Portfolio Theory with Lower Partial Moments: Asset Pricing and Performance Analysis

A thesis

submitted by

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Declaration

I do hereby declare that the work contained in the thesis entitled **Portfolio Theory with Lower Partial Moments: Asset Pricing and Performance Analysis** has been done by me, under the supervision of **Prof. N. Selvaraju**, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of degree of Doctor of Philosophy and that this work has not been submitted elsewhere for a degree.

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Certificate

It is certified that the work contained in the thesis entitled **Portfolio Theory with Lower Partial Moments: Asset Pricing and Performance Analysis** by **Dipankar Mondal** (Roll No. 146123013), a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of degree of Doctor of Philosophy, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

November 2020

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Dedicated

to

my Family

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I am dedicating this thesis to my family. I will always remain indebted to my parents for the blessings, the continuous moral support and unbounded love. I was, am and will be always grateful to my father for providing me the most valuable thing of this planet, the education. I am thankful to my wife for her continuous support and infinite love. From good to tough times, she has been always holding my hand. I am also grateful to my sister for being so supportive in every step. Finally, I would like to dedicate my thesis to my newborn little angel “Dhritisha”. Hopefully, one day, she will read and explore this thesis, and possibly make a greater contribution.

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Abstract

The main aim of the thesis is to address certain problems of portfolio selection theory with lower partial moment (LPM). A portfolio manager often relies on the traditional mean-variance (MV) portfolio model proposed by Harry Markowitz. However, due to its fundamental drawbacks and unappealing characteristics, the MV model is not suitable in the practical world. The main culprit is the risk measure, variance. Thus, as a substitute for variance, the LPM was introduced. The intuitive appealing and practical relevance make the LPM a much better risk measure than the variance.

In this thesis, we use LPM as a measure of risk and then develop the theoretical framework of portfolio management with LPM. We present various properties of LPM and discuss their practical importance in the area of investment management. We then investigate various important open problems in the mean-LPM (MLPM) portfolio theory. Some of the problems remain unsolved nearly three decades later. We analytically solve all these problems.

The second part of the thesis discusses asset pricing theory in the MLPM framework. Similar to Sharpe's CAPM, several asset pricing models have been developed in the MLPM framework. We analytically develop a new pricing model that generalizes all the existing MLPM models. Even, the classical CAPM can also be seen as a special case of our model. We further notice that one of the underlying assumptions about the existence of a risk-free asset for deriving the pricing model is unrealistic. Excluding this assumption and incorporating a practical assumption, we derive another pricing model which is analogous to the Black's zero-beta CAPM. It is also shown that, under the assumption of normally distributed returns, this model reduces to the classical zero-beta CAPM.

In the last part of the thesis, we explore the domain of performance analysis. We propose a set of desirable axioms that a performance measure should satisfy in the context of asset management. A performance measure satisfying these axioms is called *ideal*. We examine various measures whether they are ideal or not. While verifying the presence of ideal properties in LPM based measures, we found that one of the widely used ratios is not ideal. Thus, as an alternative, we propose a new ideal measure, Upside beta ratio (UBR). The performance of UBR is empirically examined through ranking risky funds for the UBR as well as for the other measures. We compare the rankings produced by all the measures and find that, in most scenarios, the funds chosen by UBR perform significantly better than the funds selected by all the other measures.

Contents

Abbreviations

MPT	Modern portfolio theory
PMPT	post-modern portfolio theory
MV	Mean-variance
Var	Variance
Std. dev.	Standard deviation
SD	Stochastic dominance
LPM	Lower partial moment
LPD	Lower partial deviation
UPM	Upper partial moment
UPD	Upper partial deviation
MLPM	Mean-lower partial moment
CLPM	Co-lower partial moment
Cov	Co-variance
VaR	Value-at-Risk
CVaR	Conditional Value-at-Risk
TFMS	Two-fund monetary separation
CAPM	Capital asset pricing model
RR	Rachev ratio
GRR	Generalized Rachev ratio
UBR	Upside beta ratio
FT	Farinelli-Tibiletti ratio
J-alpha	Jensen's alpha

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Chapter 1

Introduction

An investment portfolio is a collection of assets, such as stocks, bonds, cash, bank deposits and more, in certain proportions owned by an investor or an institution with the goal of making profit. Given the existence of numerous assets in the market for an investor to invest in, a critical question faced by the investor is to how to allocate funds among the possible investment choices. The portfolio theory basically attempts to answer the fundamental question about the optimal portfolio for the owner to hold. Prior to 1950, investment decision were driven by balance sheets and other financial analysis tools. In 1938, John Burr William authored a book called “The Theory of Investment Value” about the investment in stock market. According to this theory, the value of a stock should be equal to the present value of its future dividend stream. Since dividends are uncertain, expected value of discounted future dividend stream should be the appropriate value of a stock. William suggested that an investor should maximize the expected value of the securities in order to obtain an optimum portfolio– the portfolio with highest expected value. In order to get such portfolio, one needs to invest only in one asset– the security with highest expected return. It basically infers that the primary goal of investment is to pick a good stock and buy it at best price.

1.1 Modern portfolio theory

A major breakthrough in investment theory came in 1952 with the publication of a paper entitled “Portfolio Selection” (Markowitz, 1952). Markowitz shifted the paradigm from buying stocks at best price to buying stocks by taking risk into account. This theory is popularly known as the modern portfolio theory (MPT). The novelty of this work and its wide usage in academics and in investment world led Markowitz to win Nobel Prize in economic sciences. William Sharpe and James Tobin were also awarded the Nobel Prizes for their valuable contributions to Markowitz’s portfolio theory.

According to the MPT, an investor should look for diversification in stocks to reduce the risk. It essentially means that one should not put all eggs in one basket. The two main pillars of this theory are, risk and reward, where the risk is measured by variance (equivalently standard deviation) of return and the reward is quantified by expected return. The MPT assumes that rational investors are risk-averse, and accordingly they prefer less risk and more reward. Further, given two portfolios with the same expected return, a rational investor should choose the portfolio with lower risk. In particular, this model addresses the problem of finding investment weights in order to minimize the risk subject to a fixed expected return. The set of minimum risky portfolios forms the optimal investment frontier. Investors just need to find their own portfolios located on the optimal frontier. This approach clearly simplifies the complex portfolio selection problem. Readers intended in MPT may refer to the books by Ingersoll (1987), Huang and Litzenberger (1988), Luenberger (1997), Elton et al. (2009) and Francis and Kim (2013)

After the development of the mean-variance (MV) portfolio theory, the next step was to determine the prices of assets that can incorporate the riskiness nature intrinsic to capital market. Based on the MV theory, Sharpe (1964) developed a model, popularly known as the Capital Asset Pricing Model (CAPM). This work has made strong impact in academics as well as in finance industry. The model has been extensively used in the domain of asset pricing and decision making under uncertainty. Later, Lintner (1965) and Mossin (1966) independently derived the same pricing model. These fundamental works on capital market theory offer tremendous research possibilities on asset pricing models. The SLM (Sharpe-Lintner-Mossin) model was modified and extended several times by relaxing one or more underlying assumptions and by incorporating realistic situations. Lintner (1969) brought the concept of heterogeneous expectation on asset pricing theory and proposed a new pricing model. Brennan (1970) studied the effect of tax payment in asset pricing and developed after-tax version of CAPM. Several other market imperfections such as transactions costs, indivisibility, price information etc, have also been explored. Black (1972) observed that one of the major underlying assumptions in CAPM about the nature of risk-free rate is impractical. Thus, by abandoning that assumption, he developed a new pricing model. Merton (1973), in continuous-time framework, found another version of CAPM which is a multiperiod model, called the intertemporal CAPM. However, all these models are riven by the first two moments of the return distribution. Kraus and Litzenberger (1976) extended the CAPM by incorporating the effect of skewness in on the valuation of risky assets. The empirical evidence suggested that the three moments extension of the classical

CAPM is consistent with the investor's preferences. Fang and Lai (1997) further extended the three moments model into a four moments pricing model by including another moment, the kurtosis of return distribution. Some more interesting works related to the development of new pricing models are Kim and Zumwalt (1979), Eugene and Kenneth (1993), Pettengill et al. (1995), Acharya and Pedersen (2005) and Fama and French (2015).

After selecting various portfolios for investment, the job of a portfolio manager is to evaluate the performance of those portfolios. In order to do so, an appropriate measure needs to be developed. The measure should take both the expected return and the risk of portfolio return into consideration. Sharpe (1966) proposed a risk-adjusted performance ratio, given by the excess expected return to standard deviation of returns, to rank mutual funds. Thus, the risk-return information of an investment is collapsed into single number. It acts a performance index that ranks portfolios according to their desirability. The higher the value of the index, the more the desirable a portfolio is. This ratio is popularly known as the Sharpe ratio. Following this pioneering work, scholars proposed various other risk-adjusted performance measures to determine the efficiency of risky investments. Treynor (1965) utilized the CAPM based risk measure (beta) in performance metric and proposed reward-to-volatility ratio. Jensen (1968) too used the asset-pricing implications of CAPM to determine portfolio's performance. It is known as Jensen's alpha. Some interesting works related to these performance measures are Leland (1999), Lo (2002), Hübner (2005), Pilotte and Sterbenz (2006), Ledoit and Wolf (2008), Zakamouline and Koekebakker (2009)

Notwithstanding the popularity and acceptability, the MPT has been criticized mainly because of its choice of variance as a measure of risk. It is believed that the choice of variance is primarily due to the computation simplicity rather than any theoretical and practical justification. The primary concern is that the variance punishes upside deviation of returns in a same way as it does for downside deviation. In reality, investors mainly concern about the downside return rather the upside gains. Thus, minimizing variance is equivalent to penalizing the downside as well as the favourable upside outcomes. It is truly an undesirable characteristic of a risk measure. Further, it is observed that the MV model is appropriate when the asset returns follow a joint normal distribution or more generally an ellipsoidal distribution. The assumption of normality is clearly unrealistic as it rules out the asymmetric nature of return distributions. In fact, many empirical studies have found that common stock returns are generally not normally distributed (Singleton and Wingender, 1986; Chunhachinda et al., 1997). Under non-normality, mainly for skewed return distribution, this model fails to consider investor's preferences towards skewness. It is also observed that the MV model is not consistent with the well-established expected utility theory and stochastic dominance rules.

1.2 Beyond MPT

The fundamental drawbacks and the underlying unrealistic assumptions make the MPT impractical. The main culprit for this phenomena is nothing but the risk measure, variance. Consequently, it resulted the inception of post-modern portfolio theory (PMPT). The first step towards PMPT was also led by Harry Markowitz. Recognizing the unfavourable properties of variance, Markowitz (1970) was offered an alternative promising candidate for measuring risk, namely semivariance, which penalizes only those outcomes that fall below the expected return. Hence, the semivariance serves the actual purpose of measuring risk in the context of investment management. In fact, Markowitz argued that mean-semivariance analysis tends to produce better portfolios than those based on mean-variance (MV). Adding to this tenet, Bawa (1976) generalized the semivariance to lower partial moment (LPM) that measures the dispersion of returns below a threshold level or target return instead of just expected return. Unlike the MV model, the mean-LPM (MLPM) model is consistent with stochastic dominance (SD) rules and expected utility theory (Bawa, 1976; Fishburn, 1977; Ogryczak and Ruszczyński, 1999). Bawa concluded that the MLPM rule is theoretically superior to the MV rule because the MLPM selection rule holds true under more realistic and less restrictive assumptions on the distribution of returns. Working on MLPM optimization model, Harlow (1991) showed that MV efficient frontier lies inside the MLPM opportunity set when the distribution is non-normal. Hence, the MLPM optimal portfolios are more efficient because, for the same level of expected return, they provide less downside exposure than the MV optimal portfolios. In the case of normal distributions, both the frontiers coincide. Some relevant works on LPM can be found in Nawrocki (1999), Grootveld and Hallerbach (1999) and Unser (2000). Over the years, various other metrics of downside risk were introduced. Value at risk or VaR is widely used in banking industry to measure maximum loss for investments given a probability. It was introduced by the JP Morgan's Risk Metrics group (Morgan, 1996). Gaivoronski and Pflug (2005) described computational approach for portfolio optimization model with VaR as a measure of risk. Artzner et al. (1999) found one undesirable characteristics of VaR, and hence they proposed an alternative measure, namely, conditional value at risk or CVaR. Rockafellar and Uryasev (2000) provided a new approach for portfolio optimization which minimizes CVaR.

The development of numerous risk measures opened up the research possibilities on asset pricing with corresponding to each risk measure. The first non MV pricing model was proposed by Hogan and Warren (1974). Using semivariance as a risk measure, he developed an mean-semivariance CAPM. Following this work, a series of LPM based models were introduced. Works on the MLPM pricing model can be found in Bawa and Lindenberg (1977), Lee and Rao (1988) and Harlow and Rao (1989). Kaplanski (2004) analytically developed another downside risk asset pricing model, which is based on CVaR. De Giorgi and Post (2008) derived risk-reward CAPM analogously to the classical CAPM. Zabarankin et al. (2014) proposed CAPM with drawdown measure. Several empirical tests revealed that the downside risk based pricing models performs better than the traditional CAPM (Post and Van Vliet, 2006; Estrada, 2007; Pedersen and Hwang, 2007; Cwynar and Kazmierkiewicz, 2010).

The aura of PMPT further encouraged the development of performance measurement technique. The Sharpe ratio lost its glory because of its inappropriate measure of risk, the standard deviation. On the other hand, the downside risk based performance ratios are more favourable to the investors as downside deviations of asset returns contain the actual financial risk. Using LPM, Sortino and Price (1994) proposed a new ratio which is popularly known as Sortino ratio. The acceptance of Sortino ratio in finance community provided the boost for flourishing LPM based ratios. These are Omega ratio (Pedersen and Satchell, 2002), Kappa ratio (Kaplan and Knowles, 2004), Upside potential ratio (Sortino et al., 1999), and Farinelli-Tibiletti ratio (FT) (Farinelli and Tibiletti, 2008). Besides these, several other downside risk based ratios are developed. For example, mean-value at risk (VaR) ratio (Unser, 2000), mean-conditional value at risk (CVaR) ratio (Agarwal and Naik, 2004), Rachev ratio (Biglova et al., 2004), Calmar ratio (Young, 1991), mini-max ratio (Young, 1998), and gain-loss ratio (Dowd, 2000) are some of them. A good survey on the performance measures is provided in Zakamouline (2014).

In spite of the intuitive appeal, several computational hindrances lead to preference MPT over PMPT. The mathematical elegance and tractability of MPT is forgone in PMPT. Variance of a portfolio of assets can be expressed in terms of individual variances and covariances, while similar decomposition is not always possible for the downside risk measures (Estrada, 2008). It makes the portfolio optimization problem much more difficult. The introduction of numerous risk measures pose the problem of choosing appropriate measure. Similar issues has also been observed in the case of performance analysis. The derivation of asset pricing model with respect to different risk measures is also cumbersome. However, over the years, these issues are gradually resolved so that practitioners can utilize the PMPT. This thesis aims to addresses several important problems of PMPT.

In the following sections, the key aspects of MPT, utility theory, SD rules and portfolio theory with LPM are presented.

1.3 The mean-variance portfolio selection framework

Let $(\Omega, \mathscr{F}, \mathbb{P})$ denote a probability space and let $\mathscr{G}$ be the set of all real-valued random variables defined on the probability space such that $\mathbb{E}|R|^{\alpha} < \infty$, for all non-negative real numbers α and for all random variables $R \in \mathscr{G}$. Here, $\mathbb{E}$ is the expectation operator with respect to $\mathbb{P}$.

Suppose n different assets are available and a total amount W is available to invest over these assets. Let W_i be the investment amount on asset i, for $i = 1, 2, ..., n$, such that $W = \sum_{i=1}^{n} W_i$. The proportional amount invested on asset i is given by $w_i = \frac{W_i}{W}$. Here, w_i is called the weight of asset i and $w_i < 0$ indicates short selling on i-th asset. The short selling is a trading strategy where investors borrow an asset and then sell it later to make profit. Now, a portfolio p is a collection of different assets (say, n number of assets) in proportions $w_1, w_2, \dots, w_n$ such that $\sum_{i=1}^{n} w_i = 1$. Formally, a portfolio is defined as a weight vector $\mathbf{w}' = (w_1, w_2, ..., w_n)$.

If asset i (for $i = 1, 2, ..., n$) has random return R_i with the expected rate of return $\mathbb{E}(R_i) = \mu_i$ then the return of portfolio p is defined to be the weighted sum of individual return

$$R_p(w) = \mathbf{w}'R = w_1R_1 + w_2R_2 + ... + w_nR_n,$$

where $R' = (R_1, R_2, ..., R_n)$ is the vector of random returns. Hence, the expected rate of return of the portfolio is

$$\mu_p := \mathbb{E}(R_p) = \sum_{i=1}^{n} w_i \mathbb{E}(R_i) = \sum_{i=1}^{n} w_i \mu_i.$$

Variance is uncertainty of return or dispersion of outcome around the mean. The variance of return for ith asset is defined as

$$\sigma_i^2 := Var(R_i) = \mathbb{E}\left[R_i - \mathbb{E}(R_i)\right]^2.$$

Consequently, the variance of the portfolio is given by

$$\sigma_p^2 = \mathbb{E}\left[R_p - \mathbb{E}(R_p)\right]^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij},$$

where $\sigma_{ij} = Cov(R_i, R_j) = \mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)]$ is the covariance between the assets i and j. The portfolio variance can be expressed as

$$\sigma_p^2 = \boldsymbol{w}'\Sigma\boldsymbol{w},$$

where

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \dots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \dots & \sigma_{nn} \end{pmatrix}$$

is variance-covariance matrix. The covariance matrix is assumed to be positive definite.

1.3.1 The mean-variance optimization model

Markowitz (1952) introduced the concept of mean-variance (MV) portfolio selection criteria. He assumed that every investor is risk-averse and thus wants to minimize the variance for selecting optimal portfolio. Thus, the following optimization model is formulated:

$$\begin{cases} \underset{\boldsymbol{w}}{\text{Minimize}} & \sigma_p^2 = \boldsymbol{w}'\Sigma\boldsymbol{w} \\ \text{subject to} & \sum_{i=1}^{n} w_i \mathbb{E}(R_i) = \mu \\ & \boldsymbol{w} \in \mathscr{D}, \end{cases} \tag{1.1}$$

where $\mathscr{D}$, the set of feasible portfolios, is given by

$$\mathscr{D} = \left\{ \boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} w_i = 1, w_i \geq 0, i = 1, 2, \ldots, n \right\}.$$

It is also known as opportunity set, representing all portfolios that could be formed from a group of n assets. Fig. 1.1 depicts the opportunity set in mean-standard deviation plane. All the portfolios are lying either on or within the boundary of the opportunity set. Since the objective function σ_p is convex functional of $(w_1, w_2, \ldots, w_n)$ and the constraints are linear, problem (1.1) is a convex optimization problem. Hence, the unique optimal solution always exists.

The positive definiteness of variance-covariance matrix suffice the fact that the minimum risky frontier is convex in μ-σ space. All the minimum variance portfolios are located on this curve. The lowest possible risky portfolio is called the global minimum variance portfolio (GMVP). The upper half (starting from GMVP) of the minimum variance frontier is called efficient frontier, which comprises of the set of optimal portfolios which have either highest expected return for a given level of risk than any other portfolio in its risk class, or lowest risk than any other portfolios with same expected return.

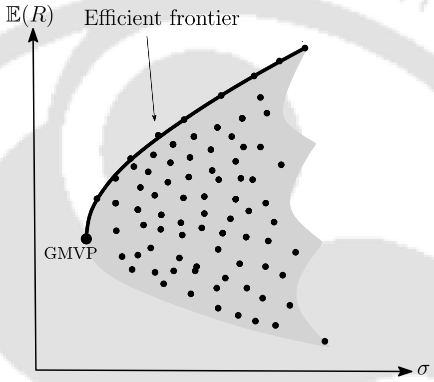


Figure 1.1: M-V Opportunity Set

Using the method of Lagrange multipliers, Merton (1972) derived the analytical solution of the above optimization problem. The optimal weight for a given level of expected return μ can be expresses by using the following formula:

$$\boldsymbol{w} = \mathbf{g} + \mathbf{h}\mu,$$

where

$$\mathbf{g} = \frac{1}{D}(B\Sigma^{-1}1 - A\Sigma^{-1}E),\ \mathbf{h} = \frac{1}{D}(C\Sigma^{-1}E - A\Sigma^{-1}1)$$

with

$$A = 1'\Sigma^{-1}E,\ B = E'\Sigma^{-1}E,\ C = 1'\Sigma^{-1}1,\ D = BC - A^2.$$

Thus, for any given expected return, the minimum variance portfolio can directly be obtained by using the above formula. This analytical solution truly simplifies the MV portfolio selection model.

The above analytical solution further yields the equation of minimum risky frontier frontier which is given by

$$\sigma^2 = \frac{C}{D}\left(\mu - \frac{A}{C}\right)^2 + \frac{1}{C}.$$

It can be seen that the minimum risky frontier is the parabola in μ-σ^2 space and the hyperbola in μ-σ plane. All the optimal portfolios for risk-averse investors are located on this curve.

