Study of Certain Classes of ψ-Hilfer Fractional Differential Equations: Qualitative Properties and Some Applications

A Thesis

Submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

by

Sunil
(Roll no. 206123107)

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI-781039, INDIA

December 2025

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DEDICATED TO MY PARENTS

Indro Devi
&
Wazir Kundu

Declaration

I do hereby declare that this thesis entitled **"Study of Certain Classes of ψ-Hilfer Fractional Differential Equations: Qualitative Properties and Some Applications"** is a presentation of my original research work carried out under the supervision of **Dr. Swaroop Nandan Bora**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

Sunil
Roll No. 206123107
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati 781039, Assam, India
December 2025

Certificate

It is to certify that the work contained in this thesis entitled **"Study of Certain Classes of ψ-Hilfer Fractional Differential Equations: Qualitative Properties and Some Applications"** has been carried out by **Mr. Sunil**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under my supervision for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

December 2025

Dr. Swaroop Nandan Bora
Professor
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati 781039, Assam, India

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Abstract

The theory of fractional differential equations has become a powerful mathematical framework for modeling systems that exhibit memory and hereditary characteristics. Among the various fractional operators, the ψ-Hilfer fractional derivative provides a unifying and flexible tool that generalizes several classical derivatives through appropriate choices of ψ and its parameters. This dissertation is devoted to the qualitative analysis of different classes of fractional differential equations involving the *ψ-Hilfer derivative*, with particular emphasis on *Ulam-type stability*. We discuss five problems pertaining to *ψ-Hilfer* fractional differential equations with the main objective of analyzing the stability of the solution in Ulam-Hyers sense.

In the first problem, we study an *abstract fractional differential equation*. Using Banach's fixed point theorem and suitable fractional inequalities, we establish sufficient conditions for the existence and stability of solutions in the sense of *Ulam–Hyers* and *Ulam–Hyers–Rassias*. Numerical results are provided to illustrate the solutions of the system for different weight functions and fractional orders. The next problem extends the analysis to *Mittag–Leffler-type stability* for ψ-Hilfer abstract fractional differential equation. The existence results are obtained via Schauder's fixed point theorem, and stability is analyzed in the sense of *Ulam–Hyers–Mittag–Leffler* and *Ulam–Hyers–Rassias–Mittag–Leffler*.

In the third problem, we consider a *neutral fractional differential equation with delay involving the ψ-Hilfer derivative*. Such models describe systems where the present state depends on both current and delayed terms. Krasnosel'skiĭ's fixed point theorem is used to prove the existence of solutions, and the stability in the sense of *Ulam–Hyers* and *Ulam–Hyers–Rassias*. Numerical simulations are presented to illustrate the behavior of delayed fractional systems. In another problem, we investigate a *coupled system of fractional differential equations*. Banach's and Schauder's fixed point theorems are applied to establish the existence and qualitative stability results of *Ulam–Hyers* and *generalized Ulam–Hyers* type. A real-world application to a *blood alcohol concentration model* is presented, and numerical results are compared with experimental data to demonstrate the effectiveness of the ψ-Hilfer fractional approach.

The fifth and final work focuses on a *system of three fractional differential equations*, each of different order. The existence and uniqueness of solutions are proved using Banach's fixed point theorem, and the stability is analyzed in the sense of *Ulam–Hyers*

and *generalized Ulam–Hyers*. As an application, a chaotic financial system is examined, and numerical experiments highlight the emergence of complex and chaotic behaviors in ψ-Hilfer fractional systems.

In summary, this dissertation develops a unified framework for the qualitative analysis and Ulam–Hyers-type stability of diverse classes of fractional differential equations under the general ψ-Hilfer fractional derivative. By addressing abstract, neutral, coupled, and system of three fractional differential equations, the work advances the understanding of how memory, delays, and interactions influence stability. The results provide a rigorous mathematical foundation for modeling and analyzing complex phenomena in science, engineering, and finance, where non-local and memory-dependent effects play a crucial role.

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CHAPTER 1

Introduction

1.1 Background and Motivation

In mathematics, few concepts have proven as universal and transformative as the derivative. It lies at the heart of Newton's laws of motion, governs the flow of heat in Fourier's theory, and underpins Maxwell's equations of electromagnetism and Schrödinger equation in quantum mechanics. It is not an exaggeration to say that almost all of the physical problems in this world can be represented through differential equations. It becomes essential to solve them to know the outcomes. Although solving the differential equation is an important aspect from the quantitative view, it is equally important to give importance to the qualitative study of differential equations. The qualitative study of dynamical systems lies at the heart of mathematical modeling and analysis. A dynamical system describes the evolution of a state over time under deterministic or stochastic rules, frequently expressed in the form of differential equations, whether integer order or fractional order. Such formulations are indispensable in understanding the underlying mechanisms of complex processes across disciplines, including engineering, biology, physics, economics, and ecology.

In practice, explicit analytic solutions to these equations are rarely attainable, particularly when models increase in dimension and incorporate nonlocal or memory-dependent effects. Consequently, the qualitative theory of differential equations has become central: rather than providing exact solutions, it investigates the structural properties of systems such as boundedness, recurrence, asymptotic behavior, and, most importantly, stability. Stability theory, in particular, evaluates the sensitivity of system trajectories to perturbations, thereby offering insights into robustness, reliability of numerical schemes, and the validity of control mechanisms.

However, the classical models as these mentioned above rely on derivatives of integer order - capturing only the instantaneous and local behavior of a system. Real-world phenomena, however, are often richer: materials retain memory of past deformations, diffusion occurs at anomalous rates, and biological processes evolve with hereditary effects. Such complexity calls for a calculus that can transcend the rigid framework of integer orders.

This motivation gave rise to *fractional calculus*, the extension of differentiation and integration to arbitrary real or even complex orders. Its origin dates back to 1695, when L'Hôpital asked Leibniz about the meaning of a derivative of order $1/2$. What began as a mathematical curiosity gradually evolved into a deep and elegant theory, enriched by the contributions of many leading mathematicians. Euler (1730) studied properties of fractional exponents, while Lagrange (1772) introduced generating functions that hinted at fractional operators. Laplace (1812) developed transform techniques that later became central to solving fractional differential equations. Fourier (1822) showed how fractional derivatives could naturally appear in heat conduction, and Liouville (1832) gave the first rigorous definition of fractional integration on the real line. Riemann (1847) extended these ideas by introducing what is now known as the Riemann–Liouville integral, Holmgren (1865) contributed to the analytical foundations of the subject, while Grünwald (1867) and Letnikov (1868) provided equivalent discrete formulations, giving concrete computational meaning to fractional derivatives. Later contributions by Laurent (1884) and Nekrasov (1888) expanded the analytical foundations of the subject.

In the early twentieth century, Hardy (1917) and Littlewood (1929) studied singular integrals and fractional operators, while Marcel Riesz (1949) linked fractional calculus with potential theory and harmonic analysis. The second half of the century witnessed a decisive shift toward systematic treatments. Oldham and Spanier (1974) published the first modern textbook on fractional calculus, followed by the influential work of Samko, Kilbas, and Marichev (1987; English edition 1993), which established rigorous definitions and properties of fractional integrals and derivatives. Miller and Ross (1993) advanced applications to differential equations, while Podlubny (1999) unified theoretical and applied perspectives, highlighting applications in physics and engineering. The works of Diethelm (2004) and Kilbas, Srivastava, and Trujillo (2006) significantly expanded the field by studying existence and uniqueness results, stability theory, and boundary value problems for fractional differential equations.

Alongside the theoretical development, various methods for solving fractional differential equations were introduced. Analytical techniques included Laplace and Fourier transforms, Mellin transforms, and series expansions involving special functions such as the Mittag-Leffler function. Fixed point theorems (Banach, Schauder, Krasnosel'skiĭ, etc.) became central tools for proving the existence and uniqueness of solutions. With the increasing complexity of real-world models, numerical methods such as fractional Euler's method, predictor–corrector schemes, and spectral methods were developed to approximate solutions of both linear and nonlinear fractional differential equations (FDEs).

Another important development is concerned with the nature of the kernels used in fractional operators. Classical derivatives such as the Riemann–Liouville and Caputo types are defined with *singular kernels*, typically of the form $(t-s)^{-\alpha}$, which naturally encode memory but pose difficulties in certain applications. In recent years, *nonsingular kernel* operators have been introduced, such as the Caputo–Fabrizio derivative (2015), which employs an exponential kernel, and the Atangana–Baleanu derivative (2016), which uses the Mittag–Leffler kernel. These new definitions aim to avoid singularities at the origin while preserving memory effects, thereby providing more flexibility for physical modeling.

As a result of these developments, FDEs emerged as a central focus of research. This field grew rapidly, not only in the analysis of existence, uniqueness, and some other qualitative properties of the solutions, but also in the design of efficient numerical methods capable of approximating solutions with memory and nonlocal terms. This evolution transformed fractional calculus from a mathematical curiosity into a mature discipline with applications ranging from viscoelasticity and rheology to anomalous diffusion, control theory, bioengineering, and finance. Today, FDEs stand at the frontier of applied mathematics, serving as both a natural extension of classical models and a powerful framework for capturing complex behaviors that cannot be described by integer-order dynamics.

In recent decades, attention has shifted toward generalized fractional operators that provide greater modeling flexibility. Among these, the *ψ-Riemann–Liouville integral* and the *ψ-Hilfer fractional derivative* have emerged as powerful tools. By introducing an auxiliary function $\psi(t)$, these operators incorporate different time scales and non-uniform dynamics into calculus. The ψ-Hilfer derivative interpolates between the ψ-Riemann–Liouville and ψ-Caputo derivatives through a parameter $\beta \in [0,1]$, thus offering a unified framework that encompasses several classical cases. This flexibility makes ψ-fractional operators especially useful in the study of nonlinear, neutral, and delay systems, as well as in the analysis of Ulam–Hyers stability and other qualitative properties of solutions.

From its origin in a speculative question to Leibniz, fractional calculus has evolved into a vibrant and indispensable area of mathematics. The development of ψ-fractional operators marks a significant stage in this evolution, enriching the theoretical foundation and broadening the range of applications of fractional differential equations.

The evolutionary landscape of fractional calculus has witnessed remarkable diversification, with mathematicians, scientists, and researchers continuously proposing innovative definitions of fractional derivatives to address the growing complexity of modern scientific challenges. While the pioneering works of Riemann–Liouville, Grünwald–Letnikov, and Caputo established the foundational pillars of fractional calculus, the contemporary mathematical community has embraced the sophisticated Hilfer fractional derivative and the more recent ψ-Hilfer derivative as powerful generalizations that unify multiple classical operators under elegant mathematical frameworks. These various definitions of fractional-order derivatives, along with their intricate properties, have been extensively documented in numerous authoritative texts and research publications [7, 12, 43, 50, 78, 80, 81], forming a rich theoretical foundation for modern applications.

The profound superiority of fractional calculus over its classical integer-order counterpart lies in its extraordinary capacity to capture the intrinsic memory and hereditary characteristics inherent in complex natural phenomena and engineered systems. Unlike traditional mathematical models that often oversimplify or entirely neglect these crucial memory effects, fractional derivatives serve as sophisticated mathematical instruments that naturally encode historical dependencies, making fractional-order models significantly more realistic and accurate representations of real-world systems. This fundamental advantage has catalyzed revolutionary applications across diverse scientific domains, including wave propagation phenomena [21, 27], viscoelastic material behavior [40, 50], advanced control systems [34, 42, 44], chaos theory and nonlinear dynamics [26, 49], sophisticated electrical circuit analysis [4, 5], biological system modeling [17, 20, 65]. Further advancements in financial market dynamics, signal and image processing, biomedical engineering, fractional-order neural networks, cryptographic systems, autonomous vehicle control, and emerging artificial intelligence applications, can also be observed.

The fundamental distinction between integer-order and fractional-order differential operators reveals a profound mathematical paradigm shift that has transformative implications for system modeling and analysis. Classical integer-order differential operators exhibit purely local characteristics, requiring only infinitesimal neighborhood information around a specific point to compute derivatives, thereby capturing instantaneous rates of change. In stark contrast, fractional differential operators possess inherently non-local properties, meaning that the fractional derivative of a function at any given point depends not merely on the local behavior, but incorporates the entire history of the function across its domain. This non-locality manifests the remarkable property that "the next state of a system depends not only on its current state, but also upon all its past states," fundamentally altering our understanding of system dynamics and introducing unprecedented complexity in both geometric interpretation and analytical solution methodologies.

This non-local nature of fractional operators enables the mathematical representation of systems exhibiting long-range temporal correlations, persistent memory effects, and hereditary properties that are ubiquitous in natural phenomena but impossible to capture with conventional integer-order models. From the molecular scale behavior of viscoelastic polymers to the macroscopic dynamics of financial markets, from the intricate patterns of biological growth to the complex propagation of acoustic waves in porous materials, fractional calculus provides the mathematical language necessary to describe these sophisticated memory-dependent processes with unprecedented accuracy and physical insight.

Furthermore, recent advances in fractional calculus have introduced variable-order fractional operators, conformable fractional derivatives, and operators with non-singular kernels, expanding the theoretical framework to accommodate even more complex phenomena involving time-varying memory effects and distributed parameter systems. These developments represent the cutting edge of mathematical modeling, offering researchers powerful tools to tackle previously intractable problems in fields ranging from quantum mechanics and field theory to epidemiology and climate science, thereby positioning fractional calculus as an indispensable mathematical foundation for twenty-first century scientific discovery and technological innovation.

1.2 Classification of Fractional Operators

Over the history of fractional calculus, a wide variety of fractional-order integral and differential operators have been introduced. The motivation for these developments stems from two main directions: (i) the need to model increasingly complex physical and engineering processes, and (ii) the attempt to address certain analytical or numerical difficulties inherent in earlier definitions. Almost all such operators can be characterized by the kernel used in their definition, and they are broadly classified into two families: operators with *singular kernels* and those with *non-singular kernels*. Since some special functions serve as indispensable parts of fractional calculus, we mention and define some of them below.

1.2.1 Special Functions in Fractional Calculus

One of the basic functions of fractional calculus is Euler's Gamma function $\Gamma(\cdot)$, which allows non-integer and even complex values [32].

Definition 1.1. *The Gamma function $\Gamma(\cdot)$ is defined by*

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} \, ds, \tag{1.1}$$

which converges for $Re(t) > 0$.

The Gamma function $\Gamma(\cdot)$ satisfies the following relations:

$$\Gamma(t+1) = t\Gamma(t), \; Re(t) > 0, \tag{1.2}$$

$$\Gamma(t+1) = t!, \; t \in \mathbb{N}. \tag{1.3}$$

The Mittag-Leffler functions have important roles in the theory of fractional calculus, which are also generalizations of exponential functions.

Definition 1.2. *The one-parameter (α) Mittag-Leffler function is given by*

$$\mathbb{E}_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+1)}, \quad t \in \mathbb{C}, \; Re(\alpha) > 0. \tag{1.4}$$

The two-parameter (α, β) *Mittag-Leffler function is given by*

$$\mathbb{E}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)}, \quad t \in \mathbb{C}, \; Re(\alpha) > 0, \; Re(\beta) > 0. \tag{1.5}$$

The solutions to fractional differential equations are often expressed not in terms of elementary functions, but through a generalization of the exponential function, such as the Mittag-Leffler function.

When $\beta = 1$, it reduces to the one-parameter Mittag-Leffler function, $\mathbb{E}_{\alpha}(t) = \mathbb{E}_{\alpha,1}(t)$. For $\alpha = 1$ and $\beta = 1$, it becomes the standard exponential function, $\mathbb{E}_{1,1}(t) = e^t$. Mittag-Leffler function naturally arises as the solution to fractional-order linear differential equations.

1.2.2 Fractional Operators with Singular Kernels

The classical and most fundamental fractional derivative operators are the Riemann–Liouville and Caputo derivatives, both defined via singular kernels of the form

$$k(t, s) = \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1.$$

These operators naturally encode long-memory effects and nonlocal behavior, making them suitable for a wide range of physical models. However, their singularity at $t = s$ poses analytical and numerical challenges.

Several other important operators also belong to the singular-kernel family. The *Hadamard derivative*, introduced in the late 19th century, employs a logarithmic kernel and is useful for problems defined on a multiplicative time scale. The *Katugampola derivative*, proposed more recently, unifies the Riemann–Liouville and Hadamard forms into a single general framework, thereby broadening the scope of applications. The *Hilfer derivative*, introduced in 2000, interpolates between the Riemann–Liouville and Caputo derivatives through an additional parameter β, and has proven effective in modeling processes where intermediate memory effects are relevant. Further extensions include the *Prabhakar derivative*, which incorporates a three-parameter Mittag–Leffler kernel, and its generalizations that capture stretched memory effects. The most flexible among these is the ψ*-Hilfer derivative*, which combines the Hilfer framework with an arbitrary increasing function ψ, allowing the modeling of systems on generalized time scales.

1.2.3 Fractional Operators with Non-singular Kernels

To circumvent the singularity-related issues, Caputo and Fabrizio [12] introduced in 2015 a derivative defined by a *non-singular kernel* of exponential type:

$$k(t, s) = \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right), \quad 0 < \alpha < 1,$$

now widely known as the *Caputo–Fabrizio derivative*. This approach eliminates the singularity and simplifies certain analytical and numerical treatments.

Atangana and Baleanu [10] further generalized this idea by introducing a derivative with a non-singular kernel of Mittag–Leffler type:

$$k(t,s) = \mathbb{E}_{\alpha}\left(-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right),$$

which is now termed the *Atangana–Baleanu derivative*. These formulations, together with other contributions in [7, 78, 80, 81], have been argued to capture more realistically phenomena involving material heterogeneity, viscoelasticity, and anomalous diffusion. However, some drawbacks have been identified: Diethelm [19] observed that the fundamental theorem of fractional calculus does not hold for these operators, while Zhang [84] pointed out that the derivative always vanishes at the initial instant, restricting admissible initial conditions.

In summary, fractional-order operators can be broadly grouped into those with singular kernels, which possess a well-established theoretical foundation but may suffer from analytical and numerical difficulties, and those with non-singular kernels, which offer computational and modeling advantages at the expense of certain mathematical properties. The proliferation of these operators - from Riemann–Liouville and Caputo to Hadamard, Katugampola, Hilfer, Prabhakar, ψ-Hilfer, Caputo–Fabrizio, and Atangana–Baleanu, etc., reflects the rich and ongoing evolution of fractional calculus as it adapts to the needs of both theory and applications.

1.3 Important Fractional Operators with Singular Kernel

The theory of fractional calculus is built upon the generalization of the standard integral and differential operators to arbitrary, non-integer orders. This section introduces the key operators that form the mathematical foundation of this thesis, progressing from the classical definitions to the highly generalized ψ-Hilfer framework.

1.3.1 Classical Fractional Operators

Building upon the fractional integral, several definitions for fractional derivatives have been proposed. The cornerstone of most fractional derivative definitions is the fractional integral. The classical form is the Riemann–Liouville fractional integral [32, 50, 87].

Definition 1.3 (Riemann–Liouville Fractional Integral). *Let $\alpha > 0$ and $u(t)$ be an integrable function. The Riemann–Liouville (RL) fractional integral of order α is defined as*

$$I_{a+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-s)^{\alpha-1}u(s)ds, \tag{1.6}$$

where $\Gamma(\cdot)$ is the Euler's Gamma function.

Definition 1.4 (Riemann–Liouville Fractional Derivative)**.** *For $n-1<\alpha<n$, the Riemann-Liouville fractional derivative of order α is defined as*

$$D_{a+}^{\alpha}u(t)=\left(\frac{d}{dt}\right)^{n}I_{a+}^{n-\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{n-\alpha-1}u(s)ds. \tag{1.7}$$

A significant drawback of the RL derivative is that its initial value problems involve fractional-order initial conditions, which often lack a clear physical interpretation.

Definition 1.5 (Caputo Fractional Derivative)**.** *To address the issue of initial conditions, Caputo introduced an alternative definition where the integer-order derivative is considered first:*

$${}^{C}D_{a+}^{\alpha}u(t)=I_{a+}^{n-\alpha}\left(\frac{d}{dt}\right)^{n}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}u^{(n)}(s)ds, \tag{1.8}$$

which is known as Caputo derivative. The key advantage of the Caputo derivative is that the initial conditions for associated differential equations are specified in terms of integer-order derivatives, which is more suitable for modeling real-world phenomena.

Definition 1.6 (Hilfer Fractional Derivative)**.** *The Hilfer derivative generalizes both the RL and Caputo derivatives. For $n-1<\alpha<n$ and $0\le\beta\le1$, it is defined as*

$${}^{H}D_{a+}^{\alpha,\beta}u(t)=I_{a+}^{\beta(n-\alpha)}\left(\frac{d}{dt}\right)^{n}I_{a+}^{(1-\beta)(n-\alpha)}u(t). \tag{1.9}$$

The parameter β is the "type" of the derivative. When $\beta=0$, the Hilfer derivative reduces to the RL derivative. When $\beta=1$, it becomes the Caputo derivative.

1.3.2 Generalized Fractional Operators

A further and powerful generalization is achieved by incorporating the function $\psi(t)$ into the definitions, leading to the ψ-Hilfer operator and its special cases [14].

Definition 1.7 (ψ-Riemann–Liouville Fractional Integral)**.** *Let $\alpha>0$, and $\psi(t)$ be an increasing, continuously differentiable function such that $\psi'(t)\neq0$. The ψ-Riemann–Liouville fractional integral of a function $u(t)$ with respect to $\psi(t)$ is defined as*

$$I_{a+}^{\alpha;\psi}u(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s)ds. \tag{1.10}$$

Definition 1.8 (ψ-Hilfer Fractional Derivative)**.** *For $n-1<\alpha<n$ and $0\le\beta\le1$, the ψ-Hilfer fractional derivative of a function $u((t)$ with respect to $\psi(t)$ is defined as*

$${}^{H}D_{a+}^{\alpha,\beta;\psi}u(t)=I_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}I_{a+}^{(1-\beta)(n-\alpha);\psi}u(t). \tag{1.11}$$

This operator unifies a vast class of fractional derivatives. By selecting specific forms for $\psi(t)$ and β, one can recover numerous well-known operators. Its main special cases are the ψ-Riemann-Liouville and ψ-Caputo derivatives.

Definition 1.9 (ψ-Riemann–Liouville Fractional Derivative)**.** *This is a special case of the ψ-Hilfer derivative with $\beta = 0$:*

$$D_{a+}^{\alpha;\psi}u(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a+}^{(n-\alpha);\psi}u(t). \tag{1.12}$$

Definition 1.10 (ψ-Caputo Fractional Derivative)**.** *This is a special case of the ψ-Hilfer derivative with $\beta = 1$:*

$${}^C D_{a+}^{\alpha;\psi}u(t) = I_{a+}^{(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t). \tag{1.13}$$

1.3.3 ψ-Hilfer Fractional Operator

da Sousa and de Oliveira [14] first presented the ψ-Hilfer fractional differentiation operator, which resulted in constructing ψ-Hilfer FDEs and prompted subsequent research. Its importance can be seen in the wide applicability of these fractional operators. ψ-Riemann–Liouville fractional integral has gained a reasonable amount of attention because of its utility to fractional differential operators like ψ-Riemann–Liouville ($D_{0+}^{\alpha;\psi}$), ψ-Caputo (${}^C D_{0+}^{\alpha;\psi}$), and ψ-Hilfer fractional derivative (${}^H D_{0+}^{\alpha\beta;\psi}$), etc., where α_1 and β, respectively, denote the order and type of operator.

Figure 1.1 illustrates the versatility of the ψ-Hilfer fractional derivative framework. By varying β, distinct types of fractional derivatives are obtained: $\beta = 1$ corresponds to the ψ-Caputo fractional derivative, $\beta = 0$ to the ψ-Riemann–Liouville fractional derivative, and $0 < \beta < 1$ yields the generalized Hilfer fractional derivative. Furthermore, selecting $\psi(t) = t$ recovers the fundamental forms of fractional derivatives, while $\psi(t) = \log(t)$ and $\psi(t) = t^\sigma$, where $\sigma \in \mathbb{R}^+$, result in the Hadamard and Katugampola-type fractional derivatives, respectively. This comprehensive generalization provides a unified framework, offering deeper insights and broader applications in fractional calculus. The ψ-Hilfer fractional derivative and ψ-Riemann–Liouville fractional integral are capable of generating a variety of different fractional operators due to their flexible kernel function (see [14]).

1.4 Some Important Definitions and Results

In this section, we present some definitions and results relevant to the thesis.

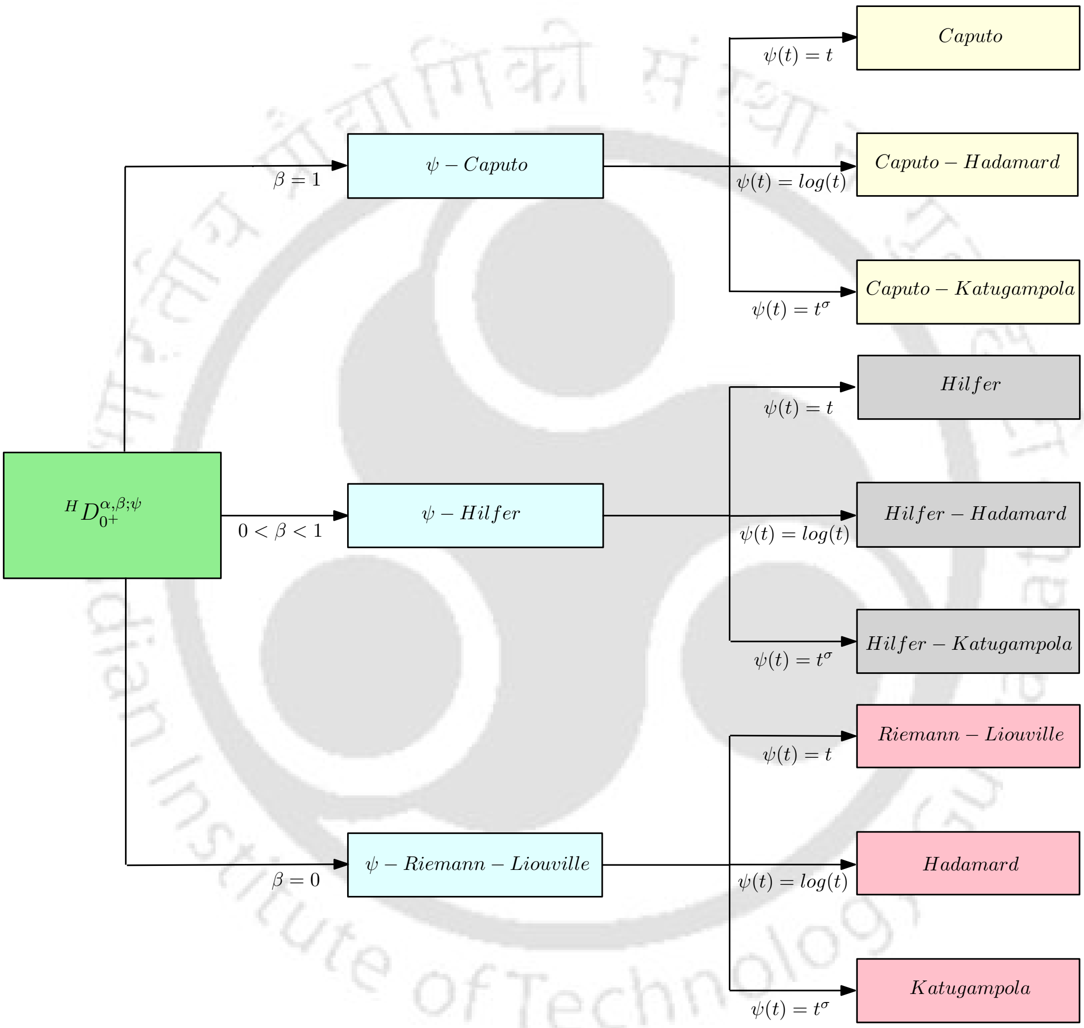


Figure 1.1: Flowchart of fractional derivatives obtained by varying parameter β and weight function $\psi(t)$ within the framework of ψ-Hilfer fractional derivative.

1.4.1 Some Functional Spaces

(i) $C(J,X)$ **Space:** Let $C(J,X)$ denote the space of all continuous functions $u : J \to X$ with the supremum norm $\|.\|_C$, i.e.,

$$\|u\|_C = \sup_{t\in J} \|u(t)\|_X. \tag{1.14}$$

(ii) $AC(J,\mathbb{C})$ **Space:** Let $AC(J,\mathbb{C})$ denote the space of all absolutely continuous functions $u : J \to \mathbb{C}$. Here, the space $AC(J,\mathbb{C})$ coincides with the space of primitives of Lebesgue summable functions, i.e.,

$$u(t) \in AC(J,\mathbb{C}) \Leftrightarrow u(t) = c + \int_a^x v(s)\,ds, \ \forall t \in J, \tag{1.15}$$

where c is some constant and v is a Lebesgue summable function, i.e., $\int_a^b v(s)\,ds < \infty$. Thus, if a function $u \in AC(J,\mathbb{C})$, then it has a summable derivative $u'(t)$ almost everywhere.

Let us denote by $AC^n(J,\mathbb{C})$, where $n = 1, 2, ...,$ the space of those functions u which have continuous derivatives up to order $(n-1)$ on J with $u^{(n-1)} \in AC(J,\mathbb{C})$.

(iii) **Weighted Space:** We denote the weighted space by $C_{1-\gamma;\psi}[a,b]$ with ψ being the weight, and for all continuous functions $u(t)$ on $(a,b]$, it is defined as

$$C_{1-\gamma;\psi}[a,b] = \{u : (a,b] \to \mathbb{R}; (\psi(t)-\psi(a))^{1-\gamma}u(t) \in C[a,b]\}, 0 \le \gamma < 1\} \tag{1.16}$$

with the norm

$$\|u(t)\|_{C_{1-\gamma;\psi}[a,b]} = \max_{t\in[a,b]} |(\psi(t)-\psi(a))^{1-\gamma}u(t)|. \tag{1.17}$$

1.4.2 Important Lemmas

Lemma 1.1. *[82, Grönwall's Inequality] Suppose that $u(t)$ and $v(t)$ are continuous real-valued functions defined on $0 \le t < T$ with $u(t) \ge 0$. Assume that u and v satisfy*

$$u(t) \le k_1 + k_2 \int_0^t u(s)v(s)\,ds \tag{1.18}$$

on $0 \le t < T$, where k_1 and k_2 are constants with $k_2 \ge 0$. Then,

$$u(t) \le k_1 \exp\left(k_2 \int_0^t v(s)\,ds\right) \tag{1.19}$$

on $0 \leq t < T$.

Lemma 1.2. *[82, Generalized Grönwall's inequality] Let* $\psi \in C^1[a,b]$ *be an increasing function with* $\psi'(t) \neq 0, \forall t \in [a,b]$. *Make the following assumptions:*
1. u *and* v *are two non-negative integrable functions,*
2. w *is a non-negative and non-decreasing continuous function on* $[a,b]$. *With* $u(t), v(t)$ *and* $w(t)$ *satisfying*

$$u(t) \leq v(t) + w(t) \int_a^t (\psi(t) - \psi(s))^{\alpha-1} u(s)\psi'(s)ds, \tag{1.20}$$

one has

$$u(t) \leq v(t) + \int_a^t \sum_{n=1}^{\infty} \frac{[w(s)\Gamma(\alpha)]^n}{\Gamma(n\alpha)} [\psi(t) - \psi(s)]^{n\alpha-1} v(s)\psi'(s)ds, \tag{1.21}$$

$\forall t \in [a,b]$.

Lemma 1.3. *[14] Let* $\alpha \in (n-1,n)$, *and* $\beta \in [0,1]$. *Then,*

$$I_{0+}^{\alpha;\psi} I_{0+}^{\beta;\psi} u(t) = I_{0+}^{\alpha+\beta;\psi} u(t). \tag{1.22}$$

Lemma 1.4. *[14] Let* $u(t) \in C^n(J,\mathbb{R})$, $\alpha \in (n-1,n)$ *and* $\beta \in [0,1]$. *Define* $\gamma = \alpha + \beta(1-\alpha)$. *Then, the following properties hold:*

$$\begin{aligned} I_{0+}^{\alpha;\psi}\,{}^{H}D_{0+}^{\alpha,\beta;\psi} u(t) =& u(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(0))^{\gamma-k}}{\Gamma(\gamma-k+1)} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n-k} \\ & \times I_{0+}^{(1-\beta)(n-\alpha);\psi} u(t), \qquad (1.23) \\ {}^{H}D_{0+}^{\alpha,\beta;\psi} I_{0+}^{\alpha;\psi} u(t) =& u(t). \qquad (1.24) \end{aligned}$$

1.4.3 Fixed Point Theorems

Another important constituent in the area of fractional differential equations is the fixed point theorems, without which it is very difficult (actually almost impossible) to study the existence and uniqueness of solutions of nonlinear differential equations. Fixed point theorems are nowadays the most widely used tool in the area of fractional differential equations. The most frequently used fixed point theorems are Banach's fixed point theorem, the nonlinear alternative of Leray-Schauder's type, Krasnosel'skiĭ's fixed point theorem, Schaefer's fixed point theorem, Schauder's fixed point theorem, Burton-Kirk's fixed point theorem, etc. Below, we state those fixed point theorems which are essential for obtaining the results in the works of the thesis.

Theorem 1.1. *If a family* $F = \{f(t)\}$ *in* $C(J,\mathbb{R})$ *is uniformly bounded and equicontinuous on* J, *then* F *has a uniformly convergent subsequence* $\{f_n(t)\}_{n=1}^{\infty}$. *If a family* $F = \{f(t)\}$

in $C(J, X)$ is uniformly bounded and equicontinuous on J, and for only $t^* \in J$, $\{f(t^*)\}$ is relatively compact, then F has a uniformly convergent subsequence $\{f_n(t)\}_{n=1}^{\infty}$.

Theorem 1.2. *[24, Banach's fixed point theorem] Let (X, d) be a non-empty complete metric space. Let $S\colon X \to X$ be a map such that, for any $u, v \in X$,*

$$d(Su, Sv) \leq kd(u, v), 0 \leq k < 1$$

holds. Then, the operator S admits a unique fixed point $u^* \in X$.

Theorem 1.3. *[24, Schauder's fixed point theorem] Let Ω be a Banach space and let S be a non-empty, closed, bounded, and convex subset of Ω. If the operator $T : S \to S$ is continuous and the image $T(S)$ is a relatively compact subset of S, then the operator T has at least one fixed point in S.*

Theorem 1.4. *[24, Krasnosel'skiĭ's fixed point theorem] Consider a Banach space X, and a subset $\Omega \subset X$ that is nonempty, convex, bounded, and closed. Define the mappings $S_1 : X \to X$ and $S_2 : \Omega \to X$ such that*

1. *S_1 is a contraction,*
2. *S_2 is completely continuous,*
3. *$S_1 z + S_2 z \in \Omega$ for all $z \in \Omega$.*

Then, there exists $z^* \in \Omega$ such that $z^* = S_1 z^* + S_2 z^*$.

