Thus, in view of the inversion formula for the Weyl transform on the Heisenberg group, we infer that

$$\sum_{\sigma\in\hat{K}} d_\sigma\ \mathrm{tr}\left(W^\lambda_\sigma(g)(\rho^\lambda_\sigma)^*(z,k_1)\right)=(2\pi)^n|\lambda|^{-n}g(z,k_1). \tag{5.1.7}$$

This completes the proof. □

5.2 Benedicks-Amrein-Berthier type theorem

For simplicity assume $\lambda=1$ and denote $\rho_\sigma(z,k)=\rho^1_\sigma(z,k)$, $W_\sigma=W^1_\sigma$. Further, throughout this section, we shall assume A is a Lebesgue measurable subset of $\mathbb{C}^n$ with finite measure. Next, we define a set of orthogonal projection operators which is core in formulating a problem analogous to Benedicks-Amrein-Berthier type theorem.

Let $\sigma\in\hat{K}$ and $\mathcal{B}_{N_\sigma}$ be an N_σ dimensional subspace of $\mathcal{H}^2_\sigma$. Then, there exists an orthonormal basis $\{\psi^\sigma_l : l\in\mathbb{N}\}$ of $\mathcal{H}^2_\sigma$ such that $\mathcal{B}_{N_\sigma}=\text{span }\{\psi^\sigma_l : 1\leq l\leq N_\sigma\}$. Define an orthogonal projection P_{N_σ} of $\mathcal{H}^2_\sigma$ onto $\mathcal{R}(P_{N_\sigma})=\mathcal{B}_{N_\sigma}$. Consider a finite subset J of

$\hat{K}$ and let $N = \max_{\sigma\in J} N_\sigma$. Now, we define a pair of orthogonal projections E_A and F_N of $L^2(G^\times)$ by

$$E_A g = \chi_{A\times K}\; g \quad \text{and} \quad W_\sigma(F_N g) = \begin{cases} P_{N_\sigma} W_\sigma(g) & \text{if } \sigma \in J, \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi_{A\times K}$ denotes the characteristic function of $A \times K$. Then, it is easy to see that $\mathcal{R}(E_A) = \{g \in L^2(G^\times) : g = g\;\chi_{A\times K}\}$ and

$$\mathcal{R}(F_N) = \{g \in L^2(G^\times) : \mathcal{R}(W_\sigma(g)) \subseteq \mathcal{B}_{N_\sigma} \text{ for } \sigma \in J \text{ and } \mathcal{R}(W_\sigma(g)) = 0 \text{ for } \sigma \notin J\}.$$

Now, we derive a key lemma that enables us to recognize $E_A F_N$ as an integral operator.

Lemma 5.2.1. *The operator $E_A F_N$ is an integral operator on $L^2(G^\times)$.*

Proof. Let $g \in L^2(G^\times)$. By the inversion formula (5.1.6), we have

$$\begin{aligned}(F_N g)(z, k_1) &= \sum_{\sigma\in\hat{K}} a_\sigma \;\mathrm{tr}\left(W_\sigma(F_N g)\rho^*_\sigma(z, k_1)\right) \\ &= \sum_{\sigma\in J} a_\sigma \;\mathrm{tr}\left(P_{N_\sigma} W_\sigma(g)\rho^*_\sigma(z, k_1)\right) \\ &= \sum_{\sigma\in J} a_\sigma \int_K \int_{\mathbb{C}^n} g(w, k_2)\;\mathrm{tr}\left(P_{N_\sigma}\rho_\sigma(w, k_2)\rho^*_\sigma(z, k_1)\right) dw dk_2,\end{aligned}$$

where $a_\sigma = (2\pi)^{-n} d_\sigma$. Hence,

$$\begin{aligned}(E_A F_N g)(z, k_1) &= \chi_{A\times K}(z, k_1)(F_N g)(z, k_1) \\ &= \sum_{\sigma\in J} a_\sigma \chi_{A\times K}(z, k_1) \int_K \int_{\mathbb{C}^n} g(w, k_2)\;\mathrm{tr}\left(P_{N_\sigma}\rho_\sigma(w, k_2)\rho^*_\sigma(z, k_1)\right) dw dk_2 \\ &= \int_K \int_{\mathbb{C}^n} g(w, k_2)\mathcal{K}((z, k_1), (w, k_2)) dw dk_2,\end{aligned}$$

where $\mathcal{K}\left((z, k_1), (w, k_2)\right) = \sum_{\sigma\in J} a_\sigma \chi_{A\times K}(z, k_1)\;\mathrm{tr}\left(P_{N_\sigma}\rho_\sigma(w, k_2)\rho^*_\sigma(z, k_1)\right).$ □

Further, the integral operator E_AF_N is a Hilbert-Schmidt operator and satisfies the following dimension condition.

