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Thermodynamic aspects and phase transition of black holes

Krishnakanta Bhattacharya

Supervisor: Dr. Bibhas Ranjan Majhi
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Thermodynamic aspects and phase transition of black holes

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Krishnakanta Bhattacharya

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Department of Physics
Indian Institute of Technology Guwahati
Guwahati - 781039, Assam, India

Statement

The work contained in the thesis entitled **"Thermodynamic aspects and phase transition of black holes"** has been carried out at the Department of Physics, Indian Institute of Technology Guwahati, India by me under the supervision of Dr. Bibhas Ranjan Majhi. The material of this thesis has not been submitted elsewhere for any other degree. Works presented in the thesis are all my own unless referenced to the contrary in the text.

(Krishnakanta Bhattacharya) Date:
Department of Physics
Indian Institute of Technology Guwahati
Guwahati - 781039, India

Disclaimer

The bibliography included in this thesis is, by no means complete but contains the ones which are consulted thoroughly by me. I apologize for inadvertently missing out some of the research papers, review articles and other scientific documents pertaining to the focus of this thesis which should also have been cited.

Certificate

It is certified that the work contained in the thesis entitled "*Thermodynamic aspects and phase transition of black holes*" by Mr. Krishnakanta Bhattacharya, a Ph.D. student of the Department of Physics, Indian Institute of Technology Guwahati is carried out under my supervision and has not been submitted elsewhere for the award of any other degree.

(Dr. Bibhas Ranjan Majhi) Date:
Department of Physics
Indian Institute of Technology Guwahati
Guwahati - 781039, India

To

My Parents (Haradhan Bhattacharya and Annakali Bhattacharya)

and

in memory of my grandmothers (Late Annapurna Bhattacharya and Late Sukeshi Acharyya)

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List of publications

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(4) **Fluctuation-dissipation relation in accelerated frames.**

Ananya Adhikari, Krishnakanta Bhattacharya, Chandramouli Chowdhury and Bibhas Ranajn Majhi

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(5) *__van der Waals criticality in AdS black holes: a phenomenological study.__

Krishnakanta Bhattacharya, Bibhas Ranjan Majhi and Saurav Samanta

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(8) *__Abbott-Deser-Tekin like conserved quantities in Lanczos-Lovelock gravity for an arbitrary diffeomorphism.__

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(10) **Thermogeometric study of van der Waals like phase transition in black holes: an alternative approach.**

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This thesis is based on the papers 1, 2, 3, 5, 6, 7 and 8, which are also marked with asterisk (*) .

Conferences/Workshops attended

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- Research Conclave, IIT Guwahati, February, 2018 .
- Black Holes: From classical to quantum gravity, IIT Gandhinagar, December, 2017 .
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- Thermogeometric description of the van der Waals like phase transition in AdS black holes, Talk, IAGRG-29 (2017), IIT Guwahati, India.

"Here on the level sand,
Between the sea and land,
What shall I build or write
Against the fall of night?

Tell me of runes to grave
That holds the bursting wave,
Or bastions to design
For longer date than mine.

Shall it be Troy or Rome
I fence against the foam,
Or my own name, to stay
When I depart for aye?"

—- *A. E. Houseman*

Abstract

Almost half-century ago, the thermodynamic aspects of black holes were discovered. Initially it was thought that the similarity between the laws of black hole (BH) mechanics near the BH horizon and the laws of thermodynamics are just mere analogy. However, this formal analogy later became robust thermodynamic correspondence and it was realized that the black hole horizons have rich microscopic structure which are responsible for the BH thermodynamics. However, the microstates of BH thermodynamics are not properly understood yet. Also, the BH thermodynamics has mostly been studied for the stationary spacetime, which allows a Killing vector. Assuming the Killing symmetry in the spacetime largely simplifies the calculations but, it does not describe a realistic scenario. In reality, a spacetime should evolve with time and, therefore, the spacetime should not have the Killing symmetry and the BH horizon should not be a Killing horizon. Presence of van der Waals type phase transition is another remarkable aspect of BH thermodynamics. So far, the van der Waals type phase transition in BH spacetime has been studied case-by-case for different spacetimes. However, the obtained results are found to be the same irrespective of any spacetime. In addition, the thermodynamic features are not well-established in the alternative theories (such as the scalar-tensor theory, f(R) theory etc.) of gravity. The present thesis addresses all these shortcomings in BH thermodynamics and BH phase transition. However, for the analysis in the scalar-tensor/ f(R) theory, we also have accounted that the spacetime is stationary and the BH horizon is a Killing horizon.

The present thesis discusses some of the unsolved issues in black hole (BH) thermodynamics. Also, it clarifies some of the major debates in this subject. Here, the following aspects of black hole thermodynamics have been addressed: (i) Obtaining a consistent and a covariant thermodynamic description for the scalar-tensor theory, which was found to be a major challenge for decades. (ii) Obtaining black hole thermodynamics in Einstein's gravity for realistic time-dependent black holes and for non-Killing horizons, as the usual way of studying the BH thermodynamics

is on the stationary spacetime that possess the Killing symmetry, which does not correspond to the realistic scenario. (iii) Exploring the possibilities of finding the microstates, which are responsible for the black hole thermodynamics. (iv) Providing a general framework to study the van der Waals phase transition in black holes, which has been studied case-by-case in different spacetimes so far. The whole analysis is presented in two parts. In the first part, the thermodynamic aspects of black holes have been discussed, whereas, in the second part, we discuss about the general framework of the van der Waals phase transition of AdS black holes.

Contents

Chapter 1

Introduction

1.1 General background

For more than two hundred years, Newton's theory of gravitation was accepted as the valid theory to describe the gravitational force. The framework, provided by Newton was considered as extremely successful to describe the motion of celestial objects. However, there were several incongruities in this theory such as: it could not explain the perihelion precession of Mercury; also, it could not explain how the gravitational force comes into play between two objects which are very far from each other and not connected via any medium. In spite of these limitations, the theory, proposed by Newton, was widely regarded as the appropriate theory for gravity as it was highly successful in describing the motion of the objects under the influence of gravity. Thus, in 1915, when Einstein came up with general relativity (GR), describing gravity in terms of the geometry of the spacetime itself, it took a while for people to accept it as the better theory for describing gravity. However, the famous experimental test during the solar eclipse of 1919, carried out by Arthur Eddington and Frank Dyson, made the theory famous overnight. A few years ago, in 2015, we have completed hundred years since Einstein proposed the theory of general relativity (GR), which is now considered as one of the most wonderful theories of physics ever proposed in the human history. Not only the theory can explain several observational phenomena, like the bending of light, perihelion precession of Mercury, gravitational lensing, gravitational redshift etc., but also the theory is sublime in its mathematical foundation. The recent discovery of gravitational waves [1] in 2016, which was also the prediction of Einstein's GR, has added another feather to the crown of this theory. Time and again, this theory has proved to be the most viable theory for gravity.

A few months after Einstein's new formulation of GR, in 1916, Karl Schwarzschild [2] found the solution of Einstein's equation for a point mass. About the same time, Johannes Droste independently arrived to the same solution. The solution had a strange behavior which, nowadays, known as the Schwarzschild radius where the Einstein's equation becomes infinite. However, this infinite singularity was later proven to be the coordinate singularity from the work of Eddington and later by Lemaitre. It was realized via Birkhoff's uniqueness theorem [3, 4] that the space-time geometry outside of a spherical, non-rotating, gravitating body is equivalent (up to the coordinate transformation) of the Schwarzschild geometry. In 1958, David Finkelstein identified the surface area of the Schwarzschild radius as the event horizon as it acts as a one way membrane and any causal curve can cross it only in one direction. This result paved the idea of black hole (BH) and the research on black holes became one of the most active areas in theoretical physics till the date. Later in 1963, Roy Kerr [5] discovered the solution of the rotating black holes–known as the Kerr black hole. Soon, it was shown that using the Penrose process [6], it is possible to extract the energy from the ergosphere of the Kerr black holes. Meanwhile, the no-hair theorem emerged [7–10], which states that the stationary black holes can be completely described by only three parameters: mass, charge and the angular momentum. For a long time, some people doubted the existence of black holes. However, the recent discovery of gravitational waves [1] abolished the doubt regarding the real existence of black hole. A year ago, in 2019, the first ever image of black hole and its surroundings was published [11], which was observed earlier by Event Horizon Telescope in 2017.

In the decade of 1970's, several remarkable works came up which added new perception to the study of black hole theory. These new results have shown the tantalizing connection of gravity with the thermodynamics. In one of his famous works [12], Hawking had shown some important results for black holes in general relativity. Firstly, Hawking described the strong rigidity theorem which states that the event horizon in the stationary spacetime is also a Killing horizon. Secondly, it also describes the topology theorem which states that the cross section of the event horizon of a black hole in four-dimensional stationary and asymptotically flat spacetime corresponds topologically to a 2-sphere if it obeys the dominant energy condition. Most importantly, this paper [12] also provides the area increase theorem of black hole horizon, which states that the horizon area of a black hole horizon always increases when the specific energy condition is satisfied. Inspired by this area theorem and from the idea to rescue the second law of thermodynamics from Wheeler's teacup gedanken experiment, Bekenstein [13, 14] realized that the black holes must have entropy and he ascribed the entropy of the black hole to be proportional to its horizon surface area. In fact, he arrived very close to obtaining the proportionality constant, which he obtained as $(1/2)\ln 2$ in the natural unit. Thereafter, the four laws of black hole mechanics were shown by Bardeen, Carter and Hawking [15], which had an astonishing similarity with the four laws of thermodynamics. However, the authors in this paper refrain themselves from claming it as the thermodynamic laws of black holes. Instead, they claimed it as an analogy with the conventional thermodynamics. In fact, this analogy became a robust correspondence with the thermodynamics when Hawking [16] revealed that the black holes can radiate when quantum effects are taken into the consideration. This radiation was later famously known as the Hawking radiation. Hawking had shown that the radiating photons are thermal in nature and the temperature of the radiating particles are the same as were predicted in [15]. Although Hawking initially tried to disprove Bekenstein's idea about black hole entropy, this work by Hawking justifies the earlier claim by Bekenstein and also fix the proportionality constant of black hole entropy with the horizon area as $1/4$ in natural unit. In the meantime, Fulling [17] Davies [18] and Unruh [19] had shown that the accelerating observer observes thermal radiation in the Minkowski vacuum, whereas the inertial observer does not. The temperature of the Unruh particles have the same form as of the Hawking temperature, except the surface gravity of black hole horizon is replaced by the acceleration of the observer. This radiation was later known as the Unruh radiation. The Unruh radiation and the Hawking radiation are equivalent on the basis of Einstein's equivalence principle. These were the stepping stone which laid the foundation of black hole thermodynamics. Since then, there have been numerous works in the direction of black hole thermodynamics and it became one of the most high-yielding domains for the theoretical physicists over the years. Later several thermodynamic features were found in black hole thermodynamics and the earlier analogies (the area of the black hole horizon as the entropy, surface gravity of the horizon as the temperature etc.) were firmly identified as the physical thermodynamic parameters of black holes.

The proof of the zeroth law and the first law of black hole thermodynamics largely depends on the Killing horizon and the Killing symmetry. Unlike the definition of the event horizon, the Killing horizon depends on the symmetry of the underlying spacetime. It was first shown by Carter [20] that the time-like Killing vector t^a is normal to the black hole horizon in a static spacetime. Whereas, for a stationary-axisymmetric black hole, there exists a Killing field of the form

$$\xi^a = t^a + \Omega_{\mathcal{H}} \phi^a \ , \tag{1.1}$$

which is normal to the horizon. Here, $\Omega_{\mathcal{H}}$ was identified as the angular velocity of the horizon. The result provided by Carter does not depend on the gravitational field equations. Later, Hawking (in [12]) proved the strong rigidity theorem which we have mentioned above. However, Hawking's result depend on the Einstein's field equation and, therefore, only valid in Einstein's gravity.

With these two versions of the Rigidity theorems (provided by Carter [20] and Hawking [12]), the two versions of zeroth law were established. Firstly by Carter [20], who argued that if the black hole is static or stationary and axisymmetric then the surface gravity κ will be constant on the black hole horizon. The second version of the zeroth law was obtained by Bardeen, Carter, and Hawking [15], where it was obtained that if the Einstein's equation holds and the energy-momentum tensor satisfies the dominant energy condition, then the surface gravity κ will be constant for any Killing horizon.

There are several expressions of the first laws in black hole thermodynamics. The "equilibrium version" of the first law of black hole depends on both the Killing symmetry and the presence of a bifurcation Killing horizon. A bifurcation Killing horizon is a two-dimensional spacelike surface, which is intersected by two Killing horizons, which are generated by the same Killing vector ξ^a. It implies that ξ^a has to vanish on the bifurcation Killing horizon. Conversely, if a Killing vector ξ^a vanishes on a two-dimensional spacelike surface, the two-dimensional surface has to be a bifurcation Killing horizon (see [21]). From the zeroth law it can be shown [22] that in the "maximally extended" spacetime representing a stationary black hole, the horizon is made of a branch of bifurcate Killing horizon if $\kappa \neq 0$. Thus, black holes are classified into two categories: one is the black hole with non-zero κ, the horizon of which is comprised of bifurcation Killing horizons and the "extremal black holes" with vanishing surface gravity.

The differential form of the first law was established as an identity connecting the changes in the black hole mass (M), horizon area (A), angular momentum (J) and the charge (Q) of a stationary black hole. Its form is given as follows:

$$\delta M = \frac{1}{8\pi}\kappa \delta A + \Omega_{\mathcal{H}} \delta J + \Phi_{\mathcal{H}} \delta Q \ . \tag{1.2}$$

In the original derivation [15] the perturbation was taken as stationary. Later, this proof has been extended for non-stationary perturbation provided that the change is evaluated at the bifurcation surface of the unperturbed black hole [23, 24]. Later, it was found that for higher curvature gravity theory [25–27] and for the Brans-Dicke theory [28], the Bekenstein formula of entropy (proportional to the horizon area) breaks down. However, for the diffeomorphism invariant theory, later Wald [24, 29] have shown that the entropy of a black hole can be obtained from the Noether charge for a stationary black hole with a Killing horizon. Unlike the Bekenstein entropy, it was found that the Wald entropy correctly provides the expression of the black hole entropy in the alternative theories of gravity: such as the $f(R)$ gravity, scalar-tensor gravity etc. In addition, it was shown that the thermodynamic laws can be obtained covariantly from the Noether charge on a Killing horizon. Another covariant formalism, to obtain the thermodynamic parameters and the first law of black hole, was obtained from the Abbott-Deser-Tekin (ADT) [30–33] current for the presence of a Killing vector in the spacetime. Thereby, it was shown that the conserved charges (or currents), such as the Noether charge and the ADT charge, plays a crucial role in defining the thermodynamic parameters and obtaining the thermodynamic laws in a covariant way

As we have discussed above, in most of the cases, the first law of black hole has been obtained for the stationary spacetimes, defined by a Killing vector and it is considered that the black hole horizon is a Killing horizon. Those proofs depend largely on the Killing symmetry of the spacetime with the following advantages. The event horizon of black hole is defined in terms of the global causal structure of the spacetime. Therefore, one has to have the causal information of the whole spacetime to identify the event horizon. On the other hand, the Killing horizon is defined where the Killing vector becomes null and, therefore, it is a quasi-local concept. Furthermore, as we have discussed earlier, from the strong rigidity theorem [12] the event horizon of black holes in stationary black holes in GR is a Killing horizon. Therefore, the analysis of the black hole thermodynamics on a Killing horizon is meaningful. If one considers the presence of a Killing vector (and the corresponding black hole horizon as the Killing horizon) in the spacetime, it greatly simplifies the calculations. Despite all these advantages with the Killing horizons, there are some serious drawbacks with it. The strong rigidity conditions has been proved only in Einstein's GR and it is still an assumption in most of other alternative theories of gravity. Moreover, the presence of the Killing vector or the Killing horizon implies that the spacetime is stationary. Therefore, for non-stationary/ time-dependent black hole, those formulations do not work. One needs to explore further in this direction to get more insight about black hole thermodynamics in non-stationary spacetime.

Phase transition is another important aspect of thermodynamics, which is also found in black hole thermodynamics as well and it has been studied for several decades. There are several types of phase transitions which are present in black hole thermodynamics. It was first introduced by Davies [34], who argued that black holes undergo a second order phase transition when it passes through a point, which known as the Davies' point, where the heat capacity diverges. However, later Kaburaki et. al. [35–38] proved that the Davies's point is not the critical point. Instead, it is a turning point, where the stability changes. Another type of black hole phase transition was found in the work of Hawking and Page [39]. It was found that a black hole in AdS space makes transition to a no-black-hole state (or radiation) at a critical temperature. In addition, the transition of black holes from a non-extremal to an extremal one is also been found out as a phase transition of black holes [38, 40–48], which is known as the extremal phase transition.

There is another type of phase transition in black hole thermodynamics, which is the van der Waals type phase transition. The presence of $P-V$ criticality was demanded earlier in some works [49, 50]. Where the P was identified as the inverse of the Hawking temperature (i.e. $1/T_H$) and V as the horizon radius (r_H). Such identification was merely due to the similarity between the $P-V$ diagram of the usual thermodynamics and the $1/T_H$ vs r_H diagram in the black hole thermodynamics. Although, it apparently seems intriguing, the main problem with this formulation was the lack of canonical definition of P and V. Later, when the cosmological constant (Λ) was taken as a variable for black holes in AdS spacetime, the usual first law of black hole thermodynamics was modified with an extra $V\delta P$ term [51–55] (we discuss the procedure of obtaining first law with Λ as a variable using Wald's formalism in this thesis). In this case, the pressure is identified as proportional to the cosmological constant and the conjugate quantity is identified as the volume. Also, the mass of the black hole is identified as the Enthalpy of the system instead of the internal energy. With this identification, it has been found that the $P-V$ diagram of black hole looks exactly similar to the usual thermodynamics implying the existence of $P-V$ criticality in black hole spacetime [56, 57]. In addition to the $P-V$ diagram in black hole spacetime, the $T-S$ [58, 59] and $Q-\Phi_H$ [60] (where Q is the charge of the black hole and Φ_H is the corresponding potential at the horizon surface) diagram are also found to indicate the van der Waals type phase transition. These van-der Waals type phase transitions in black holes have been studied case-by-case in different spacetimes. However, it has been found that the case-by-case study gives the same results (such as the behavior of the system near the critical point, critical exponents etc.) irrespective of any spacetime. Therefore, one obvious question remains whether these results can be obtained in a general way for any arbitrary spacetime. To get the answer, one has to investigate further in this direction.

The behavior of phase transition point has been studied geometrically for a long period of time. The prescription of geometry was initially introduced in thermodynamics by Gibbs [61], Carathéodory [62], Fisher [63] and Rao [64]. Later Weinhold [65] introduced the Riemannian geometry, where he defined the metric (known as the Weinhold metric) as the Hessian of the internal energy. Later Ruppeiner [66, 67] followed a slightly different approach and formulated the metric (Ruppeiner metric) as the Hessian of the entropy. In both of these works it has been claimed that the phase transition can be studied by these metric formalisms and the critical point will correspond to the curvature singularity. These claims have been verified in various systems. Later, it was found that the two formalisms, proposed by Weinhold and Ruppeiner, does not agree with each other [68]. It was claimed that this discrepancy appear due to the fact that the two metrics are not formulated in a Legendre-invariant way [69–75]. As the thermodynamic description is invariant under the Legendre transformation of the thermodynamic potential, Quevedo et.al. proposed to formulate thermogeometric metric in a Legendre-invariant way [69–75]. Their formulation is famously known as the geometrothermodynamics (GTD). Using GTD, both Davies type phase transition and the extremal phase transition has been widely explored. It is, therefore, interesting to investigate the possibility of such thermogeometric description for van der Waals type phase transition of black holes using GTD.

Despite enormous success of Einstein's GR, there are a few unsolved issues, which has lingered over the theory for decades. At least, there are two major reasons where people have felt that the Einstein's GR is not, perhaps, the ultimate theory for gravity. One reason is relatively old, which is that the Einstein's GR does not seem to be a compatible for explaining gravity at the quantum length scale. A proper quantum theory of gravity is yet to be developed. Furthermore, the theory breaks down near the singularity, be it either black hole singularity or the big bang singularity. This implies that the theory is not good enough for the description of gravity at small scale, or at high energy limit. Thus, the present theory of gravity, as proposed by Einstein, is not the ultimate one to describe gravity at the quantum limit.

The second reason stems from the current observational data. Einstein's GR cannot explain the present accelerated expansion of the Universe [76–84]. In addition, it also fails to explain the galaxy rotation curves, the inflation model etc. If one tries to explain these phenomena within the framework of Einstein's gravity, one needs to incorporate huge amounts of invisible matter and energy, which we call as the dark matter and the dark energy. Even today physicists do not have much idea about these dark matter and energy.

Both these issues, which are mentioned above, imply that Einstein's general theory of relativity has to be replaced by a more correct version. However, the more correct version of Einstein's GR is not figured out properly yet. With this hunt for a better gravity theory, there are several alternative theories of gravity, which are developed due to several different motivations. Some of those have been motivated from the theoretical viewpoints while some others are favored by the experimental results. One crucial notification in this regard, which one has to remember while forming an alternative theory of gravity is: those alternative theories might deviate remarkably from the Einstein's GR at very high gravity such as in the case of black holes and neutron stars, but those theories must not differ significantly with Einstein's theory at the weak gravity limit, such as in the solar scale regime. Thus, the framework, which has been formulated by Einstein in general relativity, cannot be ruled out completely. This is why, the alternative theories of gravity which are popular in the present time, such as the $f(R)$ gravity, scalar-tensor gravity, Lanczos-Lovelock gravity, higher derivative gravity etc., are premised upon the same concept of the spacetime geometry, which was provided by Einstein in GR.

The scalar-tensor (ST) theory of gravity has been one of the most popular among the alternative theories of gravity for a long period of time. Under the framework of this theory, the cosmological observation has been studied for the present and the earlier time [85, 86]. In addition, the theory is also favored by the string theory [87]. Moreover, the higher curvature $f(R)$ gravity can been studied as a particular type of the ST gravity. In ST theory, the gravity is mediated both by the metric tensor as well as by a scalar-field ϕ, which is non-minimally coupled with the Ricci scalar R in the action of the Jordan frame. This non-minimal coupling can be removed by a set of transformation: (i) by the conformal transformation of the metric and (ii) a re-scale in the scalar field ϕ. With the help of these two transformations one arrives to the Einstein frame where the scalar field can be separated out from the gravity, which is described only by the metric tensor in the Einstein frame. Although the theory is very popular both from the theoretical point of view as well as from the observational point of view, there have been some serious ongoing debates in this theory for decades. The first issue is: the two frames in which the ST theory is described (the Jordan and the Einstein frame), which are conformally (and thereby mathematically) equivalent, whether those are physically equivalent, or one of the frames is more physical compared to the other (see the review [88] and the references therein). The second issue is: whether covariant formalism of thermodynamic laws is possible for this theory. It is highly non-trivial because there is no unanimous conclusion regarding the expression of energy, which can play the role of internal energy in black hole thermodynamics. There are several prescriptions of defining energy in GR, such as: ADM energy [89], Hawking-Hayward quasi-local energy [90, 91], Misner-Sharp energy [92, 93], Kodama energy [94, 95], Brown-York energy [96] etc. Some of those are conformally invariant (such as the Brown-York energy [97] and some of them are not (such as the Misner-Sharp energy [98], Hawking-Hayward quasi-local energy [99–101] etc.). This makes the physicists confused regarding which of these energy is to be used for thermodynamic purpose. In addition, it has been shown earlier [28, 102], the area law of entropy provided by Bekenstein breaks down for this theory. Thus, it is required to investigate the possibility to define thermodynamic parameters and to obtain thermodynamic laws in a covariant way. Moreover, it is also worthwhile to examine whether the thermodynamic parameters are equivalent across the two frames.

A novel motivation to pursue in the direction of black hole thermodynamics came up when it interpreted the gravitational force in an unprecedented way. According to this interpretation, gravity should not be considered as a fundamental force at all. Instead, it is an emergent phenomenon, like the thermodynamics or the elasticity. This idea was predicted long ago by Sakharov [103]. However, the idea was resurrected in the work of Ted Jacobson in 1995 [104], where, accounting the physical process version of the first law on local Rindler horizon, it was shown that the Einstein's equation can be obtained as an equation of state. Therefore, it was claimed that it might not be appropriate to quantize Einstein's equation canonically as the situation will be similar to quantizing the wave equation for sound. Later T. Padmanabhan et al. contributed largely in this direction (see the review [105]). Several remarkable results came up in this direction suggesting that the gravity can be interpreted as an emergent phenomena.

From the discussions presented above, it shows that although black hole thermodynamics has been explored for a long period of time, there are still some obvious motivations for further research in this direction, which we have explored in this thesis. Since there are several indications that gravity can be viewed as an emergent phenomenon and also there are several reasons to believe that Einstein's gravity is not the ultimate theory of gravity, it is worthwhile to investigate the possibility of obtaining a consistent thermodynamic description in popular alternative theories of gravity such as the scalar-tensor gravity or the $f(R)$ gravity (which can be studied as a subclass of the ST gravity with $f'(R) = df(R)/dR = \phi$). It is also necessary to test whether the two conformally connected frames are equivalent from the thermodynamic point of view. While investigating the thermodynamic properties for the popular scalar-tensor gravity seems alluring, there are several aspects of Einstein's gravity itself which are needed to be investigated in order to have better understanding about the black hole thermodynamics in Einstein's gravity. Firstly, most of the works, which describe black hole thermodynamics, depend largely on the Killing symmetry of spacetime as we have mentioned above. This means that the spacetime is assumed to be stationary. Therefore, it will be interesting to examine whether one can obtain the thermodynamic description for a time-dependent black hole: such as the Sultana-Dyer (SD) black hole [106]. Moreover, it seems also important whether one can look beyond the Killing horizon and obtain the thermodynamic laws/quantities for the non-Killing horizons. Since the conserved currents (such as the Noether and the ADT current) define thermodynamic quantities in Einstein's gravity, it is necessary to investigate the possibility to obtain the conserved currents for non-Killing diffeomorphisms. Furthermore, the microstates, which provide the thermodynamic laws of the horizon, are not well-understood. Therefore, one can explore the possibilities to obtain the physical thermodynamic parameters of black holes (such as entropy) from the symmetry of the black hole horizon. Moreover, the van der Waals phase transition in the black holes has been studied case by case for different spacetimes. However, the results, such as the critical exponents, are found to be the same irrespective of any spacetime. This indicates that there must be a general framework, using which one can obtain all the earlier results, which are obtained case by case. Also, one is required to explore further to check whether a thermogeometric description (using GTD) is possible for van der Waals phase transition of black hole.

In the present thesis, we shall explore all these thermodynamic aspects of gravity within and beyond Einstein's GR (such as the ST gravity or $f(R)$ gravity). The discussions in the present thesis will be presented in two parts. In the first part, we shall investigate the thermodynamic aspects of Einstein's gravity and of the ST gravity, where the goal will be to obtain a consistent thermodynamic description on the black hole horizons (there are several ways of defining black hole horizon as we shall discuss later) and for time dependent cosmological black hole (SD one). In the second part, we shall discuss about the van der Waals phase transition of black hole thermodynamics. In that part, the goal will be to provide a general framework to study van der Waals criticality in black holes, which will be valid for any arbitrary spacetime that shows the van der Waals type criticality. In the following, we provide the chapter-wise overview of the thesis.

1.2 Chapter-wise overview: Outline of the thesis

Here, let us get the brief idea of the present thesis, which is based on the publications [107–113].

Chapter-2: As we have discussed above, the ST theory is described in the two frames, which are conformally connected. There are two major longstanding problems in this theory, which are not solved yet. One issue is the debate– whether the two frames are physically equivalent and the other issue is to construct a consistent thermodynamic description in the two frames. In this chapter, we study the scalar-tensor theory of gravity profoundly in the action level as well as in the thermodynamic level. We point out that the action in the two frames are not exactly conformally equivalent– a fact which is not usually mentioned in the literature. We show that the gravitational actions in the two frames differ by a total derivative term. However, we show that when the Gibbons-Hawking-York (GHY) boundary term is added to the gravitational action, then the total actions (gravitational action along with the GHY term) in the two frames are conformally equivalent. Then, we decompose the gravitational actions (in the two frames) in a bulk term as well as a total derivative surface term. Since, the surface term is a total derivative term, we show that the correct way of obtaining equations of motion is to obtain it only from the bulk part. Then, we show that in Einstein frame, the bulk term and the surface term is connected by the holographic relation– a relation which is also satisfied in Einstein-Hilbert action (or in Einstein's gravity). However, we show that the holographic relation breaks down in the Jordan frame. Which implies an equivalence in the two frames at the classical level. Then we obtain a Bianchi-like identity in the two frames of scalar-tensor theory, which helps us later to obtain conserved Noether current and potential off-shell. In the later part, we discuss the two frames from the thermodynamic point of view. We obtain the expression of entropy from the first principles using the Virasoro algebra and show the expression of entropy is conformally invariant. Then we express the action of the two frames as the free energies of the spacetime, where we obtain that the thermodynamic energy and the temperature are conformally equivalent. Again we show that the holographic relation is maintained in the Einstein frame at the thermodynamic level, while it breaks down in the Jordan frame. Finally, we establish the connection of thermodynamic quantities in the two frames from the GHY term to check whether it agrees to the analysis that we made from the gravitational actions.

Chapter-3: In chapter-2, we show that the thermodynamic parameters are conformally equivalent. However, we could not formulate a covariant expression of the thermodynamic parameters. Also, we needed to establish thermodynamic laws in a consistent way. Therefore, in this chapter, we revisit the thermodynamic aspects of the scalar-tensor theory of gravity in the Jordan and in the Einstein frame. We redefine the Lagrangian of the Jordan frame and show that the inequivalence, mentioned in the earlier chapter, goes away. In addition, this redefining of Lagrangian helps to formulate a covariant thermodynamic description in the two frames, from which we can show the connection of the thermodynamic parameters in the two frames. We establish the thermodynamic descriptions from the conserved currents and potentials by following both Wald's formalism (using the Noether current due to the diffeomorphism invariance) and the Abbott-Deser-Tekin (ADT) formalism (using the ADT current due to the Killing invariance). With the help of the conserved Noether current and potential, we define the thermodynamic quantities, which we show to be *conformally invariant*. Moreover, the defined thermodynamic quantities are shown to fit nicely in the laws of (the first and the second) black hole thermodynamics. We stretch the study of the conformal equivalence of the physical quantities in these two frames by following the ADT formalism. Our further study reveals that there is a connection between the ADT and the Noether conserved quantities, which signifies that the ADT approach provide the equivalent thermodynamic description in the two frames as obtained in Noether prescription. Our whole analysis is very general as the conserved Noether and ADT currents and potentials are formulated *off-shell* and the analysis is exempted from any prior assumption or boundary condition.

Chapter-4: Establishing thermodynamic laws in black holes largely depends on the Killing symmetry on the spacetime. This is because, when the Killing symmetry is assumed, several terms in the calculations gets simplified. Even in chapter-3, we establish the thermodynamic laws for ST gravity assuming the Killing symmetry. However, the Killing symmetry is not guaranteed unless it is assumed that the spacetime is static. But, in a realistic situation, a spacetime should not be static. Therefore, one has to look beyond Killing horizon. One of the solutions of Einstein's gravity is Sultana-Dyer (SD) spacetime, which is not static. Instead, it evolves with time. In addition, the horizon of SD black hole is not a Killing one. Therefore, the natural way of defining thermodynamic entities for the stationary ones is not applicable in the case of a time dependent spacetime. Moreover, in the literature, there exist two expressions of horizon temperature for SD black hole – one is time dependent and the other does not depend on time. To single out the correct temperature, in this chapter, we find the temperature by studying the Hawking effect in the tunneling formalism. With this approach, we obtain that the temperature is time dependent. After identifying the correct temperature, the Einstein's equations are written on the horizon and we show that this leads to the first law of thermodynamics. In this process the expressions for horizon entropy and energy, obtained earlier by explicit calculations, are being used. This provides the evidence that Einstein's equations have thermodynamic structure even for a cosmological (SD) black hole spacetime. Moreover, this study further clarifies the correctness of the expressions for the thermodynamic quantities; like temperature, entropy and internal energy.

Chapter-5: As it is mentioned earlier, to establish thermodynamic laws in non-static and realistic scenario, one has to look beyond Killing symmetry and the Killing horizon. In addition, there are several classes of non-Killing horizons defined in general relativity; such as the apparent horizon, trapped horizon etc. However, a covariant formulation of thermodynamics is not well-established for the wide class of non-Killing horizons. In chapter-3, we show that the conserved currents can elegantly formulate thermodynamic laws in a covariant way. As the conserved Noether current is already defined for any arbitrary diffeomorphism, in this chapter, we obtain the conserved Abbott-Deser-Tekin (ADT)-*like* current for the Lanczos-Lovelock gravity for a diffeomorphism vector, which defines the horizon of a spacetime and, importantly, is not necessarily a Killing vector. As the original ADT current is defined only for the presence of a background Killing vector, one cannot use it extensively for the thermodynamic description of the wide classes of non-Killing horizons which appear in gravity. On the other hand, this general approach can be utilized for those horizons. Moreover, we show that the conserved ADT current (for arbitrary diffeomorphism vector) can be written as the derivative of the two-rank anti-symmetric ADT potential. We also provide the connection of ADT potential with the conserved Noether potential from our analysis. If one assumes the diffeomorphism vector as the Killing one, the results match to the original ADT case, which is defined only for the Killing symmetry. However, we mention how the non-trivial results come for the conformal Killing vectors and other horizon defining diffeomorphism vectors.

Chapter-6: In conventional thermodynamics, the microstates which account for the thermodynamics, are well-known and well-understood. However, in black hole thermodynamics, obtaining thermodynamics from the information of microscopic configuration, is not properly known. In this chapter, we show that the diffeomorphisms, which preserve the null nature for a generic null metric very near to the null surface, provide *noncommutative* Heisenberg algebra. The analysis revels that the algebra is very general as it is obtained for a generic null surface and is applicable for any spacetime horizon. Finally using these results, the entropy of the null surface is derived in the form of the Cardy formula. Our analysis is completely *off-shell* as no equation of motion is used. Thus, the analysis in this chapter, illuminates the paradigm of "gravity as an emergent phenomenon" and this formalism can provide a possible way towards understanding the origin of gravitational entropy from the microscopic configuration of the spacetime.

Chapter-7: It is well known that interpreting the cosmological constant as the pressure, the AdS black holes behave as the van der Waals thermodynamic system. In this case, like a phase transition from vapor to liquid in a usual van der Waals system, black holes also change phases about a critical point in the P-V picture, where P is the pressure and V is the thermodynamic volume. In this chapter, considering the cosmological constant as a variable, we obtain the first law from Wald's formalism using conserved Noether current. Then we provide a general framework where we give a geometrical description of this phase transition. Defining the relevant Legendre invariant thermogeometrical metrics corresponding to the two criticality conditions, which determine the critical values of respective thermodynamical entities, we show that the critical point refers to the divergence of the Ricci scalars calculated from these metrics. The similar descriptions are also provided for the other two pictures of the van der Waals like phase transition: one is T-S and the other one is Y-X where T, S, X and Y are temperature, entropy, generalized force and generalized displacement; i.e. potential corresponding to external charge, respectively. The whole discussion is very general as no specific black hole metric is being used. Therefore, the analysis is valid for any arbitrary spacetime that shows van der Waals phase transition.

Chapter-8: Earlier, in literature, the critical exponents for the van der Waals phase transition has been studied case by case in different spacetimes. Interestingly, the critical exponents for the $P-V$ criticality in different black hole spacetimes are the same and they matches identically to the standard ones in the conventional thermodynamics. This implies that there should by a metric-independent way to obtain these critical exponents. Motivated by this fact, we give a general expression for the Helmholtz free energy near the critical point which correctly reproduces these exponents. The idea is similar to the Landau-Ginzburg model which gives a phenomenological description of the usual van der Waals phase transition. Here, two main inputs are taken into account for the analysis: (a) black holes should have van der Waals like isotherms and (b) free energy can be expressed solely as a function of thermodynamic volume and horizon temperature. This shows that the obtained form of Helmholtz free energy correctly encapsulates the features of Landau function. We also discuss the cases where the *isolated critical point* appears, accompanied by nonstandard values of critical exponents. The whole formalism is being then extended to other two criticalities, namely $Y-X$ and $T-S$ (based on the standard; i.e. non-extended phase-space), where T and S are horizon temperature and entropy, respectively while X is the potential corresponding to charge Y. We observe that for former case Gibbs free energy plays the role of Landau function, whereas in the case of later one that is played by the energy (here it is black hole mass). Our analysis shows that, although to check the existence of van der Waals phase transition relies on the explicit form of the black hole metric, the values of the critical exponents are very general.

Chapter-9: In this chapter, we provide the conclusion of the entire thesis. Also, we discuss the possible directions for future work

From the next chapter we start the main analysis of the thesis. Here we shall use geometrized unit where, $c = G = k_B = 1$. Here c is the velocity of light, G is Newtonian constant of gravitation, k_B is Boltzmann constant etc. Also, we shall use the signature of the Lorentzian manifold as {-, +, +, +} .

Part I

Thermodynamic aspects of black holes: within and beyond Einstein's gravity

Chapter 2

Inequivalence of two conformally connected frames in scalar-tensor theory: a classical analysis

[1]Before Einstein came up with his novel idea of general relativity, a scalar theory of gravity was earlier proposed by G. Nordström [114], which was considered as the major competitor to the Einstein's formulation of gravity. However, later this work was overshadowed by the seminal work of Einstein and soon Einstein's GR was unanimously accepted as the standard theory of gravitation. A few years later Kaluza [115] and Klein [116] proposed the five-dimensional theory of gravity with a compactified and constant fifth metric component. This Kaluza-Klein theory unified gravity and electromagnetism as the theory yields Einstein's equation along with Maxwell's equation in four dimensions. Later Jordan modified the fifth component of Kaluza-Klein theory as a variable which lead the gravitational constant to be a variable, as was the speculation of Paul Dirac [117]. Later Brans and Dicke adopted this idea and modified the Einstein-Hilbert action, in order to make a theory which is compatible with Mach's principle, which was left out in Einstein's gravity. The theory proposed by Brans and Dicke is known as the Brans-Dicke theory [118] (also known as the Jordan-Brans-Dicke theory), which is considered as the prototype of today's well-known scalar-tensor theory of gravity. It will be wrong if people thinks that the old idea (provided by Nordström) of a scalar gravity has been resurrected here in scalar-tensor (or Brans-Dicke) theory. In scalar-tensor gravity, the role of scalar field comes into play very non-trivially. The scalar field, in this theory, is non-minimally coupled with the Ricci scalar and the gravity is mediated both by

[1]This chapter is based on the publication [108] .

the metric tensor as well as the scalar field. Although the theory is motivated from the cosmological point of view (*i.e.* Mach's principle), the theory became more popular due to string theory. Since, in string theory, the spin-2 graviton has a spin-0 partner called dilaton, the string theory favours the scalar-tensor theory to be more appropriate for gravity instead of Einstein's GR [87]. In addition, there are several reasons which motivate to study this theory. The recent observations from type Ia supernovae [76–84] suggests the accelerated expansion of the Universe. To explain this phenomenon, one is either required to consider large amounts of exotic matter, which is called the dark matter, or one has to modify Einstein's gravity. The most popular class of alternative gravity which is suggested by the cosmology is $f(R)$ gravity, which can be expressed in terms of Brans-Dicke theory (a specific form of scalar-tensor gravity). Therefore, as it has been mentioned above, there are numerous motivations to study the scalar-tensor theory of gravity.

2.1 Two major unsolved puzzles in scalar-tensor theory for decades

The scalar-tensor theory is analyzed with respect to the two frames, one is known as Jordan frame where the conventional Lagrangian of GR gets modified with the inclusion of the scalar field ϕ. As a result, in the modified action, the Ricci scalar gets non-minimally coupled with the scalar field. This non-minimal coupling can be removed by the conformal transformation of the metric tensor along with the re-scaling of the scalar field and, by the virtue of these transformations, one arrives to the another frame, known as the Einstein frame.

Although the scalar-tensor theory is very popular for the reasons mentioned earlier, there are two major basic issues which are a matter of controversy even today. One of such controversies in literature is about the physical equivalence/inequivalence of the two frames and there are much dissonance among people in interpreting one frame as more physical than the other [119, 120] (for a review, see [88]). Moreover, since the two frames are conformally connected, people argue on whether the conformal equivalence of the actions in the two frames are merely a mathematical equivalence or this equivalence is also reflected in the dynamical [121–125] and the underlying thermodynamic aspects as well [28, 102, 126–128] (also see the recent papers [129–134], which discuss the equivalence of the two frames at the quantum level). Most of the works suggest that the two frames are equivalent in the classical level, but they are not in the quantum level [124, 130–139].

The other issue, which creates difference in opinion among people is related to the thermodynamics in the scalar-tensor theory. There are a few unsolved issues such as, what are the explicit covariant expressions of the physical quantities (energy, entropy, temperature) and how they are connected in the two frames. Although, the expressions of the entropy and the temperature are accepted [28, 126] to some extent but, there is a controversy in the expression of the energy which can be used for the thermodynamic description in this theory. Most of the existing expressions of energy (or mass) as described in literature are not conformally invariant [99–101], whereas, the expressions of the entropy and the temperature are conformally invariant. This makes the physicists more puzzled as none of the existing energies, so far, can be used together with entropy and temperature for the thermodynamic description. In this regard, the question, which remains unsolved for a long time is what is the thermodynamic approach to define the mass (or energy). Thus, in order to resolve these issues, more investigation is required to provide satisfactory answers to these questions.

2.2 Objectives of the chapter

In this chapter, we shall explore the equivalence/in-equivalence of the two frames on the various aspects. Also, many things will be highlighted that are often not mentioned in the literature. We shall cast light on the two frames in their dynamic level as well as in the thermodynamic point of view. To compare the two frames classically, the Lagrangians in the two frames require to be studied in a great detail. Besides, the Gibbons-Hawking-York boundary term will be introduced in both frames, and the role of this term in the discussion of the equivalence of the two frames at the classical level will be mentioned. Later, a comparative study of the two gravitational actions and the same of the Einstein-Hilbert one will be provided and the special properties of the latter action (especially the holographic property of the GR) will be verified so that one can compare the theory with the usual GR case and can comprehend the new aspects of this theory very distinctly. Here, we also discuss how the equations of motion are conformally connected in the two frames. Moreover, the way of obtaining the equations of motion from the bulk part will be discussed and obliterating the contribution from the surface part will be explained. In addition, we obtain a Bianchi-like identity in the scalar-tensor theory of gravity, which will help us to obtain off-shell Noether current due to diffeomorphism invariance.

To compare the two frames at the thermodynamic level, the focus will be on obtaining the entropy in the two frames. For that, the Noether current and the potential of the two frames need to be found as those are directly connected to the thermodynamic quantities. Then the Virasoro algebra technique will be introduced to obtain the entropy from the first principle and, thereby, the relation of entropy in the two frames can be found out. One of the other important features of the Einstein-Hilbert action is that it can be interpreted as the free energy of spacetime. This chapter will also highlight that aspect for the scalar-tensor theory in its two frames. From that discussion, we shall obtain the relation between the other thermodynamic quantities (the energy and the temperature) in the two frames. Once the relations between those quantities in the two frames gets determined from the analysis of the gravitational action, we shall verify all the conclusions from the Gibbons-Hawking-York (GHY) surface terms as well.

In doing all these, we shall see that the two frames are not exactly equivalent even in the classical regime, which people often claims. The explicit reason for such in-equivalence will be figured out.

2.3 Comparison of the two frames at the Lagrangian level

As mentioned earlier, there are numerous works on the equivalence/in-equivalence of the two frames in scalar-tensor theory and still no conclusive result is available. The general belief is that the two frames are classically equivalent but not from the quantum-mechanical point of view. However, from a purely classical analysis and without the consideration of the matter field, we shall see here that the two frames are not equivalent. We shall figure out the main reason of this in-equivalence. Here, we start the analysis from the action level. It will be shown, while projecting the theory from one frame to the other by the conformal transformation of the metric and rescaling of the scalar field, one neglects a total derivative term. Therefore, the usual way of saying the mathematical equivalence of the two frames is an incomplete statement. However, if one incorporates the Gibbons-Hawking-York(GHY) boundary term in the analysis, then the total action(the gravitational with the GHY term) is invariant under those transformation. Later, we shall decompose the total action in terms of the bulk part (containing only the first order derivative of the metric tensor) and the surface part (containing the second order derivative of the metric tensor) to show that the equations of motion of the fields can be obtained from the bulk part only in both the frames. Then, in the Einstein frame, we shall show that the bulk part of the Lagrangian is connected with the surface part by a relation. This relation in literature is called as the "holographic relation" [140–142]. However, we show that the holographic relation cannot be obtained in the Jordan frame, which yield that the two frames are not equivalent even in the classical regime.

2.3.1 Actions in the two frames from bird's eye view

The gravitational action of the scalar-tensor theory in the Jordan frame is given as

$$\mathcal{A} = \int d^4x\sqrt{-g}L = \int d^4x\sqrt{-g}\frac{1}{16\pi}\Big(\phi R - \frac{\omega(\phi)}{\phi}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\Big)~. \tag{2.1}$$

In this frame the scalar field ϕ is nonminimally coupled with the Ricci scalar which makes the theory highly nonlinear. With the help of the conformal transformation of the metric and the re-scaling of the scalar field ϕ, the nonminimal coupling of the field ϕ with the scalar curvature in the Jordan frame goes away and one then arrives to the mathematically equivalent picture or frame which is known as the Einstein frame. The transformations are given by

$$g_{ab} \to \tilde{g}_{ab} = \Omega^2 g_{ab}, \qquad \Omega = \sqrt{\phi}~, \tag{2.2}$$

which is known as the (Eq. (2.2)) conformal transformation. Along with the above transformation of the metric, one is also required a rescaling on the scalar field, which is given as

$$\phi \to \tilde{\phi} \text{ with } d\tilde{\phi} = \sqrt{\frac{2\omega+3}{16\pi}}\frac{d\phi}{\phi}~. \tag{2.3}$$

Now, the gravitational action in the Einstein frame, as mentioned in the literature, is given by

$$\tilde{\mathcal{A}} = \int d^4x\sqrt{-\tilde{g}}\tilde{L} = \int d^4x\sqrt{-\tilde{g}}[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - U(\tilde{\phi})]~, \tag{2.4}$$

with $U(\tilde{\phi}) = \frac{V(\phi)}{16\pi\phi^2}$. As one can see, in this frame the scalar field is not coupled with the Ricci scalar and, therefore, the gravitational part is similar to the GR case. Although, we have not incorporated the matter fields, we comment that the external matter field of the Jordan frame becomes non-minimally coupled with the scalar field ϕ in the Einstein frame.

So far, what we have discussed regarding the action of the two frames, are usually mentioned in the literature. However, we have investigated this and have found that if one takes the Lagrangian of one frame and give the transformations (2.2) and (2.3) to go another frame, it does not yield exactly the same Lagrangian of the another frame. Instead, they are related as

$$16\pi\sqrt{-\tilde{g}}\tilde{L} = 16\pi\sqrt{-g}L - 3\sqrt{-g}\Box\phi \ . \tag{2.5}$$

The last term ($3\sqrt{-g}\Box\phi$) in (2.5) appears mainly because of the fact that the Ricci-scalar is not invariant under the conformal transformation [143]. It changes as

$$\tilde{R} = \frac{1}{\phi}\Big[R + \frac{3}{2\phi^2}(\nabla_i\phi)(\nabla^i\phi) - \frac{3}{\phi}\Box\phi\Big] \ . \tag{2.6}$$

The above $\Box\phi$ term in (2.6) appears in Eq. (2.5). But, in literature, in the discussion of the scalar-tensor theory in two frames, this total derivative term of (2.5) ($3\sqrt{-g}\Box\phi = 3\partial_i(\sqrt{-g}g^{ij}\partial_j\phi)$) is completely ignored. Therefore, studying literature, it seems as if the actions in the two frames are exactly equivalent due to the transformation relations (2.2) and (2.3). However, in Eq. (2.5), we show that the actions are equivalent up to a total derivative term. Then the question arises, why people are not interested about the $\Box\phi$ term? This is probably because of the fact that since $\Box\phi$ term is a total derivative term, presence or absence of this term does not affect the dynamics of the system (as the equation of motion does not get affected due to the total derivative term). Moreover, the motivation of neglecting the $\Box\phi$ term originates from another fact. Since $\Box\phi$ term is a second order total derivative term and L already contains the first order derivative of ϕ, one faces problem while obtaining the equation of motion of ϕ as one has to fix both the scalar field ϕ and its first order derivative on the boundary (this will be explained in more detail on the next chapter where we shall consider $\Box\phi$ term into the account). In this chapter, motivated from the above-mentioned reasons from the earlier works, we shall also neglect $\Box\phi$ term in our analysis. However, as we shall see, neglecting the $\Box\phi$ term will cause several in-equivalences both at the classical level and at the thermodynamic level.

Let us now take a look at the Gibbons-Hawking-York (GHY) term in different cases, which will play an important role on the later part analysis of the present chapter. We start with the Gibbons-Hawking-York boundary action in the Einstein frame. It is given by the standard form:

$$\tilde{\mathcal{A}}_{GHY} = -\frac{1}{8\pi}\oint d^3x\sqrt{\tilde{h}}\tilde{K} \tag{2.7}$$

Here $\tilde{K} = -\tilde{\nabla}_a\tilde{N}^a = \frac{1}{\sqrt{-\tilde{g}}}\partial_a(\sqrt{-\tilde{g}}\tilde{N}^a)$, the trace of the extrinsic curvature tensor and $\tilde{N}_a$ is the unit normal to the slice, which is spacelike or timelike, depending on the type of slice chosen; i.e. $\tilde{g}_{ab}\tilde{N}^a\tilde{N}^b = \epsilon$ where $\epsilon = +1$ spacelike while $\epsilon = -1$ for timelike normals.

In the Jordan frame, the GHY surface term will be

$$\mathcal{A}_{GHY} = -\frac{1}{8\pi}\oint d^3x\sqrt{h}\phi K\ . \tag{2.8}$$

One can easily determine the relation between the K and $\tilde{K}$ as

$$\tilde{K} = \frac{1}{\Omega}K - \frac{3}{\Omega^2}N^a\partial_a\Omega\ , \tag{2.9}$$

where $K = -\nabla_a N^a$.

Now the interesting fact is that, although the actions in the two frames are not equivalent due to the conformal transformation (and the rescaling of the metric), the total action (i.e. the gravitational action along with the surface term) is exactly equivalent in the two frames i.e. $\tilde{\mathcal{A}}_{total} = \tilde{\mathcal{A}} + \tilde{\mathcal{A}}_{GHY} = \mathcal{A} + \mathcal{A}_{GHY} = \mathcal{A}_{total}$. Also, if one takes the matter field into the account, the conclusion of the above analysis will be unchanged.

2.3.2 Decomposition of the action as bulk term and surface term

Let us now have a deeper inspection of the scalar-tensor theory in the two frames. It is a well-known fact that the Einstein-Hilbert Lagrangian (without the Gibbons-Hawking-York surface term) can be separated into two parts. The first one is the quadratic part L_{quad}, containing the $\mathcal{O}(\Gamma^2)$ terms (or only the first order derivative of g_{ab}) and the second one is the total divergence term L_{sur} (containg the second order of g_{ab}) (for details see [140]). Here we introduce the same discussions for the scalar-tensor theory in the Einstein and Jordan frames and we show that one can get the similar terms in these frames as well. Like the usual one, the GHY term will not be considered. We start with the Lagrangian in the Einstein frame since it is similar to GR where this decomposition is known, and then we use the transformations to obtain the required decomposition in the Jordan frame.

Einstein frame: The Lagrangian of the scalar-tensor theory in the Einstein frame is given by (2.4), and it is connected with the Lagrangian of the Jordan frame by (2.5). Now, one can show that

$$\sqrt{-\tilde{g}}\tilde{R} = \sqrt{-\tilde{g}}\tilde{g}^{ab}(\tilde{\Gamma}^i_{ja}\tilde{\Gamma}^j_{ib} - \tilde{\Gamma}^i_{ab}\tilde{\Gamma}^j_{ij}) + \partial_c[\sqrt{-\tilde{g}}\tilde{V}^c] \tag{2.10}$$

where, $\tilde{V}^c = \tilde{g}^{ik}\tilde{\Gamma}^c_{ik} - \tilde{g}^{ck}\tilde{\Gamma}^m_{km}$. Therefore, we can write $\sqrt{-\tilde{g}}\tilde{L} = \sqrt{-\tilde{g}}\tilde{L}_{bulk} + \tilde{L}_{sur}$, where we identify the bulk term in the Einstein frame as

$$\tilde{L}_{bulk} = \frac{1}{16\pi}\tilde{g}^{ab}(\tilde{\Gamma}^i_{ja}\tilde{\Gamma}^j_{ib} - \tilde{\Gamma}^i_{ab}\tilde{\Gamma}^j_{ij}) - \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - U(\tilde{\phi}) \ ; \tag{2.11}$$

whereas the surface part is given by

$$\tilde{L}_{sur} = -\partial_c\tilde{P}^c \ , \tag{2.12}$$

where,

$$\tilde{P}^c = -\frac{1}{16\pi}\sqrt{-\tilde{g}}\tilde{V}^c = \frac{\sqrt{-\tilde{g}}}{16\pi}(\tilde{g}^{ck}\tilde{\Gamma}^i_{ki} - \tilde{g}^{ik}\tilde{\Gamma}^c_{ik}) \ . \tag{2.13}$$

Jordan frame: Now, using the transformations (2.2) and (2.3), one can get the corresponding terms in the Jordan frame. Straight forward calculations give

$$\begin{aligned}\tilde{g}^{ab}(\tilde{\Gamma}^i_{ja}\tilde{\Gamma}^j_{ib} - \tilde{\Gamma}^i_{ab}\tilde{\Gamma}^j_{ij}) &= \Omega^2 g^{ab}\sqrt{-g}[\Gamma^i_{ja}\Gamma^j_{ib} - \Gamma^i_{ab}\Gamma^j_{ij}] - 2\Omega^2 g^{ab}\sqrt{-g}\Gamma^i_{ab}(\partial_i \ln\Omega) \\ &+6\sqrt{-g}\Omega^2(\partial^i\ln\Omega)(\partial_i\ln\Omega) + 2\sqrt{-g}\Omega^2\Gamma^i_{ij}(\partial^j\ln\Omega) \ ;\end{aligned} \tag{2.14}$$

and

$$\sqrt{-\tilde{g}}\tilde{V}^c = \Omega^2\sqrt{-g}(g^{ik}\Gamma^c_{ik} - g^{ck}\Gamma^m_{km}) - 6\Omega^2\sqrt{-g}\partial^c(\ln\Omega) \ . \tag{2.15}$$

Therefore, one obtains

$$\begin{aligned}\sqrt{-\tilde{g}}\tilde{L} &= (1/16\pi)\Big[\Omega^2 g^{ab}\sqrt{-g}[\Gamma^i_{ja}\Gamma^j_{ib} - \Gamma^i_{ab}\Gamma^j_{ij}] - 2\Omega^2 g^{ab}\sqrt{-g}\Gamma^i_{ab}(\partial_i\ln\Omega) \\ &+2\sqrt{-g}\Omega^2\Gamma^i_{ij}(\partial^j\ln\Omega)\Big] - \frac{4}{16\pi}\sqrt{-g}\omega\Omega^2(\partial_i\ln\Omega)(\partial^i\ln\Omega) - \frac{V(\phi)}{16\pi\phi^2} \\ &+(1/16\pi)\Big[\partial_c[\Omega^2\sqrt{-g}(g^{ik}\Gamma^c_{ik} - g^{ck}\Gamma^m_{km})] - \partial_c[6(\Omega^2\sqrt{-g}\partial^c(\ln\Omega)]\Big] \ ,\end{aligned} \tag{2.16}$$

where one needs to use that $U(\tilde{\phi}) = V/16\pi\phi^2$. Using (2.5) one can get the action in the Jordan frame expressed in bulk and surface terms i.e. $\sqrt{-g}L = \sqrt{-g}L_{bulk} + L_{sur}$. These are given by

$$L_{bulk} = (1/16\pi)\Big[\Omega^2 g^{ab}[\Gamma^i_{ja}\Gamma^j_{ib} - \Gamma^i_{ab}\Gamma^j_{ij}] - 2\Omega^2 g^{ab}\Gamma^i_{ab}(\partial_i \ln\Omega) + 2\Omega^2\Gamma^i_{ij}(\partial^j \ln\Omega)\Big]$$
$$-\frac{4}{16\pi}\omega\Omega^2(\partial_i \ln\Omega)(\partial^i \ln\Omega) - \frac{V(\phi)}{16\pi\phi^2} \ , \tag{2.17}$$

and

$$L_{sur} = \frac{1}{16\pi}\partial_c(\Omega^2\sqrt{-g}V^c) = (1/16\pi)\partial_c[\Omega^2\sqrt{-g}(g^{ik}\Gamma^c_{ik} - g^{ck}\Gamma^m_{km})] \ , \tag{2.18}$$

where one can identify $V^c = g^{ik}\Gamma^c_{ik} - g^{ck}\Gamma^m_{km}$.

Now, at this point, one might ask what the significance is of the above discussions. The answer is given in the following two sections, where it will be shown that the equation of motion will be obtained only from the bulk part of the Lagrangian and later the holographic relation will be discussed. Since the surface part is a total derivative term, presence of this term does not affect the equation of motion. Rather, since the surface term contains the second order derivative of the metric tensor, it is in fact better to discard such term while obtaining the equation of motion by varying the action (the reader can recall the similar reasons about why people discard $\Box\phi$ term, which has been discussed earlier). In the following section, we shall describe the method of obtaining the equation of motion from the bulk part in the two frames.