Note: It may be noted that fixed point theorems enable us to establish the existence of a fixed point, thereby confirming the existence of a solution. However, in some cases, the Arzelà–Ascoli theorem is required to show that the associated operator is compact. The theorem is stated below.

Theorem 1.5. *[24, Arzelà–Ascoli Theorem] Let $F \subset C(J, X)$ be uniformly bounded and equicontinuous. If X is a Banach space and for some $t^* \in J$, the set $\{f(t^*) : f \in F\}$ is relatively compact in X, then F is relatively compact in $C(J, X)$.*

1.5 Fractional Functional Differential Equations

Differential equations are a central tool for describing evolutionary processes in the applied sciences. While classical models often rely on integer-order operators that capture only local and instantaneous dynamics, many complex processes in nature and technology are more accurately described by functional differential equations, which allow one to consider the influence of the system's prehistory or after-effects. Various classes of such equations are of fundamental importance in fields as diverse as epidemiology, electronics, automatic control, and population dynamics, where the future evolution depends not only on the present state but also on its history. This historical dependence can manifest as a continuous memory or as a response to discrete past events, known as delays. Fractional calculus has emerged as one of the best tools to characterize long-memory processes, anomalous diffusion, and long-range interactions, making the corresponding fractional differential equation models more realistic. The combination of these two concepts leads to the powerful and intricate class of *fractional functional differential equations (FFDEs)*. Their evolution is significantly more complicated, and the existence theorems for their solutions are more difficult to establish, as not all of the classical theory of differential equations can be directly applied. A particularly challenging subclass is the *neutral fractional functional differential equation*, in which the delay terms also occur in the derivative of the unknown solution, a structure that is essential for modeling many physical and engineering processes. Furthermore, many real-world systems are subject to abrupt, short-term perturbations, which are best modeled by *impulsive fractional differential equations*. These impulses can be instantaneous, representing sudden shocks, or non-instantaneous, where the change remains active over a finite time interval, as is often the case in pharmacotherapy. Since few phenomena exist in isolation, the interactions between different components of a system are often described by *coupled systems of fractional differential equations*, which are indispensable in fields like pharmacokinetics and epidemiology. Finally, from an abstract viewpoint, all of these classes of FFDEs can be studied in infinite-dimensional Banach spaces, which provides a powerful, unified framework for investigating the existence, uniqueness, and qualitative properties of their solutions using operator-theoretic methods. The increased complexity of these abstract, neutral, coupled, and impulsive systems necessitates the development of a new theoretical framework to rigorously analyze their solutions.

1.6 Ulam–Hyers Type Stability in Dynamical Systems

Stability analysis plays a vital role in understanding the qualitative behavior of fractional differential equations. Among several notions, Ulam–Hyers stability has become an effective concept for measuring how approximate solutions deviate from exact ones when small perturbations occur in the governing equation. This idea has been generalized to fractional operators, giving rise to various extensions such as *generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, generalized Ulam–Hyers–Rassias stability*, and *Ulam–Hyers–Mittag–Leffler stability*.

In recent years, numerous studies have explored these stability types for fractional and ψ-Hilfer fractional differential equations. da Sousa and de Oliveira [16] introduced a unified ψ-Hilfer framework and established corresponding stability results. Abdo et al. [2] examined Ulam–Hyers–Mittag–Leffler stability for ψ-Hilfer problems with delay, while

Ahmed et al. [3] analyzed existence and stability results for implicit fractional pantograph systems. Liu et al. [37] and Shah et al. [62] investigated existence and stability results for fractional models involving the ψ-Hilfer derivative through fixed point techniques, and Abdeljawad et al. [1] extended such results to Hilfer-type impulsive neutral systems with variable delay. Several related works have further advanced this area; for instance, da Sousa and de Oliveira [15] studied Ulam–Hyers stability for nonlinear fractional Volterra integro-differential equations, Lima et al. [36] considered ψ-Hilfer equations with impulses and delay, and Shankar and Bora [63] established generalized Ulam–Hyers–Rassias stability for fractional integro-differential systems with applications to electrical circuits. Motivated by these developments, the present thesis focuses on qualitative analysis and various forms of stability for ψ-Hilfer fractional systems, including neutral and coupled cases.

Consider the nonlinear ψ-Hilfer fractional differential equation

$$ {}^{H}D_{a^+}^{\alpha,\beta;\psi} x(t) = f(t, x(t)), \qquad t \in [a, T], \tag{1.25} $$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, and $\psi \in C^1[a, T]$ is an increasing function with $\psi'(t) \neq 0$.

Definition 1.11 (Ulam–Hyers Stability)**.** *The ψ-Hilfer fractional differential equation* (1.25) *is said to be* Ulam–Hyers stable *if there exists a constant $C_{UH} > 0$ such that for every function $y : [a, T] \to \mathbb{R}$ satisfying*

$$ \left| {}^{H}D_{a^+}^{\alpha,\beta;\psi} y(t) - f(t, y(t)) \right| \leq \varepsilon, \quad t \in [a, T], $$

for some $\varepsilon > 0$, there exists an exact solution $x(t)$ such that

$$ |y(t) - x(t)| \leq C_{UH}\, \varepsilon, \quad t \in [a, T]. $$

Definition 1.12 (Generalized Ulam–Hyers Stability)**.** *The equation* (1.25) *is said to be* generalized Ulam–Hyers stable *if there exists a continuous, strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for any $y(t)$ satisfying*

$$ \left| {}^{H}D_{a^+}^{\alpha,\beta;\psi} y(t) - f(t, y(t)) \right| \leq \varepsilon, $$

there exists an exact solution $x(t)$ for which

$$ |y(t) - x(t)| \leq \Phi(\varepsilon), \quad t \in [a, T]. $$

Definition 1.13 (Ulam–Hyers–Rassias Stability)**.** *Let $\varphi : [a, T] \to [0, \infty)$ be continuous. The equation* (1.25) *is said to be* Ulam–Hyers–Rassias stable *with respect to φ if there* *exists* $C_{UHR} > 0$ *such that for every* $y(t)$ *satisfying*

$$\left|{}^{H}D_{a^+}^{\alpha,\beta;\psi} y(t) - f(t, y(t))\right| \leq \varphi(t),$$

there exists an exact solution $x(t)$ *with*

$$|y(t) - x(t)| \leq C_{UHR}\,\varphi(t), \quad t \in [a, T].$$

Definition 1.14 (Generalized Ulam–Hyers–Rassias Stability)**.** *Let* $\varphi : [a, T] \to [0, \infty)$ *be continuous and nondecreasing, and* $\Phi : [0, \infty) \to [0, \infty)$ *be a continuous, strictly increasing function with* $\Phi(0) = 0$*. The equation* (1.25) *is said to be* generalized Ulam–Hyers–Rassias stable *if there exists a constant* $C_{GUHR} > 0$ *such that, for every function* $y(t)$ *satisfying*

$$\left|{}^{H}D_{a^+}^{\alpha,\beta;\psi} y(t) - f(t, y(t))\right| \leq \varphi(t),$$

there exists an exact solution $x(t)$ *of the equation satisfying*

$$|y(t) - x(t)| \leq C_{GUHR}\,\Phi(\varphi(t)), \qquad t \in [a, T].$$

Definition 1.15 (Ulam–Hyers–Mittag–Leffler Stability)**.** *The equation* (1.25) *is said to be* Ulam–Hyers–Mittag–Leffler stable *if there exist constants* $C_M > 0$ *such that for every* $\varepsilon > 0$ *and every* $y(t)$ *satisfying*

$$\left|{}^{H}D_{a^+}^{\alpha,\beta;\psi} y(t) - f(t, y(t))\right| \leq \varepsilon \mathbb{E}_{\alpha}\Big[(\psi(t) - \psi(a))^{\alpha}\Big],$$

there exists an exact solution $x(t)$ *such that*

$$|y(t) - x(t)| \leq C_M\,\varepsilon\,\mathbb{E}_{\alpha}\Big[(\psi(t) - \psi(a))^{\alpha}\Big], \quad t \in [a, T],$$

where $E_{\alpha}(\cdot)$ *denotes the one-parameter Mittag–Leffler function.*

Definition 1.16 (Ulam–Hyers–Rassias–Mittag–Leffler Stability)**.** *Let* $\varphi : [a, T] \to [0, \infty)$ *be continuous. The equation* (1.25) *is said to be* Ulam–Hyers–Rassias–Mittag–Leffler stable *if there exist constants* $C_{MR} > 0$ *such that for every* $y(t)$ *satisfying*

$$\left|{}^{H}D_{a^+}^{\alpha,\beta;\psi} y(t) - f(t, y(t))\right| \leq \varphi(t)\mathbb{E}_{\alpha}\Big[(\psi(t) - \psi(a))^{\alpha}\Big],$$

there exists an exact solution $x(t)$ *for which*

$$|y(t) - x(t)| \leq C_{MR}\,\varphi(t)\,\mathbb{E}_{\alpha}\Big[(\psi(t) - \psi(a))^{\alpha}\Big], \quad t \in [a, T].$$

1.7 Literature Survey

The stability analysis of solutions to fractional differential equations represents a cornerstone of modern mathematical analysis, crucial for ensuring the robustness of models that capture memory and hereditary properties inherent in complex systems across physics, engineering, and finance. The concept of Ulam–Hyers (UH) stability, and Ulam–Hyers–Rassias (UHR) stability originating from the fundamental works of Hyers, Ulam, and Rassias [54, 58] on functional equations, has been profoundly extended into the fractional calculus setting. This framework provides a powerful lens to analyze whether the existence of an approximate solution guarantees the proximity of an exact solution, a question of paramount importance for both theoretical and numerical applications. The foundational texts by Podlubny [50] and Kilbas et al. [32] established the mathematical bedrock for this field, exploring classical Riemann-Liouville and Caputo operators. However, the advent of the ψ-Hilfer fractional derivative by da Sousa and de Oliveira [14] marked a significant paradigm shift, introducing a unifying operator defined as

$$^{H}D_{a^+}^{\alpha,\beta;\psi}x(t) = I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n} I_{a^+}^{(1-\beta)(n-\alpha);\psi}x(t), \tag{1.26}$$

where $n = \lceil\alpha\rceil$. This generalization, which encapsulates a family of derivatives (e.g., Hilfer for $\psi(t) = t$, Hadamard for $\psi(t) = \log(t)$, has catalyzed a new wave of research into the existence, uniqueness, and stability of solutions for increasingly complex FDEs.

Initial investigations into ψ-Hilfer FDEs focused on establishing core stability results for scalar equations. da Sousa and de Oliveira [15] pioneered this effort, proving Ulam–Hyers–Rassias stability for nonlinear Volterra integro-differential equations using fixed point theorems. This work was swiftly followed by studies incorporating more realistic model features, such as time delays. The analysis was further extended to systems with infinite delay by Abdo et al. [2] and to impulsive systems, which model sudden state changes, in a comprehensive form by Lima et al. [36] for the problem

$$\begin{cases} ^{H}D_{0^+}^{\alpha,\beta;\psi}x(t) = f(t,x_t), & t \in (0,T]\setminus\{t_1,\dots,t_m\}, \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1,2,\dots,m, \\ I_{0^+}^{1-\gamma;\psi}x(0) = y_0,\ x(t) = h(t), & t \in [-r,0]. \end{cases} \tag{1.27}$$

Their works confirmed that fixed point methods and generalized Grönwall's inequalities remain effective tools for deriving UH and UHR stability criteria even in the presence of discontinuities.

Recent developments in abstract fractional differential systems have gained significant attention, particularly in Banach space settings. Wang et al. [73] investigated the existence of mild solutions for (k,Ψ)-Hilfer Sobolev type fractional evolution equations, demonstrating sophisticated resolvent family techniques without requiring the existence of the inverse operator. Their work extends the classical C_0-semigroup theory to fractional settings, establishing new theoretical foundations for abstract systems. Similarly, Ganesh et al. [23] developed comprehensive Hyers-Ulam-Mittag-Leffler stability results for fractional differential equations involving two Caputo derivatives, utilizing fractional Fourier transform techniques to establish stability criteria of the form

$$\begin{cases} {}^{C}D^{\alpha}x(t) + a^{C}D^{\beta}x(t) = f(t, x(t)), \quad t > 0, \\ x(0) = x_0, \quad x'(0) = x_1, \end{cases} \tag{1.28}$$

where their analysis revealed the crucial role of Mittag-Leffler functions in characterizing stability properties of multi-term fractional systems.

Concurrently, a significant branch of research has focused on FDEs subject to nonlocal boundary conditions, which often provide a more realistic description of physical processes than classical initial conditions. The works of Harikrishnan et al. [25] on pantograph equations and Ahmed et al. [3] on implicit pantograph differential equations extended the application of the ψ-Hilfer derivative to these problems. A comprehensive treatment was provided by Thaiprayoon et al. [70], who investigated a complex ψ-Hilfer implicit fractional integro-differential equation with a mixed nonlocal condition:

$$\begin{cases} {}^{H}D_{0+}^{\alpha,\rho;\psi}x(t) = f\Big(t, x(t), {}^{H}D_{0+}^{\alpha,\rho;\psi}x(t), I_{0+}^{\alpha;\psi}x(t)\Big), \quad t \in (0, T], \\ \sum_{i=1}^{m}\omega_i x(\eta_i) + \sum_{j=1}^{n}\kappa_j {}^{H}D_{0+}^{\beta_j,\rho;\psi}x(\zeta_j) + \sum_{r=1}^{k}\sigma_r I_{0+}^{\delta_r;\psi}x(\theta_r) = A. \end{cases} \tag{1.29}$$

establishing four separate types of Ulam stability. This work, alongside that of Asawasamrit et al. [9] on Caputo derivatives, highlights the mathematical intricacies of problems with nonlocal data.

The advancement in coupled fractional systems has been remarkable, with Muthaiah et al. [46] providing groundbreaking results on Ulam-Hyers stability for coupled sequential Hilfer-Hadamard fractional integro-differential systems. Their investigation of nonlinear coupled systems enhanced by nonlocal Hadamard fractional integrodifferential multipoint boundary conditions demonstrated the effectiveness of fixed point theorems in establishing existence, uniqueness, and stability properties simultaneously. Furthermore, the recent work by Mani et al. [41] on coupled Caputo-Hadamard fractional neutral differential equations with unbounded delays has opened new avenues for understanding the stability of neutral systems under coupling effects.

Neutral fractional differential equations present unique mathematical challenges that have attracted considerable research attention. Huseynov and Mahmudov [29] conducted an extensive analysis of positive fractional-order neutral time-delay systems, providing representation formulas for solutions and establishing conditions for system positivity. Their work on linear matrix coefficient systems of the form

$$\begin{cases} {}^{C}D^{\alpha}[x(t) - Cx(t-\tau)] = Ax(t) + Bx(t-\tau), & t > 0, \\ x(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{1.30}$$

established fundamental results for neutral systems with multiple delays. Complementing this, Bedi et al. [11] focused specifically on Ulam-Hyers stability of neutral delay fractional differential equations, providing sufficient conditions that ensured the stability of approximate solutions.

The study of chaotic behavior in fractional-order financial systems has emerged as a particularly exciting application area. Alzaid et al. [8] analyzed chaotic complexity in financial mathematical models using generalized Caputo fractional derivatives and established existence and uniqueness results through fixed point analysis. The model with three state variables exhibits rich dynamical behavior, with bifurcation diagrams revealing chaotic behavior across broad parameter ranges. Yang and Li [79] further advanced this field by presenting a novel fractional-order financial system considering non-constant elasticity of demand, demonstrating that such systems can exhibit diverse chaotic dynamics and periodic oscillations influenced by fractional orders and system parameters.

1.8 Research Motivation and Objectives

Despite these considerable advances as discussed above, the literature reveals distinct and critical gaps that this thesis aims to address. First, while the stability of scalar ψ-Hilfer equations is well-explored, the analysis of coupled systems of such equations remains largely underdeveloped. The dynamics of interacting components introduces cross-coupling terms that significantly complicate the stability analysis. Qian et al. [52] made important contributions by studying coupled systems of Caputo-type fractional differential equations with integral boundary conditions, establishing existence results through the Leray–Schauder alternative theorem and developing Hyers–Ulam stability conditions, but comprehensive analysis of ψ-Hilfer coupled systems remains incomplete.

Secondly, the stability theory for neutral fractional differential equations within the ψ-Hilfer framework is still nascent. Neutral differential equations, where the derivative depends on the history of the function, are crucial for modeling systems with heritage effects on the rate of change. Prabu [51] provided some initial results for ψ-Caputo fractional integro-differential equations with finite delay, but the neutral case requires more sophisticated analysis.

Thirdly, while Mittag-Leffler stability has been established for various fractional systems, the development of Ulam–Hyers–Mittag–Leffler stability theory specifically for ψ-Hilfer systems requires further advancement. The natural decay behavior characterized by Mittag–Leffler functions provides a more appropriate framework for fractional systems than classical exponential stability, yet comprehensive results combining Ulam–Hyers robustness with Mittag–Leffler decay properties remain limited. Sene [61] investigated Mittag–Leffler input stability of fractional differential systems, but the integration with Ulam–Hyers concepts for ψ-Hilfer operators requires deeper investigation.

The emerging field of generalized fractional operators has also introduced new stability challenges. Ren and Zhai [55] developed stability analysis for generalized fractional differential systems using comparison principles, while Ibrahim [30] established generalized Ulam-Hyers stability for fractional differential equations in complex Banach spaces. These developments suggest that the ψ-Hilfer framework may need further extensions to accommodate even more general classes of operators.

1.9 Outline of the Thesis

This thesis is organized in seven chapters, with Chapters 2–6 addressing a distinct class of problems related to the qualitative analysis and stability of fractional differential equations within the framework of the ψ-Hilfer fractional derivative. Chapter 7 concludes with a summary and directions for future research.

Chapter 2 presents the study of Ulam–Hyers and Ulam–Hyers–Rassias stability for *ψ-Hilfer abstract fractional differential equations* in Banach spaces. Using Banach's fixed point theorem and suitable fractional inequalities, sufficient conditions are derived to ensure the existence and stability of solutions. Numerical solutions are provided for different weight functions by varying the order and type of the ψ-Hilfer derivative.

Chapter 3 extends this framework to incorporate Ulam–Hyers–Mittag–Leffler and Ulam–Hyers–Rassias–Mittag–Leffler stability for *ψ-Hilfer abstract fractional differential equations*. Existence results are established via Schauder's fixed point theorem, leading to a refined characterization of solution behavior in fractional systems.

Chapter 4 focuses on *neutral fractional differential equations with delay*. Such equations model systems in which the present state depends on both current and delayed terms. Krasnosel'skiĭ's fixed point theorem is employed to prove existence results, while Ulam–Hyers and Ulam–Hyers–Rassias stability are established under suitable conditions. Numerical results are presented for different weight functions and fractional orders to illustrate the solution of delayed systems.

Chapter 5 investigates *coupled fractional differential equations*. Banach's and Schauder's fixed point theorems are used to derive qualitative properties and establish Ulam–Hyers and generalized Ulam–Hyers stability results. A real-world application to blood alcohol concentration modeling is presented, and numerical simulations of the ψ-Hilfer coupled system are compared with experimental data to validate the accuracy of the model.

Chapter 6 examines a *system of three fractional differential equations* with different orders of ψ-Hilfer derivative. Existence and uniqueness of solutions are obtained through Banach's fixed point theorem, followed by an analysis of Ulam–Hyers and generalized Ulam–Hyers stability. As an application, a chaotic financial system is analyzed, and numerical experiments demonstrate the emergence of complex, chaotic behavior in ψ-Hilfer fractional systems.

Chapter 7 summarizes the main contributions of the thesis and outlines prospective directions for further research. The overarching aim of this work is to develop a unified and rigorous framework for the qualitative analysis and Ulam–Hyers-type stability of diverse classes of fractional differential equations under the general ψ-Hilfer fractional derivative. The results obtained not only extend and generalize existing findings in the literature, but also provide new insights and applications to fractional modeling of real-world systems.

CHAPTER 2

On Ulam type stability of the solution to a ψ-Hilfer abstract fractional functional differential equation

In this chapter, we study the existence and uniqueness of solutions for a ψ-Hilfer abstract fractional differential equation. We then establish Ulam–Hyers and Ulam–Hyers–Rassias stability results. Furthermore, a numerical scheme is developed to illustrate the theoretical findings, and numerical solutions are presented for different fractional orders and types using three choices of the weight function.

2.1 Introduction

The concept of Ulam-Hyers and Ulam-Hyers-Rassias stability finds significant relevance in various real-world systems. For example, in electrical engineering, the stability of RLC circuits, which exhibit memory effects and time delays, can be analyzed using such stability concepts [63]. Similarly, in biological systems, such as epidemic modeling and co-dynamical systems involving diseases, e.g., cholera and COVID-19, where memory effects and environmental interactions are crucial, the stability of the solutions under small perturbations is essential to understand the disease dynamics and to predict reliable long-term outcomes [71]. These works underscore the practical utility of the theoretical results presented in this study.

da Sousa et al. [15] used the ψ-Hilfer operator in studying stability in the sense of Ulam-Hyers-Rassias for the following nonlinear FDE:

$$\begin{cases} {}^{H}D_{a^+}^{\alpha,\rho;\psi} y(t) = f\left(t, y(t), {}^{H}D_{a^+}^{\alpha,\rho;\psi} y(t)\right), \ t \in J = (a, T], \\ I_{a^+}^{1-\gamma;\psi} y(a) = y_a, \ \alpha \le \gamma = \alpha + \rho - \alpha\rho, \ T > a. \end{cases}$$

Harikrishnan et al. [25] successfully studied the existence and uniqueness of the solution to initial value problems with non-local conditions for the pantograph equation which was guided by a ψ-Hilfer fractional derivative. Ahmed et al. [3] examined the same to an implicit pantograph FDE expressed in terms of ψ-Hilfer fractional derivative, in addition to the corresponding Ulam-Hyers stability. Abdo et al. [2] additionally investigated the Ulam-Hyers-Mittag-Leffler stability for a functional FDE containing a ψ-Hilfer derivative subject to an infinite delay. Thaiprayoon et al. [70] also established the qualitative properties for a class of ψ-Hilfer implicit fractional integro-differential equation comprising a mixed non-local boundary condition, and examined the associated Ulam stability in the form of Ulam-Hyers and Ulam-Hyers-Rassias stability. Motivated by these works, Lima et al. [36] investigated the sufficient conditions for the same issue including the Ulam-Hyers stability of the solution to a fractional impulsive delay differential equation.

Taking inspiration from the above works, we are encouraged to consider the following ψ-Hilfer abstract fractional functional differential equation:

$$\begin{cases} {}^{H}D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t,u(t))] = Au(t) + Q(t,u(t)),\ t \in J = [0,a], \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where ${}^{H}D_{0+}^{\alpha,\beta;\psi}$ denotes the ψ-Hilfer fractional derivative with $\alpha \in (0,1)$ and $\beta \in (0,1]$ as the order and type, respectively, and A is a non-negative scalar. First, we look at the existence and uniqueness of the solutions which is always thought to be crucial for the analysis of complex physical systems as it provides insight into the long-term behavior of the system.

In this direction, we employ the Banach's fixed point theorem, which is a well-known tool in functional analysis, to establish the desired results. Thereafter, we explore the stability of the proposed abstract fractional functional differential equation subject to the conditions of stability in the sense of (i) Ulam-Hyers (UH), (ii) generalized Ulam-Hyers (GUH), (iii) Ulam-Hyers-Rassias (UHR), and (iv) generalized Ulam-Hyers-Rassias (GUHR). These stability conditions have significant implications in the analysis of FDEs. To showcase the applicability of the derived results, a suitable example is considered for establishing the existence and uniqueness of its solution and the corresponding stability result.

For abstract functional fractional differential equations involving the ψ-Hilfer derivative, a numerical approximation method is developed in this study. This method is essential for examining solution trajectories, particularly when it becomes challenging to obtain explicit analytical solutions. Through the approximation of these equations, which capture the memory effects and intermediate dynamics that are crucial to fractional systems, we may effectively observe the change in the behavior of the solution by adjusting different parameters. The analysis carried out here attempts to highlight the important role of the

ψ-Hilfer fractional derivative which has more capability in modeling real-world systems and underscores the utility of fractional calculus in solving complex real-life problems.

The structure of the chapter is as follows : In Section 2.2, the existence and uniqueness of solutions for the ψ-Hilfer abstract fractional differential equation are discussed. In Section 2.3, the Ulam–Hyers and Ulam–Hyers–Rassias stability results are established. In Section 2.4, a numerical scheme for the ψ-Hilfer abstract fractional differential equation is developed, and numerical solutions are presented for different fractional orders and types using three different weight functions. Finally, Section 2.5 concludes the chapter with a summary of the main results and possible directions for future research.

2.2 Existence and Uniqueness Results

Studies involving ψ-Hilfer fractional derivative have displayed worthy interest among the academic community across various disciplines. In this direction, to accomplish the same for equation (2.1), the Banach's fixed point theorem is used to establish the existence and uniqueness of the solution to the abstract functional differential equation which involves a more general fractional derivative in the form of ψ-Hilfer fractional derivative in an attempt to widen this study to a broad class of fractional derivatives.
Throughout this chapter, we take $\gamma = \alpha + \beta - \alpha\beta$ for computational ease.

Theorem 2.1. *Let $\alpha \in (0,1)$ and $\beta \in (0,1]$ denote the order and type, respectively, of the FDE given by*

$$\begin{cases} {}^{H}D_{0+}^{\alpha,\beta;\psi}\Big[u(t) - P(t,u(t))\Big] = Au(t) + Q(t,u(t)), \quad t \in J = [0,a], \\ u(0) = u_0, \end{cases} \tag{2.2}$$

where $u(t), \psi(t) \in C^1([0,a],\mathbb{R})$, $\psi(t)$ is an increasing function, and $\psi'(t) \neq 0$ for all $t \in J$. Assume $P(t,u(t))$ and $Q(t,u(t))$ to be continuous functions. Then, $u(t)$ satisfies the nonlinear ψ-Hilfer abstract fractional functional differential equation (2.2) if and only if it satisfies

$$u(t) = P(t,u(t)) + \frac{\Big[u_0 - P(0,u_0)\Big]}{\Gamma(\gamma)\Gamma(2-\gamma)} + I_{0+}^{\alpha;\psi}\Big[Au(t) + Q(t,u(t))\Big], \tag{2.3}$$

where $I_{0+}^{\alpha;\psi}$ denotes the ψ-Riemann-Liouville fractional integral of order α.

Proof. Invoking ψ-fractional operator $I_{0+}^{\alpha;\psi}$ to equation (2.2), along with Lemma 1.4, we get

$$I_{0+}^{\alpha;\psi}\Big[{}^{H}D_{0+}^{\alpha,\beta;\psi}\Big(u(t) - P(t,u(t))\Big)\Big] = I_{0+}^{\alpha;\psi}\Big[Au(t) + Q(t,u(t))\Big]$$

$$\implies u(t) - P(t,u(t)) - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0+}^{1-\gamma;\psi}[u(0) - P(0,u(0))]$$
$$= I_{0+}^{\alpha;\psi}\Big[Au(t) + Q(t,u(t))\Big]$$
$$\implies u(t) - P(t,u(t)) = \frac{\big[u_0 - P(0,u_0)\big]}{\Gamma(\gamma)\Gamma(2-\gamma)} + I_{0+}^{\alpha;\psi}\Big[Au(t) + Q(t,u(t))\Big]. \tag{2.4}$$

Thus, it leads to equation (2.3).
Now, applying the fractional operator ${}^{H}D_{0+}^{\alpha,\beta;\psi}$ to equation (2.4), we have

$${}^{H}D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t,u(t))] = {}^{H}D_{0+}^{\alpha,\beta;\psi}\frac{[u(0) - P(0,u(0))]}{\Gamma(\gamma)\Gamma(2-\gamma)}$$
$$+ {}^{H}D_{0+}^{\alpha,\beta;\psi} I_{0+}^{\alpha;\psi}\Big[Au(t) + Q(t,u(t))\Big]$$
$$\implies {}^{H}D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t,u(t))] = {}^{H}D_{0+}^{\alpha,\beta;\psi}\frac{[u_0 - P(0,u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)}$$
$$+ Au(t) + Q(t,u(t)).$$

$$\text{Since} \quad {}^{H}D_{0+}^{\alpha,\beta;\psi}\frac{u_0}{\Gamma(\gamma)\Gamma(2-\gamma)} = 0 \quad \text{and} \quad {}^{H}D_{0+}^{\alpha,\beta;\psi}\frac{P(0,u_0)}{\Gamma(\gamma)\Gamma(2-\gamma)} = 0,$$

the desired result is established, thereby concluding the proof. □

In order to establish our main result with respect to existence of the solution, the following hypothesis is required:

Hypothesis 2.1. *There exist two Lipschitz constants $l_1 > 0$ and $l_2 > 0$ satisfying Lipschitz condition in u for continuous functions P and Q, respectively, i.e.,*

$$|P(t,u_1(t)) - P(t,u_2(t))| \leq l_1|u_1(t) - u_2(t)|,$$
$$\text{and} \quad |Q(t,u_1(t)) - Q(t,u_2(t))| \leq l_2|u_1(t) - u_2(t)|.$$

For the sake of convenience, we write

$$a_1 = l_1(\psi(t) - \psi(0))^{\gamma-1}, \tag{2.5}$$
$$\text{and} \quad b_1 = \frac{p\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}(\psi(t) - \psi(0))^{\alpha+\gamma-1}, \tag{2.6}$$

with $p = A + l_2$.

Theorem 2.2. *Let Hypothesis 2.1 be valid. If*

$$a_1 + b_1 < 1, \tag{2.7}$$

then equation (2.1) admits a unique solution on J, where a_1 and b_1 are given by equations (2.5) and (2.6), respectively.

Proof. Consider the operator $S : C(J,\mathbb{R}) \to C(J,\mathbb{R})$. Then,

$$(Su)(t) = P(t,u(t)) + \frac{[u_0 - P(0,u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\Big[Au(s)+Q(s,u(s))\Big]\psi'(s)ds.$$

We have

$$\begin{aligned}
|(Su_1)(t)-(Su_2)(t)| &= \Big|\Big[P(t,u_1(t)) - P(t,u_2(t))\Big]\Big| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\Big|\Big[A(u_1(s)-u_2(s)) + Q(s,u_1(s)) - Q(s,u_2(s))\Big]\Big|\psi'(s)\,ds \\
&\le l_1|u_1(t)-u_2(t)| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}(A+l_2)|u_1(s)-u_2(s)|\psi'(s)\,ds \\
&\le l_1(\psi(t)-\psi(0))^{\gamma-1}\|u_1(t)-u_2(t)\|_{C_{1-\gamma;\psi}} \\
&+ \|u_1(t)-u_2(t)\|_{C_{1-\gamma;\psi}}\frac{p}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\gamma-1}\psi'(s)\,ds \\
&= l_1(\psi(t)-\psi(0))^{\gamma-1}\|u_1(t)-u_2(t)\|_{C_{1-\gamma;\psi}} \\
&+ \|u_1(t)-u_2(t)\|_{C_{1-\gamma;\psi}}\frac{p}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}(\psi(t)-\psi(0))^{\alpha+\gamma-1} \\
&\le \left[l_1(\psi(t)-\psi(0))^{\gamma-1} + \frac{p\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}(\psi(t)-\psi(0))^{\alpha+\gamma-1}\right]\|u_1(t)-u_2(t)\|_{C_{1-\gamma;\psi}}.
\end{aligned}$$

The above implies that

$$\|(Su_1)(t)-(Su_2)(t)\|_{C_{1-\gamma;\psi}} \quad\le\quad (a_1+b_1)\|u_1(t)-u_2(t)\|_{C_{1-\gamma;\psi}}. \tag{2.8}$$

This immediately establishes S as a contraction. Consequently, Banach's fixed point theorem ensures that S has only one fixed point which is the unique solution to equation (2.1) on J. □

2.3 Stability Analysis

We provide the definitions of different stability concepts in the sense of UH, GUH, UHR, and GUHR.

Definition 2.1. *Equation (2.1) is said to be UH stable if there exists a constant $k_G \in \mathbb{R}^+$ such that, for each $\epsilon > 0$ and for each solution $\tilde{u}$ of the inequation*

$$\left|{}^H D_{0^+}^{\alpha,\beta;\psi}[u(t) - P(t,u(t))] - Au(t) - Q(t,u(t))\right| \le \epsilon, \tag{2.9}$$

there exists a solution u to equation (2.1) with

$$|\tilde{u}(t) - u(t)| \leq k_G \epsilon,\ t \in J. \tag{2.10}$$

Definition 2.2. *Assume that $\tilde{u}$ satisfies the inequation (2.9) and u is a solution of equation (2.1). Then, equation (2.1) is said to be GUH stable if there exists a function $\Phi_G \in C(J, \mathbb{R}^+)$, with $\Phi_G(0) = 0$, which satisfies*

$$|\tilde{u}(t) - u(t)| \leq \Phi_G(\epsilon),\ t \in J. \tag{2.11}$$

Definition 2.3. *Equation (2.1) is said to be UHR stable with respect to $\Phi_G \in C(J, \mathbb{R}^+)$ if there exists a constant $k_{\Phi,G} > 0$ such that, for each $\epsilon > 0$ and for each solution $\tilde{u}$ of the inequation*

$$\left|{}^H D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t))] - Au(t) - Q(t, u(t))\right| \leq \epsilon \Phi_G(t), t \in J, \tag{2.12}$$

there exists a solution $u(t)$ of equation (2.1) satisfying

$$|\tilde{u}(t) - u(t)| \leq k_{\Phi,G} \Phi_G(t)\epsilon, t \in J. \tag{2.13}$$

Definition 2.4. *Equation (2.1) is said to be GUHR stable with respect to $\Phi_G \in C(J, \mathbb{R}^+)$ if there exists a constant $k_{\Phi,G} > 0$ such that, for each solution $\tilde{u}$ of the inequation*

$$\left|{}^H D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t))] - Au(t) - Q(t, u(t))\right| \leq \Phi_G(t), t \in J, \tag{2.14}$$

there exists a solution $u(t)$ of equation (2.1) satisfying

$$|\tilde{u}(t) - u(t)| \leq k_{\Phi,G} \Phi_G(t),\ t \in J. \tag{2.15}$$

Remark 2.1. *A function $\tilde{u} \in C^1(J, \mathbb{R})$ is a solution of (2.9) iff there exists a function $\mu \in C^1(J, \mathbb{R})$ (which depends on $\tilde{u}$) such that*
(i) $|\mu(t)| \leq \epsilon, \forall t \in J$,
(ii) ${}^H D_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t) - P(t, \tilde{u}(t))] = A\tilde{u}(t) + Q(t, \tilde{u}(t)) + \mu(t), \forall t \in J$.

