

Analytical Modeling of Flow into Open Drains in a Layered Soil Receiving Water from a Ponded Field

*A thesis submitted to the
Indian Institute of Technology Guwahati
in partial fulfillment of the requirements for the award of the degree of*

DOCTOR OF PHILOSOPHY

By

Subhadeep Chakrabarti

(Roll No: 126104012)

Under the supervision of

Gautam Barua

Professor



DEPARTMENT OF CIVIL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI-781039, ASSAM, INDIA

June 2018

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
Department of Civil Engineering
Guwahati -781039
Assam India



CERTIFICATE

This is to certify that the thesis entitled “*Analytical Modeling of Flow into Open Drains in a Layered Soil Receiving Water from a Poned Field*” submitted by Mr. Subhadeep Chakrabarti, Roll No. 126104012, to the Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy in Civil Engineering is a record of bonafide research work carried out by him under my supervision and guidance.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for award of any degree or diploma.

IIT Guwahati
June, 2018

(Gautam Barua)
Professor

Department of Civil Engineering
Indian Institute of Technology Guwahati
Guwahati-781039, Assam, India.

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
Department of Civil Engineering
Guwahati -781039
Assam India



STATEMENT

I do hereby declare that the matter embodied in the thesis is a result of research work carried out by me in the Department of Civil Engineering, Indian Institute of Technology Guwahati, Guwahati, Assam, India.

In keeping with the general practice and reporting scientific observations, due acknowledgements have been made wherever the work described is based on the findings of other investigators.

IIT Guwahati
June, 2018

(Subhadeep Chakrabarti)
Research Scholar
Department of Civil Engineering
Indian Institute of Technology Guwahati
Guwahati-781039, Assam, India.

ACKNOWLEDGEMENTS

I am deeply indebted and grateful to my supervisor Prof. Gautam Barua, without whose voracious knowledge, sage guidance and invaluable intellectual inputs, this research endeavor would not have been realized.

I would also like to thank Prof. Sudip Talukdar, Dr. Karuna Kalita and Dr. Suresh Kartha for their invaluable suggestions and insights regarding my research.

I also extend my gratitude to Dr. Ratan Sarmah, Mr. Jagadish Talukdar, Mr. Subhrangshu Purakayastha and Dr. Wazir Alam for assisting me in my work whenever the need so arose. I would also like to thank Mr. Debraj Biswas and Mr. Bhrigumani Sharma for their encouragement and support.

Lastly, I thank my parents for being accommodating and supportive of my research endeavor.

(Subhadeep Chakrabarti)

TABLE OF CONTENTS

Chapter		Page No.
	CERTIFICATE	i
	STATEMENT	ii
	ACKNOWLEDGEMENTS	iii
	TABLE OF CONTENTS	iv
	LIST OF FIGURES	ix
	LIST OF ABBREVIATIONS	xxiv
	ABSTRACT	xxvi
CHAPTER 1	INTRODUCTION, LITERATURE REVIEW AND OBJECTIVES	1
	1.1 Introduction	1
	1.2 Literature Review	6
	1.3 Objectives	11
CHAPTER 2	SEEPAGE INTO DITCH DRAINS IN A TWO-LAYERED SOIL UNDERLAIN BY AN IMPERVIOUS SUBSTRATUM	12
	2.1 Solutions for the Two-Dimensional Continuity Equation of Groundwater flow for a Homogeneous and Anisotropic Soil	12
	2.2 Mathematical Formulation and Solution	16
	2.2.1 Level of Water in the Ditches is Different and Above the Boundary between the Soil Layers	18
	2.2.1.1 Determination of Coefficients of the Steady State Solution (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 2.2)	22
	2.2.1.2 Determination of the Transient Solution Coefficients	26

[transient state solution is approximate and is applicable only when conductivity and specific storage of the layers of Fig. 2.2 satisfy $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)]

2.2.1.3	Verifications of the Proposed Solution	36
2.2.1.4	MODFLOW Verification of the Proposed Solution	40
2.2.2	Level of Water in the Ditches is Different and Below the Boundary between the Soil Layers	43
2.2.2.1	Determination of the Steady State Solution Coefficients (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 2.10)	47
2.2.2.2	Determination of the Transient Solution Coefficients [transient state solution is approximate and is applicable only when conductivity and specific storage of the layers of Fig. 2.10 satisfy $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)]	50
2.2.2.3	Verifications of the Proposed Solution	56
2.2.2.3	MODFLOW Verification of the Proposed Solution	58
2.2.3	Level of Water in the Left Ditch is Below the Boundary between the Soil Layers while Level of Water in the Right Ditch is Above the Boundary between the Soil Layers	60
2.2.3.1	Determination of the Steady State Solution Coefficients (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 2.18)	63
2.2.3.2	Determination of the Transient Solution Coefficients [transient state solution is approximate and is applicable only when conductivity and specific storage of the layers of Fig. 2.18 satisfy $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)]	65
2.2.3.3	Verifications of the Proposed Solution	69
2.2.3.4	MODFLOW Verification of the Proposed Solution	72
2.2.4	Level of Water in the Left Ditch is above the Boundary	74

between the Soil Layers while Level of Water in the Right Ditch is below the Boundary between the Soil Layers

2.2.4.1	Determination of the Steady State Solution Coefficients (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 2.26)	77
2.2.4.2	Determination of the Transient Solution Coefficients [transient state solution is approximate and is applicable only when conductivity and specific storage of the layers of Fig. 2.26 satisfy $K_{x_1} / S_{s_1} = K_{x_2} / S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)]	79
2.2.4.3	Verifications of the Proposed Solution	82
2.2.4.4	MODFLOW Verification of the Proposed Solution	86
2.3	Discussions	88
2.4	Conclusions	105
2.5	List of Notations	107
CHAPTER 3	THREE-DIMENSIONAL SEEPAGE OF PONDED WATER INTO DITCH DRAINS IN A THREE-LAYERED SOIL COLUMN UNDERLAIN BY AN IMPERVIOUS SUBSTRATUM	112
3.1	Solutions for the Three-Dimensional Continuity Equation of Groundwater flow for a Homogeneous and Anisotropic Soil	112
3.2	Mathematical Formulation and Solution	116
3.2.1	Level of Water in the Ditches is above the Boundary between the Top and the Middle Soil Layers	120
3.2.1.1	Determination of Coefficients of the Steady State Part of the Solution (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 3.4)	131
3.2.1.2	Determination of Coefficients of the Transient State Part of the Solution [transient state solution is approximate and is applicable only when directional conductivities and specific storage of the layers of Fig. 3.4 satisfy the conditions	136

$$K_{x_1} / S_{s_1} = K_{x_2} / S_{s_2} = K_{x_3} / S_{s_3}, K_{y_1} / S_{s_1} = K_{y_2} / S_{s_2} = K_{y_3} / S_{s_3}$$

and $K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0$ (but $K_{z_1} = K_{z_2} = K_{z_3} \neq 0$)

3.2.1.3	Verifications of the Proposed Solution	154
3.2.1.4	MODFLOW Verification of the Proposed Solution	158
3.2.2	Level of Water in the Ditches is below the Boundary between the Top and the Middle Soil Layers and above the Boundary between the Middle and the Bottom Soil Layers	161
3.2.2.1	Determination of Coefficients of the Steady State Part of the Solution (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 3.10)	169
3.2.2.2	Determination of Coefficients of the Transient State Part of the Solution [transient state solution is approximate and is applicable only when directional conductivities and specific storage of the layers of Fig. 3.10 satisfy the conditions $K_{x_1} / S_{s_1} = K_{x_2} / S_{s_2} = K_{x_3} / S_{s_3}, K_{y_1} / S_{s_1} = K_{y_2} / S_{s_2} = K_{y_3} / S_{s_3}$ and $K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0$ (but $K_{z_1} = K_{z_2} = K_{z_3} \neq 0$)]	176
3.2.2.3	Verifications of the Proposed Solution	188
3.2.2.4	MODFLOW Verification of the Proposed Solution	193
3.2.3	Level of Water in the Ditches is below the Boundary between the Middle Soil and the Bottom Soil Layers	195
3.2.3.1	Determination of Coefficients of the Steady State Part of the Solution (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 3.16)	204
3.2.3.2	Determination of Coefficients of the Transient State Part of the Solution [transient state solution is approximate and is applicable only when directional conductivities and specific storage of the layers of Fig. 3.16 satisfy the conditions $K_{x_1} / S_{s_1} = K_{x_2} / S_{s_2} = K_{x_3} / S_{s_3}, K_{y_1} / S_{s_1} = K_{y_2} / S_{s_2} = K_{y_3} / S_{s_3}$ and $K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0$ (but $K_{z_1} = K_{z_2} = K_{z_3} \neq 0$)]	211
3.2.3.3	Verifications of the Proposed Solution	226
3.2.3.4	MODFLOW Verification of the Proposed Solution	230
3.3	Discussions	232

3.4	Conclusions	251
3.5	List of Notations	252
CHAPTER 4	CONCLUSIONS	262
BIBLIOGRAPHY		265



LIST OF FIGURES

Figure	Title	Page No.
Fig. 2.1	General geometry of a fully penetrating ditch drainage system in a two-layered soil subjected to variable depths of ponding at the surface	16
Fig. 2.2	Ditch drainage system for a two-layered soil when height of water in the ditches is above the bottom surface of the top soil layer	18
Fig. 2.3	Variation of $Q_{top(1)} / 2Kh$ ratios with time as obtained from the proposed solution for different h and $H_1 = H_2 = 0.9h$ values when the other parameters of the flow problem are considered as $S = 100$ m, $\delta = 0$ m, $H_3 = 0.95h$ and (a) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day, $S_{s_1} = S_{s_2} = 0.001$ m ⁻¹ , and (b) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day, $S_{s_1} = S_{s_2} = 0.001$ m ⁻¹ and (c) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day and $S_{s_1} = S_{s_2} = 0.0001$ m ⁻¹	37
Fig. 2.4	Plots of $Q_{top(1)} / 2Kh$ versus $(h - H_1) / h$ as obtained from the proposed solution for different times of simulation of the system with the corresponding steady state plot as obtained from Youngs' (1994) single-layered solution of the problem for isotropic soils when the ponding depth is considered as zero and the other flow parameters of the problem are taken as $S = 100$ m, $h = 2.0$ m, $S_{s_1} = S_{s_2} = 0.0001$ m ⁻¹ and $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day	38
Fig. 2.5	Comparison of steady state hydraulic heads as obtained from the proposed solution with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.45$ m, $H_2 = 0.35$ m, $\delta = 0$ m and $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day	39
Fig. 2.6	Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils at two different times of simulation of the system when the flow parameters for the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.35$ m, $H_2 = 0.45$ m, $\varepsilon = 0.05$ m, $\delta = 0.03$ m, $K_{x_1} = K_{x_2} = 0.15$ m/day, $K_{y_1} = K_{y_2} = 0.075$ m/day and $S_{s_1} = S_{s_2} = 0.001$ m ⁻¹	40
Fig. 2.7	Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding values as obtained from MODFLOW when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.6$ m, $H_1 = 0.45$ m, $H_2 = 0.4$ m, $\delta = 0.03$ m, $\varepsilon = 0.05$ m, $K_{x_1} = 1.0$ m/day, $K_{y_1} = 0.75$ m/day, $K_{x_2} = 2.5$ m/day and	41

$$K_{y_2} = 3.5 \text{ m/day}$$

- Fig. 2.8** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.7 \text{ m}$, $H_1 = 0.6 \text{ m}$, $H_2 = 0.6 \text{ m}$, $\delta = 0.02 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $K_{x_1} = .8 \text{ m/day}$, $K_{x_2} = 1.6 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.001 \text{ m/day}$, $S_{s_1} = 0.003 \text{ m}^{-1}$ and $S_{s_2} = 0.006 \text{ m}^{-1}$ 42
- Fig. 2.9** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.7 \text{ m}$, $H_1 = 0.6 \text{ m}$, $H_2 = 0.6 \text{ m}$, $\delta = 0.02 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $K_{x_1} = 0.8 \text{ m/day}$, $K_{x_2} = 1.6 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.01 \text{ m/day}$, $S_{s_1} = 0.003 \text{ m}^{-1}$ and $S_{s_2} = 0.006 \text{ m}^{-1}$ 43
- Fig. 2.10** Ditch drainage system for a two-layered soil when height of water in the ditches is below the bottom surface of the top soil layer 44
- Fig. 2.11** Variation of $Q_{top(2)} / 2Kh$ ratios with time as obtained from the proposed solution for different h and $H_1 = H_2 = h$ values when the other parameters of the flow problem are considered as $S = 100 \text{ m}$, $\delta = 0 \text{ m}$, $H_3 = 0.95 h$ and (a) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$, and (b) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$, and (c) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.0001 \text{ m}^{-1}$ 56
- Fig. 2.12** Plots of $Q_{top(2)} / 2Kh$ versus $(h - H_1) / h$ as obtained from the proposed solution for different times of simulation of the system with the corresponding steady state plot as obtained from Youngs' (1994) single-layered solution of the problem for isotropic soils when the ponding depth is considered as zero and the other flow parameters of the problem are taken as $S = 100 \text{ m}$, $h = 2.0 \text{ m}$, $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$ and $S_{s_1} = S_{s_2} = 0.0001 \text{ m}^{-1}$ 57
- Fig. 2.13** Comparison of steady state hydraulic heads as obtained from the proposed solution with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S = 5 \text{ m}$, $h = 1 \text{ m}$, $H_1 = 0.75 \text{ m}$, $H_2 = 0.85 \text{ m}$, $\delta = 0 \text{ m}$, $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.0001 \text{ m}^{-1}$ 57
- Fig. 2.14** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils at two different times of simulation of the system when the flow 58

parameters for the problem are considered as $S = 5 \text{ m}$, $h = 1 \text{ m}$, $H_1 = 0.85 \text{ m}$, $H_2 = 0.75 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $\delta = 0.08 \text{ m}$, $K_{x_1} = K_{x_2} = 0.15 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.075 \text{ m/day}$, and $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$

- Fig. 2.15** Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding values as obtained from MODFLOW when the flow parameters of the problem are considered as $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.55 \text{ m}$, $H_1 = 0.8 \text{ m}$, $H_2 = 0.8 \text{ m}$, $\delta = 0.03 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $K_{x_1} = 3 \text{ m/day}$, $K_{y_1} = 4 \text{ m/day}$, $K_{x_2} = 1 \text{ m/day}$ and $K_{y_2} = 0.75 \text{ m/day}$ 58
- Fig. 2.16** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.55 \text{ m}$, $H_1 = 0.7 \text{ m}$, $H_2 = 0.9 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $\delta_1 = 0.05 \text{ m}$, $\delta_2 = 0.1 \text{ m}$, $\delta_3 = 0.05 \text{ m}$, $S_1 = 1.5 \text{ m}$, $S_2 = 3.5 \text{ m}$, $K_{x_1} = 1 \text{ m/day}$, $K_{x_2} = 2.5 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.001 \text{ m/day}$, $S_{s_1} = 0.003 \text{ m}^{-1}$ and $S_{s_2} = 0.0075 \text{ m}^{-1}$ 59
- Fig. 2.17** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.55 \text{ m}$, $H_1 = 0.7 \text{ m}$, $H_2 = 0.9 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $\delta_1 = 0.05 \text{ m}$, $\delta_2 = 0.1 \text{ m}$, $\delta_3 = 0.05 \text{ m}$, $S_1 = 1.5 \text{ m}$, $S_2 = 3.5 \text{ m}$, $K_{x_1} = 1 \text{ m/day}$, $K_{x_2} = 2.5 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.01 \text{ m/day}$, $S_{s_1} = 0.003 \text{ m}^{-1}$ and $S_{s_2} = 0.0075 \text{ m}^{-1}$ 59
- Fig. 2.18** Ditch drainage system for a two-layered soil where the height of water in the left ditch is below the bottom surface of the top layer and height of water in the right ditch is above it 60
- Fig. 2.19** Variation of $Q_{top(3)} / 2Kh$ ratios with time as obtained from the proposed solution for different h and $H_1 = h$ values when the other parameters of the flow problem are considered as $S = 100 \text{ m}$, $\delta = 0 \text{ m}$, $H_3 = 0.95h$ and (a) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$ and (b) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$ and (c) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.0001 \text{ m}^{-1}$ 70
- Fig. 2.20** Plots of $Q_{top(3)} / 2Kh$ versus $(h - H_1) / h$ as obtained from the proposed solution for different times of simulation of the system with the corresponding steady state plot as obtained from Youngs' (1994) single-layered solution of the problem for isotropic soils when the ponding depth is considered as zero and the other flow parameters of the problem are taken as $S = 100 \text{ m}$, $h = 2.0 \text{ m}$, $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$ and $S_{s_1} = S_{s_2} = 0.0001 \text{ m}^{-1}$ 71

- Fig. 2.21** Comparison of steady state hydraulic heads as obtained from the proposed solution with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.75$ m, $H_2 = 0.25$ m, $\delta = 0$ m, and $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day 71
- Fig. 2.22** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils at two different times of simulation of the system when the flow parameters for the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.75$ m, $H_2 = 0.25$ m, $\varepsilon = 0.05$ m, $\delta = 0.05$ m, $K_{x_1} = K_{x_2} = 0.15$ m/day, $K_{y_1} = K_{y_2} = 0.09$ m/day and $S_{s_1} = S_{s_2} = 0.001$ m⁻¹ 72
- Fig. 2.23** Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding values as obtained from MODFLOW when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.65$ m, $H_2 = 0.45$ m, $\delta = 0.05$ m, $\varepsilon = 0.05$ m, $K_{x_1} = 1.5$ m/day, $K_{y_1} = 2.5$ m/day, $K_{x_2} = 0.5$ m/day and $K_{y_2} = 0.5$ m/day 72
- Fig. 2.24** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_1 = 0.65$ m, $H_2 = 0.45$ m, $H_3 = 0.55$ m, $\varepsilon = 0.05$ m, $\delta = 0.04$ m, $K_{x_1} = 0.75$ m/day, $K_{x_2} = 2.25$ m/day, $K_{y_1} = K_{y_2} = 0.001$ m/day, $S_{s_1} = 0.005$ m⁻¹ and $S_{s_2} = 0.015$ m⁻¹ 73
- Fig. 2.25** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_2 = 0.45$ m, $H_1 = 0.65$ m, $H_3 = 0.55$ m, $\varepsilon = 0.05$ m, $\delta = 0.04$ m, $K_{x_1} = 0.75$ m/day, $K_{x_2} = 2.25$ m/day, $K_{y_1} = K_{y_2} = 0.01$ m/day, $S_{s_1} = 0.005$ m⁻¹ and $S_{s_2} = 0.015$ m⁻¹ 73
- Fig. 2.26** Ditch drainage system for a two-layered soil where the height of water in the left ditch is above the bottom surface of the top soil layer and the height of water in the right ditch is below it 74
- Fig. 2.27** Variation of $Q_{top(4)}/2Kh$ ratios with time as obtained from the proposed solution for different h and $H_2 = h$ values when the other parameters of the flow problem are considered as $S = 100$ m, $\delta = 0$ m, $H_3 = 0.95$ h 84

and (a) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day, $S_{s_1} = S_{s_2} = 0.001$ m⁻¹ and
 (b) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day, $S_{s_1} = S_{s_2} = 0.001$ m⁻¹ and (c)
 $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day, $S_{s_1} = S_{s_2} = 0.0001$ m⁻¹

- Fig. 2.28** Plots of $Q_{top(4)} / 2Kh$ versus $(h - H_1) / h$ as obtained from the proposed solution for different times of simulation of the system with the corresponding steady state plot as obtained from Youngs' (1994) single-layered solution of the problem for isotropic soils when the ponding depth is considered as zero and the other flow parameters of the problem are taken as $S = 100$ m, $h = 2.0$ m, $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day and $S_{s_1} = S_{s_2} = 0.0001$ m⁻¹ 85
- Fig. 2.29** Comparison of steady state hydraulic heads as obtained from the proposed solution with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.25$ m, $H_2 = 0.75$ m, $\delta = 0$ m and $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day 85
- Fig. 2.30** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained from Barua and Alam's (2012) single-layered transient solution of the problem for anisotropic soils at two different times of simulation of the system when the flow parameters for the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.75$ m, $H_2 = 0.25$ m, $\varepsilon = 0.2$ m, $\delta = 0.05$ m, $K_{x_1} = K_{x_2} = 0.15$ m/day, $K_{y_1} = K_{y_2} = 0.075$ m/day and $S_{s_1} = S_{s_2} = 0.001$ m⁻¹ 86
- Fig. 2.31** Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding values as obtained from MODFLOW when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.45$ m, $H_2 = 0.65$ m, $\delta = 0.04$ m, $\varepsilon = 0.05$ m, $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 1$ m/day and $K_{y_2} = 1.5$ m/day 86
- Fig. 2.32** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.45$ m, $H_2 = 0.65$ m, $\varepsilon = 0.05$ m, $\delta = 0.06$ m, $K_{x_1} = 4$ m/day, $K_{x_2} = 1$ m/day, $K_{y_1} = K_{y_2} = 0.001$ m/day, $S_{s_1} = 0.04$ m⁻¹ and $S_{s_2} = 0.01$ m⁻¹ 87
- Fig. 2.33** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.45$ m, $H_2 = 0.65$ m, $\varepsilon = 0.05$ m, $\delta = 0.06$ m, $K_{x_1} = 4$ m/day, 87

$$K_{x_2} = 1 \text{ m/day}, K_{y_1} = K_{y_2} = 0.01 \text{ m/day}, S_{s_1} = 0.04 \text{ m}^{-1} \text{ and } S_{s_2} = 0.01 \text{ m}^{-1}$$

- Fig. 2.34** Variation of steady state horizontal gradient with depth of water at the face of a ditch when the parameters of the flow problem of Fig. 2.2 are taken as $S = 5 \text{ m}$, $h = 1.5 \text{ m}$, $H_3 = 0.75 \text{ m}$, $H_1 = H_2 = 0.5 \text{ m}$, $\delta = 0 \text{ m}$ and (a) $K_{x_1} = K_{x_2} = K_{y_1} = K_{y_2} = 0.4 \text{ m/day}$ and (b) $K_{x_1} = K_{y_1} = 1 \text{ m/day}$, $K_{x_2} = K_{y_2} = 0.1 \text{ m/day}$ and (c) $K_{x_1} = K_{y_1} = 0.1 \text{ m/day}$ and $K_{x_2} = K_{y_2} = 1 \text{ m/day}$ 89
- Fig. 2.35** Plot of steady state top discharge versus the ratio of hydraulic conductivities of the top and bottom soil layers when the other parameters of the flow problem of Fig 2.10 are taken as $S = 5 \text{ m}$, $h = 1 \text{ m}$, $H_3 = 0.55 \text{ m}$, $H_1 = H_2 = 1 \text{ m}$ and $\delta = 0 \text{ m}$ 90
- Fig. 2.36** Plot of steady state top discharge values versus the thickness of the top soil layer when the other parameters of the flow problem of Fig. 2.10 are taken as $S = 5 \text{ m}$, $h = 1 \text{ m}$, $H_1 = H_2 = 1 \text{ m}$, $\delta = 0 \text{ m}$, $K_{x_1} = 1 \text{ m/day}$, $K_{y_1} = 2 \text{ m/day}$, $K_{x_2} = 0.5 \text{ m/day}$ and $K_{y_2} = 1 \text{ m/day}$ 91
- Fig. 2.37** Plot of steady state top discharge versus the thickness of the top soil layer when the other parameters of the flow problem of Fig. 2.10 are taken as $S = 5 \text{ m}$, $h = 1 \text{ m}$, $H_1 = H_2 = 1 \text{ m}$, $\delta = 0 \text{ m}$, $K_{x_1} = 0.5 \text{ m/day}$, $K_{y_1} = 1 \text{ m/day}$, $K_{x_2} = 1 \text{ m/day}$ and $K_{y_2} = 2 \text{ m/day}$ 91
- Fig. 2.38** Plots of the steady state top discharge function with horizontal distance at the surface of a two-dimensional ponded ditch drainage system when the flow parameters of the problem are taken as $S = 15 \text{ m}$, $h = 2 \text{ m}$, $H_3 = 1.35 \text{ m}$, $H_1 = H_2 = 1.0 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $\delta_1 = 0 \text{ m}$, $\delta_2 = 0.04 \text{ m}$, $\delta_3 = 0.08 \text{ m}$, $\delta_4 = 0.12 \text{ m}$, $\delta_5 = 0.08 \text{ m}$, $\delta_6 = 0.04 \text{ m}$, $\delta_7 = 0 \text{ m}$, $S_1 = 1 \text{ m}$, $S_2 = 3 \text{ m}$, $S_3 = 5 \text{ m}$, $S_4 = 10 \text{ m}$, $S_5 = 12 \text{ m}$, $S_6 = 14 \text{ m}$ and (a) $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 1 \text{ m/day}$, $K_{y_2} = 0.25 \text{ m/day}$, and (b) $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 0.5 \text{ m/day}$, $K_{y_2} = 0.125 \text{ m/day}$, and (c) $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 4 \text{ m/day}$, $K_{y_2} = 1 \text{ m/day}$, and (d) $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 8 \text{ m/day}$, $K_{y_2} = 2 \text{ m/day}$ 92
- Fig. 2.39** Plots of the steady state top discharge function with horizontal distance at the surface of a two-dimensional ponded ditch drainage system when the flow parameters of the problem are taken as $S = 15 \text{ m}$, $h = 2 \text{ m}$, $H_3 = 1.35 \text{ m}$, $H_1 = H_2 = 2.0 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $\delta_1 = 0 \text{ m}$, $\delta_2 = 0.04 \text{ m}$, $\delta_3 = 0.08 \text{ m}$, $\delta_4 = 0.12 \text{ m}$, $\delta_5 = 0.08 \text{ m}$, $\delta_6 = 0.04 \text{ m}$, $\delta_7 = 0 \text{ m}$, $S_1 = 1 \text{ m}$, $S_2 = 3 \text{ m}$, $S_3 = 5 \text{ m}$, $S_4 = 10 \text{ m}$, $S_5 = 12 \text{ m}$, $S_6 = 14 \text{ m}$ and (a) $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 1 \text{ m/day}$, $K_{y_2} = 0.25 \text{ m/day}$, and (b) $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 0.5 \text{ m/day}$, $K_{y_2} = 0.125 \text{ m/day}$, and (c) $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 4 \text{ m/day}$, $K_{y_2} = 1 \text{ m/day}$, and (d) $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 8 \text{ m/day}$, $K_{y_2} = 2 \text{ m/day}$ 93

- Fig. 2.40** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain when the flow parameters of the system (Fig. 2.2) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.65$ m, $H_1 = H_2 = 0.5$ m, $\varepsilon = 0.05$ m, $\delta = 0.05$ m, $\eta_1 = \eta_2 = 0.3$, and (a) $K_{x_1} = K_{y_1} = 0.5$ m/day, $K_{x_2} = K_{y_2} = 1$ m/day, and (b) $K_{x_1} = 0.5$ m/day, $K_{y_1} = 0.25$ m/day, $K_{x_2} = 2$ m/day, $K_{y_2} = 1$ m/day, and (c) $K_{x_1} = 0.25$ m/day, $K_{y_1} = 0.1$ m/day, $K_{x_2} = 2.5$ m/day, $K_{y_2} = 1$ m/day, and (d) $K_{x_1} = 1$ m/day, $K_{y_1} = 0.75$ m/day, $K_{x_2} = 0.5$ m/day, $K_{y_2} = 0.25$ m/day 96
- Fig. 2.41** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain when the flow parameters of the system (Fig. 2.10) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.45$ m, $H_1 = H_2 = 0.9$ m, $\varepsilon = 0.05$ m, $\eta_1 = \eta_2 = 0.3$, (a) $K_{x_1} = K_{y_1} = 0.5$ m/day, $K_{x_2} = K_{y_2} = 1$ m/day, $\delta = 0.1$ m and (b) $K_{x_1} = 1.5$ m/day, $K_{y_1} = 1$ m/day, $K_{x_2} = 0.75$ m/day, $K_{y_2} = 0.5$ m/day, $\delta = 0.1$ m 97
- Fig. 2.42** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 2.18) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.55$ m, $\varepsilon = 0.05$ m, $\delta = 0.05$ m, $\eta_1 = \eta_2 = 0.3$, $K_{x_1} = K_{x_2} = 0.5$ m/day, $K_{y_1} = K_{y_2} = 1$ m/day, and (a) $H_1 = 0.65$ m, $H_2 = 0.45$ m, and (b) $H_1 = 0.75$ m, $H_3 = 0.35$ m 98
- Fig. 2.43** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain when the flow parameters of the system (Fig. 2.10) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.3$ m, $H_1 = H_2 = 0.8$ m, $\delta = 0$ m, $\eta_1 = \eta_2 = 0.35$ and $K_{x_1} = K_{x_2} = K_{y_1} = K_{y_2} = 0.1$ m/day 99
- Fig. 2.44** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain when the flow parameters of the system (Fig. 2.10) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.3$ m, $H_1 = H_2 = 0.8$ m, $\delta = 0$ m, $\eta_1 = 0.55$, $\eta_2 = 0.35$, $K_{x_1} = K_{y_1} = 0.005$ m/day and $K_{x_2} = K_{y_2} = 0.1$ m/day 99
- Fig. 2.45** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain and the distribution of the top discharge function when the flow parameters of the system (Fig. 2.2) are taken as $S = 6$ m, $h = 2$ m, $H_3 = 1.2$ m, $H_1 = H_2 = 1$ m, $\eta_1 = 0.45$, $\eta_2 = 0.3$, $K_{x_1} = 0.5$ m/day, $K_{y_1} = 0.25$ m/day, $K_{x_2} = 2$ m/day and $K_{y_2} = 1$ m/day. The top surface ponding distributions are (a) $\delta = 0$ m and (b) $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.2$ m, $\delta_4 = 0.4$ m, $\delta_5 = 0.6$ m, $\delta_6 = 0.4$ m, $\delta_7 = 0.2$ m, $\delta_8 = 0.1$ m, $\delta_9 = 0$ m, $S_1 = 0.2$ m, $S_2 = 0.7$ m, $S_3 = 1.2$ m, $S_4 = 1.7$ m, $S_5 = 4.3$ m, $S_6 = 4.8$ m, 101

$$S_7 = 5.3 \text{ m and } S_8 = 5.8 \text{ m}$$

- Fig. 2.46** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain and the distribution of the top discharge function when the flow parameters of the system (Fig. 2.2) are taken as $S = 6 \text{ m}$, $h = 2 \text{ m}$, $H_3 = 0.9 \text{ m}$, $H_1 = H_2 = 0.6 \text{ m}$, $\eta_1 = 0.3$, $\eta_2 = 0.45$, $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 1 \text{ m/day}$, $K_{x_2} = 0.5 \text{ m/day}$ and $K_{y_2} = 0.25 \text{ m/day}$. The top surface ponding distributions are (a) $\delta = 0 \text{ m}$ and (b) $\delta_1 = 0 \text{ m}$, $\delta_2 = 0.1 \text{ m}$, $\delta_3 = 0.2 \text{ m}$, $\delta_4 = 0.4 \text{ m}$, $\delta_5 = 0.6 \text{ m}$, $\delta_6 = 0.4 \text{ m}$, $\delta_7 = 0.2 \text{ m}$, $\delta_8 = 0.1 \text{ m}$, $\delta_9 = 0 \text{ m}$, $S_1 = 0.2 \text{ m}$, $S_2 = 0.7 \text{ m}$, $S_3 = 1.2 \text{ m}$, $S_4 = 1.7 \text{ m}$, $S_5 = 4.3 \text{ m}$, $S_6 = 4.8 \text{ m}$, $S_7 = 5.3 \text{ m}$ and $S_8 = 5.8 \text{ m}$ 102
- Fig. 2.47** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain and the distribution of the top discharge function when the flow parameters of the system (Fig. 2.10) are taken as $S = 6 \text{ m}$, $h = 2 \text{ m}$, $H_3 = 1.2 \text{ m}$, $H_1 = H_2 = 2 \text{ m}$, $\eta_1 = 0.45$, $\eta_2 = 0.3$, $K_{x_1} = 0.5 \text{ m/day}$, $K_{y_1} = 0.25 \text{ m/day}$, $K_{x_2} = 2 \text{ m/day}$ and $K_{y_2} = 1 \text{ m/day}$. The top surface ponding distributions are (a) $\delta = 0 \text{ m}$ and (b) $\delta_1 = 0 \text{ m}$, $\delta_2 = 0.1 \text{ m}$, $\delta_3 = 0.2 \text{ m}$, $\delta_4 = 0.4 \text{ m}$, $\delta_5 = 0.6 \text{ m}$, $\delta_6 = 0.4 \text{ m}$, $\delta_7 = 0.2 \text{ m}$, $\delta_8 = 0.1 \text{ m}$, $\delta_9 = 0 \text{ m}$, $S_1 = 0.2 \text{ m}$, $S_2 = 0.7 \text{ m}$, $S_3 = 1.2 \text{ m}$, $S_4 = 1.7 \text{ m}$, $S_5 = 4.3 \text{ m}$, $S_6 = 4.8 \text{ m}$, $S_7 = 5.3 \text{ m}$ and $S_8 = 5.8 \text{ m}$ 103
- Fig. 2.48** Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain and the distribution of the top discharge function when the flow parameters of the system (Fig. 2.10) are taken as $S = 6 \text{ m}$, $h = 2 \text{ m}$, $H_3 = 0.9 \text{ m}$, $H_1 = H_2 = 2 \text{ m}$, $\eta_1 = 0.3$, $\eta_2 = 0.45$, $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 1 \text{ m/day}$, $K_{x_2} = 0.5 \text{ m/day}$ and $K_{y_2} = 0.25 \text{ m/day}$. The top surface ponding distributions are (a) $\delta = 0 \text{ m}$ and (b) $\delta_1 = 0 \text{ m}$, $\delta_2 = 0.1 \text{ m}$, $\delta_3 = 0.2 \text{ m}$, $\delta_4 = 0.4 \text{ m}$, $\delta_5 = 0.6 \text{ m}$, $\delta_6 = 0.4 \text{ m}$, $\delta_7 = 0.2 \text{ m}$, $\delta_8 = 0.1 \text{ m}$, $\delta_9 = 0 \text{ m}$, $S_1 = 0.2 \text{ m}$, $S_2 = 0.7 \text{ m}$, $S_3 = 1.2 \text{ m}$, $S_4 = 1.7 \text{ m}$, $S_5 = 4.3 \text{ m}$, $S_6 = 4.8 \text{ m}$, $S_7 = 5.3 \text{ m}$ and $S_8 = 5.8 \text{ m}$ 104
- Fig. 2.49** Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times when the flow parameters of Fig. 2.10 are considered as (a) $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.6 \text{ m}$, $H_1 = 0.45 \text{ m}$, $H_2 = 0.45 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $\delta = 0 \text{ m}$, $K_{x_1} = 1 \text{ m/day}$, $K_{x_2} = 2 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.05 \text{ m/day}$, $S_{s_1} = 0.0075 \text{ m}^{-1}$ and $S_{s_2} = 0.015 \text{ m}^{-1}$ and (b) $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.55 \text{ m}$, $H_1 = 0.9 \text{ m}$, $H_2 = 0.9 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $\delta = 0 \text{ m}$, $K_{x_1} = 1 \text{ m/day}$, $K_{x_2} = 2 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.05 \text{ m/day}$, $S_{s_1} = 0.0075 \text{ m}^{-1}$ and $S_{s_2} = 0.015 \text{ m}^{-1}$ 105

Fig. 3.1	General geometry of a three-dimensional ponded ditch drainage system subject to a variable ponding distribution at the surface of the soil	117
Fig. 3.2	The cross sectional view of the flow domain along the axis $x-x'$ of Fig. 3.1	118
Fig. 3.3	The cross sectional view of the flow domain along the axis $y-y'$ of Fig. 3.1	118
Fig. 3.4	Three dimensional ditch drainage system for a three-layered soil when height of water in the ditches is above the boundary between the top and the middle soil layers	121
Fig. 3.5	Comparison of steady state hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.4 with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of Fig. 3.4 are $S_1=15$ m, $S_2=5$ m, $h=1$ m, $H_1=0.4$ m, $H_5=0.6$ m, $H_6=0.8$ m, $\delta=0$ m and $K_{x_1}=K_{x_2}=K_{x_3}=K_{y_1}=K_{y_2}=K_{y_3}=K_{z_1}=K_{z_2}=K_{z_3}=1$ m/day	155
Fig. 3.6	Comparison of transient hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.4 at times $t=100$ s and 500 s with the corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem are considered as $S_1=15$ m, $S_2=5$ m, $h=1$ m, $H_1=0.25$ m, $H_5=0.35$ m, $H_6=0.8$ m, $\delta=0$ m, $K_{x_1}=K_{x_2}=K_{x_3}=K_{y_1}=K_{y_2}=K_{y_3}=1.5$ m/day, $K_{z_1}=K_{z_2}=K_{z_3}=0.75$ m/day and $S_{s_1}=S_{s_2}=S_{s_3}=0.005$ m ⁻¹	156
Fig. 3.7	. Comparison of transient hydraulic heads as obtained from the proposed solution at times $t=100$ s and 500 s with the corresponding values as obtained from Sarmah and Barua's (2017) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem (Fig. 3.4) are considered as $S_1=6$ m, $S_2=5$ m, $h=1$ m, $H_1=0.35$ m, $H_5=0.45$ m, $H_6=0.75$ m, $\delta=0.02$ m, $\varepsilon_x=\varepsilon_y=0.05$ m, $K_{x_1}=K_{x_2}=K_{x_3}=1$ m/day, $K_{y_1}=K_{y_2}=K_{y_3}=0.5$ m/day, $K_{z_1}=K_{z_2}=K_{z_3}=0.002$ m/day and $S_{s_1}=S_{s_2}=S_{s_3}=0.01$ m ⁻¹	157
Fig. 3.8	Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding results as obtained from MODFLOW when the flow parameters of Fig. 3.4 are considered as $S_1=10$ m, $S_2=5$ m, $h=1$ m, $H_1=0.35$ m, $H_5=0.45$ m, $H_6=0.75$ m, $\delta=0.03$ m, $\varepsilon_x=\varepsilon_y=0.05$ m, $K_{x_1}=1$ m/day, $K_{y_1}=2$ m/day, $K_{z_1}=0.5$ m/day, $K_{x_2}=1.2$ m/day, $K_{y_2}=1.8$ m/day, $K_{z_2}=0.8$ m/day,	159

$$K_{x_3} = 1.4 \text{ m/day}, K_{y_3} = 1.4 \text{ m/day} \text{ and } K_{z_3} = 1 \text{ m/day}$$

- Fig. 3.9** Comparison of transient hydraulic head contours as obtained from the proposed solution at times $t=100$ s and 500 s with the corresponding values as obtained from MODFLOW when the flow parameters of Fig. 3.4 are considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.35$ m, $H_5 = 0.45$ m, $H_6 = 0.75$ m, $\delta = 0.02$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = 0.75$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 1.5$ m/day, $K_{y_2} = 1$ m/day, $K_{x_3} = 0.6$ m/day, $K_{y_3} = 0.4$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.001$ m/day, $S_{s_1} = 0.01 \text{ m}^{-1}$, $S_{s_2} = 0.02 \text{ m}^{-1}$ and $S_{s_3} = 0.008 \text{ m}^{-1}$ 160
- Fig. 3.10** Three dimensional ditch drainage system for a three-layered soil where the height of water in the ditches is below the bottom boundary of the top layer and above the top boundary of the bottom layer 161
- Fig. 3.11** Comparison of steady state hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.10 with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S_1 = 15$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.7$ m, $H_5 = 0.4$ m, $H_6 = 0.8$ m, $\delta = 0$ m and $K_{x_1} = K_{x_2} = K_{x_3} = K_{y_1} = K_{y_2} = K_{y_3} = K_{z_1} = K_{z_2} = K_{z_3} = 1$ m/day 190
- Fig. 3.12** Comparison of transient hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.10 at times $t=100$ s and 500 s with the corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem are considered as $S_1 = 15$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.7$ m, $H_5 = 0.35$ m, $H_6 = 0.8$ m, $\delta = 0.03$ m, $K_{x_1} = K_{x_2} = K_{x_3} = 1.5$ m/day, $K_{y_1} = K_{y_2} = K_{y_3} = 1.5$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 1$ m/day and $S_{s_1} = S_{s_2} = S_{s_3} = 0.01 \text{ m}^{-1}$ 191
- Fig. 3.13** Comparison of transient hydraulic heads as obtained from the proposed solution at times $t=100$ s and 500 s with the corresponding values as obtained from Sarmah and Barua's (2017) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem (Fig. 3.10) are considered as $S_1 = 6$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.65$ m, $H_5 = 0.35$ m, $H_6 = 0.75$ m, $\delta = 0.04$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = K_{x_2} = K_{x_3} = 1$ m/day, $K_{y_1} = K_{y_2} = K_{y_3} = 1.5$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.002$ m/day and $S_{s_1} = S_{s_2} = S_{s_3} = 0.01 \text{ m}^{-1}$ 192
- Fig. 3.14** Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding results as obtained from MODFLOW when the flow parameters of Fig. 3.10 are considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.6$ m, $H_5 = 0.35$ m, $H_6 = 0.75$ m, 193

$$\delta = 0.05 \text{ m}, \quad \varepsilon_x = \varepsilon_y = 0.05 \text{ m}, \quad K_{x_1} = 1 \text{ m/day}, \quad K_{y_1} = 2 \text{ m/day},$$

$$K_{z_1} = 1.5 \text{ m/day}, \quad K_{x_2} = 1.2 \text{ m/day}, \quad K_{y_2} = 0.8 \text{ m/day}, \quad K_{z_2} = 0.9 \text{ m/day},$$

$$K_{x_3} = 0.5 \text{ m/day}, \quad K_{y_3} = 0.5 \text{ m/day} \text{ and } K_{z_3} = 0.6 \text{ m/day}$$

- Fig. 3.15** Comparison of transient hydraulic heads as obtained from the proposed solution at times $t=100$ s and 500 s with the corresponding values as obtained from MODFLOW when the flow parameters of Fig. 3.10 are considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.6$ m, $H_5 = 0.35$ m, $H_6 = 0.75$ m, $\delta_1 = 0.02$ m, $\delta_2 = 0.04$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $d_{x_1} = 1$ m, $d_{x_2} = 9$ m, $d_{y_1} = 1$ m, $d_{y_2} = 4$ m, $K_{x_1} = 1.5$ m/day, $K_{y_1} = 0.75$ m/day, $K_{x_2} = 2$ m/day, $K_{y_2} = 1$ m/day, $K_{x_3} = 0.75$ m/day, $K_{y_3} = 0.375$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.001$ m/day, $S_{s_1} = 0.03 \text{ m}^{-1}$, $S_{s_2} = 0.04 \text{ m}^{-1}$ and $S_{s_3} = 0.015 \text{ m}^{-1}$ 194
- Fig. 3.16** Three dimensional ditch drainage system for a three-layered soil when the height of water in the ditches is below the boundary between middle and bottom soil layers 195
- Fig. 3.17** Comparison of steady state hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.16 with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters are considered as $S_1 = 15$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.9$ m, $H_5 = 0.4$ m, $H_6 = 0.8$ m, $\delta = 0$ m and $K_{x_1} = K_{x_2} = K_{x_3} = K_{y_1} = K_{y_2} = K_{y_3} = K_{z_1} = K_{z_2} = K_{z_3} = 1$ m/day 227
- Fig. 3.18** Comparison of transient hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.16 at times $t=100$ s and 500 s with the corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem are considered as $S_1 = 15$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 1$ m, $H_5 = 0.35$ m, $H_6 = 0.8$ m, $\delta = 0.05$ m, $K_{x_1} = K_{x_2} = K_{x_3} = K_{y_1} = K_{y_2} = K_{y_3} = 2$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.8$ m/day and $S_{s_1} = S_{s_2} = S_{s_3} = 0.01 \text{ m}^{-1}$ 228
- Fig. 3.19** Comparison of transient hydraulic heads as obtained from the proposed solution at times $t=100$ s and 500 s with the corresponding values as obtained from Sarmah and Barua's (2017) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem (Fig. 3.16) are considered as $S_1 = 6$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.9$ m, $H_5 = 0.35$ m, $H_6 = 0.7$ m, $\delta = 0.06$ m, $K_{x_1} = K_{x_2} = K_{x_3} = 1$ m/day, $K_{y_1} = K_{y_2} = K_{y_3} = 1.5$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.002$ m/day and $S_{s_1} = S_{s_2} = S_{s_3} = 0.01 \text{ m}^{-1}$ 229

- Fig. 3.20** Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding results as obtained from MODFLOW where the flow parameters of Fig. 3.16 are considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.9$ m, $H_5 = 0.35$ m, $H_6 = 0.7$ m, $\delta_1 = 0.03$ m, $\delta_2 = 0.06$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $d_{x_1} = 2$ m, $d_{x_2} = 8$ m, $d_{y_1} = 1$ m, $d_{y_2} = 4$ m, $K_{x_1} = 0.8$ m/day, $K_{y_1} = 1$ m/day, $K_{z_1} = 0.5$ m/day, $K_{x_2} = 1$ m/day, $K_{y_2} = 1.5$ m/day, $K_{z_2} = 0.75$ m/day, $K_{x_3} = 1$ m/day, $K_{y_3} = 2$ m/day and $K_{z_3} = 1$ m/day 230
- Fig. 3.21** Comparison between transient hydraulic heads obtained from the proposed analytical solution and heads for the same flow problem obtained using MODFLOW model at (a) 100 s and (b) 500 s. The flow parameters of Fig. 3.16 have been considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.9$ m, $H_5 = 0.35$ m, $H_6 = 0.7$ m, $\delta_1 = 0.04$ m, $\delta_2 = 0.08$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $d_{x_1} = 2$ m, $d_{x_2} = 8$ m, $d_{y_1} = 1$ m, $d_{y_2} = 4$ m, $K_{x_1} = 0.8$ m/day, $K_{y_1} = 1.2$ m/day, $K_{z_1} = 0.001$ m/day, $K_{x_2} = 0.6$ m/day, $K_{y_2} = 0.9$ m/day, $K_{z_2} = 0.001$ m/day, $K_{x_3} = 1.2$ m/day, $K_{y_3} = 1.8$ m/day, $K_{z_3} = 0.001$ m/day, $S_{s_1} = 0.02$ m⁻¹, $S_{s_2} = 0.015$ m⁻¹ and $S_{s_3} = 0.03$ m⁻¹ 232
- Fig. 3.22** Plot of steady state top discharge versus ratio of hydraulic conductivities of the top, middle and bottom soil layers when the parameters of a ponded drainage setting are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1$ m, $\delta = 0$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_2} = K_{y_2} = K_{z_2} = K_2 = 1$ m/day 234
- Fig. 3.23** Plot of steady state top discharge values versus anisotropy ratio $K_{x_1}/K_{z_1} = K_{y_1}/K_{z_1} = K_{x_2}/K_{z_2} = K_{y_2}/K_{z_2} = K_{x_3}/K_{z_3} = K_{y_3}/K_{z_3}$ of the soil layers when the parameters of the flow problem are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1$ m, $\delta = 0$ m, $K_{z_1} = 0.5$ m/day, $K_{z_2} = 1$ m/day and $K_{z_3} = 1.5$ m/day 235
- Fig. 3.24** Plot of steady state top discharge values versus anisotropy ratio $K_{x_1}/K_{z_1} = K_{y_1}/K_{z_1} = K_{x_2}/K_{z_2} = K_{y_2}/K_{z_2} = K_{x_3}/K_{z_3} = K_{y_3}/K_{z_3}$ of the soil layers when the parameters of the flow problem are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1$ m, $\delta = 0$ m, $K_{z_1} = 1.5$ m/day, $K_{z_2} = 1$ m/day and $K_{z_3} = 0.5$ m/day 236
- Fig. 3.25** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = \eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = K_{z_1} = 1$ m/day, $K_{x_2} = K_{y_2} = K_{z_2} = 2$ m/day and $K_{x_3} = K_{y_3} = K_{z_3} = 3$ m/day 237

- Fig. 3.26** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.25$ m, $\delta_4 = 0.4$ m, $d_{x1} = 0.5$ m, $d_{x2} = 1.25$ m, $d_{x3} = 2$ m, $d_{x4} = 6$ m, $d_{x5} = 6.75$ m, $d_{x6} = 7.5$ m, $d_{y1} = 0.5$ m, $d_{y2} = 1.25$ m, $d_{y3} = 2$ m, $d_{y4} = 6$ m, $d_{y5} = 6.75$ m, $d_{y6} = 7.5$ m, $\eta_1 = \eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = K_{z_1} = 1$ m/day, $K_{x_2} = K_{y_2} = K_{z_2} = 2$ m/day and $K_{x_3} = K_{y_3} = K_{z_3} = 3$ m/day 238
- Fig. 3.27** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = \eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = K_{z_1} = 3$ m/day, $K_{x_2} = K_{y_2} = K_{z_2} = 2$ m/day and $K_{x_3} = K_{y_3} = K_{z_3} = 1$ m/day 239
- Fig. 3.28** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.25$ m, $\delta_4 = 0.4$ m, $d_{x1} = 0.5$ m, $d_{x2} = 1.25$ m, $d_{x3} = 2$ m, $d_{x4} = 6$ m, $d_{x5} = 6.75$ m, $d_{x6} = 7.5$ m, $d_{y1} = 0.5$ m, $d_{y2} = 1.25$ m, $d_{y3} = 2$ m, $d_{y4} = 6$ m, $d_{y5} = 6.75$ m, $d_{y6} = 7.5$ m, $\eta_1 = \eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = K_{z_1} = 3$ m/day, $K_{x_2} = K_{y_2} = K_{z_2} = 2$ m/day and $K_{x_3} = K_{y_3} = K_{z_3} = 1$ m/day 240
- Fig. 3.29** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = 0.35$, $\eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = 0.3$ m/day, $K_{z_1} = 0.2$ m/day, $K_{x_2} = K_{y_2} = 1.5$ m/day, $K_{z_2} = 0.5$ m/day, $K_{x_3} = K_{y_3} = 2$ m/day, $K_{z_3} = 1$ m/day 241
- Fig. 3.30** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.10) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 0.75$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = 0.35$, $\eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = 0.3$ m/day, $K_{z_1} = 0.2$ m/day, $K_{x_2} = K_{y_2} = 1.5$ m/day, $K_{z_2} = 0.5$ m/day, $K_{x_3} = K_{y_3} = 2$ m/day, $K_{z_3} = 1$ m/day 242

- Fig. 3.31** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.35$ m, $H_6 = 1.15$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = 0.35$, $\eta_2 = 0.3$, $\eta_3 = 0.4$, $K_{x_1} = K_{y_1} = 0.3$ m/day, $K_{z_1} = 0.2$ m/day, $K_{x_2} = K_{y_2} = 1.5$ m/day, $K_{z_2} = 1$ m/day, $K_{x_3} = K_{y_3} = 0.1$ m/day, $K_{z_3} = 0.05$ m/day 243
- Fig. 3.32** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.10) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 0.75$ m, $H_5 = 0.35$ m, $H_6 = 1.15$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = 0.35$, $\eta_2 = 0.3$, $\eta_3 = 0.4$, $K_{x_1} = K_{y_1} = 0.3$ m/day, $K_{z_1} = 0.2$ m/day, $K_{x_2} = K_{y_2} = 1.5$ m/day, $K_{z_2} = 0.5$ m/day, $K_{x_3} = K_{y_3} = 0.1$ m/day, $K_{z_3} = 0.05$ m/day 243
- Fig. 3.33** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 5$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.8$ m, $H_5 = 0.25$ m, $H_6 = 0.35$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = \eta_2 = 0.55$, $\eta_3 = 0.45$, $K_{x_1} = K_{y_1} = K_{z_1} = 0.005$ m/day, $K_{x_2} = K_{y_2} = K_{z_2} = 0.003$ m/day, and $K_{x_3} = K_{y_3} = K_{z_3} = 0.03$ m/day 244
- Fig. 3.34** Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 5$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.8$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = 0.55$, $\eta_2 = 0.55$, $\eta_3 = 0.45$, $K_{x_1} = K_{y_1} = 0.03$ m/day, $K_{z_1} = 0.03$ m/day, $K_{x_2} = K_{y_2} = 0.03$ m/day, $K_{z_2} = 0.03$ m/day, $K_{x_3} = K_{y_3} = 0.03$ m/day, $K_{z_3} = 0.03$ m/day 245
- Fig. 3.35** A few stream surfaces and percentage distribution of the top discharge $Q_{top(3)}$ when the flow parameters of Fig 3.16 are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.2$ m, $H_5 = 0.5$ m, $H_6 = 1$ m, $\delta = 0$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = K_{x_2} = K_{x_3} = 2$ m/day, $K_{y_1} = K_{y_2} = K_{y_3} = 2$ m/day and $K_{z_1} = K_{z_2} = K_{z_3} = 1$ m/day 246
- Fig. 3.36** A few stream surfaces and top discharge distribution in a three-dimensional ponded ditch drainage system when the parameters of Fig. 3.16 are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.2$ m, $H_5 = 0.5$ m, $H_6 = 1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = K_{y_1} = 1.5$ m/day, $K_{z_1} = 0.5$ m/day, $K_{x_2} = K_{y_2} = 2$ m/day, $K_{z_2} = 1$ m/day, $K_{x_3} = K_{y_3} = 3$ m/day, $K_{z_3} = 2$ m/day and 248

(a) $\delta = 0$ m, (b) $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.2$ m, $\delta_4 = 0.4$ m,
 $d_{x1} = 0.5$ m, $d_{x2} = 1.5$ m, $d_{x3} = 2.5$ m, $d_{x4} = 5.5$ m, $d_{x5} = 6.5$ m and
 $d_{x5} = 7.5$ m

Fig. 3.37

A few stream surfaces and percentage distribution of the top discharge $Q_{top(3)}$ when the flow parameters of Fig 3.16 are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1$ m, $H_1 = 1.2$ m, $H_5 = 0.5$ m, $H_6 = 1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = 3$ m/day, $K_{y_1} = 3$ m/day, $K_{z_1} = 2$ m/day, $K_{x_2} = 2$ m/day, $K_{y_2} = 2$ m/day, $K_{z_2} = 1$ m/day, $K_{x_3} = 1.5$ m/day, $K_{y_3} = 1.5$ m/day, $K_{z_3} = 0.5$ m/day and (a) $\delta = 0$ m and (b) $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.2$ m, $\delta_4 = 0.4$ m, $d_{x1} = 0.5$ m, $d_{x2} = 1.5$ m, $d_{x3} = 2.5$ m, $d_{x4} = 5.5$ m, $d_{x5} = 6.5$ m, $d_{x6} = 7.5$ m, $d_{y1} = 0.5$ m, $d_{y2} = 1.5$ m, $d_{y3} = 2.5$ m, $d_{y4} = 5.5$ m, $d_{y5} = 6.5$ m and $d_{y6} = 7.5$ m

250



LIST OF ABBREVIATIONS

Acad.	=Academic
Adv.	=Advances
Agri./Agric.	=Agricultural
Analyt.	=Analytical
Appl.	=Applied
Am./Amer.	=American
ASAE	=American Society of Agricultural Engineers
ASCE	=American Society of Civil Engineers
Aust.	=Australian
Biol.	=Biological
Bull.	=Bulletin
Can.	=Canadian
Comput.	=Computational
Cong.	=Congress
Dept.	=Department
Dev.	=Development
Div.	=Division
Drain.	=Drainage
Ed.	=Edition
Eng./Engg.	=Engineering
Eur.	=European
Exp.	=Experimental
FAO	=Food and Agriculture Organization
Fig.	=Figure
Geomech.	=Geomechanics
Geophys.	=Geophysical
Hydrogeol.	=Hydrogeology
Hydrol.	=Hydrology
ICID	=International Commission on Irrigation and Drainage

IDNP	=Indo-Dutch Network Project
IIT	=Indian Institute of Technology
IPTRID	=International Programme for Technology and Research in Irrigation and Drainage
Irrig.	=Irrigation
Inst.	=Institute
Int.	=International
J./Jour.	=Journal
Ltd.	=Limited
m	=Meter
Mang./Manage.	=Management
Math.	=Mathematical
Mech.	=Mechanical
No.	=Number
Numer.	=Numerical
Phys.	=Physics
Proc.	=Proceedings
Pvt.	=Private
Res.	=Research
Resour.	=Resources
s	=Second
Sci.	=Science
Soc.	=Society
Surf.	=Surface
Syst.	=Systems
Tech.	=Technology
Trans.	=Transactions
US	=United States

ABSTRACT

In this work, an effort is being made to obtain an analytical solution to the problem of groundwater flow into a network of equally spaced ditch drains in a stratified soil underlain by an impervious barrier. The problem is being solved for a few of its variants resulting from different locations of water level in the ditches. All these solutions can tackle both a constant as well as a variable ponding distribution at the surface of the soil. Also, in the first instance, solutions to the problem and its variants are obtained by assuming the flow as two-dimensional in a defined drainage space; this supposition next is been relaxed and the problem with its variants are then solved by considering all the three components of flow in the defined flow space. The separation of variables method along with necessary Fourier expansions are being used to solve all the problems considered for study – the separation of variable method for obtaining solutions to the governing groundwater flow equation corresponding to a problem and the Fourier runs for satisfying the boundary and initial conditions pertaining to the problem. The accuracy of all the solutions is being checked by comparing with the works of others for specific situations; numerical checks on them have also been carried out. All the proposed steady state solutions are new, exact and are valid for all possible variations of parameters associated with them; however, the corresponding transient solutions are approximate in nature and are strictly valid only when the directional conductivities and specific storage of a multi-layered soil satisfy certain pre-defined relations among each other. As a large number of transient ponded drainage situations can be studied even with these imposed restrictions, the proposed transient solutions are also expected to find wide use in studying time dependent behavior of these situations as well. From the study it becomes clear that ponded drainage in a stratified soil is highly influenced by the directional conductivities and specific storage of the constituent layers and that neglecting the stratifications of a layered soil column (i.e., assuming a stratified soil as a single-layered one) may lead to a serious error in reading the hydraulics of flow associated with such a system. It has also become amply clear from the study that ponded drainage of a stratified soil is pretty sensitive to the water head of the recipient drains and also the nature of the ponding distribution at the surface of the soil. A uniform depth of ponding at the surface of the soil results in mostly unequal movement of water in a ponded drainage space

with regions close to the ditch accounting for most of the flow to the drains and the regions away from the ditches accounting for a very low proportion of it into the drains. Thus, reclaiming a salt-affected soil with a uniform depth ponded drainage system would mostly lead to non-uniform cleaning of the soil profile – regions adjacent to the drains will be unnecessarily more cleaned and regions away from the drains less cleaned. However, by providing a progressively increasing ponding head towards the half-way distance between the drains, the uniformity of water movement in a ponded drainage system can be greatly improved. Another important point that has come of the study is that the presence of a very lowly conductive soil layer (like, say, the presence of plow-sole layer in a paddy field) over that of relatively more pervious layer may help considerably in improving the uniformity of water movement in a ponded drainage system, even when the soil is being subjected to a constant depth of ponding at the surface of the soil. As soils in nature are mostly stratified and heterogeneous, it is hoped that the general ditch drainage theories provided here for stratified soils prove to be important tools towards understanding of subsurface water movement into open drains under water logged conditions.

Keywords: Multi-layered soil; Hydraulic conductivity; Specific storage; Analytical solution; Ponded ditch drainage; Constant and variable ponding distribution; Steady and transient seepage

CHAPTER 1

INTRODUCTION, LITERATURE REVIEW AND OBJECTIVES

1.1 Introduction

The population of the world is expected to cross the 9 billion mark by 2050 (Ritzema 2009) and to meet this challenge, the global agriculture output must grow at about 60 percent of the production level of 2005-2007 (FAO 2012, 2013). About 3 billion people of the world – roughly around 50 percent of the world’s population – live in rural areas and about 2.5 billion out of them rely on agriculture for their daily livelihoods (FAO 2013). Also, nearly 30 percent of earth’s land is being currently used for crop production and pastures and about 70 percent of all abstracted water is being directed towards irrigation for food production (FAO 2012, 2013; United Nations, Water for Food 2013). The global area under agriculture, however, is expected to go down in future as a result of mostly wide spread urbanization of existing agricultural lands (Ausubel et al 2013); also, bringing in more freshwater into agriculture in future will be a challenge as the water demands by other competing sectors like municipality, industry and environment are also likely to increase in future as well. Thus, efforts must be directed to augment agricultural productivity per unit area of cultivable lands to meet the increasing food demand of an expanding global population (Rengasamy 2006) and to meet this objective, improved irrigation and drainage practices on existing agricultural lands is expected to play a leading part (Schultz and de Wrachien 2002; Ritzema 2016). Irrigation, as a practice for augmenting agricultural productivity, has been there for a long time now and its role in future for enhancing agricultural outputs is only expected to increase (Khan et al. 2004). Currently about 18 percent of arable and permanent cropped area throughout the world is being irrigated (ICID 2015), which accounts for as much as 40 percent of the gross agricultural outputs of the world (FAO 2003; Faures et al. 2007; Ritzema 2016). Also, the area under irrigation is expected to increase by about 6.6 percent over a period spanning from 2005/2007 to 2050, most of these being projected to be in the developing countries (Alexandratos and Bruinsma 2012). Irrigated agriculture has been found especially useful in the arid and semi-arid regions of the world, where its introduction has greatly helped in enhancing the agricultural productivity of cultivated fields (Smedema et al. 2000; Wichelns et al. 2002). Irrigation is also an essential ingredient for modern agriculture involving high yielding crop varieties as cropping of these varieties often require a very favorable soil-

water balance in the fields (Singh and Singh 1995; Carruthers et al. 1997). In India, about 34 percent of the arable land comprising of about 57 million hectares, is currently under irrigation (ICID 2003; Ritzema et al. 2008) and this coverage is expected to increase considerably in future for sustaining and supplementing the agricultural productivity of the country.

Irrigated agriculture, in spite of its many benefits in relation to crop production, has its drawbacks as well – the most notable being the causation of waterlogging and salinity in irrigated lands (Ghassemi et al. 1995; Martinez-Beltran 2002; Rhoades 1997; Wichelns et al. 2002; Martinez-Beltran and Manzur 2005; Ritzema et al. 2008; Boari et al. 2012; Ritzema 2016). It is estimated that about 10-16 percent of irrigated lands of the world is being affected by salinity and related waterlogging problems and the loss of productive land to salinity and waterlogging amounts to about 0.5 million hectares per year (Smedema 2000; Ritzema 2016). In India also, the story is no different – vast tracts of land in the country have also been reported with the problems of irrigation-induced salinity and waterlogging in different parts of the country (Wolde-Kirkos and Chawla 1994; Manjunatha et al. 2004; Ritzema et al. 2008; Ritzema 2016). It is estimated that about 8.4 million hectares of irrigated lands in the country have been afflicted with the problems of salinity and alkalinity and that 5.5 million hectares out of which have also been estimated to be waterlogged (IDNP 2002; Ritzema et al. 2008). One of the most potent ways of arresting irrigation-induced salinity and waterlogging is through introduction of subsurface drains in the irrigated fields – several studies on the subject have proven the veracity of this (Datta et al. 2000; Smedema et al. 2000; Datta and Jong 2002; Scheumann and Freisem 2002; Ayars et al. 2003; Manjunatha et al. 2004; Sharma and Gupta 2006, Ritzema et al. 2008; Ritzema 2016; Tiwari and Goel 2017 – to name a few). Drainage has its importance in diverse geo-climatic locations of the world ranging from very wet to very dry; in arid and semi-arid regions, drainage helps in preventing waterlogging and salinity buildup, in humid and semi-humid regions, drains are essential for removing excess water and in temperate regions, they can be installed for reclaiming waterlogged and salt-affected soils; thus drainage helps to create a favorable soil-air-water interface in agricultural lands in different hydro-climatic regions of the world (Pearce and Denneke 2001; Ritzema et al. 2007; Ritzema 2016).

One of the most commonly adopted methods of reclaiming a salt-affected soil is to force good quality water through it so as to remove the salt present in the soil profile to a desirable level and then collecting and draining the salt-laden water via a network of subsurface drains installed for

the purpose (Dielman 1973, Martinez-Beltran 1978, Rao and Leeds-Harrison 1991, Youngs and Leeds-Harrison 2000; Mirjat and Rose 2009; Barua and Alam 2013; Barua and Sarmah 2016). The head necessary to force water through the salt-laden soil is generally provided by introducing a ponding head at the top of the soil. Subsurface drainage is now also becoming increasingly important for maintaining proper aeration in the root zones of paddy fields (Ogino and Murashima 1993; Tabuchi 2004, Darzi-Naftchally et al. 2013; Darzi-Naftchally and Shahnazari 2014). Most importantly, controlled drainage of paddy fields has been found to greatly reduce the emissions of methane and nitrous oxide from these fields – two atmospheric trace gases which contribute greatly to the cause of global warming and ozone depletion (Nishimura et al. 2004; Shiratori et al. 2007; Qui 2009; Xiaohong et al. 2011, Zhang et al. 2011; Darzi-Naftchally and Shahnazari 2014; Yang et al. 2014 – to name a few). Subsurface drains may be buried pipelines or open ditches but open drains are mainly preferred in locations where the conductivity is low and the topography comparatively flat (Abrol et al. 1988). Ditch drains are also most suitable for drainage of lowly conductive peat soils (Stewart and Lance 1983). Ditch drains may also play a significant role in regulating the hydrologic, chemical and biological processes of a watershed and the biodiversity of a catchment may be strongly influenced due to their installation in the catchment (Youngs 1994; Bradbury and Kirby 2006; Needelman et al. 2007; Marja and Herzon 2012; Marja 2013). Also, the component of the base flow entering into a ditch drainage network in a watershed is an important hydrological entity as this information may lead to a better understanding of hydraulics of flow and nutrient dynamics of the watershed (Goswami and Kalita 2009, 2010). Thus, open drains are an important hydrological entity in field and hence due emphasis need be given to study in detail the hydraulics of flow associated with them. Most of the hydrological studies related to open drains are done by assuming the flow to be either one or two-dimensional in nature and by assuming the soil to be homogeneous and isotropic. But soils in nature are mostly stratified rather than uniform and soil stratification may greatly affect the distribution and movement of water and contaminant through them (Stephens and Heermann 1988; Sakellario-Makrantonaki 1997; Singh et al. 1999; Zhao et al. 2010; Huang et al. 2011). The conductivity contrast of the soil layers may be noticed in its extreme form in a paddy field where, very often, an extremely low conductivity top plow-sole layer can be seen to exist over relatively more conductive layer(s). Also, numerous modeling studies have categorically demonstrated that the conductivity and thickness of this

plow-sole layer plays a pivotal role in deciding overall subsurface water dynamics of a paddy field (Liu et al. 2001; Chen et al. 2002; Haung et al. 2003; Liu et al. 2005 – to suggest a few). Further, within a layer also, the conductivity of a soil may vary substantially with the direction of flow (Maasland 1957; Bazaraa et al. 1986; Braun and Kruijne 1994) and layered (Kanwar et al. 1989) or compacted soils (Dörner and Horn 2006; Petersen et al. 2008) may exhibit a higher conductivity in the horizontal direction than that in the vertical direction. However, the situation may be a reversed one for a well-structured soil, where the soil may instead exhibit a higher conductivity in the vertical direction in comparison to the horizontal direction (Bouma 1982; Barthke and Cassel 1991). It is worth noting that information on saturated hydraulic conductivity and anisotropy of the comprising layers of a stratified soil column are fundamental for successful modeling of two- and three-dimensional transport of water and contaminant in a heterogeneous porous formation (Petersen et al. 2008).

Mathematical modeling of a hydro-geological process is mostly done in two ways – analytical and numerical. Hybrid analytical and numerical models, combining the powers of both analytical and numerical models, are also becoming increasingly common now-a-days for solving complex problems involving flow and transport in porous formations and natural systems (McDermott et al. 2009; Craig and Read 2010; Wang et al. 2012; Wang et al. 2013; Morel-Seytoux 2015; Meunier et al. 2017). A mathematical model is often a much simplified representation of a real hydro-system where a set of equations developed based on some fundamental facts of nature are being solved subject to some well defined initial and boundary conditions (Wang and Anderson 1982). A numerical solution of a flow problem is an approximate algorithmic solution of a differential equation pertaining to the system at a finer level; this is being done by splitting the system into fine cells (grids) and then solving the concerned equation in each cell while taking care to see that the initial and boundary conditions pertaining to the flow problem are being satisfied at the same time (Walton 1979, 1989; Craig and Read 2010; Lewis 2013; Kuwayama and Brozović 2012). The chief advantages of a numerical procedure are its ability to handle complex irregular geometries of a flow system as well as the nonlinearities of the governing equation(s) associated with it; further, a numerical model has the added advantage of tackling highly complex initial and boundary conditions of a flow system as well (Craig and Read 2010, Lewis 2013). However, numerical models are often computationally very demanding and the really sophisticated ones often require a huge data set for their operation, a fact which may prove

to be an impediment in their applications in many hydro-geo systems. Also, being approximate in nature, the convergence and stability of a numerical method are important issues and these must be checked and addressed before such a model is being actually applied in the field. Analytical models, on the other hand, are relatively simple to operate as compared to numerical models and solve the differential equation(s) pertaining to a flow problem in an exact way. Thus, they do not have convergence and/or stability issues as do the numerical models. Even though they are less general than the numerical models in their ability to include complex flow geometries and nonlinearities of associated differential equation(s) of a flow system, nevertheless they are more amenable for theoretical purposes than numerical models for many modeling situations; also, most often, they are more flexible and offer a better physical insight of a hydro-system as compared to numerical models (Haitjema 2001, 2006; Hunt et al. 2003; Lewis 2013; Kuwayama and Brozović 2012). Also, with the advent of newer and more powerful analytical methods in the last couple of decades like the CK method (Clarkson and Kruskal 1989), homotopy analysis method (Liao 1992), Lie symmetry method (Olver 1993; Cantwel 2002), Adomian decomposition method (Adomian 1994), homotopy perturbation method (He 1999) – to name a few – the analytical methods now-a-days are also been exceedingly used to solve pretty sophisticated problems involving hydro-geo-systems as well. Further, because of the inherent mathematical exactness of analytical models, they are now also been increasingly called upon to verify complex numerical codes pertaining to various groundwater flow systems (Elfeki et al. 1997; Kacimov 1997; Haitjema 2006; Praveena et al. 2010).

In view of the importance of analytical models in understanding groundwater systems and also noting the importance of open drains for control of salinity and waterlogging in irrigated soils and further from observing what has been said before about the nature of a soil column in field being more prone to be stratified rather than being uniform, an effort is being made in this study to obtain a comprehensive analytical solution to the ponded ditch drainage problem for a heterogeneous soil by assuming first the flow to be two-dimensional only and then relaxing this assumption and solving the problem by considering all the three components of flow in a defined drainage space. To become abreast of the various works already done on ponded ditch drainage and on the use of subsurface drains for controlling salinity and waterlogging, a brief review about these will now be presented.

1.2 Literature Review

It has already been emphasized that subsurface drainage facilitated by ditch drains is proving to be extremely beneficial in reclaiming waterlogged and saline soils in areas where the conductivity of the soils is low and the topography relatively flat (Abrol et al. 1988). Leaching of a saline soil by purging it with water having minimal salinity and then proceeding to remove the washed salts with the help of a suitable ditch drainage system, has been a standard practice of cleaning salt-affected soils for quite some time now (Dielman 1973; Martinez Beltran 1978; Rao and Leeds-Harrison 2000). It is pertinent to note that over the years several ponded subsurface drainage theories have been proposed by many investigators. However, very few of them have accounted for layered soils – a situation which is most likely to be encountered in actual field locations. Kirkham (1945) developed an infinite series steady state solution for predicting flow into an array of parallel ditch drains resting on a gravel stratum and receiving water from a horizontal field of infinite extent subjected to zero or a uniform ponding depth at the surface of the soil. Yet again, Kirkham (1950) proposed a steady state solution to the ponded drainage problem when the ponding depth is negligible while the ditch drains vertically reach to an impervious stratum underlying the soil; he solved the problem by taking suitable limits to the solution of the fully penetrating auger hole problem for a homogeneous and isotropic soil (Kirkham and Van Bavel 1948). A conformal mapping solution to the steady state partially penetrating ditch drainage problem was presented by Fukuka (1957); in this analysis, the ponded depth at the surface of the soil is considered as zero and the ditches are assumed to run empty all the time. Kirkham (1960) proposed a Fourier series based solution to the ponded ditch drainage problem when the soil column is being drained by an array of equally spaced ditch drains underlain by a gravel stratum. The solution assumes the flow to be steady, the soil to be homogeneous and isotropic, the drains to be dug all the way upto the gravel stratum and soil to be subjected to a uniform ponding distribution at the surface of the soil. The steady state ditch drainage problem with unequal water levels in the adjacent ditches was analyzed by Kirkham (1965) by taking resort again to Fourier series when the drains are fully dug in a homogenous and isotropic soil overlying an impervious layer. This solution accounts for both zero as well as non-zero ponding depths at the surface of the soil. For equally spaced ditch drains running full, Warrick and Kirkham (1969) applied conformal mapping to predict seepage of drainage water from a uniformly ponded field to the filled drainage ditches. They obtained analytical solutions

for different cases of the problem, namely, when an impervious layer is present at a finite depth below the ditches and the ditches are a finite distance apart; when no impervious layer exists and the ditches are at a finite distance from each other; when an impervious layer exists but the separation between the ditches is infinite and when no impervious layer exists and the ditches happen to be at an infinite distance apart. Miyamoto and Warrick (1974) studied the effect of a partial impervious cover of a ponded field in the uniformity of movement of seepage water into a drain tube being buried at a finite distance from the top of the field. They studied the problem using a complex velocity expression derived by Warrick and Miyamoto (1974) in relation to a similar mathematical problem involving steady gas flow from a buried porous pipeline in a homogeneous soil column. Their study shows that by providing and suitably adjusting the length of an impervious cover being placed at the surface of a ponded drainage system, the uniformity of leaching of the drainage system can be considerably improved. Another conformal mapping based analytical solution to the ponded ditch drainage problem was proposed by Youngs (1994) for the situation where a ditch drain is been assumed to partially penetrate a homogeneous and isotropic soil underlain by an impervious barrier. Also, in this solution, the ditch is assumed to be of negligible width and the head to drive water to the drain is from an infinitely large ponded field with zero depth of ponding at the surface of the soil. Further, solutions are provided to the problem both for cases when the impervious barrier touches the bottom of the ditch and when the impervious barrier lies at a very large distance (theoretically infinite) from the base of the ditch. Barua and Tiwari (1995) provided a detailed Fourier series solution to the problem of steady seepage of water into a network of equally spaced parallel ditch drains receiving water from a ponded field of infinite horizontal extent, the drains being partially penetrating a homogeneous and anisotropic aquifer underlain by an impervious barrier. They obtained their solution by performing a domain discretization of one half of the flow space in between the ditches – the portion of the flow space below the bottom of the drain and up to the impervious barrier is considered as sub-domain one and the remaining region as sub-domain two – and then solving the governing equations for both these sub-domains while ensuring that all the necessary boundary and interfacial conditions get satisfied at the same time. This is a comprehensive solution as it can handle finite width and depth of drains, a uniform ponding field at the surface of the soil and also the horizontal and vertical hydraulic conductivity variations of a drained soil column. However, this solution is unable to account for

variations of horizontal and vertical conductivities with depth of a soil profile and is not valid when the ponding field at the surface is a variable one. Bereslavskii (2006) made use of the Riemann-Schwartz principle of symmetry to propose an analytical solution to the fully penetrating ditch drainage problem being defined for a homogeneous and isotropic soil. Since his solution only requires the knowledge of well-known special or elementary functions, he claims that it is computationally superior to the previous solutions based on conformal mapping. Solutions relying on conformal transformations were also proposed by Chahar and Vadodaria (2008a, 2008b, 2012) for different variants of the ditch drainage problem. These solutions, however, are limited to only single-layered soils and can account for only a uniform depth (this depth can be zero also) of ponding at the surface of the soil. Römken(2009), in his bid to study the effect of subsurface drainage on gully erosion, revisited the partially penetrating ditch drainage problem pertinent to a homogeneous isotropic soil overlying an impermeable layer using conformal mapping. In his analytical treatment, the component of flow taking place through the seepage faces of the ditches is being neglected and the component of flow through the bottom of the ditches is being only considered; further the ditch bottoms are assumed as circular in his analysis. Barua and Alam (2013) solved the transient ditch drainage flow problem for a single-layered anisotropic soil overlying an impervious stratum using the separation of variables method and Fourier series. Their solution can account for variable ponding atop the soil and unequal water level heights in the ditch drains. Afruzi et al. (2014) revisited the partially penetrating ponded ditch drainage problem for a homogenous and isotropic soil and provided analytical expressions for different variants of the problem by making use of the Schwarz-Christoffel transformation in the conformal mappings of the relevant physical and hodograph planes. Sarmah and Barua (2015) provided a steady-state solution to the partially ponded ditch drainage problem applicable to a soil with three anisotropic vertical layers and underlain by an impervious layer. This solution is being developed by taking recourse again to a domain-discretization procedure similar to the one as adopted by Barua and Tiwari (1995) while obtaining solution to a similar problem based on a single-layered soil. The main strength of this solution is its flexibility to include any depth of water level in the drains irrespective of this depth falling in the first or second or third layer. However, while this solution considers the layered nature of a soil profile, it cannot account for a variable depth of ponding at the surface of the soil nor it can account for different depth of the open drains. Barua and Sarmah (2016)

provided an analytical solution to the partially penetrating transient ponded drainage problem for a homogeneous and anisotropic soil layer underlain by an impervious substratum. In this solution, an analytical procedure is been suggested for solving the transient differential equation relevant to a ponded ditch drainage system and since this methodology is of a general nature, it can, with a little effort, may also be used to solve problems of similar nature as well. Sarmah and Tiwari (2018) revisited the transient ditch drainage problem and provided an analytical solution having considered the influence of a source/sink on the flow domain and flow characteristics.

Several studies on the uniformly ponded drainage system show that (Kirkham 1949, 1950, 1960, 1965; Luthin et al. 1965; Ortiz and Luthin 1970; Mulqueen and Kirkham 1972; Dielman 1973; Martinez Beltran 1978; Rao and Leeds-Harrison 1991; Youngs 1994; Barua and Tiwari 1995; Youngs and Leeds-Harrison 2000; Mirjat and Rose 2009; Römken 2009; Chahar and Vadodaria 2008a, 2008b, 2012; Siyal et al. 2010; Barua and Alam 2013, Sarmah and Barua 2015 – to name only a few) categorically show that the flow in such a system is mostly concentrated in areas close to the drains and only a minimal flow to the drains is being contributed from locations much away from the drains. Thus, leaching a salty soil profile with such a system would not be very efficient as regions close to the drain would then be washed quickly as compared to regions farther away from the drains. One way of overcoming this problem can be by subjecting the soil to a sequential ponding whereby ponding for the whole space is first done and then followed by progressively decreasing areas of ponding, all these areas being measured with respect to an axis located half-way between two adjacent drains (Rao and Leeds-Harrison 1991). Another way of doing this can be to follow exactly a reverse order of ponding the drainage space, namely, the ponding can be introduced at a fraction of an area in the midway region between the drains and then this area can be progressively increased till the entire region between the drains is being fully covered (Youngs and Leeds-Harrison, 2000; Mirjat et al. 2008; Mirjat and Rose 2009; Siyal et al. 2010). Still, there can be an alternate way of tackling the problem; a variable ponding distribution with progressively increasing depth of ponding towards half-way distance between the drains may be introduced in a contaminated soil with the help of a series of parallel bunds running parallel to the drains. The main advantage of this procedure over that of the other two is that the entire soil is being subjected to a ponding distribution in one single step and the soil is cleaned in one single stage itself (Barua and Alam 2013; Sarmah and Barua 2015; Barua and Sarmah 2016). All these solutions of the ponded ditch drainage problem

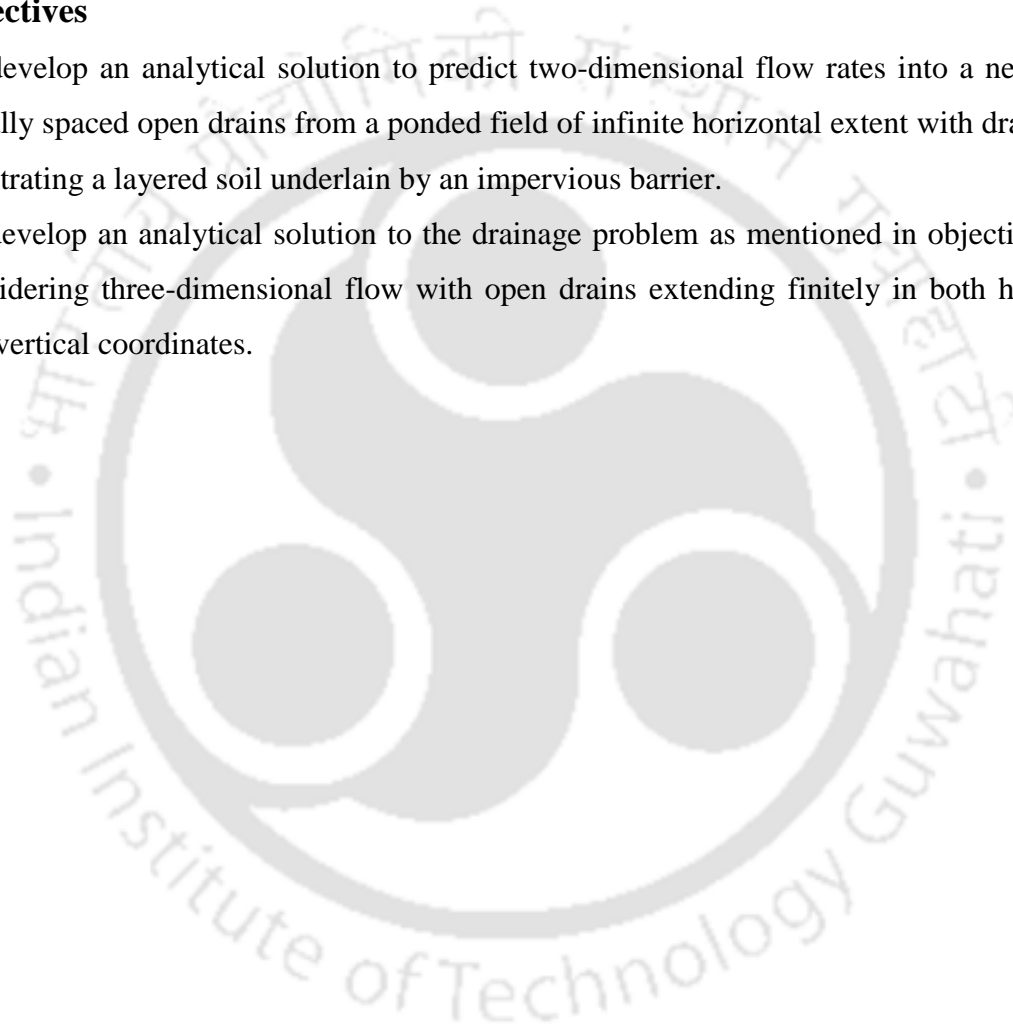
as mentioned above are based on the assumption of two-dimensional flow which intrinsically means that the ponded field at the surface of the soil is being horizontally extended to a large extent (theoretically infinite). Whereas this assumption loosely holds for large fields, it may fall shy in many real field situations where the size of a field is fixed and relatively small. Subsurface flow to a trench/stream is mostly three-dimensional in nature (Brainard and Gelhar 1991; Murdoch 1994; Meigs and Bahr 1995) and serious error in the simulation results of a model may be incurred if this aspect is not being considered in the model being developed for studying such a system. This has also been the finding of Sarmah and Barua's (2017) analytical works on the subject, where it is found that for a finite sized ponded field, subsurface flow to the ditches will mostly be three-dimensional in nature, particularly in areas close to the ditches. However, in a large field, the flow in a vertical section located further away from the Northern and Southern boundaries of a ponded ditch drainage system may show approximately two-dimensional behavior and hence can be closely modeled using such an assumption (Sarmah and Barua 2017). Nonetheless, even for this situation also, the pathlines near the drains may exhibit strong three-dimensional behavior and hence it is always better to include all the three components of flow and the relevant directional conductivity distributions while simulating such a system. Sarmah and Barua's (2017) solution for the ponded ditch drainage problem for a finite sized soil with an impervious base can handle three-dimensional flow but this solution is for a single layered soil only and hence will not be valid for a stratified soil.

From the above reviews, it is apparent that there currently does not exist any analytical solution to the ponded multi-layered ditch drainage problem by considering all the three components of flow in a finite sized drainage space even for the steady state situation. With the two-dimensional flow assumption, a steady state solution to the problem exists (Sarmah and Barua 2015) but this solution is strictly for situations where the drainage ditches are partially penetrating a ponded drainage space underlain by an impervious base and not when the drains go all the way upto the barrier. Also, most importantly, this solution is for situations where the level of water in the ditches are all same and not for the general situation of unequal water level height in the ditches. Further, a solution which can account for three-dimensional flow exists for the ponded ditch drainage problem but this solution (Sarmah and Barua 2017), as mentioned before, is not for layered soils and hence will not be able to accommodate stratification of a soil column in such a system. Further, even by making use of simplified assumptions, a transient analytical solution to

the ponded ditch drainage problem appears to be lacking for a stratified soil. Therefore, it can be concluded that further research is needed to develop new analytical solutions using various boundary and initial conditions to determine vertical and lateral seepage rates from ponded fields to open drains. In view of the same and keeping in mind that stratifications in a soil is mostly a rule rather than an exception, an effort is being made in the present study to realize the following objectives.

1.3 Objectives

- (i) To develop an analytical solution to predict two-dimensional flow rates into a network of equally spaced open drains from a ponded field of infinite horizontal extent with drains fully penetrating a layered soil underlain by an impervious barrier.
- (ii) To develop an analytical solution to the drainage problem as mentioned in objective (i) by considering three-dimensional flow with open drains extending finitely in both horizontal and vertical coordinates.



CHAPTER 2

SEEPAGE INTO DITCH DRAINS IN A TWO-LAYERED SOIL UNDERLAIN BY AN IMPERVIOUS SUBSTRATUM

In this chapter, an analytical solution is being suggested for predicting seepage into an array of ditch drains dug in a two-layered soil underlain by an impervious substratum and receiving water from a ponded field of large extent. The proposed solution can account for finite width and spacing of the drains, finite depth of the impervious substratum, anisotropy of individual layers, disparate specific yields of the layers and both a constant and variably ponded surface atop the soil layers. The proposed model for the steady state is complete but for the transient case, it is based on the assumption that the vertical hydraulic conductivity of the layers are the same in each layer and that this conductivity is of infinitesimal magnitude (but not zero) and hence much smaller than the horizontal conductivities of the layers. The accuracy of the proposed solution for the steady state case and when the flow domain is of a single layer, is been checked by reducing the model to a single-layered one by considering the hydraulic conductivities of the soil layers to be the same and then comparing the discharges as predicted by this reduced model with corresponding values obtained by other investigators through analytical or experimental means for a few drainage configurations. For the transient case also, analytical verifications of the proposed solution for a few drainage scenarios pertaining to the single layer soil have been carried out by first transforming the model to a single layer anisotropic soil and then comparing the heads obtained from it with the corresponding values obtained from the analytical works of Barua and Alam (2013). Further, numerical verifications of the developed solutions have also been done for a few two-layered ditch drainage configurations by drawing appropriate models utilizing the Processing MODFLOW (Chiang and Kinzelbach 2001) environment and then comparing the analytical outputs with their numerical equivalents.

2.1 Solutions for the Two-Dimensional Continuity Equation of Groundwater flow for a Homogeneous and Anisotropic Soil

We now obtain a few general solutions of the continuity equation which describes flow of groundwater in a saturated aquifer. We have gone along with the following assumptions

- (a) The aquifer, within each layer, has been assumed to be saturated, homogeneous and anisotropic.
- (b) The flow of water in the aquifer has been assumed to be irrotational and incompressible.
- (c) The soil matrix has been assumed to be compressible while the porosity of each soil layer has been considered as constant.
- (d) The direction of the principal hydraulic conductivities have been assumed to coincide with the horizontal and vertical directions of the flow domain.
- (e) Darcy's Law has been assumed to be valid for the analysis and
- (f) We have assumed the flow to be two-dimensional in nature.

The two-dimensional flow equation for aforementioned flow domain properties is expressed as (Bear 1972)

$$K_x \frac{\partial^2 \phi}{\partial x^2} + K_y \frac{\partial^2 \phi}{\partial y^2} = S_s \frac{\partial \phi}{\partial t}, \quad (2.1)$$

where ϕ is the hydraulic head, K_x and K_y are the hydraulic conductivities of the medium in the x - and y -directions, respectively, S_s is the specific storage of the aquifer and t is the time variable. We divide both sides of Eq. (2.1) by S_s to obtain the following simplified form

$$\left(\frac{K_x}{S_s} \right) \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{K_y}{S_s} \right) \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial t}. \quad (2.2)$$

For the steady state, the equation governing flow for the studied system can be expressed as

$$\left(\frac{K_x}{S_s} \right) \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{K_y}{S_s} \right) \frac{\partial^2 \phi}{\partial y^2} = 0. \quad (2.3)$$

Henceforth, we consider $u_p(x, y, t)$ [Eq. (2.5)] to be a solution of Eq. (2.2) and $u_c(x, y)$ [Eq. (2.13)] to be a solution of Eq. (2.3). We can easily verify that $\phi = u_p + u_c$ is also a solution of Eq. (2.2) since

$$\begin{aligned} \left(\frac{K_x}{S_s} \right) \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{K_y}{S_s} \right) \frac{\partial^2 \phi}{\partial y^2} &= \left(\frac{K_x}{S_s} \right) \frac{\partial^2 u_p}{\partial x^2} + \left(\frac{K_y}{S_s} \right) \frac{\partial^2 u_p}{\partial y^2} + \left(\frac{K_x}{S_s} \right) \frac{\partial^2 u_c}{\partial x^2} + \left(\frac{K_y}{S_s} \right) \frac{\partial^2 u_c}{\partial y^2} = \\ \left(\frac{K_x}{S_s} \right) \frac{\partial^2 u_p}{\partial x^2} + \left(\frac{K_y}{S_s} \right) \frac{\partial^2 u_p}{\partial y^2} + 0 &= \frac{\partial \phi}{\partial t}. \end{aligned} \quad (2.4)$$

We will now apply the separation of variables method (Kirkham and Powers 1972) to obtain a solution for Eq. (2.2). For that, we consider

$$u_p(x, y, t) = X(x)Y(y)T(t) \quad (2.5)$$

to be a solution of Eq. (2.2) where $X(x)$, $Y(y)$ and $T(t)$ are functions of only x , y and t , respectively. We substitute the value of u_p in Eq. (2.2). Then, we proceed to separate the different variables in the resulting equation to obtain

$$\left(\frac{K_x}{S_s} \right) \frac{X''(x)}{X(x)} + \left(\frac{K_y}{S_s} \right) \frac{Y''(y)}{Y(y)} = \frac{T'(t)}{T(t)} \quad (2.6)$$

Equating $\left(\frac{K_x}{S_s} \right) \frac{X''(x)}{X(x)} = -\alpha^2 \left(\frac{K_x}{S_s} \right)$ and $\left(\frac{K_y}{S_s} \right) \frac{Y''(y)}{Y(y)} = -\beta^2 \left(\frac{K_y}{S_s} \right)$, where α and β are

constants, and then solving the abovementioned differential equations, we obtain

$$X(x) = c_1 \sin(\alpha x) \quad (2.7)$$

and

$$Y(y) = c_2 \sin(\beta y), \quad (2.8)$$

where c_1 and c_2 are arbitrary constants. Equating each term on the left hand side of Eq. (2.6)

to the constant terms $-\alpha^2 \left(\frac{K_x}{S_s} \right)$ and $-\beta^2 \left(\frac{K_y}{S_s} \right)$ results in the following equation

$$\frac{T'(t)}{T(t)} = - \left\{ \alpha^2 \left(\frac{K_x}{S_s} \right) + \beta^2 \left(\frac{K_y}{S_s} \right) \right\} = -\lambda^2. \quad (2.9)$$

The solution for Eq. (2.9) can be expressed as

$$T(t) = c_3 \exp \left\{ - \left[\alpha^2 \left(\frac{K_x}{S_s} \right) + \beta^2 \left(\frac{K_y}{S_s} \right) \right] t \right\}, \quad (2.10)$$

where c_3 is any arbitrary constant. Substituting the solutions enumerated in Eqs. (2.7), (2.8) and (2.10) in Eq. (2.5), we get the following expression for $u_p(x, y, t)$

$$u_p(x, y, t) = E \sin(\alpha x) \sin(\beta y) \exp \left\{ - \left[\alpha^2 \left(\frac{K_x}{S_s} \right) + \beta^2 \left(\frac{K_y}{S_s} \right) \right] t \right\}, \quad (2.11)$$

where E is any arbitrary constant. The solution proposed in Eq. (2.11) is the general form of a fundamental solution of the governing partial differential equation. By varying the values of the arbitrary constants α and β , countless such solutions for the governing equation are obtained. It can be noted that the sum of all these fundamental solutions thus obtained also satisfies Eq. (2.2). Hence, a solution for the governing equation can be expressed as

$$u_p(x, y, t) = \sum_{i=1}^I \sum_{j=1}^J E_{ij} \sin(\alpha_i x) \sin(\beta_j y) \exp(-\lambda_{ij}^2 t), \quad (2.12)$$

where $\lambda_{ij}^2 = \left[\alpha_i^2 \left(\frac{K_x}{S_s} \right) + \beta_j^2 \left(\frac{K_y}{S_s} \right) \right]$, E_{ij} are arbitrary constants, i and j are summation

indices and I and J are positive integers with $I \rightarrow \infty$ and $J \rightarrow \infty$.

Analogous to $u_p(x, y, t)$, the solution of Eq. (2.3) can be expressed as

$$u_c(x, y) = B(x)C(y), \quad (2.13)$$

where $B(x)$ and $C(y)$ are functions of x and y only. Substituting the assumed value of $u_c(x, y)$ in Eq. (2.3) and then separating the variables out, we are left out with

$$\frac{B''(x)}{B(x)} + \left(\frac{K_y}{K_x} \right) \frac{C''(y)}{C(y)} = 0. \quad (2.14)$$

The second term on the left hand side of the Eq. (2.14) is equated to the product of the following constants

$$\left(\frac{K_y}{K_x} \right) \frac{C''(y)}{C(y)} = -\eta^2 \left(\frac{K_y}{K_x} \right), \quad (2.15)$$

where η is an arbitrary constant. Solving the ordinary differential equation, we have

$$C(y) = c_4 \sin(\eta y), \quad (2.16)$$

where c_4 is another arbitrary constant. From Eq. (2.15), we also obtain the following equation

$$\frac{B''(x)}{B(x)} = \eta^2 \left(\frac{K_y}{K_x} \right). \quad (2.17)$$

The solution for Eq. (2.17) is

$$B(x) = c_5 \sinh \left[\left(\sqrt{\eta^2 \left(\frac{K_y}{K_x} \right)} x \right) \right], \quad (2.18)$$

where c_5 is another arbitrary constant. Values for $B(x)$ and $C(y)$ lead to the following expression for $u_c(x, y)$

$$u_c(x, y) = A \sinh \left[\left(\sqrt{\eta^2 \left(\frac{K_y}{K_x} \right)} x \right) \right] \sin(\eta y), \quad (2.19)$$

where A is any arbitrary constant. On assigning a summation index to the constant η so as to sum up the several fundamental solutions of Eq. (2.14), we eventually arrive at the following solution

$$u_c(x, y) = \sum_{m=1}^M A_m \sinh \left[\left(\sqrt{\eta_m^2 \left(\frac{K_y}{K_x} \right)} x \right) \right] \sin(\eta_m y), \quad (2.20)$$

where A_m are arbitrary constants, m is an index and M is a positive integer with $M \rightarrow \infty$. Similarly, we can obtain other solutions for $u_c(x, y)$ as

$$u_c(x, y) = \sum_{n=1}^N A_n \cosh \left[\left(\sqrt{\eta_n^2 \left(\frac{K_y}{K_x} \right)} x \right) \right] \sin(\eta_n y), \quad (2.21)$$

$$u_c(x, y) = \sum_{p=1}^P A_p \sinh \left[\left(\sqrt{\eta_p^2 \left(\frac{K_x}{K_y} \right)} y \right) \right] \sin(\eta_p x) \quad (2.22)$$

and

$$u_c(x, y) = \sum_{q=1}^Q A_q \cosh \left[\left(\sqrt{\eta_q^2 \left(\frac{K_x}{K_y} \right)} y \right) \right] \sin(\eta_q x), \quad (2.23)$$

where A_n , A_p and A_q are arbitrary constants, n , p and q are indices and N , P and Q are positive integers with $N \rightarrow \infty$, $P \rightarrow \infty$ and $Q \rightarrow \infty$. Considering the solutions mentioned in Eqs. (2.12), (2.20), (2.21), (2.22) and (2.23), we can propose a few general solutions of Eq. (2.2) as

$$\begin{aligned}
u_p(x, y, t) = & \sum_{m=1}^M A_m \sinh \left[\left(\sqrt{\eta_m^2 \left(\frac{K_y}{K_x} \right)} x \right) \right] \sin(\eta_m y) + \sum_{p=1}^P A_p \sinh \left[\left(\sqrt{\eta_p^2 \left(\frac{K_x}{K_y} \right)} y \right) \right] \sin(\eta_p x) \\
& + \sum_{q=1}^Q A_q \cosh \left[\left(\sqrt{\eta_q^2 \left(\frac{K_x}{K_y} \right)} y \right) \right] \sin(\eta_p x) \\
& + \sum_{i=1}^I \sum_{j=1}^J E_{ij} \sin(\alpha_i x) \sin(\beta_j y) \exp(-\lambda_{ij}^2 t),
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
u_p(x, y, t) = & \sum_{q=1}^Q A_q \cosh \left[\left(\sqrt{\eta_q^2 \left(\frac{K_x}{K_y} \right)} y \right) \right] \sin(\eta_p x) \\
& + \sum_{i=1}^I \sum_{j=1}^J E_{ij} \sin(\alpha_i x) \sin(\beta_j y) \exp(-\lambda_{ij}^2 t)
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
u_p(x, y, t) = & \sum_{m=1}^M A_m \sinh \left[\left(\sqrt{\eta_m^2 \left(\frac{K_y}{K_x} \right)} x \right) \right] \sin(\eta_m y) + \sum_{q=1}^Q A_q \cosh \left[\left(\sqrt{\eta_q^2 \left(\frac{K_x}{K_y} \right)} y \right) \right] \sin(\eta_p x) \\
& + \sum_{i=1}^I \sum_{j=1}^J E_{ij} \sin(\alpha_i x) \sin(\beta_j y) \exp(-\lambda_{ij}^2 t).
\end{aligned} \tag{2.26}$$

We are now going to utilize the solutions as mentioned in Eqs. (2.24), (2.25) and (2.26) to solve the two-dimensional multilayer flow problem under study for a few ditch drainage scenarios.

2.2 Mathematical Formulation and Solution

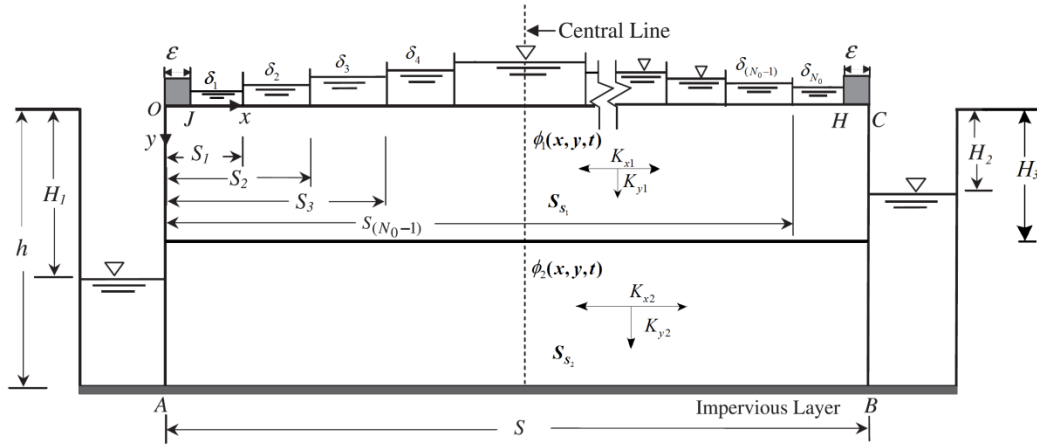


Fig. 2.1. General geometry of a fully penetrating ditch drainage system in a two-layered soil subjected to variable depths of ponding at the surface

Fig. 2.1 shows the geometry of the flow problem under study. As can be seen in the figure, the flow domain is characterized by two vertical, anisotropic soil layers with different specific storage. The bottom soil layer is underlain by an impervious stratum. A series of equally spaced parallel ditch drains (theoretically, each of infinite length) are dug in the

aquifer so that they reach the impervious stratum. The thickness of the top soil layer is taken as H_3 , while the total thickness of the flow domain is denoted by h . The spacing between the ditches is taken as S . As illustrated in Fig. 2.1, a coordinate system is assigned to the flow domain with the x -axis taken positive towards right of the origin and the y -axis is taken positive in the downward direction from the chosen origin. The level of water in the ditches is measured from the origin O . The water levels in the left and right ditches are denoted by H_1 and H_2 , respectively. The depth of water in the ditches results from an instantaneous lowering of water levels at time $t = 0$. A constant or variable distribution of ponding depths is achieved on the surface of the field, aided by a series of bunds of negligible width which run parallel to the ditch drains. The whole flow domain is considered to be fully saturated before the inception of flow. It can be seen that the ponding surface comprises of N_0 distinct segments with δ_i ($1 \leq i \leq N_0$) representing the ponding depth of the i^{th} segment. The ponding strips on the soil surface are constructed with the help of $N_0 - 1$ inner bunds of negligible width. The distance of the i^{th} inner bund from the origin has been denoted by S_i ($1 \leq i \leq N_0 - 1$).

The governing equations for the two soil layers of the flow domain based on the principle of continuity can be expressed as (Bear 1972)

$$K_{x_1} \frac{\partial^2 \phi_{1(j)}}{\partial x^2} + K_{y_1} \frac{\partial^2 \phi_{1(j)}}{\partial y^2} = S_{s_1} \frac{\partial \phi_{1(j)}}{\partial t}, \quad (2.27)$$

$$K_{x_2} \frac{\partial^2 \phi_{2(j)}}{\partial x^2} + K_{y_2} \frac{\partial^2 \phi_{2(j)}}{\partial y^2} = S_{s_2} \frac{\partial \phi_{2(j)}}{\partial t}, \quad (2.28)$$

where K_{x_1} and K_{y_1} are the directional hydraulic conductivities of the top soil layer, K_{x_2} and K_{y_2} are the directional conductivities of the bottom layer and S_{s_1} and S_{s_2} are the specific storage of the first and the second layers, respectively. The hydraulic heads in the top and bottom soil layers are denoted by $\phi_{1(j)}$ and $\phi_{2(j)}$ respectively, where the bracketed subscript j takes on the values 1, 2, 3 and 4 depending on the different boundary conditions resulting from the depth of water in the ditches. Depending on the level of water in the ditch drains, we encounter four distinct boundary value problems pertaining to the original flow problem. Before dealing with the boundary conditions specific to each case, we enumerate the common initial and boundary conditions for the two-dimensional flow problem at hand as

$$\phi_{1(j)}(x, y, t = 0) = 0, \quad 0 < x < S, \quad 0 < y \leq H_3, \quad (\text{I})$$

$$\phi_{2(j)}(x, y, t = 0) = 0, \quad 0 < x < S, \quad H_3 \leq y < h, \quad (\text{II})$$

$$\phi_{1(j)}(x, y, t > 0) = \phi_{2(j)}(x, y, t > 0), \quad 0 < x < S, \quad y = H_3, \quad (\text{IIIa})$$

$$-K_{y_1} \frac{\partial \phi_{1(j)}(x, y, t > 0)}{\partial y} = -K_{y_2} \frac{\partial \phi_{2(j)}(x, y, t > 0)}{\partial y}, \quad 0 < x < S, \quad y = H_3, \quad (\text{IIIb})$$

$$\phi_{1(j)}(x, y, t > 0) = \delta_1, \quad 0 < x < S_1, \quad y = 0, \quad (\text{IVa})$$

$$\phi_{1(j)}(x, y, t > 0) = \delta_i, \quad S_{(i-1)} < x < S_i, \quad y = 0, \quad (2 \leq i \leq N_0 - 1) \quad (IVb)$$

$$\phi_{1(j)}(x, y, t > 0) = \delta_{N_0}, \quad S_{(N_0-1)} < x < S, \quad y = 0, \quad (IVc)$$

$$-K_{y_2} \frac{\partial \phi_{2(j)}(x, y, t > 0)}{\partial y} = 0, \quad 0 < x < S, \quad y = h. \quad (V)$$

Next, we proceed to list the specific boundary conditions at the ditch drain boundaries for the four different cases of the flow problem at hand and propose analytical solutions for Eqs. (2.27) and (2.28) while taking into account all the pertinent boundary conditions.

However, we would like to point out here that the solutions that have been proposed for the transient ditch drainage problems under consideration are approximate and are valid only when the flow parameters of the flow domain satisfy certain preconditions. They can be expressed as follows

$$\frac{K_{x_1}}{S_{s_1}} = \frac{K_{x_2}}{S_{s_2}} \quad (2.29)$$

and

$$K_{y_1} = K_{y_2} \rightarrow 0 \text{ but } K_{x_1} = K_{x_2} \neq 0. \quad (2.30)$$

It is obvious that boundary condition (IIIb) will be always satisfied for any transient flow situation of Fig. 2.1 so long Eqs. (2.29) and (2.30) are valid for the concerned situation. Even though the transient solution cannot account for variation in vertical hydraulic conductivity between the soil layers, it still can accommodate a large range of disparate values as far as the other flow parameters are concerned. However, all the proposed solutions for the corresponding steady state flow problems are exact and are valid for all values of directional conductivities of the two soil layers.

2.2.1 Level of Water in the Ditches is Different and Above the Boundary between the Soil Layers

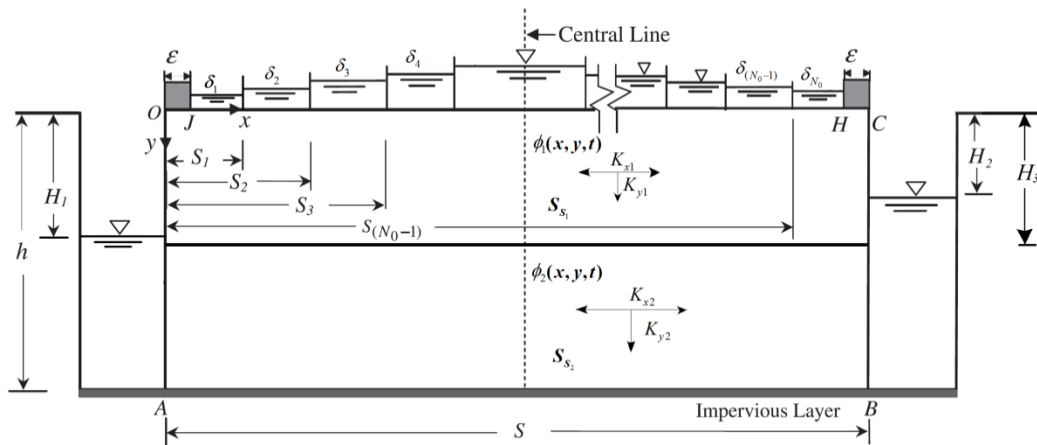


Fig. 2.2. Ditch drainage system for a two-layered soil when height of water in the ditches is above the bottom surface of the top soil layer

Fig. 2.2 illustrates the height of water, and consequently, the boundary conditions at the drainage ditches for this particular case. The boundary conditions specific to the ditch drain boundaries for this particular case are

$$\phi_{1(1)}(x, y, t > 0) = -y, \quad x = 0, \quad 0 < y \leq H_1, \quad (\text{VIa})$$

$$\phi_{1(1)}(x, y, t > 0) = -H_1, \quad x = 0, \quad H_1 \leq y \leq H_3, \quad (\text{VIb})$$

$$\phi_{2(1)}(x, y, t > 0) = -H_1, \quad x = 0, \quad H_3 \leq y < h, \quad (\text{VII})$$

$$\phi_{1(1)}(x, y, t > 0) = -y, \quad x = S, \quad 0 < y \leq H_2, \quad (\text{VIIIa})$$

$$\phi_{1(1)}(x, y, t > 0) = -H_2, \quad x = S, \quad H_2 \leq y \leq H_3, \quad (\text{VIIIb})$$

$$\phi_{2(1)}(x, y, t > 0) = -H_2, \quad x = S, \quad H_3 \leq y < h. \quad (\text{IX})$$

Taking into account the general solutions of the two-dimensional continuity equation as expressed in Eqs. (2.24), (2.25) and (2.26), we propose the following analytical solution for the flow problem at hand

$$\begin{aligned} \phi_{1(1)}(x, y, t) = & \sum_{p=1}^p B_{p(1)} \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin(N_p y) \\ & + \sum_{q=1}^q C_{q(1)} \frac{\sinh\left[\left(N_q / K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin(N_q y) \\ & + \sum_{k=1}^K E_{k(1)} \frac{\sinh\left(N_k K_1^a y\right)}{\cosh\left(N_k K_1^a H_3\right)} \sin(N_k x) \\ & + \sum_{r=1}^R D_{r(1)} \frac{\cosh\left[N_r K_1^a (H_3 - y)\right]}{\cosh\left(N_r K_1^a H_3\right)} \sin(N_r x) \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \sin(N_m x) \sin(N_n y) \exp\left[-\left(\lambda_{mn}\right)^2 t\right] \\ & + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \sin(N_u x) \sin(N_v y) \exp\left[-\left(\lambda_{uv}\right)^2 t\right], \end{aligned} \quad (2.31)$$

where

$$N_p = \left[\left(\frac{1-2p}{2} \right) \left(\frac{\pi}{H_3} \right) \right], \quad (2.32)$$

$$N_q = \left[\left(\frac{1-2q}{2} \right) \left(\frac{\pi}{H_3} \right) \right], \quad (2.33)$$

$$N_k = \left(\frac{k\pi}{S} \right), \quad (2.34)$$

$$N_r = \left(\frac{r\pi}{S} \right), \quad (2.35)$$

$$K_1^a = \sqrt{\left(\frac{K_{x_1}}{K_{y_1}} \right)}, \quad (2.36)$$

$$N_m = \left(\frac{m\pi}{S} \right), \quad (2.37)$$

$$N_n = \left[\left(\frac{1-2n}{2} \right) \frac{\pi}{h} \right], \quad (2.38)$$

$$\lambda_{mn}^2 = \left[N_m^2 \left(\frac{K_{x_1}}{S_{s_1}} \right) + N_n^2 \left(\frac{K_{y_1}}{S_{s_1}} \right) \right], \quad (2.39)$$

$$N_u = \left(\frac{u\pi}{S} \right), \quad (2.40)$$

$$N_v = \left[\left(\frac{1-2v}{2} \right) \frac{\pi}{h} \right], \quad (2.41)$$

$$\lambda_{uv}^2 = \left[N_u^2 \left(\frac{K_{x_2}}{S_{s_2}} \right) + N_v^2 \left(\frac{K_{y_2}}{S_{s_2}} \right) \right] \quad (2.42)$$

and $B_{p(1)}$, $C_{q(1)}$, $E_{k(1)}$, $D_{r(1)}$, $A_{mn(1)}$ and $Z_{uv(1)}$ are constants and

$$\begin{aligned} \phi_{2(1)}(x, y, t) = & \sum_{w=1}^W H_{w(1)} \frac{\cosh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} \sin(N_w x) \\ & - H_1 \left(\frac{S-x}{S} \right) - H_2 \left(\frac{x}{S} \right) \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \sin(N_m x) \sin(N_n y) \exp \left[-(\lambda_{mn})^2 t \right] \\ & + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \sin(N_u x) \sin(N_v y) \exp \left[-(\lambda_{uv})^2 t \right], \end{aligned} \quad (2.43)$$

where

$$N_w = \left(\frac{w\pi}{S} \right), \quad (2.44)$$

$$K_2^a = \sqrt{\left(\frac{K_{x_2}}{K_{y_2}} \right)} \quad (2.45)$$

and $H_{w(1)}$ are constants. It can be noted that in the hydraulic head expressions for both the soil layers, the double Fourier series terms represent the transient part of the solution for the flow problem under consideration. The double Fourier series term with coefficients $A_{mn(1)}$ accounts for the transient flow in the top soil layer while the series with coefficients $Z_{uv(1)}$ describes transient flow in the bottom soil layer. However, it may be observed that these two terms are written considering the whole vertical space of the flow domain and not on the basis of the individual vertical extent covered by each of the layers.

We would like to reiterate here that the solution presented here for the transient situation is valid only for situations where Eqs. (2.29) and (2.30) hold, i.e., only when $K_{x_1} / S_{s_1} = K_{x_2} / S_{s_2}$

and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$) and not otherwise; however, so far as the steady state is concerned, the proposed solution is correct for any possible combination of the flow parameters of Fig. 2.2 without any imposed constraints on them.

For our convenience, let us denote the double Fourier series

$$\sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \sin(N_m x) \sin(N_n y) = SR_1 \quad (2.46)$$

and the double Fourier series

$$\sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \sin(N_u x) \sin(N_v y) = SR_2. \quad (2.47)$$

As we proceed with our derivations for the transient case, we will ensure that the series SR_1 exists for the top soil layer only and that SR_2 exists for the bottom soil layer only; that way, as will be shown later, the coefficients $A_{mn(1)}$ and $Z_{uv(1)}$ can be readily determined by making use of the initial conditions (I) and (II).

Before deriving the values of the unknown Fourier series coefficients, we state the expressions for velocity distribution in the x - and y - directions. For this, we apply the Darcy's law to the hydraulic head functions of the top and bottom layers, i.e.,

$$V_{x1(1)} = -K_{x_1} \frac{\partial \phi_{1(1)}(x, y, t)}{\partial x}, \quad (2.48)$$

$$V_{y1(1)} = -K_{y_1} \frac{\partial \phi_{1(1)}(x, y, t)}{\partial y}, \quad (2.49)$$

$$V_{x2(1)} = -K_{x_2} \frac{\partial \phi_{2(1)}(x, y, t)}{\partial x} \quad (2.50)$$

and

$$V_{y2(1)} = -K_{y_2} \frac{\partial \phi_{2(1)}(x, y, t)}{\partial y}, \quad (2.51)$$

where $V_{x1(1)}$ and $V_{x2(1)}$ are the horizontal velocities of the top and the bottom layers and $V_{y1(1)}$ and $V_{y2(1)}$ are their respective vertical velocities. Performing the above differentiations, we get

$$\begin{aligned} V_{x1(1)} = -K_{x_1} & \left\{ \sum_{p=1}^P B_{p(1)} \frac{(N_p / K_1^a) \cosh[(N_p / K_1^a)x]}{\sinh[(N_p / K_1^a)S]} \sin(N_p y) \right. \\ & - \sum_{q=1}^Q C_{q(1)} \frac{(N_q / K_1^a) \cosh[(N_q / K_1^a)(S-x)]}{\sinh[(N_q / K_1^a)S]} \sin(N_q y) \\ & + \sum_{k=1}^K E_{k(1)} \frac{\sinh(N_k K_1^a y)}{\cosh(N_k K_1^a H_3)} N_k \cos(N_k x) \\ & \left. + \sum_{r=1}^R D_{r(1)} \frac{\cosh[N_r K_1^a (H_3 - y)]}{\cosh(N_r K_1^a H_3)} N_r \cos(N_r x) \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} N_m \cos(N_m x) \sin(N_n y) \exp[-(\lambda_{mn})^2 t] \\
& + \left. \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} N_u \cos(N_u x) \sin(N_v y) \exp[-(\lambda_{uv})^2 t] \right\}, \tag{2.52}
\end{aligned}$$

$$\begin{aligned}
V_{y1(1)} = & -K_{y_1} \left\{ \sum_{p=1}^P B_{p(1)} \frac{\sinh\left[\frac{(N_p / K_1^a) x}{(N_p / K_1^a) S}\right]}{\sinh\left[\frac{(N_p / K_1^a) S}{(N_p / K_1^a) S}\right]} N_p \cos(N_p y) \right. \\
& + \sum_{q=1}^Q C_{q(1)} \frac{\sinh\left[\frac{(N_q / K_1^a)(S-x)}{(N_q / K_1^a) S}\right]}{\sinh\left[\frac{(N_q / K_1^a) S}{(N_q / K_1^a) S}\right]} N_q \cos(N_q y) \\
& + \sum_{k=1}^K E_{k(1)} \frac{(N_k K_1^a) \cosh(N_k K_1^a y)}{\cosh(N_k K_1^a H_3)} \sin(N_k x) \\
& - \sum_{r=1}^R D_{r(1)} \frac{(N_r K_1^a) \sinh[N_r K_1^a (H_3 - y)]}{\cosh(N_r K_1^a H_3)} \sin(N_r x) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \sin(N_m x) N_n \cos(N_n y) \exp[-(\lambda_{mn})^2 t] \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \sin(N_u x) N_v \cos(N_v y) \exp[-(\lambda_{uv})^2 t] \right\}, \tag{2.53}
\end{aligned}$$

$$\begin{aligned}
V_{x2(1)} = & -K_{x_2} \left\{ \sum_{w=1}^W H_{w(1)} \frac{\cosh\left[\frac{(N_w K_2^a)(h-y)}{(N_w K_2^a)(h-H_3)}\right]}{\sinh\left[\frac{(N_w K_2^a)(h-H_3)}{(N_w K_2^a)(h-H_3)}\right]} N_w \cos(N_w x) - \left(\frac{H_2 - H_1}{S}\right) \right. \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} N_m \cos(N_m x) \sin(N_n y) \exp[-(\lambda_{mn})^2 t] \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} N_u \cos(N_u x) \sin(N_v y) \exp[-(\lambda_{uv})^2 t] \right\} \tag{2.54}
\end{aligned}$$

and

$$\begin{aligned}
V_{y2(1)} = & -K_{y_2} \left\{ - \sum_{w=1}^W H_{w(1)} \frac{(N_w K_2^a) \sinh\left[\frac{(N_w K_2^a)(h-y)}{(N_w K_2^a)(h-H_3)}\right]}{\sinh\left[\frac{(N_w K_2^a)(h-H_3)}{(N_w K_2^a)(h-H_3)}\right]} \sin(N_w x) \right. \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \sin(N_m x) N_n \cos(N_n y) \exp[-(\lambda_{mn})^2 t] \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \sin(N_u x) N_v \cos(N_v y) \exp[-(\lambda_{uv})^2 t] \right\}. \tag{2.55}
\end{aligned}$$

2.2.1.1 Determination of Coefficients of the Steady State Solution (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 2.2)

It should be noted that the hydraulic head expressions of Eqs. (2.31) and (2.43) pertaining to the first and second layers, respectively of the drainage problem of Fig. 2.2 reduce to the steady state situation when the time variable in them is allowed to go to infinity. As may be

observed, this will knock out the exponential terms of these expressions leaving behind only the steady state terms of these functions.