2.3.3 Equation of motion from the bulk term

In Einstein gravity, people obtain the equation of motion from the Einstein-Hilbert action. However, it has been shown that the bulk part of Einstein-Hilbert action is enough to obtain the equation of motion [140]. Here, we introduce the same discussion in the scalar-tensor theory as well. Not only the bulk part will provide the equation of motion, it will ascertain that the terms which we have identified as the bulk and the surface part is appropriate. Again, we start from the Einstein frame for its similarity with the Einstein-Hilbert gravity and for the better understanding of the reader.

Einstein frame: We have $\sqrt{-\tilde{g}}\tilde{L}_{bulk} = \sqrt{-\tilde{g}}\tilde{L} - \tilde{L}_{sur}$ and, hence, the arbitrary variation of the bulk term is given by

$$\delta(\sqrt{-\tilde{g}}\tilde{L}_{bulk}) = \delta(\sqrt{-\tilde{g}}\tilde{L}) - \delta(\tilde{L}_{sur}) \ . \tag{2.19}$$

Now, the variation of $\sqrt{-\tilde{g}}\tilde{L}$ is given as

$$\delta(\sqrt{-\tilde{g}}\tilde{L}) = \sqrt{-\tilde{g}}\tilde{E}_{ab}\delta\tilde{g}^{ab} + \sqrt{-\tilde{g}}\tilde{E}_{(\tilde{\phi})}\delta\tilde{\phi} + \sqrt{-\tilde{g}}\tilde{\nabla}_a\tilde{\Theta}^a(\tilde{q},\delta\tilde{q}) \ , \tag{2.20}$$

where $\tilde{q} \in \{\tilde{g}_{ab}, \tilde{\phi}\}$. The exact expressions of $\tilde{E}_{ab}$, $\tilde{E}_{(\tilde{\phi})}$ and $\tilde{\Theta}^a(\tilde{q},\delta\tilde{q})$ are given as

$$\tilde{E}_{ab} = \frac{\tilde{G}_{ab}}{16\pi} - \frac{1}{2}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} + \frac{1}{4}\tilde{g}_{ab}\tilde{\nabla}^i\tilde{\phi}\tilde{\nabla}_i\tilde{\phi} + \frac{1}{2}\tilde{g}_{ab}U(\tilde{\phi}) \ ;$$

$$\tilde{E}_{(\tilde{\phi})} = \tilde{\nabla}_a\tilde{\nabla}^a\tilde{\phi} - \frac{dU}{d\tilde{\phi}} \ ;$$

and

$$\tilde{\Theta}^a(\tilde{q},\delta\tilde{q}) = \frac{\delta\tilde{v}^a}{16\pi} - (\tilde{\nabla}^a\tilde{\phi})\delta\tilde{\phi} \ . \tag{2.21}$$

The variation of the surface term yields

$$\begin{aligned}\delta(\tilde{L}_{sur}) &= (1/16\pi)\partial_a\Big(\delta(\sqrt{-\tilde{g}}\tilde{V}^a)\Big) \\ &= \frac{1}{16\pi}\partial_a\Big[-\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{ik}\delta\tilde{g}^{ik}\tilde{V}^a + \sqrt{-\tilde{g}}\delta\tilde{v}^a + \sqrt{-\tilde{g}}(\delta\tilde{g}^{ik}\tilde{\Gamma}^a_{ik} - \delta\tilde{g}^{ak}\tilde{\Gamma}^m_{km})\Big]\end{aligned} \tag{2.22}$$

Here $\delta\tilde{v}^a = 2\tilde{P}^{ibad}\tilde{\nabla}_b\delta\tilde{g}_{id}$ and $\tilde{P}^{ibad} = \frac{\partial\tilde{R}}{\partial\tilde{R}_{ibad}} = \frac{1}{2}[\tilde{g}^{ia}\tilde{g}^{bd} - \tilde{g}^{id}\tilde{g}^{ab}]$. Usually in the literature the equations of motion for metric and $\tilde{\phi}$ are obtained from the total gravitational Lagrangian/action (2.4). But this is not conceptually correct as here one needs to fix both the metric and its first derivative at the boundary. Interestingly this does not happen for bulk term as the problematic term $\delta\tilde{v}^a$ cancels out as it appears in both the $\tilde{\Theta}^a(\tilde{q},\delta\tilde{q})$ and in $\delta(\tilde{L}_{sur})$. Then the volume integration of the total derivative terms can be written as the surface integration. Now imposing the conditions that the fields $\tilde{g}_{ab}$ and $\tilde{\phi}$ are fixed at the boundary implies $\tilde{E}_{ab} = 0$ and $\tilde{E}_{(\phi)} = 0$, which results in the known expression of the equations of motion, which are

$$\frac{\tilde{G}_{ab}}{16\pi} - \frac{1}{2}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} + \frac{1}{4}\tilde{g}_{ab}\tilde{\nabla}^i\tilde{\phi}\tilde{\nabla}_i\tilde{\phi} + \frac{1}{2}\tilde{g}_{ab}U(\tilde{\phi}) = 0 \tag{2.23}$$

and

$$\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\phi} = \frac{dU}{d\tilde{\phi}} \tag{2.24}$$

Jordan frame: In the Jordan frame, one can obtain the same result following the similar procedure. Here

$$\delta(\sqrt{-g}L_{bulk}) = \delta(\sqrt{-g}L) - \delta L_{sur} \tag{2.25}$$

In the Jordan frame, the variation of the total Lagrangian results in

$$\delta(\sqrt{-g}L) = \sqrt{-g}\Big(E_{ab}\delta g^{ab} + E_{(\phi)\delta\phi+} + \nabla_a\Theta^a(q,\delta q)\Big) \ . \tag{2.26}$$

Therefore,

$$\delta(\sqrt{-g}L_{bulk}) = \sqrt{-g}\Big(E_{ab}\delta g^{ab} + E_{(\phi)\delta\phi+} + \nabla_a\Theta^a(q,\delta q)\Big) - (1/16\pi)\delta\partial_c(\phi\sqrt{-g}V^c) \ , \tag{2.27}$$

where $q \in \{g_{ab}, \phi\}$ along with

$$E_{ab} = \frac{1}{16\pi}[\phi G_{ab} + \frac{\omega}{2\phi}\nabla_i\phi\nabla^i\phi g_{ab} - \frac{\omega}{\phi}\nabla_a\phi\nabla_b\phi + \frac{V}{2}g_{ab} - \nabla_a\nabla_b\phi + \nabla_i\nabla^i\phi g_{ab}] \ ;$$

$$E_{(\phi)} = \frac{1}{16\pi}[R + \frac{1}{\phi}\frac{d\omega}{d\phi}\nabla_i\phi\nabla^i\phi + \frac{2\omega}{\phi}\Box\phi - \frac{dV}{d\phi} - \frac{\omega}{\phi^2}\nabla_a\phi\nabla^a\phi] \ ;$$

and

$$\Theta^a(q,\delta q) = \frac{1}{16\pi}[-2g^{ab}\frac{\omega}{\phi}(\nabla_b\phi)\delta\phi + \phi\delta v^a - 2(\nabla_b\phi)p^{iabd}\delta g_{id}] \ , \tag{2.28}$$

with $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ being the Einstein tensor, $\delta v^a = 2p^{ibad}\nabla_b\delta g_{id}$ and $p^{ibad} = \frac{\partial R}{\partial R_{ibad}} = \frac{1}{2}[g^{ia}g^{bd} - g^{id}g^{ab}]$. The last term in (2.27) comes from the variation of the surface part and the rest of the previous terms come from the variation of the total Lagrangian. Like the earlier one, we can apply the same arguments here to obtain the equations of motion from only the bulk term of the total Lagrangian. The problematic total derivative term $\phi\delta v^a$ of $\Theta^a(q,\delta q)$ gets canceled from the contribution coming from the variation of the surface part. Then the volume integration of the total derivative terms can be written as the surface integration, which ultimately vanishes as the fields are kept fixed at the boundary of the surface. Then the equations of motion are obtained as

$$\phi G_{ab} + \frac{\omega}{2\phi}\nabla_i\phi\nabla^i\phi g_{ab} - \frac{\omega}{\phi}\nabla_a\phi\nabla_b\phi + \frac{V}{2}g_{ab} - \nabla_a\nabla_b\phi + \nabla_i\nabla^i\phi g_{ab} = 0 \tag{2.29}$$

and, using the above equation of g^{ab} the coefficient of $\delta\phi$ gives the equation of motion of the field ϕ as

$$\Box\phi = \frac{1}{2\omega+3}\Big(\phi\frac{dV}{d\phi} - \frac{d\omega}{d\phi}\nabla^i\phi\nabla_i\phi - 2V\Big) \tag{2.30}$$

So, one can conclude that the equations of motion can be obtained only from the bulk part of the Lagrangian. It should be mentioned here that the equation of motions of $\tilde{g}^{ab}$ and g^{ab}(or $\tilde{\phi}$ and ϕ) in the two frames are equivalent to each other by the transformation of the metrics given by (2.2) and (2.3). In other words, if the equation of motion of say, the metric tensor (or the scalar field) is given in any frame, one can obtain the equation of motion of the same field in the other frame using (2.2) and (2.3). We shall use this result in our later discussions. Thus, in the dynamic level the two frames are equivalent. However, it must be mentioned that although the equations of motion in the two frames are equivalent, the surface terms $\Theta(q,\delta q)$ and $\tilde{\Theta}(q,\delta q)$ are not equivalent to each other under the transformations (2.2) and (2.3) (this will be discussed in more detail in the next chapter). In the following, we discuss another implication of dividing the Lagrangian of the two frames into the surface and the bulk part, which is the Holographic relation.

2.3.4 Connection of the surface and the bulk terms of the action: The holographic relation

One of the most striking feature of the Einstein-Hilbert action in GR is that, the surface and the bulk part of the Lagrangian are connected to each other (for details see the original works [140, 141] for GR and [142] for Lanczos-Lovelock gravity). The basic idea is as follows: It can be shown that for the given Lagrangians, say $L_1(q,\,\partial q)$ and $L_2(q,\,\partial q,\,\partial^2 q)$ which are related as

$$L_2(q,\partial q,\partial^2 q) = L_1 - \partial\big(q\frac{\partial L_1}{\partial(\partial q)}\big), \tag{2.31}$$

the same equation of motion is produced when one extremizes the actions. For the former case one has to fix q and for the latter case one has to fix $\frac{\partial L_1}{\partial \dot{q}}$ (which is the canonical momentum) on the boundary. Remarkably, the last total derivative of (2.31) term matches with the surface part of the Lagrangian in GR when L_1 is the bulk part (which is a function of g^{ab} and ∂g^{ab}) and L_2 is the total Lagrangian (which is a function of g^{ab},∂g^{ab} and $\partial^2 g^{ab}$). This relation in literature is known as the holographic relation[140].

Here we check whether we get the same result for the scalar-tensor theory in the two frames as well.

Einstein frame: The action in the Einstein frame is quite similar to the Einstein-Hilbert action, and the Lagrangian in this frame is the function of the metric tensor $\tilde{g}^{ab}$ and its first- and second-order derivatives, but it depends only on $\tilde{\phi}$ and its first-order derivative. In spite of that, differentiation of $\sqrt{-\tilde{g}}\tilde{L}_{bulk}$ with respect to only the $\partial_c\tilde{g}_{ab}$ gives the desired result, and we do not need any term containing the differentiation of the bulk Lagrangian with respect to $\partial_i\tilde{\phi}$. This is because the total Lagrangian is not the function of the second-order derivative of $\tilde{\phi}$. From (2.31), one can say that q cannot be $\tilde{\phi}$. Otherwise, the total Lagrangian will be the function of the second-order derivative of $\tilde{\phi}$, which we cannot allow here. One can check that the holographic property is maintained here in this frame and, a straightforward calculation gives the relation between the surface and the bulk part as

$$\sqrt{-\tilde{g}}\tilde{L} = \sqrt{-\tilde{g}}\tilde{L}_{bulk} - \partial_c[\frac{\partial\sqrt{-\tilde{g}}\tilde{L}_{bulk}}{\partial\tilde{g}_{ij,c}}\tilde{g}_{ij}]. \tag{2.32}$$

Here the last term is the surface term. So, it is obvious from the above that this can be obtained from the bulk part of the Lagrangian. Thus, as is done for the Einstein-Hilbert action [140], one can get the same relations for the scalar-tensor theory in the Einstein frame as well. Therefore, like the Einstein-Hilbert case, one can correlate the bulk part of the Lagrangian $\tilde{L}_{bulk}$ with $L_1(q,\,\partial q)$ and the total gravitational Lagrangian with $L_2(q,\,\partial q,\,\partial^2 q)$ and one can conclude that in this frame the gravity is "holographic".

Jordan frame: Let us now check whether the same conclusion can be drawn for the theory in Jordan frame as well. We have already separated the Lagrangian in the Jordan frame in bulk part and the total derivative surface part (2.17) and (2.18). Here also, the total Lagrangian is the function of g^{ab} and it first- and second-order derivative, but it depends solely on ϕ and its first-order derivative. Thus, here also we can apply the same arguments and we cannot allow q to be ϕ in (2.31) to obtain the holographic relation. We perform the same steps as done in Einstein's frame and find

$$\frac{\partial\sqrt{-g}L_{bulk}}{\partial g_{ab,c}}g_{ab} = -\frac{1}{16\pi}\Omega^2\sqrt{-g}V^c + \frac{6}{16\pi}(\Omega^2\sqrt{-g}\partial^c(\ln\Omega)) \tag{2.33}$$

Note that the right-hand side of the above equation is the surface part plus an extra term. So, unlike in the Einstein frame, one cannot draw the same conclusion for the theory in the Jordan frame. Thus the holographic relation cannot be obtained in the Jordan frame when the Lagrangian is taken as L. This is a significant inequivalence of the two frames at the classical level even without considering any matter field. In the next chapter it will be discussed how one can get rid of this in-equivalence.

So far, from the analysis in the action level, we have obtained several in-equivalences in the two frames. Now we shall proceed towards the analysis from the thermodynamic point of view. It is well-known that the conserved Noether current due to the diffeomorphism plays an important role in black hole thermodynamics. In Einstein's gravity, one can obtain the off-shell (without using any equation of motion) Noether current only due to the Bianchi identity. In the following we shall obtain the Bianchi-type identity, which will help us to obtain the conserved Noether current in scalar tensor theory.

2.3.5 Generalized Bianchi identity in scalar-tensor theory of gravity

It is well known that in Einstein's gravity, one can define the conserved off-shell Noether current due to the fact that the covariant derivative of the Einstein tensor vanishes (see the project 8.1 of [144]). In the scalar-tensor theory, we are able to find out a similar identity, which helps us to formulate the off-shell Noether current.

Jordan frame: From the Eq. (2.28), we calculate $\nabla_b E^{ab}$, which is given as

$$\begin{aligned}\nabla_b E^{ab} = G^{ab}(\nabla_b\phi) - \frac{1}{2\phi}\frac{d\omega}{d\phi}(\nabla^a\phi)(\nabla^b\phi)(\nabla_b\phi) - \frac{\omega}{\phi}(\nabla^a\phi)\Box\phi \\ +\frac{1}{2}\frac{dV}{d\phi}(\nabla^a\phi) - \nabla_b\nabla^a\nabla^b\phi + \nabla^a\nabla_b\nabla^b\phi\ .\end{aligned} \tag{2.34}$$

Using $\nabla_b\nabla^a\nabla^b\phi - \nabla^a\nabla_b\nabla^b\phi = R^{ab}\nabla_b\phi$ in the above equation (2.34) and using the expression of E_ϕ from (2.28), one finally obtains,

$$\nabla_b E^{ab} = -\frac{1}{2}(\nabla^a\phi)E_{(\phi)}\ . \tag{2.35}$$

The above relation shows the explicit connection between E_{ab} and $E_{(\phi)}$ which is not intuitively expected by looking into the first two equations of eq.(2.28). This relation in turn helps us to find out the explicit value of the off-shell Noether current and potential in the Jordan frame.

Einstein frame: Proceeding similarly as in the Jordan frame analysis, here also we work under the off-shell condition and in order to define off-shell conserved quantities one needs a Bianchi-type identity in the Einstein frame. From (2.21), it is straightforward to show that a similar expression as in the Jordan frame [$i.e$(2.35)] can be obtained as follows,

$$\tilde{\nabla}_b\tilde{E}^{ab} = -\frac{1}{2}(\tilde{\nabla}^a\tilde{\phi})\Big[\tilde{\Box}\tilde{\phi} - \frac{dU}{d\tilde{\phi}}\Big] = -\frac{1}{2}(\tilde{\nabla}^a\tilde{\phi})\tilde{E}_\phi \, . \tag{2.36}$$

Now, we are ready to discuss the two frames from the thermodynamic point of view. In the following section, we obtain the off-shell Noether current in the two frames. Then we obtain the expression of the entropy from first principles, i.e. the Virasoro algebra. The detail discussions are as follows.

2.4 Entropy from Noether Current and the Noether Potential in the Two Frames

Bekenstein-Hawking formula of entropy [14] says that for a black hole in GR, the entropy is proportional to its horizon area. This was one of the earlier works that shows the connection the spacetime geometry with the gravitational thermodynamics and this connection seems more convincing with the passage of time. However, it was believed that there should be a much more general expression of entropy, of which Bekenstein-Hawking formula is just first-order approximation, and the general formula should be valid for any arbitrary dimension in any theory. For the generalization of the expression of entropy, an operative definition of the black hole entropy was essential. Meanwhile, Wald obtained a direct relation of the entropy with the Noether potential [29] and the entropy is a conserved charge when the Lagrangian is considered to be diffeomorphism invariant. But, in this method one has to put the factor of the surface gravity by hand. Later, for the Brans-Dicke theory (not the scalar-tensor one), Kang [28] formulated the entropy being proportional to the area and the scalar field ϕ from the argument of the nondecreasing surface area. It has been shown [145–148] that due to the presence of the scalar field ϕ in Brans-Dicke theory the surface area of a black hole becomes oscillatory during the dynamical evolution, violating the area law(that the horizon area of a black hole is an increasing function of time) which is valid in Einstein's gravity. Kang's prescription was to take the entropy of a black hole in this theory as $S_{BH} = \frac{1}{4}\int d^2x\sqrt{\sigma} = \frac{\phi A}{4}$, where σ is the determinant of the induced metric of the two-dimensional transverse surface and A is the area of the black hole horizon.

This section discusses about the procedure to get the Noether current and the Noether potential in the two frames. After that, using those quantities, entropy will be obtained from first principles i.e. from the central charge of the Virasoro algebra.

2.4.1 Noether current and charge

Jordan frame: The variation of the total Lagrangian in Jordan frame is shown earlier in (2.27). Due to the diffeomorphism $x^a \to x^a + \xi^a$, the off-shell change in the Lagrangian is given from (2.26) as,

$$\pounds_\xi(\sqrt{-g}L) = -2\sqrt{-g}E_{ab}\nabla^a\xi^b + \sqrt{-g}E_{(\phi)}\xi^a\nabla_a\phi + \sqrt{-g}\nabla_a\Theta^a(q, \pounds_\xi q) \ , \tag{2.37}$$

where $\pounds_\xi$ denotes the Lie variation. The LHS of the Eq. (2.37) gives

$$\begin{aligned}\pounds_\xi(\sqrt{-g}L) &= L\pounds_\xi(\sqrt{-g}) + \sqrt{-g}\pounds_\xi L) \\ &= \sqrt{-g}L\nabla_a\xi^a + \sqrt{-g}\xi^a\nabla_a L = \sqrt{-g}\nabla_a(L\xi^a) \ .\end{aligned} \tag{2.38}$$

The contribution from the RHS of (2.37) can be written as,

$$-2\sqrt{-g}\nabla_a(E^{ab}\xi_b) + 2\sqrt{-g}\xi_b\nabla_a E^{ab} + \sqrt{-g}E_{(\phi)}\xi^a\nabla_a\phi + \sqrt{-g}\nabla_a\Theta^a(q, \pounds_\xi q) \ .$$

Using the relation of (2.35) in the above expression, the whole expression reduces to a total derivative term which is given as, $\sqrt{-g}\nabla_a[-2E^{ab}\xi_b + \Theta^a(q, \pounds_\xi q)]$.
Thus, finally (2.37) gives,

$$\nabla_a[L\xi^a + 2E^{ab}\xi_b - \Theta^a(q, \pounds_\xi q)] = 0 \ . \tag{2.39}$$

Therefore, one can identify the term within the square bracket as a conserved quantity which is nothing but the Noether current due to the diffeomorphism. We denote it by J^a, where,

$$J^a = L\xi^a + 2E^{ab}\xi_b - \Theta^a(q, \pounds_\xi q) \ . \tag{2.40}$$

The above expression of J'^a can be further expressed as $J^a = \nabla_b J^{ab}$, where the anti-symmetric (off-shell) Noether potential is given as (see the appendix 2.A for detail discussion) Therefore, the antisymmetric Noether potential J^{ab}, defined as $\nabla_b J^{ab} = J^a$, turns out to be

$$J^{ab} = \frac{1}{16\pi}\Big[\phi(\nabla^a\xi^b - \nabla^b\xi^a) + 2\xi^a(\nabla^b\phi) - 2\xi^b(\nabla^a\phi)\Big] \tag{2.41}$$

Let us now obtain the Noether current and the potential in the Einstein frame.

Einstein Frame: Since, the procedure of obtaining off-shell Noether current has been discussed in details for the Jordan frame, and since we follow the same procedure in the Einstein frame, we shall not discuss it anymore in Einstein frame. The entire procedure has been discussed in the appendix 2.A.2. Here, we summarize our result for Noether current and potential as,

$$\tilde{J}^a = \tilde{L}\tilde{\xi}^a + 2\tilde{E}^{ab}\tilde{\xi}_b - \tilde{\Theta}^a(\tilde{q}, \pounds_\xi\tilde{q}) \ . \tag{2.42}$$

and,

$$\tilde{J}^{ab} = \frac{1}{16\pi}[\tilde{\nabla}^a\tilde{\xi}^b - \tilde{\nabla}^b\tilde{\xi}^a] \ . \tag{2.43}$$

Thus, we obtain the conserved off-shell Noether current and the Noether potential in the two frames. In the following section we use these expressions of the Noether currents and the Noether potentials of the two frames to obtain the entropy using the Virasoro algebra technique. One can ask why we do not follow Wald's method [29] to get the entropy, where the calculations are much simpler. The answer is: Using Wald's prescription, one can obtain the entropy for the GR cases where it is given by a quarter of the horizon surface area, and for that, one has to incorporate a factor of surface gravity by hand with the conserved Noether current to obtain it. However, in the scalar-tensor theory, we do not know the expression of the entropy, and, therefore, we cannot predict whether the inclusion of the factor of the surface gravity with the conserved Noether current gives entropy in this theory. Therefore, we want to get entropy from first principles using the Virasoro algebra technique. This method was originally introduced by Brown and Henneaux [149] and was further developed by Carlip [150] in the near horizon symmetry. We follow this technique to obtain the entropy for the scalar-tensor theory of gravity.

2.4.2 Virasoro Algebra and the Entropy

In this section, we need some important relations which are given in Appendix A of [150] and derived in Appendix B of [151]. All the following calculations in this section are done in the near horizon limit.

In this approach, we need the definition of charge and the bracket among the charges. We follow here the formalism which are given in [151]. We define,

$$Q[\xi]=\frac{1}{2}\int d\Sigma_{ab}\sqrt{\sigma}J^{ab} \tag{2.44}$$

and

$$\begin{aligned}[Q_1,Q_2] &= \int \sqrt{\sigma}d\Sigma_{ab}\Big[\xi_2^a J^b[\xi_1]-\xi_1^a J^b[\xi_2]\Big] \\ &= \int \sqrt{\sigma}d\Sigma_{ab}\Big[\xi_2^a J_1^b-\xi_1^a J_2^b\Big]\end{aligned} \tag{2.45}$$

Here, σ is the determinant of the induced 2-metric and $d\Sigma_{ab}$ is the two-dimensional surface element. The calculation of this charge and its bracket is one of the key aspects of this process. For different theories these two quantities are already calculated in [151–157] (for the analysis of Schwarzschild black string in five dimensions, see [158]). To get the explicit expressions of this bracket and the charge in this theory, we shall follow the methods prescribed in [150]. We take a set of diffeomorphism generators defined by the relation $\xi^a=T\chi^a+R_0\rho^a$, where χ^a is the timelike Killing vectors and ρ^a is orthogonal to the Killing vector. The component R_0 of the diffeomorphism generator ξ^a along the orthogonal vector ρ^a is related [as argued in [151], Eq. (12)] by $R_0=\frac{\chi^2}{\kappa\rho^2}DT$, where $D\equiv\chi^a\nabla_a$. Now, the Lie bracket of the diffeomorphism operators is given by (Eq. (26) of [151]),

$$\{\xi_1,\xi_2\}^a=\Big(T_1DT_2-T_2DT_1\Big)\chi^a-\frac{1}{\kappa}D\Big(T_1DT_2-T_2DT_1\Big)\rho^a \tag{2.46}$$

For the mentioned set of diffeomorphism generators, one can calculate (for details, see Appendix 2.B)

$$Q[\xi]=\frac{1}{16\pi}\int\sqrt{\sigma}d^2x\phi\Big(2\kappa T-\frac{1}{\kappa}D^2T\Big) \tag{2.47}$$

and

$$[Q_1, Q_2] := \frac{1}{16\pi}\int \sqrt{\sigma}d^2x\phi\Big[2\kappa(T_1DT_2 - T_2DT_1) - \frac{1}{\kappa}(T_1D^3T_2 - T_2D^3T_1)\Big] \quad (2.48)$$

Using the relation (2.46), one obtains

$$Q[\{\xi_1, \xi_2\}] = \frac{1}{16\pi}\int \sqrt{\sigma}d^2x\Big[2\kappa(T_1DT_2 - T_2DT1) - \frac{1}{\kappa}D^2(T_1DT_2 - T_2DT_1)\Big]\phi \quad (2.49)$$

Therefore, the central term, defined by the relation $K[\xi_1, \xi_2] = [Q_1, Q_2] - Q[\{\xi_1, \xi_2\}]$, is

$$K[\xi_1, \xi_2] = \frac{1}{16\pi}\int \sqrt{\sigma}d^2x\phi\frac{1}{\kappa}\Big[(DT_1)(D^2T_2) - (DT_2)(D^2T_1)\Big] \quad (2.50)$$

If one takes the Fourier modes of T as $T = \sum_m A_m T_m$ and takes the usual ansatz for T_m as given in [151],

$$T_m = \frac{1}{\kappa}\exp\Big[im\{\kappa t + g(x) + p.x_\perp\}\Big], \quad (2.51)$$

then the Fourier modes of the charges are

$$Q[\xi_m] = \frac{\int \sqrt{\sigma}d^2x\phi}{8\pi}\delta_{m,0} = \frac{A\phi}{8\pi}\delta_{m,0} \quad (2.52)$$

where we define $\int \sqrt{\sigma}d^2x = A$ as the area of the null surface. We want to comment on the fact that ϕ is the function of all coordinates in general. As the calculations in this section are near the horizon, we expand ϕ about the horizon to get $\phi(t, r, X^A) = \phi(r_H) + (r - r_H)\phi'(t, r_H, X^A) + ...$ Near the horizon only the first term contributes which is independent of all the coordinates to be consistent with the zeroth law of thermodynamics [122]. Therefore, the ϕ in Eq. (2.54) is actually the first term of the expansion of $\phi(t, r, X^A)$. We shall use this convention throughout this section. Similarly, one obtains

$$Q[\{\xi_m, \xi_n\}] = i(m - n)Q[\xi_{m+n}]. \quad (2.53)$$

and

$$K[\xi_m, \xi_n] = -\frac{im^3\phi A}{8\pi}\delta_{m+n=0} \quad (2.54)$$

From (2.49) and (2.50), collecting all the terms, we obtain

$$i[Q_m, Q_n] = (m-n)Q[\xi_{m+n}] + m^3 \frac{A\phi}{8\pi}\delta_{m+n,0} \tag{2.55}$$

Comparing the obtained relation with the stranded Virasoro algebra of charges given by

$$i[Q_m, Q_n] = (m-n)Q_{m+n} + \frac{C}{12}m^3\delta_{m+n,0}, \tag{2.56}$$

one finds

$$\frac{C}{12} = \frac{A\phi}{8\pi} \tag{2.57}$$

One already knows that the Cardy formula for entropy is given by the relation [150, 159]

$$S = 2\pi\sqrt{\frac{C\Delta}{6}} \tag{2.58}$$

where Δ is defined as $\Delta = Q_0 - \frac{C}{24}$, $Q_0 = \frac{A\phi}{8\pi}$ being the zeroth mode of charge. Therefore, the expression of entropy determined by the Cardy formula is given by

$$S = \frac{A\phi}{4} \tag{2.59}$$

We see that the expression of entropy $S = \frac{\phi A}{4}$ agrees with Kang's prescription [28]. For the Einstein frame, the calculation will be exactly similar to the usual GR case [151] as the Noether current is exactly the same in both the cases. So, in that case, $\tilde{S} = \frac{\tilde{A}}{4}$, where $\tilde{A} = \int \sqrt{\tilde{\sigma}}d^2x = \phi A$. Therefore, the entropy in both frames are equivalent under the conformal transformation.

So, using the Virasoro algebra, we find the entropy of the two frames to be equivalent. To prove the thermodynamic parameters are equivalent in the two frames, several assumptions are accounted in [102], such as the Killing vectors are the same in the two frames and the spacetime is asymptotically flat. On the contrary, we proved this without those assumptions and without including any extra prescription by hand. The equivalence of the entropy in the two frames is used later in this chapter to obtain the equivalence of the other thermodynamic quantities (the energy and the temperature) in the two frames.

Let us now explain what do we mean by "near horizon limit". The entropy is attributed to the (black hole) horizon only. Therefore, while calculating entropy, the spatial limit is always taken as $r \longrightarrow r_H$, where r_H is the black hole horizon radius. Also, the other thermodynamic parameters are defined (in the following sections) near about the BH horizon. In these cases, we consider $r \longrightarrow r_H$, and we call it as the "near horizon limit". Note, near the BH horizon, we have taken $\phi(r_H)$ as constant, which is not any assumption. As it has been argued in the literature [122], in order to make the zeroth law remain valid, one has to set $\phi(r_H)$ as constant. The thermodynamic parameters, obtained in our analysis, should be consistent with all the thermodynamic laws, including the zeroth law. This is the reason to consider $\phi(r_H)$ as constant, which also simplifies our analysis.

2.5 Comparison at the thermodynamic level

In the following part we shall find the correspondence of the thermodynamic variables between the two frames. Although in [102], it has been proved that all the thermodynamic quantities are equivalent. But, it is based on a few assumptions, such as: the two vectors ξ^a and $\tilde{\xi}^a$ are the same. However, no justification has been provided for such possibility. In the previous section we obtained the expressions of the entropy in the two frames and we proved they are equivalent in the two frames without any assumption. Now we try to develop the relation of energy and temperature in the two frames. Our approach is completely different from the others in the literature and it is not based on any assumption. Instead, our analysis will prove that $\tilde{\xi}^a = \xi^a$, justifying the earlier assumption of [102] and our results also matches to it. All the results, derived in this section, are valid in the time-independent background.

2.5.1 Action as the free energy of spacetime

One of the most interesting facts of the Einstein-Hilbert action is that it can be treated as the free energy of spacetime for stationary background (For details see [140]). The question that naturally arises is whether the same interpretation of the action is applicable for this theory or not. Firstly, we try for this theory in Einstein frame, as the action quite resembles to the Einstein-Hilbert one and then we do for the theory in Jordan frame.

Einstein frame : The gravitational action in this frame is given by (2.4). Now the equation of motion of the metric tensor in this frame, when the matter field is included, is given in (2.23), and only the zero on the right-hand side is replaced by $\frac{\tilde{T}_{ab}}{2}$, where $\tilde{T}_{ab}$ is the energy-momentum tensor of the matter field in the Einstein frame. If one rises one index of the equation of motion and then fixes the value of both indexes as zero, one obtains $\tilde{G}^0_0 = \tilde{R}^0_0 - \frac{1}{2}\tilde{R} = \frac{1}{2}\tilde{\nabla}^0\tilde{\phi}\tilde{\nabla}_0\tilde{\phi} - \frac{1}{4}\tilde{\nabla}^i\tilde{\phi}\tilde{\nabla}_i\tilde{\phi} - \frac{U}{2} + \frac{\tilde{T}^0_0}{2}$. From this relation if one replaces $\tilde{R}$ in the Lagrangian given in (2.4), one obtains

$$\tilde{L} = -\tilde{T}^0_0 + \frac{2\tilde{R}^0_0}{16\pi} - \tilde{\nabla}_0\tilde{\phi}\tilde{\nabla}^0\tilde{\phi} \tag{2.60}$$

The last term in the above equation does not contribute as we have assumed the spacetime is stationary and therefore, the field ϕ is independent of time. It should be mentioned here that the components of the energy-momentum tensor of the Einstein fame is related to the Jordan frame with the relation $\tilde{T}^b_a = \frac{T^b_a}{\phi^2}$. The stationary spacetime has a timelike Killing vector $\tilde{\chi}^a = (1, 0, 0, 0)$. Therefore, a straightforward calculation gives

$$\tilde{R}^a_j\tilde{\chi}^j = \tilde{\nabla}_b\tilde{\nabla}^a\tilde{\chi}^b = \frac{1}{\sqrt{-\tilde{g}}}\partial_b(\sqrt{-\tilde{g}}\tilde{\nabla}^a\tilde{\chi}^b), \tag{2.61}$$

where the last identity comes from the fact that $\tilde{\nabla}^a\tilde{\chi}^b$ is an antisymmetric tensor (following the relation of the Killing vector). Since $\tilde{\chi}^a$ has only the time component, one obtains

$$R^0_0 = \frac{1}{\sqrt{-\tilde{g}}}\partial_b(\sqrt{-\tilde{g}}\tilde{g}^{i0}\tilde{\Gamma}^b_{i0}) \ . \tag{2.62}$$

Note that the index b cannot be time as the spacetime is stationary. Moreover, one should integrate the Lagrangian in the four-dimensional spacetime manifold, and for that, one has to take finite range of time $(0,\tilde{\beta})$ to get a finite result as the spacetime is the stationary one. Therefore, one can express the action as

$$\tilde{\mathcal{A}} = \tilde{\beta}\int \tilde{N}\sqrt{\tilde{h}}\tilde{\rho}d^3\tilde{x} + \frac{\tilde{\beta}}{8\pi}\int d^2\tilde{x}\sqrt{\tilde{\sigma}}\tilde{N}\tilde{n}_\alpha(\tilde{g}^{i0}\tilde{\Gamma}^\alpha_{i0}) \ . \tag{2.63}$$

The last term is obtained after converting the space volume integral of $\tilde{R}^0_0$ to the surface integral. Here, $\tilde{h}$ and $\tilde{\sigma}$ are the determinant of induced 3- and 2-metric respectively, and $\tilde{N} = \sqrt{-\tilde{g}_{00}}$ is the lapse function. We have also used the relation $\tilde{T}^0_0 = -\tilde{\rho}$. Also, note that while obtaining the above result, the Euclidean/imaginary time has been taken into the account. Since the imaginary time has the periodicity (here $\tilde{\beta}$), the time-integration is done for the range $(0, \tilde{\beta})$, and then the periodicity is

identified as the inverse Hawking temperature i.e., $\tilde{T} = 1/\tilde{\beta}$. Thus, identifying the integral part of the first term of (2.63) as the energy and the whole second term as the (negative of) entropy one gets $\tilde{\mathcal{A}} \equiv \tilde{\beta}\tilde{E} - \tilde{S}$. So, one can conclude that the action can be interpreted as the free energy of spacetime. Earlier the same interpretation was given for GR as mentioned earlier and one finds that the same interpretation is applicable for this scalar-tensor theory in the Einstein frame as well.
Let us now find out whether one can give the same inference of the action of this theory in Jordan frame.

Jordan frame: The gravitational action in this frame is given by (2.1). In this case as well, one should use the equation of motion of the metric tensor (2.29). One has to raise one index and then one has to fix both indexes as zero (as done in the Einstein frame) to obtain

$$L = \frac{1}{16\pi}(2\phi R^0_0 + 16\pi\rho + \frac{2\omega}{\phi}\nabla^0\phi\nabla_0\phi - 2\nabla^0\nabla_0\phi + 2\Box\phi) \tag{2.64}$$

If we see the origin, the $\Box\phi$ term which appears in Eq. (2.5) is different from the $\Box\phi$ term appearing in Eq. (2.64). We have already mentioned how $3\sqrt{-g}\Box\phi$ term appears in Eq. (2.5) due to fact that the Ricci scalar is not invariant under the conformal transformation. On the contrary $\Box\phi$ term appears in Eq. (2.64) in the following way. The gravitational action, in the Jordan frame is given by Eq. (2.1) . Now, if we take the equation of motion of the metric tensor $E_{ab} = 0$ (given by Eq. (2.28)), raise one index and take the zero-zero component of the equation of motion (i.e. $E^0_0 = 0$), all the terms of the action (Eq. (2.1)) can be identified from the relation $E^0_0 = 0$. Then, if we keep the terms of the action (2.1) on the left hand side and the other terms on the right hand side (RHS), the terms on the RHS will be ones given in Eq. (2.64). Thus, using the equation of motion $E^0_0 = 0$, the action (2.1) is equivalent to Eq. (2.64). In this chapter, the $\Box\phi$ term of Eq. (2.5) has not been accounted. Instead, the $\Box\phi$ term of E_{ab} (last term of E_{ab} in Eq. (2.28)) has appeared in the last term of Eq. (2.64).

Now, in general, for any vector one can write

$$R^b_j(\phi\chi^j) = \nabla_a\nabla^b(\phi\chi^a) - \nabla^b\nabla_a(\phi\chi^a) \tag{2.65}$$

For this theory in stationary background, let there be the timelike Killing vector $\chi^a = (1, 0, 0, 0)$. Then from the Killing condition $\chi^a\nabla_a\phi = 0$, one finds that $\partial_0\phi = 0$. Therefore, the term $(2\omega/\phi)\nabla^0\phi\nabla_0\phi$ vanishes in (2.64). Now, the second term of

(2.65) $\nabla^b\nabla_a(\phi\chi^a) = \nabla^b(\chi^a\nabla_a\phi) - \nabla_b(\phi\nabla_a\chi^a)$ vanishes using the property of the Killing vector χ^a. So,

$$\begin{aligned}\phi R^b_j\chi^j &= \nabla_a\nabla^b(\phi\chi^a)\\ &= \nabla_a[\phi(\nabla^b\chi^a)] + (\nabla_a\chi^a)(\nabla^b\phi) + \chi^a[\nabla_a\nabla^b\phi]\end{aligned} \tag{2.66}$$

The first term in the above equation is an antisymmetric tensor which can be written in terms of a total derivative form. The second term vanishes and therefore one obtains

$$\phi R^b_j\chi^j = \frac{1}{\sqrt{-g}}\partial_a[\phi\sqrt{-g}(\nabla^b\chi^a)] + \chi^a\nabla_a\nabla^b\phi \tag{2.67}$$

As the Killing vector has only the time component, one obtains

$$\phi R^0_0 - \nabla_0\nabla^0\phi = \frac{1}{\sqrt{-g}}\partial_a[\phi\sqrt{-g}g^{0i}\Gamma^a_{i0}] \tag{2.68}$$

So, ultimately one obtains

$$\begin{aligned}\mathcal{A} &= \frac{1}{16\pi}\int\sqrt{-g}d^4x\big[16\pi\rho - \frac{2}{\sqrt{-g}}\partial_a[\phi\sqrt{-g}g^{0i}\Gamma^a_{i0}] + 2\Box\phi\big]\\ &= \frac{1}{16\pi}\int\sqrt{-g}d^4x\big[16\pi\rho - \frac{2}{\sqrt{-g}}\partial_a[\sqrt{-g}(\phi g^{0i}\Gamma^a_{i0} + g^{aj}\partial_j\phi)\big]\end{aligned} \tag{2.69}$$

As done earlier, here also the action can be treated as the free energy of the spacetime

$$\begin{aligned}\mathcal{A} &= \beta\int N\sqrt{h}d^3x\rho + \frac{\beta}{8\pi}\int d^2x\sqrt{\sigma}Nn_\alpha(\phi g^{0i}\Gamma^\alpha_{0i} + g^{\alpha j}\partial_j\phi)\\ &= \beta E - S.\end{aligned} \tag{2.70}$$

In this frame, the second integration gives (the negative of) the entropy. Again, here the time-integration is evaluated from 0 to β (time period of the of the Euclidean time) and the Hawking temperature T is identified as $T = 1/\beta$.
A few comments should be made in this regards. Earlier we have shown that the entropy in the two frames are the same. Again, one can show

$$\sqrt{\tilde{\sigma}}\tilde{N}\tilde{n}_\alpha(\tilde{g}^{i0}\tilde{\Gamma}^\alpha_{i0}) = \sqrt{\sigma}Nn_\alpha(\phi g^{i0}\Gamma^\alpha_{i0}) - \frac{1}{2}\sqrt{\sigma}Nn_\alpha g^{\alpha j}\partial_j\phi \tag{2.71}$$

Now, $\phi = \Omega^2$ is not a function of time. Also, considering a general form of a stationary metric, it can be shown that the last term on the RHS of (2.71) vanishes (since $\sqrt{\sigma}Nn_{\alpha}g^{\alpha j}$ vanishes for any j) on the horizon surface.

Earlier, using Virasoro algebra, we have shown that the entropy of the two frames are equivalent under the conformal transformation, i.e. $\tilde{S} = S$. Here, we obtain $\tilde{S} = -(\tilde{\beta}/8\pi)\int d^2\tilde{x}\tilde{s}_*$ and $S = -(\beta/8\pi)\int d^2x s_*$, where $\tilde{s}_* = \sqrt{\tilde{\sigma}}\tilde{N}\tilde{n}_{\alpha}(\tilde{g}^{i0}\tilde{\Gamma}^{\alpha}_{i0})$ and $s_* = \sqrt{\sigma}Nn_{\alpha}(\phi g^{0i}\Gamma^{\alpha}_{0i} + g^{\alpha j}\partial_j\phi)$. As we have discussed above, $\sqrt{\sigma}Nn_{\alpha}g^{\alpha j}$ vanishes on the horizon. Therefore, from (2.71), we obtain $\tilde{s}_* = s_*$. As a result, from $\tilde{S} = S$ we obtain $\tilde{\beta} = \beta$ (note that the coordinates do not change under the conformal transformation, which implies that $d^2\tilde{x} = d^2x$ and $d^3\tilde{x} = d^3x$). Again, one can also prove $\tilde{N}\sqrt{\tilde{h}}\tilde{\rho} = N\sqrt{h}\rho$, as $\tilde{N} = \sqrt{-\tilde{g}^{00}} = \sqrt{\phi}N$, $\sqrt{\tilde{h}} = \phi^{\frac{3}{2}}\sqrt{h}$ and $\tilde{\rho} = \tilde{T}^0_0 = \frac{T^0_0}{\phi^2} = \frac{\rho}{\phi^2}$. Which implies that $\tilde{E} = E$.

Therefore, from the above discussion, we obtain that the actions in the two frames, can be written as the free energy of the spacetime. In addition, we also obtain that the thermodynamic parameters are conformally equivalent in the two frames.

2.5.2 Holographic relation at the thermodynamic level

The holographic relation was obtained earlier in the action level. We found that in the Einstein frame the holographic relation holds but in the Jordan frame a total derivative term blemishes the relation. Here we try to obtain the relation in the thermodynamic level and find that a total derivative term in both the frames accounts for the entropy, and the remaining term contributes as the energy. Let us call the earlier one the surface term ($L^{(T)}_{sur}$) and the later one, the bulk term ($L^{(T)}_{bulk}$)[2]. Our aim is now to see if we can have a similar relation like in the earlier analysis.

To proceed, here one trick will be used followed from [160]. Firstly one has to calculate the variation of total Lagrangian (equivalent to $L_2(q,\, \partial q,\, \partial^2 q)$ of (2.31)) and $L^{(T)}_{(sur)}$. Then the difference of these two variations give the same for the bulk part $L^{(T)}_{bulk}$. Note that the total Lagrangians [(2.1) and (2.4)] in the two frames are functions of the metric tensor, its first derivative and the second derivative and the scalar field and its first-order derivative. Therefore, following the earlier logic of section 2.3.4, we shall consider the variation with respect to the metric tensor and its derivatives (first and second order) only.

Einstein frame : We have identified the surface term in the thermodynamic level from (2.60) as $\tilde{L}^{(T)}_{sur} = \frac{2\tilde{R}^0_0}{16\pi}$. Therefore, writing the bulk part in the thermodynamic level as $\tilde{L}^{(T)}_{bulk} = \tilde{L} - \tilde{L}^{(T)}_{sur}$ and following the calculations of Appendix A of

[2]Note, in general, that the bulk or surface terms in the two cases (the analysis in the action level and here in the thermodynamic level) are not the same.

[160] one can show,

$$\sqrt{-\tilde{g}}\tilde{L}^{(T)}_{sur} = -\partial_i\Big(\tilde{g}_{ab}\frac{\delta_E\sqrt{-\tilde{g}}\tilde{L}^{(T)}_{bulk}}{\delta_E(\partial_i\tilde{g}_{ab})} + \partial_j\tilde{g}_{ab}\frac{\partial\sqrt{-\tilde{g}}\tilde{L}^{(T)}_{bulk}}{\partial(\partial_i\partial_j\tilde{g}_{ab})}\Big)\ , \tag{2.72}$$

where $\frac{\delta_E\tilde{L}^{(T)}_{bulk}}{\delta_E(\partial_i\tilde{g}_{ab})}$ implies the Euler derivative of the bulk Lagrangian with respect to $\partial_i g_{ab}$. Thus one can conclude that in the thermodynamic level as well one can obtain the holographic relation for the theory in the Einstein frame.

Jordan frame : Here the total Lagrangian is given in (2.1) and the surface part at the thermodynamic level, from (2.64), is identified as, $L^{(T)}_{sur} = \frac{1}{16\pi}(2\phi R^0_0 - 2\nabla^0\nabla_0\phi + 2\Box\phi)$. So, the bulk part in the thermodynamic level is $L^{(T)}_{bulk} = L - L^{(T)}_{sur}$. We obtain (for detailed calculation, see the Appendix 2.C)

$$\partial_i\Big(g_{ab}\frac{\delta_E\sqrt{-g}L^{(T)}_{bulk}}{\delta_E(\partial_i g_{ab})} + \partial_j g_{ab}\frac{\partial\sqrt{-g}L^{(T)}_{bulk}}{\partial(\partial_j\partial_i g_{ab})}\Big) = -\sqrt{-g}L^{(T)}_{sur} + \frac{3}{16}\sqrt{-g}\Box\phi\ . \tag{2.73}$$

So, in the thermodynamic level one does not get the holographic relation in Jordan frame as was the case in the action level earlier. Moreover, one can check that it is the same extra term (which also appeared in (2.33)) that spoils the holographic relation. Using Eq. (2.68), it can be proved that $\sqrt{-g}L^{(T)}_{sur}$ is a total derivative term and, hence, it is the surface part of the Lagrangian at the thermodynamic level. Then the bulk part of the Lagrangian can be given as $L^{(T)}_{bulk} = L - L^{(T)}_{sur}$, where L is given in Eq. (2.1). Then, with these identifications, if we check whether the holographic relation is obtained (following what we have done in the Einstein frame in Eq. (2.72)), we see that the holographic relation in the Jordan frame cannot be obtained due to the presence of the last term in Eq. (2.73). Thus, the origin of $\Box\phi$, which appears in (2.73) is different, compared to which appeared in (2.5) and in (2.64). In fact, throughout this chapter, we have neglected the $\Box\phi$ term which appeared in (2.5). In the next chapter, we shall see that the holographic relation (both in the action level and in the thermodynamic level) breaks down due to the fact that we neglect the $\Box\phi$ term in the action of the Jordan frame (mentioned in Eq. (2.5)). If one takes the Lagrangian in the Jordan frame taking with the contribution of the $\Box\phi$ term (which we define as L' in the following chapter), these in-equivalences would not appear.

For better understanding, some additional comments are as follows. In section 2.5.1, we have shown that the gravitational action, along with the matter field, can be written in two parts when there is a timelike Killing vector in the spacetime. One part is a total derivative term, which contributes as the entropy, and the remaining bulk terms are identified as the gravitational energy times inverse-temperature. Therefore, the action, as a whole, can be considered as the free energy of the spacetime. In section 2.5.2, we have investigated whether the bulk part and the surface part, as identified in section 2.5.1, are connected with each other via holographic relation.

Note that the holographic relation in the action level (as investigated in the section 2.3.4) and the holographic relation in the thermodynamic (as investigated in the section 2.5.2) level are different in the following ways. Firstly, at the action level, the gravitational action itself can be decomposed as a bulk part and the surface part, where the bulk part contributes towards obtaining equations of motion. On the other hand, at the thermodynamic level, the bulk part and the surface part are obtained only due to the presence of the timelike Killing vector and also by using the gravitational equation of motion. In this case, the surface part contributes to entropy and the bulk part as energy times inverse temperature. Thus, the latter analysis is valid under the presence of the Killing symmetry and using the equation of motion, whereas the former case (holographic relation at the action level) does not dependent on killing symmetry or the equation of motion. Thus, L_{bulk} and $L^{(T)}_{bulk}$ are different and also $L_{sur} \neq L^{(T)}_{sur}$. The same is true in Einstein frame.

2.5.3 Relation between different thermodynamic entities from the GHY term

So far, all the calculations are done for the gravitational actions without the GHY term. It has been argued that the thermodynamic entities are invariant for the stationary background. In this section, we want to justify our arguments again from the GHY boundary term. Since, in this case, the Lagrangian in the Jordan frame is L, the proper GHY term will be given by (2.8).

We start from the GHY surface term in Einstein frame, mentioned in (2.7), which provides the antisymmetric Noether potential (see the Appendix-2.D.1)

$$\tilde{J}^{ab}_{S} = \tilde{K}(\tilde{N}^a\tilde{\xi}^b - \tilde{N}^b\tilde{\xi}^a)\ . \tag{2.74}$$

Similarly, the GHY term in the conformally connected Jordan's frame, mentioned in (2.8) yields the Noether potential (see the Appendix-2.D.2)

$$J_S^{ab} = \Omega^2 K(N^a \xi^b - N^b \xi^a) \ . \tag{2.75}$$

To develop the relation between the thermodynamic quantities, we assume that the diffeomorphism vectors $\tilde{\xi}^a$ and ξ^a of the two frames are proportional to each other with the proportionality constant being the arbitrary power of the conformal factor. Let,

$$\tilde{\xi}^a = \Omega^\alpha \xi^a \ , \tag{2.76}$$

where, $\Omega = \sqrt{\phi}$ in this theory. Unlike [102] we do not consider that the two vectors are equal. It should be mentioned here that in [102], they have made the mentioned assumption and thereby they have shown that the thermodynamic quantities are the same in the two frames. Here, in this discussion we want to fix the value of α and from that we shall show that we can get the relationship between the thermodynamic quantities of the two frames. Coincidentally, we shall show that our result matches theirs.

Using the relations (2.9) and (2.76), one can easily obtain

$$\tilde{J}_S^{ab} = \Omega^{\alpha-4} J_S^{ab} - 3(N^i \partial_i \Omega)\Omega^{\alpha-3}(N^a \xi^b - N^b \xi^a) \ . \tag{2.77}$$

Now, following the usual process, if one defines the conserved Noether charges in the Einstein and the Jordan frame as

$$\tilde{Q} = \frac{1}{2}\int d\tilde{\Sigma}_{ab} \tilde{J}_S^{ab} \ , \tag{2.78}$$

and

$$Q = \frac{1}{2}\int d\Sigma_{ab} J_S^{ab} \ , \tag{2.79}$$

then for a stationary black hole ($\xi^a = \chi^a$ and $\tilde{\xi}^a = \tilde{\chi}^a$, where χ^a and $\tilde{\chi}^a$ are the time-like Killing vectors) one obtains

$$\tilde{Q} = \Omega^\alpha Q \tag{2.80}$$

on the horizon. To achieve this one needs to use the relations $d\tilde{\Sigma}_{ab} = (\tilde{l}_a \tilde{\chi}_b - \tilde{l}_b \tilde{\chi}_a) d^2x_\perp \sqrt{\tilde{\sigma}}$, $d\Sigma_{ab} = (l_a \chi_b - l_b \chi_a) d^2 x_\perp \sqrt{\sigma}$, $\tilde{l}_a = \Omega^{-\alpha} l_a$ and $\sqrt{\tilde{\sigma}} = \Omega^2 \sqrt{\sigma}$. Here, l_a

and $\tilde{l}_a$ are the auxiliary null vectors in the two frames satisfying $\tilde{\chi}^a\tilde{l}_a = -1 = \chi^a l_a$.

From the Noether potentials (2.74) and (2.75) if one calculates the entropy on a null surface, one defines entropy in, say Jordan frame as $S = \frac{2\pi}{\kappa}Q$. One uses the similar definition in Einstein frame as well. The surface gravity κ is defined by the relation $\chi^a\nabla_a\chi^b = \kappa\chi^b$. For the transformation mentioned in (2.76) one can find $\tilde{\kappa} = \Omega^\alpha\kappa$ with $\tilde{\kappa}$ defined as $\tilde{\chi}^a\tilde{\nabla}_a\tilde{\chi}^b = \tilde{\kappa}\tilde{\chi}^b$. Therefore, from (2.80) one finds that the entropy is the same in the two frame. This implies that for any value of α, the entropy is equivalent in the two frames. So, it is not surprising that [102] got the same conclusion as the entropy is invariant in the two frames with the specific choice of $\alpha = 0$. But, for the other thermodynamic quantities one has to fix α to get the proper relation in the two frames. For instance,

$$\tilde{T} = \Omega^\alpha T\ , \tag{2.81}$$

where $T = \frac{\kappa}{2\pi}$ and so on. To fix α we take a particular example. Consider a static, spherically symmetric(SSS) metric in the Jordan frame

$$ds^2 = f_1(r)dt^2 - \frac{dr^2}{f_2(r)} - r^2(d\theta^2 + \sin^2\theta d\Phi^2) \tag{2.82}$$

Now, the GHY surface term in Jordan frame, mentioned in (2.8) gives contribution on r =const surface as

$$\begin{aligned}\mathcal{A}_{GHY}\mid_r &= -\frac{1}{8\pi}\int_{r=r_H}\phi\sqrt{h^{(r)}}K^{(r)}d^3x\\ &= -\frac{\beta\sqrt{f_1'(r_H)f_2'(r_H)}}{4}(\phi r_H^2) = -\phi\pi r_H^2 = -S\end{aligned} \tag{2.83}$$

where, $T = \frac{\kappa}{2\pi} = \frac{\sqrt{f_1'(r_H)f_2'(r_H)}}{4\pi} = \beta^{-1}$ from the definition of surface gravity has been used. We have defined ϕ times quarter of the surface area as the entropy. Here, $N_r = \frac{1}{\sqrt{|g^{rr}|}} = \frac{1}{\sqrt{f_2(r)}}$ is the normal defined in the r =const hypersurface .
Let us recall the fact mentioned in section 2.3.3 that the equations of motion are equivalent in the two frames by the transformations (2.2) and (2.3). So if the metric g_{ab} is the solution in the Jordan frame, then ϕg_{ab} is the solution in Einstein frame. Hence, the corresponding conformally connected metric tensor of (2.82) is the solution in the Einstein frame:

$$d\tilde{s}^2 = \phi(r)\Big[f_1(r)dt^2 - \frac{dr^2}{f_2(r)} - r^2(d\theta^2 + \sin^2\theta d\Phi^2)\Big]\ . \tag{2.84}$$

In the above metric, we have considered ϕ to be a function of r only, so that the metric (2.84) remains static and spherically symmetric. The same calculation (calculation of the GHY surface term (2.7) at constant r surface) in Einstein's frame leads to the result

$$\begin{aligned}\tilde{\mathcal{A}}_{GHY}\mid_r &= -\frac{1}{8\pi}\int_{r=r_H}\sqrt{\tilde{h}^{(r)}}\tilde{K}^{(r)}d^3x \\ &= -\frac{\tilde{\beta}\sqrt{f_1'(r_H)f_2'(r_H)}}{4}(\phi r_H^2) = -\tilde{S},\end{aligned} \tag{2.85}$$

where, $\tilde{N}_r = \sqrt{\frac{\phi}{f_2(r)}}$. One can verify that the two surface actions $\mathcal{A}_{GHY}$ and $\tilde{\mathcal{A}}_{GHY}$ given in (2.7) and (2.8) are invariant near the null surface as they are connected with each other by the relation (2.9) and the last term of that equation does not contribute at the r =const surface. Therefore, we can say the result of (2.83) and (2.85) are the same. From that, one can conclude that $\tilde{\beta} = \beta$. Hence, the temperature is invariant in both the frames. So, from (2.81) one gets $\alpha = 0$ which justifies the assumption of [102].