Lemma 5.2.2. *E_AF_N is a Hilbert-Schmidt operator with $\|E_AF_N\|_{HS}^2 \leq c_J m(A)N$, where $c_J = (2\pi)^n m(K)|J| \sum_{\sigma\in J} a_\sigma^2 < \infty$.*

Proof. From Lemma 5.2.1 it follows that

$$\|E_AF_N\|_{HS}^2 = \int_{G^\times}\int_{G^\times} |\mathcal{K}((z,k_1),(w,k_2))|^2 dwdk_2dzdk_1$$
$$= \int_{G^\times}\int_{G^\times} \left|\sum_{\sigma\in J} a_\sigma \chi_{A\times K}(z,k_1)\ \mathrm{tr}\left(P_{N_\sigma}\rho_\sigma(w,k_2)\rho_\sigma^*(z,k_1)\right)\right|^2 dwdk_2dzdk_1.$$

If the cardinality of J is denoted by $|J|$, from Hölder's inequality, we get

$$\|E_AF_N\|_{HS}^2 \leq |J|\sum_{\sigma\in J} a_\sigma^2 \int_{G^\times} |\chi_{A\times K}(z,k_1)|^2 \int_{G^\times} |\mathrm{tr}\left(P_{N_\sigma}\rho_\sigma(w,k_2)\rho_\sigma^*(z,k_1)\right)|^2 dwdk_2dzdk_1. \tag{5.2.1}$$

Now, we shall simplify the inner integral

$$\int_{G^\times} |\mathrm{tr}\left(P_{N_\sigma}\rho_\sigma(w,k_2)\rho_\sigma^*(z,k_1)\right)|^2 dwdk_2$$
$$= \int_{G^\times} \Big|\sum_{1\leq l\leq N_\sigma} \langle \rho_\sigma(w,k_2)\rho_\sigma^*(z,k_1)\psi_l^\sigma, \psi_l^\sigma\rangle\Big|^2 dwdk_2$$
$$= \int_{G^\times} \Big|\sum_{1\leq l\leq N_\sigma} \langle \rho_\sigma(w,k_2)\eta_l^\sigma, \psi_l^\sigma\rangle\Big|^2 dwdk_2,$$

where $\eta_l^\sigma = \rho_\sigma^*(z,k_1)\psi_l^\sigma \in \mathcal{H}_\sigma^2$. The above integral can be written in terms of Fourier-Wigner transform by

$$\int_{G^\times} \Big|\sum_{1\leq l\leq N_\sigma} \langle \rho_\sigma(w,k_2)\eta_l^\sigma, \psi_l^\sigma\rangle\Big|^2 dwdk_2 = (2\pi)^n \int_{G^\times} \Big|\sum_{1\leq l\leq N_\sigma} V_{\eta_l^\sigma}^{\psi_l^\sigma}(w,k_2)\Big|^2 dwdk_2$$
$$= (2\pi)^n \sum_{1\leq l_1,l_2\leq N_\sigma} \int_{G^\times} V_{\eta_{l_1}^\sigma}^{\psi_{l_1}^\sigma}(w,k_2)\overline{V_{\eta_{l_2}^\sigma}^{\psi_{l_2}^\sigma}(w,k_2)}dwdk_2.$$

Since,

$$\langle \eta^{\sigma}_{l_1}, \eta^{\sigma}_{l_2}\rangle = \langle \rho^*_\sigma(z,k_1)\psi^{\sigma}_{l_1}, \rho^*_\sigma(z,k_1)\psi^{\sigma}_{l_2}\rangle = \langle \psi^{\sigma}_{l_1}, \psi^{\sigma}_{l_2}\rangle = \delta_{l_1 l_2},$$

by Lemma 5.1.1, we have

$$\begin{aligned}&\int_{G^\times} \left|\mathrm{tr}\left(P_{N_\sigma}\rho_\sigma(w,k_2)\rho^*_\sigma(z,k_1)\right)\right|^2 dw dk_2\\ &= (2\pi)^n \sum_{1\le l_1,l_2\le N_\sigma} \langle \eta^{\sigma}_{l_1}, \eta^{\sigma}_{l_2}\rangle \overline{\langle \psi^{\sigma}_{l_1}, \psi^{\sigma}_{l_2}\rangle} = (2\pi)^n N_\sigma. \qquad (5.2.2)\end{aligned}$$

Thus, from (5.2.1) and (5.2.2) we get $\|E_A F_N\|^2_{HS} \le (2\pi)^n m(A)m(K)N|J| \sum_{\sigma\in J} a^2_\sigma < \infty$, where $N = \max_{\sigma\in J} N_\sigma$ as defined above. □

Let $F_N^{\perp} = I - F_N$ and A^c be the complement of A. Denote $wA = \{z \in \mathbb{C}^n : z - w \in A\}$.