Similar observations apply to inequations (2.12) and (2.14).

Theorem 2.3. *Let Hypothesis 2.1 be valid and also that $a_1 + b_1 < 1$. As a consequence, equation (2.1) is UH stable.*

Proof. Let $\tilde{u}(t)$ be a solution of inequation (2.9) so that

$$\left|{}^H D_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t) - P(t, \tilde{u}(t))] - A\tilde{u}(t) - Q(t, \tilde{u}(t))\right| \leq \epsilon. \tag{2.16}$$

Further, let $u(t)$ be the unique solution of the following problem:

$$\begin{cases} {}^H D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t,u(t))] = Au(t) + Q(t,u(t)),\ t \in J, \\ u(0) = u_0. \end{cases}$$

Applying operator $I_{0+}^{\alpha;\psi}$ to both sides of (2.16), one obtains the following:

$$\left| \tilde{u}(t) - P(t,\tilde{u}(t)) - \frac{[u_0 - P(0,u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} - I_{0+}^{\alpha;\psi}[A\tilde{u}(t) + Q(t,\tilde{u}(t))] \right| \\ \leq \epsilon \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)}. \tag{2.17}$$

Subsequently,

$$\begin{aligned} &|\tilde{u}(t) - u(t)| \\ &= \left| \tilde{u}(t) - P(t,u(t)) - \frac{[u_0 - P(0,u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} - I_{0+}^{\alpha;\psi}[Au(t) + Q(t,u(t))] \right| \\ &\leq \left| \tilde{u}(t) - P(t,\tilde{u}(t)) - \frac{[u_0 - P(0,u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} - I_{0+}^{\alpha;\psi}[A\tilde{u}(t) + Q(t,\tilde{u}(t))] \right| \\ &\quad + |P(t,\tilde{u}(t)) - P(t,u(t))| + I_{0+}^{\alpha;\psi}[A\,|\tilde{u}(t) - u(t)| + |Q(t,\tilde{u}(t)) - Q(t,u(t))|]. \end{aligned}$$

Now using equation (2.17), we get

$$\begin{aligned} |\tilde{u}(t) - u(t)| &\leq \epsilon \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + |P(t,\tilde{u}(t)) - P(t,u(t))| \\ &\quad + I_{0+}^{\alpha;\psi}[A\,|\tilde{u}(t) - u(t)| + |Q(t,\tilde{u}(t)) - Q(t,u(t))|] \\ &\leq \epsilon \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + l_1\,|\tilde{u}(t) - u(t)| + I_{0+}^{\alpha;\psi}(A + l_2)\,|\tilde{u}(t) - u(t)| \\ &\leq \epsilon \frac{(\psi(T) - \psi(0))^\alpha}{(1-l_1)\Gamma(\alpha+1)} + \frac{p}{(1-l_1)}\frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1}\,|\tilde{u}(s) - u(s)|\,\psi'(s)ds \\ &\leq \epsilon \frac{(\psi(T) - \psi(0))^\alpha}{(1-l_1)\Gamma(\alpha+1)} \left[1 + \int_0^t \sum_{n=1}^{\infty} \left(\frac{p}{1-l_1} \right)^n \frac{1}{\Gamma(n\alpha)} (\psi(t) - \psi(s))^{n\alpha-1} \psi'(s)ds \right] \\ &\leq \epsilon \frac{(\psi(T) - \psi(0))^\alpha}{(1-l_1)\Gamma(\alpha+1)} \left[1 + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\frac{p}{1-l_1} (\psi(T) - \psi(0))^\alpha \right)^n \right] \\ &\leq \epsilon \frac{(\psi(T) - \psi(0))^\alpha}{(1-l_1)\Gamma(\alpha+1)} \mathbb{E}_\alpha \left(\frac{p}{1-l_1} (\psi(T) - \psi(0))^\alpha \right), \end{aligned}$$

where $\mathbb{E}_\alpha$ is Mittag-Leffler function of one parameter. Finally, we have

$$|\tilde{u}(t) - u(t)| \leq \epsilon k_G, \tag{2.18}$$

where
$$k_G = \frac{(\psi(T)-\psi(0))^\alpha}{(1-l_1)\Gamma(\alpha+1)}\mathbb{E}_\alpha\Big(\frac{p}{1-l_1}(\psi(T)-\psi(0))^\alpha\Big).$$
Subsequently, we can conclude that equation (2.1) is UH stable. Next, considering $\Phi_G(\epsilon) = k_G\epsilon$ with $\Phi_G(0) = 0$ ensures that equation (2.1) is GUH stable. □

In order to establish UHR stability, we require the following hypothesis:

Hypothesis 2.2. *Let $\Phi_G \in C(J, \mathbb{R}^+)$ be an increasing function. Further, let there exist a constant $\lambda_\Phi > 0$ such that, for each $t \in J$, one has*
$$I_{0+}^{\alpha;\psi}\Phi_G(t) \le \lambda_\Phi \Phi_G(t). \tag{2.19}$$

Theorem 2.4. *Assume the validity of Hypotheses 2.1-2.2, and equation (2.7). Then, equation (2.1) is UHR stable.*

Proof. Consider $\tilde{u}(t)$ to be a solution of inequation (2.12) so that
$$\left|{}^H D_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t) - P(t,\tilde{u}(t))] - A\tilde{u}(t) - Q(t,\tilde{u}(t))\right| \le \epsilon\Phi_G(t), t \in J. \tag{2.20}$$
On the other hand, let $u(t)$ be the unique solution to the problem
$$\begin{cases} {}^H D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t,u(t))] = Au(t) + Q(t,u(t)),\ t \in J, \\ u(0) = u_0. \end{cases}$$
Applying operator $I_{0+}^{\alpha;\psi}$ to both sides of (2.20), the following can be obtained:
$$\begin{aligned}&\left|\tilde{u}(t) - P(t,\tilde{u}(t)) - \tfrac{[u_0 - P(0,u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} - I_{0+}^{\alpha;\psi}[A\tilde{u}(t) + Q(t,\tilde{u}(t))]\right| \\ &\le \epsilon\lambda_\Phi\Phi_G(t).\end{aligned} \tag{2.21}$$
Subsequently,
$$\begin{aligned}&|\tilde{u}(t) - u(t)| \\ &= \left|\tilde{u}(t) - P(t,u(t)) - \frac{[u_0 - P(0,u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} - I_{0+}^{\alpha;\psi}[Au(t) + Q(t,u(t))]\right| \\ &\le \left|\tilde{u}(t) - P(t,\tilde{u}(t)) - \frac{[u_0 - P(0,u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} - I_{0+}^{\alpha;\psi}[A\tilde{u}(t) + Q(t,\tilde{u}(t))]\right| \\ &\quad + |P(t,\tilde{u}(t)) - P(t,u(t))| + I_{0+}^{\alpha;\psi}[A\,|\tilde{u}(t) - u(t)| + |Q(t,\tilde{u}(t)) - Q(t,u(t))|].\end{aligned}$$
Now, using equation (2.21), we have
$$\begin{aligned}|\tilde{u}(t) - u(t)| \le& \epsilon\lambda_\Phi\Phi_G(t) + |P(t,\tilde{u}(t)) - P(t,u(t))| \\ &+ I_{0+}^{\alpha;\psi}[A\,|\tilde{u}(t) - u(t)| + |Q(t,\tilde{u}(t)) - Q(t,u(t))|].\end{aligned}$$

Using Hypothesis 2.1, we get

$$\begin{aligned}&(1-l_1)\left\|\tilde{u}(t)-u(t)\right\|_\infty\\&\quad\leq \epsilon\lambda_\Phi\Phi_G(t)+p\left\|\tilde{u}(t)-u(t)\right\|_\infty\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)ds\\&\quad\leq \epsilon\lambda_\Phi\Phi_G(t)+\frac{p}{\Gamma(\alpha+1)}\left\|\tilde{u}(t)-u(t)\right\|_\infty(\psi(T)-\psi(s))^\alpha.\end{aligned}$$

That is,

$$\begin{aligned}\left\|\tilde{u}(t)-u(t)\right\|_\infty&\leq\frac{\epsilon\lambda_\Phi\Phi_G(t)}{(1-l_1)}+\frac{p}{(1-l_1)\Gamma(\alpha+1)}\left\|\tilde{u}(t)-u(t)\right\|_\infty(\psi(T)-\psi(s))^\alpha\\&\leq\left(\frac{\Gamma(\alpha+1)}{(1-l_1)\Gamma(\alpha+1)-p(\psi(T)-\psi(s))^\alpha}\right)\epsilon\lambda_\Phi\Phi_G(t).\end{aligned}$$

Finally, we have

$$\left\|\tilde{u}(t)-u(t)\right\|_\infty\leq k_{\Phi,G}\Phi_G(t)\epsilon, t\in J, \tag{2.22}$$

where $k_{\Phi,G}=\dfrac{\Gamma(\alpha+1)\lambda_\Phi}{(1-l_1)\Gamma(\alpha+1)-p(\psi(T)-\psi(s))^\alpha}$.

Hence, equation (2.1) is UHR stable. Next, by setting $\epsilon=1$ in (2.20) and (2.22) with $\Phi_G(0)=0$, we can conclude that equation (2.1) is GUHR stable too. □

Remark 2.2. *Ulam-Hyers stability can easily be obtained from Ulam-Hyers-Rassias stability by setting $\Phi_G(t)=1$ in equations (2.12) and (2.13), and comparing them with equations (2.9) and (2.10), respectively. While Ulam-Hyers stability ensures bounded deviations under constant perturbations, Ulam-Hyers-Rassias stability extends this by accommodating state-dependent perturbations, providing a more general framework for analyzing dynamic uncertainties.*

We focus on a specific example of the ψ-Hilfer abstract fractional functional differential equation in order to relate our findings to the study of existence and stability of the solution in the Ulam sense.

Example 2.1. *Consider the ψ-Hilfer abstract fractional functional differential equation with appropriate functions $P(t,u(t))$ and $Q(t,u(t))$, and let $A=\frac{1}{10}$. This gives equation (2.1) the following form:*

$$\begin{cases}{}^H D_{a^+}^{\alpha,\beta;\psi}\left[u(t)-\dfrac{e^{-t}u(t)}{4e^t+e^{-t}}\right]=\dfrac{1}{10}u(t)+\dfrac{e^{-t}u(t)}{(9+e^{-t})(1+u(t))}, t\in[a,b],\\u(a)=u_a.\end{cases} \tag{2.23}$$

Case 1: *Let* $\alpha = \frac{2}{3}, \beta = \frac{1}{2}$ *and* $\psi(t) = t,\ t \in [0,1]$. *Then,*

$$\begin{aligned} |P(t,u_1(t)) - P(t,u_2(t))| =& \frac{1}{4e^t + e^{-t}} \left| e^{-t}u_1(t) - e^{-t}u_2(t) \right| \\ \leq& \frac{1}{5} \left| u_1(t) - u_2(t) \right|, \\ |Q(t,u_1(t)) - Q(t,u_2(t))| =& \frac{1}{9 + e^{-t}} \left| \frac{e^{-t}u_1(t)}{1+u_1(t)} - \frac{e^{-t}u_2(t)}{1+u_2(t)} \right| \\ \leq& \frac{1}{10} \left| u_1(t) - u_2(t) \right|. \end{aligned}$$

Assuming that Hypothesis 2.1 is satisfied with $l_1 = \frac{1}{5}$ *and* $l_2 = \frac{1}{10}$, *we carefully examine the validity of equation (2.23) as follows:*

$$a_1 + b_1 = \left[\frac{1}{5}(\psi(1) - \psi(0))^{\gamma-1} + \frac{1}{10}\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}(\psi(1) - \psi(0))^{\alpha+\gamma-1} \right] = 0.490641 < 1.$$

Invoking Theorem 2.2, we now establish the existence of a unique solution to (2.23) in the interval $[0,1]$.

Additionally, invoking Theorem 2.3, we also establish that equation (2.23) achieves UH stability with a suitable parameter value of

$$k_G = \frac{5}{4\Gamma(\frac{5}{3})} \mathbb{E}_{\frac{2}{3}} \left[\frac{1}{4} \right].$$

By setting $\Phi_G = t^{2/3}$ *as a continuous function, we obtain*

$$\begin{aligned} I_{0^+}^{\frac{2}{3};t} \Phi_G(t) &= \frac{1}{\Gamma(\frac{2}{3})} \int_0^t (t-s)^{\frac{-1}{3}} t^{2/3} ds \\ &\leq \frac{1}{\Gamma(\frac{2}{3})} \int_0^t (t-s)^{\frac{-1}{3}} ds \\ &\leq \frac{1}{\Gamma(\frac{5}{3})} t^{\frac{2}{3}}. \end{aligned}$$

Therefore, Hypothesis 2.2 is satisfied with $\lambda_\Phi = \frac{1}{\Gamma(\frac{5}{3})}$. *As a result of Theorem 2.4 with*

$$k_{\Phi,G} = \frac{5}{4\Gamma(\frac{5}{3}) - 1},$$

equation (2.23) is UHR stable.

Case 2: *Let* $\alpha = \frac{2}{3}, \beta = \frac{1}{2}$ *and* $\psi(t) = \ln t,\ t \in [1,e]$. *Assuming that Hypothesis 2.1 is satisfied with* $l_1 = \frac{1}{4e^2+1}$ *and* $l_2 = \frac{1}{9e+1}$, *we carefully examine the validity of condition (2.7) as follows:*

$$a_1 + b_1 = \left[\frac{1}{4e^2+1}(\ln e)^{\gamma-1} + \left(\frac{9e+11}{10(9e+1)} \right) \left(\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \right) (\ln e)^{\alpha+\gamma-1} \right]$$

$$= 0.235114 < 1.$$

Therefore, by Theorem 2.2, it can be concluded that equation (2.23) admits a unique solution in $[1, e]$.
Consequently, by Theorem 2.3, we establish that equation (2.23) is UH stable with

$$k_G = \frac{4e^2+1}{4e^2\Gamma(\frac{5}{3})}\mathbb{E}_{\frac{2}{3}}\left[\frac{(9e+11)(4e^2+1)}{40e^2(9e+1)}\right].$$

Additionally, considering $\Phi_G = \ln t^{2/3}$, *Hypothesis 2.2 is satisfied with* $\lambda_\Phi = \frac{1}{\Gamma(\frac{5}{3})}$. *As a result of Theorem 2.4, equation (2.23) is Ulam-Hyers-Rassias stable in the interval* $[1, e]$ *with*

$$k_{\Phi,G} = \frac{1}{\frac{4e^2}{4e^2+1}\Gamma(\frac{5}{3}) - \frac{9e+11}{10(9e+1)}}.$$

This example establishes the effectiveness of the assumptions as well as the results of all the obtained theorems.

2.4 Numerical Approximation

In this section, we present a numerical approximation of the solution of fractional differential equation (2.1) involving the ψ-Hilfer fractional derivative. This simulation accounts for the non-locality and memory effects inherent in fractional calculus. By refining the fractional order α and type β, the fractional formula provides a reliable approach for obtaining accurate approximations of the solution.
Using Theorem 2.1 and Definition 1.7, equation (2.3) can be rewritten as

$$\begin{aligned} u(t) =& P(t, u(t)) + \frac{[u_0 - P(0, u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t) - \psi(s))^{\alpha-1}\Big[Au(s) + Q(s, u(s))\Big]\psi'(s)ds. \end{aligned} \tag{2.24}$$

Equation (2.24) can be approximated as

$$\begin{aligned} u(t_{n+1}) =& P(t_n, u(t_n)) + \frac{[u_0 - P(0, u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{t_n} (\psi(t_n) - \psi(s))^{\alpha-1}\Big[Au(s) + Q(s, u(s))\Big]\psi'(s)ds \\ \implies u(t_{n+1}) =& P(t_n, u(t_n)) + \frac{[u_0 - P(0, u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{n}\Big[Au(t_k) + Q(t_k, u(t_k))\Big]\int_{t_{k-1}}^{t_k} (\psi(t_n) - \psi(s))^{\alpha-1}\psi'(s)ds \\ \implies u(t_{n+1}) =& P(t_n, u(t_n)) + \frac{[u_0 - P(0, u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} + \sum_{k=1}^{n}\Big[Au(t_k) + Q(t_k, u(t_k))\Big]A^{\alpha}_{n,k}, \end{aligned}$$

where $A^{\alpha}_{n,k} = \frac{1}{\Gamma(\alpha+1)}[(\psi(t_n) - \psi(t_{k-1}))^{\alpha} - (\psi(t_n) - \psi(t_k))^{\alpha}]$.
This approximation helps visualize the trajectories of the solution by varying the order and type of ψ-Hilfer fractional derivative for different choices of $\psi(t)$ function. In the following, we provide a suitable example with two different choices of $\psi(t)$.

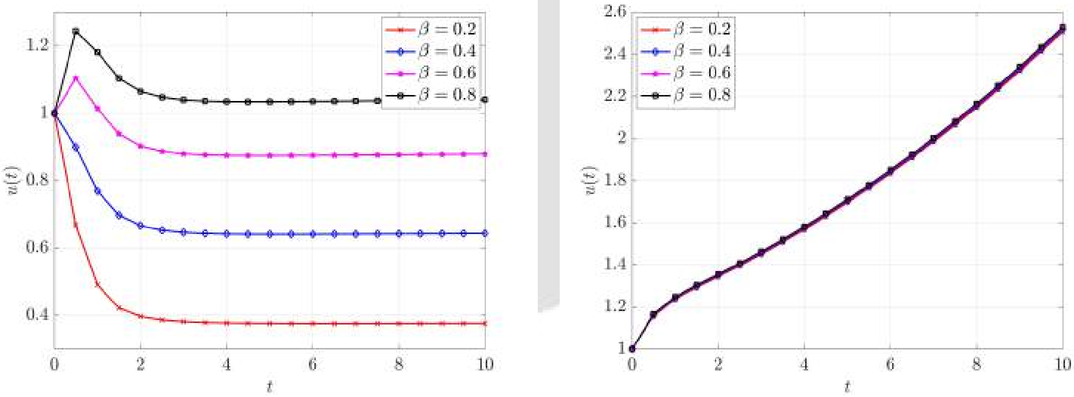
Example 2.2. *Consider the problem*

$$\begin{cases} {}^{H}D^{\alpha,\beta;\psi}_{a^+}[u(t) - \dfrac{e^{-t}u(t)}{4e^{t} + e^{-t}}] = \dfrac{1}{10}u(t) + \dfrac{e^{-t}u(t)}{(9 + e^{-t})\,(1 + u(t))}, \; t \in [a, b], \\ u(a) = u_a = 1. \end{cases} \tag{2.25}$$

The successive iterative solution of equation (2.25) can be expressed as

$$\begin{aligned} u(n+1) =& \frac{e^{-t(n)}u(n)}{4e^{t(n)} + e^{-t(n)}} + \frac{[u_0 - P(0, u_0)]}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ &+ \sum_{k=1}^{n-1} \left(\frac{u(k)}{10} + \frac{e^{-t(k)}u(k)}{(9 + e^{-t(k)})\,(1 + u(k))} \right) A^{\alpha}_{n,k}, \end{aligned} \tag{2.26}$$

where $A^{\alpha}_{n,k} = \frac{(\psi(t(n)) - \psi(t_{k+1}))^{\alpha_1} - (\psi(t(n)) - \psi(t(k)))^{\alpha_1}}{\Gamma(\alpha+1)}$.

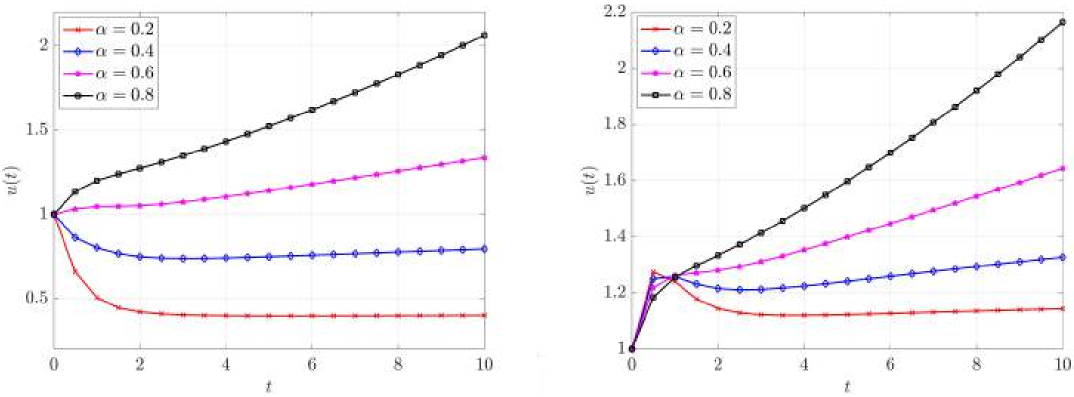


(a) $\alpha = 0.1, \psi(t) = t$ **(b)** $\alpha = 0.9, \psi(t) = t$

Figure 2.1: Fixed α and varying β

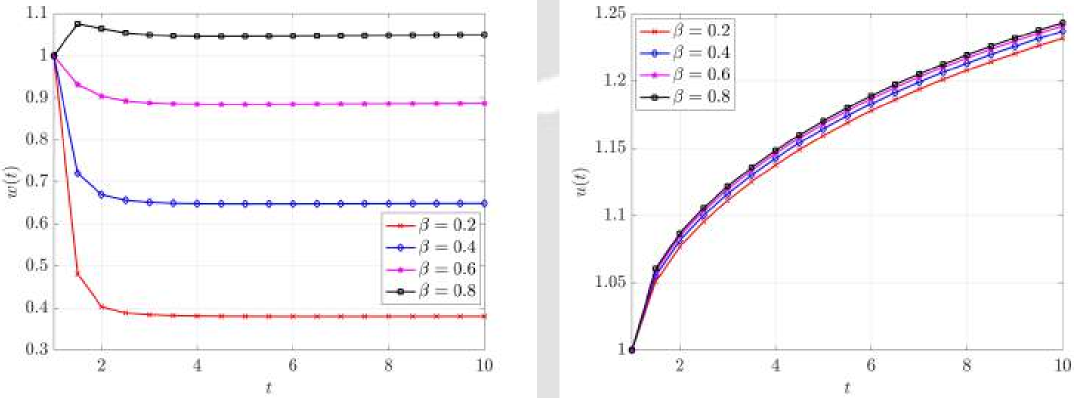
We consider the two cases: (i) $\psi(t) = t$, (ii) $\psi(t) = \log(t)$. In Figure 2.1, one can observe the solution trajectories for fixed values of α_1 at both lower (0.1) and higher (0.9) values, while varying β. Following this, for Figure 2.2 , β is kept fixed at 0.1 and 0.9, while α varies for $\psi(t) = t$. Experiments with $\psi(t) = \log(t)$ are presented with the similar consideration in Figures 2.3 and 2.4 . The findings indicate that solution trajectories are minimally impacted by α values near 1, whereas significant trajectory deviations are observed at lower α levels in Figures 2.1 and 2.3.

By highlighting the ways in which previous states impact the current behavior, the different values of α represent memory or hereditary qualities inherent to fractional-order



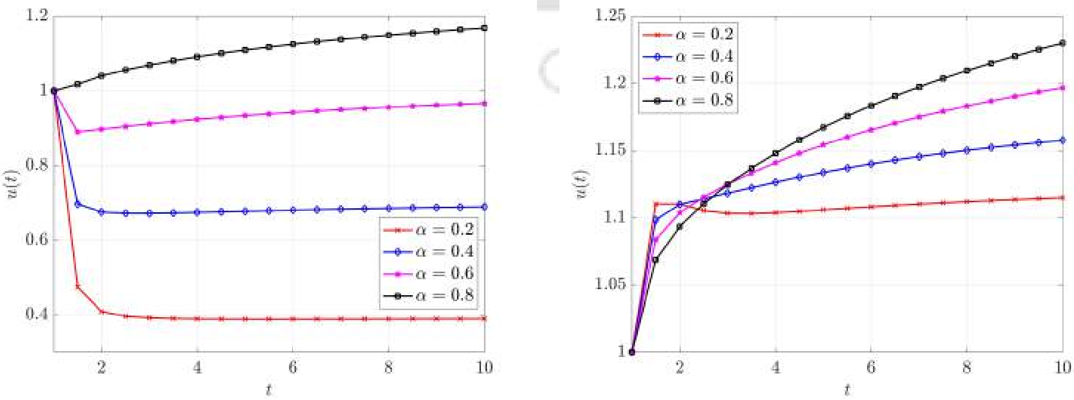
(a) $\beta = 0.1, \psi(t) = t$ **(b)** $\beta = 0.9, \psi(t) = t$

Figure 2.2: Fixed β and varying α



(a) $\alpha = 0.1, \psi(t) = \log(t)$ **(b)** $\alpha = 0.9, \psi(t) = \log(t)$

Figure 2.3: Fixed α and varying β



(a) $\beta = 0.1, \psi(t) = \log(t)$ **(b)** $\beta = 0.9, \psi(t) = \log(t)$

Figure 2.4: Fixed β and varying α

systems, providing a more comprehensive view of the solution. In particular, lower values of α introduce stronger memory effects, causing the system to deviate significantly from the classical derivative behavior. This retention of past influences is a key feature of fractional-order systems, shaping their unique dynamics.

Because of this versatility, fractional-order systems can be fine-tuned to mimic complicated dynamics unique to a given application by choosing the appropriate α values. For instance, Figures 2.3 and 2.4 demonstrate how the logarithmic weight function $\psi(t) = \log(t)$ further amplifies the impact of historical dependence, especially when α is small. This is particularly useful for applications involving diffusion processes or biological systems where changes occur over long time scales. This makes fractional-order systems extremely versatile in capturing historical dependence and long-term interactions.

Overall, these figures clearly illustrate how the interplay among α, β and the weight function $\psi(t)$ governs the dynamics of the system. By adjusting these parameters, one can model a wide range of behaviors, making fractional-order systems highly adaptable for diverse real-world applications.

2.5 Conclusion

In this chapter, we have established theoretical results with respect to the stability of the solution for the ψ-Hilfer abstract functional fractional differential equation. By leveraging the Banach's fixed point theorem and the generalized Grönwall's inequality, we have successfully demonstrated the existence, uniqueness, and stability of the solutions in the Ulam–Hyers and Ulam–Hyers–Rassias sense. The analysis highlights two distinct types of stability: Ulam–Hyers stability, which ensures bounded deviations in the solutions under constant perturbations, and Ulam–Hyers–Rassias stability, which accounts for state-dependent perturbations, providing greater flexibility in addressing dynamic uncertainties. To further illustrate the behavior of the system, we analyzed the trajectories of the solutions under varying fractional orders and types, using both linear and logarithmic forms of the weight function. This approach is particularly significant, since finding the analytical solutions for such equations under the generalized ψ-Hilfer fractional operator presents considerable challenges.

While this study focuses on Ulam–Hyers and Ulam–Hyers–Rassias stability, it has limitations in addressing highly nonlinear or chaotic behaviors. We plan to expand the stability analysis to include advanced types of stability, such as Ulam–Hyers–Mittag–Leffler stability. Additionally, applying the model to real-world data will allow for further validation and refinement, enhancing its practical applicability in various domains.

CHAPTER 3

On the Ulam-Hyers-Rassias-Mittag-Leffler stability of the solution to a ψ-Hilfer abstract fractional differential equation

This chapter devises an appropriate mathematical framework to explore the stability of the solution to ψ-Hilfer abstract fractional differential equations. Schauder's fixed point theorem serves as a cornerstone in establishing the existence of the solution for such equations. Building upon this foundation, we elegantly demonstrate the Ulam–Hyers–Mittag–Leffler stability as well as the Ulam–Hyers–Rassias–Mittag–Leffler stability pertaining to such equations. By leveraging fixed point theory and generalized Grönwall's inequality, we develop a rigorous framework that guarantees the existence and stability of the solution. This study demonstrates how resilient and consistent the solutions remain in the face of disruptions.

3.1 Introduction

The most recent studies looking at Ulam–Hyers–Mittag–Leffler (UHML) stability has profoundly advanced the field of fractional-order differential equations, providing critical insights and robust methods for analyzing the stability of complex dynamic systems [37, 62, 75]. The study of Ulam-type stability for solutions of fractional differential equations has attracted considerable attention; see, for example, [22,30,33,45,74,76,85,86], where various methodologies have been employed. For instance, Wang and Zhang [75] used a comparable approach to establish existence and uniqueness results for a Caputo-type fractional-order delay differential equation, while Ortocol and Ilea [48] applied the Picard operator method to obtain results for a fractional differential equation with delay.

Using sophisticated techniques such as fixed point method and generalized Grönwall's inequality, research endeavors such as [15, 16] addressed the uniqueness of solutions for

ψ-Hilfer FDEs, in addition to establishing the UHR stability. Liu et al. [37] analyzed the uniqueness and UHML stability of the solution to a fractional delay differential equation together with the subsequent form employing the Picard operator technique:

$$\begin{cases} {}^{H}D_{0+}^{\alpha,\beta;\psi}[z(t)] = f(t, z(t), z(g(t))),\ t \in (0, c], \\ I_{0+}^{1-\gamma;\psi} z(0^{+}) = z_{0} \in \mathbb{R}, \\ z(t) = \varphi(t), t \in [-k, 0]. \end{cases}$$

Rahima et al. [57] investigated the Ulam–Hyers–Rassias–Mittag–Leffler (UHRML) stability for ϖ-fractional partial differential equations. Shah et al. [62] examined the UHML and UHRML stability for nonlinear fractional reaction-diffusion equations with a delay. Analytical results for ABC-type pantograph equations with respect to another function [68], nonlocal hybrid inverse problems with delay [60], G-ABC implicit fractional equations with stability analysis [56], and impulsive neutral systems involving multi-term Hilfer derivatives [1] have all been the subject of recent works that have made significant contributions to the theory of generalized fractional operators of different type and stability in UHML sense.

The purpose of the present study is to address certain issues related to the existence and stability theory of real-order differential and integral operators. Inspired by the works mentioned above, we are concerned with the following ψ-Hilfer abstract fractional differential equation (ψ-HAFDE):

$$\begin{cases} {}^{H}D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t))] = Au(t) + Q(t, u(t)),\ t \in J = [0, a], \\ u(0) = u_{0}. \end{cases} \tag{3.1}$$

In Problem (3.1), ${}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}$ denotes the ψ-Hilfer fractional derivative of order $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$, with respect to a strictly increasing, continuously differentiable function $\psi : J \to \mathbb{R}$ satisfying $\psi'(t) \neq 0$ for all $t \in J$. The unknown function is $u : J \to \Omega$, where Ω is a Banach space, A is a non-negative scalar, $P : J \times \Omega \to \Omega$ and $Q : J \times \Omega \to \Omega$ are given continuous functions. The initial condition is given by $u(0) = u_0$.

The distinguished originality in this study is the creation of a general framework for the stability of the solution of abstract fractional differential equations that involve the ψ-Hilfer derivative. Unlike existing studies restricted to specific systems, the current approach addresses a broad class of abstract functional fractional differential equations. By incorporating the generalized Grönwall's inequality with Schauder's fixed point theorem, we establish the existence and several forms of stability, such as UHML and UHRML stability. To the best of our knowledge, this is one of the original instances in which this methodology has been used to address such problems in an abstract setting.

The structure of the chapter is as follows: in Section 3.2, Schauder's fixed point theorem is used to prove that solutions exist. In Section 3.3, we examine a number of concepts of stability, such as Ulam–Hyers–Mittag–Leffler stability and Ulam–Hyers–Rassias–Mittag–Leffler stability, which are supported by appropriate inequalities and integral estimates. We prove our stability results using illustrative scenarios in Section 3.4. Finally, Section 3.5 brings the study to a close by summarizing the findings.

3.2 Existence Result

In this section, we focus our investigation on a wider class of fractional derivative ${}^{H}D_{0+}^{\alpha,\beta;\psi}$ using Schauder's fixed point theorem to prove the existence of the solution for the abstract differential equation incorporated with the more general fractional operators. We employ the following notation for computational convenience:

$$[\psi(t) - \psi(0)]^{\gamma} = \psi^{\gamma}(t,0).$$

Theorem 3.1. *Let $\alpha \in (0,1)$ and $\beta \in [0,1]$ denote the order and type, respectively, of fractional differential equation (3.1). Further, let $u(t)$, $\psi(t) \in C^1(J,\mathbb{R})$ with $\psi(t)$ as an increasing function and $\psi'(t) \neq 0$ for all $t \in J$. Let $P(t,u(t))$ and $Q(t,u(t))$ be continuous functions. Then, $u(t)$ satisfies the nonlinear ψ-HAFDE (3.1) if and only if it satisfies*

$$u(t) = P(t,u(t)) + \frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0 - P(0,u_0)] + I_{0+}^{\alpha;\psi}\Big(Au(t) + Q(t,u(t))\Big). \tag{3.2}$$

Proof. Invoking ψ-fractional integral operator $I_{0+}^{\alpha;\psi}$ to equation (3.1), and from Lemma 1.4, we get

$$I_{0+}^{\alpha;\psi}\Big[{}^{H}D_{0+}^{\alpha,\beta;\psi}\Big(u(t) - P(t,u(t))\Big)\Big] = I_{0+}^{\alpha;\psi}\Big(Au(t) + Q(t,u(t))\Big).$$

Therefore,

$$\begin{aligned} u(t) - P(t,u(t)) - \frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)} I_{0+}^{(1-\beta)(1-\alpha);\psi}[u(0) - P(0,u(0))] \\ = I_{0+}^{\alpha;\psi}\Big(Au(t) + Q(t,u(t))\Big). \end{aligned}$$

From the initial condition, we get

$$u(t) = P(t,u(t)) + \frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0 - P(0,u_0)] + I_{0+}^{\alpha;\psi}\Big(Au(t) + Q(t,u(t))\Big). \tag{3.3}$$

This consequently results in equation (3.1).
Now, applying ${}^{H}D_{0+}^{\alpha,\beta;\psi}$ to equation (3.3) yields

$$\begin{aligned}{}^{H}D_{0+}^{\alpha,\beta;\psi}u(t) ={}& {}^{H}D_{0+}^{\alpha,\beta;\psi}P(t,u(t)) + {}^{H}D_{0+}^{\alpha,\beta;\psi}\left[\frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0 - P(0,u_0)]\right] \\ &+ {}^{H}D_{0+}^{\alpha,\beta;\psi}I_{0+}^{\alpha;\psi}\Big(Au(t) + Q(t,u(t))\Big),\end{aligned}$$

where, for $0 \le \gamma \le 1$, and we have ${}^{H}D_{0+}^{\alpha,\beta;\psi}\psi^{\gamma-1}(t,0) = 0$ [14]. Then,

$$^{H}D_{0+}^{\alpha,\beta;\psi}\Big[u(t) - P(t,u(t))\Big] = Au(t) + Q(t,u(t)).$$

This establishes the desired result, thereby completing the proof. □

In order to establish our main result, the following hypotheses are considered.