Next, let us see how the gravitational energies are related in the two frames. For that, let us calculate the GHY terms in all the frames. The motivation is from the fact that in the GR case the surface term is interpreted as the free energy of the spacetime($\mathcal{A}^{(GR)}_{GHY} = -S^{(GR)} + \beta^{(GR)}E^{(GR)}$) when it is calculated for all the surfaces (for details, see [161]). Note that when the GHY term is calculated for all the surfaces, generally there are four surfaces constant r, constant θ, constant ϕ and constant t. But, for SSS spacetime, the contribution from the last two surfaces vanishes as the metric components are independent of t and ϕ. The GHY surface term, when calculated collectively on all the surfaces, gives

$$\begin{aligned}\mathcal{A}_{GHY} &= \mathcal{A}_{GHY}\mid_r + \mathcal{A}_{GHY}\mid_\theta = -S - \frac{1}{8\pi}\int_\theta \phi\sqrt{h^{(\theta)}}K^{(\theta)}d^3x \\ &= -S + \frac{\beta}{2}\int_0^{r_H}\phi\sqrt{\frac{f_1}{f_2}}dr = -S + \frac{E}{T},\end{aligned} \tag{2.86}$$

where E is defined as $E = \frac{1}{2}\int_0^{r_H}\phi\sqrt{\frac{f_1}{f_2}}dr$ and the same calculation in Einstein frame gives

$$\tilde{\mathcal{A}}_{GHY} = \tilde{\mathcal{A}}_{GHY}\mid_r + \tilde{\mathcal{A}}_{GHY}\mid_\theta = -\tilde{S} - \frac{1}{8\pi}\int_{r=r_H}\sqrt{\tilde{h}^{(\theta)}}\tilde{K}^{(\theta)}d^3x$$
$$= -\tilde{S} + \frac{\tilde{\beta}}{2}\int_0^{r_H}\phi\sqrt{\frac{f_1}{f_2}}dr = -\tilde{S} + \frac{\tilde{E}}{\tilde{T}}. \tag{2.87}$$

Note, in Eq. (2.86), for r =const. surface, we have taken the spatial limit $r = r_H$. In that case, the GHY action gives $-S$. For θ =const. surface, the limit of r, in the integration of Eq. (2.86), is from 0 to r_H, in which case, the GHY action yields E/T . The same argument is also valid for Eq. (2.87). Thus, Eqs. (2.86) and (2.87) are defined near about the horizon.

Like the GR cases, we found that the GHY term has the free energy structure in both the frames. Now, one can show that two GHY actions are the same on the horizon as they are connected by the relation (2.9), and the last term of that equation does not contribute there. So we can say the results of(2.83) and (2.85) are the same. Since $\tilde{T} = T$ and $\tilde{S} = S$, one gets the equivalent expression of the energy in both frames as $\tilde{E} = E = \frac{1}{2}\int_0^{r_H}\phi\sqrt{\frac{f_1}{f_2}}dr$.

Now our analysis suggests that all the thermodynamic variables are invariant in both frames. Reference [126] argues that the expression of the temperature should be equivalent under the conformal transformation. Our arguments give identical results, although obtained in a different fashion. The literature [102] demands that all the thermodynamic variables are invariant in both the Jordan and the Einstein frame when the spacetime is asymptotically flat. To prove these, the authors made the key assumption that the vectors $\tilde{\xi}^a$ and ξ^a are the same in both the frames. There is no justification for how they took it for granted. Our analysis gives the identical results, although we have not assumed that the Killing vectors are the same in the two frames or the spacetime is asymptotically flat.

We end up the section of this chapter with the following discussions. From the analysis that we have made in the present chapter, we found that the energy in the two frames should be conformally invariant. However, we have not been able to give any covariant form of thermodynamic energy here, which has been elusive for decades. So, we cannot compare it with existing standerd expressions of energy in the literature (like ADM energy [89], Hawking-Hayward quasi-local energy [90, 91], Misner-Sharp energy [92, 93], Kodama energy [94, 95], Brown-York energy [96] *etc.*). But, we can predict whatever may be the expression, it should be conformally invariant. Interestingly, one candidate fulfils this condition on the horizon which is the Brown-York energy [96]. On the other hand, the recent works suggests that the Misner-Sharp energy [98] or the Hawking-Hayward quasilocal energy ([99], [100], [101]) and the others are not conformally invariant and, hence, they cannot be the candidates. Although, [97] suggests that the Brown York energy is conformally invariant all over the spacetime, that is not correct. The wrong statement of the mentioned paper is due to the fact that the Eq. (2.25) of [97] (the relation between the traces of the two extrinsic curvature tensors in the two frames) is incorrect. For (1+3) dimensions, we derived the correct one in (2.9). This correct one tells us that the Brown-York energy is the same only on the horizon.

Thus, in our analysis, we found that the thermodynamic quantities are conformally invariant in the two frames– which we have analyzed both from the whole action as well as from the GHY surface term. So, from the thermodynamic point of view, the two frames are equivalent for the static spherical symmetric spacetime. In the following chapter, we provide a more robust analysis on the black hole thermodynamics of the two frames, where we provide a covariant formalism to establish the thermodynamic laws in the two frames and, thereby, show that the thermodynamic parameters are exactly conformally equivalent in the two frames.

2.6 Summary and Discussions

In this chapter, we have studied the well-known scalar-tensor theory from action level as well as in the thermodynamic framework. Also, some consideration, which is often taken for granted without even proper declaration, is mentioned here.

Here, we have started analyzing the theory in the two frames on the action level. What we found is that the usually mentioned mathematical equivalence in the literature is an incomplete statement. One just neglects a total derivative term while projecting the theory from one frame to the other. It has been shown that the concern does not arise while one incorporates the GHY surface term in this theory. Which means, the usual description of the mathematical equivalence of the two frames breaks while one does not include the GHY surface term. After that, we have separated the gravitational action in each frame as a bulk part and a total derivative surface part. Later the separation is justified by obtaining the equation of motion from the bulk part of the total action. Thereafter, we endeavored to obtain the connection of the two parts of the gravitational action by the holographic relation. Unlike the GR case, it has been shown here that the holographic property is not valid in the Jordan frame. But, it continues to be valid in the Einstein one and, thereby, obtaining the inequivalence of the two frames in the action (classical) level even without the presence of the external matter field.

Afterwards, the two frames are compared at the thermodynamic level. The relation of the thermodynamic entities in the two frames were not well known. It has also been the subject of debate which form of the energy in the literature should be used to describe the thermodynamics of the system in this theory. Although, some earlier attempts has been made to solve the enigma, those are based on some assumptions. In this chapter, the entropy in the two frames has been obtained from the first principle using the Virasoro algebra technique, the first attempt to obtain entropy in this theory in that way. We have been able to prove that the entropies in the two frames are indeed equivalent. Later, we have interpreted the gravitational action as the free energy of spacetime in the two frames and it has also been shown that the identified expression of the energy and the temperature in the two frames is equivalent. The result coincides with some previous works, though obtained in a more robust way. Also, the attempt has been made to get the "holographic" relation at the thermodynamic level as we have separated out the total action in the thermodynamic level as a bulk part which was identified in terms of energy and a total derivative surface term which accounts to the entropy. The result obtained here is the same as earlier; i.e., the holographic relation is maintained in the Einstein frame, while it is defiled in the original one. We have also noticed that the same term appears in this case as well to spoil the holographic property in the Jordan frame. Finally, all the earlier obtained relation of the thermodynamic entities has been verified from the GHY surface terms. Moreover, it has been shown that the GHY surface term itself can be interpreted as the free energy of the spacetime.

From all these analysis we find that the two frames are not exactly conformally equivalent. First of all, the action of the two frames differ by a total derivative $\Box\phi$ term which is usually not even mentioned in the literature. Then, the Holographic relation (both in action level as well as in the thermodynamic level) is not valid in the Jordan frame, while it continues to be valid in the Einstein frame. Also, the charge obtained from the conserved Noether current due to the diffeomorphism is not equivalent in the two frames. It is noticeable that in all such in-equivalence, the surface part of the Lagrangian is directly involved. It indicates that there is, probably, some flaw in neglecting the $\Box\phi$ term, which we shall resolve in the following chapter. However, amid all these in-equivalences, it is clear that the thermodynamic parameters should be equivalent in the two frames. In the analysis of the present chapter, we have not been able to define thermodynamic energy in a covariant manner and also we have not established any thermodynamic law from any first principle. Therefore, in the following chapter, we shall explore not only whether the in-equivalences can be removed by accounting the $\Box\phi$ term but also we shall provide the thermodynamic laws from the conserved current approaches.

Appendix

2.A Noether current in terms of anti-symmetric Noether potential in the two frames [Eq. (2.41) and Eq. (2.43)]

2.A.1 Jordan frame [Eq. (2.41)]

The expression of $\Theta^a(q, \delta q)$ has been given in (2.28), from which one can obtain

$$\Theta^a(q, \pounds_\xi q) = -\frac{1}{16\pi}[2g^{ab}\frac{\omega}{\phi}(\nabla_b\phi)\pounds_\xi\phi - \phi\pounds_\xi v^a + 2(\nabla_b\phi)p^{iabd}\pounds_\xi g_{id}] \, , \tag{2.88}$$

where, the last two terms of Eq. (2.88) can be expressed as follows:

$$2(\nabla_b\phi)P^{iabd}\pounds_\xi g_{id} = (\nabla^d\phi)[\nabla^a\xi_d + \nabla_d\xi^a] - 2(\nabla^a\phi)(\nabla_i\xi^i) \tag{2.89}$$

and

$$\pounds_\xi v^a = 2P^{ibad}\nabla_b\pounds_\xi g_{id} = 2P^{iabd}\nabla_b\pounds_\xi g_{id} = \nabla_b\nabla^a\xi^b + \nabla_b\nabla^b\xi^a - 2\nabla^a\nabla_b\xi^b \tag{2.90}$$

Therefore,

$$\begin{aligned}
2(\nabla_b\phi)P^{iabd}\pounds_\xi g_{id} - \phi\pounds_\xi v^a = {} & (\nabla_b\phi)(\nabla^a\xi^b) + \phi\nabla_b\nabla^a\xi^b - \phi\Box\xi^a \\
& +(\nabla_b\phi)(\nabla^b\xi^a) - 2(\nabla^a\phi)(\nabla_b\xi^b) - 2\phi g^{ac}R_{kc}\xi^k,
\end{aligned} \tag{2.91}$$

where one has to use the relation $\nabla_b\nabla_d\xi_i - \nabla_d\nabla_b\xi_i = R_{ijbd}\xi^j$. The first three terms of the above equation give

$$(\nabla_b\phi)(\nabla^a\xi^b) + \phi\nabla_b\nabla^a\xi^b - \phi\Box\xi^a = \nabla_b[\phi(\nabla^a\xi^b - \nabla^b\xi^a)] + (\nabla_b\phi)(\nabla^b\xi^a) \tag{2.92}$$

Substituting (2.92) in (2.91) and using

$$2(\nabla_b\phi)(\nabla^b\xi^a) - 2(\nabla^a\phi)(\nabla_b\xi^b) = 2\nabla_b[\xi^a(\nabla^b\phi) - \xi^b(\nabla^a\phi)] + 2\xi^b\nabla_b\nabla^a\phi - 2\xi^a\Box\phi \ , \tag{2.93}$$

one obtains

$$\begin{aligned}2(\nabla_b\phi)P^{iabd}\pounds_\xi g_{id} - \phi\pounds_\xi v^a = \nabla_b[\phi(\nabla^a\xi^b - \nabla^b\xi^a) + 2\xi^a(\nabla^b\phi) - 2\xi^b(\nabla^a\phi)] \\ +2\xi^b\nabla_b\nabla^a\phi - 2\xi^a\Box\phi - 2\phi g^{ac}R_{kc}\xi^k \ .\end{aligned} \tag{2.94}$$

Using (2.94) in (2.88) one can obtain

$$\begin{aligned}\Theta^a(q,\pounds_\xi q) = -\frac{1}{16\pi}\Big[2\frac{\omega}{\phi}(\nabla^a\phi)\xi^b\nabla_b\phi + \nabla_b[\phi(\nabla^a\xi^b - \nabla^b\xi^a) + 2\xi^a(\nabla^b\phi) - 2\xi^b(\nabla^a\phi)] \\ +2\xi^b\nabla_b\nabla^a\phi - 2\xi^a\Box\phi - 2\phi g^{ac}R_{kc}\xi^k\Big] \ .\end{aligned} \tag{2.95}$$

Using the above relation (2.95) in (2.40), one can write Noether current as

$$\begin{aligned}J^a = \frac{1}{16\pi}\Big[\nabla_b[\phi(\nabla^a\xi^b - \nabla^b\xi^a) + 2\xi^a(\nabla^b\phi) - 2\xi^b(\nabla^a\phi)] + \Big\{\Big(\phi R - \frac{\omega(\phi)}{\phi}g^{ab}\nabla_a\phi\nabla_b\phi \\ -V(\phi)\Big)\xi^a + 2\frac{\omega}{\phi}(\nabla^a\phi)\xi^b(\nabla_b\phi) + 2\xi^b\nabla_b\nabla^a\phi - 2\xi^a\Box\phi - 2\phi g^{ac}R_{kc}\xi^k\Big\} + 2E^{ab}\xi_b\Big].\end{aligned} \tag{2.96}$$

In the above expression (2.96), the curly-bracketed term (the terms which are inside {}) can be as a whole as $-2E^{ab}\xi_b$ (see the expression of E^{ab} from (2.28)) and, hence, the expression of J^a is given by a total derivative of anti-symmetric Noether potential, the expression of which has been given in (2.41).

2.A.2 Einstein frame [Eq. (2.43)]

In Einstein frame, the Noether current is given by (2.42). Firstly, from (2.21), one obtains

$$\tilde{\Theta}^a(\tilde{q},\pounds_{\tilde{\xi}}\tilde{q}) = \frac{\pounds_{\tilde{\xi}}\tilde{v}^a}{16\pi} - (\tilde{\nabla}^a\tilde{\phi})\pounds_{\tilde{\xi}}\tilde{\phi} \ , \tag{2.97}$$

where, one can write

$$\pounds_\xi\tilde{v}^a = \tilde{\nabla}_b\tilde{\nabla}^a\tilde{\xi}^b + \tilde{\nabla}_b\tilde{\nabla}^b\tilde{\xi}^a - 2\tilde{\nabla}^a\tilde{\nabla}_b\tilde{\xi}^b = \tilde{\nabla}_b\tilde{\nabla}^b\tilde{\xi}^a - \tilde{\nabla}_b\tilde{\nabla}^a\tilde{\xi}^b + 2\tilde{g}^{ac}\tilde{R}_{kc}\tilde{\xi}^k \tag{2.98}$$

To obtain the last step one has to use $\tilde{\nabla}_b\tilde{\nabla}_d\tilde{\xi}_i - \tilde{\nabla}_d\tilde{\nabla}_b\tilde{\xi}_i = \tilde{R}_{ijbd}\tilde{\xi}^j$. Thus, using (2.97) (along with (2.98)) in (2.42), one can similarly obtain

$$\tilde{J}^a = \Big\{\Big[(\frac{\tilde{R}}{16\pi} - \frac{1}{2}\tilde{g}^{ij}\tilde{\nabla}_i\tilde{\phi}\tilde{\nabla}_j\tilde{\phi} - U(\tilde{\phi})\Big]\tilde{\xi}^a + (\tilde{\nabla}^a\tilde{\phi})\tilde{\xi}^b(\tilde{\nabla}_b\tilde{\phi}) - \frac{2}{16\pi}\tilde{g}^{ac}\tilde{R}_{kc}\tilde{\xi}^k\Big\}$$
$$+\frac{1}{16\pi}\tilde{\nabla}_b[\tilde{\nabla}^a\tilde{\xi}^b - \tilde{\nabla}^b\tilde{\xi}^a] + 2\tilde{E}^{ab}\tilde{\xi}_b \tag{2.99}$$

Again in the above expression (2.99), the curly-bracketed term (the terms which are inside {}) can be as a whole as $-2\tilde{E}^{ab}\tilde{\xi}_b$ (see the expression of $\tilde{E}^{ab}$ from (2.21)) and, hence, the anti-symmetric Noether potential will be given by the Eq. (2.43).

2.B Derivation of the Eqs. (2.47) and (2.48)

2.B.1 Derivation of the Eq. (2.47)

To calculate (2.44) one already knows that the Noether potential in Jordan frame is defined by (2.41). If one write it in terms of the Killing and the normal vectors using $\xi^a = T\chi^a + R\rho^a$ with $R = \frac{\chi^2}{\kappa\rho^2}DT$, one obtains

$$J^{ab} = \frac{1}{16\pi}\Big\{\phi[\frac{2\kappa}{\chi^2}(\chi^a\rho^b - \chi^b\rho^a)T - \frac{1}{\kappa\chi^2}(\chi^a\rho^b - \chi^b\rho^a)(D^2T)] + 2\xi^a\nabla^b\phi - 2\xi^b\nabla^a\phi\Big\} \tag{2.100}$$

Using the relation $d\Sigma_{ab} = -d^2x(\chi_a\rho_b - \chi_b\rho_a)\frac{|\chi|}{\rho\chi^2}$ from (A.2) of [151] one obtains

$$d\Sigma_{ab}J^{ab} = -\frac{d^2x}{16\pi}\frac{|\chi|}{\rho\chi^2}[2\rho^2\phi(2\kappa T - \frac{1}{\kappa}D^2T) + (\chi_a\rho_b - \chi_b\rho_a)(2\xi^a\nabla^b\phi - 2\xi^b\nabla^a\phi)] \tag{2.101}$$

Using $\chi^a\nabla_a\phi = 0$ and at the horizon $\rho^a\nabla_a\phi = \mathcal{O}(\chi^2)$, one gets the contributing terms at the horizon as

$$d\Sigma_{ab}J^{ab} = \frac{d^2x}{16\pi}2\phi(2\kappa T - \frac{1}{\kappa}D^2T) \tag{2.102}$$

Therefore, one can find the desired relation (2.47).

2.B.2 Derivation of Eq. (2.48)

As done in the previous part, writing ξ^a in terms of two orthogonal vectors χ^a and ρ^a, one can obtain using Eq. (2.41)

$$J^a = \frac{1}{16\pi}\Big[-\frac{1}{\kappa\chi^2}(\nabla_b\phi)\chi^a\rho^b(D^2T) - \frac{2R\kappa}{\chi^2}\rho^a\rho^b + \phi[\frac{1}{\kappa\chi^2}\rho^a(D^3T) - \frac{2\kappa}{\chi^2}\rho^a(DT)] \\ +2(T\chi^a + R\rho^a)\Box\phi - 2(\nabla^a\phi)[DT + \frac{R\kappa}{\chi^2}(\chi^2-\rho^2)] - 2(T\chi^b + R\rho^b)(\nabla_b\nabla^a\phi)\Big] \tag{2.103}$$

To calculate the Lie bracket of the charges (2.45), one needs to calculate $d\Sigma_{ab}\xi^a$, which is given by

$$d\Sigma_{ab}\xi^a = -d^2x\frac{|\chi|}{\rho\chi^2}[T\chi^2\rho_b - R\rho^2\chi_b] \tag{2.104}$$

Hence, a straightforward calculation gives

$$d\Sigma_{ab}\xi_2^a J_1^b = -\frac{d^2x}{16\pi}\frac{|\chi|}{\rho\chi^2}\Big\{(\nabla_a\phi)\Big(\frac{1}{\kappa}R_2\rho^2\rho^a(D^2T_1) - 2R_1\kappa T_2\rho^2\rho^a\Big) + T_2\rho^2\phi\Big[\frac{1}{\kappa}(D^3T_1) \\ -2\kappa(DT_1)\Big] + 2(\Box\phi)\rho^2\chi^2(R_1T_2 - T_1R_2) - 2(\nabla^b\phi)[T_2\chi^2\rho_b - R_2\rho^2\chi_b][DT_1 \\ +\frac{R_1\kappa}{\chi^2}(\chi^2-\rho^2)] - 2[T_2\chi^2\rho_b - R_2\rho^2\chi_b](T_1\chi^a + R_1\rho^a)(\nabla_a\nabla^b\phi)\Big\} \tag{2.105}$$

Now we shall take only those terms which contribute to calculate the bracket defined in (2.45) at the horizon and neglect all other terms.As χ_a is a Killing vector, $\chi^a\nabla_a\phi = 0$, and at the horizon, $\rho^a\nabla_a\phi = \mathcal{O}(\chi^2)$. Therefore, some terms vanish. Taking the nonvanishing terms and using $R = \frac{\chi^2}{\kappa\rho^2}(DT)$, one obtains

$$d\Sigma_{ab}\xi_2^a J_1^b = -\frac{d^2x}{16\pi}\frac{|\chi|}{\rho\chi^2}[T_2\rho^2\phi(\frac{1}{\kappa}D^3T_1 - 2\kappa DT_1) + 2(\Box\phi)\chi^4(T_2DT_1 - T_1DT_2) \\ -2(T_2\chi^2\rho_b - \frac{\chi^2}{\kappa}(DT_2)\chi_b)(T_1\chi^a + \frac{\chi^2\rho^a}{\kappa\rho^2}(DT_1))(\nabla_a\nabla^b\phi)] \tag{2.106}$$

At the null surface $\chi^2 = 0$. Therefore, the second term containing $\Box\phi$ does not contribute at the horizon as it is proportional to χ^2. Also, the calculation gives a few symmetric terms for the interchanging of $1 \leftrightarrow 2$. At the horizon one can prove that $\chi^a\chi_b(\nabla_a\nabla^b\phi) = \rho^a\rho_b(\nabla_a\nabla^b\phi) = 0$. Ultimately, the contributing terms to calculate the bracket (2.45) is

$$d\Sigma_{ab}\xi_2^a J_1^b = -\frac{d^2x}{16\pi}\frac{\mid \chi \mid \phi}{\rho}[2\kappa T_2(DT_1) - \frac{1}{\kappa}T_2(D^3T_1)] \\ +\mathcal{O}(\chi^2) + (1 \leftrightarrow 2 symmetric \ \ terms). \tag{2.107}$$

So, ultimately, the bracket (2.45) gives the value mentioned in (2.48)

2.C Derivation of Eq. (2.73)

The total Lagrangian in the Jordan frame is given is given by (2.1). We want the variation of the $\sqrt{-g}\phi R$ with respect to the first- and second-order derivative of the metric tensor g_{ab}, which is given by

$$\delta(\sqrt{-g}\phi R) = 2\phi\sqrt{-g}\delta(\partial_m\partial_n g_{pq})P^{pnmq} - 2\sqrt{-g}\phi\delta(\partial_m g_{nq})[P^{nbmd}\Gamma^q_{bd} \\ +P^{nbcd}\Gamma^m_{bc} + P^{imcq}\Gamma^n_{ic}], \tag{2.108}$$

where, $P_a^{bcd} = \frac{\partial R}{\partial R^a_{bcd}}$, and $P^{abcd} = \frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc})$. Now, from the straightforward calculation, one obtains (expanding the Euler derivative)

$$\partial_i\Big[g_{ab}\frac{\delta_E\sqrt{-g}L}{\delta_E(\partial_i g_{ab})} + \partial_j g_{ab}\frac{\partial\sqrt{-g}L}{\partial(\partial_i\partial_j g_{ab})}\Big] = \partial_i\Big[g_{ab}\frac{\partial\sqrt{-g}L}{\partial(\partial_i g_{ab})} - g_{ab}\partial_h\frac{\partial\sqrt{-g}L}{\partial(\partial_h\partial_i g_{ab})} \\ +\partial_j g_{ab}\frac{\partial\sqrt{-g}L}{\partial(\partial_i\partial_j g_{ab})}\Big] = \frac{3}{16\pi}\sqrt{-g}\Box\phi \tag{2.109}$$

Now, the surface term at the thermodynamic level is

$$\sqrt{-g}L_{sur} = \frac{\sqrt{-g}}{16\pi}(2\phi R^0_0 - 2\nabla^0\nabla_0\phi + 2\Box\phi) = \frac{1}{16\pi}\Big(2\partial_a(\phi\sqrt{-g}g^{0i}\Gamma^a_{i0}) + 2\sqrt{-g}\Box\phi\Big) \tag{2.110}$$

From straightforward calculation, one obtains

$$\partial_i\Big[g_{ab}\frac{\delta_E\sqrt{-g}L_{sur}}{\delta_E(\partial_i g_{ab})} + \partial_j g_{ab}\frac{\partial\sqrt{-g}L_{sur}}{\partial(\partial_i\partial_j g_{ab})}\Big] = \partial_i\Big[g_{ab}\frac{\partial\sqrt{-g}L_{sur}}{\partial(\partial_i g_{ab})} - g_{ab}\partial_h\frac{\partial\sqrt{-g}L_{sur}}{\partial(\partial_h\partial_i g_{ab})} \\ +\partial_j g_{ab}\frac{\partial\sqrt{-g}L_{sur}}{\partial(\partial_i\partial_j g_{ab})}\Big] = \sqrt{-g}L_{sur} \tag{2.111}$$

Subtracting (2.111) from (2.109), one gets the mentioned result in (2.73).

2.D Derivation of the Noether current and potential from the GHY surface term

2.D.1 Obtaining Eq. (2.74)

The GHY surface term of the action in the Einstein frame is defined in (2.7), which can be further written as

$$\tilde{\mathcal{A}}_{GHY} = \frac{1}{8\pi}\int \sqrt{-\tilde{g}}d^4x\tilde{\nabla}_a(\tilde{K}\tilde{N}^a) = \int \sqrt{-\tilde{g}}d^4x\tilde{L}_{GHY} \ , \tag{2.112}$$

where, we have identified $\tilde{L}_{GHY} = (1/8\pi)\tilde{\nabla}_a(\tilde{K}\tilde{N}^a)$. Therefore, for the diffeomorphism (in the Einstein frame) $x^a \to x^a + \tilde{\xi}^a$, the change in the GHY term will be given as

$$\pounds_{\tilde{\xi}}(\sqrt{-\tilde{g}}\tilde{L}_{GHY}) = \frac{1}{8\pi}\pounds_{\tilde{\xi}}\Big(\sqrt{-\tilde{g}}\tilde{\nabla}_a(\tilde{K}\tilde{N}^a)\Big) \ . \tag{2.113}$$

The LHS of (2.113) can be written as

$$\pounds_{\tilde{\xi}}(\sqrt{-\tilde{g}}\tilde{L}_{GHY}) = \sqrt{-\tilde{g}}\tilde{\nabla}_a(\tilde{L}_{GHY}\tilde{\xi}^a) = \frac{\sqrt{-\tilde{g}}}{8\pi}\Big(\tilde{\xi}^a\tilde{\nabla}_b(\tilde{K}\tilde{N}^b)\Big) \ , \tag{2.114}$$

and the RHS of (2.113) can be written as

$$\begin{aligned}\frac{1}{8\pi}\pounds_{\tilde{\xi}}\Big(\sqrt{-\tilde{g}}\tilde{\nabla}_a(\tilde{K}\tilde{N}^a)\Big) &= \frac{1}{8\pi}\pounds_{\tilde{\xi}}\Big(\partial_a\Big(\sqrt{-\tilde{g}}\tilde{K}\tilde{N}^a\Big)\Big) \\ &= \frac{1}{8\pi}\partial_a\Big(\pounds_{\tilde{\xi}}\Big(\sqrt{-\tilde{g}}\tilde{K}\tilde{N}^a\Big)\Big) \\ &= \frac{\sqrt{-\tilde{g}}}{8\pi}\tilde{\nabla}_a\Big[\tilde{\nabla}_b\Big(\tilde{K}\tilde{N}^a\tilde{\xi}^b\Big) - \tilde{K}\tilde{N}^b\tilde{\nabla}_b\tilde{\xi}^a\Big]\end{aligned} \tag{2.115}$$

Thus, substituting (2.114) and (2.115) in (2.113), one obtains

$$\tilde{\nabla}_a\tilde{\nabla}_b\Big[\tilde{K}(\tilde{N}^a\tilde{\xi}^b - \tilde{N}^b\tilde{\xi}^a)\Big] = 0 \ . \tag{2.116}$$

The above result is the conservation law of the Noether current due to the diffeomorphism invariance of the GHY term in the Einstein frame, i.e. $\tilde{\nabla}_a\tilde{J}^a_S = 0$. The conserved Noether current $\tilde{J}^a_S$ can be written in terms of two-rank anti-symmetric Noether potential as $\tilde{J}^a_S = \tilde{\nabla}_b\tilde{J}^{ab}_S$, where the expression of the Noether potential $\tilde{J}^{ab}_S$ (obtained from the GHY term) is given in (2.74) .

2.D.2 Obtaining Eq. (2.75)

The GHY surface term in the Jordan frame is defined in (2.8). That can be further written as

$$\mathcal{A}_{GHY} = \int d^4x\sqrt{-g}L_{GHY} = \frac{1}{8\pi}\int d^4x\sqrt{-g}\nabla_a(\phi K N^a)~, \tag{2.117}$$

where, $L_{GHY} = (1/8\pi)\nabla_a(\phi K N^a)$. Therefore, due to the diffeomorphism, the change in the GHY term will be obtained as

$$\pounds_\xi(\sqrt{-g}L_{GHY}) = \frac{1}{8\pi}\pounds_\xi\Big(\sqrt{-g}\nabla_a(\phi K N^a)\Big)~. \tag{2.118}$$

The LHS of (2.118) is given as

$$\pounds_\xi(\sqrt{-g}L_{GHY}) = \sqrt{-g}\nabla_a(L_{GHY}\xi^a) = \frac{\sqrt{-g}}{8\pi}\Big(\xi^a\nabla_b(\phi K N^b)\Big)~, \tag{2.119}$$

whereas the RHS of (2.118) yields

$$\frac{1}{8\pi}\pounds_\xi\Big(\sqrt{-g}\nabla_a(\phi K N^a)\Big) = \frac{\sqrt{-g}}{8\pi}\nabla_a\Big[\nabla_b\Big(\phi K N^a\xi^b\Big) - \phi K N^b\nabla_b\xi^a\Big]~. \tag{2.120}$$

Therefore, substituting (2.119) and (2.120) in (2.118), one obtains

$$\nabla_a\nabla_b\Big[\phi K(N^a\xi^b - N^b\xi^a)\Big] = 0~. \tag{2.121}$$

The above relation can be identified as $\nabla_a J^a_S = 0$. The Noether current due to the diffeomorphism invariance of the surface term (i.e. J^a_S) can be identified as $J^a_S = \nabla_b J^{ab}_S$, where, the expression of J^{ab}_S is provided in (2.75) .

Chapter 3

A tale of an "insignificant" term which solves the puzzles: a covariant thermodynamic description in scalar-tensor theory

3.1 Overview of the chapter

[1]It is widely known that the correspondence between the thermodynamic quantities and the spacetime geometry is not confined only to the Einstein's theory of general relativity. However, although the scalar-tensor theory is one of the most popular among the alternative theories of gravity, the underlying thermodynamic description of this theory is not yet properly developed. Some of the ambiguities were discussed earlier. We have mentioned that the scalar tensor theory can be analysed in the Jordan frame as well as in the Einstein frame. Until now there is no exact covariant expression of energy which can fit across all the thermodynamic aspects of the theory in these two frames.

In the previous chapter, we have systematically developed the arguments to prove that all the thermodynamic quantities (for example: energy, entropy, temperature) must be equivalent in the two frames, without taking any apriori assumption but, we could not formulate the exact covariant expression of the thermodynamic quantities using those methods. Therefore, we have to look for some other method which can provide us not only the covariant expression of the thermodynamic quantities, but also we obtain thermodynamic laws from first principles using those methods. It is

[1]This chapter is based on the publication [112] .

well-known that the conserved currents (like Noether current, Abbott-Deser-Tekin (ADT) current etc.) helps us to obtain thermodynamic quantities in a covariant way and also we can obtain the thermodynamic first law using those currents. In the earlier chapter we have seen that the conserved charge, obtained from the Noether current due to the diffeomorphism, are not exactly equivalent in the two frames. That is probably why people could not show the exact equivalence of the thermodynamic parameters in the two frames when those are obtained using the Noether current [102]. Moreover, in this connection, we have discussed several in-equivalences in the two frames, which indicates that the surface part of the Lagrangian in the Jordan frame is not properly defined.

In view of all these, we feel to redefine the Lagrangian in the Jordan frame in this chapter. Accounting the contribution from the $\Box\phi$ term in the Jordan frame, we show that the in-equivalences, which were mentioned earlier, can be removed. Then we formulate the thermodynamic descriptions using the conserved quantities following the two different methods. One is the Noether prescription of defining the conserved currents and the potentials due to the diffeomorphism invariance and, the other one is the ADT method of defining conserved currents in the presence of a Killing vector. In both cases, the conserved quantities will be obtained *off-shell*. Using the obtained conserved quantities, we shall define the covariant expressions of all the thermodynamic quantities, which will be shown to fit nicely in the (first and second) laws of black hole thermodynamics. Subsequently, we show that the thermodynamic quantities are exactly conformally invariant without using any prior assumptions or boundary conditions.

There are several other aspects which we discuss in this chapter in a great detail. We obtain that the conserved currents in these two approaches (Noether and ADT) are connected to each other and we shall show the explicit connection between them. In addition, the ADT potentials in two frames are shown to be related with each other in the similar fashion like the Noether counterparts, establishing the equivalence of thermodynamic quantities, defined by the ADT potentials. We also show the entropy increase theorem (or the second law of black hole mechanics) which will be shown to require a modified null energy condition in the Jordan frame. Thus, in this chapter, we provide a robust method to formulate the off-shell conserved quantities in two different approaches and resolve the ambiguities in the thermodynamic descriptions in the scalar-tensor theory which prevailed for the last few decades.

3.2 Defining a proper Lagrangian in the Jordan frame and removing the earlier in-equivalences

3.2.1 Defining the Lagrangian

In the earlier section, we have neglected the $\Box\phi$ term, as it is done in the literature. We have shown that this causes several in-equivalences, where the surface part of the Lagrangian is involved. Moreover, as we all know, finding a consistent thermodynamic relation in the scalar-tensor theory is a difficult task as it is mentioned in the literature for the reasons mentioned earlier. Although, some sporadic works like [102] have tried to formulate a consistent thermodynamic description in the two frames, those are based on several assumptions in the theory (more on that will be mentioned in the proper places later). Moreover, in our earlier analysis we have found the expressions of the thermodynamic parameters but, we could not formulate the thermodynamic laws from any first principle. In this chapter, we shall formulate the thermodynamic laws from the conserved current approaches. Since the surface term plays the major role to obtain the currents, and also since discarding $\Box\phi$ results in the in-equivalence of the surface term, here we keep this term for our analysis. In addition, other motivation will be finding the thermodynamic quantities in the two frames and establish a connection between them. Hence, for a robust study of the thermodynamic description of the scalar-tensor theory, now onward we perform our further analysis by considering the most general form of Lagrangian in the Jordan frame of the following form:

$$L' = L - (3/16\pi)\Box\phi \ . \tag{3.1}$$

Then one can check that the gravitational actions in the two frames are exactly equivalent in the two frames i.e., $\mathcal{A}' = \int d^4x\sqrt{-g}L' = \int d^4x\sqrt{-\tilde{g}}\tilde{L} = \tilde{\mathcal{A}}$.

Let us now discuss about the last term of the above equation (3.1). Note that this term is a total derivative term and contains second order time derivative of ϕ. Therefore it creates issues in obtaining the equation of motion for the following reasons. Being a second order derivative term, one needs to fix simultaneously the field and its canonical momentum at the two end points in the least action formalism. Classically, if we fix arbitrarily both the parameters ϕ as well as the first order derivative of ϕ at the two boundary points, there may not exist a classical solution for the field ϕ consistent with the boundary conditions. Moreover, in general, we prefer that the action principle must obey the composition rule. This implies that, at the intermediate point, the first order derivative of ϕ has to be continuous but not necessarily to be a smooth function. It infers that the first order derivative of ϕ remains arbitrary (for instance, see a detailed discussion in page 241 of [144]). Thus, in the classical regime, fixing both ϕ and its first order derivative simultaneously at the boundary is not admissible. Moreover, this prescription stems further problem in quantizing the theory as the simultaneous application of these two boundary conditions, indeed contradicts the uncertainty principle. It may be pointed out that this problem, however, is not new in the context of the general relativity (GR) as the same happens in case of the variations of Einstein-Hilbert action with respect to g_{ab} to obtain Einstein's equations of motion. Two ways are usually being adopted to resolve such situation. One method, of course is to discard this total derivative term, which we have done in the earlier chapter and which we are not going to repeat anymore for the reasons mentioned above. Another method is to add a judiciously chosen boundary term which cancels the unwanted terms, appearing in the variation of the original action. For example, in GR, the popular boundary term is the Gibbons-Hawking-York (GHY) boundary term. The similar can be done in the present situation as well. Addition of a precise GHY like boundary term can also be adopted in this case as well. But remember that such a choice is not unique as there may exist other term which also serves the same purpose (for GR case, see [162]).

The proper GHY surface term in this case (for the Lagrangian L') would be of the following form

$$\mathcal{A}'_{GHY} = -\frac{1}{8\pi}\oint d^3x\sqrt{h}(\phi K - \frac{3}{2}N^a\partial_a\phi) \ . \tag{3.2}$$

One can check that not only the gravitational action is equivalent in the two frames, the GHY surface term is also equivalent in the two frames as well i.e. $\mathcal{A}'_{GHY} = \tilde{\mathcal{A}}_{GHY}$. As a result, here also one can find the total action is equivalent in the two frames which means $\tilde{\mathcal{A}}_{total} = \tilde{\mathcal{A}} + \tilde{\mathcal{A}}_{GHY} = \mathcal{A}'_{total} = \mathcal{A}' + \mathcal{A}'_{GHY}$.

3.2.2 Removing the earlier in-equivalences

Let us now check whether the in-equivalences, which has been mentioned earlier, can be removed if one takes the Lagrangian as L' instead of L in the Jordan frame. As it is mentioned earlier, now the actions in the two frames are exactly equivalent. Therefore, there is no in-equivalence in the action level. We now check whether we can now establish the holographic relation in the action level.

Note that the Lagrangian L' can also be decomposed in terms of the bulk part and the surface part i.e. $\sqrt{-g}L' = \sqrt{-g}L'_{bulk} + L'_{sur}$, where the bulk part $L'_{bulk} = L_{bulk}$, is given by the relation (2.17) and the surface term will be given as

$$L'_{sur} = L_{sur} - \frac{3\sqrt{-g}}{16\pi}\Box\phi = \frac{1}{16\pi}\partial_c\Big[\sqrt{-g}\Big\{\phi(g^{ik}\Gamma^c_{ik} - g^{ck}\Gamma^m_{km}) - 3g^{cd}\partial_d\phi\Big\}\Big]~. \quad (3.3)$$

With this, one can obtain the holographic relation in the Jordan frame, which is given as

$$L'_{sur} = -\partial_c\Big(\frac{\partial\sqrt{-g}L'_{bulk}}{\partial g_{ab,c}}g_{ab}\Big) \quad (3.4)$$

Thus, we see that the holographic relation can only be established if one considers the Lagrangian in the Jordan frame as L' instead of L . This provides another motivation why we should modify the Lagrangian in the Jordan frame and not the Einstein frame (we could have modified $\tilde{L}$ so that it becomes equivalent to L, but in that case, the holographic principle will be violated in both frames). Not only in the action level, the holographic relation holds in the thermodynamic level as well when we take the Lagrangian in the Jordan frame as L'. Thus, we are able to show the holographic features of the action in both frames of the scalar-tensor theory, which were earlier supposed as the features merely in Einstein-Hilbert action. Besides, from our analysis, we found that the two frames in the scalar-tensor theory are equivalent in the classical level, provided we take the $\Box\phi$ term into the account.

In the following, we shall discuss about another decade-long issue, which is obtaining a covariant thermodynamic description in this theory. Before that, since we have modified the Lagrangian in the Jordan frame, the conserved Noether current due to the diffeomorphism invariance will also be modified, which we shall obtain in the following discussions. However, the conserved Noether current and the potential of the Einstein frame, which we have obtained in the earlier chapter, will continue to be valid in the present case as well.

3.2.3 Modified off-shell Noether current and the potential in the Jordan frame and equivalence of the conserved charge in the two frames

Since we have added extra total derivative term in the Lagrangian of the Jordan frame, the Noether current will be modified in this frame for the present case. Firstly, the arbitrary variation in the Lagrangian L' (or in the action $\mathcal{A}'$) will be obtained as follows.

$$\delta(\sqrt{-g}L') = \sqrt{-g}\Big(E_{ab}\delta g^{ab} + E_{(\phi)\delta\phi+} + \nabla_a\Theta'^a(q,\delta q)\Big) , \tag{3.5}$$

where $q \in \{g_{ab}, \phi\}$ and

$$\Theta'^a(q,\delta q) = \Theta^a(q,\delta q) - \frac{1}{16\pi}\Big\{\frac{3}{2}g^{ij}\delta g_{ij}\partial^a\phi - 3g^{ia}\partial^b\phi\delta g_{ib} + 3\partial^a(\delta\phi)\Big\} . \tag{3.6}$$

The terms within the curly brackets in the expression of $\Theta'^a(q,\delta q)$ in (3.6), are originated from the variation of the $\Box\phi$ term in L'. We show later, these extra terms in $\Theta'^a(q,\delta q)$ play crucial role in the conformal invariance of the thermodynamic quantities in the two frames.

From the above relations, we can obtain the off-shell Noether current due to the diffeomorphism following the method of the earlier chapter. If the above change in the Lagrangian (in the Eq. (3.5)) is due to the diffeomorphism $x^a \to x^a + \xi^a$, then δ will be replaced by the Lie-derivative $\pounds_\xi$. The LHS of (3.5) can be written as $\pounds_\xi(\sqrt{-g}L') = \sqrt{-g}\nabla_a(L'\xi^a)$ and the RHS of (3.5) can be written as

$$-2\sqrt{-g}\nabla_a(E^{ab}\xi_b) + 2\sqrt{-g}\xi_b\nabla_a E^{ab} + \sqrt{-g}E_{(\phi)}\xi^a\nabla_a\phi + \sqrt{-g}\nabla_a\Theta'^a(q,\pounds_\xi q) .$$

Using the relation of (2.35) in the above expression, the whole expression reduces to a total derivative term which is given as, $\sqrt{-g}\nabla_a[-2E^{ab}\xi_b + \Theta'^a(q,\pounds_\xi q)]$. Thus, the modified Noether current in the Jordan frame can be obtained as

$$J'^a = L'\xi^a + 2E^{ab}\xi_b - \Theta'^a(q,\pounds_\xi q) . \tag{3.7}$$

This Noether current can further be written in terms of total derivative of an anti-symmetric Noether potential i.e. $J'^a = \nabla_b J'^{ab}$, where the Noether potential can be obtained as (see the appendix 3.A)

$$J'^{ab} = \frac{1}{16\pi}[\nabla^a(\phi\xi^b) - \nabla^b(\phi\xi^a)] . \tag{3.8}$$

These quantities play an important role in the study of black hole thermodynamics which we shall derive by following the Iyer-Wald formalism in the subsequent section.

We end up this section with the following comments. It will be shown later that the Noether potential J'^{ab} and the boundary term Θ'^a can now be written as proportional to the same quantities in the Einstein's frame $\tilde{J}^{ab}$ and $\tilde{\Theta}^a$ respectively.

In the earlier chapter, we had taken the Lagrangian in the Jordan frame as L. In that case, J^{ab} and $\tilde{J}^{ab}$ cannot be shown to be proportional to each other. Same goes with Θ^a and $\tilde{\Theta}^a$ as well. As a result, the conserved charge (which is the Noether potential integrated over a two surface) becomes equivalent in this case when we take the Lagrangian as L' in the Jordan frame. Whereas, the equivalence of the conserved charges, in general, is absent when we consider the Lagrangian in the Jordan frame as L instead of L'. Since, the thermodynamic parameters are identified as the conserved charges (following Wald's or ADT approach), in our later discussions we are able to show that the thermodynamic parameters are conformally equivalent in the two frames. This may be another reason, why the earlier works could not show the conformal equivalence of the thermodynamic parameters in the two frames, considering the Lagrangian in the Jordan frame as L. Thus, with the modification of the Lagrangian in the Jordan frame, we resolve these inequivalences as well. The detail thermodynamic descriptions are as follows.

3.3 Thermodynamic quantities by Iyer-Wald formalism and their conformal invariance

Our analysis is in the same line of Noether prescription by Wald in GR case [29]. In this discussion one considers the diffeomorphism invariant action as $\sqrt{-g}R$ without dropping or adding a boundary term in it to define thermodynamics. Adopting the same spirit, we also do not discard or include anything in the theory. We shall find that the thermodynamic quantities are well defined and equivalent in two frames which was usually sporadically stated in earlier analysis, thereby establishing the importance of retaining the boundary term.

3.3.1 Jordan frame

The expression of the Noether current and the Noether potential in the Jordan frame are given in (3.7) and (3.8). According to the Wald's formalism we shall use the on-shell condition which is given by, $E^{ab} = 0$. Let us now take the arbitrary variation of the metric tensor and the scalar field which leaves the diffeomorphism vector ξ^a invariant (remember, $\delta\xi_a \neq 0$ in general as $\delta g_{ab} \neq 0$) and therefore the change in the conserved on-shell Noether current with respect to the variation of the fields becomes,

$$\delta(\sqrt{-g}J'^a) = \delta(\sqrt{-g}L')\xi^a - \delta[\sqrt{-g}\Theta'^a(q, \pounds_\xi q)] \ . \tag{3.9}$$

Using Eq. (2.26), we get the variation of the Noether current in terms of the boundary term Θ'^a, which is given as

$$\delta(\sqrt{-g}J'^a) = \sqrt{-g}[\nabla_i\Theta'^i(q, \delta q)]\xi^a - \delta[\sqrt{-g}\Theta'^a(q, \pounds_\xi q)] \ . \tag{3.10}$$

We shall see that this variation of the Noether current can be written in terms of the symplectic Hamiltonian density by using an identity which one can straightforwardly obtain:

$$\pounds_\xi[\sqrt{-g}\Theta'^a(q, \delta q)] = \sqrt{-g}\xi^a\nabla_i[\Theta'^i(q, \delta q)] - 2\sqrt{-g}\nabla_b[\xi^{[a}\Theta'^{b]}(q, \delta q)] \ , \tag{3.11}$$

where $A^{[a}B^{b]} = (1/2)(A^aB^b - A^bB^a)$. Using the above identity in (3.10), we obtain

$$\delta(\sqrt{-g}J'^a) = \pounds_\xi[\sqrt{-g}\Theta'^a(q, \delta q)] - \delta[\sqrt{-g}\Theta'^a(q, \pounds_\xi q)] + 2\sqrt{-g}\nabla_b[\xi^{[a}\Theta'^{b]}(q, \delta q)] \ . \tag{3.12}$$

Now define:

$$\omega^a = -\pounds_\xi[\sqrt{-g}\Theta'^a(q, \delta q)] + \delta[\sqrt{-g}\Theta'^a(q, \pounds_\xi q)] \ . \tag{3.13}$$

The significance of ω^a will be explained in a few steps later. With this definition of ω^a, one can obtain from (3.12),

$$\omega^a = -\delta(\sqrt{-g}J'^a) + 2\sqrt{-g}\nabla_b[\xi^{[a}\Theta'^{b]}(q, \delta q)]. \tag{3.14}$$

Let us now discuss the significance of ω^a. For a classical system we write, $\delta L(x_i, \dot{x}_i) = [(\frac{\partial L}{\partial x_i}) - d_t(\frac{\partial L}{\partial \dot{x}_i})]\delta x_i + d_t[p^i\delta x_i]$ where x_i is the generalized coordinate and $p^i = \frac{\partial L}{\partial \dot{x}_i}$ is the generalized momentum. The equation of motion vanishes on-shell and, the variation of the Hamiltonian due to the arbitrary variation of the the coordinates x_i is given as

$$\delta H(x_i, p^i) = \delta[p^i(d_t x_i)] - d_t[p^i(\delta x_i)] \ . \tag{3.15}$$

By comparing (3.13) and (3.15), one can identify that ω^a as the variation of the symplectic Hamiltonian density where the boundary terms in both the equations are equivalent to each other as, $\sqrt{-g}\Theta'^a(q,\delta q) \equiv p^i(\delta x_i)$ and $\sqrt{-g}\Theta'^a(q,\pounds_\xi q) \equiv p^i(d_t x_i)$.

Thus, with this identification, the total variation of the Hamiltonian can be written as (using Eq. (3.14)),

$$\delta H[\xi] = \int_c d\Sigma_a \frac{\omega^a}{\sqrt{-g}} = -\delta\int_c d\Sigma_a \nabla_b (J'^{ab}) + 2\int_c d\Sigma_a \nabla_b[\xi^{[a}\Theta'^{b]}(q,\delta q)] \ , \tag{3.16}$$

where, the integration is done on Cauchy hypersurface which we symbolize as c. $d\Sigma_a = n_a\sqrt{h}d^3x$ is the elemental surface area of the three-dimensional Cauchy hypersurface, with n_a being the normal and h being the determinant of the induced metric of the surface. Applying Stoke's law in the above equation we can reduce the 3-surface integral of above (3.16) to a 2-surface integral. We consider ξ^a is a Killing vector and the outer surface lies at asymptotic infinity ($i.e\ \partial c_\infty$). The inner surface of c is taken as a bifurcation surface $i.e\ \mathcal{H}$ which also can be depicted as the horizon of the black hole. This implies $\xi^a = 0$ at $\mathcal{H}$. Thus, from (3.16) we obtain,

$$\delta H[\xi] = -\frac{1}{2}\delta\int_{\mathcal{H}} d\Sigma_{ab}J'^{ab} + \frac{1}{2}\delta\int_{\partial c_\infty} d\Sigma_{ab}J'^{ab} - \int_{\partial c_\infty} d\Sigma_{ab}\xi^{[a}\Theta'^{b]}(q,\delta q) \ . \tag{3.17}$$

As $\xi^a = 0$, no contribution comes from the term $\xi^{[a}\Theta'^{b]}(q,\delta q)$ on $\mathcal{H}$. Moreover, as ξ^a is a Killing vector, $\delta H[\xi] = 0$. By following Wald's prescription [29], the first term on the RHS of (3.17), yields $-\frac{\kappa}{2\pi}\delta S$ with κ being the surface gravity and, the other terms result in $\delta M - \Omega_H\delta J$ (for a more rigorous discussions see [29]). Here, we define the entropy (S), the mass of the black hole (M), and the angular momentum (J) as,

$$\begin{aligned}
\delta S &= \frac{\pi}{\kappa}\delta\int_{\mathcal{H}} d\Sigma_{ab}J'^{ab} \ ; \\
\delta M &= \frac{1}{2}\int_{\partial c_\infty}[\delta(d\Sigma_{ab}J'^{ab}) - 2d\Sigma_{ab}\xi^{[a}\Theta'^{b]}(q,\delta q)]\Big|_{\xi=\xi_{(t)}} \ ; \\
\delta J &= -\frac{1}{2}\int_{\partial c_\infty}[\delta(d\Sigma_{ab}J'^{ab}) - 2d\Sigma_{ab}\xi^{[a}\Theta'^{b]}(q,\delta q)]\Big|_{\xi=\xi_{(\phi)}} \ .
\end{aligned} \tag{3.18}$$

Here, $\xi_{(t)}$ and $\xi_{(\phi)}$ are the components of the Killing vector ξ^a along t and ϕ directions respectively in the Jordan frame. So, finally from (3.17), we obtain

$$\delta M = T\delta S + \Omega_H\delta J \ , \tag{3.19}$$

where we use temperature $T = \kappa/(2\pi)$ in the above equation. We comment that eq. (3.19) is the desired form of first law of the black hole thermodynamics in the Jordan frame with the Lagrangian L'. Instead of L', if one considers the Lagrangian as L in the Jordan frame, the expression of the entropy, mass and the angular momentum of the black hole can be obtained by replacing J'^{ab} with J^{ab} and $\Theta'^{b}(q, \delta q)$ with $\Theta^{b}(q, \delta q)$ in (3.18). Let us now approach toward the Einstein frame and find out the thermodynamic quantities in that frame.

3.3.2 Einstein frame

Proceeding similarly as the analysis of the first law of thermodynamics in the Jordan frame in previous subsection, it takes hardly any computation to affirm that in the Einstein frame we get the first law of the black hole mechanics as $\delta\tilde{M} = \tilde{T}\delta\tilde{S} + \tilde{\Omega}_H\delta\tilde{J}$ and, the corresponding thermodynamic quantities are defined as

$$
\begin{aligned}
\delta\tilde{S} &= \frac{\pi}{\tilde{\kappa}}\delta\int_{\mathcal{H}} d\tilde{\Sigma}_{ab}\tilde{J}^{ab} \;; \\
\delta\tilde{M} &= \frac{1}{2}\int_{\partial c_\infty}\left[\delta(d\tilde{\Sigma}_{ab}\tilde{J}^{ab}) - 2d\tilde{\Sigma}_{ab}\tilde{\xi}^{[a}\tilde{\Theta}^{b]}(\tilde{q}, \delta\tilde{q})\right]\Big|_{\tilde{\xi}=\tilde{\xi}_{(t)}} \;; \\
\delta\tilde{J} &= -\frac{1}{2}\int_{\partial c_\infty}\left[\delta(d\tilde{\Sigma}_{ab}\tilde{J}^{ab}) - 2d\tilde{\Sigma}_{ab}\tilde{\xi}^{[a}\tilde{\Theta}^{b]}(\tilde{q}, \delta\tilde{q})\right]\Big|_{\tilde{\xi}=\tilde{\xi}_{(\phi)}} \;.
\end{aligned}
\tag{3.20}
$$

Again, in the above equation (3.20), $\tilde{\xi}_{(t)}$ and $\tilde{\xi}_{(\phi)}$ are the components of the Killing vector ξ^a along t and ϕ directions respectively in Einstein frame. Let us now compare the thermodynamic quantities obtained in the two frames.

3.3.3 Comparison of the thermodynamic quantities:

We consider the Killing vector in the Einstein frame ($\tilde{\xi^a}$) is same as in the Jordan frame i.e., $\tilde{\xi^a} = \xi^a$. The justification of taking the Killing vectors $\tilde{\xi^a} = \xi^a$ is mentioned in the previous chapter. The idea is the following. If ξ^a is a Killing vector in Jordan frame, then it must be a conformal Killing vector in Einstein frame (see [126] for a discussion on this under conformal transformation). Remember that here we are discussing the whole thermodynamics in presence of Killing vector in both frames. Therefore $\tilde{\xi^a} = \xi^a$ to be Killing one, we need to impose the condition that the conformal factor must be Lie transported along ξ^a; i.e. $\pounds_\xi\Omega^2 = 0$. Earlier the authors in [102] have addressed this issue by assuming the above condition and shown that the thermodynamic quantities are equivalent in these two frames under the condition of spacetime to be asymptotically flat.

As $\tilde{\xi^a} = \xi^a$, we obtain $\tilde{\xi_a} = \phi\xi_a$ and, the relation between the complementary null vectors in the two frames are given as $l_a = \tilde{l}_a$. Thus,

$$d\tilde{\Sigma}_{ab} = \sqrt{\tilde{\sigma}}(\tilde{\xi}_a\tilde{l}_b - \tilde{\xi}_b\tilde{l}_a)d^2x = \phi^2 d\Sigma_{ab} \ , \tag{3.21}$$

where σ and $\tilde{\sigma} = \phi^2\sigma$ are the determinant of the induced metric of the two-surface in the Jordan and Einstein frames respectively. Therefore using the above relation, it can be easily seen that, (see the appendix 3.B for detail discussion)

$$\tilde{J}^{ab} = \frac{J'^{ab}}{\phi^2} \ . \tag{3.22}$$

In the appendix 3.B, we also show that,

$$\tilde{\Theta}^a = \frac{\Theta'^a}{\phi^2} \ . \tag{3.23}$$

Using the above relations, it can be seen that $\tilde{S} = S$, $\tilde{M} = M$ and $\tilde{J} = J$ in these two frames. We comment that the equivalence of the angular velocity and the surface gravity (or the temperature) in these two frames can be shown by following the procedure as described in [102].

We want to emphasize on the fact that J^{ab} and Θ^a in the Jordan frame (when one takes the Lagrangian as L instead of L'), cannot be written as proportional to the corresponding quantities in the Einstein frame. Therefore, one cannot establish the exact equivalence of the thermodynamic quantities between the Jordan and the Einstein frame, by considering the Lagrangian L in the Jordan frame. Whereas, in our case, we show the conserved Noether potentials of the two frames are proportional to each other with the proportionality factor as ϕ^2. This implies, in our case, the conserved Noether charge is the same in two frames. We want to further emphasize that in the work of Koga and Maeda [102], assuming the spacetime to be asymptotically flat, the equivalence of the thermodynamic quantities in the two frames have been established by following the Wald's formalism. On the contrary in our work, by considering a more generalised Lagrangian L', we establish the exact equivalence of thermodynamic parameters without making any assumption or imposing boundary conditions. Therefore, in this regard our analysis is more general and implying a crucial fact that in order to explore the thermodynamic equivalence in the two frames, one needs to consider the Lagrangian as L' in the Jordan frame instead of L.

3.3.4 Connection of the derived mass with the Brown-York mass term

Above, we have defined the masses in the two frames which are conformally invariant and are compatible with the first law. In literature, there are several prescription of defining the mass but, most of them are not conformally invariant. The only candidate, which is conformally invariant in the literature, is the Brown-York (BY) mass [96] (also see [97] which discusses that the BY mass is conformally invariant but, the BY energy is not). Therefore, we investigate whether the derived expressions of mass in (3.18) and (3.20) are the same as the BY mass. Here, we do the analysis in the Einstein frame for simplicity. From the transformation relations of the quantities, the same conclusion can be drawn in the Jordan frame as well.