In order to determine the steady state coefficients of our hydraulic head functions, we now proceed to utilize boundary conditions (VIa) and (VIb) in Eq. (2.31); this leaves us with

$$\sum_{q=1}^Q C_{q(1)} \sin(N_q y) = -y, \quad 0 < y \leq H_1,$$

$$\sum_{q=1}^Q C_{q(1)} \sin(N_q y) = -H_1, \quad H_1 \leq y \leq H_3.$$

We expand the Fourier series (by letting $Q \rightarrow \infty$ in the above expressions) and evaluate $C_{q(1)}$ to obtain

$$C_{q(1)} = \left(\frac{2}{H_3} \right) \left[\int_0^{H_1} -y \sin(N_q y) dy + \int_{H_1}^{H_3} -H_1 \sin(N_q y) dy \right]. \quad (2.56)$$

On evaluating the integrals in the above expression, we get

$$C_{q(1)} = \left(\frac{2}{H_3} \right) \left\{ \left[y \frac{\cos(N_q y)}{N_q} \right]_0^{H_1} - \left[\frac{\sin(N_q y)}{N_q^2} \right]_0^{H_1} + \left[H_1 \frac{\cos(N_q y)}{N_q} \right]_{H_1}^{H_3} \right\}. \quad (2.57)$$

This, on further simplification, leads to

$$C_{q(1)} = \left(\frac{2}{H_3} \right) \left[\frac{-\sin(N_q H_1)}{N_q^2} \right]. \quad (2.58)$$

Similarly, based on boundary conditions (VIIIa) and (VIIIb), we arrive at the following relationship involving Fourier coefficients $B_{p(1)}$

$$\sum_{p=1}^P B_{p(1)} \sin(N_p y) = -y, \quad 0 < y \leq H_2,$$

$$\sum_{p=1}^P B_{p(1)} \sin(N_p y) = -H_2, \quad H_2 \leq y \leq H_3.$$

Similar to $C_{q(1)}$, we proceed to derive the following expression for the coefficients $B_{p(1)}$

$$B_{p(1)} = \left(\frac{2}{H_3} \right) \left[\frac{-\sin(N_p H_2)}{N_p^2} \right]. \quad (2.59)$$

Now, on imposing boundary conditions (IVa), (IVb) and (IVc) on Eq. (2.31), we arrive at the following relations

$$\sum_{r=1}^R D_{r(1)} \sin(N_r x) = \delta_1, \quad 0 < x < S_1, \quad y = 0,$$

$$\sum_{r=1}^R D_{r(1)} \sin(N_r x) = \delta_i, \quad S_{(i-1)} < x < S_i, (2 \leq i \leq N_0 - 1) \quad y = 0,$$

$$\sum_{r=1}^R D_{r(1)} \sin(N_r x) = \delta_{N_0}, \quad S_{(N_0-1)} < x < S, \quad y = 0.$$

The coefficients $D_{r(1)}$ can thus be evaluated by allowing R in the equations above to go to infinity and then performing a Fourier expansion in $0 < x < S$; doing so, we get

$$D_{r(1)} = \left(\frac{2}{S} \right) \left[\int_0^{S_1} \delta_1 \sin(N_r x) dx + \int_{S_1}^{S_2} \delta_2 \sin(N_r x) dx + \dots + \int_{S_{(i-1)}}^{S_i} \delta_i \sin(N_r x) dx + \dots + \int_{S_{(N_0-1)}}^S \delta_{N_0} \sin(N_r x) dx \right]. \quad (2.60)$$

Solving the integrals in Eq. (2.60), we obtain

$$D_{r(1)} = \left(\frac{2}{S} \right) \left\{ \delta_1 \left[\frac{1 - \cos(N_r S_1)}{N_r} \right] + \delta_2 \left[\frac{\cos(N_r S_1) - \cos(N_r S_2)}{N_r} \right] + \dots + \delta_i \left[\frac{\cos(N_r S_{(i-1)}) - \cos(N_r S_i)}{N_r} \right] + \dots + \delta_{N_0} \left[\frac{\cos(N_r S_{(N_0-1)}) - \cos(N_r S)}{N_r} \right] \right\}. \quad (2.61)$$

We now equate the steady state hydraulic head expressions for the top and the bottom layers at $y = H_3$ so as to satisfy boundary condition (IIIa) at time $t \rightarrow \infty$; this leads to the following relationship

$$\begin{aligned} & \sum_{p=1}^P B_{p(1)} \frac{\sinh \left[\left(N_p / K_1^a \right) x \right]}{\sinh \left[\left(N_p / K_1^a \right) S \right]} \sin(N_p H_3) + \sum_{q=1}^Q C_{q(1)} \frac{\sinh \left[\left(N_q / K_1^a \right) (S - x) \right]}{\sinh \left[\left(N_q / K_1^a \right) S \right]} \sin(N_q H_3) \\ & + \sum_{k=1}^K E_{k(1)} \frac{\sinh(N_k K_1^a H_3)}{\cosh(N_k K_1^a H_3)} \sin(N_k x) + \sum_{r=1}^R D_{r(1)} \frac{1}{\cosh(N_r K_1^a H_3)} \sin(N_r x) \\ & = \sum_{w=1}^W H_{w(1)} \frac{\cosh \left[N_w K_2^a (h - H_3) \right]}{\sinh \left[N_w K_2^a (h - H_3) \right]} \sin(N_w x) + (-H_2) \left(\frac{x}{S} \right) + (-H_1) \left(\frac{S - x}{S} \right) \end{aligned}$$

Multiplying both sides of the above expression by $\sin \left[\left(\frac{i_1 \pi}{S} \right) x \right]$ (where $i_1 = 1, 2, 3, \dots, \infty$) and

then running a Fourier series along the line $0 < x < S$, we get

$$\begin{aligned} & E_{k(1)} \tanh(N_i K_1^a H_3) - H_{w(1)} \coth \left[N_i K_2^a (h - H_3) \right] \\ & = -D_{r(1)} \operatorname{sech}(N_i K_1^a H_3) \\ & - \left(\frac{2}{S} \right) \sum_{p=1}^P B_{p(1)} \sin(N_p H_3) \int_0^S \frac{\sinh \left[\left(N_p / K_1^a \right) x \right]}{\sinh \left[\left(N_p / K_1^a \right) S \right]} \sin(N_i x) dx \\ & - \left(\frac{2}{S} \right) \sum_{q=1}^Q C_{q(1)} \sin(N_q H_3) \int_0^S \frac{\sinh \left[\left(N_q / K_1^a \right) (S - x) \right]}{\sinh \left[\left(N_q / K_1^a \right) S \right]} \sin(N_i x) dx \\ & - \left(\frac{2}{S} \right) H_2 \int_0^S \left(\frac{x}{S} \right) \sin(N_i x) dx - \left(\frac{2}{S} \right) H_1 \int_0^S \left(\frac{S - x}{S} \right) \sin(N_i x) dx, \end{aligned} \quad (2.62)$$

where $i_1 = k = r = w$, $N_{i_1} = \left(\frac{i_1 \pi}{S}\right)$ and $P = Q \rightarrow \infty$. We evaluate the integrals in Eq. (2.62) to obtain

$$\int_0^S \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin\left(N_{i_1}x\right) dx = \left[\frac{N_{i_1}^2}{N_{i_1}^2 + \left(N_p / K_1^a\right)^2}\right] \left[\frac{-\cos\left(N_{i_1}S\right)}{N_{i_1}}\right], \quad (2.63)$$

$$\int_0^S \frac{\sinh\left[\left(N_q / K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin\left(N_{i_1}x\right) dx = \left[\frac{N_{i_1}}{N_{i_1}^2 + \left(N_q / K_1^a\right)^2}\right], \quad (2.64)$$

$$\int_0^S \left(\frac{x}{S}\right) \sin\left(N_{i_1}x\right) dx = \left[\frac{-\cos\left(N_{i_1}S\right)}{N_{i_1}}\right], \quad (2.65)$$

$$\int_0^S \left(\frac{S-x}{S}\right) \sin\left(N_{i_1}x\right) dx = \left[\frac{1}{N_{i_1}}\right]. \quad (2.66)$$

Substituting the values of the integrals in Eq. (2.62), we get

$$\begin{aligned} & E_{k(1)} \tanh\left(N_{i_1} K_1^a H_3\right) - H_{w(1)} \coth\left[N_{i_1} K_2^a (h - H_3)\right] \\ &= -D_{r(1)} \operatorname{sech}\left(N_{i_1} K_1^a H_3\right) \\ & - \left(\frac{2}{S}\right) \sum_{p=1}^P B_{p(1)} \sin\left(N_p H_3\right) \left[\frac{N_{i_1}^2}{N_{i_1}^2 + \left(N_p / K_1^a\right)^2}\right] \left[\frac{-\cos\left(N_{i_1}S\right)}{N_{i_1}}\right] \\ & - \left(\frac{2}{S}\right) \sum_{q=1}^Q C_{q(1)} \sin\left(N_q H_3\right) \left[\frac{N_{i_1}}{N_{i_1}^2 + \left(N_q / K_1^a\right)^2}\right] \\ & - \left(\frac{2}{S}\right) H_2 \left[\frac{-\cos\left(N_{i_1}S\right)}{N_{i_1}}\right] - \left(\frac{2}{S}\right) H_1 \left[\frac{1}{N_{i_1}}\right], \end{aligned} \quad (2.67)$$

where $i_1 = k = r = w$ and $P = Q \rightarrow \infty$.

In order to accommodate boundary condition (IIIb) at time $t \rightarrow \infty$, we now equate the steady state hydraulic flux expressions in the y -direction for the two layers at $y = H_3$ – the following relationship emerges

$$K_{y_1} \sum_{k=1}^K E_{k(1)} \left(N_k K_1^a\right) \sin\left(N_k x\right) = K_{y_2} \sum_{w=1}^W H_{w(1)} \left[-\left(N_w K_2^a\right)\right] \sin\left(N_w x\right).$$

On simplifying the above, we get

$$K_{y_1} E_{k(1)} \left(N_k K_1^a\right) = -K_{y_2} H_{w(1)} \left(N_w K_2^a\right), \quad (2.68)$$

where $k = w$. Also, it is to be noted that $K = W \rightarrow \infty$. Coefficients $E_{k(1)}$ and $H_{w(1)}$ can thus be obtained by solving Eqs. (2.67) and (2.68) pertaining to a ditch drainage scenario.

2.2.1.2 Determination of the Transient Solution Coefficients [transient state solution is approximate and is applicable only when conductivity and specific storage of the layers of Fig. 2.2 satisfy $K_{x_1} / S_{s_1} = K_{x_1} / S_{s_1}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)]

To ascertain that SR_1 only exists for the top soil layer and also satisfies the initial condition (I) of the flow problem, we equate it to a piecewise smooth function $F_1(x, y)$ in the two spatial variables x and y , i.e.,

$$\sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \sin(N_m x) \sin(N_n y) = F_1(x, y). \quad (2.69)$$

The aforementioned piecewise smooth function may be defined as

$$\begin{aligned} F_1(x, y) = & -\sum_{p=1}^P B_{p(1)} \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin(N_p y) \\ & -\sum_{q=1}^Q C_{q(1)} \frac{\sinh\left[\left(N_q / K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin(N_q y) \\ & -\sum_{k=1}^K E_{k(1)} \frac{\sinh(N_k K_1^a y)}{\cosh(N_k K_1^a H_3)} \sin(N_k x) \\ & -\sum_{r=1}^R D_{r(1)} \frac{\cosh\left[N_r K_1^a (H_3 - y)\right]}{\cosh(N_r K_1^a H_3)} \sin(N_r x), \quad 0 < x < S, \quad 0 < y < H_3, \\ F_1(x, y) = & 0, \quad 0 < x < S, \quad H_3 < y < h. \end{aligned} \quad (2.70)$$

It can be noted that for the top soil layer of the flow domain, we have equated the transient solution at time $t = 0$ to the steady state solution. This way, the initial condition (I) pertaining to the hydraulic head in the upper layer is being satisfied; also equating it to zero in the lower soil layer ensures that the series ceases to exist for that layer.

Similarly, we equate the series SR_2 to another piecewise smooth function $F_2(x, y)$ which helps us in satisfying the initial condition (II)

$$\sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \sin(N_u x) \sin(N_v y) = F_2(x, y). \quad (2.71)$$

The function $F_2(x, y)$ can be defined as

$$\begin{aligned} F_2(x, y) = & 0, \quad 0 < x < S, \quad 0 < y < H_3, \\ F_2(x, y) = & -\sum_{w=1}^W H_{w(1)} \frac{\cosh\left[\left(N_w K_2^a\right)(h-y)\right]}{\sinh\left[\left(N_w K_2^a\right)(h-H_3)\right]} \sin(N_w x) \\ & + H_1 \left(\frac{S-x}{S}\right) + H_2 \left(\frac{x}{S}\right), \quad 0 < x < S, \quad H_3 < y < h. \end{aligned} \quad (2.72)$$

Since the functions $F_1(x, y)$ and $F_2(x, y)$ are piecewise smooth over the flow domain, the Fourier series representing them will converge at any point within the flow domain. So, we

can proceed to evaluate the coefficients $A_{mn(1)}$ and $Z_{uv(1)}$ of the two double Fourier series SR_1 and SR_2 . The expression for $A_{mn(1)}$ will be

$$A_{mn(1)} = \left(\frac{4}{Sh} \right) \left\{ - \sum_{p=1}^P B_{p(1)} \int_0^S \frac{\sinh \left[\frac{(N_p / K_1^a) x}{(N_p / K_1^a) S} \right]}{\sinh \left[\frac{(N_p / K_1^a) S}{(N_p / K_1^a) S} \right]} \sin(N_m x) dx \times \int_0^{H_3} \sin(N_p y) \sin(N_n y) dy \right. \\ - \sum_{q=1}^Q C_{q(1)} \int_0^S \frac{\sinh \left[\frac{(N_q / K_1^a) (S-x)}{(N_q / K_1^a) S} \right]}{\sinh \left[\frac{(N_q / K_1^a) S}{(N_q / K_1^a) S} \right]} \sin(N_m x) dx \times \int_0^{H_3} \sin(N_q y) \sin(N_n y) dy \\ - \sum_{k=1}^K E_{k(1)} \int_0^{H_3} \frac{\sinh(N_k K_1^a y)}{\cosh(N_k K_1^a H_3)} \sin(N_n y) dy \times \int_0^S \sin(N_k x) \sin(N_m x) dx \\ \left. - \sum_{r=1}^R D_{r(1)} \int_0^{H_3} \frac{\cosh \left[\frac{N_r K_1^a (H_3 - y)}{N_r K_1^a H_3} \right]}{\cosh(N_r K_1^a H_3)} \sin(N_n y) dy \times \int_0^S \sin(N_r x) \sin(N_m x) dx \right\}. \quad (2.73)$$

We evaluate the integrals in the above expression to obtain the following results

$$\int_0^S \frac{\sinh \left[\frac{(N_p / K_1^a) x}{(N_p / K_1^a) S} \right]}{\sinh \left[\frac{(N_p / K_1^a) S}{(N_p / K_1^a) S} \right]} \sin(N_m x) dx = \left[\frac{N_m^2}{N_m^2 + (N_p / K_1^a)^2} \right] \left[\frac{-\cos(N_m S_1)}{N_m} \right], \quad (2.74)$$

$$\int_0^{H_3} \sin(N_p y) \sin(N_n y) dz = \left(\frac{H_3}{2} \right), \quad N_p = N_n, \\ = \left[\frac{\sin(N_p H_3 - N_n H_3)}{2(N_p - N_n)} - \frac{\sin(N_p H_3 + N_n H_3)}{2(N_p + N_n)} \right], \quad N_p \neq N_n, \quad (2.75)$$

$$\int_0^S \frac{\sinh \left[\frac{(N_q / K_1^a) (S-x)}{(N_q / K_1^a) S} \right]}{\sinh \left[\frac{(N_q / K_1^a) S}{(N_q / K_1^a) S} \right]} \sin(N_m x) dx = \left[\frac{N_m}{N_m^2 + (N_q / K_1^a)^2} \right], \quad (2.76)$$

$$\int_0^{H_3} \sin(N_q y) \sin(N_n y) dz = \left(\frac{H_3}{2} \right), \quad N_q = N_n, \\ = \left[\frac{\sin(N_q H_3 - N_n H_3)}{2(N_q - N_n)} - \frac{\sin(N_q H_3 + N_n H_3)}{2(N_q + N_n)} \right], \quad N_q \neq N_n, \quad (2.77)$$

$$\int_0^S \sin(N_k x) \sin(N_m x) dx = 0, \quad N_k \neq N_m, \\ = \left(\frac{S}{2} \right), \quad N_k = N_m, \quad (2.78)$$

$$\int_0^{H_3} \frac{\sinh(N_k K_1^a y)}{\cosh(N_k K_1^a H_3)} \sin(N_n y) dy \\ = \left[\frac{N_n^2}{N_n^2 + (N_k K_1^a)^2} \right] \times \left\{ \tanh(N_k K_1^a H_3) \left[\frac{-\cos(N_n H_3)}{N_n} \right] + \left(\frac{N_k K_1^a}{N_n^2} \right) \sin(N_n H_3) \right\}, \quad (2.79)$$

$$\int_0^S \sin(N_r x) \sin(N_m x) dx = 0, \quad N_r \neq N_m,$$

$$= \left(\frac{S}{2} \right), \quad N_r = N_m \quad (2.80)$$

and

$$\int_0^{H_3} \frac{\cosh[N_r K_1^a (H_3 - y)]}{\cosh(N_r K_1^a H_3)} \sin(N_n y) dy$$

$$= \left[\frac{N_n^2}{N_n^2 + (N_r K_1^a)^2} \right] \times \left\{ \frac{1}{N_n} - \left[\frac{\cos(N_n H_3)}{N_n} \right] \left[\frac{1}{\cosh(N_r K_1^a H_3)} \right] \right\}. \quad (2.81)$$

Substituting the values of the evaluated integrals in Eq. (2.73), we have the following final expression for the coefficients $A_{mn(1)}$

$$A_{mn(1)} = \left(\frac{4}{Sh} \right) \left\{ - \sum_{p=1}^P B_{p(1)} \left[\frac{N_m^2}{N_m^2 + (N_p / K_1^a)^2} \right] \left[\frac{-\cos(N_m S_1)}{N_m} \right] \times I^{(1)} \right.$$

$$- \sum_{q=1}^Q C_{q(1)} \left[\frac{N_m}{N_m^2 + (N_q / K_1^a)^2} \right] \times I^{(2)}$$

$$- E_{k(1)} \left(\frac{S}{2} \right) \left[\frac{N_n^2}{N_n^2 + (N_k K_1^a)^2} \right]$$

$$\times \left\{ \tanh(N_k K_1^a H_3) \left[\frac{-\cos(N_n H_3)}{N_n} \right] + \left(\frac{N_k K_1^a}{N_n^2} \right) \sin(N_n H_3) \right\}$$

$$- D_{r(1)} \left(\frac{S}{2} \right) \left[\frac{N_n^2}{N_n^2 + (N_r K_1^a)^2} \right]$$

$$\times \left\{ \frac{1}{N_n} - \left[\frac{\cos(N_n H_3)}{N_n} \right] \left[\frac{1}{\cosh(N_r K_1^a H_3)} \right] \right\} \Bigg\}, \quad (2.82)$$

where $r = k = m$, $P = Q \rightarrow \infty$,

$$I^{(1)} = \left(\frac{H_3}{2} \right), \quad N_p = N_n,$$

$$= \left\{ \frac{\sin[(N_p - N_n) H_3]}{2(N_p - N_n)} - \frac{\sin[(N_p + N_n) H_3]}{2(N_p + N_n)} \right\}, \quad N_p \neq N_n \quad (2.83)$$

and

$$I^{(2)} = \left(\frac{H_3}{2} \right), \quad N_q = N_n,$$

$$= \left\{ \frac{\sin \left[(N_q - N_n) H_3 \right]}{2(N_q - N_n)} - \frac{\sin \left[(N_q + N_n) H_3 \right]}{2(N_q + N_n)} \right\}, \quad N_q \neq N_n. \quad (2.84)$$

Similarly, we arrive at the following general expression for the coefficients $Z_{uv(1)}$

$$Z_{uv(1)} = \left(\frac{4}{Sh} \right) \left\{ - \sum_{w=1}^W H_{w(1)} \int_{H_3}^h \frac{\cosh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} \sin(N_v y) dy \right.$$

$$\times \int_0^S \sin(N_u x) \sin(N_w x) dx$$

$$+ H_1 \int_0^S \left(\frac{S-x}{S} \right) \sin(N_u x) dx \int_{H_3}^h \sin(N_v y) dy$$

$$\left. + H_2 \int_0^S \left(\frac{x}{S} \right) \sin(N_u x) dx \int_{H_3}^h \sin(N_v y) dy \right\}. \quad (2.85)$$

We evaluate all the integrals in the above expression; this gives us

$$\int_0^S \sin(N_w x) \sin(N_u x) dx = 0, \quad N_w \neq N_u,$$

$$= \left(\frac{S}{2} \right), \quad N_w = N_u, \quad (2.86)$$

$$\int_{H_3}^h \frac{\cosh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} \sin(N_v y) dy =$$

$$\left[\frac{N_v^2}{N_v^2 + (N_w K_2^a)^2} \right] \left\{ \coth \left[(N_w K_2^a)(h-H_3) \right] \frac{\cos(N_v H_3)}{N_v} + \left[\frac{(N_w K_2^a)}{N_v^2} \right] \sin(N_v H_3) \right\}, \quad (2.87)$$

$$\int_0^S \left(\frac{S-x}{S} \right) \sin(N_u x) dx = \left(\frac{1}{N_u} \right), \quad (2.88)$$

$$\int_0^S \left(\frac{x}{S} \right) \sin(N_u x) dx = \left[\frac{-\cos(N_u S)}{N_u} \right] \quad (2.89)$$

and

$$\int_{H_3}^h \sin(N_v y) dy = \left[\frac{\cos(N_v H_3)}{N_v} \right]. \quad (2.90)$$

Substituting the values of the integrals in Eq. (2.85), we arrive at

$$\begin{aligned}
Z_{uv(1)} = & \left(\frac{4}{Sh} \right) \left\{ -H_{w(1)} \left(\frac{S}{2} \right) \left[\frac{N_v^2}{N_v^2 + (N_w K_2^a)^2} \right] \right. \\
& \times \left\{ \coth \left[(N_w K_2^a)(h - H_3) \right] \frac{\cos(N_v H_3)}{N_v} + \left[\frac{(N_w K_2^a)}{N_v^2} \right] \sin(N_v H_3) \right\} \\
& \left. + H_1 \left(\frac{1}{N_u} \right) \left(\frac{\cos N_v H_3}{N_v} \right) + H_2 \left(\frac{\cos N_v H_3}{N_v} \right) \left[\frac{-\cos(N_u S)}{N_u} \right] \right\}, \quad (2.91)
\end{aligned}$$

where $w = u$. We now concern ourselves with determining the transient discharge function at the top of the flow domain. We apply the Darcy's law at the surface of the soil for the hydraulic head expression $\phi_{(1)}$. This yields

$$Q_{top(1)}(t) = -K_{y_1} \int_{\varepsilon}^{S-\varepsilon} \left[\frac{\partial \phi_{(1)}(x, y, t)}{\partial y} \right]_{y=0} dx, \quad (2.92)$$

where, $Q_{top(1)}$ is the discharge rate at the surface of the soil per unit length of the ditches. We proceed to evaluate the integral in Eq. (2.92). This leads to the following expression

$$\begin{aligned}
Q_{top(1)}(t) = & -K_{y_1} \left\{ \sum_{p=1}^P B_{p(1)} K_1^a \left\{ \frac{\cosh \left[(N_p / K_1^a)(S - \varepsilon) \right] - \cosh \left[(N_p / K_1^a) \varepsilon \right]}{\sinh \left[(N_p / K_1^a) S \right]} \right\} \right. \\
& + \sum_{q=1}^Q C_{q(1)} K_1^a \left\{ \frac{\cosh \left[(N_q / K_1^a)(S - \varepsilon) \right] - \cosh \left[(N_q / K_1^a) \varepsilon \right]}{\sinh \left[(N_q / K_1^a) S \right]} \right\} \\
& + \sum_{k=1}^K E_{k(1)} K_1^a \operatorname{sech} \left(N_k K_1^a H_3 \right) \left\{ \cos(N_k \varepsilon) - \cos \left[N_k (S - \varepsilon) \right] \right\} \\
& - \sum_{r=1}^R D_{r(1)} K_1^a \tanh \left(N_r K_1^a H_3 \right) \left\{ \cos(N_r \varepsilon) - \cos \left[N_r (S - \varepsilon) \right] \right\} \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \left(\frac{N_n}{N_m} \right) \left\{ \cos(N_m \varepsilon) - \cos \left[N_m (S - \varepsilon) \right] \right\} \exp \left[-(\lambda_{mn})^2 t \right] \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \left(\frac{N_v}{N_u} \right) \left\{ \cos(N_u \varepsilon) - \cos \left[N_u (S - \varepsilon) \right] \right\} \exp \left[-(\lambda_{uv})^2 t \right] \right\}. \quad (2.93)
\end{aligned}$$

We now show that $Q_{top(1)}$ diverges in case the depth of ponding for the first ponding strip, δ_1 , happens to be non-zero, while the width of the ditch bunds ε is zero at the same time. In order to show that, we first substitute the expression for $D_{r(1)}$ from Eq. (2.61) in Eq. (2.93); this leads, after some mathematical manipulations, the term involving $D_{r(1)}$ to be of the form

$$\sum_{r=1}^R \left(\frac{2\delta_1 K_1^a}{SN_r} \right) \tanh \left(N_r K_1^a H_3 \right).$$

Since

$$\lim_{r \rightarrow \infty} \tanh(N_r K_1^a H_3) = \lim_{r \rightarrow \infty} \tanh\left[\left(\frac{r\pi}{S}\right) K_1^a H_3\right] = \lim_{\omega \rightarrow \infty} \frac{e^\omega + e^{-\omega}}{e^\omega - e^{-\omega}} = 1, \quad (2.94)$$

where $\omega = \left(\frac{r\pi}{S}\right) K_1^a H_3$, $\sum_{r=1}^R \left(\frac{2\delta_1 K_1^a}{SN_r}\right) \tanh(N_r K_1^a H_3)$ can thus be expanded for higher terms

as

$$\frac{2\delta_1}{\pi} \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots \right),$$

where, in view of Eq. (2.94), it is been assumed that $\tanh(N_r K_1^a H_3)$ reduces to 1 after the summation of first three terms of the series. But then, as can be seen, this is a diverging series; hence it can be stated that $Q_{top(1)}(t)$ diverges when $\delta_1 \neq 0$ and $\varepsilon = 0$.

Similarly, the expression for the time dependent top discharge function $Q_{topx(1)}$, which helps us in estimating the discharge entering through the top of the soil in between the origin and any location falling on the surface of the soil, can be estimated as

$$Q_{topx(1)}(x, t) = -K_{y_1} \int_{\varepsilon}^x \left[\frac{\partial \phi_{1(1)}(x, y, t)}{\partial y} \right]_{y=0} dx. \quad (2.95)$$

Solving the above integral, we obtain

$$\begin{aligned} Q_{topx(1)}(x, t) = & -K_{y_1} \left\{ \sum_{p=1}^P B_{p(1)} K_1^a \left[\frac{\cosh\left[\left(N_p/K_1^a\right)x\right] - \cosh\left[\left(N_p/K_1^a\right)\varepsilon\right]}{\sinh\left[\left(N_p/K_1^a\right)S\right]} \right] \right. \\ & + \sum_{q=1}^Q C_{q(1)} K_1^a \left[\frac{\cosh\left[\left(N_q/K_1^a\right)(S-\varepsilon)\right] - \cosh\left[\left(N_q/K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q/K_1^a\right)S\right]} \right] \\ & + \sum_{k=1}^K E_{k(1)} K_1^a \operatorname{sech}(N_k K_1^a H_3) [\cos(N_k \varepsilon) - \cos(N_k x)] \\ & - \sum_{r=1}^R D_{r(1)} K_1^a \tanh(N_r K_1^a H_3) [\cos(N_r \varepsilon) - \cos(N_r x)] \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \left(\frac{N_n}{N_m}\right) [\cos(N_m \varepsilon) - \cos(N_m x)] \exp\left[-(\lambda_{mn})^2 t\right] \\ & \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \left(\frac{N_v}{N_u}\right) [\cos(N_u \varepsilon) - \cos(N_u x)] \exp\left[-(\lambda_{uv})^2 t\right] \right\}. \quad (2.96) \end{aligned}$$

Also, the discharge per unit length through the side of the left ditch can be obtained as

$$Q_{leftside(1)}(t) = \left\{ K_{x_1} \int_0^{H_3} \left[\frac{\partial \phi_1(x, y, t)}{\partial x} \right]_{x=0} dy + K_{x_2} \int_{H_3}^h \left[\frac{\partial \phi_2(x, y, t)}{\partial x} \right]_{x=0} dy \right\}. \quad (2.97)$$

Solving the integrals in the above equation, we get

$$\begin{aligned}
Q_{leftside(1)}(t) = & K_{x_1} \left\{ \sum_{p=1}^P B_{p(1)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[(N_p / K_1^a) S \right]} \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(1)} \left(\frac{1}{K_1^a} \right) \coth \left[(N_q / K_1^a) S \right] \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(1)} \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& + \sum_{r=1}^R D_{r(1)} \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \left(\frac{N_m}{N_n} \right) \left[1 - \cos(N_n H_3) \right] \exp \left[-(\lambda_{mn})^2 t \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \left(\frac{N_u}{N_v} \right) \left[1 - \cos(N_v H_3) \right] \exp \left[-(\lambda_{uv})^2 t \right] \left. \right\} \\
& + K_{x_2} \left\{ \sum_{w=1}^W H_{w(1)} \left(\frac{1}{K_2^a} \right) - \left[\frac{(H_2 - H_1)(h - H_3)}{S} \right] \right. \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \exp \left[-(\lambda_{mn})^2 t \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \exp \left[-(\lambda_{uv})^2 t \right] \left. \right\} \quad (2.98)
\end{aligned}$$

Similarly, we evaluate the discharge per unit length through the side of the right ditch as

$$Q_{rightside(1)}(t) = \left\{ -K_{x_1} \int_0^{H_3} \left[\frac{\partial \phi_1(x, y, t)}{\partial x} \right]_{x=S} dy - K_{x_2} \int_{H_3}^h \left[\frac{\partial \phi_2(x, y, t)}{\partial x} \right]_{x=S} dy \right\}. \quad (2.99)$$

Solution of the above mentioned integrals [Eq. (2.99)] yields

$$\begin{aligned}
Q_{rightside(1)}(t) = & -K_{x_1} \left\{ \sum_{p=1}^P B_{p(1)} \left(\frac{1}{K_1^a} \right) \coth \left[(N_p / K_1^a) S \right] \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(1)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[(N_q / K_1^a) S \right]} \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(1)} \cos(N_k S) \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& + \sum_{r=1}^R D_{r(1)} \cos(N_r S) \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \left[1 - \cos(N_n H_3) \right] \exp \left[-(\lambda_{mn})^2 t \right] \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \left[1 - \cos(N_v H_3) \right] \exp \left[-(\lambda_{uv})^2 t \right] \Big\} \\
& - K_{x_2} \left\{ \sum_{w=1}^W H_{w(1)} \left(\frac{1}{K_2^a} \right) \cos(N_w S) - \left[\frac{(H_2 - H_1)(h - H_3)}{S} \right] \right. \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \exp \left[-(\lambda_{mn})^2 t \right] \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \exp \left[-(\lambda_{uv})^2 t \right] \right\}. \quad (2.100)
\end{aligned}$$

It should be noted that all these expressions for the transient state reduce to the steady state discharge expressions when t in them is allowed to increase to infinity – this, just like in the hydraulic head expressions, will remove the exponential terms in them and leave behind only the steady state terms in these expressions. Further, all the steady discharge expressions are exact for all possible combinations of parameters of Fig. 2.2 and are not dependent on Eqs. (2.29) and (2.30) [i.e., $K_{x_1}/S_{s1} = K_{x_2}/S_{s2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)] for their correctness.

Volume of water seeping through the top surface of the soil per unit length of the ditches in time T can also be estimated by integrating Eq. (2.93); the relevant expression for the same is

$$Vol_{top(1)} = \int_0^T Q_{top(1)}(t) dt, \quad (2.101)$$

where $Vol_{top(1)}$ represents the volume of water in question. Carrying out the integration in Eq. (2.101) after substituting $Q_{top(1)}(t)$ from Eq. (2.93), we get

$$\begin{aligned}
Vol_{top(1)} = & -K_{y_1} T \left\{ \sum_{p=1}^P B_{p(1)} K_1^a \left\{ \frac{\cosh \left[\left(N_p / K_1^a \right) (S - \varepsilon) \right] - \cosh \left[\left(N_p / K_1^a \right) \varepsilon \right]}{\sinh \left[\left(N_p / K_1^a \right) S \right]} \right\} \right. \\
& + \sum_{q=1}^Q C_{q(1)} K_1^a \left\{ \frac{\cosh \left[\left(N_q / K_1^a \right) (S - \varepsilon) \right] - \cosh \left[\left(N_q / K_1^a \right) \varepsilon \right]}{\sinh \left[\left(N_q / K_1^a \right) S \right]} \right\} \\
& + \sum_{k=1}^K E_{k(1)} K_1^a \operatorname{sech} \left(N_k K_1^a H_3 \right) \left\{ \cos \left(N_k \varepsilon \right) - \cos \left[N_k (S - \varepsilon) \right] \right\} \\
& \left. - \sum_{r=1}^R D_{r(1)} K_1^a \tanh \left(N_r K_1^a H_3 \right) \left\{ \cos \left(N_r \varepsilon \right) - \cos \left[N_r (S - \varepsilon) \right] \right\} \right\} \\
& - K_{y_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \left(\frac{N_n}{N_m} \right) \left\{ \cos \left(N_m \varepsilon \right) - \cos \left[N_m (S - \varepsilon) \right] \right\} \times \right. \\
& \left. \left\{ \frac{1 - \exp \left[-(\lambda_{mn})^2 T \right]}{(\lambda_{mn})^2} \right\} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \left(\frac{N_v}{N_u} \right) \left\{ \cos(N_u \varepsilon) - \cos[N_u (S - \varepsilon)] \right\} \times \\
& \left. \left[\frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right] \right\} \quad (2.102)
\end{aligned}$$

Similarly, the volume of water seeping into the left ditch drain per unit length of ditch in time T can be evaluated as

$$Vol_{leftside(1)} = \int_0^T Q_{leftside(1)}(t) dt, \quad (2.103)$$

where $Vol_{leftside(1)}$ denotes the volume of water in question. Simplifying the above equation using Eq. (2.98), we have

$$\begin{aligned}
Vol_{leftside(1)} = & K_{x_1} T \left\{ \sum_{p=1}^P B_{p(1)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh[(N_p/K_1^a)S]} [1 - \cos(N_p H_3)] \right. \\
& - \sum_{q=1}^Q C_{q(1)} \left(\frac{1}{K_1^a} \right) \coth[(N_q/K_1^a)S] [1 - \cos(N_q H_3)] \\
& + \sum_{k=1}^K E_{k(1)} \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& \left. + \sum_{r=1}^R D_{r(1)} \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \right\} \\
& + K_{x_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \left(\frac{N_m}{N_n} \right) [1 - \cos(N_n H_3)] \left[\frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right] \right\} \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \left(\frac{N_u}{N_v} \right) [1 - \cos(N_v H_3)] \left[\frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right] \left. \right\} \\
& + K_{x_2} T \left\{ \sum_{w=1}^W H_{w(1)} \left(\frac{1}{K_2^a} \right) - \left[\frac{(H_2 - H_1)(h - H_3)}{S} \right] \right\} \\
& + K_{x_2} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \left[\frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right] \right\} \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \left[\frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right] \left. \right\} \quad (2.104)
\end{aligned}$$

And water volume seeping through per unit length of the right ditch in time T can be estimated as

$$Vol_{rightside(1)} = \int_0^T Q_{rightside(1)}(t) dt. \quad (2.105)$$

Simplifying the above expression in relying on Eq. (2.100), we get

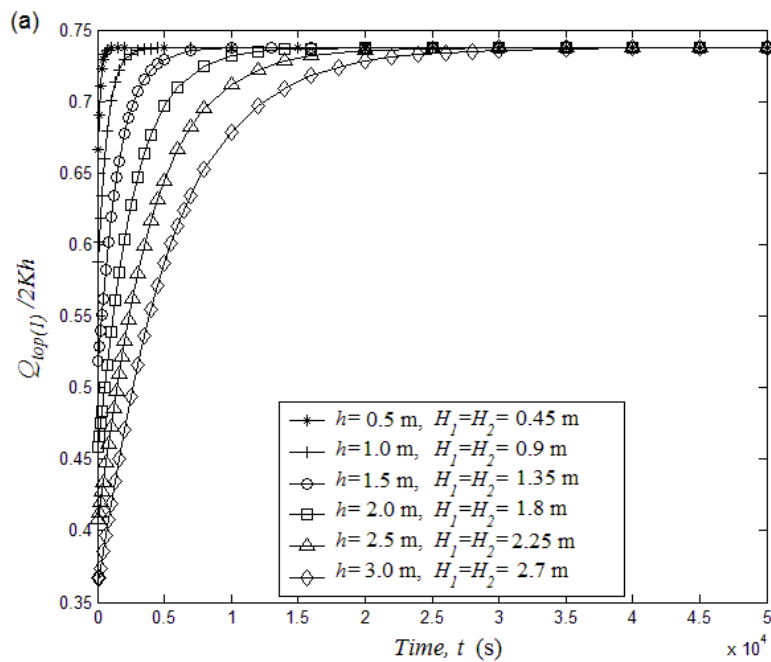
$$\begin{aligned}
Vol_{rightside(1)} = & -K_{x_1} T \left\{ \sum_{p=1}^P B_{p(1)} \left(\frac{1}{K_1^a} \right) \coth \left[\left(N_p / K_1^a \right) S \right] \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(1)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[\left(N_q / K_1^a \right) S \right]} \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(1)} \cos(N_k S) \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& \left. + \sum_{r=1}^R D_{r(1)} \cos(N_r S) \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \right\} \\
& - K_{x_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \left[1 - \cos(N_n H_3) \right] \left\{ \frac{1 - \exp \left[-(\lambda_{mn})^2 T \right]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \left[1 - \cos(N_v H_3) \right] \left\{ \frac{1 - \exp \left[-(\lambda_{uv})^2 T \right]}{(\lambda_{uv})^2} \right\} \right\} \\
& - K_{x_2} T \left\{ \sum_{w=1}^W H_{w(1)} \left(\frac{1}{K_2^a} \right) \cos(N_w S) - \left[\frac{(H_2 - H_1)(h - H_3)}{S} \right] \right\} \\
& - K_{x_2} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(1)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \left\{ \frac{1 - \exp \left[-(\lambda_{mn})^2 T \right]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(1)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \left\{ \frac{1 - \exp \left[-(\lambda_{uv})^2 T \right]}{(\lambda_{uv})^2} \right\} \right\}. \quad (2.106)
\end{aligned}$$

Before we end our theoretical analysis for the flow problem of Fig. 2.2, we would like to point out that our proposed approximate transient solution for the two-layered problem would become exact if the problem is being reduced to a single-layered one by equating the directional hydraulic conductivities and specific storage of the layers as same, i.e., by making $K_{x_1} = K_{x_2}$, $K_{y_1} = K_{y_2}$ and $S_{s_1} = S_{s_2}$. In fact, with these substitutions in place, the problem of Fig. 2.2 reduces to that of Barua and Alam's (2013) ditch drainage problem; however, it should be noted that the series involved in these two solutions of the same problem are quite different – this is understandable because Barua and Alam's solution is being inherently developed for a single-layered soil only and not for a multi-layered one whereas the solution presented here is for a two-layered soil and not for a single-layered one. Also, when our flow domain is being reduced to a single-layered soil, the hydraulic head functions for the layers [Eqs. (2.31) and (2.43)] will then anyway satisfy the interfacial conditions (IIIa) and (IIIb) for

all time t for any combination of the directional conductivities and specific storage of the drained soil. Thus, for such a situation, the requirements of Eqs. (2.29) and (2.30) [i.e., $K_{x_1} / S_{s_1} = K_{x_2} / S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)] are not necessary for the transient solution to work.

2.2.1.3 Verifications of the Proposed Solution

In an attempt to verify our proposed solution for the steady state case, a few checks are now being conducted with the experimental and theoretical results of others. In Fig. 2.3, we are showing variation of top discharge with time for a single-layered soil, where, as may be observed, the top discharges are expressed as ratios of hydraulic conductivity of soil and its thickness. The hydraulic conductivity values are assigned so as to ensure that the soil is isotropic and thereby reducing our multilayer solution to that of a single-layered one. Also, in our analysis, the ponding field above the surface of the soil is taken as zero and to make sure that the horizontal extent of the field is almost infinite, the spacing between the adjacent ditches is taken as 100 m. Further, in all the cases, the ditches are kept nearly empty by ensuring $H_1 = H_2 = 0.9h$ and $H_3 = 0.95h$. $Q_{top(t)} / 2Kh$ ratios are then obtained for different times by varying the thickness of the soil layer from 0.5 m to 3.0 m, incrementing it by 0.5 m for each plot.



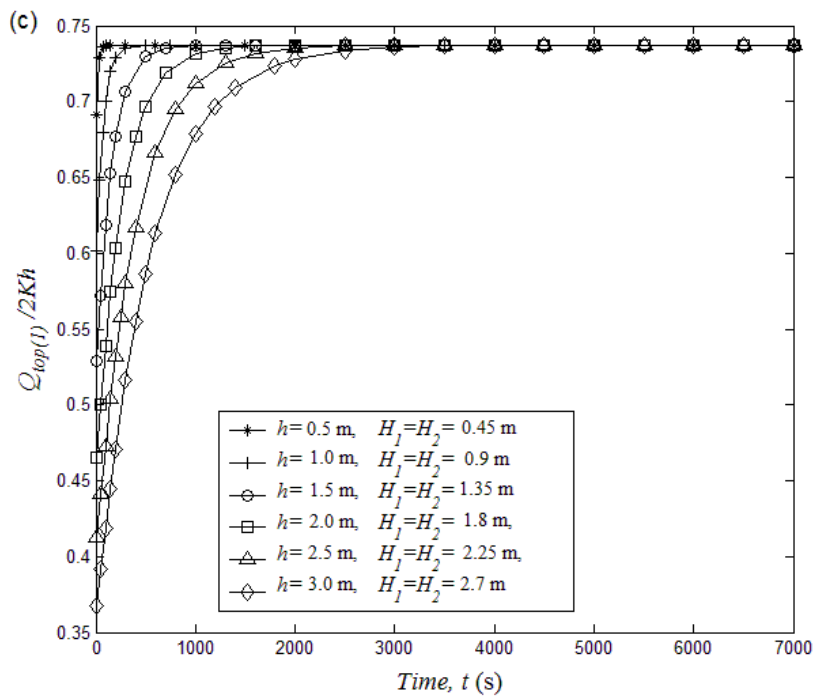
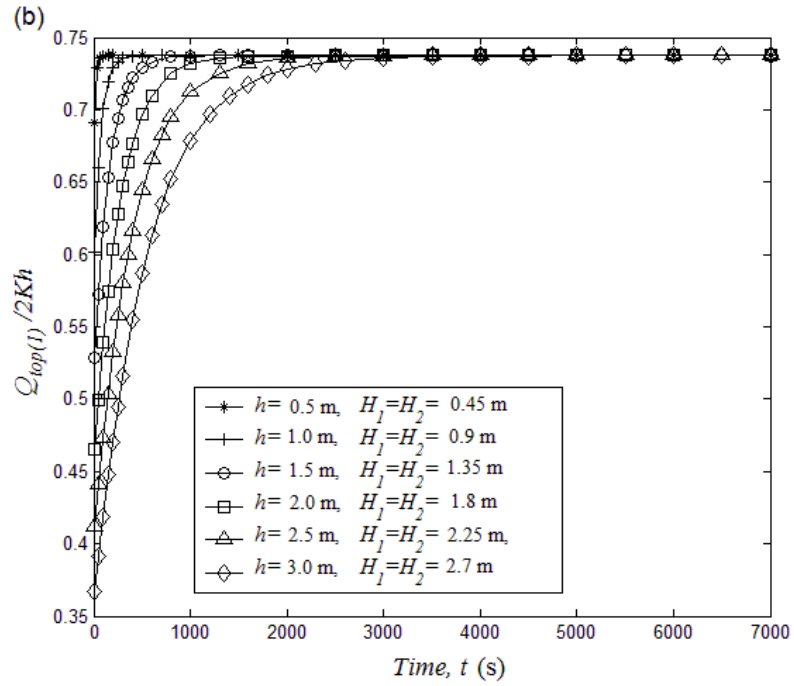


Fig. 2.3. Variation of $Q_{top(t)}/2Kh$ ratios with time as obtained from the proposed solution for different h and $H_1=H_2=0.9h$ values when the other parameters of the flow problem are considered as $S=100$ m, $\delta=0$ m, $H_3=0.95h$ and (a) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.05$ m/day, $S_{s_1}=S_{s_2}=0.001$ m⁻¹, and (b) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.5$ m/day, $S_{s_1}=S_{s_2}=0.001$ m⁻¹ and (c) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.05$ m/day and $S_{s_1}=S_{s_2}=0.0001$ m⁻¹

We observe that $Q_{top(1)} / 2Kh$ attains a steady state value of 0.737 for all the studied drainage configurations when the time variable in these ratios is allowed to increase continually. This value, as may be seen, is independent of the values of hydraulic conductivity of the soil in question. It is observed that this value is very close to the value of 0.743 as obtained from the analytical works of Fukuda (1957) for the same situation (Fig. 2.3). A slight difference in the value of this ratio vis-a vis ours is understandable as Fukuda (1957) solution is strictly valid for a drainage system with perfectly empty ditches only and not when some water level is there in them whereas in our comparison model of Fig. 2.3, the ditches, as may be noticed, are having a water depth specified as $H_1 = H_2 = 0.9h$ and are thus not totally empty. Even then, as mentioned before, our value of 0.737 is found to be in close agreement with those obtained from the analytical model of Fukuda, differing in places of decimals only. It also is close to the experimental result of 0.72 as obtained by Fukuda from his ditch drainage experimental results related to steady subsurface drainage of a ponded ditch drainage system. Plots of $Q_{top(1)} / 2Kh$ versus $(h - H_1) / h$ are also drawn for a few drainage situations of Fig. 2.2 and the same are then compared with the parallel plots obtained from Youngs' (1994) solution for the steady state case. Fig. 2.4 shows such a comparison. As may be observed, here also a perfect matching of our results with those obtained from Youngs' solution are realized thereby showing that the proposed solution is being rightly developed. It is interesting to note that as the level of water in the drains approach zero, $Q_{top(1)} / 2Kh$ ratio as obtained from Youngs' solution also approach 0.743 (Fig. 2.4) when the spacing between the drains is given a large value (here taken as 100 m).

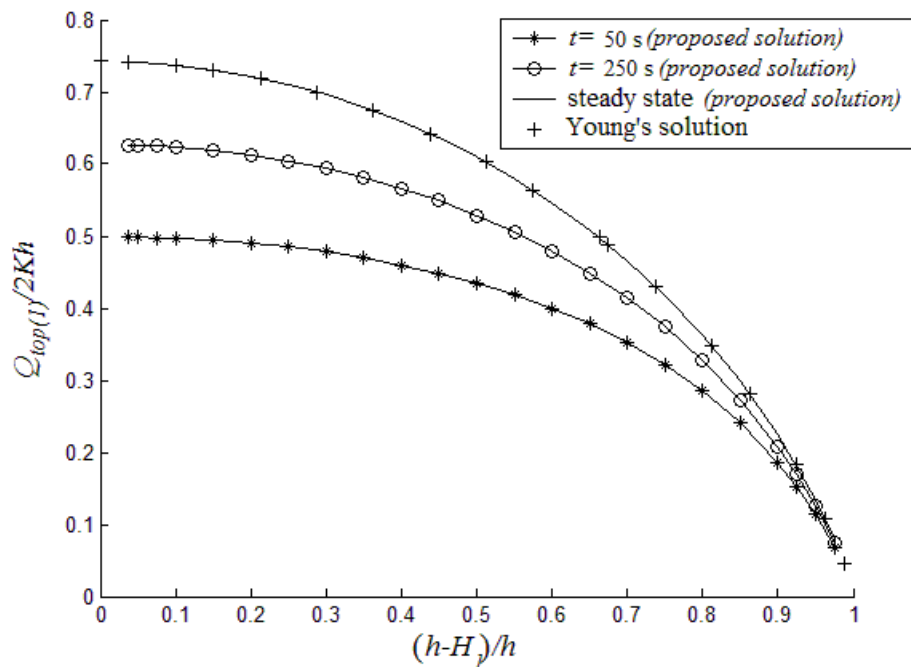


Fig. 2.4. Plots of $Q_{top(1)} / 2Kh$ versus $(h - H_1) / h$ as obtained from the proposed solution for different times of simulation of the system with the corresponding steady state plot as obtained from Youngs' (1994) single-layered solution of the problem for isotropic soils when the ponding depth is considered as zero and the other flow parameters of the problem are taken as $S = 100$ m, $h = 2.0$ m, $S_{s_1} = S_{s_2} = 0.0001 \text{ m}^{-1}$ and $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$

As a further check on our steady state solution, a comparison with Kirkham's (1965) solution is also being performed for a drainage setting as shown in Fig. 2.5. As can be seen, all our predicted contours are in perfect agreement with those obtained from Kirkham's solution, thereby verifying once again, the correctness of the developed model.

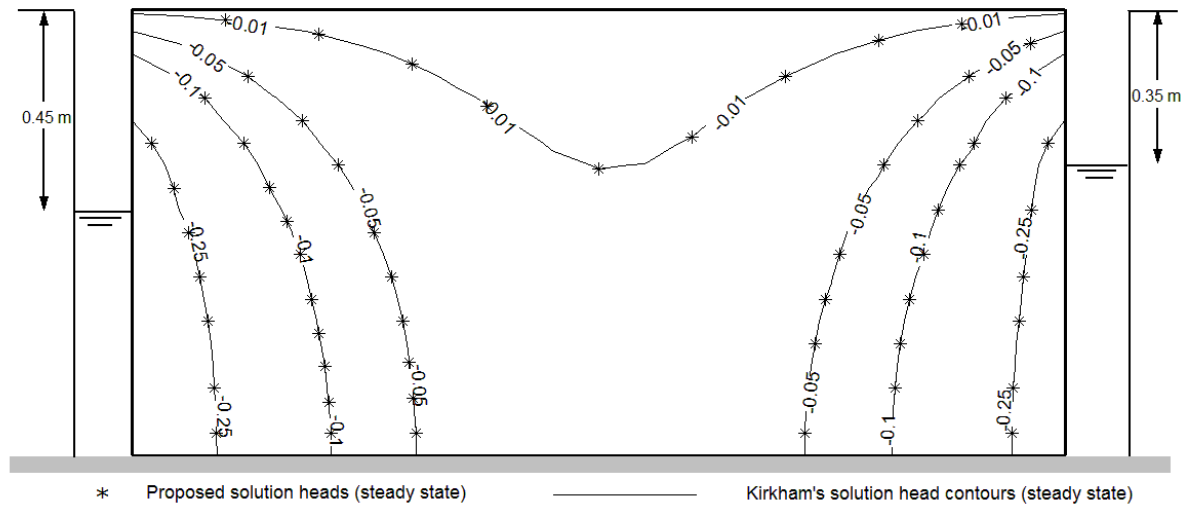


Fig. 2.5. Comparison of steady state hydraulic heads as obtained from the proposed solution with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.45$ m, $H_2 = 0.35$ m, $\delta = 0$ m and $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day

Finally, a comparison of the proposed solution for the transient situation is being carried out with the analytical works of Barua and Alam (2013) for a drainage configuration as shown in Fig. 2.6. Here also, as may be observed, the heads obtained from the developed model are matching quite closely with the ones obtained from the analytical works of Barua and Alam for both instants of time considered for study.

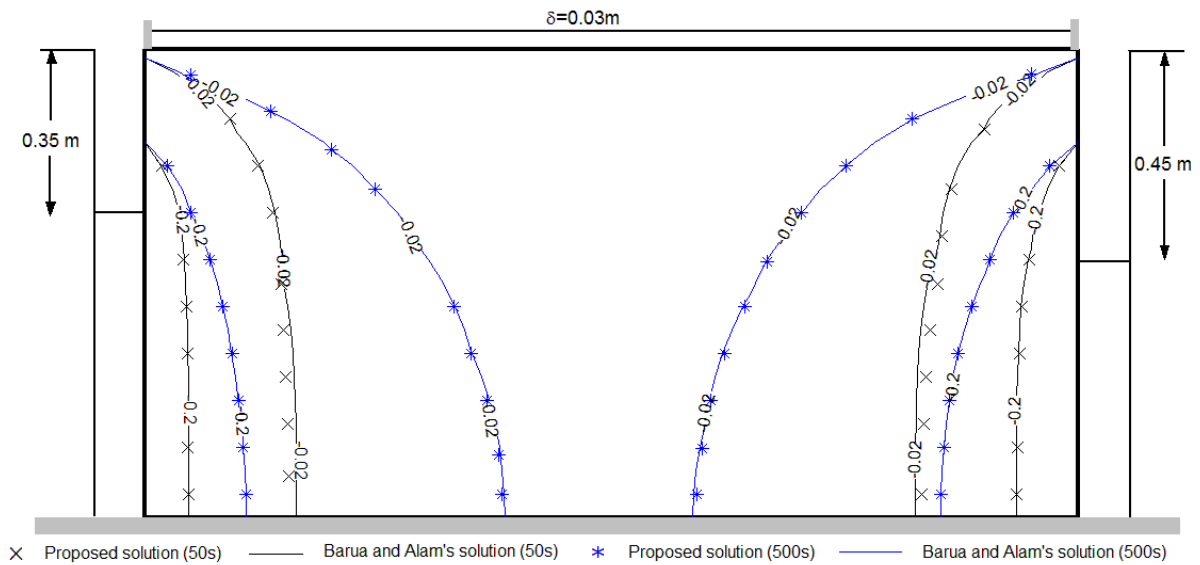


Fig. 2.6. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils at two different times of simulation of the system when the flow parameters for the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.35$ m, $H_2 = 0.45$ m, $\varepsilon = 0.05$ m, $\delta = 0.03$ m, $K_{x_1} = K_{x_2} = 0.15$ m/day, $K_{y_1} = K_{y_2} = 0.075$ m/day and $S_{s_1} = S_{s_2} = 0.001$ m⁻¹

2.2.1.4 MODFLOW Verification of the Proposed Solution

As a further check on our solution, we now carry out a numerical verification of the same utilizing the Processing MODFLOW (Chiang and Kinzelbach 2001) platform. Initially, verification is carried out for the steady state flow problem pertaining to a specific configuration of Fig. 2.2, where the flow parameters are considered as $S = 5$ m, $h = 1$ m, $H_3 = 0.6$ m, $H_1 = 0.45$ m, $H_2 = 0.40$ m, $\varepsilon = 0.05$ m, $\delta = 0.03$ m, $K_{x_1} = 1$ m/day, $K_{y_1} = 0.75$ m/day, $K_{x_2} = 2.5$ m/day and $K_{y_2} = 3.5$ m/day. A MODFLOW drainage model was created considering a cuboidal flow domain of cross sectional dimensions 15 m by 5 m and of thickness 1 m. It was subdivided into a grid mesh comprising of 151 rows, 101 columns and 22 layers. The row spacing assigned for the grid was 0.1 m, the column spacing was 0.05 m and the thickness of each vertical layer was assigned as 0.05 m. The cells of the bottommost layer, which happened to be the 22nd layer, were made inactive so as to simulate the impervious layer underlying the ponded field. The cells of the topmost layer were assigned a constant hydraulic head value of 0.03 m to replicate the ponding water depth at the top. The ditches were simulated by keeping the cells of 1st row, 151st row, 1st column and 101st column as fixed head cells for all the layers. The water level in the left ditch was simulated by assigning a value of 0 m head to the cells in the 1st column for the first layer and decreasing it by 0.05 m for every subsequent layers till it reached the 10th layer. After that, all the cells falling on the ditch was given a constant head value of -0.45 m. In a similar way, the heads for the right ditch were assigned to simulate a water level of 0.4 m in the ditch. In case

of the Northern and Southern ditch boundaries of the flow domain, a similar procedure of assigning heads is implemented for the 1st row and 151nd row, other than from the 10th layer, a head of -0.425 m, the average of -0.45 m and -0.4 m is assigned for the consequent layers. The top soil layer properties were incorporated in the model by assigning horizontal and vertical hydraulic conductivities of 1 m/day and 0.75 m/day, respectively, for all the active cells from the 1st layer to the 13th layer. From the 14th layer to the bottommost layer, the horizontal and vertical conductivities were allotted as 2.5 m/day and 3.5 m/day, respectively. With the above definition of the model in place, a MODFLOW simulation was carried out and the heads in the vertical plane passing through the 75th row were noted. This plane was considered because for the studied flow system, it was located farthest from the Northern and Southern boundaries of the model and hence was most likely to exhibit a close approximation of the two-dimensional ditch drainage scenario which had been proposed to be simulated. These heads were then compared with the ones obtained from our solution; Fig. 2.7 shows the comparison results. As can be seen, the analytically predicted heads are matching very closely with those obtained by numerical means thereby validating, once again, our developed solution for the steady state flow problem.

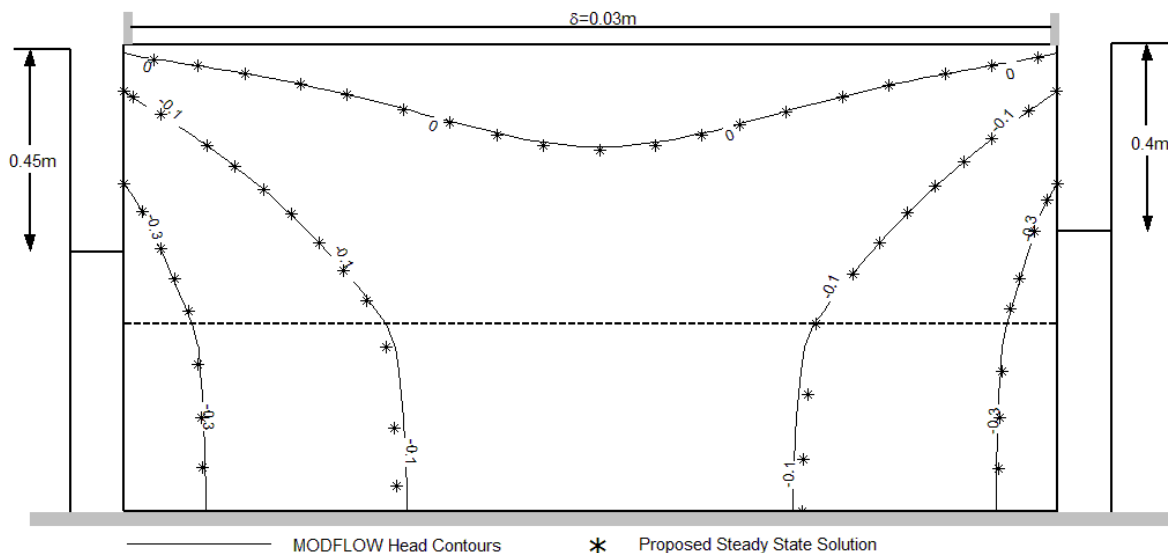


Fig. 2.7. Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding values as obtained from MODFLOW when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.6$ m, $H_1 = 0.45$ m, $H_2 = 0.4$ m, $\delta = 0.03$ m, $\varepsilon = 0.05$ m, $K_{x_1} = 1.0$ m/day, $K_{y_1} = 0.75$ m/day, $K_{x_2} = 2.5$ m/day and $K_{y_2} = 3.5$ m/day

We now propose to ascertain the veracity of the proposed transient flow model by building a MODFLOW model for a drainage scenario and then comparing the heads obtained from this model with the ones obtained from our solution for the same drainage setting. In this context, it is worth remembering that our transient solution is valid only when the conditions imposed by Eqs. (2.29) and (2.30) are being closely respected by a drainage scenario. The flow parameters for the transient ditch drainage flow problem taken into consideration for the MODFLOW simulation are as shown in Fig. 2.8. As can be seen, K_{x_1} , K_{x_2} , S_{s_1} and S_{s_2} for this drainage situation are respecting Eq. (2.29) in totality; further, as the vertical hydraulic

conductivities K_{y_1} and K_{y_2} for this situation are equal to each other and are quite close to zero (i.e., 0.001 m/day), Eq. (30) can then also be assumed to be approximately satisfied by this drainage configuration as well. The MODFLOW model was built in the same way like in the previous example. The topmost layer cells were assigned a constant hydraulic head value of 0.02 m to replicate the ponding water depth at the top. The head values in the 1st row, 151st row, 1st column and 101st column were assigned for the different layers of the model so as to simulate the depth of water in the drainage ditches. The initial value of hydraulic heads in all the active cells of the model was taken as zero. The horizontal hydraulic conductivity and specific storage values from the 1st layer till the 13th layer were assigned as 0.8 m/day and 0.003 m⁻¹, respectively. From the 14th layer to the 22nd layer, the horizontal hydraulic conductivity and specific storage values assigned to the model were 1.6 m/day and 0.006 m⁻¹, respectively. The vertical hydraulic conductivity for the whole model was kept as $K_{y_1} = K_{y_2} = 0.001$ m/day). With these inputs in place, a transient MODFLOW simulation was carried out and the numerically obtained heads on a vertical plane passing through the 75th row (as this plane lies at the furthest distance from the Northern and Southern boundaries of the model) at two different times of simulation (namely, 100 and 500 seconds, respectively) of the system were then compared with the corresponding values obtained from our solution of the drainage problem of Fig. 2.2. Fig. 2.8 shows the comparison details of the hydraulic heads for this situation at two different times of simulation of the system. As may be observed, at both the simulation times, the heads predicted by our solution are in very good agreement with the corresponding values obtained by MODFLOW thereby showing that our transient solution is also capable of giving accurate results so long as the constraints imposed by Eqs. (2.29) and (2.30) are being respected by a ponded drainage setting.

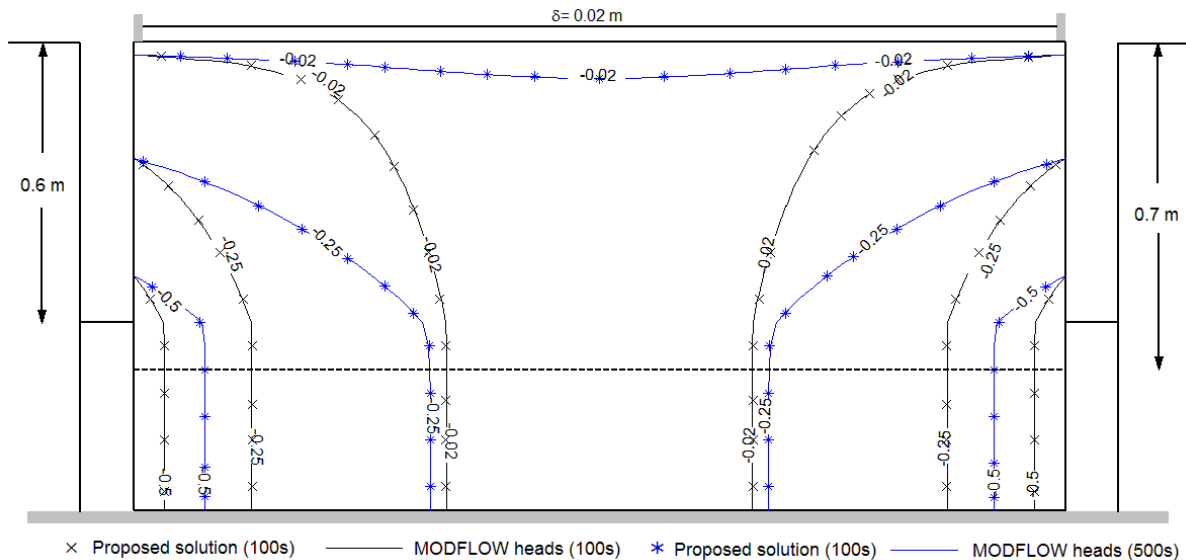


Fig. 2.8 Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.7$ m, $H_1 = 0.6$ m, $H_2 = 0.6$ m, $\delta = 0.02$ m, $\varepsilon = 0.05$ m, $K_{x_1} = .8$ m/day, $K_{x_2} = 1.6$ m/day, $K_{y_1} = K_{y_2} = 0.001$ m/day, $S_{s_1} = 0.003$ m⁻¹ and $S_{s_2} = 0.006$ m⁻¹

If we now relax the constraints imposed by Eq. (2.30) by a little bit and take $K_{y_1} = K_{y_2} = 0.01$ m/day instead of 0.001 m/day as in the previous case and the MODFLOW model is again run by keeping all the other parameters of the system same as before, then the head comparison contours would be as shown in Fig. 2.9. It is interesting to see from Fig. 2.9 that our approximate transient solution to the problem is predicting results very close to the numerically obtained values, even when Eq. (2.30) – one of the essential requirements for its validity – is been approximately satisfied, i.e., even when the vertical conductivity of the layers is relatively much above zero (i.e., $K_{y_1} = K_{y_2} = 0.01$ m/day).

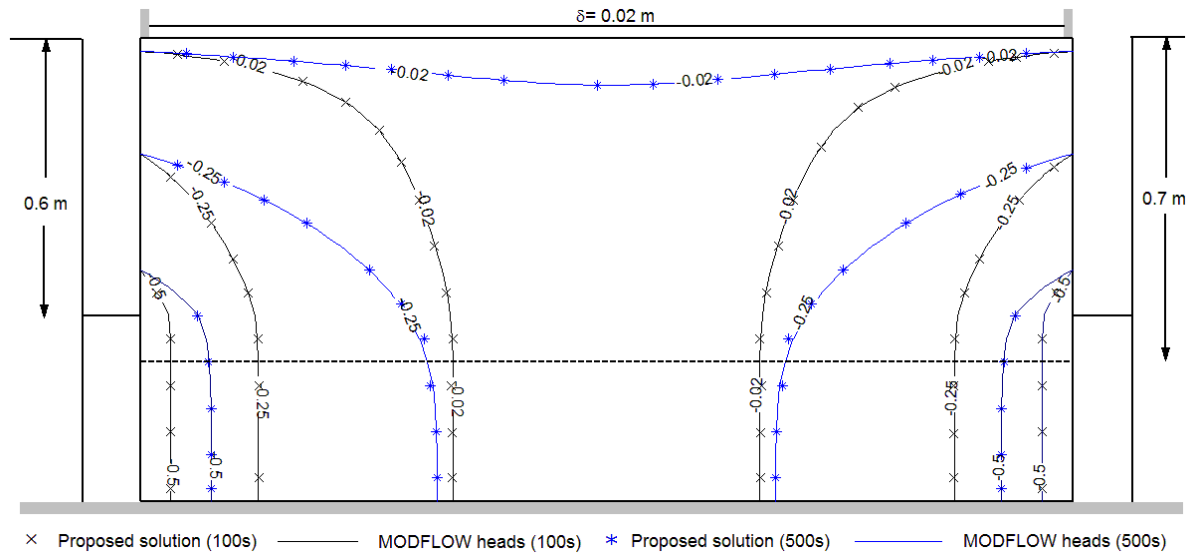


Fig 2.9. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.7$ m, $H_1 = 0.6$ m, $H_2 = 0.6$ m, $\delta = 0.02$ m, $\varepsilon = 0.05$ m, $K_{x_1} = 0.8$ m/day, $K_{x_2} = 1.6$ m/day, $K_{y_1} = K_{y_2} = 0.01$ m/day, $S_{s_1} = 0.003$ m⁻¹ and $S_{s_2} = 0.006$ m⁻¹

2.2.2 Level of Water in the Ditches is Different and Below the Boundary between the Soil Layers

Fig. 2.10 illustrates the height of water, and consequently, the boundary conditions at the drainage ditches for this particular boundary value problem.

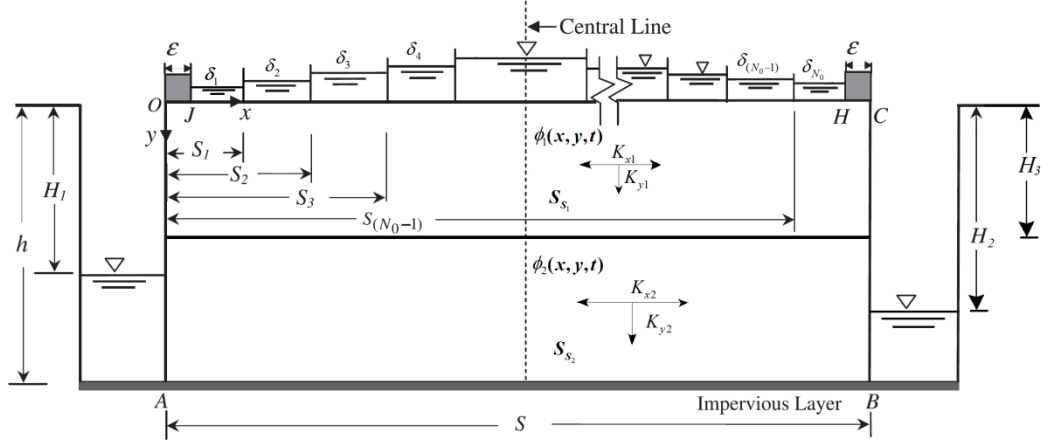


Fig. 2.10 Ditch drainage system for a two-layered soil when height of water in the ditches is below the bottom surface of the top soil layer

The boundary conditions specific to the ditch drain boundaries for this particular case are

$$\phi_{1(2)}(x, y, t > 0) = -y, \quad x = 0, \quad 0 < y \leq H_3, \quad (\text{X})$$

$$\phi_{2(2)}(x, y, t > 0) = -y, \quad x = 0, \quad H_3 \leq y \leq H_1, \quad (\text{XIa})$$

$$\phi_{2(2)}(x, y, t > 0) = -H_1, \quad x = 0, \quad H_1 \leq y < h, \quad (\text{XIb})$$

$$\phi_{1(2)}(x, y, t > 0) = -y, \quad x = S, \quad 0 < y \leq H_3, \quad (\text{XII})$$

$$\phi_{2(2)}(x, y, t > 0) = -y, \quad x = S, \quad H_3 \leq y \leq H_2, \quad (\text{XIIIa})$$

$$\phi_{2(2)}(x, y, t > 0) = -H_2, \quad x = S, \quad H_2 \leq y < h. \quad (\text{XIIIb})$$

In this case, the expressions for the hydraulic head functions for the top and the bottom layers, in view of Eqs. (2.24), (2.25) and (2.26), can be expressed as

$$\begin{aligned} \phi_{1(2)}(x, y, t) = & \sum_{p=1}^P B_{p(2)} \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin(N_p y) \\ & + \sum_{q=1}^Q C_{q(2)} \frac{\sinh\left[\left(N_q / K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin(N_q y) \\ & + \sum_{k=1}^K E_{k(2)} \frac{\sinh(N_k K_1^a y)}{\cosh(N_k K_1^a H_3)} \sin(N_k x) \\ & + \sum_{r=1}^R D_{r(2)} \frac{\cosh\left[N_r K_1^a (H_3 - y)\right]}{\cosh(N_r K_1^a H_3)} \sin(N_r x) \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \sin(N_m x) \sin(N_n y) \exp\left[-(\lambda_{mn})^2 t\right] \\ & + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \sin(N_u x) \sin(N_v y) \exp\left[-(\lambda_{uv})^2 t\right], \end{aligned} \quad (2.107)$$

where

$$N_p = \left[\left(\frac{1-2p}{2} \right) \left(\frac{\pi}{H_3} \right) \right], \quad (2.108)$$

$$N_q = \left[\left(\frac{1-2q}{2} \right) \left(\frac{\pi}{H_3} \right) \right], \quad (2.109)$$

$$N_k = \left(\frac{k\pi}{S} \right), \quad (2.110)$$

$$N_r = \left(\frac{r\pi}{S} \right), \quad (2.111)$$

$$K_1^a = \sqrt{\left(\frac{K_{x_1}}{K_{y_1}} \right)}, \quad (2.112)$$

$$N_m = \left(\frac{m\pi}{S} \right), \quad (2.113)$$

$$N_n = \left[\left(\frac{1-2n}{2} \right) \frac{\pi}{h} \right], \quad (2.114)$$

$$\lambda_{mn}^2 = \left[N_m^2 \left(\frac{K_{x_1}}{S_{s_1}} \right) + N_n^2 \left(\frac{K_{y_1}}{S_{s_1}} \right) \right], \quad (2.115)$$

$$N_u = \left(\frac{u\pi}{S} \right), \quad (2.116)$$

$$N_v = \left[\left(\frac{1-2v}{2} \right) \frac{\pi}{h} \right], \quad (2.117)$$

$$\lambda_{uv}^2 = \left[N_u^2 \left(\frac{K_{x_2}}{S_{s_2}} \right) + N_v^2 \left(\frac{K_{y_2}}{S_{s_2}} \right) \right] \quad (2.118)$$

and $B_{p(2)}$, $C_{q(2)}$, $E_{k(2)}$, $D_{r(2)}$, $A_{mn(2)}$ and $Z_{uv(2)}$ are constants and

$$\begin{aligned} \phi_{2(2)}(x, y, t) = & \sum_{w=1}^W H_{w(2)} \frac{\cosh \left[\left(N_w K_2^a \right) (h-y) \right]}{\sinh \left[\left(N_w K_2^a \right) (h-H_3) \right]} \sin(N_w x) \\ & + \sum_{i=1}^I F_{i(2)} \frac{\sinh \left[\left(N_i / K_2^a \right) x \right]}{\sinh \left[\left(N_i / K_2^a \right) S \right]} \sin \left[N_i (y-H_3) \right] \\ & + \sum_{j=1}^J G_{j(2)} \frac{\sinh \left[\left(N_j / K_2^a \right) (S-x) \right]}{\sinh \left[\left(N_j / K_2^a \right) S \right]} \sin \left[N_j (y-H_3) \right] - H_3 \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \sin(N_m x) \sin(N_n y) \exp \left[-(\lambda_{mn})^2 t \right] \\ & + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \sin(N_u x) \sin(N_v y) \exp \left[-(\lambda_{uv})^2 t \right], \end{aligned} \quad (2.119)$$

where

$$N_w = \left(\frac{w\pi}{S} \right), \quad (2.120)$$

$$N_i = \left[\left(\frac{1-2i}{2} \right) \left(\frac{\pi}{h-H_3} \right) \right], \quad (2.121)$$

$$N_j = \left[\left(\frac{1-2j}{2} \right) \left(\frac{\pi}{h-H_3} \right) \right], \quad (2.122)$$

$$K_2^a = \sqrt{\left(\frac{K_{x_2}}{K_{y_2}} \right)} \quad (2.123)$$

and $H_{w(2)}$, $F_{i(2)}$ and $G_{j(2)}$ are constants. We proceed to obtain the expressions for velocity distribution in the x - and y - directions as before. The directional velocity distribution functions, namely $V_{x1(2)}$ and $V_{y1(2)}$, for the top soil layer remain identical to the ones mentioned in Eqs. (2.52) and (2.53); however, the relevant Fourier coefficients will now be $B_{p(2)}$, $C_{q(2)}$, $E_{k(2)}$, $D_{r(2)}$, $A_{mn(2)}$ and $Z_{uv(2)}$ (the expressions for these are given in the pages to follow) instead of $B_{p(1)}$, $C_{q(1)}$, $E_{k(1)}$, $D_{r(1)}$, $A_{mn(1)}$ and $Z_{uv(1)}$ as worked out for the previous problem. Also, for the bottom layer, the directional velocity functions for this flow configuration can be written as

$$\begin{aligned} V_{x2(2)} = & -K_{x_2} \left\{ \sum_{w=1}^W H_{w(2)} \frac{\cosh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} N_w \cos(N_w x) \right. \\ & + \sum_{i=1}^I F_{i(2)} \frac{\left(N_i / K_2^a \right) \cosh \left[(N_i / K_2^a)x \right]}{\sinh \left[(N_i / K_2^a)S \right]} \sin \left[N_i (y-H_3) \right] \\ & - \sum_{j=1}^J G_{j(2)} \frac{\left(N_j / K_2^a \right) \cosh \left[(N_j / K_2^a)(S-x) \right]}{\sinh \left[(N_j / K_2^a)S \right]} \sin \left[N_j (y-H_3) \right] \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} N_m \cos(N_m x) \sin(N_n y) \exp \left[-(\lambda_{mn})^2 t \right] \\ & \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} N_u \cos(N_u x) \sin(N_v y) \exp \left[-(\lambda_{uv})^2 t \right] \right\} \quad (2.124) \end{aligned}$$

and

$$\begin{aligned} V_{y2(2)} = & -K_{y_2} \left\{ - \sum_{w=1}^W H_{w(2)} \frac{\left(N_w K_2^a \right) \sinh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} \sin(N_w x) \right. \\ & \left. + \sum_{i=1}^I F_{i(2)} \frac{\sinh \left[(N_i / K_2^a)x \right]}{\sinh \left[(N_i / K_2^a)S \right]} N_i \cos \left[N_i (y-H_3) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^J G_{j(2)} \frac{\sinh\left[\left(N_j / K_2^a\right)(S-x)\right]}{\sinh\left[\left(N_j / K_2^a\right)S\right]} N_j \cos\left[N_j(y-H_3)\right] \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \sin(N_m x) N_n \cos(N_n y) \exp\left[-\left(\lambda_{mn}\right)^2 t\right] \\
& + \left. \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \sin(N_u x) N_v \cos(N_v y) \exp\left[-\left(\lambda_{uv}\right)^2 t\right] \right\}. \tag{2.125}
\end{aligned}$$

2.2.2.1 Determination of the Steady State Solution Coefficients (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 2.10)

Substituting boundary condition (X) in Eq. (2.107), we have

$$\sum_{q=1}^Q C_{q(2)} \sin(N_q y) = -y, \quad 0 < y \leq H_3,$$

We expand the Fourier series with $Q \rightarrow \infty$ and evaluate $C_{q(2)}$ to obtain

$$C_{q(2)} = \left(\frac{2}{H_3}\right) \left[\int_0^{H_3} -y \sin(N_q y) dy \right]. \tag{2.126}$$

Evaluating the integrals in the above expression, we get

$$C_{q(2)} = \left(\frac{2}{H_3}\right) \left\{ \left[y \frac{\cos(N_q y)}{N_q} \right]_0^{H_3} - \left[\frac{\sin(N_q y)}{N_q^2} \right]_0^{H_3} \right\}. \tag{2.127}$$

The final expression for $C_{q(2)}$ is thus

$$C_{q(2)} = \left(\frac{2}{H_3}\right) \left[\frac{-\sin(N_q H_3)}{N_q^2} \right]. \tag{2.128}$$

Similarly, by making use of boundary condition (XII), we arrive at the following relationship involving the Fourier coefficients $B_{p(2)}$

$$\sum_{p=1}^P B_{p(2)} \sin(N_p y) = -y, \quad 0 < y \leq H_3,$$

Expanding the Fourier series and evaluating for $B_{p(2)}$, we get

$$B_{p(2)} = \left(\frac{2}{H_3}\right) \left[\frac{-\sin(N_p H_3)}{N_p^2} \right]. \tag{2.129}$$

The expression for $D_{r(2)}$ will be the same as that of $D_{r(1)}$ since it is evaluated based on the same boundary conditions (IVa), (IVb) and (IVc). To determine $G_{j(2)}$, we substitute boundary conditions (XIa) and (XIb) in Eq. (2.119) – this results in the following expressions

$$\sum_{j=1}^J G_{j(2)} \sin\left[N_j(y-H_3)\right] - H_3 = -y, \quad H_3 \leq y \leq H_1,$$

$$\sum_{j=1}^J G_{j(2)} \sin[N_j(y-H_3)] - H_3 = -H_1, \quad H_1 \leq y < h.$$

The coefficients $G_{j(2)}$ can thus be determined as

$$G_{j(2)} = \left(\frac{2}{h-H_3} \right) \times \left\{ \int_{H_3}^{H_1} -(y-H_3) \sin[N_j(y-H_3)] dy + \int_{H_1}^h -(H_1-H_3) \sin[N_j(y-H_3)] dy \right\}. \quad (2.130)$$

Evaluation of the above integrals gives $G_{j(2)}$ as

$$G_{j(2)} = \left(\frac{2}{h-H_3} \right) \left\{ \frac{-\sin[N_j(H_1-H_3)]}{N_j^2} \right\}. \quad (2.131)$$

Similarly, by imposing boundary conditions (XIIIa) and (XIIIb) on Eq. (2.119), we get

$$\sum_{i=1}^I F_{i(2)} \sin[N_i(y-H_3)] - H_3 = -y, \quad H_3 \leq y \leq H_2,$$

$$\sum_{i=1}^I F_{i(2)} \sin[N_i(y-H_3)] - H_3 = -H_2, \quad H_2 \leq y < h.$$

A Fourier run on the above equations gives an expression for $F_{i(2)}$ as

$$F_{i(2)} = \left(\frac{2}{h-H_3} \right) \times \left\{ \int_{H_3}^{H_2} -(y-H_3) \sin[N_i(y-H_3)] dy + \int_{H_2}^h -(H_2-H_3) \sin[N_i(y-H_3)] dy \right\}. \quad (2.132)$$

Working out the above integrals, we get the final expression for $F_{i(2)}$ as

$$F_{i(2)} = \left(\frac{2}{h-H_3} \right) \left\{ \frac{-\sin[N_i(H_2-H_3)]}{N_i^2} \right\}. \quad (2.133)$$

We now equate the steady state hydraulic head expressions for the top and the bottom layers at $y = H_3$ so as to satisfy boundary condition (IIIa) – this leads to the following relationship

$$\sum_{p=1}^P B_{p(2)} \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin\left(N_p H_3\right) + \sum_{q=1}^Q C_{q(2)} \frac{\sinh\left[\left(N_q / K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin\left(N_q H_3\right)$$

$$+ \sum_{k=1}^K E_{k(2)} \frac{\sinh\left(N_k K_1^a H_3\right)}{\cosh\left(N_k K_1^a H_3\right)} \sin\left(N_k x\right) + \sum_{r=1}^R D_{r(2)} \frac{1}{\cosh\left(N_r K_1^a H_3\right)} \sin\left(N_r x\right)$$

$$= \sum_{w=1}^W H_{w(2)} \frac{\cosh\left[N_w K_2^a (h-H_3)\right]}{\sinh\left[N_w K_2^a (h-H_3)\right]} \sin\left(N_w x\right) - H_3$$

We expand the Fourier series as in the previous problem and eventually obtain the following equation

$$\begin{aligned}
& E_{k(2)} \tanh(N_i K_1^a H_3) - H_{w(2)} \coth[N_i K_2^a (h - H_3)] \\
&= -D_{r(2)} \operatorname{sech}(N_i K_1^a H_3) - \left(\frac{2}{S}\right) H_3 \int_0^S \sin(N_i x) dx \\
& - \left(\frac{2}{S}\right) \sum_{p=1}^P B_{p(2)} \sin(N_p H_3) \int_0^S \frac{\sinh\left[\frac{(N_p/K_1^a)x}{S}\right]}{\sinh\left[\frac{(N_p/K_1^a)S}{S}\right]} \sin(N_i x) dx \\
& - \left(\frac{2}{S}\right) \sum_{q=1}^Q C_{q(2)} \sin(N_q H_3) \int_0^S \frac{\sinh\left[\frac{(N_q/K_1^a)(S-x)}{S}\right]}{\sinh\left[\frac{(N_q/K_1^a)S}{S}\right]} \sin(N_i x) dx. \tag{2.134}
\end{aligned}$$

Upon solving the integrals of the above equation, we get

$$\begin{aligned}
& E_{k(2)} \tanh(N_i K_1^a H_3) - H_{w(2)} \coth[N_i K_2^a (h - H_3)] \\
&= -D_{r(2)} \operatorname{sech}(N_i K_1^a H_3) - \left(\frac{2}{S}\right) H_3 \left[\frac{1 - \cos(N_i S)}{N_i} \right] \\
& - \left(\frac{2}{S}\right) \sum_{p=1}^P B_{p(2)} \sin(N_p H_3) \left[\frac{N_i^2}{N_i^2 + (N_p/K_1^a)^2} \right] \left[\frac{-\cos(N_i S)}{N_i} \right] \\
& - \left(\frac{2}{S}\right) \sum_{q=1}^Q C_{q(2)} \sin(N_q H_3) \left[\frac{N_i}{N_i^2 + (N_q/K_1^a)^2} \right], \tag{2.135}
\end{aligned}$$

where $i_1 = w = r = k$ and $P = Q \rightarrow \infty$. With a view to now satisfy boundary condition (IIIb), we next equate the steady state hydraulic flux expressions in the y – direction for the two soil layers at $y = H_3$ – this leads to the following relationship

$$\begin{aligned}
& -K_{y_1} \sum_{k=1}^K E_{k(2)} (N_k K_1^a) \frac{\cosh\left[\frac{(N_k K_1^a) H_3}{S}\right]}{\cosh\left[\frac{(N_k K_1^a) H_3}{S}\right]} \sin(N_k x) \\
&= K_{y_2} \sum_{w=1}^W H_{w(2)} (N_w K_2^a) \frac{\sinh\left[\frac{(N_w K_2^a)(h - H_3)}{S}\right]}{\sinh\left[\frac{(N_w K_2^a)(h - H_3)}{S}\right]} \sin(N_w x) \\
& - K_{y_2} \sum_{i=1}^I F_{i(2)} N_i \frac{\sinh\left[\frac{(N_i/K_2^a)x}{S}\right]}{\sinh\left[\frac{(N_i/K_2^a)S}{S}\right]} - K_{y_2} \sum_{j=1}^J G_{j(2)} N_j \frac{\sinh\left[\frac{(N_j/K_2^a)(S-x)}{S}\right]}{\sinh\left[\frac{(N_j/K_2^a)S}{S}\right]}.
\end{aligned}$$

After making a Fourier run in the above expression as had been done to obtain Eq. (2.134), we find

$$\begin{aligned}
& K_{y_1} (N_k K_1^a) E_{k(2)} + K_{y_2} (N_w K_2^a) H_{w(2)} \\
&= \left(\frac{2}{S}\right) K_{y_2} \sum_{i=1}^I F_{i(2)} N_i \int_0^S \frac{\sinh\left[\frac{(N_i/K_2^a)x}{S}\right]}{\sinh\left[\frac{(N_i/K_2^a)S}{S}\right]} \sin(N_i x) dx
\end{aligned}$$

$$+ \left(\frac{2}{S} \right) K_{y_2} \sum_{j=1}^J G_{j(2)} N_j \int_0^S \frac{\sinh \left[\left(N_j / K_2^a \right) (S - x) \right]}{\sinh \left[\left(N_j / K_2^a \right) S \right]} \sin \left(N_{i_1} x \right) dx. \quad (2.136)$$

Evaluating the above the integrals, we finally have

$$\begin{aligned} & K_{y_1} \left(N_k K_1^a \right) E_{k(2)} + K_{y_2} \left(N_w K_2^a \right) H_{w(2)} \\ &= \left(\frac{2}{S} \right) K_{y_2} \left\{ \sum_{i=1}^I F_{i(2)} N_i \left[\frac{-N_{i_1} \cos \left(N_{i_1} S \right)}{N_{i_1}^2 + \left(N_i / K_2^a \right)^2} \right] + \sum_{j=1}^J G_{j(2)} N_j \left[\frac{N_{i_1}}{N_{i_1}^2 + \left(N_j / K_2^a \right)^2} \right] \right\}, \end{aligned} \quad (2.137)$$

where $i_1 = k = w$ and $I = J \rightarrow \infty$.

On solving the linear equations resulting from Eqs. (2.135) and (2.137), the coefficients $E_{k(2)}$ and $H_{w(2)}$ can then thus be worked out for a chosen drainage setting of Fig. 2.10.

2.2.2.2 Determination of the Transient Solution Coefficients [transient state solution is approximate and is applicable only when conductivity and specific storage of the layers of Fig. 2.10 satisfy $K_{x_1} / S_{s_1} = K_{x_2} / S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)]

The expression for the coefficients $A_{mn(2)}$ (with the Fourier coefficients $B_{p(2)}$, $C_{q(2)}$, $E_{k(2)}$ and $D_{r(2)}$) is identical to that of $A_{mn(1)}$; however, coefficients $Z_{uv(2)}$ will now have a different expression than that of $Z_{uv(1)}$ as the steady state head distribution of the bottom layer is now different from that of the previous case. For this drainage setting, $Z_{uv(2)}$ can be expressed as

$$\begin{aligned} Z_{uv(2)} = & \left(\frac{4}{Sh} \right) \left\{ - \sum_{w=1}^W H_{w(2)} \int_{H_3}^h \frac{\cosh \left[\left(N_w K_2^a \right) (h - y) \right]}{\sinh \left[\left(N_w K_2^a \right) (h - H_3) \right]} \sin \left(N_v y \right) dy \right. \\ & \times \int_0^S \sin \left(N_u x \right) \sin \left(N_w x \right) dx \\ & - \sum_{i=1}^I F_{i(2)} \int_0^S \frac{\sinh \left[\left(N_i / K_2^a \right) x \right]}{\sinh \left[\left(N_i / K_2^a \right) S \right]} \sin \left(N_u x \right) dx \\ & \times \int_{H_3}^h \sin \left[N_i \left(y - H_3 \right) \right] \sin \left(N_v y \right) dy \\ & - \sum_{j=1}^J G_{j(2)} \int_0^S \frac{\sinh \left[\left(N_j / K_2^a \right) (S - x) \right]}{\sinh \left[\left(N_j / K_2^a \right) S \right]} \sin \left(N_u x \right) dx \\ & \times \int_{H_3}^h \sin \left[N_j \left(y - H_3 \right) \right] \sin \left(N_v y \right) dy \\ & \left. + H_3 \int_0^S \sin \left(N_u x \right) dx \int_{H_3}^h \sin \left(N_v y \right) dy \right\} \end{aligned} \quad (2.138)$$

We evaluate the integrals in the above equation and then substitute the same in Eq. (2.138) above – this gives an expression for $Z_{uv(2)}$ as

$$\begin{aligned}
Z_{uv(2)} = & \left(\frac{4}{Sh} \right) \left\{ -H_{w(2)} \left(\frac{S}{2} \right) \left[\frac{N_v^2}{N_v^2 + (N_w K_2^a)^2} \right] \right. \\
& \times \left\{ \coth \left[(N_w K_2^a)(h - H_3) \right] \frac{\cos(N_v H_3)}{N_v} + \left[\frac{(N_w K_2^a)}{N_v^2} \right] \sin(N_v H_3) \right\} \\
& - \sum_{i=1}^I F_{i(2)} \left[\frac{N_u^2}{N_u^2 + (N_i / K_2^a)^2} \right] \left[\frac{-\cos(N_u S)}{N_u} \right] \times I^{(3)} \\
& - \sum_{j=1}^J G_{j(2)} \left[\frac{N_u}{N_u^2 + (N_j / K_2^a)^2} \right] \times I^{(4)} \\
& \left. + H_3 \left[\frac{1 - \cos(N_u S)}{N_u} \right] \left[\frac{\cos(N_v H_3)}{N_v} \right] \right\}, \tag{2.139}
\end{aligned}$$

where $u = w$, $I = J \rightarrow \infty$,

$$\begin{aligned}
I^{(3)} = & \left\{ \cos(N_v H_3) \left[\frac{h - H_3}{2} + \frac{\sin(2N_v H_3)}{4N_v} \right] \right. \\
& \left. - \sin(N_v H_3) \left[\frac{\cos(2N_v H_3) - \cos(2N_v h)}{4N_v} \right] \right\}, \quad N_v = N_i, \\
= & \left[\frac{\sin(N_i h - N_i H_3 - N_v h)}{2(N_i - N_v)} + \frac{\sin(N_v H_3)}{2(N_i - N_v)} \right. \\
& \left. + \frac{\sin(N_v H_3)}{2(N_i + N_v)} - \frac{\sin(N_i h - N_i H_3 + N_v h)}{2(N_i + N_v)} \right], \quad N_i \neq N_v \tag{2.140}
\end{aligned}$$

and

$$\begin{aligned}
I^{(4)} = & \left\{ \cos(N_v H_3) \left[\frac{h - H_3}{2} + \frac{\sin(2N_v H_3) - \sin(2N_v h)}{4N_v} \right] \right. \\
& \left. - \sin(N_v H_3) \left[\frac{\cos(2N_v H_3) - \cos(2N_v h)}{4N_v} \right] \right\}, \quad N_j = N_v, \\
= & \left[\frac{\sin(N_j h - N_j H_3 - N_v h)}{2(N_j - N_v)} + \frac{\sin(N_v H_3)}{2(N_j - N_v)} \right. \\
& \left. + \frac{\sin(N_v H_3)}{2(N_j + N_v)} - \frac{\sin(N_j h - N_j H_3 + N_v h)}{2(N_j + N_v)} \right], \quad N_j \neq N_v. \tag{2.141}
\end{aligned}$$

We can easily note that the expressions for the functions $Q_{iop(2)}$, $Q_{iopx(2)}$ and $Vol_{iop(2)}$ (with the Fourier coefficients as $B_{p(2)}$, $C_{q(2)}$, $E_{k(2)}$, $D_{r(2)}$, $A_{mn(2)}$ and $Z_{uv(2)}$) will be identical to that

of functions $Q_{top(1)}$, $Q_{topx(1)}$ and $Vol_{top(1)}$ in the previous case [i.e., as given by Eqs. (2.93), (2.96) and (2.100), respectively]. However, the expressions for $Q_{leftside(2)}$, $Q_{rightside(2)}$, $Vol_{leftside(2)}$ and $Vol_{rightside(2)}$ will differ from that of the previous situation as the velocity expressions for the bottom layer are now different from that of the previous case and hence need be estimated. They are determined as

$$\begin{aligned}
Q_{leftside(2)}(t) = & K_{x_1} \left\{ \sum_{p=1}^P B_{p(2)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[\left(N_p / K_1^a \right) S \right]} \left[1 - \cos \left(N_p H_3 \right) \right] \right. \\
& - \sum_{q=1}^Q C_{q(2)} \left(\frac{1}{K_1^a} \right) \coth \left[\left(N_q / K_1^a \right) S \right] \left[1 - \cos \left(N_q H_3 \right) \right] \\
& + \sum_{k=1}^K E_{k(2)} \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh \left(N_k K_1^a H_3 \right) - 1}{\cosh \left(N_k K_1^a H_3 \right)} \right] \\
& + \sum_{r=1}^R D_{r(2)} \left(\frac{1}{K_1^a} \right) \tanh \left(N_r K_1^a H_3 \right) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \left(\frac{N_m}{N_n} \right) \left[1 - \cos \left(N_n H_3 \right) \right] \exp \left[- \left(\lambda_{mn} \right)^2 t \right] \\
& + \left. \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \left(\frac{N_u}{N_v} \right) \left[1 - \cos \left(N_v H_3 \right) \right] \exp \left[- \left(\lambda_{uv} \right)^2 t \right] \right\} \\
& + K_{x_2} \left\{ \sum_{w=1}^W H_{w(2)} \left(\frac{1}{K_2^a} \right) + \sum_{i=1}^I F_{i(2)} \left(\frac{1}{K_2^a} \right) \frac{1}{\sinh \left[\left(N_i / K_2^a \right) S \right]} \right. \\
& - \sum_{j=1}^J G_{j(2)} \left(\frac{1}{K_2^a} \right) \coth \left[\left(N_j / K_2^a \right) S \right] \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \left(\frac{N_m}{N_n} \right) \cos \left(N_n H_3 \right) \exp \left[- \left(\lambda_{mn} \right)^2 t \right] \\
& + \left. \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \left(\frac{N_u}{N_v} \right) \cos \left(N_v H_3 \right) \exp \left[- \left(\lambda_{uv} \right)^2 t \right] \right\}, \tag{2.142}
\end{aligned}$$

$$\begin{aligned}
Q_{rightside(2)}(t) = & -K_{x_1} \left\{ \sum_{p=1}^P B_{p(2)} \left(\frac{1}{K_1^a} \right) \coth \left[\left(N_p / K_1^a \right) S \right] \left[1 - \cos \left(N_p H_3 \right) \right] \right. \\
& - \sum_{q=1}^Q C_{q(2)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[\left(N_q / K_1^a \right) S \right]} \left[1 - \cos \left(N_q H_3 \right) \right] \\
& + \sum_{k=1}^K E_{k(2)} \cos \left(N_k S \right) \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh \left(N_k K_1^a H_3 \right) - 1}{\cosh \left(N_k K_1^a H_3 \right)} \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{r=1}^R D_{r(2)} \cos(N_r S) \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) [1 - \cos(N_n H_3)] \exp[-(\lambda_{mn})^2 t] \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) [1 - \cos(N_v H_3)] \exp[-(\lambda_{uv})^2 t] \Big\} \\
& - K_{x_2} \left\{ \sum_{w=1}^W H_{w(2)} \left(\frac{1}{K_2^a} \right) \cos(N_w S) + \sum_{i=1}^I F_{i(2)} \left(\frac{1}{K_2^a} \right) \coth \left[(N_i / K_2^a) S \right] \right. \\
& \quad - \sum_{j=1}^J G_{j(2)} \left(\frac{1}{K_2^a} \right) \frac{1}{\sinh \left[(N_j / K_2^a) S \right]} \\
& \quad + \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \exp[-(\lambda_{mn})^2 t] \\
& \quad \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \exp[-(\lambda_{uv})^2 t] \right\}, \tag{2.143}
\end{aligned}$$

$$\begin{aligned}
Vol_{leftside(2)} = & K_{x_1} T \left\{ \sum_{p=1}^P B_{p(2)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[(N_p / K_1^a) S \right]} [1 - \cos(N_p H_3)] \right. \\
& - \sum_{q=1}^Q C_{q(2)} \left(\frac{1}{K_1^a} \right) \coth \left[(N_q / K_1^a) S \right] [1 - \cos(N_q H_3)] \\
& + \sum_{k=1}^K E_{k(2)} \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& \left. + \sum_{r=1}^R D_{r(2)} \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \right\} \\
& + K_{x_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \left(\frac{N_m}{N_n} \right) [1 - \cos(N_n H_3)] \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \left(\frac{N_u}{N_v} \right) [1 - \cos(N_v H_3)] \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\}
\end{aligned}$$

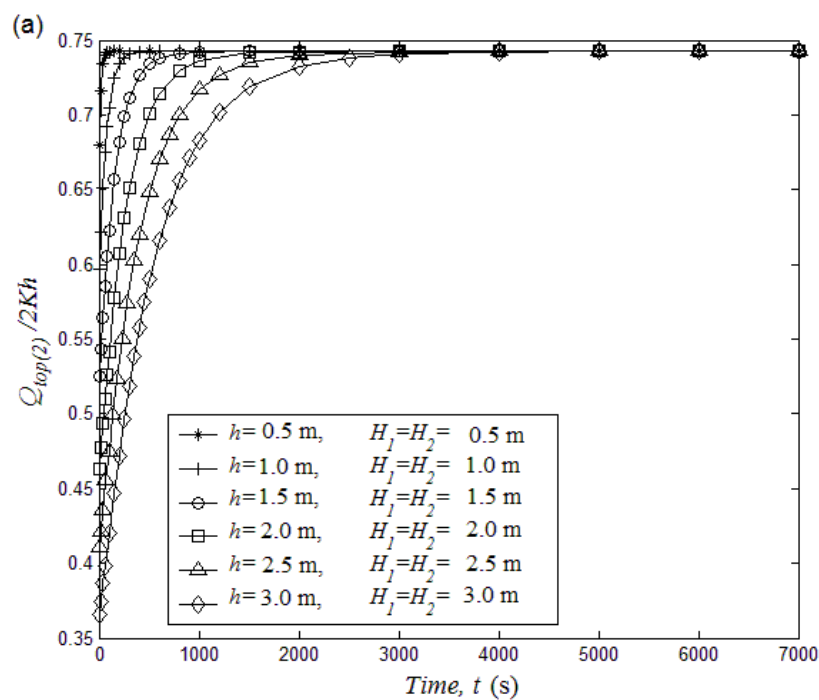
$$\begin{aligned}
& + K_{x_2} T \left\{ \sum_{w=1}^W H_{w(2)} \left(\frac{1}{K_2^a} \right) + \sum_{i=1}^I F_i \left(\frac{1}{K_2^a} \right) \operatorname{csch} \left[\left(N_i / K_2^a \right) S \right] \right. \\
& \quad \left. - \sum_{j=1}^J G_{j(2)} \left(\frac{1}{K_2^a} \right) \operatorname{coth} \left[\left(N_j / K_2^a \right) S \right] \right\} \\
& + K_{x_2} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \left\{ \frac{1 - \exp \left[-(\lambda_{mn})^2 T \right]}{(\lambda_{mn})^2} \right\} \right. \\
& \quad \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \left\{ \frac{1 - \exp \left[-(\lambda_{uv})^2 T \right]}{(\lambda_{uv})^2} \right\} \right\} \quad (2.144)
\end{aligned}$$

and

$$\begin{aligned}
Vol_{\text{rightside}(2)} = & -K_{x_1} T \left\{ \sum_{p=1}^P B_{p(2)} \left(\frac{1}{K_1^a} \right) \operatorname{coth} \left[\left(N_p / K_1^a \right) S \right] \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(2)} \left(\frac{1}{K_1^a} \right) \operatorname{csch} \left[\left(N_q / K_1^a \right) S \right] \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(2)} \cos(N_k S) \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& \left. + \sum_{r=1}^R D_{r(2)} \cos(N_r S) \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \right\} \\
& - K_{x_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \left[1 - \cos(N_n H_3) \right] \left\{ \frac{1 - \exp \left[-(\lambda_{mn})^2 T \right]}{(\lambda_{mn})^2} \right\} \right. \\
& \quad \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \left[1 - \cos(N_v H_3) \right] \left\{ \frac{1 - \exp \left[-(\lambda_{uv})^2 T \right]}{(\lambda_{uv})^2} \right\} \right\} \\
& - K_{x_2} T \left\{ \sum_{w=1}^W H_{w(2)} \left(\frac{1}{K_2^a} \right) \cos(N_w S) + \sum_{i=1}^I F_{i(2)} \left(\frac{1}{K_2^a} \right) \operatorname{coth} \left[\left(N_i / K_2^a \right) S \right] \right. \\
& \quad \left. - \sum_{j=1}^J G_{j(2)} \left(\frac{1}{K_2^a} \right) \operatorname{csch} \left[\left(N_j / K_2^a \right) S \right] \right\} \\
& - K_{x_2} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(2)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \left\{ \frac{1 - \exp \left[-(\lambda_{mn})^2 T \right]}{(\lambda_{mn})^2} \right\} \right. \\
& \quad \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(2)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \left\{ \frac{1 - \exp \left[-(\lambda_{uv})^2 T \right]}{(\lambda_{uv})^2} \right\} \right\}. \quad (2.145)
\end{aligned}$$

2.2.2.3 Verifications of the Proposed Solution

Like in the previous case, checks on the validity of the proposed solution are also being performed by comparing with other's analytical solutions for both steady as well as transient situations. MODFLOW verifications of the proposed model have also been carried out. Figs. 2.11 -2.17 show these comparisons. As can be seen from these figures, the predictions from our models are matching very closely with the corresponding analytical and numerical outputs for all the studied drainage scenarios thereby proving the validity of the proposed solution for the present case. Further, from Fig. 2.11 we see that here also $Q_{top(2)} / 2Kh$ ratio attains a value of 0.742 at large times, a value which is in close agreement with the corresponding value of 0.743 as obtained by Fukuda (1957) and Youngs' (1994) analytical models. This value can also be seen to be close to the value of 0.72 as obtained by Fukuda from his experimental results.



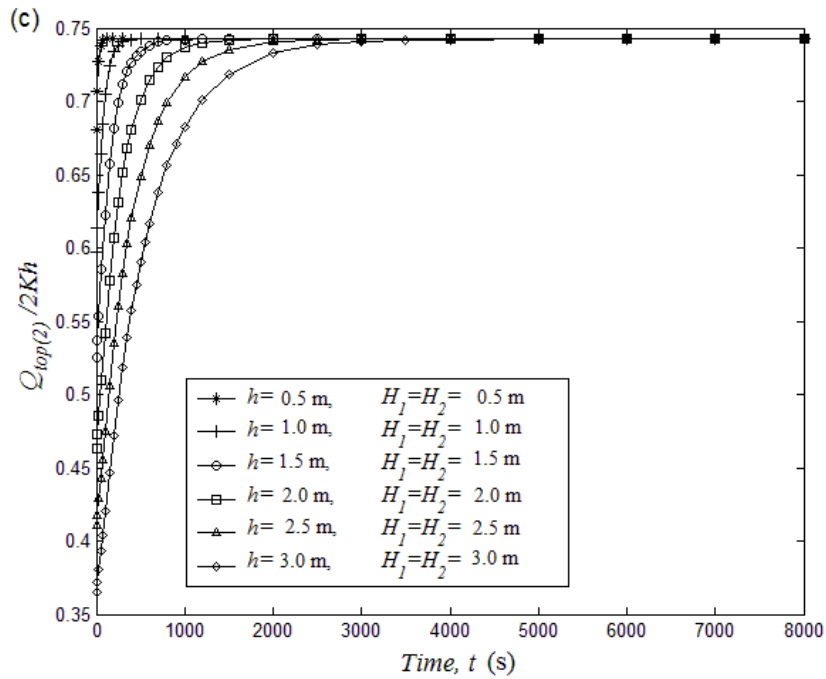
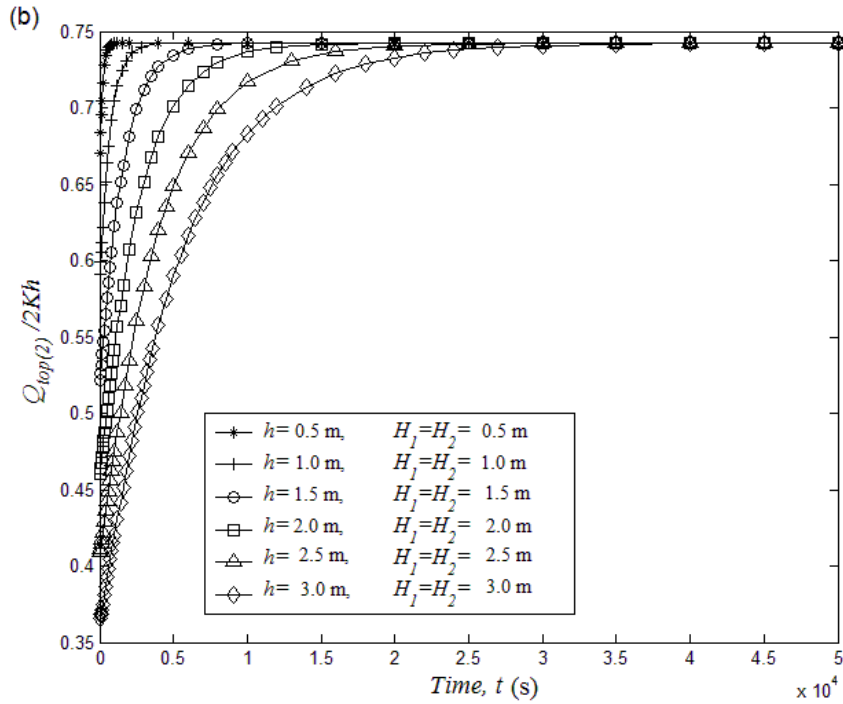


Fig. 2.11. Variation of $Q_{top(2)}/2Kh$ ratios with time as obtained from the proposed solution for different h and $H_1 = H_2 = h$ values when the other parameters of the flow problem are considered as $S = 100 \text{ m}$, $\delta = 0 \text{ m}$, $H_3 = 0.95 h$ and (a) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$, and (b) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$, and (c) $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05 \text{ m/day}$, $S_{s_1} = S_{s_2} = 0.0001 \text{ m}^{-1}$

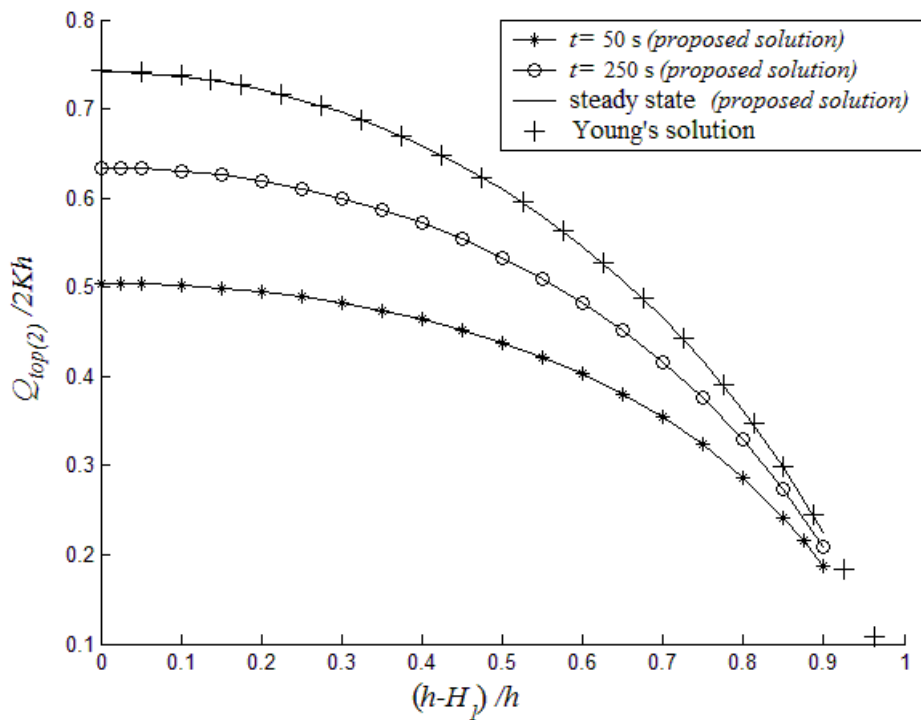


Fig. 2.12. Plots of $Q_{top(2)} / 2Kh$ versus $(h - H_1) / h$ as obtained from the proposed solution for different times of simulation of the system with the corresponding steady state plot as obtained from Youngs' (1994) single-layered solution of the problem for isotropic soils when the ponding depth is considered as zero and the other flow parameters of the problem are taken as $S = 100$ m, $h = 2.0$ m, $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day and $S_{s_1} = S_{s_2} = 0.0001$ m⁻¹

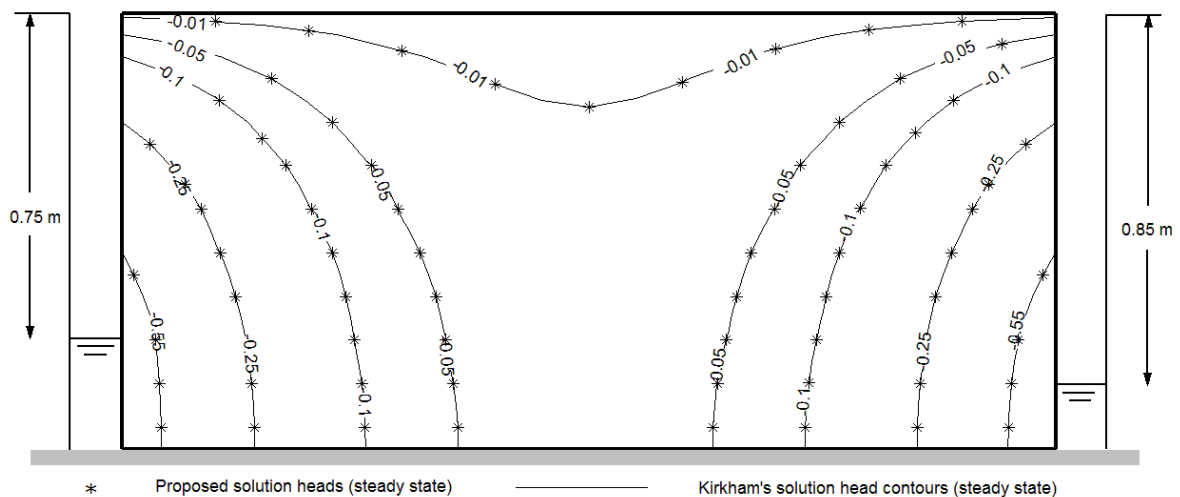


Fig. 2.13. Comparison of steady state hydraulic heads as obtained from the proposed solution with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.75$ m, $H_2 = 0.85$ m, $\delta = 0$ m, $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day, $S_{s_1} = S_{s_2} = 0.0001$ m⁻¹

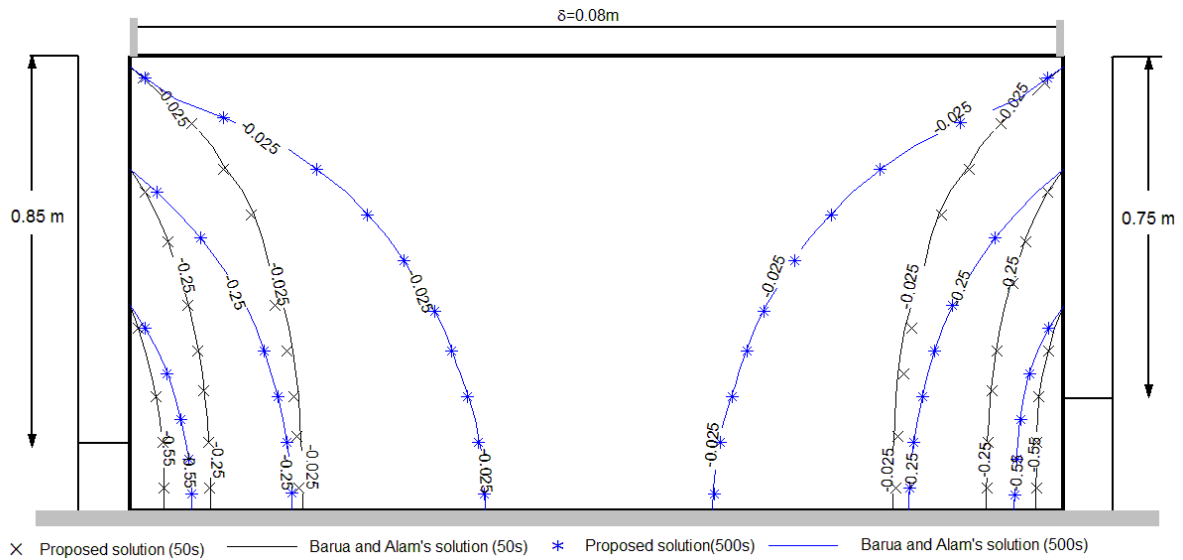


Fig. 2.14. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils at two different times of simulation of the system when the flow parameters for the problem are considered as $S = 5 \text{ m}$, $h = 1 \text{ m}$, $H_1 = 0.85 \text{ m}$, $H_2 = 0.75 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $\delta = 0.08 \text{ m}$, $K_{x_1} = K_{x_2} = 0.15 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.075 \text{ m/day}$, and $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$

2.2.2.4 MODFLOW Verification of the Proposed Solution

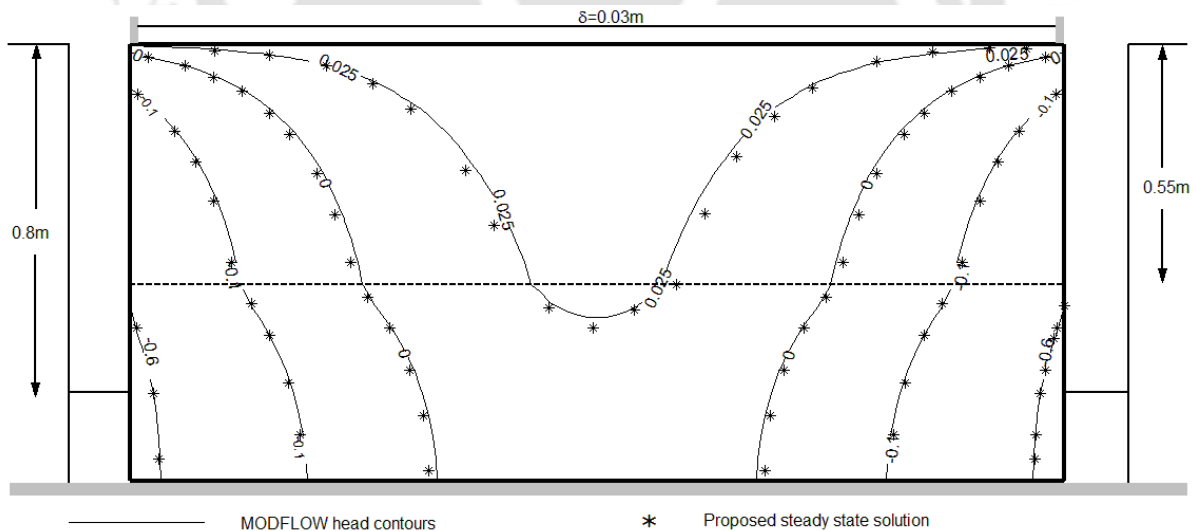


Fig 2.15. Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding values as obtained from MODFLOW when the flow parameters of the problem are considered as $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.55 \text{ m}$, $H_1 = 0.8 \text{ m}$, $H_2 = 0.8 \text{ m}$, $\delta = 0.03 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $K_{x_1} = 3 \text{ m/day}$, $K_{y_1} = 4 \text{ m/day}$, $K_{x_2} = 1 \text{ m/day}$ and $K_{y_2} = 0.75 \text{ m/day}$

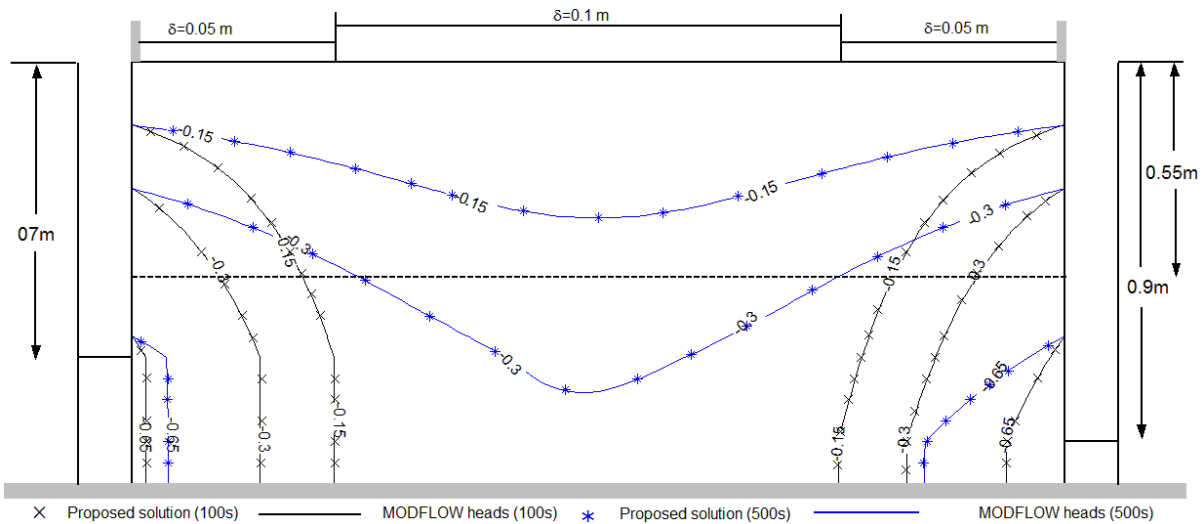


Fig 2.16. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.7$ m, $H_2 = 0.9$ m, $\varepsilon = 0.05$ m, $\delta_1 = 0.05$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.05$ m, $S_1 = 1.5$ m, $S_2 = 3.5$ m, $K_{x_1} = 1$ m/day, $K_{x_2} = 2.5$ m/day, $K_{y_1} = K_{y_2} = 0.001$ m/day, $S_{s_1} = 0.003$ m⁻¹ and $S_{s_2} = 0.0075$ m⁻¹

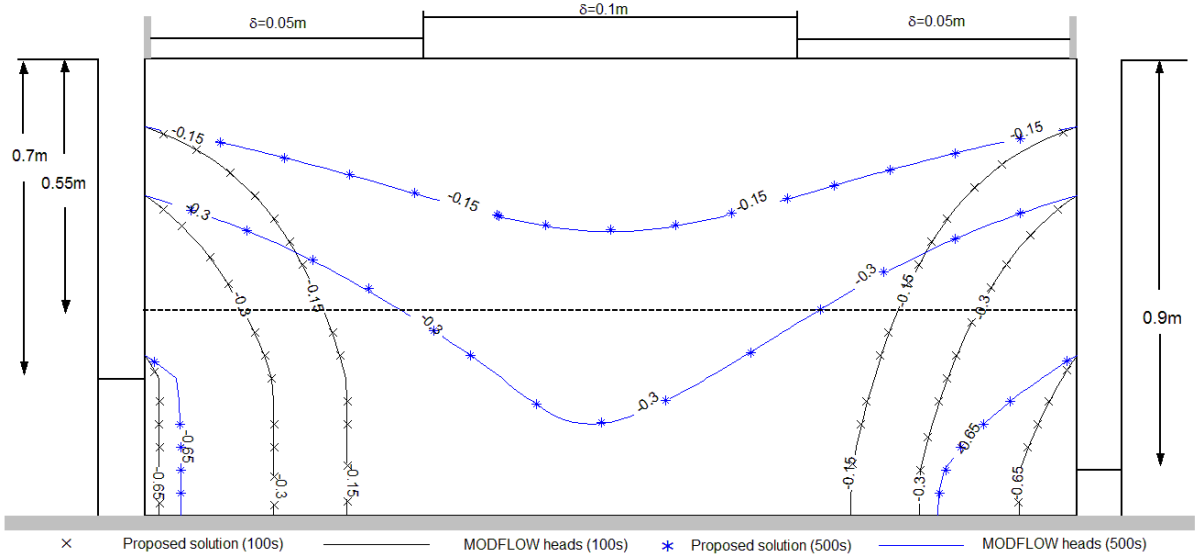


Fig 2.17. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.7$ m, $H_2 = 0.9$ m, $\varepsilon = 0.05$ m, $\delta_1 = 0.05$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.05$ m, $S_1 = 1.5$ m, $S_2 = 3.5$ m, $K_{x_1} = 1$ m/day, $K_{x_2} = 2.5$ m/day, $K_{y_1} = K_{y_2} = 0.01$ m/day, $S_{s_1} = 0.003$ m⁻¹ and $S_{s_2} = 0.0075$ m⁻¹

2.2.3 Level of Water in the Left Ditch is Below the Boundary between the Soil Layers while Level of Water in the Right Ditch is Above the Boundary between the Soil Layers

Fig. 2.18 illustrates the current flow situation under consideration, where as can be seen, the water level in the left ditch is below the top layer and the water level in the right ditch is above it.