Hypothesis 3.1. *Let $P(t,u(t))$ and $Q(t,u(t)$ be uniformly bounded continuous functions on J.*

Hypothesis 3.2. *Two Lipschitz constants $l_1 > 0$ and $l_2 > 0$ exist satisfying Lipschitz condition in u for functions P and Q, respectively, i.e.,*

$$\begin{aligned} &|P(t,u_1(t)) - P(t,u_2(t))| \le l_1|u_1(t) - u_2(t)|, \\ \textit{and}\qquad &|Q(t,u_1(t)) - Q(t,u_2(t))| \le l_2|u_1(t) - u_2(t)|.\end{aligned}$$

Theorem 3.2. *Let Hypothesis 3.1 be valid. Then, equation (3.1) admits at least one solution in J.*

Proof. Let $\Omega = \Big\{u \in C_{1-\gamma,\psi}[0,a] : \|u\|_{C_{1-\gamma,\psi}} \le M_0\Big\}$ be a non-empty, bounded, closed, and convex subset of $C_{1-\gamma,\psi}[0,a]$, with M_0 chosen such that

$$\begin{aligned} M_0 \ge{}& ||P(t,u(t))||_{C_{1-\gamma;\psi}} + \frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0 - P(0,u_0)] \\ &+ \Big[A||u(t)||_{C_{1-\gamma;\psi}} + ||Q(t,u(t))||_{C_{1-\gamma;\psi}}\Big]\frac{1}{\Gamma(\alpha+1)}\psi^{\alpha}(t,0).\end{aligned}$$

We now define the operator $S : C_{1-\gamma,\psi}[0,a] \to C_{1-\gamma,\psi}[0,a]$, where $C_{1-\gamma,\psi}[0,a]$ is the weighted Banach space endowed with the norm

$$\|u\|_{C_{1-\gamma,\psi}} := \sup_{t\in[0,a]}\Big|\psi^{1-\gamma}(t,0)u(t)\Big|.$$

Let S be defined as

$$(Su)(t) = P(t,u(t)) + \frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0 - P(0,u_0)]$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\Big(Au(s)+Q(s,u(s))\Big)ds.$$

Now, we have

$$\begin{aligned}|(Su)(t)\psi^{1-\gamma}(t,0)| =&|P(t,u(t))\psi^{1-\gamma}(t,0)|+\frac{1}{\Gamma(\gamma)}[u_0-P(0,u_0)]\\&+\frac{1}{\Gamma(\alpha)}\psi^{1-\gamma}(t,0)\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\Big|Au(s)+Q(s,u(s))\Big|ds.\end{aligned}$$

The result can be obtained step-by-step in three steps which will lead to establishing S having a fixed point. To establish the existence result via Schauder's fixed point theorem, it suffices to verify the necessary conditions. We proceed as follows.

Step 1. To establish that S is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C_{1-\gamma,\psi}$.

We have

$$\begin{aligned}&|\Big((Su_n)(t)-(Su)(t)\Big)\psi^{1-\gamma}(t,0)|\\
=&|\Big[P(t,u_n(t))-P(t,u(t))\Big]\psi^{1-\gamma}(t,0)+\frac{1}{\Gamma(\alpha)}\psi^{1-\gamma}(t,0)\\
&\times\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\Big[A(u_n(s)-u(s))+Q(s,u_n(s))-Q(s,u(s))\Big]ds|\\
\leq&|P(t,u_n(t))-P(t,u(t))|\psi^{1-\gamma}(t,0)+\frac{1}{\Gamma(\alpha)}\psi^{1-\gamma}(t,0)\\
&\times\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\Big[A|u_n(s)-u(s)|+|Q(s,u_n(s))-Q(s,u(s))|\Big]ds\\
\leq&||P(t,u_n(t))-P(t,u(t))||_{C_{1-\gamma;\psi}}+\frac{1}{\Gamma(\alpha)(\alpha+\gamma-1)}\psi^{\alpha}(t,0)\\
&\times\Big[A||u_n(t)-u(t)||_{C_{1-\gamma;\psi}}+||Q(t,u_n(t))-Q(t,u(t))||_{C_{1-\gamma;\psi}}\Big].\end{aligned}$$

By continuity of $P(t,u(t))$ and $Q(t,(u(t))$, we obtain that

$$|\Big[(Su_n)(t)-(Su)(t)\Big]\psi^{1-\gamma}(t,0)|\to 0,$$

as $u_n\to u$. This implies that S is continuous.

Step 2. To prove that S is uniformly bounded.

We have

$$\begin{aligned}|(Su)(t)\psi^{1-\gamma}(t,0)| \leq&|P(t,u(t))\psi^{1-\gamma}(t,0)|+\frac{1}{\Gamma(\gamma)}[u_0-P(0,u_0)].\\
&+\frac{\psi^{1-\gamma}(t,0)}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\Big(A|u(s)|+|Q(s,u(s))|\Big)ds\\
\leq&||P(t,u(t))||_{C_{1-\gamma;\psi}}+\frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0-P(0,u_0)]\end{aligned}$$

$$+\Big[A||u(t)||_{C_{1-\gamma;\psi}}+||Q(t,u(t))||_{C_{1-\gamma;\psi}}\Big]\frac{1}{\Gamma(\alpha+1)}\psi^{\alpha}(t,0)$$
$$=M_0,$$

and consequently,

$$||(Su)(t)||_{C_{1-\gamma;\psi}}\leq M_0.$$

Thus, S is uniformly bounded.
Step 3. To prove that S is equicontinuous and compact.
We have

$$\begin{aligned}
&|(Su)(t_1)(\psi^{1-\gamma}(t_1,0))-(Su)(t_2)(\psi^{1-\gamma}(t_2,0))|\\
=&|\Big[P(t_1,u(t_1))(\psi^{1-\gamma}(t_1,0))-P(t_2,u(t_2))(\psi^{1-\gamma}(t_2,0))\\
&+\frac{1}{\Gamma(\alpha)}\Big[(\psi^{1-\gamma}(t_1,0))\int_0^{t_1}(\psi(t_1)-\psi(s))^{\alpha-1}\Big[Au(s)+Q(s,u(s))\Big]ds\\
&-\frac{1}{\Gamma(\alpha)}\Big[(\psi^{1-\gamma}(t_2,0))\Big]\int_0^{t_2}(\psi(t_2)-\psi(s))^{\alpha-1}\Big[Au(s)+Q(s,u(s))\Big]\Big|ds\\
\leq&\Big[P(t_1,u(t_1))(\psi^{1-\gamma}(t_1,0))-P(t_2,u(t_2))(\psi^{1-\gamma}(t_2,0))\Big]\\
&+\frac{1}{\Gamma(\alpha)}\Big[(\psi^{1-\gamma}(t_1,0))\int_0^{t_1}(\psi(t_1)-\psi(s))^{\alpha-1}\Big[Au(s)+Q(s,u(s))\Big]ds\\
&-\frac{1}{\Gamma(\alpha)}\Big[(\psi^{1-\gamma}(t_2,0))\Big]\int_0^{t_2}(\psi(t_2)-\psi(s))^{\alpha-1}\Big[Au(s)+Q(s,u(s))\Big]ds\\
\leq&||P(t_1,u(t_1))-P(t_2,u(t_2))||_{C_{1-\gamma;\psi}}+\frac{1}{(\alpha+\gamma-1)\Gamma(\alpha)}\Big[A||u(t)||_{C_{1-\gamma;\psi}}\\
&+||Q(t,u(t))||_{C_{1-\gamma;\psi}}\Big]\Big[\psi^{\alpha}(t_1,0)-\psi^{\alpha}(t_2,0)\Big]ds.
\end{aligned}$$

Thus,

$$|\Big((Su)(t_1)(\psi^{1-\gamma}(t_1,0))-(Su)(t_2)(\psi^{1-\gamma}(t_2,0))\Big)|\to 0,$$

as $t_1\to t_2$.
Consequently, it is established that the family $Su:u\in\Omega$ is equicontinuous. Now using the Arzelá-Ascoli theorem 1.5, we can conclude that S is a compact operator. Thus, S has a fixed point. Hence, equation (3.1) has at least one solution on J. □

3.3 Stability Analysis

This section discusses the UHML and UHRML stabilities of equation (3.1). The concept utilized here is adapted from Rus [59] and Wang et al. [75].

Definition 3.1. *In relation to* $\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha]$*, equation* (3.1) *is considered UHML stable if there exists some* $c_{E_\alpha}>0$ *such that, for every* $\epsilon>0$ *and the solution* $\tilde{u}(t)$ *to the inequation*

$$\left|{}^{H}D_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t)-P(t,\tilde{u}(t))]-A\tilde{u}(t)-Q(t,\tilde{u}(t))\right|\leq\epsilon\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha],\quad t\in J, \tag{3.4}$$

there exists a solution $u(t)$ *of equation* (3.1) *satisfying*

$$|\tilde{u}(t)-u(t)|\leq c_{E_\alpha}\epsilon\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha],\quad t\in J. \tag{3.5}$$

Remark 3.1. *A function* $\tilde{u}\in C^1(J,\mathbb{R})$ *is a solution of inequation* (3.4) *iff there exists a function* $\zeta\in C^1(J,\mathbb{R})$ *(which depends on* $\tilde{u}$*) such that*
(i) $|\zeta(t)|\leq\epsilon\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha],\quad\forall t\in J,$
(ii) ${}^{H}D_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t)-P(t,\tilde{u}(t))]=A\tilde{u}(t)+Q(t,\tilde{u}(t))+\zeta(t),\quad\forall t\in J.$

Definition 3.2. *In relation to* $\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha]$*, equation* (3.1) *is considered UHRML stable if there exists some* $c_{E_\alpha}>0$ *such that, for every* $\epsilon>0$ *and* $\Phi(t)$*, and the solution* $\tilde{u}(t)$ *to the inequation*

$$\left|{}^{H}D_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t)-P(t,\tilde{u}(t))]-A\tilde{u}(t)-Q(t,\tilde{u}(t))\right|\leq\epsilon\Phi(t)\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha],\quad t\in J, \tag{3.6}$$

there is a solution $u(t)$ *of equation* (3.1) *satisfying*

$$|\tilde{u}(t)-u(t)|\leq c_{E_\alpha}\epsilon\Phi(t)\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha],\quad t\in J. \tag{3.7}$$

Remark 3.2. *A function* $\tilde{u}\in C^1(J,\mathbb{R})$ *is a solution of inequation* (3.6) *iff there exists a function* $\eta\in C^1(J,\mathbb{R})$ *(which depends on* $\tilde{u}$*) such that*
(i) $|\eta(t)|\leq\epsilon\Phi(t)\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha],\quad\forall t\in J,$
(ii) ${}^{H}D_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t)-P(t,\tilde{u}(t))]=A\tilde{u}(t)+Q(t,\tilde{u}(t))+\eta(t),\quad\forall t\in J.$

Theorem 3.3. *Assume that Hypotheses 3.1-3.2 are satisfied. Then, equation* (3.1) *is UHML stable on* J*.*

Proof. Let $\tilde{u}(t)$ be a solution of inequation (3.4) so that

$$\left|{}^{H}D_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t)-P(t,\tilde{u}(t))]-A\tilde{u}(t)-Q(t,\tilde{u}(t))\right|\leq\epsilon\mathbb{E}_\alpha[\psi^\alpha(t,0)],t\in J, \tag{3.8}$$

and assume $\tilde{u}(0)=u(0)=u_0$ in order that both solutions satisfy the same initial condition. Further, let $u(t)$ be the unique solution of the equation

$$\begin{cases}{}^{H}D_{0+}^{\alpha,\beta;\psi}[u(t)-P(t,u(t))]=Au(t)+Q(t,u(t)),\,t\in J,\\ u(0)=u_{0}.\end{cases}$$

Applying operator $I_{0+}^{\alpha;\psi}$ to both sides of inequation (3.8), one obtains the following:

$$\begin{aligned}&\left|\tilde{u}(t)-P(t,\tilde{u}(t))-\tfrac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0-P(0,u_0)]-I_{0+}^{\alpha;\psi}[A\tilde{u}(t)+Q(t,\tilde{u}(t))]\right|\\&\leq\epsilon\mathbb{E}_\alpha[\psi^\alpha(t,0)].\end{aligned}\tag{3.9}$$

Subsequently,

$$\begin{aligned}&|\tilde{u}(t)-u(t)|\\=&\left|\tilde{u}(t)-P(t,u(t))-\frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0-P(0,u_0)]-I_{0+}^{\alpha;\psi}[Au(t)+Q(t,u(t))]\right|\\\leq&\left|\tilde{u}(t)-P(t,\tilde{u}(t))-\frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0-P(0,u_0)]-I_{0+}^{\alpha;\psi}[A\tilde{u}(t)+Q(t,\tilde{u}(t))]\right|\\&+|P(t,\tilde{u}(t))-P(t,u(t))|+I_{0+}^{\alpha;\psi}[A|\tilde{u}(t)-u(t)|+|Q(t,\tilde{u}(t))-Q(t,u(t))|].\end{aligned}$$

Next, using equation (3.9), we get

$$\begin{aligned}&|\tilde{u}(t)-u(t)|\\\leq&\epsilon\mathbb{E}_\alpha[\psi^\alpha(t,0)]+|P(t,\tilde{u}(t))-P(t,u(t))|\\&+I_{0+}^{\alpha;\psi}[A|\tilde{u}(t)-u(t)|+|Q(t,\tilde{u}(t))-Q(t,u(t))|]\\\leq&\epsilon\mathbb{E}_\alpha[\psi^\alpha(t,0)]+l_1|\tilde{u}(t)-u(t)|+I_{0+}^{\alpha;\psi}(A+l_2)|\tilde{u}(t)-u(t)|\\\leq&\epsilon\frac{\mathbb{E}_\alpha[\psi^\alpha(t,0)]}{(1-l_1)}+\frac{A+l_2}{(1-l_1)\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}|\tilde{u}(s)-u(s)|\psi'(s)ds\\\leq&\epsilon\frac{\mathbb{E}_\alpha[\psi^\alpha(t,0)]}{(1-l_1)}+\int_0^t\left(\frac{A+l_2}{(1-l_1)}\right)^n\frac{1}{\Gamma(n\alpha)}(\psi(t)-\psi(s))^{\alpha-1}\epsilon\frac{\mathbb{E}_\alpha[\psi^\alpha(s,0)]}{(1-l_1)}\psi'(s)ds\\\leq&\epsilon\frac{\mathbb{E}_\alpha[\psi^\alpha(t,0)]}{(1-l_1)}\mathbb{E}_\alpha\left[\frac{A+l_2}{(1-l_1)}\psi^\alpha(t,0)\right]\\\leq&c_{E_\alpha}\epsilon\mathbb{E}_\alpha[(\psi(t)-\psi(0))^\alpha],\end{aligned}$$

where $c_{E_\alpha}=\frac{1}{1-l_1}\mathbb{E}_\alpha\left[\frac{A+l_2}{1-l_1}[\psi(t)-\psi(0)]^\alpha\right]$. Thus, equation (3.1) is UHML stable. □

Theorem 3.4. *Assume that Hypotheses 3.1-3.2 hold. Then, equation (3.1) is UHRML stable on J with respect to Φ if*

$$I_{0+}^{\alpha;\psi}\Phi(t)\mathbb{E}_\alpha[\psi^\alpha(t,0)]\leq\Phi(t)\mathbb{E}_\alpha[\psi^\alpha(t,0)].$$

Proof. Let $\tilde{u}(t)$ be a solution of inequation (3.6) so that

$$\left|{}^HD_{0+}^{\alpha,\beta;\psi}[\tilde{u}(t)-P(t,\tilde{u}(t))]-A\tilde{u}(t)-Q(t,\tilde{u}(t))\right|\leq\epsilon\Phi(t)\mathbb{E}_\alpha[\psi^\alpha(t,0)],t\in J,\tag{3.10}$$

and assume $\tilde{u}(0) = u(0) = u_0$ in order that both solutions satisfy the same initial condition. Further, let $u(t)$ be the unique solution of the equation

$$\begin{cases} {}^H D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t))] = Au(t) + Q(t, u(t)),\ t \in J = (0, a], \\ u(0) = u_0. \end{cases}$$

Applying operator $I_{0+}^{\alpha;\psi}$ to both sides of inequation (3.10), one obtains the following:

$$\begin{aligned} &\left|\tilde{u}(t) - P(t, \tilde{u}(t)) - \tfrac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0 - P(0, u_0)] - I_{0+}^{\alpha;\psi}[A\tilde{u}(t) + Q(t, \tilde{u}(t))]\right| \\ &\leq \epsilon\Phi(t)\mathbb{E}_\alpha[\psi^\alpha(t, 0)]. \end{aligned} \tag{3.11}$$

Subsequently,

$$\begin{aligned} &|\tilde{u}(t) - u(t)| \\ =&\left|\tilde{u}(t) - P(t, u(t)) - \frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0 - P(0, u_0)] - I_{0+}^{\alpha;\psi}[Au(t) + Q(t, u(t))]\right| \\ \leq&\left|\tilde{u}(t) - P(t, \tilde{u}(t)) - \frac{\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}[u_0 - P(0, u_0)] - I_{0+}^{\alpha;\psi}[A\tilde{u}(t) + Q(t, \tilde{u}(t))]\right| \\ &+ |P(t, \tilde{u}(t)) - P(t, u(t))| + I_{0+}^{\alpha;\psi}[A\,|\tilde{u}(t) - u(t)| + |Q(t, \tilde{u}(t)) - Q(t, u(t))|]. \end{aligned}$$

Next, using equation (3.11), we get

$$\begin{aligned} &|\tilde{u}(t) - u(t)| \\ \leq&\epsilon\Phi(t)\mathbb{E}_\alpha[\psi^\alpha(t, 0)] + |P(t, \tilde{u}(t)) - P(t, u(t))| \\ &+ I_{0+}^{\alpha;\psi}[A\,|\tilde{u}(t) - u(t)| + |Q(t, \tilde{u}(t)) - Q(t, u(t))|] \\ \leq&\epsilon\Phi(t)\mathbb{E}_\alpha[\psi^\alpha(t, 0)] + l_1\,|\tilde{u}(t) - u(t)| + I_{0+}^{\alpha;\psi}(A + l_2)\,|\tilde{u}(t) - u(t)| \\ \leq&\epsilon\Phi(t)\frac{\mathbb{E}_\alpha[\psi^\alpha(t, 0)]}{(1 - l_1)} + \frac{A + l_2}{(1 - l_1)\Gamma(\alpha)}\int_0^t (\psi(t) - \psi(s))^{\alpha-1}\,|\tilde{u}(s) - u(s)|\,\psi'(s)ds \\ \leq&\epsilon\Phi(t)\frac{\mathbb{E}_\alpha[\psi^\alpha(t, 0)]}{(1 - l_1)} \\ &+ \int_0^t \left(\frac{A + l_2}{(1 - l_1)}\right)^n \frac{1}{\Gamma(n\alpha)}(\psi(t) - \psi(s))^{\alpha-1}\epsilon\Phi(s)\frac{\mathbb{E}_\alpha[\psi^\alpha(s, 0)]}{(1 - l_1)}\psi'(s)ds \\ \leq&\epsilon\Phi(t)\frac{\mathbb{E}_\alpha[\psi^\alpha(t, 0)]}{(1 - l_1)}\mathbb{E}_\alpha\left[\frac{A + l_2}{(1 - l_1)}\psi^\alpha(t, 0)\right] \\ \leq&c_{E_\alpha}\epsilon\Phi(t)\mathbb{E}_\alpha[(\psi(t) - \psi(0))^\alpha], \end{aligned}$$

where $c_{E_\alpha} = \frac{1}{1-l_1}\mathbb{E}_\alpha\left[\frac{A+l_2}{1-l_1}[\psi(t) - \psi(0)]^\alpha\right]$. Thus, equation (3.1) is UHRML stable. □

Next, we examine the authenticity of our results through two examples in the next section.

3.4 Examples

Example 3.1. *Consider ψ-HAFDE (3.1) with $\alpha = \frac{2}{3}, \beta = \frac{1}{2}$. Thus, the following form can be obtained from equation (3.1):*

$$\begin{cases} {}^H D_{0^+}^{\frac{2}{3},\frac{1}{2};\psi}[u(t) - \frac{\cos(t)}{e^{t^2}+2}u(t-0.5)] = \frac{1}{5}u(t) + \frac{\sin(t)u(t-0.5)}{(e^{t^2}+9)(1+u(t-0.5))}, \ t \in (0,a], \\ u(0) = u_0. \end{cases} \tag{3.12}$$

Case 1: *Let $\psi(t) = t$, $t \in [0,1]$. Then, we have*

$$\begin{aligned} |P(t,u_1(t)) - P(t,u_2(t))| =& \left| \frac{\cos(t)}{e^{t^2}+2}u_1(t) - \frac{\cos(t)}{e^{t^2}+2}u_2(t) \right| \\ \leq& \frac{1}{3}|u_1(t) - u_2(t)|, \end{aligned}$$

and

$$\begin{aligned} |Q(t,u_1(t)) - Q(t,u_2(t))| =& \left| \frac{u_1(t)\sin(t)}{(e^{t^2}+9)(1+u_1(t))} - \frac{u_2(t)\sin(t)}{(e^{t^2}+9)(1+u_2(t))} \right| \\ \leq& \frac{1}{10}|u_1(t) - u_2(t)|. \end{aligned}$$

Then, $l_1 = \frac{1}{3}, l_2 = \frac{1}{10}, A + l_2 = \frac{3}{10}$. Thus,

$$c_{E_\alpha} = \frac{1}{1-l_1}\mathbb{E}_\alpha\left[\frac{A+l_2}{1-l_1}\psi^\alpha(t,0)\right] = \frac{3}{2}\mathbb{E}_{\frac{2}{3}}\left[\frac{9}{20}\right].$$

This implies that equation (3.12) is both UHML and UHRML stable.
Case 2: *Let $\psi(t) = e^t$, $t \in [0,1]$. Similarly, equation (3.12) is UHML and UHRML stable with*

$$c_{E_\alpha} = \frac{1}{1-l_1}\mathbb{E}_\alpha\left[\frac{A+l_2}{1-l_1}\psi^\alpha(t,0)\right] = \frac{3}{2}\mathbb{E}_{\frac{2}{3}}\left[\frac{9}{20}(e-1)^{\frac{2}{3}}\right].$$

Example 3.2. *Consider ψ-HAFDE (3.1) with $\alpha = \frac{2}{3}, \beta = \frac{1}{2}$. Thus, the following form can be obtained from equation (3.1):*

$$\begin{cases} {}^H D_{0^+}^{\frac{2}{3},\frac{1}{2};\psi}[u(t) - \frac{e^{-t}u(t-0.5)}{5e^t+e^{-t}}] = \frac{1}{6}u(t) + \frac{e^{-t}u(t-0.5)}{(11+e^{-t})(1+u(t-0.5))}, \ t \in (0,a], \\ u(0) = u_0. \end{cases} \tag{3.13}$$

Case 1: *Let $\psi(t) = t$, $t \in [0,1]$. Then, we have*

$$|P(t,u_1(t)) - P(t,u_2(t))| = \left| \frac{e^{-t}u_1(t)}{5e^t+e^{-t}} - \frac{e^{-t}u_2(t)}{5e^t+e^{-t}} \right|$$

$$\leq \frac{1}{6}\left|u_1(t)-u_2(t)\right|,$$

and

$$\left|Q(t,u_1(t))-Q(t,u_2(t))\right| = \left|\frac{e^{-t}u_1(t)}{(11+e^{-t})(1+u_1(t))}-\frac{e^{-t}u_2(t)}{(11+e^{-t})(1+u_2(t))}\right|$$
$$\leq \frac{1}{12}\left|u_1(t)-u_2(t)\right|.$$

Then, $l_1=\frac{1}{6}, l_2=\frac{1}{12}, A+l_2=\frac{1}{4}$. *Thus,*

$$c_{E_\alpha}=\frac{1}{1-l_1}\mathbb{E}_\alpha\left[\frac{A+l_2}{1-l_1}\psi^\alpha(t,0)\right]=\frac{6}{5}\mathbb{E}_{\frac{2}{3}}\left[\frac{3}{10}\right].$$

This implies that equation (3.13) *is both UHML and UHRML stable.*
Case 2: *Let* $\psi(t)=e^t$, $t\in[0,1]$. *Similarly, equation* (3.13) *is UHML and UHRML stable with*

$$c_{E_\alpha}=\frac{1}{1-l_1}\mathbb{E}_\alpha\left[\frac{A+l_2}{1-l_1}\psi^\alpha(t,0)\right]=\frac{6}{5}\mathbb{E}_{\frac{2}{3}}\left[\frac{3}{10}(e-1)^{\frac{2}{3}}\right].$$

3.5 Conclusion

Our study used a rigorous mathematical framework to explore the complexities of ψ-Hilfer abstract fractional differential equations. Using Schauder's fixed point theorem, we established the existence of solutions, providing a strong theoretical foundation for further investigation. Furthermore, by applying the generalized Grönwall's inequality, we derived the Ulam–Hyers–Mittag–Leffler and Ulam–Hyers–Rassias–Mittag-Leffler stability results, demonstrating the robustness of the solutions under perturbations within given parameters. A stronger Lipschitz continuity condition was required to ensure the stability in the Mittag–Leffler context.

By establishing the existence and stability of solutions, this research provides deeper insight into the dynamics and adaptability of abstract differential equations involving generalized fractional operators. These findings pave the way for future studies and applications in various scientific and engineering fields. The extension of the present study to recently developed fractional operators, including the generalized proportional Caputo and Atangana–Baleanu-type derivatives [67, 69], which provide greater versatility in modeling memory and hereditary effects, is a potential direction for further investigation. The combination of ψ-Hilfer derivatives with impulsive effects or stochastic perturbations remains largely unexplored and could significantly broaden the applicability of the results presented here.

CHAPTER 4

Solutions of Nonlinear ψ-Hilfer Neutral Fractional Differential Equations: Existence, Stability and a Numerical Perspective

In the present chapter, we investigate a nonlinear neutral fractional differential equation involving the generalized ψ-Hilfer fractional derivative of order α and type β, addressing both theoretical and numerical aspects. An existence result for the solution is established via Krasnosel'skiĭ's fixed point theorem, and sufficient conditions are derived for Ulam-Hyers and Ulam-Hyers-Rassias stability. To support these theoretical results, numerical approximations are provided for both cases of fixed β with varying α, and fixed α with varying β, using three kernel functions $\psi(t)$.

4.1 Introduction

Furati et al. [22] established the existence, uniqueness, and stability of solutions for nonlinear fractional differential equations involving the Hilfer derivative in weighted continuous function spaces. Moreover, Shankar and Bora [64] analyzed the Ulam–Hyers stability of non-instantaneous impulsive integro-differential equations using the Caputo derivative, demonstrating their results through an application to electrical circuits. In order to model financial crises, Norouzi and N'Guérékata [47] used measurements of non-compactness to examine the existence, uniqueness, and stability of solutions for a class of ψ-Hilfer fractional differential systems. Luo et al. [39] investigated the Ulam–Hyers stability of nonlinear ψ-Hilfer fractional differential equations with time-varying delays:

$$\begin{cases} {}^{H}D_{0+}^{\alpha,\beta;\psi} v(t) = A_1 v(t) + A_2 v(t-h(t)) + f(t, v(t), v(t-h(t))), t \in [0,a], \\ v(t) = \phi(t), t \in [-h, 0]. \end{cases}$$

More broadly, the stability of ψ-Hilfer fractional differential equations has been extensively studied, alongside various related classes such as integro-differential equations, impulsive systems, nonlinear Volterra equations, and systems with infinite delay, as discussed in recent surveys [15, 16, 25].

Motivated by recent advancements in the study of ψ-Hilfer fractional differential equations, particularly those addressing stability and existence results across various functional frameworks, we examine a class of nonlinear neutral fractional differential equations with finite delay, formulated as follows:

$$\begin{cases} {}^H D_{0^+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t-d(t)))] = Q(t, u(t), u(t-d(t))),\, t \in J, \\ u(t) = \phi(t), t \in [-h, 0], \end{cases} \tag{4.1}$$

with ${}^H D_{0^+}^{\alpha,\beta;\psi}$ denoting the ψ-Hilfer fractional derivative operator, $d : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi : [-h, 0] \to \mathbb{R}$ and $J = [0, a]$.

The structure of the chapter is as follows: Section 4.2 presents the existence result using Krasnosel'skiĭ's fixed point theorem. Section 4.3 discusses Ulam–Hyers and Ulam–Hyers–Rassias stability results. In Section 4.4, a numerical approximation is provided, demonstrating how the behavior of the solution changes with different kernel functions and fractional parameters. Finally, Section 4.5 concludes the work with a summary.

4.2 Existence Results for Nonlinear ψ-Hilfer Neutral Fractional Differential Equations

This section focuses on establishing the existence of solutions for a class of nonlinear neutral fractional differential equations featuring the ψ-Hilfer fractional derivative. The analysis is carried out by converting the given differential equation into an equivalent integral equation and employing Krasnosel'skiĭ's fixed point theorem.

Theorem 4.1. *Let $\alpha \in (0, 1)$, $\beta \in [0, 1]$ and $u(t)$, $\psi(t) \in C^1([0, a], \mathbb{R})$ with $\psi(t)$ as a non-decreasing function and $\psi'(t) \neq 0$. Let $P(t, u(t-d(t)))$ and $Q(t, u(t), u(t-d(t)))$ be continuous functions. Then, $u(t)$ satisfies the ψ-Hilfer nonlinear neutral fractional differential equation*

$$\begin{cases} {}^H D_{0^+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t-d(t)))] = Q(t, u(t), u(t-d(t))),\, t \in J = [0, a], \\ u(t) = \phi(t), t \in [-h, 0], \end{cases} \tag{4.2}$$

if and only if it satisfies

$$\begin{aligned} u(t) =& P(t, u(t-d(t))) + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0, u(-d(0)))] \\ &+ I_{0^+}^{\alpha;\psi}\Big[Q(t, u(t), u(t-d(t)))\Big]. \end{aligned}$$

Proof. It is straightforward to show that $u(t) = \phi(t), \forall t \in [-h, 0]$. Now, we establish the result for $t \in [0, a]$.
Invoking the ψ-fractional integral operator $I_{0+}^{\alpha;\psi}$ to equation (4.1), we obtain

$$I_{0+}^{\alpha;\psi}\Big[{}^H D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t-d(t)))\Big] = I_{0+}^{\alpha;\psi}\Big[Q(t, u(t), u(t-d(t)))\Big].$$

Therefore,

$$\begin{aligned} u(t) =& P(t, u(t-d(t))) + I_{0+}^{\alpha;\psi}\Big[Q(t, u(t), u(t-d(t)))\Big] + \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)\Gamma(1-\gamma)} \\ & \times \int_0^t (\psi(t)-\psi(s))^{-\gamma}[u(0) - P(0, u(-d(0))]\psi'(s)ds \\ \implies u(t) =& P(t, u(t-d(t))) + I_{0+}^{\alpha;\psi}\Big[Q(t, u(t), u(t-d(t)))\Big] \\ & + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0, u(-d(0))]. \end{aligned} \tag{4.3}$$

This consequently yields the integral representation of the solution to equation (4.2). Now, applying ${}^H D_{0+}^{\alpha,\beta;\psi}$ to equation (4.3) yields

$$\begin{aligned} {}^H D_{0+}^{\alpha,\beta;\psi} u(t) =& {}^H D_{0+}^{\alpha,\beta;\psi} P(t, u(t-d(t))) + {}^H D_{0+}^{\alpha,\beta;\psi}\left[\frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0, u(-d(0))]\right] \\ & + {}^H D_{0+}^{\alpha,\beta;\psi} I_{0+}^{\alpha;\psi}\Big[Q(t, u(t), u(t-d(t)))\Big], \end{aligned}$$

where ${}^H D_{0+}^{\alpha,\beta;\psi}[u(0)] = 0$ and ${}^H D_{0+}^{\alpha,\beta;\psi}[P(0, u(-d(0))] = 0$. Therefore,

$${}^H D_{0+}^{\alpha,\beta;\psi}\Big[u(t) - P(t, u(t-d(t)))\Big] = Q(t, u(t), u(t-d(t))).$$

This establishes the required result and thus concludes the proof. □

For obtaining the further desired results, we make the following hypotheses:

Hypothesis 4.1. *Let $P : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then there exists $L_P > 0$ such that*

$$|P(t, u_1(t-d(t))) - P(t, u_2(t-d(t)))| \leq L_P |u_1(t) - u_2(t)|,$$

for all $t \in J$.

Hypothesis 4.2. *Let $Q : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded function. Then there exist $M_Q > 0$ and $L_Q > 0$ such that*

$$\begin{aligned} |Q(t, u(t), u(t-d(t)))| &\leq M_Q, \\ |Q(t, u_1(t), \tilde{u}_1(t)) - Q(t, u_2(t), \tilde{u}_2(t))| &\leq L_Q\Big[|u_1(t) - u_2(t)| + |\tilde{u}_1(t) - \tilde{u}_2(t)|\Big], \end{aligned}$$

for all $t \in J$.

Hypothesis 4.3. *There exists* $r > 0$ *such that*

$$\|P\|_\infty + \frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} M_Q + |K_\gamma| \le r,$$

where $K_\gamma = \dfrac{u(0) - P(0, u(-d(0)))}{\Gamma(\gamma)\Gamma(2-\gamma)}$.

Theorem 4.2. *Suppose Hypotheses 4.1-4.3 hold. Then, the fractional differential equation (4.1) possesses at least one solution* $u \in C([-h, T], \mathbb{R})$.