We consider 2-dimensional null-hypersurface characterised by the induced metric $\tilde{\sigma}_{ab} = \tilde{g}_{ab} - \tilde{n}_a\tilde{n}_b + \tilde{u}_a\tilde{u}_b$, where $\tilde{u}^a$ and $\tilde{n}^a$ are the timelike and spacelike normals respectively. From the above expression of $\delta\tilde{M}$ in (2.21) we obtain,

$$\delta\tilde{M} = \delta\tilde{M}_{BY} - \frac{1}{8\pi}\int d^2\tilde{x}\,\delta(\sqrt{\tilde{h}}\tilde{K}^{(3)}) + \int d^2\tilde{x}[\sqrt{\tilde{h}}\tilde{n}_a\tilde{\Theta}^a(\tilde{q},\delta\tilde{q})] , \tag{3.24}$$

where, $\tilde{M}_{BY} = \frac{1}{8\pi}\int d^2\tilde{x}\tilde{N}\sqrt{\tilde{\sigma}}\tilde{k}^{(2)}$ is the expression for BY mass with $\tilde{N}$ being the lapse function and $\tilde{k}^{(2)}$ being the trace of the extrinsic curvature tensor of the null surface and $\tilde{K}^{(3)} = \tilde{\nabla}_a\tilde{n}^a$ is the trace of the extrinsic curvature tensor of the induced 3-surface characterised by the induced metric $\tilde{h}_{ab} = \tilde{g}_{ab} - \tilde{n}_a\tilde{n}_b$. The above relation (3.24) shows the explicit connection of our derived mass with the BY mass. The above relation can be further modified using eq. (12.104) of [144], which is given as

$$\delta\tilde{M} = \delta\tilde{M}_{BY} - \frac{1}{16\pi}\int d^2\tilde{x}\sqrt{\tilde{h}}\Big[\Big(\tilde{K}^{(3)}_{ab} - \tilde{K}^{(3)}\tilde{h}_{ab}\Big)\delta\tilde{h}^{ab} - \tilde{D}_i\tilde{U}^i + \tilde{n}^a(\tilde{\nabla}_a\tilde{\phi})\delta\tilde{\phi}\Big] . \tag{3.25}$$

Here, $\tilde{D}_i$ denotes the covariant derivative operator in the three-space $\tilde{h}_{ab}$ and $\tilde{U}^i = 2\tilde{n}_j\tilde{h}^i_k\delta\tilde{g}^{jk} - \tilde{n}^i\tilde{h}_{jk}\tilde{g}^{jk}$. This shows that our mass is connected with the BY mass with some additive terms.

3.4 Off-shell ADT current and potential in scalar-tensor theory

The identification of the conserved charges in GR has always been an important task for decades. There are several methods of defining the conserved charges, each with some advantages and disadvantages in its way. The ADM formalism [89] of computing the total conserved charge due to the Killing vectors has enjoyed the central attention, which holds good for the asymptotically flat spacetime. However, for the asymptotically non-flat or AdS spacetime, this approach fails.

For the asymptotically AdS solutions, a covariant method was developed by Abbott and Deser [30] to compute the conserved Killing charges asymptotically. This method was later extended by Deser and Tekin for the higher order gravity theories [31–33] which popularly known as the Abbott-Deser-Tekin (ADT) formalism. Here, we extend the ADT formalism in the scalar-tensor theory which is absent in literature. Moreover, we show the explicit connection between the off-shell Noether potential and the ADT potential and address the issue of invariance of the ADT potentials in these two frames.

For a Killing vector ξ^a, one can write the ADT current as

$$J^i_{ADT}|_{on-shell} = \delta E^{ij}\xi_j \ , \tag{3.26}$$

which indeed is a conserved quantity under the on-shell condition. Here, δE^{ij} is the linearized tensor (first order change in the equation of motion of the metric tensor due to $g^{ab} \to g^{ab} + \delta g^{ab}$). The conservation of the J^i_{ADT} follows from the fact that $\nabla_b \delta E^{ab} = 0$ on-shell (using eq (2.35)) and the property of the Killing vector (i.e. $\delta E^{ab}\nabla_a \xi_b = 0$). This conserved current we call as the ADT current. In the similar manner, the conserved on-shell ADT current in the Einstein frame can be written as, $\tilde{J}^i_{ADT}|_{on-shell} = \delta \tilde{E}^{ij}\tilde{\xi}_j$. At this stage, we urge to derive the off-shell ADT currents in order to make a more general and robust analysis. Hence, we define the off-shell ADT currents in each frame and follow the similar method as done in Einstein's gravity case [163].

3.4.1 Jordan frame

We obtain that off-shell $\delta E^{ij}\xi_j$ can be written as an anti-symmetric total derivative term added with some extra terms, where each of the extra terms is proportional to the E_{ab} i.e.,

$$\delta E^{ij}\xi_j = \nabla_j J^{ij}_{ADT} - E^{ik}h_{kj}\xi^j + \frac{1}{2}\xi^i E^{jk}h_{jk} - \frac{1}{2}\xi^j E^i_j h \ , \tag{3.27}$$

where

$$J^{ij}_{ADT} = \frac{1}{32\pi}\Big[\phi\Big(\xi^j\nabla_k h^{ki} - \xi^i\nabla_k h^{kj} + \xi_k\nabla^i h^{kj} - \xi_k\nabla^j h^{ki} + \xi^i(\nabla^j h) \\ -\xi^j(\nabla^i h) + h^{jk}\nabla_k\xi^i - h^{ik}\nabla_k\xi^j + h\nabla^{[i}\xi^{j]}\Big) + (\nabla_k\phi)\Big(\xi^j h^{ik} - \xi^i h^{jk}\Big)\Big] \ . \tag{3.28}$$

and $h_{ab} = \delta g_{ab}$ or equivalently $h^{ab} = -\delta g^{ab}$. Identifying $J^i_{ADT} = \nabla_j J^{ij}_{ADT}$ in eq.(3.27), we obtain

$$J^i_{ADT}|_{off-shell} = \delta E^{ij}\xi_j + E^{ik}h_{kj}\xi^j - \frac{1}{2}\xi^i E^{jk}h_{jk} + \frac{1}{2}\xi^j E^i_j h \ . \tag{3.29}$$

We refer our readers to the appendix 3.C for the detail derivation.

As J^{ij}_{ADT} is an anti-symmetric tensor, $\nabla_i J^i_{ADT} = 0$ even in the off-shell which imply that the off-shell ADT current is also a conserved quantity.

We now find out the conserved ADT current and potential in the Einstein frame in the following section.

3.4.2 Einstein frame

As similar to the Jordan frame, off-shell $\delta\tilde{E}^{ij}\tilde{\xi}_j$ can be written as,

$$\delta\tilde{E}^{ij}\tilde{\xi}_j = \tilde{\nabla}_j\tilde{J}^{ij}_{ADT} - \tilde{E}^{ik}\tilde{h}_{kj}\tilde{\xi}^j + \frac{1}{2}\tilde{\xi}^i\tilde{E}^{jk}\tilde{h}_{jk} - \frac{1}{2}\tilde{\xi}^j\tilde{E}^i_j\tilde{h} \ , \tag{3.30}$$

where

$$\tilde{J}^{ij}_{ADT} = \frac{1}{32\pi}\Big[\tilde{\xi}^j\tilde{\nabla}_k\tilde{h}^{ki} - \tilde{\xi}^i\tilde{\nabla}_k\tilde{h}^{kj} + \tilde{\xi}_k\tilde{\nabla}^i\tilde{h}^{kj} - \tilde{\xi}_k\tilde{\nabla}^j\tilde{h}^{ki} \\ +\tilde{\xi}^i(\tilde{\nabla}^j\tilde{h}) - \tilde{\xi}^j(\tilde{\nabla}^i\tilde{h}) + \tilde{h}^{jk}\tilde{\nabla}_k\tilde{\xi}^i - \tilde{h}^{ik}\tilde{\nabla}_k\tilde{\xi}^j + \tilde{h}\tilde{\nabla}^{[i}\tilde{\xi}^{j]}\Big] \ , \tag{3.31}$$

and $\tilde{h}_{ab} = \delta\tilde{g}_{ab}$, $\tilde{h}^{ab} = -\delta\tilde{g}^{ab}$. Therefore following the same analogy as in Jordan frame, one can define the off-shell ADT current in the Einstein frame as,

$$\tilde{J}^i_{ADT}|_{off-shell} = \delta\tilde{E}^{ij}\tilde{\xi}_j + \tilde{E}^{ik}\tilde{h}_{kj}\tilde{\xi}^j - \frac{1}{2}\tilde{\xi}^i\tilde{E}^{jk}\tilde{h}_{jk} + \frac{1}{2}\tilde{\xi}^j\tilde{E}^i_j\tilde{h} \ . \tag{3.32}$$

We refer the appendix 3.D for the detail derivation of the above equation (3.31). Following the earlier arguments, we comment that also in the Einstein frame the off-shell ADT current *i.e* $\tilde{J}^i_{ADT}$ is a conserved quantity.

3.5 Connection between conserved off-shell ADT and Noether potentials

Here, we urge to study the connection between the off-shell ADT potential and the Noether potential. For the Einstein's gravity, this connection has been studied in literature [164]. The equation (3.29) can be written as, (we drop the the subscript "off-shell" onward because all the further calculations are done off-shell)

$$\sqrt{-g}J^i_{ADT} = \delta(\sqrt{-g}E^{ij}\xi_j) - \frac{1}{2}\sqrt{-g}\xi^i E^{jk}h_{jk} \ . \qquad (3.33)$$

The above relation follows from the fact that $\delta\xi^i = 0$ and $\delta\phi = 0$ as we consider only the change due to $g_{ab} \to g_{ab} + h_{ab}$. By varying the Noether current in Jordan frame *i.e* eq.(3.7), under the change in the metric tensor, we obtain

$$\begin{aligned}\delta(\sqrt{-g}J'^i) = &2\delta(\sqrt{-g}E^{ij}\xi_j) - \sqrt{-g}\xi^i E^{jk}h_{jk} \\ &+\sqrt{-g}\xi^i\nabla_b[\Theta'^b(q,\delta q)] - \delta[\sqrt{-g}\Theta'^i(q,\pounds_\xi q)] \ .\end{aligned} \qquad (3.34)$$

Using equation (3.33), the above equation reduces to:

$$\delta(\sqrt{-g}J'^i) = 2\sqrt{-g}J^i_{ADT} + \sqrt{-g}\xi^i\nabla_b[\Theta'^b(q,\delta q)] - \delta[\sqrt{-g}\Theta'^i(q,\pounds_\xi q)] \ . \qquad (3.35)$$

As ξ^a is the Killing vector, in the above expression, therefore we use $\delta[\sqrt{-g}\Theta' i(q,\pounds_\xi q)] = \pounds_\xi[\sqrt{-g}\Theta'^i(q,\delta q)]$ (which follows from the fact that $\omega^a = 0$ in (3.13)). Using this relation in the above equation, it is straightforward to obtain

$$\sqrt{-g}J^{ij}_{ADT} = \frac{1}{2}\delta(\sqrt{-g}J'^{ij}) - \sqrt{-g}\xi^{[i}\Theta'^{j]}(q,\delta q) \ . \qquad (3.36)$$

In the Einstein frame, by following similar steps one finally obtains,

$$\sqrt{-\tilde{g}}\tilde{J}^{ij}_{ADT} = \frac{1}{2}\delta(\sqrt{-\tilde{g}}\tilde{J}^{ij}) - \sqrt{-\tilde{g}}\tilde{\xi}^{[i}\tilde{\Theta}^{j]}(\tilde{q},\delta\tilde{q}) \ . \qquad (3.37)$$

The above equations (3.36) and (3.37) show the explicit connections between the ADT and the Noether conserved quantities in the two frames.

Now we intend to show that how the ADT potentials are conformally connected in the two frames. Using (3.22) and (3.23) in the equations (3.36) and (3.37), it can be easily shown that,

$$\tilde{J}^{ij}_{ADT} = \frac{J^{ij}_{ADT}}{\phi^2} \ . \tag{3.38}$$

Thus our result is implying that the ADT potentials in the two frames are conformally connected to each other in the same manner as we obtain in the case of Noether potentials. Thus, the conserved ADT charges are invariant in the two frames. Such a prediction was given earlier in [165].

Let us now conclude this section with the following comments. In Komar's method [166] of defining the mass and angular momentum at asymptotic infinity by using the conserved Noether current, there appears an anomalous factor of 2 [167]. This anomaly can be tackled by the background subtraction method as described in [167]. Later, Wald provided an elegant solution in this context by considering the variation of the Noether current (we implemented the similar analysis in the section (3.3)) and defined the mass and the angular momentum in terms of the integrals containing the Noether potential along with the correction term (as in eq. (3.18)), which resolves the anomalous 2.

Apart from Wald's formalism, the first law can be established from the conserved ADT currents as well [168]. One can see from (3.36) and (3.37), the ADT potential consists of the Noether potential along with the same extra correction term which appears in Wald's formalism (as in eq. (3.17)). Thus, we emphasize that both ways of establishing the first law are equivalent and, therefore these two methods can be implemented alternatively according to one's convenience. For the reasons stated above, we do not include the explicit calculation of establishing the first law using the ADT formalism.

3.6 Entropy increase theorem and the modified null energy condition in Jordan frame

We analyse the entropy increase theorem in the background of this framework in order to get a complete picture of thermodynamic description of the scalar-tensor theory. Usually in GR the entropy increase theorem is established by assuming the null energy condition. But, we do not know what would be the null energy condition in the Jordan frame. Hence, one has to search for a similar energy condition which is different from the usual null energy condition. Here we show an interesting fact that the obtained similar energy condition in the Jordan frame, is proportional to the null energy condition as defined in the Einstein frame.

In this context a similar work has been done in [169], where the authors have interpreted the term at the right hand side (RHS) of eq.(5) of [169] as the stress-energy tensor of the scalar field ϕ. But, we have not adopted that approach in our analysis. In our work, using $E_{ab} = 0$ from (2.28), we obtain,

$$G_{ab} = -\frac{\omega}{2\phi^2}\nabla_i\phi\nabla^i\phi g_{ab} + \frac{\omega}{\phi^2}\nabla_a\phi\nabla_b\phi - \frac{V}{2\phi}g_{ab} + \frac{1}{\phi}\nabla_a\nabla_b\phi - \frac{1}{\phi}\nabla_i\nabla^i\phi g_{ab} \ . \quad (3.39)$$

From the above equation we cannot identify the RHS as the energy-momentum (EM) tensor of the scalar field ϕ in the Jordan frame as this is not compatible with the usual definition of the EM tensor (given as $T_{ab} = \frac{2}{\sqrt{-g}}\frac{\delta L_{matter}}{\delta g^{ab}}$). Thus, in this section, we try to provide a justifiable way to obtain the increase in the entropy by using the modified energy condition.

From (3.39), we calculate $R_{ab}l^a l^b$ (with l^a being a null vector), which is given as,

$$R_{ab}l^a l^b = \frac{\omega}{\phi^2}(l^a\nabla_a\phi)^2 + \frac{1}{\phi}l^a l^b\nabla_a\nabla_b\phi \ . \quad (3.40)$$

The first term is a positive definite for $\omega > 0$. Thus we write

$$R_{ab}l^a l^b - \frac{1}{\phi}l^a l^b\nabla_a\nabla_b\phi \geq 0 \ . \quad (3.41)$$

The expression of the entropy in the Jordan frame is given in (3.18) and using this equation our explicit calculation shows that the entropy can be written as $S = A/4$, where

$$A = \int_{\mathcal{H}} \sqrt{\sigma}\phi d^2x \ . \quad (3.42)$$

The above expression of the entropy matches Kang's prescription in [28]. Let us now find out the change in entropy along a null geodesic congruence. Hence, we calculate

$$\frac{dA}{d\lambda} = \int_{\mathcal{H}} \sqrt{\sigma}\phi d^2x\theta' \ . \quad (3.43)$$

Here, λ parametrizes the null-congruence and $\theta' = \theta^{(l)} + \frac{1}{\phi}\frac{d\phi}{d\lambda}$, where $\theta^{(l)} = \frac{1}{\sqrt{\sigma}}\frac{d\sqrt{\sigma}}{d\lambda}$

is the expansion parameter along the null vector l^a. We intend to establish in the following analysis that $\frac{dS}{d\lambda} \geq$ always, by showing $\theta' \geq 0$.

$$\frac{d\theta'}{d\lambda} = \frac{d\theta^{(l)}}{d\lambda} - \frac{1}{\phi^2}(l^a\nabla_a\phi)^2 + \frac{1}{\phi}l^al^b(\nabla_a\nabla_b\phi) \ ,$$
$$= -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}l^al^b - \frac{1}{\phi^2}(l^a\nabla_a\phi)^2 + \frac{1}{\phi}l^al^b(\nabla_a\nabla_b\phi) \ . \tag{3.44}$$

The last expression is obtained using null Raychaudhuri equation $\frac{d\theta^{(l)}}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}l^al^b$, where the null vector l^a is affinely parameterized and also is a hypersurface-orthogonal. Using (3.41) we obtain $\frac{d\theta'}{d\lambda} \leq 0$. Therefore, the prohibition of caustics demands that $\theta' \geq 0$. Thus the entropy increase theorem is established in this frame.

We now discuss that what is the significance of the condition in (3.41). Although (3.41) is an identity in the Jordan frame, here we shall prove that it corresponds to the null energy condition in the Einstein frame.

In the Einstein frame,

$$\frac{\tilde{G}_{ab}}{16\pi} = \frac{1}{2}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{4}\tilde{g}_{ab}\tilde{\nabla}^i\tilde{\phi}\tilde{\nabla}_i\tilde{\phi} - \frac{1}{2}\tilde{g}_{ab}U(\tilde{\phi}) \ . \tag{3.45}$$

The right hand side of the above equation can be identified as the stress-energy tensor $(\frac{\tilde{T}^{(\tilde{\phi})}_{ab}}{2})$ of the scalar field $\tilde{\phi}$. Thus we obtain

$$\tilde{T}^{(\tilde{\phi})}_{ab}l^al^b = (\tilde{l}^a\tilde{\nabla}_a\tilde{\phi})^2 \geq 0 \ . \tag{3.46}$$

The above equation (3.46) is the null energy condition in the Einstein frame. Due to the conformal transformation we obtain,

$$\tilde{T}^{(\tilde{\phi})}_{ab}l^al^b = \frac{1}{\phi^2\omega}\Big(\frac{2\omega+3}{16\pi}\Big)\Big[R_{ab}l^al^b - \frac{1}{\phi}l^al^b\nabla_a\nabla_b\phi\Big] \ . \tag{3.47}$$

Thus, we can conclude that the energy condition in the Jordan frame (3.41) corresponds to the null energy condition in the Einstein frame.

3.7 Summary and Discussions

In this chapter, we cast light on the two major issues, which we have mentioned in the beginning of the earlier chapter, and provide satisfactory answers to these incongruities in this theory. In the earlier chapter we have shown that the usual

Lagrangians in the two frames differ by a total derivative term due to the conformal transformation. It is common in the study of scalar-tensor theory, that most of the authors does not carefully mention that the two Lagrangians in Jordan frame and the Einstein frame are equivalent only up to a total derivative term. Although a total derivative term does not contribute to the dynamics of the system, but one must contemplate deeply before one injudiciously neglect that term in this theory while studying the thermodynamic aspects. Also, removing the $\Box\phi$ term incurs several in-equivalences, which we have witnessed in the previous chapter.

In this chapter, we have accounted the $\Box\phi$ term in the Lagrangin of the Jordan frame and we show that this surface term actually plays a crucial role to obtain the conformal equivalence of the thermodynamic quantities without imposing any assumptions and boundary conditions. Moreover, the extra term helps us to remove all the in-equivalences, which we have discussed in the earlier chapter. Here, the study of thermodynamic properties of spacetime geometry is based on the concept of conserved currents as obtained from the two different approaches such as the Noether approach and the ADT approach. All the conserved quantities are off-shell, which can be used for a generic null surface and can play a significant role in the context of the emergent gravity paradigm. At first, we obtain the off-shell Noether current and potential in both frames and, following Wald's formalism, we identify the thermodynamic quantities from the conserved Noether current. Later, we show that the identified thermodynamic quantities fit nicely in the first law and the second law of black hole mechanics. Subsequently we obtain an important result in the background of our theoretical framework that the thermodynamic parameters are conformally invariant in these two frames, if one consider the $\Box\phi$ term in the Lagrangian. Hence, at this stage, we comment that to examine the conformal invariance of the thermodynamic quantities in the two frames in the background of the scalar-tensor theory, one must not disregard the contributions from the surface term. We also emphasize that following our procedure of the inclusion of $\Box\phi$ term, one can avoid the use of any boundary condition and assumption regarding the nature of spacetime. Observing the above conclusions in Noether prescription, we are keen to verify our results using the ADT formalism in both frames of this theory. Therefore following the ADT mechanism, we obtain the conserved ADT current and the corresponding ADT potential in both the frames. Thereafter, we establish the connection of the ADT current and potential to the Noether counterparts. Moreover, we discuss the connection of the off-shell ADT currents with the off-shell Noether current and Wald's formalism. Our results strongly support that, implementing both of these standard formalisms we find that the thermodynamic descriptions and the thermodynamic quantities are invariant in the two frames in the background of the scalar-tensor theory. Our results suggest that these two approaches of finding conserved quantities and describing the first law of black hole thermodynamics are basically equivalent to each other. We hope this work will be a significant one in the thermodynamic description of the scalar-tensor theory.

Finally, we mention that in usual thermodynamics there are intensive quantities (such as temperature and pressure etc.) which do not change by conformal scalings, while there are extensive quantities (like energy) which do change under scaling. Our present situation is in contradiction with this usual understanding. This issue can be understood in the following way. In black hole thermodynamics, we cannot categorize the extensive and the intensive variables like the usual thermodynamic cases. For example, the entropy is an extensive variable and also a function of all other extensive parameters in the usual thermodynamics. But, in black hole mechanics it is not an extensive variable as it is proportional to the area of the black hole horizon. If the two black holes are combined together, then Bekenstein-Hawking area expression implies that the entropy of the combined black hole is greater than the sum of the entropy of the individual black holes. Moreover, the temperature and pressure in the usual thermodynamic case are intensive thermodynamic entities. But, in black hole thermodynamics, those two quantities are scale dependent (for instance, in the case of Schwarzschild black hole the Hawking temperature is inverse of mass of the black hole). The principle of equivalence implies that the temperature is red-shifted or blue-shifted in the same manner as of the frequency of the photons. Apart from these obvious differences with the usual thermodynamics, there are a few other facts (e.g. specific heat of Schwarzschild black hole is negative) which clearly indicates that one cannot classify the black hole thermodynamic entities as the extensive or the intensive ones. Therefore, the usual scaling argument cannot be applied here.

Although the idea of the extensive and intensive thermodynamic parameters is not compatible in BH thermodynamics, one can find that the phase space of BH thermodynamics is very much rich and most of the features of ordinary thermodynamics is also present in the BH thermodynamics as well (such as phase transition, critical phenomena, critical exponents, thermodynamic laws etc.). Therefore, it implies that the identification of thermodynamic parameters in BH thermodynamics is consistent in every aspect (although may not be categorized as the “extensive” and the “intensive” ones) and it works very well to show a robust thermodynamic feature in BH thermodynamics. However, since the BH thermodynamics differ to the normal thermodynamics in many ways (as we have mentioned above), one cannot take it for granted that all the behaviour of BH thermodynamics will be the same as of the normal thermodynamics. Instead, one has to examine properly in every aspects of BH thermodynamics.

Thus, in the present and in the earlier chapter, we have contributed towards solving several longstanding issues in scalar-tensor gravity. In addition, we have firmly extablished a consistent and covariant thermodynamic description in the two frames of this theory. We conclude this chapter with the following analysis in the context of studying $f(R)$ theory as a subclass of scalar-tensor theory.

In this chapter and in the previous chapter, we have mentioned several times that f(R) gravity can also be studied as a sub-class of scalar-tensor theory. Let us clarify this issue in more detail. The $f(R)$ gravity action can be written as (here we discuss the metric formalism of $f(R)$ theory, not the Palatini formalism)

$$\mathcal{A} = \frac{1}{2\kappa_G}\int \sqrt{-g}d^4x f(R) ~, \tag{3.48}$$

where $\kappa_G = 8\pi$. This action can be written in terms of the equivalent Brans-Dicke action in the following way. We introduce a new field ψ, and the action (3.48) can be written as dynamically equivalent action as

$$\mathcal{A} = \frac{1}{2\kappa_G}\int \sqrt{-g}d^4x[f(\psi) + f'(\psi)(R-\psi)] ~. \tag{3.49}$$

If we vary the above action with respect to ψ, we obtain the equation of motion of ψ as

$$f''(\psi)(R-\psi) = 0 ~. \tag{3.50}$$

Therefore, if $f''(\psi) \neq 0$, one obtains $\psi = R$, which makes the actions (3.48) and (3.49) to be equivalent. Redefining the fields as $\phi = f'(\psi)$ and setting

$$V(\phi) = \phi\psi(\phi) - f(\psi(\phi)) ~, \tag{3.51}$$

one obtains

$$\mathcal{A}_{BD} = \frac{1}{2\kappa_G}\int \sqrt{-g}d^4x[\phi R - V(\phi)] ~, \tag{3.52}$$

which is the Brans-Dicke action in the Jordan frame, with the Brans-Dicke parameter $\omega = 0$.

Again, the Einstein frame representation is possible for this theory by the conformal transformation of the metric and the rescaling of the scalar filed, as it has been mentioned earlier (Eq. (2.2) and Eq. (2.3)). Note that in this case, the conformal factor $\phi = f'(R)$ (from (3.50)) and the action in the Einstein frame will be given as

$$\tilde{\mathcal{A}}_{BD} = \int \sqrt{-g} d^4x [\frac{\tilde{R}}{2\kappa_G} - \frac{1}{2}\nabla^a\tilde{\phi}\nabla_a\tilde{\phi} - U(\tilde{\phi})] \ . \tag{3.53}$$

In this case, we obtain

$$\phi \equiv f'(R) = \exp\sqrt{\frac{2\kappa_G}{3}}\tilde{\phi} \ , \tag{3.54}$$

and

$$U(\tilde{\phi}) = \frac{Rf'(R) - f(R)}{2\kappa_G(f'(R))^2} \ . \tag{3.55}$$

Thus, Eqs (3.48), (3.52) and (3.53) are the different representations of the same theory [170]. Apart from these representations, one can actually find infinitely many conformal frames [171, 172]. The equivalence among these conformal frames is a matter of debate for a long time [88, 124, 173–175]. Note that $U(\tilde{\phi})$ diverges when $f'(R)$ vanishes. Also, for $f(R) = R$, one obtains $U(\tilde{\phi}) = 0$ and for $f(R) = R^2$, one obtains $U(\tilde{\phi}) = 1/8\kappa_G$. In addition, $d\phi/dR = f''(R) \neq 0$, a condition which is required to express the $f(R)$ theory as the Brans-Dicke theory (from (3.50)).

Note, for quadratic gravity, $f(R) \sim R^2$, and, one obtains $\phi \sim R$. Therefore, although it is possible to obtain solution with $R = 0$ for $f(R) \sim R^2$ gravity [176], in Einstein frame, the Ricci-flat solution is not possible as, in that case, the study of $f(R)$ as an equivalent Brans-Dicke theory will be ill-defined (for example, in that case, we cannot define $\tilde{g}_{ab} = \phi g_{ab}$ when $\phi = 0$). Thus, the solution space of $f(R)$ gravity and the equivalent Brans-Dicke theory can be different. In that case, establishing the equivalence of thermodynamics in the two frames may not be possible. However, we have not considered these situation in our thesis. We have considered that the two frames of the scalar-tensor theory are well-defined.

Above, we have shown that $f(R)$ is equivalent to the scalar-tensor theory at the classical level. However, in the quantum level, this equivalence might not hold. The (in)equivalence of the $f(R)$ gravity and the scalar-tensor gravity, at the quantum level, is still a matter of debate. While some argues it to be equivalent [133, 177], some others are against it [178]. Since our work does not provide any insight to take any stand against the debate, we do not provide any strong opinion against it. We can only say that the classical equivalence of the $f(R)$ theory and the scalar-tensor theory is not guaranteed at the quantum level, which is subject to further research.

Moreover, adding GHY surface term is more subtle for $f(R)$ theories of gravity. In this case, the GHY term is identified as [179]

$$\mathcal{A}_{GHY} \sim \oint d^3x\sqrt{h}f'(R)K \ , \tag{3.56}$$

which resembles to the GHY surface term of the Jordan frame (Eq. (2.8)) with the identification $\phi \equiv f'(R)$. Variation of the GHY surface term in the Jordan frame (2.8)) will provide the term with the coefficient $\delta\phi$. This term can be neglected saying ϕ is fixed on the boundary. Similarly, the variation of the GHY term of the $f(R)$ gravity, as given in (3.56), will incorporate term with coefficient $f''(R)\delta R$ (analogous to $\delta\phi$ of scalar-tensor theory). Therefore, in this case, not only we have to fix g_{ab} on the boundary, but also, we need to fix R on the boundary as well (i.e. $\delta R = 0$) [179]. This poses a constraint relation on the derivatives of g_{ab} and $\delta(\partial_i g_{ab})$ will not remain arbitrary on the boundary. It is argued that the Ricci scalar R, in $f(R)$ gravity, carries extra scalar degrees of freedom (just like ϕ in scalar-tensor theory). Therefore, R must be fixed at the boundary in $f(R)$ gravity [179].

Appendix

3.A Obtaining the modified expression of the Noether potential [Eq. (3.8)]

The expression of $\Theta'^a(q, \pounds_\xi q)$ can be obtained from (3.6) as

$$\Theta'^a(q, \pounds_\xi q) = \Theta^a(q, \pounds_\xi q) - \frac{1}{16\pi}\Big\{\frac{3}{2}g^{ij}\pounds_\xi g_{ij}\partial^a\phi - 3g^{ia}\partial^b\phi\pounds_\xi g_{ib} + 3\partial^a(\pounds_\xi\phi)\Big\}\,. \tag{3.57}$$

The expression of $\Theta^a(q, \pounds_\xi q)$ has been obtained earlier in (2.95). Now

$$\frac{3}{2}g^{ij}\pounds_\xi g_{ij}\partial^a\phi - 3g^{ia}\partial^b\phi\pounds_\xi g_{ib} + 3\partial^a(\pounds_\xi\phi) = 3\nabla_b[\xi^b\nabla^a\phi - \xi^a\nabla^b\phi] + 3\xi^a\Box\phi\,. \tag{3.58}$$

Using (3.57) and (3.58), we finally obtain

$$\Theta'^a(q, \pounds_\xi q) = \frac{1}{16\pi}\Big[-\nabla_b\big[\nabla^a(\phi\xi^b) - \nabla^b(\phi\xi^a)\big] - \frac{2\omega}{\phi}(\nabla^a\phi)\xi^b\nabla_b\phi - 2\xi^b\nabla_b\nabla^a\phi$$
$$-\xi^a\Box\phi + 2\phi g^{ac}R_{kc}\xi^k\Big]\,. \tag{3.59}$$

Using (3.59) and L' (from (3.1)) in (3.7) one obtains

$$J'^a = \frac{1}{16\pi}\Big[\big[\nabla^a(\phi\xi^b) - \nabla^b(\phi\xi^a)\big] + \Big\{(\phi R - \frac{\omega(\phi)}{\phi}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi))\xi^a$$
$$+2\frac{\omega}{\phi}(\nabla^a\phi)\xi^b(\nabla_b\phi) + 2\xi^b\nabla_b\nabla^a\phi - 2\xi^a\Box\phi - 2\phi g^{ac}R_{kc}\xi^k\Big\} + 2E^{ab}\xi_b\Big]. \tag{3.60}$$

One can identify the curly-bracketed term as a whole as $-2E^{ab}\xi_b$ (see the expression of E^{ab} from (2.28)) and, hence, the expression of J'^a is given by a total derivative of anti-symmetric Noether potential, the expression of which has been given in (3.8).

3.B Derivation of the Eqs. (3.22) and (3.23)

Proving (3.22) is pretty straightforward.

$$\tilde{J}^{ab} = \tilde{g}^{ai}\tilde{g}^{bj}\tilde{J}_{ij} = \tilde{g}^{ai}\tilde{g}^{bj}(\partial_a\tilde{\xi}_b - \partial_b\tilde{\xi}_a) = \frac{g^{ai}g^{bj}}{\phi^2}\Big[\partial_a(\phi\xi_b) - \partial_b(\phi\xi_a)\Big]$$
$$= \frac{1}{\phi^2}\Big[\nabla^a(\phi\xi^b) - \nabla^b(\phi\xi^a)\Big] \ . \qquad (3.61)$$

Thus, equation (3.22) is obtained.

The expression of $\tilde{\Theta}^a$ is given in (2.21). Now, $\tilde{\nabla}_b(\delta\tilde{g}_{id}) = (\partial_b\phi)\delta g_{id} + \phi\tilde{\nabla}_b(g_{id}) - \frac{g_{id}}{\phi}(\partial_b\phi)\delta\phi + g_{id}\partial_b(\delta\phi)$. Then using $\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{1}{2\phi}(\delta^a_b\partial_c\phi + \delta^a_c\partial_b\phi - g_{bc}\partial^a\phi)$ in $\tilde{\nabla}_b(g_{id})$, it requires a few steps to obtain (3.23).

3.C Derivation of the Eq. (3.28)

For $g_{ab} \to g_{ab} + h_{ab}$, the expression of $\delta G^{ij}\xi_j$ is given as [163]

$$(\delta G^{ij})\xi_j = \nabla_j F^{ij} - G^{ik}h_{kj}\xi^j + \frac{1}{2}\xi^i G^{jk}h_{jk} - \frac{1}{2}\xi^j G^i_j h \ , \qquad (3.62)$$

where, δG^{ij} denotes the linearization of the Einstein tensor. Remember, here ξ^a is a Killing vector and

$$F^{ij} = \frac{1}{2}\Big[\xi^j\nabla_k h^{ki} - \xi^i\nabla_k h^{kj} + \xi_k\nabla^i h^{kj} - \xi_k\nabla^j h^{ki} + \xi^i(\nabla^j h) - \xi^j(\nabla^i h)$$
$$+ h^{kj}\nabla_k\xi^i - h^{ki}\nabla_k\xi^j + h\nabla^{[i}\xi^{j]}\Big] \ . \qquad (3.63)$$

Now, in this frame the expression of E^{ab} has been given in (2.28) . For $g_{ab} \to g_{ab} + h_{ab}$

$$16\pi(\delta E^{ij})\xi_j = \phi[(\delta G^{ij})\xi_j] - \frac{\omega}{2\phi}h^{ij}g^{ab}(\partial_a\phi)(\partial_b\phi)\xi_j - \frac{\omega}{2\phi}\xi^i h^{ab}(\partial_a\phi)(\partial_b\phi)$$
$$+ \frac{\omega}{\phi}h^{jb}g^{ai}(\partial_a\phi)(\partial_b\phi)\xi_j - \frac{V}{2}h^{ij}\xi_j + h^{ia}\xi^b(\nabla_a\nabla_b\phi) + g^{ia}h^{jb}\xi_j(\nabla_a\nabla_b\phi)$$
$$- h^{ij}g^{ab}\xi_j(\nabla_a\nabla_b\phi) - \xi^i h^{ab}(\nabla_a\nabla_b\phi) - g^{ia}\xi^b\delta(\nabla_a\nabla_b\phi) + \xi^i g^{ab}\delta(\nabla_a\nabla_b\phi) \ . \qquad (3.64)$$

Now, we express $16\pi E^{ij} = \phi G^{ij} + \bar{E}^{ij}$ where

$$\bar{E}^{ij} = \frac{\omega}{2\phi}g^{ij}g^{ab}(\partial_a\phi)(\partial_b\phi) - \frac{\omega}{\phi}g^{ia}g^{jb}(\partial_a\phi)(\partial_b\phi) + \frac{V}{2}g^{ij} - g^{ia}g^{jb}\nabla_a\nabla_b\phi$$
$$+ g^{ij}g^{ab}\nabla_a\nabla_b\phi \ . \qquad (3.65)$$

Then, using (3.62) and (3.64) we obtain

$$16\pi(\delta E^{ij})\xi_j = \phi\nabla_j F^{ij} - 16\pi E^{ik}h_{kj}\xi^j + \frac{16\pi}{2}\xi^i E^{jk}h_{jk} - \frac{16\pi}{2}\xi^j E^i_j h + \bar{E}^{ik}h_{kj}\xi^j$$
$$-\frac{1}{2}\xi^i\bar{E}^{jk}h_{jk} + \frac{1}{2}\xi^j\bar{E}^i_j h - \frac{\omega}{2\phi}h^{ij}g^{ab}(\partial_a\phi)(\partial_b\phi)\xi_j - \frac{\omega}{2\phi}\xi^i h^{ab}(\partial_a\phi)(\partial_b\phi)$$
$$+\frac{\omega}{\phi}h^{jb}g^{ai}(\partial_a\phi)(\partial_b\phi)\xi_j - \frac{V}{2}h^{ij}\xi_j + h^{ia}\xi^b(\nabla_a\nabla_b\phi) + g^{ia}h^{jb}\xi_j(\nabla_a\nabla_b\phi)$$
$$-h^{ij}g^{ab}\xi_j(\nabla_a\nabla_b\phi) - \xi^i h^{ab}(\nabla_a\nabla_b\phi) - g^{ia}\xi^b\delta(\nabla_a\nabla_b\phi) + \xi^i g^{ab}\delta(\nabla_a\nabla_b\phi)\ . \quad (3.66)$$

Now,

$$\bar{E}^{ik}h_{kj}\xi^j - \frac{1}{2}\xi^i\bar{E}^{jk}h_{jk} + \frac{1}{2}\xi^j\bar{E}^i_j h = \frac{\omega}{2\phi}h^{ij}g^{ab}(\partial_a\phi)(\partial_b\phi)\xi_j + \frac{V}{2}h^{ij}\xi_j$$
$$-g^{ia}h^{jb}\xi_j(\nabla_a\nabla_b\phi) + h^{ij}g^{ab}\xi_j(\nabla_a\nabla_b\phi) - \frac{\omega}{\phi}h^{jb}g^{ai}(\partial_a\phi)(\partial_b\phi)\xi_j$$
$$+\frac{\omega}{2\phi}\xi^i h^{ab}(\partial_a\phi)(\partial_b\phi) + \frac{1}{2}\xi^i h^{ab}(\nabla_a\nabla_b\phi) - \frac{1}{2}g^{ia}\xi^b h(\nabla_a\nabla_b\phi)\ . \quad (3.67)$$

Substituting (3.67) in (3.66), we obtain

$$16\pi(\delta E^{ij})\xi_j = \nabla_j(\phi F^{ij}) - F^{ij}(\partial_j\phi) - 16\pi E^{ik}h_{kj}\xi^j + \frac{16\pi}{2}\xi^i E^{jk}h_{jk} - \frac{16\pi}{2}\xi^j E^i_j h$$
$$+h^{ia}\xi^b(\nabla_a\nabla_b\phi) - \frac{1}{2}\xi^i h^{ab}(\nabla_a\nabla_b\phi) - g^{ia}\xi^b\delta(\nabla_a\nabla_b\phi) + \xi^i g^{ab}\delta(\nabla_a\nabla_b\phi)$$
$$-\frac{1}{2}g^{ia}\xi^b h(\nabla_a\nabla_b\phi)\ . \quad (3.68)$$

Now,

$$\delta(\nabla_b\nabla_a\phi) = -\delta\Gamma^i_{ab}(\partial_i\phi) = -\frac{1}{2}\Big[\nabla_a h^i_b + \nabla_b h^i_a - \nabla^i h_{ab}\Big](\partial_i\phi)\ . \quad (3.69)$$

Using the above relation (3.69) with (3.63), one obtains

$$-F^{ij}(\partial_j\phi) - g^{ia}\xi^b\delta(\nabla_a\nabla_b\phi) + \xi^i g^{ab}\delta(\nabla_a\nabla_b\phi)$$
$$= \frac{1}{2}(\partial_j\phi)\Big[-\xi^i\nabla_k h^{jk} - h^{jk}(\nabla_k\xi^i) + h^{ik}(\nabla_k\xi^j) - h\nabla^{[i}\xi^{j]} + \xi^k\nabla_k h^{ij}\Big]$$
$$= \frac{1}{2}\nabla_j\Big[(\partial_k\phi)\Big(\xi^j h^{ik} - \xi^i h^{jk}\Big)\Big] + \frac{1}{2}h^{jk}\xi^i\nabla_k\nabla_j\phi + \frac{1}{2}h^{ik}(\nabla_k\xi^j)(\nabla_j\phi)$$
$$-\frac{1}{2}h(\nabla_j\phi)(\nabla^i\xi^j) - \frac{1}{2}h^{ij}\xi^k\nabla_k\nabla_j\phi\ . \quad (3.70)$$

Substituting the above relation of (3.70) in (3.68), we obtain

$$16\pi(\delta E^{ij})\xi_j = \nabla_j(\phi F^{ij}) + \frac{1}{2}\nabla_j\Big[(\partial_k\phi)\Big(\xi^j h^{ik} - \xi^i h^{jk}\Big)\Big] - 16\pi E^{ik}h_{kj}\xi^j + \frac{16\pi}{2}\xi^i E^{jk}h_{jk} - \frac{16\pi}{2}\xi^j E^i_j h + \frac{1}{2}\Big\{h^{ia}\xi^b(\nabla_a\nabla_b\phi) - g^{ia}\xi^b h(\nabla_a\nabla_b\phi) + h^{ik}(\nabla_k\xi^j)(\nabla_j\phi) - h(\nabla_j\phi)(\nabla^i\xi^j)\Big\} . \tag{3.71}$$

Using the property of the Killing vector, the terms inside the curly bracket vanish and, one obtains

$$(\delta E^{ij})\xi_j = \nabla_j J^{ij}_{ADT} - E^{ik}h_{kj}\xi^j + \frac{1}{2}\xi^i E^{jk}h_{jk} - \frac{1}{2}\xi^j E^i_j h , \tag{3.72}$$

where, the final expression of J^{ij}_{ADT} is given in (3.28).

3.D Derivation of the Eq. (3.31)

To prove (3.31), we shall follow the same procedure as in the Jordan frame. Here, let us take $\tilde{E}^{ij} = \frac{\tilde{G}^{ij}}{16\pi} + \bar{\tilde{E}}^{ij}$ with

$$\bar{\tilde{E}}^{ij} = -\frac{1}{2}\tilde{g}^{ai}\tilde{g}^{bj}(\partial_a\tilde{\phi})(\partial_b\tilde{\phi}) + \frac{1}{4}\tilde{g}^{ij}\tilde{g}^{ab}(\partial_a\tilde{\phi})(\partial_b\tilde{\phi}) + \frac{1}{2}\tilde{g}^{ij}U . \tag{3.73}$$

Therefore,

$$(\delta\tilde{E}^{ij})\tilde{\xi}_j = \frac{1}{16\pi}(\delta\tilde{G}^{ij})\tilde{\xi}_j + (\delta\bar{\tilde{E}}^{ij})\tilde{\xi}_j = \frac{1}{16\pi}\tilde{\nabla}_j\tilde{F}^{ij} - \tilde{E}^{ik}\tilde{h}_{kj}\tilde{\xi}^j + \frac{1}{2}\tilde{\xi}^i\tilde{E}^{jk}\tilde{h}_{jk} - \frac{1}{2}\tilde{\xi}^j\tilde{E}^i_j\tilde{h} + \Big\{\bar{\tilde{E}}^{ik}\tilde{h}_{kj}\tilde{\xi}^j - \frac{1}{2}\tilde{\xi}^i\bar{\tilde{E}}^{jk}\tilde{h}_{jk} + \frac{1}{2}\tilde{\xi}^j\bar{\tilde{E}}^i_j\tilde{h} + (\delta\bar{\tilde{E}}^{ij})\tilde{\xi}_j\Big\} . \tag{3.74}$$

where, the expression of $\tilde{F}^{ij}$ is similar to the expression given in (3.63) (only with tilde overhead). Detail calculations show that the terms inside the curly brackets in (3.74) vanish and, one finally obtains

$$(\delta\tilde{E}^{ij})\tilde{\xi}_j = \tilde{\nabla}_j\tilde{J}^{ij}_{ADT} - \tilde{E}^{ik}\tilde{h}_{kj}\tilde{\xi}^j + \frac{1}{2}\tilde{\xi}^i\tilde{E}^{jk}\tilde{h}_{jk} - \frac{1}{2}\tilde{\xi}^j\tilde{E}^i_j\tilde{h} , \tag{3.75}$$

where, the final expression of $\tilde{J}^{ij}_{ADT}$ is given in (3.31).

Chapter 4

Thermodynamic relations for a cosmological black hole with a conformal Killing horizon

4.1 A "realistic" black hole

[1]In the previous chapter, we have established the thermodynamics of the two frames of the scalar-tensor theory which are conformally coupled. In that case we have established the thermodynamic laws of the black holes which have a Killing horizon, or the spacetime is not evolving with respect to time. For understanding the key aspects of the black hole thermodynamics, considering the presence of a Killing horizon is a viable assumption as most of the works in this regards are premised upon the two key assumptions. One is the presence of a global Killing vector in the spacetime and another one is the asymptotic flatness of the spacetime at large spatial distance. However, in a realistic situation, a black hole should be surrounded by a local mass distribution. Therefore, at large spatial distance from the black hole, the spacetime should not be usually flat. Also, black holes are not usually time independent in a realistic thought. Therefore, one does not treat the stationary ones as the part of a cosmological scenario. Of late, many theoretical efforts have been brought to light in which experts have tried to develop the physics for a dynamical black hole with different levels of success. But, it is yet to be developed in many avenues. In addition, since the thermodynamic description of black holes largely depends on the Killing symmetry of the spacetime, a detail investigation is required to check whether it is possible to obtain a thermodynamic description in absence of

[1]This chapter is based on the publication [107] .

a Killing horizon.

In this chapter we shall discuss about the Sultana-Dyer (SD) black holes [106]. The SD metric is connected to the Schwarzschild one by a time-dependent conformal factor. The underlying motivation to work in SD spacetime is that the SD metric corresponds to the real cosmological scenario for the following reasons. Firstly, the SD metric becomes Friedmann-Lemaitre-Robertson-Walker (FLRW) metric in the asymptotic limit, and the FLRW metric describes the homogeneous, isotropic expanding universe very successfully in various cases. Moreover, the horizon of a SD black hole is not a Killing horizon, instead it is a conformal Killing one. Therefore, studying the thermodynamic property of this particular black hole is important in order to look beyond the Killing horizon to obtain a thermodynamic description. It turns out that the SD metric is a inhomogeneous and time dependent solution of general relativity (GR) in presence of two non-interacting perfect fluids. One fluid is a timelike dust and another one is a null dust. Now, as the conformal factor is time-dependent, the spacetime metric evolves with time [106]. Therefore, one should expect a different result from those calculations for the stationary cases. Recently, a few thermodynamical aspects of the SD spacetime has been discussed in [98, 155, 156, 180–182]. In this chapter the full thermodynamics of the SD black hole will be developed in a consistent way.

4.2 Objectives of the chapter

In order to obtain the thermodynamic relation for a black hole, one needs to obtain the expressions of the thermodynamic quantities (such as temperature, entropy and internal energy) very precisely. The expression of entropy has been obtained uniquely following different methods [155, 156, 180]. Also, the convenient energy expression, used to describe the thermodynamics of a black hole in evolving space time, is the Misner-Sharp energy [92]. Being spherically symmetric, the energy for the SD spacetime has been obtained easily in [98]. Therefore, as far as entropy and energy are concerned, they are uniquely determined. But, there is a discrepancy in the expression of the horizon temperature. In literature one gets two expressions of temperature for the SD black hole: one is time independent [106] while the other one is time dependent [180–182]. In spite of the fact, that the time dependent expression is favoured by the scaling argument [180], the correct one should be identified by an explicit derivation. One direct way to find the correct expression is to look at the emission spectrum from the horizon. Therefore, it is necessary to study the

Hawking effect [16] of the black hole. However, it should be mentioned that the original calculation of Hawking's procedure is not applicable here, because, in the original calculation of Hawking, the radiation spectrum was observed by evaluating the Bogoliubov coefficients of ingoing and outgoing modes. Those coefficients were defined by the boundary conditions when the spacetime is asymptotically flat. In this case, a time dependent conformal factor is added on the Schwarzschild metric which makes the spacetime to be asymptotically FLRW. Now, it has also been studied that the Bogoliubov coefficients are not conformally invariant which means that the original method of Hawking is not straightforwardly applicable to study the Hawking radiation for the SD black holes. Therefore, here the tunneling formalism [183, 184] will be used to derive the expression of temperature from the tunneling probability.

There are two principal ways of implementing the tunneling formalism, one is the Hamilton-Jacobi method [183] and the another is the null geodesic method [184] (also see [185, 186], which are more relevant in time-dependent background). Both of them are based on the semi-classical WKB approximation and they give the identical result[2]. The underlying idea of tunneling method is the formation of particle-antiparticle pairs near the event horizon. The outgoing positive energy mode is observed as Hawking radiation, whereas the ingoing negative energy mode is trapped inside the horizon. The striking feature in this case is that the SD black holes evolves with time and, therefore, there does not exist any time-like Killing vector to describe the energy of a particle. Rather, the Kodama vector [94] will be introduced to describe this energy while writing the Hamilton-Jacobi equation for the ingoing and the outgoing particles. Here, the entire analysis will be based on the null coordinates in which the ingoing and the outgoing modes are clearly separated out.

After obtaining the expression of temperature from the tunneling formalism, the next goal will be to find out the first law of thermodynamics for the SD black hole. It is now well known fact that the Einstein's equations, written on the horizon, lead to the first law [196]. This have been verified for some particular types of black hole spacetimes. Now the question is: Is the same true for the SD metric; i.e. do the Einstein's equations in this case have thermodynamic structure? The answer here is "Yes", which will be revealed clearly in the later part of the analysis. Interestingly, in this process it will be observed that the already obtained expressions of temperature, entropy and energy are well suited for getting such a relation. This provides a direct

[2]For recent progress and review on tunneling mechanism, see [187–195].

evidence of the correctness of the obtained results.

Thus, in this chapter, we shall obtain the thermodynamic relation of the time-dependent SD black hole, which has a conformal Killing horizon instead of a Killing one. The formalisms of the earlier chapter (like the Wald's formalism or the ADT formalism) is not well-defined for the spacetime having a non-Killing horizon. Therefore, the methods which we shall adopt here are different

4.3 SD metric: a brief review

The SD metric is a cosmological black hole solution of GR in presence two noninteracting perfect fluids: one is timelike and the other one is null-like as sources in the right hand side of Einstein's equations. Also, as it is mentioned earlier, the SD metric is asymptotically FLRW. It turns out that the background metric is conformal to the usual static Schwarzschild black hole metric by a time dependent conformal factor (for details see [106]). The explicit form of the metric is given by [106]:

$$ds^2 = a^2(\eta)\Big[-d\eta^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{2M}{r}(d\eta + dr)^2\Big]~. \tag{4.1}$$

The positive constant M is identified as the mass of the Schwarzschild black hole while the conformal factor is given as $a(\eta) = \eta^2$. Here η, r are the time and the radial coordinates respectively. θ and ϕ are the angular coordinates. The above form of the SD metric can be expressed in Schwarzschild like coordinates as

$$ds^2 = a^2(t,r)\Big[-F(r)dt^2 + \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]~, \tag{4.2}$$

when one uses the coordinate transformation $\eta = t + 2M\ln(r/2M - 1)$ in (4.1). In this case one has $F(r) = 1 - 2M/r$ and the conformal factor takes the following form [197]:

$$a(t,r) = \Big(t + 2M\ln\Big|\frac{r}{2M} - 1\Big|\Big)^2~. \tag{4.3}$$

It has been shown earlier [155] that the SD metric has a conformal Killing horizon at $r = 2M$ where (4.3) diverges.

The energy-momentum tensor for the source term in Einstein's equations can taken here of the form: $T^{ab} = \mu u^a u^b + \tau k^a k^b$. The first term represents the timelike dust with energy density μ and zero pressure while the last term corresponds to the null source. It has been shown in [106] that the energy density of the dust; i.e. μ is positive when we have $\eta < r(r+2M)/2M$. Moreover, in this region the outside observer will see that both fluids flow radially into the black hole. On the other hand, for late times; i.e. for $\eta > r(r+2M)/2M$ the sources become unphysical and the dust becomes superluminal. Another issue of ill-behaviour of the sources can be pointed out here in this context. In Ref. [106], it has been shown that the SD black hole is sourced by the dust and the null dust. But such an interpretation may not be correct as these fluids do not satisfy the conservation laws separately. Whatever the case may be, until another closest realistic solution, got rid of these limitations, is brought to light one can use SD black hole as a model to explore the realistic cosmological black holes. In that case it is not very important whether the matter is ill-behaving in the theory of GR [98]. Moreover, despite those limitations one cannot deny that the spacetime metric of SD black hole is one of the closest realistic solution of a cosmological black hole till date and fits nicely to the theory of expanding universe. Therefore, it is still very interesting to study in all possible ways to find different aspects. The unphysical features may be due to the fact that the solution is far from the realistic one. One can expect if one finds an exact solution that will be free of this problem. However, in the absence of the exact solutions, one needs to take a model which is close to it. With this spirit here the analysis on the SD metric will be performed.

4.4 Tunneling and Hawking temperature

To describe the Hawking radiation by tunneling mechanism, one always takes only the $(r-t)$-sector of the full metric. This is because, it has been argued that the tunneling occurs radially outward from the horizon. In addition to such hand waving argument, there is a much more concrete reason behind this. We know that the Hawking radiation is a near horizon effect. In this region, one already knew that the effective theory reduces to a two dimensional conformal theory and the main physics is driven only by the $(r-t)$-sector of the full metric. As the Hawking radiation is emitted from the horizon, one can take the two dimensional metric for the calculation of the radiation in the tunneling formalism.

To study the tunneling method, here we shall use the prescription known as the "dimensional reduction technique". Earlier, in the static background, it has been shown that the action (i.e. the Lagrangian density integrated over four volume of (1+3)-dimensional spacetime) of a scalar field near the BH horizon can be written as action of the collection of free scalar fields under the two-dimensional metric given by the $(r-t)$ sector of the whole spacetime [190, 198, 199]. Therefore, near the horizon, the effective spacetime is given only by the $(r-t)$ sector of the whole spacetime. Now, the present metric is time dependent and, hence, one has to investigate if such conclusion can also be drawn for the SD black hole. However, it has already been investigated that the same conclusion is also valid in SD spacetime as well [182]. It has been observed (in [182]) that, in this case, the near horizon effective metric is given by

$$ds^2 = a^2(t,r)\Big[-F(r)dt^2 + \tfrac{dr^2}{F(r)}\Big] \tag{4.4}$$

Let us discuss how dimensional reduction can be made in this case. If one writes the Klein-Gordon equation of a scalar field with the total SD metric (given by Eq. (4.2)), near the horizon one finds $F(r) \longrightarrow 0$ and $a(t,r) \longrightarrow \infty$. Therefore, when one takes the leading order contributing terms near the horizon, one can show that the scalar field can be expressed as a collection of independent spherical harmonics and each of the spherical harmonics obeys a new form of Klein-Gordon equation, which is given by the reduced SD metric (given by Eq. (4.4)) . Since, tunneling is a near horizon phenomenon, we can use (4.4) as the metric near the horizon. For details, see [182]. Now keeping in mind for future use we want to express the above in null coordinates which are defined as $u = t - r_*$ and $v = t + r_*$ where the tortoise coordinate r_* is given by $dr_* = dr/F$. In these coordinates (4.4) turns out to be

$$ds^2 = -\frac{a^2(t,r)F(r)}{2}\Big(dudv + dvdu\Big)\ . \tag{4.5}$$

One of the importances of null coordinates is the ingoing modes and the outgoing modes of the radiation are nicely separated out; which, as we shall observe later on, will be mostly needed in the analysis.

Now for a general background, the Klein-Gordon equation for massless particle is given by $\nabla_a\nabla^a\phi = (1/\sqrt{-g})\partial_a(\sqrt{-g}g^{ab}\partial_b\phi) = 0$. Under the background (4.5), this reduces to

$$\partial_u\partial_v\phi = 0\ . \tag{4.6}$$

Hence, the general solution of the equation takes the form $\phi(u,v) = \phi_R(u) + \phi_L(v)$. The total solution in null coordinates (u,v) is separated out as two different functions, each one is a function of single null coordinate. The subscripts R and L stand for the right and left moving respectively. Later, by calculating the momentum of the corresponding mode, it will be explicitly shown that those separated function corresponds to the outgoing (right) and the ingoing (left) modes of the Hawking radiation at the event horizon. Here lies the importance of null coordinates in the description of Hawking radiation. As the total wave function is now separated out as the functions of single variable, they should satisfy the relations:

$$\nabla_v \phi_R(u) = 0, \quad \nabla_u \phi_R(u) \neq 0, \quad \nabla_u \phi_L(v) = 0, \quad \nabla_v \phi_L(v) \neq 0 \, . \tag{4.7}$$

A general way to describe the above relations simultaneously is the equation

$$\nabla_a \phi = \pm \sqrt{-g} \epsilon_{ab} \nabla^b \phi \tag{4.8}$$

which is known as the chirality condition. For more details and usefulness of this condition, see [200]. Here the positive sign is assigned for left going mode ($\phi_L(v)$) and the negative sign is assigned for the right moving mode ($\phi_R(u)$). ϵ_{ab} is the numerical antisymmetric tensor with $\epsilon_{uv} = +1$. In (t, r_*) coordinates, under the background (4.5) the relation (4.8) reduces to

$$\partial_t \phi = \mp \partial_{r_*} \phi \tag{4.9}$$

where $\epsilon_{tr} = -1$ has been used. This is going to be the most important relation throughout our calculations. Here the negative sign is for the left moving mode while the the positive sign corresponds to the right moving mode.