1.3.2 Mean-variance portfolio selection with a risk-free asset

So far, we have seen how to select an optimal portfolio that contains only risky assets. Now, assume that a risk-free asset (T-bill, bank deposit etc.) with return r_f is available for investment, and investors wants to invest in it along with the risky assets. Here, r_f is a non-random quantity and hence $\mathbb{E}(r_f) = r_f, \sigma_f = 0$. Let R_p be the return of portfolio p consisting of a risk-free asset and a risky portfolio q in proportions w and $(1-w)$, respectively. The expected return and variance of R_p are given by

$$\mathbb{E}(R_p) = wr_f + (1-w)\mathbb{E}(R_q),$$

$$\sigma_p^2 = (1-w)^2\sigma_q^2.$$

Eliminating w from the above equations, we obtain

$$\mathbb{E}(R_p) - r_f = [\mathbb{E}(R_q) - r_f]\frac{\sigma_p}{\sigma_q}. \tag{1.2}$$

Equation (1.2) describes the linear relationship between σ_p and $\mathbb{E}(R_p)$. This relationship can be obtained for any arbitrary portfolio. Hence, the portfolios formed by a risk-free asset and some risky portfolios are located on some straight lines in the mean-risk space. The lines are illustrated in Fig. 1.2.

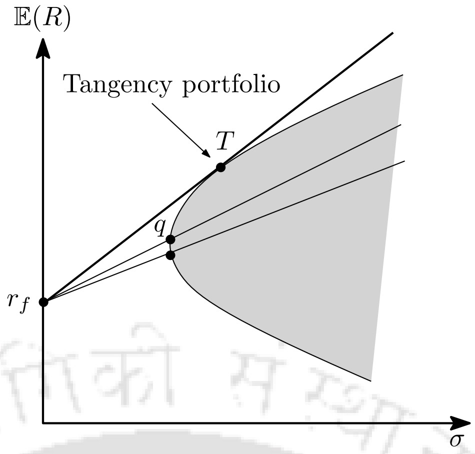


Figure 1.2: Linear efficient frontier

The portfolios located on the tangency line dominate all the other portfolios in the opportunity set. Thus, the line $\overline{r_f T}$ is the efficient frontier in this case. As a result, an investor first finds the tangency portfolio and then mixes it with the risk-free asset to obtain his desired optimal portfolio. It means that under the presence of risk-free asset, all optimal portfolios can be found just by mixing those two funds (tangency portfolio and risk-free asset). Hence, it is popularly known as two-fund monetary separation (TFMS) (Tobin, 1958). The TFMS plays a very important role in portfolio selection with a risk-free asset.

1.3.3 Single index Model

The index models are the simplest form of return generating process that used to explain asset returns. The first index model, known as the single index model, was formally developed by Sharpe (1963). The model is

$$R_{it} = \alpha_i + \beta_i R_{mt} + e_{it},$$

where R_{it} is the return of stock i at time t; R_{mt} denotes the market return at time t; α_i is the unique expected return of asset i; β_i is the sensitivity of asset i to the market movements; e_{it} is the residual returns, which are assumed independent normally distributed with mean zero and variance σ_{ei}^2, i.e., $e_{it} \sim \mathcal{N}(0, \sigma_{ei}^2)$. By adopting the standard assumptions of linear regression model, α_i and β_i can easily be estimated. This is the ordinary least square estimation.

The model has two major components: a systematic part $\alpha_i + \beta_i R_{mt}$ and an unsystematic part e_{it}. The systematic part is explained by the explanatory variable R_{mt}. The unsystematic part is unexplained by the explanatory variable. Now, the variance of R_{it} is given by

$$\sigma_{it}^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2.$$

Similarly, for a portfolio p, the total risk can be expressed as follows:

$$\sigma_{pt}^2 = \beta_p^2 \sigma_m^2 + \sigma_{ep}^2. \tag{1.3}$$

The first part of equation (1.3), $\beta_p^2 \sigma_m^2$, is called the systematic risk. It is also known as undiversifiable risk as it cannot be mitigated by diversification. The other part, σ_{ei}^2, is the unsystematic risk or diversifiable risk which can be minimized through diversification.

Following this work, researchers extended the single index model to a multi-index model that better explains the realistic scenario. The multi-index incorporates various variables to explain the asset's returns. More works related to index models can be found in Elton and Gruber (1997), Francis and Kim (2013).

1.3.4 Asset pricing models

The asset pricing theory is introduced by Sharpe (1964), Lintner (1965) and Mossin (1966). They independently developed the same asset pricing model, namely, the capital asset pricing model (CAPM). The CAPM describes that risk and expected return of an asset is linearly related. It is also known as the security market line. The model is given by the following equation

$$\mathbb{E}[R_i] - r_f = \beta_i \big(\mathbb{E}[R_m] - r_f\big),\ i = 1, 2, \ldots, n,$$

where: $\mathbb{E}(R_i)$ is the expected return of a capital asset; r_f is the risk-free rate of return; $\mathbb{E}(R_m)$ is the expected return of the market; $\beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}$ is the beta of an asset which measures the sensitivity of an asset compared to overall the market; $\mathbb{E}(R_m) - r_f$ is market premium; $\mathbb{E}(R_i) - r_f$ is the risk premium.

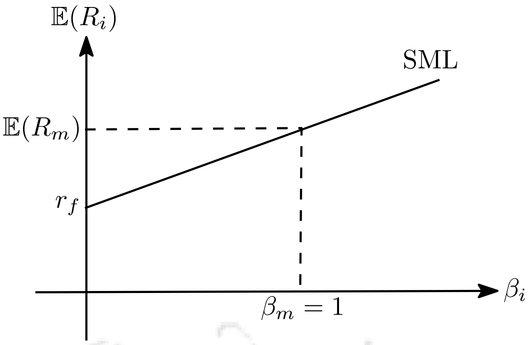


Figure 1.3: CAPM

Therefore, the CAPM states that individual's risk premium is equal to the beta times market risk premium. The CAPM is an useful tool to determine the risk-adjusted price of individual security or any portfolio. An investor can easily determine whether an asset is undervalued or overvalued, and then he can take necessary investment decisions.

Black (1972) eradicated a major deficiency of the CAPM and developed a new model known as zero-beta CAPM:

$$\mathbb{E}[R_i] - \mathbb{E}[R_z] = \beta_i \big(\mathbb{E}[R_m] - \mathbb{E}[R_z] \big),$$

where z is the zero beta portfolio. This portfolio has zero systematic risk , i.e., $\beta_z = 0$. The zero-beta CAPM is identical to the Sharpe's model except the risk-free rate is replaced by $\mathbb{E}(R_z)$. Unlike the SLM model, this model is more amenable to real world scenario.

1.3.5 Performance ratios

Portfolio manager often used risk-adjusted performance measures to rank various investment portfolios. Sharpe (1966) proposed a risk-adjusted ratio to measure fund's performance. It is given by

$$S_p = \frac{\mathbb{E}[R_p] - r_f}{\sigma_p}.$$

Geometrically, this ratio measure the slope of a straight line connecting to the risk-free rate asset and a risky portfolio in mean-standard deviation space.

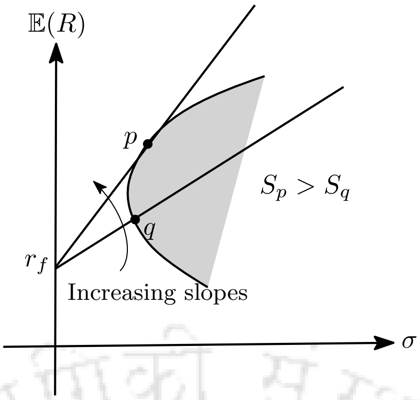


Figure 1.4: Sharpe's ranking technique

Treynor believed that beta is the better measure of asset volatility and that the beta should be used for measuring portfolio's performance. He thus proposed the following ratio:

$$T_i = \frac{\mathbb{E}(R_i) - r_f}{\beta_i}.$$

Jensen (1968) utilized CAPM to measure the performance of risky portfolios. He proposed the following risk-adjusted measure:

$$\alpha_p = \overline{R_p} - r_f - \beta_p[\overline{R_m} - r_f],$$

where $\overline{R_p}$ is the historical average returns of portfolio p. It is used to determine the excess return of a portfolio over the theoretical expected return suggested by the CAPM.

Sharpe ratio together with Jensen's alpha and Treynor ratio are widely used in practice to rank risky funds. However, all three measure does not always provide same result. The Sharpe efficient fund may not perform well in Jensen's world and vice-versa. Similarly, the top-ranked fund provided by Treynor ratio may not have good Sharpe ratio. It is expected as each ratio has different formulation and distinct risk or reward measure.

1.4 Utility theory

The concept of utility function was introduced in economics to model consumer's behaviour. It usage has evolved significantly in financial economics. This is because it takes into account investor's individual preferences. Utility is a real-valued function $U : \mathscr{X} \to \mathbb{R}$ that represents a preference relation among the elements of $\mathscr{X}$. It assigns numerical value to each alternatives and then rank them. The alternative with highest rank is preferred. In other words, the utility function basically measures the relative satisfaction that someone derives from something. If an investor A to B, then the utility of A must be greater than the utility of B. The foundation of utility theory begins with the following axioms.

1.4.1 Axioms of utility functions

(i). Monotonicity: If a person prefers A to B, then $U(A) > U(B)$; If he is indifferent between A and B, then $U(A) = U(B)$.

(ii). Transitivity: If A is preferred to B, and B is preferred to C, then A is also preferred to C. In terms of utility function, $U(A) > U(B)$ and $U(B) > U(C)$ imply $U(A) > U(C)$.

(iii). Expected utility: Decisions related to uncertain choices, i.e., risky decisions, are quantified by expected utility. If X, a random variable, describes uncertain outcomes with distribution function $F_X(x)$, then the expected utility of a person with a utility function U is given by

$$\mathbb{E}(U(X)) = \int_{\mathbb{R}} U(x) dF_X(x).$$

Risky decisions are made by maximizing expected utility. Thus, given two risky investments X and Y, X is preferred to Y if

$$\mathbb{E}(U(X)) > \mathbb{E}(U(Y)).$$

1.4.2 Types of utility functions

We know that more wealth is always more preferable than less wealth. A utility function should reflect this property. Hence, for any rational investor, $x \geq y \implies U(x) \geq U(y)$. Therefore, utility function for rational investor should be nondecreasing. If the utility function is differentiable, then this property indicates $\frac{dU}{dx} = U'(x) > 0 \; \forall \; x \in \mathbb{R}$. The term $U'(x)$ is called marginal utility which is defined as the additional utility or satisfaction that a person gains from a little change in his wealth.

Further characteristics of investor's behaviour can also be described by other properties of the utility function. We mainly describe three basic behaviours of an investor towards risk.

(i). **Risk-averse:** If an investor obtains lower expected utility from a risky investment than a risk-free investment with same expected return, then the investor is called risk-averse. This behaviour is captured by the concavity property of utility function:

$$U(\lambda x_1+(1-\lambda)x_2) > \lambda U(x_1)+(1-\lambda)U(x_2),\ \lambda \in [0,1]$$

If U is twice differentiable, then the equivalent criteria of risk-aversion is negative second order derivative, i.e.,

$$U''(x) < 0.$$

Therefore, the utility function for risk-averse investors is mathematically represented by

$$U'(x) > 0,\ U''(x) < 0$$

(ii). **Risk-loving:** In contrast to the risk aversion, risk-loving investor's expected utility from risk playing game is greater than without playing the game. This behaviour is encapsulated by the property of convexity of utility function, i.e.,

$$U(\lambda x_1+(1-\lambda)x_2) < \lambda U(x_1)+(1-\lambda)U(x_2),\ \lambda \in [0,1].$$

For twice differentiable function, the equivalent criteria is

$$U''(x) > 0.$$

For risk-seeking investor, expected utility from the risky investment is more than the risk-free investment. Therefore, the utility function exhibiting risk-loving attitude is mathematically represented by

$$U'(x) > 0,\ U''(x) > 0.$$

(iii). **Risk-neutral:** Risk-neutral investor's expected utility from risky investment is same as risk-free investment. The risk-neutral property is described by a linear utility function. Hence, for sufficiently smooth function, it translates to

$$U''(x) = 0.$$

Here, investor would be indifferent between the risky and the risk-free investments, because in both cases his/her expected utility will be the same.

1.4.3 Measure of risk-aversion

The first measure of risk aversion was developed by the economists Kenneth Arrow and John W. Pratt. It is popularly known as Arrow-Pratt measure of absolute risk aversion or co-efficient of risk aversion:

$$D(x) = -\frac{U''(x)}{U'(x)}.$$

It measures the change in risk averseness with respect to the change in wealth.

If an investor invests more wealth in risky assets as wealth increases, then it is called decreasing absolute risk aversion (DARA). Therefore, the more the wealth, the less an investor is risk-averse. Mathematically, the DARA is represented by $D'(x) < 0$. In reality, this behaviour is mostly observed. The DARA holds when

$$D'(x) = -\frac{U'(x)U'''(x) - [U''(x)]^2}{[U'(X)]^2} < 0.$$

Now, given $U'(x) > 0$, the above inequality holds only if $U'''(x) > 0$. Therefore, the DARA implies positive skewness of utility functions. Thus, nonsatiable investors who exhibit DARA have positive preference direction for third moment.

When investments in risky assets decreases as wealth increases, we call it increasing absolute risk-aversion (IARA). It indicates that as wealth increases, investors become more risk-averse. The mathematical definition of IARA is $D'(x) > 0$.

If the investments remain same as wealth increases, then the investor exhibit constant absolute risk-aversion. It is defined by $D'(x) = 0$.

Example 1.1. Some basic examples of utility functions are provided below.

(i). Quadratic utility function:

$$U(x) = ax^2 + bx + c;\ a > 0,\ b > 0$$

It is easy to see that $U'(x) > 0$ and $U''(x) > 0$. Hence, for $a > 0$ and $b > 0$, this utility function is suitable for risk-loving investor.

(ii). Exponential utility function:

$$U(x) = -\frac{e^{-ax}}{a};\ a > 0$$

We see that $U'(x) > 0$ and $U''(x) < 0$. Hence, risk-averse investors can use this utility function.

(iii). Linear utility function:

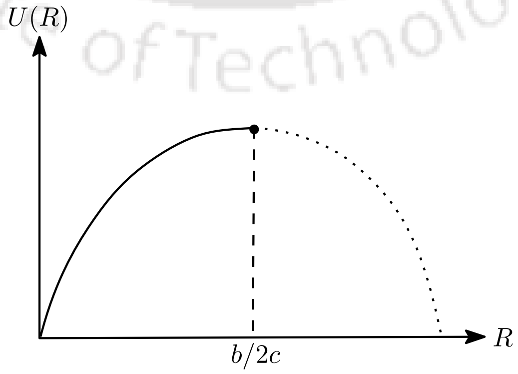
$$U(x) = ax + b;\ a > 0$$

As $U''(x) = 0$, this function captures risk-neutral behaviour.

1.4.4 MV model and utility theory

MV model is consistent with the expected utility theory if the underlying utility function is quadratic. Mostly, an investor chooses a portfolio that maximizes his expected utility. Markowitz suggested a quadratic utility function

$$U(R_p) = a + bR_p - cR_p^2$$



For this function, the marginal utility of return is

$$\frac{\partial U(R_p)}{\partial R_p} = b - 2cR_p$$

For any investors, marginal utility should always be positive and thus $R < b/2c$. For the case of $R > b/2c$, investor's marginal utility is negative, which is not desirable. It violates the assumption of nonsatiation. Thus, the above utility function is applicable with an additional restriction of $R < b/2c$. Moreover, as $U''' = 0$, the skewness preference is not captured by the quadratic utility function. As a result, this utility function is not consistent with the behaviour of decreasing absolute risk aversion.

1.5 Stochastic dominance theory

Stochastic dominance (SD) is a fundamental concept in decision theory. It is an useful tool in portfolio selection as it provides strong connection with expected utility theory. SD is a partial order ranking criteria between different random variables. While the MV framework utilizes only the first two moments of probability distributions, the SD focuses on every bit of distributions. As a result, SD efficient portfolios are sometimes more desirable than the MV efficient ones. Moreover, unlike the MV theory, the SD rules hold true for all distributions and require less restrictive as well as more realistic assumptions of investor's utility functions.

In portfolio theory, the stochastic dominance rules compare two portfolios at a time by looking their cumulative distribution functions and then determines which one of the two portfolios is preferred by an investor from a given class of utility functions. Thus, depending on the class of utility functions and the nature of distributional comparison, the SD rule can be divided into many categories. We mainly discuss the criteria for three important class of investors.

1.5.1 First order stochastic dominance

Let us denote the class of utility functions representing non-satiable investors (who prefer more to less) by

$$\mathscr{U}_1 = \{U(x) |\, U'(x) > 0, \forall x \in \mathbb{R}\}.$$

It is a set of all monotonic non-decreasing utility functions. Now we say that X dominates Y in the sense of first order SD (FSD) if any expected utility maximizing investors belonging to above class always prefer X over Y. Mathematically, it is described as follows:

$$X \succcurlyeq_1 Y \iff \mathbb{E}[U(X)] \geq \mathbb{E}[U(Y)] \; \forall \, U \in \mathscr{U}_1.$$

This criteria can also be represented in terms of the c.d.f. of X and Y as follows:

$$X \succcurlyeq_1 Y \iff F_X(t) \leq F_Y(t), \forall \, t \in \mathbb{R},$$

with strict inequality for at least one value of t.

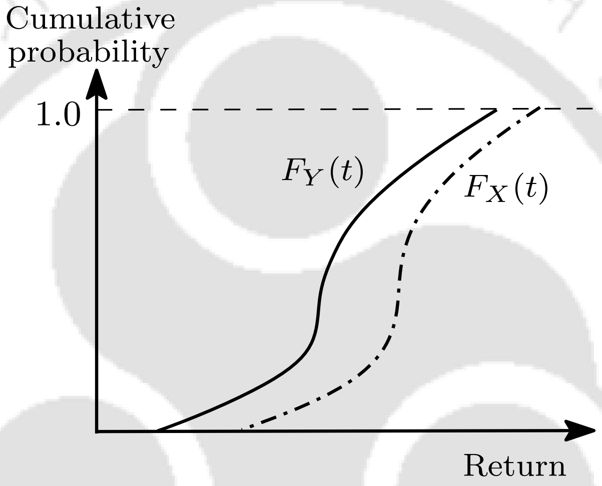


Figure 1.5: First order SD

A necessary condition for FSD is that the expected value of X (the dominating asset) is at least as high as the expected value of Y, i.e., $\mathbb{E}(X) \geq \mathbb{E}(Y)$.

1.5.2 Second order stochastic dominance

The FSD rule is not practical as it does not incorporate the notion of risk. Thus, for making decision under uncertainty, FSD is a restrictive criteria. If the notion of riskiness can be added to the FSD criteria, then the decisions under risk can be made more accurately. Let $\mathscr{U}_2$ be the class all utility functions that are non-decreasing as well as concave:

$$\mathscr{U}_2 = \{U(x) | \, U'(x) > 0, U''(x) < 0, \forall x \in \mathbb{R}\}$$

Here, the concavity represents the risk-averse behaviour. Now, X dominates Y in the sense of second order SD (SSD) if the following holds

$$X \succcurlyeq_2 Y \iff \mathbb{E}[U(X)] \geq \mathbb{E}[U(Y)],\ \forall U \in \mathscr{U}_2.$$

The equivalent c.d.f. based criteria for SSD is following,

$$X \succcurlyeq_2 Y \iff \int_{-\infty}^{t} F_X(s)ds \leq \int_{-\infty}^{t} F_Y(s)ds, \forall t \in \mathbb{R}.$$

1.5.3 Third order stochastic dominance

The SSD criteria is obviously good as it is consistent with the rational behaviour. However, it is not suitable enough for the risk-averse investors displaying skewness preference ($U''' > 0$). This behaviour is further connected with the notion of DARA. Let us denote the set of all utility functions that are non-decreasing, concave and that have positive third derivative by

$$\mathscr{U}_3 = \{U(x)|\ U'(x) > 0, U''(x) < 0, U'''(x) > 0, \forall x \in \mathbb{R}\}.$$

It is said that Y is dominated by X in the sense of third order SD (TSD) when the following holds

$$X \succcurlyeq_3 Y \iff \mathbb{E}[U(X)] \geq \mathbb{E}[U(Y)],\ \forall\ U \in \mathscr{U}_3.$$

The equivalent c.d.f. based criteria is

$$X \succcurlyeq_3 Y \iff \int_{-\infty}^{t}\int_{-\infty}^{h} F_X(s)dvds \leq \int_{-\infty}^{t}\int_{-\infty}^{h} F_Y(s)dvds, \forall t \in \mathbb{R},\ \text{and}\ \mathbb{E}(X) \geq \mathbb{E}(Y)$$

1.5.4 MV model and SD rule

MV rule is not always consistent with the SD rule especially when the underlying distributions are nonnormal. Even SD rule occasionally yield more desirable portfolios than the MV efficient portfolios. The detailed comparisons between MV and SD portfolio choice criteria can be found in Porter and Gaumnitz (1972) and Stone (1973). There are significant conflicts between SSD and MV rule. It is observed that some EV efficient portfolios are SSD (or TSD) inefficient and vice versa.