Proof. It is easy to show that $u(t) = \phi(t), \forall t \in [-h, 0]$. We now go ahead to establish the result for $t \in [0, a]$.
We proceed via decomposing the operator S into $S_1 + S_2$ as

$$(Su)(t) = \underbrace{P(t, u(t - d(t))) + K_\gamma}_{S_1 u} + \underbrace{I_{0^+}^{\alpha;\psi} Q(t, u(t), u(t - d(t)))}_{S_2 u}.$$

First, we show that S_1 is a contraction.
It is easy to follow that

$$\begin{aligned}\|S_1 u_1 - S_1 u_2\| &= |P(t, u_1(t - d(t))) - P(t, u_2(t - d(t)))| \\ &\le L_P \|u_1(t) - u_2(t)\|.\end{aligned}$$

Thus, S_1 is a contraction.
Next, we prove that S_2 is completely continuous.
For any convergent sequence $u_n \to u$ in $C(J, \mathbb{R})$, the continuity of S_2 follows from

$$\begin{aligned}|S_2 u_n(t) - S_2 u(t)| \le & \frac{1}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} L_Q \Big[|u_n(s) - u(s)| \\ & + |u_n(s - d(s)) - u(s - d(s))|\Big] ds \\ \le & \frac{2L_Q}{\Gamma(\alpha+1)} (\psi(T) - \psi(0))^\alpha \|u_n - u\|_{C(J,\mathbb{R})}.\end{aligned}$$

The right-hand side above vanishes as $n \to \infty$ since $u_n \to u$ uniformly. It implies the continuity of S_2.

Now, for all $u \in \Omega = \{u \in C(J, \mathbb{R}) : \|u\| \le r\}$,

$$\begin{aligned}\|S_2 z\|_{C(J,\mathbb{R})} &\le \frac{1}{\Gamma(\alpha)} \sup_{t \in J} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |Q(s, u(s), u(s - d(s)))| ds \\ &\le \frac{M_Q}{\Gamma(\alpha)} \sup_{t \in J} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\ &= \frac{M_Q}{\Gamma(\alpha+1)} (\psi(T) - \psi(0))^\alpha,\end{aligned}$$

where $M_Q = \sup_{t\in J} |Q(t, u(t), u(t-d(t)))| < \infty$. It implies the boundedness of S_2. Next, for any $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}
|S_2u(t_2) - S_2u(t_1)| =& |\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(\psi(t_2)-\psi(0))^{\alpha-1}Q(s,u(s),u(s-d(s)))ds\psi'(s) \\
&- \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(\psi(t_1)-\psi(0))^{\alpha-1}Q(s,u(s),u(s-d(s)))|ds\psi'(s) \\
\leq& |\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(\psi(t_2)-\psi(0))^{\alpha-1}Q(s,u(s),u(s-d(s)))ds\psi'(s) \\
&- \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(\psi(t_1)-\psi(0))^{\alpha-1}Q(s,u(s),u(s-d(s)))|ds\psi'(s) \\
&+ |\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(\psi(t_1)-\psi(0))^{\alpha-1}Q(s,u(s),u(s-d(s)))ds\psi'(s) \\
&- \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(\psi(t_1)-\psi(0))^{\alpha-1}Q(s,u(s),u(s-d(s)))|ds\psi'(s) \\
\leq& |\frac{1}{\Gamma(\alpha)}M_Q\int_0^{t_2}\Big[(\psi(t_2)-\psi(0))^{\alpha-1} - (\psi(t_1)-\psi(0))^{\alpha-1}\Big]|ds\psi'(s) \\
&+ |\frac{1}{\Gamma(\alpha)}M_Q\int_{t_1}^{t_2}(\psi(t_1)-\psi(0))^{\alpha-1}|ds\psi'(s). \\
\implies |S_2u(t_2) - S_2u(t_1)| \leq& |\frac{M_Q}{\Gamma(\alpha+1)}\Big[(\psi(t_2)-\psi(0))^{\alpha} - (\psi(t_1)-\psi(0))^{\alpha}\Big]|.
\end{aligned}$$

Thus, $|S_2u(t_2) - S_2u(t_1)| \to 0$ as $t_1 \to t_2$. It implies the equicontinuity of S_2. Finally, we establish that $S(\Omega) \subset \Omega$. For $\Omega = \{u \in C(J,\mathbb{R}) : \|u\| \leq r\}$ with

$$\begin{aligned}
\|Su(t)\| =& \|P(t,u(t-d(t))) + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0,u(-d(0))]\| \\
&+ \|I_{0^+}^{\alpha;\psi}\Big[Q(t,u(t),u(t-d(t)))\Big]\|.
\end{aligned}$$

This implies that

$$r \geq \|P\|_\infty + \frac{M_Q}{\Gamma(\alpha+1)}(\psi(T)-\psi(0))^\alpha + |K_\gamma|.$$

Therefore, we have $S(\Omega) \subseteq \Omega$.

By Krasnosel'skiĭ's fixed point theorem, Su has at least one fixed point $u \in \Omega$, which establishes the existence of at least one solution of equation (4.1). □

4.3 Ulam–Hyers and Ulam–Hyers–Rassias Stability Analysis

This section focuses on the stability behavior of solutions to the proposed nonlinear neutral fractional differential equation involving the ψ-Hilfer derivative. The stability in the sense of Ulam–Hyers and Ulam–Hyers–Rassias are investigated using adequate techniques.

Definition 4.1. *For equation (4.1) to be considered Ulam–Hyers stable, a constant $K \in \mathbb{R}^+$ must exist such that, for every $\epsilon > 0$ and for every solution y of the inequality*

$$\left|{}^H D_{0^+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t-d(t)))] - Q(t, u(t), u(t-d(t)))\right| \leq \epsilon, \tag{4.4}$$

there exists a solution $u(t)$ to equation (4.1) such that

$$|u(t) - y(t)| \leq K\epsilon,\ t \in J. \tag{4.5}$$

Remark 4.1. *Equation (4.4) admits a solution $y \in C^1(J, \mathbb{R})$ if and only if a function $\mu_1 \in C^1(J, \mathbb{R})$ exists that is dependent on y in such a way that*

(i) $|\mu_1(t)| \leq \epsilon$, *for all $t \in J$,*

(ii) ${}^H D_{0^+}^{\alpha,\beta;\psi}\Big[y(t) - P(t, y(t-d(t)))\Big] - Q(t, y(t), y(t-d(t))) + \mu_1(t) = 0, \quad \forall t \in J.$

Definition 4.2. *For equation (4.1) to be considered Ulam–Hyers–Rassias stable with respect to $\Theta(t) \in C(J, \mathbb{R}^+)$, there must exist a constant $K_\Theta > 0$ such that, for every $\epsilon > 0$ and for every solution y of the inequality*

$$\left|{}^H D_{0^+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t-d(t)))] - Q(t, u(t), u(t-d(t)))\right| \leq \epsilon\Theta(t), t \in J, \tag{4.6}$$

there exists a solution $u(t)$ of equation (4.1) such that

$$|u(t) - y(t)| \leq K_\Theta \Theta(t)\epsilon, t \in J. \tag{4.7}$$

Remark 4.2. *Equation (4.6) admits a solution $y \in C^1(J, \mathbb{R})$ if and only if a function $\mu_2 \in C^1(J, \mathbb{R})$ exists that is dependent on y in such a way that*

(i) $|\mu_2(t)| \leq \epsilon$, *for all $t \in J$,*

(ii) ${}^H D_{0^+}^{\alpha,\beta;\psi}\Big[y(t) - P(t, y(t-d(t)))\Big] - Q(t, y(t), y(t-d(t))) + \mu_2(t) = 0, \quad \forall t \in J.$

Theorem 4.3. *Suppose Hypotheses 4.1-4.3 hold, and the condition*

$$L_P + 2L_Q C_\alpha < 1 \tag{4.8}$$

is satisfied. Then, equation (4.1) is Ulam–Hyers stable.

Proof. It is quite straightforward to show that $u(t) = \phi(t), \forall t \in [-h, 0]$. Next, we establish the result for $t \in [0, a]$.
Let

$$\left|{}^H D_{0^+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t-d(t)))] - Q(t, u(t), u(t-d(t)))\right| \leq \epsilon. \tag{4.9}$$

Invoking ψ-Riemann-Liouville integral operator to both sides of inequation (4.9), we have

$$\Big|[u(t) - P(t, u(t-d(t)))] - K_\gamma - I_{0+}^{\alpha;\psi} Q(t, u(t), u(t-d(t)))\Big| \le I_{0+}^{\alpha;\psi}\epsilon,$$

which implies

$$\Big|[u(t) - P(t, u(t-d(t)))] - K_\gamma - I_{0+}^{\alpha;\psi} Q(t, u(t), u(t-d(t)))\Big| \le \epsilon \frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}.$$

Now, we have

$$\begin{aligned}
|u(t)-y(t)| \le& \Big|[u(t) - P(t, y(t-d(t)))] - K_\gamma - I_{0+}^{\alpha;\psi} Q(t, y(t), y(t-d(t)))\Big| \\
\le& \Big|[u(t) - P(t, u(t-d(t)))] - K_\gamma - I_{0+}^{\alpha;\psi} Q(t, u(t), u(t-d(t)))\Big| \\
&+ |P(t, u(t-d(t))) - P(t, y(t-d(t)))| \\
&+ \Big|I_{0+}^{\alpha;\psi} Q(t, u(t), u(t-d(t))) - I_{0+}^{\alpha;\psi} Q(t, y(t), y(t-d(t)))\Big| \\
\le& \epsilon \frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)} + |P(t, u(t-d(t))) - P(t, y(t-d(t)))| \\
&+ I_{0+}^{\alpha;\psi} |Q(t, u(t), u(t-d(t))) - Q(t, y(t), y(t-d(t)))|.
\end{aligned}$$

Subsequently, the above inequality can be written as

$$\begin{aligned}
|\delta(t)| &\le \epsilon C_\alpha + L_P|\delta(t-d)| + L_Q I_{0+}^{\alpha;\psi}\Big(|\delta(t)| + |\delta(t-d)|\Big) \\
&\le \epsilon C_\alpha + L_P\|\delta\|_\infty + L_Q I_{0+}^{\alpha;\psi}\Big(2\|\delta\|_\infty\Big),
\end{aligned}$$

where $\delta(t) = u(t) - y(t)$, $C_\alpha = \frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}$ and $\|\delta\|_\infty = \sup_{t\in[0,T]} |\delta(t)|$. Then,

$$\|\delta\|_\infty \le \frac{\epsilon C_\alpha}{1 - L_P - 2L_Q C_\alpha}.$$

The final stability constant can be considered as

$$K = \frac{C_\alpha}{1 - L_P - 2L_Q C_\alpha},$$

provided $L_P + 2L_Q C_\alpha < 1$. □

For discussing the stability in Ulam–Hyers–Rassias sense, in addition to Hypotheses 4.1-4.3, we introduce an additional hypothesis as follows:

Hypothesis 4.4. *Assume that $\Theta(t) \in C(J, \mathbb{R}^+)$ is a continuous and non-decreasing function. Further, suppose that there exists a constant $\lambda_\Theta > 0$ such that the following* *inequality is satisfied for every $t \in J$:*

$$I_{0+}^{\alpha;\psi}\Theta(t) \leq \lambda_\Theta \Theta(t). \tag{4.10}$$

Theorem 4.4. *Suppose that Hypotheses 4.1-4.4 hold, and the condition*

$$L_P + 2L_Q C_\alpha < 1 \tag{4.11}$$

is satisfied. Then, equation (4.1) is Ulam–Hyers–Rassias stable.

Proof. It is easy to show that $u(t) = \phi(t), \forall t \in [-h, 0]$. Next, we establish the result for $t \in [0, a]$.
Let

$$\left|{}^H D_{0+}^{\alpha,\beta;\psi}[u(t) - P(t, u(t - d(t)))] - Q(t, u(t), u(t - d(t)))\right| \leq \epsilon\Theta(t). \tag{4.12}$$

Applying ψ-Riemann-Liouville integral to both sides of inequation (4.12), we have

$$\left|[u(t) - P(t, u(t - d(t)))] - K_\gamma - I_{0+}^{\alpha;\psi} Q(t, u(t), u(t - d(t)))\right| \leq I_{0+}^{\alpha;\psi}\epsilon\Theta(t),$$

which gives

$$\left|[u(t) - P(t, u(t - d(t)))] - K_\gamma - I_{0+}^{\alpha;\psi} Q(t, u(t), u(t - d(t)))\right| \leq \epsilon\lambda_\Theta\Theta(t).$$

Now, we get

$$\begin{aligned}
&|u(t) - y(t)| \\
&\qquad \leq \left|[u(t) - P(t, y(t - d(t)))] - K_\gamma - I_{0+}^{\alpha;\psi} Q(t, y(t), y(t - d(t)))\right| \\
&\qquad\quad + |P(t, u(t - d(t))) - P(t, y(t - d(t)))| \\
&\qquad\quad + I_{0+}^{\alpha;\psi} |Q(t, u(t), u(t - d(t))) - Q(t, y(t), y(t - d(t)))| \\
&\qquad \leq \epsilon\lambda_\Theta\Theta(t) + |P(t, u(t - d(t))) - P(t, y(t - d(t)))| \\
&\qquad\quad + I_{0+}^{\alpha;\psi} |Q(t, u(t), u(t - d(t))) - Q(t, y(t), y(t - d(t)))|.
\end{aligned}$$

Thus, we have

$$|u(t) - y(t)| \leq \epsilon\lambda_\Theta\Theta(t) + L_P|\delta(t - d(t))| + L_Q I_{0+}^{\alpha;\psi}\Big(|\delta(t)| + |\delta(t - d(t))|\Big).$$

Therefore, the above inequality can be written as

$$\begin{aligned}
|\delta(t)| &\leq \epsilon\lambda_\Theta\Theta(t) + L_P|\delta(t - d)| + L_Q I_{0+}^{\alpha;\psi}\Big(|\delta(t)| + |\delta(t - d)|\Big) \\
&\leq \epsilon\lambda_\Theta\Theta(t) + L_P\|\delta\|_\infty + L_Q I_{0+}^{\alpha;\psi}\Big(2\|\delta\|_\infty\Big),
\end{aligned}$$

where $\delta(t) = u(t) - y(t)$, $C_\alpha = \frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}$ and $\|\delta\|_\infty = \sup_{t\in[0,T]} |\delta(t)|$.
Then, we get

$$\|\delta\|_\infty \leq \frac{\epsilon\lambda_\Theta\Theta(t)}{1 - L_P - 2L_Q C_\alpha}.$$

The final stability constant can be considered as

$$K_\Theta = \frac{C_\alpha}{1 - L_P - 2L_Q C_\alpha},$$

provided $L_P + 2L_Q C_\alpha < 1$. □

To demonstrate the practical applicability of the theoretical framework for Ulam–Hyers and Ulam–Hyers–Rassias stability, we construct a concrete example that meets the necessary conditions. This demonstrates the reliability of the proposed model and offers more insight into how solutions behave under small disruptions, in addition to demonstrating the application of the stability results.

Example 4.1. *Consider the following nonlinear ψ-Hilfer neutral fractional differential equation with constant delay:*

$$\begin{cases} {}^H D_{0^+}^{\alpha,\beta;\psi} \left[u(t) - \frac{\arctan(u(t-0.3))}{5(1+t^2)}\right] = \frac{\sin(u(t))}{10(1+|u(t-0.3)|)} + \frac{e^{-t}u(t-0.3)}{20}, t \in [a,b], \\ u(t) = \phi(t) = \frac{1}{10}\cos(2t), t \in [-h,a], \end{cases} \tag{4.13}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$, and the nonlinear terms are defined as

$$P(t, u(t-0.3)) = \frac{\arctan(u(t-0.3))}{5(1+t^2)},$$

and

$$Q(t, u(t), u(t-0.3)) = \frac{\sin(u(t))}{10(1+|u(t-0.3)|)} + \frac{e^{-t}u(t-0.3)}{20}.$$

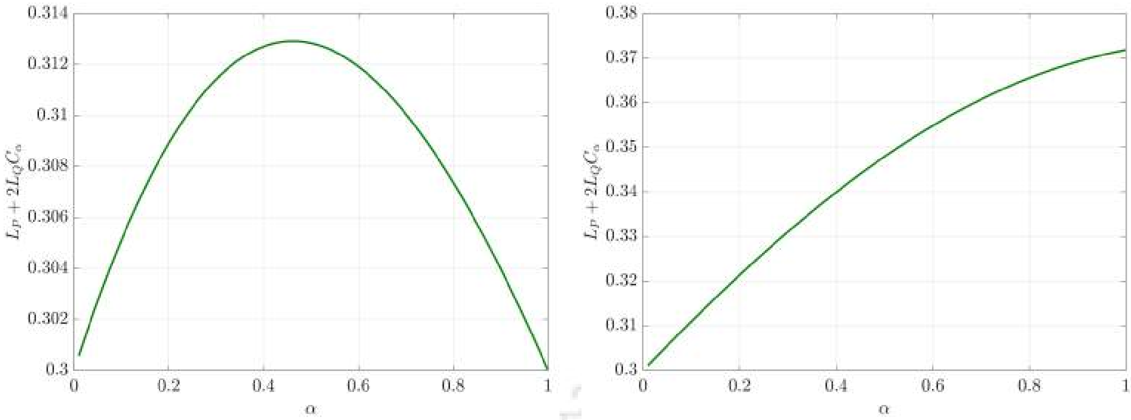
The function P satisfies the Lipschitz type condition

$$|P(t,\tilde{u}_1) - P(t,\tilde{u}_2)| \leq \frac{1}{5}|\tilde{u}_1 - \tilde{u}_2|,$$

while Q also satisfies

$$|Q(t,u_1,\tilde{u}_1) - Q(t,u_2,\tilde{u}_2)| \leq \frac{1}{20}\Big[|u_1 - u_2| + |\tilde{u}_1 - \tilde{u}_2|\Big].$$

Subsequently, for the function $\Theta(t) = (\ln t)^{\frac{2}{3}}$, the following fractional integral condition



(a) $\psi(t) = t,\ t \in [0,1]$ and $\psi(t) = \ln t,\ t \in [1,e]$

(b) $\psi(t) = e^t,\ t \in [0,1]$

Figure 4.1: Variation of the stability bound $L_P + 2L_QC_\alpha$ with respect to α for two different expressions, illustrating the influence of fractional order on stability.

is satisfied:

$$I_{0^+}^{\alpha;\psi}\Theta(t) \leq \lambda_\Theta \Theta(t),$$

where $\psi(t) = \ln t$. Through computation, we obtain the bound

$$I_{1^+}^{\alpha;\psi}\Theta(t) \leq \frac{2}{\sqrt{\pi}}(\ln t)^{\frac{2}{3}},$$

with $\lambda_\Theta = \dfrac{2}{\sqrt{\pi}}$ when $\alpha = 0.5$.

Case 1: *$\psi(t) = t$ and $t \in J = [0,1]$.*

Here, we have

$$C_\alpha = \frac{1}{\Gamma(\alpha+1)}.$$

Therefore,

$$K = \frac{1}{1 - L_P - 2L_QC_\alpha} \approx 1.4057.$$

With $K \approx 1.4057$, the system satisfies the key stability condition $L_P + 2L_QC_\alpha < 1$, confirming both Ulam–Hyers and Ulam–Hyers–Rassias stability.

Case 2: *$\psi(t) = \ln t$ and $t \in J = [1,e]$.*

Here,

$$C_\alpha = \frac{(\ln e - \ln 1)^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)}.$$

Therefore,

$$K = \frac{1}{1 - L_P - 2L_QC_\alpha} \approx 1.4057.$$

With $K \approx 1.4057$, the system satisfies the key stability condition $L_P + 2L_Q C_\alpha < 1$, confirming both Ulam–Hyers and Ulam–Hyers–Rassias stability.
Case 3: *$\psi(t) = e^t$ and $t \in J = [0, 1]$.*
In this case,

$$C_\alpha = \frac{(e-1)^\alpha}{\Gamma(\alpha+1)}.$$

Thus,

$$K = \frac{1}{1 - L_P - 2L_Q C_\alpha} \approx 1.5335.$$

With $K \approx 1.5335$, the system satisfies the key stability condition $L_P + 2L_Q C_\alpha < 1$, confirming both Ulam–Hyers and Ulam–Hyers–Rassias stability.

In all three cases above, the computed K values confirm that each system satisfies the key stability condition illustrated in Figure 4.1, which shows that $L_P + 2L_Q C_\alpha < 1$, thereby ensuring both Ulam–Hyers and Ulam–Hyers–Rassias stability.

4.4 Numerical Approximation

The objective of this section is to examine how the solutions behave when the order α and the type β of equation (4.1) are varied. A numerical scheme is proposed to approximate the solution of the fractional differential equation (4.1), which incorporates the ψ-Hilfer fractional derivative. This approach effectively captures the non-local and memory-dependent features. Through the adjustment of α and β, this method provides an accurate and flexible means of approximating the solution.

To facilitate the numerical implementation, equation (4.1) is written in a modified form as follows:

$$\begin{aligned} u(t) =& P(t, u(t-d(t))) + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0, u(-d(0)))] \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} Q(s, u(s), u(s-d(s))) ds \psi'(s). \end{aligned} \tag{4.14}$$

Equation (4.14) can be approximated as

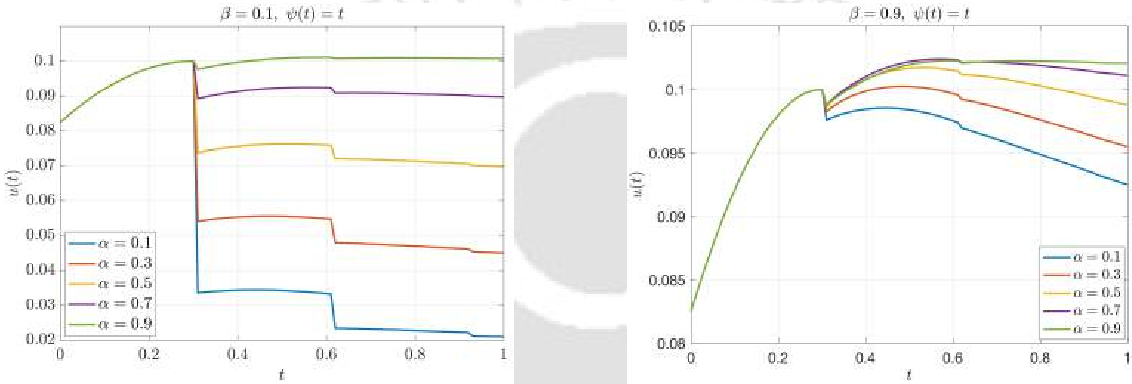
$$\begin{aligned} u(t_{n+1}) =& P(t_n, u(t_n - d(t_n))) + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0, u(-d(0)))] \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (\psi(t_n) - \psi(s))^{\alpha-1} Q(s, u(s), u(s-d(s))) ds \psi'(s) \end{aligned}$$

$$\begin{aligned} \implies u(t_{n+1}) =& P(t_n, u(t_n - d(t_n))) + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0, u(-d(0)))] \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} Q(t_k, u(t_k), u(t_k - d(t_k))) \int_{t_{k-1}}^{t_k} (\psi(t_n) - \psi(s))^{\alpha-1} ds \psi'(s) \end{aligned}$$

$$\implies u(t_{n+1}) = P(t_n, u(t_n - d(t_n))) + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0, u(-d(0))]$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} Q(t_k, u(t_k), u(t_k - d(t_k)))C^{\alpha}_{n,k},$$

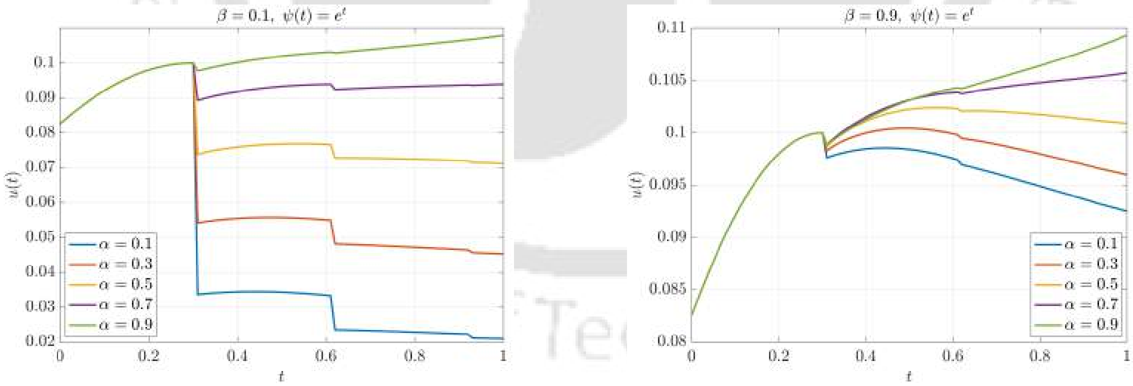
where $C^{\alpha}_{n,k} = \frac{1}{\Gamma(\alpha+1)}[(\psi(t_n) - \psi(t_{k-1}))^{\alpha} - (\psi(t_n) - \psi(t_k))^{\alpha}]$.

This numerical approximation enables the visualization of the solution trajectories by altering the order and type of the ψ-Hilfer fractional derivative, considering different forms of the function $\psi(t)$. To illustrate the method, an example is presented using three distinct choices for $\psi(t)$.



(a) $\beta = 0.1$ **(b)** $\beta = 0.9$

Figure 4.2: Fixed β and varying α for $\psi(t) = t$

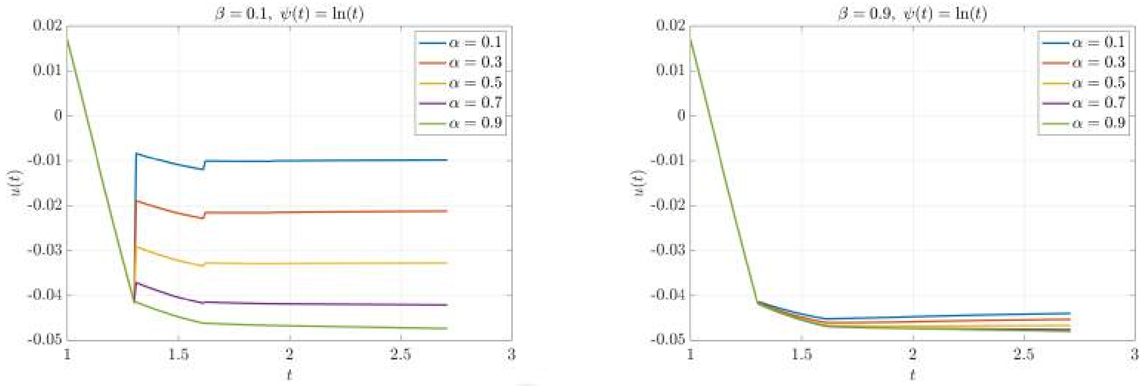


(a) $\beta = 0.1$ **(b)** $\beta = 0.9$

Figure 4.3: Fixed β and varying α for $\psi(t) = e^t$

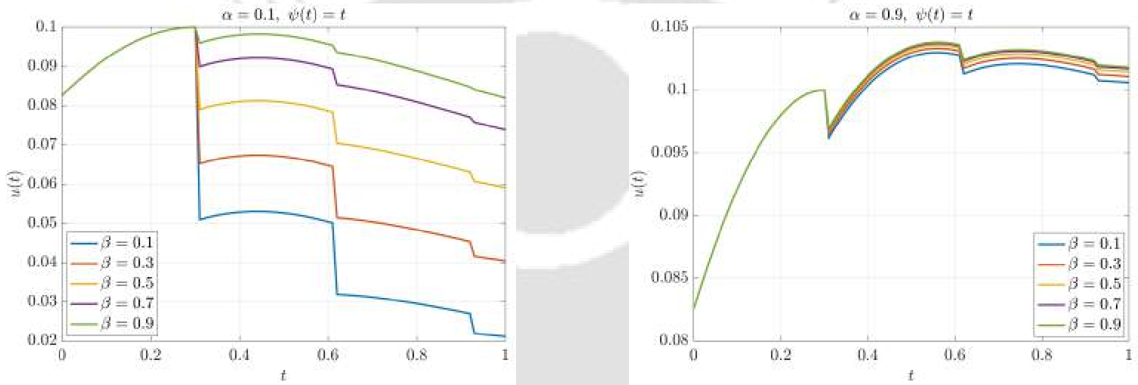
Example 4.2. *Consider the following nonlinear neutral fractional differential equation involving the Hilfer fractional derivative with respect to the function* $\psi(t)$*:*

$$\begin{cases} {}^{H}D^{\alpha,\beta;\psi}_{0^+}\left[u(t) - \frac{\arctan(u(t-0.3))}{5(1+t^2)}\right] = \frac{\sin(u(t))}{10(1+|u(t-0.3)|)} + \frac{e^{-t}u(t-0.3)}{20}, t \in [a,b], \\ u(t) = \phi(t) = \frac{1}{10}\cos(2t), t \in [-h, a]. \end{cases} \tag{4.15}$$



(a) $\beta = 0.1$ **(b)** $\beta = 0.9$

Figure 4.4: Fixed β and varying α for $\psi(t) = \ln(t)$



(a) $\alpha = 0.1$ **(b)** $\alpha = 0.9$

Figure 4.5: Fixed α and varying β for $\psi(t) = t$

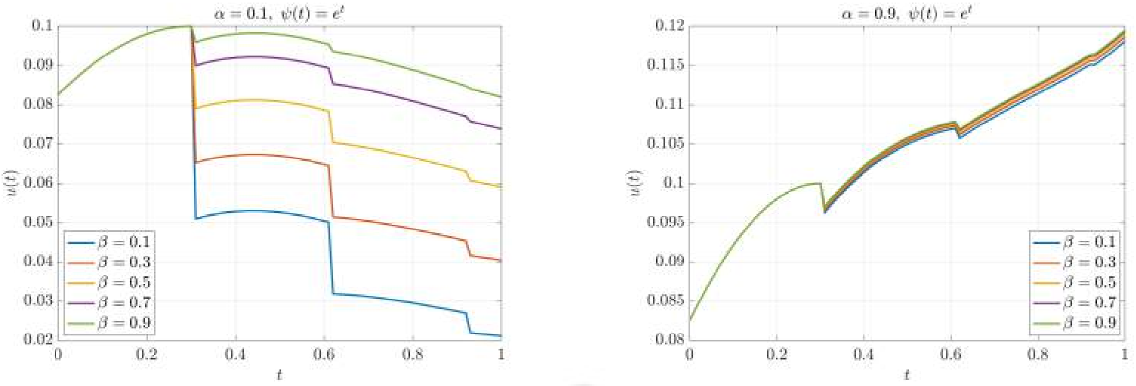
The delay term is considered a constant, given by $d(t) = 0.3$.

To approximate the solution numerically, we use the following discrete form based on fractional convolution:

$$u(t_{n+1}) = P(t_n, u(t_n - d)) + \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}[u(0) - P(0, u(-d))]$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} Q(t_k, u(t_k), u(t_k - d)) C_{n,k}^{\alpha},$$

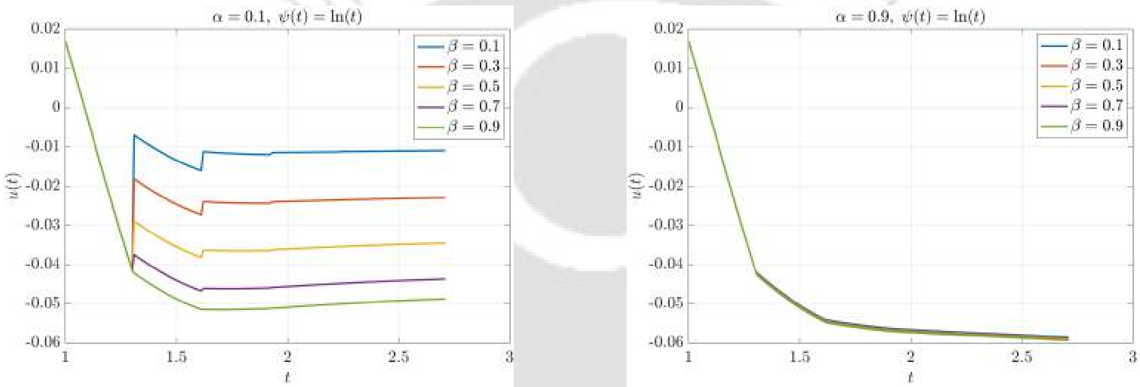
where $\gamma = \alpha + \beta - \alpha\beta$, $C_{n,k}^{\alpha} = \dfrac{(\psi(t_n) - \psi(t_{k-1}))^{\alpha} - (\psi(t_n) - \psi(t_k))^{\alpha}}{\Gamma(\alpha+1)}$, $P(t, \tilde{u}) = \dfrac{\arctan(\tilde{u})}{5(1+t^2)}$, *and* $Q(t, u, \tilde{u}) = \dfrac{\sin(u)}{10(1+|\tilde{u}|)} + \dfrac{e^{-t}\tilde{u}}{20}$. *The initial condition is given by* $\phi(t) = \dfrac{1}{10}\cos(2t)$ *on* $[-h, a]$.

Numerical results are illustrated in Figures 4.2–4.7, each consisting of two subfigures. Figures 4.2–4.4 display the approximation of $u(t)$ *for fixed values* $\beta = 0.1, 0.9$ *with varying* α, *for three different kernel functions:* $\psi(t) = t$ *over* $[0, 1]$ *(linear),* $\psi(t) = e^t$ *over*



(a) $\alpha = 0.1$ **(b)** $\alpha = 0.9$

Figure 4.6: Fixed α and varying β for $\psi(t) = e^t$



(a) $\alpha = 0.1$ **(b)** $\alpha = 0.9$

Figure 4.7: Fixed α and varying β for $\psi(t) = \ln(t)$

$[0, 1]$ *(exponential), and* $\psi(t) = \ln(t)$ *over* $[1, e]$ *(logarithmic), respectively. Figures 4.5–4.7 reverse the setting by fixing* $\alpha = 0.1, 0.9$ *while varying* β *over the same set, again for the same three respective kernels. For Figures 4.2–4.4, when* $\beta = 0.1$*, the solution curves for different* α *are clearly separated, indicating strong sensitivity to fractional order. As* β *increases to* 0.9*, the curves converge, showing that higher* β *reduces the influence of* α*. Similarly, for fixed* α *in Figures 4.5–4.7, we observe that small* α *results in more dispersed curves with respect to varying* β*, while for larger* α*, the sensitivity decreases. The logarithmic kernel (*$\psi(t) = \ln(t)$*) uniquely reverses the ordering of the curves, highlighting the influence of the weight function on the dynamics of the system. In conclusion, lower values of* α *and* β *correspond to stronger memory and hereditary effects, leading to a more significant variation in the solution profile. In contrast, higher values tend to suppress these effects, producing smoother and more regular behavior. The choice of the kernel function* ψ *significantly shapes the evolution: linear and exponential weights show conventional ordering, while the logarithmic weight accentuates early-time behavior and alters the curve hierarchy.*

4.5 Conclusion

In this study, we have established important results regarding the Ulam–Hyers and Ulam–Hyers–Rassias stability for ψ-Hilfer fractional differential equations with delay and neutral terms. Using Krasnosel'skiĭ's fixed-point theorem, we proved the existence of solutions under appropriate conditions, contributing to the theoretical understanding of fractional-order systems. The theoretical findings have been validated through a carefully constructed example, demonstrating the practical applicability of the stability results. Additionally, we presented numerical approximations of the model for both fixed β with varying α, and fixed α with varying β, under different weight functions such as linear, exponential, and logarithmic, highlighting the impact of fractional orders and kernel types on the solution behavior. These numerical results offer valuable insights into the dynamics of the system.