Next, we need to find the Hamilton-Jacobi equation for these modes. For that one has to first identify the conserved quantity which represents the energy of the particle. For static background, there is a timelike Killing vector χ^a and the corresponding conserved energy is defined as $E = -\chi^a P_a$ where P_a is the four-momentum. Since present spacetime is time dependent we can not use this definition. For evolving case the energy of a particle can be defined in terms of the Kodama vector [94]:

$$E = -K^a P_a \tag{4.10}$$

where K^a is the Kodama vector which is defined as [94, 189]

$$K^i(x) = \frac{\epsilon^{ij}}{\sqrt{-g}} \partial_j R \, . \tag{4.11}$$

In the above ϵ^{ij} is the numerical skew tensor with $\epsilon^{tr} = 1$ and here $R = ar$. The same has been used earlier in [201] to study the Hawking radiation from the apparent horizon of FRW universe. For the metric (4.5) the components of the Kodama vector, in (t, r_*) coordinates are given by

$$K^t = \frac{1}{a^2 F}\partial_{r_*} R; \quad K^{r_*} = -\frac{1}{a^2 F}\partial_t R \ . \tag{4.12}$$

It is now well known to us that Hawking radiation is the semi-classical result of quantum field theory in curved space-time. So, we can use the semi-classical WKB ansatz of the wave function i.e.

$$\phi = e^{\frac{i\mathcal{S}}{\hbar}} \ , \tag{4.13}$$

where $\mathcal{S}$ is the action. Here, we have intentionally dropped out the normalisation factor of the wave function, which would not play any significant role in the calculation. Therefore, the momentum eigenvalue, in terms of the action, turns out to be $\hat{P}_a\phi = -i\hbar(\partial\phi/\partial x^a) = P_a\phi = (\partial_a\mathcal{S})\phi$ and hence

$$P_a = \partial_a \mathcal{S} \ . \tag{4.14}$$

Earlier, we have stressed on the fact that the Hawking radiation is a near horizon event. Due to pair production, the two modes of radiation are generated. Particle with positive momentum gets out of the surface, whereas, the particle with negative momentum is trapped by the surface. Thus, the outgoing mode and the ingoing mode can be distinguished. This argument we shall use later to show that our earlier sign convention is compatible with the identification of outgoing and ingoing modes.

Now, using the explicit expressions (4.12) for the components of the Kodama vector in the definition for energy (4.10), we obtain

$$E = -\frac{1}{a^2 F}(\partial_{r_*} R)(\partial_t \mathcal{S}) + \frac{1}{a^2 F}(\partial_t R)(\partial_{r_*}\mathcal{S}) \ . \tag{4.15}$$

On the other hand by the use of the ansatz (4.13), the chirality condition (4.9) reduces to the following form:

$$\partial_t \mathcal{S} = \mp \partial_{r^*}\mathcal{S} \ . \tag{4.16}$$

For the left mode we have $\partial_t\mathcal{S} = -\partial_{r_*}\mathcal{S}$. Substituting this in (4.15) and solving for $\partial_{r^*}\mathcal{S}$ we obtain

$$\partial_{r_*}\mathcal{S} = \frac{E a^2 F}{\partial_{r_*} R + \partial_t R} \ . \tag{4.17}$$

Similarly, that for the right moving mode turns out to be

$$\partial_{r_*}\mathcal{S} = \frac{Ea^2F}{-\partial_{r_*}R + \partial_t R} \ . \tag{4.18}$$

Combining them and rewriting in (t, r) coordinates we find

$$\partial_r\mathcal{S} = \frac{Ea^2}{\pm F\partial_r R + \partial_t R} \ . \tag{4.19}$$

Here the positive sign is assigned for the left mode while the negative sign is for the right mode. In this case we have $R = ar$ where the expression for a is given by (4.3). With this, the above reduces to the following form:

$$\partial_r\mathcal{S} = \frac{E}{G(t, r)} \tag{4.20}$$

where

$$G(t, r) = \pm\frac{F}{a} \pm \frac{4M}{a^{\frac{3}{2}}} + \frac{2r}{a^{\frac{3}{2}}} \ . \tag{4.21}$$

Since Hawking radiation is due to the near horizon particle production event, we expand $G(t, r)$ about the horizon $r = r_H$:

$$G(t, r) = G(t, r_H) + (r - r_H)G'(t, r_H) + \dots \ . \tag{4.22}$$

In this case $G(t, r_H)$ is calculated from (4.21) whereas $G'(t, r_H)$ is evaluated from

$$G'(t, r) = \pm\frac{F'}{a} + \frac{2}{a^{\frac{3}{2}}} - (\pm)\frac{Fa'}{a^2} - \frac{\frac{3}{2}(2r \pm 4M)a'}{a^{\frac{5}{2}}} \ . \tag{4.23}$$

Note that at the horizon $a(t, r)$ diverges while $F(r_H) = 0$. So the leading term in the expansion (4.22) is due to the first term of (4.23). Hence, neglecting all the other terms and keeping only the leading term we obtain

$$G(t, r) \simeq \pm\frac{F'}{a}|_{r_H}(r - r_H) = \pm\frac{2\kappa}{a_H}(r - r_H) \ , \tag{4.24}$$

where $\kappa = F'(r_H)/2$ is the surface gravity of the usual Schwarzschild black hole and a_H is the value of the conformal factor at the horizon. Substitution of this in (4.20) yields

$$\mathcal{S} = \pm\frac{Ea_H}{2\kappa}\int\frac{1}{r - r_H}dr \tag{4.25}$$

with negative (positive) sign implying the right (left) mode.

This integration is well defined and real only when the initial and the final point, between which the above integration is performed, are on the same side of the horizon. But, for this case, we want to calculate the transition probability of an outgoing(or ingoing) particle, which is initially at $r < r_H$(or $r > r_H$) and finally reaches $r > r_H$(or $r < r_H$). Therefore we need an extra prescription. But before that, note that the positive (for ingoing mode) or the negative (for the ingoing mode) signs were determined by the chirality condition and after obtaining equation (4.25) we see that it was obvious for physical description. For outgoing particle (initially at $r < r_H$) $\partial_r S > 0$, which is only possible when we take the negative sign in the equation (4.25). Same argument is applicable for ingoing particle as well.

Now, the integration is performed by complex integration method. As in this case the path of integration actually passes through a singularity $r = r_H$ of the integrand, we must choose a path which avoids the singularity. The evaluation of this integration can be followed from [202] (See page 60). For outgoing particle we have

$$\int_{r_1}^{r_2} \frac{1}{r - r_H} dr = P\Big(\int_{r_1}^{r_2} \frac{dr}{r - r_H}\Big) - i\pi \ , \tag{4.26}$$

while for ingoing one

$$\int_{r_2}^{r_1} \frac{1}{r - r_H} dr = P\Big(\int_{r_2}^{r_1} \frac{dr}{r - r_H}\Big) + i\pi \ . \tag{4.27}$$

Here we have chosen the point r_1 inside the horizon while r_2 is at outside of it. The first terms in the above are the principal values of the integrals and these are real. For outgoing (ingoing) mode the contour has been chosen on the upper (lower) half plane. So, for the outgoing mode the action is given by

$$\mathcal{S}_{out} = -\frac{Ea_H}{2\kappa} \int_{r_1}^{r_2} \frac{dr}{r - r_H} = \frac{i\pi E a_H}{2\kappa} + \text{real part} \ . \tag{4.28}$$

Similarly, for ingoing mode we find

$$\mathcal{S}_{in} = \frac{Ea_H}{2\kappa} \int_{r_1}^{r_2} \frac{dr}{r - r_H} = -\frac{Ea_H}{2\kappa} \int_{r_2}^{r_1} \frac{dr}{r - r_H} = -\frac{i\pi E a_H}{2\kappa} + \text{real part} \ . \tag{4.29}$$

Substituting them in (4.13) the probabilities for emission and absorption are calculated as

$$P_{out} = |\phi_{out}|^2 = |e^{\frac{i\mathcal{S}_{out}}{\hbar}}|^2 = e^{-\frac{\pi E a_H}{\kappa\hbar}} \tag{4.30}$$

and

$$P_{in} = |\phi_{in}|^2 = e^{\frac{\pi E a_H}{\kappa \hbar}} \ . \tag{4.31}$$

Therefore the tunneling probability turns out to be

$$\Gamma = \frac{P_{out}}{P_{in}} = e^{-\frac{2\pi E a_H}{\kappa \hbar}} \ . \tag{4.32}$$

It should be noticed that we have approximated the value of $G(r,t)$ near the event horizon where only the leading order term of $G(r,t)$ has been kept. With this approximation the tunneling rate is similar to the Boltzmann factor. Had the value of $G(r,t)$ not been curtailed upto its leading order value, the calculations would end up with the extra terms, contribution of which is substantially little. So, comparing to the static case calculation, one can say that in the static case one gets exactly the Boltzmann factor for the calculation of tunneling probability, while, in non-static case only the near horizon approximation leads to the Boltzmann factor. The similar has also been done earlier (For example, see [189]). Comparing this with the Boltzmann factor $e^{-\beta E}$ where β is the inverse temperature, the temperature of the horizon is identified as

$$T = \frac{\hbar \kappa}{2\pi a_H} \ . \tag{4.33}$$

This expression is identical to what was obtained by anomaly approach [181, 182]. Also this agrees with scaling argument provided in [180].

Let us now discuss an issue in the context of the tunneling approach. In literatures one can find some ambiguities and anomalies regarding the study of Hawking radiation by tunneling methods for the stationary black holes. One of the mostly discussed among these is the coordinate dependence of the tunneling rate. If one use Schwarzschild coordinate, as we have done, the calculation predicts the value of temperature which is twice the Hawking temperature when one defines the tunneling rate as the amplitude square of the outgoing wave. To solve this anomaly, several theories were erected [183, 203–208]. One of them is mentioned in [183] where the tunneling rate is defined as the ratio of probabilities of outgoing and ingoing modes. This prescription we have adopted here in our calculations. The other one to solve this factor two problem is introducing the isotropic coordinate and the proper distance along the radial direction which is mentioned in the references [204] and [205]. In addition to them, it has been shown that if one takes into account the temporal part, then this ambiguity does not arise [188, 203]. Moreover, the coordinate dependence of tunelling rate can also be explained in an elegant way using the concept of

R and T regions by Novikov [206]. The discussion of R and T regions for spherically symmetric spacetime can be seen in [185]. Another fruitful approach we want to mention here is the following. For static black hole case, one can apply the Rindler coordinates where this problem of factor two does not arise [208]. The idea is the following. The Hawking radiation is due to the pair production near the horizon and in this region the natural coordinates can be taken as the Rindler ones. Therefore, one might say that the Rindler coordinates are the more physical coordinates while studying the Hawking radiation by the tunneling formalism of a static black hole. It should also be mentioned that the tunneling rate in Rindler coordinates is identical to that for the Schwinger mechanism [207, 208]. A comparative study of these two methods has been discussed in various literatures (for example Refs. [183, 207, 208]. A crude way of saying is: in the Schwinger mechanism, the virtual particle-anti particle pairs are separated by an electric field while in Hawking radiation these are separated by the event horizon or by the geometry of the black holes. This shows that if one discusses the tunneling method in the same footing like the Schwinger mechanism, there will not be any discrepancy in the value of the horizon temperature. But, there are some limitations of using these coordinates. Rindler coordinates can only be applied for the non-extremal black holes. Also, these coordinates are the static ones and are not defined properly for the dynamic cases. Since our metric is time dependent, it is not clear how to define the Rindler coordinates. For the above mentioned reasons, we have taken the Schwarzschid coordinates and applied the standard prescription of [183] to obtain the correct expression of temperature as one can see from (4.32).

It is known that the temperature is defined only for the thermodynamic system in equilibrium or near about equilibrium. If the system varies rapidly with time and is out of equilibrium, one cannot define temperature of the system. However, if the system changes very slowly and smoothly with time, one can use "adiabatic approximation" to claim that the system is near equilibrium and one can define temperature. Thus, while obtaining temperature for time dependent SD black hole, we have accounted adiabatic approximation. In this case, the expression of the temperature has been obtained by comparing the tunneling probability with the Boltzmann factor. However, the comparison is valid only in equilibrium thermodynamics by considering adiabatic approximation. It has been possible to use such approximation as the SD black hole is related to the known time-independent solution by a time-dependent conformal factor. This conformal factor changes very slowly with respect to time near the horizon $r \longrightarrow 2M$ (from Eq. (4.3) one finds that the conformal factor $a(t,r)$ diverges near the horizon, so the change of $a(t,r)$ near the horizon (with respect to t) can be considered very small).

Before concluding this section, let us make the following comments. Note that here temperature is time dependent as a_H depends on time. In [106], the expression was quite different and is given by the Schwarzschild temperature. The reason is as follows. The required conformal Killing vector ξ^a and the conformal factor Ω should satisfy $\xi^a\xi_a \to -1$ and $\Omega \to 1$, respectively at the null infinity. But since in the present case $\xi_a\xi^a = -a^2$ and $\Omega = a$, (see [126, 155] for details on finding the conformal Killing vector) both do not satisfy the above requirements. Therefore, calculation of the temperature based on this formalism does not give correct answer. Another point one should mention that the expression of temperature has been obtained by studying only the spherically symmetric mode. It must be remembered that Hawking effect is a near horizon phenomenon. As mentioned earlier, in this region the effective theory reduces to a two dimensional conformal theory for static as well as non-static background as SD. It has been shown in [182] that if one starts with the massive Klein-Gordon equation in four dimension, it effectively reduces to an equation which is governed by the $(r-t)$ sector of the full metric in the near horizon limit. All angular and mass terms do not contribute in this region. Since Hawking radiation is a near horizon phenomenon, we have just used this information in our calculation to derive the temperature. The idea of the dimensional reduction is like this. If one expands the Klein-Gordon equation under a spherically symmetric metric and transforms it in "tortoise" coordinate r_* and makes the partial wave decomposition, one finds that the effective radial potential, which contains the angular part, and the mass term appear with the metric coefficient. Therefore in the near horizon limit, it dies out and the full equation reduces to similar to $(1+1)$ dimensional one. More specifically, the near horizon physics can be described by an infinite collection of two dimensional fields each propagating in spacetime with the metric, given by the $(r-t)$ section of the full metric. For more discussions on how the effective theory becomes two dimensional conformal theory near the event horizon, we suggest the papers in [209], which only deal with this issue. In addition to this, it may be noted that if one calculates the tunneling rate with the full metric then also the same reduces to similar to two dimensional result in the near horizon limit. Such a discussion has already been demonstrated for static case (see the analysis around Eq. (2.31) in [183]). Here also the angular and mass terms, like the dimensional reduction technique, appear with the metric coefficient which vanishes at the horizon. So as far as the temperature is concerned, one can focus only on the spherically symmetric mode which is exactly solvable. This causes no information loss regarding temperature. If one is concerned about the radiation spectrum or the gray-body factor (the relative factor between the asymptotic radiation spectrum and the spectrum of black body radiation), then one has to take care of all the angular modes, not to lose the total information.

4.5 First law of thermodynamics from Einstein's equation

It has been observed that black holes behave like thermodynamic objects and satisfy the thermodynamic relations [15]. In the previous section we have found out the expression of temperature (T) of the SD background. Also, the expressions of entropy (S) and the energy (E) (the Misner-Sharp energy) of the black hole are already obtained by explicit calculations [182]. So, the expressions of the thermodynamic quantities (E, S, T) are now known to us for the SD background. Now, in this section we shall find out the first law of thermodynamics. It is well known that the Einstein's equations, projected on the horizon, leads to first law of black hole mechanics [196]. Which implies that the near horizon field equations of gravity behave like local thermodynamic equilibrium. In this section we want to follow the same strategy in order to find whether the same conclusion can be drawn for a more realistic and time dependent SD black hole. The steps are identical to the earlier work by Hayward [210]. The justification of presenting this discussion lies in the fact that it will give an explicit verification of the correctness of the derived thermodynamic quantities. Additionally, a new reader will find the discussion as self sufficient.

The SD metric satisfies the Einstein's equation $G_{ab} = kT_{ab}$, where $G_{ab} = R_{ab} - (1/2)g_{ab}R$ is the Einstein's tensor and T_{ab} is the energy-momentum tensor corresponding to the matter source. Here k is given by $k = 8\pi$. The expression for T_{ab} is given by [106]:

$$T_{ab} = \frac{1}{\kappa\Omega}(2g_{ab}\nabla^2\Omega - 2\nabla_a\nabla_b\Omega - 3\Omega^{-1}g_{ab}g^{mn}\nabla_m\Omega\nabla_n\Omega) \ , \qquad (4.34)$$

where in our notations, $\Omega = a(t, r)$. Using the above relation, the explicit form of the required components of the energy-momentum tensor for the present black hole

(4.2) are given by

$$kT^r_r = \frac{-2\ddot{a}}{a^3F} + \frac{5(\dot{a})^2}{a^4F} + \frac{a'F'}{a^3} - \frac{F(a')^2}{a^4}\ , \tag{4.35}$$

and

$$kT^t_t = \frac{2Fa''}{a^3} - \frac{5F(a')^2}{a^4} + \frac{a'F'}{a^3} - \frac{(\dot{a})^2}{a^4F}\ . \tag{4.36}$$

Next, the explicit form of the relevant components of Einstein's tensor under the same background turn out to be

$$G^r_r = \frac{-1+F+rF'}{r^2a^2} + \frac{3F(a')^2}{a^4} + \frac{(\dot{a})^2}{a^4F} + \frac{4Fa'}{a^3r} + \frac{F'a'}{a^3} - \frac{2\ddot{a}}{a^3F} \tag{4.37}$$

and

$$G^t_t = \frac{-1+F+rF'}{r^2a^2} + \frac{F'a'}{a^3} + \frac{4Fa'}{a^3r} + \frac{2Fa''}{a^3} - \frac{F(a')^2}{a^4} - \frac{3(\dot{a})^2}{a^4F}\ . \tag{4.38}$$

Here, we used the notations as $a' = \partial_r a$, $a'' = \partial_r^2 a$, $\dot{a} = \partial_t a$ and $\ddot{a} = \partial_t^2 a$. Using the Einstein's equation $G^r_r = kT^r_r$ one gets

$$\frac{-1}{r^2a^2} + \frac{F}{r^2a^2} + \frac{F'}{ra^2} + \frac{4Fa'}{a^3r} + \frac{4F(a')^2}{a^4} - \frac{4(\dot{a})^2}{a^4F} = 0 \tag{4.39}$$

The same expression is also obtained when one uses another Einstein's equation $G^t_t = kT^t_t$. Now, we want to evaluate the above expression at the horizon $r = 2M$. Therefore, one should examine each term and find whether the term contributes at the event horizon $r = 2M$. Since the second term is containing F at the numerator, it must vanish at the horizon. Denominators of the first and the third terms diverge as a^2 in this limit. Since $a' = (4M\sqrt{a})/(rF)$, the denominator of the fourth term diverges as $a^{5/2}$. Use of $\dot{a} = 2\sqrt{a}$ leads to the fact that the last two terms vanish as a^3F at the horizon; i.e. they are diverging. But it is interesting to note that their divergence is in the same order and, since they are opposite in sign, the collective contribution from these two terms is zero in the near horizon limit. Nevertheless, a graph has been plotted below to convince people that those two terms really do not contribute as a whole at the event horizon.

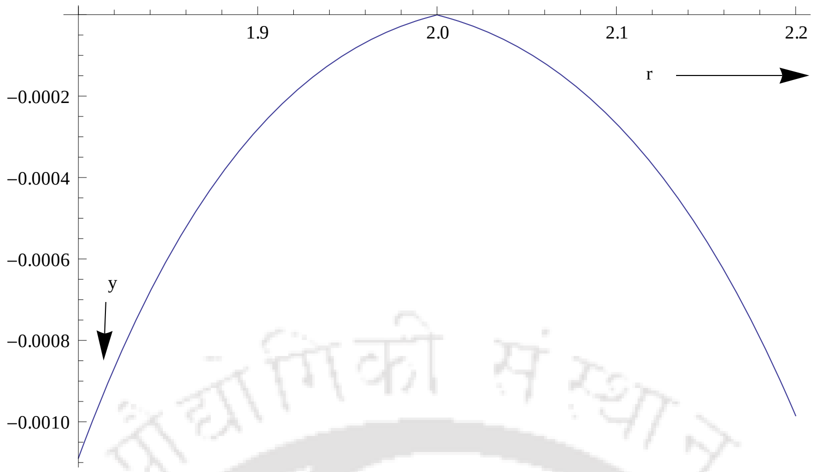

Figure 4.1: $y = \frac{4F(a')^2}{a^4} - \frac{4(\dot{a})^2}{a^4 F}$ *Vs r plot for* $t = 1 = M$.

Therefore, keeping only the dominating terms (the first and the third term which varies with a^{-2} at the horizon) in (4.39) in the near horizon limit, we obtain

$$-1 + r_H F'(r_H) = 0 \ . \tag{4.40}$$

Next multiplying each term with $d(a_H r_H)/2$ and then inserting π factor in both numerator and denominator in the second term, one obtains

$$-\frac{d(a_H r_H)}{2} + \frac{F'(r_H)}{4\pi a_H} d[\pi (r_H a_H)^2] = 0 \ . \tag{4.41}$$

Using the expressions for energy (Misner-Sharp energy) and entropy $E = (r_H a_H)/2$ and $S = \pi r_H^2 a_H^2$, respectively, derived earlier in [182] and the expression of temperature, given by (4.33) we can rewrite the above as

$$dE = TdS \ , \tag{4.42}$$

which is the first law of black hole mechanics for the SD metric.

In this section we found that the Einstein's equations, for the SD spacetime, have a thermodynamic structure like the usual cases. The radial-radial component as well as the time-time component, both leads to the same equation at the horizon which ultimately reduces to the first law of black hole mechanics. In doing this it has been observed that the derived temperature (the time dependent one, derived in this chapter), and the entropy and energy (found earlier in literature) led to the correct form of thermodynamic law. This once again is the signature of the correctness of these thermodynamic entities.

Let us now mention that the present approach of finding the first law of thermodynamics is completely equivalent to the earlier one by Hayward [210]. In the mentioned paper of Hayward, the unified form of the first law of thermodynamics has been given for a spherically symmetric background. In doing so, first he has defined two quantities: energy density and energy flux. Using them and taking into account the Misner-Sharp energy, the required result has been obtained. The main idea is as follows. The gradient of the Misner-Sharp energy, manipulated by using the Einstein's equation, when projected on the horizon leads to the unified first law in the differential form. So it is obvious that the present analysis is in same line of Hayward. But we present this not only for the completeness of the discussion, but also for a new reader who can find the discussions in a self-contained manner. Moreover, as we have mentioned earlier, the expression of temperature for the SD black hole is a long lasting problem in literature, causing confusion while giving thermodynamic description of the black hole. Introducing the tunneling formalism we have found out the expression of temperature, which turns out to be the time dependent. Now, we had to verify whether the expression was correct. To see it explicitly we presented the above analysis. Here, we had taken the Einstein's equation which we have projected near the event horizon and taking the leading order terms we have shown that the first law of thermodynamics can be achieved for the SD ones. In that process we have shown that the obtained value of the temperature fits nicely to the expression of entropy and the Misner-Sharp energy of the literature to give the first law. The analysis shows that the Einstein's equation manifests itself as a thermodynamic identity when it is projected to the horizon and this work revels the thermodynamic correspondence of Einstein's equation in a more obvious manner. Moreover, we have shown the thermodynamic structure of the SD black hole in a more explicit way with proper thermodynamic variables (energy, entropy and temperature) defined for the SD black hole. Not only this, the obtained thermodynamic expression (which is indeed the first law) has been able to solve the long lasting confusion about the exact expression of temperature and we can claim that our obtained expression of temperature is the correct one which is consistent to the other thermodynamic variables for the SD black hole given in other literature. Moreover, our analysis shows that there is a deeper connection between spacetime geometry and the thermodynamics of a black hole (or precisely thermodynamics of the horizon). In a broader picture this can have the following implications. Originally, the concept of entropy and temperature of the different kinds of horizons in general relativity were well developed, and physicists remarked that the near horizon behaviour of the field equations of gravity is like local thermodynamic equilibrium in thermodynamics (For more details, see [196]). But, the people contemplated that the spacetime geometry of a black hole is more fundamental compared to the thermodynamics and that connection of geometry with the thermodynamics might not be obtained for the more realistic cases where the spacetime is more complicated to explore and time dependent. Whereas, we have shown that in a more realistic model one can get the same connection of geometry and thermodynamics via gravitational equation. Therefore, one can conclude that the thermodynamic description of a black hole is no more less fundamental comparing to the geometric description of the black hole and the thought, that the near horizon behaviour of the field equations of gravity might appear like local thermodynamic equilibrium, is still applicable for a time dependent black hole as well[3]. Moreover, the present one implies that the gravity can be thought as a long wavelength, emergent phenomenon, and gravitational description, therefore, resembles to the equations of thermodynamics .

4.6 Summary and Discussions

In this chapter, we have tried to shed light on the thermodynamic aspects of the time dependent Sultana-Dayer(SD) black hole and thereby obtaining the informations to describe the SD black hole thermodynamically. This metric, conformally connected to the usual Schwarzschild one with a time dependent conformal factor, is asymptotically FLRW which is widely used to describe homogeneous, isotropic expanding universe. We have provided a brief review on the SD metric before starting our discussions in details. In the earlier part of our discussion in this chapter, we have bequeathed ourselves to obtain the exact expression of the temperature. We have already mentioned about the incongruity that one observes in literature in the expression of temperature. Therefore, we had to get rid of this conflicting situation and verify which expression is the correct one for this black hole, the expression in which the temperature depends on time or the expression in which it is independent of time. For that, we have studied the Hawking radiation by tunneling formalism and after overcoming all the hurdles in calculations we have proved that the expression of temperature of the SD black hole really depends on the time. Moreover, the expression of temperature that we obtained from the tunneling formalism resembles to the same, obtained by gravitational anomaly approach [182]. Also, this expres-

[3] A time dependent charged solution in the expanding universe has been given in [211].

sion of temperature accords with the scaling argument [180] as we have mentioned earlier.

In the later part of this chapter, we were prone to realise whether one can procure any information from the Einstein's equation to describe the SD black hole thermodynamically. Our inspiration was the fact that the Einstein's equation leads to the first law of thermodynamics when it is projected to the horizon, though verified only for some particular black holes. Here, in this discussion we have proved that similar expression is obtained for the time dependent SD black hole as well. We have also pointed out the fact that the already obtained expressions of entropy and energy nicely fits with our obtained value of temperature to give the first law of thermodynamics from the Einstein's equation.

To summarize, we can draw two major conclusions from the whole discussion of this chapter. Firstly, the expression of temperature depends on time and it is consistent to all other obtained expressions of the thermodynamic quantities such as entropy, energy etc. Secondly, thermodynamic description is embedded in the Einstein's equation for the time dependent SD black hole as well, implying that the thermodynamic description and the geometrical description of a black hole should be treated in equal footing.

Chapter 5

Conserved current approach beyond Killing horizon

[1]The conserved charges [29–33, 89, 166] have played a crucial role in the theory of black hole since the time when the black holes are identified as thermodynamic objects with having proper thermodynamic descriptions [14–16]. In the theory of black hole mechanics, the conserved charges often provides the physical thermodynamic quantities, such as entropy, energy, angular momentum *etc.* There are two major approaches to develop the thermodynamic description of a black hole from the conserved charges. Among them, the conserved Noether current due to the diffeomorphism has enjoyed the central attention over the years. Following the Wald's [29] prescription, one can successfully obtain the first law of black hole thermodynamics. In this approach, the entropy is defined by the conserved Noether charge on the black hole horizon, whereas the mass and the angular momentum are defined on the asymptotic infinity by the same conserved current along with a correction factor.

There is another elegant approach developed independently in General theory of Relativity (GR) for the formulation of the thermodynamic description from the conserved charges, which is popularly known as the Abbott-Deser-Tekin (abbreviated as ADT) approach [30–33, 165]. This formulation works well not only for asymptotically flat black hole spacetimes in GR, but also for the anti-deSitter (AdS) spacetimes in GR [30] as well as higher order gravity theories [31–33] (For the thermodynamic description using the ADT approach, also see the recent works [112, 163, 164, 168, 212–218]). In chapter 3, we have discussed about both the Noether and the ADT charges under the framework of the scalar-tensor gravity

[1]This chapter is based on the publication [113] .

and also have shown that the two approaches (i.e. the Noether and the ADT formalisms) are indeed equivalent. However, in that case, although the Noether current and charges are defined for arbitrary diffeomorphism, the ADT current has been defined only by assuming the presence of the Killing vector in the spacetime. In addition, it must be clear that although the Noether current is defined for any arbitrary diffeomorphism, while obtaining thermodynamic parameters from the conserved Noether charge, we again have assumed the presence of the Killing vector and the Wald/ ADT formalisms to obtain the thermodynamic quantities are valid only on the Killing horizon. Apart from this two approaches (Wald and ADT), in [219] another method has been described, which belongs to the Wald class, but with some different boundary conditions. Here, the asymptotic symmetries and conservation laws been investigated using covariant phase space method.

Before discussing the objective of the present chapter, we want to mention couple of important facts which have been observed in exploring the different features of the gravitational paradigm. This will give the motivation of the discussions in the present chapter. Soon after the observation of the thermodynamic nature of the black hole horizon, Unruh showed that an accelerated observer can find radiation in the Minkowski vacuum [19] and consequently this observer is able to associate temperature and entropy on the Rindler horizon (obtaining the entropy of the Rindler horizon using Virasoro algebra has shown recently in [152, 157]). Moreover, assigning the first law of thermodynamics on it leads to Einstein's equations of motion [104]. Later investigation reveals that such feature not only restricted to a Rindler kind of horizon, any generic null surface also incorporates thermodynamic structure (see [111, 220–223] for recent developments in this direction). Note that in the generic null case, the surface defining vector (here it is a null vector) may not be, in general, a Killing one.

All these works suggest that the thermodynamic description of a spacetime is not restricted to the Killing horizon. Instead, the thermodynamic structure is an extensive feature in the horizon thermodynamics, which is true irrespective of any specific horizon (including the Killing horizon and other non-Killing horizons such as apparent horizon, trapped horizon [224], etc. present in GR). Moreover, in the previous chapter (chapter 4), we have obtained the thermodynamic description for the SD black hole which has a time-dependent conformal Killing horizon. However, the existing approach of defining the ADT current fails to be befitting in the thermodynamics of the wide classes of non-Killing horizon. Remember, in literature, the conserved Noether current is already defined for Killing and non-Killing diffeomorphism vectors (see the project 8.1 of [144]). Therefore, the existing Noether approach can be applicable for both Killing and non-Killing horizon surfaces. On the other hand, in the existing formalism, the ADT current is only restricted to the Killing vectors which defines the horizon surface. Therefore, we must investigate whether we can go beyond the assumption of the Killing symmetry of the spacetime to obtain the conserved ADT (like) current, which can be used for the thermodynamic purposes of various non-Killing horizons in the literature. Remember, in the present chapter, we restrict ourselves only to obtain the ADT current for arbitrary diffeomorphisms. In order to obtain the thermodynamic description from the obtained conserved current it requires further investigations.

5.1 Objectives of the chapter

Our aim here is to find the conserved quantities of the gravitational theory for those diffeomorphism vectors, the vanishing of norm of which defines those wide classes of non-Killing horizons. We adopt an elegant method to formulate a quasi-local ADT-like conserved current in Lanczos-Lovelock gravity [225, 226] (often referred to as Lovelock gravity as well) for those horizon defining diffeomorphism vectors. The Lanczos-Lovelock (LL) model of gravity is an extension of Einstein's theory of gravity in higher dimension containing quadratic and higher order polynomials of the curvature tensor. In addition, like GR, the field equations contain only up to the second order derivative of the metric and the theory is free of unphysical ghosts. Moreover, a major motivation of studying the LL gravity comes from string theory. LL gravity resembles string theory inspired gravity models [227, 228]. Besides, the quadratic Gauss-Bonnet term is studied in the context of string theory with particular attention given due to its ghost free solution [228]. In summary, the LL gravity represents the higher order generalization of Einstein's gravity which is studied to examine how the effect of gravity gets modified at short distance in presence of higher order curvature terms in the action. As a result, the LL gravity is studied in several contexts of gravity including in the context of thermodynamics as well (see the review [229]).

The process which we follow here (to obtain the ADT-like current) is different from the original ADT method and will show that our current and anti-symmetric potential (we call the ADT-like potential) both reduce to the usual conserved ADT current and potential respectively, when we take the horizon defining diffeomorphism vector as the Killing one. In addition, we shall find the explicit relation between the Noether and present ADT-like anti-symmetric potentials. This relation will help to understand more about those potentials. The whole analysis is very general as the entire approach is made for LL gravity. Of course, one can easily obtain the corresponding results for GR or in Gauss-Bonnet gravity from our analysis by taking the proper limit. Moreover, the diffeomorphism vector can be chosen according to the type of the horizon surface.

Finally, we shall discuss two situations: one is the spacetime has Killing symmetry and the other one corresponds to conformal Killing symmetry of the spacetime. In that case one needs to choose the diffeomorphism vector as Killing vector (KV) and a conformal Killing vector (CKV), respectively. Interestingly, we show that both for Killing vector and conformal Killing vector, the conserved anti-symmetric potential is related to the Noether counter part in exactly same manner. However, we shall notice that the expression of the current is not the same for both cases.

In the following, we start our analysis. Firstly, we obtain the general expression of the conserved current and the potential for Lanczos-Lovelock gravity. Then we take two special cases of KVs and CKVs.

5.2 Conserved quantities for a horizon defining diffeomorphism vector

It is often proclaimed that any feasible theory of gravity must be diffeomorphism invariant. Now, since diffeomorphism is an active coordinate transformations, people often tend to refer the diffeomorphism as the "change in coordinates", which might be technically correct but, does not imply the complete scenario. The reason is: the operation diffeomorphism implies a set of two consecutive actions– a pushforward along with a pullback (see appendix B of [230] for a detailed discussion). Therefore, it is not an ordinary coordinate transformation. Now it may be noted that most of the theories in physics, which are based on Newtonian gravity or even based on special relativity, does not change its description due to the change in coordinates. In theses theories the inertial frame enjoys a special status. Whereas, GR or any other gravitational theory, described by the geometry of the spacetime, is built up in such a way that there is "no preferred set of coordinates". This implies the theory should be described in a covariant way under any general coordinate transformation, which is usually called as diffeomorphism.

As the gravity theory has such diffeomorphism symmetry, one usually finds the conserved current and the corresponding charge by the Noether prescription. It has been observed by Wald [29] that this conserved current is very useful to obtain the thermodynamic structure of gravity in presence of a Killing horizon. More precisely, the conserved Noether charge, calculated on the Killing horizon for the horizon defining Killing vector, is related to the black hole entropy. Whereas, this is related to the mass and the angular momentum of a black hole when it is evaluated at asymptotic infinity. All these state that the conserved quantities due to the diffeomorphism symmetry play a crucial role in understanding the thermodynamic description of gravity.

There is another approach developed simultaneously along with the Wald formalism. This is the ADT formalism which is developed for the presence of a Killing vector in the spacetime. This conserved current is known as the ADT current, which is mentioned in the following. Note that when the diffeomorphism vector ξ^a is a Killing one, the term $\mathcal{J}^a = (\delta E^{ab})\xi_b$ is a conserved quantity, where

$$E^{ab} = \mathcal{R}^{ab} - \frac{1}{2}g^{ab}L \ , \tag{5.1}$$

is a second rank tensor in LL gravity, which plays the analogous role to the Einstein tensor in GR. In the above Eq. (5.1), L is the Lagrangian of the LL gravity which is a general function of the Riemann tensor $R^a{}_{bcd}$ and metric tensor g^{ab} but not of the derivatives of those quantities. Also, $\mathcal{R}^{ab} = P^{acde}R^b{}_{cde}$ plays the analogous role to the Ricci tensor in GR, where $P^{abcd} = (\frac{\partial L}{\partial R_{abcd}})_{g_{ij}}$ which satisfies vanishing of covariant derivative; i.e. $\nabla_a P^{bcde} = 0$. For more information about E_{ab}, consult with the review [229]. In the above expression of $\mathcal{J}^a$, δE^{ab} represents the linearized tensor, *i.e.* the change of E^{ab} for $g_{ab} \to g_{ab} + h_{ab}$ up to first order in h_{ab}. The conservation of the quantity $\mathcal{J}^a$ (i.e., $\nabla_a \mathcal{J}^a = 0$) follows from the fact that E^{ab} satisfies the general Bianchi identity *i.e.* $\nabla_a E^{ab} = 0$ and $\nabla_a \xi_b$ is an anti-symmetric tensor as ξ^a satisfies the Killing equation. The conserved quantity now popularly known as the ADT current (for more details about ADT quantities, refer to [30–33, 165]).

Both of these currents (the Noether and the ADT) can be utilised for obtaining the thermodynamics of a Killing horizon and, they provide equivalent thermodynamic description (see [112] for an example). But in real scenario, the spacetime is not stationary and consequently, the Killing horizon ceases to exist. In this case one can have other kind of horizons, like apparent horizon, trapped horizon, conformal Killing horizon, generic null surfaces, *etc.* None of the vectors which defines the surface is a time-like Killing vector. Therefore the existing formalism for Noether and

ADT does not work in those cases.

As we have mentioned, if ξ^a is not a Killing vector, then $(\delta E^{ab})\xi_b$ is not a conserved quantity anymore. Then above way of defining quantity does not fulfill our aim. Hence we need to adopt a different approach to achieve the goal. Our idea is the following.

First, we identify the Noether current due to the diffeomorphism. Then, we take the variation of the Noether current due to the variation of the metric tensor. Afterwards, we identify the total derivative anti-symmetric part from the expression and place it along with the variation of the Noether current. Finally, the rest of the terms are identified as the conserved ADT-like current, which can be written as a total derivative of a two-rank anti-symmetric tensor, consisting of the variation of the Noether potential and other identified anti-symmetric terms. This way of approaching is very convenient, as the Noether current is known for non-Killing cases as well (see the project 8.1 of [144]). Thereafter, we shall see that our obtained ADT-like current reduces to the standard form of conserved ADT current when ξ^a is considered as a Killing vector.

We now start our analysis by considering the general LL Lagrangian $L(R^a{}_{bcd}, g^{ab})$. An arbitrary variation of the Lagrangian leads to the result

$$\delta(\sqrt{-g}\ L) = \sqrt{-g}\ E_{ab}\delta g^{ab} + \sqrt{-g}\ \nabla_a \delta v^a\ . \tag{5.2}$$

In the above we denote, $\delta v^a = 2P^{ibad}(\nabla_b \delta g_{id})$. Now, as the Lagrangian L is invariant under the diffeomorphism $x^a \to x^a + \xi^a$, we can obtain the conserved Noether current due to the diffeomorphism. Now, under the diffeomorphism, the arbitrary variation δ becomes the Lie variation and from (5.2), one can obtain $\nabla_a J^a = 0$ where,

$$J^a = 2E^{ab}\xi_b + L\xi^a - \pounds_\xi v^a\ , \tag{5.3}$$

is the conserved Noether current. Here $\pounds_\xi$ denotes the Lie derivative of a tensor along the vector ξ^a. In the above the crucial step is the first term on the right hand side of (5.2) can be expressed as total derivative term by the generalised Bianchi identity $\nabla_a E^{ab} = 0$. Note that in acheving (5.3), one does not need to use equation of motion at the operational level (for details, see Project 8.1 in page 394 of [144]). For some details step see the Appendix B of [155]. Now, this current (5.3) can be further expressed as $J^a = \nabla_b J^{ab}$ again without using equation of motion, where

$$J^{ab} = 2P^{abcd}\nabla_c \xi_d\ , \tag{5.4}$$

is the anti-symmetric Noether potential.

Let us now take an arbitrary variation of the above conserved Noether current (5.3) due to the arbitrary variation of the metric tensor $g_{ab} \to g_{ab}+h_{ab}$, which leaves the vector ξ^a invariant (*i.e.* , $\delta\xi^a = 0$, but $\delta\xi_a \neq 0$ in general due to the variation in the metric tensor). We then obtain

$$\delta(\sqrt{-g}\ J^a) = 2\delta(\sqrt{-g}E^{ab}\xi_b) - \sqrt{-g}\xi^a E^{ij}h_{ij} + \sqrt{-g}\ (\nabla_i\delta v^i)\xi^a - \delta(\sqrt{-g}\ \pounds_\xi v^a)\ . \tag{5.5}$$

To proceed further, consider the following identity

$$\pounds_\xi[\sqrt{-g}\delta v^a] = \sqrt{-g}\xi^a\nabla_i\delta v^i - 2\sqrt{-g}\nabla_b(\xi^{[a}\delta v^{b]})\ , \tag{5.6}$$

where the notation $A^{[a}B^{b]} = (1/2)(A^aB^b - A^bB^a)$ has been adopted. Using identity (5.6) in (5.5), it is straightforward to reach

$$\delta(\sqrt{-g}\ J^a) - 2\sqrt{-g}\nabla_b[\xi^{[a}\delta v^{b]}] = 2\delta(\sqrt{-g}E^{ab}\xi_b) - \sqrt{-g}\xi^a E^{ij}h_{ij} - \omega^a\ , \tag{5.7}$$

where,

$$\omega^a = \delta[\sqrt{-g}\pounds_\xi v^a] - \pounds_\xi[\sqrt{-g}\delta v^a]\ . \tag{5.8}$$

Next our aim is to collect the terms which can be expressed as covariant derivative of an anti-symmetric tensor and keep them in one side of the equality; while the rest of the terms will be kept on the other side. Note that in Eq. (5.7), the terms on the left hand side are already in the anti-symmetric form. Let us keep the first two terms on the right hand side unchanged and concentrate on ω^a. We shall see that first term of ω^a in (5.8) has an anti-symmetric part. For that purpose, re-express ω^a as

$$\begin{aligned}\omega^a = &-\pounds_\xi[\sqrt{-g}\delta v^a] + 2\sqrt{-g}\ hP^{ibad}\nabla_b\nabla_{(d}\xi_{i)} + 4\sqrt{-g}\ (\delta P^{ibad})\nabla_b\nabla_{(d}\xi_{i)}\\ &+2\sqrt{-g}\ P^{ibad}\nabla_b[\pounds_\xi h_{id}] - 4\sqrt{-g}\ P^{ibad}\delta\Gamma^l_{bd}\nabla_{(i}\xi_{l)}\ .\end{aligned} \tag{5.9}$$

The above final expression can be obtained after some involved calculation which is given in Appendix 5.A. Here, we have followed the notation $A_{(i}B_{j)} = (1/2)(A_iB_j + A_jB_i)$. Now, identifying the total derivative anti-symmetric part from the last term of ω^a from (5.9), one can have $\mathcal{K}^a = \nabla_b \mathcal{K}^{ab}$ where,

$$
\begin{aligned}
&\sqrt{-g}\ \mathcal{K}^a = \delta(\sqrt{-g}E^{ab}\xi_b) - \frac{1}{2}\sqrt{-g}\xi^a E^{ij}h_{ij} + \frac{1}{2}\pounds_\xi[\sqrt{-g}\delta v^a] \\
&-\sqrt{-g}\ hP^{ibad}\nabla_b\nabla_{(d}\xi_{i)} - 2\sqrt{-g}\ (\delta P^{ibad})\nabla_b\nabla_{(d}\xi_{i)} - \sqrt{-g}\ P^{ibad}\nabla_b[\pounds_\xi h_{id}] \\
&+\sqrt{-g}\ P^{ibad}(\nabla_b h^l_d - \nabla^l h_{bd})\nabla_{(i}\xi_{l)} - \sqrt{-g}\ P^{ibad}h^l_b\nabla_d\nabla_{(i}\xi_{l)}\ ;
\end{aligned}
\tag{5.10}
$$

and

$$
\sqrt{-g}\ \mathcal{K}^{ab} = \frac{1}{2}\delta(\sqrt{-g}\ J^{ab}) - \sqrt{-g}[\xi^{[a}\delta v^{b]}] + \sqrt{-g}\ P^{abcd}h^l_c\nabla_{(d}\xi_{l)}\ . \tag{5.11}
$$

Again, we refer our reader to the detailed calculation from Appendix 5.A.

Note, that the $\mathcal{K}^{ab}$ is an anti-symmetric tensor. Therefore, the quantity $\mathcal{K}^a$ is conserved for the diffeomorphism vector ξ^a. In the case of GR, use of equation of motion reduces (5.10) and (5.11) to the expression obtained in [231] for a ξ^a which was derived following the original approach of ADT [232]. On the contrary, the present method is different from this in which Noether prescription has played a central role. We call (5.10) and (5.11) as ADT-like conserved current and anti-symmetric potential, respectively. These are the main results of the present chapter.

A point is to be noted here that our current is much robust and more general as we have obtained the conserved current in Lanczos-Lovelock theory for a wide class of diffeomorphism vector, which defines non-Killing (and Killing as well) horizon surfaces. Also, our analysis directly shows the connection between the conserved ADT-like potential and the conserved Noether potential, which is given by (5.11). Moreover, a careful investigation shows that, in our derivation, we have not directly used the equation of motion *i.e.*, $E^{ab} = 0$. In GR, people call this procedure as an "off-shell" method. Therefore, in that sense, the obtained conserved ADT-like current and potential is off-shell ones. However, one must be cautious about the fact that the term "off-shell" is often misunderstood. The off-shell formalism of obtaining current is just operational in order to obtain the on-shell results (such as the charges). There is another point to consider. As we have not used the equation of motion in our analysis, our results can be used for those null-surfaces as well which might not be obtained from direct solution of the equation of motion.

One must note that, in our analysis, we have repeatedly used $\nabla_a P^{abcd} = 0$. Therefore, the analysis is valid for Lanczos-Lovelock gravity only. Of course, the same procedure can be extended to other theories of gravity and, in that case, the results must be modified. However, for the time being, we have concentrated only on LL gravity for the following reasons. As we know, one of the key features of GR is that it depends on the geometric properties of the Riemann tensor and the field equations are second order of the dynamical variables. Moreover, GR is ghost free. Now, it is compelling to know whether one can formulate a theory which incorporates higher order polynomials of the curvature tensor, Ricci tensor and Ricci scalar in such a way that the field equations are still second order of the dynamical variables and the theory is free of ghost. The answer is yes! and, the theory is Lanczos-Lovelock gravity [226–228]. Now, while obtaining the conserved current and the potential we do not require to use the equation of motion $E^{ab} = 0$ at the operational level but, to compute them explicitly, one has to use a particular background. If one uses the background as a solution of LL gravity, then it will be free of ghosts. However, if the background is not obtained from the LL gravity, one has to carefully check and allow only those backgrounds which does not have any ghost.

Some comments are in order about the above conserved quantities. It is well known that in $U(1)$ Yang-Mills theory due to the gauge symmetry $A_i \longrightarrow A_i + \nabla_i\theta(x)$, one obtains the conserved current as $j^a = \nabla_b[F^{ab}\theta(x)]$. When θ is a constant, it implies that $j^a = 0$ because of the equation of motion $\nabla_a F^{ab} = 0$. As a result, the conserved charge $Q = \int d^3x j^0 = \oint d^2x n_i F^{0i}\theta = 0$. The last surface integral vanishes for a compact surface which encloses the volume. But, it may not vanish for any patch of the surface, which is not a compact one.

In gravity, when there is a horizon in the spacetime, the whole spacetime is not accessible to the observer. Then the the spacelike volume has two boundaries: one is at the horizon and other one at the infinity. In this case one usually is not interested on the quantity $\int d\Sigma_a \mathcal{K}^a$. Instead the quantities of interest are the surface integral $\int d\Sigma_{ab}\mathcal{K}^{ab}$, evaluated either on the horizon or on the infinity. Both of these two-surfaces are non-compact in nature. This disconnects (and generalises) the definition of Noether charge from the integral of $\mathcal{K}^a d\Sigma_a$ over some bulk region. Here we call this as the conserved charge. For example, in the case of GR with Killing symmetry, $\int d\Sigma_{ab}\mathcal{K}^{ab}$ leads to ADT conserved quantity calculated on a particular two surface. Of course, in special cases the two integrals are related by Gauss law but the definition which we use in gravity is more general. In particular, it only depends on the integral of $dQ = \mathcal{K}^{ab} d\Sigma_{ab}$ being usefully defined and does not care about the value (zero or non-zero, positive or negative) of $\mathcal{K}^a$ over the bulk.

As mentioned above, we are interested to calculate $\mathcal{K}^{ab}$ over a two surface which is a part of a closed surface, enclosing the spacelike bulk. In this case the diffeomorphism vectors are chosen in a such a way that it defines the surface. For example, if the surface is a generic null surface defined by null vector l^a with the condition $l^a l_a = 0$, then in that case $\xi^a = l^a$. In general the expression of ξ^a is obtained by imposing certain condition by which a specific boundary is chosen. Moreover, as we have mentioned earlier, even if $\int d\Sigma_a \mathcal{K}^a = 0$ in a particular case, $\int d\Sigma_{ab}\mathcal{K}^{ab}$ is not zero in general when it is evaluated on a non-compact two-surface such as the horizon or the asymptotic infinity. Therefore our finding of $\mathcal{K}^{ab}$ in a general way is a very important one for exploring the conserved quantities on a surface which is defined not only through Killing vectors, but also by other types of vectors.

We want to emphasize once again that our analysis is not only restricted to the Killing horizon, which is defined by the vanishing norm of a Killing vector. As we have mentioned earlier, there are various types of horizons, other than the Killing one. Here we mention couple of them which appear in several discussions and have current interests in different aspects of gravity. First example is the metric which representing a generic null surface in Gaussian null coordinates (GNC) [233, 234]:

$$ds^2 = -2r\alpha du^2 + 2drdu - 2r\beta_A dx^A du + \mu_{AB} dx^A dx^B \ . \tag{5.12}$$

Note that the surface is not defined by the vanishing norm of a Killing vector. The null horizon in this spacetime (which is at $r = 0$) is defined by the vanishing norm of the null vector $l^a = (l^u, l^r, l^{x^A}) = (1, 0, \mathbf{0})$; and $l_a = (-2r\alpha, 1, -r\beta_A)$. Here, x^A represents all the transverse directions. It can be easily verified that l^a is not a Killing vector (*i.e.* $\pounds_l g_{ab} \neq 0$ in general). Therefore, the existing formalism of the ADT current cannot be used for this null surface.

Another interesting example is the case of the scalar–tensor theory. In that (*i.e.* the scalar tensor) theory, a scalar field ϕ is non-minimally coupled with gravity in the original (Jordan) frame described by the metric g_{ab}. This coupling can be removed by a conformal transformation $g_{ab} \to \tilde{g}_{ab} = \phi g_{ab}$, along with a rescaling in the scalar field [108, 112]. By virtue of these transformations, one arrives to the conformal frame, known as the Einstein frame. If there exists a Killing horizon in the Jordan frame, it is a conformal Killing horizon in the Einstein frame. It is because, a Killing vector (say χ) in Jordan frame, will be a conformal Killing vector in the Einstein frame in general. Mathematically, although $\pounds_\chi g_{ab} = 0$, we have $\pounds_\chi \tilde{g}_{ab} = (\pounds_\chi \ln\phi)\tilde{g}_{ab}$. Now, if ϕ evolves with time (*i.e.* $(\pounds_\chi \ln\phi) \neq 0$), χ will no more be Killing vector in Einstein frame and consequently, the horizon in this frame is determined by the conformal Killing vector (not by the Killing one). This is true for any two conformally connected frames. As the existing formalism of ADT current is defined for the Killing vectors, it can be used in both of the frames only with the assumption that the conformal factor ϕ is Lie-transported along χ. Since developping the thermodynamics in a conformal Killing horizon requires further investigation, we have taken the assumption in chapter 3 in order to obtain the thermodynamics in both frames. Apart from these examples, there are a lot of non-Killing horizons in gravity as we have mentioned earlier. Our approach covers all of them (of course for scalar–tensor theory, E^{ab} and P^{abcd} have to be determined accordingly). In the following, we shall discuss two cases: one is ξ^a is KV and other one is ξ^a is CKV.

5.3 Special form of diffeomorphism

Let us now discuss two popular situations. One is the spacetime has Killing symmetry and hence choose ξ^a as Killing vector. Other case is the spacetime has inherent conformal symmetry and so ξ^a can be taken as conformal Killing vector. The former case has been studied extensively in literature while the later one, so far we know, has not been dealt with in this context.

5.3.1 ξ^a is a Killing vector

If one takes ξ^a as the Killing vector, satisfying Killing equation $\nabla_a\xi_b + \nabla_b\xi_a = 0$, the obtained conserved current $\mathcal{K}^a$ and the potential $\mathcal{K}^{ab}$ (see Eqs. (5.10) and (5.11)) reduces to the conserved ADT current and the potential *i.e.*

$$\sqrt{-g}\ \mathcal{K}^a|_{KV} = \sqrt{-g}\ \mathcal{J}^a = \delta(\sqrt{-g}E^{ab}\xi_b) - \frac{1}{2}\sqrt{-g}\xi^a E^{ij}h_{ij}\ , \tag{5.13}$$

and

$$\sqrt{-g}\ \mathcal{K}^{ab}|_{KV} = \sqrt{-g}\ \mathcal{J}^{ab} = \frac{1}{2}\delta(\sqrt{-g}\ J^{ab}) - \sqrt{-g}\xi^{[a}\delta v^{b]}\ , \tag{5.14}$$

respectively. In the above, $\mathcal{J}^a$ is the expression of the usual ADT current. Exactly the same was obtained earlier in [164].

5.3.2 ξ^a is a conformal Killing vector

Considering ξ^a as a conformal Killing vector satisfying $\nabla_a \xi_b + \nabla_b \xi_a = (\psi/2) g_{ab}$, and using the identity

$$\pounds_\xi h_{ab} = \frac{\psi}{2} h_{ab} + \frac{\delta\psi}{2} g_{ab} \ , \tag{5.15}$$

which holds only for the conformal Killing vectors, the expression (5.10) reduces to

$$\begin{aligned}
&\sqrt{-g}\ \mathcal{K}^a|_{CKV} = \delta(\sqrt{-g}E^{ab}\xi_b) - \frac{1}{2}\sqrt{-g}\xi^a E^{ij} h_{ij} + \frac{1}{2}\pounds_\xi[\sqrt{-g}\delta v^a] \\
&-\frac{\sqrt{-g}}{4} h P_i^{\ bai}(\nabla_b \psi) - \frac{\sqrt{-g}}{2} g_{id}(\delta P^{ibad})(\nabla_b \psi) - \frac{\sqrt{-g}}{2} P^{ibad} h_{id}(\nabla_b \psi) \\
&-\frac{\sqrt{-g}}{2} P_i^{\ bai} \nabla_b(\delta\psi) \ ,
\end{aligned} \tag{5.16}$$

while (5.11) yields

$$\sqrt{-g}\ \mathcal{K}^{ab}|_{CKV} = \frac{1}{2}\delta(\sqrt{-g}\ J^{ab}) - \sqrt{-g}\xi^{[a}\delta v^{b]} \ . \tag{5.17}$$

Note that although the form of $\mathcal{K}^a|_{CKV}$ is not exactly the same as the $\mathcal{K}^a|_{KV}$ (which is, of course, defined for ξ^a being a Killing vector), interestingly, the conserved potential $\mathcal{K}^{ab}|_{CKV}$ is related to corresponding Noether potential in exactly identical manner as the conserved ADT current $\mathcal{K}^{ab}|_{KV}$. It is a very new finding which indicates that ADT potential for CKV can be obtained solely by the information of Noether potential and the boundary term in the variation of the action. This is completely similar to KV case.

5.3.3 Explicit evaluation of the conserved charges for a black hole spacetime

Having these conserved quantities, let us now show how to calculate these for a given metric solution of a specific gravitational theory. The charges will be calculated on a non-compact surface. For that, we define the charge as

$$\delta Q = \frac{1}{16\pi}\int_{\mathcal{S}} d\Sigma_{ab}\mathcal{K}^{ab} \ , \tag{5.18}$$

where $d\Sigma_{ab}$ is the surface element of $\mathcal{S}$.

The spacetime which we take as an example is the Sultana-Dyer (SD) [106] metric, the thermodynamics of which has been discussed in the earlier chapter. The spacetime is describes by the metric given in Eqs. (4.2) and (4.3). As we have discussed earlier, the SD metric is a time-dependent solution of general relativity (GR) in presence of time-like dust and null-like dust (for details see [106]). Moreover, as one can see from (4.2), the SD metric is conformal to the Schwarzschild metric. However, unlike the Schwarzschild metric, the SD metric is not asymptotically flat, instead asymptotically it becomes Friedmann-Lemaitre-Robertson-Walke (FLRW). Moreover, the SD metric has a conformal Killing horizon at $r_{\mathcal{H}} = 2M$ [155] and the conformal Killing horizon is located where the conformal Killing vector ξ^a of SD background becomes null *i.e.* $\xi^a\xi_a = 0$. Note, that the conformal Killing vector of SD background is given by $\xi^a = (1,0,0,0)$ and $\xi_a = a^2(t,r)(-F(r),0,0,0)$. Here we shall evaluate (5.18) at the two different surfaces: at the horizon and at the assymptotic region which is FLRW. Since the spacetime (4.2) has inherent conformal Killing vector, the expression for $\mathcal{K}_{ab}$ must be given by (5.17). In this case (5.18) can be re-expressed as

$$\delta Q = \frac{1}{32\pi}\delta\int_{\mathcal{S}} d\Sigma_{ab}J^{ab} - \frac{1}{16\pi}\int_{\mathcal{S}} d\Sigma_{ab}(\xi^{[a}\delta v^{b]}) \ . \tag{5.19}$$

In the above the first term of the right hand side is obtained in the following way. Originally, it is given by $\int d\Sigma_{ab}\delta(\sqrt{-g}J^{ab})/\sqrt{-g}$. Now the surface element is constructed as $d\Sigma_{ab} = d^2x\sqrt{\sigma}(t_ar_b - t_br_a)$ where t_a and r_a are chosen either in spacelike-timelike basis or in null-null basis, whereas σ is the determinant of the induced metric on $\mathcal{S}$. This element is also be expressed as $d\Sigma_{ab} = d^2x\sqrt{-g}(\hat{t}_a\hat{r}_b - \hat{t}_b\hat{r}_a)$ where the hated vectors just determine the direction of the actual vectors and are constant. Then, since $\delta(\hat{t}_a\hat{r}_b - \hat{t}_b\hat{r}_a) = 0$, the integration is transformed to the first term of (5.19).