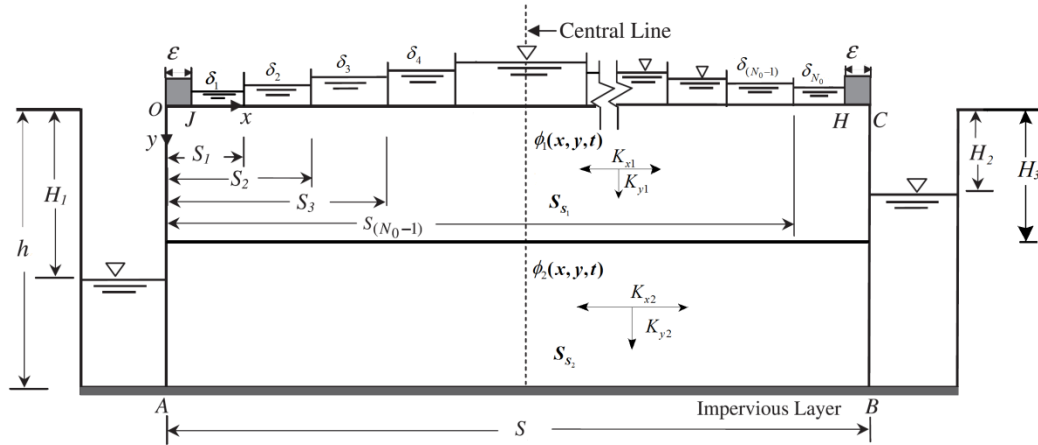


Fig. 2.18. Ditch drainage system for a two-layered soil where the height of water in the left ditch is below the bottom surface of the top layer and height of water in the right ditch is above it

The boundary conditions specific to this ditch drain boundaries can be represented as

$$\phi_{1(3)}(x, y, t > 0) = -y, \quad x = 0, \quad 0 < y \leq H_3, \quad (\text{XIV})$$

$$\phi_{2(3)}(x, y, t > 0) = -y, \quad x = 0, \quad H_3 \leq y \leq H_1, \quad (\text{XVa})$$

$$\phi_{2(3)}(x, y, t > 0) = -H_1, \quad x = 0, \quad H_1 \leq y < h, \quad (\text{XVb})$$

$$\phi_{1(3)}(x, y, t > 0) = -y, \quad x = S, \quad 0 < y \leq H_2, \quad (\text{XVIa})$$

$$\phi_{1(3)}(x, y, t > 0) = -H_2, \quad x = S, \quad H_2 < y \leq H_3, \quad (\text{XVIb})$$

$$\phi_{2(3)}(x, y, t > 0) = -H_2, \quad x = S, \quad H_3 \leq y < h. \quad (\text{XVII})$$

The hydraulic head expressions for the top soil layer and the bottom soil layer, in view of the ditch boundary conditions, can now be expressed as

$$\begin{aligned} \phi_{1(3)}(x, y, t) = & \sum_{p=1}^P B_{p(3)} \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin(N_p y) \\ & + \sum_{q=1}^Q C_{q(3)} \frac{\sinh\left[\left(N_q / K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin(N_q y) \\ & + \sum_{k=1}^K E_{k(3)} \frac{\sinh\left(N_k K_1^a y\right)}{\cosh\left(N_k K_1^a H_3\right)} \sin(N_k x) \\ & + \sum_{r=1}^R D_{r(3)} \frac{\cosh\left[N_r K_1^a (H_3 - y)\right]}{\cosh\left(N_r K_1^a H_3\right)} \sin(N_r x) \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \sin(N_m x) \sin(N_n y) \exp\left[-\left(\lambda_{mn}\right)^2 t\right] \end{aligned}$$

$$+ \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \sin(N_u x) \sin(N_v y) \exp[-(\lambda_{uv})^2 t], \quad (2.146)$$

where

$$N_p = \left[\left(\frac{1-2p}{2} \right) \left(\frac{\pi}{H_3} \right) \right], \quad (2.147)$$

$$N_q = \left[\left(\frac{1-2q}{2} \right) \left(\frac{\pi}{H_3} \right) \right], \quad (2.148)$$

$$N_k = \left(\frac{k\pi}{S} \right), \quad (2.149)$$

$$N_r = \left(\frac{r\pi}{S} \right), \quad (2.150)$$

$$K_1^a = \sqrt{\left(\frac{K_{x_1}}{K_{y_1}} \right)}, \quad (2.151)$$

$$N_m = \left(\frac{m\pi}{S} \right), \quad (2.152)$$

$$N_n = \left[\left(\frac{1-2n}{2} \right) \frac{\pi}{h} \right], \quad (2.153)$$

$$\lambda_{mn}^2 = \left[N_m^2 \left(\frac{K_{x_1}}{S_{s_1}} \right) + N_n^2 \left(\frac{K_{y_1}}{S_{s_1}} \right) \right], \quad (2.154)$$

$$N_u = \left(\frac{u\pi}{S} \right), \quad (2.155)$$

$$N_v = \left[\left(\frac{1-2v}{2} \right) \frac{\pi}{h} \right], \quad (2.156)$$

$$\lambda_{uv}^2 = \left[N_u^2 \left(\frac{K_{x_2}}{S_{s_2}} \right) + N_v^2 \left(\frac{K_{y_2}}{S_{s_2}} \right) \right] \quad (2.157)$$

and $B_{p(3)}$, $C_{q(3)}$, $E_{k(3)}$, $D_{r(3)}$, $A_{mn(3)}$ and $Z_{uv(3)}$ are constants and

$$\begin{aligned} \phi_{2(3)}(x, y, t) = & \sum_{w=1}^w H_{w(3)} \frac{\cosh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} \sin(N_w x) \\ & + \sum_{j=1}^j G_{j(3)} \frac{\sinh \left[(N_j / K_2^a)(S-x) \right]}{\sinh \left[(N_j / K_2^a)S \right]} \sin \left[N_j (y-H_3) \right] \\ & - H_3 \left(\frac{S-x}{S} \right) - H_2 \left(\frac{x}{S} \right) \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \sin(N_m x) \sin(N_n y) \exp \left[-(\lambda_{mn})^2 t \right] \end{aligned}$$

$$+ \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \sin(N_u x) \sin(N_v y) \exp[-(\lambda_{uv})^2 t], \quad (2.158)$$

where

$$N_w = \left(\frac{w\pi}{S} \right), \quad (2.159)$$

$$N_j = \left[\left(\frac{1-2j}{2} \right) \left(\frac{\pi}{h-H_3} \right) \right], \quad (2.160)$$

$$K_2^a = \sqrt{\left(\frac{K_{x_2}}{K_{y_2}} \right)} \quad (2.161)$$

and $H_{w(3)}$ and $G_{j(3)}$ are constants. The velocity functions $V_{x1(3)}$ and $V_{y1(3)}$ for the top layer remain identical to the expression mentioned in Eqs. (2.52) and (2.53); however, the Fourier coefficients to be used in this case are $B_{p(3)}$, $C_{q(3)}$, $E_{k(3)}$, $D_{r(3)}$, $A_{mn(3)}$ and $Z_{uv(3)}$ (to be defined in the pages to follow). Also, in view of Eq. (2.158), the velocity functions for the bottom layer in the x - and y - directions now work out as

$$\begin{aligned} V_{x2(3)} = & -K_{x_2} \left\{ \sum_{w=1}^W H_{w(3)} \frac{\cosh[(N_w K_2^a)(h-y)]}{\sinh[(N_w K_2^a)(h-H_3)]} N_w \cos(N_w x) - \left(\frac{H_2-H_3}{S} \right) \right. \\ & - \sum_{j=1}^J G_{j(3)} \frac{(N_j / K_2^a) \cosh[(N_j / K_2^a)(S-x)]}{\sinh[(N_j / K_2^a)S]} \sin[N_j(y-H_3)] \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} N_m \cos(N_m x) \sin(N_n y) \exp[-(\lambda_{mn})^2 t] \\ & \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} N_u \cos(N_u x) \sin(N_v y) \exp[-(\lambda_{uv})^2 t] \right\} \quad (2.162) \end{aligned}$$

and

$$\begin{aligned} V_{y2(3)} = & -K_{y_2} \left\{ - \sum_{w=1}^W H_{w(3)} \frac{(N_w K_2^a) \sinh[(N_w K_2^a)(h-y)]}{\sinh[(N_w K_2^a)(h-H_3)]} \sin(N_w x) \right. \\ & + \sum_{j=1}^J G_{j(3)} \frac{\sinh[(N_j / K_2^a)(S-x)]}{\sinh[(N_j / K_2^a)S]} N_j \cos[N_j(y-H_3)] \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \sin(N_m x) N_n \cos(N_n y) \exp[-(\lambda_{mn})^2 t] \\ & \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \sin(N_u x) N_v \cos(N_v y) \exp[-(\lambda_{uv})^2 t] \right\}. \quad (2.163) \end{aligned}$$

2.2.3.1 Determination of the Steady State Solution Coefficients (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 2.18)

Now, boundary condition (XIV) can be used to determine $C_{q(3)}$ – applying the same to Eq. (2.146), we get

$$\sum_{q=1}^Q C_{q(3)} \sin(N_q y) = -y, \quad 0 < y \leq H_3,$$

A Fourier series expansion of the above equation gives $C_{q(3)}$ as

$$C_{q(3)} = \left(\frac{2}{H_3} \right) \left[\int_0^{H_3} -y \sin(N_q y) dy \right]. \quad (2.164)$$

Solving the above integral, we find $C_{q(3)}$ as

$$C_{q(3)} = \left(\frac{2}{H_3} \right) \left[\frac{-\sin(N_q H_3)}{N_q^2} \right]. \quad (2.165)$$

Also, by introducing boundary conditions (XVIa) and (XVIb) in Eq. (2.146), we get the following two equations

$$\sum_{p=1}^P B_{p(3)} \sin(N_p y) = -y, \quad 0 < y \leq H_2,$$

$$\sum_{p=1}^P B_{p(3)} \sin(N_p y) = -H_2, \quad H_2 \leq y \leq H_3.$$

A Fourier run on the above equation thus gives $B_{p(3)}$ as

$$B_{p(3)} = \left(\frac{2}{H_3} \right) \left[\frac{-\sin(N_p H_2)}{N_p^2} \right]. \quad (2.166)$$

The expression for coefficients $D_{r(3)}$ remains the same as $D_{r(1)}$ but to get $G_{j(3)}$, boundary conditions (XVa) and (XVb) can be used in Eq. (2.158); this leads to the following expressions

$$\sum_{j=1}^J G_{j(3)} \sin[N_j (y - H_3)] - H_3 = -y, \quad H_3 \leq y \leq H_1,$$

$$\sum_{j=1}^J G_{j(3)} \sin[N_j (y - H_3)] - H_3 = -H_1, \quad H_1 \leq y < h.$$

Now, performing a Fourier series expansion on the above equations and evaluating the resulting integrals, we find

$$G_{j(3)} = \left(\frac{2}{h - H_3} \right) \left\{ \frac{-\sin[N_j (H_1 - H_3)]}{N_j^2} \right\}. \quad (2.167)$$

In order that head equality [boundary condition (IIIa)] happens at $y = H_3$, we must have

$$\sum_{p=1}^P B_{p(3)} \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin(N_p H_3) + \sum_{q=1}^Q C_{q(3)} \frac{\sinh\left[\left(N_q / K_1^a\right)(S - x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin(N_q H_3)$$

$$\begin{aligned}
& + \sum_{k=1}^K E_{k(3)} \frac{\sinh(N_k K_1^a H_3)}{\cosh(N_k K_1^a H_3)} \sin(N_k x) + \sum_{r=1}^R D_{r(3)} \frac{1}{\cosh(N_r K_1^a H_3)} \sin(N_r x) \\
& = \sum_{w=1}^W H_{w(3)} \frac{\cosh[N_w K_2^a (h - H_3)]}{\sinh[N_w K_2^a (h - H_3)]} \sin(N_w x) - H_3 \left(\frac{S-x}{S} \right) - H_2 \left(\frac{x}{S} \right).
\end{aligned}$$

From the above, we must have then (using Fourier expansion)

$$\begin{aligned}
& E_{k(3)} \tanh(N_i K_1^a H_3) - H_{w(3)} \coth[N_i K_2^a (h - H_3)] \\
& = -D_{r(3)} \operatorname{sech}(N_i K_1^a H_3) - \left(\frac{2}{S} \right) H_3 \int_0^S \left(\frac{S-x}{S} \right) \sin(N_i x) dx \\
& - \left(\frac{2}{S} \right) H_2 \int_0^S \left(\frac{x}{S} \right) \sin(N_i x) dx \\
& - \left(\frac{2}{S} \right) \sum_{p=1}^P B_{p(3)} \sin(N_p H_3) \int_0^S \frac{\sinh\left[\frac{(N_p/K_1^a)x}{S}\right]}{\sinh\left[\frac{(N_p/K_1^a)S}{S}\right]} \sin(N_i x) dx \\
& - \left(\frac{2}{S} \right) \sum_{q=1}^Q C_{q(3)} \sin(N_q H_3) \int_0^S \frac{\sinh\left[\frac{(N_q/K_1^a)(S-x)}{S}\right]}{\sinh\left[\frac{(N_q/K_1^a)S}{S}\right]} \sin(N_i x) dx
\end{aligned} \tag{2.168}$$

Solving the above integrals, we finally get

$$\begin{aligned}
& E_{k(3)} \tanh(N_i K_1^a H_3) - H_{w(3)} \coth[N_i K_2^a (h - H_3)] \\
& = -D_{r(3)} \operatorname{sech}(N_i K_1^a H_3) - \left(\frac{2}{S} \right) H_2 \left[\frac{-\cos(N_i S)}{N_i} \right] - \left(\frac{2}{S} \right) H_3 \left(\frac{1}{N_i} \right) \\
& - \left(\frac{2}{S} \right) \sum_{p=1}^P B_{p(3)} \sin(N_p H_3) \left[\frac{N_i^2}{N_i^2 + (N_p/K_1^a)^2} \right] \left[\frac{-\cos(N_i S)}{N_i} \right] \\
& - \left(\frac{2}{S} \right) \sum_{q=1}^Q C_{q(3)} \sin(N_q H_3) \left[\frac{N_i}{N_i^2 + (N_q/K_1^a)^2} \right],
\end{aligned} \tag{2.169}$$

where $k = w = r = i$ and $P = Q \rightarrow \infty$. Also, to achieve flux equality [boundary condition (IIIb)] at $y = H_3$, we must have the relation

$$\begin{aligned}
& -K_{y_1} \sum_{k=1}^K E_{k(3)} (N_k K_1^a) \sin(N_k x) = \\
& K_{y_2} \sum_{w=1}^W H_{w(3)} (N_w K_2^a) \sin(N_w x) - K_{y_2} \sum_{j=1}^J G_{j(3)} N_j \frac{\sinh\left[\frac{(N_j/K_2^a)(S-x)}{S}\right]}{\sinh\left[\frac{(N_j/K_2^a)S}{S}\right]}.
\end{aligned}$$

The above equation with the aid of Fourier series gives us

$$K_{y_1} (N_k K_1^a) E_{k(3)} + K_{y_2} (N_w K_2^a) H_{w(3)} =$$

$$\left(\frac{2}{S}\right) K_{y_2} \sum_{j=1}^J G_{j(3)} N_j \int_0^S \frac{\sinh\left[\left(N_j/K_2^a\right)(S-x)\right]}{\sinh\left[\left(N_j/K_2^a\right)S\right]} \sin(N_i x) dx, \quad (2.170)$$

which, upon simplification, gives us the relation

$$K_{y_1} \left(N_k K_1^a\right) E_{k(3)} + K_{y_2} \left(N_w K_2^a\right) H_{w(3)} = \left(\frac{2}{S}\right) K_{y_2} \sum_{j=1}^J G_{j(3)} N_j \left[\frac{N_i}{N_i^2 + \left(N_j/K_2^a\right)^2} \right], \quad (2.171)$$

where $k = w = i_1$ and $J \rightarrow \infty$. Equations (2.169) and (2.171) can now be solved simultaneously to evaluate the coefficients $E_{k(3)}$ and $H_{w(3)}$.

2.2.3.2 Determination of the Transient Solution Coefficients [transient state solution is approximate and is applicable only when conductivity and specific storage of the layers of Fig. 2.18 satisfy $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)]

The expression for $A_{mn(3)}$ shall be identical to the one for $A_{mn(1)}$ [Eq. (2.73)] but the relevant Fourier coefficients now will be $B_{p(3)}$, $C_{q(3)}$, $E_{k(3)}$ and $D_{r(3)}$ and not the constants mentioned in the $A_{mn(1)}$ expression. For $Z_{uv(3)}$, there will be a new expression since the steady state hydraulic head expression for the bottom soil layer is now different than that of the previous cases. For this case, it works out as

$$\begin{aligned} Z_{uv(3)} = & \left(\frac{4}{Sh}\right) \left\{ -\sum_{w=1}^W H_{w(3)} \int_{H_3}^h \frac{\cosh\left[\left(N_w K_2^a\right)(h-y)\right]}{\sinh\left[\left(N_w K_2^a\right)(h-H_3)\right]} \sin(N_v y) dy \times \int_0^S \sin(N_u x) \sin(N_w x) dx \right. \\ & - \sum_{j=1}^J G_{j(3)} \int_0^S \frac{\sinh\left[\left(N_j/K_2^a\right)x\right]}{\sinh\left[\left(N_j/K_2^a\right)S\right]} \sin(N_u x) dx \\ & \times \int_{H_3}^h \sin\left(N_j(y-H_3)\right) \sin(N_v y) dy \\ & + H_3 \int_0^S \left(\frac{S-x}{S}\right) \sin(N_u x) dx \int_{H_3}^h \sin(N_v y) dy \\ & \left. + H_2 \int_0^S \left(\frac{x}{S}\right) \sin(N_u x) dx \int_{H_3}^h \sin(N_v y) dy \right\}. \quad (2.172) \end{aligned}$$

Solving the above integrals, we finally get

$$\begin{aligned} Z_{uv(3)} = & \left(\frac{4}{Sh}\right) \left\{ -H_{w(3)} \left(\frac{S}{2}\right) \left[\frac{N_v^2}{N_v^2 + \left(N_w K_2^a\right)^2} \right] \right. \\ & \left. \times \left\{ \coth\left[\left(N_w K_2^a\right)(h-H_3)\right] \frac{\cos(N_v H_3)}{N_v} + \left[\frac{\left(N_w K_2^a\right)}{N_v^2} \right] \sin(N_v H_3) \right\} \right\} \end{aligned}$$

$$\begin{aligned}
& -\sum_{j=1}^J G_{j(3)} \left[\frac{N_u}{N_u^2 + (N_j/K_2^a)^2} \right] \times I^{(4)} \\
& + H_3 \left(\frac{1}{N_u} \right) \left(\frac{\cos N_v H_3}{N_v} \right) + H_2 \left(\frac{\cos N_v H_3}{N_v} \right) \left[\frac{-\cos(N_u S)}{N_u} \right], \tag{2.173}
\end{aligned}$$

where $w=u$, $J \rightarrow \infty$ and the integral $I^{(4)}$ is the same as that mentioned in Eq. (2.137). Here also, the expressions of $Q_{top(3)}$, $Q_{topx(3)}$ and $Vol_{top(3)}$ will be similar to those of the previous cases but the Fourier constants now are $B_{p(3)}$, $C_{q(3)}$, $E_{k(3)}$, $D_{r(3)}$, $A_{mn(3)}$ and $Z_{uv(3)}$ and not the constants pertaining to the earlier cases. Also, due to the difference in the velocity expressions of the bottom layer in comparison to the previous cases, the expressions for $Q_{leftside(3)}$, $Q_{rightside(3)}$, $Vol_{leftside(3)}$ and $Vol_{rightside(3)}$ will be different from those of the previous cases – for this drainage configuration, they turn out as

$$\begin{aligned}
Q_{leftside(3)}(t) = & K_{x_1} \left\{ \sum_{p=1}^P B_{p(3)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[(N_p/K_1^a) S \right]} \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(3)} \left(\frac{1}{K_1^a} \right) \coth \left[(N_q/K_1^a) S \right] \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(3)} \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& + \sum_{r=1}^R D_{r(3)} \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \left(\frac{N_m}{N_n} \right) \left[1 - \cos(N_n H_3) \right] \exp \left[-(\lambda_{mn})^2 t \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \left(\frac{N_u}{N_v} \right) \left[1 - \cos(N_v H_3) \right] \exp \left[-(\lambda_{uv})^2 t \right] \left. \right\} \\
& + K_{x_2} \left\{ \sum_{w=1}^W H_{w(3)} \left(\frac{1}{K_2^a} \right) - \sum_{j=1}^J G_{j(3)} \left(\frac{1}{K_2^a} \right) \coth \left[(N_j/K_2^a) S \right] - \left[\frac{(H_2 - H_3)(h - H_3)}{S} \right] \right. \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \exp \left[-(\lambda_{mn})^2 t \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \exp \left[-(\lambda_{uv})^2 t \right] \left. \right\}, \tag{2.174}
\end{aligned}$$

$$\begin{aligned}
Q_{rightside(3)}(t) = & -K_{x_1} \left\{ \sum_{p=1}^P B_{p(3)} \left(\frac{1}{K_1^a} \right) \coth \left[(N_p/K_1^a) S \right] \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(3)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[(N_q/K_1^a) S \right]} \left[1 - \cos(N_q H_3) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^K E_{k(3)} \cos(N_k S) \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& + \sum_{r=1}^R D_{r(3)} \cos(N_r S) \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) [1 - \cos(N_n H_3)] \exp[-(\lambda_{mn})^2 t] \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) [1 - \cos(N_v H_3)] \exp[-(\lambda_{uv})^2 t] \Big\} \\
& - K_{x_2} \left\{ \sum_{w=1}^W H_{w(3)} \left(\frac{1}{K_2^a} \right) \cos(N_w S) - \sum_{j=1}^J G_{j(3)} \left(\frac{1}{K_2^a} \right) \frac{1}{\sinh[(N_j / K_2^a) S]} \right. \\
& \quad \left. - \left[\frac{(H_2 - H_3)(h - H_3)}{S} \right] \right. \\
& \quad + \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \exp[-(\lambda_{mn})^2 t] \\
& \quad \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \exp[-(\lambda_{uv})^2 t] \right\}, \tag{2.175}
\end{aligned}$$

$$\begin{aligned}
Vol_{leftside(3)} = & K_{x_1} T \left\{ \sum_{p=1}^P B_{p(3)} \left(\frac{1}{K_1^a} \right) \operatorname{csch}[(N_p / K_1^a) S] [1 - \cos(N_p H_3)] \right. \\
& - \sum_{q=1}^Q C_{q(3)} \left(\frac{1}{K_1^a} \right) \operatorname{coth}[(N_q / K_1^a) S] [1 - \cos(N_q H_3)] \\
& + \sum_{k=1}^K E_{k(3)} \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& \left. + \sum_{r=1}^R D_{r(3)} \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \right\} \\
& + K_{x_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \left(\frac{N_m}{N_n} \right) [1 - \cos(N_n H_3)] \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \left(\frac{N_u}{N_v} \right) [1 - \cos(N_v H_3)] \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\} \\
& + K_{x_2} T \left\{ \sum_{w=1}^W H_{w(3)} \left(\frac{1}{K_2^a} \right) - \sum_{j=1}^J G_{j(3)} \left(\frac{1}{K_2^a} \right) \operatorname{coth}[(N_j / K_2^a) S] \right. \\
& \left. - \left[\frac{(H_2 - H_3)(h - H_3)}{S} \right] \right\}
\end{aligned}$$

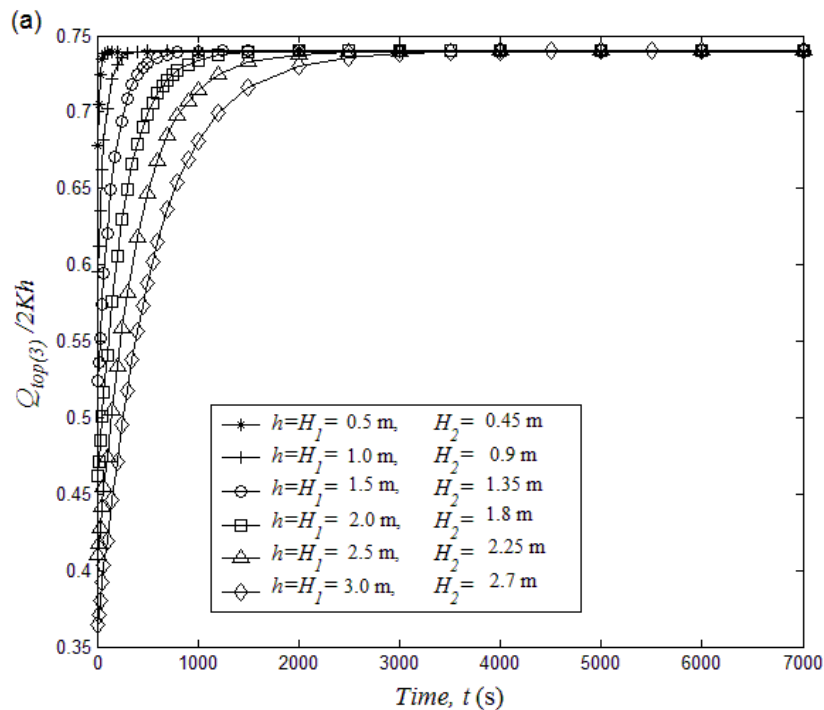
$$\begin{aligned}
& + K_{x_2} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\} \quad (2.176)
\end{aligned}$$

and

$$\begin{aligned}
Vol_{rightside(3)} = & -K_{x_1} T \left\{ \sum_{p=1}^P B_{p(3)} \left(\frac{1}{K_1^a} \right) \coth \left[(N_p / K_1^a) S \right] \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(3)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[(N_q / K_1^a) S \right]} \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(3)} \cos(N_k S) \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& \left. + \sum_{r=1}^R D_{r(3)} \cos(N_r S) \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \right\} \\
& - K_{x_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \left[\frac{N_m \cos(N_m S)}{N_n} \right] \left[1 - \cos(N_n H_3) \right] \right. \\
& \times \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \\
& + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \left[\frac{N_u \cos(N_u S)}{N_v} \right] \left[1 - \cos(N_v H_3) \right] \\
& \left. \times \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\} \\
& - K_{x_2} T \left\{ \sum_{w=1}^W H_{w(3)} \left(\frac{1}{K_2^a} \right) \cos(N_w S) - \sum_{j=1}^J G_{j(3)} \left(\frac{1}{K_2^a} \right) \frac{1}{\sinh \left[(N_j / K_2^a) S \right]} \right. \\
& \left. - \left[\frac{(H_2 - H_3)(h - H_3)}{S} \right] \right\} \\
& - K_{x_2} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(3)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(3)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\}. \quad (2.177)
\end{aligned}$$

2.2.3.3 Verifications of the Proposed Solution

Here also, a few checks are being carried out to ascertain the accuracy of the proposed model for the drainage setting as shown in Fig. 2.18. Figs. 2.19 - 2.22 shows analytical comparison of our solution with the analytical works of Fukuda (1957), Youngs (1994), Kirkham (1965) and Barua and Alam (2013) for a few drainage situations. As may be observed, predictions from our model for this drainage setting can also be seen to be matching quite accurately with the corresponding values obtained from the analytical models of Fukuda (1957), Youngs (1994), Kirkham (1965) and Barua and Alam (2013) for both steady as well as transient situations. Further, Figs. 2.23, 2.24 and 2.25 also show that predictions from our model agree very closely with the corresponding values obtained by numerical means (MODFLOW). In addition to that, we can also observe from Fig. 2.19 that the $Q_{top(3)}/2Kh$ ratio here also approaches 0.739 as the time of simulation of the system is allowed to increase continually; in this context, it should be noted that this value is also quite close to that obtained by Fukuda from his experimental studies conducted for similar drainage settings.



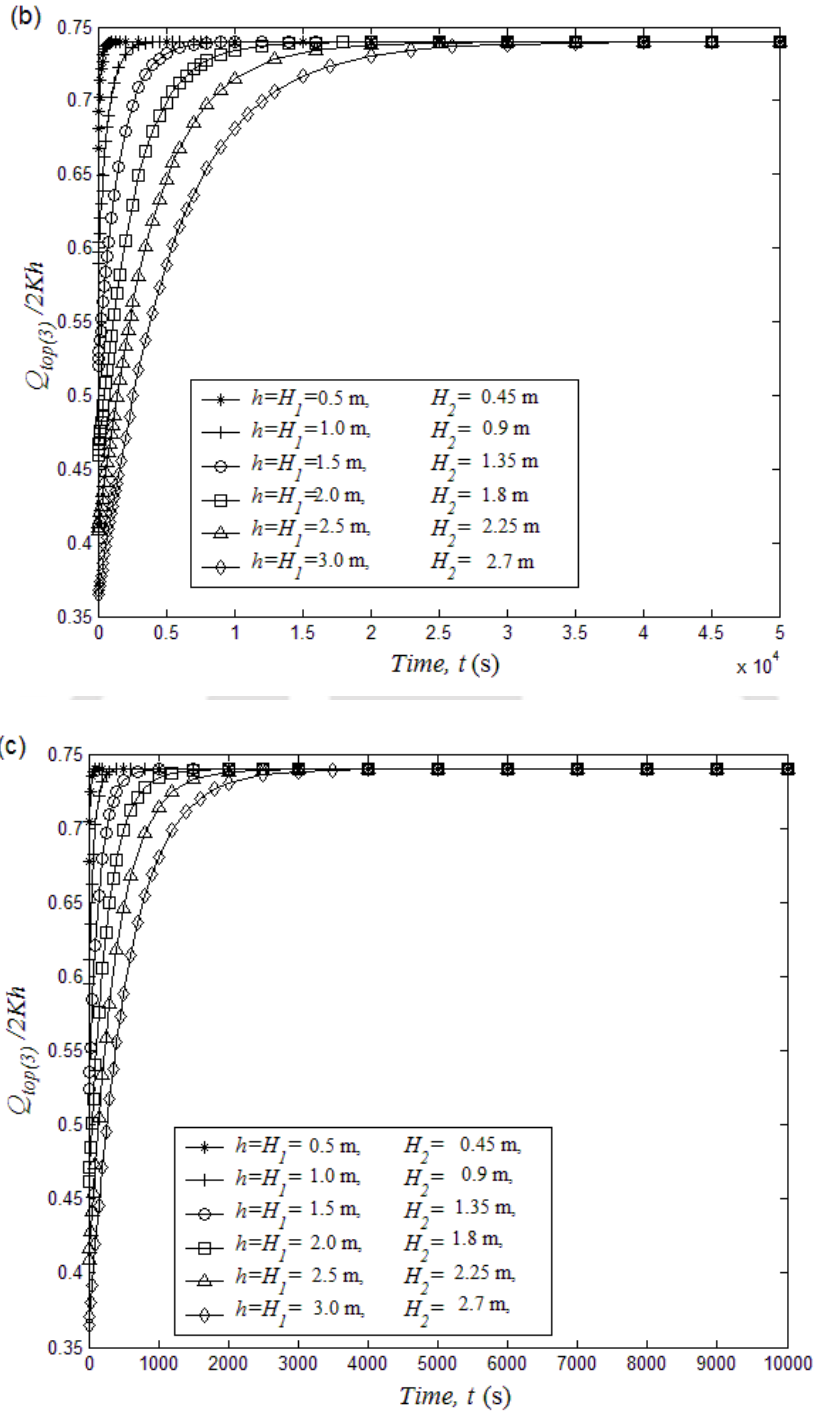


Fig. 2.19. Variation of $Q_{top(3)}/2Kh$ ratios with time as obtained from the proposed solution for different h and $H_1=h$ values when the other parameters of the flow problem are considered as $S=100$ m, $\delta=0$ m, $H_3=0.95h$ and (a) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.05$ m/day, $S_{s_1}=S_{s_2}=0.001$ m⁻¹ and (b) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.5$ m/day, $S_{s_1}=S_{s_2}=0.001$ m⁻¹ and (c) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.05$ m/day, $S_{s_1}=S_{s_2}=0.0001$ m⁻¹

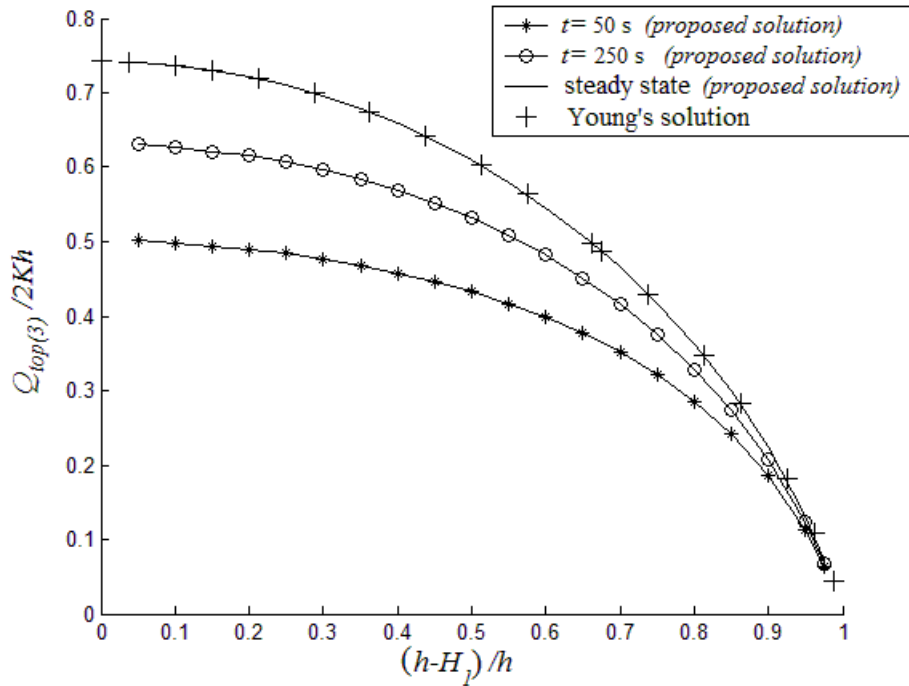


Fig. 2.20. Plots of $Q_{top(3)}/2Kh$ versus $(h-H_1)/h$ as obtained from the proposed solution for different times of simulation of the system with the corresponding steady state plot as obtained from Youngs' (1994) single-layered solution of the problem for isotropic soils when the ponding depth is considered as zero and the other flow parameters of the problem are taken as $S = 100$ m, $h = 2.0$ m, $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day and $S_{s_1} = S_{s_2} = 0.0001$ m⁻¹

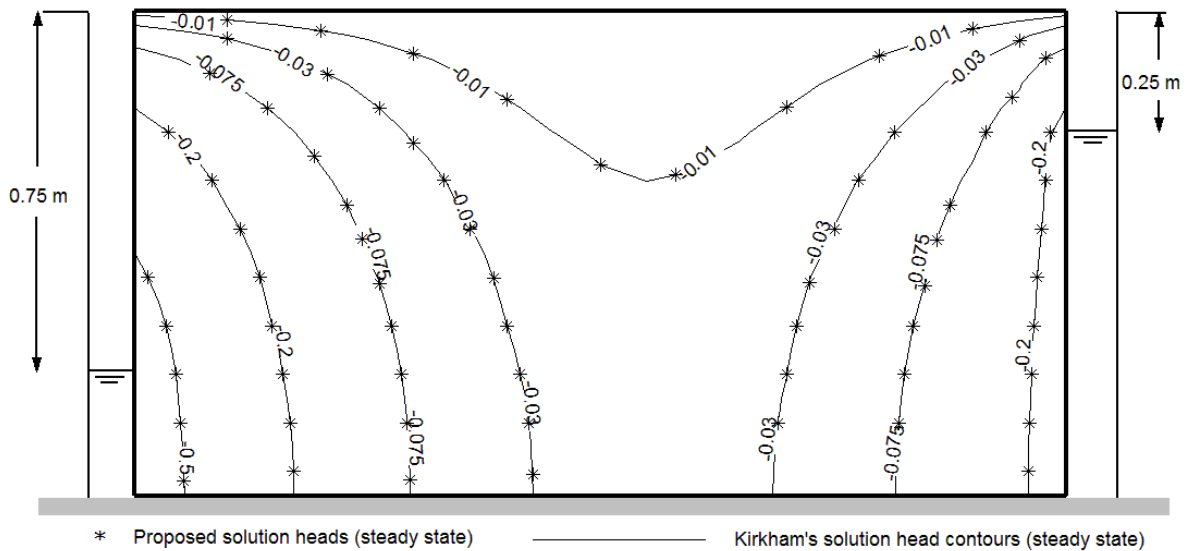


Fig. 2.21. Comparison of steady state hydraulic heads as obtained from the proposed solution with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.75$ m, $H_2 = 0.25$ m, $\delta = 0$ m, and $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day

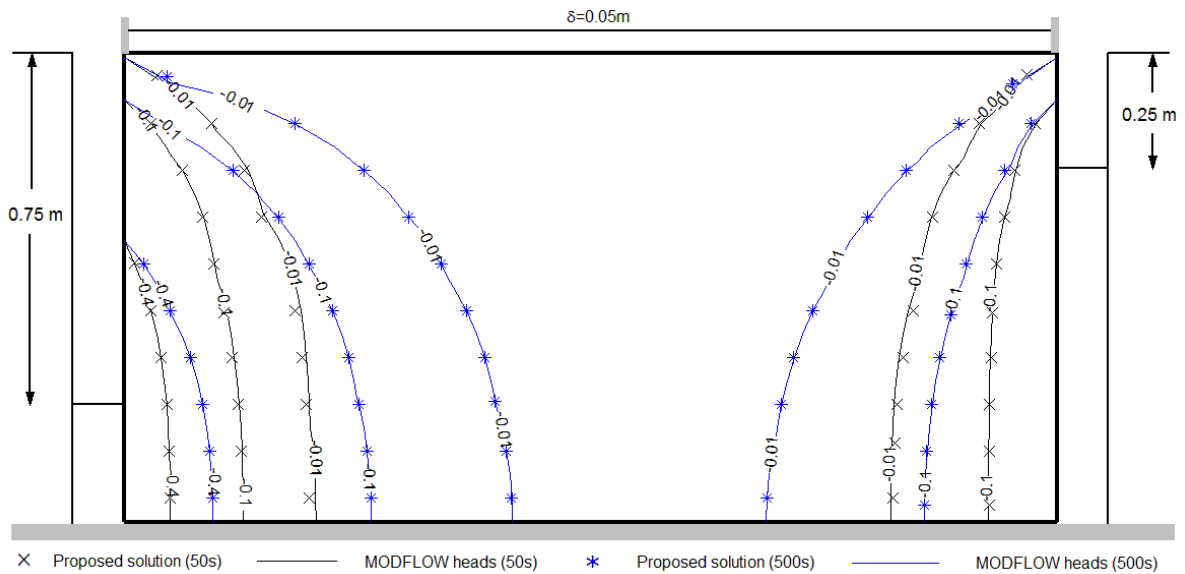


Fig. 2.22. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils at two different times of simulation of the system when the flow parameters for the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.75$ m, $H_2 = 0.25$ m, $\varepsilon = 0.05$ m, $\delta = 0.05$ m, $K_{x_1} = K_{x_2} = 0.15$ m/day, $K_{y_1} = K_{y_2} = 0.09$ m/day and $S_{s_1} = S_{s_2} = 0.001$ m⁻¹

2.2.3.4 MODFLOW Verification of the Proposed Solution

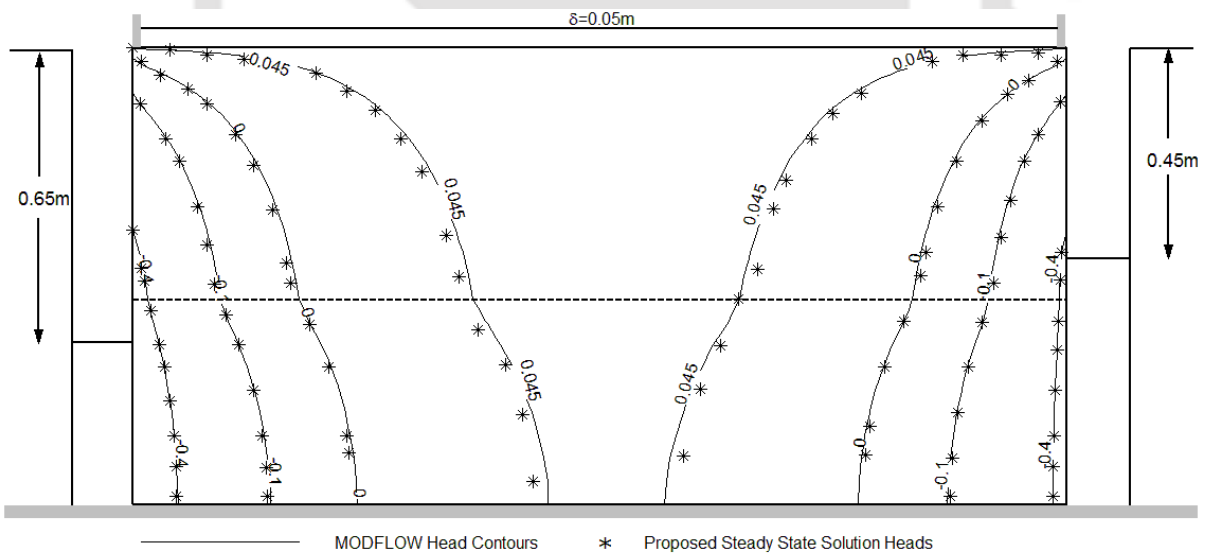


Fig. 2.23. Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding values as obtained from MODFLOW when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.65$ m, $H_2 = 0.45$ m, $\delta = 0.05$ m, $\varepsilon = 0.05$ m, $K_{x_1} = 1.5$ m/day, $K_{y_1} = 2.5$ m/day, $K_{x_2} = 0.5$ m/day and $K_{y_2} = 0.5$ m/day

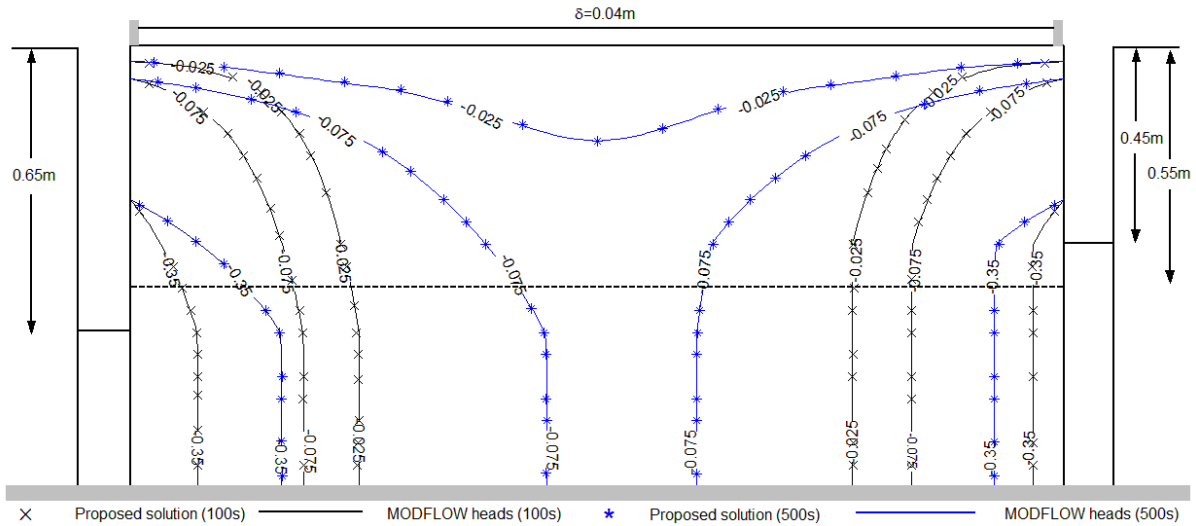


Fig 2.24. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0\text{ m}$, $h = 1.0\text{ m}$, $H_1 = 0.65\text{ m}$, $H_2 = 0.45\text{ m}$, $H_3 = 0.55\text{ m}$, $\varepsilon = 0.05\text{ m}$, $\delta = 0.04\text{ m}$, $K_{x_1} = 0.75\text{ m/day}$, $K_{x_2} = 2.25\text{ m/day}$, $K_{y_1} = K_{y_2} = 0.001\text{ m/day}$, $S_{s_1} = 0.005\text{ m}^{-1}$ and $S_{s_2} = 0.015\text{ m}^{-1}$

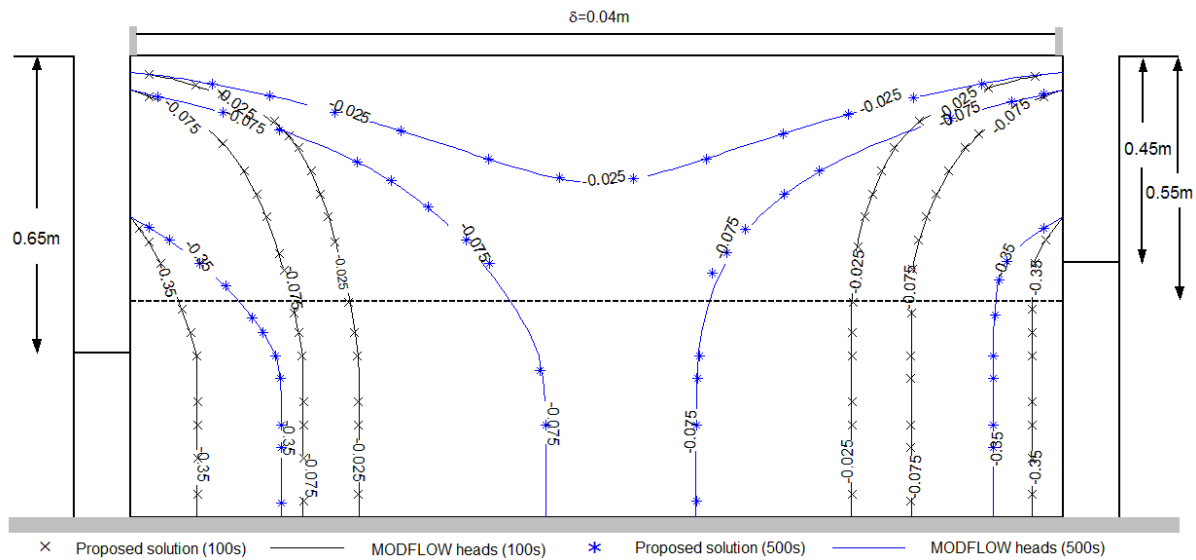


Fig 2.25. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0\text{ m}$, $h = 1.0\text{ m}$, $H_2 = 0.45\text{ m}$, $H_1 = 0.65\text{ m}$, $H_3 = 0.55\text{ m}$, $\varepsilon = 0.05\text{ m}$, $\delta = 0.04\text{ m}$, $K_{x_1} = 0.75\text{ m/day}$, $K_{x_2} = 2.25\text{ m/day}$, $K_{y_1} = K_{y_2} = 0.01\text{ m/day}$, $S_{s_1} = 0.005\text{ m}^{-1}$ and $S_{s_2} = 0.015\text{ m}^{-1}$

2.2.4 Level of Water in the Left Ditch is Above the Boundary Between the Soil Layers while Level of Water in the Right Ditch is Below the Boundary Between the Soil Layers

Fig. 2.26 illustrates the height of water, and consequently, the boundary conditions at the drainage ditches for this particular boundary value problem.

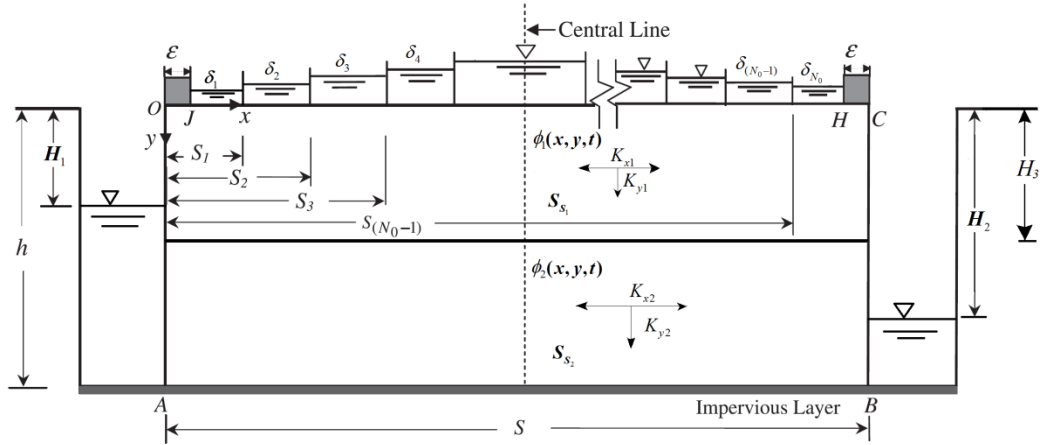


Fig. 2.26. Ditch drainage system for a two-layered soil where the height of water in the left ditch is above the bottom surface of the top soil layer and the height of water in the right ditch is below it

The boundary conditions specific to the ditch drain boundaries for this particular case are

$$\phi_{1(4)}(x, y, t > 0) = -y, \quad x = 0, \quad 0 < y \leq H_1, \quad (\text{XVIIIa})$$

$$\phi_{1(4)}(x, y, t > 0) = -H_1, \quad x = 0, \quad H_1 \leq y \leq H_3, \quad (\text{XVIIIb})$$

$$\phi_{2(4)}(x, y, t > 0) = -H_1, \quad x = 0, \quad H_3 \leq y < h, \quad (\text{XIX})$$

$$\phi_{1(4)}(x, y, t > 0) = -y, \quad x = S, \quad 0 < y \leq H_3, \quad (\text{XX})$$

$$\phi_{2(4)}(x, y, t > 0) = -y, \quad x = S, \quad H_3 \leq y \leq H_2, \quad (\text{XXIa})$$

$$\phi_{2(4)}(x, y, t > 0) = -H_2, \quad x = S, \quad H_2 \leq y < h. \quad (\text{XXIb})$$

The hydraulic head functions for the top and bottom layers for this case can be expressed as

$$\begin{aligned} \phi_{1(4)}(x, y, t) = & \sum_{p=1}^P B_{p(4)} \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin(N_p y) \\ & + \sum_{q=1}^Q C_{q(4)} \frac{\sinh\left[\left(N_q / K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin(N_q y) \\ & + \sum_{k=1}^K E_{k(4)} \frac{\sinh\left(N_k K_1^a y\right)}{\cosh\left(N_k K_1^a H_3\right)} \sin(N_k x) \\ & + \sum_{r=1}^R D_{r(4)} \frac{\cosh\left[N_r K_1^a (H_3 - y)\right]}{\cosh\left(N_r K_1^a H_3\right)} \sin(N_r x) \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \sin(N_m x) \sin(N_n y) \exp\left[-\left(\lambda_{mn}\right)^2 t\right] \end{aligned}$$

$$+ \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \sin(N_u x) \sin(N_v y) \exp[-(\lambda_{uv})^2 t], \quad (2.178)$$

where

$$N_p = \left[\left(\frac{1-2p}{2} \right) \left(\frac{\pi}{H_3} \right) \right], \quad (2.179)$$

$$N_q = \left[\left(\frac{1-2q}{2} \right) \left(\frac{\pi}{H_3} \right) \right], \quad (2.180)$$

$$N_k = \left(\frac{k\pi}{S} \right), \quad (2.181)$$

$$N_r = \left(\frac{r\pi}{S} \right), \quad (2.182)$$

$$K_1^a = \sqrt{\left(\frac{K_{x_1}}{K_{y_1}} \right)}, \quad (2.183)$$

$$N_m = \left(\frac{m\pi}{S} \right), \quad (2.184)$$

$$N_n = \left[\left(\frac{1-2n}{2} \right) \frac{\pi}{h} \right], \quad (2.185)$$

$$\lambda_{mn}^2 = \left[N_m^2 \left(\frac{K_{x_1}}{S_{s_1}} \right) + N_n^2 \left(\frac{K_{y_1}}{S_{s_1}} \right) \right], \quad (2.186)$$

$$N_u = \left(\frac{u\pi}{S} \right), \quad (2.187)$$

$$N_v = \left[\left(\frac{1-2v}{2} \right) \frac{\pi}{h} \right], \quad (2.188)$$

$$\lambda_{uv}^2 = \left[N_u^2 \left(\frac{K_{x_2}}{S_{s_2}} \right) + N_v^2 \left(\frac{K_{y_2}}{S_{s_2}} \right) \right] \quad (2.189)$$

and $B_{p(4)}$, $C_{q(4)}$, $E_{k(4)}$, $D_{r(4)}$, $A_{mn(4)}$ and $Z_{uv(4)}$ are constants and

$$\begin{aligned} \phi_{2(4)}(x, y, t) = & \sum_{w=1}^W H_{w(4)} \frac{\cosh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} \sin(N_w x) \\ & + \sum_{i=1}^I F_{i(4)} \frac{\sinh \left[(N_i / K_2^a)x \right]}{\sinh \left[(N_i / K_2^a)S \right]} \sin \left[N_i (y-H_3) \right] - H_1 \left(\frac{S-x}{S} \right) - H_3 \left(\frac{x}{S} \right) \\ & + \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \sin(N_m x) \sin(N_n y) \exp \left[-(\lambda_{mn})^2 t \right] \\ & + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \sin(N_u x) \sin(N_v y) \exp \left[-(\lambda_{uv})^2 t \right], \end{aligned} \quad (2.190)$$

where

$$N_w = \left(\frac{w\pi}{S} \right), \quad (2.191)$$

$$N_i = \left[\left(\frac{1-2i}{2} \right) \left(\frac{\pi}{h-H_3} \right) \right], \quad (2.192)$$

$$K_2^a = \sqrt{\left(\frac{K_{x_2}}{K_{y_2}} \right)} \quad (2.193)$$

and $H_{w(4)}$ and $F_{i(4)}$ are constants. Also, like in the last two cases, the expressions for the directional velocity functions $V_{x1(4)}$ and $V_{y1(4)}$ for the top layer in this drainage situation will also be similar to that of the first case, but here again the Fourier coefficients required to be used in these expressions should be $B_{p(4)}$, $C_{q(4)}$, $E_{k(4)}$, $D_{r(4)}$, $A_{mn(4)}$ and $Z_{uv(4)}$ and not the relevant Fourier coefficient of the earlier cases. Further, in view of Eq. (2.190), the velocity functions for the bottom layer in the x - and y - directions for this case work out as

$$\begin{aligned} V_{x2(4)} = -K_{x_2} \left\{ \sum_{w=1}^W H_{w(4)} \frac{\cosh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} N_w \cos(N_w x) \right. \\ + \sum_{i=1}^I F_{i(4)} \frac{(N_i / K_2^a) \cosh \left[(N_i / K_2^a)x \right]}{\sinh \left[(N_i / K_2^a)S \right]} \sin \left[N_i (y-H_3) \right] \\ - \left(\frac{H_3-H_1}{S} \right) + \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} N_m \cos(N_m x) \sin(N_n y) \exp \left[-(\lambda_{mn})^2 t \right] \\ \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} N_u \cos(N_u x) \sin(N_v y) \exp \left[-(\lambda_{uv})^2 t \right] \right\}, \quad (2.194) \end{aligned}$$

and

$$\begin{aligned} V_{y2(4)} = -K_{y_2} \left\{ - \sum_{w=1}^W H_{w(4)} \frac{(N_w K_2^a) \sinh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} \sin(N_w x) \right. \\ + \sum_{i=1}^I F_{i(4)} \frac{\sinh \left[(N_i / K_2^a)x \right]}{\sinh \left[(N_i / K_2^a)S \right]} N_i \cos \left[N_i (y-H_3) \right] \\ + \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \sin(N_m x) N_n \cos(N_n y) \exp \left[-(\lambda_{mn})^2 t \right] \\ \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \sin(N_u x) N_v \cos(N_v y) \exp \left[-(\lambda_{uv})^2 t \right] \right\}. \quad (2.195) \end{aligned}$$

2.2.4.1 Determination of the Steady State Solution Coefficients (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 2.26)

Substituting boundary conditions (XVIIIa) and (XVIIIb) in Eq. (2.173), we have

$$\sum_{q=1}^Q C_{q(4)} \sin(N_q y) = -y, \quad 0 < y \leq H_1,$$

$$\sum_{q=1}^Q C_{q(4)} \sin(N_q y) = -H_1, \quad H_1 \leq y \leq H_3.$$

Hence, as in the first case, the final expression for $C_{q(4)}$ comes out to be

$$C_{q(4)} = \left(\frac{2}{H_3} \right) \left[\frac{-\sin(N_q H_1)}{N_q^2} \right]. \quad (2.196)$$

Similarly, boundary condition (XX) yields

$$\sum_{p=1}^P B_{p(4)} \sin(N_p y) = -y, \quad 0 < y \leq H_3.$$

Expanding the Fourier series and evaluating for $B_{p(4)}$, we get

$$B_{p(4)} = \left(\frac{2}{H_3} \right) \left[\frac{-\sin(N_p H_3)}{N_p^2} \right]. \quad (2.197)$$

The expression for $D_{r(4)}$ remains the same as in $D_{r(1)}$. To determine the coefficients $F_{i(4)}$, we substitute boundary conditions (XXIa) and (XXIb) in Eq. (2.173); this results in

$$\sum_{i=1}^I F_{i(4)} \sin[N_i (y - H_3)] - H_3 = -y, \quad H_3 \leq y \leq H_2,$$

$$\sum_{i=1}^I F_{i(4)} \sin[N_i (y - H_3)] - H_3 = -H_2, \quad H_2 \leq y < h.$$

Running a Fourier series in the concerned domain and evaluating the resulting integrals, we get

$$F_{i(4)} = \left(\frac{2}{h - H_3} \right) \left\{ \frac{-\sin[N_i (H_2 - H_3)]}{N_i^2} \right\}. \quad (2.198)$$

As before, we equate the steady state hydraulic head expressions for the top layer and the bottom layer at $y = H_3$ so as to satisfy boundary condition (IIIa). In this case, it leads to the following relationship

$$\sum_{p=1}^P B_{p(4)} \frac{\sinh\left[\left(N_p / K_1^a\right)x\right]}{\sinh\left[\left(N_p / K_1^a\right)S\right]} \sin(N_p H_3) + \sum_{q=1}^Q C_{q(4)} \frac{\sinh\left[\left(N_q / K_1^a\right)(S-x)\right]}{\sinh\left[\left(N_q / K_1^a\right)S\right]} \sin(N_q H_3)$$

$$+ \sum_{k=1}^K E_{k(4)} \frac{\sinh\left(N_k K_1^a H_3\right)}{\cosh\left(N_k K_1^a H_3\right)} \sin(N_k x) + \sum_{r=1}^R D_{r(4)} \frac{1}{\cosh\left(N_r K_1^a H_3\right)} \sin(N_r x)$$

$$= \sum_{w=1}^W H_{w(4)} \frac{\cosh\left[N_w K_2^a (h - H_3)\right]}{\sinh\left[N_w K_2^a (h - H_3)\right]} \sin(N_w x) - H_1 \left(\frac{S-x}{S} \right) - H_3 \left(\frac{x}{S} \right).$$

We eventually obtain the following equation after some mathematical manipulations

$$\begin{aligned}
& E_{k(4)} \tanh(N_i K_1^a H_3) - H_{w(4)} \coth[N_i K_2^a (h - H_3)] \\
&= -D_{r(4)} \operatorname{sech}(N_i K_1^a H_3) - \left(\frac{2}{S}\right) H_3 \int_0^S \left(\frac{x}{S}\right) \sin(N_i x) dx \\
& - \left(\frac{2}{S}\right) H_1 \int_0^S \left(\frac{S-x}{S}\right) \sin(N_i x) dx \\
& - \left(\frac{2}{S}\right) \sum_{p=1}^P B_{p(4)} \sin(N_p H_3) \int_0^S \frac{\sinh\left[\frac{(N_p/K_1^a)x}{S}\right]}{\sinh\left[\frac{(N_p/K_1^a)S}{S}\right]} \sin(N_i x) dx \\
& - \left(\frac{2}{S}\right) \sum_{q=1}^Q C_{q(4)} \sin(N_q H_3) \int_0^S \frac{\sinh\left[\frac{(N_q/K_1^a)(S-x)}{S}\right]}{\sinh\left[\frac{(N_q/K_1^a)S}{S}\right]} \sin(N_i x) dx. \tag{2.199}
\end{aligned}$$

Solving the above integrals, we get

$$\begin{aligned}
& E_{k(4)} \tanh(N_i K_1^a H_3) - H_{w(4)} \coth[N_i K_2^a (h - H_3)] = -D_{r(4)} \operatorname{sech}(N_i K_1^a H_3) \\
& + \left(\frac{2}{S}\right) H_3 \left[\frac{\cos(N_i S)}{N_i}\right] - \left(\frac{2}{S}\right) \left(\frac{H_1}{N_i}\right) + \left(\frac{2}{S}\right) \sum_{p=1}^P B_{p(4)} \sin(N_p H_3) \left[\frac{N_i \cos(N_i S)}{N_i^2 + (N_p/K_1^a)^2}\right] \\
& - \left(\frac{2}{S}\right) \sum_{q=1}^Q C_{q(4)} \sin(N_q H_3) \left[\frac{N_i}{N_i^2 + (N_q/K_1^a)^2}\right], \tag{2.200}
\end{aligned}$$

where $k = r = w = i$ and $P = Q \rightarrow \infty$. Also, to satisfy boundary condition (IIIb), the following relationship should be met

$$\begin{aligned}
& -K_{y_1} \sum_{k=1}^K E_{k(4)} (N_k K_1^a) \sin(N_k x) = \\
& K_{y_2} \sum_{w=1}^W H_{w(4)} (N_w K_2^a) \sin(N_w x) - K_{y_2} \sum_{i=1}^I F_{i(4)} N_i \frac{\sinh\left[\frac{(N_i/K_2^a)x}{S}\right]}{\sinh\left[\frac{(N_i/K_2^a)S}{S}\right]}.
\end{aligned}$$

As in the previous problems, we arrive at the following equation

$$\begin{aligned}
& K_{y_1} (N_k K_1^a) E_{k(4)} + K_{y_2} (N_w K_2^a) H_{w(4)} = \\
& \left(\frac{2}{S}\right) K_{y_2} \sum_{i=1}^I F_{i(4)} N_i \int_0^S \frac{\sinh\left[\frac{(N_i/K_2^a)x}{S}\right]}{\sinh\left[\frac{(N_i/K_2^a)S}{S}\right]} \sin(N_i x) dx. \tag{2.201}
\end{aligned}$$

The integrals in the above equation have already been evaluated. Substituting their values in Eq. (2.201), we get

$$K_{y_1} (N_k K_1^a) E_{k(4)} + K_{y_2} (N_w K_2^a) H_{w(4)} = \left(\frac{2}{S}\right) K_{y_2} \sum_{i=1}^I F_{i(4)} N_i \left[\frac{-N_i \cos(N_i S)}{N_i^2 + (N_i/K_2^a)^2}\right], \tag{2.202}$$

where $k = w = i$ and $I \rightarrow \infty$. Now, on solving Eqs. (2.200) and (2.202) simultaneously, the coefficients $E_{k(4)}$ and $H_{w(4)}$ can then be evaluated.

2.2.4.2 Determination of the Transient Solution Coefficients [transient state solution is approximate and is applicable only when conductivity and specific storage of the layers of Fig. 2.26 satisfy $K_{x_1} / S_{s_1} = K_{x_2} / S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$)]

As before, $A_{m(4)}$ has an identical expression with $A_{m(1)}$ but again the concerned Fourier coefficients in it would be different from that of the previous cases (i.e., these coefficients now would be $B_{p(4)}$, $C_{q(4)}$, $E_{k(4)}$ and $D_{r(4)}$). In this case, the final expression for the coefficient $Z_{uv(4)}$ can be expressed as

$$\begin{aligned}
 Z_{uv(4)} = & \left(\frac{4}{Sh} \right) \left\{ - \sum_{w=1}^W H_{w(4)} \int_{H_3}^h \frac{\cosh \left[(N_w K_2^a)(h-y) \right]}{\sinh \left[(N_w K_2^a)(h-H_3) \right]} \sin(N_v y) dy \right. \\
 & \times \int_0^S \sin(N_u x) \sin(N_w x) dx \\
 & - \sum_{i=1}^I F_{i(4)} \int_0^S \frac{\sinh \left[(N_i / K_2^a)x \right]}{\sinh \left[(N_i / K_2^a)S \right]} \sin(N_u x) dx \\
 & \times \int_{H_3}^h \sin \left[N_i (y - H_3) \right] \sin(N_v y) dy \\
 & + H_1 \int_0^S \left(\frac{S-x}{S} \right) \sin(N_u x) dx \int_{H_3}^h \sin(N_v y) dy \\
 & \left. + H_3 \int_0^S \left(\frac{x}{S} \right) \sin(N_u x) dx \int_{H_3}^h \sin(N_v y) dy \right\}. \quad (2.203)
 \end{aligned}$$

Solving the above integrals, we get

$$\begin{aligned}
 Z_{uv(4)} = & \left(\frac{4}{Sh} \right) \left\{ -H_{w(4)} \left(\frac{S}{2} \right) \left[\frac{N_v^2}{N_v^2 + (N_w K_2^a)^2} \right] \times \right. \\
 & \left\{ \coth \left[(N_w K_2^a)(h-H_3) \right] \frac{\cos(N_v H_3)}{N_v} + \left[\frac{(N_w K_2^a)}{N_v^2} \right] \sin(N_v H_3) \right\} \\
 & - \sum_{i=1}^I F_{i(4)} \left[\frac{N_u^2}{N_u^2 + (N_i / K_2^a)^2} \right] \left[\frac{-\cos(N_u S)}{N_u} \right] \times I^{(3)} \\
 & \left. + H_1 \left(\frac{1}{N_u} \right) \left(\frac{\cos N_v H_3}{N_v} \right) + H_3 \left(\frac{\cos N_v H_3}{N_v} \right) \left[\frac{-\cos(N_u S)}{N_u} \right] \right\}, \quad (2.204)
 \end{aligned}$$

where $u_1 = w$, $I \rightarrow \infty$ and the integral $I^{(3)}$ is the same as that mentioned in Eq. (2.136). The expressions for $Q_{top(4)}$, $Q_{topx(4)}$ and $Vol_{top(4)}$ will remain the same as in Eqs. (2.93), (2.96) and (2.100), respectively except that here also we need to use the Fourier coefficients specific to this case (i.e., $B_{p(4)}$, $C_{q(4)}$, $E_{k(4)}$, $D_{r(4)}$, $A_{m(4)}$ and $Z_{w(4)}$) only and not the coefficients

pertaining to the earlier cases. $Q_{leftside(4)}$, $Q_{rightside(4)}$, $Vol_{leftside(4)}$ and $Vol_{rightside(4)}$ for this case work out as

$$\begin{aligned}
Q_{leftside(4)}(t) = & K_{x_1} \left\{ \sum_{p=1}^P B_{p(4)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[\left(N_p / K_1^a \right) S \right]} \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(4)} \left(\frac{1}{K_1^a} \right) \coth \left[\left(N_q / K_1^a \right) S \right] \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(4)} \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& + \sum_{r=1}^R D_{r(4)} \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \left(\frac{N_m}{N_n} \right) \left[1 - \cos(N_n H_3) \right] \exp \left[-(\lambda_{mn})^2 t \right] \\
& + \left. \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \left(\frac{N_u}{N_v} \right) \left[1 - \cos(N_v H_3) \right] \exp \left[-(\lambda_{uv})^2 t \right] \right\} \\
& + K_{x_2} \left\{ \sum_{w=1}^W H_{w(4)} \left(\frac{1}{K_2^a} \right) - \sum_{j=1}^J G_{j(4)} \left(\frac{1}{K_2^a} \right) \coth \left[\left(N_j / K_2^a \right) S \right] \right. \\
& - \left. \left[\frac{(H_3 - H_1)(h - H_3)}{S} \right] \right. \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \exp \left[-(\lambda_{mn})^2 t \right] \\
& + \left. \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \exp \left[-(\lambda_{uv})^2 t \right] \right\}, \tag{2.205}
\end{aligned}$$

$$\begin{aligned}
Q_{rightside(4)}(t) = & -K_{x_1} \left\{ \sum_{p=1}^P B_{p(4)} \left(\frac{1}{K_1^a} \right) \coth \left[\left(N_p / K_1^a \right) S \right] \left[1 - \cos(N_p H_3) \right] \right. \\
& - \sum_{q=1}^Q C_{q(4)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[\left(N_q / K_1^a \right) S \right]} \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(4)} \cos(N_k S) \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& + \sum_{r=1}^R D_{r(4)} \cos(N_r S) \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \\
& + \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \left[1 - \cos(N_n H_3) \right] \exp \left[-(\lambda_{mn})^2 t \right] \\
& + \left. \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \left[1 - \cos(N_v H_3) \right] \exp \left[-(\lambda_{uv})^2 t \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& -K_{x_2} \left\{ \sum_{w=1}^W H_{w(4)} \left(\frac{1}{K_2^a} \right) \cos(N_w S) - \sum_{j=1}^J G_{j(4)} \left(\frac{1}{K_2^a} \right) \frac{1}{\sinh[(N_j / K_2^a) S]} \right. \\
& \quad \left. - \left[\frac{(H_3 - H_1)(h - H_3)}{S} \right] \right. \\
& \quad + \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \exp[-(\lambda_{mn})^2 t] \\
& \quad \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \exp[-(\lambda_{uv})^2 t] \right\}, \tag{2.206}
\end{aligned}$$

$$\begin{aligned}
Vol_{leftside(4)} = & K_{x_1} T \left\{ \sum_{p=1}^P B_{p(4)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh[(N_p / K_1^a) S]} [1 - \cos(N_p H_3)] \right. \\
& - \sum_{q=1}^Q C_{q(4)} \left(\frac{1}{K_1^a} \right) \coth[(N_q / K_1^a) S] [1 - \cos(N_q H_3)] \\
& + \sum_{k=1}^K E_{k(4)} \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& \left. + \sum_{r=1}^R D_{r(4)} \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \right\} \\
& + K_{x_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \left(\frac{N_m}{N_n} \right) [1 - \cos(N_n H_3)] \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \left(\frac{N_u}{N_v} \right) [1 - \cos(N_v H_3)] \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\} \\
& + K_{x_2} T \left\{ \sum_{w=1}^W H_{w(4)} \left(\frac{1}{K_2^a} \right) - \sum_{j=1}^J G_{j(4)} \left(\frac{1}{K_2^a} \right) \coth[(N_j / K_2^a) S] \right. \\
& \quad \left. - \left[\frac{(H_3 - H_1)(h - H_3)}{S} \right] \right\} \\
& + K_{x_2} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \right. \\
& \quad \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\} \tag{2.207}
\end{aligned}$$

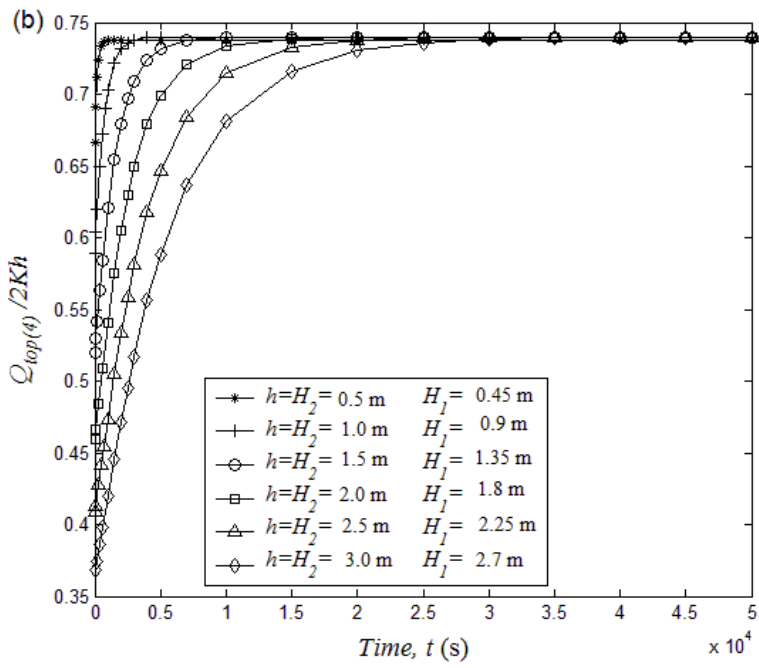
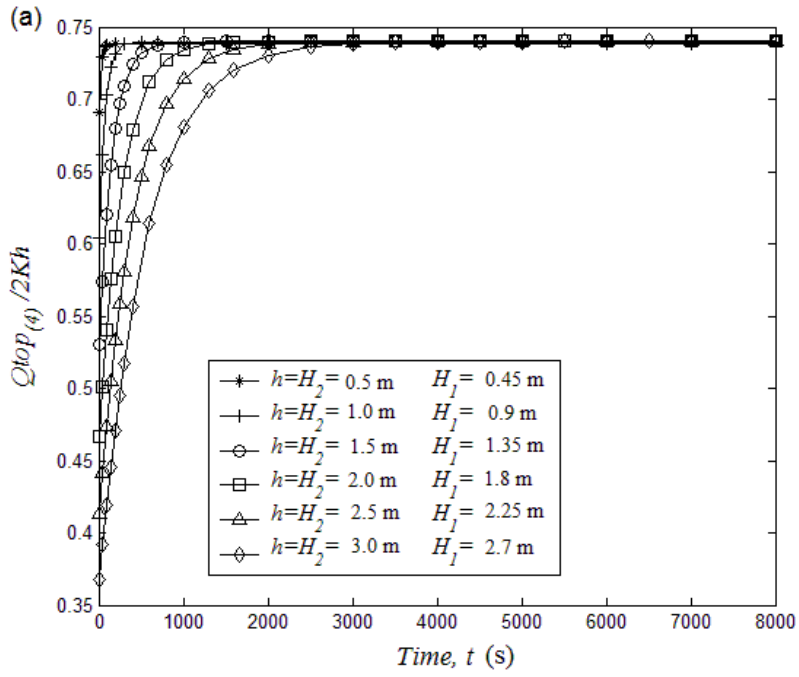
and

$$Vol_{rightside(4)} = -K_{x_1} T \left\{ \sum_{p=1}^P B_{p(4)} \left(\frac{1}{K_1^a} \right) \coth[(N_p / K_1^a) S] [1 - \cos(N_p H_3)] \right\}$$

$$\begin{aligned}
& - \sum_{q=1}^Q C_{q(4)} \left(\frac{1}{K_1^a} \right) \frac{1}{\sinh \left[(N_q / K_1^a) S \right]} \left[1 - \cos(N_q H_3) \right] \\
& + \sum_{k=1}^K E_{k(4)} \cos(N_k S) \left(\frac{1}{K_1^a} \right) \left[\frac{\cosh(N_k K_1^a H_3) - 1}{\cosh(N_k K_1^a H_3)} \right] \\
& + \sum_{r=1}^R D_{r(4)} \cos(N_r S) \left(\frac{1}{K_1^a} \right) \tanh(N_r K_1^a H_3) \left. \vphantom{\sum_{q=1}^Q} \right\} \\
& - K_{x_1} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \left[1 - \cos(N_n H_3) \right] \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \left[1 - \cos(N_v H_3) \right] \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\} \\
& - K_{x_2} T \left\{ \sum_{w=1}^W H_{w(4)} \left(\frac{1}{K_2^a} \right) \cos(N_w S) - \sum_{j=1}^J G_{j(4)} \left(\frac{1}{K_2^a} \right) \frac{1}{\sinh \left[(N_j / K_2^a) S \right]} \right. \\
& \left. - \left[\frac{(H_3 - H_1)(h - H_3)}{S} \right] \right\} \\
& - K_{x_2} \left\{ \sum_{m=1}^M \sum_{n=1}^N A_{mn(4)} \cos(N_m S) \left(\frac{N_m}{N_n} \right) \cos(N_n H_3) \left\{ \frac{1 - \exp[-(\lambda_{mn})^2 T]}{(\lambda_{mn})^2} \right\} \right. \\
& \left. + \sum_{u=1}^U \sum_{v=1}^V Z_{uv(4)} \cos(N_u S) \left(\frac{N_u}{N_v} \right) \cos(N_v H_3) \left\{ \frac{1 - \exp[-(\lambda_{uv})^2 T]}{(\lambda_{uv})^2} \right\} \right\}. \quad (2.208)
\end{aligned}$$

2.2.4.3 Verifications of the Proposed Solution

Like in the previous cases, here also we perform a few checks to test the correctness of our solution. Figs. 2.27 – 2.30 illustrate that the predictions from our model could successfully capture the predictions obtained from other well established analytical models for a few tested drainage situations thereby proving that the proposed analytical solution for the flow problem of Fig. 2.26 has been rightly developed. Also, as can be observed from Fig. 2.27, here also the $Q_{top(4)} / 2Kh$ ratio attains a steady state value of 0.739 which is in close agreement with the corresponding values obtained from the analytical and experimental works of Fukuda (1957). Further, from Fig. 2.28 we see that the predictions from our solution at large times (steady state) are matching perfectly with those obtained from Youngs' (1994) solution – this is the way it should be since Youngs solution is for the steady state only. Also, Figs. 2.31, 2.32 and 2.33 show that the predictions from our model agree very well with the corresponding ones obtained via numerical simulations (MODFOW) for the studied drainage situations; this further adds to the veracity of our solution for the concerned drainage case of Fig. 2.26.



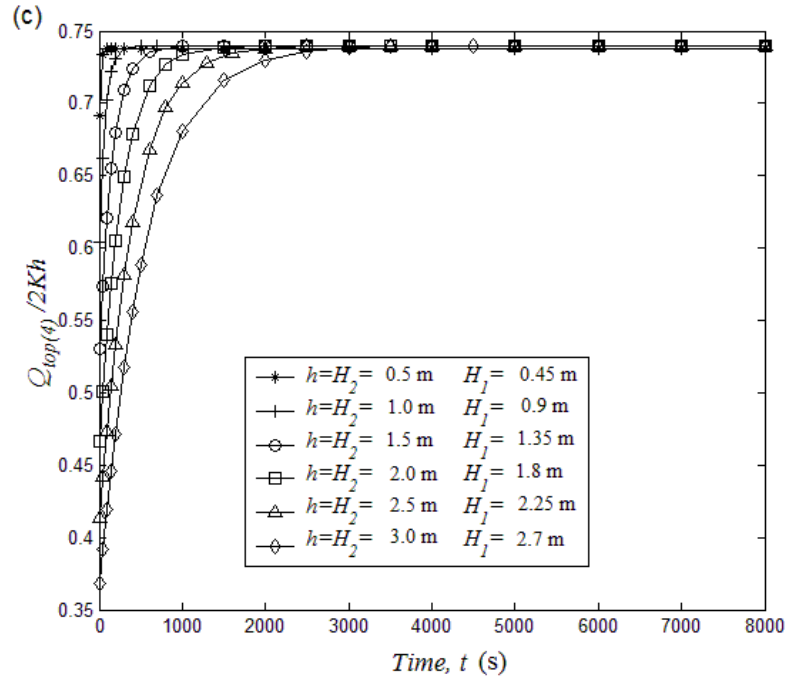


Fig. 2.27. Variation of $Q_{top(t)}/2Kh$ ratios with time as obtained from the proposed solution for different h and $H_2=h$ values when the other parameters of the flow problem are considered as $S=100 \text{ m}$, $\delta=0 \text{ m}$, $H_3=0.95 h$ and (a) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.05 \text{ m/day}$, $S_{s_1}=S_{s_2}=0.001 \text{ m}^{-1}$ and (b) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.5 \text{ m/day}$, $S_{s_1}=S_{s_2}=0.001 \text{ m}^{-1}$ and (c) $K_{x_1}=K_{y_1}=K_{x_2}=K_{y_2}=0.05 \text{ m/day}$, $S_{s_1}=S_{s_2}=0.0001 \text{ m}^{-1}$

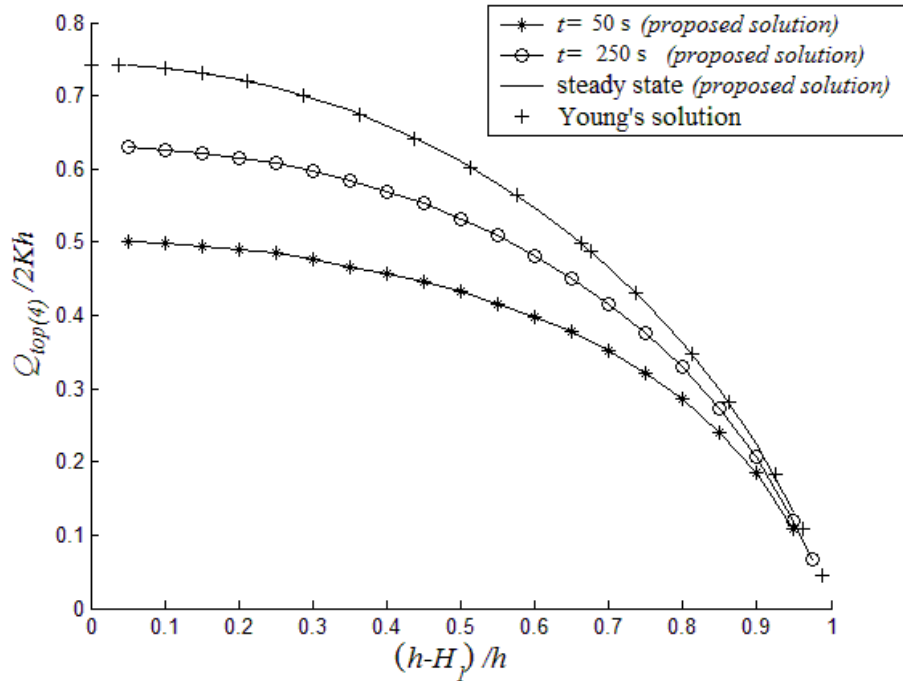


Fig. 2.28. Plots of $Q_{top(4)} / 2Kh$ versus $(h - H_1) / h$ as obtained from the proposed solution for different times of simulation of the system with the corresponding steady state plot as obtained from Youngs' (1994) single-layered solution of the problem for isotropic soils when the ponding depth is considered as zero and the other flow parameters of the problem are taken as $S = 100$ m, $h = 2.0$ m, $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.05$ m/day and $S_{s_1} = S_{s_2} = 0.0001$ m⁻¹

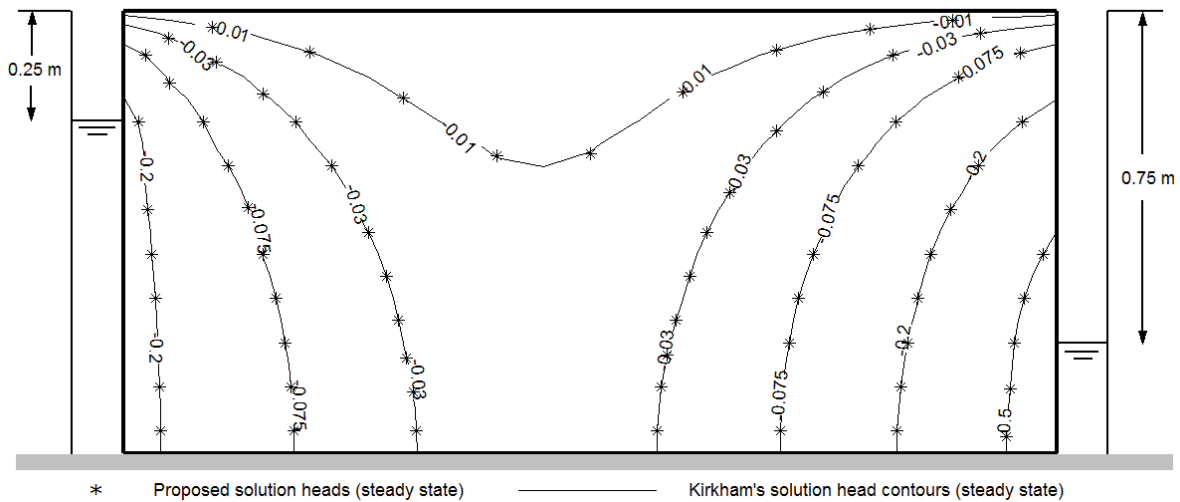


Fig. 2.29. Comparison of steady state hydraulic heads as obtained from the proposed solution with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S = 5$ m, $h = 1$ m, $H_1 = 0.25$ m, $H_2 = 0.75$ m, $\delta = 0$ m and $K_{x_1} = K_{y_1} = K_{x_2} = K_{y_2} = 0.5$ m/day

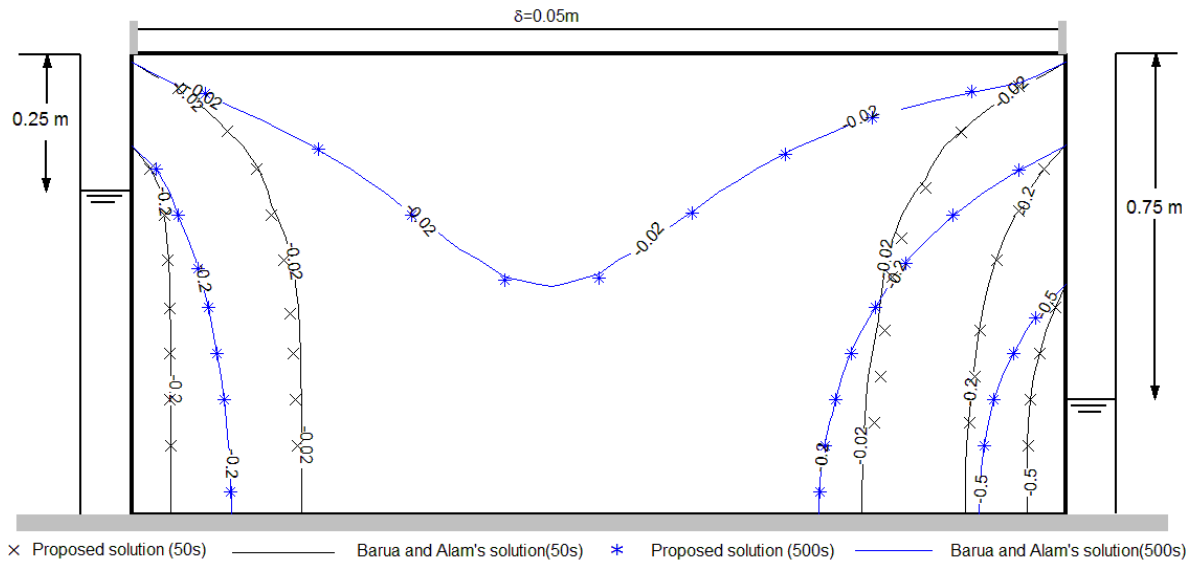


Fig. 2.30. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained from Barua and Alam's (2012) single-layered transient solution of the problem for anisotropic soils at two different times of simulation of the system when the flow parameters for the problem are considered as $S = 5 \text{ m}$, $h = 1 \text{ m}$, $H_1 = 0.75 \text{ m}$, $H_2 = 0.25 \text{ m}$, $\varepsilon = 0.2 \text{ m}$, $\delta = 0.05 \text{ m}$, $K_{x_1} = K_{x_2} = 0.15 \text{ m/day}$, $K_{y_1} = K_{y_2} = 0.075 \text{ m/day}$ and $S_{s_1} = S_{s_2} = 0.001 \text{ m}^{-1}$

2.2.4.4 MODFLOW Verification of the Proposed Solution

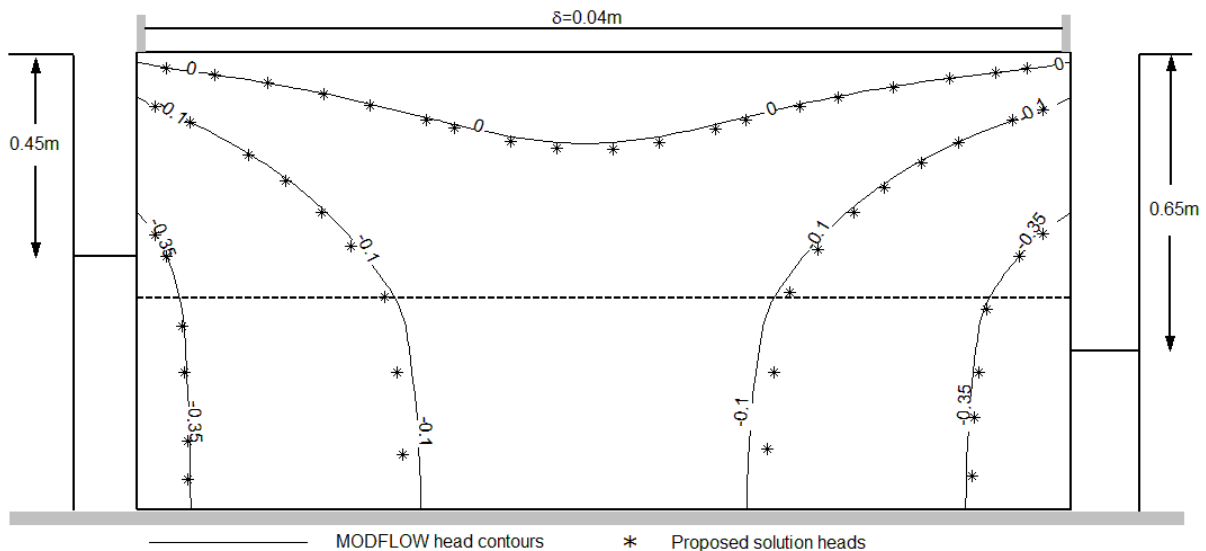


Fig. 2.31. Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding values as obtained from MODFLOW when the flow parameters of the problem are considered as $S = 5.0 \text{ m}$, $h = 1.0 \text{ m}$, $H_3 = 0.55 \text{ m}$, $H_1 = 0.45 \text{ m}$, $H_2 = 0.65 \text{ m}$, $\delta = 0.04 \text{ m}$, $\varepsilon = 0.05 \text{ m}$, $K_{x_1} = 2 \text{ m/day}$, $K_{y_1} = 0.5 \text{ m/day}$, $K_{x_2} = 1 \text{ m/day}$ and $K_{y_2} = 1.5 \text{ m/day}$

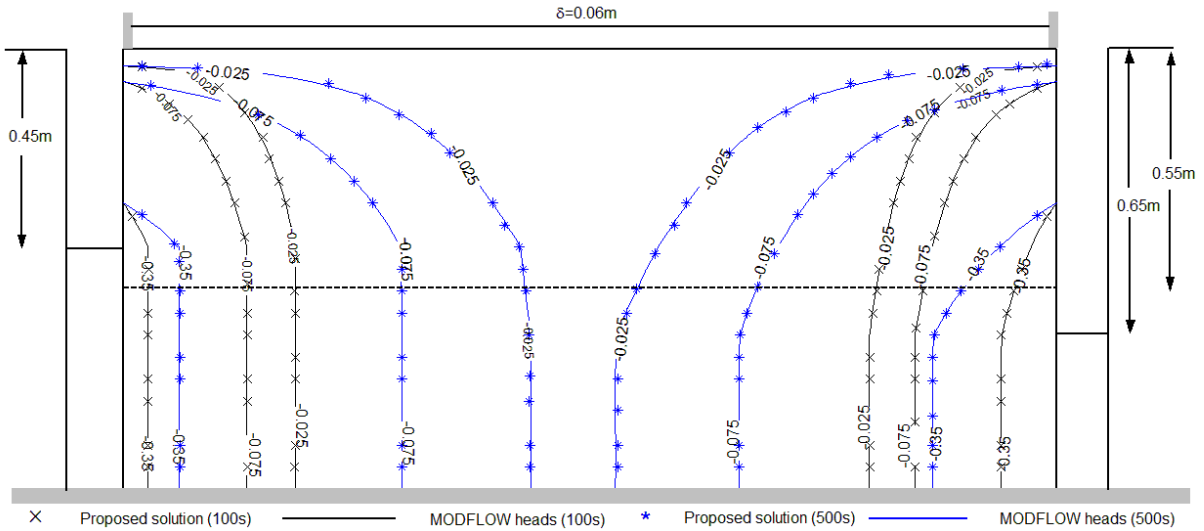


Fig 2.32. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.45$ m, $H_2 = 0.65$ m, $\varepsilon = 0.05$ m, $\delta = 0.06$ m, $K_{x_1} = 4$ m/day, $K_{x_2} = 1$ m/day, $K_{y_1} = K_{y_2} = 0.001$ m/day, $S_{s_1} = 0.04$ m⁻¹ and $S_{s_2} = 0.01$ m⁻¹

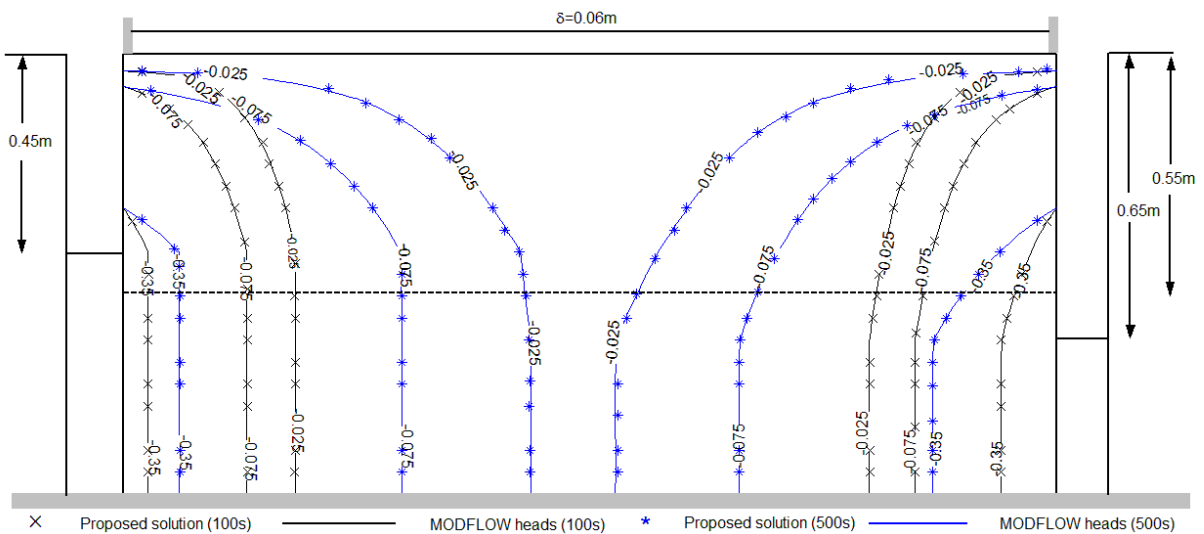
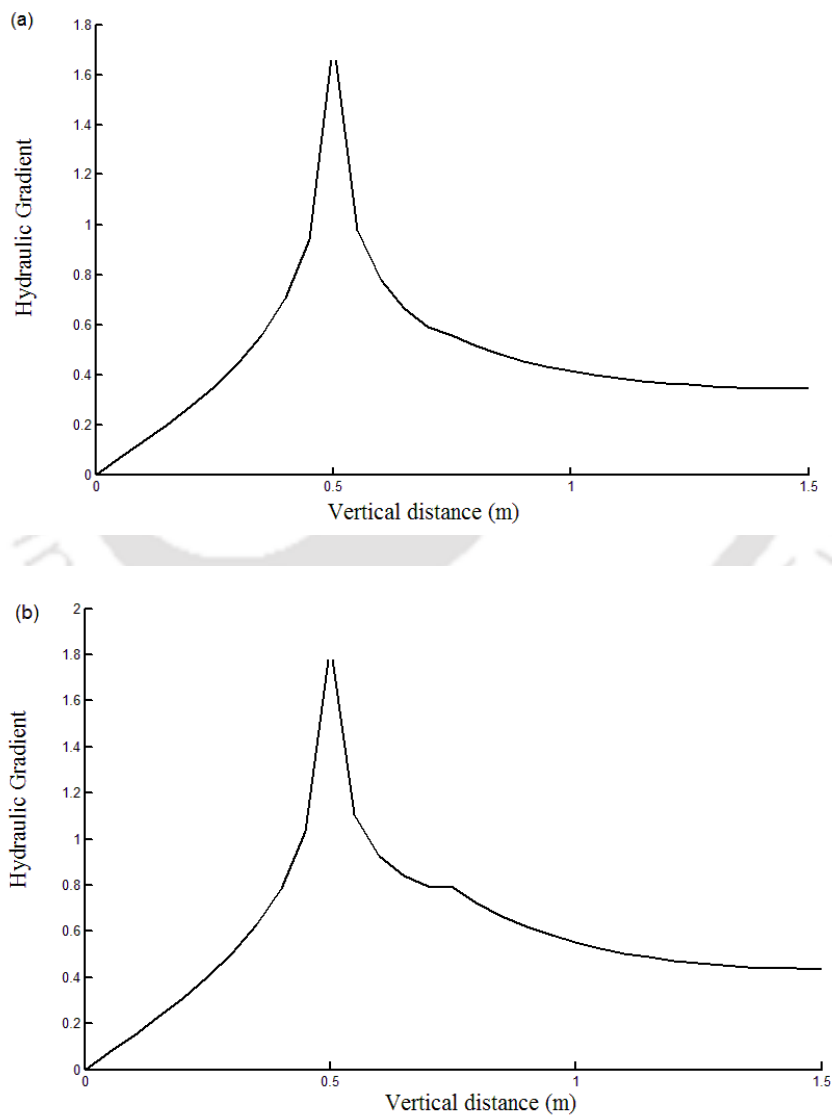


Fig 2.33. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times of simulation of the system when the flow parameters of the problem are considered as $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.45$ m, $H_2 = 0.65$ m, $\varepsilon = 0.05$ m, $\delta = 0.06$ m, $K_{x_1} = 4$ m/day, $K_{x_2} = 1$ m/day, $K_{y_1} = K_{y_2} = 0.01$ m/day, $S_{s_1} = 0.04$ m⁻¹ and $S_{s_2} = 0.01$ m⁻¹

2.3 Discussions

In this section, we will try to analyze a few flow situations of Figs 2.2, 2.10, 2.18 and 2.26 by making use of the proposed analytical models for the same. We first investigate the steady state hydraulic gradient distribution at the face of a ditch of a two-layered ditch drainage system for different conductivity variations of the layers. The information regarding exit gradients at the face of a ditch/trench receiving water from a flooded field is important as it throws light on the possible locations where gully formation at the bank of the ditch/trench is likely to occur (Römken 2009, 2010). From Fig. 2.34, we see that the maximum gradient occurs at a location where the water level in a ditch meets the soil surface of a layered ponded drainage system. The analytical works of Barua and Sarmah (2016) on the single-layered ponded drainage problem has also led to a similar conclusion. Also, as can be seen from Fig. 2.34, the increase in the directional conductivities of the top layer has a tendency to shift the falling limb of the hydraulic curve upward and reverse is the trend when the conductivity of the bottom layer is made to increase over that of the top layer.



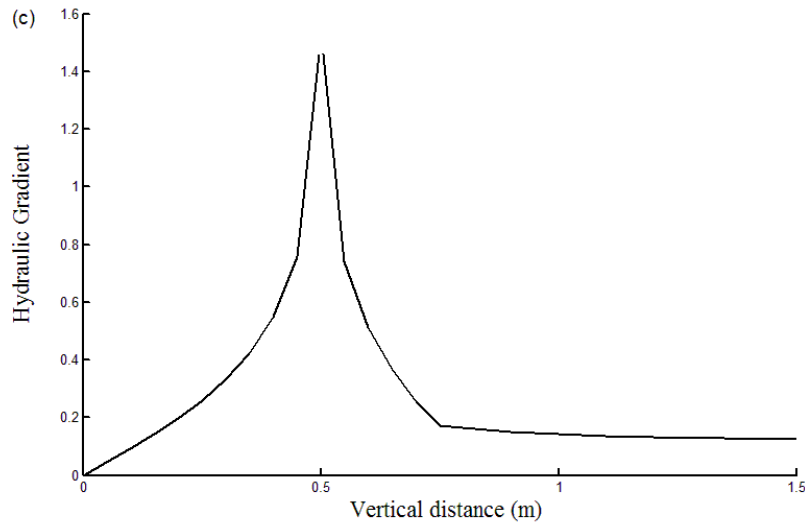


Fig 2.34. Variation of steady state horizontal gradient with depth of water at the face of a ditch when the parameters of the flow problem of Fig. 2.2 are taken as $S=5$ m, $h=1.5$ m, $H_3=0.75$ m, $H_1=H_2=0.5$ m, $\delta=0$ m and (a) $K_{x_1}=K_{x_2}=K_{y_1}=K_{y_2}=0.4$ m/day and (b) $K_{x_1}=K_{y_1}=1$ m/day, $K_{x_2}=K_{y_2}=0.1$ m/day and (c) $K_{x_1}=K_{y_1}=0.1$ m/day and $K_{x_2}=K_{y_2}=1$ m/day

We next investigate how the directional conductivity differences of the layers affect the steady state top discharge of a two-layered ponded drainage system. Towards realizing that, we consider two drainage examples as shown in Fig 2.35. For the first example, we consider the drainage parameters of Fig 2.10 as shown in Fig. 2.35 and then note the top discharge variation for this scenario when the directional conductivities of the bottom layer – being expressed as ratios with respect to the directional conductivities of the top layers – are allowed to progressively increase from 0.0625 to 1 m/day (i.e., $\frac{K_{x_2}}{K_{x_1}} = \frac{K_{y_2}}{K_{y_1}} = 0.0625$,

0.125, 0.25, 0.5 and 1). For this simulation, K_{x_1} and K_{y_1} are taken as 4 and 1 m/day, respectively. Fig 2.35 shows the variation of top discharge with change in hydraulic conductivities of the bottom layer for the studied drainage example; as can be seen, the top discharge, expectedly, increases with the increase in directional conductivities of the bottom layer and for this scenario, this variation is close to linear. However, for the studied example, when the situation is reversed (i.e., when K_{x_2} and K_{y_2} are taken as 4 and 1 m/day ,

respectively and conductivity ratio $\frac{K_{x_1}}{K_{x_2}} = \frac{K_{y_1}}{K_{y_2}}$ is then made to increase from 0.0625 to 1)

then also the top discharge can be seen to increase with the increase in conductivities of the top layer but now this change is much more prominent as compared to the former case, particularly for situations where the conductivity ratio is low. This, thus tells that, among other factors remaining the same, the top discharge is more sensitive to conductivities of the top layer in comparison to conductivities of the bottom layer.

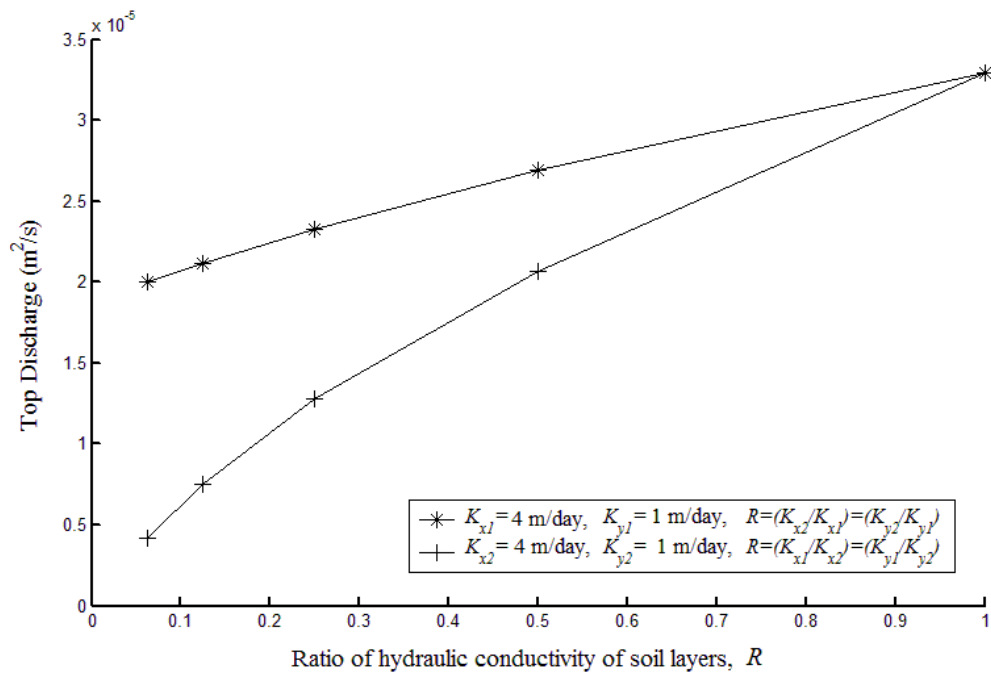


Fig. 2.35. Plot of steady state top discharge versus the ratio of hydraulic conductivities of the top and bottom soil layers when the other parameters of the flow problem of Fig 2.10 are taken as $S = 5 \text{ m}$, $h = 1 \text{ m}$, $H_3 = 0.55 \text{ m}$, $H_1 = H_2 = 1 \text{ m}$ and $\delta = 0 \text{ m}$

We also proceed to evaluate the variation of steady state top discharge with change in thickness of the constituent soil layers by keeping the combined thickness of the soil layers as same at the same time; Figs. 2.36 and 2.37 show such a variation for a typical drainage setting of Fig. 2.10. As can be seen, for this drainage setting with higher directional conductivities of the top layer as compared to that of the bottom layer, the top discharge increases almost linearly with the increase in depth of the top soil layer. However, if the situation is reversed and the conductivities of the bottom layer are made higher than that of the top layer, the top discharge can then be seen to follow a reverse trend, decreasing now almost linearly with the increase in thickness of the top layer for the considered drainage situation.

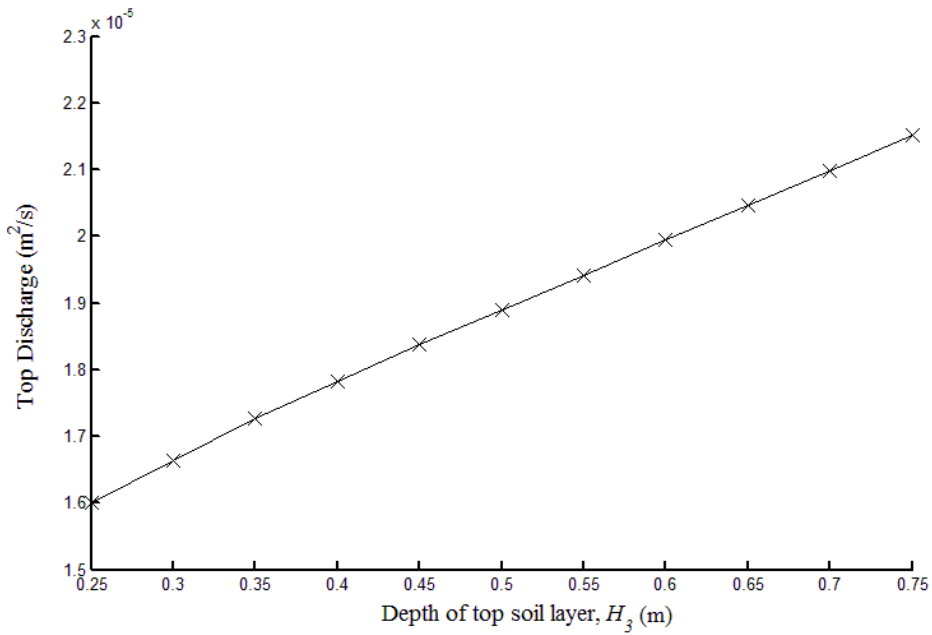


Fig. 2.36. Plot of steady state top discharge values versus the thickness of the top soil layer when the other parameters of the flow problem of Fig. 2.10 are taken as $S = 5$ m, $h = 1$ m, $H_1 = H_2 = 1$ m, $\delta = 0$ m, $K_{x_1} = 1$ m/day, $K_{y_1} = 2$ m/day, $K_{x_2} = 0.5$ m/day and $K_{y_2} = 1$ m/day

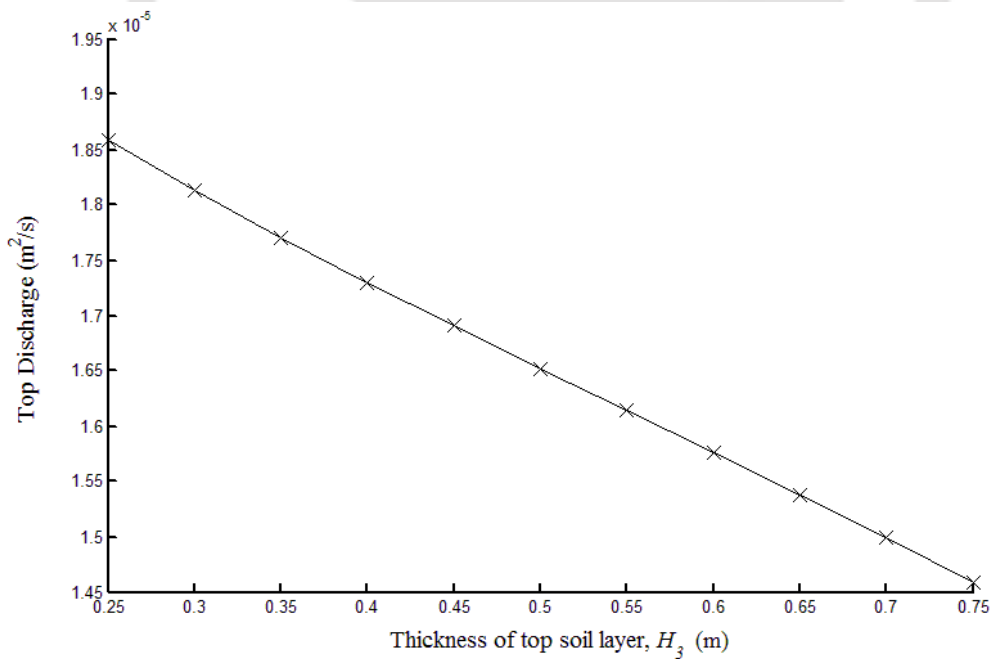


Fig. 2.37. Plot of steady state top discharge versus the thickness of the top soil layer when the other parameters of the flow problem of Fig. 2.10 are taken as $S = 5$ m, $h = 1$ m, $H_1 = H_2 = 1$ m, $\delta = 0$ m, $K_{x_1} = 0.5$ m/day, $K_{y_1} = 1$ m/day, $K_{x_2} = 1$ m/day and $K_{y_2} = 2$ m/day

We next study the surface distribution of discharge of a two-layered ponded drainage system under steady state. We consider different combinations of directional hydraulic conductivities of the constituent soil layers. A particular configuration of Fig. 2.2 is considered with the flow parameters as mentioned in Fig. 2.38. From the figure, it can be seen that the top discharge function – expressed as a percentage of the total steady state top discharge – becomes more evenly distributed at the surface of the soil with the increase in directional conductivities of the lower layer, provided the other parameters of the concerned drainage system remain the same. Thus, the presence of a more conductive soil layer below a relatively less pervious layer automatically helps in achieving a more uniform distribution of flow in a typical two-layered ponded drainage space. Also, when the upper soil layer has considerably higher hydraulic conductivity than the bottom layer, then the top discharge function curves indicate that bulk of the discharge to a ditch comes from regions close to the ditch and that considerably less discharge contribution to the drains is being made from areas lying further away from the drains. Similar behavior of the top discharge function could also be observed if the ponded system of Fig. 2.38 is allowed to operate with the ditches running empty. From this analysis, it is clear that to reclaim a salty two-layered soil using a uniformly ponded system, considerably more water need to be provided for efficient cleaning of areas lying further away from the ditches, particularly if the directional conductivities of the bottom layer are less.