CHAPTER 5

Ulam–Hyers Stability Analysis of a Coupled ψ-Hilfer Fractional System with Application to Blood Alcohol Dynamics

In this chapter, we investigate a coupled system of ψ-Hilfer fractional differential equations. First, the existence and uniqueness result is established using Banach's fixed point theorem, and an existence result is further proved by employing Schauder's fixed point theorem. Moreover, the Ulam-Hyers and generalized Ulam-Hyers stability of the system are explored under suitable conditions. A numerical scheme is also developed to approximate the solution of the ψ-Hilfer coupled system. Finally, our results are applied to a real-world problem, modeling the blood alcohol concentration in the human body. Comparing the numerical solutions for various weight functions with experimental data and the classical integer-order model, we demonstrate the improved accuracy and flexibility of the fractional framework.

5.1 Introduction

Beyond the theoretical analysis, the application of fractional models to real-world problems has gained considerable momentum. The non-local nature of fractional derivatives makes them particularly well-suited for modeling systems with memory, such as those found in biology, medicine, and engineering [66, 72]. A compelling example is the modeling of blood alcohol concentration (BAC). The classical integer-order model, while providing a baseline approximation, often fails to precisely capture the complex, time-dependent dynamics of alcohol absorption and metabolism in the human body [38]. The development of fractional models offers a promising avenue to address these limitations. By introducing a fractional order, a model can be fine-tuned to more accurately represent the observed experimental data, leading to a better understanding and potentially more reliable applications, such as in the development of breathalyzers and other diagnostic tools. Our present work builds upon this foundation by providing a rigorous analysis of a coupled ψ-Hilfer fractional system and validating our findings through a numerical simulation of blood alcohol dynamics.

Motivated by the growing interest in such generalized systems, the current work focuses on a nonlinear coupled system involving ψ-Hilfer FDE on the interval $J = [0, T]$:

$$\begin{cases} {}^H D_{0+}^{\alpha,\beta;\psi} u(t) = P(t, u(t), v(t)), \quad t \in J, \\ {}^H D_{0+}^{\alpha,\beta;\psi} v(t) = Q(t, u(t), v(t)), \quad t \in J, \\ I_{0+}^{1-\gamma;\psi} u(0) = u_0, \\ I_{0+}^{1-\gamma;\psi} v(0) = v_0, \end{cases} \tag{5.1}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta - \alpha\beta$. We assume that $\psi \in C^1(J, \mathbb{R})$ is strictly increasing with $\psi'(t) \neq 0$ for all $t \in J$. Let $E = C_{1-\gamma;\psi}[J, \mathbb{R}]$ denote the weighted space of continuous functions endowed with the norm

$$\|u\|_E = \sup_{t \in J} \left| (\psi(t) - \psi(0))^{1-\gamma} u(t) \right|. \tag{5.2}$$

In the context of the BAC model, we show that the ψ-Hilfer fractional system captures the absorption and elimination processes more accurately when compared to the classical integer-order model. Our simulations reveal that different fractional orders and weight functions significantly influence the fidelity of the model to experimental data, thereby justifying the use of ψ-Hilfer operators for physiological modeling.

The structure of the chapter is organized in the following manner: Section 5.2 presents the existence and uniqueness theorems using fixed point theory. Section 5.3 addresses the Ulam-Hyers and generalized Ulam-Hyers stability results. Section 5.4 provides a numerical scheme and applies the model to BAC data. At the end, the key findings are highlighted in Section 5.5.

5.2 Existence and Uniqueness of the Solution

The existence and uniqueness of solutions to the coupled system of ψ-Hilfer FDEs defined on the interval $J = [0, T]$ is investigated in this section. To achieve this, the problem is initially reformulated as an equivalent system of integral equations, that serves as the foundation for our analysis. For brevity in the forthcoming analysis, we introduce the notations

$$\Lambda_\psi^\gamma(t, 0) := (\psi(t) - \psi(0))^\gamma, \qquad \Theta_\psi^\gamma(t, s) := (\psi(t) - \psi(s))^\gamma \psi'(s).$$

Lemma 5.1. *The system (5.1) is equivalent to the following coupled system of integral equations:*

$$\begin{cases} u(t) = \dfrac{\Lambda_{\psi}^{\gamma-1}(t,0)}{\Gamma(\gamma)} u_0 + \mathcal{I}_{0+}^{\alpha;\psi} P(t,u(t),v(t))(t), \\ v(t) = \dfrac{\Lambda_{\psi}^{\gamma-1}(t,0)}{\Gamma(\gamma)} v_0 + \mathcal{I}_{0+}^{\alpha;\psi} Q(t,u(t),v(t))(t). \end{cases} \tag{5.3}$$

Proof. Applying the ψ-Riemann-Liouville fractional integral operator $\mathcal{I}_{0+}^{\alpha;\psi}$ to the first two equations of system (5.1), and using Lemma 1.4, we have

$$I_{0+}^{\alpha;\psi}\, {}^{H}D_{0+}^{\alpha,\beta;\psi} u(t) = u(t) - \frac{\Lambda_{\psi}^{\gamma-1}(t,0)}{\Gamma(\gamma)} \mathcal{I}_{0+}^{1-\gamma;\psi} u(0).$$

We immediately obtain the system (5.3). □

5.2.1 Existence via Banach's Fixed Point Theorem

To proceed with establishing the existence and uniqueness theorem for our problem, we make the following assumption.

Hypothesis 5.1. *The functions $P, Q : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the Lipschitz conditions: there exist constants $L_P, L_Q > 0$ such that for all $t \in J$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$,*

$$|P(t,u_1,v_1) - P(t,u_2,v_2)| \le L_P(|u_1 - u_2| + |v_1 - v_2|),$$
$$|Q(t,u_1,v_1) - Q(t,u_2,v_2)| \le L_Q(|u_1 - u_2| + |v_1 - v_2|).$$

With this, we now state and establish the main result as follows.

Theorem 5.1. *Let Hypothesis 5.1 hold. If*

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} (\psi(T) - \psi(0))^{\alpha} (L_P + L_Q) < 1, \tag{5.4}$$

then the system (5.1) possesses a unique solution $(u,v) \in E \times E$.

Proof. Define the operator $\mathcal{T} : E \times E \to E \times E$:

$$\mathcal{T}(u,v)(t) = \left(\mathcal{T}_1(u,v)(t), \mathcal{T}_2(u,v)(t)\right),$$

with

$$\mathcal{T}_1(u,v)(t) = \frac{\Lambda_{\psi}^{\gamma-1}(t,0)}{\Gamma(\gamma)} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \Theta_{\psi}^{\alpha-1}(t,s) P(s,u(s),v(s))\, ds,$$

and

$$\mathcal{T}_2(u,v)(t) = \frac{\Lambda_\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)} v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \Theta_\psi^{\alpha-1}(t,s) Q(s,u(s),v(s))\, ds.$$

Let $(u_1,v_1),(u_2,v_2) \in E \times E$. Then, for $t \in J$, we obtain

$$\begin{aligned}
&\Lambda_\psi^{1-\gamma}(t,0)|\mathcal{T}_1(u_1,v_1)(t) - \mathcal{T}_1(u_2,v_2)(t)| \\
&\leq \Lambda_\psi^{1-\gamma}(t,0)\frac{1}{\Gamma(\alpha)} \int_0^t \Theta_\psi^{\alpha-1}(t,s)\, |P(s,u_1(s),v_1(s)) - P(s,u_2(s),v_2(s))|\, ds \\
&\leq \Lambda_\psi^{1-\gamma}(t,0)\frac{L_P}{\Gamma(\alpha)} \int_0^t \Theta_\psi^{\alpha-1}(t,s)\, (|u_1(s) - u_2(s)| + |v_1(s) - v_2(s)|)\, ds \\
&\leq \Lambda_\psi^{1-\gamma}(t,0)\frac{L_P}{\Gamma(\alpha)} (\|u_1 - u_2\|_E + \|v_1 - v_2\|_E) \int_0^t \Theta_\psi^{\alpha-1}(t,s)\Lambda_\psi^{\gamma-1}(s,0)ds.
\end{aligned}$$

Use of the identity for the ψ-Riemann-Liouville fractional integral of a power function given by

$$\frac{1}{\Gamma(\alpha)} \int_0^t \Theta_\psi^{\alpha-1}(t,s)\Lambda_\psi^{\gamma-1}(s,0)ds = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\Lambda_\psi^{\alpha+\gamma-1}(t,0)$$

yields

$$\begin{aligned}
&\Lambda_\psi^{1-\gamma}(t,0)|\mathcal{T}_1(u_1,v_1)(t) - \mathcal{T}_1(u_2,v_2)(t)| \\
&\leq L_P\left(\|u_1 - u_2\|_E + \|v_1 - v_2\|_E\right)\Lambda_\psi^{1-\gamma}(t,0)\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\Lambda_\psi^{\alpha+\gamma-1}(t,0) \\
&= L_P\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(\psi(t) - \psi(0))^\alpha \left(\|u_1 - u_2\|_E + \|v_1 - v_2\|_E\right).
\end{aligned}$$

Taking the supremum over $t \in J$, we obtain

$$\begin{aligned}
&\|\mathcal{T}_1(u_1,v_1) - \mathcal{T}_1(u_2,v_2)\|_E \\
&\leq L_P\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(\psi(T) - \psi(0))^\alpha \left(\|u_1 - u_2\|_E + \|v_1 - v_2\|_E\right).
\end{aligned}$$

Let $K = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(\psi(T) - \psi(0))^\alpha$. Then, we have

$$\|\mathcal{T}_1(u_1,v_1) - \mathcal{T}_1(u_2,v_2)\|_E \leq L_P K\left(\|u_1 - u_2\|_E + \|v_1 - v_2\|_E\right).$$

Similarly, for $\mathcal{T}_2$, we obtain the following:

$$\|\mathcal{T}_2(u_1,v_1) - \mathcal{T}_2(u_2,v_2)\|_E \leq L_Q K\left(\|u_1 - u_2\|_E + \|v_1 - v_2\|_E\right).$$

Now, consider the norm in the product space $E \times E$ given by

$$\begin{aligned}
&\|\mathcal{T}(u_1, v_1) - \mathcal{T}(u_2, v_2)\|_{E\times E} \\
&= \|\mathcal{T}_1(u_1, v_1) - \mathcal{T}_1(u_2, v_2)\|_E + \|\mathcal{T}_2(u_1, v_1) - \mathcal{T}_2(u_2, v_2)\|_E \\
&\leq (L_P K + L_Q K) (\|u_1 - u_2\|_E + \|v_1 - v_2\|_E) \\
&= (L_P + L_Q) K \|(u_1 - u_2, v_1 - v_2)\|_{E\times E}.
\end{aligned}$$

So, the Lipschitz constant for $\mathcal{T}$ is $L = (L_P + L_Q)K = (L_P + L_Q)\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(\psi(T) - \psi(0))^\alpha$.

By assumption (5.4), we have $L < 1$. Consequently, we can conclude that $\mathcal{T}$ is a contraction on the Banach space $E \times E$, and hence, by Banach's fixed point theorem, there exists a unique fixed point $(u, v) \in E \times E$ as the unique solution of the system (5.1). □

5.2.2 Existence via Schauder's Fixed Point Theorem

Next, the existence of at least one solution to the system (5.1) using Schauder's fixed point theorem is established. We make the following hypothesis to obtain the result.

Hypothesis 5.2. *The functions $P, Q : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, and there exist continuous functions $h_1, h_2 : J \to [0, \infty)$ and a constant $M > 0$ such that, for all $t \in J$ and $|x|, |y| \leq M$, the following inequalities hold:*

$$|P(t, u, v)| \leq h_1(t),$$
$$|Q(t, u, v)| \leq h_2(t).$$

Theorem 5.2. *Assume that Hypothesis 5.2 holds and that $\psi \in C^1(J, \mathbb{R})$ is strictly increasing with $\psi'(t) \neq 0$ on J. Then, the system (5.1) has at least one solution $(u, v) \in E \times E$.*

Proof. Define the operator $\mathcal{T} : E \times E \to E \times E$:

$$\mathcal{T}(u, v)(t) = (\mathcal{T}_1(u, v)(t), \mathcal{T}_2(u, v)(t)),$$

with

$$\mathcal{T}_1(u, v)(t) = \frac{\Lambda_\psi^{\gamma-1}(t, 0)}{\Gamma(\gamma)} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \Theta_\psi^{\alpha-1}(t, s) P(s, u(s), v(s))\, ds,$$

and

$$\mathcal{T}_2(u, v)(t) = \frac{\Lambda_\psi^{\gamma-1}(t, 0)}{\Gamma(\gamma)} v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \Theta_\psi^{\alpha-1}(t, s) Q(s, u(s), v(s))\, ds.$$

We verify the conditions of Schauder's fixed point theorem.

Step 1: Selection of Subset $\mathcal{B}_\psi$.

Let $\mathcal{B}_\psi$ be the closed, bounded, and convex subset of $E \times E$ defined by

$$\mathcal{B}_\psi := \{(u, v) \in E \times E : \|x\|_E \leq M, \|y\|_E \leq M\},$$

where $M > 0$ is chosen such that

$$M \geq \max\left\{\frac{|u_0|}{\Gamma(\gamma)} + \frac{\|h_1\|_\infty}{\Gamma(\alpha+1)}\Lambda_\psi^{\alpha+1-\gamma}(T,0), \frac{|v_0|}{\Gamma(\gamma)} + \frac{\|h_2\|_\infty}{\Gamma(\alpha+1)}\Lambda_\psi^{\alpha+1-\gamma}(T,0)\right\}.$$

Step 2: Invariance of $\mathcal{B}_\psi$ under $\mathcal{T}$.

For any $(u, v) \in \mathcal{B}_\psi$, we estimate $\mathcal{T}_1(u, v)$ in the norm of E:

$$|\Lambda_\psi^{1-\gamma}(t,0)\mathcal{T}_1(u,v)(t)| \leq \frac{|u_0|}{\Gamma(\gamma)} + \frac{\|h_1\|_\infty}{\Gamma(\alpha+1)}\Lambda_\psi^{\alpha+1-\gamma}(t,0).$$

Taking the supremum over $t \in J$ yields

$$\|\mathcal{T}_1(u,v)\|_E \leq \frac{|u_0|}{\Gamma(\gamma)} + \frac{\|h_1\|_\infty}{\Gamma(\alpha+1)}\Lambda_\psi^{\alpha+1-\gamma}(T,0) \leq M.$$

Similarly,

$$\|\mathcal{T}_2(u,v)\|_E \leq \frac{|v_0|}{\Gamma(\gamma)} + \frac{\|h_2\|_\infty}{\Gamma(\alpha+1)}\Lambda_\psi^{\alpha+1-\gamma}(T,0) \leq M.$$

Thus, $\mathcal{T}(\mathcal{B}_\psi) \subset \mathcal{B}_\psi$.

Step 3: Continuity of $\mathcal{T}$.

Consider a sequence (u_n, v_n) in $\mathcal{B}_\psi$ that follows

$$(u_n, v_n) \to (u, v) \quad \text{in} \quad E \times E.$$

Since P and Q are continuous and (u_n, v_n) converges uniformly to (u, v), it is clear that

$$P(t, u_n(t), v_n(t)) \to P(t, u(t), v(t)),$$
$$Q(t, u_n(t), v_n(t)) \to Q(t, u(t), v(t)),$$

uniformly on J.

Now, consider $\mathcal{T}_1(u_n, v_n)$:

$$|\Lambda_\psi^{1-\gamma}(t,0)(\mathcal{T}_1(u_n,v_n)(t) - \mathcal{T}_1(u,v)(t))|$$
$$\leq \frac{\Lambda_\psi^{1-\gamma}(t,0)}{\Gamma(\alpha)}\int_0^t \Theta_\psi^{\alpha-1}(t,s)|P(s,u_n(s),v_n(s)) - P(s,u(s),v(s))|\,ds.$$

By uniform convergence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$|P(s, u_n(s), v_n(s)) - P(s, u(s), v(s))| < \varepsilon \quad \forall s \in J.$$

Thus,

$$\|\mathcal{T}_1(u_n, v_n) - \mathcal{T}_1(u, v)\|_E \leq \frac{\varepsilon}{\Gamma(\alpha+1)} \Lambda_\psi^{\alpha+1-\gamma}(T, 0).$$

Similarly,

$$\|\mathcal{T}_2(u_n, v_n) - \mathcal{T}_2(u, v)\|_E \leq \frac{\varepsilon}{\Gamma(\alpha+1)} \Lambda_\psi^{\alpha+1-\gamma}(T, 0).$$

Hence, $\mathcal{T}(u_n, v_n) \to \mathcal{T}(u, v)$ in $E \times E$, establishing its continuity.

Step 4: Compactness of $\mathcal{T}$.

We show that $\mathcal{T}$ is compact via the Arzelà–Ascoli theorem. From **Step 2**, it follows that $\mathcal{T}(\mathcal{B}_\psi)$ is uniformly bounded. Moreover, to verify equicontinuity, let $t_1, t_2 \in J$ with $t_1 < t_2$. We estimate

$$\begin{aligned} &|\Lambda_\psi^{1-\gamma}(t_2, 0)\mathcal{T}_1(u, v)(t_2) - \Lambda_\psi^{1-\gamma}(t_1, 0)\mathcal{T}_1(u, v)(t_1)| \\ &\leq \frac{\|h_1\|_\infty}{\Gamma(\alpha+1)} \left| \Lambda_\psi^{\alpha+1-\gamma}(t_2, 0) - \Lambda_\psi^{\alpha+1-\gamma}(t_1, 0) \right|. \end{aligned} \tag{5.5}$$

By the continuity of ψ, the right-hand side in (5.5) tends to zero as $t_2 \to t_1$. The same holds for $\mathcal{T}_2$ too.

Therefore, $\mathcal{T}(\mathcal{B}_\psi)$ is relatively compact in $E \times E$; so $\mathcal{T}$ is a compact operator. Since $\mathcal{T}$ is continuous and $\mathcal{B}_\psi \subset E \times E$ is nonempty, closed, bounded, and convex with $\mathcal{T}(\mathcal{B}_\psi) \subset \mathcal{B}_\psi$, Schauder's fixed point theorem guarantees the existence of $(u, v) \in \mathcal{B}_\psi$ such that $\mathcal{T}(u, v) = (u, v)$. Consequently, system (5.1) admits at least one solution in $E \times E$. □

5.3 Stability Results

In this section, we delve into the UH and GUH stability of the coupled system of nonlinear ψ-Hilfer FDEs. Stability analysis is crucial for understanding how small perturbations in the equations or initial conditions affect the solutions.

Definition 5.1. *The coupled ψ-Hilfer fractional system*

$$\begin{cases} {}^H D_{0^+}^{\alpha,\beta;\psi} u(t) = P(t, u(t), v(t)), & t \in [0, T], \\ {}^H D_{0^+}^{\alpha,\beta;\psi} v(t) = Q(t, u(t), v(t)), & t \in [0, T], \end{cases} \tag{5.6}$$

with initial conditions

$$I_{0^+}^{1-\gamma;\psi} u(0^+) = u_0, \quad \mathcal{I}_{0^+}^{1-\gamma;\psi} v(0^+) = v_0, \tag{5.7}$$

where $\gamma = \alpha + \beta - \alpha\beta$, is said to be Ulam-Hyers stable in $E \times E$ if there exists a constant $C_u = (c_1, c_2) > 0$ such that, for some $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and for all approximate solutions $(\tilde{u}, \tilde{v}) \in E \times E$ satisfying

$$\begin{cases} \left| {}^{H}D_{0+}^{\alpha,\beta;\psi}\tilde{u}(t) - P(t, \tilde{u}(t), \tilde{v}(t)) \right| \leq \epsilon_1, & t \in [0, T], \\ \left| {}^{H}D_{0+}^{\alpha,\beta;\psi}\tilde{v}(t) - Q(t, \tilde{u}(t), \tilde{v}(t)) \right| \leq \epsilon_2, & t \in [0, T], \end{cases} \tag{5.8}$$

there exists a unique exact solution $(u, v) \in E \times E$ of system (5.1) such that

$$\|(\tilde{u}, \tilde{v}) - (u, v)\|_{E \times E} \leq C_u \epsilon. \tag{5.9}$$

Definition 5.2. *The coupled system is said to be generalized Ulam-Hyers stable in $E \times E$ if there exists a continuous function*

$$\Phi \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+), \quad \Phi(0, 0) = (0, 0),$$

such that, for every $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and for all approximate solutions $(\tilde{u}, \tilde{v}) \in E \times E$ satisfying the inequalities in Definition 5.1, there exists an exact solution $(u, v) \in E \times E$ of the system with

$$\|(\tilde{u}, \tilde{v}) - (u, v)\|_{E \times E} \leq \Phi(\epsilon). \tag{5.10}$$

Remark 5.1. *The coupled ψ-Hilfer fractional system (5.1) admits an approximate solution $(\tilde{u}, \tilde{v}) \in E \times E$ if and only if there exist perturbation functions $\mu_1, \mu_2 \in C([0, T], \mathbb{R})$ such that*

(i) $|\mu_1(t)| \leq \epsilon_1$ and $|\mu_2(t)| \leq \epsilon_2$, for all $t \in [0, T]$,

(ii) for $t \in [0, T]$, we have

$$\begin{cases} {}^{H}D_{0+}^{\alpha,\beta;\psi}\tilde{u}(t) = P(t, \tilde{u}(t), \tilde{v}(t)) + \mu_1(t), \\ {}^{H}D_{0+}^{\alpha,\beta;\psi}\tilde{v}(t) = Q(t, \tilde{u}(t), \tilde{v}(t)) + \mu_2(t). \end{cases}$$

Theorem 5.3. *Assume that Hypothesis 5.1 holds and condition (5.4) is satisfied. Then, the coupled ψ-Hilfer fractional system (5.1) is both Ulam–Hyers stable and generalized Ulam–Hyers stable in $E \times E$.*

Proof. Let (u, v) denote the exact solution of the system and let $(\tilde{u}, \tilde{v}) \in E \times E$ be an approximate solution satisfying

$$\left| {}^{H}D_{0+}^{\alpha,\beta;\psi}\tilde{u}(t) - P(t, \tilde{u}(t), \tilde{v}(t)) \right| \leq \epsilon_1,$$
$$\left| {}^{H}D_{0+}^{\alpha,\beta;\psi}\tilde{v}(t) - Q(t, \tilde{u}(t), \tilde{v}(t)) \right| \leq \epsilon_2,$$

with $\mathcal{I}_{0+}^{1-\gamma;\psi}\tilde{u}(0^+) = u_0$ and $\mathcal{I}_{0+}^{1-\gamma;\psi}\tilde{v}(0^+) = v_0$.

By Remark 5.1, there exist perturbation functions $\mu_1, \mu_2 \in C([0,T],\mathbb{R})$ such that

$$\begin{cases} {}^H D_{0+}^{\alpha,\beta;\psi}\tilde{u}(t) = P(t,\tilde{u}(t),\tilde{v}(t)) + \mu_1(t), \\ {}^H D_{0+}^{\alpha,\beta;\psi}\tilde{v}(t) = Q(t,\tilde{u}(t),\tilde{v}(t)) + \mu_2(t), \end{cases} \tag{5.11}$$

with $|\mu_i(t)| \le \epsilon_i$ for $i = 1,2$ and all $t \in [0,T]$.

Applying the fractional integral operator $\mathcal{I}_{0+}^{\alpha;\psi}$ to (5.11), we obtain

$$\tilde{u}(t) = \frac{\Lambda_\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}u_0 + \mathcal{I}_{0+}^{\alpha;\psi}P(t,\tilde{u}(t),\tilde{v}(t)) + \mathcal{I}_{0+}^{\alpha;\psi}\mu_1(t), \tag{5.12}$$

$$\tilde{v}(t) = \frac{\Lambda_\psi^{\gamma-1}(t,0)}{\Gamma(\gamma)}v_0 + \mathcal{I}_{0+}^{\alpha;\psi}Q(t,\tilde{u}(t),\tilde{v}(t)) + \mathcal{I}_{0+}^{\alpha;\psi}\mu_2(t). \tag{5.13}$$

Subtracting the above two from the corresponding integral equations (5.3) of the exact solution leads, respectively, to the estimates

$$|u(t) - \tilde{u}(t)| \le \mathcal{I}_{0+}^{\alpha;\psi}\,|P(t,u(t),v(t)) - P(t,\tilde{u}(t),\tilde{v}(t))| + \mathcal{I}_{0+}^{\alpha;\psi}|\mu_1(t)|,$$
$$|v(t) - \tilde{v}(t)| \le \mathcal{I}_{0+}^{\alpha;\psi}\,|Q(t,u(t),v(t)) - Q(t,\tilde{u}(t),\tilde{v}(t))| + \mathcal{I}_{0+}^{\alpha;\psi}|\mu_2(t)|.$$

Using the Lipschitz continuity of P and Q, the following integral inequality is obtained:

$$\begin{aligned} &|u(t) - \tilde{u}(t)| \\ &\le \frac{L_P}{\Gamma(\alpha)}\int_0^t \Theta_\psi^{\alpha-1}(t,s)\Big(|u(s) - \tilde{u}(s)| + |v(s) - \tilde{v}(s)|\Big)\,ds + \frac{\epsilon_1}{\Gamma(\alpha)}\int_0^t \Theta_\psi^{\alpha-1}(t,s)ds. \end{aligned} \tag{5.14}$$

Multiplying (5.14) by $(\psi(t) - \psi(0))^{1-\gamma}$ and then taking suprema yields

$$\|u - \tilde{u}\|_E \le \frac{L_P\Lambda_\psi^{\alpha+1-\gamma}(T,0)}{\Gamma(\alpha+1)}\Big(\|u - \tilde{u}\|_E + \|v - \tilde{v}\|_E\Big) + \frac{\epsilon_1\Lambda_\psi^{\alpha+1-\gamma}(T,0)}{\Gamma(\alpha+1)}.$$

Proceeding analogously for v, we obtain the coupled estimate

$$\left(1 - \frac{(L_P + L_Q)\Lambda_\psi^{\alpha+1-\gamma}(T,0)}{\Gamma(\alpha+1)}\right)\|(\tilde{u},\tilde{v}) - (u,v)\|_{E\times E} \le \frac{(\epsilon_1+\epsilon_2)\Lambda_\psi^{\alpha+1-\gamma}(T,0)}{\Gamma(\alpha+1)}.$$

It follows that the exact Ulam-Hyers stability constant is

$$C_u = \frac{\Lambda_\psi^{\alpha+1-\gamma}(T,0)}{\Gamma(\alpha+1) - (L_P + L_Q)\Lambda_\psi^{\alpha+1-\gamma}(T,0)}. \tag{5.15}$$

Therefore, the system (5.1) is Ulam-Hyers stable with the above Ulam-Hyers stability constant. Moreover, since the above estimate can be put in the form

$$\|(\tilde{u},\tilde{v})-(u,v)\|_{E\times E}\leq \Phi(\epsilon),$$

with $\Phi(\epsilon)=C_u\epsilon$ and $\Phi(0,0)=(0,0)$, the system is also generalized Ulam-Hyers stable. □

Example 5.1. *Consider the following coupled system of ψ-Hilfer FDEs:*

$$\begin{cases} {}^H D_{0+}^{\alpha,\beta;\psi}u(t)=\dfrac{e^{-t}}{2}+\dfrac{1}{4}u(t)+\dfrac{1}{6}v(t), \\ {}^H D_{0+}^{\alpha,\beta;\psi}v(t)=\sqrt{t}+\dfrac{1}{8}u(t)+\dfrac{1}{5}v(t), \end{cases} \quad t\in J=[0,1], \tag{5.16}$$

subject to the initial conditions

$$I_{0+}^{1-\gamma;\psi}u(0)=0, \qquad I_{0+}^{1-\gamma;\psi}v(0)=0,$$

where the weight function is chosen as $\psi(t)=t$, with

$$\alpha=\tfrac{2}{3}, \qquad \beta=\tfrac{1}{3}, \qquad \gamma=\alpha+\beta(1-\alpha).$$

Define

$$P(t,u,v)=\frac{e^{-t}}{2}+\frac{1}{4}u+\frac{1}{6}v, \qquad Q(t,u,v)=\sqrt{t}+\frac{1}{8}u+\frac{1}{5}v.$$

For any $t\in J$ and $u_1,u_2,v_1,v_2\in\mathbb{R}$,

$$|P(t,u_1,v_1)-P(t,u_2,v_2)|\leq \tfrac{1}{4}|u_1-u_2|+\tfrac{1}{6}|v_1-v_2|,$$
$$|Q(t,u_1,v_1)-Q(t,u_2,v_2)|\leq \tfrac{1}{8}|u_1-u_2|+\tfrac{1}{5}|v_1-v_2|.$$

Adding these inequalities gives

$$|P(t,u_1,v_1)-P(t,u_2,v_2)|+|Q(t,u_1,v_1)-Q(t,u_2,v_2)| \leq \tfrac{9}{20}\Big(|u_1-u_2|+|v_1-v_2|\Big).$$

Thus, P and Q are globally Lipschitz with constant $L=\frac{9}{20}$. For $\psi(t)=t$, $T=1$, and $\gamma=\frac{2}{3}+\frac{1}{3}(1-\frac{2}{3})=\frac{7}{9}$, the Banach contraction condition is

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\Lambda_{\psi}^{\alpha}(T,0)L=\frac{\Gamma(\frac{7}{9})}{\Gamma(\frac{13}{9})}\cdot\frac{9}{20}<1.$$

Since this inequality is satisfied, hence, by Banach's fixed point theorem, the system admits a unique solution on J. Assume $(\tilde{u}(t), \tilde{v}(t))$ is an approximate solution satisfying

$$\left| {}^{H}D_{0+}^{\alpha,\beta;\psi}\tilde{u}(t) - P(t, \tilde{u}(t), \tilde{v}(t)) \right| \leq \varepsilon_1,$$
$$\left| {}^{H}D_{0+}^{\alpha,\beta;\psi}\tilde{v}(t) - Q(t, \tilde{u}(t), \tilde{v}(t)) \right| \leq \varepsilon_2,$$

for all $t \in [0,1]$. Since $\psi(t) = t$ is continuous and non-decreasing on $[0,1]$, the Ulam-Hyers stability result is applicable. Hence, system (5.16) admits a unique solution $(u(t), v(t))$ which is both Ulam-Hyers and generalized Ulam-Hyers stable. Moreover, the associated stability constant is explicitly given by

$$C_u = \frac{\Lambda_{\psi}^{\alpha+1-\gamma}(T,0)}{\Gamma(\alpha+1) - (L_P + L_Q)\,\Lambda_{\psi}^{\alpha+1-\gamma}(T,0)} = \frac{1}{\Gamma(5/3) - \frac{9}{20}}.$$

5.4 Numerical Approximation and Application to Blood Alcohol Dynamics

In this section, we demonstrate the applicability of the proposed coupled ψ-Hilfer fractional differential model to a real-world biological process: the dynamics of blood alcohol concentration (BAC). The primary objective is to illustrate the superior performance and flexibility of the fractional framework compared to the classical integer-order model.

5.4.1 Classical Integer-Order Model

The standard approach for modeling alcohol concentration in the body is based on first-order kinetic reactions, which describe the alcohol concentrations $A(t)$ in the stomach and $B(t)$ in the blood, as follows [38]:

$$\frac{dA(t)}{dt} = -k_1 A(t), \quad A(0) = 245.8769,$$
$$\frac{dB(t)}{dt} = k_1 A(t) - k_2 B(t), \quad B(0) = 0.$$

Here, k_1 represents the absorption rate of alcohol from the stomach into the bloodstream, and k_2 denotes the elimination rate of alcohol from the blood. The experimental data for BAC reported in [38] are reproduced in Table 5.1.

Time (min)	0	10	20	30	45	80	90	110	170
Experiment (mg/L)	0	150	200	160	130	70	60	40	20

Table 5.1: Experimental data for blood alcohol concentration (mg/L) from [38].

For this model, the best-fit parameters are $k_1 = 0.109456$ min^{-1} and $k_2 = 0.017727$ min^{-1}, which yield closed-form analytical solutions for $A(t)$ and $B(t)$. Figure 5.1 compares the model predictions with the experimental data. While the classical ODE system provides a reasonable approximation, discrepancies are observed; particularly during the absorption phase (10–20 minutes) and around the peak concentration.

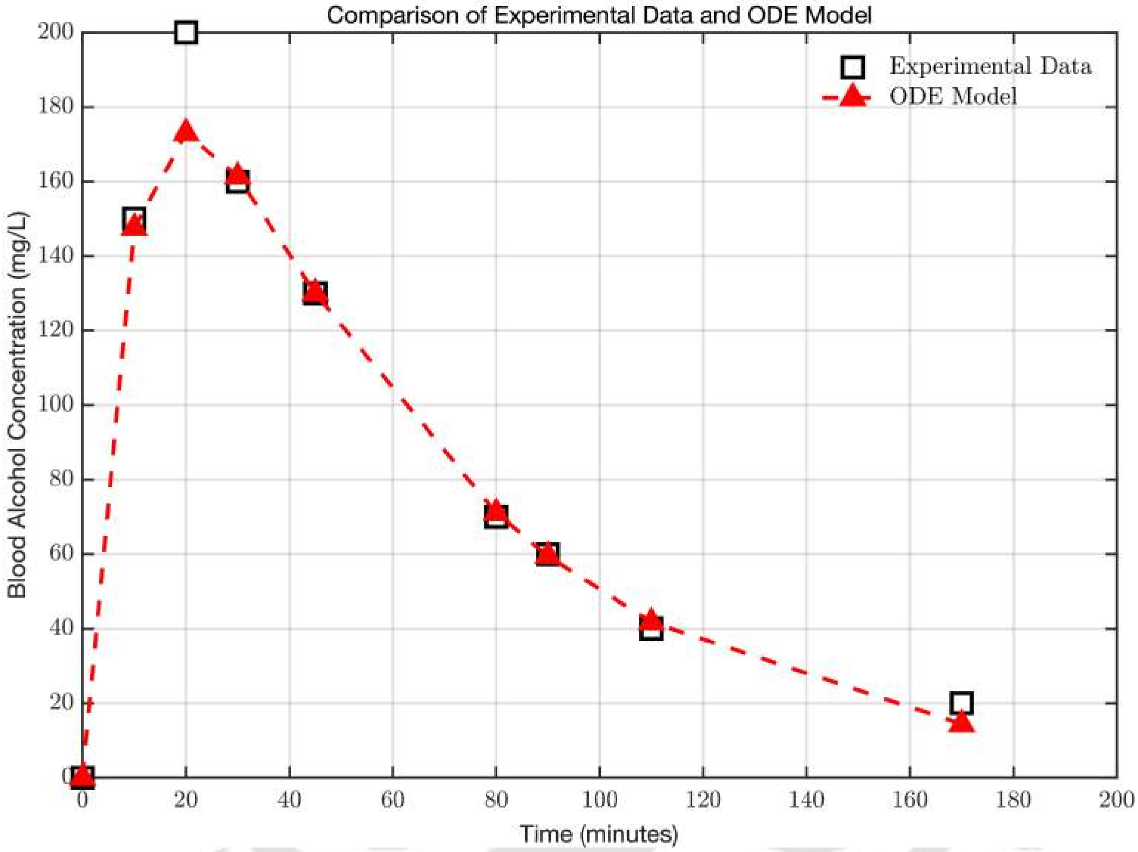


Figure 5.1: Comparison of experimental BAC data with predictions from the classical integer-order ODE model.