First let us calculate (5.19) on the conformal Killing horizon. In this case we choose r^a as the conformal Killing vector ξ^a of the spacetime and t^a is constructed to be the auxilary null vector l^a, such that $l.\xi = -1$. Then one finds that no contribution comes from the second integral of (5.19) as ξ^2 and ξ_a both vanish on the horizon. The first term of (5.19) is evaluated in the following way. Using the definition of (5.4), for GR one obtains $J^{ab} = \nabla^a\xi^b - \nabla^b\xi^a$ which by conformal Killing equation $\nabla_a\xi_b + \nabla_b\xi_b = \Omega^2 g_{ab}$ yields $J^{ab} = \Omega^2 g_{ab} - 2\nabla_b\xi_a$ with $\Omega^2 = \nabla_a\xi^a/2$. In presence of conformal Killing vector, one can define a surface gravity κ as $\nabla_a\xi^2 = -2\kappa\xi_a$ which is constant on the horizon [235]. Using these, the first integration is evaluated to be as

$$
\begin{aligned}
\frac{1}{32\pi}\int_{\mathcal{H}} d\Sigma_{ab}J^{ab} &= -\frac{1}{16\pi}\int_{\mathcal{H}} d^2x\sqrt{\sigma}\xi_a l_b(-2\nabla^b\xi^a + \Omega^2 g^{ab}) \\
&= \frac{1}{16\pi}\int_{\mathcal{H}} d^2x\sqrt{\sigma}(l^a\nabla_a\xi^2 + \Omega^2) = \frac{1}{16\pi}\int_{\mathcal{H}} d^2x\sqrt{\sigma}(2\kappa + \Omega^2)\ , \qquad (5.20)
\end{aligned}
$$

where in the last step $\nabla_a\xi^2 = -2\kappa\xi_a$ and $\xi \cdot l = -1$ have been used. For SD metric, one finds $\Omega^2 = 4/\sqrt{a}$ and then

$$
\delta Q_{\mathcal{H}} = \frac{1}{16\pi}\int_{\mathcal{H}} d^2x\sqrt{\sigma}(2\kappa + \frac{4}{\sqrt{a}})\ . \qquad (5.21)
$$

Since κ is a constant and $a(t,r)$ is independent of the angular coordinates, whereas the integration is performed on the angular coordinates, (5.21) implies

$$
\delta Q_{\mathcal{H}} = \frac{1}{2}r^2_{\mathcal{H}}a^2_{\mathcal{H}}(\kappa + \frac{2}{\sqrt{a_{\mathcal{H}}}})\ . \qquad (5.22)
$$

Further as $a_{\mathcal{H}}$ diverges, we consider only the leading order contribution which provides

$$
\delta Q_{\mathcal{H}} = \delta(\frac{\kappa}{2\pi}\pi a^2_{\mathcal{H}}r^2_{\mathcal{H}})\ . \qquad (5.23)
$$

Although, here we have obtained the charge at the horizon, we cannot concretely say what physical quantity the charge implies. However, from the known result of [182], we can identify $\pi a^2_{\mathcal{H}}r^2_{\mathcal{H}}$ as the entropy of the SD black hole. Thus, we can say that entropy is given in terms of our defined charge as $S = \frac{2\pi}{\kappa}Q_{\mathcal{H}}$.

Now, we shall calculate the conserved charge in the asymptotic region on a r =constant and t =constant surface. We denote this asymptotic two-surface as ∂c. In this case, we write the elemental surface area of two-surface $d\Sigma_{ab}$ in terms of timelike and spacelike normals as $d\Sigma_{ab} = (n_a s_b - n_b s_a)\sqrt{\sigma}d^2x$, with $n^a n_a = -1$ and $s^a s_a = 1$. The normals are found to be

$$
n^a = (\frac{1}{a\sqrt{F}}, 0, 0, 0);\quad s^a = (0, \frac{\sqrt{F}}{a}, 0, 0)\ . \qquad (5.24)
$$

With this, the first integral of (5.19) yields

$$
\frac{1}{32\pi}\delta\int_{\partial c} d\Sigma_{ab}J^{ab} = \frac{1}{2}[Ma^2 + 4Mra\sqrt{a}]\ . \qquad (5.25)
$$

The second integral of (5.19) provides $-\frac{1}{16\pi}\int_{\partial c} d\Sigma_{ab}(\xi^{[a}\delta v^{b]}) = \frac{1}{16\pi}\int_{\partial c}\sqrt{\sigma}a^2\delta v^r$. From the definition of δv^a, we further obtain in this case $\delta v^r = -g^{rr}g^{tt}\nabla_r\delta g_{tt}$. Now, to compute δg_{tt}, one has remember that the SD spacetime is asymptotically FLRW where the metric component $g_{tt} \to g_{tt} = -a^2$. Therefore, $\delta g_{tt} = \frac{2Ma^2}{r}$. Using this, finally one can obtain $\delta v^r = \frac{1}{a^4}[\frac{2Ma^2}{r^2} - \frac{4Ma^2}{r^3 F} - \frac{32M^2a^{\frac{3}{2}}}{r^2 F}]$. Then the second integration yields

$$-\frac{1}{16\pi}\int_{\partial c} d\Sigma_{ab}(\xi^{[a}\delta v^{b]}) = \frac{1}{2}[Ma^2 - \frac{2Ma^2}{rF} - \frac{16M^2a^{\frac{3}{2}}}{F}] \ . \tag{5.26}$$

Therefore, finally we obtain

$$\delta Q_{\partial c} = \left[Ma^2 + 2Mra^{\frac{3}{2}} - \frac{Ma^2}{rF} - \frac{8M^2a^{\frac{3}{2}}}{F}\right] . \tag{5.27}$$

Here, we have shown the procedure how to compute those charges in the two regions. However, for the present moment, we cannot identify (using any first principle) what macroscopic quantities of a black hole (like mass, angular momentum, entropy, etc) those charges imply. In the following section, we shall discuss that if someone tries to obtain the first law following the ADT approach for the conformal Killing horizon with the charges and potential that we have obtained, one has to deal with several non-trivial terms. Since the original ADT approach is based on the Killing symmetry, those term vanishes. Presently, we cannot say what the extra terms, which appear for the conformal Killing vectors, contribute to. Therefore, using the original formalism, we cannot identify mass, angular momentum *etc.* for the conformal Killing horizon. This topic is now under investigation.

5.4 Summary and outlook

Defining thermodynamical entities and establishing the thermodynamic laws for a non-Killing dynamical horizon has been a major challenge in gravity. Several physicists (including us) are trying to obtain the thermodynamic first law for a generic null horizon or for a conformal Killing horizon. What has been understood so far is that one has to start developing everything from the very basics. As it is well-known, the root of the black hole thermodynamics lies in the conserved current in the theory because, all the physical thermodynamic parameters can be obtained from the conserved current. Therefore, it is very much needed to generalize the conserved quantities for a non-Killing symmetry.

It appears that the ADT approach is a very successful method to obtain the thermodynamic description from the conserved currents without any ambiguity like anomalous two factor in Komar case [166]. But till now all attempt has been done for existence of a Killing vector in the spacetime. Since in reality we need to encounter non-Killing situation, it is now necessary to obtain the conserved quantities for any diffeomorphism vector. So far it has been successfully done for Noether conserved quantities. In this chapter, we have formulated an elegant approach to define an ADT-like current for a horizon defining diffeomorphism vector in Lanczos-Lovelock gravity as the original ADT approach is valid only for the Killing vectors.

Here, we have taken the general LL Lagrangian. Then we have varied the Lagrangian to obtain the Noether current due to the diffeomorphism. Thereafter, our purpose was to obtain the ADT-like current for non-Killing diffeomorphism vector. We have defined the conserved current as the derivative of a two-rank anti-symmetric conserved potential. Then we have taken two limits: ξ^a as a Killing vector, and then ξ^a as a conformal Killing vector. For Killing vector case, the entire result reduces to the original ADT currents and the potential, defined for the Killing vectors. We obtain even more interesting result for the conformal Killing vectors. In this case, the expression of conserved potential is exactly similar to the ADT potential. However, the expression of the conserved current differs. Following our method, one can show the connection of the conserved potential with the conserved Noether potential in each case.

As we have emphasized several times, the entire analysis is very general in every aspects. The currents and the potentials are defined for wide class of diffeomorphism vectors, which define horizon surfaces. Also, the analysis has been made for the general Lanczos-Lovelock gravity, form which one can obtain the corresponding results for the GR or for any other higher order gravity, and yet nowhere the equation of motion has been used.

Before concluding this chapter, we want to mention an important future aspect of our analysis. Till now we are familiar with Wald's approach [29] to find the first law of thermodynamics for a Killing vector. Now the question is: Can we follow the similar prescription to find a reasonable form of first law in the case for a null surface which is not accompanied by a Killing symmetry. Probably, our most general relation $\mathcal{K}^a = \nabla_b \mathcal{K}^{ab}$ with the quantities are given by equations (5.10) and (5.11), inherently captures that general structure. To get a feel of the generalization, let us first consider the Killing situation. In this case $\mathcal{K}^a$ and $\mathcal{K}^{ab}$ are given by (5.13) and (5.14), respectively. On-shell (i.e. using equation of motion) $\mathcal{K}^a$ vanishes and then using the steps of Wald, one obtains the first law of black hole mechanics. This is all known. But, for a conformal Killing vector the situation is a bit different. For simplicity, we consider GR theory with conformal symmetry. Here $P^{abcd} = (1/2)(g^{ac}g^{bd} - g^{ad}g^{bc})$, and then for on-shell condition, the relation $\mathcal{K}^a = \nabla_b \mathcal{K}^{ab}$ reduces to the following form

$$\begin{aligned} &\frac{1}{2}\delta(\sqrt{-g}\ J^a) - \sqrt{-g}\nabla_b[\xi^{[a}\delta v^{b]}] \\ &= \frac{\sqrt{-g}}{4}\Big[\psi\nabla_i h^{ai} + h(\nabla^a\psi) - \psi(\nabla^a h) - h^{ai}\nabla_i\psi\Big]\ , \end{aligned} \tag{5.28}$$

where, $\psi = \nabla_i \xi^i$. This clearly indicates that the terms on the right hand side of the above equation contributes to thermodynamic quantities in this case. Note that they vanish for the Killing situation. Hence it is evident that for most general case the usual definition of thermodynamic quantities (like entropy is given by the Noether charge calculated on the horizon) can not be taken as general one. Therefore it would be interesting to see how the terms like those appear on the right hand side of (5.28) modify the definition of several thermodynamic quantities. Till now we do not have any conclusive statement. The investigations are going on in this direction and the work is under progress.

Appendix

5.A Derivation of the Eqs. (5.9), (5.10) and (5.11)

We know that

$$\pounds_\xi v^a = 4p^{ibad}\nabla_b\nabla_{(i}\xi_{d)} \ . \tag{5.29}$$

Thus,

$$\delta[\sqrt{-g}\pounds_\xi v^a] = 2\sqrt{-g}\ hP^{ibad}\nabla_b\nabla_{(i}\xi_{d)} + 4\sqrt{-g}\ (\delta P^{ibad})\nabla_b\nabla_{(i}\xi_{d)}$$
$$+4\sqrt{-g}\ P^{ibad}\delta\{\nabla_b\nabla_{(i}\xi_{d)}\} \ . \tag{5.30}$$

Now, a detailed calculation shows that

$$4P^{ibad}\delta\{\nabla_b\nabla_{(i}\xi_{d)}\} = 2P^{ibad}\nabla_b[\pounds_\xi h_{id}] - 4P^{ibad}\delta\Gamma^l_{bd}\nabla_{(i}\xi_{l)}. \tag{5.31}$$

Replacing (5.30) and (5.31) in (5.8) one can obtain (5.9).

Now,

$$4P^{ibad}\delta\Gamma^l_{bd}\nabla_{(i}\xi_{l)} = 2P^{ibad}[\nabla_b h^l_d - \nabla^l h_{bd}]\nabla_{(i}\xi_{l)} + 2\nabla_b[P^{abic}h^l_c\nabla_{(i}\xi_{l)}]$$
$$-2P^{ibad}h^l_b\nabla_d\nabla_{(i}\xi_{l)} \tag{5.32}$$

Thus, from (5.7) one can obtain

$$\delta(\sqrt{-g}\ J^a) - 2\sqrt{-g}\nabla_b[\xi^{[a}\delta v^{b]}] = 2\delta(\sqrt{-g}E^{ab}\xi_b) - \sqrt{-g}\xi^a E^{ij}h_{ij} + \pounds_\xi[\sqrt{-g}\delta v^a]$$
$$-2\sqrt{-g}\ hP^{ibad}\nabla_b\nabla_{(d}\xi_{i)} - 4\sqrt{-g}\ (\delta P^{ibad})\nabla_b\nabla_{(d}\xi_{i)} - 2\sqrt{-g}\ P^{ibad}\nabla_b[\pounds_\xi h_{id}]$$
$$+2\sqrt{-g}\ P^{ibad}[\nabla_b h^l_d - \nabla^l h_{bd}]\nabla_{(i}\xi_{l)} - 2\sqrt{-g}\ P^{ibad}h^l_b\nabla_d\nabla_{(i}\xi_{l)} + 2\sqrt{-g}\ \nabla_b[P^{abic}h^l_c\nabla_{(i}\xi_{l)}] \ . \tag{5.33}$$

Replacing the last term on the left hand side, one can obtain the desired results given in (5.10) and (5.11).

Chapter 6

Towards the microscopic origin of black hole thermodynamics, a possible way: symmetry of the null surface and the algebra of the associated charges

6.1 Prologue

[1]The works of Bekenstein and Hawking [14–16] led to the conclusion that the black holes are the thermodynamic objects which have entropy, proportional to the surface area of the horizon. However, later it was shown that the entropy and temperature can be can be associated with any null surface in general relativity [221] (see also [222, 236]). It is an important observation in the context of "gravity as an emergent phenomenon" as a general null surface is not a solution of any equation of the spacetime (for an extensive review on this direction, see [105]). It reveals that the thermodynamic property is not restricted only to Einstein's gravity. Earlier, we have obtained the thermodynamic description in the scalar-tensor gravity. That description is also applicable for higher curvature $f(R)$ gravity as well. Therefore, our earlier discussions also reveal the existence of the thermodynamic description beyond Einstein's graivity.

Although, the black hole thermodynamics is similar to usual classical thermodynamics in many ways, there are several differences as well, which make it a rather

[1]This chapter is based on the publication [111] .

less-understood phenomenon and motivates to explore it (the black hole thermodynamics) further. In classical thermodynamics, the thermodynamic description stems from the underlying microstates of the system. However, in black hole thermodynamics, we do not have proper idea about the microstates which yields the horizon thermodynamics. Moreover, in case of black holes, the concept of entropy and temperature comes into the picture only when one takes the quantum effect into the account. Therefore, it should be quite natural that the quantum microstates give rise to the Bekenstein-Hawking entropy. However, the absence of a consistent quantum theory of gravity has made it even more difficult to comprehend the possible microscopic degrees of freedom which contribute to the entropy.

It has earlier been found that the commutator algebra among the charges associated with the diffeomorphisms, that preserves symmetries of the spacetime in the asymptotic region, generally leads to the Virasoro algebra [237] with a central charge, which was firstly shown by Brown and Henneaux [149]. One can obtain the entropy of a black hole from this central charge (of Virasoro algebra) using the Cardy formula [238], which was shown in [239]. This work of Brown and Henneaux [149] was further developed by Carlip [150]. Some recent works [108, 151–156, 223] show that the Bekenstein-Hawking entropy can also be obtained when one explores the near horizon symmetry by imposing some suitable fall-off conditions. In the earlier chapter (Chapter- 2), we have obtained the entropy for the ST gravity using the Virasoro algebra for the Killing horizon. Therefore, in this method, one can connect the entropy with the configuration and the associated symmetry of the surface. In this case, due to the specific choice of horizon preserving symmetry, one arrives to a subset (which preserves the location of the null surface) from the whole set of diffeomorphisms. Therefore, roughly speaking, some of the original gauge degrees of freedom (which could have been eleminated if we had allowed the whole set of diffeomorphisms) can now be considered as being upgraded as the physical degrees of freedom (as we have allowed only the subset of diffeomorphisms), which attribute the entropy. However, the near horizon structure and the symmetry algebra should be studied for a generic null surface in a more rigorous manner in order to get more understanding about the microscopic degrees of freedom of the null surface which is associated with the null-horizon/BH thermodynamics.

The above discussion motivates us for the following discussions in this chapter. Here we present the discussion under the framework of a generic null surface. This is the generalization of the earlier work [157] for the Killing horizon in the Rindler spacetime. It was shown [157] that the charges corresponding to the Killing horizon structure preserving conditions for the Rindler metric exhibits *noncommutative* type algebra. Since we know, in general, the null surface is not a solution of any equation of motion of the spacetime and also shows the thermodynamic features, it would be interesting to investigate such possibility for a general null-surface. If the same result is obtained, it would imply the results are very general and may have some deeper significance to play a crucial role in revealing the quantum nature of gravitational theories. Moreover, since null surface is a local concept, the present analysis will enlighten the "emergent nature of gravity" for the following reasons. A local horizon (Rindler horizon) can attribute temperature and entropy. In addition, the first law of thermodynamics leads to the Einstein's equation of motion for such a horizon [104] implying that the gravity emerges from the more fundamental theory like thermodynamics. Therefore one would be interested to investigate the null surface in this context to see the generality of these concepts.

6.2 Objectives of the chapter

Here we shall consider the similar boundary conditions as taken in some earlier works [152, 157, 223] and compute two different Noether potentials for the two components of the diffeomorphism vectors and obtain the conserved charges along those directions. The choice of the boundary conditions on metric coefficients are based on the fact that the structure of the null boundary does not change after perturbation near the null surface. More precisely, the only condition we impose is the location of null surface must not change. Moreover, this is imposed only on the time-radial sector of the metric – no condition on the angular or the transverse part. In this sense, it is much weaker condition compared to the other choices, existing in literature [108, 154–156] (see [240] for a complete list of references), which are imposed on the full metric. We show that the algebra between these charges, in a different basis, exhibits non-commutative algebra. Moreover, as we shall show, following the Sugawara construction [237] one can also obtain the entropy from the Cardy formula. The whole analysis is done on a general null surface, which is not a solution of any equation of motion and, hence, all the results are *off-shell*.

Notations: The Latin letters a, b, etc. stand for all spacetime indices while A, B, etc. take transverse (or angular) coordinates only. Moreover, we also use $\mathbf{0}$ which imply a zero vector in transverse directions.

6.3 Near horizon symmetry, charges, brackets and the algebra

Due to the seminal work of Bondi, Metzner and Sachs (BMS) [241–243] and also of Brown and Henneaux [149], it is known that one can define non-local conserved charges in any theory in which local gauge symmetry is present. Here, in the context of general relativity, we have dealt with diffeomorphism as the local gauge symmetry and have defined the charges and brackets on a null surface. Instead of getting the Heisenberg algebra (which was the case for many earlier works; such as [244–253]), here we show that our set-up defines non-commutative Heisenberg algebra near the null horizon. In the following, we have briefly described about the Gaussian null coordinates (GNC) in which we have made our whole analysis. Thereafter, we have defined the charges by fixing the boundary value of the metric perturbation caused by the diffeomorphism. Subsequently, the near horizon algebra of the charges will be studied.

6.3.1 Null metric in Gaussian null coordinates (GNC)

The motivation of this chapter is to understand the near-horizon behaviour of an arbitrary null surface. In contemplation of that goal, we shall briefly discuss about some precursory constructions in this section. To describe the neighborhood of a null-hypersurface, there exists a preferable choice of a set of adapted coordinate system called the *Gaussian Null Coordinates (GNC)*. For more details on how the metrics is constructed can be found in [233, 234]. The metric is given as

$$ds^2 = -2r\alpha du^2 + 2drdu - 2r\beta_A dx^A du + \mu_{AB}dx^A dx^B \ , \tag{6.1}$$

where, $r = 0$ corresponds to the null surface. The same has also been addressed earlier in [254]. Besides, the metric components α, β_A and μ_{AB} are the smooth functions of all the coordinates u, r and x^A and μ_{AB} is invertible. On the null surface, one can attribute a set of null vectors. One we choose as $l^a = (l^u, l^r, l^{x^A}) = (1, 0, \mathbf{0})$, and the other one, the complementary null vector is chosen as $k^a = (0, -1, \mathbf{0})$, so that $g_{ab}l^a k^b = -1$ is satisfied. The covariant components of these two vectors are given as: $l_a = (-2r\alpha, 1, -r\beta_A)$ and $k_a = (-1, 0, \mathbf{0})$. Note, l^a is a "Killing-like" (not exactly a Killing vector, as one can verify easily) vector having the property that it becomes a null vector on the (null) horizon. Nevertheless, as we shall argue later, l^a does not define a Killing horizon. On the contrary, k^a is a null vector throughout the spacetime. Furthermore, l^a can be thought of as the future-outgoing null vector and k^a as the future-ingoing null vector as those two are related as $l^a k_a = -1$. With this set of the null vectors, one can give a covariant definition of the elemental surface area of the null $(d-2)$-hypersurface (with d is the dimension of the whole spacetime) as $d\Sigma_{ab} = -\sqrt{\sigma} d^{d-2}x^A (l_a k_b - l_b k_a)$, where, σ is the determinant of the induced metric of the $(d-2)$-hypersurface. For the present case, one can check that the explicit form of the elemental surface area can be determined as $d\Sigma_{ur} = -\sqrt{\mu} d^{d-2}x^A$ whereas, $d\Sigma_{uA} = \mathcal{O}(r)$ and $d\Sigma_{rA} = \mathbf{0}$. In the later part of our analysis, we have defined the charges and the brackets which will be computed on the close $(d-2)$-hypersurface on the neighborhood of a null surface. Therefore, the finite non-zero contribution in those calculations will come from only the hypersurface which is transverse to the u and r directions i.e., only from the term containing $d\Sigma_{ur}$.

As we have mentioned several times in this chapter, the metric (6.1) in general is not chosen as a solution of equation of motion of gravitational field governed by a particular gravity theory. In our case, the theory will be taken to be diffeomorphism invariant and the charge for this symmetry, which we shall consider later, is for General theory of relativity (this is for simplicity, but one can in principle consider any other diffeomorphism invariant gravity theory as well). Of course, a choice can be done for the metric parameters α, β_A and μ_{AB} using the equations of motion for g_{ab}; which is a subset of all allowed null surfaces. Here we are not making any such further restriction on them. So present analysis does not use any information of gravitational field equations of motion and, hence, the metric (6.1) in general may not be a solution of a particular gravity theory, *i.e.* we are considering the whole set of null surfaces. Therefore, we call this as an off-shell analysis.

6.3.2 Set-up: null surface preserving diffeomorphisms and charges

In this section, we shall calculate the charges for the diffeomorphism $x^a \to x^a + \xi^a$ which are chosen by the null surface preserving condition, which means *the location of the null surface is unchanged*. The diffeomorphism vector ξ^a will be shown to have two components. Considering each component as individual vector, we shall have two different charges. The brackets among them, in order to study the algebra between the Fourier modes of the charges, will be calculated.

Our aim in this chapter is to study the algebra of the charges which are defined only for the diffeomorphism of the $u-r$ sector of the spacetime (6.1). For that we fix the transverse component of ξ^a as zero, i.e., $\xi^A = 0$. Now, the other two components of ξ^a can be determined by the fact that the two components of the metric, g_{rr} and g_{ur}, do not change along ξ^a and, therefore, are Lie-transported. Hence, from the two conditions $\pounds_\xi g_{rr} = 0$ and $\pounds_\xi g_{ur} = 0$, one can find the non-zero components of the vector ξ^a (see the Appendix 6.A for detailed derivation):

$$\xi^u = F(u, x^A); \qquad \xi^r = -r\partial_u F; \qquad \xi^A = 0 \; ; \tag{6.2}$$

and, hence, the covariant components are

$$\xi_u = r\partial_u F + \bar{\alpha} F; \qquad \xi_r = F; \qquad \xi_A = \bar{\beta}_A F \; ; \tag{6.3}$$

where $\bar{\alpha} = -2r\alpha$ and $\bar{\beta}_A = -r\beta_A$. Here F is, for the moment, an unknown function which depends only on u and transverse coordinates x^A. Note that the obtained diffeomorphism vector becomes null on $r = 0$. Moreover, it is easy to verify that $\pounds_\xi g_{uu} = \mathcal{O}(r)$, which vanishes on the null surface. Therefore, ξ^i is a Killing vector near the horizon for only the $u - r$ sector of the spacetime. Also, note that the chosen conditions are like gauge only for this sector of the metric (6.1), expressed in the particular coordinates. Now, we define two vectors $\xi^\pm$, which are along the components of the vector ξ^a with the definition $\xi^+ = (0, \xi^r, \mathbf{0})$ and $\xi^- = (\xi^u, 0, \mathbf{0})$. Afterwards, we shall define all the charges and the brackets along these two vectors, ξ^+ and ξ^-.

Before entering into the calculation, let us spend some time on understanding the meaning of our imposed boundary conditions. To construct a null metric in d (spacetime) dimension, one needs $d(d-1)/2$ independent metric components. The metric (6.1) in Gaussian null coordinates accords to that and $r = 0$ implies a generic null surface. The main feature of a null surface is that it acts as a one way membrane in the spacetime and it blocks information from the other side. It is worthy to note that the blockage of the information is determined only by the $u - r$ sector, while the angular part of the metric does not play any role in it. Therefore, as long as one is concerned with the null features of the surface, one can only focus on the $u - r$ sector of the metric. Hence, the diffeomorphism vector has been formed in such a way that it preserves g_{uu}, g_{ur} and g_{rr} and, thereby, acting as the isometry in this sector which governs the characteristics of the null surface. The other components of the metric (for instance g_{rA}) might change along the diffeomorphism but, it does not affect the null characteristics or the location of the null surface (which is $r = 0$). Note that initially, the null surface at $r = 0$ was the induced metric of the angular coordinates (i.e. μ_{AB}) and, finally after diffeomorphism, one is again left with the induced metric of angular coordinates as the $drdx^A$ part of the metric vanishes (because, although metric component g_{rA} is non-zero under these conditions, but at $r = 0$ it does not contribute, as $dr = 0$, and hence again we obtain null surface). It must be emphasised that our condition is much weaker than treating the whole metric as diffeomorphism invariant near the null surface and sufficient to make the null character invariant. Here we shall show that this has an interesting feature. Moreover, it must be mentioned that same boundary condition was adopted earlier in various cases. For instance, finding the entropy associated to null surface in the context of Virasoro algebra [223]. Also similar one plays an important role in hydrodynamics of gravity [255] and membrane paradigm - horizon Bondi-Metzner-Sachs symmetry [256]. All these indicates that ξ^a is not a Killing vector for the full metric. Rather ξ^a appears as the Killing vector for the $u-r$ sector only. If we have imposed the condition that the whole spacetime is unaltered due to the arbitrary diffeomorphism (*i.e.* by considering $\pounds_\xi g_{uA} = \pounds_\xi g_{rA} = \pounds_\xi g_{AB} = 0$ as well along with our conditions for the $u-r$ sector), then ξ^a would have been a Killing vector. But, as has been mentioned earlier, that is a more strong condition to be imposed on the spacetime, whereas our condition is the minimum requirement to keep null structure invariant and certainly a much weaker one.

To calculate the charges and construct the algebra, one has to calculate the components of the Noether potential due to the diffeomorphism symmetry. This is, for General theory of Relativity (GR), given by the anti-symmetric tensor:

$$J^{ab} = \frac{1}{16\pi}[\nabla^a \xi^b - \nabla^b \xi^a] \, . \tag{6.4}$$

For proceeding further, let us discuss why the above diffeomorphims can be applied in this form of Noether potential. It is very much well known that the Einstein-Hilbert action is invariant under any diffeomorphism $x^a \to x^a + \xi^a$. Therefore, one can obtain a conserved Noether current J^a from the Noether's theorem due to the mentioned diffeomorphism invariance of the action. As J^a is conserved (i.e. $\nabla_a J^a = 0$), one can write J^a as $J^a = \nabla_b J^{ab}$. For the GR case, the Noether potential is given above. The conserved Noether charge is then defined as $Q = \int d\Sigma_a J^a$ over a bulk three-volume transverse to a (usually timelike) congruence. The integral can be further expressed in terms of the surface integral (using Gauss's law) as $Q = (1/2)\oint d\Sigma_{ab} J^{ab}$, where the surface encloses the three-volume and is compact. If there is a horizon in the spacetime, the compact surface in Gauss's law consists of the horizon and the asymptotic infinity. Usually, the horizon part is related to horizon entropy. More precisely, Wald showed that if one calculates this on the horizon for a timelike Killing vector and multiply it by $2\pi/\kappa$ (κ is the surface gravity), this turns out to be the entropy of black hole (see [24, 229]). With this idea we define our charge as $Q = (1/2)\int d\Sigma_{ab}J^{ab}$ where the calculation will be done on the horizon (here it is on the null surface). The important point is that in this definition till now ξ^a is completely arbitrary which is reflected from the fact of the symmetry of the action under any diffeomorphism.

Now, to relate this charge to some physical quantity, one has to choose ξ^a using particular condition. One such condition is that due to the diffeomorphism $x^a \to x^a + \xi^a$, all the metric components are invariant. Then that particular choice of ξ^a is known as the Killing vector. This is usually followed extensively and a subset of all possible allowed diffeomorphism which keeps the action invariant. But there is no hard and fast rule to choose them in this way; it is merely a particular choice of ξ^a. As we explained earlier, in our definition of charge, ξ is completely arbitrary. Keeping this in mind, we provide another choice of them by imposing the condition that the null metric remains null. As explained earlier, we find that for this the sufficient condition is it leaves the $u - r$ sector of the metric invariant. Precise meaning of such condition is– it is the minimum criterion for the location of the null surface being unchanged. Such a choice has been taken earlier and it has interesting connection with horizon entropy [223] and fluid-gravity correspondence [255]. In literature, there is another important choice of boundary condition which keeps the asymptotic form of the metric invariant. This set of boundary conditions gives rise to the superrotation and supertranslation algebra [257]. As already mentioned, the charge is defined for any ξ^a and since there is no unique way to choose this, one has the freedom to find ξ^a by different boundary condition. Of course, the choice should be such that it leads to a meaningful quantity. With this spirit, we have adopted another method to fix the gauge. We choose those particular ξ^a vectors for which only the location of the null surface is unchanged. Moreover this keeps the null surface as null again. We found that our choice is the minimal condition on the metric coefficients which does not violate the nature of the surface at $r = 0$ (it remains null under these boundary conditions). In this sense the present one is weaker condition than the earlier one. This symmetry is very much significant in the context of GR as the obtained diffeomorphism gives rise to the entropy [223] of the null horizon. Therefore, our method of fixing the gauge has no connection with those which results in the superrotation and supetranslation algebra; both are different in nature. As far as the motivation is concerned, here we shall show that some of the gauge degrees of freedom, which raises to the true degrees of freedom due to the horizon preserving conditions and attribute to the entropy. Moreover, the conditions are imposed by the physical reason that the null surface remains null.

Here, only J^{ur} is needed to be computed to calculate the charge which is given as

$$J^{ur} = \frac{1}{16\pi}[\partial_r\xi^r - \partial_u\xi^u + \bar{\beta}^A\partial_A\xi^u + (\partial_r\bar{\alpha} - \bar{\beta}^A\partial_r\bar{\beta}_A)\xi^u] \ . \tag{6.5}$$

Now, if we calculate the potential along ξ^+ and ξ^- with the definition $J^{ur}_{(\pm)} = J^{ur}[\xi^{\pm}]$, the expressions will be given as

$$J^{ur}_{(+)} = -\frac{\partial_u F}{16\pi} \ , \tag{6.6}$$

and

$$J^{ur}_{(-)} = -\frac{1}{16\pi}[\partial_u F + 2\alpha F] + \mathcal{O}(r) \ . \tag{6.7}$$

The calculation of the corresponding charges is very straightforward, which is defined as [151]:

$$Q^{\pm} = (1/2)\int_{\mathcal{H}} d\Sigma_{ab} J^{(ab)}_{(\pm)} \ , \tag{6.8}$$

where $\mathcal{H}$ stands for the fact that the integration has to be evaluated on the null surface $r = 0$ for the metric (6.1).

Let us now comment on one more fact which shall be followed throughout the analysis. We define charge as the surface integral of the Noether potential which has to be evaluated on the mentioned null surface. Now, the metric components α, β and the determinant of μ_{ij} can be expanded in Taylor series as a function of r. For example, $\alpha(r,u,x^A) = \alpha_0(u,x^A) + r\alpha_0'(u,x^A) + \ldots$. So, at the null surface, the leading order terms of these quantities, such as α_0, β_0, μ_0, will contribute. But, for the sake of convenience, we shall keep them as α, β and μ. However, a prudent reader must understand that those are actually the leading order terms when they appear on the charges and later on the brackets of the charges which shall be calculated on the horizon.

Following the definition of the charge (6.8), one can obtain

$$Q^+ = \frac{1}{16\pi}\int \sqrt{\mu} d^{d-2}x^A (\partial_u F) \ , \tag{6.9}$$

and
$$Q^- = \frac{1}{16\pi}\int \sqrt{\mu} d^{d-2}x^A[\partial_u F + 2\alpha F] \ . \tag{6.10}$$

Now we need to obtain the bracket among these charges. In literature there is no unique way to define it. Among various definitions [258, 259], we shall use here the following expression:

$$[Q_1, Q_2] := \int_{\mathcal{H}} d\Sigma_{ab}[\xi_2^a J_1^b - \xi_1^a J_2^b] \tag{6.11}$$

which was obtained in [151]. Here, we have denoted $J_1^a \equiv J^a[\xi_1]$; i.e. J_1^a is calculated for the diffeomorphism ξ_1^a and so on. Subsequently, this has been used in several contexts (see, [108, 152–157, 223] for example). The connection with other ways of defining the bracket has been discussed in [157, 258]. It is obvious from the above that one needs to calculate the components of current J^a which is related to J^{ab} by $J^a_{(\pm)} = \nabla_b J^{ab}_{(\pm)}$. The explicit form of the currents are given as

$$J^r_{(+)} = \frac{1}{16\pi}[\partial_u^2 F + (\partial_u F)\partial_u(\ln\sqrt{\mu})] + \mathcal{O}(r) \ , \tag{6.12}$$

and, similarly,

$$J^r_{(-)} = \frac{1}{16\pi}\Big[\partial_u^2 F + [2\alpha + \partial_u(\ln\sqrt{\mu})]\partial_u F + [2\partial_u\alpha + 2\alpha\partial_u(\ln\sqrt{\mu})]F\Big] + \mathcal{O}(r) \ , \tag{6.13}$$

where, we have used the fact that $J^{rA}_{(\pm)} = \mathcal{O}(r)$. The relevant brackets, calculated by using (6.11), are found to be:

$$\begin{aligned}[Q_1^+, Q_2^+] &= \int d\Sigma_{ur}[\xi_2^u J^r_{1(+)} - \xi_2^r J^u_{1(+)}] - (1 \leftrightarrow 2)\\ &= \frac{1}{16\pi}\int \sqrt{\mu} d^{d-2}x^A[(F_1\partial_u^2 F_2 - F_2\partial_u^2 F_1) + (\partial_u \ln\sqrt{\mu})(F_1\partial_u F_2 - F_2\partial_u F_1)] \ ;\end{aligned} \tag{6.14}$$

$$\begin{aligned}[Q_1^-, Q_2^-] &= \int d\Sigma_{ur}[\xi_2^u J^r_{1(-)} - \xi_2^r J^u_{1(-)}] - (1 \leftrightarrow 2)\\ &= \frac{1}{16\pi}\int \sqrt{\mu} d^{d-2}x^A[(F_1\partial_u^2 F_2 - F_2\partial_u^2 F_1) + (2\alpha + \partial_u \ln\sqrt{\mu})(F_1\partial_u F_2 - F_2\partial_u F_1)] \ ;\end{aligned} \tag{6.15}$$

and

$$[Q_1^+, Q_2^-] = \int d\Sigma_{ur}[\xi_2^u J^r_{1(+)} - \xi_2^r J^u_{1(+)}] - [\xi_1^u J^r_{2(-)} - \xi_1^r J^u_{2(-)}]$$
$$= \frac{1}{16\pi}\int \sqrt{\mu} d^{d-2}x^A[(F_1\partial_u^2 F_2 - F_2\partial_u^2 F_1) + 2\alpha F_1\partial_u F_2$$
$$+(2\partial_u\alpha + 2\alpha\partial_u \ln\sqrt{\mu})F_1F_2] \ . \tag{6.16}$$

Once again, let us remind that α and μ are the leading order contributions of the same quantities in the above equations (6.14), (6.15) and (6.16). In the above three equations, we need not put the value of $J^u_{1(\pm)}$ and $J^u_{2(\pm)}$ as those are multiplied with ξ_2^r and ξ_1^r respectively, which are $\mathcal{O}(r)$. Next we need to find the Fourier modes of the above charges and brackets by using the Fourier decomposition of the function F. In this process one needs to perform the integrations.

As a mindful reader has already noticed, the integrations of the equations (6.9), (6.10), (6.14), (6.15) and (6.16) are not exactly obtainable in the required form as the explicit form of the functions α and μ are not exactly known. But, one is needed to solve those for the sake of obtaining a compact near-horizon algebra. Some earlier works have tried to compute similar integrations under some assumptions without giving the physical explanations of those assumptions. For example, in [223] it has been presumed that α and the transverse metric coefficients μ_{AB} are independent of the coordinate u and a particular transverse coordinate, which does not correspond to any physical situation. However, in our work we shall show that considering the leading order contributions of α and μ as independent of *only one* coordinate u is enough for the purpose. Although it is assumed that μ is independent of u, does not require that all the components of μ_{AB} to be independent of u – only the determinant of the metric is needed to fulfil that criterion. Since we can set the "restricted liberty" on μ_{AB} that the components may be the functions of u, therefore, it can be shown that the horizon, we are interested in, is not a Killing horizon. A more general discussion, whether l^a defines a Killing vector, is tested on the Appendix 6.B. In this sense, the present condition is much weaker than that taken in [223]. Under these weaker restrictions Eqs. (6.14), (6.15) and (6.16) reduce to the following forms:

$$[Q_1^+, Q_2^+] = \frac{1}{16\pi}\int \sqrt{\mu} d^{d-2}x^A[(F_1\partial_u^2 F_2 - F_2\partial_u^2 F_1) \ ; \tag{6.17}$$

$$[Q_1^-, Q_2^-] = \frac{1}{16\pi}\int \sqrt{\mu} d^{d-2}x^A[(F_1\partial_u^2 F_2 - F_2\partial_u^2 F_1) + 2\alpha(F_1\partial_u F_2 - F_2\partial_u F_1)] \ ; \quad (6.18)$$

and, the last one

$$[Q_1^+, Q_2^-] = \frac{1}{16\pi}\int \sqrt{\mu} d^{d-2}x^A[(F_1\partial_u^2 F_2 - F_2\partial_u^2 F_1) + 2\alpha F_1\partial_u F_2] \ . \quad (6.19)$$

Before going into the next step, let us mention the possible meaning of the condition: μ is independent of u. Consider an apparent horizon, which is a marginally trapped surface [224] and, hence, is determined by the conditions of the expansion parameters $\Theta^{(l)} = q^{ab}\nabla_a l_b = 0$ and $\Theta^{(k)} = q^{ab}\nabla_a k_b < 0$, where $q_{ab} = g_{ab} + l_a k_b + l_b k_a$. Now for the present metric (6.1), $\Theta^{(l)} = \partial_u \ln\sqrt{\mu}$ and $\Theta^{(k)} = -\partial_r \ln\sqrt{\mu}$. This implies, near the null surface $r = 0$, $\Theta^{(l)}$ vanishes, provided μ is independent of u. Also one can note that the other condition $\Theta^{(k)} < 0$ is automatically satisfied. In this sense, the condition, μ is independent of u, implies that the near null surface geometry is a particular class of null surface, known as apparent horizon. This was not mentioned in the earlier work [223]. However, the physical implication of assuming α as independent of u is still left as an open problem.

In the following section we shall calculate the Fourier-modes of the charges and the brackets where we shall use these information to get the explicit values of the charges and the brackets.

6.3.3 Algebra of the charges

Now, to calculate the Fourier modes of the charges and the brackets, we first define the Fourier modes of F as

$$F_m = (1/a)\exp[im(au + p_A x^A)] \ , \quad (6.20)$$

where, m and p_A include all the positive and negative integers. Also, the periodicity of the coordinate u has been accounted, given by $R = 2\pi/a$ with a being a constant. Later it will be shown that this periodicity will help us to execute these intractable integrations. The detailed method of solving the integrations of the Fourier modes of the charges and the brackets is described in the Appendix 6.C by taking the leading order term of μ and α as independent of u on the horizon. From (6.9) and (6.10), using the key results of (6.47) and (6.48), we find:

$$Q_m^+ = 0 \ ; \qquad Q_m^- = \frac{C}{2}\delta_{m,0} \ ; \quad (6.21)$$

and (6.17), (6.18), (6.19) yield

$$[Q^+_m, Q^+_n] := 0 \; ; \quad [Q^-_m, Q^-_n] := -iCm\delta_{m+n,0} \; ;$$
$$[Q^+_m, Q^-_n] := -i\frac{C}{2}m\delta_{m+n,0} \; ; \tag{6.22}$$

where $C = \alpha A/4\pi a$ with A being the transverse surface area, as defined in the Appendix 6.C. The expressions of the Fourier modes of the charges and the brackets look exactly similar to the same of the earlier work [157] in the Rindler spacetime. Here we show these are much more general – valid even for a generic null surface in the Gaussian null coordinates.

Now let us discuss the underlying significance of the above results. For that we need go in a new basis. This is similar to the earlier work [157]. We make our choice as follows:

$$P_0 = Q^+_0 + Q^-_0 \, , \qquad P_m = \mathbb{A}Q^+_{-m} + \mathbb{B}Q^-_{-m}(\text{with } m \neq 0) \; ,$$
$$X_m = \mathbb{C}Q^+_m + \mathbb{D}Q^-_m \; , \tag{6.23}$$

where, the coefficients $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ have some freedom to take particular values. Here, we show that for some compulsive choice of these coefficients, we get the non-commutative algebra near the null horizon.

Case 1: For the choice $\mathbb{A} = \mathbb{B} = \pm 1/(Cm)$ and $\mathbb{C} = \mathbb{D} = \mp 1/2$ one gets the anti-commutation algebra between X_n and P_n, which is given as follows.

$$[X_m, X_n] = -\frac{i(m-n)C}{4}\delta_{m+n,0} \; ;$$
$$[P_m, P_n] = \frac{4i}{(m-n)C}\delta_{m+n,0} \; ; \qquad [X_m, P_n] = i\delta_{m,n} \; . \tag{6.24}$$

Case 2: If one chooses $\mathbb{A} = -1/(Cm) \pm (\sqrt{1+1/m^2})/C - (1/m \pm \sqrt{1+1/m^2})$, $\mathbb{B} = 1/m \pm \sqrt{1+1/m^2}$, $\mathbb{C} = -(1+1/C)$ and $\mathbb{D} = 1$, one gets

$$[X_m, X_n] = \frac{i}{2}(m-n)\delta_{m+n,0} = [P_m, P_n] \; ;$$
$$[X_m, P_n] = i\delta_{m,n} \; . \tag{6.25}$$

Case 3: Another choice is being made as follows. $\mathbb{A} = -2/m$, $\mathbb{B} = 2/m$, $\mathbb{C} =$

$-(1+1/C)$ and $\mathbb{D}=1$. Then the brackets are

$$[X_m, X_n] = \frac{i}{2}(m-n)\delta_{m+n,0} \ ; \quad [P_m, P_n] = 0 \ ;$$
$$[X_m, P_n] = i\delta_{m,n} \ . \tag{6.26}$$

Case 4: Lastly, we show that another choice is possible which is as follows. $\mathbb{A} = 2/(mC) - m/2$, $\mathbb{B} = m/2$, $\mathbb{C} = 1$ and $\mathbb{D} = -1$. Then

$$[X_m, X_n] = 0 \ ; \quad [P_m, P_n] = \frac{i}{2}(m-n)\delta_{m+n,0} \ ;$$
$$[X_m, P_n] = i\delta_{m,n} \ . \tag{6.27}$$

These anti-commutation relations near the null-horizon, shown in the above equations (6.24), (6.25), (6.26) and (6.27), are the key results of our analysis. Note, the same was obtained earlier for the Rindler metric [157] which is a subclass of (6.1) for specific constraint on these metric coefficients. As was for the prior case, the results shown above illustrates the "restricted" non-commutative algebra since the non-commutativity prevails only for the condition $m+n=0$ and not for any arbitrary m or n.

Before going to the next stage, let us now discuss the necessity of the present analysis. Rindler frame is adopted by an uniformly accelerated observer on a Minkowski spacetime. Therefore, even if the metric coefficients are coordinate dependent, the spacetime is still inherently flat. Note, in that case (Rindler), only metric coefficients corresponding to $(u-r)$ sector depend on coordinates while the transverse ones are constant as this sector is a flat plane. Although, equivalence principle accounts the role of accelerated frame in exploring gravity; but in some situation all features of the true curved spacetimes do not emerge from this simple model. For example, at the classical level the motion of a particle is non-trivially effected in presence of curved background and also in the case of Hawking radiation, which is a quantum phenomenon, the emitted spectrum is modified by the grey body factor. On the other hand, in Rindler spacetime, one does not account these – the emitted spectrum is purely Planckian in nature. These suggest that one needs to incorporate the true curvature in spacetime metric to know more about gravity.

It has been a general belief that the local nature of gravity can be always interesting. Now, locally an observer can adopt a null surface, metric coefficients of which is spacetime dependent in a general scenario. Considering such situation, one can write a d dimensional metric (6.1) in coordinates adapted to this null surface.

In this case, only one restriction is that, on the null surface, the induced metric is only the $(d-2)$ dimensional transverse surface. These indicate that the metric coefficients are not necessarily restricted to a Rindler one. Accelerated (or Rindler) frames can be regarded as a subset of particular kind. Another example can be more illuminating. The near horizon structure of a Kerr black hole, is not exactly a Rindler one [260]. It has a form similar to our present metric (6.1) with some more restrictions on the coefficients. Therefore, any feature related to Rindler metric is not guaranteed by other null cases and, hence, those must be tested in order to find the generality of them. Therefore, it is necessary to investigate the validity of the earlier result of [157] in the case of GNC.

In our present analysis, we have taken the most general metric (in GNC) for a null surface (see [233, 234] for the construction of the metric) whose metric coefficients are functions of coordinates. Moreover, the metric coefficients can depend on timelike coordinates as well. This is the crucial difference between the Rindler form and our present form. In that sense our situation is much more general. As we have shown that one can formulate the non-commutative Heisenberg algebra for the metric in GNC, the result will hold for any null surface– including the horizon of a Kerr black hole or any other Rindler horizon as well. This indicates that the obtained property has a wide generality.

Here, we have made a particular type of linear combination of our primary charges (see Eq. (6.23)). The primary goal is to get an algebra between X_m and P_m such that they satisfy the Heisenberg algebra *i.e.* $[X_m, P_n] = i\delta_{m,n}$. Motivation for such aim is the following. In quantum mechanics, position and the corresponding momentum do not commute which plays a huge role to get a quantum description of any system. Since, till date, no successful quantum description of horizon exists in literature, it may be worthy to think in this direction. Therefore we demanded such algebra as a starting point. Now given this, we found that there are four possible choices of the coefficients $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$. In all cases, the algebra is not the usual one. We have obtained three types of anti-commutativity. Case 1 and case 2 represents the anti-commutativity in both X_m and P_m. Case 3 represents the anti-commutativity in X_m only, whereas the P_m commutes. Lastly, case 4 represents the anti-commutativity in P_m only, where X_m commutes. In this respect the choices are not arbitrary.

Let us now understand why we have decomposed ξ^a in terms of specific ξ^+ and ξ^- vectors. The reason is the following. Remember, the spacetime (6.1) represents a generic null surface at $r = 0$. Interestingly, the same surface is also represented by

$u =$ constant. So on the null surface, we have two normals: one is along r direction and other one is along u direction. Both of them are null vectors on this surface. This is consistent with the general feature of null surface: one can always find an auxiliary null vector corresponding to the null vector which defines the location of the null surface. Hence, in the present situation, we can choose a null-null basis to analyze the properties of the surface. Here one is along r direction and other one is along u direction and any tensorial quantity can be decomposed in these basis. This is the main motivation of our present decomposition of ξ^a vector as null-null basis is quite natural to examine the behavior of null surfaces. We are interested to see the algebra of the charges which are defined along these two directions (*i.e.* along ξ^+ and ξ^-). Moreover, as we shall address in the conclusion, individually ξ^+ and ξ^- preserve the null surface structure like ξ^a. In this sense, the division of ξ^a along ξ^+ and ξ^- is unique.

However, one can choose other basis like, $\xi^1 = (b\xi^u, a\xi^r, \mathbf{0})$ and $\xi^2 = (d\xi^u, c\xi^r, \mathbf{0})$ with $a + c = 1$ and $b + d = 1$. In that case the algebra of the charges along these directions can be derived from (6.22). The new charges in terms of the earlier ones are

$$Q^1_m = aQ^+_m + bQ^-_m \ ;$$
$$Q^2_m = cQ^+_m + dQ^-_m \ . \tag{6.28}$$

So using (6.22), one finds the algebra among the above charges as

$$\begin{aligned}
\Big[Q^1_m, Q^1_n\Big] &= \Big(- ab - b^2\Big) iCm\delta_{m+n,0} \ ; \\
\Big[Q^1_m, Q^2_n\Big] &= \Big(- \frac{a}{2} + ab - \frac{3b}{2} + b^2\Big) iCm\delta_{m+n,0} \ ; \\
\Big[Q^2_m, Q^2_n\Big] &= \Big(- 2 + 3b + a - ab - b^2\Big) iCm\delta_{m+n,0} \ .
\end{aligned} \tag{6.29}$$

Note that the above will reduce to (6.22), for $a = 1$ and $b = 0$. Here we discuss for different values of a, b; i.e. the basis can be different from r and u directions. In this case again one can show that the combination like (6.23) satisfies similar non-commutative Heisenberg algebra in the following manner.

We define,

$$
\begin{aligned}
P_0 &= Q_0^1 + Q_0^2 = Q_0^+ + Q_0^- \ , \\
P_m &= \tilde{A} Q_{-m}^1 + \tilde{B} Q_{-m}^2 = \mathbb{A} Q_{-m}^+ + \mathbb{B} Q_{-m}^- \ , \\
X_m &= \tilde{C} Q_m^1 + \tilde{D} Q_m^2 = \mathbb{C} Q_m^+ + \mathbb{D} Q_m^- \ .
\end{aligned} \tag{6.30}
$$

In this case,

$$
\begin{aligned}
\mathbb{A} &= \tilde{A} a + \tilde{B} c \ , \qquad \mathbb{B} = \tilde{A} b + \tilde{B} d \ , \\
\mathbb{C} &= \tilde{C} a + \tilde{D} c \ , \qquad \mathbb{D} = \tilde{C} b + \tilde{D} d \ .
\end{aligned} \tag{6.31}
$$

The same anti-commutative Heisenberg algebra between X_m and P_m can be obtained in this case as well for the four sets of choices of $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ once one demands, like earlier, $[X_m, P_n] = i\delta_{m,n}$. The coefficients $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ can be obtained by solving (6.31), which is given as

$$
\begin{aligned}
\tilde{A} &= \frac{\mathbb{A} d - \mathbb{B} c}{ad - bc} \ , \qquad \tilde{B} = \frac{\mathbb{A} b - \mathbb{B} a}{bc - ad} \ , \\
\tilde{C} &= \frac{\mathbb{C} d - \mathbb{D} c}{ad - bc} \ , \qquad \tilde{D} = \frac{\mathbb{D} a - \mathbb{C} b}{ad - bc} \ .
\end{aligned} \tag{6.32}
$$

Thus, similar to Case 1, if we want to obtain the same algebra between X_m and P_m as given in (6.24), the choices of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ will be:

$$
\begin{aligned}
\tilde{A} &= \pm \frac{1}{mC} \frac{c-d}{bc-ad} \ , \qquad \tilde{B} = \pm \frac{1}{mC} \frac{a-b}{ad-bc} \ , \\
\tilde{C} &= \mp \frac{1}{2} \frac{c-d}{bc-ad} \ , \qquad \tilde{D} = \mp \frac{1}{2} \frac{a-b}{ad-bc} \ .
\end{aligned} \tag{6.33}
$$

To obtain the algebra between X_m and P_m as given in (6.25) (Case 2), the choices of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ will be:

$$
\begin{aligned}
\tilde{A} &= \frac{-\frac{1}{m}(\frac{d}{C} + c + d) \pm (\sqrt{1 + \frac{1}{m^2}})(\frac{d}{C} - c - d)}{ad - bc} \ , \\
\tilde{B} &= \frac{-\frac{1}{m}(\frac{b}{C} + a + b) \pm (\sqrt{1 + \frac{1}{m^2}})(\frac{b}{C} - a - b)}{bc - ad} \ , \\
\tilde{C} &= -\frac{c + d + \frac{d}{C}}{ad - bc} \ , \qquad \tilde{D} = \frac{a + b + \frac{b}{C}}{ad - bc} \ .
\end{aligned} \tag{6.34}
$$

Again, similar to case 3, we obtain the algebra between X_m and P_m as given in

(6.26) for the following choices of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$.

$$\tilde{A} = -\frac{2}{m}\frac{c+d}{ad-bc}~, \quad \tilde{B} = -\frac{2}{m}\frac{a+b}{bc-ad}~,$$
$$\tilde{C} = -\frac{c+d+\frac{d}{C}}{ad-bc}~, \quad \tilde{D} = \frac{a+b+\frac{b}{C}}{ad-bc}~. \tag{6.35}$$

Lastly, similar to case 4, we obtain the algebra between X_m and P_m as given in (6.27) for the following choices of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$.

$$\tilde{A} = \frac{\frac{2d}{mC} - \frac{m}{2}(c+d)}{ad-bc}~, \quad \tilde{B} = \frac{\frac{2b}{mC} - \frac{m}{2}(a+b)}{bc-ad}~,$$
$$\tilde{C} = \frac{c+d}{ad-bc}~, \quad \tilde{D} = -\frac{a+b}{ad-bc}~. \tag{6.36}$$

Note, for $a = 1$ and $b = 0$, we obtain $\tilde{A} = \mathbb{A}$, $\tilde{B} = \mathbb{B}$, $\tilde{C} = \mathbb{C}$ and $\tilde{D} = \mathbb{D}$ in each case which corresponds to our earlier result. This analysis indicates the results are indeed basis independent.

So far, to define X_m and P_m, we have taken the linear combinations of the Fourier modes of the charges. Let us now take the combination of the charges according to the Sugawara construction [237] to define the new generators in the following manner:

$$J^{\pm}_m = \frac{1}{2C}\sum_p Q^{\pm}_{m-p}Q^{\pm}_p + imQ^{\pm}_m~. \tag{6.37}$$

Below, we get the interesting algebra of the newly defined generators as follows

$$[J^+_m, J^+_n] = 0~;$$
$$i[J^-_m, J^-_n] = (m-n)J^-_{m+n} + m^3 C\delta_{m+n,0}~;$$
$$[J^+_m, J^-_n] = \frac{m^2}{2}Q^-_{m+n} - i\frac{m^3 C}{2}\delta_{m+n,0}~; \tag{6.38}$$

The second relation in the above defines the Virasoro algebra, with the central charge($\tilde{C}$) is defined as $\tilde{C} = 12C$. Also, note that in this case $J^-_0 = (1/2C)(Q^-_0)^2$ and, hence, we define the entropy of the null surface as $S = (2\pi a/\alpha)Q^-_0 = (2\pi a/\alpha)\sqrt{2CJ^-_0}$. Therefore one can write the entropy in terms of the Cardy "like" formula as

$$S = 2\pi\sqrt{\frac{\tilde{C}J^-_0 a^2}{6\alpha^2}}~. \tag{6.39}$$

Thus, the obtained results for the Rindler horizon of [157] can be further extended for a more general null horizon. Moreover, since the null surface is a local concept, the above expression for entropy can be described as a *local* form of Cary formula.

In the original work of Cardy [238], the relation between the entropy and the Virasoro algebra via Cardy's formula was obtained due to the conformal symmetry in $(1+1)$ dimensional Minkowski spacetime. The finding of Virasoro like algebra in the case of black holes in the near horizon regime [150, 261] may be a deeper fact. In Cardy's work things happen at the quantum level while in gravity this is obtained at the classical computation. Therefore this similarity is not so obvious. Usually, application of the standard Cardy formula leads to the entropy which matches with that of horizon. Therefore, the validity of Cardy formula in curved spacetime although not clear but it works well.

In the present case, we adopted a reverse approach. Here the standard expression of the entropy has been taken into account. Then considering the algebra among the charges for null case, the central charge has been identified. It has been observed that combination of these information leads to the Cardy like expression (6.39). This we call a local version of Cardy formula for the following reasons. First of all, the choice of the null metric is a local concept as locally an observer can always perceive a null surface adapted to its frame. Secondly, the boundary conditions, which have been imposed on the metric coefficients, are local in nature as they are satisfied near the null surface. In addition, the validity of the local version of Cardy formula is may be due to the fact that one can always obtain a locally inertial frame even in the curved manifold. A similar expression of Cardy's formula, in the context of a null surface, has also been obtained in [223] as well in a different method. However, both the results (ours and of [223]) imply the astonishing robustness of Cardy's formula which was not predicted earlier.