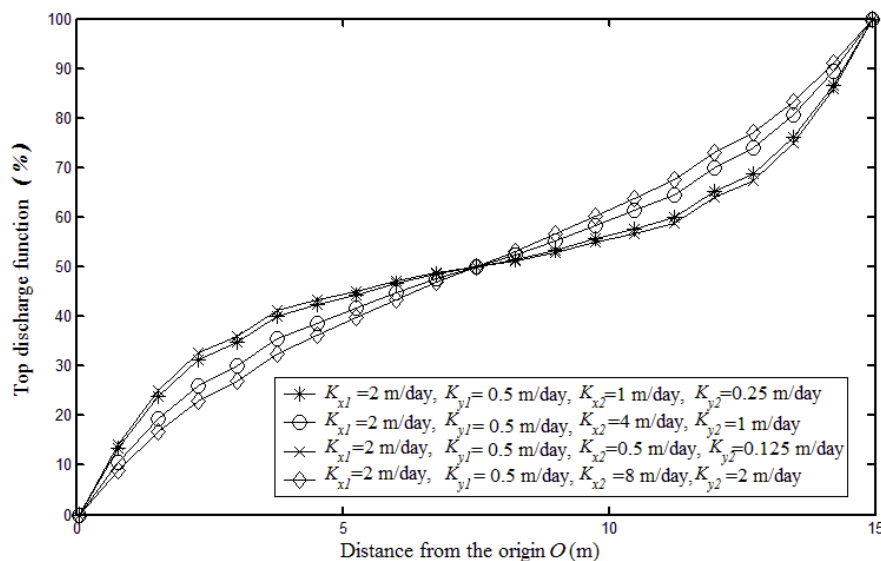


Fig. 2.38. Plots of the steady state top discharge function with horizontal distance at the surface of a two-dimensional ponded ditch drainage system when the flow parameters of the problem are taken as $S = 15$ m, $h = 2$ m, $H_3 = 1.35$ m, $H_1 = H_2 = 1.0$ m, $\varepsilon = 0.05$ m, $\delta_1 = 0$ m, $\delta_2 = 0.04$ m, $\delta_3 = 0.08$ m, $\delta_4 = 0.12$ m, $\delta_5 = 0.08$ m, $\delta_6 = 0.04$ m, $\delta_7 = 0$ m, $S_1 = 1$ m, $S_2 = 3$ m, $S_3 = 5$ m, $S_4 = 10$ m, $S_5 = 12$ m, $S_6 = 14$ m and (a) $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 1$ m/day, $K_{y_2} = 0.25$ m/day, and (b) $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 0.5$ m/day, $K_{y_2} = 0.125$ m/day, and (c) $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 4$ m/day, $K_{y_2} = 1$ m/day, and (d) $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 8$ m/day, $K_{y_2} = 2$ m/day

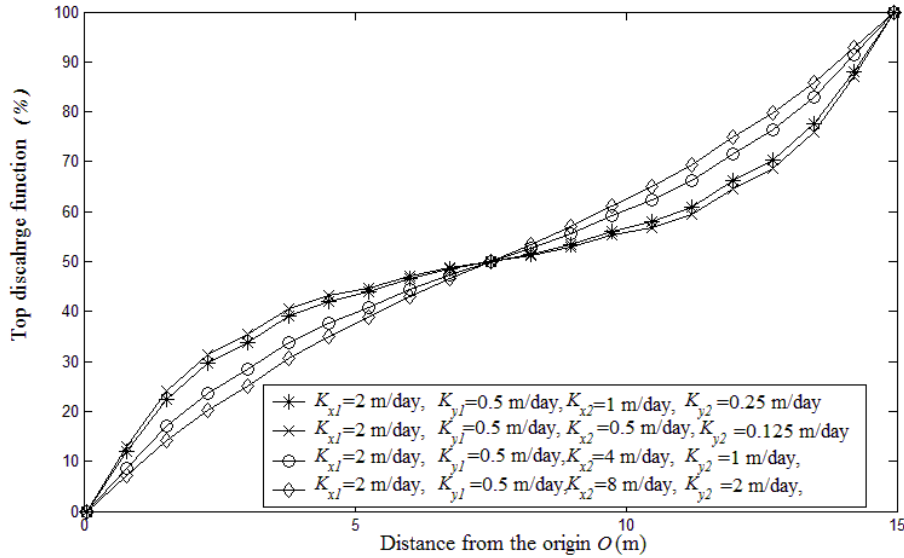
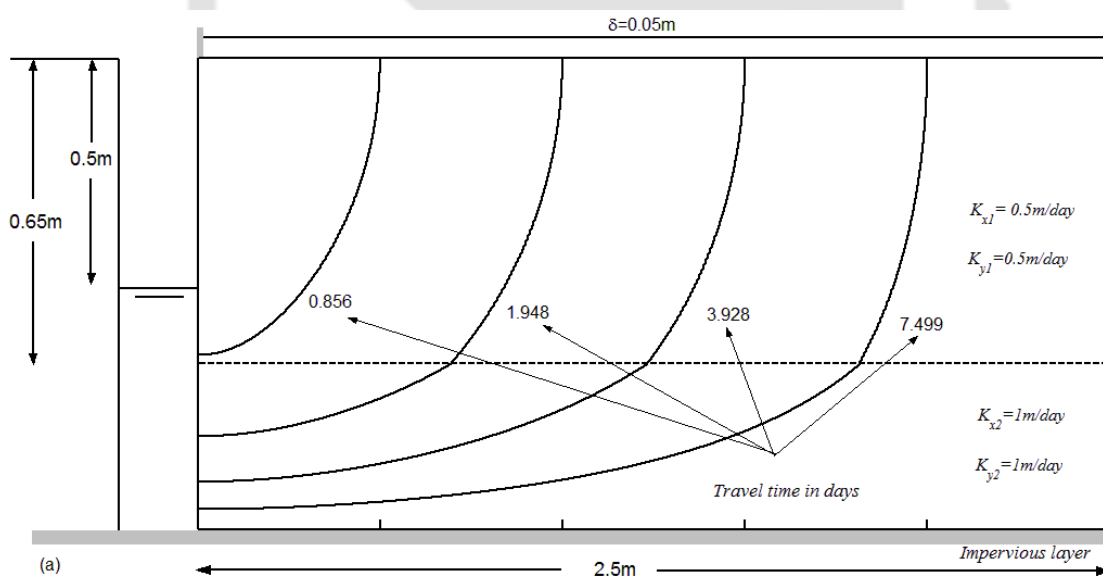
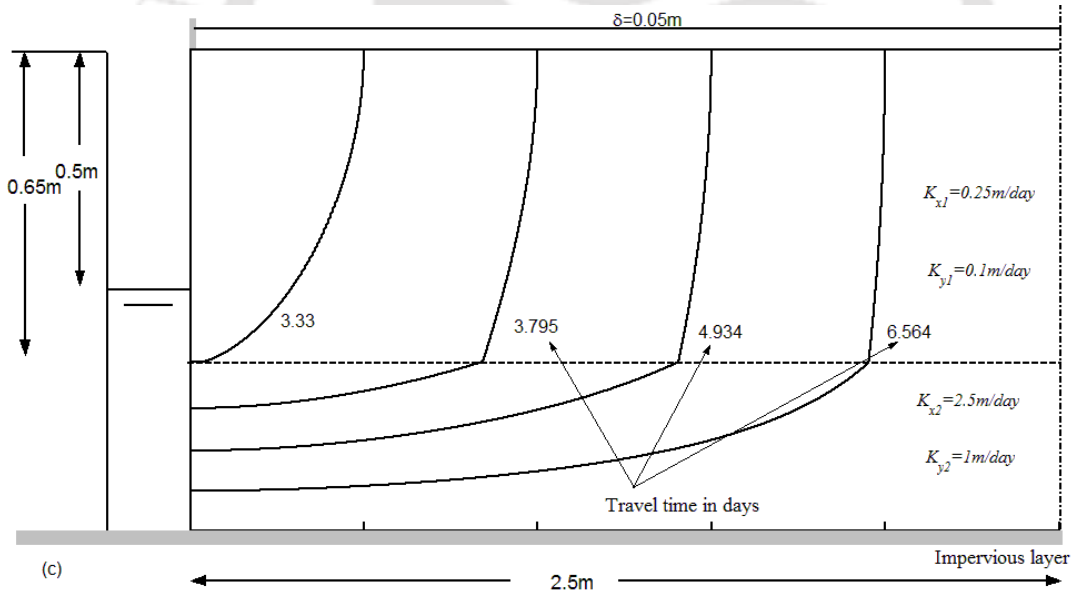
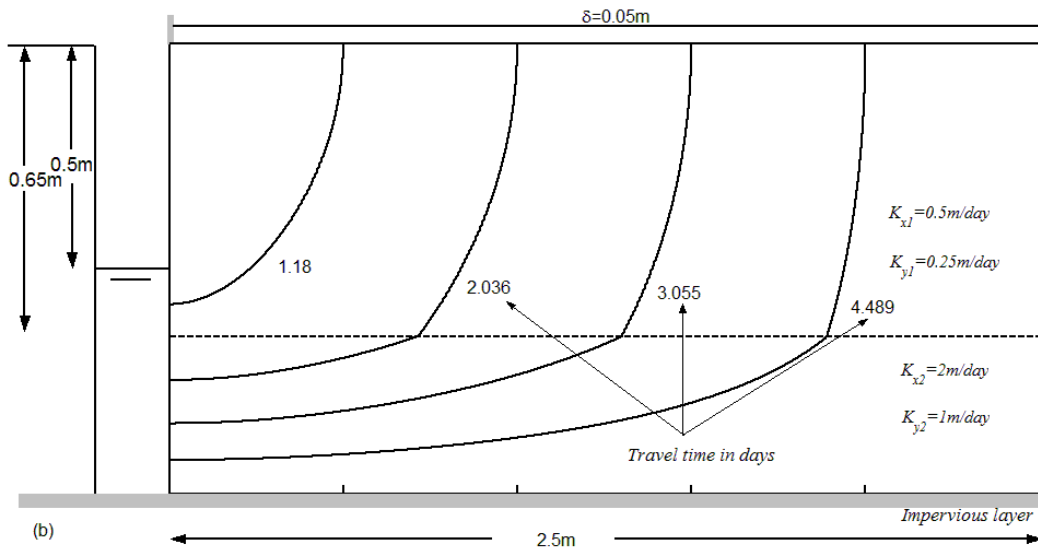


Fig. 2.39. Plots of the steady state top discharge function with horizontal distance at the surface of a two-dimensional ponded ditch drainage system when the flow parameters of the problem are taken as $S = 15$ m, $h = 2$ m, $H_3 = 1.35$ m, $H_1 = H_2 = 2.0$ m, $\varepsilon = 0.05$ m, $\delta_1 = 0$ m, $\delta_2 = 0.04$ m, $\delta_3 = 0.08$ m, $\delta_4 = 0.12$ m, $\delta_5 = 0.08$ m, $\delta_6 = 0.04$ m, $\delta_7 = 0$ m, $S_1 = 1$ m, $S_2 = 3$ m, $S_3 = 5$ m, $S_4 = 10$ m, $S_5 = 12$ m, $S_6 = 14$ m and (a) $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 1$ m/day, $K_{y_2} = 0.25$ m/day, and (b) $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 0.5$ m/day, $K_{y_2} = 0.125$ m/day, and (c) $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 4$ m/day, $K_{y_2} = 1$ m/day, and (d) $K_{x_1} = 2$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 8$ m/day, $K_{y_2} = 2$ m/day

We now study the steady state travel time distribution in a two-layered ditch drainage system for a few drainage situations of Figs. 2.2, 2.10 and 2.18. We start with some simulations (Figs. 2.40 and 2.41) where we are keeping the water level of the right and left trenches as the same. For a uniformly ponded situation this naturally will make the flow system symmetrical with respect to a centroidal axis passing halfway between and parallel to the drains. Thus, for such situations we need to consider only half of the flow domain as one half will be simply the mirror image of the other. For estimating the travel time, we have followed a procedure as proposed by Grove et al. (1970). In this method, the flow space is first divided into grids of any desired size and then using the velocity distribution functions V_{x1} , V_{y1} , V_{x2} and V_{y2} , a particle at a location (x,y) in the i^{th} soil layer of the flow domain at time t is then traced to its new location $(x + \Delta x, y + \Delta y)$ in time $t + \Delta t$ by making use of the expressions $\Delta x = V_{x1} \Delta t / \eta_i$ and $\Delta y = V_{y1} \Delta t / \eta_i$, where η_i is the porosity of the i^{th} soil layer. All the pathlines of Figs. 2.40, 2.41, 2.42, 2.43 and 2.44 have been drawn by making use of this methodology. It should be noted that since we are dealing with steady state flow here, all the pathlines shown in these figures are also their respective streamlines. From Figs. 2.40 and 2.41, it is clear that the travel time of a water particle in a layered ponded drainage system is highly sensitive to

the directional conductivities of the constituent layers and its point of origin at the surface of the soil. Increase in directional conductivities of the layers, expectedly, tend to lower the travel times and decrease in conductivity of the layers tend to enhance it, irrespective of whether the level of water in the ditches are above or below the boundary of two layers. Further, with the increase in horizontal hydraulic conductivity of the layers, the streamlines tend to get flattened out and with the decrease in horizontal conductivity of the layers, they tend to get straightened out, this effect being more pronounced for pathlines originating from surficial locations further away from the ditches. Higher values of horizontal hydraulic conductivities of the layers cause the travel times along streamlines far away from the ditch boundaries to decrease considerably while lower values of vertical conductivities of the layers cause the travel times to increase noticeably, particularly for water particles traversing along streamlines originating from surficial locations close to the drains. Also, when the vertical hydraulic conductivity of the upper soil layer is much less as compared to that of the lower soil layer, the streamlines in the upper layer are almost vertical; however, they may show wide deviation as they move in the lower layer on their way to the recipient ditches. It may also be noticed that the streamlines tend to unbuckle when they enter from the top to the bottom layer when the horizontal hydraulic conductivity of the top layer is relatively less than that of the bottom layer. These observations are in general agreement with those made by Sarmah and Barua (2015) while analyzing flow behavior in a partially penetrating ponded ditch drainage system.





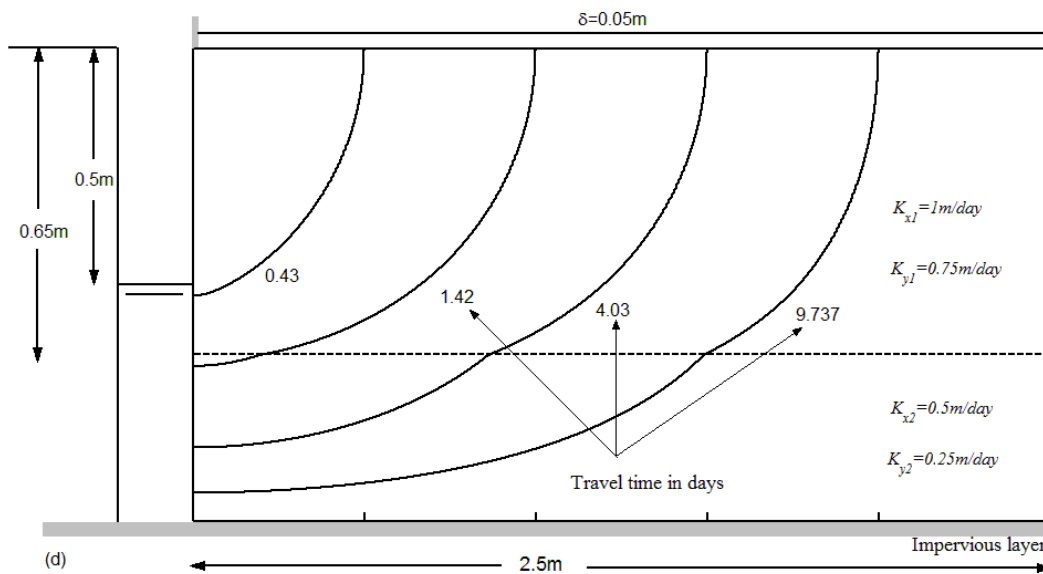


Fig. 2.40. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain when the flow parameters of the system (Fig. 2.2) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.65$ m, $H_1 = H_2 = 0.5$ m, $\varepsilon = 0.05$ m, $\delta = 0.05$ m, $\eta_1 = \eta_2 = 0.3$, and (a) $K_{x1} = K_{y1} = 0.5$ m/day, $K_{x2} = K_{y2} = 1$ m/day, and (b) $K_{x1} = 0.5$ m/day, $K_{y1} = 0.25$ m/day, $K_{x2} = 2$ m/day, $K_{y2} = 1$ m/day, and (c) $K_{x1} = 0.25$ m/day, $K_{y1} = 0.1$ m/day, $K_{x2} = 2.5$ m/day, $K_{y2} = 1$ m/day, and (d) $K_{x1} = 1$ m/day, $K_{y1} = 0.75$ m/day, $K_{x2} = 0.5$ m/day, $K_{y2} = 0.25$ m/day

Fig. 2.41 shows that, if the level of water in the ditches is taken below the boundary between the layers, then also the trends for streamlines would remain more or less the same like in the previous cases where the water level of the ditches is kept above the boundary level. However, as may be observed in Fig 2.42, even a small change in the water level heights of the ditches may bring about noticeable changes in the travel times, particularly for water particles starting their journey from locations further away from the ditches. Also, as can be seen, an increase in water level height in the ditches is making the travel times to rise and a decrease in water level is causing them to fall. This is understandable since, considering all other flow parameters to remain constant, an increase in the level of water in a ditch results in a relatively lesser head being available for driving water particles to the ditch and converse is the case when the water level in a ditch falls.

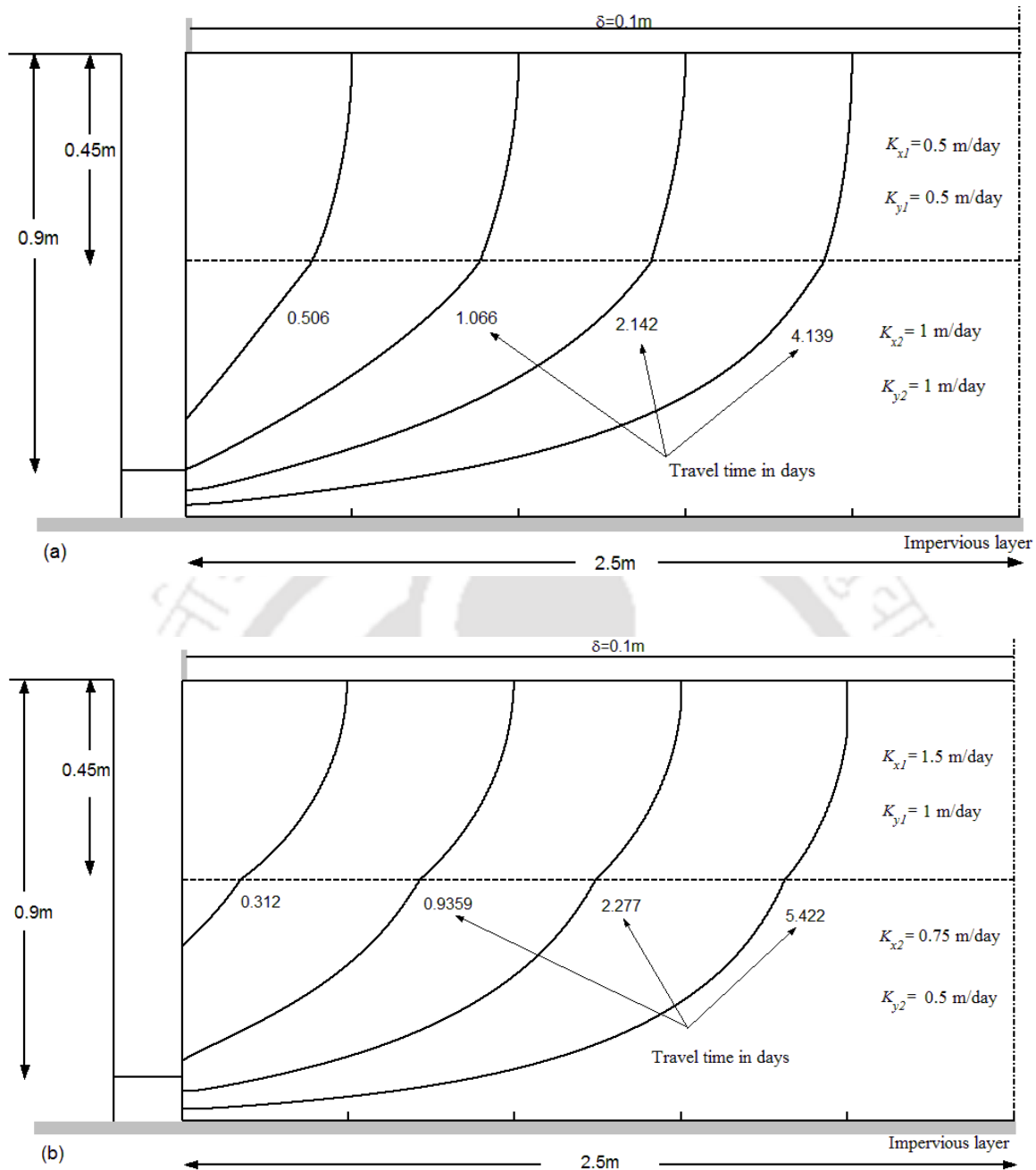


Fig. 2.41. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain when the flow parameters of the system (Fig. 2.10) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.45$ m, $H_1 = H_2 = 0.9$ m, $\varepsilon = 0.05$ m, $\eta_1 = \eta_2 = 0.3$, (a) $K_{x1} = K_{y1} = 0.5$ m/day, $K_{x2} = K_{y2} = 1$ m/day, $\delta = 0.1$ m and (b) $K_{x1} = 1.5$ m/day, $K_{y1} = 1$ m/day, $K_{x2} = 0.75$ m/day, $K_{y2} = 0.5$ m/day, $\delta = 0.1$ m

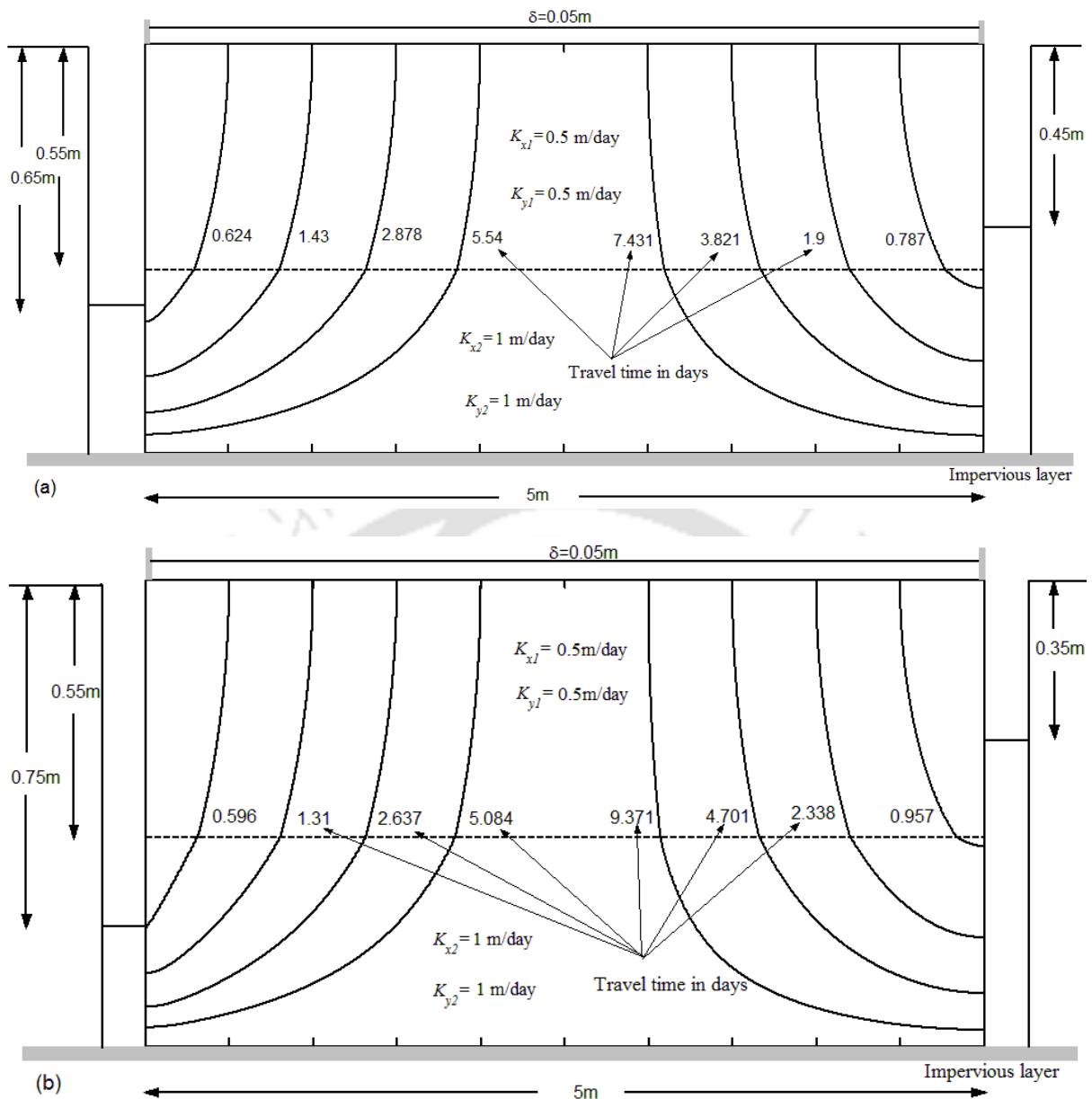


Fig. 2.42. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 2.18) are taken as $S = 5\text{ m}$, $h = 1\text{ m}$, $H_3 = 0.55\text{ m}$, $\varepsilon = 0.05\text{ m}$, $\delta = 0.05\text{ m}$, $\eta_1 = \eta_2 = 0.3$, $K_{x1} = K_{x2} = 0.5\text{ m/day}$, $K_{y1} = K_{y2} = 1\text{ m/day}$, and (a) $H_1 = 0.65\text{ m}$, $H_2 = 0.45\text{ m}$, and (b) $H_1 = 0.75\text{ m}$, $H_3 = 0.35\text{ m}$

The presence of a muddy and a plow sole layer in ponded paddy fields may greatly restrict water movement in these fields (Chen et al. 2002; Huang et al. 2003; Liu et al. 2005). Figs. 2.43 and 2.44 substantiates this observation where it can be seen that the time of travel of water particles in a ponded paddy field to subsurface drains may be extremely high in the presence of a lowly conductive top soil layer, especially for regions closer to the ditches. In this context, it is worth noting that drainage of paddy fields are now getting increasingly important not only for enhancing productivity from these fields (Darzi-Naftchali and Shahnazari 2014) but also for having a check on the emission of methane from paddy

environments (Nishimura et al. 2004; Qiu 2009; Yang et al. 2014; Yvon-Durocher et al. 2014), a greenhouse gas which contribute greatly to global warming.

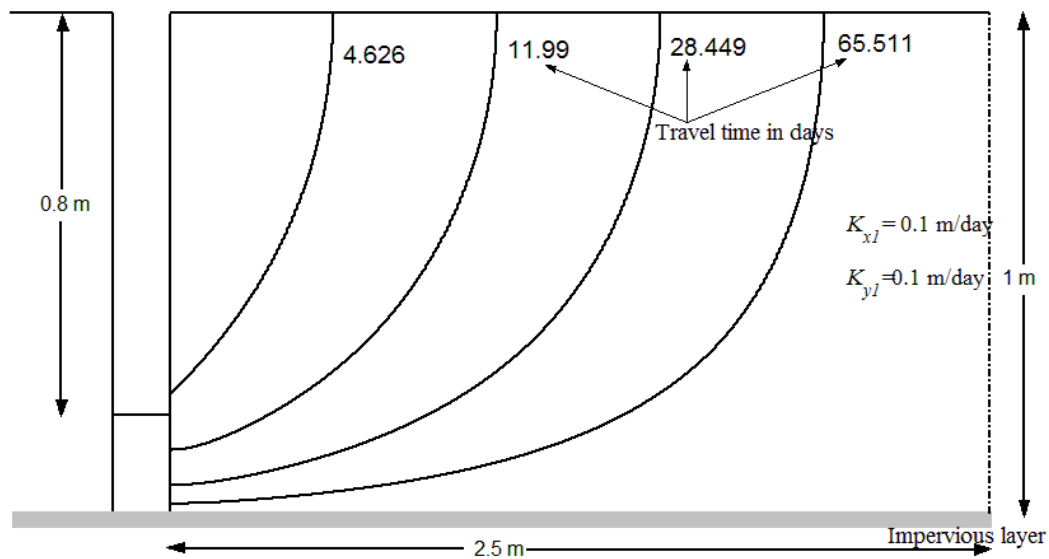


Fig. 2.43. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain when the flow parameters of the system (Fig. 2.10) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.3$ m, $H_1 = H_2 = 0.8$ m, $\delta = 0$ m, $\eta_1 = \eta_2 = 0.35$ and $K_{x_1} = K_{x_2} = K_{y_1} = K_{y_2} = 0.1$ m/day

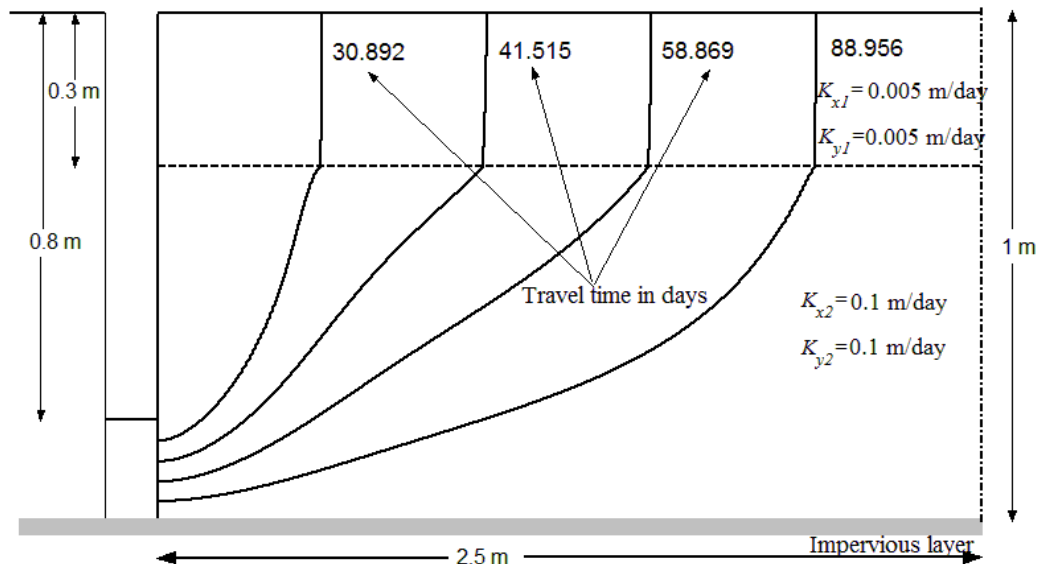
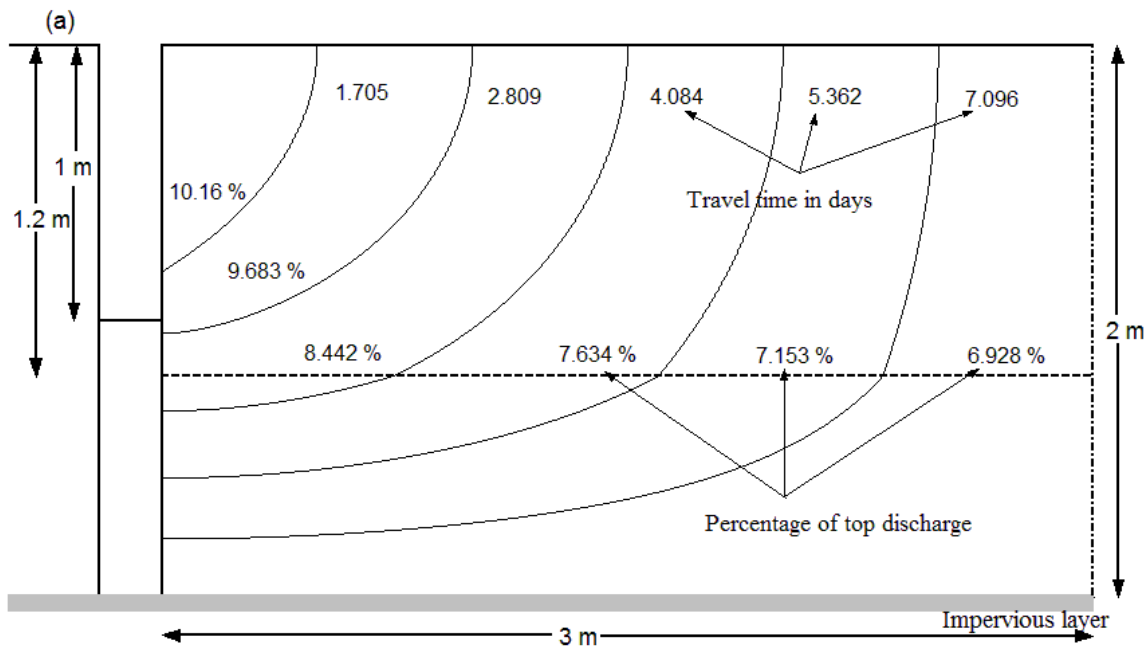


Fig. 2.44. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain when the flow parameters of the system (Fig. 2.10) are taken as $S = 5$ m, $h = 1$ m, $H_3 = 0.3$ m, $H_1 = H_2 = 0.8$ m, $\delta = 0$ m, $\eta_1 = 0.55$, $\eta_2 = 0.35$, $K_{x_1} = K_{y_1} = 0.005$ m/day and $K_{x_2} = K_{y_2} = 0.1$ m/day

From Figs. 2.45, 2.46, 2.47 and 2.48, it can be seen that the effect of a variable ponding field on the uniformity of travel times and streamline distribution of a two-layered ponded drainage system is sensitive to the directional conductivities of the layers and is more pronounced if the directional conductivities of the bottom layer are relatively lower than that of the top layer. Figs. 2.45, 2.46, 2.47 and 2.48 also clearly show that the distribution of the streamlines in a ponded drainage space can be markedly changed just by playing with ponding distribution on its surface and/or adjusting the water level heights of the drains. Thus, by merely providing an appropriate ponding field specific to a drainage situation, considerable improvement in uniformity of cleaning of a salt-affected soil may be achieved in comparison to cleaning of the same soil by utilizing a uniform ponded drainage system. This is in line with the findings of Barua and Alam (2013) and Barua and Sarmah (2016) who also obtained similar results from their analytical works related to the ponded ditch drainage system.



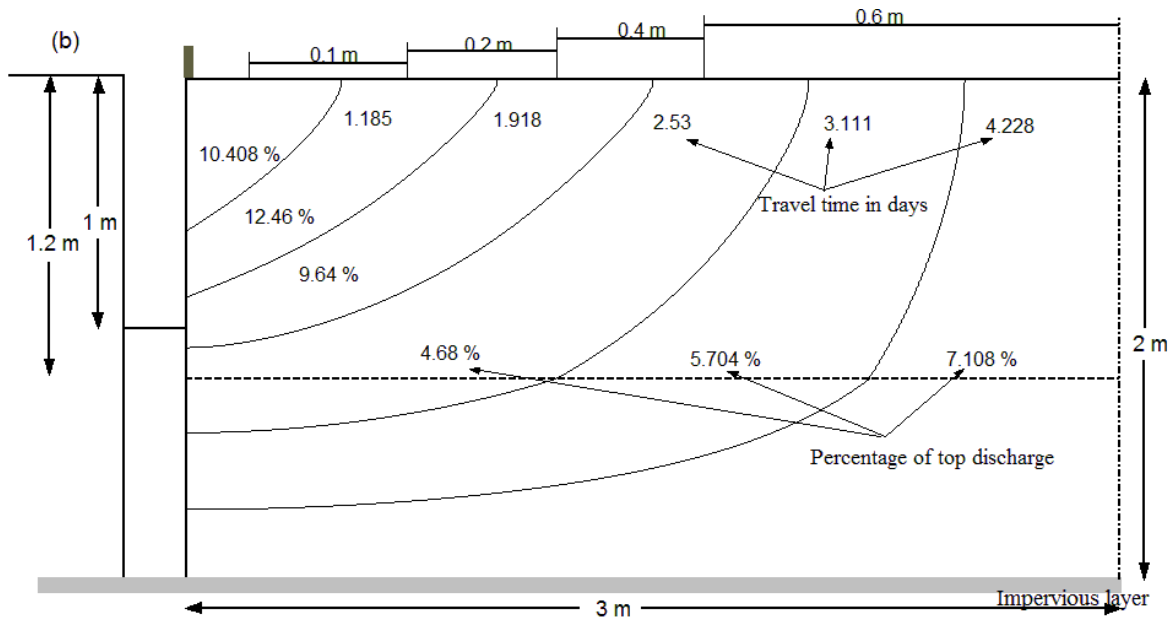
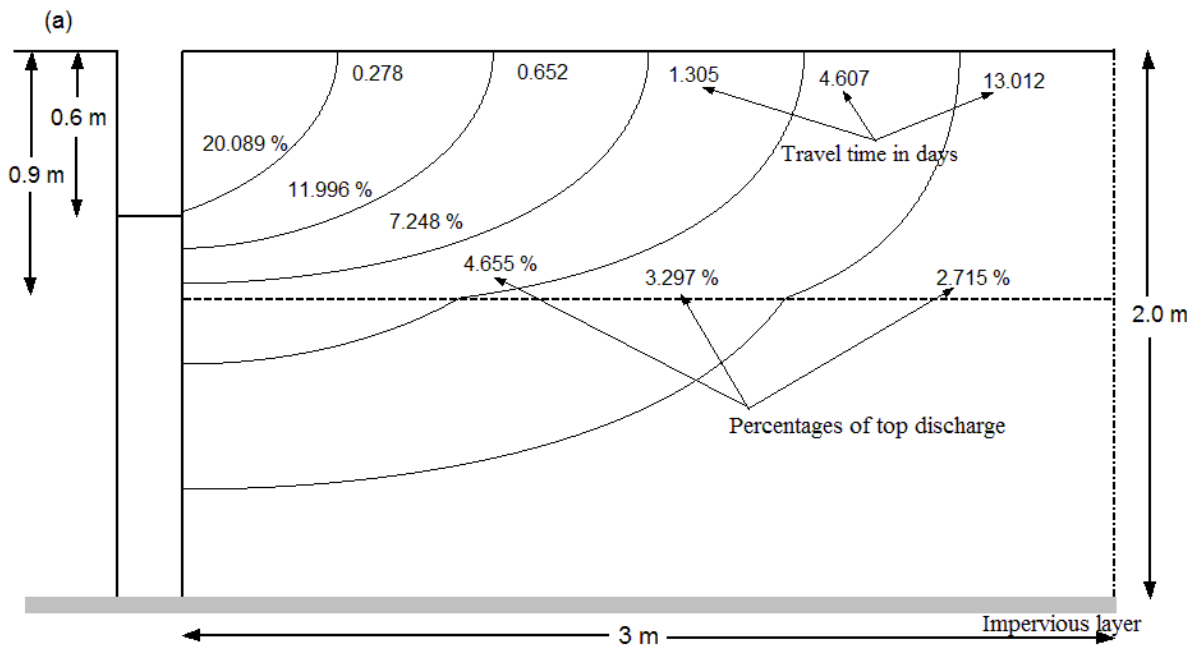


Fig. 2.45. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain and the distribution of the top discharge function when the flow parameters of the system (Fig. 2.2) are taken as $S = 6$ m, $h = 2$ m, $H_3 = 1.2$ m, $H_1 = H_2 = 1$ m, $\eta_1 = 0.45$, $\eta_2 = 0.3$, $K_{x_1} = 0.5$ m/day, $K_{y_1} = 0.25$ m/day, $K_{x_2} = 2$ m/day and $K_{y_2} = 1$ m/day. The top surface ponding distributions are (a) $\delta = 0$ m and (b) $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.2$ m, $\delta_4 = 0.4$ m, $\delta_5 = 0.6$ m, $\delta_6 = 0.4$ m, $\delta_7 = 0.2$ m, $\delta_8 = 0.1$ m, $\delta_9 = 0$ m, $S_1 = 0.2$ m, $S_2 = 0.7$ m, $S_3 = 1.2$ m, $S_4 = 1.7$ m, $S_5 = 4.3$ m, $S_6 = 4.8$ m, $S_7 = 5.3$ m and $S_8 = 5.8$ m



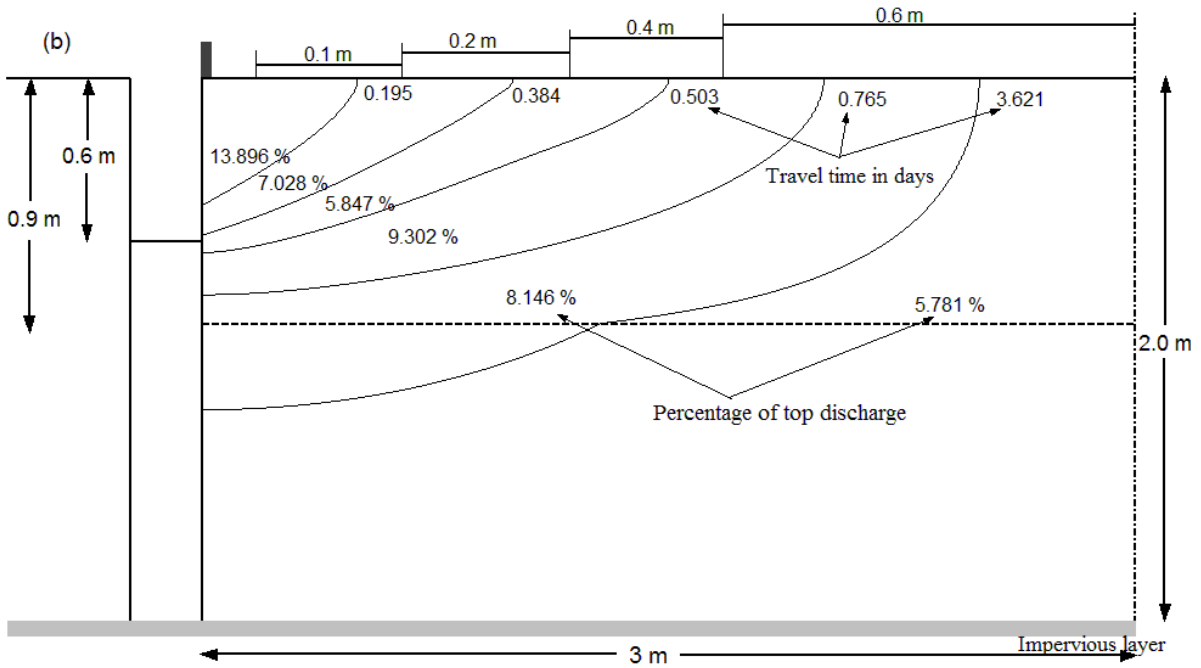
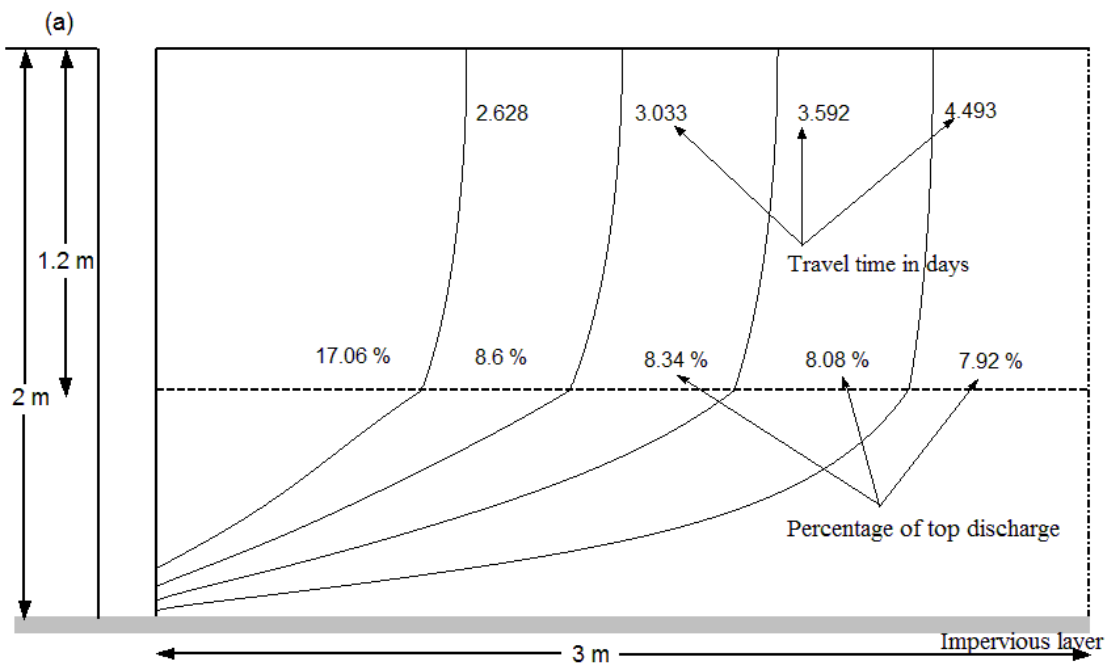


Fig. 2.46. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain and the distribution of the top discharge function when the flow parameters of the system (Fig. 2.2) are taken as $S = 6$ m, $h = 2$ m, $H_3 = 0.9$ m, $H_1 = H_2 = 0.6$ m, $\eta_1 = 0.3$, $\eta_2 = 0.45$, $K_{x_1} = 2$ m/day, $K_{y_1} = 1$ m/day, $K_{x_2} = 0.5$ m/day and $K_{y_2} = 0.25$ m/day. The top surface ponding distributions are (a) $\delta = 0$ m and (b) $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.2$ m, $\delta_4 = 0.4$ m, $\delta_5 = 0.6$ m, $\delta_6 = 0.4$ m, $\delta_7 = 0.2$ m, $\delta_8 = 0.1$ m, $\delta_9 = 0$ m, $S_1 = 0.2$ m, $S_2 = 0.7$ m, $S_3 = 1.2$ m, $S_4 = 1.7$ m, $S_5 = 4.3$ m, $S_6 = 4.8$ m, $S_7 = 5.3$ m and $S_8 = 5.8$ m



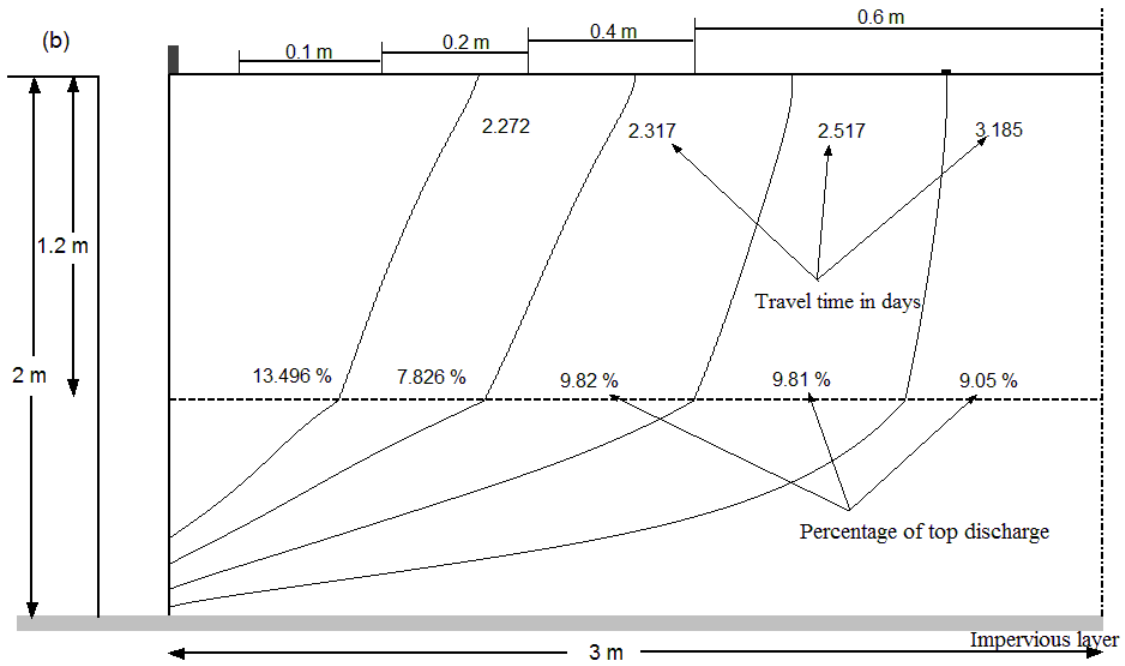
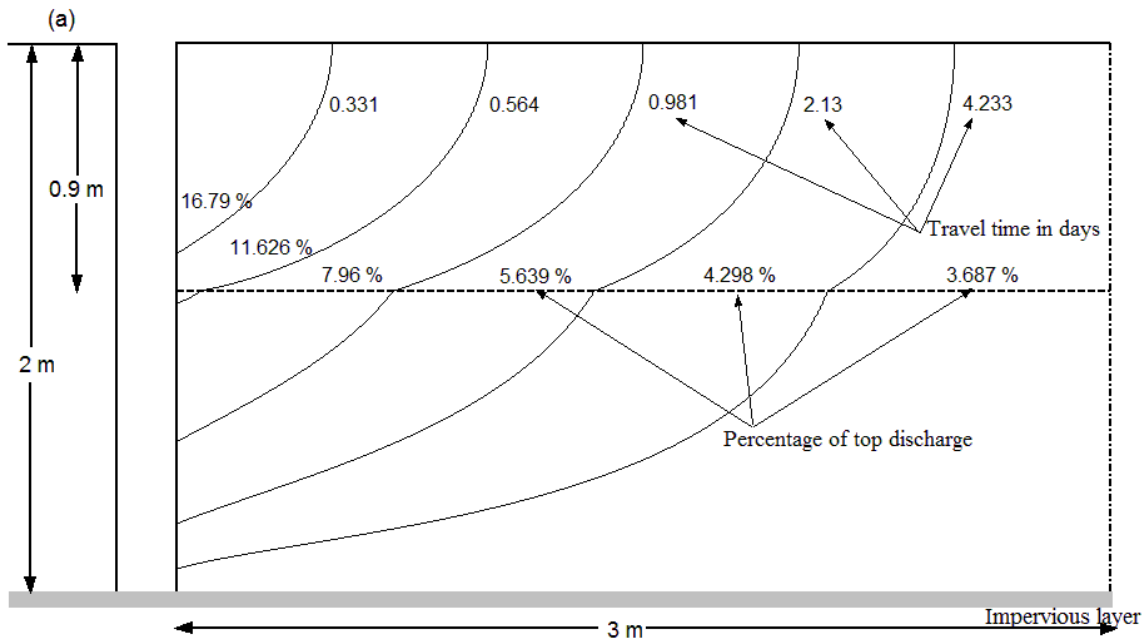


Fig. 2.47. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain and the distribution of the top discharge function when the flow parameters of the system (Fig. 2.10) are taken as $S = 6$ m, $h = 2$ m, $H_3 = 1.2$ m, $H_1 = H_2 = 2$ m, $\eta_1 = 0.45$, $\eta_2 = 0.3$, $K_{x_1} = 0.5$ m/day, $K_{y_1} = 0.25$ m/day, $K_{x_2} = 2$ m/day and $K_{y_2} = 1$ m/day. The top surface ponding distributions are (a) $\delta = 0$ m and (b) $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.2$ m, $\delta_4 = 0.4$ m, $\delta_5 = 0.6$ m, $\delta_6 = 0.4$ m, $\delta_7 = 0.2$ m, $\delta_8 = 0.1$ m, $\delta_9 = 0$ m, $S_1 = 0.2$ m, $S_2 = 0.7$ m, $S_3 = 1.2$ m, $S_4 = 1.7$ m, $S_5 = 4.3$ m, $S_6 = 4.8$ m, $S_7 = 5.3$ m and $S_8 = 5.8$ m



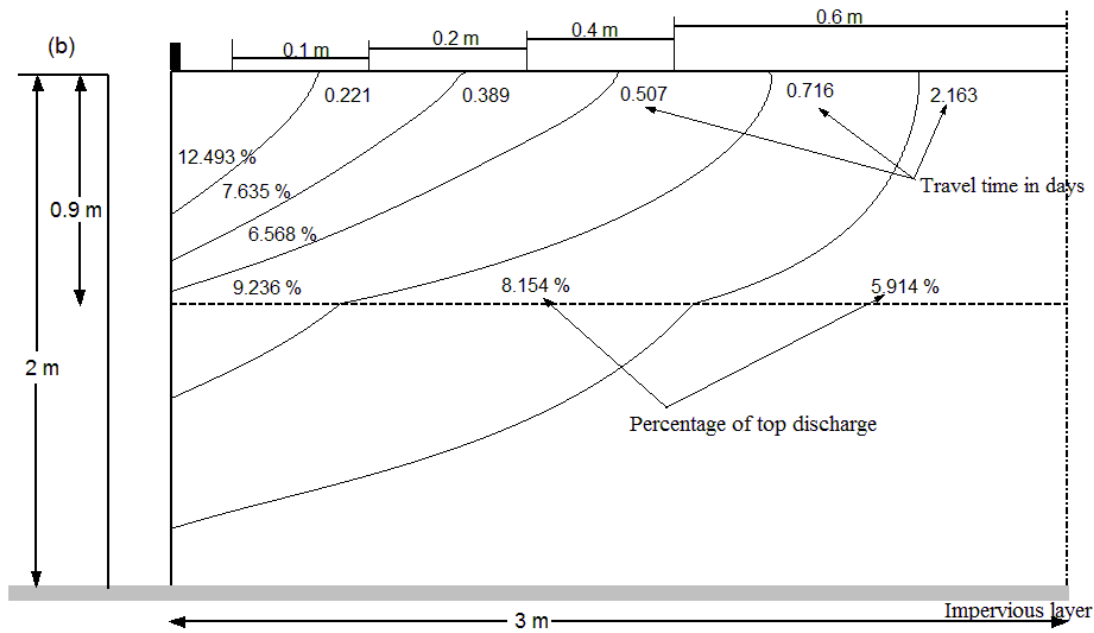


Fig. 2.48. Travel times (in days) of water particles starting from the surface of a two-layered ponded ditch drainage system to a recipient drain and the distribution of the top discharge function when the flow parameters of the system (Fig. 2.10) are taken as $S = 6$ m, $h = 2$ m, $H_3 = 0.9$ m, $H_1 = H_2 = 2$ m, $\eta_1 = 0.3$, $\eta_2 = 0.45$, $K_{x_1} = 2$ m/day, $K_{y_1} = 1$ m/day, $K_{x_2} = 0.5$ m/day and $K_{y_2} = 0.25$ m/day. The top surface ponding distributions are (a) $\delta = 0$ m and (b) $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.2$ m, $\delta_4 = 0.4$ m, $\delta_5 = 0.6$ m, $\delta_6 = 0.4$ m, $\delta_7 = 0.2$ m, $\delta_8 = 0.1$ m, $\delta_9 = 0$ m, $S_1 = 0.2$ m, $S_2 = 0.7$ m, $S_3 = 1.2$ m, $S_4 = 1.7$ m, $S_5 = 4.3$ m, $S_6 = 4.8$ m, $S_7 = 5.3$ m and $S_8 = 5.8$ m

It has already been stated that the proposed transient solutions for the ditch drainage problem is an approximate one, valid only when directional conductivities and specific storage of the layers satisfy the relation $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2}$ and the condition $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$) at the same time. However, MODFLOW comparison results for the drainage situations of Fig. 2.49 shows that even if the vertical hydraulic conductivities of the layers are taken much higher than zero, then also our transient solution may give pretty accurate results, provided the relation $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2}$ is being strictly respected while making use of this solution. This is mainly true for regions lying close to the ditches. It should be noted that analytical solutions are frequently been made use of to test the veracity of numerical codes and since our transient solutions for the different variations of the ponded drainage problem considered for study will give accurate results so long the assumptions inherent in them are being held true, these solutions are then may also be utilized for that purpose as well. We would like to emphasize once again that though our transient solution is approximate by nature [which becomes exact when the requirements $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2}$ and $K_{y_1} = K_{y_2} \rightarrow 0$ (but $K_{y_1} = K_{y_2} \neq 0$) are being satisfied by a drainage scenario], our steady state solutions for the different variants of the two-dimensional ponded ditch problem are exact and are valid for any combinations of the flow parameters specific to these problems.

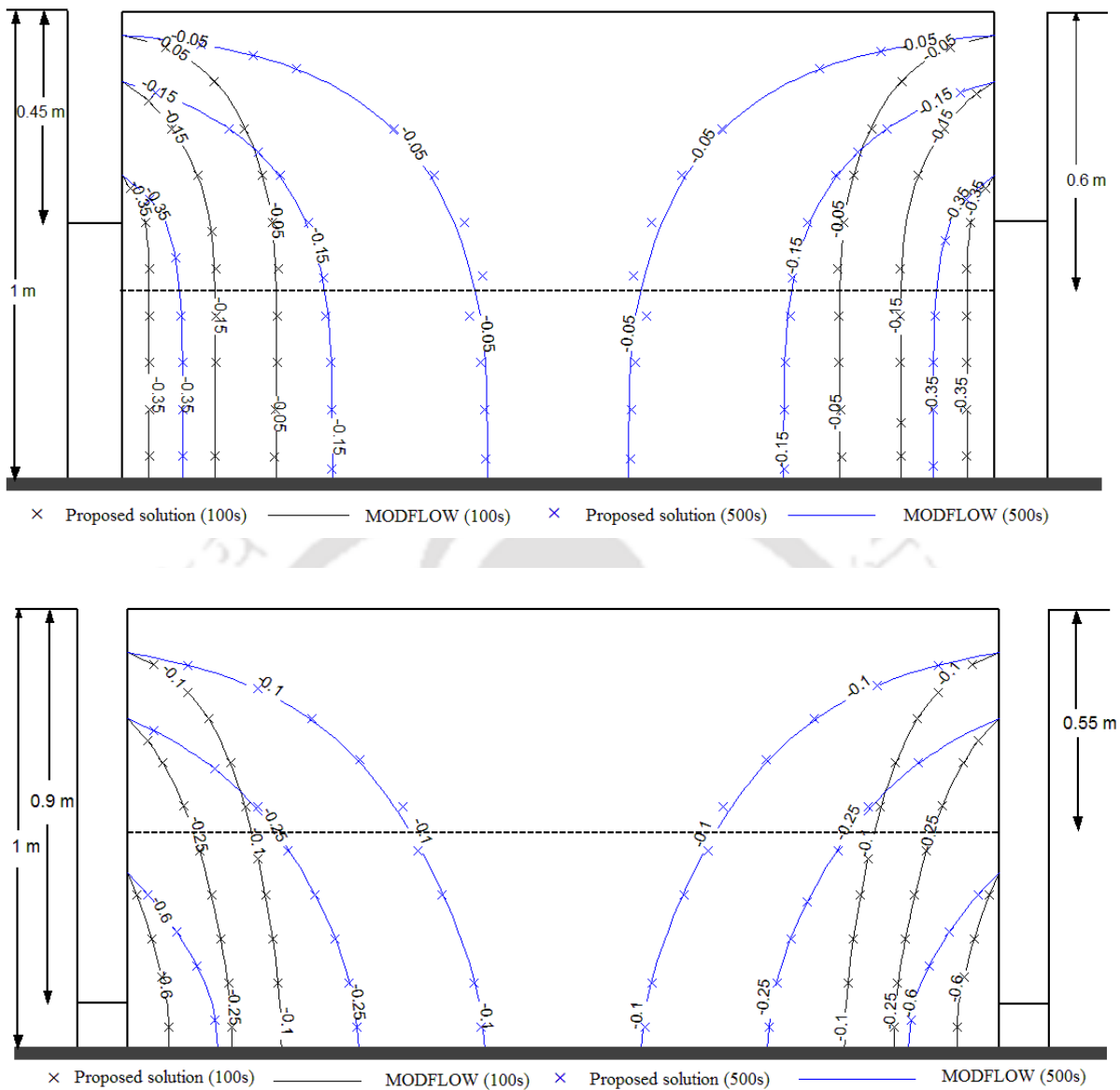


Fig 2.49. Comparison of transient hydraulic heads as obtained from the proposed solution with corresponding values as obtained by MODFLOW at two different times when the flow parameters of Fig. 2.10 are considered as (a) $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.6$ m, $H_1 = 0.45$ m, $H_2 = 0.45$ m, $\varepsilon = 0.05$ m, $\delta = 0$ m, $K_{x_1} = 1$ m/day, $K_{x_2} = 2$ m/day, $K_{y_1} = K_{y_2} = 0.05$ m/day, $S_{s_1} = 0.0075$ m⁻¹ and $S_{s_2} = 0.015$ m⁻¹ and (b) $S = 5.0$ m, $h = 1.0$ m, $H_3 = 0.55$ m, $H_1 = 0.9$ m, $H_2 = 0.9$ m, $\varepsilon = 0.05$ m, $\delta = 0$ m, $K_{x_1} = 1$ m/day, $K_{x_2} = 2$ m/day, $K_{y_1} = K_{y_2} = 0.05$ m/day, $S_{s_1} = 0.0075$ m⁻¹ and $S_{s_2} = 0.015$ m⁻¹

2.4 Conclusions

Analytical solutions have been worked out for predicting flow into a network of equally spaced ditch drains fully penetrating a two-layered ponded soil underlain by an impervious barrier when the level of water in both the ditches is above the boundary between soil layers, when it is below the boundary between the layers, when the level of water in the left ditch is below the aforementioned boundary and the right ditch above it and finally when the level of

water in the left ditch is above the boundary between the layers and the right ditch below it. The separation of the variables technique in conjunction with Fourier analysis has been made use of to solve all the problems considered in this study. All the developed steady state solutions are exact and are valid for all possible combinations of the flow parameters associated with them; however, the corresponding transient solutions to these problems are strictly correct only when the horizontal hydraulic conductivities and specific storage of the constituent layers obey the relation $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2}$ and the vertical conductivities of layers are equal and tend to zero at the same time (i.e., $K_{y_1} = K_{y_2} \rightarrow 0$). As the existing analytical solution to the two-layered ponded ditch drainage problem cannot account for unequal water level heights of the ditches as do the present solutions, all the developed solutions, even for the steady state, can be taken as new. The correctness of all the developed solutions have been extensively checked by comparing with the analytical works of others for specific situations; further, MODFLOW checks for all these solutions have also been performed. The study shows that the hydraulics of flow around a full penetrating ponded ditch drainage system in a two-layered soil is widely influenced, among other factors, by the directional conductivities of the constituent soil layers and the ponding field imposed on the surface of the soil. A higher horizontal conductivity of the layers tends to flatten the flow lines and a higher vertical conductivity of the layers tends to straighten them. A higher water level in a drainage ditch causes comparatively less water to flow into it in comparison to a situation when the level of water in the ditch is relatively low. Thus, considering all other drainage parameters to remain the same, the flow into a ditch in a ponded drainage space can be modulated by just changing the level of water in a ditch. This is an important observation since it provides a way for controlling the amount of flow to the drains of a ponded ditch drainage system. It has also come out of the study that the presence of a plow-sole layer in a paddy field greatly inhibits the movement of ponded water through the field and that considerable time may be required by a water particle to reach a subsurface drain in these fields. It should be noted that drainage of paddy fields are now getting increasing important owing to the ability of subsurface drains in bringing down emission of methane – a greenhouse gas – from these environments. It has also become amply clear from the study that ponded drainage of a stratified soil with a uniform ponding depth at the surface of the soil would lead to non-uniform movement of water into the soil – regions close to the drains will have a much higher share of flow through them as compared to regions away from the ditches. Thus, cleaning of a soil profile via a ponded drainage system with a uniform ponding field will lead to unequal cleaning of the soil profile with regions close to the drains over-washed and regions further away from the drains under-washed. However, as analysis of our analytical solutions have shown, by suitable adjusting the water level heights in the drains and/or by introducing an appropriate ponding field at the surface of the soil specific to a drainage requirement, considerable improvement in the uniformity of water movement as well as time of travel of water particles in a stratified ponded drainage space can be brought about. Thus, it is hoped that the solutions proposed here would prove to be more useful in the design of subsurface drains for reclaiming salty and waterlogged soils than the available solutions on the subject since, unlike the available drainage models on the subject, the proposed solutions, apart from handling soil stratifications and a varying ponding field at the

surface of the soil, can also accommodate unequal water level heights in the drains as well. Hence, these solutions are expected to be more versatile in their applications in drainage design than the existing solutions of the ponded drainage problem.

As mentioned at the beginning of the chapter, the main limitation of the analytical model that has been developed is that it cannot account for unsaturated flow, nor the general variation of the hydraulic conductivities of a soil profile in spatial and temporal dimensions. The other limitation of the developed solutions is that the drains have been assumed to be dug all the way up to an impervious barrier which, in actual field situations, is hardly the case. It should also be noted that we have already assumed the inner bunds at the surface of the soil to be of infinitesimally small thickness so as to reduce the surface boundary to strictly a Dirichlet boundary for mathematical convenience. This, of course, is a limitation of our analytical models as the inner bunds will be having some finite thickness even though this thickness will be much smaller in comparison to the spacing between the open drains.

2.5 List of Notations

The following notations are used in this chapter

$B_{p(1)}, C_{q(1)}, E_{k(1)}, D_{r(1)}, H_{w(1)}, A_{mn(1)}, Z_{uv(1)}, B_{p(2)}, C_{q(2)}, E_{k(2)}, D_{r(2)}, H_{w(2)}, F_{i(2)}, G_{j(2)}, A_{mn(2)}, Z_{uv(2)}, B_{p(3)}, C_{q(3)}, E_{k(3)}, D_{r(3)}, H_{w(3)}, G_{j(3)}, A_{mn(3)}, Z_{uv(3)}, B_{p(4)}, C_{q(4)}, E_{k(4)}, D_{r(4)}, H_{w(4)}, F_{i(4)}, A_{mn(4)}, Z_{uv(4)} = \text{constants with } p = 1, 2, 3, \dots, \infty, q = 1, 2, 3, \dots, \infty, k = 1, 2, 3, \dots, \infty, r = 1, 2, 3, \dots, \infty, w = 1, 2, 3, \dots, \infty, i = 1, 2, 3, \dots, \infty, j = 1, 2, 3, \dots, \infty, m = 1, 2, 3, \dots, \infty, n = 1, 2, 3, \dots, \infty, u = 1, 2, 3, \dots, \infty \text{ and } v = 1, 2, 3, \dots, \infty;$

h = depth up to the impervious layer as measured from the surface of the soil [L];

H_1 = depth of water in the left ditch as measured from the surface of the soil [L];

H_2 = depth of water in the right ditch as measured from the surface of the soil [L];

H_3 = depth of the top soil layer as measured from the surface of the soil [L];

K_{x_1} = horizontal hydraulic conductivity of the top soil layer as in Fig. 2.1 [LT^{-1}];

K_{y_1} = vertical hydraulic conductivity of the top soil layer as in Fig. 2.1 [LT^{-1}];

K_{x_2} = horizontal hydraulic conductivity of the bottom soil layer as in Fig. 2.1 [LT^{-1}];

K_{y_2} = vertical hydraulic conductivity of the bottom soil layer as in Fig. 2.1 [LT^{-1}];

$(K_1^a)^2 = (K_{x_1}/K_{y_1})$ anisotropy ratio of the top soil layer (dimensionless);

$(K_2^a)^2 = (K_{x_1}/K_{y_1})$ anisotropy ratio of the bottom soil layer (dimensionless);

$P, Q, K, R, W, I, J, M, N, U, V$ = number of terms to be summed in the infinite series solutions, 1, 2, 3,....

$$N_i = \left[(1-2i)\pi/2(h-H_3) \right] \text{ with } i = 1, 2, 3, \dots, \infty;$$

$$N_{i_1} = (i_1\pi/S) \text{ with } i_1 = 1, 2, 3, \dots, \infty;$$

$$N_j = \left[(1-2j)\pi/2(h-H_3) \right] \text{ with } j = 1, 2, 3, \dots, \infty;$$

$$N_k = (k\pi/S) \text{ with } k = 1, 2, 3, \dots, \infty;$$

$$N_m = (m\pi/S) \text{ with } m = 1, 2, 3, \dots, \infty;$$

$$N_n = \left[(1-2n)\pi/2h \right] \text{ with } n = 1, 2, 3, \dots, \infty;$$

$$N_p = \left[(1-2p)\pi/2H_3 \right] \text{ with } p = 1, 2, 3, \dots, \infty;$$

$$N_q = \left[(1-2q)\pi/2H_3 \right] \text{ with } q = 1, 2, 3, \dots, \infty;$$

$$N_r = (r\pi/S) \text{ with } r = 1, 2, 3, \dots, \infty;$$

$$N_u = (u\pi/S) \text{ with } u = 1, 2, 3, \dots, \infty;$$

$$N_v = \left[(1-2v)\pi/2h \right] \text{ with } v = 1, 2, 3, \dots, \infty;$$

$$N_w = (w\pi/S) \text{ with } w = 1, 2, 3, \dots, \infty;$$

N_0 = number of divisions of the ponding surface at the top of the soil (Fig. 2.1);

$Q_{leftside(1)}$ = discharge through the side of the left ditch for Fig. 2.2 problem [L^3T^{-1}];

$Q_{leftside(2)}$ = discharge through the side of the left ditch for Fig. 2.10 problem [L^3T^{-1}];

$Q_{leftside(3)}$ = discharge through the side of the left ditch for Fig. 2.18 problem [L^3T^{-1}];

$Q_{leftside(4)}$ = discharge through the side of the left ditch for Fig. 2.26 problem [L^3T^{-1}];

$Q_{rightside(1)}$ = discharge through the side of the right ditch for Fig. 2.2 problem [L^3T^{-1}];

$Q_{rightside(2)}$ = discharge through the side of the right ditch for Fig. 2.10 problem [L^3T^{-1}];

$Q_{\text{rightside}(3)}$ = discharge through the side of the right ditch for Fig. 2.18 problem [L^3T^{-1}];

$Q_{\text{rightside}(4)}$ = discharge through the side of the right ditch for Fig. 2.26 problem [L^3T^{-1}];

$Q_{\text{top}(1)}$ = discharge through the top surface for the flow problem of Fig. 2.2 [L^3T^{-1}];

$Q_{\text{top}(2)}$ = discharge through the top surface for the flow problem of Fig. 2.10 [L^3T^{-1}];

$Q_{\text{top}(3)}$ = discharge through the top surface for the flow problem of Fig. 2.18 [L^3T^{-1}];

$Q_{\text{top}(4)}$ = discharge through the top surface for the flow problem of Fig. 2.26 [L^3T^{-1}];

$Q_{\text{topx}(1)}$ = top discharge function for the flow problem of Fig. 2.2;

$Q_{\text{topx}(2)}$ = top discharge function for the flow problem of Fig. 2.10;

$Q_{\text{topx}(3)}$ = top discharge function for the flow problem of Fig. 2.18;

$Q_{\text{topx}(4)}$ = top discharge function for the flow problem of Fig. 2.26;

S = spacing between any two adjacent ditches for the flow problem of Fig. 2.1 [L];

S_i = horizontal distance of the i^{th} inner bund from the origin 'O' for the flow problem of Fig. 2.1 [L];

S_{s_1} = specific storage of the top soil layer [L^{-1}];

S_{s_2} = specific storage of the bottom soil layer [L^{-1}];

t = time variable [T];

$V_{x1(1)}$ = horizontal velocity distribution for the top layer of Fig. 2.2 [LT^{-1}];

$V_{x1(2)}$ = horizontal velocity distribution for the top layer of Fig. 2.10 [LT^{-1}];

$V_{x1(3)}$ = horizontal velocity distribution for the top layer of Fig. 2.18 [LT^{-1}];

$V_{x1(4)}$ = horizontal velocity distribution for the top layer of Fig. 2.26 [LT^{-1}];

$V_{x2(1)}$ = horizontal velocity distribution for the bottom layer of Fig. 2.2 [LT^{-1}];

$V_{x2(2)}$ = horizontal velocity distribution for the bottom layer of Fig. 2.10 [LT^{-1}];

$V_{x2(3)}$ = horizontal velocity distribution for the bottom layer of Fig. 2.18 [LT^{-1}];

$V_{x2(4)}$ = horizontal velocity distribution for the bottom layer of Fig. 2.26 [LT^{-1}];

$V_{y1(1)}$ = vertical velocity distribution for the top layer of Fig. 2.2 [LT^{-1}];

$V_{y1(2)}$ = vertical velocity distribution for the top layer of Fig. 2.10 [LT^{-1}];

$V_{y1(3)}$ = vertical velocity distribution for the top layer of Fig. 2.18 [LT^{-1}];

$V_{y1(4)}$ = vertical velocity distribution for the top layer of Fig. 2.26 [LT^{-1}];

$V_{y2(1)}$ = vertical velocity distribution for the bottom layer of Fig. 2.2 [LT^{-1}];

$V_{y2(2)}$ = vertical velocity distribution for the bottom layer of Fig. 2.10 [LT^{-1}];

$V_{y2(3)}$ = vertical velocity distribution for the bottom layer of Fig. 2.18 [LT^{-1}];

$V_{y2(4)}$ = vertical velocity distribution for the bottom layer of Fig. 2.26 [LT^{-1}];

$Vol_{leftside(1)}$ = volume of water seeping through the side faces of the left ditch for the flow problem represented by Fig. 2.2 [L^3];

$Vol_{leftside(2)}$ = volume of water seeping through the side faces of the left ditch for the flow problem represented by Fig. 2.10 [L^3];

$Vol_{leftside(3)}$ = volume of water seeping through the side faces of the left ditch for the flow problem represented by Fig. 2.18 [L^3];

$Vol_{leftside(4)}$ = volume of water seeping through the side faces of the left ditch for the flow problem represented by Fig. 2.26 [L^3];

$Vol_{rightside(1)}$ = volume of water seeping through the side faces of the right ditch for the flow problem represented by Fig. 2.2 [L^3];

$Vol_{rightside(2)}$ = volume of water seeping through the side faces of the right ditch for the flow problem represented by Fig. 2.10 [L^3];

$Vol_{rightside(3)}$ = volume of water seeping through the side faces of the right ditch for the flow problem represented by Fig. 2.18 [L^3];

$Vol_{rightside(4)}$ = volume of water seeping through the side faces of the right ditch for the flow problem represented by Fig. 2.26 [L^3];

$Vol_{top(1)}$ = volume of water seeping through the top soil surface of Fig. 2.2 [L^3];

$Vol_{top(2)}$ = volume of water seeping through the top soil surface of Fig. 2.10 [L³];

$Vol_{top(3)}$ = volume of water seeping through the top soil surface of Fig. 2.18 [L³];

$Vol_{top(4)}$ = volume of water seeping through the top soil surface of Fig. 2.26 [L³];

x = horizontal coordinate [L];

y = vertical coordinate [L];

$\phi_{1(1)}$ = hydraulic head distribution for the top soil layer for the flow problem of Fig. 2.2 [L];

$\phi_{1(2)}$ = hydraulic head distribution for the top soil layer for the flow problem of Fig. 2.10 [L];

$\phi_{1(3)}$ = hydraulic head distribution for the top soil layer for the flow problem of Fig. 2.18 [L];

$\phi_{1(4)}$ = hydraulic head distribution for the top soil layer for the flow problem of Fig. 2.26 [L];

$\phi_{2(1)}$ = hydraulic head distribution for the bottom soil layer for the flow problem of Fig. 2.2 [L];

$\phi_{2(2)}$ = hydraulic head distribution for the bottom soil layer for the flow problem of Fig. 2.10 [L];

$\phi_{2(3)}$ = hydraulic head distribution for the bottom soil layer for the flow problem of Fig. 2.18 [L];

$\phi_{2(4)}$ = hydraulic head distribution for the bottom soil layer for the flow problem of Fig. 2.26 [L];

η_1 = porosity of the top soil layer in Fig. 2.1 (dimensionless);

η_2 = porosity of the bottom soil layer in Fig. 2.1 (dimensionless);

δ_i = ponding depth of the i^{th} strip at the soil surface of Fig. 2.1 [L];

ε = width of the ditch banks of Fig. 2.1 [L];

$$(\lambda_{mm})^2 = \left[N_m^2 \left(K_{x_1} / S_{s_1} \right) + N_n^2 \left(K_{y_1} / S_{s_1} \right) \right];$$

$$(\lambda_{uv})^2 = \left[N_u^2 \left(K_{x_2} / S_{s_2} \right) + N_v^2 \left(K_{y_2} / S_{s_2} \right) \right];$$

CHAPTER 3

THREE-DIMENSIONAL SEEPAGE OF PONDED WATER INTO DITCH DRAINS IN A THREE-LAYERED SOIL COLUMN UNDERLAIN BY AN IMPERVIOUS SUBSTRATUM

In this chapter, an analytical solution is being suggested for predicting three-dimensional seepage into ditch drains through a soil column comprising of three distinct vertical soil layers and underlain by an impervious barrier. The drains are being fed by a distributed ponding water head introduced at the surface of the soil column. As in the case of the previous two-dimensional ditch drainage problem, the solutions proposed here for the three variants of the three-dimensional ponded ditch drainage problem are also exact and valid for any configuration of the flow parameters inherent in these problems for the steady state; however, for the transient state, these solutions are strictly valid only when the directional conductivities and specific storage of the layers obey certain rules (as specified in the text) and not otherwise. All the derived solutions are being checked for their correctness by comparing with analytical solutions of others for specific situations; numerical checks on these solutions are also been carried out for a few drainage scenarios by making use of the Processing MODFLOW (Chiang and Kinzelbach 2001) environment.

3.1 Solutions for the Three-Dimensional Continuity Equation of Groundwater flow for a Homogeneous and Anisotropic Soil

We first proceeded to obtain a few general solutions pertaining to the continuity equation which describes three-dimensional flow of groundwater in a saturated flow domain. For the flow problem in question, we have made the following assumptions

- (a) The aquifer, within each layer, has been assumed to be saturated, homogeneous and anisotropic.
- (b) The flow of water in the aquifer has been assumed to be irrotational and incompressible.
- (c) The soil matrix has been assumed to be compressible while the porosity of each soil layer has been considered as constant.
- (d) The direction of the principal hydraulic conductivities have been assumed to coincide with the horizontal and vertical directions of the flow domain.
- (e) Darcy's Law has been assumed to be valid for the analysis and
- (f) We have assumed the flow to be three-dimensional in nature.

The concerned governing equation for the aforementioned groundwater flow condition can be expressed as (Bear 1972)

$$K_x \frac{\partial^2 \phi}{\partial x^2} + K_y \frac{\partial^2 \phi}{\partial y^2} + K_z \frac{\partial^2 \phi}{\partial z^2} = S_s \frac{\partial \phi}{\partial t}, \quad (3.1)$$

where ϕ is the hydraulic head, K_x , K_y and K_z are the hydraulic conductivities of the medium in the x -, y - and z -directions, respectively and S_s is the specific storage of the aquifer.

Dividing both sides of Eq. (3.1) by S_s , the continuity equation is reduced to

$$\left(\frac{K_x}{S_s}\right)\frac{\partial^2\phi}{\partial x^2} + \left(\frac{K_y}{S_s}\right)\frac{\partial^2\phi}{\partial y^2} + \left(\frac{K_z}{S_s}\right)\frac{\partial^2\phi}{\partial z^2} = \frac{\partial\phi}{\partial t}. \quad (3.2)$$

Let us consider $u_p(x, y, z, t)$ [Eq. (3.5)] to be a solution of Eq. (3.2) and $u_c(x, y, z)$ [Eq. (3.14)] to be a solution of the steady state portion of Eq. (3.2), that is

$$\left(\frac{K_x}{S_s}\right)\frac{\partial^2\phi}{\partial x^2} + \left(\frac{K_y}{S_s}\right)\frac{\partial^2\phi}{\partial y^2} + \left(\frac{K_z}{S_s}\right)\frac{\partial^2\phi}{\partial z^2} = 0. \quad (3.3)$$

It is evident that $\phi = u_p + u_c$ happens to be a solution of the Eq. (3.2) since

$$\begin{aligned} &\left(\frac{K_x}{S_s}\right)\frac{\partial^2\phi}{\partial x^2} + \left(\frac{K_y}{S_s}\right)\frac{\partial^2\phi}{\partial y^2} + \left(\frac{K_z}{S_s}\right)\frac{\partial^2\phi}{\partial z^2} = \\ &\left(\frac{K_x}{S_s}\right)\frac{\partial^2 u_p}{\partial x^2} + \left(\frac{K_y}{S_s}\right)\frac{\partial^2 u_p}{\partial y^2} + \left(\frac{K_z}{S_s}\right)\frac{\partial^2 u_p}{\partial z^2} + \left(\frac{K_x}{S_s}\right)\frac{\partial^2 u_c}{\partial x^2} + \left(\frac{K_y}{S_s}\right)\frac{\partial^2 u_c}{\partial y^2} + \left(\frac{K_z}{S_s}\right)\frac{\partial^2 u_c}{\partial z^2} = \\ &\left(\frac{K_x}{S_s}\right)\frac{\partial^2 u_p}{\partial x^2} + \left(\frac{K_y}{S_s}\right)\frac{\partial^2 u_p}{\partial y^2} + \left(\frac{K_z}{S_s}\right)\frac{\partial^2 u_p}{\partial z^2} + 0 = \frac{\partial\phi}{\partial t}. \end{aligned} \quad (3.4)$$

Employing the separation of variables method (Kirkham and Powers 1972), we proceed to obtain a solution for Eq. (3.2). We conveniently assume that

$$u_p(x, y, z, t) = X(x)Y(y)Z(z)T(t) \quad (3.5)$$

is a solution of Eq. (3.2), where $X(x)$, $Y(y)$, $Z(z)$ and $T(t)$ are functions of only x , y , z and t , respectively. Substituting the expression for u_p in Eq. (3.2) and separating the variables out, we get

$$\left(\frac{K_x}{S_s}\right)\frac{X''(x)}{X(x)} + \left(\frac{K_y}{S_s}\right)\frac{Y''(y)}{Y(y)} + \left(\frac{K_z}{S_s}\right)\frac{Z''(z)}{Z(z)} = \frac{T'(t)}{T(t)}. \quad (3.6)$$

$$\text{Equating } \left(\frac{K_x}{S_s}\right)\frac{X''(x)}{X(x)} = -\alpha^2\left(\frac{K_x}{S_s}\right), \left(\frac{K_y}{S_s}\right)\frac{Y''(y)}{Y(y)} = -\beta^2\left(\frac{K_y}{S_s}\right) \text{ and } \left(\frac{K_z}{S_s}\right)\frac{Z''(z)}{Z(z)} = -\gamma^2\left(\frac{K_z}{S_s}\right),$$

where α , β and γ are constants, and then solving the abovementioned equations, we arrive at

$$X(x) = c_1 \sin(\alpha x), \quad (3.7)$$

$$Y(y) = c_2 \sin(\beta y) \quad (3.8)$$

and

$$Z(z) = c_3 \sin(\gamma z), \quad (3.9)$$

where c_1 , c_2 and c_3 are arbitrary constants. Equating each successive term on the left hand side of Eq. (3.6) to the constant terms $-\alpha^2\left(\frac{K_x}{S_s}\right)$, $-\beta^2\left(\frac{K_y}{S_s}\right)$ and $-\gamma^2\left(\frac{K_z}{S_s}\right)$, respectively,

we get

$$\frac{T'(t)}{T(t)} = -\left[\alpha^2\left(\frac{K_x}{S_s}\right) + \beta^2\left(\frac{K_y}{S_s}\right) + \gamma^2\left(\frac{K_z}{S_s}\right)\right] = -\lambda^2. \quad (3.10)$$

The solution for Eq. (3.10) can be expressed as

$$T(t) = c_4 \exp \left\{ - \left[\alpha^2 \left(\frac{K_x}{S_s} \right) + \beta^2 \left(\frac{K_y}{S_s} \right) + \gamma^2 \left(\frac{K_z}{S_s} \right) \right] t \right\}, \quad (3.11)$$

where c_4 is any arbitrary constant. Substituting Eqs. (3.7), (3.8), (3.9) and (3.11) in Eq. (3.5), we get $u_p(x, y, z, t)$ as

$$u_p(x, y, z, t) = E \sin(\alpha x) \sin(\beta y) \sin(\gamma z) \exp \left\{ - \left[\alpha^2 \left(\frac{K_x}{S_s} \right) + \beta^2 \left(\frac{K_y}{S_s} \right) + \gamma^2 \left(\frac{K_z}{S_s} \right) \right] t \right\}, \quad (3.12)$$

where E is any arbitrary constant. The solution proposed in Eq. (3.12) is the general form of a fundamental solution of the governing partial differential equation. By varying the values of the arbitrary constants α , β and γ , we can obtain countless such solutions for the governing equation. As the sum of all these solutions will also satisfy Eq. (3.2), a solution of the governing equation can thus be expressed as

$$u_p(x, y, z, t) = \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K E_{ijk} \sin(\alpha_i x) \sin(\beta_j y) \sin(\gamma_k z) \exp[-\lambda_{ijk}^2 t], \quad (3.13)$$

where $\lambda_{ijk}^2 = \left[\alpha^2 \left(\frac{K_x}{S_s} \right) + \beta^2 \left(\frac{K_y}{S_s} \right) + \gamma^2 \left(\frac{K_z}{S_s} \right) \right]$, E_{ijk} are arbitrary constants, i, j and k are

summation indices and I, J and K ($I, J, K \rightarrow \infty$) are positive integers. We next proceed with the assumption that the solution of Eq. (3.3) can be expressed as

$$u_c(x, y, z) = B(x)C(y)D(z), \quad (3.14)$$

where $B(x)$, $C(y)$ and $D(z)$ happen to be functions of x , y and z only. Substituting the assumed value of $u_c(x, y, z)$ in Eq. (3.3) and rearranging the variables, we get

$$\frac{B''(x)}{B(x)} + \left(\frac{K_y}{K_x} \right) \frac{C''(y)}{C(y)} + \left(\frac{K_z}{K_x} \right) \frac{D''(z)}{D(z)} = 0. \quad (3.15)$$

The second and third terms on the left hand side of Eq. (3.15) are equated to the following constants

$$\left(\frac{K_y}{K_x} \right) \frac{C''(y)}{C(y)} = -\eta^2 \left(\frac{K_y}{K_x} \right) \quad (3.16)$$

and

$$\left(\frac{K_z}{K_x} \right) \frac{D''(z)}{D(z)} = -\mu^2 \left(\frac{K_z}{K_x} \right), \quad (3.17)$$

where η and μ are arbitrary constants. Solving the abovementioned ordinary differential equations, we get

$$C(y) = c_5 \sin(\eta y) \quad (3.18)$$

and

$$D(z) = c_6 \sin(\mu z), \quad (3.19)$$

where c_5 and c_6 are arbitrary constants. In case we substitute the second and third terms in the left hand side expression of Eq. (3.15) with $-\eta^2 \left(\frac{K_y}{K_x} \right)$ and $-\mu^2 \left(\frac{K_z}{K_x} \right)$, then we get

$$\frac{B''(x)}{B(x)} = \eta^2 \left(\frac{K_y}{K_x} \right) + \mu^2 \left(\frac{K_z}{K_x} \right). \quad (3.20)$$

The solution for Eq. (3.20) can be expressed as

$$B(x) = c_7 \sinh \left[\left(\sqrt{\eta^2 \left(\frac{K_y}{K_x} \right) + \mu^2 \left(\frac{K_z}{K_x} \right)} \right) x \right], \quad (3.21)$$

where c_7 is another arbitrary constant. On substituting Eqs. (3.18), (3.19) and (3.21) in Eq. (3.14), we obtain

$$u_c(x, y, z) = A \sinh \left[\left(\sqrt{\eta^2 \left(\frac{K_y}{K_x} \right) + \mu^2 \left(\frac{K_z}{K_x} \right)} \right) x \right] \sin(\eta y) \sin(\mu z), \quad (3.22)$$

where A is any arbitrary constant. Assigning summation indices to the constants η and μ and then summing up several fundamental solutions of Eq. (3.15), we can have another solution of the differential equation as

$$u_c(x, y, z) = \sum_{m=1}^M \sum_{n=1}^N A_{mn} \sinh \left[\left(\sqrt{\eta_m^2 \left(\frac{K_y}{K_x} \right) + \mu_n^2 \left(\frac{K_z}{K_x} \right)} \right) x \right] \sin(\eta_m y) \sin(\mu_n z), \quad (3.23)$$

where A_{mn} are arbitrary constants, m and n are indices and M and N ($M, N \rightarrow \infty$) are positive integers. In a similar fashion, after dividing Eq. (3.3) by K_y and proceeding to

equate $\left(\frac{K_x}{K_y} \right) \frac{B''(x)}{B(x)} = -\eta^2 \left(\frac{K_x}{K_y} \right)$ and $\left(\frac{K_x}{K_y} \right) \frac{D''(z)}{D(z)} = -\mu^2 \left(\frac{K_x}{K_y} \right)$, we attain the following

alternate expression for $u_c(x, y, z)$

$$u_c(x, y, z) = \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{m_1 n_1} \sinh \left[\left(\sqrt{\eta_{m_1}^2 \left(\frac{K_x}{K_y} \right) + \mu_{n_1}^2 \left(\frac{K_z}{K_y} \right)} \right) y \right] \sin(\eta_{m_1} x) \sin(\mu_{n_1} z). \quad (3.24)$$

Analogously, the following solutions for $u_c(x, y, z)$ can also be obtained

$$u_c(x, y, z) = \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} A_{m_2 n_2} \sinh \left[\left(\sqrt{\eta_{m_2}^2 \left(\frac{K_x}{K_z} \right) + \mu_{n_2}^2 \left(\frac{K_y}{K_z} \right)} \right) z \right] \sin(\eta_{m_2} x) \sin(\mu_{n_2} y) \quad (3.25)$$

and

$$u_c(x, y, z) = \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} A_{m_3 n_3} \cosh \left[\left(\sqrt{\eta_{m_3}^2 \left(\frac{K_x}{K_z} \right) + \mu_{n_3}^2 \left(\frac{K_y}{K_z} \right)} \right) z \right] \sin(\eta_{m_3} x) \sin(\mu_{n_3} y). \quad (3.26)$$

Thus, in view of Eqs. (3.13), (3.23), (3.24), (3.25) and (3.26), a few general solutions of Eq. (3.2) can be expressed as

$$\begin{aligned}
u_p(x, y, z, t) = & \sum_{m=1}^M \sum_{n=1}^N A_{mn} \sinh \left[\left(\sqrt{\eta_m^2 \left(\frac{K_y}{K_x} \right) + \mu_n^2 \left(\frac{K_z}{K_x} \right)} \right) x \right] \sin(\eta_m y) \sin(\mu_n z) \\
& + \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{m_1 n_1} \sinh \left[\left(\sqrt{\eta_{m_1}^2 \left(\frac{K_x}{K_y} \right) + \mu_{n_1}^2 \left(\frac{K_z}{K_y} \right)} \right) y \right] \sin(\eta_{m_1} x) \sin(\mu_{n_1} z) \\
& + \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} A_{m_2 n_2} \sinh \left[\left(\sqrt{\eta_{m_2}^2 \left(\frac{K_x}{K_z} \right) + \mu_{n_2}^2 \left(\frac{K_y}{K_z} \right)} \right) z \right] \sin(\eta_{m_2} x) \sin(\mu_{n_2} y) \\
& + \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} A_{m_3 n_3} \cosh \left[\left(\sqrt{\eta_{m_2}^2 \left(\frac{K_x}{K_z} \right) + \mu_{n_2}^2 \left(\frac{K_y}{K_z} \right)} \right) z \right] \sin(\eta_{m_2} x) \sin(\mu_{n_2} y) \\
& + \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K E_{ijk} \sin(\alpha_i x) \sin(\beta_j y) \sin(\gamma_k z) \exp[-\lambda_{ijk}^2 t], \tag{3.27}
\end{aligned}$$

$$\begin{aligned}
u_p(x, y, z, t) = & \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} A_{m_3 n_3} \cosh \left[\left(\sqrt{\eta_{m_2}^2 \left(\frac{K_x}{K_z} \right) + \mu_{n_2}^2 \left(\frac{K_y}{K_z} \right)} \right) z \right] \sin(\eta_{m_2} x) \sin(\mu_{n_2} y) \\
& + \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K E_{ijk} \sin(\alpha_i x) \sin(\beta_j y) \sin(\gamma_k z) \exp[-\lambda_{ijk}^2 t] \tag{3.28}
\end{aligned}$$

and

$$\begin{aligned}
u_p(x, y, z, t) = & \sum_{m=1}^M \sum_{n=1}^N A_{mn} \sinh \left[\left(\sqrt{\eta_m^2 \left(\frac{K_y}{K_x} \right) + \mu_n^2 \left(\frac{K_z}{K_x} \right)} \right) x \right] \sin(\eta_m y) \sin(\mu_n z) \\
& + \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{m_1 n_1} \sinh \left[\left(\sqrt{\eta_{m_1}^2 \left(\frac{K_x}{K_y} \right) + \mu_{n_1}^2 \left(\frac{K_z}{K_y} \right)} \right) y \right] \sin(\eta_{m_1} x) \sin(\mu_{n_1} z) \\
& + \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} A_{m_3 n_3} \cosh \left[\left(\sqrt{\eta_{m_2}^2 \left(\frac{K_x}{K_z} \right) + \mu_{n_2}^2 \left(\frac{K_y}{K_z} \right)} \right) z \right] \sin(\eta_{m_2} x) \sin(\mu_{n_2} y) \\
& + \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K E_{ijk} \sin(\alpha_i x) \sin(\beta_j y) \sin(\gamma_k z) \exp[-\lambda_{ijk}^2 t]. \tag{3.29}
\end{aligned}$$

Eqs. (3.27), (3.28) and (3.29) will now be made use of to solve the three different variants of the three-dimensional multi-layered ponded ditch drainage problem considered in the present study.

3.2 Mathematical Formulation and Solution

Fig. 3.1 shows an aquifer comprising of three vertical, anisotropic soil layers. As can be seen, the flow domain is assumed to be a finite horizontal rectangular ponded field, $S_1 \times S_2$ in horizontal extent, being drained by vertical ditch drains on its four sides. The vertical

thickness of the flow domain, which is underlain by an impervious stratum, is considered to be h . The depth of the top of the middle soil layer from the ponded surface is denoted by H_5 while the depth of the top of the bottom layer from the ponded surface is represented as H_6 . The water level is assumed to be the same for all the ditch drains and is denoted by H_1 . A specific coordinate system, with the origin located at O , is assigned to the flow domain. As has been illustrated in Fig. 3.1, the x -axis is taken positive towards the Northern boundary while the y -axis is taken positive towards the Eastern boundary. The z -axis is taken to be positive in the vertically downward direction from the origin. The implementation of the variable ponding pyramid atop the field is achieved by introducing a network of thin bunds (theoretically of zero width) at the surface of the soil. N_0 denotes the number of ponding strips over the surface of soil and δ_i ($1 \leq i \leq N_0$) denotes the ponding depth of the i^{th} strip. Ditch bunds of width ε_x and ε_y are provided along the edges of the field so as to ensure that the ponded water does not get directly rolled over to the ditches from the field. The distances of the i^{th} bund, as measured from the origin O , are taken as d_{xi} and d_{yi} ($1 \leq i \leq (2N_0 - 2)$) in the x - and y -directions, respectively. When $N_0 = 1$, there are no inner bunds on the surface of the aquifer and the ponding depth is then uniform. For a better clarity of the studied flow situation, sectional views along two orthogonal axes (namely along $x-x'$ and $y-y'$ of Fig. 3.1) are also provided (Figs. 3.2 and 3.3).

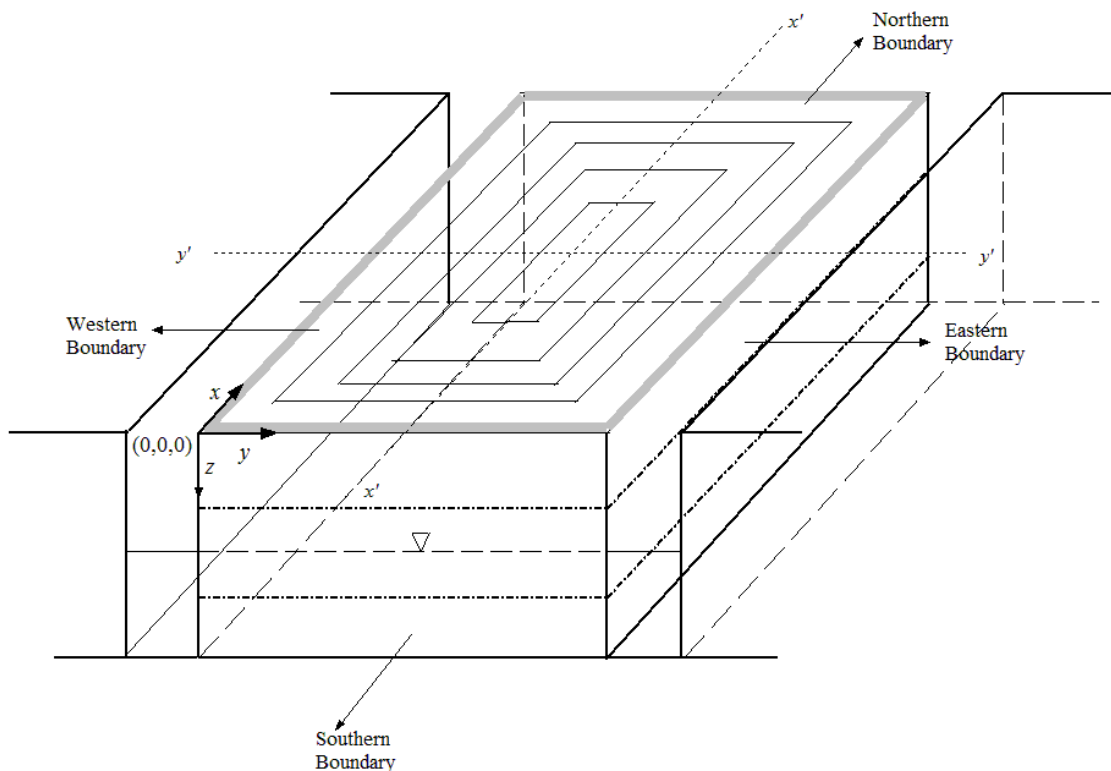


Fig. 3.1. General geometry of a three-dimensional ponded ditch drainage system subject to a variable ponding distribution at the surface of the soil

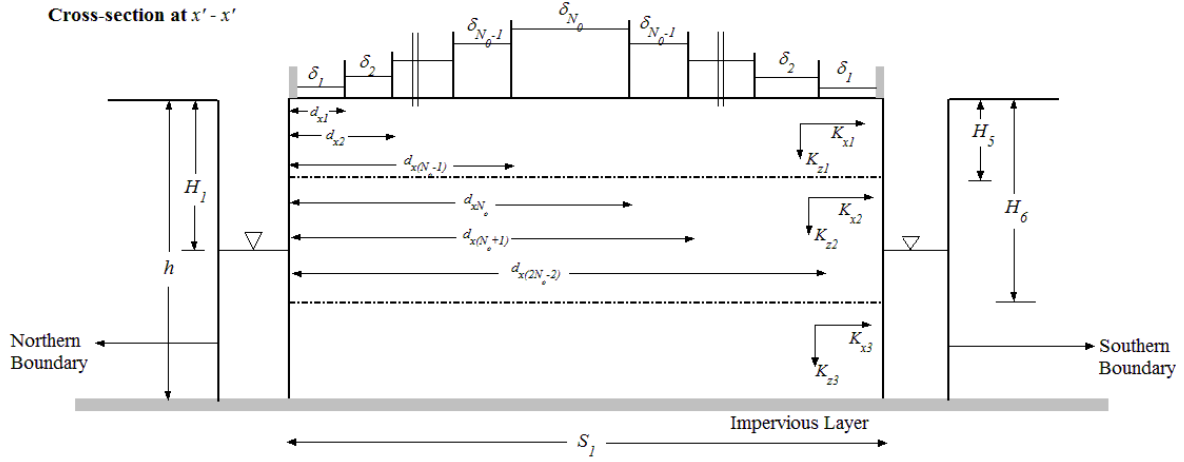


Fig 3.2. The cross sectional view of the flow domain along the axis $x - x'$ of Fig. 3.1

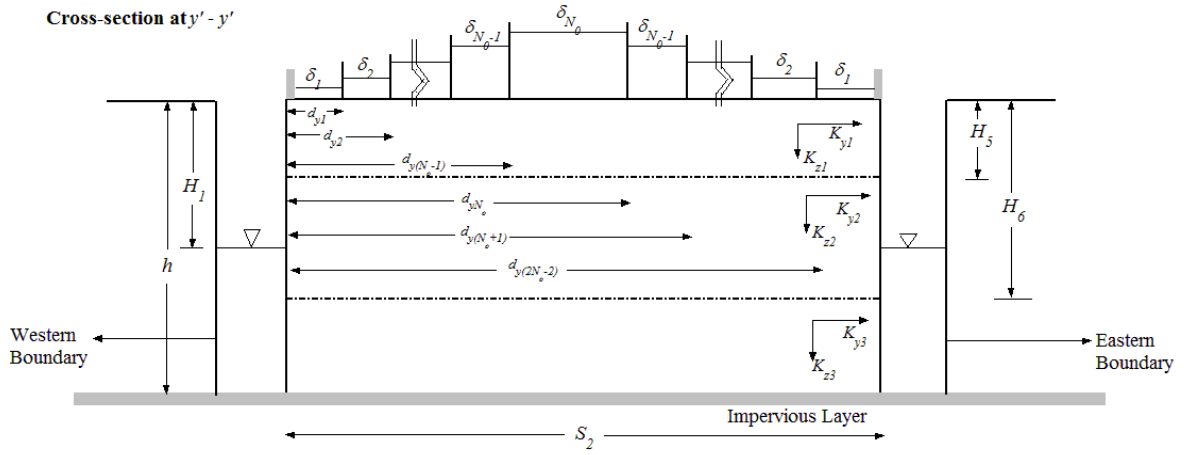


Fig 3.3. The cross sectional view of the flow domain along the axis $y - y'$ of Fig. 3.1

The governing equations for the three soil layers of the flow domain based on the principle of continuity can be expressed as

$$K_{x_1} \frac{\partial^2 \phi_{1(j)}}{\partial x^2} + K_{y_1} \frac{\partial^2 \phi_{1(j)}}{\partial y^2} + K_{z_1} \frac{\partial^2 \phi_{1(j)}}{\partial z^2} = S_{s_1} \frac{\partial \phi_{1(j)}}{\partial t}, \quad (3.30)$$

$$K_{x_2} \frac{\partial^2 \phi_{2(j)}}{\partial x^2} + K_{y_2} \frac{\partial^2 \phi_{2(j)}}{\partial y^2} + K_{z_2} \frac{\partial^2 \phi_{2(j)}}{\partial z^2} = S_{s_2} \frac{\partial \phi_{2(j)}}{\partial t} \quad (3.31)$$

and

$$K_{x_3} \frac{\partial^2 \phi_{3(j)}}{\partial x^2} + K_{y_3} \frac{\partial^2 \phi_{3(j)}}{\partial y^2} + K_{z_3} \frac{\partial^2 \phi_{3(j)}}{\partial z^2} = S_{s_3} \frac{\partial \phi_{3(j)}}{\partial t}, \quad (3.32)$$

where K_{x_1} , K_{y_1} and K_{z_1} , K_{x_2} , K_{y_2} and K_{z_2} and K_{x_3} , K_{y_3} and K_{z_3} are the directional conductivities of the top, middle and bottom layers, respectively and S_{s_1} , S_{s_2} and S_{s_3} the specific storage of these layers. The hydraulic heads of the top, middle and bottom soil layers are denoted by $\phi_{1(j)}$, $\phi_{2(j)}$ and $\phi_{3(j)}$, respectively where the subscript j takes the values 1, 2 or

3 depending on the position of water level in the drains (this will be clearer in the pages to follow) and t is the time variable.

Whereas all the steady state solutions proposed for the considered boundary value problems are exact for all possible combinations of the flow parameters, the corresponding transient solutions, like in the solutions of the two-dimensional ditch drainage problems of the previous chapter, are strictly valid only when the directional conductivities and specific storage of the layers satisfy certain relations among each other; they are as listed below.

$$\frac{K_{x_1}}{S_{s_1}} = \frac{K_{x_2}}{S_{s_2}} = \frac{K_{x_3}}{S_{s_3}}, \quad (3.33)$$

$$\frac{K_{y_1}}{S_{s_1}} = \frac{K_{y_2}}{S_{s_2}} = \frac{K_{y_3}}{S_{s_3}} \quad (3.34)$$

and

$$K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0 \text{ but } K_{z_1} = K_{z_2} = K_{z_3} \neq 0. \quad (3.35)$$

However, if the three soil layers have the same corresponding directional conductivities and the same specific storage (i.e., if $K_{x_1} = K_{x_2} = K_{x_3}$, $K_{y_1} = K_{y_2} = K_{y_3}$, $K_{z_1} = K_{z_2} = K_{z_3}$ and $S_{s_1} = S_{s_2} = S_{s_3}$) – which essentially means a single layer anisotropic soil – our transient solution will then be applicable for all possible combinations of parameters of Fig. 3.1 and the requirements [it is to be noted for these situations Eqs. (3.33) and (3.34) and $K_{z_1} = K_{z_2} = K_{z_3}$ would be automatically satisfied] $K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0$ for its validity would then be not needed.

Depending on the level of water in the ditch drains, we encounter three distinct boundary value problems pertaining to the flow problem. Before dealing with the boundary conditions specific to each case, we enumerate the common initial and boundary conditions for the three-dimensional flow problem at hand; they are as expressed below.

$$\phi_{1(j)}(x, y, z, t = 0) = 0, \quad 0 < x < S_1, \quad 0 < y < S_2, \quad 0 < z \leq H_5, \quad (I)$$

$$\phi_{2(j)}(x, y, z, t = 0) = 0, \quad 0 < x < S_1, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_6, \quad (II)$$

$$\phi_{3(j)}(x, y, z, t = 0) = 0, \quad 0 < x < S_1, \quad 0 < y < S_2, \quad H_6 \leq z < h, \quad (III)$$

$$\begin{aligned} \phi_{1(j)}(x, y, z, t > 0) = \\ \phi_{2(j)}(x, y, z, t > 0), \quad 0 < x < S_1, \quad 0 < y < S_2, \quad z = H_5, \end{aligned} \quad (IVa)$$

$$\begin{aligned} -K_{z_1} \frac{\partial \phi_{1(j)}(x, y, z, t > 0)}{\partial z} = \\ -K_{z_2} \frac{\partial \phi_{2(j)}(x, y, z, t > 0)}{\partial z}, \quad 0 < x < S_1, \quad 0 < y < S_2, \quad z = H_5, \end{aligned} \quad (IVb)$$

$$\begin{aligned} \phi_{2(j)}(x, y, z, t > 0) = \\ \phi_{3(j)}(x, y, z, t > 0), \quad 0 < x < S_1, \quad 0 < y < S_2, \quad z = H_6, \end{aligned} \quad (Va)$$

$$-K_{z_2} \frac{\partial \phi_{2(j)}(x, y, z, t > 0)}{\partial z} =$$

$$-K_{z_3} \frac{\partial \phi_{3(j)}(x, y, z, t > 0)}{\partial z}, \quad 0 < x < S_1, \quad 0 < y < S_2, \quad z = H_6, \quad (\text{Vb})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_1, \quad 0 < x < S_1, \quad 0 < y < d_{y1}, \quad z = 0, \quad (\text{VIa})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_1, \quad 0 < x < S_1, \quad d_{y(2N_0-2)} < y < S_2, \quad z = 0, \quad (\text{VIb})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_1, \quad 0 < x < d_{x1}, \quad d_{y1} < y < d_{y(2N_0-2)}, \quad z = 0, \quad (\text{VIc})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_1, \quad d_{x(2N_0-2)} < x < S_1, \quad d_{y1} < y < d_{y(2N_0-2)}, \quad z = 0, \quad (\text{VIId})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_i, \quad d_{x(i-1)} < x < d_{x(2N_0-i)}, \quad d_{y(i-1)} < y < d_{yi}, \quad z = 0, \quad (\text{VIe})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_i, \quad d_{x(i-1)} < x < d_{x(2N_0-i)}, \quad d_{y(2N_0-i-1)} < y < d_{y(2N_0-i)}, \quad z = 0, \quad (\text{VIIf})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_i, \quad d_{x(i-1)} < x < d_{xi}, \quad d_{yi} < y < d_{y(2N_0-i-1)}, \quad z = 0, \quad (\text{VIg})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_i, \quad d_{x(2N_0-i-1)} < x < d_{x(2N_0-i)}, \quad d_{yi} < y < d_{y(2N_0-i-1)}, \quad z = 0, \quad (\text{VIh})$$

$$\phi_{1(j)}(x, y, z, t > 0) = \delta_{N_0}, \quad d_{x(N_0-1)} < x < d_{xN_0}, \quad d_{y(N_0-1)} < y < d_{yN_0}, \quad z = 0, \quad (\text{VIi})$$

$$-K_{z_3} \frac{\partial \phi_{3(j)}(x, y, z, t > 0)}{\partial z} = 0, \quad 0 < x < S_1, \quad 0 < y < S_2, \quad z = h. \quad (\text{VII})$$

We will now make an effort to obtain solution to the first of our problems arising out of the situation where the water level of all the drains lies in the top layer only both for cases when the soil is subjected to a constant as well as variable ponding distribution at the surface of the soil.

3.2.1 Level of Water in the Ditches is above the Boundary between the Top and the Middle Soil Layers

Fig. 3.4 provides a representation of the transient ditch drainage problem under consideration where the water level in the ditches is above the boundary between the top soil layer and middle soil layer of the flow domain.