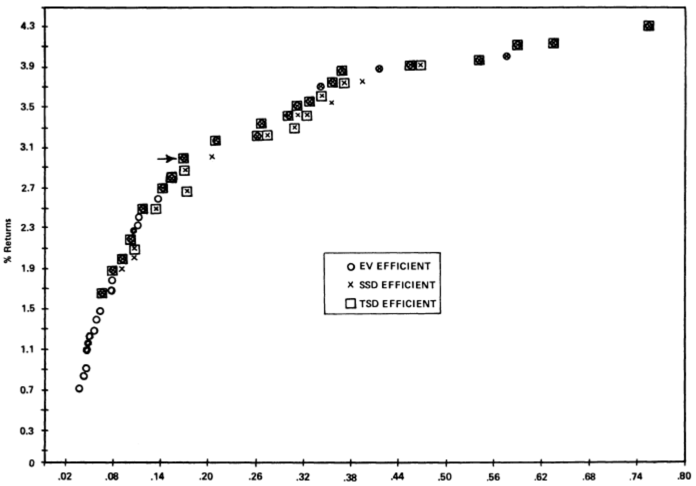


Figure 1.6: MV vs. SD rule

However, the differences are less (more) in the range of high (low) mean and high (low) variance of return (see Fig. 1.6 [1]). It implies that the conflict is not severe in the case of low risk-averse investor's portfolio choice problem. In contrast, the choice of a highly risk-averse investor is inconsistent with the maximization of expected utility. The obvious reason is that the variance is not an appropriate measure of risk.

1.6 Downside risk measure

In MPT, the investment risk is measured by variance, or equivalently standard deviation, that quantifies the variability of returns around the mean. In contrast to MPT, PMPT makes clear distinction between good and bad return, and then measures risk by penalizing only the bad returns. It better reflects what we perceive as risk in investment world. According to Grootveld and Hallerbach (1999), "*A risk concept in which undesirable downside fluctuations are separated from desirable upside fluctuations better matches investors' intuition about risk than the variance*".

In addition, PMPT recognizes that risk should be tied to each investor's desired target return and that the return above the target level should not represent the risk. The return below the target level contains the actual risk and the return above the target level is the potential to gain high return. The target return is also refereed to as minimum acceptable return (MAR) or threshold level. An investor desires to earn at least this return. The farther the return falls below the target level, the higher the

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[1] The figure is taken from Porter and Gaumnitz (1972)

risk of loss. Sine the risk measures in PMPT concentrates only the downside outcomes or the returns below some threshold level, they are called downside risk measures. Some important downside risk metrics are disccused below.

1.6.1 Safety first

The safety first criteria, formulated by Roy (1952), is probably the first downside risk measure in investment literature. He defined risk as a shortfall probability which measures the chances of the return falling below some target level, called minimum acceptable return, τ:

$$\mathbb{P}(R < \tau).$$

The main idea behind this measure is to compute the possibility of getting return below some target return which is the safety return for an investor. It mainly indicates the left-hand-side area of return distribution. It is the probability of downside outcomes that measures the likelihood of failure to meet the MAR. It basically quantifies the uncertainty of loss.

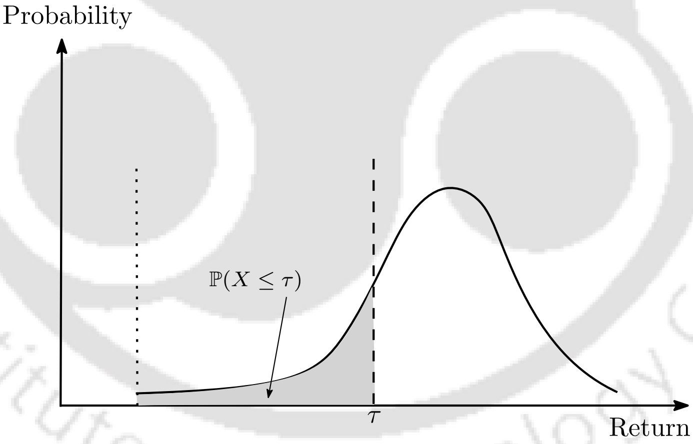


Figure 1.7: Shortfall probability

1.6.2 Semivariance

The semivariance is defined as

$$\sigma_-^2 = \mathbb{E}\left[(R - \mathbb{E}(R)^2 \mathbb{1}_{\{R \leq \mathbb{E}(R)\}}\right].$$

It penalizes only those outcomes that fall below the mean return. Thus, it is called one sided variance. Under normal return distribution, minimization of semivariance is equivalent to minimization of variance as

$$\sigma^2 = 2\sigma_-^2.$$

It follows from the symmetric nature of normal distribution. However, for non-symmetric distributions, semivariance is better measure of risk than the variance. In fact, Markowtiz argued that mean-semivariance analysis tend to produce better portfolios than those based on mean-variance.

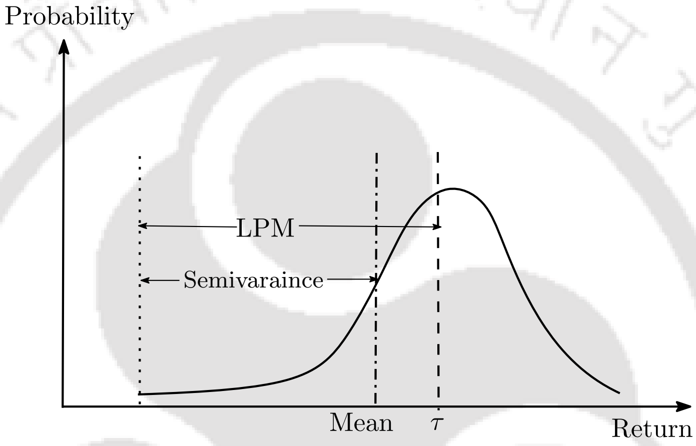


Figure 1.8: Semivariance and LPM

1.6.3 Lower partial moment

Bawa (1975) and Fishburn (1977) introduced generalized the semivariance to lower partial moment (LPM) that measures the dispersion of returns below a threshold level or target return instead of just expected return. The α-th order LPM of a random variable $R \in \mathcal{G}$ about a fixed target τ is defined as

$$\mathrm{LPM}_{\alpha}(\tau;R) = \mathbb{E}\left[(\tau - R)^{\alpha}\mathbb{1}_{\left\{\tau \geq R\right\}}\right],$$

Consequently, the lower partial deviation (LPD) of order n (> 0) is given by

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R) = \left(\mathbb{E}\left[(\tau - R)^{\alpha}\mathbb{1}_{\left\{\tau \geq R\right\}}\right]\right)^{\frac{1}{\alpha}}.$$

Here, returns below the target level τ are only taken into account to measure the risk. Hence, τ is called disaster level or minimum acceptable return (MAR). It can be easily seen that the safety first measure and the semivariance are special cases of LPM. The safety first measure is obtained by setting $\alpha = 0$. On the other hand, for $\alpha = 2$ and $\tau = \mathbb{E}(R)$, LPM is identical to the semivariance. Moreover, by varying the values of τ and α, we can obtain various new downside risk measures.

It is clear that LPM penalizes only those return that falls below some threshold level. As a result, minimizing LPM is equivalent to minimizing the un-favourable returns as well as the negative skewness. Hence, the LPM based portfolio selection model takes into account the skewness of return distribution.

1.6.4 VaR

VaR is defined as the smallest value $\tau \in \mathbb{R}$ such that the probability of loss exceeding τ is at most $1-\varepsilon$ over a given time horizon.

$$\text{VaR}_{\varepsilon}(R) = \text{Inf}\{\tau : \mathbb{P}(-R \geq \tau) \leq 1-\varepsilon\}$$

The VaR only measure the worst-case loss with a probability. But, it completely disregards loss beyond the wort-case loss. Hence, for heavy-tailed distribution, VaR underestimates the actual risk.

1.6.5 CVaR

In order to overcome the issue of VaR, Artzner et al. (1999) proposed alternative risk measure called CVaR or expected shortfall (ES). It estimates the average loss beyond VaR. The CVaR is defined as

$$\text{CVaR}_{\varepsilon}(R) = \mathbb{E}[-R \mid -R \geq \text{VaR}_{\varepsilon}(R)].$$

For heavy-tailed distribution, it is the appropriate measure of downside risk. In addition, the CVaR has better theoretical properties than the VaR. The CVaR supports the diversification principle, whereas the VaR does not.

Among all the above mentioned downside risk measures, this thesis mainly focuses on LPM, because, in one side, it provides strong connection with the expected utility theory and the SD rules, and on the other hand it furnishes variety of investment preferences through its parameters. In the following segments, we discuss the key contributions on the domain of portfolio theory with LPM.

1.7 Portfolio theory with LPM

1.7.1 Mean-LPM model and utility theory

Fishburn (1977) established the connection between LPM and expected utility theory. He presented the idea of mean-risk dominance model, where the risk is measured as below-target deviation. The risk measure, according to Fishburn, is

$$\rho(R) = \mathbb{E}[\phi(\tau - R)\mathbb{1}_{\tau \geq R}],$$

where $\phi(r)$ is non-negative non-decreasing function for $r \geq 0$ with $\phi(0) = 0$

Now, a portfolio with return R_1 dominates another portfolio with return R_2 in the sense of mean-risk dominance rule if and only if

$$\mathbb{E}(R_1) \geq \mathbb{E}(R_2) \text{ and } \rho(R_1) \leq \rho(R_2)$$

with at least one strict inequality. The connection between the mean-risk model and utility theory is presented in the following theorem.

THEOREM 1.1. *The mean-LPM utility model is congruent with the expected utility model in the sense that*

$$G(\mathbb{E}(R_1), \rho(R_1)) > G(\mathbb{E}(R_2), \rho(R_2))$$

if and only if

$$\mathbb{E}(U(R_1)) > \mathbb{E}(U(R_2))$$

with G increasing in $\mu(R) = \mathbb{E}(R)$*, decreasing in* $\rho(R)$*, and with* $U(\tau) = \tau$ *and* $U(\tau + 1) = \tau + 1$*. Then, there is a positive constant k such that*

$$U(R) = \begin{cases} R & R \geq \tau \\ R - k\phi(\tau - x) & R \leq \tau \end{cases} \tag{1.4}$$

where $\phi(R)$ *is non-negative non-decreasing function for* $R \geq 0$ *with* $\phi(0) = 0$.

Therefore, the mean-LPM model is congruent with the expected utility model, where the utility functions can be written in the following form:

$$U(R) = \begin{cases} R & R \geq \tau \\ R - k(\tau - R)^{\alpha}, & R \leq \tau \end{cases}$$

The degree α reflects investors' risk taking attitude below the target level τ as the degree determines the nature of utility function described above. It is also noted that all the functions are linear above τ and hence it depicts risk-neutral behaviour above τ. For $\alpha = 0, 1$, the utility curves are linear below τ, which suits the risk-neutral investor. Since the curves are concave for $\alpha > 1$, investors choosing $\alpha > 1$ are risk-averse below τ. On the other hand, the risk-seeking attitude is captured by setting $0 < \alpha < 1$.

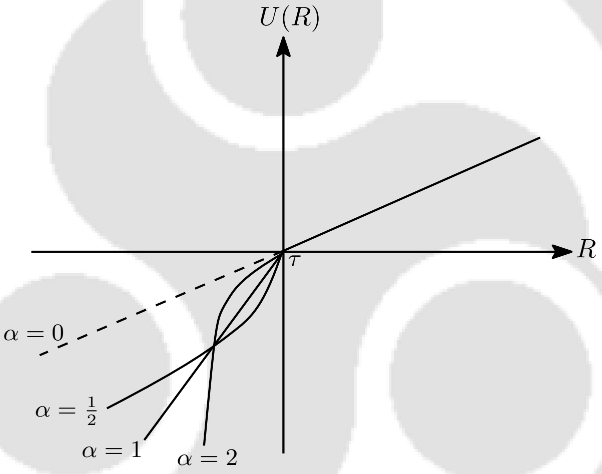


Figure 1.9: Plots of $U(R)$ for different values of α

The corresponding expected utility is given by

$$\mathbb{E}(U(R)) = \mathbb{E}(R) - k\,\mathrm{LPM}_{\alpha}(\tau; R).$$

Thus, for any given value of k, the indifference curves in mean-LPM_{α} space will be parallel straight lines.

1.7.2 UPM-LPM model and utility theory

Holthausen (1981) introduced reward-risk model, where risk is measured by ρ and reward is determined as deviations above some target level. The reward measure is defined below:

$$\Pi(R) = \mathbb{E}[\Psi(R-\tau)],$$

where $\Psi(R)$, for $R \geq 0$, is a nonnegative nondecreasing function of R. It mainly measures the upside deviation of return distribution. Hence, the Π-ρ model captures both the upside and the downside deviations. The consistency of this model with expected utility theory is established below.

THEOREM 1.2. *The Π-ρ utility model is congruent with the expected utility model in the sense that*

$$V(\Pi(R_1), \rho(R_1)) > V(\Pi(R_2), \rho(R_2))$$

if and only if

$$\mathbb{E}(U(R_1)) > \mathbb{E}(U(R_2))$$

with V increasing in π and decreasing in ρ, and with $U(\tau) = 0$ and $U(\tau+1) = 1$. Then, there is a positive constant k such that

$$U(R) = \begin{cases} \frac{\Psi(R-\tau)}{\Psi(1)} & R \geq \tau \\ -k\phi(\tau - R) & R \leq \tau \end{cases} \tag{1.5}$$

where $\phi(R)$ and $\Psi(R)$ both are non-negative non-decreasing functions for $R \geq 0$ with $\phi(0) = 0$ and $\Psi(0) = 0$.

When $\Psi(R-\tau) = (R-\tau)^{\gamma}$, for $\gamma \geq 0$, the reward measure becomes upper partial moment (UPM):

$$\text{UPM}_{\gamma}(\tau; R) = \mathbb{E}\left[(R-\tau)^{\gamma}\mathbb{1}_{\{R>\tau\}}\right].$$

The corresponding root-UPM or upper partial deviation (UPD) is

$$\text{UPM}_{\gamma}^{1/\gamma}(\tau; R) = \left(\mathbb{E}\left[(R-\tau)^{\gamma}\mathbb{1}_{\{R>\tau\}}\right]\right)^{\frac{1}{\gamma}}.$$

In contrast to the LPM, the UPM measures upside potential of asset returns. Therefore, the UPM-LPM model is congruent with the expected utility model, where the utility functions can be written as follows:

$$U(R) = \begin{cases} (R-\tau)^{\gamma} & R \geq \tau \\ -k(\tau-R)^{\alpha} & R \leq \tau \end{cases}$$

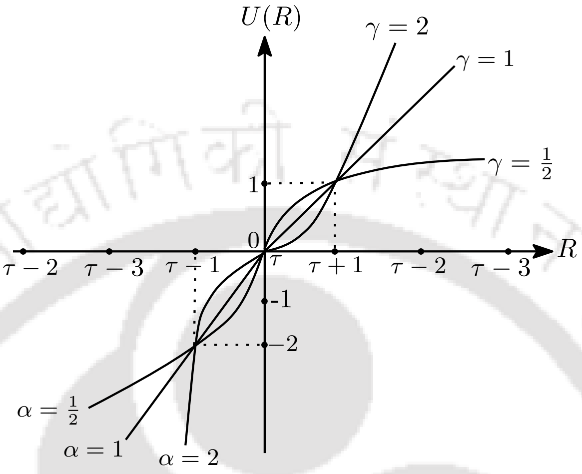


Figure 1.10: Plots of $U(R)$ for different values of α and γ

The values of γ and α reflects the risk-taking behaviour of an investor. For example, if $\alpha > 1$ and $\gamma > 1$, then the investor is risk-averse below τ and risk-seeking above τ. In other words, the investor is downside risk-averse and upside potential-seeking. On the other hand, for $\alpha < 1$ and $\gamma < 1$, the investor is downside risk-seeking and upside potential-averse. For $\alpha = 1, \gamma = 1$, the investor exhibits risk-neutral behaviour.

1.7.3 Stochastic dominance and LPM

Porter (1974) first established the consistency between SSD rules and mean-semivariance (MSV) selection criteria with the semivariance calculated around a fixed point.

Theorem 1.3. *Except for portfolios with identical mean and semivariance, every portfolio that is efficient by the MSV criterion is also efficient by the SSD criterion.*

Hence, with the one possible exception noted, the MSV efficient set is a subset of the SSD efficient set for any value of τ. Ogryczak and Ruszczyński (1999) showed that mean−semideviation (square root of the semivariance) trade-off model is consistent with the SSD provided that the trade-off co-efficient is bounded by 1. Thus, the optimal solution of the problem

$$\text{Max}\left\{\mathbb{E}(R) - \lambda\ \text{LPM}_2^{1/2}(\tau;R) : 0 < \lambda \leq 0,\ R \in \mathscr{Q}\right\}$$

is efficient under SSD rules. Here, $\mathscr{Q}$ is the feasible set of random variables denoting possible the risky alternatives.

Bawa extended this result to the case of general MLPM framework and showed that the MLPM rule with different values of α is consistent with different order of SD criteria (Bawa, 1975; Bawa, 1976). In particular, for any fixed τ, the MLPM$_\alpha$ dominance rule is consistent with the popular SD rule of order $\alpha+1$ for $\alpha = 0,1,2$. These results are summarized by the following theorem:

THEOREM 1.4. *(i) R_1 is preferred to R_2 for all utility functions in $\mathscr{U}_\alpha, \alpha = 1,2$, if and only if*

$$\text{LPM}_{\alpha-1}(\tau;R_1) \leq \text{LPM}_{\alpha-1}(\tau;R_2)\ \forall\ \tau \in \mathbb{R}$$

with strict inequality for some τ;

(ii) R_1 is preferred to R_2 for all utility functions in $\mathscr{U}_3$ if and only if

$$\mathbb{E}(R_2) \geq \mathbb{E}(R_2)$$

and

$$\text{LPM}_2(\tau;R_1) \leq \text{LPM}_2(\tau;R_2)\ \forall\ \tau \in \mathbb{R}$$

with strict inequality for some τ.

Therefore, for $\alpha \geq 0$, the MLPM$_\alpha$ selection rule produces first order stochastically dominant portfolios. The SSD is consistent with the MLPM rule with $\alpha \geq 1$. In addition, the TSD results will be reached for $\alpha \geq 2$. Thus, LPM$_1$ is valid for all the risk averse utility functions $(U' > 0, U'' < 0)$ while LPM$_2$ is consistent with all the risk-averse functions displaying skewness preferences $(U' > 0, U'' < 0, U''' > 0)$. Hence, an investor who prefers more to less, exhibits risk averseness and displays decreasing absolute risk aversion (DARA) will choose LPM$_\alpha$ with $\alpha \geq 2$.

1.8 Portfolio selection models with LPM

1.8.1 The MLPM selection models

Similar to the Markowitz's MV model, the MLPM portfolio allocation problem can be formulated by the following non-linear optimization model

$$\begin{cases} \underset{\boldsymbol{w}}{\text{Minimize}} & \text{LPM}_{\alpha}(\tau; R_p) \\ \text{subject to} & \sum_{i=1}^{n} w_i \mathbb{E}[R_i] = \mu \\ & \boldsymbol{w} \in \mathscr{D} \end{cases} \tag{1.6}$$

As shown by Bawa (1975), LPM_{α} is a convex functional of $\boldsymbol{w}$ and hence it is a convex optimization problem. As a result, a unique optimal solution always exists. Yet, there exist some difficulties that make this problem slightly cumbersome. Variance of a portfolio of assets can be expressed in terms of individual variances and covariances. Similar decomposition in the case of LPM is not possible. Thus, it is very difficult to find analytical solution of mean-LPM optimization model. However, over the years, various algorithms have been developed to solve this problem efficiently. Some important results related to the mean-LPM model are presented below.

(i). Harlow (1991) describes the following optimization model as a tool for portfolio selection. The nonlinear MLPM optimization problem is represented as:

$$\begin{cases} \underset{\boldsymbol{w}}{\text{Minimize}} & \text{LPM}_{\alpha}(\tau; R_p) \equiv \frac{1}{n-1} \sum_{t=1}^{n} \left[(\tau - R_p)^{\alpha} \mathbb{1}_{\{R_p \leq \tau\}} \right] \\ \text{subject to} & \sum_{i=1}^{n} w_i \mathbb{E}(R_i) = \mu \\ & \boldsymbol{w} \in \mathscr{D} \end{cases}$$

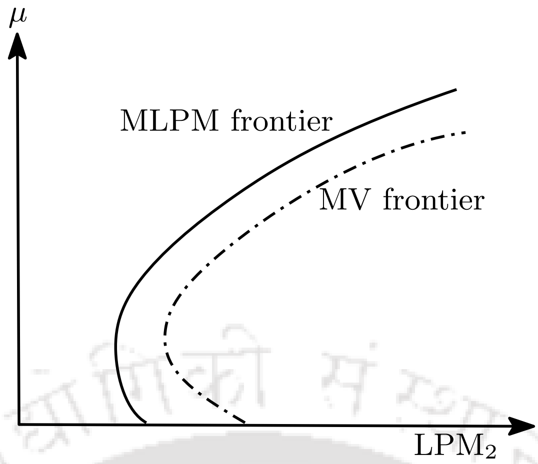


Figure 1.11: MV vs MLPM$_2$

The optimal solution for each μ forms the set of MLPM efficient portfolios. Harlow showed that the MV efficient set lies inside the MLPM opportunity set when historical returns are not jointly normal distribution. It means downside risk framework can provide more downside protection while preserving the same or greater expected return. Hence, the MLPM$_2$ portfolios are more efficient than MV portfolios.