Proposition 5.2.3. *Let A be a measurable subset of $\mathbb{C}^n$ of finite Lebesgue measure. Then, the projection $E_A \cap F_N = 0$.*

Proof. Assume towards a contradiction that there exists a non-zero function g in $\mathcal{R}(E_A \cap F_N)$. Then $\mathcal{R}(W_\sigma(g)) \subseteq \mathcal{B}_{N_\sigma}$ for $\sigma \in J$ and $\mathcal{R}(W_\sigma(g)) = 0$ for $\sigma \in \hat{K} \setminus J$. Consider $A_0 = \{z \in A : \exists$ a positive measure set $K_z \subseteq K$ with $g(z,k) \ne 0, \forall k \in K_z\}$. Then $0 < m(A_0) < \infty$. Let $g_0(z,k) = \chi_{A_0}(z)g(z,k)$. Thus $g = g_0$ a.e. and hence $g_0 \in \mathcal{R}(E_A \cap F_N)$. Choose $s \in \mathbb{N}$ such that $s > 2c_J m(A_0)N$. Now, we construct an increasing sequence of sets $\{A_l : l = 1, \ldots, s\}$. Using Lemma 3.2.3 with $\epsilon = \frac{1}{2c_J N}$, $B_0 = A_0$ and $B = A_{l-1}$, there exists $w_l \in \mathbb{C}^n$ such that

$$m(A_{l-1}) < m(A_{l-1} \cup w_l A_0) < m(A_{l-1}) + \frac{1}{2c_J N}.$$

Denote $A_l = A_{l-1} \cup w_l A_0$. Then from (3.2.2), we get

$$\dim \mathcal{R}(E_{A_s} \cap F_N) \le c_J m(A_s)N < \left\{m(A_0) + \frac{s}{2c_J N}\right\} c_J N < s. \qquad (5.2.3)$$

On the other hand, we construct $s+1$ linearly independent functions in the space $\mathcal{R}(E_{A_s}\cap F_N)$, after verifying $\mathcal{R}(F_N)$ is a twisted translation invariant space.

Let $g_l(z,k)=e^{\frac{i}{2}Im(z.\bar{w}_l)}g_0(z-w_l,k)$. Then for $\eta^\sigma\in\mathcal{H}^2_\sigma$ and $p>N_\sigma$, where $\sigma\in J$, we have

$$\begin{aligned}\langle W_\sigma(g_l)\eta^\sigma,\psi_p^\sigma\rangle &=\int_{G^\times} g_l(z,k)\langle\rho_\sigma(z,k)\eta^\sigma,\psi_p^\sigma\rangle dzdk\\ &=\int_{G^\times} e^{\frac{i}{2}Im(z.\bar{w}_l)}g_0(z-w_l,k)\langle\rho_\sigma(z,k)\eta^\sigma,\psi_p^\sigma\rangle dzdk\\ &=\int_{G^\times} e^{\frac{i}{2}Im(z.\bar{w}_l)}g_0(z,k)\langle\rho_\sigma(z+w_l,k)\eta^\sigma,\psi_p^\sigma\rangle dzdk.\end{aligned}$$

Since $\rho_\sigma(z,k)\rho_\sigma(k^{-1}w,\mathsf{e})=e^{\frac{i}{2}Im(z.\bar{w})}\rho_\sigma(z+w,k)$, where e is the identity element in K, we get

$$\begin{aligned}\langle W_\sigma(g_l)\eta^\sigma,\psi_p^\sigma\rangle &=\int_{G^\times} g_0(z,k)\langle\rho_\sigma(z,k)\rho_\sigma(k^{-1}w_l,\mathsf{e})\eta^\sigma,\psi_p^\sigma\rangle dzdk\\ &=\int_{G^\times} g_0(z,k)\langle\rho_\sigma(z,k)\zeta^\sigma,\psi_p^\sigma\rangle dzdk\\ &=\langle W(g_0)\zeta^\sigma,\psi_p^\sigma\rangle=0.\end{aligned}$$