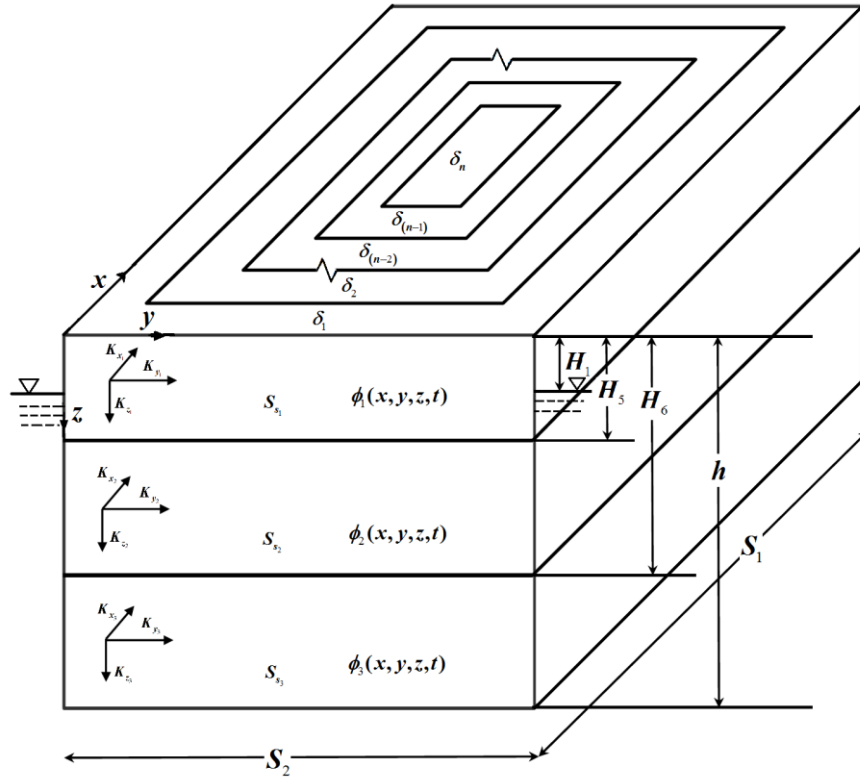


Fig. 3.4. Three dimensional ditch drainage system for a three-layered soil when height of water in the ditches is above the boundary between the top and the middle soil layers

The boundary conditions specific to the ditch drain boundaries for this particular case are

$$\phi_{1(l)}(x, y, z, t > 0) = -z, \quad x = 0, \quad 0 < y < S_2, \quad 0 < z \leq H_1, \quad \text{(VIIIa)}$$

$$\phi_{1(l)}(x, y, z, t > 0) = -H_1, \quad x = 0, \quad 0 < y < S_2, \quad H_1 \leq z \leq H_5, \quad \text{(VIIIb)}$$

$$\phi_{2(l)}(x, y, z, t > 0) = -H_1, \quad x = 0, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_6, \quad \text{(IX)}$$

$$\phi_{3(l)}(x, y, z, t > 0) = -H_1, \quad x = 0, \quad 0 < y < S_2, \quad H_6 \leq z < h, \quad \text{(X)}$$

$$\phi_{1(l)}(x, y, z, t > 0) = -z, \quad x = S_1, \quad 0 < y < S_2, \quad 0 < z \leq H_1, \quad \text{(XIa)}$$

$$\phi_{1(l)}(x, y, z, t > 0) = -H_1, \quad x = S_1, \quad 0 < y < S_2, \quad H_1 < z \leq H_5, \quad \text{(XIb)}$$

$$\phi_{2(l)}(x, y, z, t > 0) = -H_1, \quad x = S_1, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_6, \quad \text{(XII)}$$

$$\phi_{3(l)}(x, y, z, t > 0) = -H_1, \quad x = S_1, \quad 0 < y < S_2, \quad H_6 \leq z < h, \quad \text{(XIII)}$$

$$\phi_{1(l)}(x, y, z, t > 0) = -z, \quad 0 < x < S_1, \quad y = 0, \quad 0 < z \leq H_1, \quad \text{(XIVa)}$$

$$\phi_{1(l)}(x, y, z, t > 0) = -H_1, \quad 0 < x < S_1, \quad y = 0, \quad H_1 < z \leq H_5, \quad \text{(XIVb)}$$

$$\phi_{2(l)}(x, y, z, t > 0) = -H_1, \quad 0 < x < S_1, \quad y = 0, \quad H_5 \leq z \leq H_6, \quad \text{(XV)}$$

$$\phi_{3(l)}(x, y, z, t > 0) = -H_1, \quad 0 \leq x \leq S_1, \quad y = 0, \quad H_6 \leq z < h, \quad \text{(XVI)}$$

$$\phi_{1(l)}(x, y, z, t > 0) = -z, \quad 0 < x < S_1, \quad y = S_2, \quad 0 < z \leq H_1, \quad \text{(XVIIa)}$$

$$\phi_{1(1)}(x, y, z, t) = -H_1, \quad 0 < x < S_1, \quad y = S_2, \quad H_1 < z \leq H_5, \quad (\text{XVIIIb})$$

$$\phi_{2(1)}(x, y, z, t) = -H_1, \quad 0 < x < S_1, \quad y = S_2, \quad H_5 \leq z \leq H_6, \quad (\text{XVIII})$$

$$\phi_{3(1)}(x, y, z, t) = -H_1, \quad 0 \leq x \leq S_1, \quad y = S_2, \quad H_6 \leq z < h. \quad (\text{XIX})$$

Taking into account the general solutions of the three-dimensional continuity equation as expressed in Eqs. (3.27), (3.28) and (3.29), we propose the following analytical solution for the flow problem at hand

$$\begin{aligned} \phi_{1(1)}(x, y, z, t) = & \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \sin(N_{p_1} y) \sin(N_{q_1} z) \\ & + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \sin(N_{p_2} y) \sin(N_{q_2} z) \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{p_3} x) \sin(N_{q_3} z) \\ & + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{p_4} x) \sin(N_{q_4} z) \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \frac{\sinh(\lambda_{kl} z)}{\cosh(\lambda_{kl} H_5)} \sin(N_k x) \sin(N_l y) \\ & + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \frac{\cosh[\lambda_{uv} (H_5 - z)]}{\cosh(\lambda_{uv} H_5)} \sin(N_u x) \sin(N_v y) \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right], \end{aligned} \quad (3.36)$$

where

$$N_{p_1} = \left(\frac{p_1 \pi}{S_2} \right), \quad (3.37)$$

$$N_{q_1} = \left[\left(\frac{1-2q_1}{2} \right) \frac{\pi}{H_5} \right], \quad (3.38)$$

$$N_{p_2} = \left(\frac{p_2 \pi}{S_2} \right), \quad (3.39)$$

$$N_{q_2} = \left[\left(\frac{1-2q_2}{2} \right) \frac{\pi}{H_5} \right], \quad (3.40)$$

$$N_{p_3} = \left(\frac{p_3 \pi}{S_1} \right), \quad (3.41)$$

$$N_{q_3} = \left[\left(\frac{1-2q_3}{2} \right) \frac{\pi}{H_5} \right], \quad (3.42)$$

$$N_{p_4} = \left(\frac{p_4 \pi}{S_1} \right), \quad (3.43)$$

$$N_{q_4} = \left[\left(\frac{1-2q_4}{2} \right) \frac{\pi}{H_5} \right], \quad (3.44)$$

$$N_k = \left(\frac{k \pi}{S_1} \right), \quad (3.45)$$

$$N_l = \left(\frac{l \pi}{S_2} \right), \quad (3.46)$$

$$N_u = \left(\frac{u \pi}{S_1} \right), \quad (3.47)$$

$$N_v = \left(\frac{v \pi}{S_2} \right), \quad (3.48)$$

$$(\lambda_{p_1 q_1})^2 = (N_{p_1}^2 K_{xy}^{a1} + N_{q_1}^2 K_{xz}^{a1}), \quad (3.49)$$

$$(\lambda_{p_2 q_2})^2 = (N_{p_2}^2 K_{xy}^{a1} + N_{q_2}^2 K_{xz}^{a1}), \quad (3.50)$$

$$(\lambda_{p_3 q_3})^2 = \left[(N_{p_3}^2 / K_{xy}^{a1}) + N_{q_3}^2 K_{yz}^{a1} \right], \quad (3.51)$$

$$(\lambda_{p_4 q_4})^2 = \left[(N_{p_4}^2 / K_{xy}^{a1}) + N_{q_4}^2 K_{yz}^{a1} \right], \quad (3.52)$$

$$(\lambda_{kl})^2 = \left[(N_k^2 / K_{xz}^{a1}) + (N_l^2 / K_{yz}^{a1}) \right], \quad (3.53)$$

$$(\lambda_{uv})^2 = \left[(N_u^2 / K_{xz}^{a1}) + (N_v^2 / K_{yz}^{a1}) \right], \quad (3.54)$$

$$K_{xy}^{a1} = \left(\frac{K_{y_1}}{K_{x_1}} \right), \quad (3.55)$$

$$K_{xz}^{a1} = \left(\frac{K_{z_1}}{K_{x_1}} \right), \quad (3.56)$$

$$K_{yz}^{a1} = \left(\frac{K_{z_1}}{K_{y_1}} \right), \quad (3.57)$$

$$N_{l_1} = \left(\frac{l_1 \pi}{S_1} \right), \quad (3.58)$$

$$N_{m_1} = \left(\frac{m_1 \pi}{S_2} \right), \quad (3.59)$$

$$N_{n_1} = \left[\left(\frac{1-2n_1}{2} \right) \frac{\pi}{h} \right], \quad (3.60)$$

$$N_{l_2} = \left(\frac{l_2 \pi}{S_1} \right), \quad (3.61)$$

$$N_{m_2} = \left(\frac{m_2 \pi}{S_2} \right), \quad (3.62)$$

$$N_{n_2} = \left[\left(\frac{1-2n_2}{2} \right) \frac{\pi}{h} \right], \quad (3.63)$$

$$N_{l_3} = \left(\frac{l_3 \pi}{S_1} \right), \quad (3.64)$$

$$N_{m_3} = \left(\frac{m_3 \pi}{S_2} \right), \quad (3.65)$$

$$N_{n_3} = \left[\left(\frac{1-2n_3}{2} \right) \frac{\pi}{h} \right], \quad (3.66)$$

$$(\lambda_{l_1 m_1 n_1})^2 = \left[N_{l_1}^2 \left(\frac{K_{x_1}}{S_{s_1}} \right) + N_{m_1}^2 \left(\frac{K_{y_1}}{S_{s_1}} \right) + N_{n_1}^2 \left(\frac{K_{z_1}}{S_{s_1}} \right) \right], \quad (3.67)$$

$$(\lambda_{l_2 m_2 n_2})^2 = \left[N_{l_2}^2 \left(\frac{K_{x_2}}{S_{s_2}} \right) + N_{m_2}^2 \left(\frac{K_{y_2}}{S_{s_2}} \right) + N_{n_2}^2 \left(\frac{K_{z_2}}{S_{s_2}} \right) \right], \quad (3.68)$$

$$(\lambda_{l_3 m_3 n_3})^2 = \left[N_{l_3}^2 \left(\frac{K_{x_3}}{S_{s_3}} \right) + N_{m_3}^2 \left(\frac{K_{y_3}}{S_{s_3}} \right) + N_{n_3}^2 \left(\frac{K_{z_3}}{S_{s_3}} \right) \right] \quad (3.69)$$

and $B_{p_1 q_1(1)}$, $C_{p_2 q_2(1)}$, $D_{p_3 q_3(1)}$, $F_{p_4 q_4(1)}$, $E_{kl(1)}$, $Q_{uv(1)}$, $A_{l_1 m_1 n_1(1)}$, $R_{l_2 m_2 n_2(1)}$ and $W_{l_3 m_3 n_3(1)}$ are constants and

$$\begin{aligned} \phi_{2(1)}(x, y, z, t) = & \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \frac{\cosh[\lambda_{i_1 j_1}(z - H_5)]}{\cosh[\lambda_{i_1 j_1}(H_6 - H_5)]} \sin(N_{i_1} x) \sin(N_{j_1} y) \\ & + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \frac{\cosh[\lambda_{i_2 j_2}(H_6 - z)]}{\cosh[\lambda_{i_2 j_2}(H_6 - H_5)]} \sin(N_{i_2} x) \sin(N_{j_2} y) - H_1 \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right], \quad (3.70) \end{aligned}$$

where

$$N_{i_1} = \left(\frac{i_1 \pi}{S_1} \right), \quad (3.71)$$

$$N_{j_1} = \left(\frac{j_1 \pi}{S_2} \right), \quad (3.72)$$

$$N_{i_2} = \left(\frac{i_2 \pi}{S_1} \right), \quad (3.73)$$

$$N_{j_2} = \left(\frac{j_2 \pi}{S_2} \right), \quad (3.74)$$

$$(\lambda_{i_1 j_1})^2 = \left[(N_{i_1}^2 / K_{xz}^{a2}) + (N_{j_1}^2 / K_{yz}^{a2}) \right], \quad (3.75)$$

$$(\lambda_{i_2 j_2})^2 = \left[(N_{i_2}^2 / K_{xz}^{a2}) + (N_{j_2}^2 / K_{yz}^{a2}) \right], \quad (3.76)$$

$$K_{xz}^{a2} = \left(\frac{K_{z_2}}{K_{x_2}} \right), \quad (3.77)$$

$$K_{yz}^{a2} = \left(\frac{K_{z_2}}{K_{y_2}} \right) \quad (3.78)$$

and $G_{i_1 j_1(1)}$ and $H_{i_2 j_2(1)}$ are constants and

$$\begin{aligned} \phi_{3(1)}(x, y, z, t) = & \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \frac{\cosh \left[\lambda_{i_3 j_3} (h-z) \right]}{\sinh \left[\lambda_{i_3 j_3} (h-H_6) \right]} \sin(N_{i_3} x) \sin(N_{j_3} y) - H_1 \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right], \end{aligned} \quad (3.79)$$

where

$$N_{i_3} = \left(\frac{i_3 \pi}{S_1} \right), \quad (3.80)$$

$$N_{j_3} = \left(\frac{j_3 \pi}{S_2} \right), \quad (3.81)$$

$$(\lambda_{i_3 j_3})^2 = \left[(N_{i_3}^2 / K_{xz}^{a3}) + (N_{j_3}^2 / K_{yz}^{a3}) \right], \quad (3.82)$$

$$K_{xz}^{a3} = \left(\frac{K_{z_3}}{K_{x_3}} \right), \quad (3.83)$$

$$K_{yz}^{a3} = \begin{pmatrix} K_{z_3} \\ K_{y_3} \end{pmatrix} \quad (3.84)$$

and $P_{i_3 j_3 (1)}$ are constants.

It can be noted that in the hydraulic head expressions for all the soil layers, the three triple Fourier series terms are common. They represent the transient part of the solution for the flow problem under consideration. Thus, the transient portion of the solution can be written as

$$\begin{aligned} \phi_T = & \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1 (1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2 (1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3 (1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right]. \end{aligned} \quad (3.85)$$

We can also duly note that at time $t = 0$, ϕ_T can be expressed as

$$\begin{aligned} \phi_T = & \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1 (1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2 (1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3 (1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z). \end{aligned} \quad (3.86)$$

For our convenience, let us denote the three triple Fourier series as

$$\sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1 (1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) = SR_1, \quad (3.87)$$

$$\sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2 (1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) = SR_2 \quad (3.88)$$

and

$$\sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3 (1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) = SR_3. \quad (3.89)$$

It should be noted here that SR_1 exists only for the top soil layer and is zero for the other two layers; similarly, SR_2 exists only for the middle layer and is zero in the other two layers and SR_3 is non-zero only for the bottom layer and is zero elsewhere. We would like to again reiterate that the proposed transient solution for the flow problem of Fig. 3.4 will be valid only when Eqs. (3.33) and (3.34) and (3.35) are being satisfied by a drainage setting of Fig. 3.4 and not otherwise, i.e., only when the constraints $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2} = K_{x_3}/S_{s_3}$, $K_{y_1}/S_{s_1} = K_{y_2}/S_{s_2} = K_{y_3}/S_{s_3}$ and $K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0$ but $K_{z_1} = K_{z_2} = K_{z_3} \neq 0$ are being satisfied by a drainage scenario of Fig. 3.3. However, so far as the steady state solution is concerned, it is valid for all possible drainage combinations of drainage parameters of Fig. 3.4.

Before we proceed to work out the Fourier coefficients relevant to the problem, it is imperative that we state the expressions for velocity distribution in the flow domain in the x -, y - and z - directions.

Since the hydraulic head expressions for the three soil layers have already been evaluated, the velocity distributions in the x -, y - and z - directions can be worked out for each layer using the Darcy's law; thus, we have

$$V_{x1(1)} = -K_{x_1} \frac{\partial \phi_{1(1)}(x, y, z, t)}{\partial x}, \quad (3.90)$$

$$V_{y1(1)} = -K_{y_1} \frac{\partial \phi_{1(1)}(x, y, z, t)}{\partial y}, \quad (3.91)$$

$$V_{z1(1)} = -K_{z_1} \frac{\partial \phi_{1(1)}(x, y, z, t)}{\partial z}, \quad (3.92)$$

$$V_{x2(1)} = -K_{x_2} \frac{\partial \phi_{2(1)}(x, y, z, t)}{\partial x}, \quad (3.93)$$

$$V_{y2(1)} = -K_{y_2} \frac{\partial \phi_{2(1)}(x, y, z, t)}{\partial y}, \quad (3.94)$$

$$V_{z2(1)} = -K_{z_2} \frac{\partial \phi_{2(1)}(x, y, z, t)}{\partial z}, \quad (3.95)$$

$$V_{x3(1)} = -K_{x_3} \frac{\partial \phi_{3(1)}(x, y, z, t)}{\partial x}, \quad (3.96)$$

$$V_{y3(1)} = -K_{y_3} \frac{\partial \phi_{3(1)}(x, y, z, t)}{\partial y}, \quad (3.97)$$

and

$$V_{z3(1)} = -K_{z_3} \frac{\partial \phi_{3(1)}(x, y, z, t)}{\partial z}, \quad (3.98)$$

where $V_{xi(1)}$, $V_{yi(1)}$ and $V_{zi(1)}$ ($i = 1, 2$ and 3) are the velocity distributions in the layers in the x -, y - and z - directions, respectively.

Performing the above differentiations using the relevant head expressions, we finally get the velocity expressions for the layers as

$$\begin{aligned} V_{x1(1)} = & -K_{x_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1q_1(1)} \frac{\lambda_{p_1q_1} \cosh(\lambda_{p_1q_1} x)}{\sinh(\lambda_{p_1q_1} S_1)} \sin(N_{p_1} y) \sin(N_{q_1} z) \right. \\ & - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2q_2(1)} \frac{\lambda_{p_2q_2} \cosh[\lambda_{p_2q_2} (S_1 - x)]}{\sinh(\lambda_{p_2q_2} S_1)} \sin(N_{p_2} y) \sin(N_{q_2} z) \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3q_3(1)} \frac{\sinh(\lambda_{p_3q_3} y)}{\sinh(\lambda_{p_3q_3} S_2)} N_{p_3} \cos(N_{p_3} x) \sin(N_{q_3} z) \\ & \left. + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4q_4(1)} \frac{\sinh[\lambda_{p_4q_4} (S_2 - y)]}{\sinh(\lambda_{p_4q_4} S_2)} N_{p_4} \cos(N_{p_4} x) \sin(N_{q_4} z) \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \frac{\sinh(\lambda_{kl} z)}{\cosh(\lambda_{kl} H_5)} N_k \cos(N_k x) \sin(N_l y) \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \frac{\cosh[\lambda_{uv}(H_5 - z)]}{\cosh(\lambda_{uv} H_5)} N_u \cos(N_u x) \sin(N_v y) \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \cos(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \cos(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \cos(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\}, \quad (3.99)
\end{aligned}$$

$$\begin{aligned}
V_{y1(1)} = & -K_{y_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} N_{p_1} \cos(N_{p_1} y) \sin(N_{q_1} z) \right. \\
& + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \frac{\sinh[\lambda_{p_2 q_2}(S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} N_{p_2} \cos(N_{p_2} y) \sin(N_{q_2} z) \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \frac{\lambda_{p_3 q_3} \cosh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{p_3} x) \sin(N_{q_3} z) \\
& - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \frac{\lambda_{p_4 q_4} \cosh[\lambda_{p_4 q_4}(S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{p_4} x) \sin(N_{q_4} z) \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \frac{\sinh(\lambda_{kl} z)}{\cosh(\lambda_{kl} H_5)} \sin(N_k x) N_l \cos(N_l y) \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \frac{\cosh[\lambda_{uv}(H_5 - z)]}{\cosh(\lambda_{uv} H_5)} \sin(N_u x) N_v \cos(N_v y) \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) N_{m_1} \cos(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) N_{m_2} \cos(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) N_{m_3} \cos(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\}, \quad (3.100)
\end{aligned}$$

$$\begin{aligned}
V_{z1(1)} = & -K_{z_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \sin(N_{p_1} y) N_{q_1} \cos(N_{q_1} z) \right. \\
& + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \frac{\sinh[\lambda_{p_2 q_2}(S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \sin(N_{p_2} y) N_{q_2} \cos(N_{q_2} z)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{p_3} x) N_{q_3} \cos(N_{q_3} z) \\
& + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{p_4} x) N_{q_4} \cos(N_{q_4} z) \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \frac{\lambda_{kl} \cosh(\lambda_{kl} z)}{\cosh(\lambda_{kl} H_5)} \sin(N_k x) \sin(N_l y) \\
& - \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \frac{\lambda_{uv} \sinh[\lambda_{uv} (H_5 - z)]}{\cosh(\lambda_{uv} H_5)} \sin(N_u x) \sin(N_v y) \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) N_{n_1} \cos(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) N_{n_2} \cos(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) N_{n_3} \cos(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\}, \quad (3.101)
\end{aligned}$$

$$\begin{aligned}
V_{x2(1)} = -K_{x_2} & \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \frac{\cosh[\lambda_{i_1 j_1} (z - H_5)]}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} N_{i_1} \cos(N_{i_1} x) \sin(N_{j_1} y) \right. \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \frac{\cosh[\lambda_{i_2 j_2} (H_6 - z)]}{\cosh[\lambda_{i_2 j_2} (H_6 - H_5)]} N_{i_2} \cos(N_{i_2} x) \sin(N_{j_2} y) \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \cos(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \cos(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& \left. + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \cos(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \right\}, \quad (3.102)
\end{aligned}$$

$$\begin{aligned}
V_{y2(1)} = -K_{y_2} & \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \frac{\cosh[\lambda_{i_1 j_1} (z - H_5)]}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} \sin(N_{i_1} x) N_{j_1} \cos(N_{j_1} y) \right. \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \frac{\cosh[\lambda_{i_2 j_2} (H_6 - z)]}{\cosh[\lambda_{i_2 j_2} (H_6 - H_5)]} \sin(N_{i_2} x) N_{j_2} \cos(N_{j_2} y) \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) N_{m_1} \cos(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& \left. + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) N_{m_2} \cos(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \right\}
\end{aligned}$$

$$+ \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) N_{m_3} \cos(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\}, \quad (3.103)$$

$$\begin{aligned} V_{z2(1)} = & -K_{z_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \frac{\lambda_{i_1 j_1} \sinh[\lambda_{i_1 j_1} (z - H_5)]}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} \sin(N_{i_1} x) \sin(N_{j_1} y) \right. \\ & - \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \frac{\lambda_{i_2 j_2} \sinh[\lambda_{i_2 j_2} (H_6 - z)]}{\cosh[\lambda_{i_2 j_2} (H_6 - H_5)]} \sin(N_{i_2} x) \sin(N_{j_2} y) \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) N_{n_1} \cos(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) N_{n_2} \cos(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & \left. + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) N_{n_3} \cos(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \right\}, \quad (3.104) \end{aligned}$$

$$\begin{aligned} V_{x3(1)} = & -K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \frac{\cosh[\lambda_{i_3 j_3} (h - z)]}{\sinh[\lambda_{i_3 j_3} (h - H_6)]} N_{i_3} \cos(N_{i_3} x) \sin(N_{j_3} y) \right. \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \cos(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \cos(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & \left. + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \cos(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \right\}, \quad (3.105) \end{aligned}$$

$$\begin{aligned} V_{y3(1)} = & -K_{y_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \frac{\cosh[\lambda_{i_3 j_3} (h - z)]}{\sinh[\lambda_{i_3 j_3} (h - H_6)]} \sin(N_{i_3} x) N_{j_3} \cos(N_{j_3} y) \right. \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) N_{m_1} \cos(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) N_{m_2} \cos(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & \left. + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) N_{m_3} \cos(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \right\} \quad (3.106) \end{aligned}$$

and

$$\begin{aligned} V_{z3(1)} = & -K_{z_3} \left\{ - \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \frac{\lambda_{i_3 j_3} \sinh[\lambda_{i_3 j_3} (h - z)]}{\sinh[\lambda_{i_3 j_3} (h - H_6)]} \sin(N_{i_3} x) \sin(N_{j_3} y) \right. \\ & \left. + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) N_{n_1} \cos(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) N_{n_2} \cos(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) N_{n_3} \cos(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \}. \quad (3.107)
\end{aligned}$$

3.2.1.1 Determination of Coefficients of the Steady State Part of the Solution (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 3.4)

Before determining the coefficients for the transient terms in the solution, let us first determine the coefficients of the steady state terms of the solution. Introducing boundary conditions (VIIIa) and (VIIIb) in Eq. (3.36), we have the following relationships involving the coefficients $C_{p_2 q_2(1)}$

$$\begin{aligned}
\sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \sin(N_{p_2} y) \sin(N_{q_2} z) &= -z, & x=0, & 0 < y < S_2, & 0 < z \leq H_1, \\
\sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \sin(N_{p_2} y) \sin(N_{q_2} z) &= -H_1, & x=0, & 0 < y < S_2, & H_1 \leq z \leq H_5.
\end{aligned}$$

Expanding the double Fourier series, we have

$$C_{p_2 q_2(1)} = \left(\frac{2}{S_2}\right) \left(\frac{2}{H_5}\right) \times \int_0^{S_2} \sin(N_{p_2} y) dy \times \left[\int_0^{H_1} -z \sin(N_{q_2} z) dz + \int_{H_1}^{H_5} -H_1 \sin(N_{q_2} z) dz \right]. \quad (3.108)$$

After evaluating the integrals in the above expression, the final expression for $C_{p_2 q_2(1)}$ works out as

$$C_{p_2 q_2(1)} = \left(\frac{2}{S_2}\right) \left(\frac{2}{H_5}\right) \left[\frac{1 - \cos(N_{p_2} S_2)}{N_{p_2}} \right] \left[\frac{-\sin(N_{q_2} H_1)}{N_{q_2}} \right]. \quad (3.109)$$

Similarly, by considering boundary conditions (XIa) and (XIb), we obtain the following expression for the coefficients $B_{p_1 q_1(1)}$

$$B_{p_1 q_1(1)} = \left(\frac{2}{S_2}\right) \left(\frac{2}{H_5}\right) \left[\frac{1 - \cos(N_{p_1} S_2)}{N_{p_1}} \right] \left[\frac{-\sin(N_{q_1} H_1)}{N_{q_1}} \right]. \quad (3.110)$$

Analogously, boundary conditions (XIVa) and (XIVb) yield the following equation for $F_{p_4 q_4(1)}$

$$F_{p_4 q_4(1)} = \left(\frac{2}{S_1}\right) \left(\frac{2}{H_5}\right) \left[\frac{1 - \cos(N_{p_4} S_1)}{N_{p_4}} \right] \left[\frac{-\sin(N_{q_4} H_1)}{N_{q_4}} \right], \quad (3.111)$$

while, from boundary conditions (XVIIa) and (XVIIb), we have the expression for $D_{p_3 q_3(1)}$ as

$$D_{p_3 q_3(1)} = \left(\frac{2}{S_1}\right) \left(\frac{2}{H_5}\right) \left[\frac{1 - \cos(N_{p_3} S_1)}{N_{p_3}} \right] \left[\frac{-\sin(N_{q_3} H_1)}{N_{q_3}} \right]. \quad (3.112)$$

Now applying boundary conditions (VIa), (VIb), (VIc), (VI d), (VIe), (VIf), (VIg), (VIh) and (VIi) to Eq. (3.36), we get

$$\begin{aligned}
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_1, & 0 < x < S_1, & & 0 < y < d_{y1}, & & z = 0, \\
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_1, & 0 < x < S_1, & & d_{y(2N_0-2)} < y < S_2, & & z = 0, \\
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_1, & 0 < x < d_{x1}, & & d_{y1} < y < d_{y(2N_0-2)}, & & z = 0, \\
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_1, & d_{x(2N_0-2)} < x < S_1, & & d_{y1} < y < d_{y(2N_0-2)}, & & z = 0, \\
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_i, & d_{x(i-1)} < x < d_{x(2N_0-i)}, & & d_{y(i-1)} < y < d_{yi}, & & z = 0, \\
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_i, & d_{x(i-1)} < x < d_{x(2N_0-i)}, & & d_{y(2N_0-i-1)} < y < d_{y(2N_0-i)}, & & z = 0, \\
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_i, & d_{x(i-1)} < x < d_{xi}, & & d_{yi} < y < d_{y(2N_0-i-1)}, & & z = 0, \\
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_i, & d_{x(2N_0-i-1)} < x < d_{x(2N_0-i)}, & & d_{yi} < y < d_{y(2N_0-i-1)}, & & z = 0, \\
& & & & & & 2 \leq i \leq (N_0 - 1), \\
\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) &= \delta_{N_0}, & d_{x(N_0-1)} < x < d_{xN_0}, & & d_{y(N_0-1)} < y < d_{yN_0}, & & z = 0.
\end{aligned}$$

We now make a double Fourier run in $0 < x < S_1$, $0 < y < S_2$ for the above equations; this yields an expression for $Q_{uv(1)}$ as

$$\begin{aligned}
Q_{uv(1)} &= \left(\frac{2}{S_1} \right) \left(\frac{2}{S_2} \right) \times \\
&\left\{ \delta_1 \left[\int_0^{S_1} \int_0^{d_{x1}} \sin(N_u x) \sin(N_v y) dx dy + \int_0^{S_1} \int_{d_{y(2N_0-2)}}^{S_2} \sin(N_u x) \sin(N_v y) dx dy \right. \right. \\
&+ \left. \int_0^{d_{x1}} \int_0^{d_{y(2N_0-2)}} \sin(N_u x) \sin(N_v y) dx dy + \int_{d_{x(2N_0-2)}}^{S_1} \int_{d_{y1}}^{d_{y(2N_0-2)}} \sin(N_u x) \sin(N_v y) dx dy \right] \\
&+ \sum_{i=2}^{I=(N_0-1)} \delta_i \left[\int_{d_{x(i-1)}}^{d_{x(2N_0-i)}} \int_{d_{y(i-1)}}^{d_{yi}} \sin(N_u x) \sin(N_v y) dx dy + \int_{d_{x(i-1)}}^{d_{x(2N_0-i)}} \int_{d_{y(2N_0-i-1)}}^{d_{y(2N_0-i)}} \sin(N_u x) \sin(N_v y) dx dy + \right. \\
&\left. \int_{d_{x(i-1)}}^{d_{xi}} \int_{d_{yi}}^{d_{y(2N_0-i-1)}} \sin(N_u x) \sin(N_v y) dx dy + \int_{d_{x(2N_0-i-1)}}^{d_{x(2N_0-i)}} \int_{d_{yi}}^{d_{y(2N_0-i-1)}} \sin(N_u x) \sin(N_v y) dx dy \right] \\
&+ \left. \delta_{N_0} \int_{d_{x(N_0-1)}}^{d_{xN_0}} \int_{d_{y(N_0-1)}}^{d_{yN_0}} \sin(N_u x) \sin(N_v y) dx dy \right\}. \tag{3.113}
\end{aligned}$$

On evaluating the integrals, we get

$$\begin{aligned}
Q_{uv(1)} = & \left(\frac{2}{S_1} \right) \left(\frac{2}{S_2} \right) \times \\
& \left\{ \delta_1 \left[\frac{1 - \cos(N_u S_1)}{N_u} \right] \left[\frac{1 - \cos(N_v d_{y1})}{N_v} + \frac{\cos(N_v d_{y(2N_0-2)}) - \cos(N_v S_2)}{N_v} \right] + \right. \\
& \left[\frac{\cos(N_v d_{y1}) - \cos(N_v d_{y(2N_0-2)})}{N_v} \right] \times \\
& \left. \left[\frac{1 - \cos(N_u d_{x1})}{N_u} + \frac{\cos(N_u d_{x(2N_0-2)}) - \cos(N_u S_1)}{N_u} \right] \right\} \\
& + \sum_{i=2}^{i=N_0-1} \delta_i \left\{ \left[\frac{\cos(N_u d_{x(i-1)}) - \cos(N_u d_{x(2N_0-i)})}{N_u} \right] \times \right. \\
& \left[\frac{\cos(N_v d_{y(i-1)}) - \cos(N_v d_{yi})}{N_v} + \frac{\cos(N_v d_{y(2N_0-i-1)}) - \cos(N_v d_{y(2N_0-i)})}{N_v} \right] + \\
& \left[\frac{\cos(N_v d_{yi}) - \cos(N_v d_{y(2N_0-i-1)})}{N_v} \right] \times \\
& \left. \left[\frac{\cos(N_u d_{x(i-1)}) - \cos(N_u d_{xi})}{N_u} + \frac{\cos(N_u d_{x(2N_0-i-1)}) - \cos(N_u d_{x(2N_0-i)})}{N_u} \right] \right\} \\
& + \delta_{N_0} \left\{ \left[\frac{\cos(N_u d_{x(N_0-1)}) - \cos(N_u d_{xN_0})}{N_u} \right] \times \left[\frac{\cos(N_v d_{y(N_0-1)}) - \cos(N_v d_{yN_0})}{N_v} \right] \right\}. \quad (3.114)
\end{aligned}$$

In order to satisfy boundary condition (IVa), we next equate the expressions for hydraulic head for the top and middle soil layers at $z = H_5$; this results in the following equation

$$\begin{aligned}
& \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \sin(N_{p_1} y) \sin(N_{q_1} H_5) \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \\
& + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \sin(N_{p_2} y) \sin(N_{q_2} H_5) \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \sin(N_{p_3} x) \sin(N_{q_3} H_5) \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \\
& + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \sin(N_{p_4} x) \sin(N_{q_4} H_5) \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \sin(N_k x) \sin(N_l y) \tanh(\lambda_{kl} H_5) \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \sin(N_u x) \sin(N_v y) \operatorname{sech}(\lambda_{uv} H_5) \\
& = \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \sin(N_{i_1} x) \sin(N_{j_1} y) \operatorname{sech}[\lambda_{i_1 j_1} (H_6 - H_5)] \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \sin(N_{i_2} x) \sin(N_{j_2} y) - H_1.
\end{aligned}$$

Now, making a Fourier run in $0 < x < S_1$ and $0 < y < S_2$ by multiplying both sides of above equation by $\sin\left[\left(\frac{u_1 \pi}{S_1}\right)x\right] \times \sin\left[\left(\frac{v_1 \pi}{S_2}\right)y\right]$, we get

$$\begin{aligned}
& E_{kl(1)} \tanh(\lambda_{kl} H_5) - G_{i_1 j_1(1)} \frac{1}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} - H_{i_2 j_2(1)} + Q_{uv(1)} \frac{1}{\cosh(\lambda_{uv} H_5)} \\
& = -\left(\frac{2}{S_1}\right) \left(\frac{2}{S_2}\right) \left\{ H_1 \int_0^{S_1} \sin(N_{u_1} x) dx \int_0^{S_2} \sin(N_{v_1} y) dy \right. \\
& \quad + \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \sin(N_{q_1} H_5) \int_0^{S_2} \sin(N_{p_1} y) \sin(N_{v_1} y) dy \times \\
& \quad \int_0^{S_1} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \sin(N_{u_1} x) dx \\
& \quad + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \sin(N_{q_2} H_5) \int_0^{S_2} \sin(N_{p_2} y) \sin(N_{v_1} y) dy \times \\
& \quad \int_0^{S_1} \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \sin(N_{u_1} x) dx \\
& \quad + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \sin(N_{q_3} H_5) \int_0^{S_1} \sin(N_{p_3} x) \sin(N_{u_1} x) dx \times \\
& \quad \int_0^{S_2} \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{v_1} y) dy \\
& \quad + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \sin(N_{q_4} H_5) \int_0^{S_1} \sin(N_{p_4} x) \sin(N_{u_1} x) dx \times \\
& \quad \left. \int_0^{S_2} \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{v_1} y) dy \right\}, \tag{3.115}
\end{aligned}$$

where $N_{u_1} = \left(\frac{u_1 \pi}{S_1}\right)$, $N_{v_1} = \left(\frac{v_1 \pi}{S_2}\right)$, $u_1 = k = u = i_1 = i_2$ and $v_1 = l = v = j_1 = j_2$.

On evaluating the integrals in the above equation, we finally arrive at the following result

$$\begin{aligned}
& E_{kl(1)} \tanh(\lambda_{kl} H_5) - G_{i_1 j_1(1)} \frac{1}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} - H_{i_2 j_2(1)} + Q_{uv(1)} \frac{1}{\cosh(\lambda_{uv} H_5)} \\
&= -\left(\frac{2}{S_1}\right) \left(\frac{2}{S_2}\right) H_1 \left[\frac{1 - \cos(N_{u_1} S_1)}{N_{u_1}} \right] \left[\frac{1 - \cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
&- \left(\frac{2}{S_1}\right) \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \sin(N_{q_1} H_5) \left[\frac{N_{u_1}^2}{N_{u_1}^2 + \lambda_{p_1 q_1}^2} \right] \left[\frac{-\cos(N_{u_1} S_1)}{N_{u_1}} \right] \\
&- \left(\frac{2}{S_1}\right) \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \sin(N_{q_2} H_5) \left[\frac{N_{u_1}}{N_{u_1}^2 + \lambda_{p_2 q_2}^2} \right] \\
&- \left(\frac{2}{S_2}\right) \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \sin(N_{q_3} H_5) \left[\frac{N_{v_1}^2}{N_{v_1}^2 + \lambda_{p_3 q_3}^2} \right] \left[\frac{-\cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
&- \left(\frac{2}{S_2}\right) \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \sin(N_{q_4} H_5) \left[\frac{N_{v_1}}{N_{v_1}^2 + \lambda_{p_4 q_4}^2} \right], \tag{3.116}
\end{aligned}$$

where $u_1 = k = u = i_1 = i_2 = p_3 = p_4$, $v_1 = l = v = j_1 = j_2 = p_1 = p_2$ and $Q_1 = Q_2 = Q_3 = Q_4 \rightarrow \infty$. Also, to tackle boundary condition (IVb), we now equate the steady hydraulic flux expressions for the top and the middle layers at $z = H_5$; this yields the following relationship

$$\begin{aligned}
& -K_{z_1} \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \lambda_{kl} \sin(N_k x) \sin(N_l y) = \\
& K_{z_2} \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \sin(N_{i_2} x) \sin(N_{j_2} y) \lambda_{i_2 j_2} \tanh[\lambda_{i_2 j_2} (H_6 - H_5)].
\end{aligned}$$

Simplifying the above equation by again making a Fourier run in $0 < x < S_1$ and $0 < y < S_2$, we get

$$K_{z_1} \lambda_{kl} E_{kl(1)} + K_{z_2} \lambda_{i_2 j_2} H_{i_2 j_2(1)} \tanh[\lambda_{i_2 j_2} (H_6 - H_5)] = 0, \tag{3.117}$$

where $k = i_2$ and $l = j_2$. It also needs to be noted that $K = L = I_2 = J_2 \rightarrow \infty$. Also, the head equality of the middle and the bottom layer [boundary condition (Va)] at $z = H_6$ yields the relation

$$\begin{aligned}
& \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \sin(N_{i_2} x) \sin(N_{j_2} y) \operatorname{sech}[\lambda_{i_2 j_2} (H_6 - H_5)] + \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \sin(N_{i_1} x) \sin(N_{j_1} y) \\
&= \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \sin(N_{i_3} x) \sin(N_{j_3} y) \coth[\lambda_{i_3 j_3} (h - H_6)].
\end{aligned}$$

Carrying out again a double Fourier series expansion in $0 < x < S_1$ and $0 < y < S_2$ as before and simplifying the resultant integrals, we get

$$G_{i_1 j_1(1)} + H_{i_2 j_2(1)} \operatorname{sech}[\lambda_{i_2 j_2} (H_6 - H_5)] - P_{i_3 j_3(1)} \coth[\lambda_{i_3 j_3} (h - H_6)] = 0, \tag{3.118}$$

where $i_1 = i_2 = i_3$ and $j_1 = j_2 = j_3$. Further, equating the steady flux expressions for the middle layer and bottom layer at $z = H_6$ [so as to satisfy boundary condition (Vb)] gives us

$$\begin{aligned}
& -K_{z_2} \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} G_{i_1 j_1(1)} \sin(N_{i_1} x) \sin(N_{j_1} y) \lambda_{i_1 j_1} \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] \\
& = K_{z_3} \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \sin(N_{i_3} x) \sin(N_{j_3} y) \lambda_{i_3 j_3}
\end{aligned}$$

On simplifying the above equation, we get

$$K_{z_2} G_{i_1 j_1(1)} \lambda_{i_1 j_1} \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] + K_{z_3} P_{i_3 j_3(1)} \lambda_{i_3 j_3} = 0, \quad (3.119)$$

where $i_1 = i_3$ and $j_1 = j_3$.

The system of linear equations resulting from Eqs. (3.116), (3.117), (3.118) and (3.119) can now be solved to evaluate coefficients $E_{kl(1)}$, $G_{i_1 j_1(1)}$, $H_{i_2 j_2(1)}$ and $P_{i_3 j_3(1)}$ specific to a drainage configuration of Fig. 3.4. Thus, our steady state problem pertaining to Fig. 3.4 stands solved. As mentioned before, this solution is exact and is valid for all possible combination of flow parameters of Fig. 3.4.

3.2.1.2 Determination of Coefficients of the Transient State Part of the Solution [transient state solution is approximate and is applicable only when directional conductivities and specific storage of the layers of Fig. 3.4 satisfy the conditions $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2} = K_{x_3}/S_{s_3}$, $K_{y_1}/S_{s_1} = K_{y_2}/S_{s_2} = K_{y_3}/S_{s_3}$ and $K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0$ (but $K_{z_1} = K_{z_2} = K_{z_3} \neq 0$)]

To ascertain that the triple Fourier series SR_l only exists in the top soil layer of the aquifer, we ensure that the sine bases associated with the series disappear for the middle layer and the bottom layer. To achieve this, we equate it with the negative of steady state solution for the top soil layer, while for the rest of the flow domain, we equate it with zero. This way, the initial condition (I) for the flow domain is also satisfied. Thus, from Eqs. (3.36) and (3.87), we get

$$\begin{aligned}
& \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) = \\
& - \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \sin(N_{p_1} y) \sin(N_{q_1} z) \\
& - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \sin(N_{p_2} y) \sin(N_{q_2} z) \\
& - \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{p_3} x) \sin(N_{q_3} z) \\
& - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{p_4} x) \sin(N_{q_4} z) \\
& - \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \frac{\sinh(\lambda_{kl} z)}{\cosh(\lambda_{kl} H_5)} \sin(N_k x) \sin(N_l y)
\end{aligned}$$

$$-\sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \frac{\cosh[\lambda_{uv}(H_5 - z)]}{\cosh(\lambda_{uv}H_5)} \sin(N_u x) \sin(N_v y),$$

$$0 < x < S_1,$$

$$0 < y < S_2,$$

$$0 < z < H_5,$$

$$\sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) = 0,$$

$$0 < x < S_1,$$

$$0 < y < S_2,$$

$$H_5 < z < H_6,$$

$$\sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) = 0,$$

$$0 < x < S_1,$$

$$0 < y < S_2,$$

$$H_6 < z < h.$$

Running a triple Fourier series in the specified domain, an expression for evaluation of the coefficients $A_{l_1 m_1 n_1(1)}$ can be expressed as

$$\begin{aligned} A_{l_1 m_1 n_1(1)} = & \left(\frac{8}{S_1 S_2 h} \right) \left\{ - \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \int_0^{S_1} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \sin(N_{l_1} x) dx \times \right. \\ & \int_0^{S_2} \sin(N_{p_1} y) \sin(N_{m_1} y) dy \times \int_0^{H_5} \sin(N_{q_1} z) \sin(N_{n_1} z) dz \\ & - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \int_0^{S_1} \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \sin(N_{l_1} x) dx \times \\ & \int_0^{S_2} \sin(N_{p_2} y) \sin(N_{m_1} y) dy \times \int_0^{H_5} \sin(N_{q_2} z) \sin(N_{n_1} z) dz \\ & - \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \int_0^{S_2} \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{m_1} y) dy \times \\ & \int_0^{S_1} \sin(N_{p_3} x) \sin(N_{l_1} x) dx \times \int_0^{H_5} \sin(N_{q_3} z) \sin(N_{n_1} z) dz \\ & - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \int_0^{S_2} \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{m_1} y) dy \times \\ & \int_0^{S_1} \sin(N_{p_4} x) \sin(N_{l_1} x) dx \times \int_0^{H_5} \sin(N_{q_4} z) \sin(N_{n_1} z) dz \\ & - \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \int_0^{H_5} \frac{\sinh(\lambda_{kl} z)}{\cosh(\lambda_{kl} H_5)} \sin(N_{n_1} z) dz \times \\ & \int_0^{S_1} \sin(N_k x) \sin(N_{l_1} x) dx \times \int_0^{S_2} \sin(N_l y) \sin(N_{m_1} y) dy \\ & \left. - \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \int_0^{H_5} \frac{\cosh[\lambda_{uv}(H_5 - z)]}{\cosh(\lambda_{uv} H_5)} \sin(N_{n_1} z) dz \times \right\} \end{aligned}$$

$$\int_0^{S_1} \sin(N_u x) \sin(N_{l_1} x) dx \times \int_0^{S_2} \sin(N_v y) \sin(N_{m_1} y) dy. \quad (3.120)$$

Evaluating the above integrals, we get the final expression for $A_{l_1 m_1 n_1(1)}$ as

$$\begin{aligned} A_{l_1 m_1 n_1(1)} = & \left(\frac{8}{S_1 S_2 h} \right) \left\{ - \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \left[\frac{N_{l_1}^2}{N_{l_1}^2 + \lambda_{p_1 q_1}^2} \right] \left[\frac{-\cos(N_{l_1} S_1)}{N_{l_1}} \right] \times \left(\frac{S_2}{2} \right) \times I^{(7)} \right. \\ & - \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \left[\frac{N_{l_1}}{N_{l_1}^2 + \lambda_{p_2 q_2}^2} \right] \times \left(\frac{S_2}{2} \right) \times I^{(8)} \\ & - \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \left[\frac{N_{m_1}^2}{N_{m_1}^2 + \lambda_{p_3 q_3}^2} \right] \left[\frac{-\cos(N_{m_1} S_2)}{N_{m_1}} \right] \times \left(\frac{S_1}{2} \right) \times I^{(9)} \\ & - \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \left[\frac{N_{m_1}}{N_{m_1}^2 + \lambda_{p_4 q_4}^2} \right] \times \left(\frac{S_1}{2} \right) \times I^{(10)} \\ & - E_{kl(1)} \left[\frac{N_{n_1}^2}{N_{n_1}^2 + \lambda_{kl}^2} \right] \left(\frac{S_1}{2} \right) \left(\frac{S_2}{2} \right) \times \\ & \left[\frac{-\cos(N_{n_1} H_5)}{N_{n_1}} \tanh(\lambda_{kl} H_5) + \frac{\lambda_{kl}}{N_{n_1}^2} \sin(N_{n_1} H_5) \right] \\ & - Q_{uv(1)} \left[\frac{N_{n_1}^2}{N_{n_1}^2 + \lambda_{uv}^2} \right] \left(\frac{S_1}{2} \right) \left(\frac{S_2}{2} \right) \times \\ & \left. \left[\frac{1}{N_{n_1}} - \frac{\cos(N_{n_1} H_5)}{N_{n_1}} \times \frac{1}{\cosh(\lambda_{uv} H_5)} \right] \right\}, \quad (3.121) \end{aligned}$$

where $N_{m_1} = N_{p_1}$, $N_{m_1} = N_{p_2}$, $N_{l_1} = N_{p_3}$, $N_{l_1} = N_{p_4}$, $N_k = N_{l_1}$, $N_l = N_{n_1}$, $N_u = N_{l_1}$, $N_v = N_{m_1}$, $Q_1 = Q_2 = Q_3 = Q_4 \rightarrow \infty$ and

$$\begin{aligned} I^{(7)} = & \left(\frac{H_5}{2} \right), & N_{q_1} = N_{n_1}, \\ = & \left\{ \frac{\sin \left[(N_{q_1} - N_{n_1}) H_5 \right]}{2(N_{q_1} - N_{n_1})} - \frac{\sin \left[(N_{q_1} + N_{n_1}) H_5 \right]}{2(N_{q_1} + N_{n_1})} \right\}, & N_{q_1} \neq N_{n_1}, \end{aligned} \quad (3.122)$$

$$\begin{aligned} I^{(8)} = & \left(\frac{H_5}{2} \right), & N_{q_2} = N_{n_1}, \\ = & \left\{ \frac{\sin \left[(N_{q_2} - N_{n_1}) H_5 \right]}{2(N_{q_2} - N_{n_1})} - \frac{\sin \left[(N_{q_2} + N_{n_1}) H_5 \right]}{2(N_{q_2} + N_{n_1})} \right\}, & N_{q_2} \neq N_{n_1}, \end{aligned} \quad (3.123)$$

$$I^{(9)} = \left(\frac{H_5}{2} \right), \quad N_{q_3} = N_{n_1},$$

$$= \left\{ \frac{\sin \left[(N_{q_3} - N_{n_1}) H_5 \right]}{2(N_{q_3} - N_{n_1})} - \frac{\sin \left[(N_{q_3} + N_{n_1}) H_5 \right]}{2(N_{q_3} + N_{n_1})} \right\}, \quad N_{q_3} \neq N_{n_1} \quad (3.124)$$

and

$$I^{(10)} = \left(\frac{H_5}{2} \right), \quad N_{q_4} = N_{n_1},$$

$$= \left\{ \frac{\sin \left[(N_{q_4} - N_{n_1}) H_5 \right]}{2(N_{q_4} - N_{n_1})} - \frac{\sin \left[(N_{q_4} + N_{n_1}) H_5 \right]}{2(N_{q_4} + N_{n_1})} \right\}, \quad N_{q_4} \neq N_{n_1}. \quad (3.125)$$

Also, we need to ensure that the series SR_2 exists solely for the second layer and is zero elsewhere (i.e., in the top and bottom layers); further, it must accommodate initial condition (II) at the same time. Thus, for SR_2 , we must have the following relations

$$\sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) = 0,$$

$$0 < x < S_1, \quad 0 < y < S_2, \quad 0 < z < H_5,$$

$$\sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) =$$

$$- \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \frac{\cosh \left[\lambda_{i_1 j_1} (z - H_5) \right]}{\cosh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right]} \sin(N_{i_1} x) \sin(N_{j_1} y)$$

$$- \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \frac{\cosh \left[\lambda_{i_2 j_2} (H_6 - z) \right]}{\cosh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right]} \sin(N_{i_2} x) \sin(N_{j_2} y) - (-H_1),$$

$$0 < x < S_1, \quad 0 < y < S_2, \quad H_5 < z < H_6,$$

$$\sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) = 0,$$

$$0 < x < S_1, \quad 0 < y < S_2, \quad H_6 < z < h.$$

$R_{l_2 m_2 n_2(1)}$ can thus be worked by running a triple Fourier series over the defined flow space; the expression for the same can be written as

$$R_{l_2 m_2 n_2(1)} = \left(\frac{8}{S_1 S_2 h} \right) \left\{ - \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \int_{H_5}^{H_6} \frac{\cosh \left[\lambda_{i_1 j_1} (z - H_5) \right]}{\cosh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right]} \sin(N_{n_2} z) dz \times \right.$$

$$\int_0^{S_1} \sin(N_{i_1} x) \sin(N_{l_2} x) dx \times \int_0^{S_2} \sin(N_{j_1} y) \sin(N_{m_2} y) dy$$

$$- \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \int_{H_5}^{H_6} \frac{\cosh \left[\lambda_{i_2 j_2} (H_6 - z) \right]}{\cosh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right]} \sin(N_{n_2} z) dz \times$$

$$\int_0^{S_1} \sin(N_{i_2} x) \sin(N_{l_2} x) dx \times \int_0^{S_2} \sin(N_{j_2} y) \sin(N_{m_2} y) dy$$

$$+H_1 \int_0^{S_1} \sin(N_{l_2} x) dx \times \int_0^{S_2} \sin(N_{m_2} y) dy \times \int_{H_5}^{H_6} \sin(N_{n_2} z) dz \left. \right\}. \quad (3.126)$$

On evaluating the above integrals, we find $R_{l_2 m_2 n_2(1)}$ as

$$\begin{aligned} R_{l_2 m_2 n_2(1)} = & \left(\frac{8}{S_1 S_2 h} \right) \left\{ -G_{i_1 j_1(1)} \left(\frac{S_1}{2} \right) \left(\frac{S_2}{2} \right) \times \left[\frac{N_{n_2}^2}{N_{n_2}^2 + \lambda_{i_1 j_1}^2} \right] \times \right. \\ & \left[-\frac{\cos(N_{n_2} H_6)}{N_{n_2}} + \left(\frac{\lambda_{i_1 j_1}}{N_{n_2}^2} \right) \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \sin(N_{n_2} H_6) \right. \\ & \left. \left. + \frac{\cos(N_{n_2} H_5)}{N_{n_2}} \frac{1}{\cosh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right]} \right] \right. \\ & - H_{i_2 j_2(1)} \left(\frac{S_1}{2} \right) \left(\frac{S_2}{2} \right) \times \left[\frac{N_{n_2}^2}{N_{n_2}^2 + \lambda_{i_2 j_2}^2} \right] \times \\ & \left[\frac{\cos(N_{n_2} H_5)}{N_{n_2}} - \frac{\cos(N_{n_2} H_6)}{N_{n_2}} \operatorname{sech} \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] + \right. \\ & \left. \left(\frac{\lambda_{i_2 j_2}}{N_{n_2}^2} \right) \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \sin(N_{n_2} H_5) \right] \\ & \left. + H_1 \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \right\}, \quad (3.127) \end{aligned}$$

where $i_1 = i_2 = l_2$ and $j_1 = j_2 = m_2$. We again need to make certain that the triple Fourier series SR_3 exists solely for the bottom soil layer and is zero in the other two layers; further it must tackle initial condition (III) at the same time. Thus, the following relations must be satisfied concurrently, namely

$$\begin{aligned} & \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) = 0, \\ & 0 < x < S_1, \quad 0 < y < S_2, \quad 0 < z < H_6, \\ & \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) = 0, \\ & 0 < x < S_1, \quad 0 < y < S_2, \quad H_5 < z < H_6, \\ & \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) = \\ & - \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \frac{\cosh \left[\lambda_{i_3 j_3} (h - z) \right]}{\sinh \left[\lambda_{i_3 j_3} (h - H_6) \right]} \sin(N_{i_3} x) \sin(N_{j_3} y) - (-H_1), \\ & 0 < x < S_1, \quad 0 < y < S_2, \quad H_6 < z < h. \end{aligned}$$

The Fourier coefficients $W_{l_3 m_3 n_3(1)}$ can thus be worked out as

$$W_{l_3 m_3 n_3(1)} = \left(\frac{8}{S_1 S_2 h} \right) \left\{ - \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \int_{H_6}^h \frac{\cosh \left[\lambda_{i_3 j_3} (h-z) \right]}{\sinh \left[\lambda_{i_3 j_3} (h-H_6) \right]} \sin(N_{n_3} z) dz \times \right. \\ \left. \int_0^{S_1} \sin(N_{i_3} x) \sin(N_{l_3} x) dx \times \int_0^{S_2} \sin(N_{j_3} y) \sin(N_{m_3} y) dy \right. \\ \left. + H_1 \int_0^{S_1} \sin(N_{l_3} x) dx \times \int_0^{S_2} \sin(N_{m_3} y) dy \times \int_{H_6}^h \sin(N_{n_3} z) dz \right\}. \quad (3.128)$$

The above integrals, upon simplification, give $W_{l_3 m_3 n_3(1)}$ as

$$W_{l_3 m_3 n_3(1)} = \left(\frac{8}{S_1 S_2 h} \right) \left\{ -P_{i_3 j_3(1)} \left(\frac{S_1}{2} \right) \left(\frac{S_2}{2} \right) \left[\frac{N_{n_3}^2}{N_{n_3}^2 + \lambda_{i_3 j_3}^2} \right] \times \right. \\ \left. \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \coth \left[\lambda_{i_3 j_3} (h-H_6) \right] + \left(\frac{\lambda_{i_3 j_3}}{N_{n_3}^2} \right) \sin(N_{n_3} H_6) \right] \right. \\ \left. + H_1 \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \right\}, \quad (3.129)$$

where $i_3 = l_3$ and $j_3 = m_3$.

We now evaluate the expressions for the discharges through the different faces of the flow domain under consideration when the top of the field is being subjected to a variable ponding distribution as shown in Figs. 3.2 and 3.3. The expression for discharge of water entering the flow domain from the top of the field, $Q_{top(1)}(t)$, can be written as

$$Q_{top(1)}(t) = -K_{z_1} \int_{\varepsilon_x}^{S_1 - \varepsilon_x} \int_{\varepsilon_y}^{S_2 - \varepsilon_y} \left(\frac{\partial \phi_{1(1)}}{\partial z} \right)_{z=0} dx dy \quad (3.130)$$

On simplifying the above integral, we get $Q_{top(1)}(t)$ as

$$Q_{top(1)}(t) = -K_{z_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \left(\frac{N_{q_1}}{\lambda_{p_1 q_1}} \right) \left\{ \frac{\cosh \left[\lambda_{p_1 q_1} (S_1 - \varepsilon_x) \right] - \cosh \left(\lambda_{p_1 q_1} \varepsilon_x \right)}{\sinh \left(\lambda_{p_1 q_1} S_1 \right)} \right\} \times \right. \\ \left. \left\{ \frac{\cos(N_{p_1} \varepsilon_y) - \cos \left[N_{p_1} (S_2 - \varepsilon_y) \right]}{N_{p_1}} \right\} \right. \\ \left. + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \left(\frac{N_{q_2}}{\lambda_{p_2 q_2}} \right) \left\{ \frac{\cosh \left[\lambda_{p_2 q_2} (S_1 - \varepsilon_x) \right] - \cosh \left(\lambda_{p_2 q_2} \varepsilon_x \right)}{\sinh \left(\lambda_{p_2 q_2} S_1 \right)} \right\} \times \right. \\ \left. \left\{ \frac{\cos(N_{p_2} \varepsilon_y) - \cos \left[N_{p_2} (S_2 - \varepsilon_y) \right]}{N_{p_2}} \right\} \right\}$$

$$\begin{aligned}
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \left(\frac{N_{q_3}}{\lambda_{p_3 q_3}} \right) \left\{ \frac{\cosh[\lambda_{p_3 q_3} (S_2 - \varepsilon_y)] - \cosh(\lambda_{p_3 q_3} \varepsilon_y)}{\sinh(\lambda_{p_3 q_3} S_2)} \right\} \times \\
& \left\{ \frac{\cos(N_{p_3} \varepsilon_x) - \cos[N_{p_3} (S_1 - \varepsilon_x)]}{N_{p_3}} \right\} \\
& + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \left(\frac{N_{q_4}}{\lambda_{p_4 q_4}} \right) \left\{ \frac{\cosh[\lambda_{p_4 q_4} (S_2 - \varepsilon_y)] - \cosh(\lambda_{p_4 q_4} \varepsilon_y)}{\sinh(\lambda_{p_4 q_4} S_2)} \right\} \times \\
& \left\{ \frac{\cos(N_{p_4} \varepsilon_x) - \cos[N_{p_4} (S_1 - \varepsilon_x)]}{N_{p_4}} \right\} \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \left[\frac{\lambda_{kl}}{\cosh(\lambda_{kl} H_5)} \right] \left\{ \frac{\cos(N_k \varepsilon_x) - \cos[N_k (S_1 - \varepsilon_x)]}{N_k} \right\} \times \\
& \left\{ \frac{\cos(N_l \varepsilon_y) - \cos[N_l (S_2 - \varepsilon_y)]}{N_l} \right\} \\
& - \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \lambda_{uv} \tanh(\lambda_{uv} H_5) \left\{ \frac{\cos(N_u \varepsilon_x) - \cos[N_u (S_1 - \varepsilon_x)]}{N_u} \right\} \times \\
& \left\{ \frac{\cos(N_v \varepsilon_y) - \cos[N_v (S_2 - \varepsilon_y)]}{N_v} \right\} \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{n_1} \left\{ \frac{\cos(N_{l_1} \varepsilon_x) - \cos[N_{l_1} (S_1 - \varepsilon_x)]}{N_{l_1}} \right\} \times \\
& \left\{ \frac{\cos(N_{m_1} \varepsilon_y) - \cos[N_{m_1} (S_2 - \varepsilon_y)]}{N_{m_1}} \right\} \times \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{n_2} \left\{ \frac{\cos(N_{l_2} \varepsilon_x) - \cos[N_{l_2} (S_1 - \varepsilon_x)]}{N_{l_2}} \right\} \times \\
& \left\{ \frac{\cos(N_{m_2} \varepsilon_y) - \cos[N_{m_2} (S_2 - \varepsilon_y)]}{N_{m_2}} \right\} \times \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{n_3} \left\{ \frac{\cos(N_{l_3} \varepsilon_x) - \cos[N_{l_3} (S_1 - \varepsilon_x)]}{N_{l_3}} \right\} \times \\
& \left\{ \frac{\cos(N_{m_3} \varepsilon_y) - \cos[N_{m_3} (S_2 - \varepsilon_y)]}{N_{m_3}} \right\} \times \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\}. \tag{3.131}
\end{aligned}$$

Similarly, the top discharge function $Q_{top(1)}^f(x, y, t)$ for any arbitrary point (x, y) at the flow domain can be evaluated as

$$Q_{top(1)}^f(x, y, t) = -K_{z_1} \int_{\varepsilon_x}^x \int_{\varepsilon_y}^y \left(\frac{\partial \phi_{(1)}}{\partial z} \right)_{z=0} dx dy \quad (3.132)$$

Solution of the above integral yields $Q_{top(1)}^f(x, y, t)$ as

$$\begin{aligned} Q_{top(1)}^f(x, y, t) = & -K_{z_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \left(\frac{N_{q_1}}{\lambda_{p_1 q_1}} \right) \left[\frac{\cosh(\lambda_{p_1 q_1} x) - \cosh(\lambda_{p_1 q_1} \varepsilon_x)}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \right\} \times \\ & \left[\frac{\cos(N_{p_1} \varepsilon_y) - \cos(N_{p_1} y)}{N_{p_1}} \right] \\ & + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \left(\frac{N_{q_2}}{\lambda_{p_2 q_2}} \right) \left\{ \frac{\cosh[\lambda_{p_2 q_2} (S_1 - \varepsilon_x)] - \cosh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \right\} \times \\ & \left[\frac{\cos(N_{p_2} \varepsilon_y) - \cos(N_{p_2} y)}{N_{p_2}} \right] \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \left(\frac{N_{q_3}}{\lambda_{p_3 q_3}} \right) \left[\frac{\cosh(\lambda_{p_3 q_3} y) - \cosh(\lambda_{p_3 q_3} \varepsilon_y)}{\sinh(\lambda_{p_3 q_3} S_2)} \right] \times \\ & \left[\frac{\cos(N_{p_3} \varepsilon_x) - \cos(N_{p_3} x)}{N_{p_3}} \right] \\ & + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \left(\frac{N_{q_4}}{\lambda_{p_4 q_4}} \right) \left\{ \frac{\cosh[\lambda_{p_4 q_4} (S_2 - \varepsilon_y)] - \cosh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \right\} \times \\ & \left[\frac{\cos(N_{p_4} \varepsilon_x) - \cos(N_{p_4} x)}{N_{p_4}} \right] \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \left[\frac{\lambda_{kl}}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{\cos(N_k \varepsilon_x) - \cos(N_k x)}{N_k} \right] \times \\ & \left[\frac{\cos(N_l \varepsilon_y) - \cos(N_l y)}{N_l} \right] \\ & - \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \lambda_{uv} \tanh(\lambda_{uv} H_5) \left[\frac{\cos(N_u \varepsilon_x) - \cos(N_u x)}{N_u} \right] \times \\ & \left[\frac{\cos(N_v \varepsilon_y) - \cos(N_v y)}{N_v} \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{n_1} \left[\frac{\cos(N_{l_1} \varepsilon_x) - \cos(N_{l_1} x)}{N_{l_1}} \right] \times \\
& \left[\frac{\cos(N_{m_1} \varepsilon_y) - \cos(N_{m_1} y)}{N_{m_1}} \right] \times \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{n_2} \left[\frac{\cos(N_{l_2} \varepsilon_x) - \cos(N_{l_2} x)}{N_{l_2}} \right] \times \\
& \left[\frac{\cos(N_{m_2} \varepsilon_y) - \cos(N_{m_2} y)}{N_{m_2}} \right] \times \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{n_3} \left[\frac{\cos(N_{l_3} \varepsilon_x) - \cos(N_{l_3} x)}{N_{l_3}} \right] \times \\
& \left[\frac{\cos(N_{m_3} \varepsilon_y) - \cos(N_{m_3} y)}{N_{m_3}} \right] \times \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \}. \tag{3.133}
\end{aligned}$$

Here, we need to note that $Q_{top(1)}^f$ happens to diverge when it is being calculated exactly at a location separating two unequal ponding depths at the top of the field. To demonstrate the same, we first substitute $Q_{uv(1)}$ of Eq. (3.114) in Eq. (3.133); then we will have, after some mathematical simplifications, a term like

$$\left(\frac{4}{S_1 S_2} \right) \sqrt{\left(\frac{K_{x_1}}{K_{z_1}} \right)} \sum_{u=1}^U \sum_{v=1}^V \delta_1 \left(\frac{1}{N_u N_v^2} \right) \sqrt{\left[1 + \frac{N_v^2}{N_u^2} \left(\frac{K_{y_1}}{K_{x_1}} \right) \right]} \tanh \left\{ \left[\sqrt{N_u^2 \left(\frac{K_{x_1}}{K_{z_1}} \right) + N_v^2 \left(\frac{K_{y_1}}{K_{z_1}} \right)} \right] H_5 \right\} \times \\
\cos^2(N_u d_{x_1}) \cos(N_v y) \cos(N_v d_{y(2N_0-2)})$$

It is easily discernable that for any specified value of v , both $\sqrt{\left[1 + \frac{N_v^2}{N_u^2} \left(\frac{K_{y_1}}{K_{x_1}} \right) \right]}$ and

$\tanh \left\{ \left[\sqrt{N_u^2 \left(\frac{K_{x_1}}{K_{z_1}} \right) + N_v^2 \left(\frac{K_{y_1}}{K_{z_1}} \right)} \right] H_5 \right\}$ tend to 1 when we allow u , and consequently N_u , to

increase sans any upper limit. This can be mathematically represented as

$$\lim_{u \rightarrow \infty} \sqrt{\left[1 + \left(\frac{S_1 N_v}{u \pi} \right)^2 \left(\frac{K_{y_1}}{K_{x_1}} \right) \right]} = 1$$

and

$$\lim_{u \rightarrow \infty} \tanh \left\{ \left[\sqrt{\left(\frac{u \pi}{S_1} \right)^2 \left(\frac{K_{x_1}}{K_{z_1}} \right) + N_v^2 \left(\frac{K_{y_1}}{K_{z_1}} \right)} \right] H_5 \right\} = 1.$$

If we assume that $\tanh \left\{ \left[\sqrt{N_u^2 \left(\frac{K_{x_1}}{K_{z_1}} \right) + N_v^2 \left(\frac{K_{y_1}}{K_{z_1}} \right)} \right] H_5 \right\}$ and $\sqrt{1 + \frac{N_v^2 \left(\frac{K_{y_1}}{K_{x_1}} \right)}{N_u^2}}$ reach

approximately 1 after the first four terms of u for a specified value of v , then the rest of the aforementioned series, after some mathematical manipulations, can be expressed as

$$\left(\frac{4\delta_1}{\pi S_2} \right) \sqrt{\left(\frac{K_{x_1}}{K_{z_1}} \right)} \sum_{u=5}^{U \rightarrow \infty} \left(\frac{1}{u N_v^2} \right) \cos^2(u\pi\alpha) \cos(N_v y) \cos(N_v d_{y(2N_0-2)}),$$

where $\alpha = d_{x_1} / S_1$ ($0 < \alpha < 1$). It is to be noted that $\cos^2(u\pi\alpha)$ will lie between 0 and 1 for all possible values of u [i.e. $0 \leq \cos^2(u\pi\alpha) \leq 1, u \in \{5, 6, 7, \dots\}$]. Thus, there can be two possibilities for a chosen α , namely $\cos^2(u\pi\alpha) \neq 0$ for any value of $u \in \{5, 6, 7, \dots\}$ or $\cos^2(u\pi\alpha) = 0$ for a subset of positive integral values represented as $\{u_{n_1}, u_{n_2}, u_{n_3}, \dots\}$ which belongs to the set $u \in \{5, 6, 7, \dots\}$. For the first case, if we conveniently assume that $M_{\min} =$ minimum of $\cos^2(u\pi\alpha)$ for any $u \in \{5, 6, 7, \dots\}$, then we can write

$$M_{\min} \left(\frac{4\delta_1}{\pi S_2} \right) \left(\frac{1}{N_v^2} \right) \cos(N_v y) \cos(N_v d_{y(2N_0-2)}) \sum_{u=5}^{U \rightarrow \infty} \left(\frac{1}{u} \right) < \left(\frac{4\delta_1}{\pi S_2} \right) \left(\frac{1}{N_v^2} \right) \cos(N_v y) \cos(N_v d_{y(2N_0-2)}) \times \sum_{u=5}^{U \rightarrow \infty} \left[\frac{\cos^2(u\pi\alpha)}{u} \right].$$

However, since the series $\sum_{u=5}^{U \rightarrow \infty} \left(\frac{1}{u} \right)$ diverges, we can safely infer that so does the series on the right hand side of the inequality.

Now, we need to prove the divergence of the series for the second case, where $\cos^2(u\pi\alpha) = 0$ for $u \in \{u_{n_1}, u_{n_2}, u_{n_3}, \dots\}$. For our convenience, we divide the series into two parts as follows

$$\left(\frac{4\delta_1}{\pi S_2} \right) \left(\frac{1}{N_v^2} \right) \cos(N_v y) \cos(N_v d_{y(2N_0-2)}) \sum_{u_{ni}}^{U \rightarrow \infty} \left[\frac{\cos^2(u\pi\alpha)}{u} \right] + \left(\frac{4\delta_1}{\pi S_2} \right) \left(\frac{1}{N_v^2} \right) \cos(N_v y) \cos(N_v d_{y(2N_0-2)}) \sum_{u=5}^{U \rightarrow \infty} \left[\frac{\cos^2(u\pi\alpha)}{u} \right],$$

where the summation index u of the second term of the series belongs to the set $\{5, 6, 7, \dots\} \setminus \{u_{n_1}, u_{n_2}, u_{n_3}, \dots\} = u \in \{5, 6, 7, \dots\} \setminus \{u_{n_1}, u_{n_2}, u_{n_3}, \dots\} = \{u_{p_1}, u_{p_2}, u_{p_3}, \dots\}$ (say). If we now take $M'_{\min} =$ minimum value of $\cos^2(u\pi\alpha)$ for all $u \in \{u_{p_1}, u_{p_2}, u_{p_3}, \dots\}$, we will then have the inequality

$$M'_{\min} \left(\frac{4\delta_1}{\pi S_2} \right) \left(\frac{1}{N_v^2} \right) \cos(N_v y) \cos(N_v d_{y(2N_0-2)}) \sum_{u=5}^{U \rightarrow \infty} \left(\frac{1}{u} \right) < \left(\frac{4\delta_1}{\pi S_2} \right) \left(\frac{1}{N_v^2} \right) \cos(N_v y) \cos(N_v d_{y(2N_0-2)}) \times \sum_{u=5}^{U \rightarrow \infty} \left[\frac{\cos^2(u\pi\alpha)}{u} \right].$$

But even in this case, the series in the right hand side of the above

inequality diverges. Hence, we can state with conviction that $Q_{top(1)}^f$ diverges at this bund location. In an analogous fashion, it can be proved that $Q_{top(1)}^f$ happens to diverge at other inner bund locations too where they divide two unequal ponding depths at the surface of the soil.

The expression for discharge through the Northern ditch face of the flow domain can be expressed as

$$Q_{North(1)}(t) = K_{x_1} \int_0^{S_2} \int_0^{H_5} \left(\frac{\partial \phi_{1(1)}}{\partial x} \right)_{x=0} dydz + K_{x_2} \int_0^{S_2} \int_{H_5}^{H_6} \left(\frac{\partial \phi_{2(1)}}{\partial x} \right)_{x=0} dydz + K_{x_3} \int_0^{S_2} \int_{H_6}^h \left(\frac{\partial \phi_{3(1)}}{\partial x} \right)_{x=0} dydz. \quad (3.134)$$

Solving the above integrals, we get $Q_{North(1)}(t)$ as

$$Q_{North(1)}(t) = K_{x_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \left[\frac{\lambda_{p_1 q_1}}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \left[\frac{1 - \cos(N_{p_1} S_2)}{N_{p_1}} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \lambda_{p_2 q_2} \coth(\lambda_{p_2 q_2} S_1) \left[\frac{1 - \cos(N_{p_2} S_2)}{N_{p_2}} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \left(\frac{N_{p_3}}{\lambda_{p_3 q_3}} \right) \left[\frac{\cosh(\lambda_{p_3 q_3} S_2) - 1}{\sinh(\lambda_{p_3 q_3} S_2)} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \left(\frac{N_{p_4}}{\lambda_{p_4 q_4}} \right) \left[\frac{\cosh(\lambda_{p_4 q_4} S_2) - 1}{\sinh(\lambda_{p_4 q_4} S_2)} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \left(\frac{N_k}{\lambda_{kl}} \right) \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_l S_2)}{N_l} \right] + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \left(\frac{N_u}{\lambda_{uv}} \right) \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_v S_2)}{N_v} \right] + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times$$

$$\begin{aligned}
& \exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \Big\} \\
& +K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \left(\frac{N_{i_1}}{\lambda_{i_1 j_1}}\right) \tanh\left[\lambda_{i_1 j_1} (H_6 - H_5)\right] \left[\frac{1 - \cos(N_{j_1} S_2)}{N_{j_1}}\right] \right. \\
& \quad + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \left(\frac{N_{i_2}}{\lambda_{i_2 j_2}}\right) \tanh\left[\lambda_{i_2 j_2} (H_6 - H_5)\right] \left[\frac{1 - \cos(N_{j_2} S_2)}{N_{j_2}}\right] \\
& \quad + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}}\right] \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}}\right] \times \\
& \quad \exp\left[-\left(\lambda_{l_1 m_1 n_1}\right)^2 t\right] \\
& \quad + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}}\right] \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}}\right] \times \\
& \quad \exp\left[-\left(\lambda_{l_2 m_2 n_2}\right)^2 t\right] \\
& \quad + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}}\right] \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}}\right] \times \\
& \quad \left. \exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \right\} \\
& +K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \left(\frac{N_{i_3}}{\lambda_{i_3 j_3}}\right) \left[\frac{1 - \cos(N_{j_3} S_2)}{N_{j_3}}\right] \right. \\
& \quad + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}}\right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}}\right] \times \\
& \quad \exp\left[-\left(\lambda_{l_1 m_1 n_1}\right)^2 t\right] \\
& \quad + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}}\right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}}\right] \times \\
& \quad \exp\left[-\left(\lambda_{l_2 m_2 n_2}\right)^2 t\right] \\
& \quad + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}}\right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}}\right] \times \\
& \quad \left. \exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \right\}
\end{aligned}$$

$$\exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \}. \quad (3.135)$$

Also, the discharge expression, $Q_{South(1)}(t)$, for the Southern ditch face of the flow domain can be expressed as

$$\begin{aligned} Q_{South(1)}(t) = & -K_{x_1} \int_0^{S_2} \int_0^{H_5} \left(\frac{\partial \phi_{1(1)}}{\partial x} \right)_{x=S_1} dydz - K_{x_2} \int_0^{S_2} \int_{H_5}^{H_6} \left(\frac{\partial \phi_{2(1)}}{\partial x} \right)_{x=S_1} dydz \\ & - K_{x_3} \int_0^{S_2} \int_0^h \left(\frac{\partial \phi_{3(1)}}{\partial x} \right)_{x=S_1} dydz. \end{aligned} \quad (3.136)$$

Evaluation of these integrals gives $Q_{South(1)}(t)$ as

$$\begin{aligned} Q_{South(1)}(t) = & K_{x_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \lambda_{p_1 q_1} \coth(\lambda_{p_1 q_1} S_1) \left[\frac{1 - \cos(N_{p_1} S_2)}{N_{p_1}} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\ & - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \left[\frac{\lambda_{p_2 q_2}}{\sinh(\lambda_{p_2 q_2} S_1)} \right] \left[\frac{1 - \cos(N_{p_2} S_2)}{N_{p_2}} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \left[\frac{N_{p_3} \cos(N_{p_3} S_1)}{\lambda_{p_3 q_3}} \right] \left[\frac{\cosh(\lambda_{p_3 q_3} S_2) - 1}{\sinh(\lambda_{p_3 q_3} S_2)} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\ & + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \left[\frac{N_{p_4} \cos(N_{p_4} S_1)}{\lambda_{p_4 q_4}} \right] \left[\frac{\cosh(\lambda_{p_4 q_4} S_2) - 1}{\sinh(\lambda_{p_4 q_4} S_2)} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \left[\frac{N_k \cos(N_k S_1)}{\lambda_{kl}} \right] \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_l S_2)}{N_l} \right] \\ & + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \left[\frac{N_u \cos(N_u S_1)}{\lambda_{uv}} \right] \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_v S_2)}{N_v} \right] \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\ & \exp\left[-\left(\lambda_{l_1 m_1 n_1}\right)^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\ & \exp\left[-\left(\lambda_{l_2 m_2 n_2}\right)^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \end{aligned}$$

$$\begin{aligned}
& \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
& + K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \left[\frac{N_{i_1} \cos(N_{i_1} S_1)}{\lambda_{i_1 j_1}} \right] \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{j_1} S_2)}{N_{j_1}} \right] \right. \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \left[\frac{N_{i_2} \cos(N_{i_2} S_1)}{\lambda_{i_2 j_2}} \right] \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{j_2} S_2)}{N_{j_2}} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \times \\
& \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \\
& \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \\
& \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
& + K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \left[\frac{N_{i_3} \cos(N_{i_3} S_1)}{\lambda_{i_3 j_3}} \right] \left[\frac{1 - \cos(N_{j_3} S_2)}{N_{j_3}} \right] \right. \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times
\end{aligned}$$

$$\exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \left. \right\}. \quad (3.137)$$

The discharge expression, $Q_{East(1)}(t)$, pertaining to the Eastern ditch face of the flow domain can be expressed as

$$\begin{aligned} Q_{East(1)}(t) = & K_{y_1} \int_0^{S_1} \int_0^{H_5} \left(\frac{\partial \phi_{1(1)}}{\partial y} \right)_{y=0} dx dz + K_{y_2} \int_0^{S_1} \int_{H_5}^{H_6} \left(\frac{\partial \phi_{2(1)}}{\partial y} \right)_{y=0} dx dz \\ & + K_{y_3} \int_0^{S_1} \int_{H_6}^h \left(\frac{\partial \phi_{3(1)}}{\partial y} \right)_{y=0} dx dz. \end{aligned} \quad (3.138)$$

Evaluation of these integrals yields $Q_{East(1)}(t)$ as

$$\begin{aligned} Q_{East(1)}(t) = & K_{y_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \left(\frac{N_{p_1}}{\lambda_{p_1 q_1}} \right) \left[\frac{\cosh(\lambda_{p_1 q_1} S_1) - 1}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\ & + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \left(\frac{N_{p_2}}{\lambda_{p_2 q_2}} \right) \left[\frac{\cosh(\lambda_{p_2 q_2} S_1) - 1}{\sinh(\lambda_{p_2 q_2} S_1)} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \left[\frac{\lambda_{p_3 q_3}}{\sinh(\lambda_{p_3 q_3} S_2)} \right] \left[\frac{1 - \cos(N_{p_3} S_1)}{N_{p_3}} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\ & - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \lambda_{p_4 q_4} \coth(\lambda_{p_4 q_4} S_2) \left[\frac{1 - \cos(N_{p_4} S_1)}{N_{p_4}} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \left(\frac{N_l}{\lambda_{kl}} \right) \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_k S_1)}{N_k} \right] \\ & + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \left(\frac{N_v}{\lambda_{uv}} \right) \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_u S_1)}{N_u} \right] \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{m_1} \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\ & \exp\left[-\left(\lambda_{l_1 m_1 n_1}\right)^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{m_2} \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\ & \exp\left[-\left(\lambda_{l_2 m_2 n_2}\right)^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{m_3} \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \end{aligned}$$

$$\begin{aligned}
& \left[\exp - (\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
+ K_{y_2} & \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \left(\frac{N_{j_1}}{\lambda_{i_1 j_1}} \right) \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_1} S_1)}{N_{i_1}} \right] \right. \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \left(\frac{N_{j_2}}{\lambda_{i_2 j_2}} \right) \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_2} S_1)}{N_{i_2}} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{m_1} \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \times \\
& \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \exp \left[- (\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{m_2} \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \times \\
& \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \exp \left[- (\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{m_3} \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \times \\
& \left. \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \exp \left[- (\lambda_{l_3 m_3 n_3})^2 t \right] \right\} \\
+ K_{y_3} & \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \left(\frac{N_{j_3}}{\lambda_{i_3 j_3}} \right) \left[\frac{1 - \cos(N_{i_3} S_1)}{N_{i_3}} \right] \right. \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{m_1} \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp \left[- (\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{m_2} \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp \left[- (\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{m_3} \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times
\end{aligned}$$

$$\exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \left. \right\}. \quad (3.139)$$

Finally, the discharge expression, $Q_{West(1)}(t)$, related to the Western ditch face of the flow domain can be represented as

$$\begin{aligned} Q_{West(1)}(t) = & -K_{y_1} \int_0^{S_1} \int_0^{H_5} \left(\frac{\partial \phi_{1(1)}}{\partial y} \right)_{y=S_2} dx dz - K_{y_2} \int_0^{S_1} \int_0^{H_5} \left(\frac{\partial \phi_{2(1)}}{\partial y} \right)_{y=S_2} dx dz \\ & - K_{y_3} \int_0^{S_1} \int_0^{H_6} \left(\frac{\partial \phi_{3(1)}}{\partial y} \right)_{y=S_2} dx dz. \end{aligned} \quad (3.140)$$

By carrying out these integrals, we find $Q_{West(1)}(t)$ as

$$\begin{aligned} Q_{West(1)}(t) = & K_{y_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(1)} \left[\frac{N_{p_1} \cos(N_{p_1} S_2)}{\lambda_{p_1 q_1}} \right] \left[\frac{\cosh(\lambda_{p_1 q_1} S_1) - 1}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\ & + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(1)} \left[\frac{N_{p_2} \cos(N_{p_2} S_2)}{\lambda_{p_2 q_2}} \right] \left[\frac{\cosh(\lambda_{p_2 q_2} S_1) - 1}{\sinh(\lambda_{p_2 q_2} S_1)} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(1)} \lambda_{p_3 q_3} \coth(\lambda_{p_3 q_3} S_2) \left[\frac{1 - \cos(N_{p_3} S_1)}{N_{p_3}} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\ & - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(1)} \left[\frac{\lambda_{p_4 q_4}}{\sinh(\lambda_{p_4 q_4} S_2)} \right] \left[\frac{1 - \cos(N_{p_4} S_1)}{N_{p_4}} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(1)} \left[\frac{N_l \cos(N_l S_2)}{\lambda_{kl}} \right] \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_k S_1)}{N_k} \right] \\ & + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(1)} \left[\frac{N_v \cos(N_v S_2)}{\lambda_{uv}} \right] \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_u S_1)}{N_u} \right] \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\ & \exp\left[-\left(\lambda_{l_1 m_1 n_1}\right)^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\ & \exp\left[-\left(\lambda_{l_2 m_2 n_2}\right)^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \end{aligned}$$

$$\begin{aligned}
& \exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \Big\} \\
& +K_{y_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(1)} \left[\frac{N_{j_1} \cos(N_{j_1} S_2)}{\lambda_{i_1 j_1}} \right] \tanh\left[\lambda_{i_1 j_1} (H_6 - H_5)\right] \left[\frac{1 - \cos(N_{i_1} S_1)}{N_{i_1}} \right] \right. \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(1)} \left[\frac{N_{j_2} \cos(N_{j_2} S_2)}{\lambda_{i_2 j_2}} \right] \tanh\left[\lambda_{i_2 j_2} (H_6 - H_5)\right] \left[\frac{1 - \cos(N_{i_2} S_1)}{N_{i_2}} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \times \\
& \left. \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \exp\left[-\left(\lambda_{l_1 m_1 n_1}\right)^2 t\right] \right. \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \times \\
& \left. \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \exp\left[-\left(\lambda_{l_2 m_2 n_2}\right)^2 t\right] \right. \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \times \\
& \left. \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \right\} \\
& +K_{y_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(1)} \left[\frac{N_{j_3} \cos(N_{j_3} S_2)}{\lambda_{i_3 j_3}} \right] \left[\frac{1 - \cos(N_{i_3} S_1)}{N_{i_3}} \right] \right. \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(1)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp\left[-\left(\lambda_{l_1 m_1 n_1}\right)^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(1)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp\left[-\left(\lambda_{l_2 m_2 n_2}\right)^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(1)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times
\end{aligned}$$

$$\exp\left[-\left(\lambda_{l_3 m_3 n_3}\right)^2 t\right] \}. \quad (3.141)$$

3.2.1.3 Verifications of the Proposed Solution

We now proceed to check the veracity of the proposed solution by comparing with relevant analytical, experimental and numerical outputs for a few specified drainage situations of Fig. 3.4. It should be noted that all our transient hydraulic head expressions of Eqs. (3.36), (3.70) and (3.79) reduce to that of the steady state if the time variable in these equations is allowed to extend to infinity. As may be observed, this will make the exponential terms to vanish from these equations leaving behind only the steady state portions of the hydraulic head functions. We would also like to point out that if one of the horizontal dimensions of Fig. 3.4 is taken much larger (theoretically infinite) than the vertical and the other horizontal dimension then flow in a vertical section located half-way between the boundaries lying at the extremities of this long dimension will closely be two-dimensional in nature; thus, the three-dimensional solution provided here for the drainage problem of Fig. 3.4 can also be applied, after appropriate modification, for predicting two-dimensional flow to a ponded ditch drainage system in a stratified soil as well. Adopting such a procedure, we next determine the $Q_{top(1)}/2Kh$ ratio at a vertical section located 500 m from the Northern and Southern boundaries of the flow domain for a drainage situation with the flow parameters of Fig. 3.4 taken as $S_1 = 1000$ m, $S_2 = 100$ m, $h = 3$ m, $H_1 = 2.55$ m, $H_5 = 2.7$ m, $H_6 = 2.85$ m, $\delta = 0$ m and $K_x = K_y = K_z = 0.05$ m/day. From our solution, we find this ratio as 0.7197 for the considered flow situation. For a parallel situation but with the drains running totally empty, this value from Fukuda's (1957) and Youngs' (1994) solutions, works out as 0.743 and 0.742, respectively. In this context it should be noted that the water level in our computation for the concerned flow situation is taken as 2.55 m from the top and not 3 m since the proposed drainage solution of Fig. 3.4 is only valid for situations where the level of water in the ditches is above the bottom of the first layer and not when this level is below it. Even then, as may be observed, our $Q_{top(1)}/2Kh$ ratio for the concerned situation is matching very closely with the ones obtained from Fukuda's and Youngs' solutions thereby showing the correctness of our proposed solution. Further, from his experimental results, Fukuda (1957) found this ratio as 0.72, which again can be seen to agree very closely with the value predicted from our analytical solution. Thus, this matching of our result with the experimentally observed results of Fukuda can also be treated as an experimental verification of our solution.

As a further check of our analytical solution, we again compare our steady state hydraulic heads for a two-dimensional flow situation of Fig. 3.4 with the corresponding results obtained from the analytical works of Kirkham (1965) when the flow parameters are as shown in Fig. 3.5. Here also, like in the previous case, we have first reduced our model to a two-dimensional situation by considering a section half-way in between the Northern and the Southern boundaries of the flow domain and then studying the head distribution in this

section. As can be seen from Fig. 3.5, the steady state hydraulic heads as predicted by our reduced two-dimensional model are in a good agreement with those obtained from Kirkham's solution for the studied drainage scenario thereby showing once again the rightness of the proposed solution.

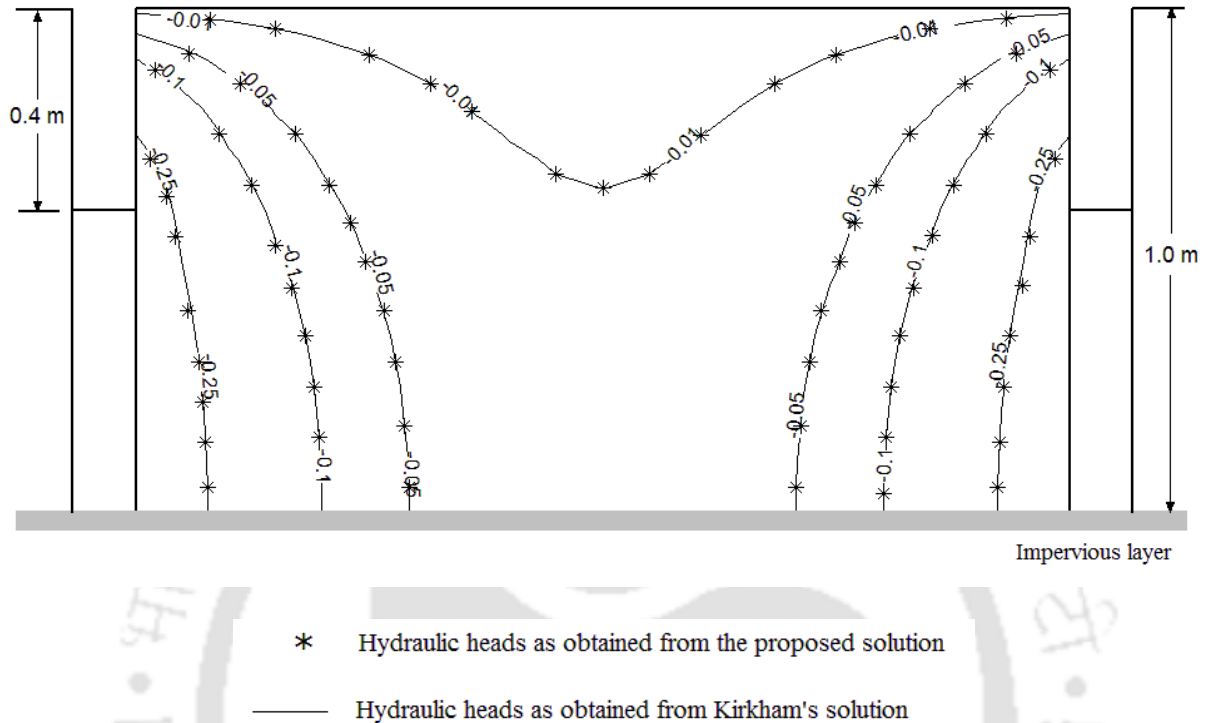


Fig. 3.5. Comparison of steady state hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.4 with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of Fig. 3.4 are $S_1 = 15$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.4$ m, $H_5 = 0.6$ m, $H_6 = 0.8$ m, $\delta = 0$ m and $K_{x_1} = K_{x_2} = K_{x_3} = K_{y_1} = K_{y_2} = K_{y_3} = K_{z_1} = K_{z_2} = K_{z_3} = 1$ m/day

As mentioned earlier, the transient solution of Fig. 3.4 as provided here would remain valid provided Eqs. (3.33), (3.34) and (3.35) are being concurrently satisfied by a flow situation; however, if the three-layered soil is being reduced to a single layer one, these conditions are then not needed for the validity of this solution. Thus, if $K_{x_1} = K_{x_2} = K_{x_3}$, $K_{y_1} = K_{y_2} = K_{y_3}$, $K_{z_1} = K_{z_2} = K_{z_3}$ and $S_{s_1} = S_{s_2} = S_{s_3}$, the transient solution is then exact for all possible combinations of the parameters of Fig. 3.4. As for such a situation, Barua and Alam (2013) have already provided an analytical solution to the problem for a two-dimensional flow domain, this solution then can be compared with the reduced two-dimensional solution of our problem of Fig. 3.4; Fig. 3.6 shows such a comparison for a particular flow setting of Fig. 3.4. As may be seen, the hydraulic heads predicted by our model for both the simulation times are in close agreement with the ones obtained from the analytical works of Barua and Alam (2013) for the concerned drainage situation thereby providing us with a further verification of the proposed solution. At this stage, we would like to add that Sarmah and

Barua (2017) obtained an analytical solution for the flow problem of Fig. 3.4 for the single-layered soil and hence our reduced solution for such a situation can then also be compared with this solution; in Fig. 3.7 such a comparison is being shown for a drainage scenario of Fig. 3.4 for two times of simulation of the system, namely for $t=100$ s and 500 s, respectively. As may be observed, the hydraulic heads as predicted by our solution are in perfect agreement with those obtained from the analytical works of Sarmah and Barua (2017), for both the times of simulation of the studied systems. All these further reinforce the fact that our solution for the flow situation of Fig. 3.4 is being correctly developed.

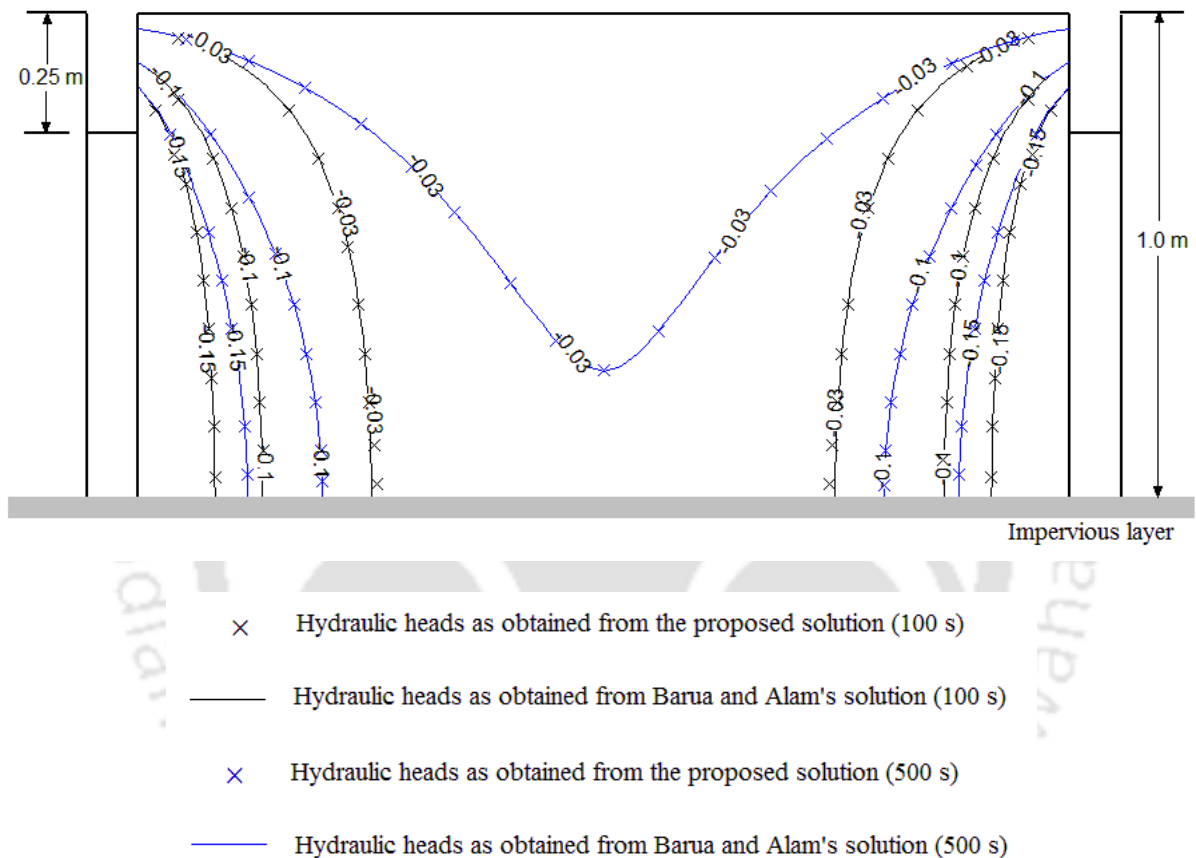


Fig. 3.6. Comparison of transient hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.4 at times $t=100$ s and 500 s with the corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem are considered as $S_1=15$ m, $S_2=5$ m, $h=1$ m, $H_1=0.25$ m, $H_5=0.35$ m, $H_6=0.8$ m, $\delta=0$ m, $K_{x_1}=K_{x_2}=K_{x_3}=K_{y_1}=K_{y_2}=K_{y_3}=1.5$ m/day, $K_{z_1}=K_{z_2}=K_{z_3}=0.75$ m/day and $S_{s_1}=S_{s_2}=S_{s_3}=0.005$ m⁻¹

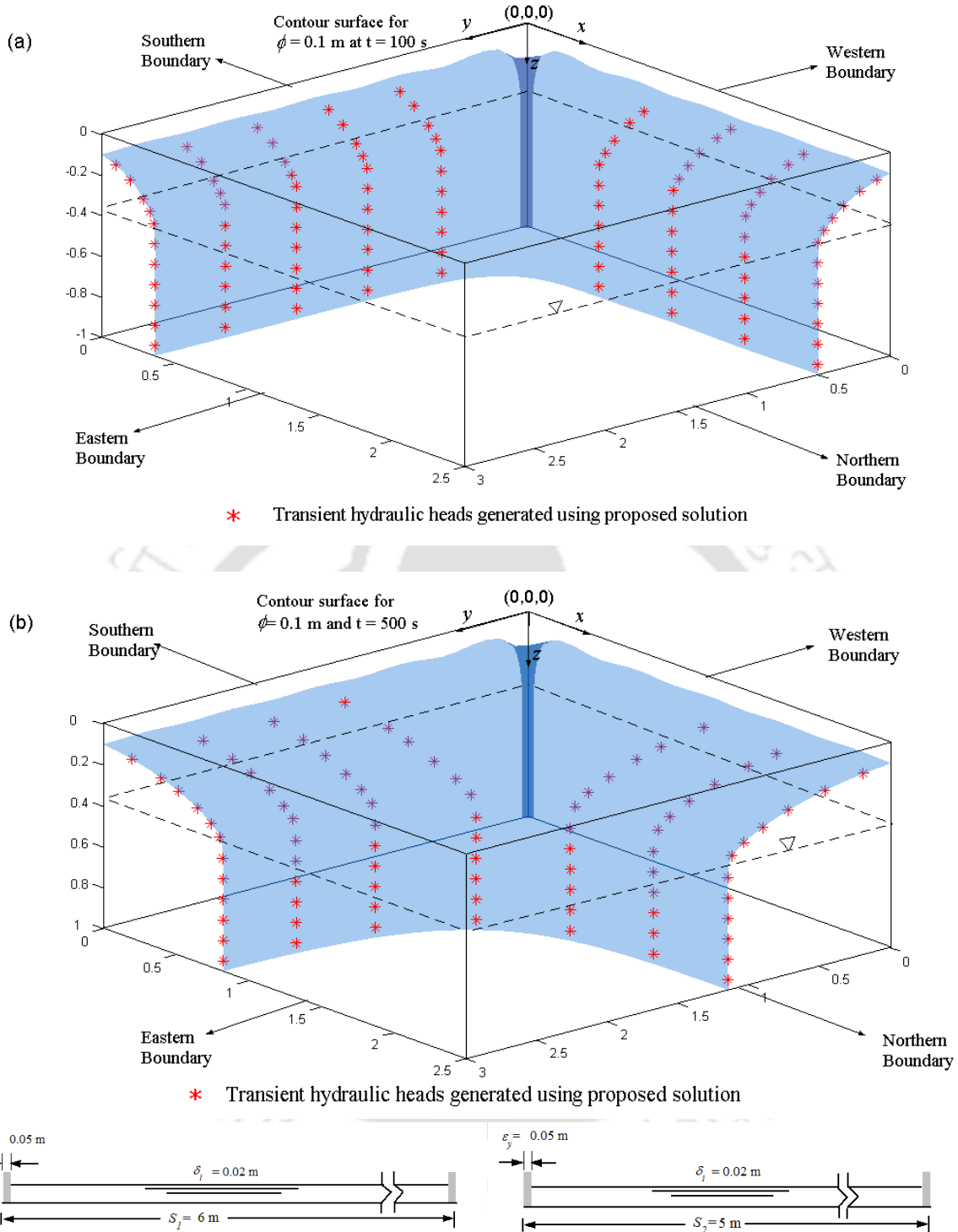


Fig 3.7. Comparison of transient hydraulic heads as obtained from the proposed solution at times $t = 100$ s and 500 s with the corresponding values as obtained from Sarmah and Barua's (2017) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem (Fig. 3.4) are considered as $S_1 = 6$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.35$ m, $H_5 = 0.45$ m, $H_6 = 0.75$ m, $\delta = 0.02$ m, $\epsilon_x = \epsilon_y = 0.05$ m, $K_{x_1} = K_{x_2} = K_{x_3} = 1$ m/day, $K_{y_1} = K_{y_2} = K_{y_3} = 0.5$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.002$ m/day and $S_{s_1} = S_{s_2} = S_{s_3} = 0.01$ m⁻¹

3.2.1.4 MODFLOW Verifications of the Proposed Solution

As a further check of our analytical model for both steady as well as transient states, numerical models were also drawn for two typical drainage settings of Fig. 3.4 utilizing the Processing MODFLOW environment (Chiang and Kinzelbach 2001). To perform a MODFLOW simulation of the flow situation of Fig. 3.4, a flow domain of $10\text{ m} \times 5\text{ m} \times 1\text{ m}$ was first considered and then it was subdivided into a network of grids containing 101 rows, 101 columns and 22 layers. A row spacing of 0.1 m was considered while the column spacing for the grid was kept at 0.05 m. The thickness of each vertical layer of the grid was assigned as 0.05 m. For simulating the impervious layer underlying the flow domain, all the cells of the 22nd layer were kept inactive. The cells of the 1st row, 101st row, 1st column and 101st column were kept as fixed head cells to model the ditch drains. The cells of the topmost layer were made as fixed head cells and assigned a value of 0.03 m to replicate the constant ponding atop the field. All the other cells in the model were assigned as active and the head value for them was kept as 0 m. The water level in the ditches were simulated by assigning a 0 m head to cells in the 1st column, 101st column, 1st row and 101st row and then proceeding to decrease it by 0.05 for every next layer encountered in the vertical direction till it reached the 8th layer where a head value of -0.35 m was assigned. The horizontal and vertical hydraulic conductivities of the cells of the first layer and extending upto the 10th layer were assigned values of 0.5 m/day and 2 m/day, respectively and the anisotropy ratio of these cells was assigned a value of 0.5; all these provided the conductivity information for the first soil layer of the model. The middle soil layer was made up of MODFLOW layers starting from the 11th layer and extending up to the 16th layer. For these layers, the horizontal and vertical hydraulic conductivities were inputted as 1.8 m/day and 0.8 m/day, respectively, and the anisotropy ratio as 0.6666. To simulate the conductivities of the bottom soil layer, a value of 1.4 m/day was assigned for the horizontal hydraulic conductivity and a value of 1 m/day for the vertical hydraulic conductivity to all the MODFLOW layers starting from the 17th layer and up to the 22nd layer while maintaining an anisotropy ratio of unity in these layers. With these settings in place, a steady state MODFLOW run was carried and the hydraulic head contours obtained for a few numerical figures compared with the corresponding contours obtained from our proposed solution. Fig. 3.8 shows such a comparison. As may be observed, here also the predictions from our solution are matching very closely with the corresponding MODFLOW results thereby providing a numerical verification to the steady state part of our solution.

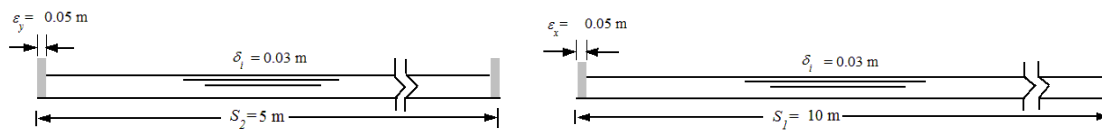
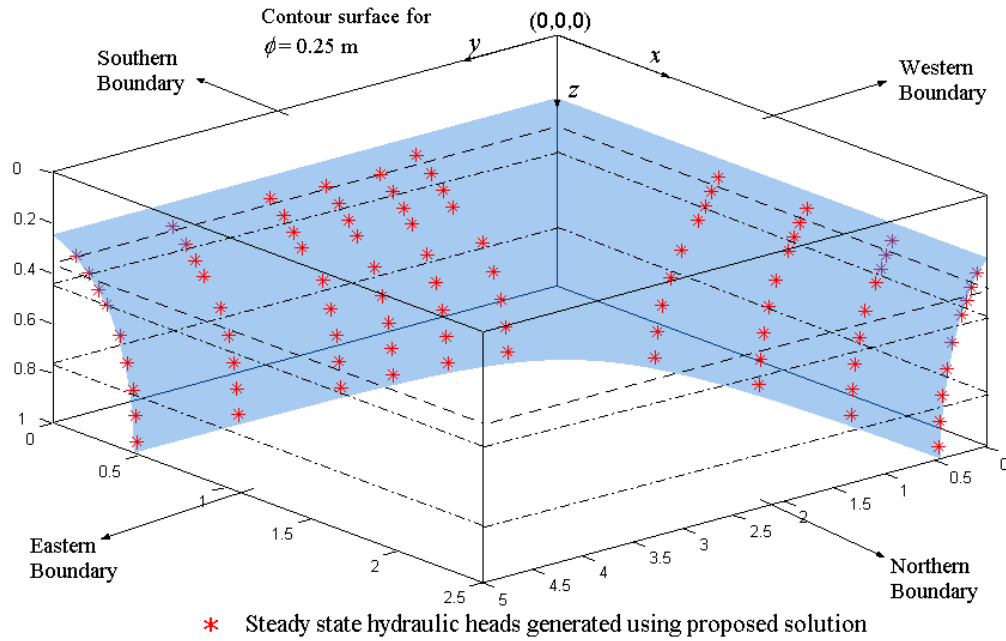
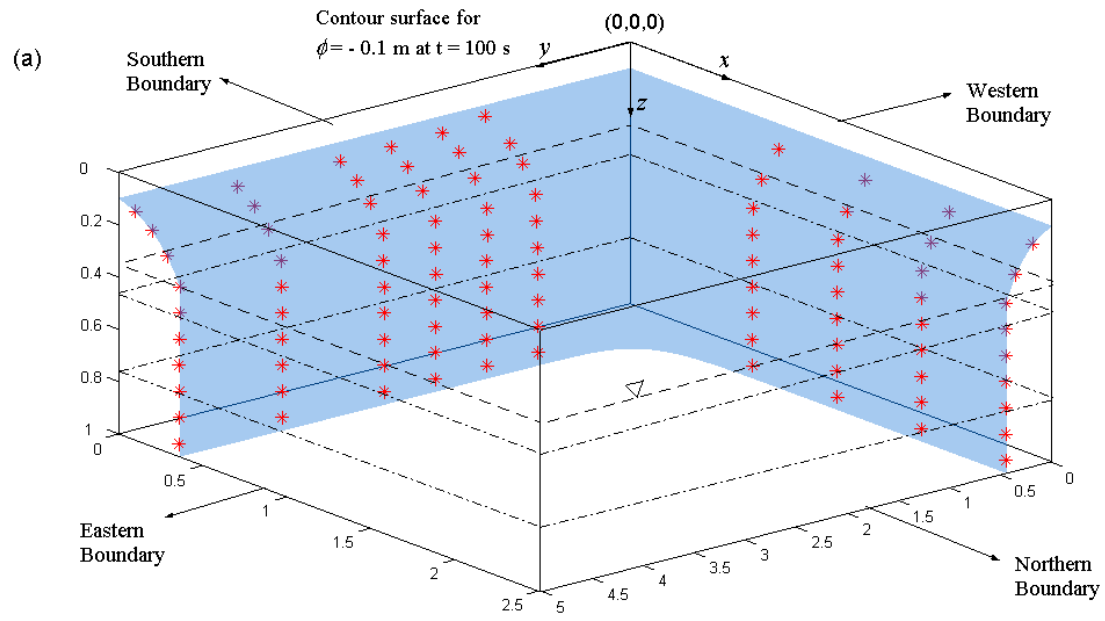
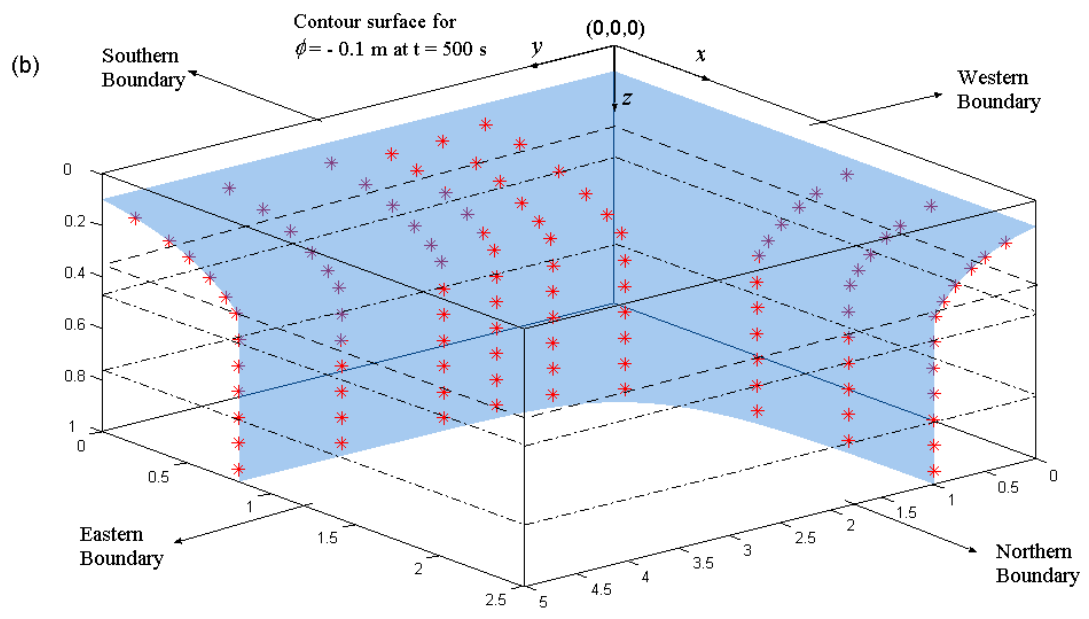


Fig 3.8. Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding results as obtained from MODFLOW when the flow parameters of Fig. 3.4 are considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.35$ m, $H_5 = 0.45$ m, $H_6 = 0.75$ m, $\delta = 0.03$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = 1$ m/day, $K_{y_1} = 2$ m/day, $K_{z_1} = 0.5$ m/day, $K_{x_2} = 1.2$ m/day, $K_{y_2} = 1.8$ m/day, $K_{z_2} = 0.8$ m/day, $K_{x_3} = 1.4$ m/day, $K_{y_3} = 1.4$ m/day and $K_{z_3} = 1$ m/day

In order to numerically verify our transient solution, a MODFLOW model was also constructed with the flow parameters as shown in Fig. 3.9 using the same methodology as has been described for the steady state model; here however, in order to simulate the initial condition, an additional input in the form of 0 m head was also imposed to all the active cells of the model in Processing MODFLOW interface. Further, as our transient solution requires a flow setting of Fig. 3.4 to satisfy Eqs. (3.33), (3.34) and (3.35) for its validity for a stratified soil, the directional conductivities and specific storage of the layers in the MODFLOW model were chosen the way as has been shown in Fig. 3.10. With these inputs in place, a transient MODFLOW run was carried out and the hydraulic head contours of -0.1 m strength were observed at two times of simulation of the system, namely 100 s and 500 s. These contours were then compared (Fig. 3.9) with the ones obtained from our analytical solution for the same flow setting; as can be seen, for this drainage setting also our solution could mimic quite accurately the heads as predicted by MODFLOW for both the simulation times of the system thereby proving that the proposed transient solution, within the restriction imposed by Eqs. (3.33) to (3.35), is also correctly developed.



* Transient hydraulic heads generated using proposed solution



* Transient hydraulic heads generated using proposed solution

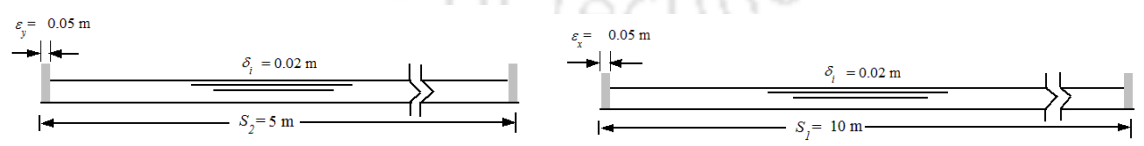


Fig 3.9. Comparison of transient hydraulic head contours as obtained from the proposed solution at times $t = 100$ s and 500 s with the corresponding values as obtained from MODFLOW when the flow parameters of Fig. 3.4 are considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.35$ m, $H_5 = 0.45$ m, $H_6 = 0.75$ m, $\delta = 0.02$ m, $\epsilon_x = \epsilon_y = 0.05$ m, $K_{x_1} = 0.75$ m/day, $K_{y_1} = 0.5$ m/day, $K_{x_2} = 1.5$ m/day, $K_{y_2} = 1$ m/day, $K_{x_3} = 0.6$ m/day, $K_{y_3} = 0.4$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.001$ m/day, $S_{s_1} = 0.01$ m⁻¹, $S_{s_2} = 0.02$ m⁻¹ and $S_{s_3} = 0.008$ m⁻¹

3.2.2 Level of Water in the Ditches is below the Boundary between the Top and the Middle Soil Layers and above the Boundary between the Middle and the Bottom Soil Layers

The flow problem for the present configuration is as shown in Fig. 3.10 where the water level in the ditches is now seen to be located in the middle soil layer in between the base of the top layer and the top boundary of the bottom layer.