(ii). Estrada (2008) proposed mean-semivariance portfolio optimization model with slightly different expression of cosemivariance and semivariance. He used symmetric cosemivariance measure instead of using Markowitz's cosemivariance measure. The cosemivariance between asset i and j with respect to benchmark τ is defined as

$$\bar{\bar{\sigma}}_{ij,\tau} = \mathbb{E}\left[(R_i - \tau, 0)\mathbb{1}_{\{R_i \le \tau\}}(R_j - \tau, 0)\mathbb{1}_{\{R_j \le \tau\}}\right].$$

He further argued that the semivariance of a portfolio with respected to a benchmark τ can be approximated with the expression

$$\bar{\bar{\sigma}}^2_{p,\tau} \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \bar{\bar{\sigma}}_{ij,\tau}$$

Thus the symmetric cosemivariance matrix can be used in the same way the covariance matrix is used in solving of mean-variance optimization problems. As a result, similar to the MV model, MSV optimization problem can be solved easily. The MSV model formulated by

Estrada is presented below:

$$\begin{cases} \underset{w}{\text{Minimize}} & \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \bar{\bar{\sigma}}_{ij,\tau} \\ \text{subject to} & \sum_{i=1}^{n} w_i \mathbb{E}(R_i) = \mu \\ & \boldsymbol{w} \in \mathscr{D} \end{cases}$$

1.8.2 UPM-LPM selection model

Cumova and Nawrocki (2014) used both upside partial moment and lower partial moment (UPM/LPM) to maximize investor's utility. They did not consider the upside deviation from the benchmark return to be risk and therefore label it as return potential. The reason is that most investors are risk-averse below the target return and risk-seeking above the benchmark return. Thus, they proposed the following model:

$$\begin{cases} \underset{w}{\text{Minimize}} & \text{LPM}_{\alpha}(\tau; R_p) \\ \underset{w}{\text{Maximize}} & \text{UPM}_{\gamma}(R_p;\tau) \\ \text{subject to} & \boldsymbol{w} \in \mathscr{D}, \end{cases}$$

They further simplified the above multi-objective optimization as a minimization of a downside risk (LPM) for a fixed upside potential.

$$\begin{cases} \underset{w}{\text{Minimize}} & \text{LPM}_{\alpha}(R_p \tau) \\ \text{subject to} & \text{UPM}_{\gamma}(R_p;\tau) = b \\ & \boldsymbol{w} \in \mathscr{D} \end{cases}$$

1.8.3 MLPM Efficient frontier and two-fund separation

Solving the MLPM optimization problem for various μ, and then plotting optimal values in mean-$\text{LPM}_{\alpha}^{1/\alpha}$ space produces a curve called minimum $\text{LPM}_{\alpha}^{1/\alpha}$ frontier. Upper part of this curve is called MLPM efficient frontier. For any level of $\mu = \mu^*$, the portfolio lies on the frontier is most desirable among all the portfolios with the same expected return μ^*. Empirically, it is observed that the frontier is convex in μ-$\text{LPM}_{\alpha}^{1/\alpha}$ space. For example, see the works of Harlow and Rao (1989), Pedersen and

Satchell (2002), Brogan and Stidham (2008), Estrada (2008).

With additional assumption, the MLPM portfolio selection problem simplifies further under the presence of a unique risk-free asset. When the target rate is equal to the risk-free rate or the expected return, the efficient frontier becomes a straight line. Thus, the two-fund separation of MV portfolio theory easily obtains in this framework.

THEOREM 1.5 (Bawa and Lindenberg (1977), Lee and Rao (1988))**.** *Under the presence of a risk-free asset, the optimal portfolio choice in* $\mathrm{MLPM}_\alpha(\tau)$ *framework for the cases of* $\alpha = 1,2$ *admits linear separation between the risk-free asset and a unique tangency portfolio of risky assets either if* $\tau = r_f$ *or if* $\tau = \mu$.

1.9 Asset pricing with LPM

Similar to the Sharpe's nobel prize winning CAPM, several asset pricing models are developed in the MLPM framework. Using Sharpe's methodology, Bawa and Lindenberg (1977) analytically developed a new model with LPM, where the target rate is risk-free rate. The pricing model is summarized in the following theorem.

THEOREM 1.6 (Bawa and Lindenberg (1977))**.** *Under the standard assumptions, if all investors evaluates portfolios in a MLPM framework, then the market equilibrium prices satisfy the following relationship, with* $\alpha = 1,2$,

$$\mathbb{E}[R_i] - r_f = \beta_{i;r_f}^{\mathrm{MLPM}_\alpha}\left(\mathbb{E}[R_m] - r_f\right),\ i = 1,2,\ldots,n,$$

where

$$\beta_{i;r_f}^{\mathrm{MLPM}_\alpha} = \frac{\mathrm{CLPM}_\alpha(r_f, r_f; R_m, R_i)}{\mathrm{LPM}_\alpha(r_f; R_m)},$$

the generalized MLPM beta and

$$\mathrm{CLPM}_\alpha(r_f, r_f; R_m, R_i) = \mathbb{E}\left[\left(r_f - R_m\right)^{\alpha-1}\left(r_f - R_i\right)\mathbb{1}_{\left\{r_f > R_m\right\}}\right],$$

the co-lower partial moment of order α *between the market portfolio m and the asset i.*

Lee and Rao (1988) proposed another MLPM model, where the target return is measured by expected return,

$$\mathbb{E}[R_i]-r_f=\beta_{i;\mathbb{E}(.)}^{\mathrm{MLPM}_\alpha}\left(\mathbb{E}[R_m]-r_f\right),\ i=1,2,\ldots,n,$$

where

$$\beta_{i;\mathbb{E}(.)}^{\mathrm{MLPM}_\alpha}=\frac{\mathrm{CLPM}_\alpha(\mathbb{E}(R_m),\mathbb{E}(R_i);R_m,R_i)}{\mathrm{LPM}_\alpha(\mathbb{E}(R_m);R_m)}$$

with

$$\mathrm{CLPM}_\alpha\left(\mathbb{E}[R_m],\mathbb{E}[R_i];R_m,R_i\right)=\mathbb{E}\left[\left(\mathbb{E}[R_m]-R_m\right)^{\alpha-1}\left(\mathbb{E}[R_i]-R_i\right)\mathbb{1}_{\left\{\mathbb{E}[R_m]>R_m\right\}}\right].$$

Nantell and Price (1979) conducted a detail analytical comparison between the classical CAPM and LPM based CAPM. It is found that that both the pricing models are identical under normally distributed returns. Price et al. (1982) presented some interesting analytical as well as empirical results. They showed that there are significant systemic differences in the two models when the return distributions are not normal. Estrada (2007) observed that empirical results clearly supports the downside risk based CAPM over the traditional one. Some other interesting works on this direction can be found in Galagedera (2007), Pedersen and Hwang (2007) and Nurjannah et al. (2012), .

1.10 Performance ratios with LPM

The Sharpe ratio is highly debated because of its inappropriate measure of risk, the standard deviation. The fundamental drawbacks of standard deviation make this ratio less reliable as a measure of portfolio's performance. As an alternative, Sortino and Price (1994) proposed a new ratio, where variance is replaced by LPM. The ratio is popularly known as Sortino ratio:

$$\mathrm{Sor}(\tau)=\frac{\mathbb{E}[R]-\tau}{\mathrm{LPM}_2^{1/2}(\tau;R)}.$$

Since the Sortino ratio penalizes only downside outcomes, it is intuitively much more appealing than the Sharpe ratio. Kaplan and Knowles (2004) generalized this ratio and termed it Kappa ratio:

$$\kappa(\tau)=\frac{\mathbb{E}[R]-\tau}{\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R)}.$$

For $\alpha=2$, it is identical to the Sortino ratio.

In order to reflect both the upside and downside behaviours in asset ranking, Sortino et al. (1999) used UPD of order 1 and LPD of order 2 as measures of reward and risk, respectively and proposed the following ratio known as upside potential ratio (UPR):

$$\text{UPR}(\tau;\alpha,\gamma) = \frac{\text{UPM}_1(\tau;R)}{\text{LPM}_2^{1/2}(\tau;R)}.$$

In order to capture all the nuisances of risk preferences, Farinelli and Tibiletti (2008) generalized the UPR to FT ratio that allows the values of γ and α to be any positive real number,

$$\text{FT}(\tau;\alpha,\gamma) = \frac{\text{UPM}_\gamma^{1/\gamma}(\tau;R)}{\text{LPM}_\alpha^{1/\alpha}(\tau;R)}.$$

1.11 Research gap

There is no doubt that the MLPM theory has more theoretical and practical relevance than the MPT. However, there are several important issues which lead to preference variance over LPM. The issues are discussed below one by one.

(i). **Convexity of efficient frontier:** We begin with the issue of MLPM efficient frontier. Unlike the case of MV efficient frontier, the convexity of the MLPM frontier is not analytically shown. The positive definitiveness of variance-covariance matrix is sufficient for the convexity of mean-standard deviation efficient frontier. However, similar sufficient condition for the convexity of mean-LPD efficient frontier is unknown. The convexity is a very important fact as it guarantees the existence of various fundamental results such as two-fund monetary separation (TFMS) and capital market equilibrium relationship.

(ii). **Two-fund separation:** The second issue is related to the two-fund monetary separation (TFMS). For the case of MLPM model, the TFMS is not guaranteed except for the two special targets, the riskless rate and mean return. It means for arbitrary values of τ, the existence of separation is not known. This issue has remained unsolved for more than three decades. As a result, non-separation occurs while using arbitrary targets and thereby several pitfalls arise in MLPM portfolio management. Brogan and Stidham (2008) showed that under the non-separation, an investor who mistakenly assumes separation holds may face excess portfolio risk. They also proposed a conjecture that describes the shape of MLPM efficient frontier more accurately. However, the conjecture has not been analytically proved yet.

(iii). **CAPM:** Another important issue is about asset pricing model in the MLPM framework. Hogan and Warren (1974) proposed a downside asset pricing model by using semivariance as a measure of risk. Bawa and Lindenberg (1977) generalized the semivariance to LPM and developed MLPM CAPM, where the target return is equal to risk-free rate. Lee and Rao (1988) developed another MLPM CAPM, where the target is the expected return of the portfolio whose risk is being measured. This is because these two targets permit linear separation which guarantees the existence of unique market equilibrium. Since no other such target (if any) is available in the literature, pricing models for other targets have not been developed.

(iv). **Realistic CAPM:** All the MLPM CAPMs are developed by adopting the standard assumptions of Sharpe-Lintner-Mossin (SLB) model (Sharpe, 1964; Lintner, 1965; Mossin, 1966). However, one of the assumptions is really unrealistic. This assumption is that the investors can lend and borrow any amount at a single risk-free rate. If the lending and borrowing rates are different, the MV CAPM and the MLPM CAPMs will be invalid. In order to overcome the deficiency, Black (1972) developed the zero-beta CAPM. However, in the MLPM framework, no such alternative is known yet.

(v). **Performance ratio:** Risk-adjusted performance ratios are widely used to rank risky portfolios. For this purpose, numerous ratios have been developed in the literature. A major question in this context is: Which of these measures are appropriate in the context of portfolio management. Cherny and Madan (2008) proposed a set of axioms to characterise performance measures for cash flows generated by writing options. Measures satisfying these axioms are termed indexes of acceptability. On the other hand, by defining desirable properties of risk and reward measures, Rachev et al. (2008) axiomatically defined coherent ratios to measure portfolio's performance. These axioms mainly emphasize on the characteristics of reward and risk measures rather than the performance measures. Thus, it is important to identify the most desirable properties that an performance ratio should satisfy.

This thesis aims to address all these issues. In addition to that, new asset pricing models and new performance measures are developed.

1.12 Thesis outline

The thesis contains seven chapters, of which, Chapter 1 and Chapter 7 present introduction and conclusions, respectively. Based on different problems and their significance, the main contributions are split into five different chapters (Chapter 2 to Chapter 6). The contributions of these chapter are summarized below point by point.

(i). **Chapter 2:** This chapter first presents various mathematical properties of LPM and discusses their practical interpretations in the context of portfolio management. In addition, we solve two long-standing open problems and a conjecture. First, we analytically prove an open problem—the convexity of efficient frontier in a mean-LPD space. Second, the Brogan-Stidham conjecture, that describes the behaviours of investment curve, is analytically proved. Then, we discuss the second open problem related to the two-fund monetary separation. We show that there exists a unique unique generalized family of target returns that guarantees the MLPM separation.

(ii). **Chapter 3:** In this chapter, we derive a generalized CAPM which reduces to the existing MLPM CAPMs and the traditional CAPM for special cases. In addition, we derive a risk-adjusted capital budgeting criteria. Finally, we present some interesting empirical behaviours of generalized MLPM beta and LPM based $alpha$.

(iii). **Chapter 4:** In this chapter, we analytically develop a new MLPM asset pricing model without the assumption of unique risk-free asset. It is also shown that this model is valid for realistic situations. Furthermore, the model specializes to Black's zero-beta CAPM for normally distributed return.

(iv). **Chapter 5:** This chapter proposes a set of desirable axioms to characterize performance measures in the context of portfolio management. A performance measure consistent with the axioms is called ideal. We examine whether the LPM based performance ratios are ideal. In addition, other popular ratios are examined.

(v). **Chapter 6:** While verifying the presence of ideal properties in LPM based measures, we found that one of the LPM based performance measures, Farinelli-Tibiletti (FT) ratio (Farinelli and Tibiletti, 2008), which captures potential-seeking behaviour, is not ideal. It violates a very important property of portfolio theory, the diversification. As an alternative, we propose a new ideal performance measure, upside beta ratio (UBR). To examine its performance, we rank mutual funds for UBR and other four performance measures– Sharpe, Sortino, FT and Jensen's alpha, and then we compare the rankings of UBR with the rankings of other ratios. In addition, the performance of top-ranked funds are compared through back-testing and out-of-sample analysis. Our findings reveal that the UBR performs significantly better than the other ratios in most scenarios. Finally, in order to check robustness of the new measure, a parameter sensitivity analysis is presented.

(vi). **Chapter 7:** This chapter presents concluding remarks along with future direction of research.

1.13 Conclusion

This chapter reviewed and discussed the important aspects of modern portfolio theory (MPT) as well as post-modern portfolio theory. We also depicted the issues of MPT and then described how the downside risk based model has emerged as an alternative. In particular, the LPM is widely utilized as a replacement for the classical risk measure, variance. This is because the LPM has the intuitive appealing and a number of desirable characteristics. The following table 1.1 summarizes the key differences between variance and LPM.

Table 1.1: Variance vs LPM

Subject	Variance	LPM
Treatment	Identical treatment for both the upside and downside deviations	Only penalizes the downside deviations
Tail events	Fails to capture	Able to capture
Suitable distribution	Normal / Elliptical / Symmetric	For all distribution
Stochastic dominance	Not consistent	Always consistent
Choice of target return	Not permissible	Provides the choice of fixing desired target level
Degree of risk-taking attitude	Only second order	Varies with the values of n
Optimization	Easy to solve (Quadratic optimization provides the analytical solution)	Not easy to solve (numerical optimization needed)

In spite of having several advantages over the MPT, the LPM based portfolio theory has some serious drawbacks with regard to the computational part. This thesis aims to address all those issues and help investors to utilize LPM more efficiently. In the following table 1.2, we briefly presents the main issues of portfolio theory with LPM and our remediation.

Table 1.2: Issues and it's remediation of LPM based portfolio theory

Subject	Issue	Remediation
Convexity	Assumed and utilized by Bawa and Lindenberg (1977) and Lee and Rao (1988) but no analytical proof available so far	Provide an analytical proof
Brogan-Stidham Conjecture	Proposed by Brogan and Stidham (2008) but not yet solved	Solve the conjecture
Two-fund separation	Separation is not guaranteed other than two special targets	Show existence and uniqueness of a generalized target, and hence, infinitely many targets will be available
CAPM	Developed only for two targets	Develop for the generalized target
CAPM without risk-free asset	No such model is developed in MLPM framework	Develop a new pricing model without risk-free asset
Performance measure	So many performance measure exist but which are really useful	Provide three important properties to classify "ideal" performance measure
FT ratio	Does not support diversification	Propose an alternative measure that supports diversification as well as other properties of ideal measure

Chapter 2

Mean-LPM Portfolio Theory

Measuring portfolio risk and reward is the primary and the most important step in investment management. Thus, before applying arbitrary measures to quantify risk and reward, every investor should need to know how good the measures are themselves. Over the years, several theories have been proposed to investigate desirable properties of a risk measure. Stone (1973) introduced a general class of risk measures that contains several the popular risk measures (variance, semivariance, mean absolute deviation, shortfall probability) as special cases. Kijima and Ohnishi (1993) provided an axiomatic definition of a financial risk measure− four desirable properties that a risk measure should satisfy. The properties are non-negativity, positive homogeneity, sub-additivity and shift-invariance. Artzner et al. (1999) proposed another axiomatic definition of risk measure, in which the properties are− normalized condition, monotonicity, positive homogeneity and sub-additivity. A risk measure satisfying these properties is called "*coherent*". After this work, there has been considerable development in the area of risk and reward measurement. Some interesting works towards this direction can be found in Föllmer and Schied (2002), Acerbi (2002), Biglova et al. (2004) and Rockafellar et al. (2006). Connecting the link with expected utility theory, De Giorgi (2005) proposed some appropriate properties for both the risk and the reward measures. Rachev et al. (2008) also presented various properties for risk, reward and performance measures. It is very difficult for a measure to satisfy all the properties developed in the literature. Yet, they should have some minimal and practically relevant properties. This chapter presents various properties of lower partial deviation (LPD) (the root of LPM) as a measure of portfolio risk. The practical implications of all the properties are also discussed.

It is true that many practitioners prefer variance over LPM due to its theoretical and computational issues. One of the issues is the convexity of mean-LPD (MLPD) efficient frontier. We know that the positive definitiveness of variance-covariance matrix is sufficient for the convexity of mean-standard deviation efficient frontier. On contrary, similar simple and sufficient condition for the convexity of MLPD efficient frontier is not available. The convexity is a very important fact as it guarantees the existence of two-fund separation and capital equilibrium relationship. In this chapter, we provide an analytically proof for this convexity.

The MV portfolio selection model under the presence of a risk-free asset becomes simpler as the efficient frontier is linear. In contrast, the MLPD efficient frontier is not always linear. The actual shape, in this case, depends on the shape of investment curves which are formed by the affine combinations of a risk-free asset and some risky portfolios. Brogan and Stidham (2008) empirically observed some behaviours and conjectured that the curves are always convex and the convexity approaches linearity as mean return tends to infinity. We prove the conjecture analytically.

The target return makes the LPM more attractive to the investors as it allows them to set their own benchmark levels or minimum acceptable return. Although the LPM provides infinitely many choices, mainly two targets (the risk-free rate and the expected return) have gained popularity. This is because these targets admit linear separation or TFMS which further ensures the existence of a unique market equilibrium in the mean-LPM (MLPM) framework. However, in the MLPM framework, the separation is not ensured except the two special targets. As a result, non-separation occurs for using arbitrary targets, and thereby several pitfalls arise in MLPM portfolio management. Brogan and Stidham (2008) described that, under the non-separation, an investor may face excess portfolio risk in the MLPM framework. Moreover, they described that the MLPM portfolio optimization problem becomes more complex due to the non-separation.

There are some important works in the literature related to the MLPM separation. Bawa and Lindenberg (1977) first showed that the TFMS holds when the target equals risk-free rate. By setting risk-free rate as a target level, they further derived a capital market equilibrium relationship similar to Sharpe's capital asset pricing model (CAPM) (Sharpe, 1964). Lee and Rao (1988) found another candidate for the separation, the expected return of the portfolio whose risk is being measured, and then developed corresponding asset pricing equation. Grootveld and Hallerbach (1999) nicely described analytical and empirical differences between variance and LPM, and showed that the separation is preserved for these two special targets. De Giorgi et al. (2011) developed a sufficient condition for the TFMS in a general risk-reward space and showed that the condition holds for those two targets in the MLPM framework. Since all these papers did not consider the possibility of other such target returns, a natural question to ask is: which other targets, if any, allow the separation? This chapter addresses this problem and provides a complete solution.

The chapter is structured as follows. Section 2.1 describes the properties of LPM and discusses their practical implications. In Section 2.2, we formate the MLPM portfolio optimization model and then prove the convexity of the minimum risky frontier. In Section 2.3, portfolio selection model with a risk-free asset is formulated and various properties of investment curves are described. Section 2.4 investigates the linearity property of investment curves to obtain the separation. We develop a unique generalized family of target returns that makes all investment curves linear and then present the theorem of generalized two-fund separation.