Thus, $\mathcal{R}(W_\sigma(g_l))\subseteq\mathcal{B}_{N_\sigma}$ for $\sigma\in J$. Similarly, for $\sigma\notin J$, it can be shown that $\mathcal{R}(W_\sigma(g_l))=0$. Since $A_m=A_0\cup w_1A_0\cup\cdots\cup w_mA_0$ and $g_l=0$ on $(w_lA_0)^c\times K$, we have $E_{A_m}g_l=g_l$ for $l=0,1,\ldots,m$. Furthermore, $E_{A_m\setminus A_{m-1}}g_l=0$ for $l=0,\ldots,m-1$ and by the definition of A_0, it follows that $E_{A_m\setminus A_{m-1}}g_m\neq0$ on a set of positive measure. Therefore, it shows that g_m is not a linear combination of $g_0,\ldots,g_{m-1}$. Hence, $g_0,\ldots,g_s$ are $s+1$ linearly independent functions in $\mathcal{R}(E_{A_s}\cap F_N)$ that contradicts (5.2.3). This completes the proof. □

This leads to the following version of Benedicks-Amrein-Berthier theorem for the Weyl transform.

Proposition 5.2.4. *Let $g \in L^1(G^\times)$ and $\{(z,k) \in G^\times : g(z,k) \neq 0\} \subseteq A \times K$, where $m(A) < \infty$. Suppose J be a finite subset of $\hat{K}$. If $W_\sigma(g)$ is a finite rank operator for each $\sigma \in J$ and $W_\sigma(g) = 0$ for $\sigma \in \hat{K} \setminus J$, then $g = 0$.*

If $g \in L^1(G^\times)$, by the Plancherel theorem (5.1.3), the assumed rank condition implies $g \in L^2(G^\times)$. Further, for $g \in L^2(G^\times)$, proof of Proposition 5.2.4 follows from Proposition 5.2.3.

In the Heisenberg motion group, in terms of Fourier transform, the above result takes the following form.

Theorem 5.2.5. *Let $f \in L^1(G)$ and $\{(z,t,k) \in G : g(z,t,k) \neq 0\} \subseteq A \times \mathbb{R} \times K$, where $m(A) < \infty$. For each $\lambda \in \mathbb{R}^*$, consider a finite subset J_λ of $\hat{K}$. If for each $\lambda \in \mathbb{R}^*$, $\hat{f}(\lambda,\sigma)$ has finite rank for $\sigma \in J_\lambda$ and $\hat{f}(\lambda,\sigma) = 0$ for $\sigma \in \hat{K} \setminus J_\lambda$, then $f = 0$.*

Remark 5.2.6. *Notice that, for the Heisenberg motion group, the condition* (1.2.1) *for QUP is equivalent to*

$$\int_{\mathbb{R}\setminus\{0\}} \left(\sum_{\sigma \in \hat{K}} d_\sigma \ rank\ \hat{f}(\lambda,\sigma) \right) |\lambda|^n d\lambda < \infty. \tag{5.2.4}$$

Therefore, the rank condition in Theorem 5.2.5 will not satisfy (5.2.4), *and hence Theorem 5.2.5 improves the QUP in that perspective. Further, the assumption in Theorem 5.2.5 that, for each $\lambda \in \mathbb{R}^*$, $\hat{f}(\lambda,\sigma) = 0$ except finitely many σ, looks natural in view of* (5.2.4).

As a consequence of Proposition 5.2.4, we obtain the following analogue result in terms of the Fourier-Wigner decomposition. For this, we recall the Fourier-Wigner decomposition. Let $g \in L^2(G^\times)$. By Proposition 5.1.2, we get $g = \bigoplus_{\sigma \in \hat{K}} g_\sigma$.

Proposition 5.2.7. *Let $g \in L^2(G^\times)$ and $\{(z,k) \in G^\times : g_\sigma(z,k) \neq 0\} \subseteq A_\sigma \times K$, where $m(A_\sigma) < \infty$, whenever $\sigma \in \hat{K}$. If $W_\sigma(g)$ is a finite rank operator for each σ, then $g = 0$.*

Proof. For $\varphi, \psi \in \mathcal{H}_\sigma^2$ we have

$$\begin{aligned}\langle W_\sigma(g)\varphi, \psi\rangle &= \int_K \int_{\mathbb{C}^n} g(z,k)\,\langle \rho_\sigma(z,k)\varphi, \psi\rangle\, dzdk \\ &= \int_K \int_{\mathbb{C}^n} g_\sigma(z,k)\,\langle \rho_\sigma(z,k)\varphi, \psi\rangle\, dzdk \\ &= \langle W_\sigma(g_\sigma)\varphi, \psi\rangle\,.\end{aligned}$$