Let us now comment on one important difference of our work from that of Carlip [150]. Although both ideas are equivalent, there is a subtle difference in the two approaches. In Carlip's approach, the algebra of the charge is obtained due to the asymptotic Killing symmetry of the Killing horizon. In that case, the diffeomorphism vector is an asymptotic Killing vector near the horizon for the *full* spacetime and, consequently, the different Fourier modes of the same charge lead to the Virasoro algebra:

$$i[Q_m, Q_n] = (m-n)Q_{m+n} + \frac{C}{12}m^3\delta_{m+n,0} \ , \tag{6.40}$$

where C is known as the central charge, which is connected to the entropy by the Cardy's formula. On the other hand, our present null surface is not necessarily a

Killing horizon and the diffeomorphism vectors are chosen to asymptotic Killing for only $(u-r)$ sector. Hence these are not Killing for the full spacetime. Consequently, the algebra of the corresponding charges are not central extended Virasoro algebra. Rather we find that a non-linear combination of them satisfies Virasoro algebra. Moreover, we study the algebra of two different charges Q^+ and Q^-, which are defined along radial and u directions both of which correspond to null vectors on the null surface.

Remember that the original Virasoro algebra is the bracket among the charges of fields corresponding to the conformal symmetry in $(1+1)$ dimensional Minkowski spacetime. It comes after the quantization of the fields on this spacetime which also acquires this conformal symmetry. So such an algebra is a pure consequence of quantum field theory. On the contrary, in gravity similar type of algebra is purely classical result. In our present case, the specific combination X_m and P_m of the base charges Q^+_m and Q^-_m leads to the interesting (anti-commutative) Heisenberg algebra. As it is well known that the commutation relation $[X, Y] = i\hbar$ originally comes from quantum mechanics. This is due to the inherent uncertainty in measurement of two observables X and Y simultaneously and it acts as a building block for the quantum nature of a system. So it is evident that both Virasoro and Heisenberg algebra are related to quantum nature of a system while their structure is different from each other.

6.4 Conclusions and outlook

The essence of any quantum theory is the commutation relations among the associate quantities present in the theory. Unfortunately, as mentioned earlier in numerous occasions, the quantum theory of gravity is yet to be developed in a consistent manner. Though the recently developed theoretical frameworks, namely the string theory and the loop quantum gravity, is currently trying to formulate a persuasive theory in the realm of quantum gravity, a convincing theory is not developed yet. In this circumstance, people are trying to find out different ways which can shed some light on the quantum nature of gravity. We want to mention that some similar works [244–253] are going on along the same route of this work where the bracket algebra (formulated in the same spirit as of the commutation relation in quantum mechanics) between the conserved charges are being studied. Earlier, the similar non-commutative algebra on the neighborhood of the null surface in the Rindler background. Therefore, this work is the generalization to the earlier one which justifies the acceptability of the previous results in a much more general scenario. However, as shown in our analysis, when the spacetime is described using GNC, lots of complexities emerges in the picture.

Here, we have imposed the boundary conditions of the metric perturbation and have identified the set of diffeomorphism vectors ξ^i that preserves the null horizon structure. After that, we have computed the conserved charges along the components of the diffeomorphism vector and also have defined the brackets of the charges. Later on, the Fourier modes of the charges and brackets have been computed. On that course, we have mentioned the challenges that one faces to get a compact algebra and have mentioned the procedure to overcome it. Also, it has been clarified that the surface, which we are dealing with, is not a Killing surface even with our assumptions. As one can easily point out, the algebra shown by the brackets of the charges is insignificant. Later, we were keen to know, whether these algebras show any physical importance in different basis of the charges. Interestingly, the answer we have found is *"Yes!"*. As we have shown, one can define four different basis of charges for which one gets noncommutative Heisenberg algebra. However, the noncommutativity exists only for some particular choices of the Fourier modes. Therefore, one can say the algebra can be categorized as a *"restricted"* class of noncommutative Heisenberg algebra. Thereafter, we have defined Sugawara like construction of the conserved charges and it has been shown that one class of the generators of charges satisfy the Virasoro algebra and the entropy of the surface can be determined from the zeroth mode of charges using the Cardy formula. It reveals the thermodynamic connection of the conserved charges which arise due to the symmetry conditions of the horizon configuration.

As we said, the charges Q^+ and Q^- are defined in this work corresponds to ξ^+ and ξ^-, respectively. Now if we consider only ξ^+, it changes our metric in the following way: $\delta_{\xi^+}g_{ur} = -\partial_u F$, $\delta_{\xi^+}g_{rr} = 0$ and $\delta_{\xi^+}g_{uu} = -2r\partial_u^2 F - r(\partial_r\bar{\alpha})(\partial_u F) = \mathcal{O}(r)$; while ξ^- changes as $\delta_{\xi^-}g_{ur} = \partial_u F$, $\delta_{\xi^-}g_{rr} = 0$ and $\delta_{\xi^-}g_{uu} = 2\bar{\alpha}\partial_u F + (\partial_u\bar{\alpha})F = \mathcal{O}(r)$. This shows that the location of the null surface does not change under any one of the diffeomorphism vectors as the null surface is determined by the condition $l^2 = g_{ab}l^a l^b = g_{uu} = 0$ (note, here only u component of l^a is non-zero) and for both cases the leading order correction to δg_{uu} is $\mathcal{O}(r)$. It implies the individual vector also respects our original imposed condition to find ξ^u and ξ^r.

As mentioned earlier, the study of these algebras on a null surface might enlighten the obscure quantum nature of gravity as the noncommutativity between the charges (the origin of which is solely determined by the configuration of the null surface) is apparent. It is true that we cannot exactly say whether these quantities, manifesting the noncommutativity, correspond to the physical observables. Nevertheless, our analysis provides the glimpse on the quantum nature that arises due to the configuration of the spacetime. Moreover, the null surface is a local concept and, in general, is not the solution of any equation of motion of the spacetime. Also, it shows all the thermodynamic features. Therefore, in the context of emergent gravity paradigm, where gravity is considered as the outcome of more fundamental interactions (i.e., thermodynamics), the null surface is highly regarded. Hence, this analysis is important in the theory of emergent gravity as well.

Let us now comment on the connections and the comparisons with the contemporary works. As mentioned earlier, this type of algebra has also been obtained in string theory [262]. In that case, the spacetime is itself noncommutative. Moreover, the works which are alike to this one [244–253], show similar results but, those analysis are confined to three dimensions and the algebra which they show is the usual Heisenberg one. On the contrary, our analysis is valid in any dimensions of the spacetime. Moreover, our algebra casts more light on the quantum nature of the surface as it is noncommutative.

Some open issues are needed to be taken care. In our calculation, the assumption is considered to be as α and μ are independent of coordinate u. Although it appears to be "OK" as we have shown that this does not lead to $r = 0$ null surface as Killing horizon. In that sense, the present analysis reflects the properties of a generic null surface. But it would be interesting to see if such happens without invoking any assumption as input. Another point has to be noted is that the main results in this chapter depends on the specific choice of the Fourier modes of the unknown function F, given by (6.20), seems to be "ad-hoc" as there is no such, in general, clear periodicity in the coordinates for the metric (6.1). But what is important is that such a choice works well. So one has to look for the justification for this choice. More importantly, we need to see if this is an unique one or there is other choice. It must be mentioned that the same has also been adopted earlier. To tell more on this, in our work, we have taken the periodicity of the u coordinate which is essential for obtaining the compact values of the Fourier charges and brackets. The time coordinate in a Lorentzian manifold, in general, has a periodicity in the Euclidean space. The Schwarzschild metric or even the Rindler metric can be the examples for that. The idea of utilizing the periodicity of the time coordinate in the Fourier mode of the generating function (in our case F_m) was followed from the earlier work [258] and was used in several works [151–156], which was later followed by many other works [108, 223, 263–265]. Recently, the same idea has been used for the same metric in GNC as well [223] while obtaining the Virasoro algebra, where the periodicity of u has been used. So, our assumption of periodicity in the u coordinate is not a bizarre or unfamiliar one. This is widely used in several works in this line. In our case (or in [223]), the periodicity in u is imposed and it is taken to be the same of retarded time coordinate in the Rindler metric. It is because under the certain assumptions in the near null limit, the metric in GNC resembles to the Rindler one. Periodicity of Euclidean timelike coordinate is common in the discussion of thermodynamics of horizon [266]. Since we are also interested to thermodynamics, it is quite natural to consider such input in this case as well. Such an analysis is semi-classical in nature. Therefore, it can be considered as a mere limitation of the analysis and certainly not as a drawback. Moreover, as mentioned earlier, people find it reasonable to go with this limitation and we too found the same in our work as well. Our intention here is to explore the more deeper meaning of the imposed boundary condition within the existing approaches. But certainly, there is option to improve this.

Our proposal – presence of *noncommutative Heisenberg hair* on the horizon – in this and in the previous work [157] implicates that the surface structure the horizon is much "richer" than what was predicted by the earlier works. The whole result is general and can be fitted in any theory of gravity.

Appendix

6.A Obtaining ξ^a in Eq. (6.2)

Let us consider the first condition $\pounds_\xi g_{rr} = 0$, which implies $\nabla_r \xi_r = 0$. Hence

$$g_{ri}\nabla_r \xi^i = 0 \ . \tag{6.41}$$

One can write (6.41) further as

$$\partial_r \xi^u + \Gamma^u_{rr}\xi^r + \Gamma^u_{ur}\xi^u = 0 \ . \tag{6.42}$$

Since $\Gamma^u_{rr} = \Gamma^u_{ur} = 0$ (see [234]), one can obtain the expression of ξ^u given in the Eq. (6.2).

We now take the second condition, $\pounds_\xi g_{ru} = 0$. It implies $g_{ui}\nabla_r \xi^i + g_{ri}\nabla_u \xi^i = 0$, which further results in

$$g_{uu}[\partial_r \xi^u + \Gamma^u_{ri}\xi^i] + g_{ur}[\partial_r \xi^r + \Gamma^r_{ri}\xi^i] + g_{ru}[\partial_u \xi^u + \Gamma^u_{ui}\xi^i] + g_{uA}[\partial_r \xi^A + \Gamma^A_{ri}\xi^i] = 0 \ . \tag{6.43}$$

If one substitutes the values of ξ^u and of all Γ's from [234], the final expression of (6.43) reduces to $\partial_r \xi^r + \partial_u F = 0$, from which one can obtain the expression of ξ^r given in Eq. (6.2).

6.B Killing horizon?

In our analysis, we have taken α and μ being independent of the coordinate u only on the null surface, as the leading order contribution of these quantities are considered to be independent of u. Besides, the null surface $r = 0$ in our case is defined by the vanishing of the norm of the vector $l^a = (\partial/\partial u)^a$. So the natural question comes in one's mind: Do the imposed conditions lead to the fact that the null surface, in our analysis, is a Killing horizon? However, we shall show that l^a does not define a Killing horizon.

For our present metric (6.1), one can check that

$$\pounds_l g_{ab} = \partial_u g_{ab} \ . \tag{6.44}$$

At the first glance, one might be tempted to remark that our assumption, that the leading order contribution of α and μ are independent of u, might lead to the fact that l^a is a Killing vector on the null surface. However, this is not the case. It must be noted that not all the metric components are needed to be independent of u and therefore right hand side of the above does not vanish for all values of indices a and b. Let us explain the reason little elaborately. As it has been mentioned earlier, we have taken the determinant of transverse metric μ being independent of u; not all the components of μ_{AB}. In this situation one can analytically verify that μ can be independent of u even though μ_{AB} depends on it. Secondly, l^a does not satisfy the Frobenius theorem as it is not hypersurface orthogonal all over the spacetime. For this reasons, we can say that our assumptions do not imply the null surface, defined by $l^2 = 0$, to be a Killing one.

One might be interested to seek if there is any other possible Killing vector, satisfying the Frobenius theorem, whose vanishing norm defines our $r = 0$ null horizon as a Killing one. The answer will be "No". We have taken α and μ to be independent of u only and not of any other coordinate. Therefore, we cannot imagine that the Killing vector to be a linear sum of two vectors (as is the case of a Kerr black hole, where the metric admits two Killing vector, and the horizon surface is determined by the linear combination of the two Killing vectors). We have checked that $l^a = (\partial/\partial u)^a$ is not a Killing vector and it does not define any Killing horizon. Therefore, any vector proportional to l^a will not define a Killing horizon as well.

To sum up, the null surface is not assumed to be a Killing one when the assumption – α and μ are independent of u – is taken. In that sense the whole analysis predicts the properties of a generic null surface.

6.C To prove (6.21) and (6.22)

The Fourier modes of the integrations, (6.9), (6.10), (6.17), (6.18) and (6.19), which are needed to be evaluated to obtain the Fourier modes of the charges and of the brackets in the desired form, are highly daunting task as the functional form of α and μ are not exactly known. In an earlier attempt [223], as we have mentioned in our main analysis, assumed that α and the transverse metric coefficients μ_{AB} are independent of u and a particular transverse coordinate. Here, we find that a much weaker restriction can be imposed. It will be shown that taking the leading order terms of α and μ (not the transverse metric coefficients) as independent of *only u coordinate is enough* to get the near horizon algebra. In our case, the periodicity condition of u helps to get the desired results even if the functional form of α and μ are not known. We have also found out the physical interpretation of μ being independent of u which is mentioned earlier. In the next Appendix, it will be shown that this assumption does not imply a Killing horizon near to the null surface.

Now, while calculating the Fourier modes from the Eqs. (6.9), (6.10), (6.17), (6.18) and (6.19) with the Fourier mode of F as mentioned earlier, we mainly encounter two integrations, i.e., $I_1 \sim \int \sqrt{\mu} d^{d-2}x^A e^{i(m+n)(au+p_A x^A)}$ and $I_2 \sim \int \sqrt{\mu} d^{d-2}x^A \alpha e^{i(m+n)(au+p_A x^A)}$. To solve these two integrations, we have used the periodicity condition on the coordinate u with the fact that the periodicity is independent of any coordinate. To get the value of the integration I_1, let us start with the fact that

$$\int_0^R du \int d^{d-2}x^A \sqrt{\mu} e^{i(m+n)(au+p_A x^A)} = R \int \sqrt{\mu} d^{d-2}x^A \delta_{m+n,0} \,. \tag{6.45}$$

In the above we have used the result $\int_0^R du e^{i(m+n)au} = R\delta_{m+n,0}$ and $R = 2\pi/a$. The next step is to take derivative with respect to R on the both sides of (6.45). From the left hand side we get

$$\begin{aligned}\int \sqrt{\mu} d^{d-2}x^A \frac{\partial}{\partial R} \int_0^R du e^{i(m+n)(au+p_A x^A)} &= \int \sqrt{\mu} d^{d-2}x^A e^{i(m+n)(aR+p_A x^A)} \\ &= \int \sqrt{\mu} d^{d-2}x^A e^{i(m+n)p_A x^A} \,, \end{aligned} \tag{6.46}$$

where, in the second equality $\frac{\partial}{\partial b}[\int_a^b f(x)dx] = f(b)$ has been used and the final result is obtained after substitution of $R = 2\pi/a$. Whereas, the derivative with respect to R on the right hand side of (6.45) gives $\int \sqrt{\mu} d^{d-2}x^A \delta_{m+n,0}$. This implies

$$\int \sqrt{\mu} d^{d-2}x^A e^{i(m+n)(au+p_A x^A)} = A\delta_{m+n,0} \tag{6.47}$$

where $A = \int \sqrt{\mu} d^{d-2}x^A$ is the transverse surface area.

The second integration I_2 can be performed following the previous method of I_1 and considering α as independent of u while performing the integration of u. One can similarly obtain

$$\int \sqrt{\mu} d^{d-2}x^A \alpha e^{i(m+n)(au+p_A x^A)} = \alpha A \delta_{m+n,0} \ . \tag{6.48}$$

Using the results of (6.47) and (6.48), the Fourier modes of the charges and the brackets of the charges can be obtained which are given in (6.21) and (6.22).

Part II

van der Waals type phase transition of black holes: a general framework

Chapter 7

Thermogeometrical study of van der Waals type phase transition in black hole

7.1 Phase transitions in black hole thermodynamics

[1]In the earlier chapters, we have discussed several features which are present in black hole thermodynamics. Black hole phase transition is another phenomenon which strengthens the analogy of GR with thermodynamics. In the present and in the following chapters we shall discuss phase transition in black hole thermodynamics.

There are several types of black hole phase transitions which have been found in black hole thermodynamics. One among those phase transitions was introduced firstly by Davies [34] and subsequently followed by many other researchers [267–270]. Davies suggested that a black hole goes through a second order phase transition when it passes through a point (Davies' point) where the heat capacity becomes infinitely discontinuous. Another group argued that a second order phase transition indeed takes place when a non-extremal black hole transforms to an extremal one and the extremal limit was identified as a critical point [38, 40–47]. Recently, a general framework to study the extremal phase transition has been provided in [48]. In addition Hawking and Page [39] have found a first order phase transition, known as the Hawking-Page (HP) phase transition. HP transition states that a black hole solution in AdS space makes a transition to a no-black hole solution (or radiation) at a critical temperature.

There is another line of studying the black hole phase transition. In the AdS

[1]This chapter is based on the publication [109] .

space, when the cosmological constant is regarded as the thermodynamic pressure [51–55], it was found that the $P - V$ diagram of black holes in the extended phase space looks similar to the $P - V$ diagram of the van der Waals fluid system in conventional thermodynamics [56, 57], which implies a van der Waals phase transition in the black hole thermodynamics. Interestingly, for black holes, the van dar Waals phase transition can be found not only in the phase space of P and V, but also can be found in the phase space diagram of T vs. S [58, 59, 271–275] and Q vs. Φ [60] as well.

In the present and the following chapter, we shall present a general framework to study all these types of van der Waals phase transition in black hole thermodynamics. Several critical phenomena (especially the Davies' type) has been studied extensively under the geometrothermodynamics (GTD) [69–75] , by which a thermogeometric description of criticality has been provided [271–273, 276–280]. We provide a detailed review of GTD later in our discussion. In this chapter, we firstly obtain the thermodynamic first law in the extended phase space, obtained by considering the cosmological constant as the thermodynamic pressure. Then we adopt the geometrothermodynamics (GTD) formalism to provide the thermogeometrical description to all van der Waals type phase transition present in black hole thermodynamics. The analysis in this chapter is valid for all the spacetimes which shows the van der Waals criticality.

7.2 Extended phase space and thermodynamic first law using Wald's formalism

The presence of $P - V$ criticality in black hole spacetime was demanded earlier in [49, 50] where P was identified as the inverse of the Hawking temperature ($i.e.$ $1/T_H$) and V as the horizon radius ($i.e.$ r_H). This analysis mainly relies on the similarity of the graph of the $1/T_H$ and the graph of P in van der Waals phase transition. Although the analysis apparently is intriguing, its main limitation was the lack of a proper canonical definition of P and V variables.

Later on, the when the cosmological constant (Λ) was taken as a variable for the BH's in AdS spacetime, the extended version of first law [51–55] was obtained and the canonical definition of P and V was provided where, the pressure P was identified as $-\Lambda/8\pi$ and its conjugate quantity is interpreted as the thermodynamic volume (denoted by V). Later, it was found [56, 57, 110, 281] that this identification is consistent throughout in BH thermodynamics and with this identification (of pressure and volume in BH thermodynamics in extended phase space) it was found that the phase space diagram (i.e. the $P-V$ diagram), the $P-T$ diagram, the equations of state, the behavior of the Gibbs free energy– all are very similar to the van der Waals fluid system in ordinary thermodynamics. In addition, the van der Waals criticality was also found, where the two independent conditions coincide $i.e.$ $(\partial P/\partial V)_{T,Y_i} = 0 = (\partial^2 P/\partial V^2)_{T,Y_i}$. On the following, we mention the procedure of obtaining thermodynamic first law using Wald's approach in the extended phase space, taking Λ as a variable.

The first law in the extended phase space is not exactly similar to the non-extended phase space. Here we shall see that M, the mass of the black hole, is interpreted as the enthalpy of the system instead of the internal energy. In addition, defining thermodynamic volume is not very straightforward as we shall see here. For the simplicity of the calculation we shall not incorporate the charge in the proof of the first law. Since we shall follow Wald's formalism, at first, we have to properly identify the Noether current and potential due to the diffeomorphism invariance for the present case. At the operational level of obtaining the Nother current and potential, we shall follow the off-shell formulation. After obtaining the Noether current and potential, Wald's approach is applied on-shell, as normally people do.

The action of a AdS black hole with cosmological constant is given as (in GR)

$$A = \int d^4x\sqrt{-g}L = \frac{1}{16\pi}\int \sqrt{-g}(R-2\Lambda). \tag{7.1}$$

To obtain the equation of motion and the boundary term one has to take the variation of the total action. Here it leads to the result

$$\begin{aligned}\delta(\sqrt{-g}L) = \frac{1}{16\pi}[&\sqrt{-g}(G_{ab}+\Lambda g_{ab})\delta g^{ab} \\ &+\sqrt{-g}\nabla_a\delta v^a - 2\sqrt{-g}\delta\Lambda],\end{aligned} \tag{7.2}$$

where $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ is the well known Einstein tensor and $\delta v^a = 2P^{ibad}\nabla_b(\delta g_{id})$ with $P^{abcd} = \partial R/\partial R_{abcd} = \frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc})$. Here, we have considered the fact that Λ is also a variable. For the diffeomorphism symmetry $x^a \to x^a + \xi^a$ the operation of δ is replaced by the Lie-derivative $\pounds_\xi$, with $\pounds_\xi g^{ab} = -(\nabla^a\xi^b + \nabla^b\xi^a)$. Using the Bianchi identity $\nabla_a G^a_b = 0$, one can obtain

$$\pounds_\xi(\sqrt{-g}L) = \frac{\sqrt{-g}}{16\pi}[-2\nabla_a(G^a_b\xi^b) + \nabla_a\pounds_\xi v^a - 2\nabla_a(\xi^a\Lambda)] , \tag{7.3}$$

where we have used $\Lambda g_{ab}\pounds_\xi g^{ab} - 2\pounds_\xi\Lambda = -2\Lambda\nabla_a\xi^a - 2\xi^a\nabla_a\Lambda = -2\nabla_a(\xi^a\Lambda)$ since Λ is a scalar. Also, for the diffeomorphism symmetry, the left hand side is given by $\pounds_\xi(\sqrt{-g}L) = \sqrt{-g}\nabla_a(L\xi^a)$. Therefore, finally one obtains

$$\sqrt{-g}\nabla_a[L\xi^a + \frac{1}{16\pi}(2G^a_b\xi^b + 2\Lambda\xi^a - \pounds_\xi v^a)] = 0, \tag{7.4}$$

where one can obtain the explicit expression $\pounds_\xi v^a = \nabla_b\nabla^a\xi^b + \Box\xi^a - 2\nabla_a\nabla_b\xi^b$. So, the conserved Noether current for the diffeomorphism symmetry can be identified as

$$J^a = L\xi^a + \frac{1}{16\pi}(2G^a_b\xi^b + 2\Lambda\xi^a - \pounds_\xi v^a). \tag{7.5}$$

Using the results $2G^a_b\xi^b + R\xi^a = 2R^a_j\xi^j = 2[\nabla_b\nabla^a\xi^b - \nabla^a\nabla_b\xi^b]$ and the earlier mentioned expression of $\pounds_\xi v^a$, the above equation (eq. (7.5)) yields $J^a = \nabla_b J^{ab} = \frac{1}{16\pi}\nabla_b[\nabla^a\xi^b - \nabla^b\xi^a]$, thereby obtaining the anti-symmetric Noether potential as

$$J^{ab} = \frac{1}{16\pi}[\nabla^a\xi^b - \nabla^b\xi^a]. \tag{7.6}$$

Note, although the Noether current J^a depends on the cosmological constant Λ, the Noether potential is independent of it. Also notice that the expression of the Noether potential does not change when one takes Λ as a pure constant or even when one does not take Λ in the theory. Later we shall show that the entropy and the energy of the black hole system directly relate to the Noether potential. Therefore, it can be concluded that those quantities are exempted from any change due to Λ. Also notice that to obtain the above expression we never took the help of Einstein's equations of motion. So it is an *off-shell* result.

On-shell one finds $G^a_b\xi^b = -\Lambda\xi^a$ from the equation of motion $G_{ab} + \Lambda g_{ab} = 0$, yielding the on-shell Noether current (from (7.5)) as

$$J^a = L\xi^a - \frac{\pounds_\xi v^a}{16\pi}, \tag{7.7}$$

which implies

$$\delta(\sqrt{-g}J^a) = \delta(\sqrt{-g}L)\xi^a + \sqrt{-g}L\delta\xi^a - \frac{\delta(\sqrt{-g}\pounds_\xi v^a)}{16\pi}, \tag{7.8}$$

where δ represents arbitrary field variation (here the variation of the metric tensor g^{ab} and the cosmological constant Λ) that does not affect the vector ξ^a. So, $\delta\xi^a = 0$, but $\delta\xi_a \neq 0$, then using (7.2) we obtain

$$\delta(\sqrt{-g}J^a) = \frac{\xi^a}{16\pi}[\sqrt{-g}(G_{ij} + \Lambda g_{ij})\delta g^{ij}$$
$$+\sqrt{-g}\nabla_i \delta v^i - 2\sqrt{-g}\delta\Lambda] - \frac{\delta(\sqrt{-g}\pounds_\xi v^a)}{16\pi} \tag{7.9}$$

Let us calculate the above variation on-shell. which gives

$$\delta(\sqrt{-g}J^a) = \frac{\sqrt{-g}\xi^a}{16\pi}[\nabla_i \delta v^i - 2\delta\Lambda] - \frac{\delta(\sqrt{-g}\pounds_\xi v^a)}{16\pi} \, . \tag{7.10}$$

Using straightforward calculation one can obtain $\pounds_\xi(\sqrt{-g}\delta v^a) = -2\sqrt{-g}\nabla_b(\xi^{[a}\delta v^{b]}) + \sqrt{-g}\xi^a \nabla_b \delta v^b$ with $A^{[a}B^{b]} = \frac{1}{2}(A^a B^b - A^b B^a)$. Using this relation in (7.10) one obtains

$$\delta(\sqrt{-g}J^a) = \frac{1}{16\pi}[\pounds_\xi(\sqrt{-g}\delta v^a) - \delta(\sqrt{-g}\pounds_\xi v^a)$$
$$+2\sqrt{-g}\nabla_b(\xi^{[a}\delta v^{b]}) - 2\sqrt{-g}\xi^a \delta\Lambda]. \tag{7.11}$$

Let us now denote

$$\omega^a = -\frac{1}{16\pi}\pounds_\xi(\sqrt{-g}\delta v^a) + \frac{1}{16\pi}\delta(\sqrt{-g}\pounds_\xi v^a). \tag{7.12}$$

The significance of ω^a will be explained later in our discussions. Therefore, using this convention one obtains

$$\delta(\sqrt{-g}J^a) = -\omega^a + \frac{2\sqrt{-g}}{16\pi}[\nabla_b(\xi^{[a}\delta v^{b]}) - \xi^a \delta\Lambda], \tag{7.13}$$

which implies that

$$\omega^a = -\delta(\sqrt{-g}J^a) + \frac{2\sqrt{-g}}{16\pi}[\nabla_b(\xi^{[a}\delta v^{b]}) - \xi^a \delta\Lambda]. \tag{7.14}$$

To realize all the things properly let us take the refuge of the classical mechanics. From the classical calculations one can obtain $\delta L(q,\dot{q}) = [(\frac{\partial L}{\partial q}) - d_t(\frac{\partial L}{\partial \dot{q}})]\delta q + d_t[p\delta q]$. where the first term is the equation of motion, that vanishes on-shell and the last term is the temporal derivative of the boundary term, which let us denote as v. Let us now adopt the conventions

$$
\begin{aligned}
v(\delta q) &= p\delta q, \\
v(\dot{q}) &= p\dot{q}.
\end{aligned}
\tag{7.15}
$$

Using this convention, the variation of the Hamiltonian can be written as:

$$
\delta H(q,p) = \delta[p(d_t q)] - d_t[p(\delta q)] = \delta[v(\dot{q})] - d_t[v(\delta q)]. \tag{7.16}
$$

Let us now try to realize the physical significance of the term ω^a. But before that, to compare ω^a with the classical term, let us take the one-to-one correspondence in the following way.

Let the metric tensor g^{ab} corresponds to q in classical mechanics. The arbitrary variation of the metric tensor δg^{ab} corresponds to δq and the Lie-derivative of the metric tensor $\pounds_\xi g^{ab}$ corresponds to $\dot{q}$. Following this convention one can write

$$
\begin{aligned}
\sqrt{-g}\pounds_\xi v^a &\equiv v(\dot{q}), \\
\sqrt{-g}\delta v^a &\equiv v(\delta q).
\end{aligned}
\tag{7.17}
$$

So, using the above mentioned convention in (7.17) if one compares (7.12) and (7.16) one actually finds (apart from the normalization factor $\frac{1}{16\pi}$) ω^a corresponds to $\delta\mathcal{H}$, where $\mathcal{H}$ is the Hamiltonian density. So, variation of the total Hamiltonian is given as

$$
\begin{aligned}
\delta H[\xi] &= \delta \int_c d\Sigma_a \frac{\omega^a}{\sqrt{-g}} \\
&= -\int_c d\Sigma_a \nabla_b (J^{ab}) + \frac{2}{16\pi}\int_c d\Sigma_a [\nabla_b(\xi^{[a}\delta v^{b]}) \\
&\quad -\xi^a \delta\Lambda],
\end{aligned}
\tag{7.18}
$$

where c is the Cauchy surface. To obtain the last step we use $J^a = \nabla_b J^{ab}$ in (7.14), where J^{ab} is the anti-symmetric Noether potential. Here $d\Sigma_a = n_a\sqrt{h}d^3y$ is the infinitesimal surface area of three dimensional hypersurface with h being the determinant of 3-metric and n_a being the normal to the surface. Let us now replace the cosmological constant Λ by the pressure. In case of AdS black hole, the pressure is identified as $P = -\Lambda/8\pi$. After converting the volume integral to the surface integral, one can write (7.18) as

$$\delta H[\xi] = -\frac{1}{2}\delta\int_{\mathcal{H}} d\Sigma_{ab}J^{ab} + \frac{1}{2}\delta\int_{\partial c_\infty} d\Sigma_{ab}J^{ab}$$
$$-\frac{1}{16\pi}\int_{\partial c_\infty} d\Sigma_{ab}\xi^{[a}\delta v^{b]} + \delta P\int_c d\Sigma_a\xi^a~, \tag{7.19}$$

where the new surface integrations are to be done on a bifurcation surface $\mathcal{H}$ and at 2-dimensional boundary of c at asymptotic infinity (i.e., ∂c_∞). On the bifurcation surface $\mathcal{H}$, taking ξ^a as a timelike Killing vector, one must have $\xi^a = 0$. So no contribution comes from the term containing $\xi^{[a}\delta v^{b]}$. In this situation $\delta H[\xi] = 0$, although $H[\xi]$ might not be zero. Then the first term on the right hand side can be identified as $-\frac{\kappa}{2\pi}\delta S$ from the Wald's prescription with κ as the surface gravity. The second and the third term as a whole contributes as $\delta M - \Omega_H\delta J$ (see [29] for rigorous discussion).

Now, let us concentrate on the last term; i.e. $\int_c d\Sigma_a\xi^a$ of (7.19). The mentioned integral can be written further as $\int_{\mathcal{H}}\sqrt{h}d^3yn_a\xi^a - \int_\infty\sqrt{h}d^3yn_a\xi^a$ where the first term is calculated at the horizon and the second term is evaluated at the asymptotic boundary. One finds that the first integral gives a finite result, whereas, the second term appears as a diverging one. To remove this divergence one needs to adopt the regularization procedure. Usually in the literature there are two prescriptions. One is adding a counter term in the action such that its contribution removes the divergence. Another one is to use the background subtraction method. In this case the background contribution removes the divergence and we get a finite volume. The addition of the extra term can be justified due to the fact that one can always introduce a total derivative term along with the actual Lagrangian as the governing dynamics is unaltered or due to the fact that the Noether potential (J^{ab}) is not uniquely determined (one can include arbitrary anti-symmetric tensor with it, when the divergence of that arbitrary term vanishes). Therefore, one has the freedom to include a term in the Lagrangian or in the Noether potential such that the mentioned divergence at the infinity can be removed. Hence, considering the covariant definition of the volume V as

$$V = -\int_{\mathcal{H}}\sqrt{h}d^3yn_a\xi^a + \int_\infty\sqrt{h}d^3y[n_a\xi^a - (n_a\xi^a)_{BG}]~, \tag{7.20}$$

where, the term containing $(n_a\xi^a)_{BG}$ is considered as the "background contribution" to obliterate the divergence, we interpret the last term of (7.19) as $-V\delta P$. This volume, in literature, is usually called the thermodynamic volume. If one calculates (7.20), for example, in Reissner–Nordstrom AdS (RNAdS) black hole (we introduce RNADS black hole later) with the horizon radius r_+, one obtains $V = 4\pi r_+^3/3$, which we shall use later. The pressure is given purely by the consmological constant: $P = -\Lambda/8\pi$. Similar prescriptions have been adopted in [51, 282].

Therefore, from (7.19) one obtains the desired result

$$\delta M = T\delta S + V\delta P + \Omega_H \delta J \ , \tag{7.21}$$

as one can identify $T = \frac{\kappa}{2\pi}$ being the black hole temperature. Thus the relation obtained in this analysis is the first law for the AdS black hole with varying cosmological constant. Here we have not incorporated any hair in this theory for the simplicity of the calculations. As we mentioned earlier, inclusion of hair gives the well-known contribution and, ultimately, the final expression while considering all the hairs is given as

$$dM = TdS + VdP + \sum_i X_i dY_i \ , \tag{7.22}$$

where $X = \{\Omega, \phi\}$ and $Y = \{J, Q\}$ with Ω, ϕ, J and Q being the angular velocity, electric potential, angular momentum and electric charge (that includes all the charges due to the symmetry as well as the hair in the system) respectively. Let us make one important comment in this regard. If we compare the above equation (7.22) with the thermodynamic law $dH = TdS + VdP + \mu dN$ (where H is the enthalpy, μ is the chemical potential and N is the number of particles of the thermodynamic system), we find that the mass of the black hole is identified as the enthalpy of the BH system, instead of the internal energy. On the contrary, as we have seen earlier, the mass of the black hole in non-extended phase space is identified with the internal energy of the system. Remember in the usual thermodynamics, the enthalpy gives the measure of energy to create the system (i.e., the internal energy) added with the amount required to establish the pressure and the volume of that system. Thus, the role of black hole mass is also changed in the description of the black hole thermodynamics in the extended phase space.

We end up this section with the following comments. The fact, that the Noether potential is independent of the cosmological constant, has earlier been studied in [283]. But, in that case, the cosmological constant (Λ) has been taken as a proper constant and, consequently, it does not appear in the potential. But, in our case, we have taken the variation of the cosmological constant along with the metric tensor to derive the first-law of the AdS black holes in the extended phase space. We have shown although the Noether current depends on the cosmological constant, the potential does not depend on it even for the off-shell case. In this regard, the interesting point to be noted is that the expression of the entropy and the mass-energy in the extended phase space is identical to the non-extended-phase space as the expression of the Noether-potential is the same in both the cases.

Another important comment. In the earlier chapter, when we have discussed the methods of obtaining thermodynamic first law from the conserved currents in scalar-tensor theory, we have discussed not only the Wald formalism, but also the ADT formalism as well. We have also shown that the two methods are equivalent. Following the same steps, one can show that the two methods will be equivalent in extended phase space as well. Thus, (7.21) can also be obtained alternatively by the ADT approach. For that, the reader can follow [168], which has explicitly obtained (7.21) using the ADT approach.

7.3 Thermodynamic geometry in a Legendre invariant way: a brief review

The prescription of geometry was earlier incorporated in thermodynamics by Gibbs [61], Carathéodory [62], Fisher [63] and Rao [64]. Later Weinhold [65] and Ruppeiner [66, 67] introduced Riemannian metrics in the thermodynamic phase space to study the black hole phase transition. The Weinhold metric was formulated as the Hessian of the internal energy and the Ruppeiner one is defined as the negative of the Hessian of the entropy. In fact, the Ruppeiner metric is conformal to the Weinhold metric with the conformal factor as the inverse temperature. Since in gravity the gravitational interaction results in the curvature in the spacetime, the same motivation is utilized while formulating thermogeometry *i.e.* the thermodynamic interaction is described here by the curvature of the thermodynamic phase space and the critical points are identified as the point where the Ricci-scalar diverges [45, 68, 284, 285]. It was found later that the Weinhold metric and the Ruppeiner metric are not consistent with each other in some cases. For example [68], in the case of the Kerr black hole, the Weinhold metric is flat and the Ruppeiner metric gives the divergence of the Ricci-scalar only at the extremal limit. Both of these metrics does not imply a phase transition as predicted by Davies where the heat capacity diverges. Moreover, recently in a general formulation, we have shown that the Weinhold metric cannot predict the extremal phase transition of black holes, whereas the Ruppeiner metric successfully can predict that [48]. The discrepancy in these two formulations is believed to appear due to that fact that both the Weinhold and the Ruppeiner metrics are not formulated in a Legendre-invariant way [69–75]. Since the thermodynamics is invariant due to the Legendre transformation, Quevedo *et al.* suggested that Legendre-invariance must be maintained while formulating a thermo-geometric metric and provided the prescription to formulate it in a Legendre-invariant way. Later Quevedo's formulation was widely accepted and it was shown that at the critical point, the Ricci scalar of the Legendre-invariant metric diverges [271–273, 276–280]. Here, in the unified picture, we follow Quevedo's formalism. In the following, we mention the method to obtain the Legendre-invariant thermo-geometric metric.

To formulate a thermo-geometric metric in a Legendre-invariant way, one first needs to define a thermodynamic phase space $\mathcal{T}$. The coordinates of $\mathcal{T}$ are defined as $Z^A = \left(\Phi, \tilde{X}^a, \tilde{P}^a\right)$. Here Φ can be a thermodynamic potential or some other thermodynamic parameter, $\tilde{X}^a$ are the variables on which Φ depends and $\tilde{P}^a = \partial\Phi/\partial\tilde{X}^a$ are the conjugate variables. Now, the Legendre transformation in $\mathcal{T}$ is given by the following set of transformation

$$(\Phi, \tilde{X}^a, \tilde{P}^a) \rightarrow (\Phi', \tilde{X}'^a, \tilde{P}'^a) \tag{7.23}$$

$$\Phi = \Phi' - \delta_{ab}\tilde{X}'^a\tilde{P}'^b \ , \tilde{X}^a = -\tilde{P}'^a \ , \tilde{P}^a = \tilde{X}'^a \ . \tag{7.24}$$

The canonical contact structure on the phase space $\mathcal{T}$ is defined by the fundamental Gibbs 1-form $\Theta = d\Phi - \sum_{a,b}\delta_{ab}\tilde{P}^a d\tilde{X}^b$, which is Legendre invariant for the given transformation in (7.24). One can now define several Legendre-invariant thermo-geometric metricS in $\mathcal{T}$ and the choice of those metrics IS not unique [69]. Here we discuss a particular type of Legendre-invariant metric having the following structure

$$G = \Theta^2 + \left(\lambda \sum_{a,b} \xi_{ab}\tilde{P}^a\tilde{X}^b\right)\left(\sum_{c,d}\eta_{cd}d\tilde{P}^c d\tilde{X}^d\right) \ . \tag{7.25}$$

One can verify that the metric (7.25) is invariant under the Legendre transformation (7.24). For simplicity in the calculation we have taken $\lambda = 1$, ξ_{ab} =diag$(1, 1, ...1)$ and η_{ab} =diag$(-1, 1, ...1)$. The space of equilibrium $\mathcal{E}$ is defined by the mapping $\varphi : \mathcal{E} \longrightarrow \mathcal{T}$ with the constraint of the thermodynamic relation $d\phi = \sum_i \tilde{P}^i d\tilde{X}^i$. Then further expanding $d\tilde{P}^i = \sum_j (d^2\Phi/d\tilde{X}^i d\tilde{X}^j) d\tilde{X}^j$ one obtains the expression of the thermo-geometric metric in the equilibrium space as

$$g = \varphi^*(G) = \Big(\sum_a \tilde{X}^a \frac{\partial \Phi}{\partial \tilde{X}^a}\Big)\Big(\sum_{c,d,e} \eta_{cd} \frac{d^2\Phi}{d\tilde{X}^c d\tilde{X}^e} d\tilde{X}^d d\tilde{X}^e\Big) \ . \tag{7.26}$$

Here, φ^* is the pullback of φ . Also, note that while obtaining the metric (7.26) we have explicitly used $\lambda = 1$ and ξ_{ab} =diag$(1, 1, ...1)$.

The above metric (7.26), which has been defined in the equilibrium phase space, is the general form of the metric that we have used in our analysis. In the following section we explicitly present the form of thermo-geometric metric for different types of van der Waals criticalities (such as the $P-V$ criticality, $T-S$ criticality, $Y-X$ criticality *etc.*).

7.4 P-V criticality in thermogeometric picture

In the usual thermodynamics, the critical point is the extreme point of the vaporization curve (in the P-T phase diagram of the van der Waals gas). The gas above the critical temperature cannot be liquefied and, therefore, only one state exists which is the gaseous state. Mathematically, the critical point of the van der Waals gas system represents a particular point in the P-V diagram of the critical isotherm where the tangent vanishes as well as the maxima and the minima of the P vs V curve meet to form a inflection point. Therefore, at the critical point the two independent conditions coincide; i.e. $P_V = 0 = P_{VV}$.

The same discussions can be applied for the black holes as well because it has been found that the P-V diagram of the AdS black holes at constant temperature and charges looks identical to that for van der Waals gas system when one interprets the cosmological constant as pressure (P) (See [56] and for the review see [57]). Consequently, the critical conditions are obtained by the following set of relations

$$\begin{aligned} P_V &= \Big(\frac{\partial P}{\partial V}\Big)_{T,Y_i}\Big|_c = 0 \ ; \\ P_{VV} &= \Big(\frac{\partial^2 P}{\partial V^2}\Big)_{T,Y_i}\Big|_c = 0 \ . \end{aligned} \tag{7.27}$$

In this section, we want to obtain the geometrical interpretation of those conditions in terms of the thermogeometrical metric. We have already mentioned that the critical point refers to the singular behavior of the scalar curvature of the thermogeometrical metric. Here, our aim is to investigate such possibility for the P-V

criticality of the AdS black holes. Let us now find out the relevant thermogeometrical metric in a Legendre invariant way.

7.4.1 The first condition: $P_V = 0$

For the first condition, we found out the proper thermodynamic quantity to start with is the Helmholtz free energy F which is the function of V, T and Y_i as shown later. The underlying motivation of choosing the Helmholtz free energy as the appropriate quantity will be understood from the main analysis. Since in this case the black hole mass M is identified as the enthalpy [51], the internal energy is given by $E = M - PV$ and the free energy is defined as

$$F = E - TS = M - PV - TS \ . \tag{7.28}$$

The first law of the black holes under the AdS background with the role of the cosmological constant as the pressure is given by the expression (7.22). Using this and (7.28) one gets

$$dF = -PdV - SdT + \sum_i X_i dY_i \ , \tag{7.29}$$

i.e. F is function of V, T and Y_i. Therefore, the conjugate quantities corresponding to these variables are

$$P = -\Big(\frac{\partial F}{\partial V}\Big)_{T,Y_i} \equiv -F_V; \quad S = -\Big(\frac{\partial F}{\partial T}\Big)_{V,Y_i} \equiv -F_T; \quad X_i = \Big(\frac{\partial F}{\partial Y_i}\Big)_{T,V} \equiv F_{Y_i} \ . \tag{7.30}$$

With the above relations, the criticality conditions (7.27) are given as

$$\begin{aligned} \Big(\frac{\partial P}{\partial V}\Big)_{T,Y_i}\Big|_c &= -\Big(\frac{\partial^2 F}{\partial V^2}\Big)_{T,Y_i}\Big|_c = 0 \ ; \\ \Big(\frac{\partial^2 P}{\partial V^2}\Big)_{T,Y_i}\Big|_c &= -\Big(\frac{\partial^3 F}{\partial V^3}\Big)_{T,Y_i}\Big|_c = 0 \ . \end{aligned} \tag{7.31}$$

Note, in this case our thermodynamic potential is the function of both the extensive variables (V, Y_i) and the intensive variable (T). We define a thermodynamic phase space $\mathcal{T}$ with the coordinates $Z^A = \{F, \tilde{X}^a, \tilde{P}^a\}$ where $\tilde{X}^a = \{V, T, Y_i\}$ are the thermodynamic variables and $\tilde{P}^a = \{F_V = -P, F_T = -S, F_{Y_i} = X_i\}$ are the the conjugate variables. Note, the conjugate variables are the functions of the thermodynamic variables $\tilde{X}^a$. The methods of obtianing the thermogeometric metric in a

Legendre invariant way has earlier been mentioned in section 7.3. If we identify Φ of section 7.3 as F, we obtain the thermogeometrical metric as the following.

$$G_1^{(PV)} = \Big(dF - \sum_{ab} \delta_{ab} \tilde{P}^a d\tilde{X}^b\Big)^2 + \lambda\Big(\sum_{ab} \xi_{ab} \tilde{P}^a \tilde{X}^b\Big)\Big(\sum_{cd} \eta_{cd} d\tilde{P}^c d\tilde{X}^d\Big) , \tag{7.32}$$

where η_{cd} =diag(-1,...,1), λ is arbitrary Legendre invariant function of $\tilde{X}^a$ and ξ_{ab} is arbitrary constant diagonal function. Again, for the sake of simplicity, we take $\lambda = 1$, and ξ_{ab}=diag(1,...,1). Thus, the simplified form can be written as

$$G_1^{(PV)} = \theta_F^2 + (-PV - ST + \sum_i X_i Y_i)(dPdV - dSdT + \sum_i dX_i dY_i) . \tag{7.33}$$

Note, while forming the metric we multiply the variables with their conjugate ones to get the dimensionally consistent result. One can check the above metric is Legendre invariant from the definition of the Legendre transformation (7.24). The metric when induced on $\mathcal{E}$ by means of $\mathcal{G}_1^{(PV)} = \varphi_F^*(G_1^{(PV)})$, it yields

$$\begin{aligned}\mathcal{G}_1^{(PV)} = (-PV - ST + \sum_i X_i Y_i)[-F_{VV} dV^2 + F_{TT} dT^2 + \sum_{i,j} F_{Y_i Y_j} dY_i dY_j \\ +2\sum_i F_{TY_i} dT dY_i] ,\end{aligned} \tag{7.34}$$

where in the above we use the fact that the conjugate variables are the functions of the thermodynamic variables $\tilde{X}^a$. Now, one can verify that the exact nature of divergence of the Ricci scalar on F_{VV} turns out to be

$$R_1^{(PV)}|_{max.diver.} \sim \mathcal{O}(\frac{1}{F_{VV}^2}) . \tag{7.35}$$

This can be observed from the explicit expression of the Ricci-scalar for a single charged metric. Although, for the presence of the multiple charges, the exact form of it can not be given but, the nature is exactly the same like the simple one. For the presence of the single charge, the expression of the scalar curvature is explicitly provided in Appendix 7.A (see (7.60)). The metric coefficients can be identified as

$$\begin{aligned} f(V,T,Y) &= F_{VV}(VF_V + TF_T + YF_Y) ; \quad g(V,T,Y) = F_{TT}(VF_V + TF_T + YF_Y) ; \\ A(V,T,Y) &= F_{YY}(VF_V + TF_T + YF_Y) ; \quad h(V,T,Y) = F_{TY}(VF_V + TF_T + YF_Y) . \end{aligned} \tag{7.36}$$

Now the expression (7.60) tells that the Ricci scalar diverges as $\mathcal{O}(1/f^2)$. As f is given by the first equation of (7.36), one obtains (7.35). Since P is given by the first relation of (7.30), we have $\partial P/\partial V = -F_{VV}$. Therefore, one of the two conditions of determining the critical point of the phase transition leads to the vanishing of F_{VV} and, consequently, the calculated Ricci-scalar diverges. Later, we shall give an explicit example to make more elaborative comments on these discussions. Therefore, we conclude that the vanishing of the tangent condition is equivalent to the divergence of the Ricci scalar of the thermogeometry in F picture and this is a covariantly invariant statement as the quantity is a scalar. Below we shall concentrate on the other condition.

7.4.2 The second condition: $P_{VV} = 0$

To get the geometrical interpretation of the second condition (point of inflection) let us take pressure as a function of the volume, the temperature and the charges, i.e., $P = P(V, T, Y_i)$. It implies

$$dP = \left(\frac{\partial P}{\partial V}\right)_{T,Y_i} dV + \left(\frac{\partial P}{\partial T}\right)_{V,Y_i} dT + \sum_i \left(\frac{\partial P}{\partial Y_i}\right)_{V,T} dY_i = P_V dV + P_T dT + \sum_i P_{Y_i} dY_i. \tag{7.37}$$

Here also we define the thermodynamic phase space $\mathcal{T}$ with the coordinates $Z^A = \{P, \tilde{X}^a, \tilde{P}^a\}$ where $\tilde{X}^a = \{V, T, Y_i\}$ are the variables and $\tilde{P}^a = \{P_V, P_T, P_{Y_i}\}$ are the corresponding conjugate quantities of those variables. As done earlier, we again choose a metric properly in the θ_P invariant form with $\theta_P = dP - P_V dV - P_T dT - \sum_i P_{Y_i} dY_i$. Again we consider the subspace $\mathcal{E}$ having the coordinates $\tilde{X}^a$. Also we assume a smooth mapping $\varphi_P : \mathcal{E} \to \mathcal{T}$ and the subspace $\mathcal{E}$ is called the space of equilibrium states if $\varphi_P^*(\theta_P) = 0$. Let us take the same definition of (7.25) to form the metric with Φ being replaced by P. In this case it yields:

$$G_2^{(PV)} = \theta_P^2 + (VP_V + TP_T + \sum_i Y_i P_{Y_i})(-dV dP_V + dT dP_T + \sum_j dY_j dP_{Y_j}). \tag{7.38}$$

The metric is Legendre invariant according to the definition (7.24) with F being replaced by P. For $\mathcal{G}_2^{(PV)} = \varphi_P^*(G_2^{(PV)})$ one obtains from (7.26)

$$\mathcal{G}_2^{(PV)} = (VP_V + TP_T + \sum_i Y_i P_{Y_i})(-P_{VV}dV^2 + P_{TT}dT^2 + \sum_{j,k} P_{Y_jY_k}dY_j dY_k + 2\sum_l P_{TY_l}dTdY_l) \ . \quad (7.39)$$

The metric is identical in form like the previous one (see (7.34)). Here also, by the earlier argument, the nature of divergence of the Ricci scalar on P_{VV} is $R_2^{(PV)} \sim \mathcal{O}(1/P_{VV}^2)$. When the van der Waals system reaches the critical point, then the condition of the point of inflection gives $(\partial^2 P)/(\partial V^2) = P_{VV} = 0$, implying the Ricci scalar for this metric diverges.

In the above, we have presented the explicit form of two Legendre invariant thermogeometrical metrics: One is in F-picture and the other one is in the P-picture. The first one describes the condition $\partial P/\partial V = 0$ as the divergence of the corresponding Ricci-scalar while the other one gives the same result for $\partial^2 P/\partial V^2 = 0$. Hence, we conclude that the van der Waals like critical point of a black hole can be equivalently analyzed by properly defined thermogeometry.

Let us now make a comment on the above analysis. When we are saying that the Ricci scalar diverges at the critical point, we are assuming that the numerator does not vanish at this point in all cases. If such situation arises, then one has to be careful. Since it is 0/0 form, may be L'Hospital's rule can help to conclude. We shall take an explicit case where such situation would not arise. Let us discuss for the simplest example. Consider the Reissner–Nordstrom AdS (RNAdS) metric:

$$ds^2 = -\tilde{f}(r)dt^2 + \frac{dr^2}{\tilde{f}(r)} + r^2 d\Omega^2 \ , \quad (7.40)$$

with $\tilde{f}(r) = 1 - (2M/r) + (Q^2/r^2) + (r^2/l^2)$ where, M, Q and l being the mass, the electric charge ($\equiv Y$) and the AdS curvature radius (related to the cosmological constant as $\Lambda = -3/l^2$) respectively. The conjugate variables for the invariant θ_F picture can be written in terms of the thermodynamic variables as

$$P = \frac{T}{2CV^{\frac{1}{3}}} - \frac{1}{8\pi C^2 V^{\frac{2}{3}}} + \frac{Q^2}{8\pi C^4 V^{\frac{4}{3}}} \ , \quad S = \pi C^2 V^{\frac{2}{3}} \ ,$$
$$F = \frac{CV^{\frac{1}{3}}}{2} - \pi T C^2 V^{\frac{2}{3}} + \frac{Q^2 V^{-\frac{1}{3}}}{2C} \ , \quad \phi = \frac{Q}{CV^{\frac{1}{3}}} \ , \quad (7.41)$$

with the relation of the event horizon radius $r_+ = CV^{\frac{1}{3}}$, where $C = (3/4\pi)^{\frac{1}{3}}$. Since in this case $F_{TT} = 0 = F_{TQ}$, the thermogeometrical metric (7.34) reduces to

$$\mathcal{G}^{(PV)}_{1(RN)} = -f(V,Q)dV^2 + A(V,Q)dQ^2 \ , \tag{7.42}$$

which is a two dimensional metric. Now, the calculation of the Ricci scalar becomes very straight forward, which is given by the relation

$$R^{(PV)}_{1(RN)} = \frac{1}{2f^2A^2}[f(f_QA_Q - A_V^2) + A\{f_Q^2 - f_VA_V - 2f(f_{QQ} - A_{VV})\}] \ . \tag{7.43}$$

It is obvious that the Ricci-scalar diverges as $\mathcal{O}(1/f^2)$. But, one should verify whether the numerator vanishes at the critical point. The critical values of the thermodynamic quantities can be obtained satisfying the two criticality conditions (i.e., $F_{VV} = P_{VV} = 0$). One can find the numerator (i.e., $A(f_Q^2 - f_VA_V)$) of the Ricci-scalar gives small finite nonzero result for the critical values. But, not surprisingly, the total expression of the scalar curvature diverges at the critical point again.

For the geometrical description of the point of inflection of an RN-AdS black hole in the invariant θ_P picture the metric (7.39) becomes

$$\mathcal{G}^{(PV)}_{2(RN)} = -h(V,Q)dV^2 + w(V,Q)dQ^2 \ , \tag{7.44}$$

with

$$h = P_{VV}(VP_V + TP_T + QP_Q) \ ; \quad w = P_{QQ}(VP_V + TP_T + QP_Q) \ . \tag{7.45}$$

The above one is again a two-dimensional metric as the other metric coefficients containing P_{TT} and P_{TQ} vanish in this case. Calculation of the Ricci scalar is again very straightforward yielding

$$R^{(PV)}_{2(RN)} = \frac{1}{2h^2w^2}[h(h_Qw_Q - w_V^2) + w\{h_Q^2 - h_Vw_V - 2h(h_{QQ} - w_{VV})\}] \ . \tag{7.46}$$

Again the scalar curvature diverges as $\mathcal{O}(1/h^2)$. Here also, one can verify whether the numerator vanishes at the critical point. One can find that for the critical values, the numerator of the Ricci-scalar (i.e., $w(h_Q^2 - h_Vw_V)$) gives small finite nonzero result. But, the total expression of the scalar curvature diverges.

In this section, we have elaborately described the procedure to provide the geometric description of the black hole criticality at the extended phase space. Besides the P-V criticality, as we have mentioned earlier, there are also the T-S criticality and the Y-X criticality in the non-extended phase space. In the following sections we shall find whether the previous arguments of the P-V criticality in the extended phase space is also applicable for these two criticality of black holes in the non-extended phase space as well.

7.5 The T-S criticality

Recently, there are many works that deals with the $T-S$ criticality [58, 59, 273] of black holes. The criticality, as said, comes due to the fact that the $P-V$ and the $T-S$ space are dual. The expression of the pressure and the temperature comes from the same relation of black hole temperature [58] and the criticality conditions also looks alike both in $P-V$ and $T-S$ pictures. In the $T-S$ phase space the conditions are $(\partial T/\partial S)_{Y_i} = 0 = (\partial^2 T/\partial S^2)_{Y_i}$. In this case, one expresses T as a function of S and Y_i. We shall show here that our method, presented in the earlier section to give the geometrical description of the critical point, can also be used for this criticality as well.