2.1 Properties of LPD

The purpose of this section is to see the efficiency of LPD as a measure of risk. We present various properties of LPD. The properties have realistic economical interpretation and proper relevance to the practical portfolio management.

2.1.1 Zero risk condition

An investor fixes a target level with the aim of earning higher return than the target rate. Hence, loss occurs only when portfolio returns fall below the MAR. If a portfolio always generates higher return than the target rate, then there is no loss and thus risk of the portfolio should be zero. We show that LPD of a portfolio is zero if and only if returns of the portfolio exceed target return almost surely.

THEOREM 2.1. $R \geq \tau$ *a.s.* $\iff$ $\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R) = 0$.

PROOF. Let us define $Z = (\tau - R)^{\alpha} \mathbb{1}_{\{\tau \geq R\}}$. Since Z is a non-negative random variable, we obtain

$$\begin{aligned} \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R) = 0 &\iff \mathbb{E}(Z) = 0 \\ &\iff \mathbb{P}(Z = 0) = 1 \\ &\iff \mathbb{P}(R \geq \tau) = 1. \end{aligned}$$

Thus, $\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R) = 0$ if and only if $R \geq \tau$ almost surely.

■

2.1.2 Asset monotonicity

Suppose that a portfolio R_1 always generates higher return than the returns of portfolio R_2. It is easy to see that R_2 has lower possibility to beat a target return, i.e., $\mathbb{P}(R_2 \leq \tau) \leq \mathbb{P}(R_1 \leq \tau)$. Moreover, it is unrealistic to imagine why should an investor invest in an asset that has always lower return than another asset. Thus, in this sense, R_2 should be more risky than R_1.

THEOREM 2.2. *For any two random returns* $R_1, R_2 \in \mathscr{G}$, $R_1(\omega) \geq R_2(\omega)\ \forall\ \omega \in \Omega \implies \mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_1) \leq \mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_2)$.

PROOF. We can see that

$$\begin{aligned}
R_1 \geq R_2 &\implies (\tau - R_1) \leq (\tau - R_2) \\
&\implies (\tau - R_1)\mathbb{1}_{\{\tau \geq R_1\}} \leq (\tau - R_2)\mathbb{1}_{\{\tau \geq R_1\}} \leq (\tau - R_2)\mathbb{1}_{\{\tau \geq R_2\}} \\
&\implies (\tau - R_1)^\alpha \mathbb{1}_{\{\tau \geq R_1\}} \leq (\tau - R_2)^\alpha \mathbb{1}_{\{\tau \geq R_2\}} \\
&\implies \mathbb{E}\left[(\tau - R_1)^\alpha \mathbb{1}_{\{\tau \geq R_1\}}\right] \leq \mathbb{E}\left[(\tau - R_2)^\alpha \mathbb{1}_{\{\tau \geq R_2\}}\right] \\
&\implies \mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_1) \leq \mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_2).
\end{aligned}$$

■

2.1.3 Second order stochastic dominance (SSD) consistency

R_1 stochastically dominates R_2 in second order ($R_1 \succcurlyeq_2 R_2$) if and only if $\mathbb{E}(U(R_1)) \geq \mathbb{E}(U(R_2))$ for any non-decreasing and concave utility function U. Thus, any risk-averse expected utility maximizing investor prefers R_1 over R_2. Hence, risk of R_2 should be higher than the risk of R_1. We show that LPD with $\alpha \geq 1$ ensures the consistency with SSD.

THEOREM 2.3. *For any* $R_1, R_2 \in \mathscr{G}$, $R_1 \succcurlyeq_2 R_2 \implies \mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_1) \leq \mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_2)\ \forall \alpha \geq 1$.

PROOF. We know that

$$R_1 \succcurlyeq_\alpha R_2 \iff F_{R_1}^\alpha(\tau) \leq F_{R_2}^\alpha(\tau)\ \forall\ \alpha \geq 1,$$

where

$$F_{R_1}^{\alpha+1}(\tau) = \int_{-\infty}^{\tau} F_{R_1}^\alpha(t)\, dt,\ \alpha \geq 1$$

and

$$F^1_{R_1}(\tau) = F_{R_1}(\tau) = \mathbb{P}(R_1 \leq \tau).$$

As shown by Ogryczak and Ruszczyński (2001), the LPD can be written as

$$\mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_1) = \left(\alpha! F^{\alpha+1}_{R_1}(\tau)\right)^{\frac{1}{\alpha}}, \tag{2.1}$$

For $n \geq 1$,

$$\begin{aligned} R_1 \succcurlyeq_2 R_2 &\iff F^2_{R_1}(\tau) \leq F^2_{R_2}(\tau) \\ &\implies F^{\alpha+1}_{R_1}(\tau) \leq F^{\alpha+1}_{R_2}(\tau) \\ &\implies \left(\alpha! F^{\alpha+1}_{R_1}(\tau)\right)^{\frac{1}{\alpha}} \leq \left(\alpha! F^{\alpha+1}_{R_2}(\tau)\right)^{\frac{1}{\alpha}} \\ &\implies \mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_1) \leq \mathrm{LPM}_\alpha^{1/\alpha}(\tau; R_2) \end{aligned}$$

■

2.1.4 Target monotonicity

Target monotonicity indicates that risk of a portfolio increases as target rate grows. It basically implies that investor setting higher threshold return should face higher risk. In the following theorem, we show that LPD of a portfolio increases with the increase of target return.

THEOREM 2.4. *For any* τ_1, τ_2 *and* $R \in \mathscr{G}$, $\tau_1 \geq \tau_2 \implies \mathrm{LPM}_\alpha^{1/\alpha}(\tau_1; R) \geq \mathrm{LPM}_\alpha^{1/\alpha}(\tau_2; R)$.

PROOF. We have

$$\begin{aligned} \tau_1 \geq \tau_2 &\implies (\tau_1 - R) \geq (\tau_2 - R) \\ &\implies (\tau_1 - R)\mathbb{1}_{\{\tau_1 \geq R\}} \geq (\tau_2 - R)\mathbb{1}_{\{\tau_1 \geq R\}} \geq (\tau_2 - R)\mathbb{1}_{\{\tau_2 \geq R\}} \\ &\implies (\tau_1 - R)^\alpha \mathbb{1}_{\{\tau_1 \geq R\}} \geq (\tau_2 - R)^\alpha \mathbb{1}_{\{\tau_2 \geq R\}} \\ &\implies \mathbb{E}\left[(\tau_1 - R)^\alpha \mathbb{1}_{\{\tau_1 \geq R\}}\right] \geq \mathbb{E}\left[(\tau_2 - R)^\alpha \mathbb{1}_{\{\tau_2 \geq R\}}\right] \\ &\implies \mathrm{LPM}_\alpha^{1/\alpha}(\tau_1; R) \geq \mathrm{LPM}_\alpha^{1/\alpha}(\tau_2; R). \end{aligned}$$

■

REMARK 2.1. The converse of above theorem is not completely true. Suppose that a portfolio with random return $R = (4,5,3)$ with probability $\mathbb{P} = (1/3,1/3,1/3)$ and target rates $\tau_1 = 2$, $\tau_2 = 3$. We can see that $\text{LPM}_\alpha^{1/\alpha}(\tau_1;R) = \text{LPM}_\alpha^{1/\alpha}(\tau_2;R) = 0$ but $\tau_1 \ngtr \tau_2$. This behaviour is clearly visible from Fig. 2.1, where $\text{LPM}_\alpha^{1/\alpha}(R;\tau)$ is plotted against τ. However, the converse holds in strict inequality sense. This fact is presented in the following theorem.

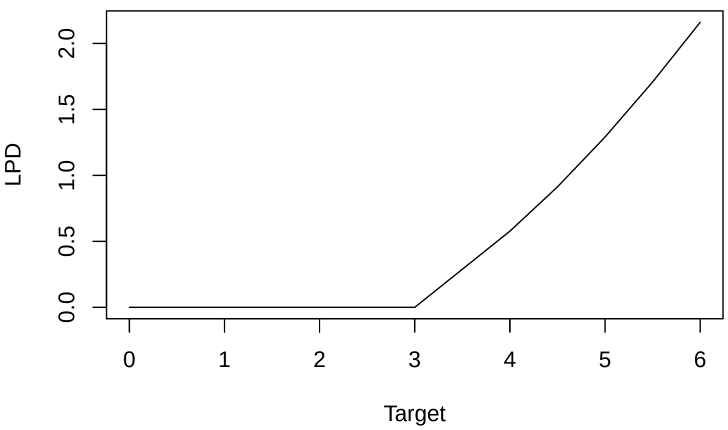


Figure 2.1: LPD as a function of target return

THEOREM 2.5. *For any* τ_1, τ_2 *and* $R \in \mathscr{G}$, $\text{LPM}_\alpha^{1/\alpha}(\tau_1;R) > \text{LPM}_\alpha^{1/\alpha}(\tau_2;R) \implies \tau_1 > \tau_2$.

PROOF. Let us assume $\tau_1 \leq \tau_2$. Using equation (2.1), we obtain

$$\begin{aligned}
&\text{LPM}_\alpha^{1/\alpha}(\alpha,\tau_1;R) - \text{LPM}_\alpha^{1/\alpha}(\alpha,\tau_2;R) > 0\\
\implies &F_R^{\alpha+1}(\tau_1) - F_R^{\alpha+1}(\tau_2) > 0\\
\implies &\int_{-\infty}^{\tau_1} F_R^\alpha(t)\;dt - \int_{-\infty}^{\tau_2} F_R^\alpha(t)\;dt > 0\\
\implies &\int_{\tau_1}^{\tau_2} F_R^\alpha(t)\;dt < 0.
\end{aligned}$$

It is a contradiction as $F_R^\alpha(.)$ is a non-negative function. Thus, $\tau_1 > \tau_2$. ■

2.1.5 Shift invariance

If we add (or subtract) same constant to (from) both the asset return and the target return, then the relative profit or loss $R-\tau$ remains unchanged. We know that risk is associated only with the potential loss. Since the loss random variable remains same after the translation, the overall risk should not be changed. The LPD clearly satisfies this property, i.e.,

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau+\lambda;R+\lambda)=\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R)\ \forall\,\lambda\in\mathbb{R}.$$

2.1.6 Positive homogeneity

The homogeneity property indicates that risk should proportionally increase to both the portfolio size and target portfolio size simultaneously. Let us consider a risky portfolio with return R and a risk-free asset with fixed return τ. Suppose that an investor invests λ dollar in R with the aim of earning return at least higher than the risk-free rate. The dollar return of the investment can be determined by converting the returns into amounts, λR and $\lambda\tau$. We can easily observe that the relative profit or loss $\lambda R-\lambda\tau=\lambda(R-\tau)$ proportionally increases with the invested money λ. Since both quantities are scaled down by a same factor, the risk should scale down proportionally. The LPD always satisfies this property:

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\tau;\lambda R)=\lambda\ \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R)\ \forall\,\lambda>0.$$

2.1.7 Diversification

Diversification is a strategy to mitigate risk by investing in various financial assets. According to Artzner et al. (1999), the diversification principle can mathematically be represented by sub-additivity of risk measure. The sub-additivity describes that the risk of any two portfolios together cannot be larger than adding the two risks separately. However, Föllmer and Schied (2002) described that the actual essence of diversification in the context of portfolio management is encapsulated by convexity of risk measure. The convexity ensures that overall risk of a portfolio is less than the weighted average risk of assets that constitute the portfolio. As a result, exposure of a portfolio to one particular asset can be reduced by investing in numerous assets. The LPD is sub-additive as well as convex. The proofs are presented below.

THEOREM 2.6 (Sub-additivity)**.** *For any two random variables* $R_1,R_2\in\mathscr{G}$ *and a non-negative target* *rate* $\tau \geq 0$,

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;\, R_1+R_2) \leq \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;\, R_1)+\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;\, R_2).$$

PROOF. The LPM of R_1+R_2 is given by

$$\mathrm{LPM}_{\alpha}(\tau;R_1+R_2) = \mathbb{E}\big[(\tau-R_1-R_2)^{\alpha}\mathbb{1}_{\{\tau>R_1+R_2\}}\big].$$

Since LPM is a non-decreasing function of τ and $\tau \geq 0$,

$$\mathrm{LPM}_{\alpha}(\tau;R_1+R_2) \leq \mathrm{LPM}_{\alpha}(2\tau;R_1+R_2). \tag{2.2}$$

We can write

$$\mathrm{LPM}_{\alpha}(2\tau;R_1+R_2) = \mathbb{E}\left[\Big((\tau-R_1)+(\tau-R_2)\Big)^{\alpha}\mathbb{1}_{\{(\tau-R_1)+(\tau-R_2)>0\}}\right]. \tag{2.3}$$

For any two random variables $R_1, R_2 \in \mathscr{G}$, it can easily be shown that

$$X_1\mathbb{1}_{\{X_1+Y_1>0\}} \leq X_1\mathbb{1}_{\{X_1>0\}} \text{ and } Y_1\mathbb{1}_{\{X_1+Y_1>0\}} \leq Y_1\mathbb{1}_{\{Y_1>0\}}.$$

We thus have

$$X_1\mathbb{1}_{\{X_1+Y_1>0\}} + Y_1\mathbb{1}_{\{X_1+Y_1>0\}} \leq X_1\mathbb{1}_{\{X_1>0\}} + Y_1\mathbb{1}_{\{Y_1>0\}}. \tag{2.4}$$

Setting $X_1 = (\tau-R_1)$ and $Y_1 = (\tau-R_2)$, we get

$$[(\tau-R_1)+(\tau-R_2)]\mathbb{1}_{\{(\tau-R_1)+(\tau-R_2)>0\}} \leq (\tau-R_1)\mathbb{1}_{\{(\tau-R_1)>0\}} + (\tau-R_2)\mathbb{1}_{\{(\tau-R_2)>0\}}.$$

Since both sides are nonegative,

$$\begin{aligned}&\mathbb{E}\left[\Big((\tau-R_1)+(\tau-R_2)\Big)^{\alpha}\mathbb{1}_{\{(\tau-R_1)+(\tau-R_2)>0\}}\right]\\ &\leq \mathbb{E}\left[\Big((\tau-R_1)\mathbb{1}_{\{(\tau-R_1)>0\}} + (\tau-R_2)\mathbb{1}_{\{(\tau-R_2)>0\}}\Big)^{\alpha}\right].\end{aligned} \tag{2.5}$$

The Minkowski's inequality for two random variables X_1 and X_2 is

$$\left(\mathbb{E}\left[\left|X_1+X_2\right|^n\right]\right)^{\frac{1}{\alpha}} \leq \left(\mathbb{E}\left[\left|X_1\right|^\alpha\right]\right)^{\frac{1}{\alpha}} + \left(\mathbb{E}\left[\left|X_2\right|^\alpha\right]\right)^{\frac{1}{n}}.$$

We have

$$\begin{aligned}&\left(\mathbb{E}\left[\left((\tau-R_1)\mathbb{1}_{\{(\tau-R_1)>0\}}+(\tau-R_2)\mathbb{1}_{\{(\tau-R_2)>0\}}\right)^\alpha\right]\right)^{\frac{1}{\alpha}}\\ &\leq \left(\mathbb{E}\left[(\tau-R_1)^\alpha\mathbb{1}_{\{(\tau-R_1)>0\}}\right]\right)^{\frac{1}{\alpha}} + \left(\mathbb{E}\left[(\tau-R_2)^\alpha\mathbb{1}_{\{(\tau-R_2)>0\}}\right]\right)^{\frac{1}{\alpha}} \qquad (2.6)\end{aligned}$$

Now (2.3), (2.5) and (2.6) yields,

$$\mathrm{LPM}_\alpha^{1/\alpha}(2\tau;R_1+R_2) \leq \mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_1)+\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_2) \qquad (2.7)$$

Finally (2.2) and (2.7) gives,

$$\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_1+R_2) \leq \mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_1)+\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_2).$$

■

THEOREM 2.7 (Convexity). *For any* $R_1, R_2 \in \mathscr{G}$ *and* $\tau \in \mathbb{R}$,

$$\mathrm{LPM}_\alpha^{1/\alpha}(\tau;\lambda R_1+(1-\lambda)R_2) \leq \lambda\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_1)+(1-\lambda)\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_2).$$

PROOF. For any $R_1, R_2 \in \mathscr{G}$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}\mathrm{LPM}_\alpha^{1/\alpha}(\tau;\lambda R_1+(1-\lambda)R_2) &= \mathrm{LPM}_\alpha^{1/\alpha}(0;\lambda(R_1-\tau)+(1-\lambda)(R_2-\tau))\\ &\leq \mathrm{LPM}_\alpha^{1/\alpha}(0;\lambda(R_1-\tau))+\mathrm{LPM}_\alpha^{1/\alpha}(0;\lambda(R_2-\tau)) \text{ [Using sub-additivity]}\\ &= \lambda\mathrm{LPM}_\alpha^{1/\alpha}(0;R_1-\tau)+(1-\lambda)\mathrm{LPM}_\alpha^{1/\alpha}(0;R_2-\tau)\\ &= \lambda\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_1)+(1-\lambda)\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_2)\end{aligned}$$

■

REMARK 2.2. It is noted that the sub-additivity may not hold for $\tau < 0$. Let us consider a pair of random variables R_1 and R_2 with joint pmf given in the following table:

	$R_2(\omega_1) = -1$	$R_2(\omega_2) = -2$	$R_2(\omega_3) = -3$
$R_1(\omega_1) = +1$	$1/3$	0	0
$R_1(\omega_2) = +2$	0	$1/3$	0
$R_1(\omega_3) = -3$	0	0	$1/3$

Setting $\tau = -2$ and $\alpha = 2$, we obtain

$$\mathrm{LPM}_2^{1/2}(R_1; -2) = \mathrm{LPM}_2^{1/2}(R_2; -2) = \frac{1}{\sqrt{3}}.$$

On the other hand,

$$\mathrm{LPM}_2^{1/2}(R_1 + R_2; -2) = \frac{4}{\sqrt{3}} > \mathrm{LPM}_2^{1/2}(R_1; -2) + \mathrm{LPM}_2^{1/2}(R_2; -2).$$

Therefore, $\mathrm{LPM}_\alpha^{1/\alpha}$ is not sub-additive for $\tau < 0$. However, no sensible investor would choose negative target return.

2.2 Efficient frontier without risk-free asset

Let us assume that LPM_α of a risky portfolio is non-zero and finite. Accordingly, target set is defined as $\mathscr{T} = \{\tau \in \mathbb{R} : \mathrm{LPM}_\alpha(\tau; R_p) \neq 0\}$. Now the portfolio allocation problem for downside risk-averse investors can be formulated as

$$\begin{cases} \underset{\boldsymbol{w}}{\text{Minimize}} & \mathrm{LPM}_\alpha(\tau; R_p) \\ \text{subject to} & \sum_{i=1}^{n} w_i \mathbb{E}[R_i] = \mu \\ & \boldsymbol{w} \in \mathscr{D}, \end{cases} \tag{2.8}$$

where $\mathscr{D} = \{(w_1, w_2, \ldots, w_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} w_i = 1\}$ is the opportunity set or the feasible region of all portfolios.

An investor finds his optimal portfolio $\boldsymbol{w}$ such that LPM_n is minimized for his desired expected return $\mu = \mu^*$. As shown by Bawa (1976), $\text{LPM}_\alpha(\tau;\boldsymbol{w}'R)$ is a convex functional of W. Hence, problem (2.8) is a convex optimization problem and thereby its optimal solution always exists.

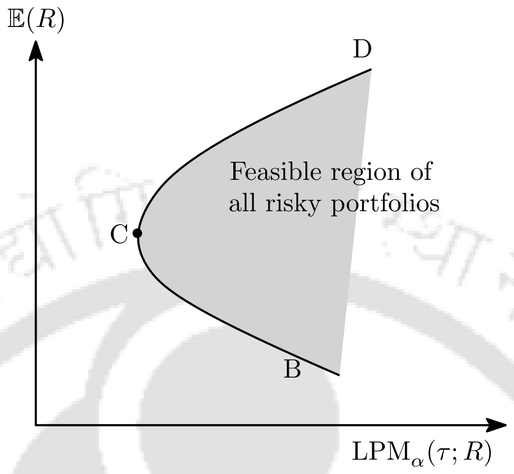


Figure 2.2: Minimum LPM_α frontier

Solving this problem for various μ and then plotting the optimal values of LPM_α into mean-LPM_α space produces the minimum LPM frontier, denoted as $\text{LPM}_\alpha(\mu)$ (see the curve BCD in Fig. 2.2):

$$\text{LPM}_\alpha(\mu) \equiv \text{Min}\left\{\text{LPM}_\alpha(\tau;\boldsymbol{w}'R) \;\middle|\; \mathbb{E}(\boldsymbol{w}'R) = \mu, \boldsymbol{w} \in \mathscr{D}\right\}.$$

Bawa proved that $\text{LPM}_\alpha(\mu)$ is convex function of μ and strictly increasing for $\mu \geq \breve{\mu}$, where $\text{LPM}_\alpha(\breve{\mu}) = \underset{\mu}{\text{Min}}\,\text{LPM}_\alpha(\mu)$. It means that the minimum LPM portfolio has expected return $\breve{\mu}$ (see curve CD in Fig. 2.2). Similar proofs of convexity for the minimum semivaraince frontier and for the minimum shortfall frontier are provided by Hogan and Warren (1972) and Bertsimas et al. (2004), respectively.