Hence for $\sigma_o \in \hat{K}$, $\mathcal{R}(W_{\sigma_o}(g_{\sigma_o})) = \mathcal{R}(W_{\sigma_o}(g))$ be a finite dimensional subspace of $\mathcal{H}_{\sigma_o}^2$ and $\mathcal{R}(W_\sigma(g_{\sigma_o})) = 0$ for $\sigma(\neq \sigma_o) \in \hat{K}$. Thus, by Proposition 5.2.4 we get $g_{\sigma_o} = 0$. Since $\sigma_o \in \hat{K}$ is arbitrary, we infer that $g = 0$. □

Remark 5.2.8. (**a**). *For $U(n)$-bi-invariant function, the rank condition in Proposition 5.2.7 is obviously true. Thus support condition is enough for the conclusion. In dimension one, it can argue by the fact that each Fourier-Wigner piece will be of the form $\tilde{\bar{g}}_\alpha = \tilde{\bar{g}}_\alpha \times \Phi_{\alpha,\alpha}$, where $\tilde{\bar{g}}_\alpha(z) = \bar{g}_\alpha(-z)$, which is real analytic. Hence it cannot be supported on a set of finite measure.*

(**b**). *After a close examination of the utility of $U(n)$ to obtain the decomposition of $L^2(G^\times)$ as in Proposition 5.1.2, we observed that $U(n)$-invariance is nevermore used except while realizing the irreducible action of metaplectic repression μ_λ on P_m^λ. If we consider a compact subgroup K of $U(n)$ which makes $(\mathbb{H}^n \rtimes K, K)$ a Gelfand pair, then P_m^λ will be decomposed into finitely many irreducible pieces according to the metaplectic representation μ_λ of K. To avoid further complexity in the calculation, we have preferred to prove the results for the Gelfand pair $(\mathbb{H}^n \rtimes U(n), U(n))$ instead of $(\mathbb{H}^n \rtimes K, K)$.*

5.2.1 Strong annihilating pair

Now, we shall define strong annihilating pair for the Weyl transform on the Heisenberg motion group.

Definition 5.2.9. *For each $\sigma \in \hat{K}$, let A_σ be a measurable subset of $\mathbb{C}^n$ and S_σ be a closed subspace of $\mathcal{H}^2_\sigma$. By abuse of notations, denote $A = (A_\sigma)_{\sigma\in\hat{K}}$ and $S = (S_\sigma)_{\sigma\in\hat{K}}$. We say that the pair (A, S) is a* strong annihilating pair *for the Weyl transform, if there exist positive numbers $C_\sigma = C_\sigma(A_\sigma, S_\sigma)$ such that for every $g \in L^2(\mathbb{C}^n \times K)$,*

$$\|g\|_2^2 \leq \sum_{\sigma\in\hat{K}} C_\sigma \left(\|g_\sigma\|^2_{L^2(A^c_\sigma \times K)} + \|P^\perp_{S_\sigma} W^\lambda_\sigma(g)\|^2_{HS} \right), \tag{5.2.5}$$

where g_σ's are the orthogonal pieces of g, according to Proposition 5.1.2, and P_{S_σ} is the projection of $\mathcal{H}^2_\sigma$ onto S_σ.

For $\sigma_o \in \hat{K}$, let S_{σ_o} be a finite dimensional subspace of $\mathcal{H}^2_{\sigma_o}$ and A_{σ_o} be any subset of $\mathbb{C}^n$ with finite measure. Then, recall the set of projections $E_{A_{\sigma_o}} g(z,k) = \chi_{A_{\sigma_o}}(z) g(z,k)$ and $W^\lambda_{\sigma_o}(F_{S_{\sigma_o}} g) = P_{S_{\sigma_o}} W^\lambda_{\sigma_o}(g)$, $W^\lambda_\sigma(F_{S_{\sigma_o}} g) = 0$ for $\sigma \neq \sigma_o$. Now, $E_{A_{\sigma_o}} F_{S_{\sigma_o}}$ is a compact operator and $E_{A_{\sigma_o}} \cap F_{S_{\sigma_o}} = 0$. Therefore, we must have $\|E_{A_{\sigma_o}} F_{S_{\sigma_o}}\| < 1$. Since $W^\lambda_{\sigma_o}(F^\perp_{S_{\sigma_o}} g) = P^\perp_{S_{\sigma_o}} W^\lambda_{\sigma_o}(g)$ and $W^\lambda_\sigma(F^\perp_{S_{\sigma_o}} g) = W^\lambda_\sigma(g)$ for $\sigma \neq \sigma_o$, by Lemma 3.2.10, there exists $\tilde{C}_{\sigma_o} = \tilde{C}_{\sigma_o}(A_{\sigma_o}, S_{\sigma_o}) > 0$ such that