Let us first concentrate on the first condition. Here the relevant thermodynamic potential, as will be justified by the main analysis, is the internal energy $E(S, Y_i)$. The first law of the black hole (in the non-extended phase space) is given as $dE = TdS + \sum_i X_i dY_i$, where E is identified as the black hole mass M. Hence, the thermodynamic variables are $\tilde{X}^a = \{S, Y_i\}$. Also, the conjugate variables are $\tilde{P}^a = \{T, X_i\}$ with $T = E_S$ and $X_i = E_{Y_i}$. The thermodynamic phase space $\mathcal{T}$ has the coordinates $Z^a = \{E, \tilde{X}^a, \tilde{P}^a\}$. The Legendre invariant metric, following the definition of (7.25), can be written as

$$G_1^{(TS)} = \theta_E^2 + (TS + \sum_i X_i Y_i)(-dTdS + \sum_j dX_j dY_j), \tag{7.47}$$

with the invariant $\theta_E = TdS + \sum_i X_i dY_i$. Also, we assume a subspace $\mathcal{E}$ with the coordinates $\tilde{X}^a$. Let there be a smooth mapping $\varphi_E : \mathcal{E} \to \mathcal{T}$ and the subspace $\mathcal{E}$ is called the subspace of equilibrium states for the condition $\varphi_E^*(\theta_E) = 0$. Now, the metric induced on $\mathcal{E}$ by means of $\mathcal{G}_1^{(TS)} = \varphi_E^*(G_1^{(TS)})$ can be written as

$$\mathcal{G}_1^{(TS)} = -f'(S, Y_i)dS^2 + \sum_{ij} g'_{ij}(S, Y_i)dY_i dY_j, \tag{7.48}$$

with

$$f'(S,Y_i) = E_{SS}(SE_S + \sum_i Y_i E_{Y_i}) \ ; \quad g'_{ij}(S,Y_i) = E_{Y_iY_j}(SE_S + \sum_k Y_k E_{Y_k}) \ . \quad (7.49)$$

For this multi-dimensional metric, if one calculates the scalar curvature it diverges at the critical point as E_{SS} vanishes at that point. Also, like the earlier case, the maximum divergence of the Ricci-scalar is proportional to the square inverse of E_{SS} i.e., $R_1^{(TS)}|_{max.diver.} \sim \mathcal{O}(1/E_{SS}^2)$. For better understanding we assume the presence of the single charge rather then multiple ones. Taking this liberty, the metric gets the form:

$$\mathcal{G}_1^{(TS)} = -f'(S,Y)dS^2 + g'(S,Y)dY^2 \ , \quad (7.50)$$

with $f'(S,Y) = E_{SS}(SE_S + YE_Y)$ and $g'(S,Y) = E_{YY}(SE_S + YE_Y)$. Again the scalar curvature can be calculated for this two dimensional thermogeometrical metric very straight-forwardly, which is given as

$$R_1^{(TS)} = \frac{1}{2f'^2 g'^2}[f'(f'_Y g'_Y - g'^2_S) + g'(f'^2_Y - f'_S g'_S - 2f'(f'_{YY} - g'_{SS}))]. \quad (7.51)$$

The above shows that $R_1^{(TS)} \sim \mathcal{O}(1/f'^2)$. Now $\partial T/\partial S = M_{SS} = 0$ at the critical point. So, the Ricci-scalar diverges for this metric at the critical point. Thus, one condition gives the singular Ricci scalar at the critical point. Let us look at the other condition of the T-S criticality.

As done earlier, to give the geometric interpretation of the second condition, one has to express the temperature as a function of the entropy S and the charges Y_i, thereby obtaining $dT = (\partial T/\partial S)_{Y_i} dS + \sum_i (\partial T/\partial Y_i)_S dY_i = T_S dS + \sum_i T_{Y_i} dY_i$. We define the thermodynamic phase space $\mathcal{T}$ with coordinates $Z^a = \{T, \tilde{X}^a, \tilde{P}^a\}$, with $\tilde{X}^a = \{S, Y_i\}$ being the thermodynamic variables and $\tilde{P}^a = \{T_S, T_{Y_i}\}$ being the corresponding conjugate variables. The thermogeometrical metric can be defined as

$$G_2^{(TS)} = \theta_T^2 + (ST_S + \sum_i Y_i T_{Y_i})(-dSdT_S + \sum_j dY_j dT_{Y_j}), \quad (7.52)$$

which is Legendre invariant with $\theta_T = dT - T_S dS - \sum_i T_{Y_i} dY_i$. Let there be the subspace $\mathcal{E}$ with a smooth mapping $\varphi_T : \mathcal{E} \to \mathcal{T}$ and the subspace $\mathcal{E}$ is called the space of equilibrium states for $\varphi_T^*(\theta_T) = 0$. So, the metric induced on $\mathcal{E}$ by the relation $\mathcal{G}_2^{(TS)} = \varphi_T^*(G_2^{(TS)})$ can be written in the form

$$\mathcal{G}_2^{(TS)} = -h'(S,Y_i)dS^2 + \sum_{jk} k'_{jk}(S,Y_i)dY_jdY_k \ , \tag{7.53}$$

with

$$h'(S,Y_i) = (ST_S + \sum_i Y_iT_{Y_i})T_{SS} \ ; \quad k'_{jk}(S,Y_i) = (ST_S + \sum_i Y_iT_{Y_i})T_{Y_jY_k} \ . \tag{7.54}$$

The metric is identical to the previous one. Therefore, the earlier discussion is befitting in this case as well. So, the scalar curvature diverges at the critical point for this set up as well with the maximum divergence being proportional to the inverse square of T_{SS}, i.e., $R_2^{(TS)}|_{max.diver.} \sim \mathcal{O}(1/T_{SS}^2)$.

Thus we can see that the two criticality conditions in the T-S picture of the non-extended phase space corresponds to the two singular scalar curvature. In other words, one can say that the T-S criticality of the black holes can be obtained when the scalar curvature diverges simultaneously in the two pictures.

7.6 Y-X criticality

Besides the $P-V$ and the $T-S$ criticality, for the black holes there are also the $Y-X$ criticality as shown in some recent works (for eg. [60]). This criticality is also described in the non-extended phase space and the conditions, like the earlier ones, are $(\partial Y_i/\partial X_i) = 0 = (\partial^2 Y_i/\partial X_i^2)$. The appropriate quantity to start with is the Gibbs free energy to form the thermogeometrical metric for the first condition. The Gibbs free energy of the black holes is defined as $G = E - TS - \sum_i X_iY_i$ (see [286, 287]), where $E = M$ in this case. Using the first law $dE = TdS + \sum_i X_idY_i$, one obtains $dG = -SdT - \sum_i Y_idX_i$. The thermodynamic phase space $\mathcal{T}$ can be defined by the coordinates $Z^a = \{G, \tilde{X}^a, \tilde{P}^a\}$ with $\tilde{X}^a = \{T, X\}$ is the thermodynamic variables and $\tilde{P}^a = \{G_T = -S, G_{X_i} = -Y_i\}$ is the corresponding conjugate variables. Also, we assume the subspace $\mathcal{E}$ with the smooth mapping $\varphi_G : \mathcal{E} \to \mathcal{T}$ and $\mathcal{E}$ is called the space of equilibrium states for $\varphi_G^*(\theta_G) = 0$. Like our previous cases, we want to define a thermogeometrical metric in a Legendre-invariant way (induced on $\mathcal{E}$ for $\varphi_G^*(\theta_G) = 0$) which is given as

$$\mathcal{G}_1^{(YX)} = -\sum_{ij} f''_{ij}(T,X_i)dX_idX_j + g''(T,X_i)dT^2 \ , \tag{7.55}$$

with

$$f''_{ij}(T, X_i) = G_{X_iX_j}(-TS - \sum_k Y_k X_k) \ ; \ \ \gamma''(T, X_i) = G_{TT}(-TS - \sum_k Y_k X_k) \ . \tag{7.56}$$

At the critical point, since $G_{X_iX_i} = -\partial Y_i/\partial X_i$, we have vanishing of $G_{X_iX_i}$. Therefore, like the previous cases, the scalar curvature corresponding to this metrics becomes singular and the maximum divergence of the scalar curvature is proportional to the inverse square of $G_{X_iX_i}$ i.e., $R_1^{(YX)}|_{max.diver.} \sim \mathcal{O}(1/G^2_{X_iX_i})$.

To give the geometric interpretation of the second condition we express the charge Y_i as a function the existing potential X_i and temperature T i.e., $dY_i = \sum_j Y_{iX_j} dX_j + Y_{iT}dT$ with $Y_{iX_j} = (\partial Y_i/\partial X_j)_T$ and $Y_{iT} = (\partial Y_i/\partial T)_{X_i}$. The thermodynamic phase space $\mathcal{T}$ is defined by the coordinates $Z^a = \{Y_i, \tilde{X}^a, \tilde{P}^a\}$ with $\tilde{X}^a = \{X_j, T\}$ and $\tilde{P}^a = \{Y_{iX_j}, Y_{iT}\}$. The Legendre-invariant metric is given as (which is induced on $\mathcal{E}$ and $\mathcal{E}$ is defined as the same way as done earlier)

$$\mathcal{G}_2^{(YX)} = -\sum_{i,j} h''_{mij}(T, X_i) dX_i dX_j + k''_m(T, X_i) dT^2, \tag{7.57}$$

with

$$\begin{aligned} h''_{mij}(T, X_i) &= Y_{mX_iX_j}(\sum_k Y_{mX_k} X_k + Y_{mT} T) \ ; \\ k''_m(T, X_i) &= Y_{mTT}(\sum_k Y_{mX_k} X_k + Y_{mT} T) \ . \end{aligned} \tag{7.58}$$

Now, the metric elements $h''_{iii}(T, X_i)$ vanishes at the critical point as $Y_{iX_iX_i} = 0$ at that concerned point. Therefore, like the earlier cases, the scalar curvature diverges at that point with the maximum divergence being proportional to the inverse square of $Y_{iX_iX_i}$ i.e., $R_2^{(YX)}|_{max.diver.} \sim \mathcal{O}(1/Y^2_{iX_iX_i})$. So, at the critical point, the condition $(\partial^2 Y_i/\partial X_i^2)_T = 0$ yields the divergence of the Ricci scalar.

Thus we see, for each criticality we can obtain two sets of metrics and the corresponding Ricci-scalars. In every situation, the critical point is determined by the divergence of these scalars. For the $P-V$ criticality has been discussed in extended phase-space; whereas others are in usual phase-space. This is the consequence of the original description of criticalities in different situations.

7.7 Conclusions

Since long, physicists are keen to know whether all the features of the conventional thermodynamics is present in black hole thermodynamics as well. One of the conspicuous outcomes of the analysis resulted in the conclusion that the critical behavior of the usual thermodynamic system can also be found in the black hole mechanics as well. To fit the criticality properly in the black hole thermodynamics, it became important to describe the criticality in terms of the geometry. In this spirit, Weinhold and later Ruppeiner appeared with the idea of the formation of the thermogeometrical metric to describe the thermal interaction in terms of the curvature of the geometry. It was later found that, in several cases, these two formulations do not provide the same result. This inconsistency is believed to appear because of the fact that the two metrics are not formulated in a Legendre-invariant way. Later, Quevedo et al. were able to formulate the thermogeometrical metric in a Legendre invariant way to describe the phase transition in a geometrical way. In the study of the black hole criticality, one can find $P-V$ criticality, $T-S$ criticality and also the $Y-X$ criticality as mentioned earlier. There had been made multiple isolated attempts to explain those criticality in a geometrical way. Those attempts mainly highlight the criticality as the point of diverging heat capacity rather than one as a point of inflection and so far, as we know, no work could explain all these criticality in a unified way.

In this work, we have given the geometrical description of the critical point as a point of inflection and the whole work is independent of any particular spacetime. Moreover, the analysis can explain all the criticalities in a unified way. Besides, we have followed the recent trend of defining the metric in a Legendre invariant way to avoid the inconsistency in the result. For the $P-V$ criticality, we have taken the Helmholtz free energy (F) in the extended phase space and have defined our thermogeometrical metric in the invariant θ_F picture. It has been shown that the scalar curvature diverges at the critical point while satisfying one criticality condition i.e. $(\partial P/\partial V)_{T,Y} = 0$. For another condition, i.e., the condition of the point of inflection, the pressure is expressed as a function of volume, temperature and the charges and the metric has been defined in invariant θ_P picture. In this case as well one gets the singular behavior of the scalar curvature at the critical point. Later, it has been shown that the previous discussions can further be extended for the other criticalities ($T-S$ and $Y-X$) as well. For the $T-S$ criticality, the Helmholtz free energy is replaced by the internal energy (E) in the non-extended phase space and the metric is defined in invariant θ_E picture to get the singular behavior of the thermogeometrical metric at the critical point. The second metric for this criticality is made on the invariant θ_T picture. It has been shown that the scalar curvature in that picture also diverges at the critical point of the $T-S$ criticality. The same procedure has been followed for the $Y-X$ criticality as well where the two metrics were formulated in θ_G (G being the Gibbs free energy) and θ_Y picture and the Ricci-scalar corresponding to these metrics diverges at the critical point.

Here we want to mention the fact that in the most of the works people takes the thermodynamic potential as the function of the extensive variables to form the thermogeometrical metric. For the ordinary system in the thermodynamics, the fundamental equation comes from the internal energy which is the function of the extensive variables only. But, for the non-ordinary system such as black holes, the fundamental equation might come from the thermodynamic potential which is a function of both the extensive and the intensive variables as we have seen here. Although, [70] only discusses this issue in a brief, but it has also taken the thermodynamic potential as the function of only the extensive variables. Also, [71] takes the thermodynamic potential as the function of both the extensive and the intensive variables (see eq. 6.14) [2]. We want to mention it clearly that taking the thermodynamic potential as the function of the extensive variables has nothing to do with Legendre-invariance. Moreover, apart from defining the thermogeometrical metric in the thermodynamic potential picture (such as in F, E, G etc.), one can also define it in thermodynamic variable picture (such as in S picture which is done in [70, 73]). Here, we have done the same to describe the point of inflection. But unlike in S-picture, the thermodynamic variables in our cases are both the intensive and the extensive variables. Again, it has no connection with the Legendre invariance, the only condition which we have given the utmost importance in our analysis.

Let us make some additional comments. In the $T-S$ and the $Y-X$ cases, the critical point is determined by the conditions in which $\Lambda = -3/l^2$ is kept constant (where l is the AdS curvature radius). Therefore, one is not needed to go to the extended phase space for these cases. The usual first law is well enough to describe the criticality. However, it should be remembered that this criticality can be described only for the AdS black holes and not for the asymptotically flat black holes as the critical temperature at the asymptotic flat limit ($l \to \infty$) vanishes and the charge and the entropy diverges (for an explicit example see Eq.(3.14) of [59]). Therefore,

[2] Also we find [45], where the Ruppeiner metric is written in T, J coordinates (see eqn. (21))

the AdS nature of the black hole is required. On the other hand, the arguments made in the sections 7.4.1 and 7.4.2 are not applicable for the asymptotically flat black holes. The reason is in these cases cosmological constant $\Lambda = 0$ and hence there is no pressure term in first law of thermodynamics (note pressure is defined by the relation $P = -\Lambda/8\pi$). Due to the absence of the pressure term, the question of presence of the $P - V$ criticality does not arise.

Another very interesting point may be worth mentioning. There are some works [288, 289] which discuss the thermodynamic instability of the black holes and also argues how the thermodynamic instability is connected to the dynamical instability. For the normal (non-extended) phase space, there is a prescription which says that the thermodynamic and dynamical instability implies $\mathcal{B} < 0$ under certain conditions (for details and the definition of $\mathcal{B}$ see [289]). The quantity $\mathcal{B}$ is called "canonical energy". Now it has been proved that if there exists a turning point (see [289] for the definition) in the system, there must be a thermodynamic instability on one side of that turning point. For the present case, note that the critical point is not an extremum rather a point of inflection and, hence, it is not a turning point. Therefore, we can not apply this theorem to discuss whether there exists any thermodynamic instability in the current scenario. So one needs to go back to the basic condition (i.e. the condition on $\mathcal{B}$) to explore such possibilities. Now remember that the $P-V$ criticality appears in the extended phase space. Therefore in this case, one needs to construct the proper quantity like $\mathcal{B}$ from first principles; i.e. one needs to start from scratch and all these prescriptions should be revamped. Of course, the cases like $T-S$ and $Y-X$ can be discussed in the already existing framework by explicit use of the particular black hole metric. Considering it to be a wonderful and sizable future work, we have kept aside the discussion of the dynamic and the thermodynamic stability for the future and have focused ourselves for the geometrical description of the critical point in a metric independent way.

We provide some additional comments in the following on why we feel two metrics are required for the thermogeometric description of each type of criticality. Note, in thermogeometric description, the condition of criticality corresponds to the divergence of the scalar curvature of a properly constructed thermogeometrical metric. Also, there should be a one-to-one correspondence between the criticality condition and the divergence of a Ricci scalar. Therefore, if the Ricci scalar diverges for more than one thermodynamic condition (one criticality condition along with some additional condition), then the thermogeometry is not considered as properly constructed. For Davies type criticality, we have only one critical condition, which is $C_V = 0$ (where C_V is the specific heat at constant volume). In that case, the thermogeometric description is provided in terms of a thermogeometrical metric, Ricci scalar of which diverges when $C_V = 0$. However, in the van der Waals type phase transition, we have two independent conditions of criticality. Both of the conditions are required to be satisfied simultaneously. However, we feel one Ricci scalar of a thermogeometrical metric should not diverge for multiple criticality conditions, as we may not be able to comprehend whether both the criticality conditions are satisfied simultaneously. Therefore, we construct two different Legendre-invariant metrics, scalar curvature of which diverge simultaneously at the critical point. Thereby, we are confirm that both the criticality conditions are satisfied and this reflects on our thermogeometric description.

To sum up, we estimate the critical point as a point where the Ricci scalar of the two different pictures diverge simultaneously. Most importantly, we have geometrically described the critical point in terms of the point of inflection which is completely new. The analysis must give more inputs in both van der Waals like phase transition for AdS black holes as well as geometrothermodynamics.

Appendix

7.A Expression for Ricci scalar

Consider the following metric:

$$ds^2 = -f(V,T,Y)dV^2 + g(V,T,Y)dT^2 + 2h(V,T,Y)dTdY + A(V,T,Y)dY^2 \,. \tag{7.59}$$

The Ricci scalar is found out to be (calculated by Mathematica 11)

$$\begin{aligned}
R = \frac{1}{2f^2\left(h^2 - Ag\right)^2}\Big[\Big\{ & A\Big(g_Y^2 + (A_T - 2h_Y)g_T\Big) + h\Big(- g_Y(2h_Y + A_T) + A_Y g_T + 4h_Y h_T \\
& -2A_T h_T + 2h(g_{YY} - 2h_{TY} + A_{TT})\Big) + g\Big(A_T^2 + A_Y(g_Y - 2h_T) - 2A(g_{YY} - 2h_{TY} + A_{TT})\Big)\Big\} f^2 \\
& + f[-4(f_{TY} - h_{VV})h^3 + (-h_V^2 + 2f_Y g_Y + 2A_T f_T + 2Af_{TT} - 3A_V g_V - 2Ag_{VV})h^2 \\
& -A(g_Y f_T + 2h_T f_T + f_Y g_T - 4g_V h_V)h + A^2(f_T g_T - g_V^2) + g^2\Big(- A_V^2 + A_Y f_Y - 2A(f_{YY} \\
& -A_{VV})\Big) - g\Big\{2(f_{TT} - g_{VV})A^2 + \Big(3h_V^2 - 2h_Y f_T + A_T f_T + f_Y(g_Y - 2h_T) - 4hf_{TY} - A_V g_V \\
& +4hh_{VV}\Big)A + h\Big(f_Y(2h_Y + A_T) + A_Y f_T - 4A_V h_V - 2h(f_{YY} - A_{VV})\Big)\Big\}] \\
& +(Ag - h^2)\Big(g(f_Y^2 - A_V f_V) + A(f_T^2 - f_V g_V) + h(2f_V h_V - 2f_Y f_T)\Big)\Big]
\end{aligned} \tag{7.60}$$

Chapter 8

A general phenomenological description to obtain critical exponents in van der Waals phase transition of black holes

8.1 Universality of critical exponents in different spacetime

[1]In the earlier chapter, we have discussed that the van der Waals type of phase transitions are observed in black hole thermodynamics and also we have provided the thermogeometrical description of that criticality in an unified way. In the present chapter, the goal is to obtain the critical exponents of van der Waals criticality of black holes from a general framework. In most of the works, the van der Waals criticality in black holes is studied for a particular spacetime and the critical exponents are obtained. This criticality was first shown for charged AdS black holes [56] and then was obtained in several other spacetimes. Some examples are: charged and rotating black holes in higher dimensional spacetime [290], charged topological black hole [291], quantum corrected black hole [292], dilaton black hole [293, 294], Gauss-Bonnet black hole [295, 296] and also for black holes in Lovelock gravity [297–299] quasi-topological gravity [300] and conformal gravity [301]. However, although these analysis [56, 57, 290–308] have been made in different spacetimes, the critical exponents are the same irrespective of the spacetimes. Therefore, intuitively one can

[1]This chapter is based on the publication [110] .

expect that the values of the critical exponents do not depend on the spacetime and there must be a method to obtain the critical exponents in a metric-independent way. Following Landau's approach, in this chapter, we shall provide a metric-independent phenomenological description to obtain the critical exponents for van der Waals type phase transitions in black hole.

In usual thermodynamics, critical phenomena are thoroughly analyzed topics. There it is well known that critical exponents of completely dissimilar processes, like liquid-vapour and ferromagnet to paramagnet phase transition, can be exactly same (see [309, 310] for details). The reason is that near the phase transition point, upto certain order, van der Waals equation for fluid and Weiss equation for magnet are same. Taking this hint, Landau made a brilliant guess about a common free energy for all second order phase transition. The beauty of this approach is that, the expression of energy is obtained from general symmetry consideration without invoking any microscopic detail. This theory plays a central role in the study of critical phenomena because the values of critical exponents can be calculated easily from this Landau free energy (also known as *Landau function*).

The remarkable aspect of black hole criticality (van der Waals type) is that its similarity with the criticality in conventional thermodynamics is not only qualitative but also quantitative. The critical exponents of both systems (liquid vapour and black hole) are exactly same. This suggests that a free energy for black hole can be proposed without detail consideration of the spacetime. In the present chapter we explore this possibility. We find that an expression for Helmholtz free energy can be given from general consideration alone where only assumption is the existence of van der Waals type phase transition.

Recently, within the domain of van der Waals type phase transition some points known as *isolated* critical points have been found where critical exponents have different values [297, 298, 300, 311, 312]. These values are somewhat unusual because they do not satisfy the scaling laws. In this chapter we show that appropriate Helmholtz free energy can be constructed which will give the correct values of the exponents for the isolated critical points. The whole formalism is then used to study $Y-X$ and $T-S$ criticality in non-extended phase space. In the former case Gibbs free energy and in the later case internal energy (for black holes, it is the mass) serve the purpose. Reassuringly in all cases we get the correct values of the critical exponents. Thus we show critical phenomena can be described correctly by three energy functions for three different cases. (i) For $P-V$ criticality it is Helmholtz free energy, (ii) for $Y-X$ criticality it is Gibbs free energy and (iii) for $T-S$

criticality it is internal energy. These are the Landau functions of this type of black hole phase transition.

In the following, we present the discussions for each van der Waals type criticality in black hole thermodynamics. The only input that we need here is that the black hole spacetime shows the van der Waals criticality with the specific criticality conditions (for example, $(\partial P/\partial V)_{T_c} = 0 = (\partial^2 P/\partial V^2)_{T_c}$ for the $P - V$ criticality of black holes). There is no other information required about the spacetime in our analysis. In that sense, the whole analysis is very general and can be applicable in any spacetime which shows the van der Waals phase transition. This demonstrates that if there is a van der Waals type phase transition in a black hole in any arbitrary spacetime, the values of critical exponents must be the same with another one in a different spacetime and these values can be calculated without any further assumption about the spacetime. Although, intuitively it is expected, but it is not explicitly shown.[2]

8.2 $P - V$ criticality

For AdS black holes, with varying cosmological constant, the first law of thermodynamics is obtained earlier in (7.22), where we have shown that the mass of the black hole plays the role of enthalpy of the black hole thermodynamic system. Also, we have obtained the thermogeometric description of first condition of criticality (7.31), which described in the F-picture, where F, the free energy of the system, is defined in (7.28). Here we shall see that F also plays the role of Landau function for the $P - V$ criticality of black holes. From (7.29), it can be mentioned that F is function of independent variables T, V and Y_i. However, it has been observed that the AdS black holes show van der Waals type feature when Y_i's are kept constant [56]. Since we are interested in this issue, we assume that there exists this type of phase transition and hence one can consider $F = F(V, T)$.

Now, the thermodynamic quantities, which shows the singular behaviour on the

[2]Readers can also follow another work [281] in the similar context, which has obtained critical exponents in a different way.

critical point, are related by the critical exponents $(\alpha, \beta, \gamma, \delta)$ as [295] [3]:

$$\begin{aligned}
&(P - P_c) \sim (V - V_c)^{\delta} \ ; \\
&(V_g - V_l) \sim (T - T_c)^{\beta} \ ; \\
&C_V \sim (T - T_c)^{-\alpha} \ ; \\
&K_T \sim (T - T_c)^{-\gamma} \ .
\end{aligned} \tag{8.1}$$

Here, we have denoted P_c, V_c and T_c as the critical values of pressure, volume and temperature of BH respectively. In usual thermodynamics V_g and V_l are identified as the volume of the gaseous state and of the liquid state of the van der Waals fluid system. In the present case, those are interpreted as the volume of the large BH and the volume of the small BH respectively [56]; i.e. $V_g \equiv$ volume of the large BH and $V_l \equiv$ volume of the small BH. Moreover, in the above, C_V and K_T are the specific heat at constant volume and isothermal compressibility, respectively. Since we are interested in the behavior of the AdS black holes near the critical point, let us expand the Helmholtz free energy around this:

$$\begin{aligned}
&F(V,T) = a_{00} + a_{10}(V - V_c) + a_{01}(T - T_c) \\
&+a_{11}(V - V_c)(T - T_c) + a_{02}(T - T_c)^2 \\
&+a_{21}(V - V_c)^2(T - T_c) + a_{12}(V - V_c)(T - T_c)^2 \\
&+a_{03}(T - T_c)^3 + a_{40}(V - V_c)^4 + a_{31}(V - V_c)^3(T - T_c) \\
&+a_{22}(V - V_c)^2(T - T_c)^2 + a_{13}(V - V_c)(T - T_c)^3 \\
&+a_{04}(T - T_c)^4 + \dots \ .
\end{aligned} \tag{8.2}$$

In the above a_{ij} is i^{th} order the partial derivative of F with respect to V and j^{th} order partial derivation of F with respect to T which must be evaluated at the critical point. Note that, coefficients a_{20} and a_{30} are absent due to the conditions, presented in (7.31). We shall observe that retaining upto $\mathcal{O}(T - T_c)^2$ in $(T - T_c)$, $\mathcal{O}(V - V_c)^4$ in $(V - V_c)$ and $\mathcal{O}(V - V_c)^2(T - T_c)$ while ignoring the other terms will be sufficient for determining the critical exponents. This implies, as will be clear later, the free energy is sensitive upto order $(T - T_c)^2$ as $(V - V_c) \sim (T - T_c)^{1/2}$. The same also happens for the usual van der Waals gas system (see the discussion

[3]Usually the critical exponents are defined in terms of specific volume. But since in the case of AdS black holes, the thermodynamic volume V and the specific volume are proportional to each other, we introduce the definitions in terms of V to keep the analysis simple.

after Eq. (4.37) of [310]). In this case our free energy takes the following form:

$$F = a_{00} + a'_{10}v + a'_{01}t + a'_{11}vt + a'_{02}t^2 + a'_{21}v^2t + a'_{40}v^4 , \tag{8.3}$$

where we have used

$$t = \frac{T}{T_c} - 1; \quad v = \frac{V}{V_c} - 1 . \tag{8.4}$$

In the above the constant coefficients are rescaled quantities of those appearing in (8.2) by the factors like V_c and T_c. Since explicit form of these are irrelevant for our present discussion, we are not bothering about their actual values.

With this near critical expression of F, let us now find the critical exponents of the theory. For that, let us note

$$\left(\frac{\partial F}{\partial v}\right)_t = a'_{10} + a'_{11}t + 2a'_{21}vt + 4a'_{40}v^3 . \tag{8.5}$$

Since, the thermodynamic pressure can be defined from the Helmholtz free energy as $P = -(\partial F/\partial V)_T = -(1/V_c)\,(\partial F/\partial v)_t$, one obtains the expression of the pressure in the following form

$$P = P_c + a''_{11}t + a''_{21}vt + a''_{04}v^3 , \tag{8.6}$$

where the double primed constants are again rescaled ones of the earlier constants. This nomenclature will be adopted in later analysis also and hence from now we shall not mention this explicitly. Defining $\pi = P/P_c$, one gets Eq. (8.6) in a more useful form as

$$\pi = 1 + a'''_{11}t + a'''_{21}vt + a'''_{04}v^3 . \tag{8.7}$$

For the isothermal curves (fixed t) in the $P - V$ diagram, it has to satisfy Maxwell construction $\oint vdp = 0$ (as it has been already assumed that the AdS black hole performs van der Waals type phase transition), where $p = P/P_c - 1 = \pi - 1$. Now from (8.7) $dp = d\pi = (a'''_{21}t + 3a'''_{04}v^2)dv$, giving

$$\int_{v_l}^{v_g} v(a'''_{21}t + 3a'''_{04}v^2)dv = 0 , \tag{8.8}$$

where v_l and v_g correspond to the volumes of the black hole at two phases. The above Eq. (8.8) is valid for all the (non-critical) isotherms, which are near about the critical point. Let us now focus on those isotherms which are very near to the critical one. If one considers a particular non-critical isotherm, then t is constant and $v \to 0$ for the curve and, therefore, the second term in (8.8) can be neglected in those cases. Thereafter, performing the integration and equating it with zero on the right hand side implies $v_g = -v_l$. Remember, here v_g and v_l are calculated from the critical point, i.e., from $v = 0$.

Now since both at v_l and v_g the pressure is same, one can express (8.7) terms of v_l and v_g, which yields

$$\pi = 1 + a'''_{11}t + a'''_{21}v_g t + a'''_{04}v_g^3 \; ;$$
$$\pi = 1 + a'''_{11}t + a'''_{21}v_l t + a'''_{04}v_l^3 \; . \tag{8.9}$$

Since $v_g = -v_l$, writing v_l in terms of v_g and then subtracting the above two equations of (8.9), one obtains

$$a'''_{21}v_g t + a'''_{04}v_g^3 = 0 \; . \tag{8.10}$$

Since $v_g \neq 0$, one obtains $v_g \sim (T - T_c)^{1/2}$, which implies

$$(V_g - V_l) \sim (T - T_c)^{1/2} \; . \tag{8.11}$$

So we find $\beta = 1/2$. This also implies that $v \sim (T - T_c)^{1/2}$ at the leading order which was mentioned earlier to write Eq. (8.3). Setting $t = 0$ in the expression of π in (8.7) yields $\pi - 1 = c''_2 v^3$, which implies

$$(P - P_c) \sim (V - V_c)^3 \; , \tag{8.12}$$

and therefore we obtain $\delta = 3$.

Let us now find out the relation between the specific heat at constant volume and temperature. It is by definition $C_V = T(\partial S/\partial T)_V$ and therefore using (7.30) one obtains $C_V = -T\left(\partial^2 F/\partial T^2\right)_V = -(T/T_c^2)\left(\partial^2 F/\partial t^2\right)_v$. From (8.3), one can straightforwardly obtain $C_V \sim (1+t)a'_{02}$. This implies that near the critical point it is dominated by a constant; i.e.

$$C_V \sim const \; , \tag{8.13}$$

and hence $\alpha = 0$. The isothermal compressibility is defined as $K_T = -(1/V)(\partial P/\partial V)_T^{-1}$. From (8.6) one obtains

$$K_T \sim (T - T_c)^{-1} \ , \tag{8.14}$$

at $V = V_c$. This means $\gamma = 1$.

Note, that the obtained values of the critical exponents i.e., $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$ and $\delta = 3$ obeys the scaling laws of the ordinary thermodynamic systems. Those are as follows:

$$\begin{aligned}
&\alpha + 2\beta + \gamma = 2 \ , \\
&\alpha + \beta(\delta + 1) = 2 \ , \\
&\gamma(\delta + 1) = (2 - \alpha)(\delta - 1) \ , \\
&\gamma = \beta(\delta - 1) \ .
\end{aligned} \tag{8.15}$$

It must be mentioned that these are values for critical exponents one obtains earlier for the AdS black holes by considering the explicit form of the metrics. Here we see that if any black hole metric exhibit the van der Waals like isotherms and the free energy near critical point has the form (7.31), the phase transition must be associated with the above exponents.

Having the standard values of the critical exponents for AdS black holes, let us now concentrate on an exceptional case. There are a few works [297, 298, 300, 311, 312] that describes about the *isolated* critical point in Lovelock gravity, which is characterized by a different set of critical exponents. Most of these works find the isolated critical point in 3rd-order Lovelock gravity . Presence of isolated critical point for any order Lovelock gravity has been, however, discussed in [297]. Moreover, the critical exponents, which are unlike that of the van der Waals system, do not obey all the scaling laws mentioned in above Eq. (8.15).

On the isolated critical point, the $P-V$ curves of various isotherms coincide (but not cross each other); i.e. they just touch each other. Therefore, for the non-critical (as well as critical) isotherms, the conditions of coincidence at the isolated singular point are given as

$$\begin{aligned}
&\Big(\frac{\partial P}{\partial T}\Big)_V\Big|_c = 0 \ , \\
&\Big(\frac{\partial}{\partial T}\Big(\frac{\partial P}{\partial V}\Big)_T\Big)_V\Big|_c = 0 \ .
\end{aligned} \tag{8.16}$$

The above two relations are coming from the fact that the pressure and the tangent of the $P-V$ curves (i.e., $(\partial P/\partial V)_T$) are same on the isolated critical point for the various isotherms; or in other words, they are independent of the temperature. Note, that isolated critical point is a thermodynamic singular point as well [298] and therefore $(\partial P/\partial V)_T = (\partial P/\partial T)_V = 0$. So in this case, in addition to the conditions (7.31), the isotherms must satisfy (8.16) at the critical point.

As mentions above, the isolated critical point is obtained by the four conditions mentioned in (7.31) and (8.16). Therefore, the free energy can be expanded in the following form

$$F = a_{00} + a'_{10}v + a'_{01}t + a'_{02}t^2 + a'_{12}vt^2 + a'_{03}t^3 \\ +a'_{22}v^2t^2 + a'_{31}v^3t + a'_{13}vt^3 + a'_{40}v^4 + a'_{04}t^4 \ . \tag{8.17}$$

Due to the four conditions of the isolated critical point, the term containing v^2, v^3, vt and v^2t do not appear here. As we shall show later, in this case $v \sim t$ and so the terms upto $\mathcal{O}(v^4)$ are kept in the above which are sufficient to extract the required information. Straight forwardly, one obtains the expansion of the pressure near about the isolated critical point as

$$p = a'''_{12}t^2 + a'''_{22}vt^2 + a'''_{31}v^2t + a'''_{13}t^3 + a'''_{04}v^3 \ , \tag{8.18}$$

Note, that in this case the equal area law for Maxwell construction is redefined as

$$\int_{v_l}^{v_g} v(a'''_{22}t^2 + 2a'''_{31}vt + 3a'''_{04}v^2)dv = 0 \ . \tag{8.19}$$

Applying the same argument as earlier, one can conclude $v_l = -v_g$. Also, the Eq. (8.10) gets modified here as

$$a'''_{22}v_gt^2 + a'''_{04}v_g^3 = 0 \ , \tag{8.20}$$

which implies

$$(V_g - V_l) \sim (T - T_c) \ . \tag{8.21}$$

Therefore one finds $\tilde{\beta} = 1$.

The relation between the pressure and the volume can be obtained setting $t = 0$

in Eq. (8.18), which yields

$$(P - P_c) = (V - V_c)^3 \ , \tag{8.22}$$

implying $\tilde{\delta} = 3$ as the earlier case. In this case, the variation of the specific heat with the temperature is the same as earlier, i.e. $C_V \sim (1 + t)a'_{02}$. Thus

$$C_V \sim const \ , \tag{8.23}$$

near the critical point and hence $\tilde{\alpha} = 0$. Also, the isothermal compressibility $K_T = -(1/V)(\partial P/\partial V)_T^{-1}$ can be obtained from (8.18), which will yield

$$K_T \sim (T - T_c)^{-2} \ , \tag{8.24}$$

at $V = V_c$. This means $\tilde{\gamma} = 2$ in this case.

Note, for the isolated critical point ($\tilde{\alpha} = 0$, $\tilde{\beta} = 1$, $\tilde{\gamma} = 2$ and $\tilde{\delta} = 3$) all the scaling laws of (8.15) are not satisfied. We found that only $\tilde{\gamma} = \tilde{\beta}(\tilde{\delta} - 1)$ remains valid. It implies that out of these three critical exponents (i.e., $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\delta}$) only two are independent.

8.3 $Y - X$ criticality

As it is well known, the $Y - X$ and the $T - S$ criticalities take place in the non-extended phase space, where the cosmological constant is treated as a true constant in the theory. In this section, we shall find out the critical exponents of the thermodynamic quantities which become singular at the critical point. Those quantities are supposed to follow the following relations in terms of the critical exponents [271, 313].

$$\begin{aligned}
&(Y - Y_c) \sim (X - X_c)^{\delta} \ ; \\
&(X - X_c) \sim (T - T_c)^{\beta} \ ; \\
&C_Y \sim (T - T_c)^{-\alpha} \ ; \\
&K_T \sim (T - T_c)^{-\gamma} \ .
\end{aligned} \tag{8.25}$$

In the earlier chapter, we have shown that the first criticality condition can be describe thermogeometrically in the G picture, where G is the Gibbs free energy and is a function of T and X_i, i.e., $G = G(T, X_i)$. Here, we take is as the Landau function. For the sake of generality in the analysis, we have started with the presence of multiple charges in the theory. However, when one discusses the criticality in the picture of a particular charge Y and its corresponding potential X, the other charges and the corresponding potentials are kept constant. Therefore, from here on we shall keep only X and Y, while the other charges will be suppressed. That will not create any loss of generality. Now, the criticality conditions of the $Y - X$ criticality are written as

$$\begin{aligned}\Big(\frac{\partial Y}{\partial X}\Big)_T\Big|_c &= -\Big(\frac{\partial^2 G}{\partial X^2}\Big)_T\Big|_c = 0\ ;\\ \Big(\frac{\partial^2 Y}{\partial X^2}\Big)_T\Big|_c &= -\Big(\frac{\partial^3 G}{\partial X^3}\Big)_T\Big|_c = 0\ .\end{aligned} \tag{8.26}$$

To provide a general prescription of determining the critical exponents for this criticality, we expand the Gibbs free energy about the critical point:

$$\begin{aligned}G(T,X) &= b_{00} + b_{10}(T-T_c) + b_{01}(X-X_c)\\ &+b_{11}(T-T_c)(X-X_c) + b_{20}(T-T_c)^2\\ &+b_{21}(T-T_c)^2(X-X_c) + b_{12}(T-T_c)(X-X_c)^2\\ &+b_{30}(T-T_c)^3 + b_{31}(T-T_c)^3(X-X_c)\\ &+b_{22}(T-T_c)^2(X-X_c)^2 + b_{13}(T-T_c)(X-X_c)^3\\ &+b_{04}(X-X_c)^4 + b_{40}(T-T_c)^4 \dots\ .\end{aligned} \tag{8.27}$$

Following the same arguments as after Eq. (8.2), here also the coefficients b_{02} and b_{03} are absent in the above expansion due to the criticality conditions.

It will be shown later that near the critical point, $(X - X_c) \sim (T - T_c)^{1/3}$. So, in the expansion of $G(T,X)$, keeping $\mathcal{O}(T - T_c)$ in $(T - T_c)$ and $\mathcal{O}(X - X_c)^4$ in $(X - X_c)$ will be enough to serve our purpose. One can, however, keep the higher order terms which will not make impact in the main analysis. Therefore, keeping the relevant terms in the expansion of G in terms of t and $x = \frac{X}{X_c} - 1$, one obtains

$$G = b_{00} + b'_{10}t + b'_{01}x + b'_{11}tx + b'_{04}x^4\ . \tag{8.28}$$

Now, $Y = (\partial G/\partial X)_T = (1/X_c)(\partial G/\partial x)_t$, which implies

$$Y = Y_c + b''_{11}t + b''_{04}x^3\ , \tag{8.29}$$

and so

$$(Y - Y_c) \sim (X - X_c)^3 , \tag{8.30}$$

for $T = T_c$. Therefore, we get $\delta = 3$.

Also, from (8.29) we get

$$(X - X_c) \sim (T - T_c)^{1/3} \text{ when } Y = Y_c , \tag{8.31}$$

which implies $\beta = 1/3$.

Now, the entropy is given as $S = -(\partial G/\partial T)_X = -(1/T_c)(\partial G/\partial t)_x$, which is found out to be

$$\begin{aligned} S &= b'''_{10} + b'''_{11} x , \\ &= b'''_{10} + b'''_{11} \left(\frac{Y - Y_c - b''_{11} t}{b''_{04}} \right)^{1/3} , \end{aligned} \tag{8.32}$$

To obtain the last step, we have used the Eq. (8.29). It is needed to be done because we want to obtain the expression of the specific heat C_Y using the relation $C_Y = T(\partial S/\partial T)_Y = (1+t)(\partial S/\partial t)_Y$ from (8.32). Ultimately, one can get $C_Y = (1+t)t^{-2/3}$ when $Y = Y_c$. Taking the leading order contribution, we obtain

$$C_Y \sim (T - T_c)^{-2/3} \text{ when } Y = Y_c , \tag{8.33}$$

which implies $\alpha = 2/3$. Now, $K_T = -(1/X)(\partial X/\partial Y)$. Therefore, from Eq. (8.29) one obtains at $Y = Y_c$

$$K_T \sim t^{-\frac{2}{3}} , \tag{8.34}$$

implying $\gamma = 2/3$.

Let us mention that the obtained values of the critical exponents in this picture ($\alpha = 2/3$, $\beta = 1/3$, $\gamma = 2/3$ and $\delta = 3$) also obeys the scaling laws mentioned in (8.15). These are again the same which were obtained by explicit use of black hole metric.

8.4 $T - S$ criticality

In literature one can found a number of papers that discusses the $T - S$ criticality. However, few have mentioned the critical exponents in this criticality picture. Only a few [59] mention that

$$C_Y \sim (T - T_c)^{-\alpha} , \tag{8.35}$$

and the value of α is expected to be the same as the $Y - X$ criticality picture. This relation can be obtained starting from the internal energy. The criticality condition for the $T - S$ criticality has been mentioned in the earlier chapter. In keeping with the arguments provided in the earlier sections of this chapter, here we take the internal energy as the Landau function, which is actually the mass of the black hole in this picture and a function of entropy and the charges. In the following, we have considered the presence of a single charge (Y) to determine C_Y, as the other charges can be taken as constants while obtaining C_Y and, hence, all other charges ($Y's$) and the corresponding potentials ($X's$) can be suppressed in the theory. In this situation, the criticality conditions are given as

$$\begin{aligned} \left(\frac{\partial T}{\partial S}\right)_Y\Big|_c &= 0 = \left(\frac{\partial^2 E}{\partial S^2}\right)_Y\Big|_c , \\ \left(\frac{\partial^2 T}{\partial S^2}\right)_Y\Big|_c &= 0 = \left(\frac{\partial^3 E}{\partial S^3}\right)_Y\Big|_c \end{aligned} \tag{8.36}$$

From the analogy of (8.3) and (8.28), proceeding in identical way the internal energy can be expanded near the critical point in terms of $s = S/S_c - 1$ and $y = Y/Y_c - 1$ as

$$E = c_{00} + c_{01}s + c_{10}y + c_{11}ys + c_{21}y^2s + c_{12}ys^2 + c_{04}s^4 . \tag{8.37}$$

Again, using $T = (\partial E/\partial S)_Y$ one obtains

$$t = b'_{11}y + b'_{21}y^2 + b'_{12}ys + b'_{04}s^3 . \tag{8.38}$$

Now as $C_Y = T(\partial T/\partial S)_Y^{-1}$, from Eq. (8.38), one obtains $C_Y \sim (1 + t)t^{-2/3}$ when $y = 0$. Taking the leading order contribution near the critical point

$$C_Y \sim (T - T_c)^{-2/3} , \tag{8.39}$$

which implies, $\alpha = 2/3$.

8.5 Conclusions

There are many similarities between black hole and usual thermodynamic systems. Recent studies show that if one interprets cosmological constant as a pressure term, black hole phase transition and liquid vapour transition are identical in nature. The critical exponents of both the systems are same and near the critical point equations of states are also same.

In usual thermodynamics, some common features of all second order phase transitions motivated Landau to construct a common energy function describing the behaviour of dissimilar systems near the critical point. This method is extremely powerful because structure of Landau energy can be given from general symmetry consideration without considering underlying statistical behaviour of molecules.

In this chapter we followed the same spirit to study the black hole phase transition in extended space from energy point of view. We showed that, if one accepts the existence of van der Waals type phase transition for a black hole, it is possible to give an expression of Helmholtz free energy near the critical point. This Helmholtz energy can then be used to find the critical exponents. This naturally explains the universality of critical exponents. An interesting point is, within this framework, non-standard values of exponents for isolated critical point can be described.

Black hole phase transition in non-extended space has also been studied. The suitable energy function for $X-Y$ criticality is Gibbs energy and for $T-S$ criticality it is internal energy. The appropriate expressions of both these functions were obtained from general arguments and then critical exponents were calculated. Thus our approach to obtain the values of exponents is reminiscent of Landau's approach to explain the universality of critical phenomena.

It needs to be mentioned that our whole analysis based on two inputs: (a) the AdS black holes perform van der Waals like phase transition with a critical point and (b) there exists first law of thermodynamics for the black holes such that the Landau functions can be expressed as relevant variables. What we found is that these set of Landau functions correctly reproduce the known critical exponents. This implies that if any system has the above features, that must associated with these sets of exponents near the critical point. Therefore our analysis gives a strong reason for such universality. Hope this will shed some light in the black hole phase transition paradigm.

We end up this chapter with the following comments. In our analysis, we have checked how the black hole changes its phase due to its change in macroscopic parameters. With the identification of black hole mass as the enthalpy, cosmological constant as pressure, its conjugate quantity as volume, surface gravity of BH horizon as the temperature, area of the BH horizon as the entropy etc., we show that the black hole system behaves exactly as a van der Waals gas system and shows the van der Waals type criticality. However, it is well-known that black hole thermodynamics is different from ordinary thermodynamics in several ways. Some of the major differences have been mentioned in section 3.7 of chapter 3. Therefore, one cannot take it for granted that all the thermodynamic behaviours of black holes will be the same as of the ordinary thermodynamics. In chapters 7 and 8, we have explored this van der Waals phase transition in AdS black holes to find the similarities and differences with the normal thermodynamics. Also, we have provided the unified picture to study the van der Waals criticality in AdS black holes.

However, in the references [314] and [315], the authors study the thermodynamic system, which mimics gravity at certain limit (study of analogue gravity). For example, in [315], the authors discuss that microscopic quantum description of a black hole is analogous to "an overpacked self-sustained Bose-condensate of N weakly-interacting soft gravitons, which obeys the rules of 't Hoofts large-N physics" . In the paper, the authors derive an effective Landau-Ginzburg Lagrangian for the condensate and show that it serves as the black hole analogue of 't Hooft's planar limit. In [314], the event horizon of a black hole has been studied as a quantum phase transition of the vacuum of space-time, which is analogous to the liquid-vapor critical point of Bose fluid. Thus our approaches and motivations are different compared to the references [314, 315].

Chapter 9

Conclusions and scopes for future work

9.1 Conclusions

The major aim of this thesis is to shed some light on the unexplored areas of the black hole thermodynamics. Although black hole thermodynamics has been studied for decades, we have mentioned earlier that there are several motivations for further study along this direction. Especially, in this thesis, we have focused ourselves on the following aspects of black hole thermodynamics: (i) obtaining a consistent and covariant thermodynamic description for the scalar-tensor theory, which is probably the most potential among all the alternative theories of gravity from both the theoretical and observational point of view, (ii) obtaining black hole thermodynamics in Einstein's gravity for more realistic black holes and for non-Killing horizons, (iii) exploring the internal degrees of freedom of the horizon surface, which can give rise to black hole thermodynamics, (iv) providing a general framework to study the van der Waals phase transition in black holes. In the following, the conclusions of the thesis work are provided.

In the second chapter, we have tested the mathematical (in)-equivalence of the Jordan frames and the Einstein frame in the action level. Then we have tested the (in)-equivalence at the thermodynamic level. We obtain the following results:

- The actions of ST gravity, which are taken usually in the two frames, are not exactly equivalent due to the conformal transformation. Instead, those differ by a total derivative term $\Box\phi$. Thus, there remains an built-in in-equivalence of the two frames by neglecting the $\Box\phi$ term in the action level.

- Although the actions in the two frames are not exactly equivalent (differ by a $\Box\phi$ term), the total action (gravitational action along with the GHY term) is equivalent under the conformal ternasformation.

- The unresolved in-equivalence of the two frames in the action level reflects on the holographic relations. The holographic relations (at the action level and at the thermodynamic level) are maintained in the Einstein frame, but it breaks down in the Jordan frame.

- The mentioned in-equivalence in the two frames in the action level also results in the in-equivalence of the conserved Noether currents and potential (due to the diffeomorphism invariance) of the two frames.

- Although there are those in-equivalences of the two frames in the action level, arising due to neglecting the contribution of the $\Box\phi$ term in the action level, we find that the two frames are dynamically and thermodynamically equivalent.

In the third chapter, we have taken the contribution of the $\Box\phi$ term in the action of the Jordan frame and, thereby, we reach to the following conclusions from our analysis.

- If one takes the contribution of $\Box\phi$ term in the action of the Jordan frame, one find that the in-equivalences mentioned in chapter 2 go away.

- With the modified action and following the Wald's approach, one can obtain a consistent and a covariant thermodynamic first law for this theory in the two frames. In addition, one can show that the thermodynamic parameters are exactly equivalent in the two frames without considering any prior assumption.

- We find that the thermodynamic first law and the thermodynamic parameters can also be obtained from the ADT current as we have found that the Wald's approach and the ADT approach are equivalent.

- In addition, we obtain the generalized second law for this theory in the two frames and show that the entropy always increases.

Thus, the discussions presented in the second and the third chapter resolves a long-standing issue of the ST gravity(obtaining a proper covariant thermodynamic description of the ST gravity) and also discuss some key issues regarding the equivalence/in-equivalence of the two conformally connected frames.

In the fourth chapter, we have obtained the correct horizon temperature of the time dependent SD black hole. Earlier, people had debate whether the temperature of this black hole is at all time-dependent. We confirm it by introducing the tunneling formalism near the horizon surface of this black hole. In addition, we find that the expression of temperature fits nicely with the expression of other thermodynamic parameters to give a thermodynamic first law for this black hole, which implies the correctness of the obtained expression of the temperature. Moreover, the horizon of the SD black hole is not the Killing horizon. Instead it is a conformal Killing horizon with the conformal factor being time-dependent. Thus, in this chapter, we move one step ahed towards obtaining the thermodynamic description for the large classes of non-Killing horizons defined in GR.

In the fifth chapter, we have generalized the ADT current for any arbitrary diffeomorphism invariance. The original expression of the ADT current is valid for the presence of the Killing invariance of the spacetime and, therefore, it cannot be used to define the thermodynamic parameters and the thermodynamic first law for the non-Killing horizons. As we have seen earlier, the Noether and the ADT current is widely used to obtain the thermodynamic first law and to define the thermodynamic parameters in the presence of a Killing horizon. The Noether current has already been obtained for any arbitrary diffeomorphism invariance of the theory, whereas the original ADT current is obtained only for the Killing invariance of the theory. Therefore, the ADT current was required to be obtained for any arbitrary diffeomorphism, which defines the black hole horizon and not necessarily those are needed to be Killing vectors. We find that our obtained ADT-like current for arbitrary diffeomorphism invariance gives the ADT current for the Killing diffeomorphism. However, we find that non-trivial results come for other non-Killing diffeomorphism vectors, such as the conformal Killing vectors.

In chapter 6, we have investigated one of the possible approaches to find out the microstates of black hole thermodynamics. We find that the diffeomorphism symmetry, which preserves the location of a generic null-hypersurface, gives rise to the (non-commutative Heisenberg) quantum algebra among the conserved charges. In addition, we also obtain the Bekenstein entropy from the Sugawara construction of the charges, which implies that some of the the microscopic gauge degrees of freedom of the null horizon surface (which preserves the location of the null-surface) raises to the true degrees of freedom of the horizon thermodynamics.

In the second part of the thesis we provide a general framework to study the van der Waals phase transition in black holes and, thereby, obtain the earlier results in a general way which has earlier been obtained case by case in different spacetimes. In chapter 7, we have obtained the thermogeometric description of the van der Waals phase transition of black hole. In formulating the thermodynamic geometry, we have followed the Legendre-invariant approach of obtaining thermogeometrical metric, which has been formulated by Quevedo and collaborators. We show that the critical point can be described by the simultaneous divergence of two Ricci scalars obtained from two independent thermogeometrical metrics. In addition, we also show the procedure of obtaining the thermodynamic first law (using Wald's formalism) in the extended phase space, where the pressure is identified in terms of the cosmological constant in the AdS spacetime. The whole analysis is general and applicable for any spacetime which shows the van der Waals phase transition.

In chapter 8, we have obtained all the critical exponents of the van der Waals phase transition of black holes in a general way. Earlier, the critical exponents of the van der Waals phase transition in black holes has been studied case by case for different black holes. However, the results has been obtained as the same irrespective of any spacetime. Here we have provided a phenomenological model, very similar to the Landau-Ginzburg model, which gives all the values of critical exponents in $P-V$, $T-S$ and $Y-X$ criticalities. In addition, our model also provides the values of the critical exponents for isolated critical points as well. We found that in $P-V$ criticality, the Helmholtz free energy plays the role of the Landau function, whereas in $T-S$ and $Y-X$ criticalities, the same role is played by the Gibbs free energy and the internal energy (which is the black hole mass) respectively. Our analysis suggests that the van der Waals criticality is a general phenomena in black hole thermodynamics. It is no more required to study these phenomena case-by-case. Instead, those results can be obtained in a general way from the formulation which we have provided in this thesis.

Thus, the analysis presented in the thesis illuminates some of the unexplored areas of the black hole (BH) thermodynamics. In addition, it also resolves some of the major debates in this subject (BH thermodynamics). Moreover, it also generalizes some of the features of BH thermodynamics which is studied case by case in different spacetimes. Thus, we expect, the discussions presented in the thesis can be very useful in this subject. In the following, we provide the areas where people can explore further for the better understanding of the subject.

9.2 Scopes for future work

9.2.1 Obtaining thermodynamic laws in a covariant way for non-Killing horizons

The first law of black hole thermodynamics can be obtained covariantly using Wald's formalism or the ADT current formalism. However, both these approaches (Wald and ADT) depend to a large degree on the Killing symmetry and works only for the Killing horizon so far. If someone imposes Killing symmetry in the spacetime, it greatly simplifies the analysis. However, if one thinks of a general situation, a spacetime need not necessarily have a Killing symmetry. But, then the question arises how one can covariantly define thermodynamic laws (and parameters) for any arbitrary horizon, which are not the Killing ones. The Noether current for arbitrary diffeomorphism is already known and, in this thesis, we have obtained the generalized expression of the ADT-like current for arbitrary diffeomorphism. In the future, one can investigate the possibility of obtaining the covariant form of the first law of black hole mechanics for several classes of non-Killing horizons which are defined in GR such as the conformal Killing horizon, apparent horizon, trapped surface, generic null surface *etc.*

9.2.2 Generalization of various phenomena in black hole phase transition

Black hole phase transition is another phenomenon which strengthens the analogy of GR with thermodynamics. There are several types of black hole phase transitions as we have discussed earlier; such as Davies' type, van der Waals type in extended phase space, Hawking-Page transition, non extremal to extremal phase transition *etc.* Usually people study these phase transitions case by case in different spacetimes. In this thesis, we have shown that those results can be can be obtained for any arbitrary spacetime in a general way for van der Waals type phase transition. Also, the same generalization has been done by us for the extremal phase transitions [48]. However, there are still several results in black hole phase transition which are yet to be obtained in a general way. Some of those are: providing thermogeometric description of Hawking-Page transition, proving $P_cV_c/T_c = \text{const.}$ (c in the subscript stands for the critical value) in a general way for black holes in extended phase space, examining dynamic and thermodynamic stability in extended phase space *etc.*

9.2.3 Aspects of alternative theories of gravity

There are several motivations for studying various alternative theories of gravity such as $f(R)$ gravity, Lanczos-Lovelock gravity, scalar-tensor theory of gravity *etc.* Similar to Einstein's theory of gravity there are several aspects in alternative theories of gravity as well which are yet to be discovered. Those features are important in order to have a proper understanding of those theories. Some of those aspects are the conserved charges and their properties, thermodynamic aspects, fluid-gravity analogy, similarity and differences with the Einstein's gravity *etc.* In this thesis, we have found out some of the important features of scalar-tensor gravity. However, still there remains a large unexplored area where one can explore in future.

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[232] We find that our expressions of $\mathcal{B}^{ab}$ (our notation $\mathcal{K}^{ab}$) and $\mathcal{C}^a$ (in this paper it is denoted by $\mathcal{K}^a$) are to be a bit different to what has been obtained in the Eqs. (61) and (62) of [231]. We feel that this is because there is typo in this paper. If one derives them following [231], the correct expressions will be $\mathcal{B}^{ab} = (\tilde{\nabla}_c K^{abdc})\xi_d - \frac{1}{2}K^{abdc}\tilde{\nabla}_{[c}\xi_{d]}$ and $\mathcal{C}^a = \mathcal{T}^{ab}\xi_b + \tilde{\nabla}_c\Big[K^{adbc}\tilde{\nabla}_{(d}\xi_{b)}\Big] - \Big(K^{acbd}\tilde{\nabla}_d\tilde{\nabla}_c + \frac{1}{2}K^{aecd}\tilde{R}^b_{ecd}\Big)\xi_b$ with the conventions, $A_{[i}B_{j]} = \frac{1}{2}(A_iB_j - A_jB_i)$ and $A_{(i}B_{j)} = \frac{1}{2}(A_iB_j + A_jB_i)$. Here $K^{abcd} = \frac{1}{2}[\tilde{g}^{ad}H^{bc} + \tilde{g}^{bc}H^{ad} - \tilde{g}^{ac}H^{bd} - \tilde{g}^{bd}H^{ac}]$ with $H^{ab} = h^{ab} - \frac{1}{2}\tilde{g}^{ab}h$ and $\tilde{g}_{ab}$ is the background metric tensor in their case. These coincide with our findings (5.10) and (5.11) for GR theory on imposition of the Einstein's equations of motion.

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