As the risk is quantified ultimately in terms of $\text{LPM}_\alpha^{1/\alpha}$, the convexity of minimum $\text{LPM}_\alpha^{1/\alpha}$ frontier is needed. Thus, an optimization problem for $\text{LPM}_\alpha^{1/\alpha}$ can be defined as

$$\begin{cases} \underset{\boldsymbol{w}}{\text{Minimize}} & \text{LPM}_\alpha^{1/\alpha}(\tau;R_p) \\ \text{subject to} & \sum_{i=1}^{n} w_i \mathbb{E}[R_i] = \mu \\ & \boldsymbol{w} \in \mathscr{D}. \end{cases} \tag{2.9}$$

Since, α is just a fixed value, the set of optimal solutions for (2.9) is identical to the set of solutions for the problem 2.8. Thus, the minimum $\mathrm{LPM}_\alpha^{1/\alpha}$ frontier (denoted as $\mathrm{LPM}_\alpha^{1/\alpha}(\mu)$) is easily obtained by taking the α-th root of $\mathrm{LPM}_\alpha(\mu)$. This means if we put the optimal values obtained from (2.8) into mean-$\mathrm{LPM}_\alpha^{1/\alpha}$ space, we get the minimum LPD frontier or the efficient frontier.

The convexity of $\mathrm{LPM}_\alpha(\mu)$ does not ensure the convexity of $\mathrm{LPM}_\alpha^{1/\alpha}(\mu)$. The convexity of $\mathrm{LPM}_\alpha^{1/\alpha}(\mu)$ is a very important fact as it guarantees the existence of unique tangency line which further ensures the TFMS and the capital market equilibrium relationship. Thus, by assuming that the convexity of $\mathrm{LPM}_\alpha^{1/\alpha}(\mu)$ holds true, Bawa and Lindenberg (1977) described that the TFMS can be obtained in the mean-LPD space. Then, by using the Sharpe's methodology (Sharpe, 1964), they developed MLPM asset pricing model which is analogus to the classical CAPM. Similar assumption was also adopted by Lee and Rao (1988) to develop another MLPM pricing model. Thus, the convexity of $\mathrm{LPM}_\alpha^{1/\alpha}(\mu)$ is a fundamental assumption to obtain the TFMS and the equilibrium relationships in the mean-LPD space. In the following, we first prove that $\mathrm{LPM}_\alpha^{1/\alpha}(\tau;\boldsymbol{w}'R)$ is a convex functional of weight vector $\boldsymbol{w}$, and then using it we finally show the convexity of $\mathrm{LPM}_\alpha^{1/\alpha}(\mu)$.

LEMMA 2.1. $\mathrm{LPM}_\alpha^{1/\alpha}(\tau;\boldsymbol{w}'R)$ *is a convex functional of* $\boldsymbol{w}$.

PROOF. We define

$$f(\boldsymbol{w}) = \mathrm{LPM}_\alpha^{1/\alpha}(\tau;\boldsymbol{w}'R) = \left(\mathbb{E}\left[(\tau-\boldsymbol{w}'R)^\alpha \mathbb{1}_{\{\tau\geq \boldsymbol{w}'R\}}\right]\right)^{\frac{1}{\alpha}}.$$

For any $\lambda \in [0,1]$,

$$f(\lambda\boldsymbol{w}_1+(1-\lambda)\boldsymbol{w}_2) = \left(\mathbb{E}\left[\left(\tau-(\lambda\boldsymbol{w}_1+(1-\lambda)\boldsymbol{w}_2)'R\right)^\alpha \mathbb{1}_{\{\tau\geq(\lambda\boldsymbol{w}_1+(1-\lambda)\boldsymbol{w}_2)'R\}}\right]\right)^{\frac{1}{\alpha}}.$$

Now

$$\begin{aligned}
&\mathbb{E}\left[\left(\tau-(\lambda\boldsymbol{w}_1+(1-\lambda)\boldsymbol{w}_2)'R\right)^\alpha \mathbb{1}_{\{\tau\geq(\lambda\boldsymbol{w}_1+(1-\lambda)\boldsymbol{w}_2)'R\}}\right]\\
&=\mathbb{E}\left[\left(\lambda(\tau-\boldsymbol{w}_1'R)+(1-\lambda)(\tau-\boldsymbol{w}_2'R)\right)^\alpha \mathbb{1}_{\{\tau\geq(\lambda\boldsymbol{w}_1'R+(1-\lambda)\boldsymbol{w}_2'R)\}}\right]\\
&\leq\mathbb{E}\left[\left(\lambda(\tau-\boldsymbol{w}_1'R)\mathbb{1}_{\{\lambda(\tau-\boldsymbol{w}_1'R)\geq 0\}}+(1-\lambda)(\tau-\boldsymbol{w}_2'R)\mathbb{1}_{\{(1-\lambda)(\tau-\boldsymbol{w}_2'R)\geq 0\}}\right)^\alpha\right]
\end{aligned}$$

The last line follows from inequality (2.4). Using the Minkowski's inequality, we can easily obtain

$$
\begin{aligned}
& f(\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2) \\
\leq & \left(\mathbb{E}\left[\left(\lambda(\tau - \boldsymbol{w}_1'R)\mathbb{1}_{\left\{\lambda(\tau-\boldsymbol{w}_1'R)\geq 0\right\}} + (1-\lambda)(\tau - \boldsymbol{w}_2'R)\mathbb{1}_{\left\{(1-\lambda)(\tau-\boldsymbol{w}_2'R)\geq 0\right\}}\right)^{\alpha}\right]\right)^{\frac{1}{\alpha}} \\
\leq & \left(\mathbb{E}\left[(\lambda(\tau - \boldsymbol{w}_1'R))^{n}\mathbb{1}_{\left\{\lambda(\tau-\boldsymbol{w}_1'R)\geq 0\right\}}\right]\right)^{\frac{1}{\alpha}} + \left(\mathbb{E}\left[((1-\lambda)(\tau - \boldsymbol{w}_1'R))^{\alpha}\mathbb{1}_{\left\{(1-\lambda)(\tau-\boldsymbol{w}_1'R)\geq 0\right\}}\right]\right)^{\frac{1}{\alpha}} \\
= & \lambda \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \boldsymbol{w}_1'R) + (1-\lambda)\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \boldsymbol{w}_2'R) \\
= & \lambda f(\boldsymbol{w}_1) + (1-\lambda) f(\boldsymbol{w}_2).
\end{aligned}
$$

Therefore, for any $\lambda \in [0,1]$, $f(\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2) \leq \lambda f(\boldsymbol{w}_1) + (1-\lambda) f(\boldsymbol{w}_2)$. Hence, $f(\boldsymbol{w})$ is a convex functional of $\boldsymbol{w}$. ■

THEOREM 2.8. *The minimum* $\mathrm{LPM}_{\alpha}^{1/\alpha}$ *frontier is convex.*

PROOF. Let us define $\mathscr{M}_{\mu} = \left\{\boldsymbol{w} \in \mathscr{D} \mid \mathbb{E}(\boldsymbol{w}'R) = \mu\right\}$. Thus, the minimum $\mathrm{LPM}_{\alpha}^{1/\alpha}$ frontier (or the optimal solutions) as a function of μ can be written as

$$L(\mu) = \underset{\boldsymbol{w} \in \mathscr{M}_{\mu}}{\mathrm{Min}} \ \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \boldsymbol{w}'R).$$

Let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be any two frontier portfolios with mean μ_1 and μ_2, respectively. Hence, $\mathbb{E}[(\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2)'R] = \lambda \mathbb{E}[\boldsymbol{w}_1'R] + (1-\lambda)\mathbb{E}[\boldsymbol{w}_2'R] = \lambda\mu_1 + (1-\lambda)\mu_2$. It implies

$$\lambda \boldsymbol{w}_1 + (1-\lambda)\boldsymbol{w}_2 \in \mathscr{M}_{\lambda\mu_1 + (1-\lambda)\mu_2}. \tag{2.10}$$

Moreover,

$$L(\mu_1) = \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \boldsymbol{w}_1'R), \ L(\mu_2) = \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \boldsymbol{w}_2'R). \tag{2.11}$$

Now for any $\lambda \in [0,1]$, we have

$$
\begin{aligned}
L(\lambda\mu_1 + (1-\lambda)\mu_2) = & \underset{\boldsymbol{w} \in \mathscr{M}_{\lambda\mu_1+(1-\lambda)\mu_2}}{\mathrm{Min}} \ \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \boldsymbol{w}'R) \\
\leq & \ \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \lambda \boldsymbol{w}_1' + (1-\lambda)\boldsymbol{w}_2') \ \text{[from 2.10]} \\
\leq & \ \lambda \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \boldsymbol{w}_1'R) + (1-\lambda)\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; \boldsymbol{w}_2'R) \ \text{[from the convexity of } \mathrm{LPM}_{\alpha}^{1/\alpha}\text{]} \\
= & \ \lambda L(\mu_1) + (1-\lambda_1)L(\mu_2) \ \text{[from 2.11]}
\end{aligned}
$$

Therefore, $L(\mu)$ is a convex function of μ. ■

2.3 Efficient frontier with a risk-free asset

Suppose that a risk-free asset with return r_f is available. We now include the risk free asset to the opportunity set formed by n risky assets. Assume that w be the proportion of initial wealth invested in the risk-free asset and $\boldsymbol{w}'_q = (w_1, w_2, ..., w_n)$ be the weights for the risky portfolio q such that $R_q = \sum\limits_{i=1}^{n} w_i R_i$ and $\sum_{i=1}^{n} w_i = 1$. The new opportunity set is now determined by $(w, \boldsymbol{w}_q)$, and hence the portfolio selection problem becomes

$$\begin{cases} \underset{(w,\boldsymbol{w}_q)}{\text{Minimize}} & \text{LPM}_\alpha(\tau; R_p) \\ \text{subject to} & w r_f + (1-w)\sum\limits_{i=1}^{n} w_i \mu_i = \mu \\ & \boldsymbol{w}_q \in \mathscr{D}. \end{cases}$$

Here, $1 - w > 0$ as no investor would like to short-sell the risk-free asset, assuming that the mean of the risky portfolio exceeds the risk-free rate. All the feasible portfolios are now determined by the linear combinations of the risk free asset and some risky portfolio. These combinations form an investment curve in a mean-risk space. According to Brogan and Stidham (2008), an investment curve represents all affine combinations of a risk-free asset and some arbitrary risky portfolio.

The shape of efficient frontier purely depends on the structure of investment curves. In the mean-standard deviation framework, all the curves are always linear or straight lines and thereby the efficient frontier is perfectly linear. On the other hand, in the mean-LPM$_\alpha^{1/\alpha}$ space, the scenario is different– the curves are not always linear. The actual shape of efficient frontier (under the presence of a risk-free asset) can be determined by investigating the properties of investment curves. In the following segment, various properties of investment curves are presented.

2.3.1 Properties of investment curves

Let R_p denote the affine combinations of r_f and R_q, i.e.,

$$R_p = w r_f + (1-w) R_q.$$

The expected return of R_p is

$$\mu_p = wr_f + (1-w)\mu_q. \tag{2.12}$$

As described by Brogan and Stidham (2008), from the above the relations, one can easily obtain

$$R_p = r_f + \frac{R_q - r_f}{\mu_q - r_f}(\mu_p - r_f).$$

Thus, the investment curve in mean-LPM$_\alpha$ space is given by

$$\begin{aligned}\mathrm{LPM}_\alpha(\tau;R_p) &= \mathbb{E}\left[(\tau - R_p)^\alpha \mathbb{1}_{\{\tau>R_p\}}\right]\\ &= \mathbb{E}\left[\left\{\tau - r_f + X(\mu_p - r_f)\right\}^\alpha \mathbb{1}_{\left\{\tau>r_f - X(\mu_p - r_f)\right\}}\right]\\ &:= L(\mu_p),\end{aligned}$$

where $X = \frac{r_f - R_q}{\mu_q - r_f}$. Therefore, LPM$_\alpha(\tau;r_p)$ is a function of μ_p for a given risky portfolio with return R_q and for the given parameters α and τ.

Brogan and Stidham (2008) empirically observed interesting behaviours of investment curves. They conjectured that every curve is convex for $\alpha \geq 1$ and approaches linearity as required mean return tends to infinity. In this segment, we provide theoretical proof of the conjecture. In addition, two additional properties (convergence and monotonicity) are described.

LEMMA 2.2. *For all target* τ,

$$L'(\mu_p) = \frac{\partial \mathrm{LPM}_\alpha}{\partial \mu_p} = \alpha \mathbb{E}\left[(\tau - R_p)^{\alpha-1}\mathbb{1}_{\{\tau>R_p\}}X\right] \ \forall\, \alpha \geq 1$$

and

$$L''(\mu_p) = \frac{\partial^2 \mathrm{LPM}_\alpha}{\partial \mu_p^2} = \alpha(\alpha-1)\mathbb{E}\left[(\tau - R_p)^{\alpha-2}\mathbb{1}_{\{\tau>R_p\}}X^2\right] \ \forall\, \alpha \geq 2.$$

PROOF. Let us define $g : \mathbb{R}\times\Omega \to \mathbb{R}$ such that $g(\mu_p,\omega) := (\tau - R_p(\omega))^\alpha \mathbb{1}_{\tau>R_p(\omega)}$. $g(\mu_p,\omega)$ is a measurable function. It is also integrable for all μ_p. Now for all $\omega \in \Omega$ and $\alpha \geq 1$, $\frac{\partial g}{\partial \mu_p}$ exists and

$$\left|\frac{\partial g}{\partial \mu_p}\right| = \left|\alpha(\tau - R_p)^{\alpha-1}\mathbb{1}_{\{\tau>R_p\}}X\right| \leq \left|\alpha\tau^{\alpha-1}X\right| := h(\omega).$$

Moreover, $\mathbb{E}|h(\omega)| < \infty$ as $\mu_q < \infty$. Thus, using differentiation under the integral sign, we get $\frac{\partial \text{LPM}_\alpha}{\partial \mu_p} = \alpha \mathbb{E}\left[(\tau - R_p)^{\alpha-1} \mathbb{1}_{\{\tau > R_p\}} X\right]$ for all $\alpha \geq 1$. Similarly it can be easily shown that $\frac{\partial^2 \text{LPM}_\alpha}{\partial \mu_p^2} = \alpha(\alpha-1)\mathbb{E}\left[(\tau - R_p)^{\alpha-2} \mathbb{1}_{\{\tau > R_p\}} X^2\right]$ for all $\alpha \geq 2$. ■

We now prove the Brogan-Stidham conjecture. The proof of the conjecture is divided into two parts. We first prove the convexity and then show the convergence.

THEOREM 2.9 (Convexity)**.** *All investment curves are convex in mean-*$\text{LPM}_\alpha^{1/\alpha}$ *space.*

PROOF. For $\alpha = 1$, since $\text{LPM}_\alpha \equiv \text{LPM}_\alpha^{1/\alpha}$, the convexity follows from Theorem 1 of Brogan and Stidham (2008). In order to show the convexity for $\alpha \geq 2$, we need to prove $\frac{\partial^2 \text{LPM}_\alpha^{1/\alpha}(\tau; R_p)}{\partial \mu_p^2} \geq 0$.

It is easy to see that for $\alpha \geq 1$,

$$\frac{\partial \text{LPM}_\alpha^{1/\alpha}(\tau; R_p)}{\partial \mu_p} = \frac{\mathbb{E}\left[(\tau - R_p)^{\alpha-1} \mathbb{1}_{\{\tau > R_p\}} X\right]}{\left(\mathbb{E}\left[(\tau - R_p)^{\alpha} \mathbb{1}_{\{\tau > R_p\}}\right]\right)^{\frac{\alpha-1}{\alpha}}}. \tag{2.13}$$

For $\alpha \geq 2$, we have

$$\frac{\partial^2 \text{LPM}_\alpha^{1/\alpha}(\tau; R_p)}{\partial \mu_p^2} = (\alpha - 1) \frac{\mathbb{E}\left[(\tau - R_p)^{\alpha-2} \mathbb{1}_{\{\tau > R_p\}} X^2\right] \mathbb{E}\left[(\tau - R_p)^{\alpha} \mathbb{1}_{\{\tau > R_p\}}\right] - \left(\mathbb{E}\left[(\tau - R_p)^{\alpha-1} \mathbb{1}_{\{\tau > R_p\}} X\right]\right)^2}{\left(\mathbb{E}\left[(\tau - R_p)^{\alpha} \mathbb{1}_{\{\tau > R_p\}}\right]\right)^{\frac{2\alpha-1}{\alpha}}}. \tag{2.14}$$

Since $\mathbb{P}(R_p < \tau) \neq 0$, the denominator is always greater than zero. Hence, we need to show that the numerator is non-negative. Let $Y_1 = (\tau - R_p)^{\frac{\alpha-2}{2}} \mathbb{1}_{\{\tau > R_p\}} X$ and $Y_2 = (\tau - R_p)^{\frac{\alpha}{2}} \mathbb{1}_{\{\tau > R_p\}}$ be two random variables. The Cauchy-Schwartz inequality yields

$$\mathbb{E}[Y_1^2]\mathbb{E}[Y_2^2] \geq (\mathbb{E}[Y_1 Y_2])^2,$$

which implies

$$\mathbb{E}\left[(\tau - R_p)^{\alpha-2} \mathbb{1}_{\{\tau > R_p\}} X^2\right] \mathbb{E}\left[(\tau - R_p)^{\alpha} \mathbb{1}_{\{\tau > R_p\}}\right] \geq \left(\mathbb{E}\left[(\tau - R_p)^{\alpha-1} \mathbb{1}_{\{\tau > R_p\}} X\right]\right)^2.$$

Hence $\frac{\partial^2 \mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_p)}{\partial \mu_p^2} \geq 0$. Therefore, investment curves are convex in mean-LPM$_\alpha^{1/\alpha}$ plane. ■

THEOREM 2.10 (Convergence)**.** *Every investment curve approaches linearity as the mean return tends to infinity.*

PROOF. Since an investment curve is convex in mean-LPM$_\alpha^{1/\alpha}$ space, slopes of the curve are non-decreasing as μ_p increases. Hence, to show the curves are linear in limit, we need to show that the slopes converge as $\mu_p \to \infty$, i.e.,

$$\lim_{\mu_p \to \infty} \frac{\partial \mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_p)}{\partial \mu_p} < \infty.$$

Since $\mu_q > r_f > 0$ and $w > 0$, equation (2.12) gives $\mu_p > r_f$. Suppose $Y_{\mu_p} = \left(\frac{\tau-R_p}{\mu_p}\right)^\alpha \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}}$. As $\mu_p \to \infty$, $Y_{\mu_p} \to \left(\frac{r_f-R_q}{\mu_q-r_f}\right)^\alpha \mathbb{1}_{\{r_f>R_q\}}$.

Since $\mu_p > r_f$,

$$\begin{aligned}
|Y_{\mu_p}| &= \left|\left(\frac{\tau-R_p}{\mu_p}\right)^\alpha \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}}\right| \\
&\leq \left|\left(\frac{\tau-R_p}{\mu_p}\right)^\alpha\right| \\
&= \left(\left|\frac{\tau-r_f}{\mu_p} + X\left(1-\frac{r_f}{\mu_p}\right)\right|\right)^\alpha \\
&\leq \left(\left|\frac{\tau-r_f}{r_f}\right| + 2|X|\right)^\alpha := Z.
\end{aligned}$$

It is easy to see that $\mathbb{E}|Z| < \infty$. Using the dominated convergence theorem, we obtain

$$\lim_{\mu_p \to \infty} \mathbb{E}[Y_{\mu_p}] = \mathbb{E}\left[\lim_{\mu_p \to \infty} Y_{\mu_p}\right] = \mathbb{E}[X^\alpha \mathbb{1}_{\{r_f>R_q\}}].$$

Similarly,

$$\lim_{\mu_p \to \infty} \mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha-1} \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}} X\right] = \mathbb{E}[X^\alpha \mathbb{1}_{\{r_f>R_q\}}].$$

Now, equation (2.13) becomes

$$\frac{\partial \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R_p)}{\partial \mu_p} = \frac{\mathbb{E}\left[(\tau-R_p)^{\alpha-1}\mathbb{1}_{\{\tau>R_p\}}X\right]}{\left(\mathbb{E}\left[(\tau-R_p)^{\alpha}\mathbb{1}_{\{\tau>R_p\}}\right]\right)^{\frac{\alpha-1}{\alpha}}}$$

$$= \frac{\mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha-1}\mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}}X\right]}{\left(\mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha}\mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}}\right]\right)^{\frac{\alpha-1}{\alpha}}}.$$

Therefore

$$\lim_{\mu_p\to\infty}\frac{\partial \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R_p)}{\partial \mu_p} = \frac{1}{\mu_q-r_f}\mathrm{LPM}_{\alpha}^{1/\alpha}(r_f;R_q) < \infty. \tag{2.15}$$

■

Since the investment curves are convex and slopes of those are convergent, maximum slope is the convergence point $\frac{1}{\mu_q-r_f}\mathrm{LPM}_{\alpha}^{1/\alpha}(r_f;r_q)$. Interestingly, the maximum slop does not depend on the target return τ.