$$\|g\|_2^2 \leq \tilde{C}_{\sigma_o} \left(\|g\|^2_{L^2(A^c_{\sigma_o} \times K)} + d_{\sigma_o} \|P^\perp_{S_{\sigma_o}} W^\lambda_{\sigma_o}(g)\|^2_{HS} + \sum_{\sigma\neq\sigma_o} d_\sigma \|W^\lambda_\sigma(g)\|^2_{HS} \right),$$

for all $g \in L^2(\mathbb{C}^n \times K)$. In particular, for any $g_{\sigma_o} \in V^\lambda_{\sigma_o}$ we have

$$\|g_{\sigma_o}\|_2^2 \leq C_{\sigma_o} \left(\|g_{\sigma_o}\|^2_{L^2(A^c_{\sigma_o} \times K)} + \|P^\perp_{S_{\sigma_o}} W^\lambda_{\sigma_o}(g_{\sigma_o})\|^2_{HS} \right), \tag{5.2.6}$$

where $C_{\sigma_o} = d_{\sigma_o} \tilde{C}_{\sigma_o}$. For any $g \in L^2(\mathbb{C}^n \times K)$, by Proposition 5.1.2, $g = \bigoplus_{\sigma\in\hat{K}} g_\sigma$. Since σ_o is arbitrary, from (5.2.6) we can conclude that (A, S) is a strong annihilating pair, whenever A_σ has finite measure and dimension of S_σ is finite for each $\sigma \in \hat{K}$.

Remark 5.2.10. *Consider the hypothesis of Proposition 5.2.4. There exist two large classes of functions of which one satisfies the support condition, and the other satisfies* *the rank condition. However, in Proposition 5.2.7, it is not clear which functions will fulfill such a support condition. In other words, whether the assumption of finite support condition in each piece is strong enough for the conclusion of Proposition 5.2.7. We know this is true for the* $U(n)$*-bi-invariant functions. However, we reached out to a quantitative estimate* (5.2.5, 5.2.6) *of Proposition 5.2.7, which is true for all square integrable functions, irrespective of their support.*

Bibliography

[1] M. L. Agranovsky, V. V. Volchkov and L. A. Zalcman, *Conical uniqueness sets for the spherical Radon transform,* Bull. London Math. Soc. 31 (1999), no. 2, 231-236.

[2] W. O. Amrein and A. M. Berthier, *On support properties of L^p-functions and their Fourier transforms,* J. Funct. Anal. 24 (1977), no. 3, 258-267.

[3] G. E. Andrews, R. Askey and R. Roy, *Special Functions,* Cambridge University Press, Cambridge, 1999.

[4] D. H. Armitage, *Cones on which entire harmonic functions can vanish,* Proc. Roy. Irish Acad. Sect. A 92 (1992), no. 1, 107-110.

[5] D. Arnal and J. Ludwig, *Q.U.P. and Paley-Wiener properties of unimodular, especially nilpotent, Lie groups,* Proc. Amer. Math. Soc. 125 (1997), no. 4, 1071-1080.

[6] D. B. Babot, *Heisenberg uniqueness pairs in the plane. Three parallel lines,* Proc. Amer. Math. Soc. 141 (2013), no. 11, 3899-3904.

[7] S. Bagchi, *Heisenberg uniqueness pairs corresponding to a finite number of parallel lines,* Adv. Math. 325 (2018), 814-823.

[8] M. Benedicks, *On Fourier transforms of functions supported on sets of finite Lebesgue measure,* J. Math. Anal. Appl. 106 (1985), no. 1, 180-183.

[9] C. Benson, J. Jenkins and G. Ratcliff, *Bounded K-spherical functions on Heisenberg groups,* J. Funct. Anal. 105 (1992), no. 2, 409-443.

[10] A. Bonami and B. Demange, *A survey on uncertainty principles related to quadratic forms,* Collect. Math. 2006, Vol. Extra, 1-36.