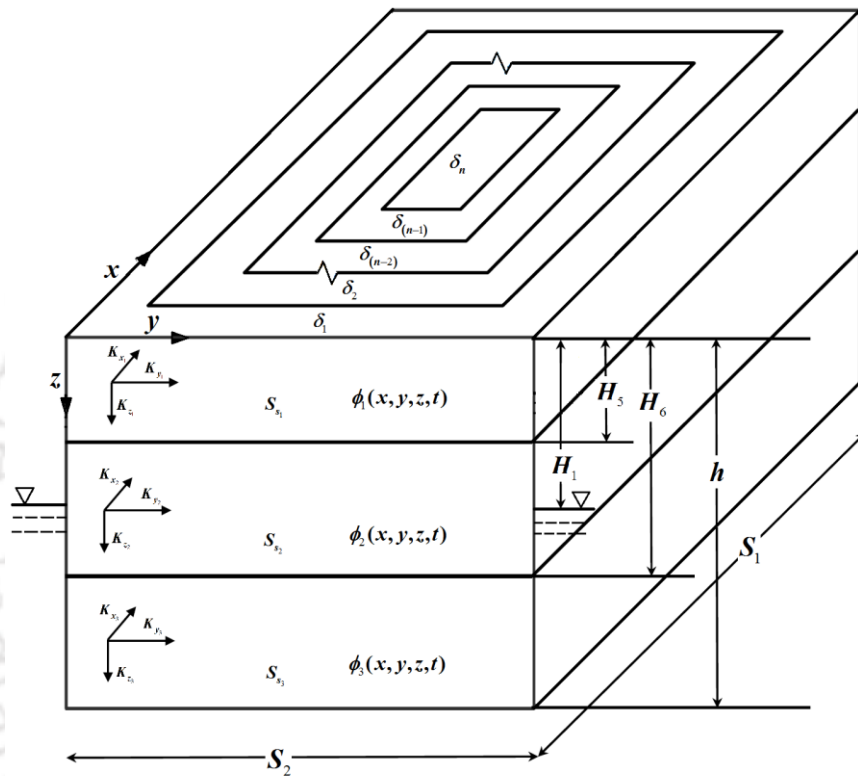


Fig. 3.10. Three dimensional ditch drainage system for a three-layered soil where the height of water in the ditches is below the bottom boundary of the top layer and above the top boundary of the bottom layer

The boundary conditions specific to this drainage situation can be represented as

$$\phi_{1(2)}(x, y, z, t > 0) = -z, \quad x = 0, \quad 0 < y < S_2, \quad 0 < z \leq H_5, \quad (\text{XX})$$

$$\phi_{2(2)}(x, y, z, t > 0) = -z, \quad x = 0, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_1, \quad (\text{XXIa})$$

$$\phi_{2(2)}(x, y, z, t > 0) = -H_1, \quad x = 0, \quad 0 < y < S_2, \quad H_1 \leq z \leq H_6, \quad (\text{XXIb})$$

$$\phi_{3(2)}(x, y, z, t > 0) = -H_1, \quad x = 0, \quad 0 < y < S_2, \quad H_6 \leq z < h, \quad (\text{XXII})$$

$$\phi_{1(2)}(x, y, z, t > 0) = -z, \quad x = S_1, \quad 0 < y < S_2, \quad 0 < z \leq H_5, \quad (\text{XXIII})$$

$$\phi_{2(2)}(x, y, z, t > 0) = -z, \quad x = S_1, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_1, \quad (\text{XXIVa})$$

$$\phi_{2(2)}(x, y, z, t > 0) = -H_1, \quad x = S_1, \quad 0 < y < S_2, \quad H_1 \leq z \leq H_6, \quad (\text{XXIVb})$$

$$\phi_{3(2)}(x, y, z, t > 0) = -H_1, \quad x = S_1, \quad 0 < y < S_2, \quad H_6 \leq z < h, \quad (\text{XXV})$$

$$\phi_{1(2)}(x, y, z, t > 0) = -z, \quad 0 < x < S_1, \quad y = 0, \quad 0 < z \leq H_5, \quad (\text{XXVI})$$

$$\phi_{2(2)}(x, y, z, t > 0) = -z, \quad 0 < x < S_1, \quad y = 0, \quad H_5 \leq z \leq H_1, \quad (\text{XXVIIa})$$

$$\phi_{2(2)}(x, y, z, t > 0) = -H_1, \quad 0 < x < S_1, \quad y = 0, \quad H_1 \leq z \leq H_6, \quad (\text{XXVIIb})$$

$$\phi_{3(2)}(x, y, z, t > 0) = -H_1, \quad 0 \leq x \leq S_1, \quad y = 0, \quad H_6 \leq z < h, \quad (\text{XXVIII})$$

$$\phi_{1(2)}(x, y, z, t > 0) = -z, \quad 0 < x < S_1, \quad y = S_2, \quad 0 < z \leq H_5, \quad (\text{XXIX})$$

$$\phi_{2(2)}(x, y, z, t) = -z, \quad 0 < x < S_1, \quad y = S_2, \quad H_5 \leq z \leq H_1, \quad (\text{XXXa})$$

$$\phi_{2(2)}(x, y, z, t) = -H_1, \quad 0 < x < S_1, \quad y = S_2, \quad H_1 \leq z \leq H_6, \quad (\text{XXXb})$$

$$\phi_{3(2)}(x, y, z, t) = -H_1, \quad 0 \leq x \leq S_1, \quad y = S_2, \quad H_6 \leq z < h. \quad (\text{XXXI})$$

Taking into account the general solutions of the three-dimensional continuity equation as expressed in Eqs. (3.27), (3.28) and (3.29), the hydraulic head expressions for this boundary value problem can be expressed

$$\begin{aligned} \phi_{1(2)}(x, y, z, t) = & \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(2)} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \sin(N_{p_1} y) \sin(N_{q_1} z) \\ & + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(2)} \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \sin(N_{p_2} y) \sin(N_{q_2} z) \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(2)} \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{p_3} x) \sin(N_{q_3} z) \\ & + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(2)} \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{p_4} x) \sin(N_{q_4} z) \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(2)} \frac{\sinh(\lambda_{kl} z)}{\cosh(\lambda_{kl} H_5)} \sin(N_k x) \sin(N_l y) \\ & + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(2)} \frac{\cosh[\lambda_{uv} (H_5 - z)]}{\cosh(\lambda_{uv} H_5)} \sin(N_u x) \sin(N_v y) \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right], \end{aligned} \quad (3.142)$$

where

$$N_{p_1} = \left(\frac{p_1 \pi}{S_2} \right), \quad (3.143)$$

$$N_{q_1} = \left[\left(\frac{1-2q_1}{2} \right) \frac{\pi}{H_5} \right], \quad (3.144)$$

$$N_{p_2} = \left(\frac{p_2 \pi}{S_2} \right), \quad (3.145)$$

$$N_{q_2} = \left[\left(\frac{1-2q_2}{2} \right) \frac{\pi}{H_5} \right], \quad (3.146)$$

$$N_{p_3} = \left(\frac{p_3 \pi}{S_1} \right), \quad (3.147)$$

$$N_{q_3} = \left[\left(\frac{1-2q_3}{2} \right) \frac{\pi}{H_5} \right], \quad (3.148)$$

$$N_{p_4} = \left(\frac{p_4 \pi}{S_1} \right), \quad (3.149)$$

$$N_{q_4} = \left[\left(\frac{1-2q_4}{2} \right) \frac{\pi}{H_5} \right], \quad (3.150)$$

$$N_k = \left(\frac{k \pi}{S_1} \right), \quad (3.151)$$

$$N_l = \left(\frac{l \pi}{S_2} \right), \quad (3.152)$$

$$N_u = \left(\frac{u \pi}{S_1} \right), \quad (3.153)$$

$$N_v = \left(\frac{v \pi}{S_2} \right), \quad (3.154)$$

$$(\lambda_{p_1 q_1})^2 = (N_{p_1}^2 K_{xy}^{a1} + N_{q_1}^2 K_{xz}^{a1}), \quad (3.155)$$

$$(\lambda_{p_2 q_2})^2 = (N_{p_2}^2 K_{xy}^{a1} + N_{q_2}^2 K_{xz}^{a1}), \quad (3.156)$$

$$(\lambda_{p_3 q_3})^2 = \left[(N_{p_3}^2 / K_{xy}^{a1}) + N_{q_3}^2 K_{yz}^{a1} \right], \quad (3.157)$$

$$(\lambda_{p_4 q_4})^2 = \left[(N_{p_4}^2 / K_{xy}^{a1}) + N_{q_4}^2 K_{yz}^{a1} \right], \quad (3.158)$$

$$(\lambda_{kl})^2 = \left[(N_k^2 / K_{xz}^{a1}) + (N_l^2 / K_{yz}^{a1}) \right], \quad (3.159)$$

$$(\lambda_{uv})^2 = \left[(N_u^2 / K_{xz}^{a1}) + (N_v^2 / K_{yz}^{a1}) \right], \quad (3.160)$$

$$K_{xy}^{a1} = \left(\frac{K_{y_1}}{K_{x_1}} \right), \quad (3.161)$$

$$K_{xz}^{a1} = \left(\frac{K_{z_1}}{K_{x_1}} \right), \quad (3.162)$$

$$K_{yz}^{a1} = \left(\frac{K_{z_1}}{K_{y_1}} \right), \quad (3.163)$$

$$N_{l_1} = \left(\frac{l_1 \pi}{S_1} \right), \quad (3.164)$$

$$N_{m_1} = \left(\frac{m_1 \pi}{S_2} \right), \quad (3.165)$$

$$N_{n_1} = \left[\left(\frac{1-2n_1}{2} \right) \frac{\pi}{h} \right], \quad (3.166)$$

$$N_{l_2} = \left(\frac{l_2 \pi}{S_1} \right), \quad (3.167)$$

$$N_{m_2} = \left(\frac{m_2 \pi}{S_2} \right), \quad (3.168)$$

$$N_{n_2} = \left[\left(\frac{1-2n_2}{2} \right) \frac{\pi}{h} \right], \quad (3.169)$$

$$N_{l_3} = \left(\frac{l_3 \pi}{S_1} \right), \quad (3.170)$$

$$N_{m_3} = \left(\frac{m_3 \pi}{S_2} \right), \quad (3.171)$$

$$N_{n_3} = \left[\left(\frac{1-2n_3}{2} \right) \frac{\pi}{h} \right], \quad (3.172)$$

$$(\lambda_{l_1 m_1 n_1})^2 = \left[N_{l_1}^2 \left(\frac{K_{x_1}}{S_{s_1}} \right) + N_{m_1}^2 \left(\frac{K_{y_1}}{S_{s_1}} \right) + N_{n_1}^2 \left(\frac{K_{z_1}}{S_{s_1}} \right) \right], \quad (3.173)$$

$$(\lambda_{l_2 m_2 n_2})^2 = \left[N_{l_2}^2 \left(\frac{K_{x_2}}{S_{s_2}} \right) + N_{m_2}^2 \left(\frac{K_{y_2}}{S_{s_2}} \right) + N_{n_2}^2 \left(\frac{K_{z_2}}{S_{s_2}} \right) \right], \quad (3.174)$$

$$(\lambda_{l_3 m_3 n_3})^2 = \left[N_{l_3}^2 \left(\frac{K_{x_3}}{S_{s_3}} \right) + N_{m_3}^2 \left(\frac{K_{y_3}}{S_{s_3}} \right) + N_{n_3}^2 \left(\frac{K_{z_3}}{S_{s_3}} \right) \right] \quad (3.175)$$

and $B_{p_1 q_1(2)}$, $C_{p_2 q_2(2)}$, $D_{p_3 q_3(2)}$, $F_{p_4 q_4(2)}$, $E_{kl(2)}$, $Q_{uv(2)}$, $A_{l_1 m_1 n_1(2)}$, $R_{l_2 m_2 n_2(2)}$ and $W_{l_3 m_3 n_3(2)}$ are constants and

$$\begin{aligned} \phi_{2(2)}(x, y, z, t) = & \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \frac{\cosh \left[\lambda_{i_1 j_1} (z - H_5) \right]}{\cosh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right]} \sin(N_{i_1} x) \sin(N_{j_1} y) \\ & + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \frac{\cosh \left[\lambda_{i_2 j_2} (H_6 - z) \right]}{\cosh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right]} \sin(N_{i_2} x) \sin(N_{j_2} y) \end{aligned}$$

$$\begin{aligned}
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \sin(N_{p_5} y) \sin[N_{q_5}(z - H_5)] \\
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \frac{\sinh[\lambda_{p_6 q_6}(S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \sin(N_{p_6} y) \sin[N_{q_6}(z - H_5)] \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \sin(N_{p_7} x) \sin[N_{q_7}(z - H_5)] \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \frac{\sinh[\lambda_{p_8 q_8}(S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \sin(N_{p_8} x) \sin[N_{q_8}(z - H_5)] - H_5 \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp[-(\lambda_{l_1 m_1 n_1})^2 t] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp[-(\lambda_{l_2 m_2 n_2})^2 t] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp[-(\lambda_{l_3 m_3 n_3})^2 t], \quad (3.176)
\end{aligned}$$

where

$$N_{i_1} = \left(\frac{i_1 \pi}{S_1} \right), \quad (3.177)$$

$$N_{j_1} = \left(\frac{j_1 \pi}{S_2} \right), \quad (3.178)$$

$$N_{i_2} = \left(\frac{i_2 \pi}{S_1} \right), \quad (3.179)$$

$$N_{j_2} = \left(\frac{j_2 \pi}{S_2} \right), \quad (3.180)$$

$$N_{p_5} = \left(\frac{p_5 \pi}{S_2} \right), \quad (3.181)$$

$$N_{q_5} = \left[\left(\frac{1-2q_5}{2} \right) \left(\frac{\pi}{H_6 - H_5} \right) \right], \quad (3.182)$$

$$N_{p_6} = \left(\frac{p_6 \pi}{S_2} \right), \quad (3.183)$$

$$N_{q_6} = \left[\left(\frac{1-2q_6}{2} \right) \left(\frac{\pi}{H_6 - H_5} \right) \right], \quad (3.184)$$

$$N_{p_7} = \left(\frac{p_7 \pi}{S_1} \right), \quad (3.185)$$

$$N_{q_7} = \left[\left(\frac{1-2q_7}{2} \right) \left(\frac{\pi}{H_6 - H_5} \right) \right], \quad (3.186)$$

$$N_{p_8} = \left(\frac{p_8 \pi}{S_1} \right), \quad (3.187)$$

$$N_{q_8} = \left[\left(\frac{1-2q_8}{2} \right) \left(\frac{\pi}{H_6 - H_5} \right) \right], \quad (3.188)$$

$$(\lambda_{i_1 j_1})^2 = \left[(N_{i_1}^2 / K_{xz}^{a2}) + (N_{j_1}^2 / K_{yz}^{a2}) \right], \quad (3.189)$$

$$(\lambda_{i_2 j_2})^2 = \left[(N_{i_2}^2 / K_{xz}^{a2}) + (N_{j_2}^2 / K_{yz}^{a2}) \right], \quad (3.190)$$

$$(\lambda_{p_5 q_5})^2 = N_{p_5}^2 K_{xy}^{a2} + N_{q_5}^2 K_{xz}^{a2}, \quad (3.191)$$

$$(\lambda_{p_6 q_6})^2 = N_{p_6}^2 K_{xy}^{a2} + N_{q_6}^2 K_{xz}^{a2}, \quad (3.192)$$

$$(\lambda_{p_7 q_7})^2 = \left[(N_{p_7}^2 / K_{xy}^{a2}) + N_{q_7}^2 K_{yz}^{a2} \right], \quad (3.193)$$

$$(\lambda_{p_8 q_8})^2 = \left[(N_{p_8}^2 / K_{xy}^{a2}) + N_{q_8}^2 K_{yz}^{a2} \right], \quad (3.194)$$

$$K_{xz}^{a2} = \left(\frac{K_{z_2}}{K_{x_2}} \right), \quad (3.195)$$

$$K_{yz}^{a2} = \left(\frac{K_{z_2}}{K_{y_2}} \right), \quad (3.196)$$

$$K_{xy}^{a2} = \left(\frac{K_{y_2}}{K_{x_2}} \right) \quad (3.197)$$

and $G_{i_1 j_1(2)}$, $H_{i_2 j_2(2)}$, $I_{p_5 q_5(2)}$, $J_{p_6 q_6(2)}$, $K_{p_7 q_7(2)}$ and $L_{p_8 q_8(2)}$ are constants and

$$\begin{aligned} \phi_{3(2)}(x, y, z, t) = & \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(2)} \frac{\cosh[\lambda_{i_3 j_3}(h-z)]}{\sinh[\lambda_{i_3 j_3}(h-H_6)]} \sin(N_{i_3} x) \sin(N_{j_3} y) - H_1 \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right], \end{aligned} \quad (3.198)$$

where

$$N_{i_3} = \left(\frac{i_3 \pi}{S_1} \right), \quad (3.199)$$

$$N_{j_3} = \left(\frac{j_3 \pi}{S_2} \right), \quad (3.200)$$

$$(\lambda_{i_3 j_3})^2 = \left[(N_{i_3}^2 / K_{xz}^{a3}) + (N_{j_3}^2 / K_{yz}^{a3}) \right], \quad (3.201)$$

$$K_{xz}^{a3} = \left(\frac{K_{z_3}}{K_{x_3}} \right), \quad (3.202)$$

$$K_{yz}^{a3} = \left(\frac{K_{z_3}}{K_{y_3}} \right) \quad (3.203)$$

and $P_{i_3 j_3(2)}$ are constants.

As can be seen, the hydraulic head expressions for the top and the bottom layers for this drainage situation are similar to those of the previous case. This thus means that the Darcian velocity functions expressions for the top and the bottom layers for this flow situation (i.e., $V_{x1(2)}$, $V_{y1(2)}$, $V_{z1(2)}$, $V_{x3(2)}$, $V_{y3(2)}$ and $V_{z3(2)}$) would also be similar to the corresponding functions of the previous case (i.e., to $V_{x1(1)}$, $V_{y1(1)}$, $V_{z1(1)}$, $V_{x3(1)}$, $V_{y3(1)}$ and $V_{z3(1)}$); however, it must be noted the relevant Fourier coefficients to be used in these expressions are now $B_{p_1 q_1(2)}$, $C_{p_2 q_2(2)}$, $D_{p_3 q_3(2)}$, $F_{p_4 q_4(2)}$, $E_{kl(2)}$, $Q_{uv(2)}$, $A_{i_1 m_1 n_1(2)}$, $R_{l_2 m_2 n_2(2)}$ and $W_{l_3 m_3 n_3(2)}$ instead of $B_{p_1 q_1(1)}$, $C_{p_2 q_2(1)}$, $D_{p_3 q_3(1)}$, $F_{p_4 q_4(1)}$, $E_{kl(1)}$, $Q_{uv(1)}$, $A_{i_1 m_1 n_1(1)}$, $R_{l_2 m_2 n_2(1)}$ and $W_{l_3 m_3 n_3(1)}$ and $P_{i_3 j_3(2)}$, $A_{i_1 m_1 n_1(2)}$, $R_{l_2 m_2 n_2(2)}$ and $W_{l_3 m_3 n_3(2)}$ instead of $P_{i_3 j_3(1)}$, $A_{i_1 m_1 n_1(1)}$, $R_{l_2 m_2 n_2(1)}$ and $W_{l_3 m_3 n_3(1)}$, respectively. As the hydraulic head function of the middle layer for the drainage situation of Fig. 3.10 is different from that of the previous case, naturally then, the directional velocity expressions for this layer would also be different from those corresponding to the previous case; for this problem, they work out as

$$\begin{aligned} V_{x2(2)} = & -K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \frac{\cosh[\lambda_{i_1 j_1}(z - H_5)]}{\cosh[\lambda_{i_1 j_1}(H_6 - H_5)]} N_{i_1} \cos(N_{i_1} x) \sin(N_{j_1} y) \right. \\ & + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \frac{\cosh[\lambda_{i_2 j_2}(H_6 - z)]}{\cosh[\lambda_{i_2 j_2}(H_6 - H_5)]} N_{i_2} \cos(N_{i_2} x) \sin(N_{j_2} y) \\ & + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \frac{\lambda_{p_5 q_5} \cosh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \sin(N_{p_5} y) \sin[N_{q_5}(z - H_5)] \\ & - \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \frac{\lambda_{p_6 q_6} \cosh[\lambda_{p_6 q_6}(S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \sin(N_{p_6} y) \sin[N_{q_6}(z - H_5)] \\ & + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} N_{p_7} \cos(N_{p_7} x) \sin[N_{q_7}(z - H_5)] \\ & \left. + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \frac{\sinh[\lambda_{p_8 q_8}(S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} N_{p_8} \cos(N_{p_8} x) \sin[N_{q_8}(z - H_5)] \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{l_1} \cos(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{l_2} \cos(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{l_3} \cos(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\}, \quad (3.204)
\end{aligned}$$

$$\begin{aligned}
V_{y2(2)} = -K_{y_2} & \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \frac{\cosh[\lambda_{i_1 j_1}(z-H_5)]}{\cosh[\lambda_{i_1 j_1}(H_6-H_5)]} \sin(N_{i_1} x) N_{j_1} \cos(N_{j_1} y) \right. \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \frac{\cosh[\lambda_{i_2 j_2}(H_6-z)]}{\cosh[\lambda_{i_2 j_2}(H_6-H_5)]} \sin(N_{i_2} x) N_{j_2} \cos(N_{j_2} y) \\
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} N_{p_5} \cos(N_{p_5} y) \sin[N_{q_5}(z-H_5)] \\
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \frac{\sinh[\lambda_{p_6 q_6}(S_1-x)]}{\sinh(\lambda_{p_6 q_6} S_1)} N_{p_6} \cos(N_{p_6} y) \sin[N_{q_6}(z-H_5)] \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \frac{\lambda_{p_7 q_7} \cosh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \sin(N_{p_7} x) \sin[N_{q_7}(z-H_5)] \\
& - \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \frac{\lambda_{p_8 q_8} \cosh[\lambda_{p_8 q_8}(S_2-y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \sin(N_{p_8} x) \sin[N_{q_8}(z-H_5)] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} \sin(N_{l_1} x) N_{m_1} \cos(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} \sin(N_{l_2} x) N_{m_2} \cos(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} \sin(N_{l_3} x) N_{m_3} \cos(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\} \quad (3.205)
\end{aligned}$$

and

$$\begin{aligned}
V_{z2(2)} = -K_{z_2} & \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \frac{\lambda_{i_1 j_1} \sinh[\lambda_{i_1 j_1}(z-H_5)]}{\cosh[\lambda_{i_1 j_1}(H_6-H_5)]} \sin(N_{i_1} x) \sin(N_{j_1} y) \right. \\
& - \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \frac{\lambda_{i_2 j_2} \sinh[\lambda_{i_2 j_2}(H_6-z)]}{\cosh[\lambda_{i_2 j_2}(H_6-H_5)]} \sin(N_{i_2} x) \sin(N_{j_2} y) \\
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \sin(N_{p_5} y) N_{q_5} \cos[N_{q_5}(z-H_5)] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \frac{\sinh[\lambda_{p_6 q_6}(S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \sin(N_{p_6} y) N_{q_6} \cos[N_{q_6}(z - H_5)] \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \sin(N_{p_7} x) N_{q_7} \cos[N_{q_7}(z - H_5)] \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \frac{\sinh[\lambda_{p_8 q_8}(S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \sin(N_{p_8} x) N_{q_8} \cos[N_{q_8}(z - H_5)] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} \sin(N_{l_1} x) \sin(N_{m_1} y) N_{n_1} \cos(N_{n_1} z) \exp[-(\lambda_{l_1 m_1 n_1})^2 t] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} \sin(N_{l_2} x) \sin(N_{m_2} y) N_{n_2} \cos(N_{n_2} z) \exp[-(\lambda_{l_2 m_2 n_2})^2 t] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} \sin(N_{l_3} x) \sin(N_{m_3} y) N_{n_3} \cos(N_{n_3} z) \exp[-(\lambda_{l_3 m_3 n_3})^2 t] \Big\}, \quad (3.206)
\end{aligned}$$

where $V_{x(2)}$, $V_{y(2)}$ and $V_{z(2)}$ are the directional velocity expressions for the middle layer of the flow problem of Fig. 3.10 along the x -, y - and z - directions, respectively.

3.2.2.1 Determination of Coefficients of the Steady State Part of the Solution (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 3.10)

Introducing boundary conditions (XX) in Eq. (3.142), we get a relationship involving the coefficients $C_{p_2 q_2(2)}$ as

$$\sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(2)} \sin(N_{p_2} y) \sin(N_{q_2} z) = -z, \quad x = 0, \quad 0 < y < S_2, \quad 0 < z \leq H_5.$$

Thus, $C_{p_2 q_2(2)}$ can be determined by performing a double Fourier run in $0 < y < S_2$ and $0 < z \leq H_5$; after carrying out the concerned integrals, we find $C_{p_2 q_2(2)}$ as

$$C_{p_2 q_2(2)} = \left(\frac{2}{S_2} \right) \left(\frac{2}{H_5} \right) \left[\frac{1 - \cos(N_{p_2} S_2)}{N_{p_2}} \right] \left[\frac{-\sin(N_{q_2} H_5)}{N_{q_2}} \right]. \quad (3.207)$$

Similarly, imposition of boundary condition (XXIII) on Eq. (3.142) gives

$$\sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(2)} \sin(N_{p_1} y) \sin(N_{q_1} z) = -z, \quad x = S_1, \quad 0 < y < S_2, \quad 0 < z \leq H_5,$$

from which $B_{p_1 q_1(2)}$ works out as

$$B_{p_1 q_1(2)} = \left(\frac{2}{S_2} \right) \left(\frac{2}{H_5} \right) \left[\frac{1 - \cos(N_{p_1} S_2)}{N_{p_1}} \right] \left[\frac{-\sin(N_{q_1} H_5)}{N_{q_1}} \right]. \quad (3.208)$$

Also, application of boundary condition (XXVI) to Eq. (3.142) leads to the relation

$$\sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(2)} \sin(N_{p_4} x) \sin(N_{q_4} z) = -z, \quad 0 < x < S_1, \quad y = 0, \quad 0 < z \leq H_5.$$

Solving the concerned integrals, we find the Fourier coefficients $F_{p_4q_4(2)}$ as

$$F_{p_4q_4(2)} = \left(\frac{2}{S_1} \right) \left(\frac{2}{H_5} \right) \left[\frac{1 - \cos(N_{p_4} S_1)}{N_{p_4}} \right] \left[\frac{-\sin(N_{q_4} H_5)}{N_{q_4}} \right]. \quad (3.209)$$

Similarly, from boundary condition (XXIX), we get

$$\sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3q_3(2)} \sin(N_{p_3} x) \sin(N_{q_3} z) = -z, \quad 0 < x < S_1, \quad y = S_2, \quad 0 < z \leq H_5$$

and consequently $D_{p_3q_3(2)}$ as

$$D_{p_3q_3(2)} = \left(\frac{2}{S_1} \right) \left(\frac{2}{H_5} \right) \left[\frac{1 - \cos(N_{p_3} S_1)}{N_{p_3}} \right] \left[\frac{-\sin(N_{q_3} H_5)}{N_{q_3}} \right]. \quad (3.210)$$

It is to be noted that $Q_{uv(2)}$ will have an identical expression as $Q_{uv(1)}$ in the previous problem since the boundary condition at the top of the field is the same as in the previous problem. Also, by applying boundary conditions (XXIa) and (XXIb) to Eq. (3.176), we get

$$\sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6q_6(2)} \sin(N_{p_6} y) \sin[N_{q_6} (z - H_5)] = -(z - H_5), \quad x = 0, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_1,$$

$$\sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6q_6(2)} \sin(N_{p_6} y) \sin[N_{q_6} (z - H_5)] = -(H_1 - H_5), \quad x = 0, \quad 0 < y < S_2, \quad H_1 \leq z \leq H_6.$$

Performing a double Fourier run in the concerned domains, we get the relevant integrals for evaluating $J_{p_6q_6(2)}$ as

$$J_{p_6q_6(2)} = \left(\frac{2}{S_2} \right) \left(\frac{2}{H_6 - H_5} \right) \int_0^{S_2} \sin(N_{p_6} y) dy \times$$

$$\left[\int_{H_5}^{H_1} -(z - H_5) \sin[N_{q_6} (z - H_5)] dz + \int_{H_1}^{H_6} -(H_1 - H_5) \sin[N_{q_6} (z - H_5)] dz \right]. \quad (3.211)$$

Solving the above integrals, we get

$$J_{p_6q_6(2)} = \left(\frac{2}{S_2} \right) \left(\frac{2}{H_6 - H_5} \right) \left[\frac{1 - \cos(N_{p_6} S_2)}{N_{p_6}} \right] \left\{ \frac{-\sin[N_{q_6} (H_1 - H_5)]}{N_{q_6}} \right\}. \quad (3.212)$$

Similarly, from boundary conditions (XXIVa) and (XXIVb) and Eq. (3.176), we have

$$\sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5q_5(2)} \sin(N_{p_5} y) \sin[N_{q_5} (z - H_5)] = -(z - H_5), \quad x = S_1, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_1,$$

$$\sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5q_5(2)} \sin(N_{p_5} y) \sin[N_{q_5} (z - H_5)] = -(H_1 - H_5), \quad x = S_1, \quad 0 < y < S_2, \quad H_1 \leq z \leq H_6.$$

The final expression for $I_{p_5q_5(2)}$ is obtained as

$$I_{p_5q_5(2)} = \left(\frac{2}{S_2} \right) \left(\frac{2}{H_6 - H_5} \right) \left[\frac{1 - \cos(N_{p_5} S_2)}{N_{p_5}} \right] \left\{ \frac{-\sin[N_{q_5} (H_1 - H_5)]}{N_{q_5}} \right\}. \quad (3.213)$$

Also, from boundary conditions (XXVIIa) and (XXVIIb) and Eq. (3.176), we get

$$\sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \sin(N_{p_8} x) \sin(N_{q_8} (z - H_5)) = -(z - H_5), \quad 0 < x < S_1, \quad y = 0, \quad H_5 \leq z \leq H_1,$$

$$\sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \sin(N_{p_8} x) \sin(N_{q_8} (z - H_5)) = -(H_1 - H_5), \quad 0 < x < S_1, \quad y = 0, \quad H_1 \leq z \leq H_6.$$

Performing the necessary Fourier expansions, we get $L_{p_8 q_8(2)}$ as

$$L_{p_8 q_8(2)} = \left(\frac{2}{S_1} \right) \left(\frac{2}{H_6 - H_5} \right) \left[\frac{1 - \cos(N_{p_8} S_1)}{N_{p_8}} \right] \left\{ \frac{-\sin[N_{q_8} (H_1 - H_5)]}{N_{q_8}} \right\}. \quad (3.214)$$

Further, by incorporating boundary conditions (XXXa) and (XXXb) in Eq. (3.176) and carrying out the necessary Fourier expansion on the resultant equations, we get $K_{p_7 q_7(2)}$ as

$$K_{p_7 q_7(2)} = \left(\frac{2}{S_1} \right) \left(\frac{2}{H_6 - H_5} \right) \left[\frac{1 - \cos(N_{p_7} S_1)}{N_{p_7}} \right] \left\{ \frac{-\sin[N_{q_7} (H_1 - H_5)]}{N_{q_7}} \right\}. \quad (2.215)$$

Also, by equating the hydraulic heads for the top and the middle layers at $z = H_5$ [boundary condition (IVa)], we find

$$\begin{aligned} & \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(2)} \sin(N_{p_1} y) \sin(N_{q_1} H_5) \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \\ & + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(2)} \sin(N_{p_2} y) \sin(N_{q_2} H_5) \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(2)} \sin(N_{p_3} x) \sin(N_{q_3} H_5) \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \\ & + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(2)} \sin(N_{p_4} x) \sin(N_{q_4} H_5) \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(2)} \sin(N_k x) \sin(N_l y) \tanh(\lambda_{kl} H_5) \\ & + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(2)} \sin(N_u x) \sin(N_v y) \frac{1}{\cosh(\lambda_{uv} H_5)} \\ & = \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \sin(N_{i_1} x) \sin(N_{j_1} y) \frac{1}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} \\ & + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \sin(N_{i_2} x) \sin(N_{j_2} y) - H_5. \end{aligned}$$

Performing a double Fourier expansion in $0 < x < S_1$ and $0 < y < S_2$ in the above equations, we get the expression

$$E_{kl(2)} \tanh(\lambda_{kl} H_5) - G_{i_1 j_1(2)} \frac{1}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} - H_{i_2 j_2(2)} + Q_{uv(2)} \frac{1}{\cosh(\lambda_{uv} H_5)}$$

$$\begin{aligned}
&= -\left(\frac{2}{S_1}\right)\left(\frac{2}{S_2}\right)\left\{H_5 \int_0^{S_1} \sin(N_{u_1} x) dx \int_0^{S_2} \sin(N_{v_1} y) dy \right. \\
&+ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(2)} \sin(N_{q_1} H_5) \int_0^{S_2} \sin(N_{p_1} y) \sin(N_{v_1} y) dy \int_0^{S_1} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \sin(N_{u_1} x) dx \\
&+ \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(2)} \sin(N_{q_2} H_5) \int_0^{S_2} \sin(N_{p_2} y) \sin(N_{v_1} y) dy \times \\
&\int_0^{S_1} \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \sin(N_{u_1} x) dx \\
&+ \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(2)} \sin(N_{q_3} H_5) \int_0^{S_1} \sin(N_{p_3} x) \sin(N_{u_1} x) dx \int_0^{S_2} \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{v_1} y) dy \\
&+ \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(2)} \sin(N_{q_4} H_5) \int_0^{S_1} \sin(N_{p_4} x) \sin(N_{u_1} x) dx \times \\
&\left. \int_0^{S_2} \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{v_1} y) dy \right\}. \tag{3.216}
\end{aligned}$$

Solving the above integrals, we finally arrive at the following equation

$$\begin{aligned}
&E_{kl(2)} \tanh(\lambda_{kl} H_5) - G_{i_1 j_1(2)} \frac{1}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} - H_{i_2 j_2(2)} + Q_{uv(2)} \frac{1}{\cosh(\lambda_{uv} H_5)} \\
&= -\left(\frac{4}{S_1 S_2}\right) H_5 \left[\frac{1 - \cos(N_{u_1} S_1)}{N_{u_1}} \right] \left[\frac{1 - \cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
&- \left(\frac{2}{S_1}\right) \sum_{q_1=1}^{Q_1} B_{p_1 q_1(2)} \sin(N_{q_1} H_5) \left[\frac{N_{u_1}^2}{N_{u_1}^2 + \lambda_{p_1 q_1}^2} \right] \left[\frac{-\cos(N_{u_1} S_1)}{N_{u_1}} \right] \\
&- \left(\frac{2}{S_1}\right) \sum_{q_2=1}^{Q_2} C_{p_2 q_2(2)} \sin(N_{q_2} H_5) \left[\frac{N_{u_1}}{N_{u_1}^2 + \lambda_{p_2 q_2}^2} \right] \\
&- \left(\frac{2}{S_2}\right) \sum_{q_3=1}^{Q_3} D_{p_3 q_3(2)} \sin(N_{q_3} H_5) \left[\frac{N_{v_1}^2}{N_{v_1}^2 + \lambda_{p_3 q_3}^2} \right] \left[\frac{-\cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
&- \left(\frac{2}{S_2}\right) \sum_{q_4=1}^{Q_4} F_{p_4 q_4(2)} \sin(N_{q_4} H_5) \left[\frac{N_{v_1}}{N_{v_1}^2 + \lambda_{p_4 q_4}^2} \right], \tag{3.217}
\end{aligned}$$

where $u_1 = k = u = i_1 = i_2 = p_3 = p_4$, $v_1 = l = v = j_1 = j_2 = p_1 = p_2$ and $Q_1 = Q_2 = Q_3 = Q_4 \rightarrow \infty$.

Also, flux equality at $z = H_5$ [boundary condition (IVb)] gives us the relation

$$-K_{z_1} \sum_{k=1}^K \sum_{l=1}^L E_{kl(2)} \lambda_{kl} \sin(N_k x) \sin(N_l y) =$$

$$\begin{aligned}
& -K_{z_2} \left\{ -\sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \sin(N_{i_2} x) \sin(N_{j_2} y) \lambda_{i_2 j_2} \tanh[\lambda_{i_2 j_2} (H_6 - H_5)] \right. \\
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} N_{q_5} \sin(N_{p_5} y) \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \\
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} N_{q_6} \sin(N_{p_6} y) \frac{\sinh[\lambda_{p_6 q_6} (S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} N_{q_7} \sin(N_{p_7} x) \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \\
& \left. + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} N_{q_8} \sin(N_{p_8} x) \frac{\sinh[\lambda_{p_8 q_8} (S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \right\}
\end{aligned}$$

from which, using again a Fourier expansion in $0 < x < S_1$ and $0 < y < S_2$, gives us the relation

$$\begin{aligned}
& \left(\frac{K_{z_1}}{K_{z_2}} \right) E_{kl(2)} \lambda_{kl} + H_{i_2 j_2(2)} \lambda_{i_2 j_2} \tanh[\lambda_{i_2 j_2} (H_6 - H_5)] = \\
& \left(\frac{4}{S_1 S_2} \right) \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} N_{q_5} \int_0^{S_2} \sin(N_{p_5} y) \sin(N_{q_5} y) dy \int_0^{S_1} \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \sin(N_{u_1} x) dx \\
& + \left(\frac{4}{S_1 S_2} \right) \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} N_{q_6} \int_0^{S_2} \sin(N_{p_6} y) \sin(N_{q_6} y) dy \times \\
& \int_0^{S_1} \frac{\sinh[\lambda_{p_6 q_6} (S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \sin(N_{u_1} x) dx \\
& + \left(\frac{4}{S_1 S_2} \right) \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} N_{q_7} \int_0^{S_1} \sin(N_{p_7} x) \sin(N_{q_7} x) dx \int_0^{S_2} \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \sin(N_{v_1} y) dy \\
& + \left(\frac{4}{S_1 S_2} \right) \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} N_{q_8} \int_0^{S_1} \sin(N_{p_8} x) \sin(N_{q_8} x) dx \times \\
& \int_0^{S_2} \frac{\sinh[\lambda_{p_8 q_8} (S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \sin(N_{v_1} y) dy. \tag{3.218}
\end{aligned}$$

Solving the above integrals, we finally get the equation

$$\begin{aligned}
& \left(\frac{K_{z_1}}{K_{z_2}} \right) E_{kl(2)} \lambda_{kl} + H_{i_2 j_2(2)} \lambda_{i_2 j_2} \tanh[\lambda_{i_2 j_2} (H_6 - H_5)] = \\
& \left(\frac{2}{S_1} \right) \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} N_{q_5} \left[\frac{N_{u_1}^2}{N_{u_1}^2 + \lambda_{p_5 q_5}^2} \right] \left[\frac{-\cos(N_{u_1} S_1)}{N_{u_1}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2}{S_1} \right) \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} N_{q_6} \left[\frac{N_{u_1}}{N_{u_1}^2 + \lambda_{p_6 q_6}^2} \right] \\
& + \left(\frac{2}{S_2} \right) \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} N_{q_7} \left[\frac{N_{v_1}^2}{N_{v_1}^2 + \lambda_{p_7 q_7}^2} \right] \left[\frac{-\cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
& + \left(\frac{2}{S_2} \right) \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} N_{q_8} \left[\frac{N_{v_1}}{N_{v_1}^2 + \lambda_{p_8 q_8}^2} \right], \tag{3.219}
\end{aligned}$$

where $u_1 = k = i_2 = p_7 = p_8$, $v_1 = l = j_2 = p_5 = p_6$ and $Q_5 = Q_6 = Q_7 = Q_8 \rightarrow \infty$. In the same way, equating the hydraulic head expressions of the middle and bottom layers at $z = H_6$ [boundary condition (Va)] gives us the relation

$$\begin{aligned}
& \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \sin(N_{i_1} x) \sin(N_{j_1} y) \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \sin(N_{i_2} x) \sin(N_{j_2} y) \frac{1}{\cosh[\lambda_{i_2 j_2} (H_6 - H_5)]} \\
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \sin(N_{p_5} y) \sin[N_{q_5} (H_6 - H_5)] \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \\
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \sin(N_{p_6} y) \sin[N_{q_6} (H_6 - H_5)] \frac{\sinh[\lambda_{p_6 q_6} (S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \sin(N_{p_7} x) \sin[N_{q_7} (H_6 - H_5)] \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \sin(N_{p_8} x) \sin[N_{q_8} (H_6 - H_5)] \frac{\sinh[\lambda_{p_8 q_8} (S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} - H_5 \\
& = \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(2)} \sin(N_{i_3} x) \sin(N_{j_3} y) \coth[\lambda_{i_3 j_3} (h - H_6)] - H_1.
\end{aligned}$$

Performing a double Fourier expansion in $0 < x < S_1$ and $0 < y < S_2$ on the above equation will now yield the expression

$$\begin{aligned}
& -G_{i_1 j_1(2)} - H_{i_2 j_2(2)} \operatorname{sech}[\lambda_{i_2 j_2} (H_6 - H_5)] + P_{i_3 j_3(2)} \coth[\lambda_{i_3 j_3} (h - H_6)] = \\
& \left(\frac{4}{S_1 S_2} \right) \left[\sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \sin[N_{q_5} (H_6 - H_5)] \times \right. \\
& \quad \int_0^{S_2} \sin(N_{p_5} y) \sin(N_{v_1} y) dy \int_0^{S_1} \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \sin(N_{u_1} x) dx \\
& \quad \left. + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \sin[N_{q_6} (H_6 - H_5)] \times \right.
\end{aligned}$$

$$\begin{aligned}
& \int_0^{S_2} \sin(N_{p_5} y) \sin(N_{v_1} y) dy \int_0^{S_1} \frac{\sinh[\lambda_{p_6 q_6} (S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \sin(N_{u_1} x) dx \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \sin[N_{q_7} (H_6 - H_5)] \times \\
& \int_0^{S_1} \sin(N_{u_1} x) \sin(N_{p_7} x) dx \int_0^{S_2} \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \sin(N_{v_1} y) dy \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \sin[N_{q_8} (H_6 - H_5)] \times \\
& \int_0^{S_1} \sin(N_{u_1} x) \sin(N_{p_8} x) dx \int_0^{S_2} \frac{\sinh[\lambda_{p_8 q_8} (S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \sin(N_{v_1} y) dy \\
& + (H_1 - H_5) \int_0^{S_1} \sin(N_{u_1} x) dx \int_0^{S_2} \sin(N_{v_1} y) dy \Big]. \tag{3.220}
\end{aligned}$$

Solving the above integrals, we get

$$\begin{aligned}
& -G_{i_1 j_1(2)} - H_{i_2 j_2(2)} \operatorname{sech}[\lambda_{i_2 j_2} (H_6 - H_5)] + P_{i_3 j_3(2)} \operatorname{coth}[\lambda_{i_3 j_3} (h - H_6)] = \\
& + \left(\frac{2}{S_1}\right) \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \sin[N_{q_5} (H_6 - H_5)] \left[\frac{N_{u_1}^2}{N_{u_1}^2 + \lambda_{p_5 q_5}^2} \right] \left[\frac{-\cos(N_{u_1} S_1)}{N_{u_1}} \right] \\
& + \left(\frac{2}{S_1}\right) \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \sin[N_{q_6} (H_6 - H_5)] \left[\frac{N_{u_1}}{N_{u_1}^2 + \lambda_{p_6 q_6}^2} \right] \\
& + \left(\frac{2}{S_2}\right) \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \sin[N_{q_7} (H_6 - H_5)] \left[\frac{N_{v_1}^2}{N_{v_1}^2 + \lambda_{p_7 q_7}^2} \right] \left[\frac{-\cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
& + \left(\frac{2}{S_2}\right) \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \sin[N_{q_8} (H_6 - H_5)] \left[\frac{N_{v_1}}{N_{v_1}^2 + \lambda_{p_8 q_8}^2} \right] \\
& + \left(\frac{4}{S_1 S_2}\right) (H_1 - H_5) \left[\frac{1 - \cos(N_{u_1} S_1)}{N_{u_1}} \right] \left[\frac{1 - \cos(N_{v_1} S_2)}{N_{v_1}} \right] \tag{3.221}
\end{aligned}$$

where $u_1 = p_7 = p_8 = i_1 = i_2 = i_3$, $v_1 = p_5 = p_6 = j_1 = j_2 = j_3$ and $Q_5 = Q_6 = Q_7 = Q_8 \rightarrow \infty$. Also, boundary condition (Vb) leads us to the relation

$$\begin{aligned}
& -K_{z_2} \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \sin(N_{i_1} x) \sin(N_{j_1} y) \lambda_{i_1 j_1} \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] \\
& = -K_{z_3} \left[-\sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(2)} \lambda_{i_3 j_3} \sin(N_{i_3} x) \sin(N_{j_3} y) \right].
\end{aligned}$$

Simplifying the above equation, we get

$$\left(\frac{K_{z_2}}{K_{z_3}}\right) G_{i_1 j_1(2)} \lambda_{i_1 j_1} \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] + P_{i_3 j_3(2)} \lambda_{i_3 j_3} = 0 \tag{3.222}$$

where $i_1 = i_3$ and $j_1 = j_3$. Eqs. (3.217), (3.219), (3.221) and (3.222) can now be solved simultaneously to evaluate the Fourier coefficients $E_{kl(2)}$, $G_{i_1 j_1(2)}$, $H_{i_2 j_2(2)}$ and $P_{i_3 j_3(2)}$ corresponding to a ditch drainage scenario of Fig. 3.10; thus, our boundary value problem of Fig. 3.10 corresponding to the steady state situation stands solved. We would again like to emphasize here that our steady state solution is exact for all possible combination of the flow parameters of Fig. 3.10 and does not depend on any predefined conditions for its validity.

3.2.2.2 Determination of Coefficients of the Transient State Part of the Solution [transient state solution is approximate and is applicable only when directional conductivities and specific storage of the layers of Fig. 3.10 satisfy the conditions $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2} = K_{x_3}/S_{s_3}$, $K_{y_1}/S_{s_1} = K_{y_2}/S_{s_2} = K_{y_3}/S_{s_3}$ and $K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0$ (but $K_{z_1} = K_{z_2} = K_{z_3} \neq 0$)]

Since the head expression of the steady state solution for the top soil layer remains identical (but with the exception that the Fourier coefficients to be used now are $B_{p_1 q_1(2)}$, $C_{p_2 q_2(2)}$, $D_{p_3 q_3(2)}$, $F_{p_4 q_4(2)}$, $E_{kl(2)}$ and $Q_{uv(2)}$ and not the corresponding coefficients of the previous problem) to that of the previous problem, the coefficients of the triple Fourier series $A_{l_1 m_1 n_1(2)}$ will also have a similar expression as that of $A_{l_1 m_1 n_1(1)}$ (Eq. 3.121). The same is also true for the coefficients $W_{l_3 m_3 n_3(2)}$ associated with the bottom layer since for this layer also the steady state head expression for the present problem is identical (again with the exception that the coefficients to be used for this expression are now $P_{i_3 j_3(2)}$ instead of $P_{i_3 j_3(1)}$). However, as the head expression for the middle layer is now different from that of the previous problem, the coefficients $R_{l_2 m_2 n_2(2)}$ for this drainage setting (i.e., for the flow problem of Fig. 3.10) will have a different expression from that of the previous problem; an expression for the same, in view of initial condition (II), can be written as

$$\begin{aligned} & \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) = 0, \\ & 0 < x < S_1, \quad 0 < y < S_2, \quad 0 < z < H_5, \\ & \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \\ & = - \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \frac{\cosh[\lambda_{i_1 j_1} (z - H_5)]}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} \sin(N_{i_1} x) \sin(N_{j_1} y) \\ & \quad - \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \frac{\cosh[\lambda_{i_2 j_2} (H_6 - z)]}{\cosh[\lambda_{i_2 j_2} (H_6 - H_5)]} \sin(N_{i_2} x) \sin(N_{j_2} y) \\ & \quad - \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \sin(N_{p_5} y) \sin[N_{q_5} (z - H_5)] \end{aligned}$$

$$\begin{aligned}
& - \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \frac{\sinh \left[\lambda_{p_6 q_6} (S_1 - x) \right]}{\sinh \left(\lambda_{p_6 q_6} S_1 \right)} \sin \left(N_{p_6} y \right) \sin \left[N_{q_6} (z - H_5) \right] \\
& - \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \frac{\sinh \left(\lambda_{p_7 q_7} y \right)}{\sinh \left(\lambda_{p_7 q_7} S_2 \right)} \sin \left(N_{p_7} x \right) \sin \left[N_{q_7} (z - H_5) \right] \\
& - \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \frac{\sinh \left[\lambda_{p_8 q_8} (S_2 - y) \right]}{\sinh \left(\lambda_{p_8 q_8} S_2 \right)} \sin \left(N_{p_8} x \right) \sin \left[N_{q_8} (z - H_5) \right] - (-H_5),
\end{aligned}$$

$$0 < x < S_1,$$

$$0 < y < S_2,$$

$$H_5 < z < H_6,$$

$$\sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} \sin \left(N_{l_2} x \right) \sin \left(N_{m_2} y \right) \sin \left(N_{n_2} z \right) = 0,$$

$$0 < x < S_1,$$

$$0 < y < S_2,$$

$$H_6 < z < h.$$

This allows us to evaluate the coefficients $R_{l_2 m_2 n_2(2)}$ by running a triple Fourier series in the concerned domain; the required expression thus for the determination of the same can be expressed as

$$\begin{aligned}
R_{l_2 m_2 n_2(2)} = & \left(\frac{8}{S_1 S_2 h} \right) \left\{ - \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \int_{H_5}^{H_6} \frac{\cosh \left[\lambda_{i_1 j_1} (z - H_5) \right]}{\cosh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right]} \sin \left(N_{n_2} z \right) dz \times \right. \\
& \int_0^{S_1} \sin \left(N_{i_1} x \right) \sin \left(N_{l_2} x \right) dx \times \int_0^{S_2} \sin \left(N_{j_1} y \right) \sin \left(N_{m_2} y \right) dy \\
& - \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \int_{H_5}^{H_6} \frac{\cosh \left[\lambda_{i_2 j_2} (H_6 - z) \right]}{\cosh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right]} \sin \left(N_{n_2} z \right) dz \times \\
& \int_0^{S_1} \sin \left(N_{i_2} x \right) \sin \left(N_{l_2} x \right) dx \times \int_0^{S_2} \sin \left(N_{j_2} y \right) \sin \left(N_{m_2} y \right) dy \\
& - \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \int_0^{S_1} \frac{\sinh \left(\lambda_{p_5 q_5} x \right)}{\sinh \left(\lambda_{p_5 q_5} S_1 \right)} \sin \left(N_{l_2} x \right) dx \times \\
& \int_0^{S_2} \sin \left(N_{p_5} y \right) \sin \left(N_{m_2} y \right) dy \int_{H_5}^{H_6} \sin \left[N_{q_5} (z - H_5) \right] \sin \left(N_{n_2} z \right) dz \\
& - \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \int_0^{S_1} \frac{\sinh \left[\lambda_{p_6 q_6} (S_1 - x) \right]}{\sinh \left(\lambda_{p_6 q_6} S_1 \right)} \sin \left(N_{l_2} x \right) dx \times \\
& \int_0^{S_2} \sin \left(N_{p_6} y \right) \sin \left(N_{m_2} y \right) dy \int_{H_5}^{H_6} \sin \left[N_{q_6} (z - H_5) \right] \sin \left(N_{n_2} z \right) dz \\
& - \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \int_0^{S_2} \frac{\sinh \left(\lambda_{p_7 q_7} y \right)}{\sinh \left(\lambda_{p_7 q_7} S_2 \right)} \sin \left(N_{m_2} y \right) dy \times \\
& \int_0^{S_1} \sin \left(N_{p_7} x \right) \sin \left(N_{l_2} x \right) dx \int_{H_5}^{H_6} \sin \left[N_{q_7} (z - H_5) \right] \sin \left(N_{n_2} z \right) dz
\end{aligned}$$

$$\begin{aligned}
& - \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \int_0^{S_2} \frac{\sinh \left[\lambda_{p_8 q_8} (S_2 - y) \right]}{\sinh \left(\lambda_{p_8 q_8} S_2 \right)} \sin \left(N_{m_2} y \right) dy \times \\
& \int_0^{S_1} \sin \left(N_{p_8} x \right) \sin \left(N_{l_2} x \right) dx \int_{H_5}^{H_6} \sin \left[N_{q_8} (z - H_5) \right] \sin \left(N_{n_2} z \right) dz \\
& + H_5 \int_0^{S_1} \sin \left(N_{l_2} x \right) dx \times \int_0^{S_2} \sin \left(N_{m_2} y \right) dy \times \int_{H_5}^{H_6} \sin \left(N_{n_2} z \right) dz \left. \right\}.
\end{aligned} \tag{3.223}$$

Simplifying the above integrals, we finally get $R_{l_2 m_2 n_2(2)}$ as

$$\begin{aligned}
R_{l_2 m_2 n_2(2)} &= \left(\frac{8}{S_1 S_2 h} \right) \left\{ -G_{i_1 j_1(2)} \left(\frac{S_1}{2} \right) \left(\frac{S_2}{2} \right) \left[\frac{N_{n_2}^2}{N_{n_2}^2 + \lambda_{i_1 j_1}^2} \right] \times \right. \\
& \left[-\frac{\cos \left(N_{n_2} H_6 \right)}{N_{n_2}} + \left(\frac{\lambda_{i_1 j_1}}{N_{n_2}^2} \right) \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \sin \left(N_{n_2} H_6 \right) \right. \\
& \left. + \frac{\cos \left(N_{n_2} H_5 \right)}{N_{n_2}} \frac{1}{\cosh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right]} \right] \\
& - H_{i_2 j_2(2)} \left(\frac{S_1}{2} \right) \left(\frac{S_2}{2} \right) \left[\frac{N_{n_2}^2}{N_{n_2}^2 + \lambda_{i_2 j_2}^2} \right] \times \\
& \left[\frac{\cos \left(N_{n_2} H_5 \right)}{N_{n_2}} - \frac{\cos \left(N_{n_2} H_6 \right)}{N_{n_2}} \frac{1}{\cosh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right]} + \right. \\
& \left. \left(\frac{\lambda_{i_2 j_2}}{N_{n_2}^2} \right) \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \sin \left(N_{n_2} H_5 \right) \right] \\
& - \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \left[\frac{N_{l_2}^2}{N_{l_2}^2 + \lambda_{p_5 q_5}^2} \right] \left\{ \frac{-\cos \left(N_{l_2} S_1 \right)}{N_{l_2}} \right\} \left(\frac{S_2}{2} \right) \times \\
& \left\{ \frac{\sin \left[N_{n_2} H_6 - N_{q_5} (H_6 - H_5) \right] - \sin \left(N_{n_2} H_5 \right)}{2 \left(N_{n_2} - N_{q_5} \right)} \right. \\
& \left. + \frac{\sin \left(N_{n_2} H_5 \right) - \sin \left[N_{n_2} H_6 + N_{q_5} (H_6 - H_5) \right]}{2 \left(N_{n_2} + N_{q_5} \right)} \right\} \\
& - \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \left[\frac{N_{l_2}^2}{N_{l_2}^2 + \lambda_{p_6 q_6}^2} \right] \left(\frac{S_2}{2} \right) \times \\
& \left\{ \frac{\sin \left[N_{n_2} H_6 - N_{q_6} (H_6 - H_5) \right] - \sin \left(N_{n_2} H_5 \right)}{2 \left(N_{n_2} - N_{q_6} \right)} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{\sin(N_{n_2} H_5) - \sin[N_{n_2} H_6 + N_{q_6} (H_6 - H_5)]}{2(N_{n_2} + N_{q_6})} \Bigg\} \\
& - \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \left[\frac{N_{m_2}^2}{N_{m_2}^2 + \lambda_{p_7 q_7}^2} \right] \left[\frac{-\cos(N_{m_2} S_2)}{N_{m_2}} \right] \left(\frac{S_1}{2} \right) \times \\
& \left\{ \frac{\sin[N_{n_2} H_6 - N_{q_7} (H_6 - H_5)] - \sin(N_{n_2} H_5)}{2(N_{n_2} - N_{q_7})} \right. \\
& + \frac{\sin(N_{n_2} H_5) - \sin[N_{n_2} H_6 + N_{q_7} (H_6 - H_5)]}{2(N_{n_2} + N_{q_7})} \Bigg\} \\
& - \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \left[\frac{N_{m_2}}{N_{m_2}^2 + \lambda_{p_8 q_8}^2} \right] \left(\frac{S_1}{2} \right) \times \\
& \left\{ \frac{\sin[N_{n_2} H_6 - N_{q_8} (H_6 - H_5)] - \sin(N_{n_2} H_5)}{2(N_{n_2} - N_{q_8})} \right. \\
& + \frac{\sin(N_{n_2} H_5) - \sin[N_{n_2} H_6 + N_{q_8} (H_6 - H_5)]}{2(N_{n_2} + N_{q_8})} \Bigg\} \\
& + H_5 \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \Bigg\}, \quad (3.224)
\end{aligned}$$

where $l_2 = p_7 = p_8$ and $m_2 = p_5 = p_6$ and $Q_5 = Q_6 = Q_7 = Q_8 \rightarrow \infty$. The discharge and volume expressions pertaining to this problem can also be determined by following exactly the same procedures as have been done in the previous problem. The expressions for $Q_{top(2)}(t)$ and $Q_{top(2)}^f(x, y, t)$ shall remain identical to $Q_{top(1)}(t)$ and $Q_{top(1)}^f(x, y, t)$ of the previous problem [Eqs. (3.131) and (3.133)]. However, the Fourier coefficients are to be $B_{p_1 q_1(2)}$, $C_{p_2 q_2(2)}$, $D_{p_3 q_3(2)}$, $F_{p_4 q_4(2)}$, $E_{kl(2)}$, $Q_{uv(2)}$, $A_{l_1 m_1 n_1(2)}$, $R_{l_2 m_2 n_2(2)}$ and $W_{l_3 m_3 n_3(2)}$ in place of $B_{p_1 q_1(1)}$, $C_{p_2 q_2(1)}$, $D_{p_3 q_3(1)}$, $F_{p_4 q_4(1)}$, $E_{kl(1)}$, $Q_{uv(1)}$, $A_{l_1 m_1 n_1(1)}$, $R_{l_2 m_2 n_2(1)}$ and $W_{l_3 m_3 n_3(1)}$. The discharge expressions for the four different ditch faces for the current problem now work out as

$$\begin{aligned}
Q_{North(2)}(t) = & K_{x_1} \left\{ \sum_{p_1=1}^P \sum_{q_1=1}^{Q_1} B_{p_1 q_1(2)} \left[\frac{\lambda_{p_1 q_1}}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \left[\frac{1 - \cos(N_{p_1} S_2)}{N_{p_1}} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\
& - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(2)} \lambda_{p_2 q_2} \coth(\lambda_{p_2 q_2} S_1) \left[\frac{1 - \cos(N_{p_2} S_2)}{N_{p_2}} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(2)} \left(\frac{N_{p_3}}{\lambda_{p_3 q_3}} \right) \left[\frac{\cosh(\lambda_{p_3 q_3} S_2) - 1}{\sinh(\lambda_{p_3 q_3} S_2)} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(2)} \left(\frac{N_{p_4}}{\lambda_{p_4 q_4}} \right) \left[\frac{\cosh(\lambda_{p_4 q_4} S_2) - 1}{\sinh(\lambda_{p_4 q_4} S_2)} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(2)} \left(\frac{N_k}{\lambda_{kl}} \right) \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_l S_2)}{N_l} \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(2)} \left(\frac{N_u}{\lambda_{uv}} \right) \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_v S_2)}{N_v} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
& + K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \left(\frac{N_{i_1}}{\lambda_{i_1 j_1}} \right) \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] \left[\frac{1 - \cos(N_{j_1} S_2)}{N_{j_1}} \right] \right. \\
& \quad + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \left(\frac{N_{i_2}}{\lambda_{i_2 j_2}} \right) \tanh[\lambda_{i_2 j_2} (H_6 - H_5)] \left[\frac{1 - \cos(N_{j_2} S_2)}{N_{j_2}} \right] \\
& \quad + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \left[\frac{\lambda_{p_5 q_5}}{\sinh(\lambda_{p_5 q_5} S_1)} \right] \left[\frac{1 - \cos(N_{p_5} S_2)}{N_{p_5}} \right] \times \\
& \quad \left. \left[\frac{1 - \cos[N_{q_5} (H_6 - H_5)]}{N_{q_5}} \right] \right\} \\
& - \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \lambda_{p_6 q_6} \coth(\lambda_{p_6 q_6} S_1) \left[\frac{1 - \cos(N_{p_6} S_2)}{N_{p_6}} \right] \times \\
& \quad \left[\frac{1 - \cos[N_{q_6} (H_6 - H_5)]}{N_{q_6}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7q_7(2)} \left(\frac{N_{p_7}}{\lambda_{p_7q_7}} \right) \left\{ \frac{1 - \cos[N_{q_7}(H_6 - H_5)]}{N_{q_7}} \right\} \left[\frac{\cosh(\lambda_{p_7q_7} S_2) - 1}{\sinh(\lambda_{p_7q_7} S_2)} \right] \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8q_8(2)} \left(\frac{N_{p_8}}{\lambda_{p_8q_8}} \right) \left\{ \frac{1 - \cos[N_{q_8}(H_6 - H_5)]}{N_{q_8}} \right\} \left[\frac{\cosh(\lambda_{p_8q_8} S_2) - 1}{\sinh(\lambda_{p_8q_8} S_2)} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1m_1n_1(2)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \quad \exp\left[-(\lambda_{l_1m_1n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2m_2n_2(2)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \quad \exp\left[-(\lambda_{l_2m_2n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3m_3n_3(2)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \quad \exp\left[-(\lambda_{l_3m_3n_3})^2 t\right] \Big\} \\
& + K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3j_3(2)} \left(\frac{N_{i_3}}{\lambda_{i_3j_3}} \right) \left[\frac{1 - \cos(N_{j_3} S_2)}{N_{j_3}} \right] \right. \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1m_1n_1(2)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \quad \exp\left[-(\lambda_{l_1m_1n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2m_2n_2(2)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \quad \exp\left[-(\lambda_{l_2m_2n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3m_3n_3(2)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \quad \left. \exp\left[-(\lambda_{l_3m_3n_3})^2 t\right] \right\}, \tag{3.225}
\end{aligned}$$

$$\begin{aligned}
Q_{South(2)}(t) = & K_{x_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1q_1(2)} \lambda_{p_1q_1} \coth(\lambda_{p_1q_1} S_1) \left[\frac{1 - \cos(N_{p_1} S_2)}{N_{p_1}} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\
& - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2q_2(2)} \left[\frac{\lambda_{p_2q_2}}{\sinh(\lambda_{p_2q_2} S_1)} \right] \left[\frac{1 - \cos(N_{p_2} S_2)}{N_{p_2}} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3q_3(2)} \left[\frac{N_{p_3} \cos(N_{p_3} S_1)}{\lambda_{p_3q_3}} \right] \left[\frac{\cosh(\lambda_{p_3q_3} S_2) - 1}{\sinh(\lambda_{p_3q_3} S_2)} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\
& + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4q_4(2)} \left[\frac{N_{p_4} \cos(N_{p_4} S_1)}{\lambda_{p_4q_4}} \right] \left[\frac{\cosh(\lambda_{p_4q_4} S_2) - 1}{\sinh(\lambda_{p_4q_4} S_2)} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(2)} \left[\frac{N_k \cos(N_k S_1)}{\lambda_{kl}} \right] \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_l S_2)}{N_l} \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(2)} \left[\frac{N_u \cos(N_u S_1)}{\lambda_{uv}} \right] \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_v S_2)}{N_v} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1m_1n_1(2)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\
& \exp \left[-(\lambda_{l_1m_1n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2m_2n_2(2)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\
& \exp \left[-(\lambda_{l_2m_2n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3m_3n_3(2)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \\
& \left. \exp \left[-(\lambda_{l_3m_3n_3})^2 t \right] \right\} \\
& + K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1j_1(2)} \left[\frac{N_{i_1} \cos(N_{i_1} S_1)}{\lambda_{i_1j_1}} \right] \tanh[\lambda_{i_1j_1} (H_6 - H_5)] \left[\frac{1 - \cos(N_{j_1} S_2)}{N_{j_1}} \right] \right. \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2j_2(2)} \left[\frac{N_{i_2} \cos(N_{i_2} S_1)}{\lambda_{i_2j_2}} \right] \tanh[\lambda_{i_2j_2} (H_6 - H_5)] \left[\frac{1 - \cos(N_{j_2} S_2)}{N_{j_2}} \right] \\
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5q_5(2)} \lambda_{p_5q_5} \coth(\lambda_{p_5q_5} S_1) \left[\frac{1 - \cos(N_{p_5} S_2)}{N_{p_5}} \right] \times
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{1 - \cos[N_{q_5}(H_6 - H_5)]}{N_{q_5}} \right\} \\
& - \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \left[\frac{\lambda_{p_6 q_6}}{\sinh(\lambda_{p_6 q_6} S_1)} \right] \left[\frac{1 - \cos(N_{p_6} S_2)}{N_{p_6}} \right] \times \\
& \left\{ \frac{1 - \cos[N_{q_6}(H_6 - H_5)]}{N_{q_6}} \right\} \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \left[\frac{N_{p_7} \cos(N_{p_7} S_1)}{\lambda_{p_7 q_7}} \right] \left\{ \frac{1 - \cos[N_{q_7}(H_6 - H_5)]}{N_{q_7}} \right\} \times \\
& \left[\frac{\cosh(\lambda_{p_7 q_7} S_2) - 1}{\sinh(\lambda_{p_7 q_7} S_2)} \right] \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \left[\frac{N_{p_8} \cos(N_{p_8} S_1)}{\lambda_{p_8 q_8}} \right] \left\{ \frac{1 - \cos[N_{q_8}(H_6 - H_5)]}{N_{q_8}} \right\} \times \\
& \left[\frac{\cosh(\lambda_{p_8 q_8} S_2) - 1}{\sinh(\lambda_{p_8 q_8} S_2)} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \times \\
& \left\{ \frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right\} \times \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \times \\
& \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \times \\
& \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \\
& + K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(2)} \left[\frac{N_{i_3} \cos(N_{i_3} S_1)}{\lambda_{i_3 j_3}} \right] \left[\frac{1 - \cos(N_{j_3} S_2)}{N_{j_3}} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Bigg\}, \tag{3.226}
\end{aligned}$$

$$\begin{aligned}
Q_{East(2)}(t) = & K_{y_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(2)} \left(\frac{N_{p_1}}{\lambda_{p_1 q_1}} \right) \left[\frac{\cosh(\lambda_{p_1 q_1} S_1) - 1}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\
& + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(2)} \left(\frac{N_{p_2}}{\lambda_{p_2 q_2}} \right) \left[\frac{\cosh(\lambda_{p_2 q_2} S_1) - 1}{\sinh(\lambda_{p_2 q_2} S_1)} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(2)} \left[\frac{\lambda_{p_3 q_3}}{\sinh(\lambda_{p_3 q_3} S_2)} \right] \left[\frac{1 - \cos(N_{p_3} S_1)}{N_{p_3}} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\
& - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(2)} \lambda_{p_4 q_4} \coth(\lambda_{p_4 q_4} S_2) \left[\frac{1 - \cos(N_{p_4} S_1)}{N_{p_4}} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(2)} \left(\frac{N_l}{\lambda_{kl}} \right) \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_k S_1)}{N_k} \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(2)} \left(\frac{N_v}{\lambda_{uv}} \right) \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_u S_1)}{N_u} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{m_1} \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{m_2} \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{m_3} \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \\
& \quad \left. \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \right\} \\
& + K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \left(\frac{N_{j_1}}{\lambda_{i_1 j_1}} \right) \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_1} S_1)}{N_{i_1}} \right] \right. \\
& \quad + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \left(\frac{N_{j_2}}{\lambda_{i_2 j_2}} \right) \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_2} S_1)}{N_{i_2}} \right] \\
& \quad + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \left(\frac{N_{p_5}}{\lambda_{p_5 q_5}} \right) \left[\frac{\cosh(\lambda_{p_5 q_5} S_1) - 1}{\sinh(\lambda_{p_5 q_5} S_1)} \right] \left\{ \frac{1 - \cos[N_{q_5} (H_6 - H_5)]}{N_{q_5}} \right\} \\
& \quad + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \left(\frac{N_{p_6}}{\lambda_{p_6 q_6}} \right) \left[\frac{\cosh(\lambda_{p_6 q_6} S_1) - 1}{\sinh(\lambda_{p_6 q_6} S_1)} \right] \left\{ \frac{1 - \cos[N_{q_6} (H_6 - H_5)]}{N_{q_6}} \right\} \\
& \quad + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \left[\frac{\lambda_{p_7 q_7}}{\sinh(\lambda_{p_7 q_7} S_2)} \right] \left\{ \frac{1 - \cos[N_{q_7} (H_6 - H_5)]}{N_{q_7}} \right\} \times \\
& \quad \left[\frac{1 - \cos(N_{p_7} S_1)}{N_{p_7}} \right] \\
& \quad - \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \lambda_{p_8 q_8} \coth(\lambda_{p_8 q_8} S_2) \left\{ \frac{1 - \cos[N_{q_8} (H_6 - H_5)]}{N_{q_8}} \right\} \times \\
& \quad \left[\frac{1 - \cos(N_{p_8} S_1)}{N_{p_8}} \right] \\
& \quad + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{m_1} \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \times \\
& \quad \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& \quad + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{m_2} \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \times \\
& \quad \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{m_3} \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \\
& \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
& + K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(2)} \left(\frac{N_{j_3}}{\lambda_{i_3 j_3}} \right) \left[\frac{1 - \cos(N_{i_3} S_1)}{N_{i_3}} \right] \right. \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{m_1} \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] + \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{m_2} \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] + \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{m_3} \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_3}} \right] \times \\
& \left. \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \right\} \tag{3.227}
\end{aligned}$$

and

$$\begin{aligned}
Q_{West(2)}(t) = & K_{y_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(2)} \left[\frac{N_{p_1} \cos(N_{p_1} S_2)}{\lambda_{p_1 q_1}} \right] \left[\frac{\cosh(\lambda_{p_1 q_1} S_1) - 1}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\
& + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(2)} \left[\frac{N_{p_2} \cos(N_{p_2} S_2)}{\lambda_{p_2 q_2}} \right] \left[\frac{\cosh(\lambda_{p_2 q_2} S_1) - 1}{\sinh(\lambda_{p_2 q_2} S_1)} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(2)} \lambda_{p_3 q_3} \coth(\lambda_{p_3 q_3} S_2) \left[\frac{1 - \cos(N_{p_3} S_1)}{N_{p_3}} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\
& - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(2)} \left[\frac{\lambda_{p_4 q_4}}{\sinh(\lambda_{p_4 q_4} S_2)} \right] \left[\frac{1 - \cos(N_{p_4} S_1)}{N_{p_4}} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(2)} \left[\frac{N_l \cos(N_l S_2)}{\lambda_{kl}} \right] \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_k S_1)}{N_k} \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(2)} \left[\frac{N_v \cos(N_v S_2)}{\lambda_{uv}} \right] \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_u S_1)}{N_u} \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \} \\
& + K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(2)} \left[\frac{N_{j_1} \cos(N_{j_1} S_2)}{\lambda_{i_1 j_1}} \right] \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_1} S_1)}{N_{i_1}} \right] \right. \\
& \quad + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(2)} \left[\frac{N_{j_2} \cos(N_{j_2} S_2)}{\lambda_{i_2 j_2}} \right] \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_2} S_1)}{N_{i_2}} \right] \\
& \quad + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(2)} \left[\frac{N_{p_5} \cos(N_{p_5} S_2)}{\lambda_{p_5 q_5}} \right] \left[\frac{\cosh(\lambda_{p_5 q_5} S_1) - 1}{\sinh(\lambda_{p_5 q_5} S_1)} \right] \left\{ \frac{1 - \cos[N_{q_5} (H_6 - H_5)]}{N_{q_5}} \right\} \\
& \quad + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(2)} \left[\frac{N_{p_6} \cos(N_{p_6} S_2)}{\lambda_{p_6 q_6}} \right] \left[\frac{\cosh(\lambda_{p_6 q_6} S_1) - 1}{\sinh(\lambda_{p_6 q_6} S_1)} \right] \left\{ \frac{1 - \cos[N_{q_6} (H_6 - H_5)]}{N_{q_6}} \right\} \\
& \quad + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(2)} \lambda_{p_7 q_7} \coth(\lambda_{p_7 q_7} S_2) \left\{ \frac{1 - \cos[N_{q_7} (H_6 - H_5)]}{N_{q_7}} \right\} \times \\
& \quad \left[\frac{1 - \cos(N_{p_7} S_1)}{N_{p_7}} \right] \\
& \quad - \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(2)} \left[\frac{\lambda_{p_8 q_8}}{\sinh(\lambda_{p_8 q_8} S_2)} \right] \left\{ \frac{1 - \cos[N_{q_8} (H_6 - H_5)]}{N_{q_8}} \right\} \times \\
& \quad \left[\frac{1 - \cos(N_{p_8} S_1)}{N_{p_8}} \right] \\
& \quad + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \times
\end{aligned}$$

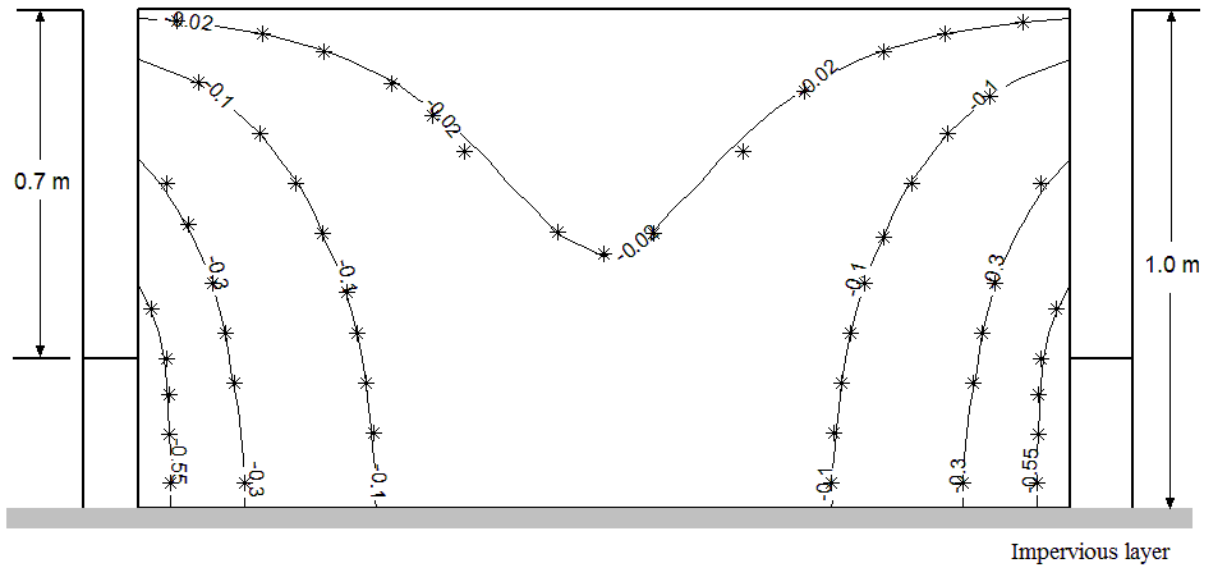
$$\begin{aligned}
& \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \times \\
& \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \\
& \left. \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \right\} \\
& + K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(2)} \left[\frac{N_{j_3} \cos(N_{j_3} S_2)}{\lambda_{i_3 j_3}} \right] \left[\frac{1 - \cos(N_{i_3} S_1)}{N_{i_3}} \right] \right. \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(2)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(2)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(2)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \left. \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \right\}, \tag{3.228}
\end{aligned}$$

where $Q_{North(2)}$, $Q_{South(2)}$, $Q_{East(2)}$ and $Q_{West(2)}$ are the discharge expressions pertaining to the Northern, Southern, Eastern and Western boundaries of the flow problem of Fig. 3.10.

3.2.2.3 Verifications of the Proposed Solution

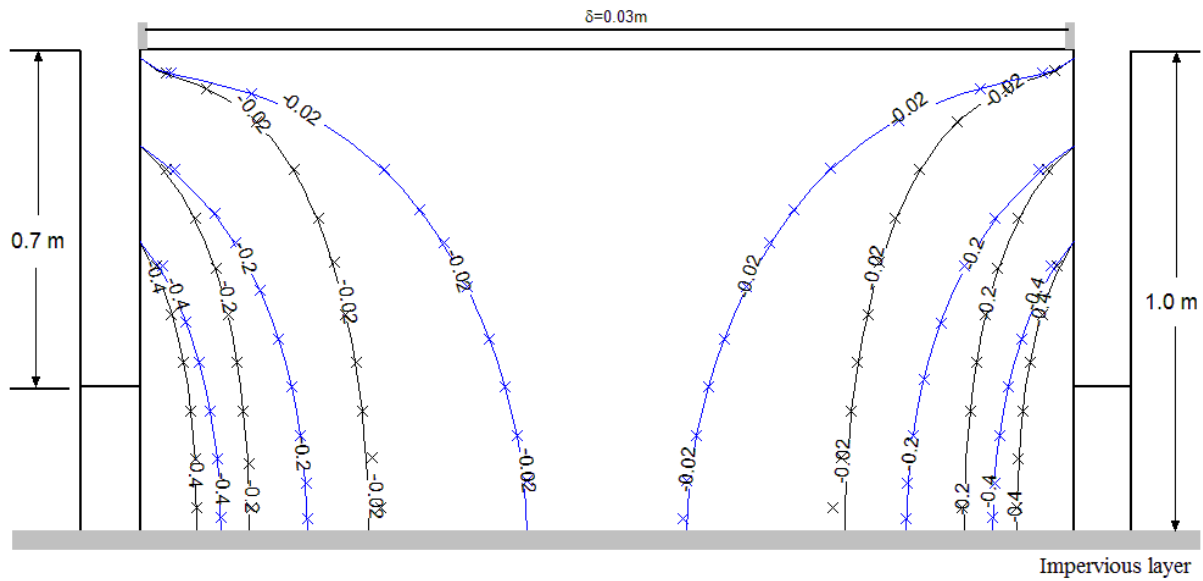
Like in the previous solution, here also a battery of checks has been carried out to determine the validity of the proposed solution both for the steady as well as transient situations. Further, the current three-dimensional solution can also be made applicable for a two-dimensional situation by following exactly a similar procedure as has been mentioned in the

previous problem. The $Q_{top(2)}/2Kh$ ratio for a two-dimensional flow situation located at a section 500 m mid-way between the Northern and Southern boundaries of Fig. 3.10 is now working out as 0.732 when the other flow parameters of Fig. 3.10 are taken as $S_1 = 1000$ m, $S_2 = 100$ m, $h = 3$ m, $H_1 = 2.7$ m, $H_5 = 2.55$ m, $H_6 = 2.85$ m, $\delta = 0$ m and $K_x = K_y = K_z = 0.05$ m/day. As this ratio from Fukuda's (1957) and Youngs' (1994) solutions for the same flow setting are turning out to be 0.743 and 0.742, respectively, values quite close to the one predicted by our model, the developed solution thus can be considered as correctly developed. It is worth noting that Fukuda found this value experimentally to be 0.72 and the close matching of this value with our predicted value can then also be taken as an experimental verification of the developed solution. Figs. 3.12, 3.13 and 3.14 also show comparison of our solution with the analytical works of Kirkham (1965), Barua and Alam (2013) and Sarmah and Barua (2017) for a few drainage settings of Fig. 3.10. As can be seen, in all these cases, the predictions from our solution are found to match closely with the corresponding values of others, thereby showing that the proposed solution is a correct one. Further, like in the previous model, MODFLOW models were also drawn and checks were carried out on the proposed solution for both steady as well as transient state of simulations of the system; Figs. 3.15 and 3.16 show comparisons of numerically obtained heads with the ones obtained by our solution when the parameters of Fig. 3.10 are taken as shown. Here also, our analytically obtained heads can be seen to be matching very closely with the corresponding values obtained by numerical means, thereby providing us with a further verification about the validity of the proposed solution.



- * Hydraulic heads as obtained from the proposed solution
- Hydraulic heads as obtained from Kirkham's solution

Fig. 3.11. Comparison of steady state hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.10 with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters of the problem are considered as $S_1 = 15$ m , $S_2 = 5$ m , $h = 1$ m , $H_1 = 0.7$ m , $H_5 = 0.4$ m , $H_6 = 0.8$ m , $\delta = 0$ m and $K_{x_1} = K_{x_2} = K_{x_3} = K_{y_1} = K_{y_2} = K_{y_3} = K_{z_1} = K_{z_2} = K_{z_3} = 1$ m/day



- × Hydraulic heads as obtained from the proposed solution (100 s)
- Hydraulic heads as obtained from Barua and Alam's solution (100 s)
- × Hydraulic heads as obtained from the proposed solution (500 s)
- Hydraulic heads as obtained from Barua and Alam's solution (500 s)

Fig. 3.12. Comparison of transient hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1 / 2$) between the Northern and the Southern boundaries of Fig. 3.10 at times $t=100$ s and 500 s with the corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem are considered as $S_1 = 15$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.7$ m, $H_5 = 0.35$ m, $H_6 = 0.8$ m, $\delta = 0.03$ m, $K_{x_1} = K_{x_2} = K_{x_3} = 1.5$ m/day, $K_{y_1} = K_{y_2} = K_{y_3} = 1.5$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 1$ m/day and $S_{s_1} = S_{s_2} = S_{s_3} = 0.01$ m⁻¹

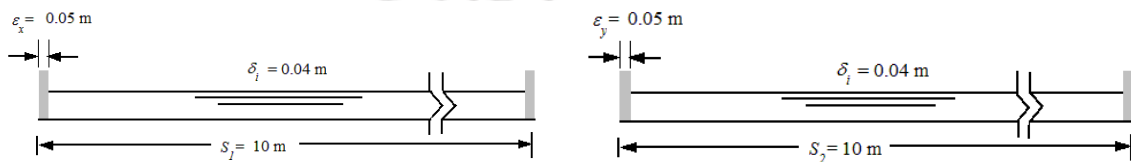
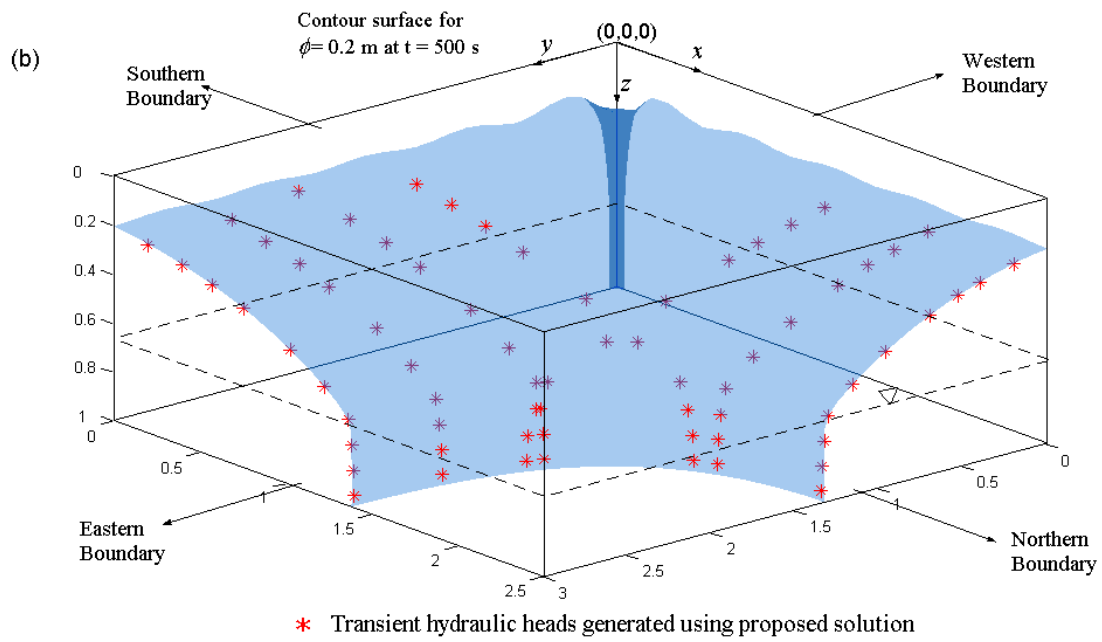
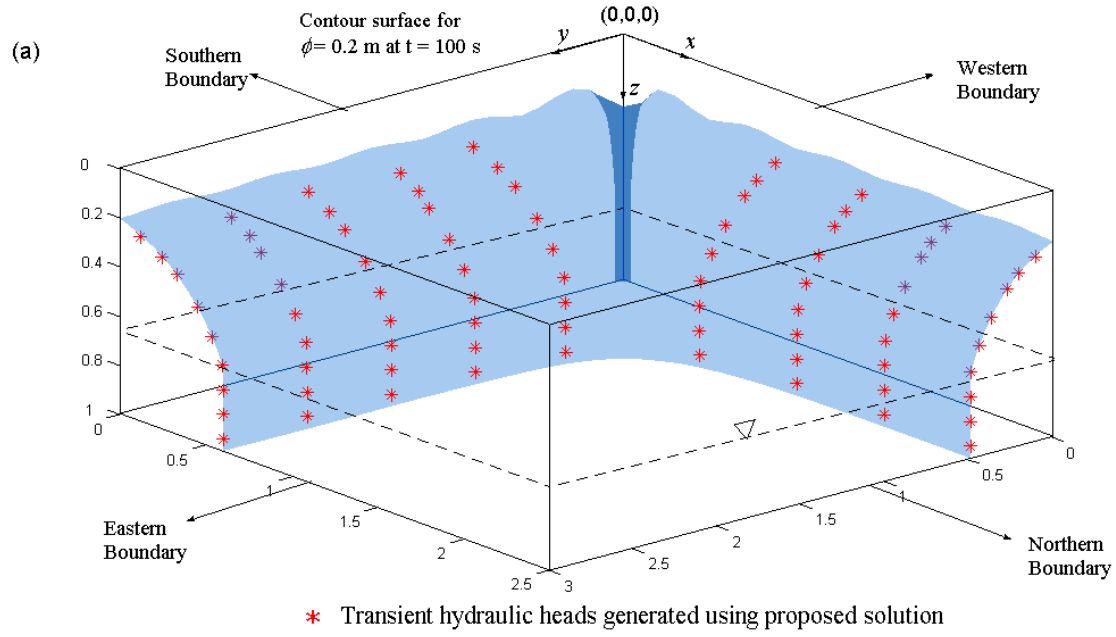


Fig 3.13. Comparison of transient hydraulic heads as obtained from the proposed solution at times $t = 100$ s and 500 s with the corresponding values as obtained from Sarmah and Barua's (2017) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem (Fig. 3.10) are considered as $S_1 = 6$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.65$ m, $H_5 = 0.35$ m, $H_6 = 0.75$ m, $\delta = 0.04$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = K_{x_2} = K_{x_3} = 1$ m/day, $K_{y_1} = K_{y_2} = K_{y_3} = 1.5$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.002$ m/day and $S_{s_1} = S_{s_2} = S_{s_3} = 0.01$ m⁻¹

3.2.2.4 MODFLOW Verifications of the Proposed Solution

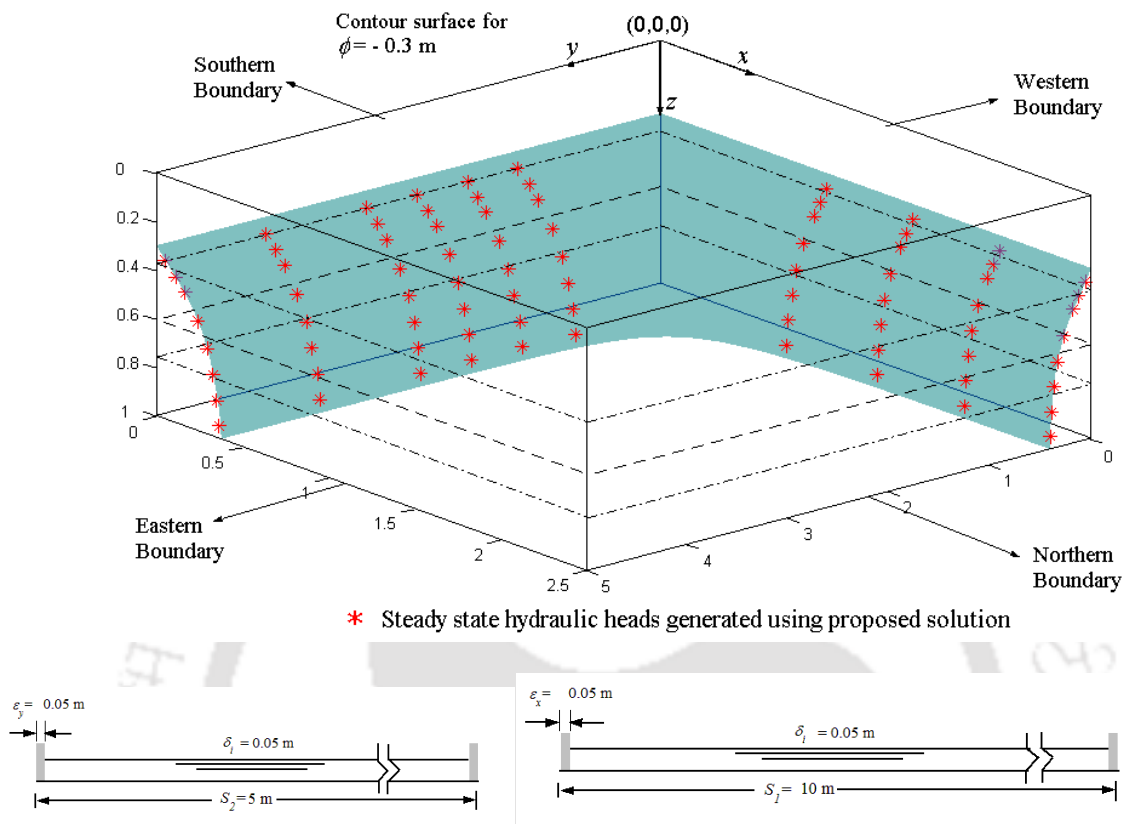


Fig. 3.14. Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding results as obtained from MODFLOW when the flow parameters of Fig. 3.10 are considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.6$ m, $H_5 = 0.35$ m, $H_6 = 0.75$ m, $\delta = 0.05$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = 1$ m/day, $K_{y_1} = 2$ m/day, $K_{z_1} = 1.5$ m/day, $K_{x_2} = 1.2$ m/day, $K_{y_2} = 0.8$ m/day, $K_{z_2} = 0.9$ m/day, $K_{x_3} = 0.5$ m/day, $K_{y_3} = 0.5$ m/day and $K_{z_3} = 0.6$ m/day

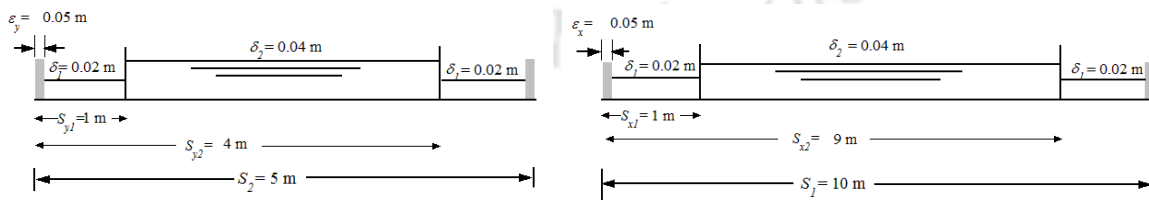
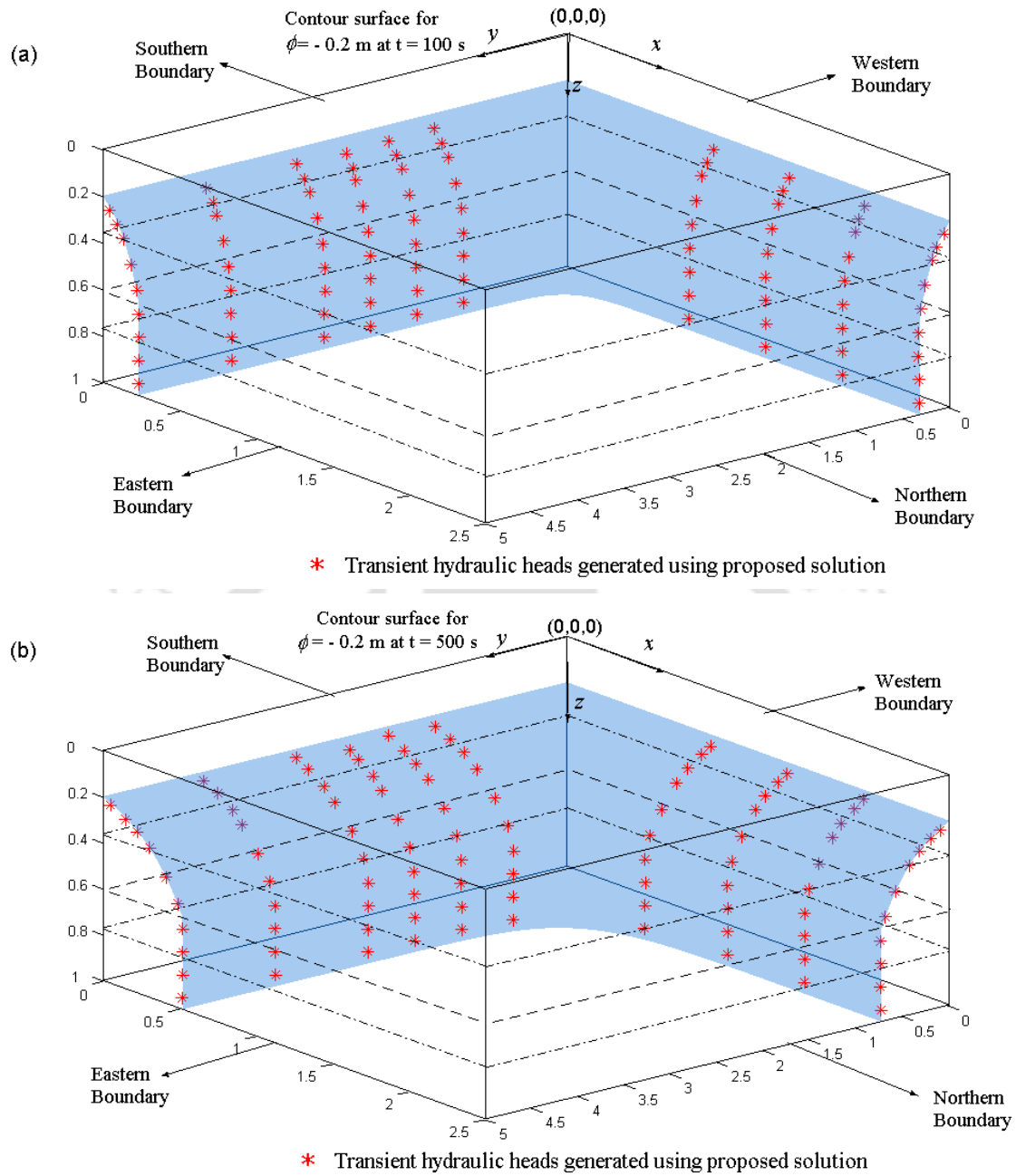


Fig 3.15. Comparison of transient hydraulic heads as obtained from the proposed solution at times $t=100$ s and 500 s with the corresponding values as obtained from MODFLOW when the flow parameters of Fig. 3.10 are considered as $S_1=10$ m, $S_2=5$ m, $h=1$ m, $H_1=0.6$ m, $H_5=0.35$ m, $H_6=0.75$ m, $\delta_1=0.02$ m, $\delta_2=0.04$ m, $\epsilon_x=\epsilon_y=0.05$ m, $d_{x1}=1$ m, $d_{x2}=9$ m, $d_{y1}=1$ m, $d_{y2}=4$ m, $K_{x1}=1.5$ m/day, $K_{y1}=0.75$ m/day, $K_{x2}=2$ m/day, $K_{y2}=1$ m/day, $K_{x3}=0.75$ m/day, $K_{y3}=0.375$ m/day, $K_{z1}=K_{z2}=K_{z3}=0.001$ m/day, $S_{s1}=0.03$ m⁻¹, $S_{s2}=0.04$ m⁻¹ and $S_{s3}=0.015$ m⁻¹

3.2.3 Level of Water in the Ditches is below the Boundary between the Middle Soil Layer and the Bottom Soil Layer

We have now come to the last of the cases of the drainage problem of Fig. 3.1 where the level of water in the ditches lies in the bottom layer. Fig. 3.16 shows such a configuration where the level of water can be seen to be below the top horizontal boundary of the bottom layer. Like before, the soil is being introduced to a variable ponding distributing at the top of the soil with the help of a network of inner bunds.

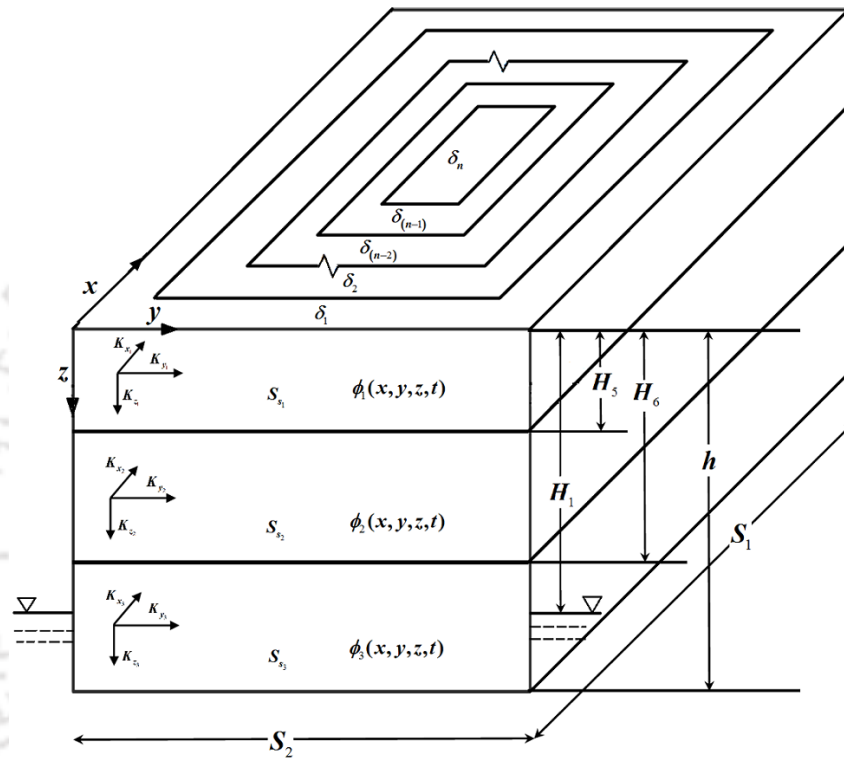


Fig. 3.16. Three dimensional ditch drainage system for a three-layered soil when the height of water in the ditches is below the boundary between middle and bottom soil layers

The boundary conditions specific to this problem can be represented as

$$\phi_{1(3)}(x, y, z, t > 0) = -z, \quad x = 0, \quad 0 < y < S_2, \quad 0 < z \leq H_5, \quad (\text{XXXII})$$

$$\phi_{2(3)}(x, y, z, t > 0) = -z, \quad x = 0, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_6, \quad (\text{XXXIII})$$

$$\phi_{3(3)}(x, y, z, t > 0) = -z, \quad x = 0, \quad 0 < y < S_2, \quad H_6 \leq z \leq H_1, \quad (\text{XXXIVa})$$

$$\phi_{3(3)}(x, y, z, t > 0) = -H_1, \quad x = 0, \quad 0 < y < S_2, \quad H_1 \leq z < h, \quad (\text{XXXIVb})$$

$$\phi_{1(3)}(x, y, z, t > 0) = -z, \quad x = S_1, \quad 0 < y < S_2, \quad 0 < z \leq H_5, \quad (\text{XXXV})$$

$$\phi_{2(3)}(x, y, z, t > 0) = -z, \quad x = S_1, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_6, \quad (\text{XXXVI})$$

$$\phi_{3(3)}(x, y, z, t > 0) = -z, \quad x = S_1, \quad 0 < y < S_2, \quad H_6 \leq z \leq H_1, \quad (\text{XXXVIIa})$$

$$\phi_{3(3)}(x, y, z, t > 0) = -H_1, \quad x = S_1, \quad 0 < y < S_2, \quad H_1 \leq z < h, \quad (\text{XXXVIIb})$$

$$\phi_{1(3)}(x, y, z, t > 0) = -z, \quad 0 < x < S_1, \quad y = 0, \quad 0 < z \leq H_5, \quad (\text{XXXVIII})$$

$$\phi_{2(3)}(x, y, z, t > 0) = -z, \quad 0 < x < S_1, \quad y = 0, \quad H_5 \leq z \leq H_6, \quad (\text{XII})$$

$$\phi_{3(3)}(x, y, z, t > 0) = -z, \quad 0 \leq x \leq S_1, \quad y = 0, \quad H_6 \leq z \leq H_1, \quad (\text{XLa})$$

$$\phi_{3(3)}(x, y, z, t > 0) = -H_1, \quad 0 \leq x \leq S_1, \quad y = 0, \quad H_1 \leq z < h, \quad (\text{XLb})$$

$$\phi_{1(3)}(x, y, z, t > 0) = -z, \quad 0 < x < S_1, \quad y = S_2, \quad 0 < z \leq H_5, \quad (\text{XLI})$$

$$\phi_{2(3)}(x, y, z, t) = -z, \quad 0 < x < S_1, \quad y = S_2, \quad H_5 \leq z \leq H_6, \quad (\text{XLII})$$

$$\phi_{3(3)}(x, y, z, t) = -z, \quad 0 \leq x \leq S_1, \quad y = S_2, \quad H_6 \leq z \leq H_1, \quad (\text{XLIIIa})$$

$$\phi_{3(3)}(x, y, z, t) = -H_1, \quad 0 \leq x \leq S_1, \quad y = S_2, \quad H_1 \leq z < h. \quad (\text{XLIIIb})$$

The hydraulic head expressions for the top and middle layers will now be identical to those in the previous flow problem; however, for the bottom layer, the head function, in view of Eqs. (3.27), (3.28) and (3.29), will be different. The head expressions for this flow problem can be expressed as

$$\begin{aligned} \phi_{1(3)}(x, y, z, t) = & \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(3)} \frac{\sinh(\lambda_{p_1 q_1} x)}{\sinh(\lambda_{p_1 q_1} S_1)} \sin(N_{p_1} y) \sin(N_{q_1} z) \\ & + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(3)} \frac{\sinh[\lambda_{p_2 q_2} (S_1 - x)]}{\sinh(\lambda_{p_2 q_2} S_1)} \sin(N_{p_2} y) \sin(N_{q_2} z) \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3(3)} \frac{\sinh(\lambda_{p_3 q_3} y)}{\sinh(\lambda_{p_3 q_3} S_2)} \sin(N_{p_3} x) \sin(N_{q_3} z) \\ & + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4(3)} \frac{\sinh[\lambda_{p_4 q_4} (S_2 - y)]}{\sinh(\lambda_{p_4 q_4} S_2)} \sin(N_{p_4} x) \sin(N_{q_4} z) \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(3)} \frac{\sinh(\lambda_{kl} z)}{\cosh(\lambda_{kl} H_5)} \sin(N_k x) \sin(N_l y) \\ & + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(3)} \frac{\cosh[\lambda_{uv} (H_5 - z)]}{\cosh(\lambda_{uv} H_5)} \sin(N_u x) \sin(N_v y) \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right], \end{aligned} \quad (3.229)$$

where

$$N_{p_1} = \left(\frac{p_1 \pi}{S_2} \right), \quad (3.230)$$

$$N_{q_1} = \left[\left(\frac{1-2q_1}{2} \right) \frac{\pi}{H_5} \right], \quad (3.231)$$

$$N_{p_2} = \left(\frac{p_2 \pi}{S_2} \right), \quad (3.232)$$

$$N_{q_2} = \left[\left(\frac{1-2q_2}{2} \right) \frac{\pi}{H_5} \right], \quad (3.233)$$

$$N_{p_3} = \left(\frac{p_3 \pi}{S_1} \right), \quad (3.234)$$

$$N_{q_3} = \left[\left(\frac{1-2q_3}{2} \right) \frac{\pi}{H_5} \right], \quad (3.235)$$

$$N_{p_4} = \left(\frac{p_4 \pi}{S_1} \right), \quad (3.236)$$

$$N_{q_4} = \left[\left(\frac{1-2q_4}{2} \right) \frac{\pi}{H_5} \right], \quad (3.237)$$

$$N_k = \left(\frac{k \pi}{S_1} \right), \quad (3.238)$$

$$N_l = \left(\frac{l \pi}{S_2} \right), \quad (3.239)$$

$$N_u = \left(\frac{u \pi}{S_1} \right), \quad (3.240)$$

$$N_v = \left(\frac{v \pi}{S_2} \right), \quad (3.241)$$

$$(\lambda_{p_1 q_1})^2 = (N_{p_1}^2 K_{xy}^{a1} + N_{q_1}^2 K_{xz}^{a1}), \quad (3.242)$$

$$(\lambda_{p_2 q_2})^2 = (N_{p_2}^2 K_{xy}^{a1} + N_{q_2}^2 K_{xz}^{a1}), \quad (3.243)$$

$$(\lambda_{p_3 q_3})^2 = \left[(N_{p_3}^2 / K_{xy}^{a1}) + N_{q_3}^2 K_{yz}^{a1} \right], \quad (3.244)$$

$$(\lambda_{p_4 q_4})^2 = \left[(N_{p_4}^2 / K_{xy}^{a1}) + N_{q_4}^2 K_{yz}^{a1} \right], \quad (3.245)$$

$$(\lambda_{kl})^2 = \left[(N_k^2 / K_{xz}^{a1}) + (N_l^2 / K_{yz}^{a1}) \right], \quad (3.246)$$

$$(\lambda_{uv})^2 = \left[(N_u^2 / K_{xz}^{a1}) + (N_v^2 / K_{yz}^{a1}) \right], \quad (3.247)$$

$$K_{xy}^{a1} = \left(\frac{K_{y_1}}{K_{x_1}} \right), \quad (3.248)$$

$$K_{xz}^{a1} = \left(\frac{K_{z_1}}{K_{x_1}} \right), \quad (3.249)$$

$$K_{yz}^{a1} = \begin{pmatrix} K_{z_1} \\ K_{y_1} \end{pmatrix}, \quad (3.250)$$

$$N_{l_1} = \left(\frac{l_1 \pi}{S_1} \right), \quad (3.251)$$

$$N_{m_1} = \left(\frac{m_1 \pi}{S_2} \right), \quad (3.252)$$

$$N_{n_1} = \left[\left(\frac{1-2n_1}{2} \right) \frac{\pi}{h} \right], \quad (3.253)$$

$$N_{l_2} = \left(\frac{l_2 \pi}{S_1} \right), \quad (3.254)$$

$$N_{m_2} = \left(\frac{m_2 \pi}{S_2} \right), \quad (3.255)$$

$$N_{n_2} = \left[\left(\frac{1-2n_2}{2} \right) \frac{\pi}{h} \right], \quad (3.256)$$

$$N_{l_3} = \left(\frac{l_3 \pi}{S_1} \right), \quad (3.257)$$

$$N_{m_3} = \left(\frac{m_3 \pi}{S_2} \right), \quad (3.258)$$

$$N_{n_3} = \left[\left(\frac{1-2n_3}{2} \right) \frac{\pi}{h} \right], \quad (3.259)$$

$$(\lambda_{l_1 m_1 n_1})^2 = \left[N_{l_1}^2 \left(\frac{K_{x_1}}{S_{s_1}} \right) + N_{m_1}^2 \left(\frac{K_{y_1}}{S_{s_1}} \right) + N_{n_1}^2 \left(\frac{K_{z_1}}{S_{s_1}} \right) \right], \quad (3.260)$$

$$(\lambda_{l_2 m_2 n_2})^2 = \left[N_{l_2}^2 \left(\frac{K_{x_2}}{S_{s_2}} \right) + N_{m_2}^2 \left(\frac{K_{y_2}}{S_{s_2}} \right) + N_{n_2}^2 \left(\frac{K_{z_2}}{S_{s_2}} \right) \right], \quad (3.261)$$

$$(\lambda_{l_3 m_3 n_3})^2 = \left[N_{l_3}^2 \left(\frac{K_{x_3}}{S_{s_3}} \right) + N_{m_3}^2 \left(\frac{K_{y_3}}{S_{s_3}} \right) + N_{n_3}^2 \left(\frac{K_{z_3}}{S_{s_3}} \right) \right] \quad (3.262)$$

and $B_{p_1 q_1(3)}$, $C_{p_2 q_2(3)}$, $D_{p_3 q_3(3)}$, $F_{p_4 q_4(3)}$, $E_{kl(3)}$, $Q_{uv(3)}$, $A_{l_1 m_1 n_1(3)}$, $R_{l_2 m_2 n_2(3)}$ and $W_{l_3 m_3 n_3(3)}$ are constants and

$$\begin{aligned} \phi_{2(3)}(x, y, z, t) = & \sum_{i=1}^{I_1} \sum_{j=1}^{J_1} G_{i_j(3)} \frac{\cosh[\lambda_{i_j}(z - H_5)]}{\cosh[\lambda_{i_j}(H_6 - H_5)]} \sin(N_i x) \sin(N_j y) \\ & + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(3)} \frac{\cosh[\lambda_{i_2 j_2}(H_6 - z)]}{\cosh[\lambda_{i_2 j_2}(H_6 - H_5)]} \sin(N_{i_2} x) \sin(N_{j_2} y) \end{aligned}$$

$$\begin{aligned}
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(3)} \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \sin(N_{p_5} y) \sin[N_{q_5} (z - H_5)] \\
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} \frac{\sinh[\lambda_{p_6 q_6} (S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \sin(N_{p_6} y) \sin[N_{q_6} (z - H_5)] \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(3)} \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \sin(N_{p_7} x) \sin[N_{q_7} (z - H_5)] \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} \frac{\sinh[\lambda_{p_8 q_8} (S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \sin(N_{p_8} x) \sin[N_{q_8} (z - H_5)] - H_5 \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp[-(\lambda_{l_1 m_1 n_1})^2 t] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp[-(\lambda_{l_2 m_2 n_2})^2 t] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp[-(\lambda_{l_3 m_3 n_3})^2 t], \quad (2.263)
\end{aligned}$$

where

$$N_{i_1} = \left(\frac{i_1 \pi}{S_1} \right), \quad (3.264)$$

$$N_{j_1} = \left(\frac{j_1 \pi}{S_2} \right), \quad (3.265)$$

$$N_{i_2} = \left(\frac{i_2 \pi}{S_1} \right), \quad (3.266)$$

$$N_{j_2} = \left(\frac{j_2 \pi}{S_2} \right), \quad (3.267)$$

$$N_{p_5} = \left(\frac{p_5 \pi}{S_2} \right), \quad (3.268)$$

$$N_{q_5} = \left[\left(\frac{1-2q_5}{2} \right) \left(\frac{\pi}{H_6 - H_5} \right) \right], \quad (3.269)$$

$$N_{p_6} = \left(\frac{p_6 \pi}{S_2} \right), \quad (3.270)$$

$$N_{q_6} = \left[\left(\frac{1-2q_6}{2} \right) \left(\frac{\pi}{H_6 - H_5} \right) \right], \quad (3.271)$$

$$N_{p_7} = \left(\frac{p_7 \pi}{S_1} \right), \quad (3.272)$$

$$N_{q_7} = \left[\left(\frac{1-2q_7}{2} \right) \left(\frac{\pi}{H_6 - H_5} \right) \right], \quad (3.273)$$

$$N_{p_8} = \left(\frac{p_8 \pi}{S_1} \right), \quad (3.274)$$

$$N_{q_8} = \left[\left(\frac{1-2q_8}{2} \right) \left(\frac{\pi}{H_6 - H_5} \right) \right], \quad (3.275)$$

$$(\lambda_{i_1 j_1})^2 = \left[(N_{i_1}^2 / K_{xz}^{a2}) + (N_{j_1}^2 / K_{yz}^{a2}) \right], \quad (3.276)$$

$$(\lambda_{i_2 j_2})^2 = \left[(N_{i_2}^2 / K_{xz}^{a2}) + (N_{j_2}^2 / K_{yz}^{a2}) \right], \quad (3.277)$$

$$(\lambda_{p_5 q_5})^2 = N_{p_5}^2 K_{xy}^{a2} + N_{q_5}^2 K_{xz}^{a2}, \quad (3.278)$$

$$(\lambda_{p_6 q_6})^2 = N_{p_6}^2 K_{xy}^{a2} + N_{q_6}^2 K_{xz}^{a2}, \quad (3.279)$$

$$(\lambda_{p_7 q_7})^2 = \left[(N_{p_7}^2 / K_{xy}^{a2}) + N_{q_7}^2 K_{yz}^{a2} \right], \quad (3.280)$$

$$(\lambda_{p_8 q_8})^2 = \left[(N_{p_8}^2 / K_{xy}^{a2}) + N_{q_8}^2 K_{yz}^{a2} \right], \quad (3.281)$$

$$K_{xz}^{a2} = \left(\frac{K_{z_2}}{K_{x_2}} \right), \quad (3.282)$$

$$K_{yz}^{a2} = \left(\frac{K_{z_2}}{K_{y_2}} \right), \quad (3.283)$$

$$K_{xy}^{a2} = \left(\frac{K_{y_2}}{K_{x_2}} \right) \quad (3.284)$$

and $G_{i_1 j_1(2)}$, $H_{i_2 j_2(2)}$, $I_{p_5 q_5(2)}$, $J_{p_6 q_6(2)}$, $K_{p_7 q_7(2)}$ and $L_{p_8 q_8(2)}$ are constants and

$$\begin{aligned} \phi_{3(3)}(x, y, z, t) = & \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \frac{\cosh[\lambda_{i_3 j_3}(h-z)]}{\sinh[\lambda_{i_3 j_3}(h-H_6)]} \sin(N_{i_3} x) \sin(N_{j_3} y) \\ & + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \frac{\sinh(\lambda_{p_9 q_9} x)}{\sinh(\lambda_{p_9 q_9} S_1)} \sin(N_{p_9} y) \sin[N_{q_9}(z-H_6)] \\ & + \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10} q_{10}(3)} \frac{\sinh[\lambda_{p_{10} q_{10}}(S_1-x)]}{\sinh(\lambda_{p_{10} q_{10}} S_1)} \sin(N_{p_{10}} y) \sin[N_{q_{10}}(z-H_6)] \\ & + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11} q_{11}(3)} \frac{\sinh(\lambda_{p_{11} q_{11}} y)}{\sinh(\lambda_{p_{11} q_{11}} S_2)} \sin(N_{p_{11}} x) \sin[N_{q_{11}}(z-H_6)] \\ & + \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12} q_{12}(3)} \frac{\sinh[\lambda_{p_{12} q_{12}}(S_2-y)]}{\sinh(\lambda_{p_{12} q_{12}} S_2)} \sin(N_{p_{12}} x) \sin[N_{q_{12}}(z-H_6)] - H_6 \end{aligned}$$

$$\begin{aligned}
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} \sin(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} \sin(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right], \quad (3.285)
\end{aligned}$$

where

$$N_{i_3} = \left(\frac{i_3 \pi}{S_1} \right), \quad (3.286)$$

$$N_{j_3} = \left(\frac{j_3 \pi}{S_2} \right), \quad (3.287)$$

$$N_{p_9} = \left(\frac{p_9 \pi}{S_2} \right), \quad (3.288)$$

$$N_{q_9} = \left[\left(\frac{1-2q_9}{2} \right) \frac{\pi}{h-H_6} \right], \quad (3.289)$$

$$N_{p_{10}} = \left(\frac{p_{10} \pi}{S_2} \right), \quad (3.290)$$

$$N_{q_{10}} = \left[\left(\frac{1-2q_{10}}{2} \right) \frac{\pi}{h-H_6} \right], \quad (3.291)$$

$$N_{p_{11}} = \left(\frac{p_{11} \pi}{S_1} \right), \quad (3.292)$$

$$N_{q_{11}} = \left[\left(\frac{1-2q_{11}}{2} \right) \frac{\pi}{h-H_6} \right], \quad (3.293)$$

$$N_{p_{12}} = \left(\frac{p_{12} \pi}{S_1} \right), \quad (3.294)$$

$$N_{q_{12}} = \left[\left(\frac{1-2q_{12}}{2} \right) \frac{\pi}{h-H_6} \right], \quad (3.295)$$

$$(\lambda_{i_3 j_3})^2 = \left[(N_{i_3}^2 / K_{xz}^{a3}) + (N_{j_3}^2 / K_{yz}^{a3}) \right], \quad (3.296)$$

$$(\lambda_{p_9 q_9})^2 = N_{p_9}^2 K_{xy}^{a3} + N_{q_9}^2 K_{xz}^{a3}, \quad (3.297)$$

$$(\lambda_{p_{10} q_{10}})^2 = N_{p_{10}}^2 K_{xy}^{a3} + N_{q_{10}}^2 K_{xz}^{a3}, \quad (3.298)$$

$$(\lambda_{p_{11} q_{11}})^2 = \left[(N_{p_{11}}^2 / K_{xy}^{a3}) + N_{q_{11}}^2 K_{yz}^{a3} \right], \quad (3.299)$$

$$(\lambda_{p_{12} q_{12}})^2 = \left[(N_{p_{12}}^2 / K_{xy}^{a3}) + N_{q_{12}}^2 K_{yz}^{a3} \right], \quad (3.300)$$

$$K_{xz}^{a3} = \left(\frac{K_{z_3}}{K_{x_3}} \right), \quad (3.301)$$

$$K_{yz}^{a3} = \left(\frac{K_{z_3}}{K_{y_3}} \right), \quad (3.302)$$

$$K_{xy}^{a3} = \left(\frac{K_{y_3}}{K_{x_3}} \right) \quad (3.303)$$

and $P_{i_3 j_3(3)}$, $M_{p_9 q_9(3)}$, $N_{p_{10} q_{10}(3)}$, $U_{p_{11} q_{11}(3)}$ and $V_{p_{12} q_{12}(3)}$ are constants. Also, the Darcian velocity expressions for the top layer, $V_{x1(3)}$, $V_{y1(3)}$ and $V_{z1(3)}$ will be identical to those as mentioned in Eqs. (3.99), (3.100), (3.101) for the first problem; however, the Fourier coefficients to be used now in these expressions are $B_{p_1 q_1(3)}$, $C_{p_2 q_2(3)}$, $D_{p_3 q_3(3)}$, $F_{p_4 q_4(3)}$, $E_{kl(3)}$, $Q_{uv(3)}$, $A_{l_1 m_1 n_1(3)}$, $R_{l_2 m_2 n_2(3)}$ and $W_{l_3 m_3 n_3(3)}$ and not the corresponding coefficients pertaining to the first problem. Similarly, Darcian velocity expressions for the middle layer, represented by $V_{x2(3)}$, $V_{y2(3)}$ and $V_{z2(3)}$ will be identical to the expressions mentioned in Eqs. (3.204), (3.205) and (3.206). But again, it should be noted that the Fourier coefficients to be used now for these velocity functions are $G_{i_1 j_1(3)}$, $H_{i_2 j_2(3)}$, $I_{p_5 q_5(3)}$, $J_{p_6 q_6(3)}$, $K_{p_7 q_7(3)}$, $L_{p_8 q_8(3)}$, $A_{l_1 m_1 n_1(3)}$, $R_{l_2 m_2 n_2(3)}$ and $W_{l_3 m_3 n_3(3)}$ and not the corresponding coefficients pertaining to the second problem. But, for the bottom layer, the velocity distribution functions, $V_{x3(3)}$, $V_{y3(3)}$ and $V_{z3(3)}$, along x -, y - and z -directions, respectively for the present case may be expressed as

$$\begin{aligned} V_{x3(3)} = & -K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \frac{\cosh[\lambda_{i_3 j_3}(h-z)]}{\sinh[\lambda_{i_3 j_3}(h-H_6)]} N_{i_3} \cos(N_{i_3} x) \sin(N_{j_3} y) \right. \\ & + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \frac{\lambda_{p_9 q_9} \cosh(\lambda_{p_9 q_9} x)}{\sinh(\lambda_{p_9 q_9} S_1)} \sin(N_{p_9} y) \sin[N_{q_9}(z-H_6)] \\ & - \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10} q_{10}(3)} \frac{\lambda_{p_{10} q_{10}} \cosh[\lambda_{p_{10} q_{10}}(S_1-x)]}{\sinh(\lambda_{p_{10} q_{10}} S_1)} \sin(N_{p_{10}} y) \sin[N_{q_{10}}(z-H_6)] \\ & + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11} q_{11}(3)} \frac{\sinh(\lambda_{p_{11} q_{11}} y)}{\sinh(\lambda_{p_{11} q_{11}} S_2)} N_{p_{11}} \cos(N_{p_{11}} x) \sin[N_{q_{11}}(z-H_6)] \\ & + \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12} q_{12}(3)} \frac{\sinh[\lambda_{p_{12} q_{12}}(S_2-y)]}{\sinh(\lambda_{p_{12} q_{12}} S_2)} N_{p_{12}} \cos(N_{p_{12}} x) \sin[N_{q_{12}}(z-H_6)] \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} N_{l_1} \cos(N_{l_1} x) \sin(N_{m_1} y) \sin(N_{n_1} z) \exp[-(\lambda_{l_1 m_1 n_1})^2 t] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} N_{l_2} \cos(N_{l_2} x) \sin(N_{m_2} y) \sin(N_{n_2} z) \exp[-(\lambda_{l_2 m_2 n_2})^2 t] \end{aligned}$$

$$+ \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} N_{l_3} \cos(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\}, \quad (3.304)$$

$$\begin{aligned} V_{y3(3)} = & -K_{y_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \frac{\cosh\left[\lambda_{i_3 j_3} (h-z)\right]}{\sinh\left[\lambda_{i_3 j_3} (h-H_6)\right]} \sin(N_{i_3} x) N_{j_3} \cos(N_{j_3} y) \right. \\ & + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \frac{\sinh(\lambda_{p_9 q_9} x)}{\sinh(\lambda_{p_9 q_9} S_1)} N_{p_9} \cos(N_{p_9} y) \sin[N_{q_9} (z-H_6)] \\ & + \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10} q_{10}(3)} \frac{\sinh[\lambda_{p_{10} q_{10}} (S_1-x)]}{\sinh(\lambda_{p_{10} q_{10}} S_1)} N_{p_{10}} \cos(N_{p_{10}} y) \sin[N_{q_{10}} (z-H_6)] \\ & + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11} q_{11}(3)} \frac{\lambda_{p_{11} q_{11}} \cosh(\lambda_{p_{11} q_{11}} y)}{\sinh(\lambda_{p_{11} q_{11}} S_2)} \sin(N_{p_{11}} x) \sin[N_{q_{11}} (z-H_6)] \\ & - \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12} q_{12}(3)} \frac{\lambda_{p_{12} q_{12}} \cosh[\lambda_{p_{12} q_{12}} (S_2-y)]}{\sinh(\lambda_{p_{12} q_{12}} S_2)} \sin(N_{p_{12}} x) \sin[N_{q_{12}} (z-H_6)] \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} \sin(N_{l_1} x) N_{m_1} \cos(N_{m_1} y) \sin(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} \sin(N_{l_2} x) N_{m_2} \cos(N_{m_2} y) \sin(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\ & + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} \sin(N_{l_3} x) N_{m_3} \cos(N_{m_3} y) \sin(N_{n_3} z) \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \Big\}, \quad (3.305) \end{aligned}$$

$$\begin{aligned} V_{z3(3)} = & -K_{z_3} \left\{ - \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \frac{\lambda_{i_3 j_3} \sinh[\lambda_{i_3 j_3} (h-z)]}{\sinh[\lambda_{i_3 j_3} (h-H_6)]} \sin(N_{i_3} x) \sin(N_{j_3} y) \right. \\ & + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \frac{\sinh(\lambda_{p_9 q_9} x)}{\sinh(\lambda_{p_9 q_9} S_1)} \sin(N_{p_9} y) N_{q_9} \cos[N_{q_9} (z-H_6)] \\ & + \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10} q_{10}(3)} \frac{\sinh[\lambda_{p_{10} q_{10}} (S_1-x)]}{\sinh(\lambda_{p_{10} q_{10}} S_1)} \sin(N_{p_{10}} y) N_{q_{10}} \cos[N_{q_{10}} (z-H_6)] \\ & + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11} q_{11}(3)} \frac{\sinh(\lambda_{p_{11} q_{11}} y)}{\sinh(\lambda_{p_{11} q_{11}} S_2)} \sin(N_{p_{11}} x) N_{q_{11}} \cos[N_{q_{11}} (z-H_6)] \\ & + \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12} q_{12}(3)} \frac{\sinh[\lambda_{p_{12} q_{12}} (S_2-y)]}{\sinh(\lambda_{p_{12} q_{12}} S_2)} \sin(N_{p_{12}} x) N_{q_{12}} \cos[N_{q_{12}} (z-H_6)] \\ & + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} \sin(N_{l_1} x) \sin(N_{m_1} y) N_{n_1} \cos(N_{n_1} z) \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\ & + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} \sin(N_{l_2} x) \sin(N_{m_2} y) N_{n_2} \cos(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \Big\} \end{aligned}$$

$$+ \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} W_{l_2 m_2 n_2(3)} \sin(N_{l_2} x) \sin(N_{m_2} y) N_{n_2} \cos(N_{n_2} z) \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \}. \quad (3.306)$$

3.2.3.1 Determination of the Steady State Solution Coefficients (steady state solution is exact and valid for all possible arrangement of parameters of Fig. 3.16)

As mentioned before, all our transient head expressions will reduce to steady state if the time variable in them is allowed to extend to infinity; as may be observed, this will make the exponential terms in these expressions to go to zero leaving behind only the steady state parts of head functions. The Fourier coefficients $B_{p_1 q_1(3)}$, $C_{p_2 q_2(3)}$, $D_{p_3 q_3(3)}$ and $F_{p_4 q_4(3)}$ have the same expressions as in the previous case for $B_{p_1 q_1(2)}$, $C_{p_2 q_2(2)}$, $D_{p_3 q_3(2)}$ and $F_{p_4 q_4(2)}$ and their expressions are as mentioned in Eqs. (3.208), (3.207), (3.210) and (3.209), respectively. The expression for the coefficient $Q_{uv(3)}$ will also remain the same as that of $Q_{uv(2)}$ since boundary conditions atop the field remain the same.