In addition to the above properties, we find one more interesting behaviour when target rate is below the risk-free rate. This property is presented below.

THEOREM 2.11 (Monotonicity). *For* $0\le\tau\le r_f$, *all investment curves are strictly increasing in mean-*$\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R_p)$ *space for all* $\mu_p > r_f$.

PROOF. When $R_p<\tau$, $r_f+\frac{R_q-r_f}{\mu_q-r_f}(\mu_p-r_f)<\tau$. Now $\frac{R_q-r_f}{\mu_q-r_f}(\mu_p-r_f)<0$ because $\tau-r_f\le 0$. Since $\mu_p-r_f>0$ and $\mu_q-r_f>0$, $R_q-r_f<0$. That means $\tau>R_p$ implies $r_f-R_q>0$ when $\tau\le r_f$ and $\mu_p>r_f$. Thus

$$L'(\mu_p)=\frac{\alpha}{\mu_q-r_f}\mathbb{E}\left[(\tau-R_p)^{\alpha-1}(r_f-R_q)\mathbb{1}_{\{\tau>R_p\}}\right]>0.$$

We have $\frac{\partial \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R_p)}{\partial \mu_p}=\frac{L'(\mu_p)}{\left(\mathbb{E}\left[(\tau-R_p)^{\alpha}\mathbb{1}_{\{\tau>R_p\}}\right]\right)^{\frac{\alpha-1}{\alpha}}}$. 6 Since $L'(\mu_p)>0$ for $\tau\le r_f$ and $\mathbb{P}(R_p<\tau)\ne 0$, $\frac{\partial \mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R_p)}{\partial \mu_p}>0$. ■

Three investment curves are shown in Fig. 2.3. Each curve is convex and the convexity gradually approaches perfect linearity as the expected return increases. Hence, eventually, every curve becomes a straight line. Furthermore, the strict convexity gradually approaches linearity as the investment curve rotates from A to B and to C. This is because the maximum slope increases as the curve rotates. Hence, the curve becomes more straight and less curvy.

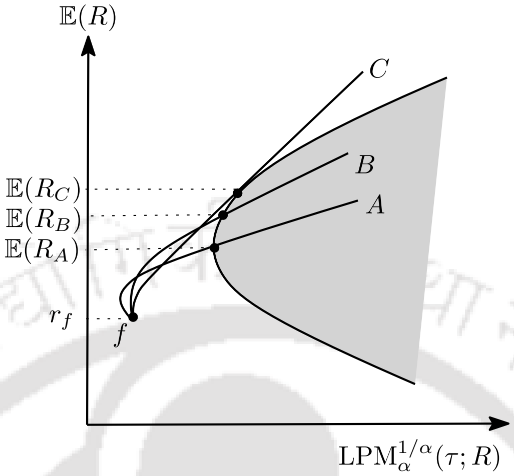


Figure 2.3: Investment curves

Unlike in the MV framework, the whole tangency curve is not the efficient frontier in this space. The reason is that the investment curves are not linear and they may intersect each other in mean-$\mathrm{LPM}_\alpha^{1/\alpha}$ space. The efficient frontier is now formed by all the investment curves (see Theorem 2 of Brogan and Stidham (2008)). Hence, similar to the investment curves, the new efficient frontier (curve fC in Fig. 2.4) is convex and the convexity converges to linearity as mean return tends to infinity.

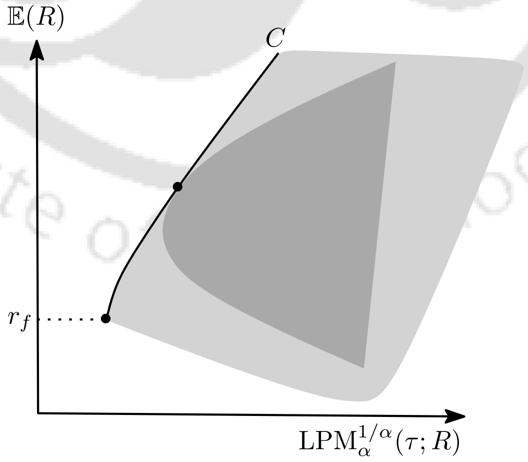
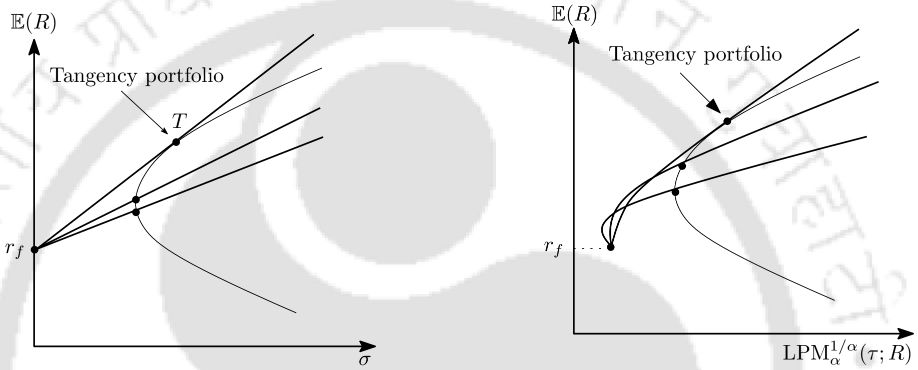


Figure 2.4: Efficient frontier with a risk-free asset

2.4 Two-fund monetary separation

The two-fund monetary separation (TFMS) holds if every investor can create his optimal portfolio just by holding the risk-free asset (one fund) and a unique risky portfolio (another fund). It is guaranteed if we can find a unique investment curve which is tangent to the efficient frontier. As a result, an investor first finds the tangency portfolio and then mixes the risk-free asset with it in a certain proportion to find his optimal portfolio. It is easily achieved when all the investment curves are perfectly linear.



(a) MV framework (b) MLPM framework

Figure 2.5: Investment curves in different spaces

As we have discussed earlier that, unlike in mean-standard deviation space, the investment curves are not always linear in mean-LPM$_\alpha^{1/\alpha}$ space (see Fig. 2.5). In the above section, we have shown various properties of the curves. We have proved that, regardless of the target levels, an investment curve is convex and approaches linearity as mean return goes to infinity (see Theorem 3.1 and Theorem 3.2). Therefore, the curves are not perfectly linear for all targets. Due to the non-linearity, a unique tangency curve $\overline{r_f T}$ as described in Fig. 2.5a, may not be obtained in the MLPM framework. The reason is that, unlike the MV efficient frontier, the MLPM efficient frontier is formed by all the investment curves (see Theorem 2 of Brogan and Stidham (2008)). Thus, the linear separation is not guaranteed for arbitrary targets. Bawa and Lindenberg (1977) shown that the curves are perfectly linear for $\tau = r_f$ and thereby the separation is guaranteed for this target. Later, Harlow and Rao (1989) described that the separation can also be obtained for $\tau = \mu$. The existence of other such targets is still unknown. Our goal is to find such targets.

2.4.1 Linearity of investment curve

It is pretty clear that the linear separation is guaranteed when all the investment curves are perfectly linear. Thus, we need to know when the curves are linear in mean-$\mathrm{LPM}_\alpha^{1/\alpha}$ space. A necessary and sufficient condition for the linearity is presented below.

The target rates can be distinguished into two classes: dynamic and fixed. It is dynamic when τ is a function of $\mathbb{E}[R]$. This is because the target rate changes as the distribution of R varies. The target is a fixed quantity if $\tau = t$ such that $t \in \mathbb{R}$. We now consider the target as

$$\tau_\lambda = \lambda \mathbb{E}[R] + t,$$

where $\lambda \in \mathbb{R}$. Hence, τ_λ is fixed for $\lambda = 0$ and dynamic for $\lambda \neq 0$.

THEOREM 2.12 (Linearity criteria). *An investment curve formed by a risk-free asset with return r_f and an arbitrary risky portfolio q with random return R_q is linear in mean-$\mathrm{LPM}_\alpha^{1/\alpha}(\tau_\lambda;)$ space if and only if*

$$\mathrm{LPM}_\alpha^{1/\alpha}(\lambda \mathbb{E}[R_q] + t; R_q) - \mathrm{LPM}_\alpha^{1/\alpha}(\lambda r_f + t; r_f) = \mathrm{LPM}_\alpha^{1/\alpha}(\lambda \mathbb{E}[R_q] + (1-\lambda) r_f; R_q) \tag{2.16}$$

for all λ, t.

PROOF. **Necessity:** Let us assume that an investment curve is linear and it originates from P_1 and passes through P_2 in mean$-\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R)$ space, where $P_1 = \left(r_f, \mathrm{LPM}_\alpha^{1/\alpha}(\lambda r_f + t; r_f)\right)$ and $P_2 = \left(\mathbb{E}[R_q], \mathrm{LPM}_\alpha^{1/\alpha}(\lambda \mathbb{E}[R_q] + t; R_q)\right)$. Therefore, the slope of line $\overline{P_1 P_2}$ is given by

$$\frac{\mathrm{LPM}_\alpha^{1/\alpha}(\lambda \mathbb{E}[R_q] + t; R_q) - \mathrm{LPM}_\alpha^{1/\alpha}(\lambda r_f + t; r_f)}{\mathbb{E}[R_q] - r_f} \tag{2.17}$$

We have proved that slopes of an investment curve in mean-$\mathrm{LPM}_\alpha^{1/\alpha}(t;R)$ space converge to $\frac{\mathrm{LPM}_\alpha^{1/\alpha}(r_f;R_q)}{\mathbb{E}[R_q]-r_f}$ (see equation 2.15 in Theorem 2.10). Similarly, it can be easily shown that slopes of the curve in mean$-\mathrm{LPM}_\alpha^{1/\alpha}(\tau_\lambda;R)$ space converge to

$$\frac{\mathrm{LPM}_\alpha^{1/\alpha}(\lambda \mathbb{E}[R_q] + (1-\lambda) r_f; R_q)}{\mathbb{E}[R_q] - r_f}.$$

Hence, we can write

$$\frac{\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+t;R_q)-\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda r_f+t;r_f)}{\mathbb{E}[R_q]-r_f}=\frac{\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+(1-\lambda)r_f;R_q)}{\mathbb{E}[R_q]-r_f} \tag{2.18}$$

$$\implies \mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+t;R_q)-\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda r_f+t;r_f)=\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+(1-\lambda)r_f;R_q).$$

Sufficiency: We assume that relation (2.16) holds true. Hence, equation (2.18) also holds as $\mathbb{E}[R_q]-r_f>0$. Since the curve is convex, converging point of the slopes must be the maximum slope. Now, equation (2.16) indicates that the maximum slope of a convex curve originating from the point P_1 and passing through the point P_2 is the same as the slope of line $\overline{P_1P_2}$ joining those two points. This is possible if and only if the convex curve is identical to the line $\overline{P_1P_2}$. ■

If a target return satisfies the condition 2.16 for all q, then all the investment curves must be linear in the mean$-\mathrm{LPM}_{\alpha}^{1/\alpha}$ space. We now try to find all those targets that satisfies the above criteria.

2.4.2 Generalized target for linearity: existence and uniqueness

We observe that the relation (2.16) holds when $t=(1-\lambda)r_f$. Hence, the curve must be linear if $\tau=\lambda\mathbb{E}[R]+(1-\lambda)r_f$. For the sake of illustration, we present the following derivation.

The random and expected return of a portfolio p consists of a risk-free asset and a risky portfolio q are

$$R_p=wr_f+(1-w)R_q \tag{2.19}$$

and

$$\mathbb{E}[R_p]=wr_f+(1-w)\mathbb{E}[R_q], \tag{2.20}$$

respectively. We define the target rate $\tau_{\lambda,p}=\lambda\mathbb{E}[R_p]+(1-\lambda)r_f$. Thus, the risk of portfolio p can be determined as

$$\mathrm{LPM}_{\alpha}(\tau_{\lambda,p};R_p)=\mathbb{E}\left[\left(\tau_{\lambda,p}-R_p\right)^{\alpha}\mathbb{1}_{\left\{\tau_{\lambda,p}>R_p\right\}}\right]. \tag{2.21}$$

We note that

$$\begin{aligned}\tau_{\lambda,p} - R_p &= \lambda\mathbb{E}[R_p] + (1-\lambda)r_f - R_p\\ &= \lambda w r_f + \lambda(1-w)\mathbb{E}[R_q] + (1-\lambda)r_f - w r_f - (1-w)R_q\\ &= w(\lambda r_f - r_f) + (1-\lambda)r_f + (1-w)(\lambda\mathbb{E}[R_q] - R_q)\\ &= (1-\lambda)(r_f - w r_f) + (1-w)(\lambda\mathbb{E}[R_q] - R_q)\\ &= (1-w)\big[(1-\lambda)r_f + \lambda\mathbb{E}[R_q] - R_q\big]\\ &= (1-w)[\tau_{\lambda,q} - R_q].\end{aligned}$$

Now for $(1-w) > 0$, equation (2.21) yields

$$\mathrm{LPM}_\alpha(\tau_{\lambda,p}; R_p) = (1-w)^\alpha \mathrm{LPM}_\alpha(\tau_{\lambda,q}; R_q),$$

which implies

$$\mathrm{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,p}; R_p) = (1-w)\mathrm{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,q}; R_q). \tag{2.22}$$

Using equations (2.20) and (2.22), we obtain the equation of MLPM capital allocation line (CAL)

$$\mathbb{E}[R_p] = r_f + \left[\frac{\mathbb{E}[R_q] - r_f}{\mathrm{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,q}; R_q)}\right] \mathrm{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,p}; R_p). \tag{2.23}$$

Therefore, the linear combinations of a risk-free asset and some risky portfolio lie along a straight line in mean-$\mathrm{LPM}_\alpha^{1/\alpha}(\tau; R)$ space with $\tau = \lambda\mathbb{E}[R] + (1-\lambda)r_f$. Since equation (2.23) is derived for arbitrary portfolio q, it holds for all other risky portfolios. Thus, for all $\alpha \geq 1$, every investment curve is linear if $\tau = \lambda\mathbb{E}[R] + (1-\lambda)r_f$. Therefore, besides the targets $\mathbb{E}[R]$ and r_f, there exist infinitely many other targets that admit linearity.

The next important question is that whether $\left\{\lambda\mathbb{E}[R] + (1-\lambda)r_f : \lambda \in \mathbb{R}\right\}$ is the unique family of targets to obtain the linearity. In other words, we want to know whether this is the complete set of targets that permit the linearity. This question is addressed in the following segment.

In order to address the problem of uniqueness, we divide the following discussion into two parts: for $\alpha > 1$ and for $\alpha = 1$. Let us begin with the first case, $\alpha > 1$.

LEMMA 2.3. *If R is a non-constant random variable and $\alpha > 1$, then*

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R)-\tau$$

is a strictly decreasing function of τ for all $\tau \in \mathscr{T}$.

PROOF. Define $f(\tau)=\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R)-\tau$. The first order derivative of f with respect to τ is

$$f'(\tau)=\frac{\mathbb{E}[Z^{\alpha-1}]}{\Big(\mathbb{E}[Z^{\alpha}]\Big)^{\frac{\alpha-1}{\alpha}}}-1,$$

where $Z=(\tau-R)\mathbb{1}_{\tau>R}$. The Lyapunov's inequality for any random variable X is

$$\Big(\mathbb{E}[|X|^{r}]\Big)^{\frac{1}{r}}\leq\Big(\mathbb{E}[|X|^{s}]\Big)^{\frac{1}{s}} \tag{2.24}$$

for $0<r<s<\infty$. Equality occurs if and only if X is a constant random variable. Since Z is a non-constant and non-negative random variable with $\mathbb{E}(Z)>0$, the inequality (2.24) yields

$$\Big(\mathbb{E}[Z^{\alpha-1}]\Big)^{\frac{1}{\alpha-1}}<\Big(\mathbb{E}[Z^{\alpha}]\Big)^{\frac{1}{\alpha}}.$$

This implies

$$\frac{\mathbb{E}[Z^{\alpha-1}]}{\Big(\mathbb{E}[Z^{\alpha}]\Big)^{\frac{\alpha-1}{\alpha}}}<1.$$

Hence $f'(\tau)<0$ for all $\tau\in\mathscr{T}$. Thus, $f(\tau)$ is a strictly decreasing function. ■

THEOREM 2.13. *All investment curves are linear in mean-$\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R)$ space for $\alpha>1$ if and only if $\tau=\lambda\mathbb{E}[R]+(1-\lambda)r_f$.*

PROOF. **Necessity:** Suppose an investment curve, formed by the risk-free asset with return r_f and an arbitrary risky portfolio with random return R_q, is linear. We need to show that τ $(=\lambda\mathbb{E}[R]+t)$ is equal to $\lambda\mathbb{E}[R]+(1-\lambda)r_f$, i.e., $t=(1-\lambda)r_f$. Let us assume $t\neq(1-\lambda)r_f$. We divide the proof into two possible cases: (I) $t<(1-\lambda)r_f$ and (II) $t>(1-\lambda)r_f$.

Case I: It is easy to see that $t<(1-\lambda)r_f \implies \lambda\mathbb{E}[R_q]+t<\lambda\mathbb{E}[R_q]+(1-\lambda)r_f$. As $\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau;R_q)$

is a strictly increasing function of τ,

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+t;R_q) < \mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+(1-\lambda)r_f;R_q).$$

As the curve is linear, Theorem 2.12 yields

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+t;R_q) = \mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+(1-\lambda)r_f;R_q).$$

This is a contradiction.

Case II: We see that $t > (1-\lambda)r_f \implies \lambda\mathbb{E}[R_q]+t > \lambda\mathbb{E}[R_q]+(1-\lambda)r_f$. Using Lemma 2.3, we obtain

$$\begin{aligned}&\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+t;R_q)-(\lambda\mathbb{E}[R_q]+t)\\ &< \mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+(1-\lambda)r_f;R_q)-[\lambda\mathbb{E}[R_q]+(1-\lambda)r_f].\end{aligned}$$

It implies

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+t;R_q)-\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda r_f+t;r_f) < \mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+(1-\lambda)r_f;R_q).$$

Since the curve is linear, Theorem 2.12 gives

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+t;R_q)-\mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda r_f+t;r_f) = \mathrm{LPM}_{\alpha}^{1/\alpha}(\lambda\mathbb{E}[R_q]+(1-\lambda)r_f;R_q).$$

It is also a contradiction.

Therefore, we conclude that

$$t=(1-\lambda)r_f.$$

Sufficiency: It directly follows from Theorem 2.12. ■

We have thus shown that, for $\alpha > 1$, $\left\{\lambda\mathbb{E}[R]+(1-\lambda)r_f : \lambda\in\mathbb{R}\right\}$ is the only family that makes all the investment curves perfectly linear. However, for $\alpha = 1$, it is not the unique family. As an evidence, we present the following example.

EXAMPLE 2.1. Suppose return distributions of an efficient portfolio A is given as follows:

Table 2.1: Return distribution

R_A	5%	1%	7%	3%
Probability	0.25	0.25	0.25	0.25

Let us assume $r_f = 2\%$, $\lambda = 5$ and $t = 1$. Thus, the target rate is now $\tau = 0.05\mathbb{E}[R_A] + 1$. Although $t \neq (1-\lambda)r_f$, the relation (2.16) is satisfied and thereby the investment curve I_A formed by the risky portfolio A and the risk-free asset is linear. Hence, for $\alpha = 1$, the target $\tau = \lambda\mathbb{E}(R_A) + (1-\lambda)r_f$ is not unique to obtain the linearity.

The uniqueness of $\tau = \lambda\mathbb{E}[R] + (1-\lambda)r_f$ for $\alpha = 1$ can be achieved by imposing some restriction on the distribution of asset returns. This result is presented below.

LEMMA 2.4. *If R is a non-constant random variable such that $\mathbb{P}(\tau_* > R) < 1$ and $\tau_1 < \tau_* < \tau_2$ for all $\tau^*, \tau_1, \tau_2 \in \mathscr{T}$, then*

$$\mathrm{LPM}_1(\tau_2;R) - \tau_2 < \mathrm{LPM}_1(\tau^*;R) - \tau^* < \mathrm{LPM}_1(\tau_1;R) - \tau_1.$$

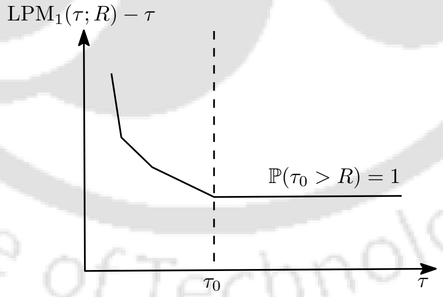


Figure 2.6: Convexity

PROOF. Define $g(\tau) = \mathrm{LPM}_1(\tau;R) - \tau$. $g(\tau)$ is a convex function of τ. Since $g'(\tau) = \mathbb{P}(\tau > R) - 1 \leq 0$, g is a decreasing function of τ. Moreover, g is strictly decreasing for all τ such that $\mathbb{P}(\tau > R) < 1$.