5.4.2 ψ-Hilfer Fractional Model and Numerical Approximation

We generalize the classical system by incorporating the ψ-Hilfer fractional derivative, which effectively captures the nonlocal and memory-dependent dynamics of biological processes. The coupled fractional model is formulated as

$$ {}^{H}D_{0+}^{\alpha,\beta;\psi}u(t) = -k_1 u(t), \tag{5.17} $$

$$ {}^{H}D_{0+}^{\alpha,\beta;\psi}v(t) = k_1 u(t) - k_2 v(t), \tag{5.18} $$

where $u(t)$ and $v(t)$ denote the fractional analogues of $A(t)$ and $B(t)$, respectively. The parameters include the fractional order $\alpha \in (0,1)$, the type $\beta \in [0,1]$, and a strictly increasing weight function $\psi(t)$. The associated nonlocal initial conditions are

$$I_{0^+}^{1-\gamma;\psi} u(t)\Big|_{t=0} = u_0, \quad I_{0^+}^{1-\gamma;\psi} v(t)\Big|_{t=0} = v_0, \tag{5.19}$$

where $\gamma = \alpha + \beta - \alpha\beta$.

Using the fractional integral operator, the coupled system (5.17)–(5.18) admits the equivalent integral form

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) \Big(\psi(t) - \psi(s)\Big)^{\alpha-1} \Big(- k_1 u(s)\Big)\, ds, \tag{5.20}$$

$$v(t) = v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) \Big(\psi(t) - \psi(s)\Big)^{\alpha-1} \Big(k_1 u(s) - k_2 v(s)\Big)\, ds. \tag{5.21}$$

For the numerical approximation, we divide the interval $[0, T]$ into N sub-intervals with step size $h = T/N$ and denote the grid points by $t_n = nh$, $n = 0, 1, \ldots, N$. Applying a finite difference discretization to the above integral form, we obtain the iterative scheme

$$u(t_{n+1}) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n+1} \Big(- k_1 u(t_k)\Big)\, C^{(\alpha)}_{n+1,k}, \tag{5.22}$$

$$v(t_{n+1}) = v_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n+1} \Big(k_1 u(t_k) - k_2 v(t_k)\Big)\, C^{(\alpha)}_{n+1,k}, \tag{5.23}$$

where the fractional weights for the ψ-Hilfer kernel are defined as

$$C^{(\alpha)}_{n+1,k} = \frac{1}{\Gamma(\alpha+1)} \Big[\Big(\psi(t_{n+1}) - \psi(t_{k-1})\Big)^{\alpha} - \Big(\psi(t_{n+1}) - \psi(t_k)\Big)^{\alpha} \Big],$$

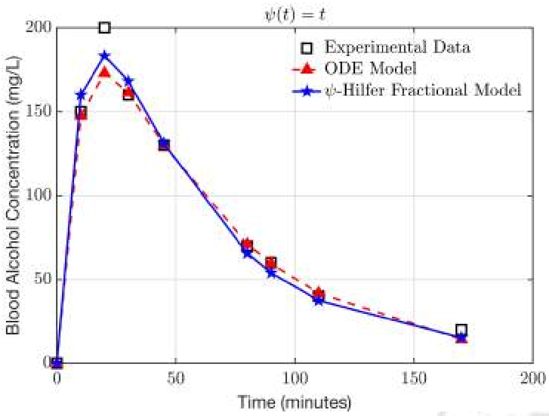
$k = 1, 2, \ldots, n+1.$

The recursive relations (5.22)–(5.23) generate the approximate solutions $u(t_n)$ and $v(t_n)$ step by step, and thus provide a practical numerical method for studying the dynamics of the coupled ψ-Hilfer fractional system. We examine the performance of the fractional model under different choices of the weight function $\psi(t)$. For comparison, the classical ODE model yields SSE = 775.1745. The optimal parameters and corresponding error indices - sum of squared errors (SSE) - for the fractional cases are produced in Table 5.2. Figure 5.2 illustrates the comparison of experimental BAC data with numerical solutions

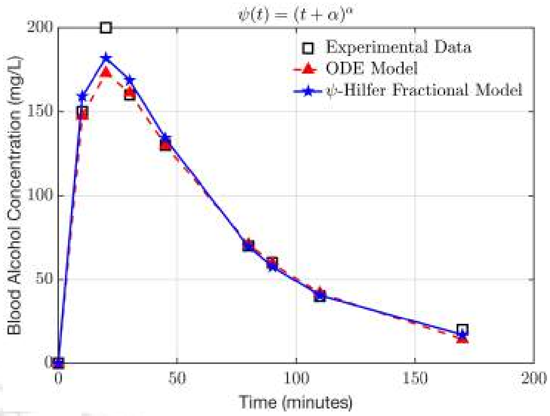
Choice of $\psi(t)$	α	k_1	k_2	a_0	a_1	u_0	SSE
$\psi(t) = t$	0.9342	0.036414	0.100443	–	–	938.3509	536.1602
$\psi(t) = (t+\alpha)^{\alpha}$	0.9342	0.052688	0.103654	–	–	751.0538	521.6371
$\psi(t) = a_0 + a_1 t$	0.9342	0.032274	0.097205	0	1.0571	991.085	486.4897

Table 5.2: Optimal parameters and performance metrics for different choices of $\psi(t)$.

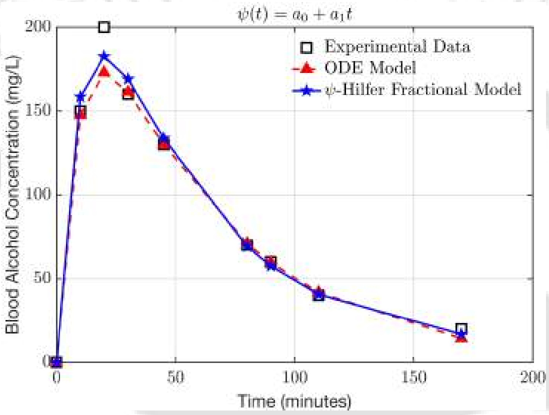
of the fractional model under three choices of $\psi(t)$.



(a) $\psi(t) = t$



(b) $\psi(t) = (t+\alpha)^\alpha$



(c) $\psi(t) = a_0 + a_1 t$

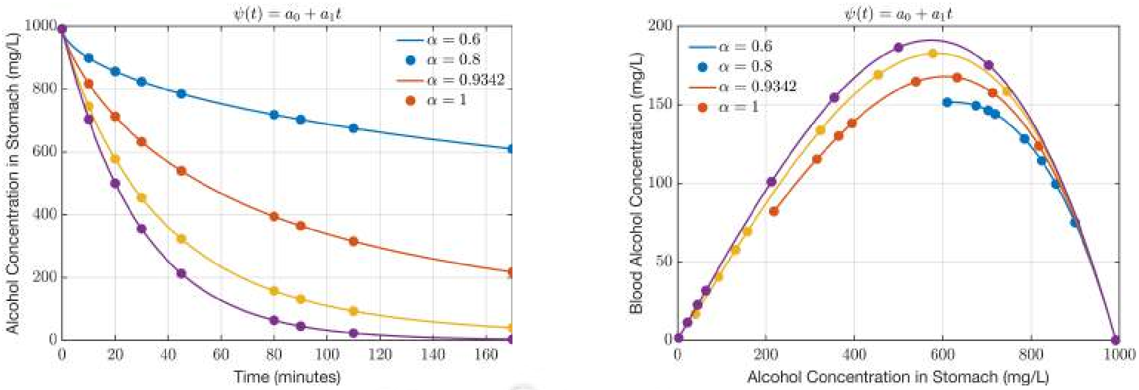
Figure 5.2: Comparison of experimental BAC data with numerical solutions of the fractional model for different weight functions $\psi(t)$.

5.4.3 Discussion of Numerical Result

The comparative analysis highlights the advantages of the proposed fractional framework over the classical integer-order model. As shown in Figure 5.1, the ODE model reproduces the overall behavior of BAC, but fails to accurately capture the absorption phase (10–20 minutes) and the peak concentration region, leading to noticeable discrepancies. These limitations arise from the local and memory less nature of the integer-order formulation.

In contrast, the ψ-Hilfer fractional model (Figure 5.2) demonstrates a significantly improved fit across all phases of blood alcohol dynamics. The incorporation of fractional order α and the weight function $\psi(t)$ allows the system to account for memory effects and heterogeneous dynamics inherent in biological processes. Among the considered choices of ψ, the generalized linear form $\psi(t) = a_0 + a_1 t$ yields the closest agreement with experimental data, as reflected by the lowest SSE in Table 5.2.

Figure 5.3 provides further insight into the internal dynamics of the fractional model. Figure 5.3(a) illustrates how smaller values of α slow down the rate of alcohol absorption



(a) Alcohol concentration in stomach for different fractional orders α

(b) Phase plot for different fractional orders α

Figure 5.3: Numerical results of the ψ-Hilfer fractional alcohol model with $\psi(t) = a_0 + a_1 t$, showing the influence of the fractional order α on absorption and transfer dynamics.

from the stomach, producing a more gradual decline in concentration that captures long-term memory behavior. Figure 5.3(b) presents the corresponding blood–stomach phase trajectories, where decreasing α leads to smoother and delayed transfer of alcohol between compartments. These results reflect physiologically realistic kinetics, showing that the ψ-Hilfer derivative effectively models the memory-driven interplay between absorption and elimination.

Previous studies have employed alternative fractional operators such as the Caputo, Caputo–Fabrizio, or Atangana–Baleanu derivatives [53, 72]. Some of these approaches introduced different fractional orders for each compartment to enhance flexibility. While this can yield additional degrees of freedom, it often complicates the theoretical analysis and parameter estimation. In the present work, a unified fractional order α is applied to both compartments, ensuring analytical consistency and interpretability while still achieving superior accuracy compared to the classical ODE model.

Overall, Figures 5.2 and 5.3 confirm that the ψ-Hilfer formulation offers a coherent and robust extension of the standard alcohol absorption model. It provides a physiologically meaningful representation of memory-driven effects, leading to improved correspondence with experimental observations and reduced model error metrics.

5.5 Conclusion

In this chapter, we studied a coupled system of ψ-Hilfer fractional differential equations. We first demonstrated the existence and uniqueness results using Banach's contraction principle and further proved the existence of solutions via Schauder's fixed point theorem, providing a solid theoretical foundation. We then investigated the stability properties by establishing both Ulam–Hyers and generalized Ulam–Hyers stability, demonstrating that the solutions remain robust under small perturbations and controlled deviations.

To demonstrate the practical relevance of our results, we developed a numerical method to approximate the solution of a ψ-Hilfer coupled system. As an application, we modeled blood alcohol concentration in the human body and compared the numerical outcomes with the experimental data and ODE model. The results showed that the fractional model provides a better fit to real data in comparison to the classical integer-order model. Moreover, the numerical simulations of the alcohol concentration in the stomach and the corresponding blood–stomach phase relation demonstrated that smaller fractional orders result in slowed absorption and delayed transfer dynamics, which is indicative of the memory-driven nature of the process.

CHAPTER 6

Memory-Driven Financial Chaos: Qualitative and Numerical Perspectives via the ψ–Hilfer Derivative

In this chapter, we consider a system of three ψ-Hilfer fractional differential equations. Within an appropriate functional framework, we establish the existence and uniqueness of solutions using fixed point techniques and analyze the stability of the system in the sense of Ulam–Hyers. The comprehensive numerical study is carried out for fractional financial chaotic model, revealing rich dynamical behaviors including chaotic, transitional, and stable under different parameter regimes and choices of the weight function ψ.

6.1 Introduction

Financial markets are fundamentally nonlinear, sensitive to small fluctuations, and prone to abrupt structural changes. Traditional integer-order models often fail to capture the persistent volatility, structural memory, and hereditary effects observed in real financial data. These limitations became especially evident during the 2008 global financial crisis during which classical economic models proved inadequate in predicting systemic contagion and sudden market transitions. This has motivated the development of more realistic mathematical frameworks capable of capturing long-range dependence and complex interactions within financial systems [6, 28, 31, 77, 83].

Since exact solutions of nonlinear fractional systems are rarely obtainable, numerical methods are essential [18, 35]. Fractional systems with generalized kernels require numerical schemes that preserve the non-local integral structure of the operator. In this regard, convolution-type methods based on the ψ-Riemann–Liouville integral formulation provide an effective computational approach and are well-suited for simulating complex memory-driven dynamics such as those arising in financial models.

Fractional chaotic systems with three and four state variables have been widely studied, exhibiting rich dynamical behaviors such as bifurcations, multistability, and transitions to chaos. Fractional financial models, in particular, have demonstrated that introducing memory can alter stability regions, dampen or enhance chaotic behavior, and fundamentally change long-term economic predictions [6, 28, 31]. Motivated by these developments, we study a ψ-Hilfer fractional system with three state variables, and as an application of our numerical framework, we apply this system to a nonlinear financial chaotic model involving the interaction among interest rate, investment demand, and price index.

The structure of this chapter is as follows: Section 6.2 establishes existence and uniqueness results utilizing Banach's fixed point theorem. Section 6.3 presents Ulam–Hyers stability results for ψ-Hilfer fractional system. Section 6.4 presents a numerical scheme adapted to the ψ-Hilfer operator and applies it to a chaotic financial model under different weight functions. Section 6.5 concludes the chapter.

6.2 Existence Result for ψ-Hilfer Fractional System

Let $J = [a, b]$ be a finite interval on $\mathbb{R}$. The space of all continuous functions $\mathcal{F}$ on J is denoted by $C(J, \mathbb{R})$. Let $\psi \in C^1(J, \mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in J$. Let $C(J, \mathbb{R}^3)$ denote the Banach space of continuous functions $\mathcal{X}(t) = (u(t), v(t), w(t))^\top$, endowed with the norm

$$\|\mathcal{X}\| = \max_{t\in J} |u(t)| + \max_{t\in J} |v(t)| + \max_{t\in J} |w(t)|. \tag{6.1}$$

We consider a ψ-Hilfer fractional system with non-local initial conditions

$$\begin{cases} {}^{H}D_{a+}^{\alpha_1,\beta_1;\psi} u(t) = f_1\big(t, u(t), v(t), w(t)\big), & t \in J = [a, b], \\ {}^{H}D_{a+}^{\alpha_2,\beta_2;\psi} v(t) = f_2\big(t, u(t), v(t), w(t)\big), & t \in J, \\ {}^{H}D_{a+}^{\alpha_3,\beta_3;\psi} w(t) = f_3\big(t, u(t), v(t), w(t)\big), & t \in J, \\ I_{a+}^{1-\gamma_1;\psi} u(t)\Big|_{t=a} = c_1,\ I_{a+}^{1-\gamma_2;\psi} v(t)\Big|_{t=a} = c_2,\ I_{a+}^{1-\gamma_3;\psi} w(t)\Big|_{t=a} = c_3, \end{cases} \tag{6.2}$$

where ${}^{H}D_{a+}^{\alpha_i,\beta_i;\psi}$ is the ψ-Hilfer derivative of order α_i, and type β_i with $I_{a+}^{1-\gamma_i;\psi}$ the ψ-Riemann–Liouville integral of order $1 - \gamma_i$; and $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, for $i = 1, 2, 3$. Here, $f_i : J \times \Omega \times \Omega \times \Omega \to \Omega$ are continuous functions and $\Omega \subset \mathbb{R}$ is a real Banach space.

We impose the following hypotheses.

Hypothesis 6.1. *Each function $f_i : J \times \Omega \times \Omega \times \Omega \to \Omega$, $i = 1, 2, 3$, is continuous and satisfies a Lipschitz condition in the state variables, i.e., there exist constants $L_i > 0$ such* *that*

$$\begin{aligned}|f_i(t,u(t),v(t),w(t)) - f_i(t,\hat{u}(t),\hat{v}(t),\hat{w}(t))| \\ \leq L_i[\|u-\hat{u}\| + \|v-\hat{v}\| + \|w-\hat{w}\|].\end{aligned}$$

Hypothesis 6.2. *Each f_i is bounded on $J \times \Omega^3$, i.e., there exist $M_i > 0$ such that*

$$|f_i(t,u(t),v(t),w(t))| \leq M_i, \qquad \forall t \in J, \text{and } u,v,w \in \Omega.$$

Theorem 6.1. *Assume Hypotheses 6.1-6.2 to hold and define*

$$\Lambda = \sum_{i=1}^{3} \frac{L_i}{\Gamma(\alpha_i+1)}\big(\psi(b)-\psi(a)\big)^{\alpha_i}.$$

If $\Lambda < 1$, then system (6.2) admits a unique solution on J.

Proof. Using the definition of the ψ-Hilfer derivative, system (6.2) is equivalent to the coupled Volterra integral system

$$\begin{aligned}\mathcal{X}_i(t) =& \frac{(\psi(t)-\psi(a))^{\gamma_i-1}}{\Gamma(\gamma_i)}c_i \\ &+ \frac{1}{\Gamma(\alpha_i)}\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1} f_i(\xi,u(\xi),v(\xi),w(\xi))d\xi, \quad i=1,2,3.\end{aligned} \tag{6.3}$$

Define the operator $\mathcal{T}_i : \Omega \to \Omega$ by

$$\begin{aligned}(\mathcal{T}\mathcal{X})_i(t) =& \frac{(\psi(t)-\psi(a))^{\gamma_i-1}}{\Gamma(\gamma_i)}c_i \\ &+ \frac{1}{\Gamma(\alpha_i)}\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1} f_i(\xi,u(\xi),v(\xi),w(\xi))d\xi.\end{aligned}$$

Let $\mathcal{X}i, \mathcal{Y}i \in \Omega$. For each $i = 1,2,3$ and $t \in J$. Then,

$$\begin{aligned}&|(\mathcal{T}\mathcal{X})_i(t) - (\mathcal{T}\mathcal{Y})_i(t)| \\ &\leq \frac{1}{\Gamma(\alpha_i)}\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}|f_i(\xi,\mathcal{X}(\xi)) - f_i(\xi,\mathcal{Y}(\xi))|\,d\xi \\ &\leq \frac{L_i}{\Gamma(\alpha_i)}\|\mathcal{X}-\mathcal{Y}\| \int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}d\xi.\end{aligned}$$

We know

$$\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}d\xi = \frac{(\psi(t)-\psi(a))^{\alpha_i}}{\Gamma(\alpha_i+1)/\Gamma(\alpha_i)}. \tag{6.4}$$

Hence,

$$|(\mathcal{T}\mathcal{X})_i(t) - (\mathcal{T}\mathcal{Y})_i(t)| \leq \frac{L_i}{\Gamma(\alpha_i + 1)}(\psi(t) - \psi(a))^{\alpha_i}\|\mathcal{X}_i - \mathcal{Y}_i\|.$$

Taking the supremum over $t \in J$ and summing over i gives

$$\|\mathcal{T}\mathcal{X} - \mathcal{T}\mathcal{Y}\| \leq \Lambda\|\mathcal{X} - \mathcal{Y}\|. \tag{6.5}$$

Since $\Lambda < 1$, $\mathcal{T}$ is a contraction. Thus, by Banach's fixed point theorem, $\mathcal{T}$ admits a unique fixed point $\mathcal{X}^*$, which is the unique solution of system (6.2). □

6.3 Ulam–Hyers Stability

Definition 6.1. *System (6.2) is said to be* Ulam–Hyers stable *if there exists a constant $C_U > 0$ such that, for every $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) > 0$ and for every function $\mathcal{Y} = (\tilde{u}, \tilde{v}, \tilde{w}) \in \Omega^3$ satisfying*

$$\left|{}^{H}D_{a+}^{\alpha_i,\beta_i;\psi}\mathcal{Y}_i(t) - f_i\big(t, \tilde{u}(t), \tilde{v}(t), \tilde{w}(t)\big)\right| \leq \epsilon_i, \qquad i = 1, 2, 3, \tag{6.6}$$

there exists a unique exact solution $\mathcal{X} = (u, v, w) \in \Omega^3$ of system (6.2) satisfying

$$|\mathcal{Y}_i(t) - \mathcal{X}_i(t)| \leq C_U\,\epsilon_i, \qquad t \in J,\ i = 1, 2, 3.$$

Lemma 6.1. *A function $\mathcal{Y} \in \Omega^3$ satisfies the inequation (6.6) if and only if there exist continuous functions $g_i \in C(J, \Omega)$, $i = 1, 2, 3$, such that*

1. $|g_i(t)| \leq \epsilon_i$ *for all* $t \in J$,
2. ${}^{H}D_{a+}^{\alpha_i,\beta_i;\psi}\mathcal{Y}_i(t) = f_i\big(t, \tilde{u}(t), \tilde{v}(t), \tilde{w}(t)\big) + g_i(t), \quad i = 1, 2, 3.$

Theorem 6.2. *Assume Hypothesis 6.1 to hold and $\Lambda < 1$. Then, system (6.2) is Ulam–Hyers stable.*

Proof. Let $\mathcal{X} = (u, v, w)$ be the unique exact solution and let $\mathcal{Y} = (\tilde{u}, \tilde{v}, \tilde{w})$ be an approximate solution satisfying

$${}^{H}D_{a+}^{\alpha_i,\beta_i;\psi}\mathcal{Y}_i(t) = f_i\big(t, \tilde{u}(t), \tilde{v}(t), \tilde{w}(t)\big) + g_i(t). \tag{6.7}$$

Applying the ψ-Riemann–Liouville integral operator $I_{a+}^{\alpha_i;\psi}$ to both sides of equation (6.7), we have

$$\mathcal{Y}_i(t) = \frac{c_i}{\Gamma(\gamma_i)}(\psi(t) - \psi(a))^{\gamma_i - 1} + I_{a+}^{\alpha_i;\psi} f_i\big(t, \tilde{u}(t), \tilde{v}(t), \tilde{w}(t)\big) + I_{a+}^{\alpha_i;\psi} g_i(t). \tag{6.8}$$

For the exact solution $\mathcal{X}_i(t)$, we have

$$\mathcal{X}_i(t) = \frac{c_i}{\Gamma(\gamma_i)}(\psi(t)-\psi(a))^{\gamma_i-1} + I_{a+}^{\alpha_i;\psi} f_i(t,u(t),v(t),w(t)). \tag{6.9}$$

Now

$$|\mathcal{Y}_i(t)-\mathcal{X}_i(t)| \le \left| I_{a+}^{\alpha_i;\psi} f_i\Big(t,\tilde{u}(t),\tilde{v}(t),\tilde{w}(t)\Big) - f_i(t,u(t),v(t),w(t))\Big)\right| + \left| I_{a+}^{\alpha_i;\psi} g_i(t)\right|.$$

Using the Hypothesis 6.1, we have

$$\begin{aligned}|\mathcal{Y}_i(t)-\mathcal{X}_i(t)| \le{}& \frac{L_i}{\Gamma(\alpha_i)}\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}\sum_{j=1}^{3}|\mathcal{Y}_j(\xi)-\mathcal{X}_j(\xi)|\,d\xi\\ &+\frac{\epsilon_i}{\Gamma(\alpha_i)}\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}\,d\xi.\end{aligned}$$

Now,

$$E(t) = |\tilde{u}(t)-u(t)| + |\tilde{v}(t)-v(t)| + |\tilde{w}(t)-w(t)|.$$

Thus,

$$\begin{aligned}E(t) \le{}& \sum_{i=1}^{3}\frac{L_i}{\Gamma(\alpha_i)}\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}E(\xi)\,d\xi\\ &+\sum_{i=1}^{3}\frac{\epsilon_i}{\Gamma(\alpha_i)}\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}\,d\xi.\end{aligned}$$

Since

$$\frac{1}{\Gamma(\alpha_i)}\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}\,d\xi = \frac{(\psi(t)-\psi(a))^{\alpha_i}}{\Gamma(\alpha_i+1)},$$

we obtain

$$E(t) \le \sum_{i=1}^{3} L_i\int_a^t \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha_i-1}E(\xi)\,d\xi + \sum_{i=1}^{3}\frac{\epsilon_i(\psi(t)-\psi(a))^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

Applying the generalized Grönwall's inequality, we find that, for all $t \in J$,

$$E(t) \le \left(\sum_{i=1}^{3}\frac{\epsilon_i(\psi(t)-\psi(a))^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\mathbb{E}_\alpha\left[\sum_{i=1}^{3}L_i(\psi(t)-\psi(a))^{\alpha_i}\right],$$

where $\mathbb{E}_\alpha(\cdot)$ denotes the Mittag–Leffler function. Since $\mathbb{E}_\alpha(t)$ is bounded for finite t, there exists a constant $C_U > 0$ (depending on L_i, α_i, and ψ, but not on ϵ_i) such that

$$E(t) \le C_U\sum_{i=1}^{3}\epsilon_i, \qquad t \in J.$$

Consequently,

$$|\mathcal{Y}_i(t) - \mathcal{X}_i(t)| \leq C_U \, \epsilon_i, \qquad i = 1, 2, 3, \ t \in J. \tag{6.10}$$

Hence the system (6.2) is Ulam–Hyers stable. □

6.4 Application to a Financial Chaotic Model and Numerical Solution

The financial system is inherently dynamic and exhibits complex interactions among variables such as interest rate, investment demand, and price index. Classical models, although useful, often fail to capture the long-range dependence and memory effects observed in real financial data. To address this limitation, we consider a fractional-order extension of the well-known financial chaotic system using the ψ-Hilfer fractional derivative. This framework not only generalizes existing integer- and fractional-order formulations, but also introduces a flexible weight function $\psi(t)$ that enables the analysis of diverse memory structures and nonlocal behaviors within financial dynamics.

6.4.1 Model: Classical Origin and ψ–Hilfer Extension

The financial chaotic model describes interactions among the interest rate $u(t)$, investment demand $v(t)$, and price index $w(t)$. The classical (integer-order) form is

$$\begin{aligned} \dot{u}(t) &= w(t) + (v(t) - \kappa_1)u(t), \\ \dot{v}(t) &= 1 - \kappa_2 v(t) - u^2(t), \\ \dot{w}(t) &= -u(t) - \kappa_3 w(t), \end{aligned} \tag{6.11}$$

where $\kappa_1, \kappa_2, \kappa_3 > 0$ are system parameters controlling the internal adjustment rates [13, 31]. This model exhibits complex dynamics, including chaotic attractors, making it a natural candidate for studying memory effects in nonlinear finance.

To incorporate nonlocal and hereditary behavior, we extend system (6.11) by replacing the classical derivative with the ψ-Hilfer fractional derivative. Let $\psi : [0, T] \to \mathbb{R}$ be a strictly increasing C^1 function with $\psi'(t) > 0$. For fractional orders $0 < \alpha_i < 1$ and a type parameter $\beta_i \in [0, 1]$, define

$$\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i, \quad i = 1, 2, 3.$$

The resulting ψ-Hilfer fractional chaotic model is given by

$$\begin{aligned} {}^H D_{0^+}^{\alpha_1,\beta_1;\psi} u(t) &= w(t) + (v(t) - \kappa_1)u(t), \\ {}^H D_{0^+}^{\alpha_2,\beta_2;\psi} v(t) &= 1 - \kappa_2 v(t) - u^2(t), \\ {}^H D_{0^+}^{\alpha_3,\beta_3;\psi} w(t) &= -u(t) - \kappa_3 w(t), \end{aligned} \tag{6.12}$$

with the associated nonlocal initial conditions

$$I_{0^+}^{1-\gamma_i;\psi}\mathcal{X}_i(t)\Big|_{t=0} = c_i, \qquad \mathcal{X}_i = u, v, w, \quad \text{for} = 1, 2, 3, \text{ respectively.}$$

The nonlinear function ψ acts as a generalized time scale, controlling how past states influence the present. Typical choices such as $\psi(t) = t$, $\psi(t) = \log(1+t)$, and $\psi(t) = e^t - 1$ model different intensities of memory and nonlocal scaling effects.

6.4.2 Integral Formulation and Numerical Scheme

Applying the ψ-Riemann–Liouville fractional integral operator $I_{0^+}^{\alpha_i;\psi}$ to system (6.12), we obtain the equivalent integral form

$$\begin{aligned}\mathcal{X}_i(t) =& \frac{(\psi(t)-\psi(0))^{\gamma_i-1}}{\Gamma(\gamma_i)}c_i \\ &+ \frac{1}{\Gamma(\alpha_i)}\int_0^t \psi'(\xi)\,(\psi(t)-\psi(\xi))^{\alpha_i-1}\, f_i\Big(\xi, u(\xi), v(\xi), w(\xi)\Big)\,d\xi, \end{aligned} \tag{6.13}$$

where c_i is the non-local initial condition and

$$f_1 = w + (v-\kappa_1)u, \quad f_2 = 1 - \kappa_2 v - u^2, \quad f_3 = -u - \kappa_3 w.$$

For numerical computation, let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform mesh with step size $h = T/N$. We approximate the fractional integral in (6.13) using discrete convolution weights derived from the ψ-kernel:

$$\begin{aligned}&\frac{1}{\Gamma(\alpha_i)}\int_0^{t_n}\psi'(\xi)\,(\psi(t_n)-\psi(\xi))^{\alpha_i-1} f_i(\xi, u(\xi), v(\xi), w(\xi))\,d\xi \\ &\qquad\approx \sum_{k=1}^{n} C_{n,k}^{(\alpha_i)} f_i(t_k, u_k, v_k, w_k), \quad \text{where}\end{aligned} \tag{6.14}$$

$$C_{n,k}^{(\alpha_i)} = \frac{1}{\Gamma(\alpha_i+1)}\Big[\Big(\psi(t_n)-\psi(t_{k-1})\Big)^{\alpha_i} - \Big(\psi(t_n)-\psi(t_k)\Big)^{\alpha_i}\Big], \quad k = 1, \ldots, n. \tag{6.15}$$

Thus, we can write

$$\begin{aligned}u(t_n) &= \frac{(\psi(t_n)-\psi(0))^{\gamma_1-1}}{\Gamma(\gamma_1)}c_1 + \frac{1}{\Gamma(\alpha_1)}\sum_{k=1}^{n} C_{n,k}^{(\alpha_1)}\,[w(t_k) + (v(t_k)-\kappa_1)u(t_k)], \\ v(t_n) &= \frac{(\psi(t_n)-\psi(0))^{\gamma_2-1}}{\Gamma(\gamma_2)}c_2 + \frac{1}{\Gamma(\alpha_2)}\sum_{k=1}^{n} C_{n,k}^{(\alpha_2)}\,[1 - \kappa_2 v(t_k) - u^2(t_k)], \\ w(t_n) &= \frac{(\psi(t_n)-\psi(0))^{\gamma_3-1}}{\Gamma(\gamma_3)}c_3 + \frac{1}{\Gamma(\alpha_3)}\sum_{k=1}^{n} C_{n,k}^{(\alpha_3)}\,[-u(t_k) - \kappa_3 w(t_k)].\end{aligned}$$

This scheme generalizes the discretization of the ψ-Hilfer fractional system.

6.4.3 Numerical Results and Discussion

In this subsection, we present and analyze the numerical results obtained for the ψ-Hilfer fractional financial chaotic model using the integral-type numerical scheme developed earlier. The system describes the evolution of the interest rate $u(t)$, investment demand $v(t)$, and price index $w(t)$, which together form a nonlinear macroeconomic interaction. All simulations use the same initial conditions $c_1 = 1$, $c_2 = 2$, and $c_3 = 0.9$ to ensure a consistent comparison. The fractional structure introduces memory effects, allowing present values to depend on the entire history of the system through the ψ-Hilfer operator.