Now from boundary condition (XXXIII) and Eq. (3.263), we have the following relationship

$$\sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} \sin(N_{p_6} y) \sin[N_{q_6} (z - H_5)] = -(z - H_5), \quad x = 0, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_6.$$

Thus, the coefficients $J_{p_6 q_6(3)}$ can be determined by making a Fourier run in the concerned domain; the resulting equations works out as

$$J_{p_6 q_6(3)} = \left(\frac{2}{S_2}\right) \left(\frac{2}{H_6 - H_5}\right) \int_0^{S_2} \sin(N_{p_6} y) dy \times \left[\int_{H_5}^{H_6} -(z - H_5) \sin(N_{q_6} z - N_{q_6} H_5) dz \right]. \quad (3.307)$$

Solving the above integrals, we get

$$J_{p_6 q_6(3)} = \left(\frac{2}{S_2}\right) \left(\frac{2}{H_6 - H_5}\right) \left[\frac{1 - \cos(N_{p_6} S_2)}{N_{p_6}} \right] \left\{ \frac{-\sin[N_{q_6} (H_6 - H_5)]}{N_{q_6}} \right\}. \quad (3.308)$$

Similarly, by applying boundary condition (XXXVI) in Eq. (3.263), we get

$$\sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(3)} \sin(N_{p_5} y) \sin[N_{q_5} (z - H_5)] = -(z - H_5), \quad x = S_1, \quad 0 < y < S_2, \quad H_5 \leq z \leq H_6.$$

Fourier expansion in the relevant domain gives the final expression for $I_{p_5 q_5(3)}$ as

$$I_{p_5 q_5(3)} = \left(\frac{2}{S_2}\right) \left(\frac{2}{H_6 - H_5}\right) \left[\frac{1 - \cos(N_{p_5} S_2)}{N_{p_5}} \right] \left\{ \frac{-\sin[N_{q_5} (H_6 - H_5)]}{N_{q_5}} \right\}. \quad (3.309)$$

Also, from boundary condition (XIL) and Eq. (3.263), we obtain

$$\sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} \sin(N_{p_8} x) \sin[N_{q_8} (z - H_5)] = -(z - H_5), \quad 0 < x < S_1, \quad y = 0, \quad H_5 \leq z \leq H_6.$$

Solving for the coefficients $L_{p_8 q_8(3)}$ via Fourier expansion, we get

$$L_{p_8 q_8(3)} = \left(\frac{2}{S_1}\right) \left(\frac{2}{H_6 - H_5}\right) \left[\frac{1 - \cos(N_{p_8} S_1)}{N_{p_8}} \right] \left\{ \frac{-\sin[N_{q_8} (H_6 - H_5)]}{N_{q_8}} \right\}. \quad (3.310)$$

Similarly, by incorporating boundary condition (XLII) in Eq. (3.263) and performing the necessary Fourier expansion, we obtain $K_{p_7q_7(3)}$ as

$$K_{p_7q_7(3)} = \left(\frac{2}{S_1} \right) \left(\frac{2}{H_6 - H_5} \right) \left[\frac{1 - \cos(N_{p_7} S_1)}{N_{p_7}} \right] \left\{ \frac{-\sin[N_{q_7} (H_6 - H_5)]}{N_{q_7}} \right\}. \quad (3.311)$$

Implementation of boundary conditions (XXXIVa) and (XXXIVb) in Eq. (3.285) lead to the following relations

$$\begin{aligned} \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} \sin(N_{p_{10}} y) \sin[N_{q_{10}} (z - H_6)] &= -(z - H_6), \\ x = 0, \quad 0 < y < S_2, \quad H_6 \leq z \leq H_1, \\ \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} \sin(N_{p_{10}} y) \sin[N_{q_{10}} (z - H_6)] &= -(H_1 - H_6), \\ x = 0, \quad 0 < y < S_2, \quad H_1 \leq z < h. \end{aligned}$$

Solving for $N_{p_{10}q_{10}(3)}$ utilizing Fourier expansion, we get

$$N_{p_{10}q_{10}(3)} = \left(\frac{2}{S_2} \right) \left(\frac{2}{h - H_6} \right) \int_0^{S_2} \sin(N_{p_{10}} y) dy \times \left\{ \int_{H_6}^{H_1} -(z - H_6) \sin[N_{q_{10}} (z - H_6)] dz + \int_{H_1}^h -(H_1 - H_6) \sin[N_{q_{10}} (z - H_6)] dz \right\}. \quad (3.312)$$

Carrying out the above integrals, we find expression for $N_{p_{10}q_{10}(3)}$ as

$$N_{p_{10}q_{10}(3)} = \left(\frac{2}{S_2} \right) \left(\frac{2}{h - H_6} \right) \left[\frac{1 - \cos(N_{p_{10}} S_2)}{N_{p_{10}}} \right] \left\{ \frac{-\sin[N_{q_{10}} (H_1 - H_6)]}{N_{q_{10}}} \right\}. \quad (3.313)$$

Also, substitution of boundary conditions (XXXVIIa) and (XXXVIIb) in Eq. (3.285) give us the relations

$$\begin{aligned} \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9q_9} \sin(N_{p_9} y) \sin[N_{q_9} (z - H_6)] &= -(z - H_6), \\ x = 0, \quad 0 < y < S_2, \quad H_6 \leq z \leq H_1, \\ \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9q_9} \sin(N_{p_9} y) \sin[N_{q_9} (z - H_6)] &= -(H_1 - H_6), \\ x = 0, \quad 0 < y < S_2, \quad H_1 \leq z < h. \end{aligned}$$

By performing a Fourier expansion in the concerned domain and solving the ensuing integrals, we get $M_{p_9q_9(3)}$ as

$$M_{p_9q_9(3)} = \left(\frac{2}{S_2} \right) \left(\frac{2}{h - H_6} \right) \left[\frac{1 - \cos(N_{p_9} S_2)}{N_{p_9}} \right] \left\{ \frac{-\sin[N_{q_9} (H_1 - H_6)]}{N_{q_9}} \right\}. \quad (3.314)$$

Next, we implement boundary conditions (XLa) and (XLb) in Eq. (3.285); this gives us the following relations

$$\sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}} \sin(N_{p_{12}} x) \sin(N_{q_{12}} (z - H_6)) = -(z - H_6),$$

$$0 < x < S_1, \quad y = 0, \quad H_6 \leq z \leq H_1,$$

$$\sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}} \sin(N_{p_{12}} x) \sin(N_{q_{12}} (z - H_6)) = -(H_1 - H_6),$$

$$0 < x < S_1, \quad y = 0, \quad H_1 \leq z < h.$$

Solving for $V_{p_{12}q_{12}(3)}$ via Fourier expansion, we get

$$V_{p_{12}q_{12}(3)} = \left(\frac{2}{S_1} \right) \left(\frac{2}{h - H_6} \right) \left[\frac{1 - \cos(N_{p_{12}} S_1)}{N_{p_{12}}} \right] \left\{ \frac{-\sin[N_{q_{12}} (H_1 - H_6)]}{N_{q_{12}}} \right\}. \quad (3.315)$$

Similarly, from boundary conditions (XLIIIa) and (XLIIIb) and Eq. (3.285), we find $U_{p_{11}q_{11}(3)}$ as

$$U_{p_{11}q_{11}(3)} = \left(\frac{2}{S_1} \right) \left(\frac{2}{h - H_6} \right) \left[\frac{1 - \cos(N_{p_{11}} S_1)}{N_{p_{11}}} \right] \left\{ \frac{-\sin[N_{q_{11}} (H_1 - H_6)]}{N_{q_{11}}} \right\}. \quad (3.316)$$

Now, by equating the hydraulic heads of the top and the middle layer at $z = H_5$ [boundary condition (IVa)], we get the relation

$$\begin{aligned} & \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1q_1(3)} \sin(N_{p_1} y) \sin(N_{q_1} H_5) \frac{\sinh(\lambda_{p_1q_1} x)}{\sinh(\lambda_{p_1q_1} S_1)} \\ & + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2q_2(3)} \sin(N_{p_2} y) \sin(N_{q_2} H_5) \frac{\sinh[\lambda_{p_2q_2} (S_1 - x)]}{\sinh(\lambda_{p_2q_2} S_1)} \\ & + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3q_3(3)} \sin(N_{p_3} x) \sin(N_{q_3} H_5) \frac{\sinh(\lambda_{p_3q_3} y)}{\sinh(\lambda_{p_3q_3} S_2)} \\ & + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4q_4(3)} \sin(N_{p_4} x) \sin(N_{q_4} H_5) \frac{\sinh[\lambda_{p_4q_4} (S_2 - y)]}{\sinh(\lambda_{p_4q_4} S_2)} \\ & + \sum_{k=1}^K \sum_{l=1}^L E_{kl(3)} \sin(N_k x) \sin(N_l y) \tanh(\lambda_{kl} H_5) \\ & + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(3)} \sin(N_u x) \sin(N_v y) \frac{1}{\cosh(\lambda_{uv} H_5)} \\ & = \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1j_1(3)} \sin(N_{i_1} x) \sin(N_{j_1} y) \frac{1}{\cosh[\lambda_{i_1j_1} (H_6 - H_5)]} \\ & + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2j_2(3)} \sin(N_{i_2} x) \sin(N_{j_2} y) - H_5. \end{aligned}$$

Performing a double Fourier expansion in $0 < x < S_1$ and $0 < y < S_2$ in the above equations and solving the integrals that emerge, we get the expression

$$\begin{aligned}
& E_{kl(3)} \tanh(\lambda_{kl} H_5) - G_{i_1 j_1(3)} \frac{1}{\cosh[\lambda_{i_1 j_1} (H_6 - H_5)]} - H_{i_2 j_2(3)} + Q_{uv(3)} \frac{1}{\cosh(\lambda_{uv} H_5)} \\
&= -\left(\frac{4}{S_1 S_2}\right) H_5 \left[\frac{1 - \cos(N_{u_1} S_1)}{N_{u_1}} \right] \left[\frac{1 - \cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
&\quad - \left(\frac{2}{S_1}\right) \sum_{q_1=1}^{Q_1} B_{p_1 q_1(3)} \sin(N_{q_1} H_5) \left[\frac{N_{u_1}^2}{N_{u_1}^2 + \lambda_{p_1 q_1}^2} \right] \left[\frac{-\cos(N_{u_1} S_1)}{N_{u_1}} \right] \\
&\quad - \left(\frac{2}{S_1}\right) \sum_{q_2=1}^{Q_2} C_{p_2 q_2(3)} \sin(N_{q_2} H_5) \left[\frac{N_{u_1}}{N_{u_1}^2 + \lambda_{p_2 q_2}^2} \right] \\
&\quad - \left(\frac{2}{S_2}\right) \sum_{q_3=1}^{Q_3} D_{p_3 q_3(3)} \sin(N_{q_3} H_5) \left[\frac{N_{v_1}^2}{N_{v_1}^2 + \lambda_{p_3 q_3}^2} \right] \left[\frac{-\cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
&\quad - \left(\frac{2}{S_2}\right) \sum_{q_4=1}^{Q_4} F_{p_4 q_4(3)} \sin(N_{q_4} H_5) \left[\frac{N_{v_1}}{N_{v_1}^2 + \lambda_{p_4 q_4}^2} \right], \tag{3.317}
\end{aligned}$$

where $u_1 = p_3 = p_4 = k = i_1 = i_2 = u$, $v_1 = p_1 = p_2 = l = j_1 = j_2 = v$ and $Q_1 = Q_2 = Q_3 = Q_4 \rightarrow \infty$.

Equating the fluxes of the top and the middle layers at $z = H_5$ yields

$$\begin{aligned}
& -K_{z_1} \sum_{k=1}^K \sum_{l=1}^L E_{kl(3)} \lambda_{kl} \sin(N_k x) \sin(N_l y) = \\
& -K_{z_2} \left\{ -\sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(3)} \sin(N_{i_2} x) \sin(N_{j_2} y) \lambda_{i_2 j_2} \tanh[\lambda_{i_2 j_2} (H_6 - H_5)] \right. \\
& \quad + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(3)} N_{q_5} \sin(N_{p_5} y) \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \\
& \quad + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} N_{q_6} \sin(N_{p_6} y) \frac{\sinh[\lambda_{p_6 q_6} (S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \\
& \quad + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(3)} N_{q_7} \sin(N_{p_7} x) \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \\
& \quad \left. + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} N_{q_8} \sin(N_{p_8} x) \frac{\sinh[\lambda_{p_8 q_8} (S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \right\}.
\end{aligned}$$

From the above, we get

$$\begin{aligned}
& \left(\frac{K_{z_1}}{K_{z_2}} \right) E_{kl} \lambda_{kl(3)} + H_{i_2 j_2(3)} \lambda_{i_2 j_2} \tanh[\lambda_{i_2 j_2} (H_6 - H_5)] = \\
& \left(\frac{2}{S_1} \right) \sum_{q_5=1}^{Q_5} I_{p_5 q_5(3)} N_{q_5} \left[\frac{N_{u_1}^2}{N_{u_1}^2 + \lambda_{p_5 q_5}^2} \right] \left[\frac{-\cos(N_{u_1} S_1)}{N_{u_1}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2}{S_1} \right) \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} N_{q_6} \left[\frac{N_{u_1}}{N_{u_1}^2 + \lambda_{p_6 q_6}^2} \right] \\
& + \left(\frac{2}{S_2} \right) \sum_{q_7=1}^{Q_7} K_{p_7 q_7(3)} N_{q_7} \left[\frac{N_{v_1}^2}{N_{v_1}^2 + \lambda_{p_7 q_7}^2} \right] \left[\frac{-\cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
& + \left(\frac{2}{S_2} \right) \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} N_{q_8} \left[\frac{N_{v_1}}{N_{v_1}^2 + \lambda_{p_8 q_8}^2} \right], \tag{3.318}
\end{aligned}$$

where $u_1 = k = i_2 = p_7 = p_8$, $v_1 = l = j_2 = p_5 = p_6$ and $Q_5 = Q_6 = Q_7 = Q_8 \rightarrow \infty$. Equating the hydraulic heads of the middle and the bottom layer at $z = H_6$ [boundary condition (Va)], we get

$$\begin{aligned}
& \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(3)} \sin(N_{i_1} x) \sin(N_{j_1} y) \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2(3)} \sin(N_{i_2} x) \sin(N_{j_2} y) \frac{1}{\cosh[\lambda_{i_2 j_2} (H_6 - H_5)]} \\
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(3)} \sin(N_{p_5} y) \sin[N_{q_5} (H_6 - H_5)] \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \\
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} \sin(N_{p_6} y) \sin[N_{q_6} (H_6 - H_5)] \frac{\sinh[\lambda_{p_6 q_6} (S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(3)} \sin(N_{p_7} x) \sin[N_{q_7} (H_6 - H_5)] \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} \sin(N_{p_8} x) \sin[N_{q_8} (H_6 - H_5)] \frac{\sinh[\lambda_{p_8 q_8} (S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} - H_5 \\
& = \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \sin(N_{i_3} x) \sin(N_{j_3} y) \coth[\lambda_{i_3 j_3} (h - H_6)] - H_6.
\end{aligned}$$

A double Fourier run on the above equation gives us the relation

$$\begin{aligned}
& -G_{i_1 j_1(3)} - H_{i_2 j_2(3)} \operatorname{sech}[\lambda_{i_2 j_2} (H_6 - H_5)] + P_{i_3 j_3(3)} \coth[\lambda_{i_3 j_3} (h - H_6)] = \\
& \left(\frac{4}{S_1 S_2} \right) \left\{ \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5(3)} \sin[N_{q_5} (H_6 - H_5)] \times \right. \\
& \quad \int_0^{S_2} \sin(N_{p_5} y) \sin(N_{v_1} y) dy \int_0^{S_1} \frac{\sinh(\lambda_{p_5 q_5} x)}{\sinh(\lambda_{p_5 q_5} S_1)} \sin(N_{u_1} x) dx \\
& \quad \left. + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} \sin[N_{q_6} (H_6 - H_5)] \times \right.
\end{aligned}$$

$$\begin{aligned}
& \int_0^{S_2} \sin(N_{p_5} y) \sin(N_{v_1} y) dy \int_0^{S_1} \frac{\sinh[\lambda_{p_6 q_6} (S_1 - x)]}{\sinh(\lambda_{p_6 q_6} S_1)} \sin(N_{u_1} x) dx \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(3)} \sin[N_{q_7} (H_6 - H_5)] \times \\
& \int_0^{S_1} \sin(N_{u_1} x) \sin(N_{p_7} x) dx \int_0^{S_2} \frac{\sinh(\lambda_{p_7 q_7} y)}{\sinh(\lambda_{p_7 q_7} S_2)} \sin(N_{v_1} y) dy \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} \sin[N_{q_8} (H_6 - H_5)] \times \\
& \int_0^{S_1} \sin(N_{u_1} x) \sin(N_{p_8} x) dx \int_0^{S_2} \frac{\sinh[\lambda_{p_8 q_8} (S_2 - y)]}{\sinh(\lambda_{p_8 q_8} S_2)} \sin(N_{v_1} y) dy \\
& + (H_6 - H_5) \int_0^{S_1} \sin(N_{u_1} x) dx \int_0^{S_2} \sin(N_{v_1} y) dy \Big\}. \tag{3.319}
\end{aligned}$$

Solving the integrals, we finally get

$$\begin{aligned}
& -G_{i_1 j_1(3)} - H_{i_2 j_2(3)} \operatorname{sech}[\lambda_{i_2 j_2} (H_6 - H_5)] + P_{i_3 j_3(3)} \operatorname{coth}[\lambda_{i_3 j_3} (h - H_6)] = \\
& \left(\frac{2}{S_1} \right) \sum_{q_5=1}^{Q_5} I_{p_5 q_5(3)} \sin[N_{q_5} (H_6 - H_5)] \left[\frac{N_{u_1}^2}{N_{u_1}^2 + \lambda_{p_5 q_5}^2} \right] \left[\frac{-\cos(N_{u_1} S_1)}{N_{u_1}} \right] \\
& + \left(\frac{2}{S_1} \right) \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} \sin[N_{q_6} (H_6 - H_5)] \left[\frac{N_{u_1}}{N_{u_1}^2 + \lambda_{p_6 q_6}^2} \right] \\
& + \left(\frac{2}{S_2} \right) \sum_{q_7=1}^{Q_7} K_{p_7 q_7(3)} \sin[N_{q_7} (H_6 - H_5)] \left[\frac{N_{v_1}^2}{N_{v_1}^2 + \lambda_{p_7 q_7}^2} \right] \left[\frac{-\cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
& + \left(\frac{2}{S_2} \right) \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} \sin[N_{q_8} (H_6 - H_5)] \left[\frac{N_{v_1}}{N_{v_1}^2 + \lambda_{p_8 q_8}^2} \right] \\
& + \left(\frac{4}{S_1 S_2} \right) (H_6 - H_5) \left[\frac{1 - \cos(N_{u_1} S_1)}{N_{u_1}} \right] \left[\frac{1 - \cos(N_{v_1} S_2)}{N_{v_1}} \right], \tag{3.320}
\end{aligned}$$

where $u_1 = p_7 = p_8 = i_1 = i_2 = i_3$, $v_1 = p_5 = p_6 = j_1 = j_2 = j_3$ and $Q_5 = Q_6 = Q_7 = Q_8 \rightarrow \infty$. Finally, equating the fluxes at $z = H_6$ [boundary condition (Vb)], we get the following relation

$$\begin{aligned}
& -K_{z_2} \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1(3)} \sin(N_{i_1} x) \sin(N_{j_1} y) \lambda_{i_1 j_1} \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] \\
& = -K_{z_3} \left\{ - \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \lambda_{i_3 j_3} \sin(N_{i_3} x) \sin(N_{j_3} y) \right. \\
& \quad \left. + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} N_{q_9} \sin(N_{p_9} y) \frac{\sinh(\lambda_{p_9 q_9} x)}{\sinh(\lambda_{p_9 q_9} S_1)} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} N_{q_{10}} \sin(N_{p_{10}} y) \frac{\sinh[\lambda_{p_{10}q_{10}}(S_1 - x)]}{\sinh(\lambda_{p_{10}q_{10}} S_1)} \\
& + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11}q_{11}(3)} N_{q_{11}} \sin(N_{p_{11}} x) \frac{\sinh(\lambda_{p_{11}q_{11}} y)}{\sinh(\lambda_{p_{11}q_{11}} S_2)} \\
& + \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}(3)} N_{q_{12}} \sin(N_{p_{12}} x) \frac{\sinh[\lambda_{p_{12}q_{12}}(S_2 - y)]}{\sinh(\lambda_{p_{12}q_{12}} S_2)} \Big\}.
\end{aligned}$$

As before, by carrying out a double Fourier expansion in the relevant domain, we arrive at the following equation

$$\begin{aligned}
& \left(\frac{S_1 S_2}{4} \right) \left\{ \left(\frac{K_{z_2}}{K_{z_3}} \right) G_{i_1 j_1(3)} \lambda_{i_1 j_1} \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] + P_{i_3 j_3(3)} \lambda_{i_3 j_3} \right\} = \\
& \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} N_{q_9} \int_0^{S_2} \sin(N_{v_1} y) \sin(N_{p_9} y) dy \int_0^{S_1} \frac{\sinh(\lambda_{p_9 q_9} x)}{\sinh(\lambda_{p_9 q_9} S_1)} \sin(N_{u_1} x) dx \\
& + \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} N_{q_{10}} \int_0^{S_2} \sin(N_{v_1} y) \sin(N_{p_{10}} y) dy \int_0^{S_1} \frac{\sinh[\lambda_{p_{10}q_{10}}(S_1 - x)]}{\sinh(\lambda_{p_{10}q_{10}} S_1)} \sin(N_{u_1} x) dx \\
& + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11}q_{11}(3)} N_{q_{11}} \int_0^{S_1} \sin(N_{p_{11}} x) \sin(N_{u_1} x) dx \int_0^{S_2} \frac{\sinh(\lambda_{p_{11}q_{11}} y)}{\sinh(\lambda_{p_{11}q_{11}} S_2)} \sin(N_{v_1} y) dy \\
& + \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}(3)} N_{q_{12}} \int_0^{S_1} \sin(N_{p_{12}} x) \sin(N_{u_1} x) dx \int_0^{S_2} \frac{\sinh[\lambda_{p_{12}q_{12}}(S_2 - y)]}{\sinh(\lambda_{p_{12}q_{12}} S_2)} \sin(N_{v_1} y) dy.
\end{aligned} \tag{3.321}$$

Solving the above integrals, we finally get the expression

$$\begin{aligned}
& \left(\frac{K_{z_2}}{K_{z_3}} \right) G_{i_1 j_1(3)} \lambda_{i_1 j_1} \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] + P_{i_3 j_3(3)} \lambda_{i_3 j_3} = \\
& \left(\frac{4}{S_1 S_2} \right) \left\{ \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} N_{q_9} \left(\frac{S_2}{2} \right) \left[\frac{N_{u_1}^2}{N_{u_1}^2 + \lambda_{p_9 q_9}^2} \right] \left[\frac{-\cos(N_{u_1} S_1)}{N_{u_1}} \right] \right. \\
& \quad + \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} N_{q_{10}} \left(\frac{S_2}{2} \right) \left[\frac{N_{u_1}}{N_{u_1}^2 + \lambda_{p_{10}q_{10}}^2} \right] \\
& \quad + \sum_{q_{11}=1}^{Q_{11}} U_{p_{11}q_{11}(3)} N_{q_{11}} \left(\frac{S_1}{2} \right) \left[\frac{N_{v_1}^2}{N_{v_1}^2 + \lambda_{p_{11}q_{11}}^2} \right] \left[\frac{-\cos(N_{v_1} S_2)}{N_{v_1}} \right] \\
& \quad \left. + \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}(3)} N_{q_{12}} \left(\frac{S_1}{2} \right) \left[\frac{N_{v_1}}{N_{v_1}^2 + \lambda_{p_{12}q_{12}}^2} \right] \right\},
\end{aligned} \tag{3.322}$$

where $u_1 = p_{11} = p_{12} = i_1 = i_3$, $v_1 = p_9 = p_{10} = j_1 = j_3$ and $Q_9 = Q_{10} = Q_{11} = Q_{12} \rightarrow \infty$. Eqs. (3.317), (3.318), (3.320) and (3.322) can be used to determine all the coefficients of the steady state part of the solution pertaining to any ditch drainage scenario of Fig. 3.16. We would like to

again emphasize here that our steady state solution for the flow problem of Fig. 3.16 is exact and is valid for all possible combination of parameters of the problem.

3.2.3.2 Determination of Coefficients for the Transient Part of the Solution [transient state solution is approximate and is applicable only when directional conductivities and specific storage of the layers of Fig. 3.16 satisfy the conditions $K_{x_1}/S_{s_1} = K_{x_2}/S_{s_2} = K_{x_3}/S_{s_3}$, $K_{y_1}/S_{s_1} = K_{y_2}/S_{s_2} = K_{y_3}/S_{s_3}$ and $K_{z_1} = K_{z_2} = K_{z_3} \rightarrow 0$ (but $K_{z_1} = K_{z_2} = K_{z_3} \neq 0$)]

Since the steady state hydraulic head expression for the top soil layer remains identical to that of the previous problem, the expression for evaluation of the Fourier coefficient $A_{l_1 m_1 n_1(3)}$ will be identical to that of the one as given in Eq. (3.121) except that now we need to use the Fourier coefficients $B_{p_1 q_1(3)}$, $C_{p_2 q_2(3)}$, $D_{p_3 q_3(3)}$, $F_{p_4 q_4(3)}$, $E_{kl(3)}$ and $Q_{uv(3)}$ instead of the coefficients mentioned in this equation. Similarly, the expression for $R_{l_2 m_2 n_2(3)}$ will be identical to that of $R_{l_2 m_2 n_2(2)}$ of the previous problem owing to the similarity of the steady state hydraulic head function of the present problem with that of the previous one but here also care must be exercised to ensure that the Fourier coefficients pertaining to this flow problem (i.e., $G_{i_1 j_1(3)}$, $H_{i_2 j_2(3)}$, $I_{p_5 q_5(3)}$, $J_{p_6 q_6(3)}$, $K_{p_7 q_7(3)}$ and $L_{p_8 q_8(3)}$) be only used while analyzing transient flow behavior of the drainage situation of Fig. 3.16. The steady state head expression for the bottom layer is now different as compared to the solutions of the previous two problems and hence, naturally, the coefficients $W_{l_3 m_3 n_3(3)}$ will now not have a similar expression as $W_{l_3 m_3 n_3(1)}$ and $W_{l_3 m_3 n_3(2)}$. An equation describing $W_{l_3 m_3 n_3(3)}$, in view of initial condition (III), can be expressed as

$$\begin{aligned} & \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) = 0, \\ & 0 < x < S_1, \quad 0 < y < S_2, \quad 0 < z < H_6, \\ & \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) = 0, \\ & 0 < x < S_1, \quad 0 < y < S_2, \quad H_5 < z < H_6, \\ & \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} \sin(N_{l_3} x) \sin(N_{m_3} y) \sin(N_{n_3} z) = \\ & - \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \frac{\cosh[\lambda_{i_3 j_3}(h-z)]}{\sinh[\lambda_{i_3 j_3}(h-H_6)]} \sin(N_{i_3} x) \sin(N_{j_3} y) \\ & - \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \frac{\sinh(\lambda_{p_9 q_9} x)}{\sinh(\lambda_{p_9 q_9} S_1)} \sin(N_{p_9} y) \sin[N_{q_9}(z-H_6)] \\ & - \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10} q_{10}(3)} \frac{\sinh[\lambda_{p_{10} q_{10}}(S_1-x)]}{\sinh(\lambda_{p_{10} q_{10}} S_1)} \sin(N_{p_{10}} y) \sin[N_{q_{10}}(z-H_6)] \end{aligned}$$

$$\begin{aligned}
& - \sum_{p_{11}=1}^{P_1} \sum_{q_{11}=1}^{Q_1} U_{p_{11}q_{11}(3)} \frac{\sinh(\lambda_{p_{11}q_{11}} y)}{\sinh(\lambda_{p_{11}q_{11}} S_2)} \sin(N_{p_{11}} x) \sin[N_{q_{11}}(z - H_6)] \\
& - \sum_{p_{12}=1}^{P_2} \sum_{q_{12}=1}^{Q_2} V_{p_{12}q_{12}(3)} \frac{\sinh[\lambda_{p_{12}q_{12}}(S_2 - y)]}{\sinh(\lambda_{p_{12}q_{12}} S_2)} \sin(N_{p_{12}} x) \sin[N_{q_{12}}(z - H_6)] - (-H_6),
\end{aligned}$$

$$0 < x < S_1,$$

$$0 < y < S_2,$$

$$H_6 < z < h.$$

Performing a triple Fourier run in the concerned space, we get an expression for $W_{l_3 m_3 n_3(3)}$ as

$$\begin{aligned}
W_{l_3 m_3 n_3(3)} = & \left(\frac{8}{S_1 S_2 h} \right) \left\{ - \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \int_{H_6}^h \frac{\cosh[\lambda_{i_3 j_3}(h-z)]}{\sinh[\lambda_{i_3 j_3}(h-H_6)]} \sin(N_{n_3} z) dz \times \right. \\
& \int_0^{S_1} \sin(N_{i_3} x) \sin(N_{j_3} x) dx \times \int_0^{S_2} \sin(N_{j_3} y) \sin(N_{m_3} y) dy \\
& - \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \int_0^{S_1} \frac{\sinh(\lambda_{p_9 q_9} x)}{\sinh(\lambda_{p_9 q_9} S_1)} \sin(N_{l_3} x) dx \times \\
& \int_0^{S_2} \sin(N_{p_9} y) \sin(N_{m_3} y) dy \int_{H_6}^h \sin[N_{q_9}(z - H_6)] \sin(N_{n_3} z) dz \\
& - \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} \int_0^{S_1} \frac{\sinh[\lambda_{p_{10}q_{10}}(S_1 - x)]}{\sinh(\lambda_{p_{10}q_{10}} S_1)} \sin(N_{l_3} x) dx \times \\
& \int_0^{S_2} \sin(N_{p_{10}} y) \sin(N_{m_3} y) dy \int_{H_6}^h \sin[N_{q_{10}}(z - H_6)] \sin(N_{n_3} z) dz \\
& - \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11}q_{11}(3)} \int_0^{S_2} \frac{\sinh(\lambda_{p_{11}q_{11}} y)}{\sinh(\lambda_{p_{11}q_{11}} S_2)} \sin(N_{m_3} y) dy \times \\
& \int_0^{S_1} \sin(N_{p_{11}} x) \sin(N_{l_3} x) dx \int_{H_6}^h \sin[N_{q_{11}}(z - H_6)] \sin(N_{n_3} z) dz \\
& - \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}(3)} \int_0^{S_2} \frac{\sinh[\lambda_{p_{12}q_{12}}(S_2 - y)]}{\sinh(\lambda_{p_{12}q_{12}} S_2)} \sin(N_{m_3} y) dy \times \\
& \int_0^{S_1} \sin(N_{p_{12}} x) \sin(N_{l_3} x) dx \int_{H_6}^h \sin[N_{q_{12}}(z - H_6)] \sin(N_{n_3} z) dz \\
& \left. + H_6 \int_0^{S_1} \sin(N_{l_3} x) dx \times \int_0^{S_2} \sin(N_{m_3} y) dy \times \int_{H_6}^h \sin(N_{n_3} z) dz \right\}. \tag{3.323}
\end{aligned}$$

Evaluating the above integrals, we get

$$W_{l_3 m_3 n_3(3)} = \left(\frac{8}{S_1 S_2 h} \right) \left\{ -P_{i_3 j_3(3)} \left(\frac{S_1}{2} \right) \left(\frac{S_2}{2} \right) \left[\frac{N_{n_3}^2}{N_{n_3}^2 + \lambda_{i_3 j_3}^2} \right] \times \right.$$

$$\begin{aligned}
& \left\{ \frac{\cos(N_{n_3} H_6)}{N_{n_3}} \coth \left[\lambda_{i_3 j_3} (h - H_6) \right] + \left(\frac{\lambda_{i_3 j_3}}{N_{n_3}^2} \right) \sin(N_{n_3} H_6) \right\} \\
& - \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \left[\frac{N_{l_3}^2}{N_{l_3}^2 + \lambda_{p_9 q_9}^2} \right] \left[\frac{-\cos(N_{l_3} S_1)}{N_{l_3}} \right] \left(\frac{S_2}{2} \right) \times \\
& \left\{ \frac{\sin \left[N_{n_3} h - N_{q_9} (h - H_6) \right] - \sin(N_{n_3} H_6)}{2(N_{n_3} - N_{q_9})} + \right. \\
& \left. \frac{\sin(N_{n_3} H_6) - \sin \left[N_{n_3} h + N_{q_9} (h - H_6) \right]}{2(N_{n_3} + N_{q_9})} \right\} \\
& - \sum_{q_{10}=1}^{Q_{10}} N_{p_{10} q_{10}(3)} \left[\frac{N_{l_3}^2}{N_{l_3}^2 + \lambda_{p_{10} q_{10}}^2} \right] \left(\frac{S_2}{2} \right) \times \\
& \left\{ \frac{\sin \left[N_{n_3} h - N_{q_{10}} (h - H_6) \right] - \sin(N_{n_3} H_6)}{2(N_{n_3} - N_{q_{10}})} + \right. \\
& \left. \frac{\sin(N_{n_3} H_6) - \sin \left[N_{n_3} h + N_{q_{10}} (h - H_6) \right]}{2(N_{n_3} + N_{q_{10}})} \right\} \\
& - \sum_{q_{11}=1}^{Q_{11}} U_{p_{11} q_{11}(3)} \left[\frac{N_{l_3}^2}{N_{l_3}^2 + \lambda_{p_{11} q_{11}}^2} \right] \left[\frac{-\cos(N_{l_3} S_2)}{N_{l_3}} \right] \left(\frac{S_1}{2} \right) \times \\
& \left\{ \frac{\sin \left[N_{n_3} h - N_{q_{11}} (h - H_6) \right] - \sin(N_{n_3} H_6)}{2(N_{n_3} - N_{q_{11}})} + \right. \\
& \left. \frac{\sin(N_{n_3} H_6) - \sin \left[N_{n_3} h + N_{q_{11}} (h - H_6) \right]}{2(N_{n_3} + N_{q_{11}})} \right\} \\
& - \sum_{q_{12}=1}^{Q_{12}} V_{p_{12} q_{12}(3)} \left[\frac{N_{l_3}^2}{N_{l_3}^2 + \lambda_{p_{12} q_{12}}^2} \right] \left(\frac{S_1}{2} \right) \times \\
& \left\{ \frac{\sin \left[N_{n_3} h - N_{q_{12}} (h - H_6) \right] - \sin(N_{n_3} H_6)}{2(N_{n_3} - N_{q_{12}})} + \right. \\
& \left. \frac{\sin(N_{n_3} H_6) - \sin \left[N_{n_3} h + N_{q_{12}} (h - H_6) \right]}{2(N_{n_3} + N_{q_{12}})} \right\} \\
& + H_6 \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \left. \right\}. \tag{3.324}
\end{aligned}$$

The expressions for $Q_{top(3)}(t)$ and $Q_{top(3)}^f(x, y, t)$ remain the same as derived before for $Q_{top(1)}(t)$ and $Q_{top(1)}^f(x, y, t)$ [Eqs. (3.131) and (3.133)]; however, the Fourier coefficients needed to be used in these expressions should correspond to the present problem only (i.e. $B_{p_1q_1(3)}$, $C_{p_2q_2(3)}$, $D_{p_3q_3(3)}$, $F_{p_4q_4(3)}$, $E_{kl(3)}$, $Q_{uv(3)}$, $A_{l_1m_1n_1(3)}$, $R_{l_2m_2n_2(3)}$ and $W_{l_3m_3n_3(3)}$) and not to those associated with the earlier problems. Further, for this case also, it can be shown in a similar way as has been shown for the first problem that $Q_{top(3)}^f(x, y, t)$ will diverge when being calculated exactly at an inner bund separating two dissimilar ponding depths at the surface of the soil. Also, the discharge expressions for the four ditch faces of the current problem can be written as

$$\begin{aligned}
Q_{North(3)}(t) = & K_{x_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1q_1(3)} \left[\frac{\lambda_{p_1q_1}}{\sinh(\lambda_{p_2q_2} S_1)} \right] \left[\frac{1 - \cos(N_{p_1} S_2)}{N_{p_1}} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\
& - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2q_2(3)} \left[\frac{\lambda_{p_2q_2}}{\tanh(\lambda_{p_2q_2} S_1)} \right] \left[\frac{1 - \cos(N_{p_2} S_2)}{N_{p_2}} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3q_3(3)} \left(\frac{N_{p_3}}{\lambda_{p_3q_3}} \right) \left[\frac{\cosh(\lambda_{p_3q_3} S_2) - 1}{\sinh(\lambda_{p_3q_3} S_2)} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\
& + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4q_4(3)} \left(\frac{N_{p_4}}{\lambda_{p_4q_4}} \right) \left[\frac{\cosh(\lambda_{p_4q_4} S_2) - 1}{\sinh(\lambda_{p_4q_4} S_2)} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(3)} \left(\frac{N_k}{\lambda_{kl}} \right) \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_l S_2)}{N_l} \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(3)} \left(\frac{N_u}{\lambda_{uv}} \right) \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_v S_2)}{N_v} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1m_1n_1(3)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_1m_1n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2m_2n_2(3)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_2m_2n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3m_3n_3(3)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \\
& \quad \left. \exp \left[-(\lambda_{l_3m_3n_3})^2 t \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& +K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1}^{(3)} \left(\frac{N_{i_1}}{\lambda_{i_1 j_1}} \right) \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{j_1} S_2)}{N_{j_1}} \right] \right. \\
& + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2}^{(3)} \left(\frac{N_{i_2}}{\lambda_{i_2 j_2}} \right) \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{j_2} S_2)}{N_{j_2}} \right] \\
& + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5}^{(3)} \left[\frac{\lambda_{p_5 q_5}}{\sinh(\lambda_{p_5 q_5} S_1)} \right] \left[\frac{1 - \cos(N_{p_5} S_2)}{N_{p_5}} \right] \times \\
& \left. \left\{ \frac{1 - \cos[N_{q_5} (H_6 - H_5)]}{N_{q_5}} \right\} \right. \\
& - \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6}^{(3)} \left[\frac{\lambda_{p_6 q_6}}{\tanh(\lambda_{p_6 q_6} S_1)} \right] \left[\frac{1 - \cos(N_{p_6} S_2)}{N_{p_6}} \right] \times \\
& \left. \left\{ \frac{1 - \cos[N_{q_6} (H_6 - H_5)]}{N_{q_6}} \right\} \right. \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7}^{(3)} \left(\frac{N_{p_7}}{\lambda_{p_7 q_7}} \right) \left\{ \frac{1 - \cos[N_{q_7} (H_6 - H_5)]}{N_{q_7}} \right\} \left[\frac{\cosh(\lambda_{p_7 q_7} S_2) - 1}{\sinh(\lambda_{p_7 q_7} S_2)} \right] \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8}^{(3)} \left(\frac{N_{p_8}}{\lambda_{p_8 q_8}} \right) \left\{ \frac{1 - \cos[N_{q_8} (H_6 - H_5)]}{N_{q_8}} \right\} \left[\frac{\cosh(\lambda_{p_8 q_8} S_2) - 1}{\sinh(\lambda_{p_8 q_8} S_2)} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1}^{(3)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2}^{(3)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3}^{(3)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \left. \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \right\} \\
& + K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3}^{(3)} \left(\frac{N_{i_3}}{\lambda_{i_3 j_3}} \right) \left[\frac{1 - \cos(N_{j_3} S_2)}{N_{j_3}} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9 (3)} \left[\frac{\lambda_{p_9 q_9}}{\sinh(\lambda_{p_9 q_9} S_1)} \right] \left[\frac{1 - \cos(N_{p_9} S_2)}{N_{p_9}} \right] \times \\
& \left\{ \frac{1 - \cos[N_{q_9} (h - H_6)]}{N_{q_9}} \right\} \\
& - \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10} q_{10} (3)} \left[\frac{\lambda_{p_{10} q_{10}}}{\tanh(\lambda_{p_{10} q_{10}} S_1)} \right] \left[\frac{1 - \cos(N_{p_{10}} S_2)}{N_{p_{10}}} \right] \times \\
& \left\{ \frac{1 - \cos[N_{q_{10}} (h - H_6)]}{N_{q_{10}}} \right\} \\
& + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11} q_{11} (3)} \left(\frac{N_{p_{11}}}{\lambda_{p_{11} q_{11}}} \right) \left\{ \frac{1 - \cos[N_{q_{11}} (h - H_6)]}{N_{q_{11}}} \right\} \times \\
& \left[\frac{\cosh(\lambda_{p_{11} q_{11}} S_2) - 1}{\sinh(\lambda_{p_{11} q_{11}} S_2)} \right] \\
& + \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12} q_{12} (3)} \left(\frac{N_{p_{12}}}{\lambda_{p_{12} q_{12}}} \right) \left\{ \frac{1 - \cos[N_{q_{12}} (h - H_6)]}{N_{q_{12}}} \right\} \times \\
& \left[\frac{\cosh(\lambda_{p_{12} q_{12}} S_2) - 1}{\sinh(\lambda_{p_{12} q_{12}} S_2)} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1 (3)} N_{l_1} \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2 (3)} N_{l_2} \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3 (3)} N_{l_3} \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Bigg\}, \tag{3.325}
\end{aligned}$$

$$Q_{South(3)}(t) = K_{x_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1 (3)} \lambda_{p_1 q_1} \coth(\lambda_{p_1 q_1} S_1) \left[\frac{1 - \cos(N_{p_1} S_2)}{N_{p_1}} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right\}$$

$$\begin{aligned}
& - \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2 (3)} \left[\frac{\lambda_{p_2 q_2}}{\sinh(\lambda_{p_2 q_2} S_1)} \right] \left[\frac{1 - \cos(N_{p_2} S_2)}{N_{p_2}} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3 (3)} \left[\frac{N_{p_3} \cos(N_{p_3} S_1)}{\lambda_{p_3 q_3}} \right] \left[\frac{\cosh(\lambda_{p_3 q_3} S_2) - 1}{\sinh(\lambda_{p_3 q_3} S_2)} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\
& + \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4 (3)} \left[\frac{N_{p_4} \cos(N_{p_4} S_1)}{\lambda_{p_4 q_4}} \right] \left[\frac{\cosh(\lambda_{p_4 q_4} S_2) - 1}{\sinh(\lambda_{p_4 q_4} S_2)} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(3)} \left[\frac{N_k \cos(N_k S_1)}{\lambda_{kl}} \right] \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_l S_2)}{N_l} \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(3)} \left[\frac{N_u \cos(N_u S_1)}{\lambda_{uv}} \right] \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_v S_2)}{N_v} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1 (3)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2 (3)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3 (3)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
& + K_{x_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1 (3)} \left[\frac{N_{i_1} \cos(N_{i_1} S_1)}{\lambda_{i_1 j_1}} \right] \tanh[\lambda_{i_1 j_1} (H_6 - H_5)] \left[\frac{1 - \cos(N_{j_1} S_2)}{N_{j_1}} \right] \right. \\
& \quad + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2 (3)} \left[\frac{N_{i_2} \cos(N_{i_2} S_1)}{\lambda_{i_2 j_2}} \right] \tanh[\lambda_{i_2 j_2} (H_6 - H_5)] \left[\frac{1 - \cos(N_{j_2} S_2)}{N_{j_2}} \right] \\
& \quad + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5 (3)} \left[\frac{\lambda_{p_5 q_5}}{\tanh(\lambda_{p_5 q_5} S_1)} \right] \left[\frac{1 - \cos(N_{p_5} S_2)}{N_{p_5}} \right] \times \\
& \quad \left. \left[\frac{1 - \cos[N_{q_5} (H_6 - H_5)]}{N_{q_5}} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} \left[\frac{\lambda_{p_6 q_6}}{\sinh(\lambda_{p_6 q_6} S_1)} \right] \left[\frac{1 - \cos(N_{p_6} S_2)}{N_{p_6}} \right] \times \\
& \left\{ \frac{1 - \cos[N_{q_6} (H_6 - H_5)]}{N_{q_6}} \right\} \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(3)} \left(\frac{N_{p_7}}{\lambda_{p_7 q_7}} \right) \cos(N_{p_7} S_1) \left[\frac{\cosh(\lambda_{p_7 q_7} S_2) - 1}{\sinh(\lambda_{p_7 q_7} S_2)} \right] \times \\
& \left\{ \frac{1 - \cos[N_{q_7} (H_6 - H_5)]}{N_{q_7}} \right\} \\
& + \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} \left(\frac{N_{p_8}}{\lambda_{p_8 q_8}} \right) \cos(N_{p_8} S_1) \left[\frac{\cosh(\lambda_{p_8 q_8} S_2) - 1}{\sinh(\lambda_{p_8 q_8} S_2)} \right] \times \\
& \left\{ \frac{1 - \cos[N_{q_8} (H_6 - H_5)]}{N_{q_8}} \right\} \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \\
& \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \\
& \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \\
& \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \\
& + K_{x_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \left[\frac{N_{i_3} \cos(N_{i_3} S_1)}{\lambda_{i_3 j_3}} \right] \left[\frac{1 - \cos(N_{j_3} S_2)}{N_{j_3}} \right] \right. \\
& \left. + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \left[\frac{\lambda_{p_9 q_9}}{\tanh(\lambda_{p_9 q_9} S_1)} \right] \left[\frac{1 - \cos(N_{p_9} S_2)}{N_{p_9}} \right] \right\} \times
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{1 - \cos \left[N_{q_9} (h - H_6) \right]}{N_{q_9}} \right\} \\
& - \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} \left[\frac{\lambda_{p_{10}q_{10}}}{\sinh(\lambda_{p_{10}q_{10}} S_1)} \right] \left[\frac{1 - \cos(N_{p_{10}} S_2)}{N_{p_{10}}} \right] \times \\
& \left\{ \frac{1 - \cos \left[N_{q_{10}} (h - H_6) \right]}{N_{q_{10}}} \right\} \\
& + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11}q_{11}(3)} \left(\frac{N_{p_{11}}}{\lambda_{p_{11}q_{11}}} \right) \cos(N_{p_{11}} S_1) \left\{ \frac{1 - \cos \left[N_{q_{11}} (h - H_6) \right]}{N_{q_{11}}} \right\} \times \\
& \left[\frac{\cosh(\lambda_{p_{11}q_{11}} S_2) - 1}{\sinh(\lambda_{p_{11}q_{11}} S_2)} \right] \\
& + \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}(3)} \left(\frac{N_{p_{12}}}{\lambda_{p_{12}q_{12}}} \right) \cos(N_{p_{12}} S_1) \left\{ \frac{1 - \cos \left[N_{q_{12}} (h - H_6) \right]}{N_{q_{12}}} \right\} \times \\
& \left[\frac{\cosh(\lambda_{p_{12}q_{12}} S_2) - 1}{\sinh(\lambda_{p_{12}q_{12}} S_2)} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} N_{l_1} \cos(N_{l_1} S_1) \left[\frac{1 - \cos(N_{m_1} S_2)}{N_{m_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} N_{l_2} \cos(N_{l_2} S_1) \left[\frac{1 - \cos(N_{m_2} S_2)}{N_{m_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} N_{l_3} \cos(N_{l_3} S_1) \left[\frac{1 - \cos(N_{m_3} S_2)}{N_{m_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Bigg\}, \tag{3.326}
\end{aligned}$$

$$\begin{aligned}
Q_{East(3)}(t) = K_{y_1} & \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(3)} \left(\frac{N_{p_1}}{\lambda_{p_1 q_1}} \right) \left[\frac{\cosh(\lambda_{p_1 q_1} S_1) - 1}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right. \\
& + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2(3)} \left(\frac{N_{p_2}}{\lambda_{p_2 q_2}} \right) \left[\frac{\cosh(\lambda_{p_2 q_2} S_1) - 1}{\sinh(\lambda_{p_2 q_2} S_1)} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3 (3)} \left[\frac{\lambda_{p_3 q_3}}{\sinh(\lambda_{p_3 q_3} S_2)} \right] \left[\frac{1 - \cos(N_{p_3} S_1)}{N_{p_3}} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\
& - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4 (3)} \left[\frac{\lambda_{p_4 q_4}}{\tanh(\lambda_{p_4 q_4} S_2)} \right] \left[\frac{1 - \cos(N_{p_4} S_1)}{N_{p_4}} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(3)} \left(\frac{N_l}{\lambda_{kl}} \right) \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_k S_1)}{N_k} \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(3)} \left(\frac{N_v}{\lambda_{uv}} \right) \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_u S_1)}{N_u} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1 (3)} N_{m_1} \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2 (3)} N_{m_2} \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3 (3)} N_{m_3} \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
& + K_{y_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1 (3)} \left(\frac{N_{j_1}}{\lambda_{i_1 j_1}} \right) \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_1} S_1)}{N_{i_1}} \right] \right. \\
& \quad + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2 (3)} \left(\frac{N_{j_2}}{\lambda_{i_2 j_2}} \right) \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_2} S_1)}{N_{i_2}} \right] \\
& \quad + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5 (3)} \left(\frac{N_{p_5}}{\lambda_{p_5 q_5}} \right) \left[\frac{\cosh(\lambda_{p_5 q_5} S_1) - 1}{\sinh(\lambda_{p_5 q_5} S_1)} \right] \times \\
& \quad \left. \left[\frac{1 - \cos \left[N_{q_5} (H_6 - H_5) \right]}{N_{q_5}} \right] \right\} \\
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6 (3)} \left(\frac{N_{p_6}}{\lambda_{p_6 q_6}} \right) \left[\frac{\cosh(\lambda_{p_6 q_6} S_1) - 1}{\sinh(\lambda_{p_6 q_6} S_1)} \right] \times
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{1 - \cos [N_{q_6} (H_6 - H_5)]}{N_{q_6}} \right\} \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7 (3)} \left[\frac{\lambda_{p_7 q_7}}{\sinh (\lambda_{p_7 q_7} S_2)} \right] \left\{ \frac{1 - \cos [N_{q_7} (H_6 - H_5)]}{N_{q_7}} \right\} \times \\
& \left[\frac{1 - \cos (N_{p_7} S_1)}{N_{p_7}} \right] \\
& - \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8 (3)} \left[\frac{\lambda_{p_8 q_8}}{\tanh (\lambda_{p_8 q_8} S_2)} \right] \left\{ \frac{1 - \cos [N_{q_8} (H_6 - H_5)]}{N_{q_8}} \right\} \times \\
& \left[\frac{1 - \cos (N_{p_8} S_1)}{N_{p_8}} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1 (3)} N_{m_1} \left[\frac{1 - \cos (N_{l_1} S_1)}{N_{l_1}} \right] \times \\
& \left[\frac{\cos (N_{n_1} H_5) - \cos (N_{n_1} H_6)}{N_{n_1}} \right] \times \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2 (3)} N_{m_2} \left[\frac{1 - \cos (N_{l_2} S_1)}{N_{l_2}} \right] \times \\
& \left[\frac{\cos (N_{n_2} H_5) - \cos (N_{n_2} H_6)}{N_{n_2}} \right] \times \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3 (3)} N_{m_3} \left[\frac{1 - \cos (N_{l_3} S_1)}{N_{l_3}} \right] \times \\
& \left[\frac{\cos (N_{n_3} H_5) - \cos (N_{n_3} H_6)}{N_{n_3}} \right] \times \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
& + K_{y_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3 (3)} \left(\frac{N_{j_3}}{\lambda_{i_3 j_3}} \right) \left[\frac{1 - \cos (N_{i_3} S_1)}{N_{i_3}} \right] \right. \\
& + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9 (3)} \left(\frac{N_{p_9}}{\lambda_{p_9 q_9}} \right) \left[\frac{\cosh (\lambda_{p_9 q_9} S_1) - 1}{\sinh (\lambda_{p_9 q_9} S_1)} \right] \times \\
& \left. \left. \left[\frac{1 - \cos [N_{q_9} (h - H_6)]}{N_{q_9}} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} \left(\frac{N_{p_{10}}}{\lambda_{p_{10}q_{10}}} \right) \left[\frac{\cosh(\lambda_{p_{10}q_{10}} S_1) - 1}{\sinh(\lambda_{p_{10}q_{10}} S_1)} \right] \times \\
& \left\{ \frac{1 - \cos[N_{q_{10}}(h - H_6)]}{N_{q_{10}}} \right\} \\
& + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11}q_{11}(3)} \left[\frac{\lambda_{p_{11}q_{11}}}{\sinh(\lambda_{p_{11}q_{11}} S_2)} \right] \left\{ \frac{1 - \cos[N_{q_{11}}(h - H_6)]}{N_{q_{11}}} \right\} \times \\
& \left[\frac{1 - \cos(N_{p_{11}} S_1)}{N_{p_{11}}} \right] \\
& - \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}(3)} \left[\frac{\lambda_{p_{12}q_{12}}}{\tanh(\lambda_{p_{12}q_{12}} S_2)} \right] \left\{ \frac{1 - \cos[N_{q_{12}}(h - H_6)]}{N_{q_{12}}} \right\} \times \\
& \left[\frac{1 - \cos(N_{p_{12}} S_1)}{N_{p_{12}}} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} N_{m_1} \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp\left[-(\lambda_{l_1 m_1 n_1})^2 t\right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} N_{m_2} \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp\left[-(\lambda_{l_2 m_2 n_2})^2 t\right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} N_{m_3} \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \exp\left[-(\lambda_{l_3 m_3 n_3})^2 t\right] \left. \right\} \tag{3.327}
\end{aligned}$$

and

$$Q_{West(3)}(t) = K_{y_1} \left\{ \sum_{p_1=1}^{P_1} \sum_{q_1=1}^{Q_1} B_{p_1 q_1(3)} \left[\frac{N_{p_1} \cos(N_{p_1} S_2)}{\lambda_{p_1 q_1}} \right] \left[\frac{\cosh(\lambda_{p_1 q_1} S_1) - 1}{\sinh(\lambda_{p_1 q_1} S_1)} \right] \left[\frac{1 - \cos(N_{q_1} H_5)}{N_{q_1}} \right] \right\}$$

$$\begin{aligned}
& + \sum_{p_2=1}^{P_2} \sum_{q_2=1}^{Q_2} C_{p_2 q_2 (3)} \left[\frac{N_{p_2} \cos(N_{p_2} S_2)}{\lambda_{p_2 q_2}} \right] \left[\frac{\cosh(\lambda_{p_2 q_2} S_1) - 1}{\sinh(\lambda_{p_2 q_2} S_1)} \right] \left[\frac{1 - \cos(N_{q_2} H_5)}{N_{q_2}} \right] \\
& + \sum_{p_3=1}^{P_3} \sum_{q_3=1}^{Q_3} D_{p_3 q_3 (3)} \left[\frac{\lambda_{p_3 q_3}}{\tanh(\lambda_{p_3 q_3} S_2)} \right] \left[\frac{1 - \cos(N_{p_3} S_1)}{N_{p_3}} \right] \left[\frac{1 - \cos(N_{q_3} H_5)}{N_{q_3}} \right] \\
& - \sum_{p_4=1}^{P_4} \sum_{q_4=1}^{Q_4} F_{p_4 q_4 (3)} \left[\frac{\lambda_{p_4 q_4}}{\sinh(\lambda_{p_4 q_4} S_2)} \right] \left[\frac{1 - \cos(N_{p_4} S_1)}{N_{p_4}} \right] \left[\frac{1 - \cos(N_{q_4} H_5)}{N_{q_4}} \right] \\
& + \sum_{k=1}^K \sum_{l=1}^L E_{kl(3)} \left[\frac{N_l \cos(N_l S_2)}{\lambda_{kl}} \right] \left[\frac{\cosh(\lambda_{kl} H_5) - 1}{\cosh(\lambda_{kl} H_5)} \right] \left[\frac{1 - \cos(N_k S_1)}{N_k} \right] \\
& + \sum_{u=1}^U \sum_{v=1}^V Q_{uv(3)} \left[\frac{N_v \cos(N_v S_2)}{\lambda_{uv}} \right] \tanh(\lambda_{uv} H_5) \left[\frac{1 - \cos(N_u S_1)}{N_u} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1 (3)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{1 - \cos(N_{n_1} H_5)}{N_{n_1}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2 (3)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{1 - \cos(N_{n_2} H_5)}{N_{n_2}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3 (3)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{1 - \cos(N_{n_3} H_5)}{N_{n_3}} \right] \times \\
& \quad \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \Big\} \\
& + K_{y_2} \left\{ \sum_{i_1=1}^{I_1} \sum_{j_1=1}^{J_1} G_{i_1 j_1 (3)} \left[\frac{N_{j_1} \cos(N_{j_1} S_2)}{\lambda_{i_1 j_1}} \right] \tanh \left[\lambda_{i_1 j_1} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_1} S_1)}{N_{i_1}} \right] \right. \\
& \quad + \sum_{i_2=1}^{I_2} \sum_{j_2=1}^{J_2} H_{i_2 j_2 (3)} \left[\frac{N_{j_2} \cos(N_{j_2} S_2)}{\lambda_{i_2 j_2}} \right] \tanh \left[\lambda_{i_2 j_2} (H_6 - H_5) \right] \left[\frac{1 - \cos(N_{i_2} S_1)}{N_{i_2}} \right] \\
& \quad + \sum_{p_5=1}^{P_5} \sum_{q_5=1}^{Q_5} I_{p_5 q_5 (3)} \left[\frac{N_{p_5} \cos(N_{p_5} S_2)}{\lambda_{p_5 q_5}} \right] \left[\frac{\cosh(\lambda_{p_5 q_5} S_1) - 1}{\sinh(\lambda_{p_5 q_5} S_1)} \right] \times \\
& \quad \left. \left[\frac{1 - \cos \left[N_{q_5} (H_6 - H_5) \right]}{N_{q_5}} \right] \right\}
\end{aligned}$$

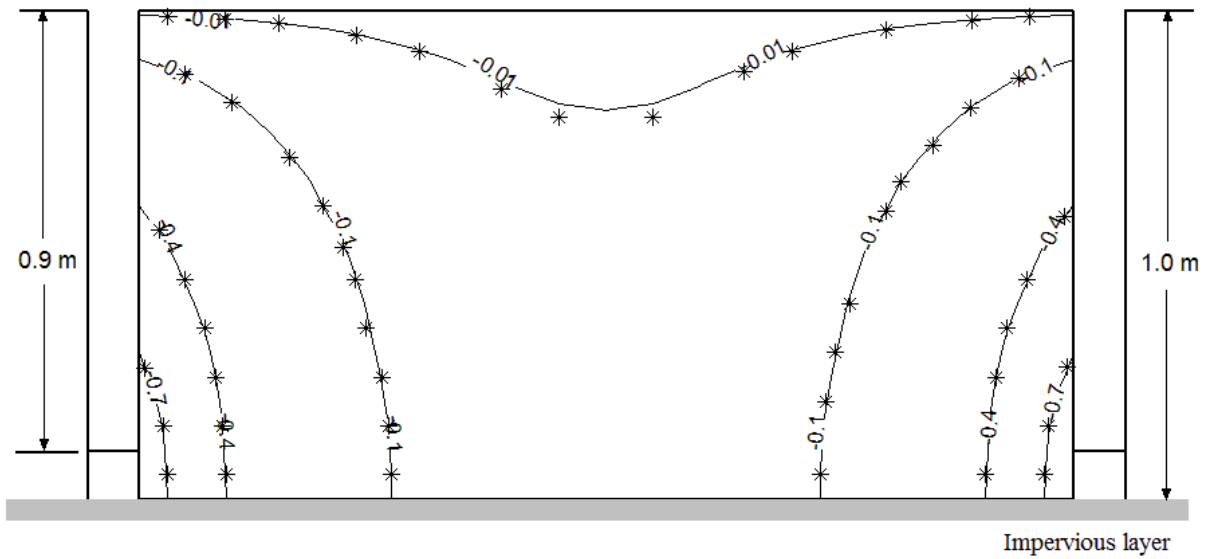
$$\begin{aligned}
& + \sum_{p_6=1}^{P_6} \sum_{q_6=1}^{Q_6} J_{p_6 q_6(3)} \left[\frac{N_{p_6} \cos(N_{p_6} S_2)}{\lambda_{p_6 q_6}} \right] \left[\frac{\cosh(\lambda_{p_6 q_6} S_1) - 1}{\sinh(\lambda_{p_6 q_6} S_1)} \right] \times \\
& \quad \left\{ \frac{1 - \cos[N_{q_6} (H_6 - H_5)]}{N_{q_6}} \right\} \\
& + \sum_{p_7=1}^{P_7} \sum_{q_7=1}^{Q_7} K_{p_7 q_7(3)} \lambda_{p_7 q_7} \coth(\lambda_{p_7 q_7} S_2) \left\{ \frac{1 - \cos[N_{q_7} (H_6 - H_5)]}{N_{q_7}} \right\} \times \\
& \quad \left[\frac{1 - \cos(N_{p_7} S_1)}{N_{p_7}} \right] \\
& - \sum_{p_8=1}^{P_8} \sum_{q_8=1}^{Q_8} L_{p_8 q_8(3)} \left[\frac{\lambda_{p_8 q_8}}{\sinh(\lambda_{p_8 q_8} S_2)} \right] \left\{ \frac{1 - \cos[N_{q_8} (H_6 - H_5)]}{N_{q_8}} \right\} \times \\
& \quad \left[\frac{1 - \cos(N_{p_8} S_1)}{N_{p_8}} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \times \\
& \quad \left[\frac{\cos(N_{n_1} H_5) - \cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \exp[-(\lambda_{l_1 m_1 n_1})^2 t] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \times \\
& \quad \left[\frac{\cos(N_{n_2} H_5) - \cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \exp[-(\lambda_{l_2 m_2 n_2})^2 t] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \\
& \quad \left[\frac{\cos(N_{n_3} H_5) - \cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \exp[-(\lambda_{l_3 m_3 n_3})^2 t] \\
& + K_{y_3} \left\{ \sum_{i_3=1}^{I_3} \sum_{j_3=1}^{J_3} P_{i_3 j_3(3)} \left[\frac{N_{j_3} \cos(N_{j_3} S_2)}{\lambda_{i_3 j_3}} \right] \left[\frac{1 - \cos(N_{i_3} S_1)}{N_{i_3}} \right] \right. \\
& \quad \left. + \sum_{p_9=1}^{P_9} \sum_{q_9=1}^{Q_9} M_{p_9 q_9(3)} \left[\frac{N_{p_9} \cos(N_{p_9} S_2)}{\lambda_{p_9 q_9}} \right] \left[\frac{\cosh(\lambda_{p_9 q_9} S_1) - 1}{\sinh(\lambda_{p_9 q_9} S_1)} \right] \times \right.
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{1 - \cos \left[N_{q_9} (h - H_6) \right]}{N_{q_9}} \right\} \\
& + \sum_{p_{10}=1}^{P_{10}} \sum_{q_{10}=1}^{Q_{10}} N_{p_{10}q_{10}(3)} \left[\frac{N_{p_{10}} \cos(N_{p_{10}} S_2)}{\lambda_{p_{10}q_{10}}} \right] \left[\frac{\cosh(\lambda_{p_{10}q_{10}} S_1) - 1}{\sinh(\lambda_{p_{10}q_{10}} S_1)} \right] \times \\
& \left\{ \frac{1 - \cos \left[N_{q_{10}} (h - H_6) \right]}{N_{q_{10}}} \right\} \\
& + \sum_{p_{11}=1}^{P_{11}} \sum_{q_{11}=1}^{Q_{11}} U_{p_{11}q_{11}(3)} \left[\frac{\lambda_{p_{11}q_{11}}}{\tanh(\lambda_{p_{11}q_{11}} S_2)} \right] \left\{ \frac{1 - \cos \left[N_{q_{11}} (h - H_6) \right]}{N_{q_{11}}} \right\} \times \\
& \left[\frac{1 - \cos(N_{p_{11}} S_1)}{N_{p_{11}}} \right] \\
& - \sum_{p_{12}=1}^{P_{12}} \sum_{q_{12}=1}^{Q_{12}} V_{p_{12}q_{12}(3)} \left[\frac{\lambda_{p_{12}q_{12}}}{\sinh(\lambda_{p_{12}q_{12}} S_2)} \right] \left\{ \frac{1 - \cos \left[N_{q_{12}} (h - H_6) \right]}{N_{q_{12}}} \right\} \times \\
& \left[\frac{1 - \cos(N_{p_{12}} S_1)}{N_{p_{12}}} \right] \\
& + \sum_{l_1=1}^{L_1} \sum_{m_1=1}^{M_1} \sum_{n_1=1}^{N_1} A_{l_1 m_1 n_1(3)} N_{m_1} \cos(N_{m_1} S_2) \left[\frac{1 - \cos(N_{l_1} S_1)}{N_{l_1}} \right] \left[\frac{\cos(N_{n_1} H_6)}{N_{n_1}} \right] \times \\
& \exp \left[-(\lambda_{l_1 m_1 n_1})^2 t \right] \\
& + \sum_{l_2=1}^{L_2} \sum_{m_2=1}^{M_2} \sum_{n_2=1}^{N_2} R_{l_2 m_2 n_2(3)} N_{m_2} \cos(N_{m_2} S_2) \left[\frac{1 - \cos(N_{l_2} S_1)}{N_{l_2}} \right] \left[\frac{\cos(N_{n_2} H_6)}{N_{n_2}} \right] \times \\
& \exp \left[-(\lambda_{l_2 m_2 n_2})^2 t \right] \\
& + \sum_{l_3=1}^{L_3} \sum_{m_3=1}^{M_3} \sum_{n_3=1}^{N_3} W_{l_3 m_3 n_3(3)} N_{m_3} \cos(N_{m_3} S_2) \left[\frac{1 - \cos(N_{l_3} S_1)}{N_{l_3}} \right] \left[\frac{\cos(N_{n_3} H_6)}{N_{n_3}} \right] \times \\
& \exp \left[-(\lambda_{l_3 m_3 n_3})^2 t \right] \left. \right\}, \tag{3.328}
\end{aligned}$$

where $Q_{North(3)}$, $Q_{South(3)}$, $Q_{East(3)}$ and $Q_{West(3)}$ are the discharge expressions related to the Northern, Southern, Eastern and Western boundaries of the flow problem of Fig. 3.16.

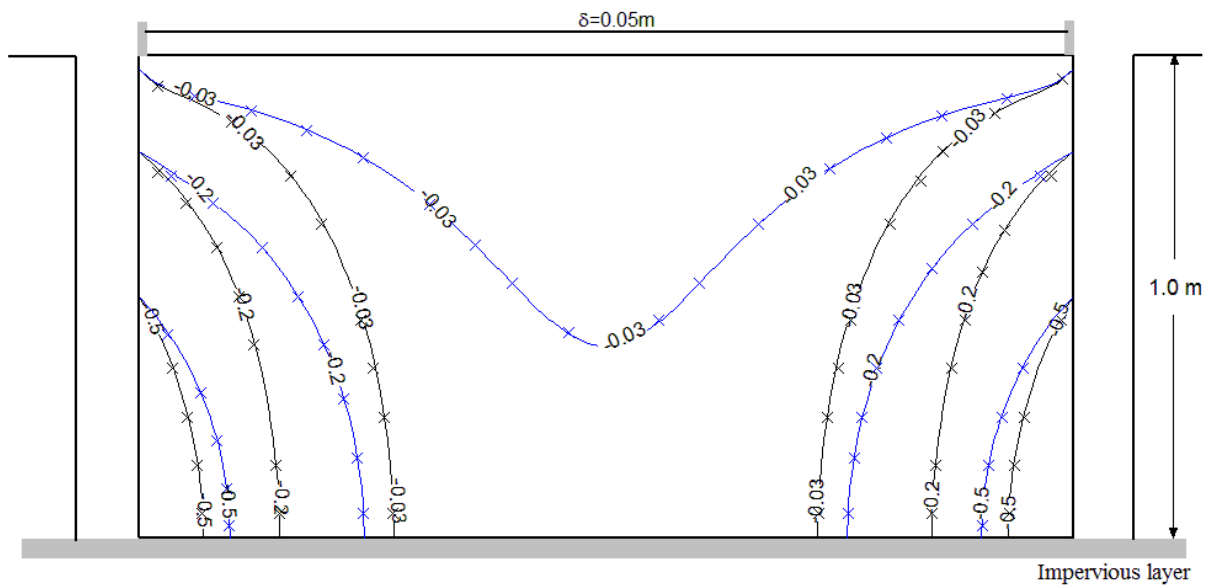
3.2.3.3 Verifications of the Proposed Solution

The validity of the current solution has also been extensively tested, like in the previous two solutions, by making comparisons with the works of others for a few drainage situations of Fig. 3.16. Also, numerical checks on the proposed solution for both steady as well as transient states of simulation of the system have also been carried out for a drainage setting of Fig. 3.16 utilizing the Processing MODFLOW platform. Figs. 3.17, 3.18, 3.19, 3.20 and 3.21 show these comparisons. As can be seen, in all the tested situations, our predictions are found to match closely with their analytical and numerical counterparts, thereby establishing that our solution for the drainage problem of Fig. 3.16 has been correctly developed. It is worth noting here that, like in the previous two solutions, here also the solution can be closely reduced to a two-dimensional flow situation in the $y-z$ plane by assigning a large separation distance (theoretically infinite) between the Northern and Southern boundaries of the model and then considering a $y-z$ section located further away from both these boundaries. Considering such a reduced two-dimensional model, the steady state $Q_{top(3)} / 2Kh$ ratio at a vertical section located mid-way between the Northern and Southern boundaries of Fig. 3.17 has been worked out as 0.744 when the flow parameters of the problem are taken as $S_1 = 1000$ m, $S_2 = 100$ m, $h = 3$ m, $H_1 = 3$ m, $H_5 = 2$ m, $H_6 = 2.5$ m, $\delta = 0$ m and $K_{x_1} = K_{x_2} = K_{x_3} = K_{y_1} = K_{y_2} = K_{y_3} = K_{z_1} = K_{z_2} = K_{z_3} = 0.05$ m/day. For the same drainage configuration, as mentioned before, this ratio has been predicted as 0.743 and 0.742, respectively by Fukuda's (1957) and Youngs' (1994) solutions – values very close to our predicted value of 0.744. Also, Fukuda (1957) found this ratio as 0.72 from his experimental results; thus, the close matching of this result with our value of 0.744 can also be taken as an experimental verification of our proposed solution for the flow problem of Fig. 3.16. In this context, we would like to point out that for the drainage configuration of Fig. 3.16, simulating a drainage system with the drains running totally empty is a possibility since for this system we have assumed the water level of the drains to be lying below the upper boundary of the bottom layer. However, simulating a drainage system with the ditches running totally empty cannot be done using our solutions for the previous two configurations of the system as in both these systems, the water level in the ditches is assumed to lie either in the top or middle layer of the stratified soil and not the bottom layer. Nevertheless, simulating a system with the ditches running nearly empty (but not totally empty) is still possible with our previous solutions since we can always take the thickness of the bottom and/or the middle layer to be appreciably small (but not totally zero and that is what we have done in the previous cases) while imitating such a system.



* Hydraulic heads as obtained from the proposed solution
 — Hydraulic heads as obtained from Kirkham's solution

Fig. 3.17. Comparison of steady state hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.16 with the corresponding values as obtained from Kirkham's (1965) single-layered steady state solution of the problem for isotropic soils when the flow parameters are considered as $S_1 = 15$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.9$ m, $H_5 = 0.4$ m, $H_6 = 0.8$ m, $\delta = 0$ m and $K_{x_1} = K_{x_2} = K_{x_3} = K_{y_1} = K_{y_2} = K_{y_3} = K_{z_1} = K_{z_2} = K_{z_3} = 1$ m/day



- × Hydraulic heads as obtained from the proposed solution (100 s)
- Hydraulic heads as obtained from Barua and Alam's solution (100 s)
- × Hydraulic heads as obtained from the proposed solution (500 s)
- Hydraulic heads as obtained from Barua and Alam's solution (500 s)

Fig. 3.18. Comparison of transient hydraulic heads as obtained from the proposed solution at a vertical cross-section located half-way (i.e., at $S_1/2$) between the Northern and the Southern boundaries of Fig. 3.16 at times $t=100$ s and 500 s with the corresponding values as obtained from Barua and Alam's (2013) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem are considered as $S_1=15$ m, $S_2=5$ m, $h=1$ m, $H_1=1$ m, $H_5=0.35$ m, $H_6=0.8$ m, $\delta=0.05$ m, $K_{x_1}=K_{x_2}=K_{x_3}=K_y=K_{y_1}=K_{y_2}=K_{y_3}=2$ m/day, $K_{z_1}=K_{z_2}=K_{z_3}=0.8$ m/day and $S_{s_1}=S_{s_2}=S_{s_3}=0.01$ m⁻¹

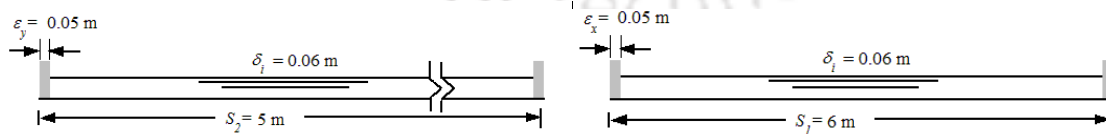
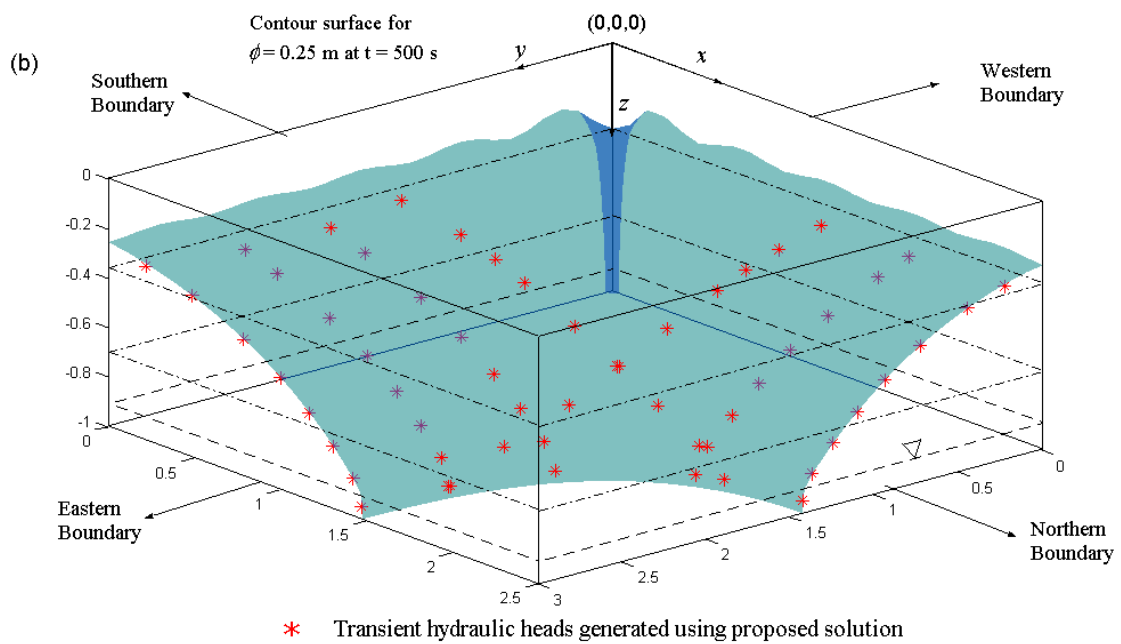
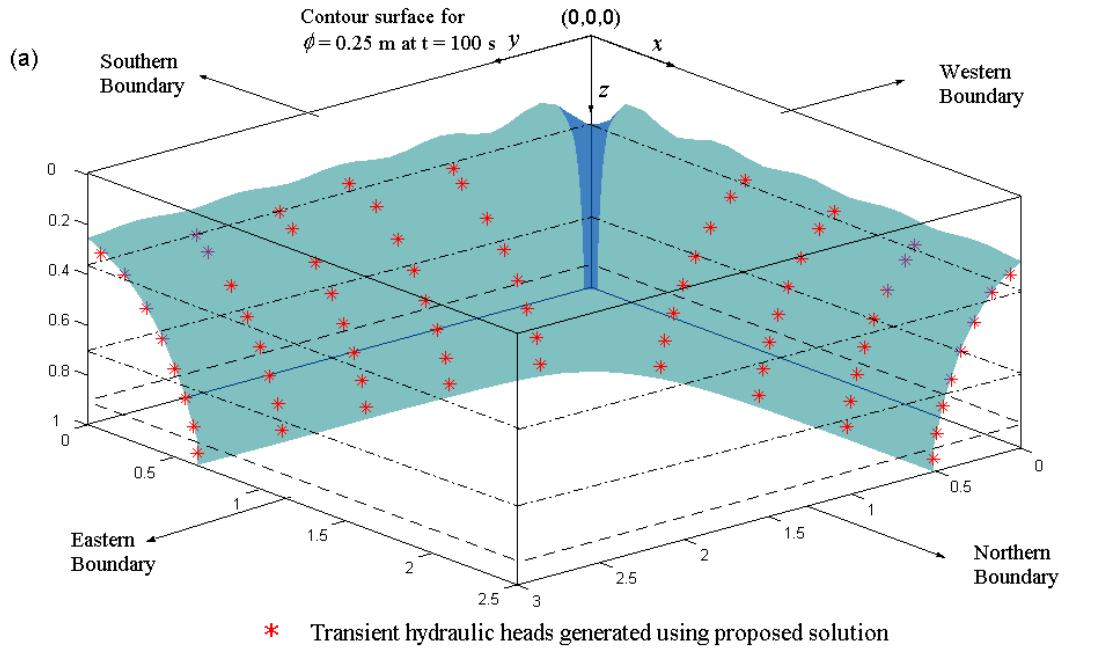


Fig 3.19. Comparison of transient hydraulic heads as obtained from the proposed solution at times $t = 100$ s and 500 s with the corresponding values as obtained from Sarmah and Barua's (2017) single-layered transient solution of the problem for anisotropic soils when the flow parameters of the problem (Fig. 3.16) are considered as $S_1 = 6$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.9$ m, $H_5 = 0.35$ m, $H_6 = 0.7$ m, $\delta = 0.06$ m, $K_{x_1} = K_{x_2} = K_{x_3} = 1$ m/day, $K_{y_1} = K_{y_2} = K_{y_3} = 1.5$ m/day, $K_{z_1} = K_{z_2} = K_{z_3} = 0.002$ m/day and $S_{s_1} = S_{s_2} = S_{s_3} = 0.01$ m⁻¹

3.2.3.4 MODFLOW Verifications of the Proposed Solution

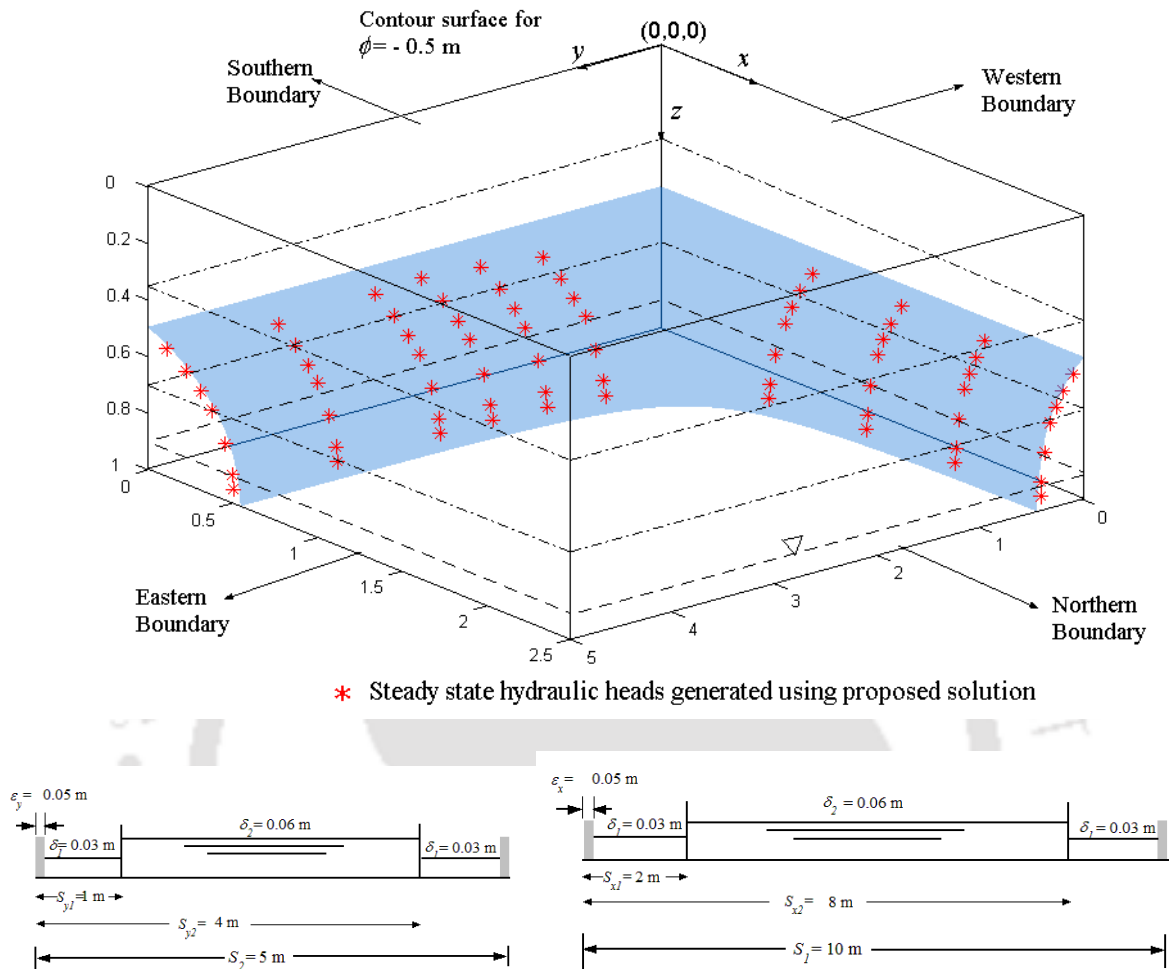
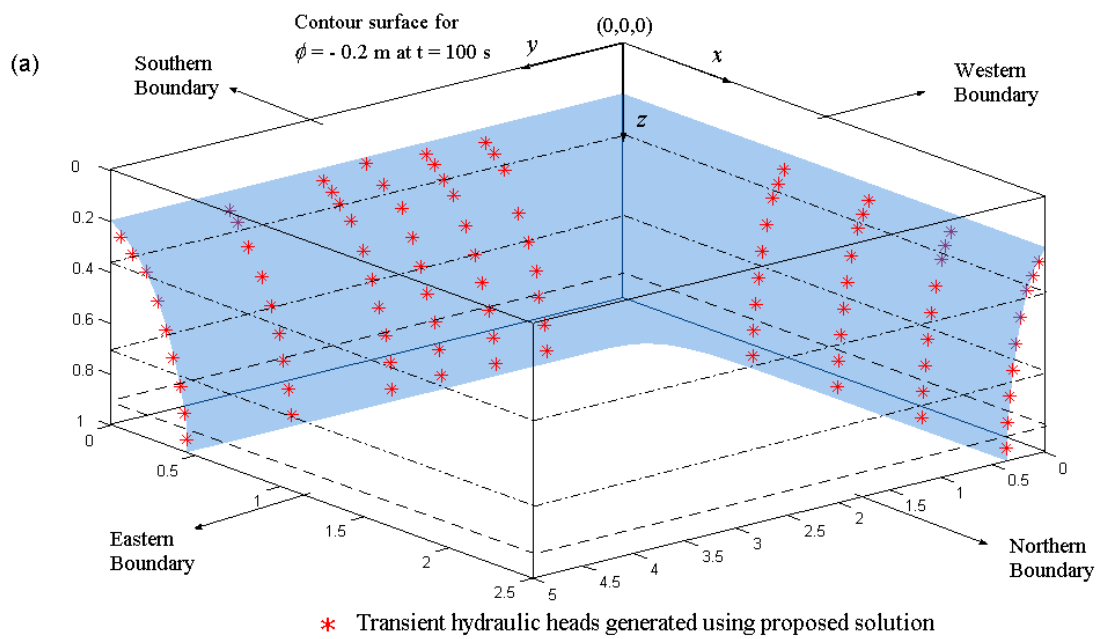


Fig. 3.20. Comparison of steady state hydraulic heads as obtained from the proposed solution with corresponding results as obtained from MODFLOW where the flow parameters of Fig. 3.16 are considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.9$ m, $H_5 = 0.35$ m, $H_6 = 0.7$ m, $\delta_1 = 0.03$ m, $\delta_2 = 0.06$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $d_{x_1} = 2$ m, $d_{x_2} = 8$ m, $d_{y_1} = 1$ m, $d_{y_2} = 4$ m, $K_{x_1} = 0.8$ m/day, $K_{y_1} = 1$ m/day, $K_{z_1} = 0.5$ m/day, $K_{x_2} = 1$ m/day, $K_{y_2} = 1.5$ m/day, $K_{z_2} = 0.75$ m/day, $K_{x_3} = 1$ m/day, $K_{y_3} = 2$ m/day and $K_{z_3} = 1$ m/day



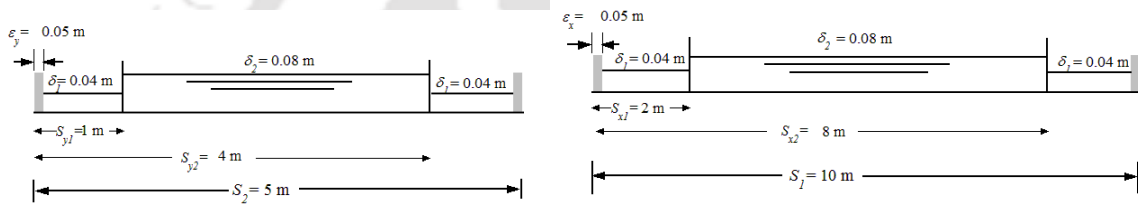
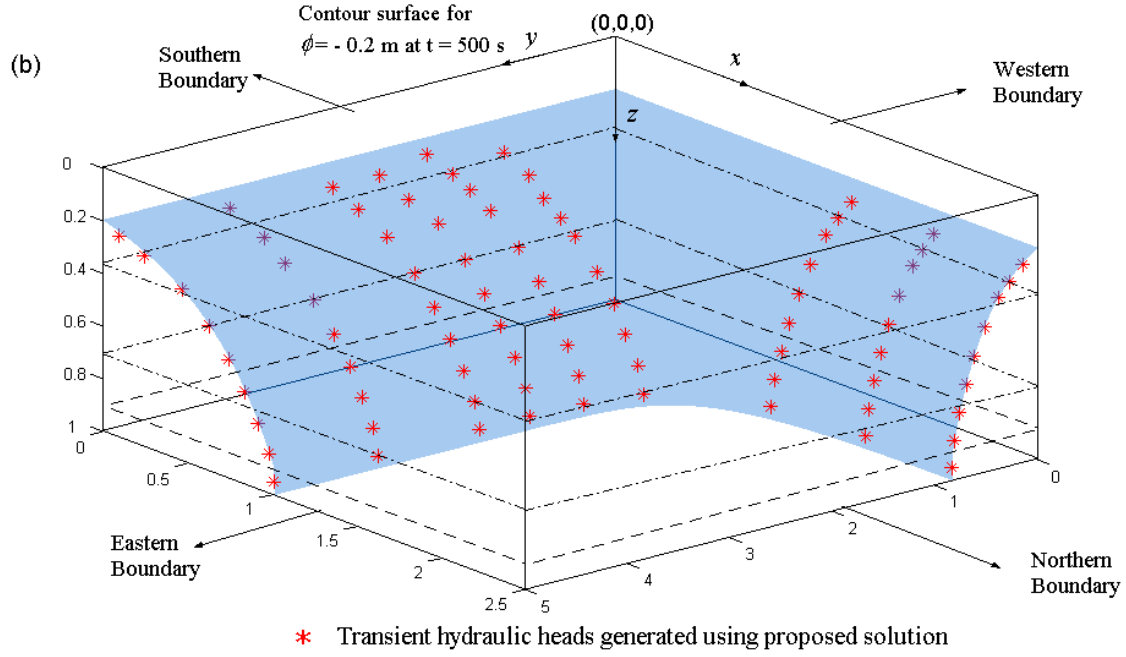
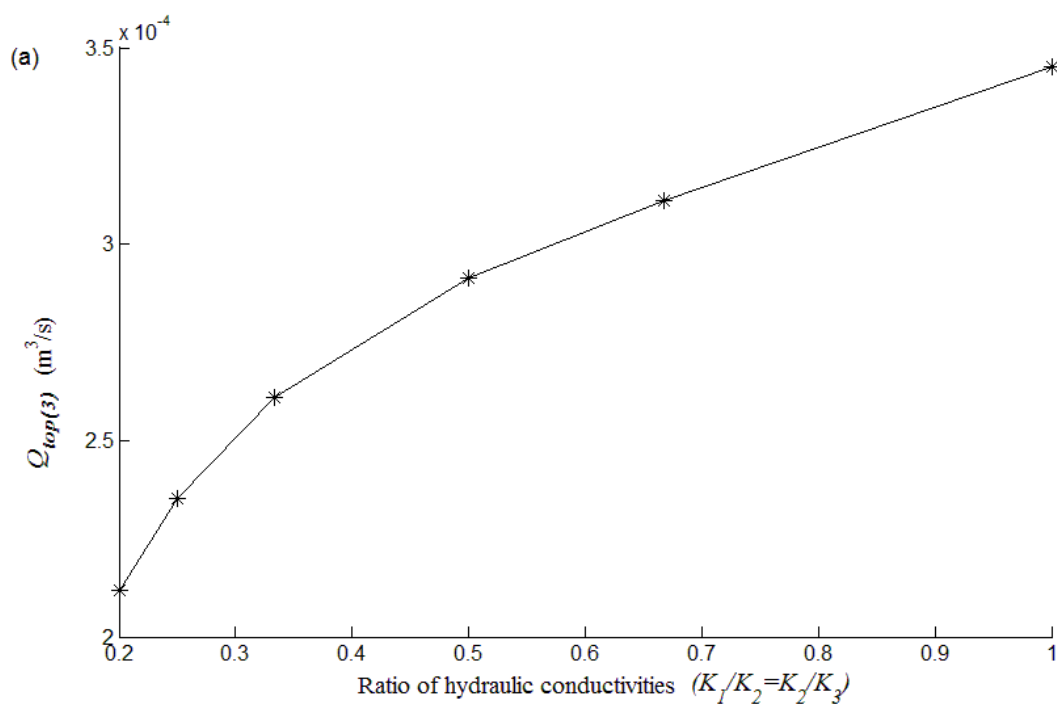


Fig 3.21. Comparison between transient hydraulic heads obtained from the proposed analytical solution and heads for the same flow problem obtained using MODFLOW model at (a) 100 s and (b) 500 s. The flow parameters of Fig. 3.16 have been considered as $S_1 = 10$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.9$ m, $H_5 = 0.35$ m, $H_6 = 0.7$ m, $\delta_1 = 0.04$ m, $\delta_2 = 0.08$ m, $\epsilon_x = \epsilon_y = 0.05$ m, $d_{x1} = 2$ m, $d_{x2} = 8$ m, $d_{y1} = 1$ m, $d_{y2} = 4$ m, $K_{x1} = 0.8$ m/day, $K_{y1} = 1.2$ m/day, $K_{z1} = 0.001$ m/day, $K_{x2} = 0.6$ m/day, $K_{y2} = 0.9$ m/day, $K_{z2} = 0.001$ m/day, $K_{x3} = 1.2$ m/day, $K_{y3} = 1.8$ m/day, $K_{z3} = 0.001$ m/day, $S_{s1} = 0.02$ m⁻¹, $S_{s2} = 0.015$ m⁻¹ and $S_{s3} = 0.03$ m⁻¹

3.3 Discussions

We will now make use of our developed solutions to study a few three-dimensional ponded situations. Firstly, we look into the variance of the top discharge for a flow configuration depending on the hydraulic conductivities of the three soil layers of the flow domain. For this purpose, we have considered a situation where the ditches are running dry and the soil layers are isotropic. Since the soil layers have been considered to be isotropic, for our convenience, we can denote the hydraulic conductivities of the top, middle and bottom soil layers as $K_{x1} = K_{y1} = K_{z1} = K_1$, $K_{x2} = K_{y2} = K_{z2} = K_2$ and $K_{x3} = K_{y3} = K_{z3} = K_3$, respectively. Keeping the hydraulic conductivity of the middle soil layer as 1 m/day ($K_{x2} = K_{y2} = K_{z2} = K_2 = 1$ m/day), we vary the hydraulic conductivities of the other soil layers so that $K_1 / K_2 = K_2 / K_3$ and evaluate the top discharges for the flow situations that emerge; Figs. 3.22(a) and

3.22(b) show variations of top discharge with conductivity ratio $K_1 / K_2 = K_2 / K_3$ for such scenarios when the other flow parameters of Fig. 3.16 are taken as has been mentioned. As can be seen from these figures, the top discharge for the studied situations is varying mostly non-linearly with the increase of conductivity ratio $K_1 / K_2 = K_2 / K_3$ when this ratio is less than unity but for higher values of this ratio, the top discharge can be observed to follow approximately a linear trend with the increase of this ratio. It should be noted a similar observation has been obtained regarding variation of top discharge with conductivity ratio while studying two-dimensional ponded drainage scenarios with our solutions developed in the previous chapter.



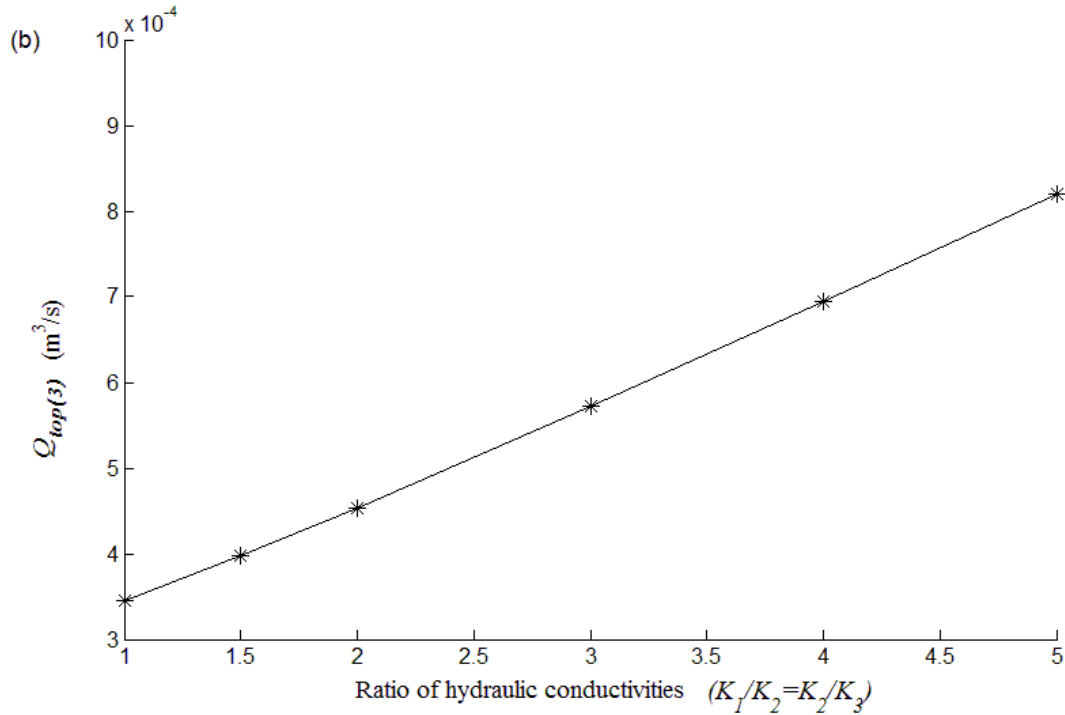


Fig. 3.22. Plot of steady state top discharge versus ratio of hydraulic conductivities of the top, middle and bottom soil layers when the parameters of a ponded drainage setting are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1$ m, $\delta = 0$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_2} = K_{y_2} = K_{z_2} = K_2 = 1$ m/day

Next, we study the relationship between the top discharge and the anisotropy of the soil layers of the flow domain. Towards realizing that, we consider a particular flow configuration of Fig. 3.16 where the soil layers are of equal thickness and the ditches are running dry and where the vertical hydraulic conductivities of the layers are taken as a progressively increasing one with depth with K_{z_1} , K_{z_2} and

K_{z_3} as 0.5 m/day, 1 m/day and 1.5 m/day, respectively. We then vary the anisotropy ratio (i.e., $\frac{K_{x_1}}{K_{z_1}} = \frac{K_{y_1}}{K_{z_1}} = \frac{K_{x_2}}{K_{z_2}} = \frac{K_{y_2}}{K_{z_2}} = \frac{K_{x_3}}{K_{z_3}} = \frac{K_{y_3}}{K_{z_3}}$) of the soil layers from 0.2 to 4 and note the

variation of the top discharge for the corresponding situations. Fig. 3.23 shows such a variation. As may be observed, for the considered flow scenarios, the top discharge is more sensitive to change in the anisotropy ratio of the layers at low values of this ratio and the slope of the discharge variation curve tends to decrease with the increase in the anisotropy ratio of the soil layers. If we, however, reverse the order of vertical hydraulic conductivity of the layers (i.e., take $K_{z_1} = 1.5$ m/day, $K_{z_2} = 1$ m/day and $K_{z_3} = 0.5$ m/day) and then proceed to trace the variation of top discharge with anisotropy ratio of the layers for the above flow scenarios, the top discharge can then be seen [Fig. (3.24)] to be much more affected with the change in anisotropy ratio of the layers as compared to the previous situation. Thus, a mere change in the vertical conductivity of the layers alone may result in a considerable change in

the top discharge of a multi-layered ponded ditch drainage system, particularly if the vertical conductivity of the top layer is much higher than the vertical conductivities of the lower layers.

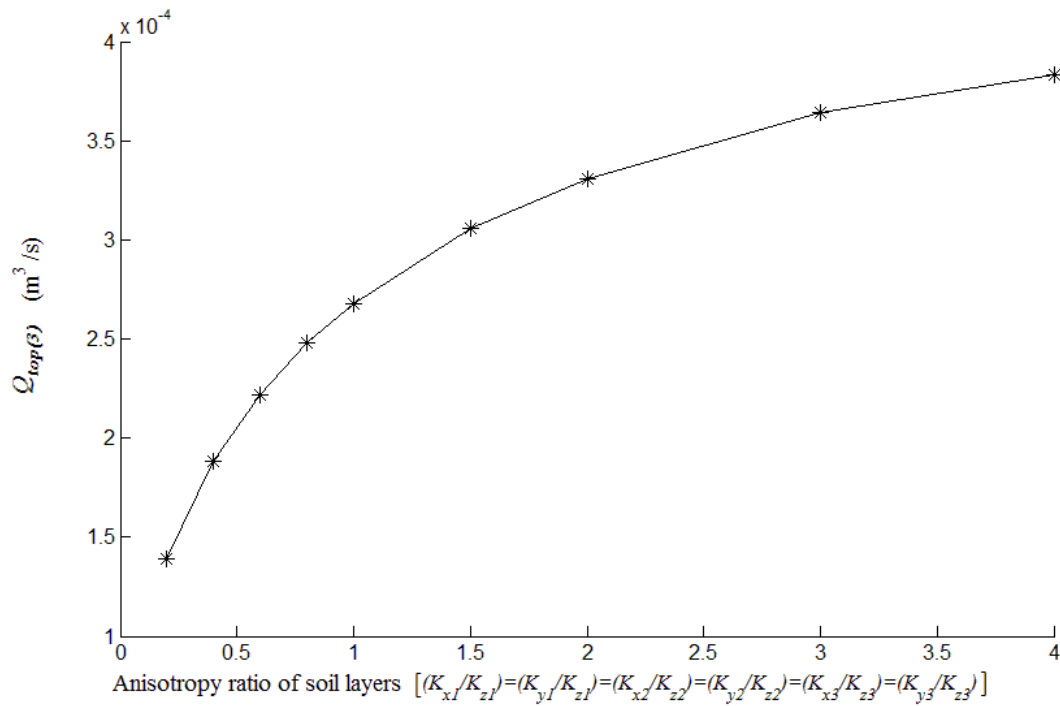


Fig. 3.23. Plot of steady state top discharge values versus anisotropy ratio $K_{x_1}/K_{z_1} = K_{y_1}/K_{z_1} = K_{x_2}/K_{z_2} = K_{y_2}/K_{z_2} = K_{x_3}/K_{z_3} = K_{y_3}/K_{z_3}$ of the soil layers when the parameters of the flow problem are taken as $S_1 = 8 \text{ m}$, $S_2 = 8 \text{ m}$, $h = 1.5 \text{ m}$, $H_1 = 1.5 \text{ m}$, $H_5 = 0.5 \text{ m}$, $H_6 = 1 \text{ m}$, $\delta = 0 \text{ m}$, $K_{z_1} = 0.5 \text{ m/day}$, $K_{z_2} = 1 \text{ m/day}$ and $K_{z_3} = 1.5 \text{ m/day}$

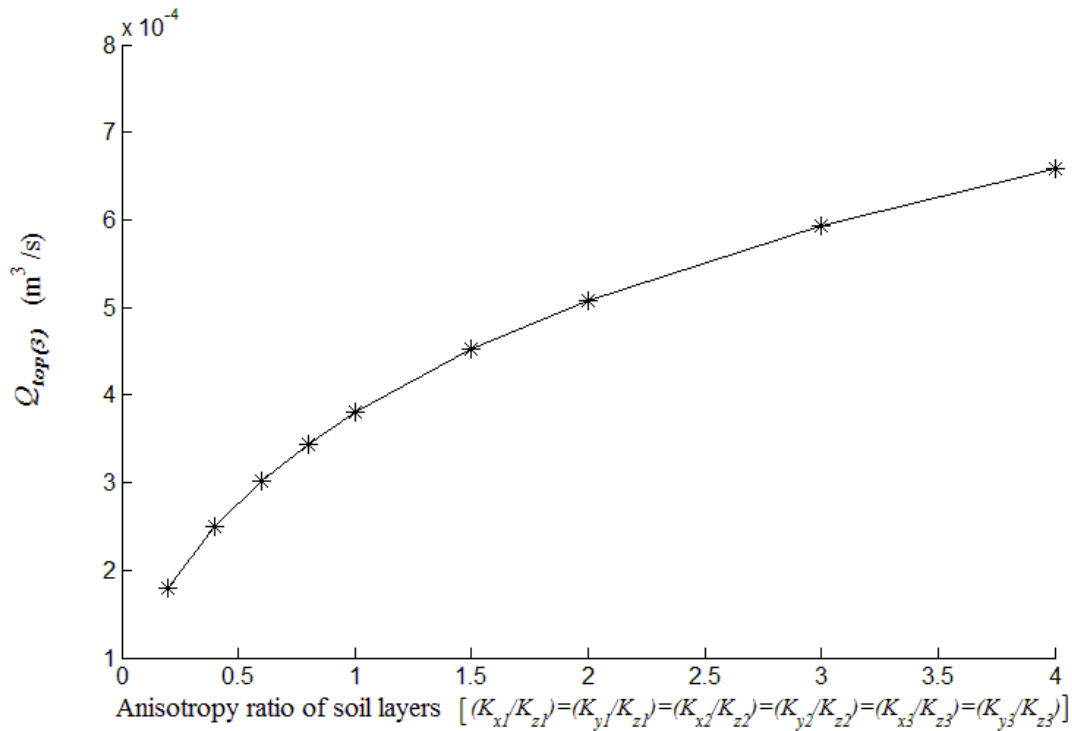


Fig. 3.24. Plot of steady state top discharge values versus anisotropy ratio $K_{x_1}/K_{z_1} = K_{y_1}/K_{z_1} = K_{x_2}/K_{z_2} = K_{y_2}/K_{z_2} = K_{x_3}/K_{z_3} = K_{y_3}/K_{z_3}$ of the soil layers when the parameters of the flow problem are taken as $S_1 = 8 \text{ m}$, $S_2 = 8 \text{ m}$, $h = 1.5 \text{ m}$, $H_1 = 1.5 \text{ m}$, $H_5 = 0.5 \text{ m}$, $H_6 = 1 \text{ m}$, $\delta = 0 \text{ m}$, $K_{z_1} = 1.5 \text{ m/day}$, $K_{z_2} = 1 \text{ m/day}$ and $K_{z_3} = 0.5 \text{ m/day}$

Next, we now investigate the travel times of water particles in their journeys starting from the surface of the soil to the recipient ditches of a few steady state three-dimensional ponded ditch drainage scenarios. We again make use of Grove et al.'s (1970) procedures to do the same as has been done in the previous chapter while estimating travel times of water particles for a few two-dimensional ponded drainage situations. From Figs 3.25 and 3.27, we see that when hydraulic conductivities of the soil layers decrease with depth then the travel times of water particles from regions away from the ditch faces are considerably higher than the corresponding values for drainage situations where the hydraulic conductivities of the soil layers increase with depth. However, travel times of water particles closer to the ditches are generally not affected to the same extent when the vertical conductivity distributions of the layers get reversed. Also, it can be inferred from the Figs 3.27 and 3.28 that implementing a staggered variable ponding atop the flow domain, in situations where the hydraulic conductivities of the constituent soil layers decrease in the vertically downward direction, leads to a drastic decrease in the travel times of water particles originating from central locations of the field and traveling all the way up to the recipient ditches. On the other hand, Figs 3.25 and 3.26 seem to indicate that adopting a similar measure results in a much lesser decrease in travel times from similar locations when the hydraulic conductivities of soil layers tend to increase with depth.