[11] F. Canto-Martín, H. Hedenmalm and A. Montes-Rodríguez, *Perron-Frobenius operators and the Klein-Gordon equation,* J. Eur. Math. Soc. (JEMS) 16 (2014), no. 1, 31-66.

[12] D. C. Chang and I. Markina, *Geometric analysis on quaternion $\mathbb{H}$-type groups,* J. Geom. Anal. 16 (2006), no. 2, 265-294.

[13] A. Chattopadhyay, D. K. Giri and R. K. Srivastava, *Uniqueness of the Fourier transform on certain Lie groups,* arXiv:1607.03832.

[14] L. Chen and J. Zhao, *Weyl transform and generalized spectrogram associated with quaternion Heisenberg group,* Bull. Sci. Math. 136 (2012), no. 2, 127-143.

[15] L. J. Corwin and F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications,* Cambridge Studies in Advanced Mathematics, 18, Cambridge University Press, Cambridge, 1990.

[16] C. F. Dunkl, *Boundary value problems for harmonic functions on the Heisenberg group,* Canad. J. Math. 38 (1986), no. 2, 478-512.

[17] A. Fernández-Bertolin, K. Gröchenig and P. Jaming, *From Heisenberg uniqueness pairs to properties of the Helmholtz and Laplace equations,* J. Math. Anal. Appl. 469 (2019), no. 1, 202-219.

[18] M. Filaseta and T.-Y. Lam, *On the irreducibility of the generalized Laguerre polynomials,* Acta Arit. 105 (2002), 177-182.

[19] G. B. Folland and A. Sitaram, *The uncertainty principle: a mathematical survey,* J. Fourier Anal. Appl. 3 (1997), 207-238.

[20] D. K. Giri and R. Rawat, *Heisenberg uniqueness pairs for the hyperbola,* Bull. Lond. Math. Soc., doi: 10.1112/blms.12391 (to appear).

[21] D. K. Giri and R. K. Srivastava, *Heisenberg uniqueness pairs for some algebraic curves in the plane,* Adv. Math. 310 (2017), 993-1016.

[22] P. C. Greiner, *Spherical harmonics on the Heisenberg group,* Canad. Math. Bull. 23 (1980), no. 4, 383-396.

[23] K. Gröchenig and P. Jaming, *The Cramér-Wold theorem on quadratic surfaces and Heisenberg uniqueness pairs,* J. Inst. Math. Jussieu 19 (2020), no. 1, 117-135.

[24] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 28, Springer-Verlag, Berlin, 1994.

[25] H. Hedenmalm and A. Montes-Rodríguez, *Heisenberg uniqueness pairs and the Klein-Gordon equation,* Ann. of Math. (2) 173 (2011), no. 3, 1507-1527.

[26] H. Hedenmalm and A. Montes-Rodríguez, *The Klein-Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps,* J. Eur. Math. Soc. (JEMS) 22 (2020), no. 6, 1703-1757.

[27] H. Hedenmalm and A. Montes-Rodríguez, *The Klein-Gordon equation, the Hilbert transform, and Gauss-type maps: H^∞ approximation,* J. Anal. Math. (2020) (to appear).

[28] W. Heisenberg, *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik,* Z. Physik 43 (1927), 172-198.

[29] P. Jaming, *Nazarov's uncertainty principles in higher dimension,* J. Approx. Theory 149 (2007), no. 1, 30-41.

[30] P. Jaming and K. Kellay, *A dynamical system approach to Heisenberg uniqueness pairs,* J. Anal. Math. 134 (2018), no. 1, 273-301.

[31] N. Lev, *Uniqueness theorems for Fourier transforms,* Bull. Sci. Math. 135 (2011), no. 2, 134-140.

[32] C. C. Moore and J. A. Wolf, *Square integrable representations of nilpotent groups,* Trans. Amer. Math. Soc. 185 (1973), 445-462.

[33] E. K. Narayanan and P. K. Ratnakumar, *Benedicks' theorem for the Heisenberg group,* Proc. Amer. Math. Soc. 138 (2010), no. 6, 2135-2140.

[34] F. L. Nazarov, *Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type,* (Russian) Algebra i Analiz 5 (1993), no. 4, 3-66; translation in St. Petersburg Math. J. 5 (1994), no. 4, 663-717.

[35] S. Parui and R. P. Sarkar, *Beurling's theorem and $L^p - L^q$ Morgan's theorem for step two nilpotent Lie groups,* Publ. Res. Inst. Math. Sci. 44 (2008), no. 4, 1027-1056.