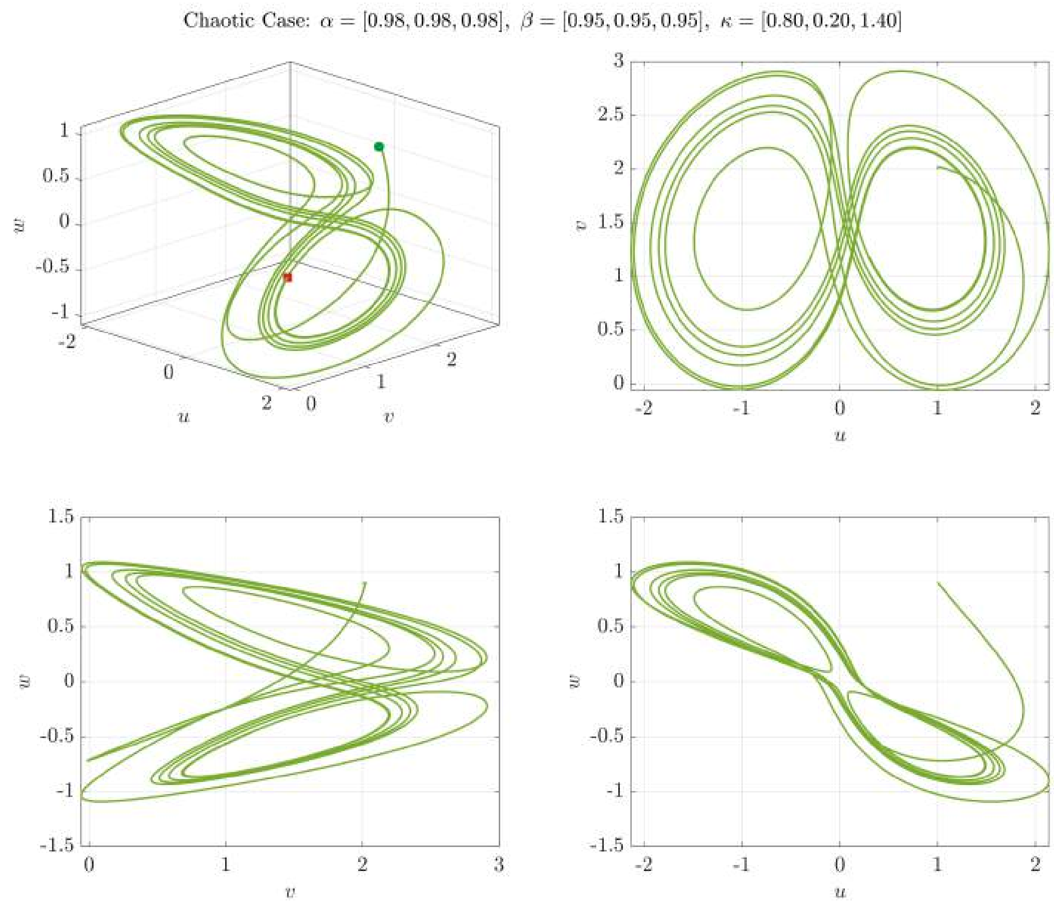


Figure 6.1: Chaotic dynamics behavior in the linear case ($\psi(t) = t$)

To study the influence of different memory structures, three choices of the weight function are considered: the linear case $\psi(t) = t$, the exponential case $\psi(t) = e^t - 1$, and the logarithmic case $\psi(t) = 10\log(1 + t)$. For each ψ, three dynamical regimes are investigated: chaotic, transitional, and stable. In the figures, we use the notation $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$ and $\kappa = (\kappa_1, \kappa_2, \kappa_3)$. Figures 6.1–6.9 display the

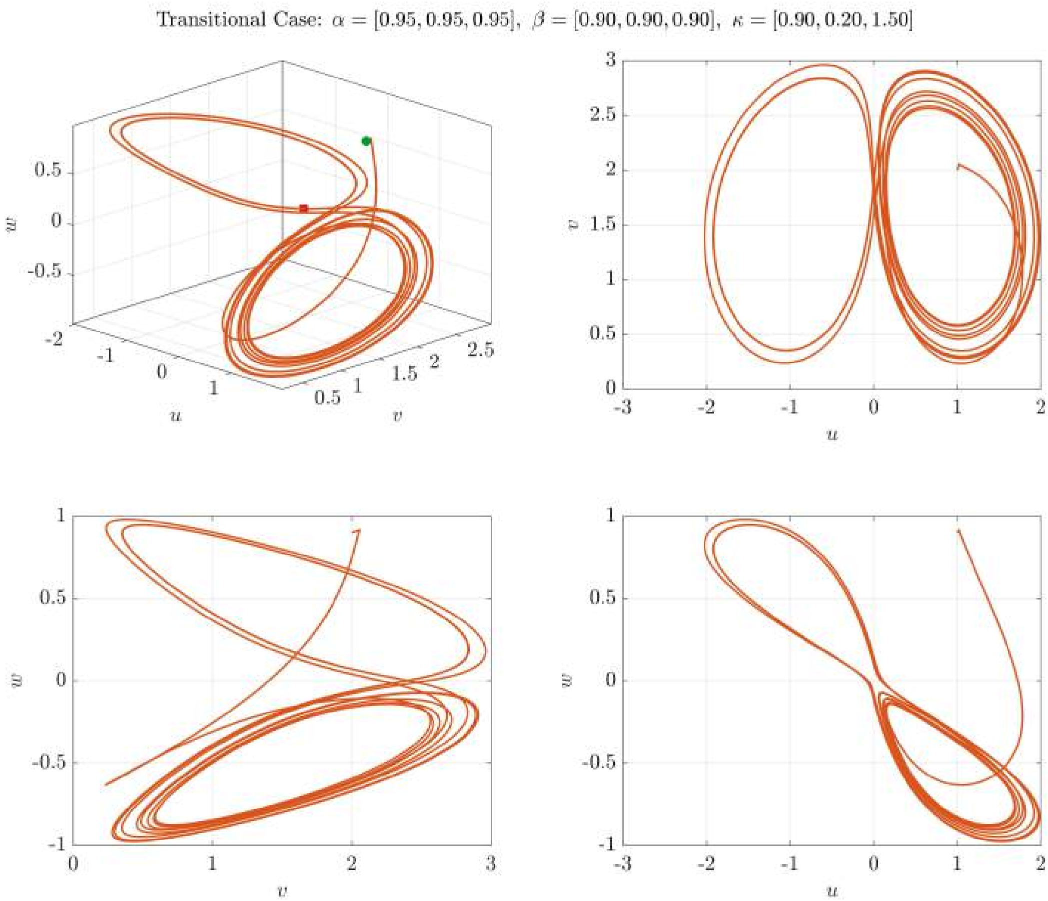


Figure 6.2: Transitional dynamics behavior in the linear case ($\psi(t) = t$)

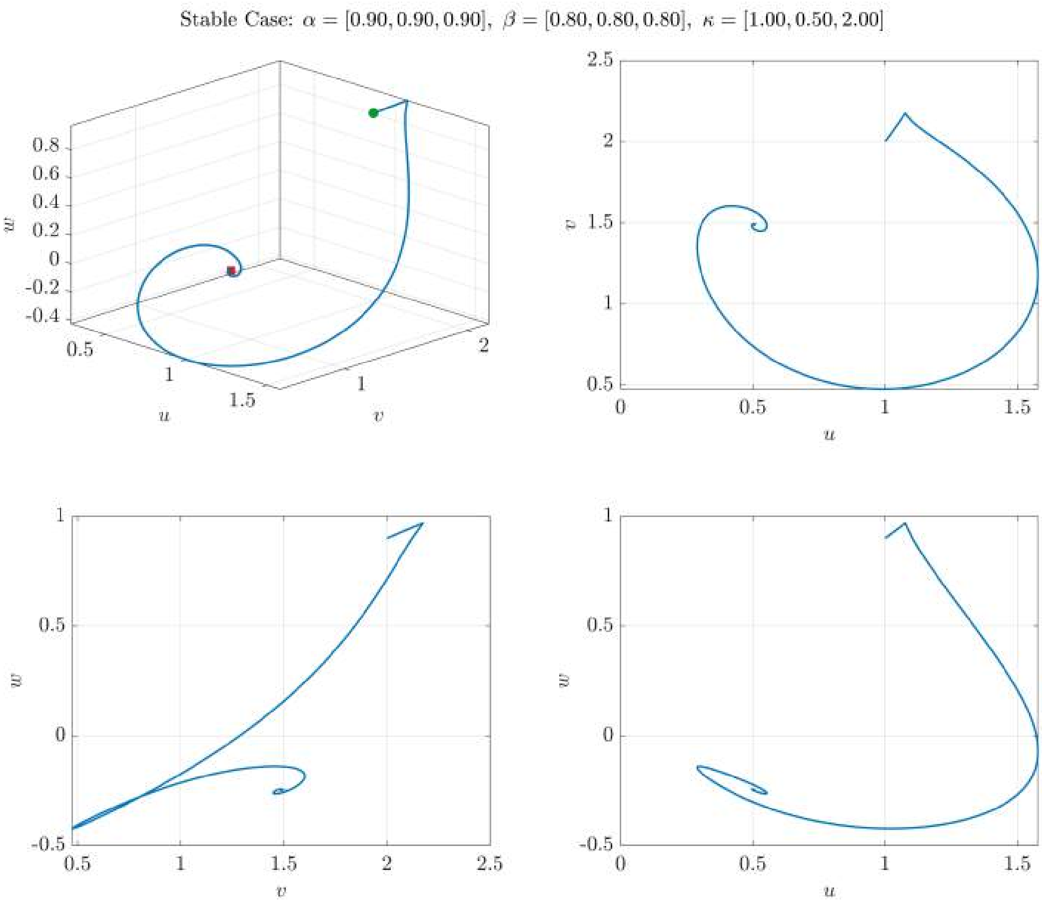


Figure 6.3: Stable dynamics behavior in the linear case ($\psi(t) = t$)

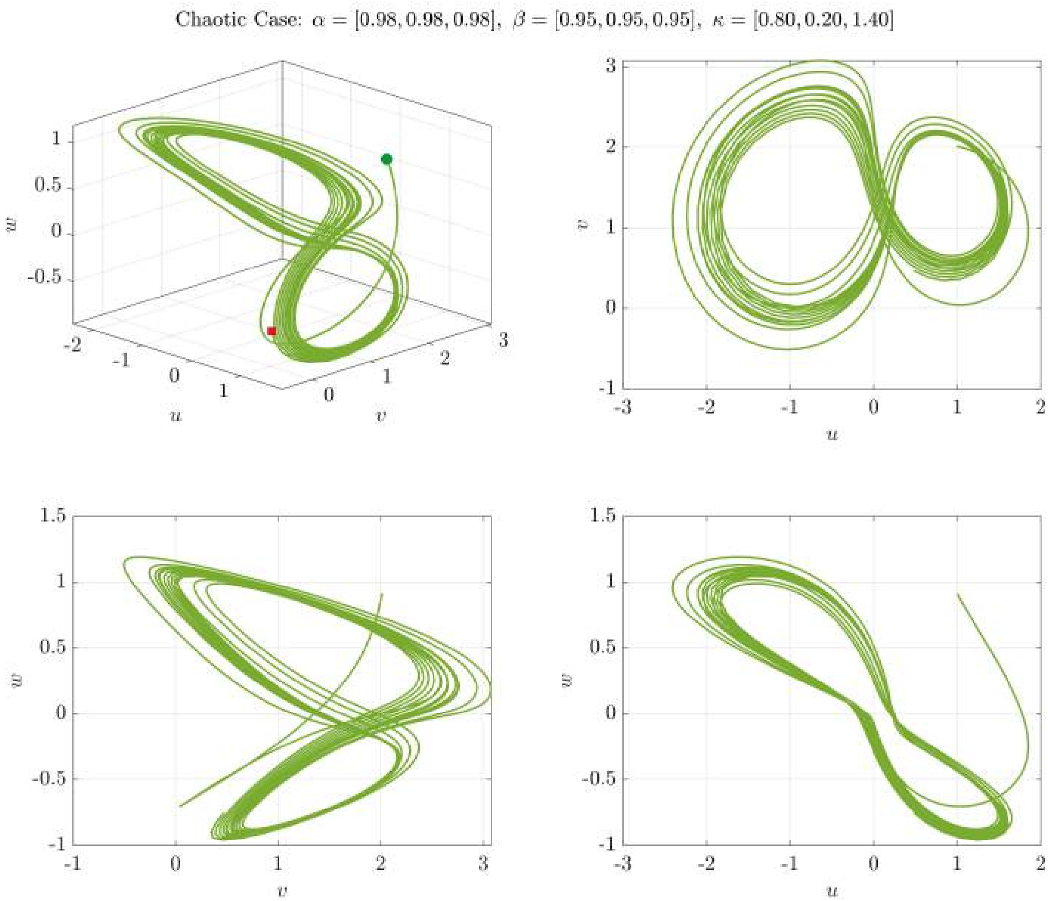


Figure 6.4: Chaotic dynamics behavior in the exponential case ($\psi(t) = e^t - 1$)

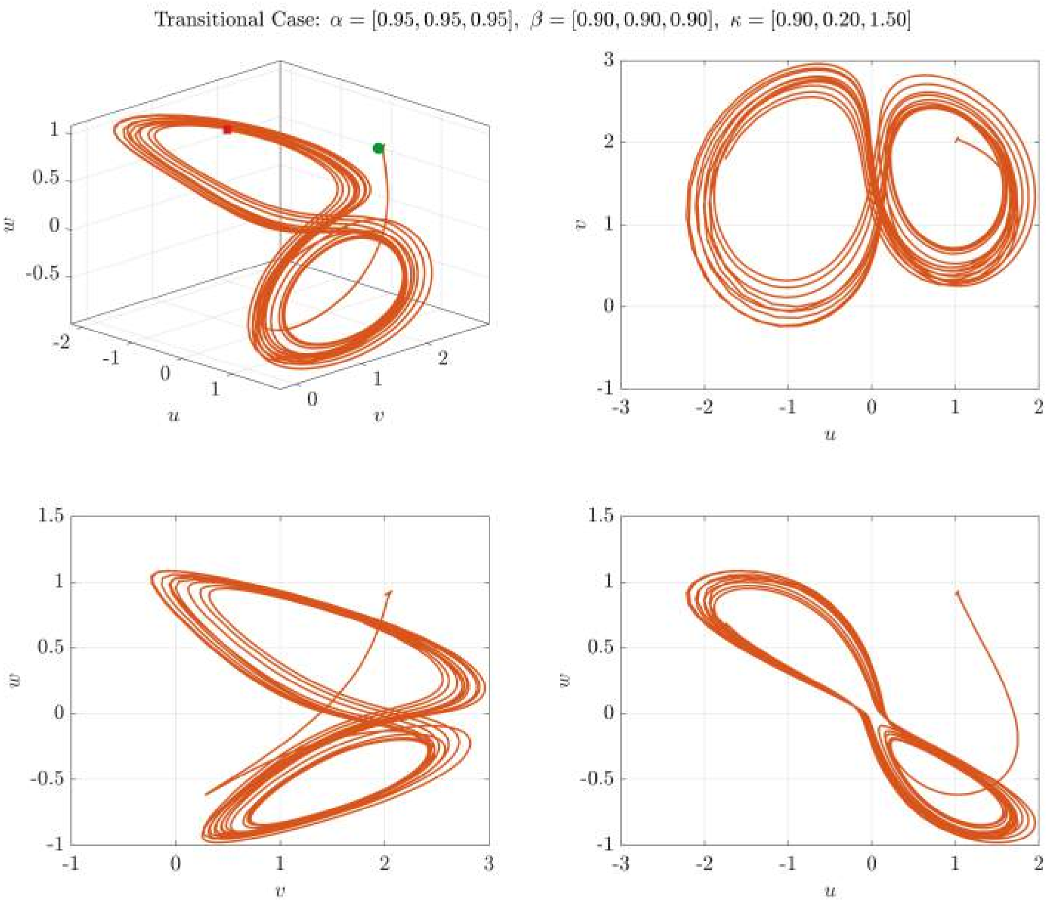


Figure 6.5: Transitional dynamics behavior for the exponential case ($\psi(t) = e^t - 1$)

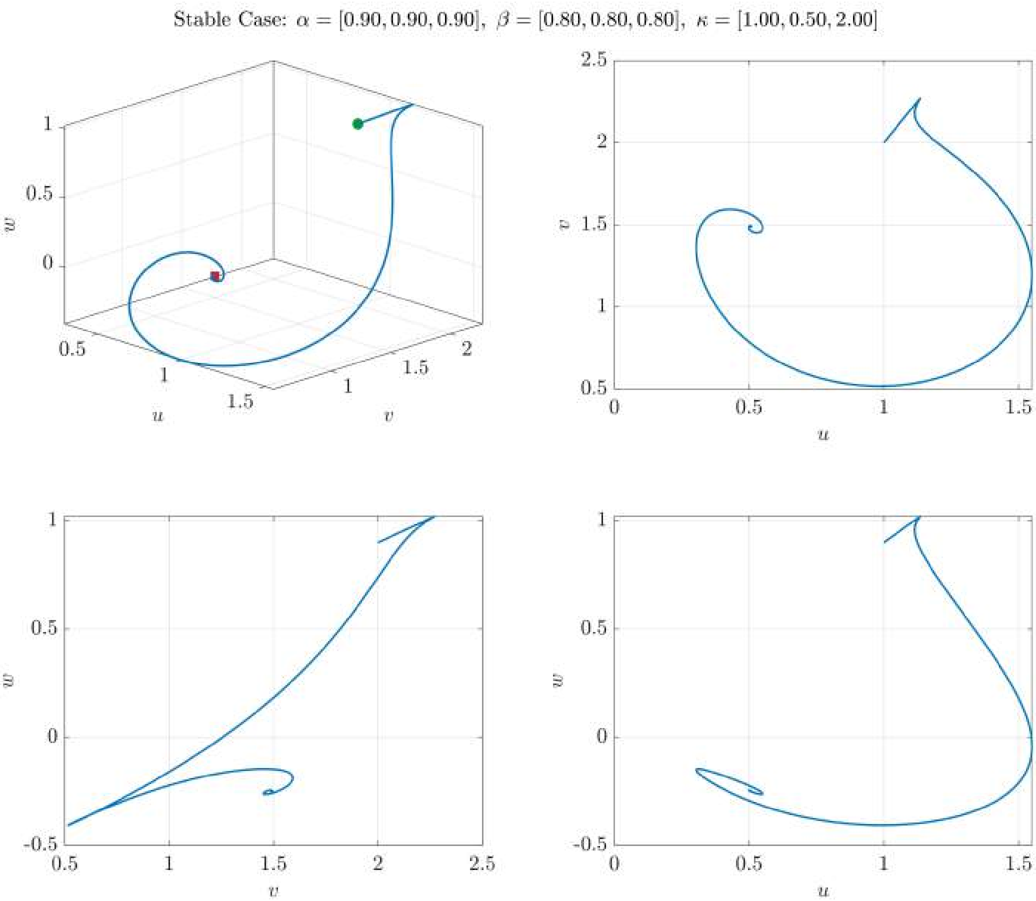


Figure 6.6: Stable dynamics behavior in the exponential case ($\psi(t) = e^t - 1$)

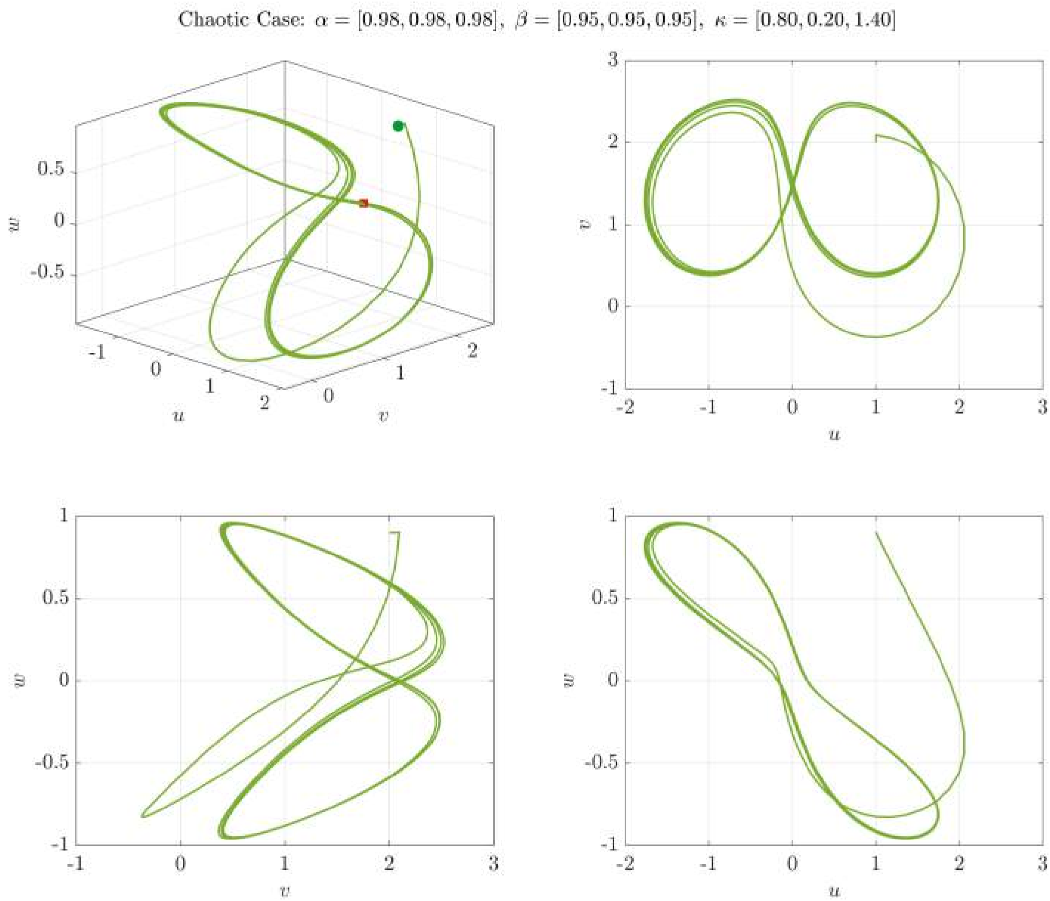


Figure 6.7: Chaotic dynamics behavior in the logarithmic case ($\psi(t) = 10 \log(1 + t)$)

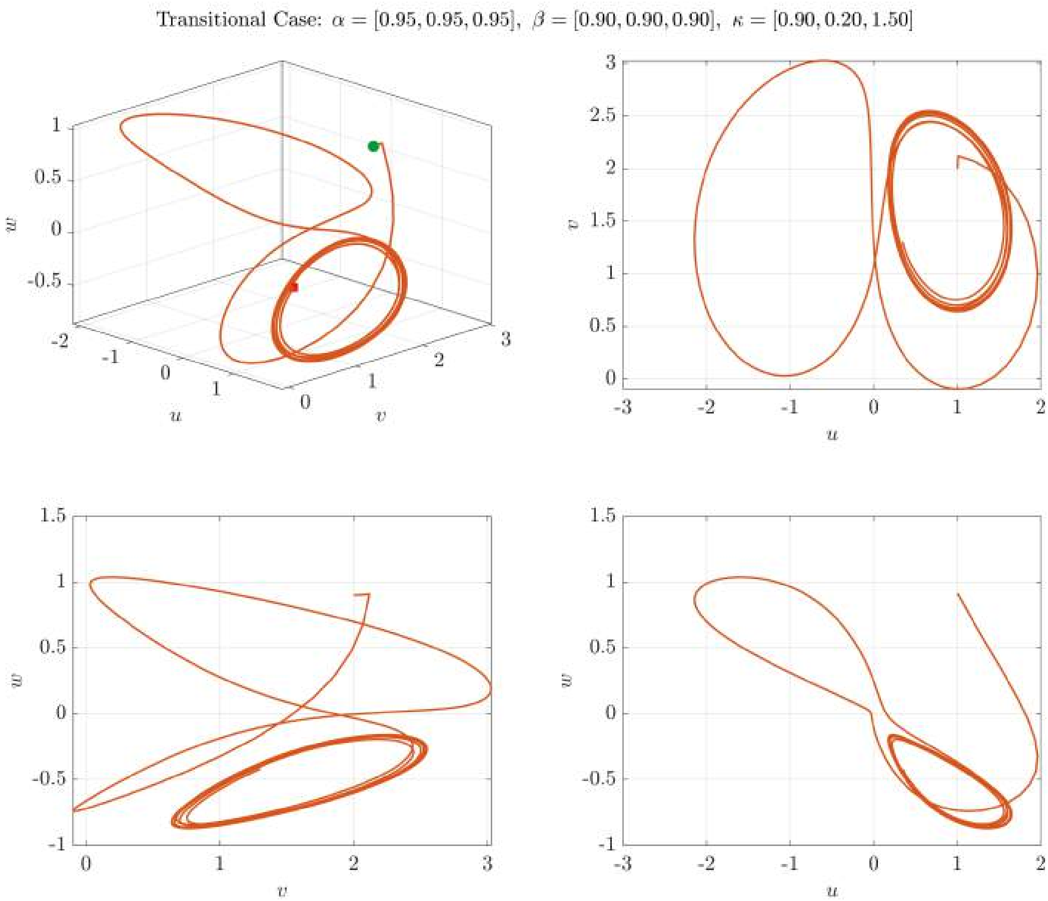


Figure 6.8: Transitional dynamics behavior in the logarithmic case ($\psi(t) = 10\log(1 + t)$)

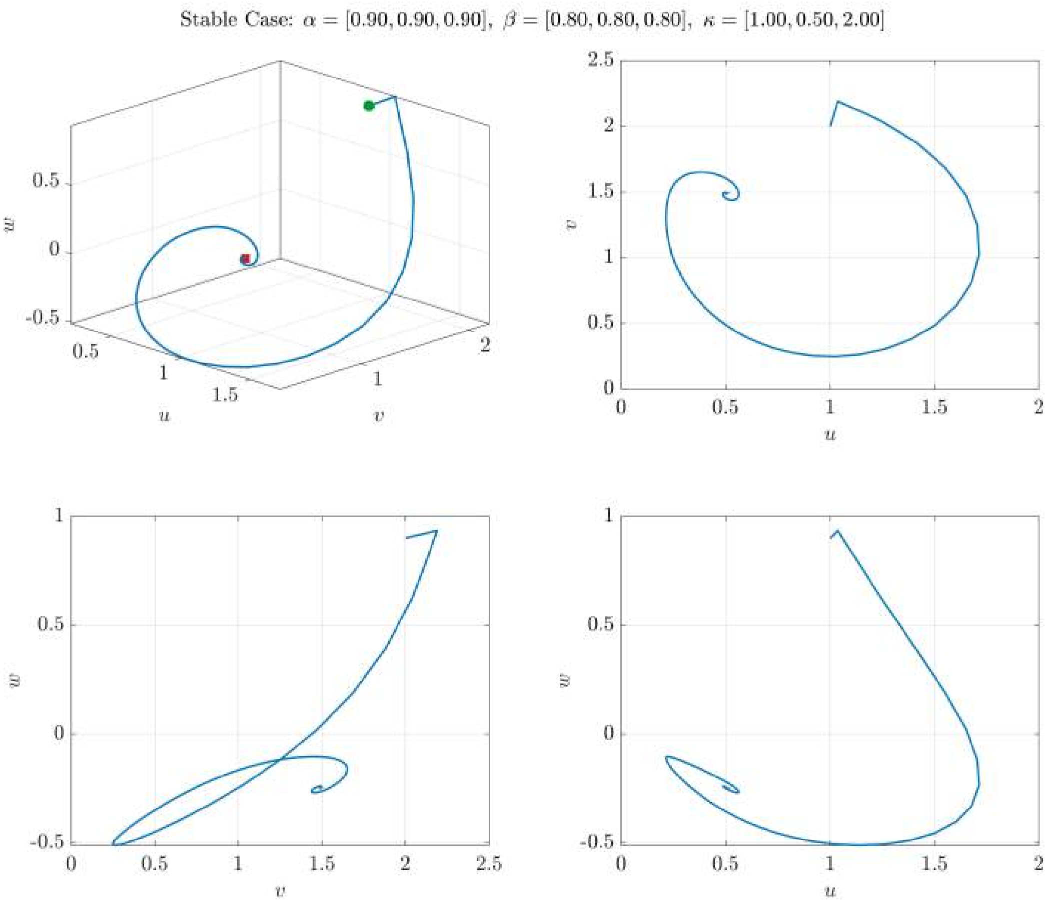


Figure 6.9: Stable dynamics behavior in the logarithmic case ($\psi(t) = 10 \log(1 + t)$)

corresponding results, where each figure contains a four–panel representation consisting of the 3D phase portrait of (u, v, w) along with the three 2D projections.

(i) Linear weight: $\psi(t) = t$ (Figures 6.1–6.3). In the linear case, the ψ-Hilfer derivative behaves similarly like the classical Hilfer fractional derivative with a uniform time scale. The results show a clear progression from fully chaotic motion to weak oscillations and finally to stable convergence. In the chaotic regime (Figure 6.1), all three variables oscillate irregularly: the interest rate u fluctuates strongly, the investment demand v exhibits irregular jumps, and the price index w responds sensitively to these fluctuations. As parameters are adjusted, the system moves into a transitional regime (Figure 6.2), where oscillations persist, but with more regularity. Finally, in Figure 6.3, all trajectories converge to a stable equilibrium, indicating that the uniform memory effect smoothens the system dynamics and encourages stabilization.

(ii) Exponential weight: $\psi(t) = e^t - 1$ (Figures 6.4–6.6). The exponential weight induces a rapidly growing memory influence, amplifying the importance of recent history. As a consequence, the chaotic regime in Figure 6.4 is more pronounced: small changes in u or v lead to large responses in w, and the full system displays high sensitivity to initial conditions. In the transitional case (Figure 6.5), the trajectories exhibit periodic or quasiperiodic oscillations with noticeable amplitude variations. As parameters move toward the stable regime (Figure 6.6), the strong memory effect forces the system to settle more rapidly, and all variables converge toward a stable equilibrium. The exponential memory structure thus intensifies both chaos and stabilization depending on parameter values.

(iii) Logarithmic weight: $\psi(t) = 10\log(1 + t)$ (Figures 6.7–6.9). The logarithmic function grows slowly, producing a gently increasing memory effect. This results in delayed but sustained dynamical interactions: chaotic behavior emerges in Figure 6.7, but with smoother and less abrupt fluctuations compared to the exponential case. The transitional regime (Figure 6.8) shows mild oscillations, where the investment demand and price index evolve with more regular patterns. In the stable regime (Figure 6.9), the system converges gradually, indicating that the logarithmic memory distributes influence more evenly over time, leading to smoother long-term behavior.

The numerical experiments reveal that the interplay among fractional order α_i, type parameter β_i, and system coefficients $(\kappa_1, \kappa_2, \kappa_3)$ strongly influences the qualitative dynamics. Lower fractional orders produce stronger memory effects, often enhancing stability, while higher orders allow the system to respond more sharply, leading to chaotic or transitional behavior. The coefficients determine how quickly the financial variables feed back into each other, thereby shaping the transition between stability and chaos.

Overall, the results demonstrate that the ψ-Hilfer financial fractional formulation offers a more flexible and realistic modeling framework than classical integer-order or standard fractional-order models. By adjusting the weight function ψ, one can modulate how past information influences current dynamics, allowing for a richer spectrum of behavior including smooth stabilization, strong oscillations, and fully developed chaos. This makes the model particularly suitable for capturing complex financial phenomena exhibiting memory, feedback loops, and sensitivity to historical data.

6.5 Conclusion

A system of three ψ-Hilfer fractional differential equations was investigated to model the dynamic feedback among interest rate, investment demand, and price index. Existence and uniqueness of solutions were established using the Banach's fixed point theorem. The system was further shown to be Ulam–Hyers stable.

Numerical simulations, performed under different weight functions ψ and parameter configurations, demonstrated versatile behavior ranging from stability to chaos. The results reveal that the ψ-Hilfer derivative effectively captures non-locality and memory effects, making it a valuable tool for analyzing complex financial dynamics.

CHAPTER 7

Conclusions and Future Scope

7.1 Conclusions

The present thesis is devoted to the qualitative analysis and stability theory of different classes of fractional differential equations involving the ψ-Hilfer fractional derivative. The study develops a unified mathematical framework connecting existence, uniqueness, and various notions of Ulam–type stability for an abstract, a neutral, a coupled, and a system of three fractional differential equation(s). These works build on the existence results and Ulam-type stability results for more generalized fractional operators across different fractional systems, and are further supported by numerical solutions together with discussions of some relevant real-world applications.

In the Chapter 2, the focus is on ψ-*Hilfer abstract fractional differential equations*, where the existence and uniqueness of solutions were established using Banach's fixed point theorem. Ulam–Hyers and Ulam–Hyers–Rassias stabilities were analyzed in a weighted functional setting, and numerical results illustrated how the fractional order and the weight function ψ influence the solution behavior.

The analysis was then extended to incorporate *Ulam–Hyers–Mittag–Leffler-type stabilities*, providing a refined stability characterization for ψ-Hilfer abstract fractional differential equations in Chapter 3. The existence results were obtained via Schauder's fixed point theorem, and sufficient conditions were derived to guarantee the Ulam–Hyers–Mittag–Leffler and Ulam–Hyers–Rassias–Mittag–Leffler stabilities.

The Chapter 4 examined ψ-*Hilfer neutral fractional differential equations with delay*, motivated by systems in which the current state depends on both present and past states. The existence results were obtained through Krasnosel'skiĭ's fixed point theorem, and Ulam–Hyers as well as Ulam–Hyers–Rassias stabilities results were established. Numerical illustrations demonstrated the influence of delay, order, and weight functions on the solution behavior.

Subsequently, a *coupled fractional differential system* involving the ψ-Hilfer derivative was analyzed in Chapter 5. Existence and uniqueness results were established using the Banach's and Schauder's fixed point theorems. Ulam–Hyers and generalized Ulam–Hyers stability results were also obtained. An application of the developed numerical scheme to a blood alcohol concentration model was presented, demonstrating that ψ-Hilfer fractional-order dynamics provided a more accurate and flexible description of physiological processes with memory. Numerical results were compared with experimental data and with the corresponding ODE model, validating the proposed approach.

In Chapter 6, a *fractional system of three differential equations* governed by the ψ-Hilfer derivative was investigated. Existence, uniqueness, and Ulam–Hyers stability results were established. As an application of the proposed numerical scheme, a fractional financial chaotic model was examined to demonstrate the complexity and sensitivity of fractional systems. The numerical simulations showed transitions from chaotic dynamics to stable behavior under varying coefficients, fractional orders, and weight functions.

Overall, this thesis presents a rigorous and unified stability theory for ψ-Hilfer fractional differential equations, encompassing multiple system types and a broad range of stability notions, including Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, generalized Ulam–Hyers–Rassias stability, Ulam–Hyers–Mittag–Leffler stability, and Ulam–Hyers–Rassias–Mittag–Leffler stability. The results generalize existing findings and offer new insights into modeling real-world processes characterized by nonlocality, hereditary effects, and complex temporal interactions.

7.2 Future Scope

Collectively, these findings pave the way for further research and applications in various scientific and engineering fields. Extending the current framework to recently developed fractional operators, such as the generalized proportional Caputo and Atangana–Baleanu-type derivatives, could provide greater versatility in representing memory and hereditary phenomena. Moreover, the combination of ψ-Hilfer derivatives with impulsive effects, stochastic perturbations, or control mechanisms remains an open and promising area. The analytical and numerical insights developed in this thesis can also be adapted to study ψ-Hilfer fractional partial differential equations, stochastic models, and hybrid fractional systems, thereby enriching both theoretical foundations and interdisciplinary applications of fractional calculus.

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Published and Communicated Papers

Based on the works carried out in this thesis, the following articles are published/communicated:

1. Sunil Kundu and Swaroop Nandan Bora, On Ulam type stability of the solution to a ψ-Hilfer abstract fractional functional differential equation, *Physica Scripta*, 100(4):045235, 2025. https://doi.org/10.1088/1402-4896/adbdfb

2. Sunil Kundu and Swaroop Nandan Bora, On the Ulam-Hyers-Mittag-Leffler stability of the solution to a ψ-Hilfer abstract fractional differential equation, *International Journal of Theoretical Physics*, 64(7), 190 (2025). https://doi.org/10.1007/s10773-025-06055-w

3. Sunil Kundu and Swaroop Nandan Bora, Solutions of nonlinear ψ-Hilfer neutral fractional differential equations: Existence, stability and a numerical perspective (Communicated).

4. Sunil Kundu and Swaroop Nandan Bora, Ulam–Hyers stability analysis of a coupled ψ-Hilfer fractional system with application to blood alcohol dynamics (Communicated).

5. Sunil Kundu and Swaroop Nandan Bora, Memory-driven financial chaos: Qualitative and numerical perspectives via the ψ–Hilfer derivative (Communicated).