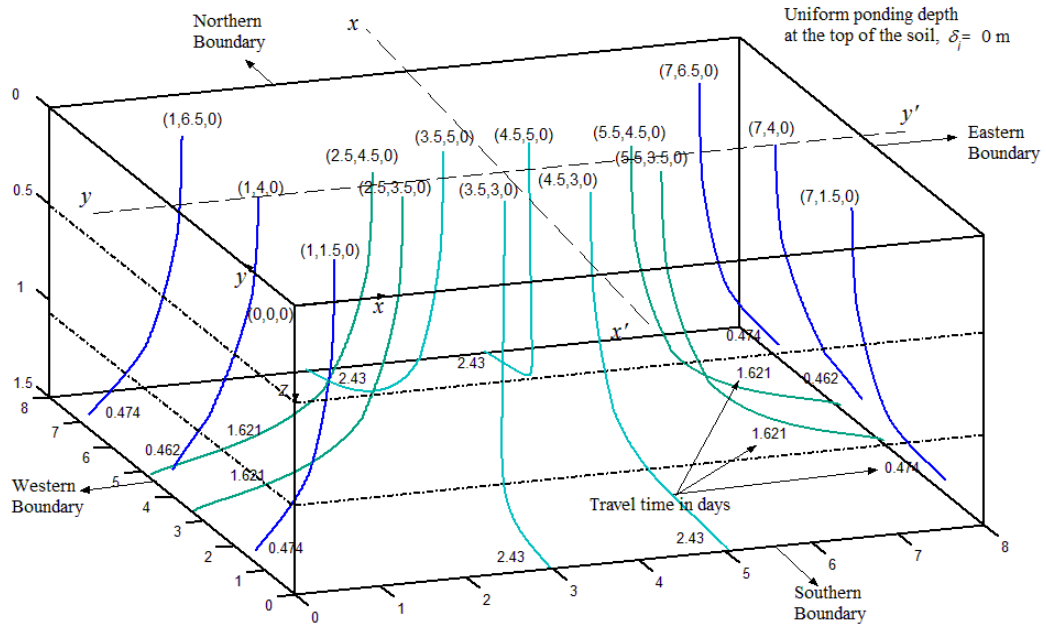
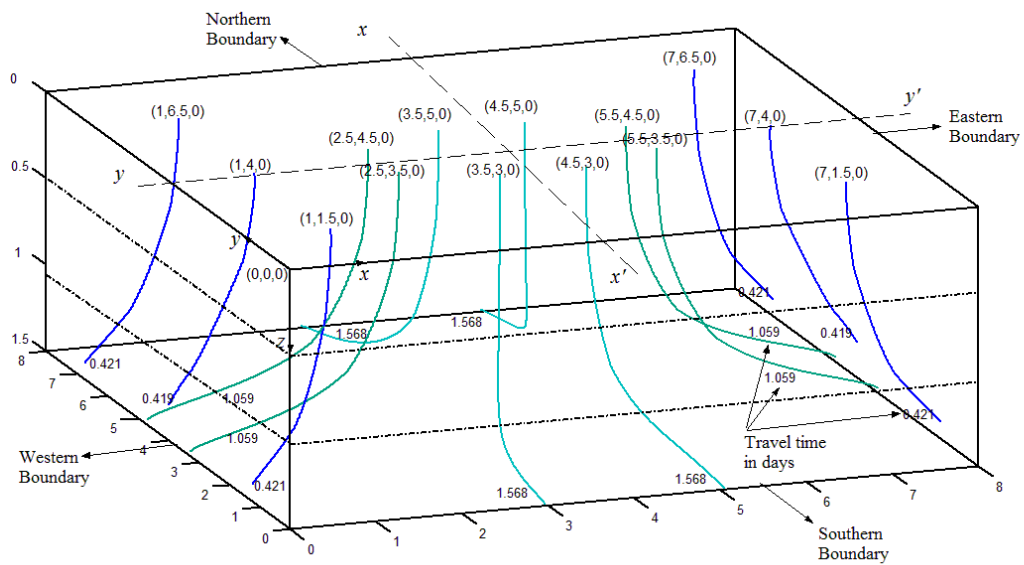


Fig. 3.25. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = \eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = K_{z_1} = 1$ m/day, $K_{x_2} = K_{y_2} = K_{z_2} = 2$ m/day and $K_{x_3} = K_{y_3} = K_{z_3} = 3$ m/day



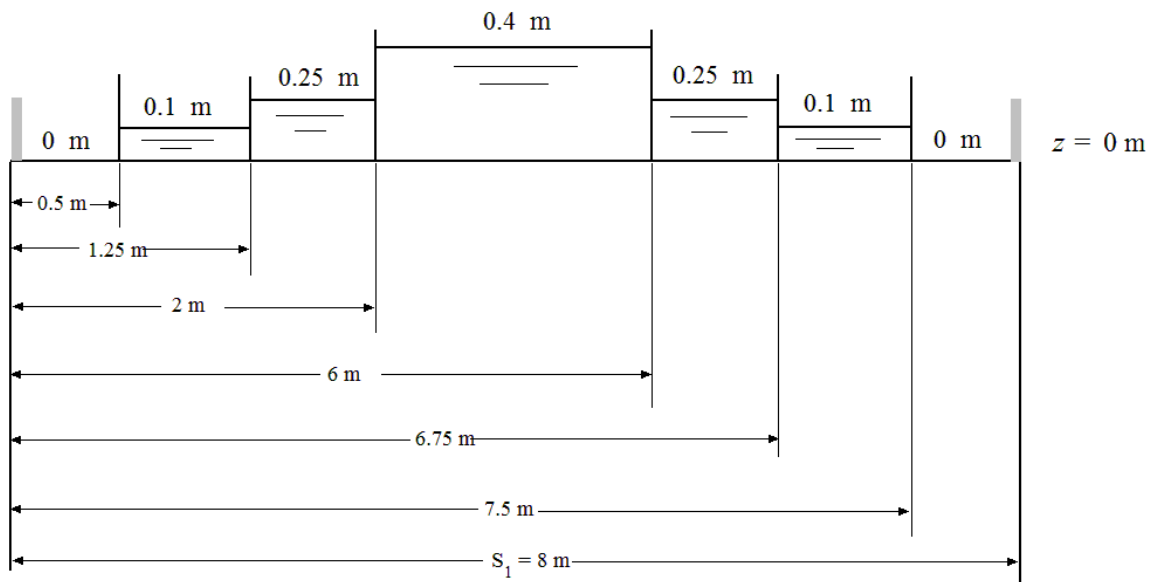
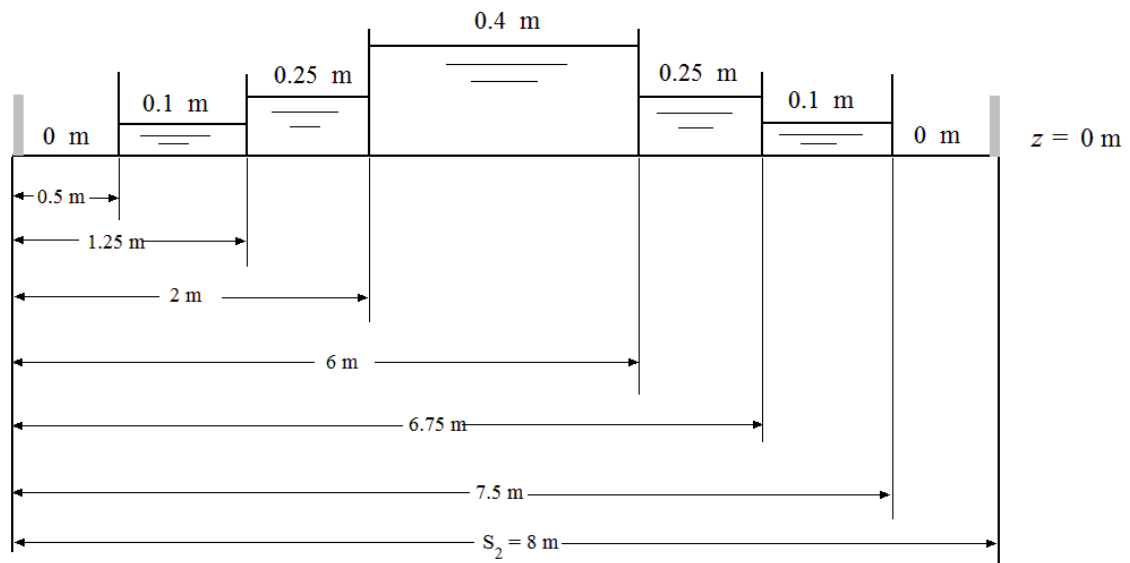


Fig. 3.26. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.25$ m, $\delta_4 = 0.4$ m, $d_{x1} = 0.5$ m, $d_{x2} = 1.25$ m, $d_{x3} = 2$ m, $d_{x4} = 6$ m, $d_{x5} = 6.75$ m, $d_{x6} = 7.5$ m, $d_{y1} = 0.5$ m, $d_{y2} = 1.25$ m, $d_{y3} = 2$ m, $d_{y4} = 6$ m, $d_{y5} = 6.75$ m, $d_{y6} = 7.5$ m, $\eta_1 = \eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = K_{z_1} = 1$ m/day, $K_{x_2} = K_{y_2} = K_{z_2} = 2$ m/day and $K_{x_3} = K_{y_3} = K_{z_3} = 3$ m/day

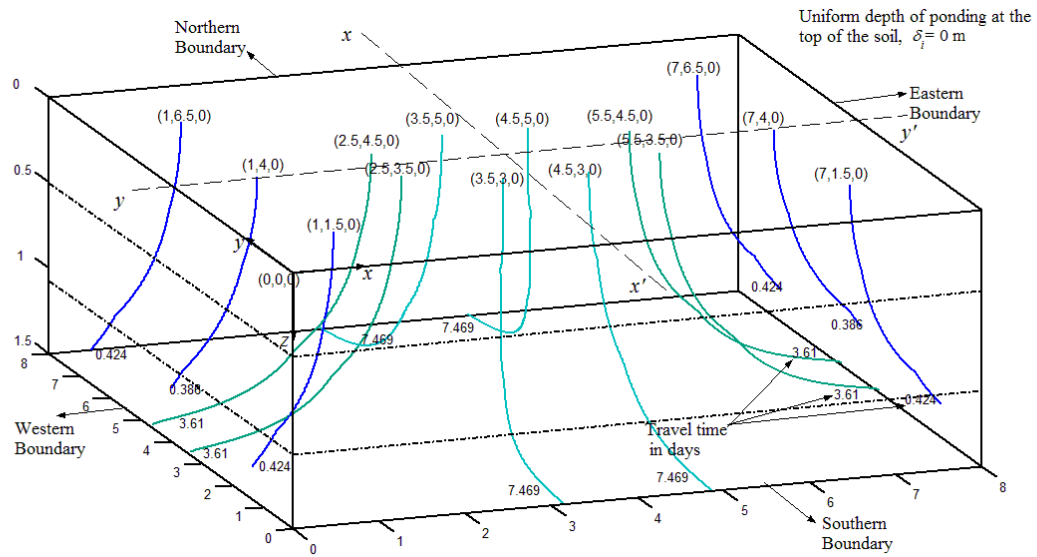
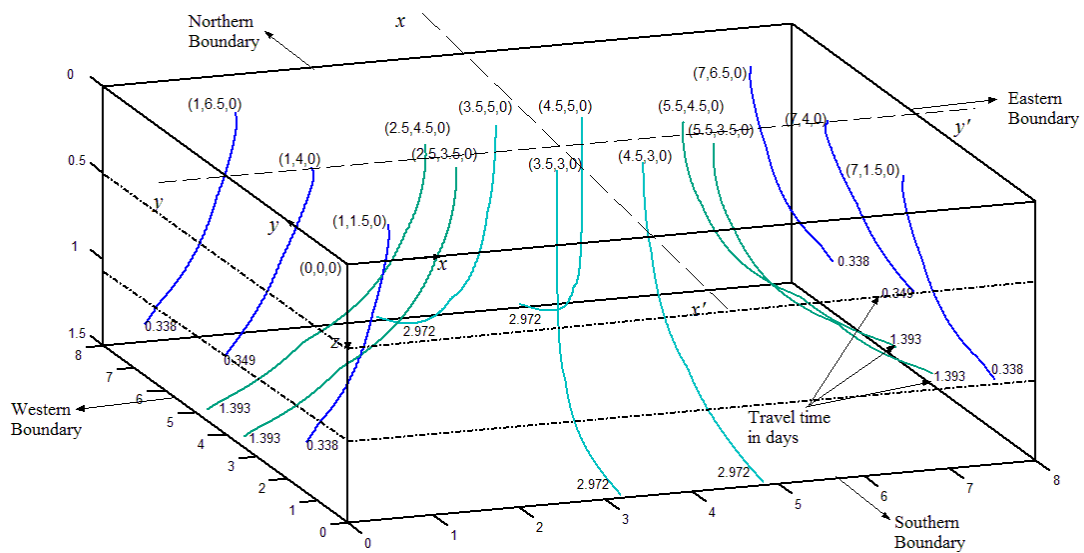


Fig. 3.27. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8 \text{ m}$, $S_2 = 8 \text{ m}$, $h = 1.5 \text{ m}$, $H_1 = 1.5 \text{ m}$, $H_5 = 0.5 \text{ m}$, $H_6 = 1.1 \text{ m}$, $\varepsilon_x = \varepsilon_y = 0.05 \text{ m}$, $\delta = 0 \text{ m}$, $\eta_1 = \eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = K_{z_1} = 3 \text{ m/day}$, $K_{x_2} = K_{y_2} = K_{z_2} = 2 \text{ m/day}$ and $K_{x_3} = K_{y_3} = K_{z_3} = 1 \text{ m/day}$



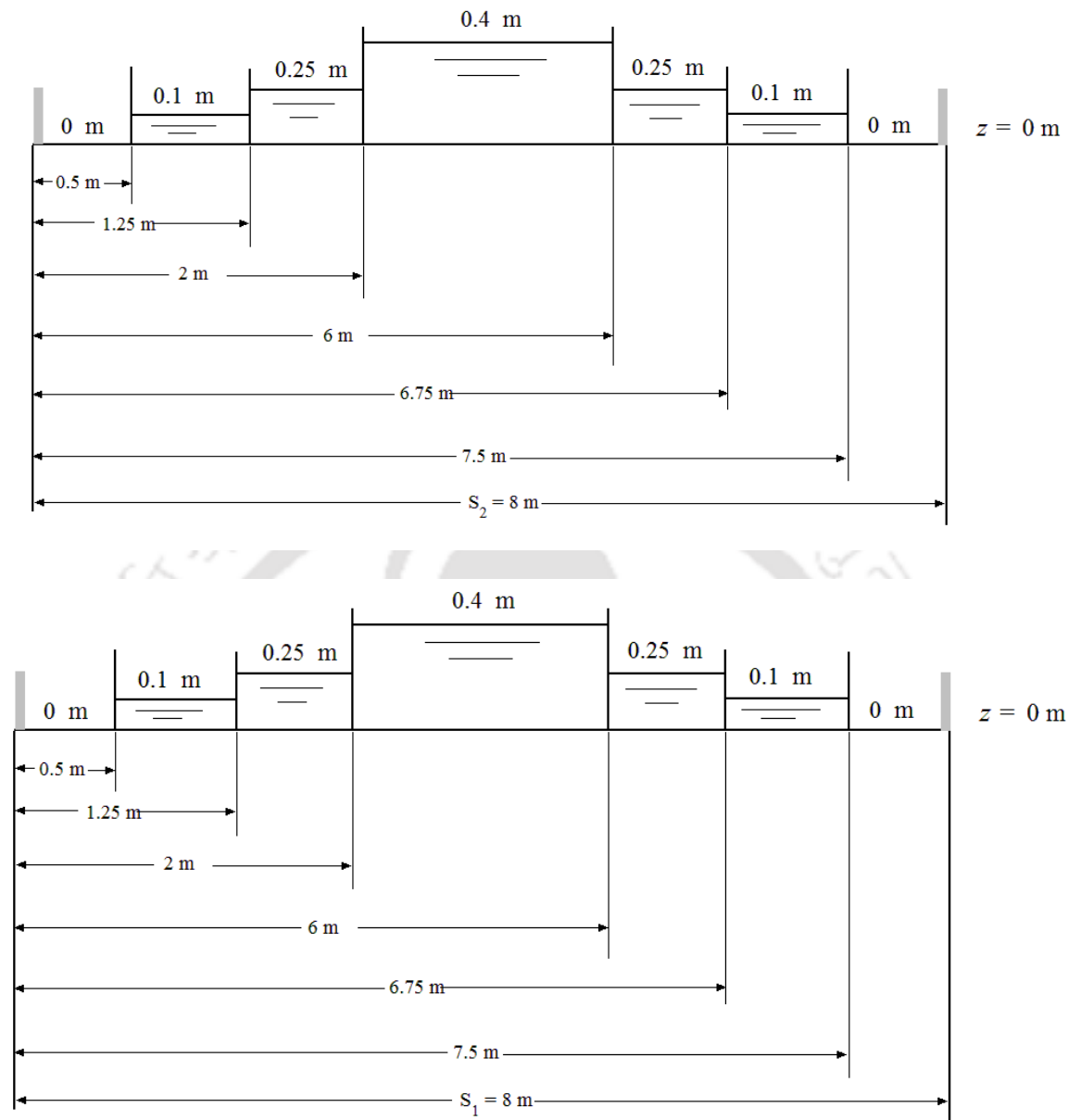


Fig. 3.28. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.25$ m, $\delta_4 = 0.4$ m, $d_{x1} = 0.5$ m, $d_{x2} = 1.25$ m, $d_{x3} = 2$ m, $d_{x4} = 6$ m, $d_{x5} = 6.75$ m, $d_{x6} = 7.5$ m, $d_{y1} = 0.5$ m, $d_{y2} = 1.25$ m, $d_{y3} = 2$ m, $d_{y4} = 6$ m, $d_{y5} = 6.75$ m, $d_{y6} = 7.5$ m, $\eta_1 = \eta_2 = \eta_3 = 0.3$, $K_{x1} = K_{y1} = K_{z1} = 3$ m/day, $K_{x2} = K_{y2} = K_{z2} = 2$ m/day and $K_{x3} = K_{y3} = K_{z3} = 1$ m/day

Further, as can be inferred from Figs. 3.29 and 3.30, the level of water in the drains plays a crucial role in determining travel times of water particles in a three-dimensional ponded ditch drainage system – the travel times of the particles to a ditch for the same ponded drainage

scenario increase considerably when the water head in the ditch is allowed to increase; this is all the more prevalent if the comparison is being made with respect to a situation where the ditch is running dry. This finding is in tune with that observed for the two-dimensional ponded drainage situation where, as have been previously stated, among other factors remaining the same the flow to a ditch drain may be greatly influenced by the level of water in it. It can be also inferred from the pathlines in Fig. 3.30 that water from regions closer to the ditch faces in a ponded layered drainage system may mostly traverse only on the top layers of the soil before being discharged to the collecting drains.

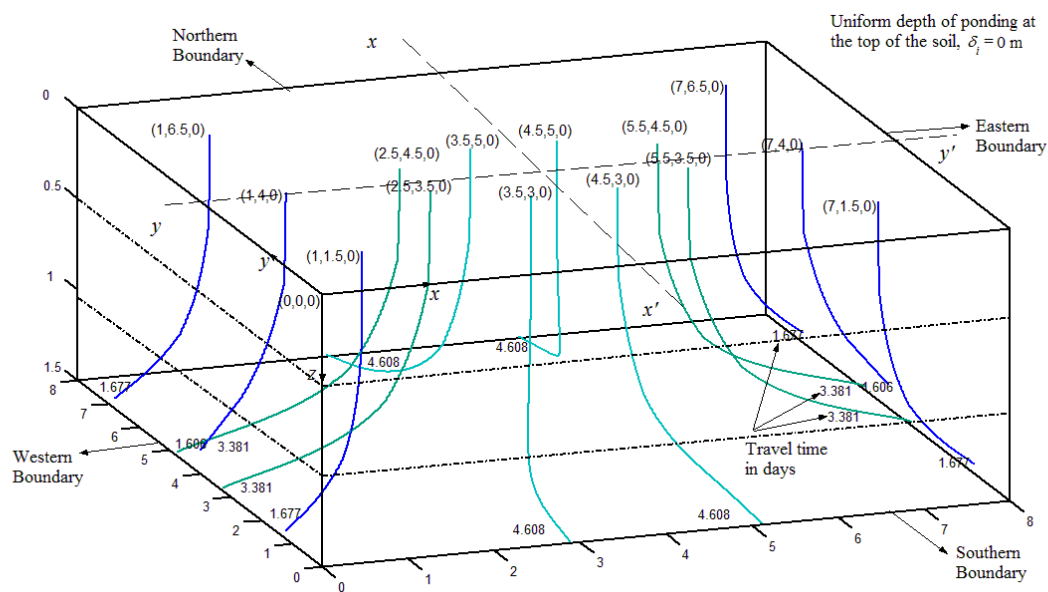


Fig. 3.29. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 1.5$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = 0.35$, $\eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = 0.3$ m/day, $K_{z_1} = 0.2$ m/day, $K_{x_2} = K_{y_2} = 1.5$ m/day, $K_{z_2} = 0.5$ m/day, $K_{x_3} = K_{y_3} = 2$ m/day, $K_{z_3} = 1$ m/day

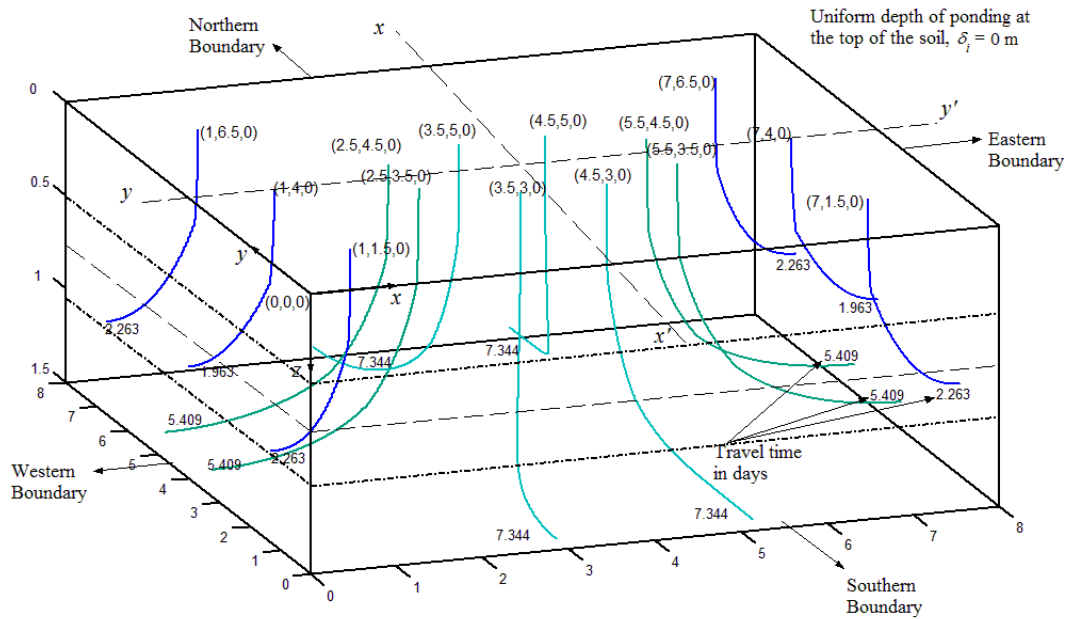


Fig. 3.30. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.10) are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1.5$ m, $H_1 = 0.75$ m, $H_5 = 0.5$ m, $H_6 = 1.1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = 0.35$, $\eta_2 = \eta_3 = 0.3$, $K_{x_1} = K_{y_1} = 0.3$ m/day, $K_{z_1} = 0.2$ m/day, $K_{x_2} = K_{y_2} = 1.5$ m/day, $K_{z_2} = 0.5$ m/day, $K_{x_3} = K_{y_3} = 2$ m/day, $K_{z_3} = 1$ m/day

Next, we consider two drainage situations where the thickness and the directional conductivities of the middle soil layer are much higher than the other two layers. As can be seen from Figs. 3.31 and 3.32, the effect of the level of water in the ditches on the travel times is not very significant for these cases. Also, it may be mentioned that very miniscule amount of water actually gets infiltrated into the third soil layer if the conductivities and thickness of the bottom layer are relatively much lower than the corresponding values of the upper layers.

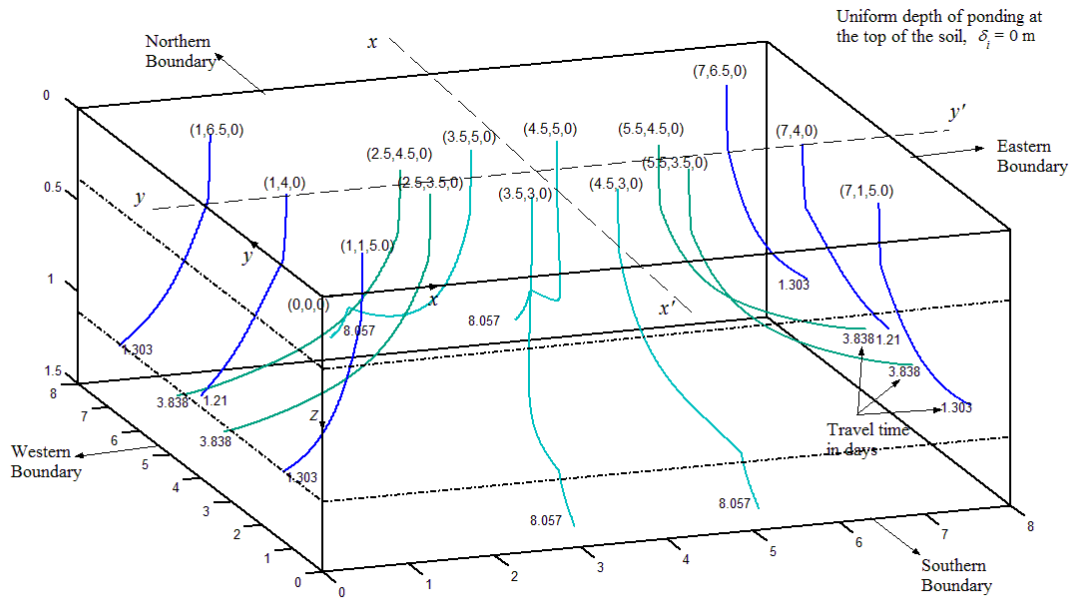


Fig. 3.31. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 8 \text{ m}$, $S_2 = 8 \text{ m}$, $h = 1.5 \text{ m}$, $H_1 = 1.5 \text{ m}$, $H_5 = 0.35 \text{ m}$, $H_6 = 1.15 \text{ m}$, $\varepsilon_x = \varepsilon_y = 0.05 \text{ m}$, $\delta = 0 \text{ m}$, $\eta_1 = 0.35$, $\eta_2 = 0.3$, $\eta_3 = 0.4$, $K_{x_1} = K_{y_1} = 0.3 \text{ m/day}$, $K_{z_1} = 0.2 \text{ m/day}$, $K_{x_2} = K_{y_2} = 1.5 \text{ m/day}$, $K_{z_2} = 1 \text{ m/day}$, $K_{x_3} = K_{y_3} = 0.1 \text{ m/day}$, $K_{z_3} = 0.05 \text{ m/day}$

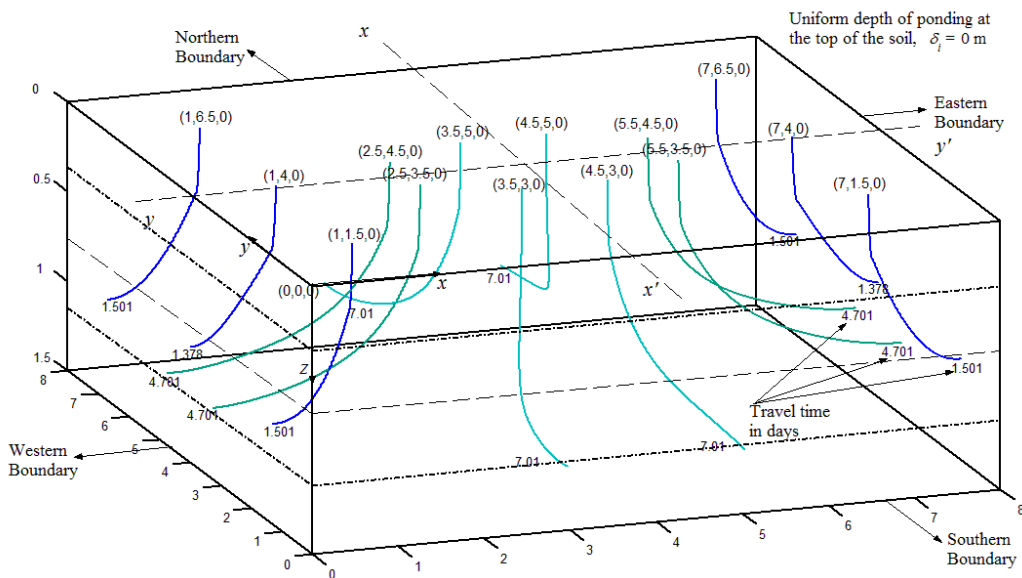


Fig. 3.32. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.10) are taken as $S_1 = 8 \text{ m}$, $S_2 = 8 \text{ m}$, $h = 1.5 \text{ m}$, $H_1 = 0.75 \text{ m}$, $H_5 = 0.35 \text{ m}$, $H_6 = 1.15 \text{ m}$, $\varepsilon_x = \varepsilon_y = 0.05 \text{ m}$, $\delta = 0 \text{ m}$, $\eta_1 = 0.35$, $\eta_2 = 0.3$, $\eta_3 = 0.4$, $K_{x_1} = K_{y_1} = 0.3 \text{ m/day}$, $K_{z_1} = 0.2 \text{ m/day}$, $K_{x_2} = K_{y_2} = 1.5 \text{ m/day}$, $K_{z_2} = 0.5 \text{ m/day}$, $K_{x_3} = K_{y_3} = 0.1 \text{ m/day}$, $K_{z_3} = 0.05 \text{ m/day}$

The presence of muddy and plow sole layers in paddy fields may greatly inhibit movement of infiltrating water in these fields (Chen et al. 2002; Huang et al. 2003; Liu et al. 2005) and their effect on the hydraulics of a two-dimensional ponded drainage system has already been touched upon in the previous chapter. In the flow situations of Figs. 3.33 and 3.34, an attempt is being made to study how the presence or absence of a plow sole layer in a ponded paddy field affects the movement of drainage water in a three-dimensional ditch drainage system. As may be observed from these figures, here also, like in the two-dimensional ponded situations, the occurrence of muddy and plow sole layers close to the surface of a drained field may greatly extend the travel times of water particles to the drains, particularly for those originating from surficial locations close to the drains. However, the presence of these layers does not seem to affect the travel times of particles starting from areas close to the centre of the field in a major way.

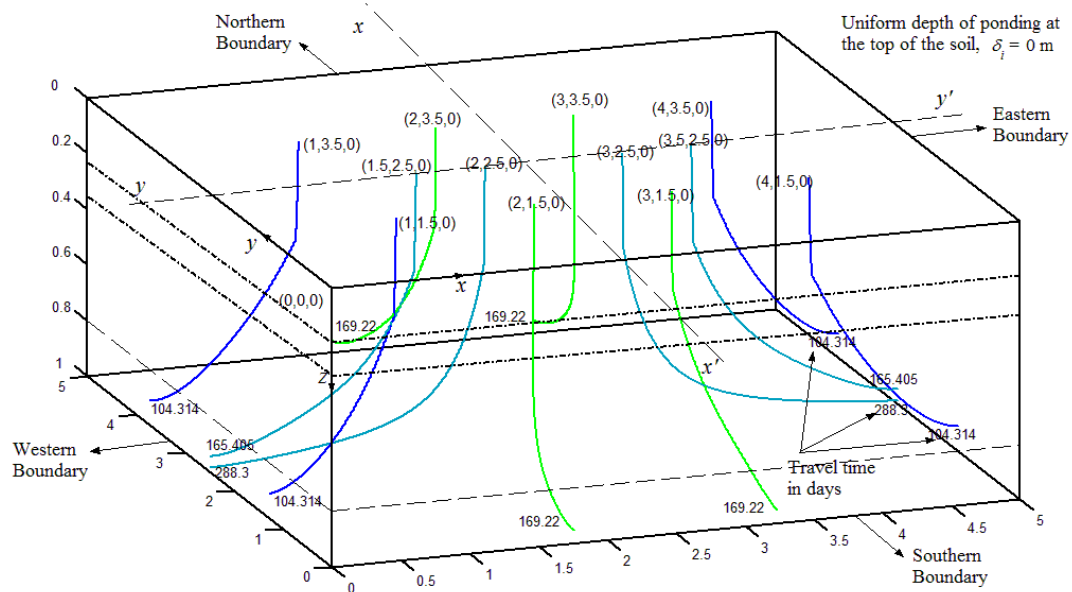


Fig. 3.33. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 5$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.8$ m, $H_5 = 0.25$ m, $H_6 = 0.35$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = \eta_2 = 0.55$, $\eta_3 = 0.45$, $K_{x_1} = K_{y_1} = K_{z_1} = 0.005$ m/day, $K_{x_2} = K_{y_2} = K_{z_2} = 0.003$ m/day, and $K_{x_3} = K_{y_3} = K_{z_3} = 0.03$ m/day

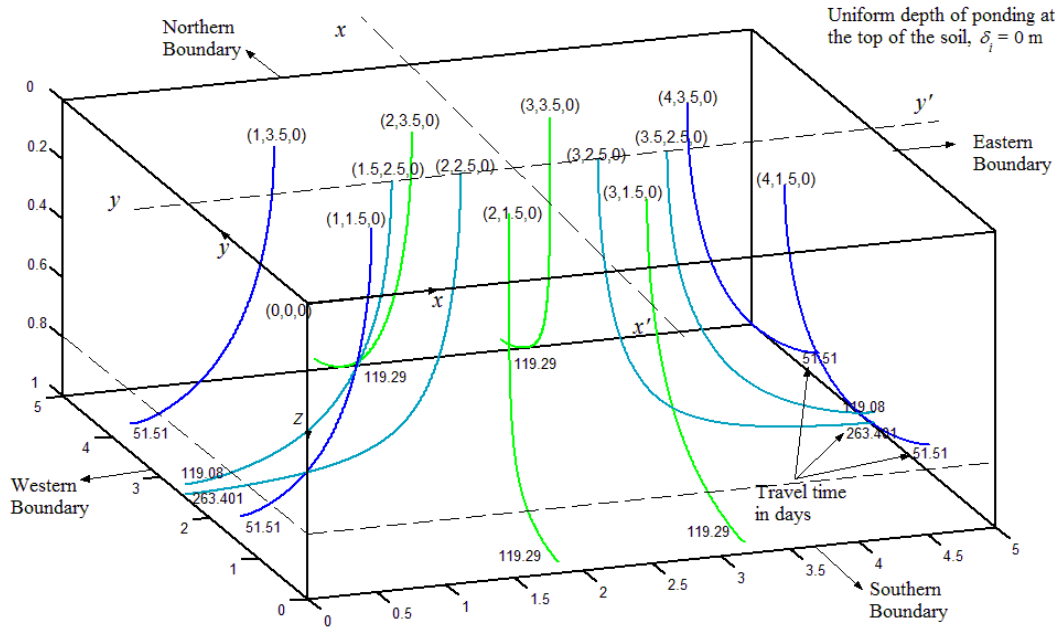


Fig. 3.34. Travel times (in days) of water particles starting from the surface of a three-dimensional ponded ditch drainage system to recipient drains when the flow parameters of the system (Fig. 3.16) are taken as $S_1 = 5$ m, $S_2 = 5$ m, $h = 1$ m, $H_1 = 0.8$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $\delta = 0$ m, $\eta_1 = 0.55$, $\eta_2 = 0.55$, $\eta_3 = 0.45$, $K_{x_1} = K_{y_1} = 0.03$ m/day, $K_{z_1} = 0.03$ m/day, $K_{x_2} = K_{y_2} = 0.03$ m/day, $K_{z_2} = 0.03$ m/day, $K_{x_3} = K_{y_3} = 0.03$ m/day, $K_{z_3} = 0.03$ m/day

We next study the top discharge distributions for a few three-dimensional ditch drainage scenarios of Fig. 3.16, where now the top discharges within predefined surficial locations are being expressed as a percentage of the total discharge through the top surface of the soil. A few stream surfaces corresponding to these situations have also been plotted [(Figs. 3.35, 3.36(a), 3.36(b), 3.37(a) and 3.37(b)] to illustrate in detail the flow patterns for each of these cases. From Figs 3.35, 3.36(a) and 3.37(a), we can conclude that top discharge distribution in a three-dimensional ponded ditch drainage system is profoundly impacted by the layeredness of a soil profile and that neglecting this stratification may lead to an erroneous understanding of movement of drainage water through such a soil column. Also from Figs. 3.36(b) and 3.37(b), we see that introduction of a gradually increasing ponding pyramid towards the centre of a three-dimensional ponded ditch drainage system may result in a considerable improvement in the uniformity of water movement in such a system as compared to a situation where the ponding head at the surface of the field is kept as a constant. This finding is in tune with what has been observed in a two-dimensional ponded ditch drainage system as well, where also the introduction of a progressively increasing ponding head towards half-way distance between the drains at the surface of the soil has been found to improve the uniformity of water movement in such a system to a great extent as compared to a situation where the surface of the soil is being imposed with a constant depth of ponding. These examples also seem to suggest that when the hydraulic conductivities of the upper soil layer is lower than that of the other soil layers, uniformity of top discharge distribution of a three-

dimensional drainage system is more in comparison to the case where the top soil layer has the same or larger hydraulic conductivity values than the lower layers – this is been observed to be true even when such a system is being imposed with a uniform depth of ponding at the surface of the soil.

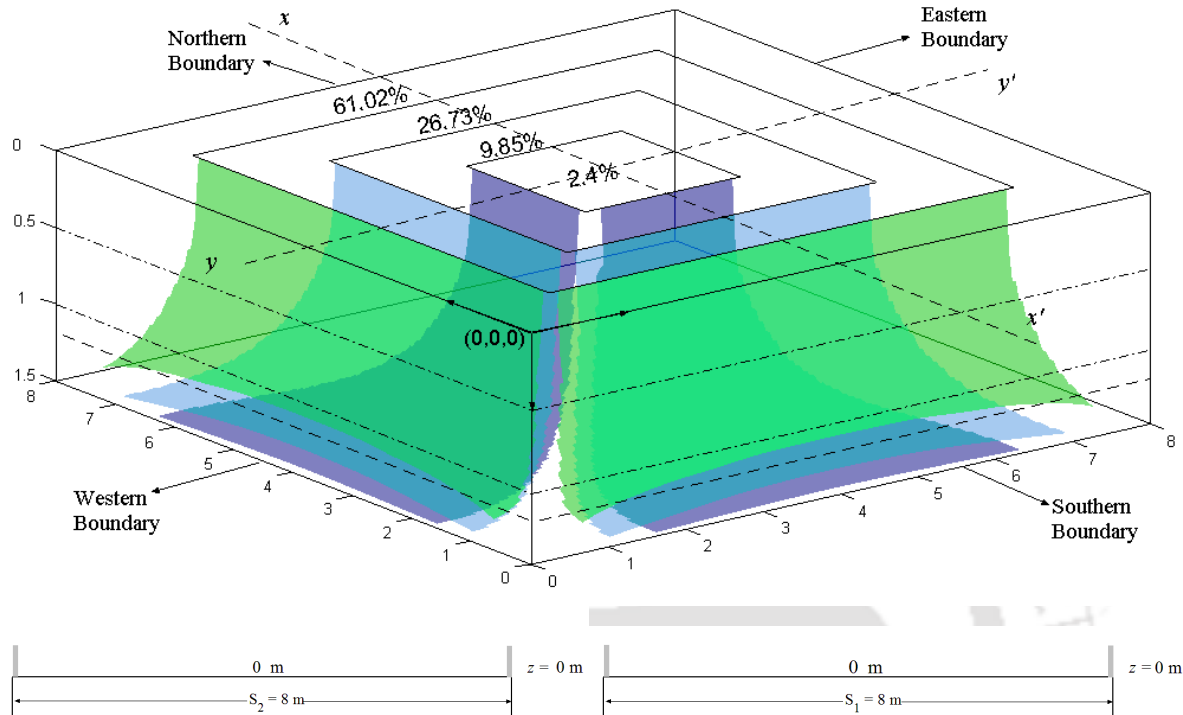
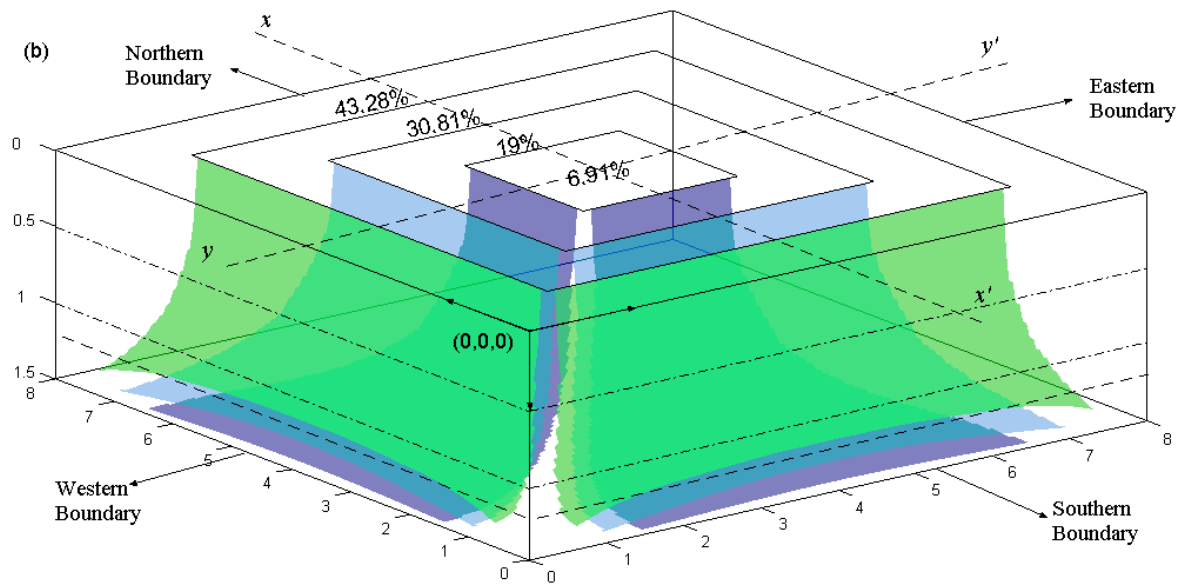
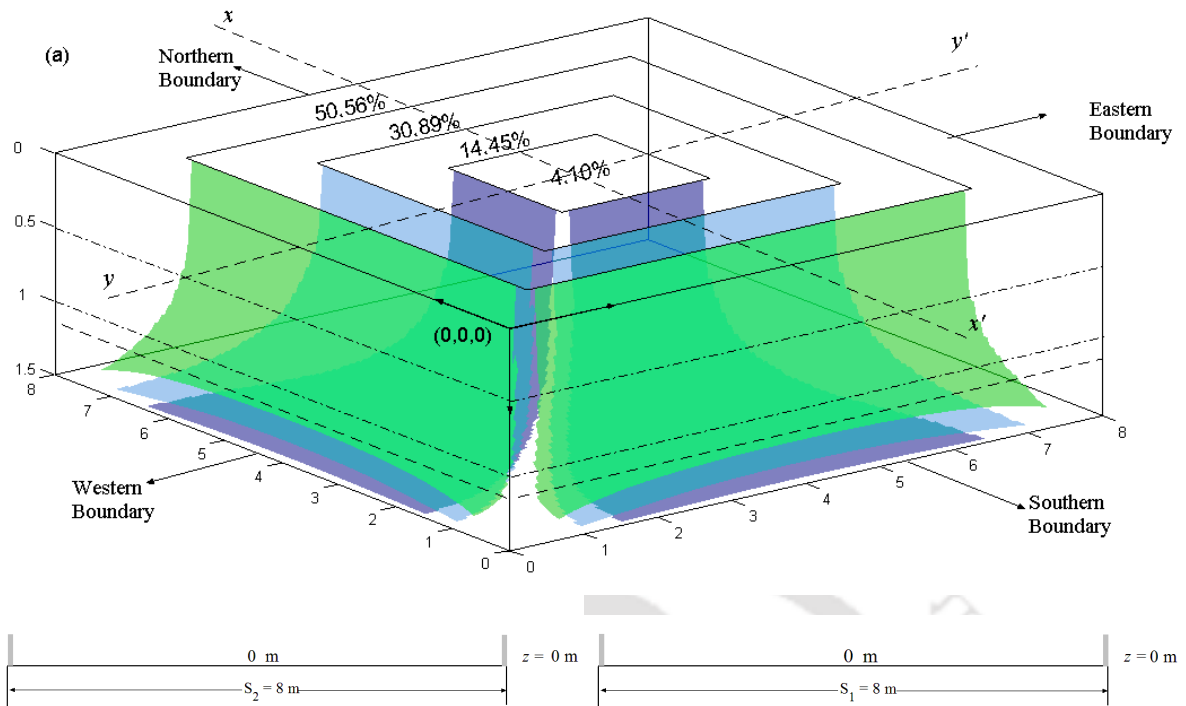


Fig. 3.35. A few stream surfaces and percentage distribution of the top discharge $Q_{top(3)}$ when the flow parameters of Fig 3.16 are taken as $S_1 = 8 \text{ m}$, $S_2 = 8 \text{ m}$, $h = 1.5 \text{ m}$, $H_1 = 1.2 \text{ m}$, $H_5 = 0.5 \text{ m}$, $H_6 = 1 \text{ m}$, $\delta = 0 \text{ m}$, $\varepsilon_x = \varepsilon_y = 0.05 \text{ m}$, $K_{x_1} = K_{x_2} = K_{x_3} = 2 \text{ m/day}$, $K_{y_1} = K_{y_2} = K_{y_3} = 2 \text{ m/day}$ and $K_{z_1} = K_{z_2} = K_{z_3} = 1 \text{ m/day}$



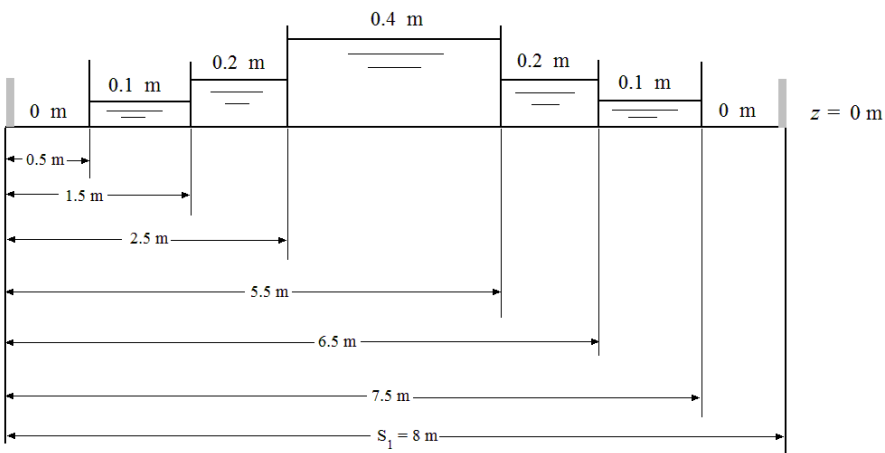
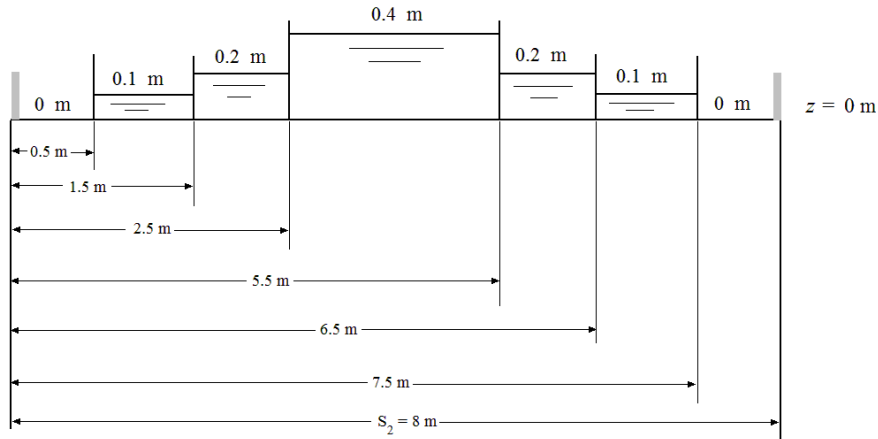
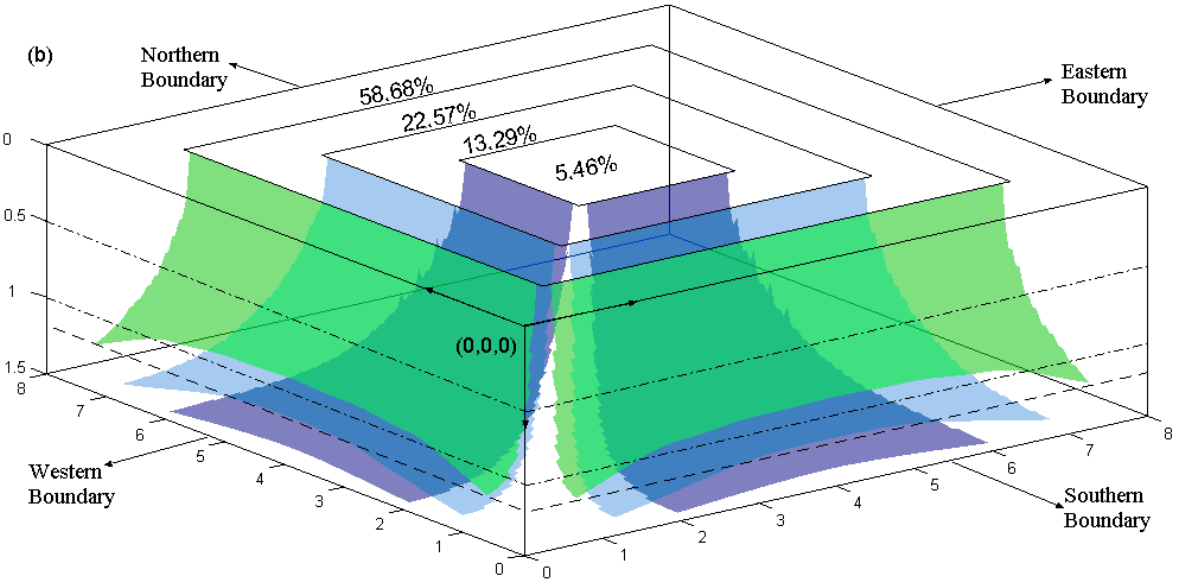
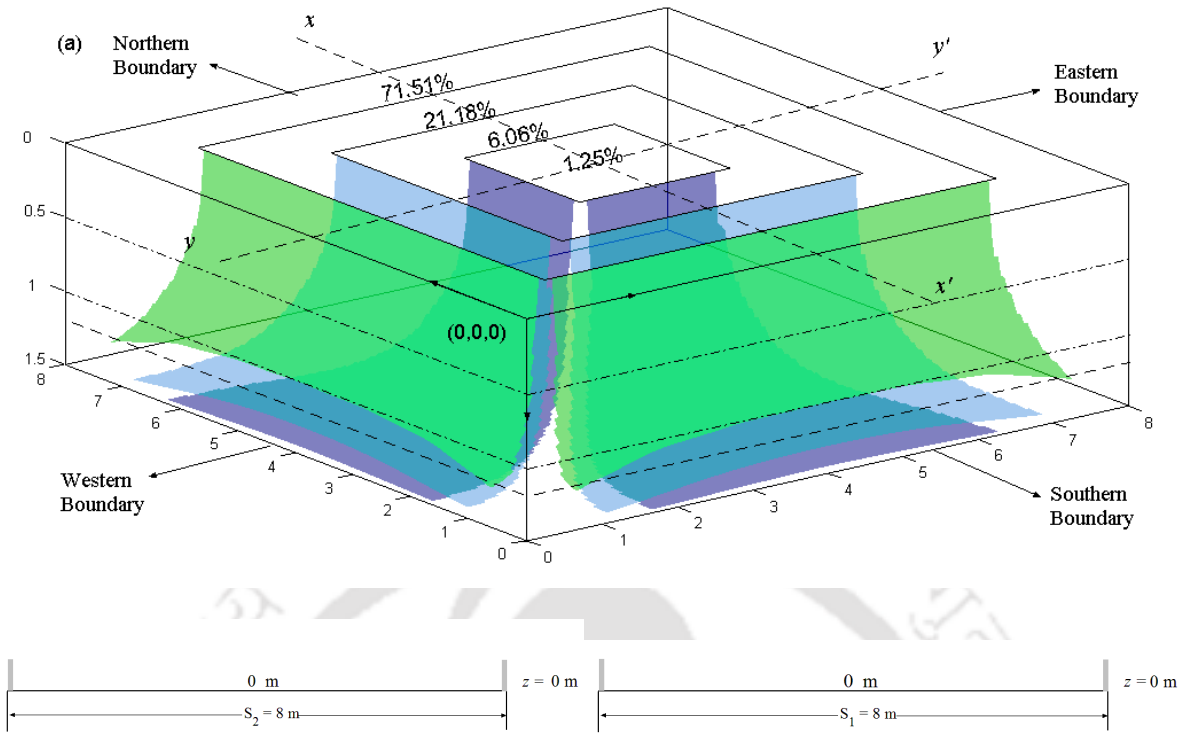


Fig. 3.36. A few stream surfaces and percentage distribution of the top discharge $Q_{top(3)}$ when the flow parameters of Fig 3.16 are taken as $S_1 = 8 \text{ m}$, $S_2 = 8 \text{ m}$, $h = 1 \text{ m}$, $H_1 = 1.2 \text{ m}$, $H_5 = 0.5 \text{ m}$, $H_6 = 1 \text{ m}$, $\varepsilon_y = \varepsilon_x = 0.05 \text{ m}$, $K_{x_1} = 1.5 \text{ m/day}$, $K_{y_1} = 1.5 \text{ m/day}$, $K_{z_1} = 0.5 \text{ m/day}$, $K_{x_2} = 2 \text{ m/day}$, $K_{y_2} = 2 \text{ m/day}$, $K_{z_2} = 1 \text{ m/day}$, $K_{x_3} = 3 \text{ m/day}$, $K_{y_3} = 3 \text{ m/day}$, $K_{z_3} = 2 \text{ m/day}$ and (a) $\delta = 0 \text{ m}$ and (b) $\delta_1 = 0 \text{ m}$, $\delta_2 = 0.1 \text{ m}$, $\delta_3 = 0.2 \text{ m}$, $\delta_4 = 0.4 \text{ m}$, $d_{x_1} = 0.5 \text{ m}$, $d_{x_2} = 1.5 \text{ m}$, $d_{x_3} = 2.5 \text{ m}$, $d_{x_4} = 5.5 \text{ m}$, $d_{x_5} = 6.5 \text{ m}$, $d_{x_6} = 7.5 \text{ m}$, $d_{y_1} = 0.5 \text{ m}$, $d_{y_2} = 1.5 \text{ m}$, $d_{y_3} = 2.5 \text{ m}$, $d_{y_4} = 5.5 \text{ m}$, $d_{y_5} = 6.5 \text{ m}$ and $d_{y_6} = 7.5 \text{ m}$



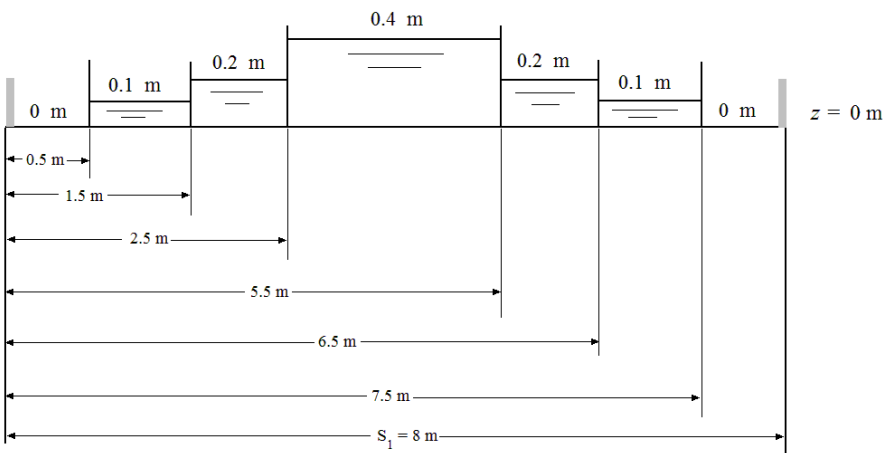
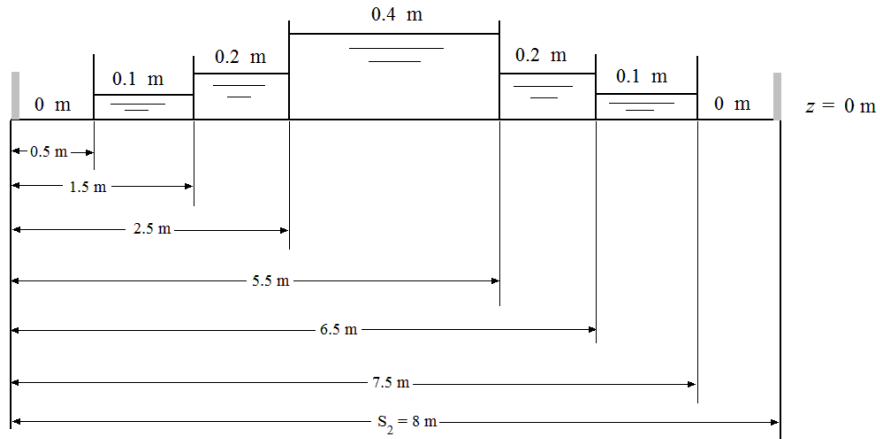


Fig. 3.37. A few stream surfaces and percentage distribution of the top discharge $Q_{top(3)}$ when the flow parameters of Fig 3.16 are taken as $S_1 = 8$ m, $S_2 = 8$ m, $h = 1$ m, $H_1 = 1.2$ m, $H_5 = 0.5$ m, $H_6 = 1$ m, $\varepsilon_x = \varepsilon_y = 0.05$ m, $K_{x_1} = 3$ m/day, $K_{y_1} = 3$ m/day, $K_{z_1} = 2$ m/day, $K_{x_2} = 2$ m/day, $K_{y_2} = 2$ m/day, $K_{z_2} = 1$ m/day, $K_{x_3} = 1.5$ m/day, $K_{y_3} = 1.5$ m/day, $K_{z_3} = 0.5$ m/day and (a) $\delta = 0$ m and (b) $\delta_1 = 0$ m, $\delta_2 = 0.1$ m, $\delta_3 = 0.2$ m, $\delta_4 = 0.4$ m, $d_{x_1} = 0.5$ m, $d_{x_2} = 1.5$ m, $d_{x_3} = 2.5$ m, $d_{x_4} = 5.5$ m, $d_{x_5} = 6.5$ m, $d_{x_6} = 7.5$ m, $d_{y_1} = 0.5$ m, $d_{y_2} = 1.5$ m, $d_{y_3} = 2.5$ m, $d_{y_4} = 5.5$ m, $d_{y_5} = 6.5$ m and $d_{y_6} = 7.5$ m

3.4 Conclusions

Analytical solutions have been worked out for predicting groundwater seepage into a network of subsurface ditch drains in a three-layered soil, the soil being assumed to be underlain by an impermeable barrier at a finite distance from the surface of the soil. Solutions have been obtained for three different cases of the problem, namely, when the water level in the drains lie in the top soil layer, next when it lies in the middle layer and finally when the water level in the drains lies in the bottom layer. In all these solutions, it has been assumed that the level of water in the drains remains same in all the drains. Further, all these solutions can handle both a uniform as well as a non-uniform ponding distribution at the surface of the soil. The separation of variables method along with appropriate Fourier expansions has been made use of to solve all these problems. These solutions are all new; however, whereas the steady state solutions are valid for all possible combination of the flow parameters associated with them, the corresponding transient solutions are strictly accurate only when the directional conductivities and specific storage of the soil layers respect some predefined relations among each other. From the study, it becomes clear that subsurface drainage of a ponded finite size stratified soil is mostly three-dimensional in nature and is highly dependent on the soil hydraulic properties of the constituent layers of the soil. However, for a large sized ponded field, flow in a vertical section located far away from the boundaries may be closely approximated using the two-dimensional flow assumption but even here the pathlines may exhibit strong three-dimensional characteristics in regions close to the drains. The volume of water captured by a ditch is quite sensitive to the level of water in it – a low water head in it would result in comparatively much higher water flow into it than when the level of water in it is high. An increase in the horizontal hydraulic conductivity of the layers tends to flatten the streamlines and an increase in the vertical conductivity of the layers tends to make them straight. The presence of a plow sole layer in a ponded paddy field may greatly impede movement of subsurface water into drains in such a field and the journey times of water particles from the surface of the soil to the drains may be quite high in such a drainage setting, particularly for water particles moving from surficial locations located further away from the ditches. In this context, it is worth noting that controlled drainage of paddy fields are now getting increasing importance since drainage not only helps in maintaining a proper air-water balance in these fields but also helps in having a check on the emission of methane from these environments – a potent greenhouse which global warming potential is only next to carbon dioxide as measured with respect to mass (Qiu 2009; Xiaohong et al. 2011, Zhang et al. 2011; Darzi-Naftchally and Shahnazari 2014; Yang et al. 2014 – to mention a few).

It has also come out of the study that, similar to the two-dimensional ponded drainage situations, flow in three-dimensional ponded drainage system with a constant ponding depth at the surface of the soil is also mostly restricted to areas close to the drains with minimal contribution of drainage taking place from areas located further away from the drains. However, by providing a progressively increasing ponding distribution towards the centre of the field, considerable improvement on the uniformity of water flow can also be brought about in a three-dimensional ponded ditch drainage system as well in comparison to a drainage scenario when the field is being imposed with a constant ponding depth at the

surface of the soil. Also, similar to two-dimensional ponded drainage situations, for three-dimensional ponded drainage situations as well the very presence of a very lowly conductive layer close to the surface may greatly help in improving the uniformity of water movement in such settings; however, if the situation is reversed and the drains are installed in soils where the conductivity of the layers are now decreasing with depth, then a variable ponding field at the surface of such a soil column is generally required to achieve a desired degree of uniformity of subsurface water movement to drains in such soils. Also, it has come out from the study that the travel times of water particles in a ponded ditch three-dimensional drainage system are pretty sensitive to the directional conductivities of the layers and an increase in these values may lower the travel times substantially, particularly for water particles traversing from locations close to the ditches. As soils in nature are mostly stratified and the groundwater flow to drains in most of the field settings is mostly three-dimensional in nature, it is hoped that the new and comprehensive analytical solutions proposed here would lead to better design of subsurface drains for reclaiming waterlogged and salt affected soils than designs based on existing solutions developed by taking recourse to more stringent assumptions.

As mentioned at the beginning of the chapter, the main limitation of the three-dimensional model that we have proposed is that it can neither account for unsaturated flow and nor the general spatio-temporal variation of the hydraulic conductivities of a soil profile. Also, in case the solutions developed, the drains have been assumed to be dug all the way up to an impervious barrier which need not be true for actual field situations. It should also be noted that we have already assumed the inner bunds at the surface of the soil to be of infinitesimally small thickness so as to reduce the surface boundary to strictly a Dirichlet boundary for mathematical convenience. However, in actual field conditions, the inner bunds will be having some finite thickness even though this thickness will be much smaller in comparison to the spacing between the open drains. Thus, even this assumption induces limitations in our three-dimensional analytical models.

3.5 List of Notations

The following notations are used in this chapter

$A_{l_1 m_1 n_1(1)}, A_{l_1 m_1 n_1(2)}, A_{l_1 m_1 n_1(3)}, B_{p_1 q_1(1)}, B_{p_1 q_1(2)}, B_{p_1 q_1(3)}, C_{p_2 q_2(1)}, C_{p_2 q_2(2)}, C_{p_2 q_2(3)}, D_{p_3 q_3(1)}, D_{p_3 q_3(2)}, D_{p_3 q_3(3)}, E_{kl(1)}, E_{kl(2)}, E_{kl(3)}, F_{p_4 q_4(1)}, F_{p_4 q_4(2)}, F_{p_4 q_4(3)}, G_{i_1 j_1(1)}, G_{i_1 j_1(2)}, G_{i_1 j_1(3)}, H_{i_1 j_1(1)}, H_{i_1 j_1(2)}, H_{i_1 j_1(3)}, I_{p_5 q_5(2)}, I_{p_5 q_5(3)}, J_{p_6 q_6(2)}, J_{p_6 q_6(3)}, K_{p_7 q_7(2)}, K_{p_7 q_7(3)}, L_{p_8 q_8(2)}, L_{p_8 q_8(3)}, M_{p_9 q_9(3)}, N_{p_{10} q_{10}(3)}, P_{i_3 j_3(1)}, P_{i_3 j_3(2)}, P_{i_3 j_3(3)}, Q_{uv(1)}, Q_{uv(2)}, Q_{uv(3)}, R_{l_2 m_2 n_2(1)}, R_{l_2 m_2 n_2(2)}, R_{l_2 m_2 n_2(3)}, U_{p_{11} q_{11}(3)}, V_{p_{12} q_{12}(3)}, W_{l_3 m_3 n_3(1)}, W_{l_3 m_3 n_3(2)}, W_{l_3 m_3 n_3(3)} = \text{constants with } i_1 = 1, 2, 3, \dots, \infty, i_2 = 1, 2, 3, \dots, \infty, i_3 = 1, 2, 3, \dots, \infty, j_1 = 1, 2, 3, \dots, \infty, j_2 = 1, 2, 3, \dots, \infty, j_3 = 1, 2, 3, \dots, \infty, k = 1, 2, 3, \dots, \infty, l = 1, 2, 3, \dots, \infty, l_1 = 1, 2, 3, \dots, \infty, l_2 = 1, 2, 3, \dots, \infty, l_3 = 1, 2, 3, \dots, \infty, m_1 = 1, 2, 3, \dots, \infty, m_2 = 1, 2, 3, \dots, \infty, m_3 = 1, 2, 3, \dots, \infty, n_1 = 1, 2, 3, \dots, \infty, n_2 = 1, 2, 3, \dots, \infty, n_3 = 1, 2, 3, \dots, \infty, p_1 = 1, 2, 3,$

$\dots, \infty, p_2 = 1, 2, 3, \dots, \infty, p_3 = 1, 2, 3, \dots, \infty, p_4 = 1, 2, 3, \dots, \infty, p_5 = 1, 2, 3, \dots, \infty, p_6 = 1, 2, 3, \dots, \infty,$
 $p_7 = 1, 2, 3, \dots, \infty, p_8 = 1, 2, 3, \dots, \infty, p_9 = 1, 2, 3, \dots, \infty, p_{10} = 1, 2, 3, \dots, \infty,$
 $p_{11} = 1, 2, 3, \dots, \infty, p_{12} = 1, 2, 3, \dots, \infty, q_1 = 1, 2, 3, \dots, \infty, q_2 = 1, 2, 3, \dots, \infty, q_3 = 1, 2, 3, \dots, \infty,$
 $q_4 = 1, 2, 3, \dots, \infty, q_5 = 1, 2, 3, \dots, \infty, q_6 = 1, 2, 3, \dots, \infty, q_7 = 1, 2, 3, \dots, \infty, q_8 = 1, 2, 3, \dots, \infty,$
 $q_9 = 1, 2, 3, \dots, \infty, q_{10} = 1, 2, 3, \dots, \infty, q_{11} = 1, 2, 3, \dots, \infty, q_{12} = 1, 2, 3, \dots, \infty, u = 1, 2, 3, \dots, \infty,$
 $v = 1, 2, 3, \dots, \infty,$

d_{xi}, d_{yi} = distances of the i^{th} inner bund from the origin O as illustrated in Figs. 3.1 [L];

h = depth up to the impervious layer as measured from the surface of the soil [L];

H_1 = depth of water in the ditches as measured from the surface of the soil [L];

H_5 = depth of the top soil layer as measured from the surface of the soil [L];

H_6 = depth of the middle soil layer as measured from the surface of the soil [L];

$I_1, I_2, I_3, J_1, J_2, J_3, K, L, L_1, L_2, L_3, M_1, M_2, M_3, N_1, N_2, N_3, P_1, P_2, P_3, P_4,$
 $P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}, Q_{11}, Q_{12}, U,$
 V = number of terms to be summed in the infinite series solutions, 1, 2, 3,....

K_{x_1} = horizontal hydraulic conductivity of the top soil layer in the North-South direction as in Fig. 3.1 [LT^{-1}];

K_{x_2} = horizontal hydraulic conductivity of the middle soil layer in the North-South direction as in Fig. 3.1 [LT^{-1}];

K_{x_3} = horizontal hydraulic conductivity of the bottom soil layer in the North-South direction as in Fig. 3.1 [LT^{-1}];

K_{y_1} = horizontal hydraulic conductivity of the top soil layer in the East-West direction as in Fig. 3.1 [LT^{-1}];

K_{y_2} = horizontal hydraulic conductivity of the middle soil layer in the East-West direction as in Fig. 3.1 [LT^{-1}];

K_{y_3} = horizontal hydraulic conductivity of the bottom soil layer in the East-West direction as in Fig. 3.1 [LT^{-1}];

K_{z_1} = vertical hydraulic conductivity of the top soil layer in Fig. 3.1 [LT^{-1}];

K_{z_2} = vertical hydraulic conductivity of the middle soil layer in Fig. 3.1 [LT^{-1}];

K_{z_3} = vertical hydraulic conductivity of the bottom soil layer in Fig. 3.1 [LT^{-1}];

$K_{xy}^{a1} = (K_{y_1}/K_{x_1})$ anisotropy ratio of the top soil layer (dimensionless);

$K_{xy}^{a2} = (K_{y_2}/K_{x_2})$ anisotropy ratio of the middle soil layer (dimensionless);

$K_{xy}^{a3} = (K_{y_3}/K_{x_3})$ anisotropy ratio of the bottom soil layer (dimensionless);

$K_{xz}^{a1} = (K_{z_1}/K_{x_1})$ anisotropy ratio of the top soil layer (dimensionless);

$K_{xz}^{a2} = (K_{z_2}/K_{x_2})$ anisotropy ratio of the middle soil layer (dimensionless);

$K_{xz}^{a3} = (K_{z_3}/K_{x_3})$ anisotropy ratio of the bottom soil layer (dimensionless);

$K_{yz}^{a1} = (K_{z_1}/K_{y_1})$ anisotropy ratio of the top soil layer (dimensionless);

$K_{yz}^{a2} = (K_{z_2}/K_{y_2})$ anisotropy ratio of the middle soil layer (dimensionless);

$K_{yz}^{a3} = (K_{z_3}/K_{y_3})$ anisotropy ratio of the bottom soil layer (dimensionless);

$N_{i_1} = (i_1\pi/S_1)$ with $i_1 = 1, 2, 3, \dots, \infty$;

$N_{i_2} = (i_2\pi/S_1)$ with $i_2 = 1, 2, 3, \dots, \infty$;

$N_{i_3} = (i_3\pi/S_1)$ with $i_3 = 1, 2, 3, \dots, \infty$;

$N_{j_1} = (j_1\pi/S_2)$ with $j_1 = 1, 2, 3, \dots, \infty$;

$N_{j_2} = (j_2\pi/S_2)$ with $j_2 = 1, 2, 3, \dots, \infty$;

$N_{j_3} = (j_3\pi/S_2)$ with $j_3 = 1, 2, 3, \dots, \infty$;

$N_k = (k\pi/S_1)$ with $k = 1, 2, 3, \dots, \infty$;

$N_l = (l\pi/S_2)$ with $l = 1, 2, 3, \dots, \infty$;

$N_{l_1} = (l_1\pi/S_1)$ with $l_1 = 1, 2, 3, \dots, \infty$;

$N_{l_2} = (l_2\pi/S_1)$ with $l_2 = 1, 2, 3, \dots, \infty$;

$N_{l_3} = (l_3\pi/S_1)$ with $l_3 = 1, 2, 3, \dots, \infty$;

$N_{m_1} = (m_1\pi/S_2)$ with $m_1 = 1, 2, 3, \dots, \infty$;

$$N_{m_2} = (m_2\pi/S_2) \text{ with } m_2 = 1, 2, 3, \dots, \infty;$$

$$N_{m_3} = (m_3\pi/S_2) \text{ with } m_3 = 1, 2, 3, \dots, \infty;$$

$$N_{n_1} = [(1-2n_1)\pi/2h] \text{ with } n_1 = 1, 2, 3, \dots, \infty;$$

$$N_{n_2} = [(1-2n_2)\pi/2h] \text{ with } n_2 = 1, 2, 3, \dots, \infty;$$

$$N_{n_3} = [(1-2n_3)\pi/2h] \text{ with } n_3 = 1, 2, 3, \dots, \infty;$$

$$N_{p_1} = (p_1\pi/S_2) \text{ with } p_1 = 1, 2, 3, \dots, \infty;$$

$$N_{p_2} = (p_2\pi/S_2) \text{ with } p_2 = 1, 2, 3, \dots, \infty;$$

$$N_{p_3} = (p_3\pi/S_1) \text{ with } p_3 = 1, 2, 3, \dots, \infty;$$

$$N_{p_4} = (p_4\pi/S_1) \text{ with } p_4 = 1, 2, 3, \dots, \infty;$$

$$N_{p_5} = (p_5\pi/S_2) \text{ with } p_5 = 1, 2, 3, \dots, \infty;$$

$$N_{p_6} = (p_6\pi/S_2) \text{ with } p_6 = 1, 2, 3, \dots, \infty;$$

$$N_{p_7} = (p_7\pi/S_1) \text{ with } p_7 = 1, 2, 3, \dots, \infty;$$

$$N_{p_8} = (p_8\pi/S_1) \text{ with } p_8 = 1, 2, 3, \dots, \infty;$$

$$N_{p_9} = (p_9\pi/S_2) \text{ with } p_9 = 1, 2, 3, \dots, \infty;$$

$$N_{p_{10}} = (p_{10}\pi/S_2) \text{ with } p_{10} = 1, 2, 3, \dots, \infty;$$

$$N_{p_{11}} = (p_{11}\pi/S_1) \text{ with } p_{11} = 1, 2, 3, \dots, \infty;$$

$$N_{p_{12}} = (p_{12}\pi/S_1) \text{ with } p_{12} = 1, 2, 3, \dots, \infty;$$

$$N_{q_1} = [(1-2q_1)\pi/2H_5] \text{ with } q_1 = 1, 2, 3, \dots, \infty;$$

$$N_{q_2} = [(1-2q_2)\pi/2H_5] \text{ with } q_2 = 1, 2, 3, \dots, \infty;$$

$$N_{q_3} = [(1-2q_3)\pi/2H_5] \text{ with } q_3 = 1, 2, 3, \dots, \infty;$$

$$N_{q_4} = [(1-2q_4)\pi/2H_5] \text{ with } q_4 = 1, 2, 3, \dots, \infty;$$

$$N_{q_5} = \left[(1 - 2q_5) \pi / 2 (H_6 - H_5) \right] \text{ with } q_5 = 1, 2, 3, \dots, \infty;$$

$$N_{q_6} = \left[(1 - 2q_6) \pi / 2 (H_6 - H_5) \right] \text{ with } q_6 = 1, 2, 3, \dots, \infty;$$

$$N_{q_7} = \left[(1 - 2q_7) \pi / 2 (H_6 - H_5) \right] \text{ with } q_7 = 1, 2, 3, \dots, \infty;$$

$$N_{q_8} = \left[(1 - 2q_8) \pi / 2 (H_6 - H_5) \right] \text{ with } q_8 = 1, 2, 3, \dots, \infty;$$

$$N_{q_9} = \left[(1 - 2q_9) \pi / 2 (h - H_6) \right] \text{ with } q_9 = 1, 2, 3, \dots, \infty;$$

$$N_{q_{10}} = \left[(1 - 2q_{10}) \pi / 2 (h - H_6) \right] \text{ with } q_{10} = 1, 2, 3, \dots, \infty;$$

$$N_{q_{11}} = \left[(1 - 2q_{11}) \pi / 2 (h - H_6) \right] \text{ with } q_{11} = 1, 2, 3, \dots, \infty;$$

$$N_{q_{12}} = \left[(1 - 2q_{12}) \pi / 2 (h - H_6) \right] \text{ with } q_{12} = 1, 2, 3, \dots, \infty;$$

$$N_u = (u\pi/S_1) \text{ with } u = 1, 2, 3, \dots, \infty;$$

$$N_v = (v\pi/S_2) \text{ with } v = 1, 2, 3, \dots, \infty;$$

$$N_{u_1} = (u_1\pi/S_1) \text{ with } u_1 = 1, 2, 3, \dots, \infty;$$

$$N_{v_1} = (v_1\pi/S_2) \text{ with } v_1 = 1, 2, 3, \dots, \infty;$$

N_0 = number of divisions of the ponding surface at the top of the soil (Figs. 3.1, 3.2 and 3.3);

$Q_{East(1)}$ = discharge through the Eastern ditch face for the flow problem of Fig. 3.4 [L^3T^{-1}];

$Q_{East(2)}$ = discharge through the Eastern ditch face for the flow problem of Fig. 3.10 [L^3T^{-1}];

$Q_{East(3)}$ = discharge through the Eastern ditch face for the flow problem of Fig. 3.16 [L^3T^{-1}];

$Q_{North(1)}$ = discharge through the Northern ditch face for the flow problem of Fig. 3.4 [L^3T^{-1}];

$Q_{North(2)}$ = discharge through the Northern ditch face for the flow problem of Fig. 3.10 [L^3T^{-1}];

$Q_{North(3)}$ = discharge through the Northern ditch face for the flow problem of Fig. 3.16 [L^3T^{-1}];

$Q_{South(1)}$ = discharge through the Southern face of the flow problem in Fig. 3.4 [L^3T^{-1}];

$Q_{South(2)}$ = discharge through the Southern face of the flow problem in Fig. 3.10 [L^3T^{-1}];

$Q_{South(3)}$ = discharge through the Southern face of the flow problem in Fig. 3.16 [L^3T^{-1}];

$Q_{top(1)}$ = discharge through the top surface of the flow problem in Fig. 3.4 [L^3T^{-1}];

$Q_{top(2)}$ = discharge through the top surface of the flow problem in Fig. 3.10 [L^3T^{-1}];

$Q_{top(3)}$ = discharge through the top surface of the flow problem in Fig. 3.16 [L^3T^{-1}];

$Q_{top(1)}^f$ = top discharge function for the flow problem of Fig. 3.4;

$Q_{top(2)}^f$ = top discharge function for the flow problem of Fig. 3.10;

$Q_{top(3)}^f$ = top discharge function for the flow problem of Fig. 3.16;

$Q_{West(1)}$ = discharge through the Southern face of the flow problem in Fig. 3.4 [L^3T^{-1}];

$Q_{West(2)}$ = discharge through the Southern face of the flow problem in Fig. 3.10 [L^3T^{-1}];

$Q_{West(3)}$ = discharge through the Southern face of the flow problem in Fig. 3.16 [L^3T^{-1}];

S_1 = the horizontal extent of the flow domain in the North-South direction in Fig. 3.1 [L];

S_2 = the horizontal extent of the flow domain in the East-West direction in Fig. 3.1 [L];

S_{s_1} = specific storage of the top soil layer [L^{-1}];

S_{s_2} = specific storage of the middle soil layer [L^{-1}];

S_{s_3} = specific storage of the bottom soil layer [L^{-1}];

t = time variable [T];

$V_{x1(1)}$ = horizontal velocity distribution in the North-South direction for the top soil layer of Fig. 3.4 [LT^{-1}];

$V_{x1(2)}$ = horizontal velocity distribution in the North-South direction for the top soil layer of Fig. 3.10 [LT^{-1}];

$V_{x1(3)}$ = horizontal velocity distribution in the North-South direction for the top soil layer of Fig. 3.16 [LT^{-1}];

$V_{x2(1)}$ = horizontal velocity distribution in the North-South direction for the middle soil layer of Fig. 3.4 [LT^{-1}];

$V_{x2(2)}$ = horizontal velocity distribution in the North-South direction for the middle soil layer of Fig. 3.10 [LT^{-1}];

$V_{x2(3)}$ = horizontal velocity distribution in the North-South direction for the middle soil layer of Fig. 3.16 [LT^{-1}];

$V_{x3(1)}$ = horizontal velocity distribution in the North-South direction for the bottom soil layer of Fig. 3.4 [LT^{-1}];

$V_{x3(2)}$ = horizontal velocity distribution in the North-South direction for the bottom soil layer of Fig. 3.10 [LT^{-1}];

$V_{x3(3)}$ = horizontal velocity distribution in the North-South direction for the bottom soil layer of Fig. 3.16 [LT^{-1}];

$V_{y1(1)}$ = horizontal velocity distribution in the East-West direction for the top soil layer of Fig. 3.4 [LT^{-1}];

$V_{y1(2)}$ = horizontal velocity distribution in the East-West direction for the top soil layer of Fig. 3.10 [LT^{-1}];

$V_{y1(3)}$ = horizontal velocity distribution in the East-West direction for the top soil layer of Fig. 3.16 [LT^{-1}];

$V_{y2(1)}$ = horizontal velocity distribution in the East-West direction for the middle soil layer of Fig. 3.4 [LT^{-1}];

$V_{y2(2)}$ = horizontal velocity distribution in the East-West direction for the middle soil layer of Fig. 3.10 [LT^{-1}];

$V_{y2(3)}$ = horizontal velocity distribution in the East-West direction for the middle soil layer of Fig. 3.16 [LT^{-1}];

$V_{y3(1)}$ = horizontal velocity distribution in the East-West direction for the bottom soil layer of Fig. 3.4 [LT^{-1}];

$V_{y3(2)}$ = horizontal velocity distribution in the East-West direction for the bottom soil layer of Fig. 3.10 [LT^{-1}];

$V_{y3(3)}$ = horizontal velocity distribution in the East-West direction for the bottom soil layer of Fig. 3.16 [LT^{-1}];

$V_{z1(1)}$ = vertical velocity distribution for the top layer of Fig. 3.4 [LT^{-1}];

$V_{z1(2)}$ = vertical velocity distribution for the top layer of Fig. 3.10 [LT^{-1}];

$V_{z1(3)}$ = vertical velocity distribution for the top layer of Fig. 3.16 [LT^{-1}];

$V_{z2(1)}$ = vertical velocity distribution for the middle layer of Fig. 3.4 [LT^{-1}];

$V_{z2(2)}$ = vertical velocity distribution for the middle layer of Fig. 3.10 [LT^{-1}];

$V_{z2(3)}$ = vertical velocity distribution for the middle layer of Fig. 3.16 [LT^{-1}];

$V_{z3(1)}$ = vertical velocity distribution for the bottom layer of Fig. 3.4 [LT^{-1}];

$V_{z3(2)}$ = vertical velocity distribution for the bottom layer of Fig. 3.10 [LT^{-1}];

$V_{z3(3)}$ = vertical velocity distribution for the bottom layer of Fig. 3.16 [LT^{-1}];

x = horizontal coordinate along the North-South direction [L];

y = horizontal coordinate along the East-West direction [L];

z = vertical coordinate [L];

$\phi_{1(1)}$ = hydraulic head distribution for the top soil layer for the flow problem of Fig. 3.4 [L];

$\phi_{1(2)}$ = hydraulic head distribution for the top soil layer for the flow problem of Fig. 3.10 [L];

$\phi_{1(3)}$ = hydraulic head distribution for the top soil layer for the flow problem of Fig. 3.17 [L];

$\phi_{2(1)}$ = hydraulic head distribution for the middle soil layer for the flow problem of Fig. 3.4 [L];

$\phi_{2(2)}$ = hydraulic head distribution for the middle soil layer for the flow problem of Fig. 3.10 [L];

$\phi_{2(3)}$ = hydraulic head distribution for the middle soil layer for the flow problem of Fig. 3.17 [L];

$\phi_{3(1)}$ = hydraulic head distribution for the bottom soil layer for the flow problem of Fig. 3.4 [L];

$\phi_{3(2)}$ = hydraulic head distribution for the bottom soil layer for the flow problem of Fig. 3.10
[L];

$\phi_{3(3)}$ = hydraulic head distribution for the bottom soil layer for the flow problem of Fig. 3.17
[L];

η_1 = porosity of the top soil layer (dimensionless);

η_2 = porosity of the middle soil layer (dimensionless);

η_3 = porosity of the bottom soil layer (dimensionless);

δ_i = ponding depth of the i^{th} strip at the soil surface for the flow problems of Figs. 3.4, 3.10 and 3.17 [L];

ε_x = width of the ditch banks in the North-South direction for the flow problem of Fig. 3.1
[L];

ε_y = width of the ditch banks in the East-West direction for the flow problem of Fig. 3.1 [L];

$$(\lambda_{i_1 j_1})^2 = \left[\left(N_{i_1}^2 / K_{xz}^{a2} \right) + \left(N_{j_1}^2 / K_{yz}^{a2} \right) \right];$$

$$(\lambda_{i_2 j_2})^2 = \left[\left(N_{i_2}^2 / K_{xz}^{a2} \right) + \left(N_{j_2}^2 / K_{yz}^{a2} \right) \right];$$

$$(\lambda_{i_3 j_3})^2 = \left[\left(N_{i_3}^2 / K_{xz}^{a3} \right) + \left(N_{j_3}^2 / K_{yz}^{a3} \right) \right];$$

$$(\lambda_{kl})^2 = \left[\left(N_k^2 / K_{xz}^{a1} \right) + \left(N_l^2 / K_{yz}^{a1} \right) \right];$$

$$(\lambda_{l_1 m_1 n_1})^2 = \left[N_{l_1}^2 \left(K_{x_1} / S_{s_1} \right) + N_{m_1}^2 \left(K_{y_1} / S_{s_1} \right) + N_{n_1}^2 \left(K_{z_1} / S_{s_1} \right) \right];$$

$$(\lambda_{l_2 m_2 n_2})^2 = \left[N_{l_2}^2 \left(K_{x_2} / S_{s_2} \right) + N_{m_2}^2 \left(K_{y_2} / S_{s_2} \right) + N_{n_2}^2 \left(K_{z_2} / S_{s_2} \right) \right];$$

$$(\lambda_{l_3 m_3 n_3})^2 = \left[N_{l_3}^2 \left(K_{x_3} / S_{s_3} \right) + N_{m_3}^2 \left(K_{y_3} / S_{s_3} \right) + N_{n_3}^2 \left(K_{z_3} / S_{s_3} \right) \right];$$

$$(\lambda_{p_1 q_1})^2 = N_{p_1}^2 K_{xy}^{a1} + N_{q_1}^2 K_{xz}^{a1};$$

$$(\lambda_{p_2 q_2})^2 = N_{p_2}^2 K_{xy}^{a1} + N_{q_2}^2 K_{xz}^{a1};$$

$$(\lambda_{p_3 q_3})^2 = \left[\left(N_{p_3}^2 / K_{xy}^{a1} \right) + N_{q_3}^2 K_{yz}^{a1} \right];$$

$$(\lambda_{p_4q_4})^2 = \left[(N_{p_4}^2 / K_{xy}^{a1}) + N_{q_4}^2 K_{yz}^{a1} \right];$$

$$(\lambda_{p_5q_5})^2 = N_{p_5}^2 K_{xy}^{a2} + N_{q_5}^2 K_{xz}^{a2};$$

$$(\lambda_{p_6q_6})^2 = N_{p_6}^2 K_{xy}^{a2} + N_{q_6}^2 K_{xz}^{a2};$$

$$(\lambda_{p_7q_7})^2 = \left[(N_{p_7}^2 / K_{xy}^{a2}) + N_{q_7}^2 K_{yz}^{a2} \right];$$

$$(\lambda_{p_8q_8})^2 = \left[(N_{p_8}^2 / K_{xy}^{a2}) + N_{q_8}^2 K_{yz}^{a2} \right];$$

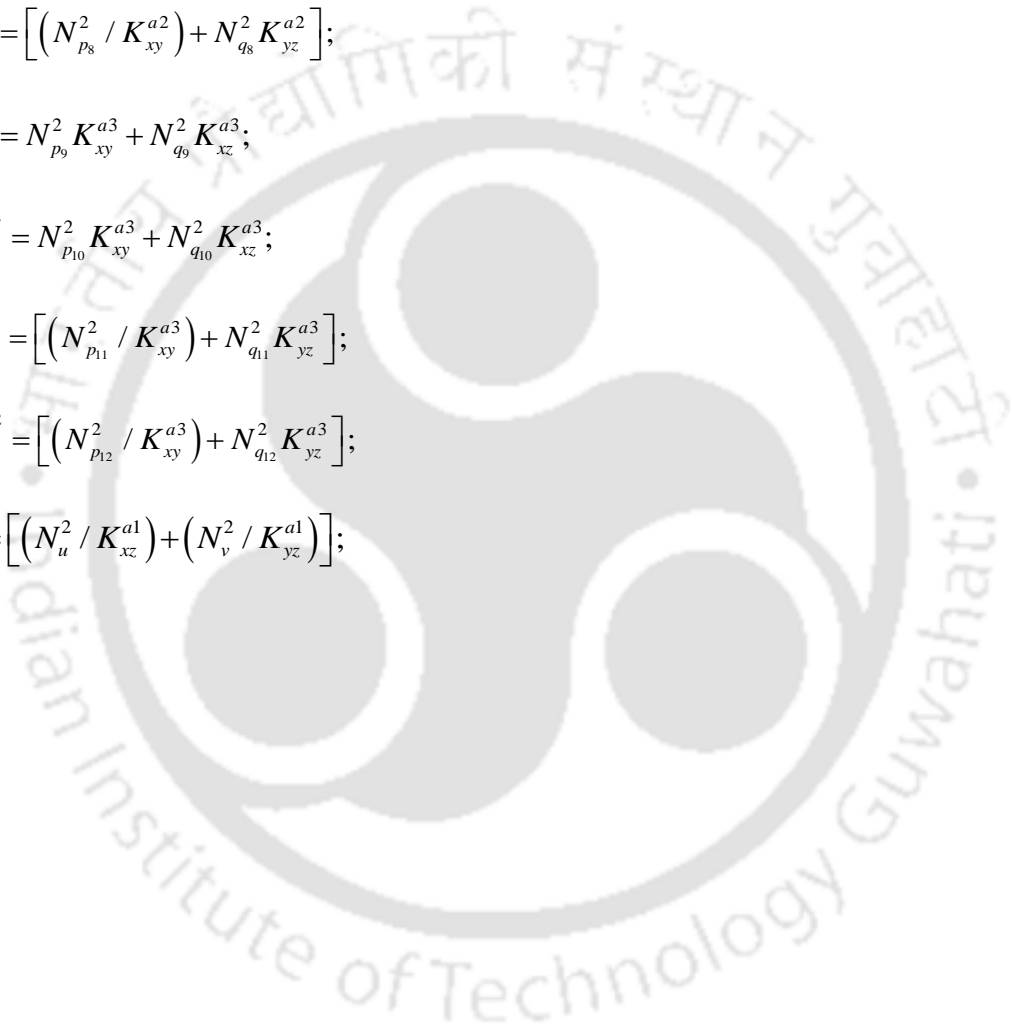
$$(\lambda_{p_9q_9})^2 = N_{p_9}^2 K_{xy}^{a3} + N_{q_9}^2 K_{xz}^{a3};$$

$$(\lambda_{p_{10}q_{10}})^2 = N_{p_{10}}^2 K_{xy}^{a3} + N_{q_{10}}^2 K_{xz}^{a3};$$

$$(\lambda_{p_{11}q_{11}})^2 = \left[(N_{p_{11}}^2 / K_{xy}^{a3}) + N_{q_{11}}^2 K_{yz}^{a3} \right];$$

$$(\lambda_{p_{12}q_{12}})^2 = \left[(N_{p_{12}}^2 / K_{xy}^{a3}) + N_{q_{12}}^2 K_{yz}^{a3} \right];$$

$$(\lambda_{uv})^2 = \left[(N_u^2 / K_{xz}^{a1}) + (N_v^2 / K_{yz}^{a1}) \right];$$



CHAPTER 4

CONCLUSIONS

In this study, a few analytical solutions have been worked out for studying subsurface drainage of stratified soils under ponded conditions. Solutions for a few variants of the problem have been first obtained by assuming the flow in a ponded drainage space as two-dimensional and later on this assumption has been dropped and the problem has been tackled by considering all the three components of flow in a chosen drainage space. The outcomes of the study have already been provided in details in the earlier chapters, but for ready reference, we are now again running through the salient outcomes of the study as under.

1. The ponded drainage problem for a three-layered soil has been solved for a few variants of the problem by assuming the flow as two-dimensional in a ponded drainage space. These solutions can account for both a uniform as well as a variable ponding field at the surface of the soil. The separation of variables technique along with appropriate Fourier expansions has been utilized to obtain solutions to all these problems. The steady state solutions are all exact and new and can handle both equal as well as unequal water level heights in the drains. However, their transient counterparts are strictly valid only when the horizontal hydraulic conductivities and specific storage of the constituent layers of a stratified soil column obey a pre-defined relation among each other. The validity of all these solutions has also been checked by comparing with the analytical works for a few simplified drainage scenarios; also, numerical checks on them have also been carried out utilizing the processing MODFLOW environment.

The study shows that flow in a ponded ditch system in a layered soil is highly sensitive to the directional conductivities of the constituent layers, the level of water in the ditches and the imposed ponding distribution on the surface of the soil. A higher horizontal conductivity of the layers tends to wound the pathlines and a higher vertical conductivity of the layers tends to unwound them. A lower water level in a ditch causes more water to flow into the ditch as compared to a situation when the water level in the ditch is high. It has also come out of the study that ponded drainage with a uniform ponding head at the surface of the soil is mostly restricted to areas close to the drains with relatively much lesser contribution of water coming to drains from locations located further away from the drains. However, if the imposed ponding head is a variable one with increasing depth of ponding towards the semi-spacing distance

between the drains, then the uniformity of water movement in a two-dimensional ponded drainage space may be considerably improved. Another point that has come of the study is that the mere presence of a very low conductive top soil layer over the bottom layer of a two-layered ponded drainage system may lead to significant improvement in the uniformity of water movement in the drainage space of such a system even when the surface ponding distribution over such a drainage setting is kept as uniform. This is an important observation since, as mentioned before, this type of hydraulic scenarios may be frequently encountered in the drainage of paddy fields, where the hydraulic conductivity of the top soil layer may be substantially lower than that of the layers below it.

2. Analytical solutions in the form of infinite series have also been obtained for the ponded ditch drainage problem for a stratified soil for different locations of water level in the drains without resorting to the two-dimensional flow assumption as has been made for the earlier solutions. Here also, like in the previous solutions, the separation of variables method along with appropriate multi-dimensional Fourier expansions has been made use to solve all the three variants of the three-dimensional-multi-layered ponded problem considered for study. All these solutions are new; however, whereas the steady-state solutions are valid for all possible combinations of the flow parameters associated with them, the corresponding transient solutions are valid only when the directional conductivities and specific storage of the layers of a stratified soil satisfy some well defined relations among each other.

From the study, it has become clear that ponded drainage of a finite-sized field with stratifications is mostly three dimensional in nature and that the soil hydraulic properties of the constituent layers of a stratified soil have a noticeable influence on the hydraulics of flow associated with such a system. The three-dimensional affect is more pronounced in areas close to the recipient ditches where, depending on a flow situation, curvature of the flow lines can be quite appreciable. However, if the length of the drainage lines on one side is much longer than on a side orthogonal to it, then the flow in a vertical section located further away from the longer boundaries of the flow domain may be closely approximated (i.e., without introducing appreciable error) using the two-dimensional flow assumption. It is also observed that a higher horizontal conductivity of the layers has a tendency to flatten the pathlines and a higher vertical conductivity of the layers has a tendency to make them more vertical. This is because an increase in the horizontal and lateral conductivities of a layer, allowing all other factors to remain the

same, results in a proportionate increase in the components of the velocity vector in these directions in relation to its vertical component thereby causing the pathlines to tilt more towards the drains when measured with respect to its initial state. The situation, however, gets reversed with the increase in the vertical conductivity of a layer as in such a situation, the share of the vertical component of the velocity vector gets proportionally increased with respect to the other two components of the velocity vector thereby causing the pathlines to straighten up within the layer. In this context, it should also be noted that, just like the two-dimensional ponded case, here also the very existence of a very lowly conductive surface soil layer – say, the presence of a plow-sole layer in a paddy field – may greatly improve the uniformity of water movement in such soils even when the imposed ponding head over their surface is a uniform one. This is due to the fact that when the conductivity of the top layer is low, the water particles are not getting readily infiltrated into the soil even from closer locations to the drains thereby resulting in relatively less volume of water to seep through a surface area close to the ditches as compared to seepage through the same area when the conductivity of the top layer is high. Thus, the conductivity contrasts of a multi-layered soil alone may have a significant effect on the overall dynamics of ground water movement of a ponded ditch drainage system in a stratified soil. Further, like in the two-dimensional situations, three-dimensional ponded drainage in a stratified soil with a uniform ponding field is mostly restricted to areas close to the drains with minimal seepage taking place in regions away from the drains. However, by providing a progressively increasing ponding distribution away from the ditches and towards the centre of the ponding field, a considerable improvement on the uniformity of water movement in a stratified ponded space can be brought about in comparison to a situation where the imposed ponding head over the field is a uniform one. As the three-dimensional solutions provided to the multi-layered ponded drainage problem for its different variants are quite comprehensive in nature, it is hoped that these solutions will lead to a better understanding of flow behavior in such systems as compared to available solutions to the problem based on more restrictive assumptions.

BIBLIOGRAPHY

- Abrol, I. P., Yadav, J. S. P., and Massoud, F. I. (1988). "Salt-affected soils and their management." *FAO Soils Bulletin*, Soil Resources, Management and Conservation Service, FAO Land and Water Development Division, 39, 131.
- Adomian, G. (1994). *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, 1994.
- Afruzi, A., Nazemi, A.H., Sadraddini, A.A., (2014). "Steady-state subsurface drainage of ponded fields by rectangular ditch drains." *Irrig. and Drain.*, 63, 668-681.
- Alexandratos, N., and Bruinsma, J. (2012). "World agriculture towards 2030/2050: the 2012 revision." ESA Working paper No. 12-03. Rome, FAO.
- Ausubel, J. H., Wernick, I. K., and Waggoner, P. E. (2013). "Peak farmland and the prospect for land sparing." *Popul. Dev. Rev.*, 38(Supplement), 221-242.
- Ayars, J. E., Christen, E. W., and Hornbuckle, J. W. (2006). "Controlled drainage for improved water management in arid regions irrigated agriculture." *Agric. Water Manage.*, 86, 128-139.
- Barua, G., and Alam, W. (2013). "An analytical solution for predicting transient seepage into ditch drains from a ponded field." *Adv. in Water Resour.*, 52, 78-92.
- Barua, G., and Sarmah, R. (2016). "An analytical solution for predicting transient seepage into partially penetrating ditch drains receiving water from a ponded field." *Acta Geophys.*, 64(1), 149-205.
- Barua, G., and Tiwari, K. N. (1995). "Analytical solutions of seepage into ditches from ponded fields." *J. Irrig. Drain. Eng.*, 121(6), 396-404
- Barua, G., and Tiwari, K. N. (1996a). "Ditch drainage theories for homogeneous anisotropic soil." *J. Irrig. Drain. Eng.*, 122(5), 276 -285.
- Barua, G., and Tiwari, K. N. (1996b). "Theories of ditch drainage in layered anisotropic soil." *J. Irrig. Drian. Eng.*, 122(6), 321-330.
- Bathke, G. R., and Cassel, D. K. (1991). "Anisotropic variation of profile characteristics and saturated hydraulic conductivity in an ultisol landscape." *Soil. Sci. Soc. Sm. J.*, 55, 333-339.
- Bazaraa, A. S., Abdel-Dayem, M. S., Amer, A., and Willardson, L.S. (1986). "Artesian and anisotropic effect of drainage spacing." *J. Irrig. Drain. Eng.*, 112(1), 55-64.
- Bear, J. (1972). *Dynamics of Fluids in Porous Media*. Elsevier, New York, USA.

- Bear, J. (1979). *Hydraulics of Groundwater*. McGraw-Hill, New York, USA.
- Bereslavskii, E. N. (2006). "Groundwater flow to a system of drainage canals." *Water Resour.*, 33(4), 417-420.
- Boari, F., Pace, B., Todorovic, M., Palma, E. D., and Cantore, V. (2012). "Effect of water regime and salinity on artichoke yield." *Italian J. of Agronomy*, 7(1), 58-63.
- Bouma, J. (1982). "Measuring the hydraulic conductivity of soil horizons with continuous macropores." *Soil. Sci. Soc. Sm. J.*, 46, 438-441.
- Bradbury, R. B., Kirby, W. B. (2006). "Farmland birds and resource protection in the UK: Cross-cutting solutions for multi-functional farming?" *Biol. Conserv.*, 129(4), 530-542.
- Brainard, E. C., and Gelhar, L. W. (1991). "Influence of Vertical Flow on Ground-Water Transport." *Water Resour. Res.*, 29(5), 693-701.
- Braun, H. M. H., and Kruijne, R. (1994). *Drainage Principles and Applications*. ILRI Publication 16, International Institute of Land Reclamation and Improvement, Wageningen, The Netherlands, 77-110.
- Cantwell, B. J. (2002). *Introduction to Symmetry Analysis*. Cambridge University Press.
- Carruthers, I., Rosegrant, M. W., and Seckler, D. (1997). "Irrigation and food security in the 21st. century." *Irrigation and Drainage Systems*, 11, 83-101.
- Chahar, B. R., and Vadodaria, G. P. (2008a). "Steady subsurface drainage of homogeneous soil by ditches." *Proc. ICE Water Management*, 161(WM6), 303-11.
- Chahar, B. R., and Vadodaria, G. P. (2008b). "Drainage of ponded surface by an array of ditches." *J. Irrig. Drain. Eng.*, 134(6), 815-23.
- Chahar, B. R., and Vadodaria, G. P. (2010). "Optimal spacing in an array of fully penetrating ditches for subsurface drainage." *J. Irrig. Drain. Eng.*, 136(1), 63-67.
- Chahar, B. R., and Vadodaria, G. P. (2012). "Steady subsurface drainage of ponded surface by an array of parallel ditches." *J. Hydrologic Eng.*, 17(8), 895-908.

- Chen, S. K., Liu, C. W. (2002). "Analysis of water movement in paddy rice fields (I) experimental studies." *J. of Hydrol.*, 260, 206-215.
- Chiang, W., and Kinzelbach, W. (2001). *3D-Groundwater Modeling with PMWIN: A Simulation System for Modeling Groundwater Flow and Pollution*, Berlin, Springer-Verlag.
- Clarkson, P. A., and Kruskal, M.D. (1989). "New similiarity reductions of the Boussinesq equation." *J. Math. Phys.*, 30(10), 2201-2213.
- Craig, J. R., and Read, W. W. (2010). "The future of analytical solution methods for groundwater flow and transport simulation." XVIII International Conference on Water Resources, Barcelona
- CSSRI, (2011). "Vision-2030, CSSRI Perspective Plan." Central Soil Salinity Research Institute, Indian Council of Agricultural Research, Karnal, India.
- Darzi-Naftchali, A., Mirlatifi, S. M., Shahnazari, A., Ejlali, F., and Mahdian, M. H. (2013). "Effect of subsurface drainage on water balance and water table in poorly drained paddy fields." *Agric. Water Manage.*, 130, 61-68.
- Darzi-Naftchali, A., and Shahnazari, A. (2014). "Influence of subsurface drainage on the productivity of poorly drained paddy fields." *Eur. J. Agron.*, 56, 1-8.
- Datta, K. K., and de Jong, C. (2002). "Adverse effect of Waterlogging and soil salinity on crop and land productivity in northwest region of Haryana, India." *Agric. Water Manage.*, 57, 223-238.
- Datta, K. K., de Jong, C., and Sing, O. P. (2000). "Reclaiming salt-affected land drainage in Haryana, India." *Agric. Water Manage.*, 46, 55-71.
- Datta, K. K., Tewari, L., and Toshi, P. K. (2004). "Impact of subsurface drainage on improvement of crop production and farm income in northwest India." *Irrig. Drain. Sys.*, 18, 43-56.
- Dielman, P. J. (1973). "Reclamation of salt-affected soils in Iraq." Publication no. 11, International Institute for Land Reclamation and Improvement, Wageningen, The Netherlands.

- Dörner, J., and Horn, R. (2006). "Anisotropy of pore functions in structured stagnic luvisols in the Weichselian moraine region in N Germany." *J. Plant Nutr. Soil Sci.*, 169, 213-220.
- Elfeki, A. M. M., Uffink, G. J. M., and Barends, F. B. J. (1997). *Goundwater Contaminnant Transport-Impact of Heterogeneous Characterization: A New View on Dispersion*, A.A. Balkema, Rotterdam, The Netherlands
- FAO (Food and Agricultural Organisation of United Nations), (2003). <http://www.fao.org/ag/magazine/0303sp3.htm>.
- FAO (Food and Agricultural Organisation of United Nations), (2012). <http://www.fao.org/docrep/016/i3027e/i3027e.pdf>.
- FAO (Food and Agricultural Organisation of United Nations), (2013). <http://www.fao.org/docrep/018/i3434e/i3434e.pdf>.
- Faures, J. M., Hoozeveld, J., and Bruinsma, J. (2002). *The FAO irrigated area forecast for 2030*, FAO, Rome, Italy.
- Faures, J. M., Svendsen, M., Turrall, H. (2007). "Reinventing irrigation. In: Molden, D. (Ed.), *Water for Food, Water for Life: A Comprehensive Assessment of Water Management in Agriculture*." Earthscan and International Water Management Institute, London, Colombo (Chapter9).
- Fukuda, H. (1957). "Underdrainage into ditches in soil overlying an impervious substratum." *Trans. Am. Geophys. Union*, 38(5), 730-39.
- Ghassemi, F., Jakeman, A. J., and Nix, H. A. (1995). "Salinisation of land and water resource: human causes, extent, management and case studies." University of New South Wales Press, Sydney.
- Goswami, D., and Kalita, P. K. (2009). "Simulation of base-flow and tile-flow for storm events in a subsurface drained watershed." *Biosystems Eng.*, 102, 227-235. doi:10.1016/j.biosystemseng.2008.11.004.
- Goswami, D., and Kalita, P. K. (2010). "Modeling and simulation of base-flow to drainage ditches during low-flow period." *Water Resour. Manage.*, 24, 173-191. doi:10.1007/s11269-009-9443-0

- Grove, D.B., Beatam, W.A., and Sower, F.B. (1970). "Fluid travel time between a recharging well pair in an aquifer having a uniform regional flow field." *Water Resour. Res.*, 6(5), 1404-1410.
- Guitjens, J. C., and Luthin, J. N. (1965). "Viscous model study of drain spacing on sloping land and comparison with mathematical solution." *Water Resour. Res.*, 1(4), 523-530.
- Haitjema, H. M. (2001). "Selecting MODFLOW cell sizes for accurate flow fields." *Ground Water*, 39(6), 931-938.
- Haitjema, H. M. (2006). "The role of hand calculations in ground water flow modeling" *Ground Water*, 44(1), 102-105.
- Huang, H. C., Liu, C. W., Chen, S. K., and Chen, J. S. (2003). "Analysis of percolation and seepage through paddy bunds." *J. Hydrol.*, 284, 13-25.
- He, J. H., (1999). "Homotopy perturbation technique." *Comput. Methods Appl. Mech. Engrg.*, 178, 257-262.
- Hendry, M. J. (1982). "Hydraulic conductivity of a glacial till in Alberta." *Ground Water*, 20(2), 162-169.
- Huang, M., Barbour, S. L., Elshorbagy, A., Zettl, J. D., and Si, B. C. (2011). "Infiltration and drainage processes in multi-layered coarse soils." *Can. J. Soil Sci.*, 91, 169-183.
- ICID (International Commission on Irrigation and Drainage), (2003). "Important data of ICID Member Countries." *Int. Comm. Irrig. Drain.*, database on website: <http://www.icid.org>.
- IDNP (Indo-Dutch Network Project), (2002). "Recommendations on Waterlogging and Salinity Control Based on Pilot Area Drainage Research." Indo-Dutch Network Project, Centr. Soil Sal. Res. Inst., Karnal, India, 56 pp.
- Ilyinsky, N. B., and Kacimov, A. R. (1992). "Problems of seepage to empty ditch and drain." *Water Resour. Res.*, 28(3), 871-77.
- Kacimov, A. R. (1997). "Optimization of seepage rate through a triangular core." *Int. J. Numer. Analyt. Methods Geomech.*, 21, 443-451.
- Kacimov, A. R. (2006). "Seepage to a drainage ditch and optimization of its shape." *J. Irrig. Drain. Eng.*, 132(6), 619-22.

- Kanwar, R. S. (1989). "Effect of tillage systems on the variability of soil-water tensions and soil-water content." *Transactions of the ASAE* 32, 605-610.
- Khan, S., Tariq, R., Yuanlai, C., and Blackwell, J. (2004). "Can irrigation be sustainable?" *New directions for a diverse planet: Proceedings of the 4th International Crop Science Congress*, Brisbane, Australia, 26 Sep - 1 Oct 2004.
- Kirkham, D. (1950). "Seepage into ditches in the case of a plane water table and an impervious substratum." *Trans. Am. Geophys. Union*, 31(3), 425-30.
- Kirkham, D. (1957). "Theory of land drainage: The ponded water case." In: Luthin JN, editor, *Drainage of agricultural lands*, Vol. 7, Madison, Wisconsin.
- Kirkham, D. (1960). "Seepage into ditches in the case of a plane water table overlying a gravel substratum." *Trans. Am. Geophys. Union*, 65(4), 1267-72.
- Kirkham, D. (1965). "Seepage of leaching water into drainage ditches of unequal water level height." *J. Hydrol.*, 3, 207-24.
- Kirkham, D., and Powers, W. L. (1972). *Advanced Soil Physics*. Wiley-Interscience, New York.
- Kirkham, D., Toksoz, S., and van Der Ploeg, R. R. (1974). "Steady flow to drains and wells." In: Schilfgaard JV, editor, *Drainage for agriculture*, Am. Soc. of Agron., Vol. 17, Madison, Wisconsin.
- Kirkham, D., and Van Bavel, C. H. M. (1948). "Theory of seepage into auger holes." *Soil Sci. Soc. Am. Proc.*, 13, 75-82.
- Kirkham, D., Van der Ploeg, R. R., and Horton, R. (1997). "Potential theory for dual-depth subsurface drainage of ponded land." *Water Resour. Res.*, 33: 1643-1654.
- Kroger, R., Cooper, C. M., and Moore, M. T. (2008). "A preliminary study of an alternative controlled drainage strategy in surface drainage ditches: low-grade weirs." *Agric. Water Management*, 95, 678-684.
- Kuwayama, Y., and Brozović, N. (2012). "Analytical hydrologic models and the design of policy instruments for groundwater-quality management." *Hydrogeol. J.*, 20(5), 957-972.

- Lewis, K.C. (2013). "Forgotten merits of the analytic viewpoint." *Eos*, 94(7), American Geophysical Union, 71-72
- Liao, S. J. (1992). "A second-order approximate analytical solution of a simple pendulum by the process analysis method." *J. Applied Mechanics*, 14, 1173-1191.
- Liu, C. W., Tan, C. H., and Huang, C. C. (2005). "Determination of the magnitudes and values for groundwater recharge from Taiwan's paddy field." *Paddy Water Environ.*, 00, 1-6.
- Liu, S., Xingguo, M., Haibin, L., Gongbing, P., and Alan, R. (2001). "Spatial variation of soil moisture in China: Geostatistical characterization." *J. Meteorol. Soc. Japan*, 79, 555-574.
- Maasland, M. (1957). "Soil anisotropy and land drainage." In Luthin, J.N., *Drainage of Agricultural Lands*. Am. Soc. Agron., Madison, Wisconsin, 216-285.
- McDermott, C. I., Walsh, R., Mettier, R., Kosakowski, G., and Kolditz, O. (2009). "Hybrid analytical and finite element numerical modeling of mass and heat transport in fractured rocks with matrix diffusion." *Comput Geosci.*, 13(3), 349-361.
- Manjunatha, M. V., Oosterbaan, R. J., Gupta, S. K., Rajkumar, H., and Jansen, H. (2004). "Performance of subsurface drains for reclaiming waterlogged saline under rolling topography in Tungabhadra irrigation project in India." *Agric. Water Management*, 69, 69-82.
- Marja, R. (2013). *The relationships between farmland birds, land use and landscape structure in Northern Europe*, University of Tartu Press.
- Marja, R., and Herzon, I. (2012). "The importance of drainage ditches for farmland birds in agricultural landscapes in the Baltic countries: does field type matter?" *Ornis Fennica* 89(3), 170.
- Martinez-Beltran, J. (1978). *Drainage and reclamation of salt affected soils in the Bardenas area, Spain*. ILRI Publication 24, Wageningen, The Netherlands.
- Martinez-Beltran, J. (2002). "World food summit", Food and Agriculture Organization of the United Nations, 10-13th June, 2002 (<http://www.fao.org/worldfoodsummit/english/newsroom/focus/focus1.htm>).
- Martinez-Beltran, J., and Manzur, C. L. (2005). "Overview of salinity problems in the world and FAO strategies to address the problem." *Proceedings of the international salinity forum*, Riverside, California, April 2005, 311-313.

- Meigs, L. C., and Bahr, J. M. (1995). "Three-dimensional groundwater flow near narrow surface water bodies." *Water Resour. Res.*, 31(12), 3299-3307.
- Meunier, F., Draye, X., Vanderborght, J., and Javaux, M. (2017). "A hybrid analytical-numerical method for solving water flow equations in root hydraulic architectures." *Appl Math Model.*, 52, 648-663.
- Mirjat, M. S., and Rose, D. A. (2009). "Streamline pattern and salt leaching through progressive flooding between subsurface drains." *Irrig. and Drain.*, Wiley, 58, 199–208.
- Mirjat, M. S., Rose, D. A., and Adey, M. A. (2008). "Desalination by zone leaching: laboratory investigations in a model sand-tank." *Aust. J. Soil Res.*, 46(2), 91–100.
- Miyamoto, S., and Warrick, A. W. (1974). "Two-Dimensional Displacement into or from Water-Filled Ditches." *Soil Sci. Soc. of Am. J.*, 38(5), 723-727.
- Morel-Seytoux., H. J. (2015). "Analytical solutions using integral formulations and their coupling with numerical approaches." *Ground Water.*, 53(5), 810-818.
- Murdoch, L. C. (1994). "Transient analysis of an interceptor trench." *Water Resour. Res.*, 30(11), 3023-3031
- Mohanty, B. P., Kanwar, R. S., and Everts, C. J. (1994). "Comparison of saturated hydraulic conductivity measurement methods for a glacial till soil." *Soil Sci. Soc. Am. J.*, 58, 672-677.
- Mulquen, J., and Kirkham, D. (1972). "Leaching of a surface layer of sodium chloride into tile drains in a sand tank model." *Soil Sci. Soc. Amer Proc.*, 36, 3-8.
- Muskat, M. (1937). *The flow of homogeneous fluids through porous media*. McGraw-Hill, New York. (second printing by J. W. Edwards, Ann Arbor, Michigan, 1946, p.763).
- Needelman, B. A., Kleinman, P. J. A., Allen, A. L., and Strock, J. S. (2007). "Managing agricultural drainage ditches for water quality protection." *Journal of Soil and Water Conservation*, 62(4), 171-178.
- Nield, S. P., Townley, L. R., and Barr, A. D. (1994). "A framework for quantitative analysis of surface water-groundwater interaction: Flow geometry in a vertical section." *Water Resour. Res.*, 30(8), 2461-2475.

- Odhiambo, L. O., and Murty, V. V. N. (1996). "Modeling water balance components in relation to field layout in lowland paddy fields I. Model development." *Agric. Water Management*, 30, 185-199.
- Ogino, Y., and Murashima, K. (1993), Subsurface drainage system of large size paddies for crop diversification in Japan, International commission on irrigation and drainage. Fifteenth Congress, The Hague.
- Olver, P. J. (1993). *Applications of Lie Groups to Differential Equations*. Springer.
- Pearce, G., and Denecke, H. W. (2001). "Drainage and sustainability." IPTRID Issues Paper No. 3.FAO/IPTRID, Rome.
- Ortiz, J., and Luthin, J. N., (1970). "Movement of salts in ponded anisotropic soils." *J. Irrig. Drain. Div. Amer. Soc. Civil Eng.*, 96(3), 257-264.
- Praveena, S. M., Abdullah, M. H., Aris, A. Z., and Bidin, K. (2010). "Groundwater Solution Techniques: Environmental Applications." *J. Water Resource and Protection*, 2, 8-13.
- Petersen, S.O., Schjøning, P., Thomsen, I. K., and Christensen, B. T. (2008). "Nitrous oxide evolution from structurally intact soil as influenced by tillage and soil water content." *Soil Biol. Biochem.*, 40, 967-977.
- Qiu, J. (2009). "China cuts methane emissions from rice fields." *Nature News*, doi:10.1038/news.2009.833.
- Rao, K. V. G. K., and Leeds-Harrison, P. B. (1991). "Desalination with subsurface drainage." *Agric. Water Management*, 19, 303-11.
- Rengasamy, P. (2006). "World salinization with emphasis on Australia." *J. Exp. Botany*, 57(5), 1017-1023.
- Rhoades, J. D. (1997). "Sustainability of Irrigation: An Overview of Salinity Problems and Control Strategies." *Proc. 1997 Annual Conference of the Canadian Water Resources Association*, Footprints of Humanity: Reflections on Fifty Years of Water Resources Developments, Lethbridge, Alberta, Canada, June 3-6, 1997.
- Ritzema, H. P. (2016). "Drain for Gain: Managing salinity in irrigated lands – A review." *Agric. Water Management*, 176, 18-28.
- Ritzema, H. P., Satyanarayana, T. V., Raman, S., and Boonstra, J. (2008). "Subsurface drainage to combat water logging and salinity in irrigated lands in India: Lessons learned in farmer's field." *Agric. Water Management*, 95, 179-189.

- Römken, M. J. M. (2009). "Estimating seepage and hydraulic potentials near incised ditches in a homogeneous, isotropic aquifer." *Earth Surf. Process. Landforms*, 34, 1903–1914.
- Sakellariou-Makrantonaki, M. (1997). "Water drainage in layered soils. Laboratory experiments and numerical simulation." *Water Resour. Manage.*, 11, 437-444.
- Sarmah, R., and Barua, G. (2015). "Hydraulics of a partially penetrating ditch drainage system in a layered soil receiving water from a ponded field." *J. Irrig. Drain. Eng.*, 141(8), 04015001.
- Sarmah, R., and Barua, G. (2017). "Analysis of three-dimensional transient seepage into ditch drains from a ponded field." *Sadhana*, 42(5), 769-793.
- Sarmah, R., and Tiwary, S. (2018). "A two-dimensional transient analytical solution for a ponded ditch drainage system under the influence of source/sink." *J. of Hydrol.*, 558, 205-213.
- Scheumann, W., and Freisem, C. (2002). "The role of drainage for sustainable agriculture." *J. of Applied Irrigation Science*, 37(1), 33-61.
- Schultz, B., and Wrachein, D. D., (2002). "Irrigation and drainage systems research and development in the 21st century." *Irrig. And Drain.*, 51, 311-327.
- Sharma, D. P., and Gupta, S. K. (2006). "Subsurface drainage for reversing degradation of waterlogged saline lands." *Land Degradation and Development*, 17(6), 605-614.
- Sharma, R. C., Rao, B. R. M., and Saxena, R. K. (2004). "Salt Affected soils in India- Current Assessment." *In: Advances in Sodic Land Reclamation*, International Conference on Sustainable Management of Sodic Lands, Lucknow, India, 1-26.
- Shiratori, Y., Watanabe, H., Furukawa, Y., Tsuruta, H., and Inubushi, K. (2007). "Effectiveness of a subsurface drainage system in poorly drained paddy fields on reduction of methane emissions." *Soil Science and Plant Nutrition*, 53(4), 387-400.
- Singh, J., and Singh, J. P. (1995). "Land degradation and economic sustainability." *Ecological Economic*, 15(1), 77-86.
- Singh, P.K., Singh, O.P., Jaiswal, C. S., and Chauhan, H. S. (1999). "Subsurface drainage of a three layered soil with slowly permeable top layer." *Agric. Water Manage.*, 42, 97-109.
- Siyal, A. A., Skaggs, T. H., and van Genuchten, M. Th. (2010). "Reclamation of saline soils by partial ponding: simulations for different soils." *Vadose Zone J.*, 9, 486-495.

- Smedema, L. K., Abdel-Dayem, S., and Ochs, W. J. (2000). "Drainage and agricultural development." *Irrig. Drain. Syst.*, 14, 223–235.
- Smedema, L. K., and Rycroft, D. W. (1983). "Land drainage: planning and design of Agricultural Drainage Systems." Batsford, London, p.376.
- Stephens, D. B., and Heermann, S. (1988). "Dependence of Anisotropy on saturation in a stratified sand." *Water Resour. Res.*, 24(5), 770-778.
- Stewart, A. J., Lance, A. N. (1983). "Moor-draining: a review of impacts on land use." *J. Environ. Manage.*, 17 (1), 81–99.
- Stibinger, J. (2009). "Terrain experimental measurement of saturated hydraulic conductivity on paddy fields in Taoyuan (Taiwan) during the cycle of flooded period." *Agricultura. Tropica. et subtropica.*, 42(2), 82-89.
- Tabuchi, T. (2004). "Improvement of paddy field drainage for mechanization." *Paddy Water Environ.*, 2004, 2:5-10, doi:10.1007/s10333-004-0034-7.
- Tiwary, P., and Goel, A. (2017). "An overview of impact of subsurface drainage project studies on salinity management in developing countries." *Appl. Water. Sci.*, 7, 569-580.
- United Nations (<http://www.un.org/apps/news/story.asp?NewsID=45165#.U79xREDv-eE>).
- Van Hoorn, J. W., and Van Alphen, J. G. (1994). "Salinity control." In H.P. Ritzema, ed. *Drainage Principles and Applications*, 2nd edition. ILRI Publication 16, Wageningen, The Netherlands, 533-600.
- Wang, J. F., and Anderson, M. P. (1982). *Introduction to Groundwater Modeling*, Freeman, San Francisco, CA, 237 pp.
- Walton, W.C. (1979). "Progress in analytical groundwater modeling." *J. of Hydrol.*, 43, 149–159.
- Walton, W. C. (1989). *Analytical Ground Water Modeling*, Lewis Publishers, Chelsea, Michigan.
- Warrick, A. W., and Kirkham, D. (1969). "Two-dimensional seepage of ponded water to full ditch drains." *Water Resour. Res.*, 5(3), 685-93.
- Water for Food (2013). <http://waterforfood.nebraska.edu/>.

- Wichelns, D., Cone, D., and Stuhr, G. (2002). "Evaluating the impact of irrigation and drainage policies on agricultural sustainability." *Irrig. Drain. Syst.*, 16, 1–14.
- Wild, A. (2003). *Soils, land and food: managing the land during the twenty-first century*, Cambridge, UK, Cambridge University Press.
- Wolde-Kirkos, A. T., and Chowla, A. S. (1994). "Seepage from canal to asymmetric drainages." *J. Irrig. Drain. Eng.*, 120(5), 949-956.
- Xiaohong, Z., Jia, H., and Junxin, C. (2011), "Study on Mitigation Strategies of Methane Emission from Rice Paddies in the Implementation of Ecological Agriculture." *Energy Procedia* 5, 2474-2480.
- Yang, J., Ma, W., Chen, D., Holmen, A., and Davis, B. H. (2014). "Fischer–Tropsch synthesis: A review of the effect of CO conversion on methane selectivity." *Applied Catalysis A: General*, 470, 250-260.
- Youngs, E. G. (1982). "Calculations of ponded water drainage for flow regions of various geometries to demonstrate effect of disturbed soil-zone shape on drain performance." *J. Agric. Eng. Res.*, 27. 441-54.
- Youngs, E. G. (1994). "Seepage to ditches from a ponded surface." *J. Hydrol.*, 161, 145-54.
- Youngs, E. G., and Leeds-Harrison, R. B. (2000). "Improving efficiency of desalinization with subsurface drainage." *J. Irrig. and Drain. Eng.*, 126(6), 375-80.
- Yvon-Durocher, G., Allen, A. P., Bastviken, D., Conrad, R., Gudasz, C., St-Pierre, A., Thanh-Duc, N., and Giorgio, P. A. del (2014). "Methane fluxes show consistent temperature dependence across microbial to ecosystem scale." *Nature News*, doi:10.1038/nature13164.
- Zaslavsky, D. (1979). "Drainage for salt leaching." In: Wesseling J, editor. *Proceedings of the international drainage workshop*, International Institute for Land Reclamation and Improvement, Vol 25, Wageningen.
- Zhang, W., Yu, Y., Huang, Y., Li, T., and Wang, P. (2011). "Modeling methane emissions from irrigated rice cultivation in China from 1960 to 2050." *Global Change Biology*, 17, 3511–3523.
- Zhao, P., Shao, M., and Melegy, A. A. (2010). "Soil water distribution and movement in layered soils of a dam farmland." *Water Resour. Manage.*, 24, 3871-3883.
- Zheng, C., Bradbury, K. R., and Anderson, M. P. (1988a). "Role of interceptor ditches in limiting the spread of contaminants in ground water." *Ground Water*, 26(6), 734-742.

Zheng, C., Wang, H. F., Anderson, M. P., and Bradbury, K. R. (1988b). "Analysis of interceptor ditches for control of ground water pollution." *J. Hydrol.*, 98, 67-81.