[36] S. Parui and S. Thangavelu, *On theorems of Beurling and Hardy for certain step two nilpotent groups,* Integral Transforms Spec. Funct. 20 (2009), no. 1-2, 127-145.

[37] V. Pati and A. Sitaram, *Some questions on integral geometry on Riemannian manifolds,* Ergodic theory and harmonic analysis (Mumbai, 1999), Sankhyā Ser. A 62 (2000), no. 3, 419-424.

[38] F. J. Price and A. Sitaram, *Functions and their Fourier transforms with supports of finite measure for certain locally compact groups,* J. Funct. Anal. 79 (1988), no. 1, 166-182.

[39] S. K. Ray, *Uncertainty principles on two step nilpotent Lie groups,* Proc. Indian Acad. Sci. Math. Sci. 111 (2001), no. 3, 293-318.

[40] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag, New York-Berlin, 1980.

[41] R. P. Sarkar and S. Thangavelu, *An analogue of the Wiener Tauberian theorem for the Heisenberg motion group,* J. Indian Inst. Sci. 87 (2007), no. 4, 467-474.

[42] S. Sen, *Segal-Bargmann transform and Paley-Wiener theorems on Heisenberg motion groups*, Adv. Pure Appl. Math. 7 (2016), no. 1, 13-28.

[43] A. Sitaram, M. Sundari and S. Thangavelu, *Uncertainty principles on certain Lie groups,* Proc. Indian Acad. Sci. Math. Sci. 105 (1995), no. 2, 135-151.

[44] P. Sjölin, *Heisenberg uniqueness pairs and a theorem of Beurling and Malliavin,* Bull. Sci. Math. 135 (2011), no. 2, 125-133.

[45] P. Sjölin, *Heisenberg uniqueness pairs for the parabola,* J. Fourier Anal. Appl. 19 (2013), no. 2, 410-416.

[46] C. D. Sogge, *Oscillatory integrals and spherical harmonics,* Duke Math. J. 53 (1986), no. 1, 43-65.

[47] R. K. Srivastava, *Real analytic expansion of spectral projections and extension of the Hecke-Bochner identity,* Israel J. Math. 200 (2014), no. 1, 171-192.

[48] R. K. Srivastava, *Non-harmonic cones are sets of injectivity for the twisted spherical means on* $\mathbb{C}^n$, Trans. Amer. Math. Soc. 368 (2016), no. 3, 1941-1957.

[49] R. K. Srivastava, *Non-harmonic cones are Heisenberg uniqueness pairs for the Fourier transform on* $\mathbb{R}^n$, J. Fourier Anal. Appl. 24 (2018), no. 6, 1425-1437.

[50] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces,* Princeton Mathematical Series, No. 32. Princeton University Press (1971).

[51] M. Sugiura, *Unitary representations and harmonic analysis*, North-Holland Mathematical Library, 44, North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1990.

[52] S. Thangavelu, *Harmonic analysis on the Heisenberg group,* Prog. Math., 159, Birkhuser, Boston, 1998.

[53] S. Thangavelu, *An introduction to the uncertainty principle,* Prog. Math., 217, Birkhauser, Boston, 2004.

[54] M. K. Vemuri, *Benedicks theorem for the Weyl transform*, J. Math. Anal. Appl. 452 (2017), no. 1, 209-217.

[55] F. J. G. Vieli, *A uniqueness result for the Fourier transform of measures on the sphere,* Bull. Aust. Math. Soc. 86 (2012), no. 1, 78-82.

[56] F. J. G. Vieli, *A uniqueness result for the Fourier transform of measures on the paraboloid,* Matematicki Vesnik 67 (2015), no. 1, 52-55.

[57] G. N. Watson, *A treatise on the theory of Bessel functions,* second edition, Cambridge University Press, Cambridge, 1944.

List of communicated papers

1. A. Chattopadhyay, S. Ghosh, D. K. Giri and R. K. Srivastava, *Heisenberg uniqueness pairs on the Euclidean spaces and the motion group,* **C. R. Math. Acad. Sci. Paris 358 (2020), no. 3, 365-377**.

2. S. Ghosh and R. K. Srivastava, *Heisenberg uniqueness pairs for the Fourier transform on the Heisenberg group,* **Bull. Sci. Math. 166 (2021), 102941, 23 pp**.

3. S. Ghosh and R. K. Srivastava, *Benedicks-Amrein-Berthier theorem for the Heisenberg motion group and quaternion Heisenberg group,* arXiv:1904.04023.