We choose τ_0 such that $\mathbb{P}(\tau_0 > R) = 1$ and for all $\tau < \tau_0$, $\mathbb{P}(\tau > R) < 1$. Hence g is strictly decreasing for all $\tau < \tau_0$ and constant for all $\tau > \tau_0$. $g(\tau)$ is described in Fig. 2.6.

Since $\mathbb{P}(\tau^* > R) < 1$ and g is convex, $\tau^* < \tau_0$. Hence, $\tau_1 < \tau^*$ implies $f(\tau^*) < f(\tau_1)$. Similarly, for $\tau^* < \tau_2$, $f(\tau_2) < f(\tau^*)$. Thus,

$$f(\tau_2) < f(\tau^*) < f(\tau_1).$$

■

THEOREM 2.14. *If R_q is the random return of a risky portfolio q such that $\mathbb{P}(\lambda\mathbb{E}[R_q] + (1-\lambda)r_f > R_q) < 1$, the investment curve formed by the portfolio q is linear in mean-*$\mathrm{LPM}_1(\tau; R)$ *space if and only if $\tau = \lambda\mathbb{E}[R] + (1-\lambda)r_f$.*

PROOF. Proof is similar to Theorem 2.13. ■

2.4.3 Generalized separation

We have shown that all the investment curves are linear in mean-$\mathrm{LPM}_\alpha^{1/\alpha}(\tau; R)$ space if $\tau = \lambda\mathbb{E}[R] + (1-\lambda)r_f$. Hence, the optimal investment curve must be a tangent line to the efficient frontier of risky assets. Thus, the optimal investment line (see $\overline{r_f TD}$ in Fig. 2.7) is formed by affine combinations of the risk-free asset and tangency portfolio T in the MLPM framework.

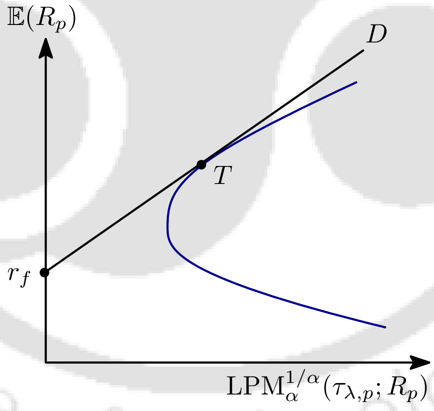


Figure 2.7: Two-fund septation

Now the two-fund separation of classical portfolio theory can easily be transferred to this framework. This result is formally explained by the following theorem that generalizes Theorem 2 of Bawa and Lindenberg (1977) and Theorem 1 of Lee and Rao (1988).

THEOREM 2.15 (Generalized MLPM separation)**.** *Under the presence of a risk-free asset, the optimal portfolio choice in* $\mathrm{MLPM}_\alpha(\tau)$ *framework admits linear separation between the risk-free asset* *and a unique tangency portfolio of risky assets either if (i)* $\alpha = 1$ *and* $\tau = \lambda\mathbb{E}[R] + (1-\lambda)r_f$ *or if and only if (ii)* $\alpha > 1$ *and* $\tau = \lambda\mathbb{E}[R] + (1-\lambda)r_f$*.*

2.5 Concluding remarks

In this chapter, we have presented various mathematical properties of LPD and then discussed their practical interpretations. The desirability of the properties truly justify the efficiency of LPD as a measure of portfolio risk. In addition to that, a couple of long-standing open problems in the area of MLPM portfolio management have been addressed successfully. First, the convexity of efficient frontier has been analytically shown. Then, the Brogan-Stidham conjecture related to the behaviours of investment curves is proved. Finally, the problem of TFMS has been solved completely. The TFMS was obtained in the MLPM framework for two special targets, risk-free rate and expected return. The question of which other targets, if any, permit the separation was not answered in the last three decades. It has been solved by developing a unique generalized family of target returns that allows the TFMS. In particular, all the targets are just affine combinations of the risk-free rate and the mean return. Thus, one can obtain infinitely many other targets, besides the specific two (risk-free rate and mean return), that ensure the TFMS. As a result, investors have so many choices to set their desired threshold levels that admit the separation. In addition, because of the uniqueness, investors can easily detect whether their chosen targets permit the separation or not. Our developments will surely help to remove the barriers of choosing target returns to a great extent, and to encourage practitioners to use LPM in investment management.

Chapter 3

A Generalized Asset Pricing Model

The CAPM is developed under the framework where risk is measured by variance. We have described that the variance, as a measure of risk, has several drawbacks. Hence, the beta in the traditional CAPM is also not appropriate measure of systematic risk. Lower partial moments (LPM), on the other hand, only penalizes the downside deviations of returns. Hence, the MLPM framework is intuitively much more appealing than the MV model. This encouraged researchers to develop several CAPMs in the MLPM framework. Hogan and Warren (1974) proposed a downside asset pricing model by using semivariance as a measure of risk. Bawa and Lindenberg (1977) generalized the semivariance to LPM and developed MLPM CAPM, where the target return is equal to risk-free rate. Lee and Rao (1988) developed another MLPM CAPM, where the target is the expected return of the portfolio whose risk is being measured. In this chapter, we develop a generalized version of MLPM CAPM which contains all the above mentioned MLPM CAPMs as special cases. In fact, the classical MV CAPM is identical to the generalized CAPM when the underlying distribution is normal. Using the new pricing model, we also derive an acceptance criteria for undertaking risky projects. Finally, we present some interesting empirical behaviours of new beta and LPM based *alpha*.

This chapter is organized as follows. In section 3.1, we mention underlying assumptions to develop market equilibrium model. Section 3.2 defines market portfolio and then presents the equation of generalized capital market line (G-CML). We also show that the G-CML is incidental to the MV CML for normally distributed returns. In section 3.3, we analytically derive a generalized asset pricing equation which specializes to several other pricing models including the traditional MV CAPM. We further derive a certainty equivalent pricing formula and then develop an acceptance criteria for capital investment projects. Section 3.5 presents empirical experiments to investigate the empirical behaviours of generalized beta and LPM based *alpha*. Section 3.6 provides concluding remarks.

3.1 Underlying assumptions

Asset pricing theory is developed on the basis of some assumptions that not only provide mathematical soundness but also articulate the logical implications of practical portfolio analysis. The assumptions with reference to MLPM portfolio analysis are listed below.

(i). All investors are downside risk-averse and maximize their expected utility over a one-period planning horizon.

(ii). Investors find their optimal portfolios solely on the basis of the mean and LPD of return.

(iii). The mean and LPD of return associated with all the portfolios are finite numbers that exist and can be estimated or measured.

(iv). All capital assets are infinitely divisible, meaning that fractions of shares can be bought or sold.

(v). All investors are price takers, not price makers.

(vi). Markets are ideal. There are no taxes and transactions costs

(vii). Investors can lend and borrow unlimited amounts at a single risk-free interest rate.

(viii). Investors can sell short any amount of any share.

(ix). Capital markets are perfect− all information is freely and instantly available, no margin requirements exist, and investors have unlimited opportunities to borrow, lend, or sell assets short.

(x). Investors all have homogeneous expectations over the same one-period investment horizon.

3.2 MLPM capital market line

We have assumed that every investor finds his/her portfolio solely on the basis of mean and LPM of return. Thus, everyone is a mean-LPM$_\alpha$ optimizer as described in the previous chapter. It is also assumed that there is a unique risk-free rate r_f at which borrowing and lending can be done. Now, if the target rate is set as $\tau = \lambda \mathbb{E}(R) + (1-\lambda) r_f$, then the linear separation or TFMS easily obtains. In this situation, every investor will first choose same risk fund, the tangency portfolio, and then mixes it with risk-free asset in desired proportions to find his optimal portfolio. Some investor, who are defensive, will put high percentage of their wealth in the risk-free asset. In contrast, the aggressive investors will have high proportion of the risky fund. However, all the optimal choices are located on the line joining the risk-free rate and the risky fund in mean-LPM$_\alpha^{1/\alpha}$ space.

Now the question is, in reality what must that risky fund be ? The answer is market portfolio. The market portfolio is a portfolio consisting of all assets available in the market, with each asset weighted in the proportions to its market value relative to the market value of all assets. If w_i be the weight of asset i in the market portfolio, then

$$w_i = \frac{\text{total market value of asset } i}{\text{total market value of all assets in the market}}.$$

Accordingly, the market return is defined as weighted average of the returns of all assets in the market. In equilibrium, supply equals demand for every asset, investors optimal portfolio choices are aggregated into the market portfolio.

3.2.1 The capital market line

As discussed above, the market portfolio, denoted m, is the unique risky portfolio located on the tangency point to the efficient frontier of risky assets. Thus, the efficient set is a just single line originating from the risk-free point and passing through the market portfolio m in mean$-$LPM$_\alpha^{1/\alpha}$ space (see Fig. 3.1). This is the MLPM capital market line. The equation of MLPM capital market line (CML) is given by

$$\mathbb{E}[R_p] = r_f + \left(\frac{\mathbb{E}[R_m] - r_f}{\text{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,m}; R_m)} \right) \text{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,p}; R_p), \tag{3.1}$$

where $\tau_{\lambda,p} = \lambda \mathbb{E}[R_p] + (1-\lambda) r_f$.

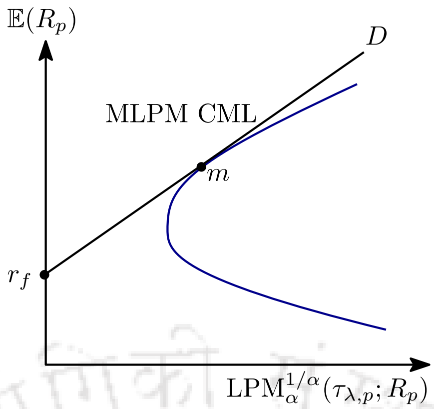


Figure 3.1: Generalized capital market line

We note that the MLPM CML, represented by (3.1), is identical to the equation of MV CML except the measure of risk, standard deviation (σ), is replaced by $\mathrm{LPM}_\alpha^{1/\alpha}$. However, both the CMLs are identical when asset returns follow multivariate normal distribution. This result is presented below.

THEOREM 3.1. *If* $(R_1, R_2, \ldots, R_n)$ *follows multivariate normal distribution, the MV CML*

$$\mathbb{E}[R_p] - r_f = \left(\mathbb{E}[R_m] - r_f\right)\frac{\sigma_p}{\sigma_m} \tag{3.2}$$

is identical to the generalized MLPM CML

$$\mathbb{E}[R_p] - r_f = \left(\mathbb{E}[R_m] - r_f\right)\frac{\mathrm{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,p}; R_p)}{\mathrm{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,m}; R_m)} \tag{3.3}$$

for all $\alpha \geq 1$.

PROOF. Since $(R_1, R_2, \ldots, R_n)$ follows multivariate normal distribution, $R_m \sim \mathcal{N}(\mathbb{E}[R_m], \sigma_m^2)$. Then for $\alpha = 1$,

$$\begin{aligned}\mathrm{LPM}_1(\tau; R_m) &= (\tau - \mathbb{E}[R_m])N(\theta_m) + \sigma_m \varphi(\theta_m)\\ &= \sigma_m\Big[\theta_m N(\theta_m) + \varphi(\theta_m)\Big],\end{aligned}$$

where $\theta_m = \frac{\tau - \mathbb{E}[R_m]}{\sigma_m}$, and $N(.)$ and $\varphi(.)$ are the CDF and PDF of standard normal random variable, respectively.

Similarly for $\alpha = 2$,

$$\begin{aligned}\mathrm{LPM}_2(\tau;R_m) &= (\tau-\mathbb{E}[R_m])^2 N(\theta_m) + \sigma_m^2 N(\theta_m) + \sigma_m(\tau-\mathbb{E}[R_m])\varphi(\theta_m)\\ &= \sigma_m^2\Big[(\theta_m^2+1)N(\theta_m)+\theta_m\varphi(\theta_m)\Big]\end{aligned}$$

and for $\alpha = 3$,

$$\begin{aligned}\mathrm{LPM}_3(\tau;R_m) &= (\tau-\mathbb{E}[R_m])^3 N(\theta_m) + 3\sigma_m^2(\tau-\mathbb{E}[R_m])N(\theta_m) + 2\sigma_m^3\varphi(\theta_m) + \sigma_m(\tau-\mathbb{E}[R_m])^2\varphi(\theta_m)\\ &= \sigma_m^3\Big[\theta_m^3 N(\theta_m) + 3\theta_m N(\theta_m) + 2\varphi(\theta_m) + \theta_m^2\varphi(\theta_m)\Big]\\ &= \sigma_m^3\Big[(\theta_m^3+3\theta_m)N(\theta_m) + (\theta_m^2+2)\varphi(\theta_m)\Big]\end{aligned}$$

Similarly, for any $\alpha \geq 1$, we obtain

$$\begin{aligned}\mathrm{LPM}_\alpha(\tau;R_m) = \sigma_m^\alpha\Big[&\big(A_0\theta_m^\alpha + A_1\theta_m^{\alpha-1} + A_2\theta_m^{\alpha-2} + \cdots + A_{\alpha-1}\theta_m\big)N(\theta_m) +\\ &\big(B_0\theta_m^{\alpha-1} + B_1\theta_m^{\alpha-1} + B_2\theta_m^{\alpha-2} + B_3\theta_m^{\alpha-3} + \cdots + B_{\alpha-1}\theta_m\big)\varphi(\theta_m)\Big],\end{aligned}$$

where A_i and $B_i \in \mathbb{R}$ for $i = 0,1,2,\ldots,\alpha-1$. The above equation can be written as

$$\mathrm{LPM}_\alpha(\tau;R_m) = \sigma_m^\alpha\Big[P_1(\theta_m)N(\theta_m) + P_2(\theta_m)\varphi(\theta_m)\Big],$$

where P_1 and P_2 are polynomial functions of θ_m with degrees α and $\alpha-1$, respectively.

The properties of multivariate normal distribution yields $R_p \sim \mathcal{N}(\mathbb{E}[R_p], \sigma_p^2)$. Thus,

$$\mathrm{LPM}_\alpha(\tau;R_p) = \sigma_p^\alpha\Big[P_1(\theta_p)N(\theta_p) + P_2(\theta_p)\varphi(\theta_p)\Big],$$

where $\theta_{\lambda,p} = \frac{\tau_{\lambda,p}-\mathbb{E}[R_p]}{\sigma_p}$. Now, we have

$$\frac{\mathrm{LPM}_\alpha(\tau_{\lambda,p};R_p)}{\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)} = \frac{\sigma_p^\alpha\Big[P_1(\theta_{\lambda,p})N(\theta_{\lambda,p}) + P_2(\theta_{\lambda,p})\varphi(\theta_{\lambda,p})\Big]}{\sigma_m^\alpha\Big[P_1(\theta_{\lambda,m})N(\theta_{\lambda,m}) + P_2(\theta_{\lambda,m})\varphi(\theta_{\lambda,m})\Big]}, \tag{3.4}$$

where $\theta_{\lambda,p}=\frac{\tau_{\lambda,p}-\mathbb{E}[R_p]}{\sigma_p}=(1-\lambda)\frac{r_f-\mathbb{E}[R_p]}{\sigma_p}$ and $\theta_{\lambda,m}=(1-\lambda)\frac{r_f-\mathbb{E}[R_m]}{\sigma_m}$. Using MV CML (3.2), we get

$$\theta_{\lambda,p}=(1-\lambda)\frac{r_f-\mathbb{E}[R_p]}{\sigma_p}=(1-\lambda)\frac{r_f-\mathbb{E}[R_m]}{\sigma_m}=\theta_{\lambda,m}.$$

Hence, equation (3.4) gives

$$\frac{\mathrm{LPM}_\alpha(\tau_{\lambda,p};R_p)}{\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)}=\frac{\sigma_p^\alpha}{\sigma_m^\alpha},$$

which implies

$$\frac{\mathrm{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,p};R_p)}{\mathrm{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,m};R_m)}=\frac{\sigma_p}{\sigma_m}$$

Therefore MV CML and generalized MLPM CML is identical. ■

3.3 The generalized pricing model

In this section, we present a generalized version of CAPM in the MLPM framework. We use the preceding results and the graphic cum analytical technique as described in Sharpe (1964) to derive the generalized CAPM.

THEOREM 3.2 (Generalized MLPM CAPM)**.** *Under the standard assumptions, if all investors find their optimal portfolios solely on the basis of mean and* $\mathrm{LPM}_\alpha^{1/\alpha}$ *of return, then the market equilibrium prices satisfy the following relationship, with* $\alpha \geq 1$,

$$\mathbb{E}[R_i]-r_f=\beta_{i;\lambda}^{\mathrm{MLPM}_\alpha}\left(\mathbb{E}[R_m]-r_f\right),\ i=1,2,\ldots,n, \tag{3.5}$$

where

$$\beta_{i;\lambda}^{\mathrm{MLPM}_\alpha}=\frac{\mathrm{CLPM}_\alpha(\tau_{\lambda,m},\tau_{\lambda,i};R_m,R_i)}{\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)}, \tag{3.6}$$

the generalized MLPM beta (G-beta) with $\tau_{\lambda,p}=\lambda\mathbb{E}[R_p]+(1-\lambda)r_f$ *and*

$$\mathrm{CLPM}_\alpha(\tau_{\lambda,m},\tau_{\lambda,i};R_m,R_i)=\mathbb{E}\left[\left(\tau_{\lambda,m}-R_m\right)^{\alpha-1}\left(\tau_{\lambda,i}-R_i\right)\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}\right], \tag{3.7}$$

the generalized co-lower partial moment of order α *between the market portfolio m and the asset i.*

PROOF. Consider a portfolio p containing an arbitrary asset i and the market portfolio m in the proportions of w_i and $(1-w_i)$, respectively. the portfolio p is located somewhere on the curve imm'

(see Fig. 3.2). The expected return and the risk of the portfolio p are

$$\mathbb{E}\big(R_p\big) = w_i\mathbb{E}\big(R_i\big) + (1-w_i)\mathbb{E}\big(R_m\big)$$

and

$$\mathrm{LPM}_{\alpha}^{1/\alpha}\big(\tau_{\lambda,p};R_p\big) = \mathbb{E}\bigg[\big(\tau_{\lambda,p} - R_p\big)^{\alpha}\mathbb{1}_{\{\tau_{\lambda,p}>R_p\}}\bigg],$$

where $\tau_{\lambda,p} = \lambda\mathbb{E}(R_m) + (1-\lambda)r_f$. As w_i varies, these values trace out a curve *imm′* as described in the figure. The curve *imm′* is convex because the opportunity set of two asset portfolio is convex in μ_p-$\mathrm{LPM}_{\alpha}^{1/\alpha}$ space.

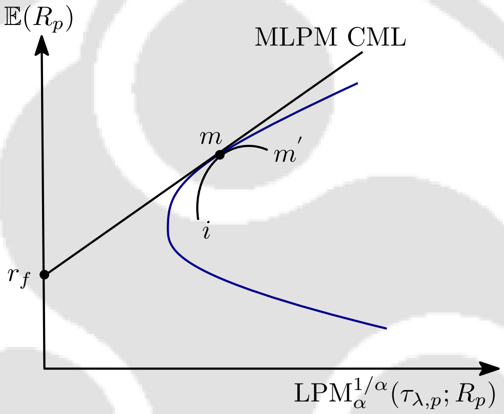


Figure 3.2: Generalized capital market line

Since

$$\frac{\partial\mathbb{E}(R_p)}{\partial w_i} = \mathbb{E}(R_i) - \mathbb{E}(R_m)$$

and

$$\frac{\partial\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau_{\lambda,p};R_p)}{\partial w_i} = \frac{\mathbb{E}\Big[(\tau_{\lambda,p} - R_p)^{\alpha-1}\{(\tau_{\lambda,i} - R_i) - (\tau_{\lambda,m} - R_m)\}\mathbb{1}_{\{\tau_{\lambda,p}>R_p\}}\Big]}{\Big(\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau_{\lambda,p};R_p)\Big)^{\alpha-1}},$$

the slope of *imm′* at m is given by,

$$\frac{\partial\mathbb{E}(R_p)}{\partial\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau_{\lambda,p};R_p)}\bigg|_{w_i=0} = \frac{\Big[\mathbb{E}(R_i) - \mathbb{E}(R_m)\Big]\Big(\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau_{\lambda,m};R_m)\Big)^{\alpha-1}}{\mathbb{E}\Big[(\tau_{\lambda,m} - R_m)^{\alpha-1}\{(\tau_{\lambda,i} - R_i) - (\tau_{\lambda,m} - R_m)\}\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big]}.$$

We note that the tangent line $r_f m$ is the MLPM CML whose slope is $\frac{\mathbb{E}[R_m]-r_f}{\text{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,m};R_m)}$. Now, in equilibrium, at $w_i=0$, the slope of curve imm' is same as the slope of the CML. Equating the slope of $r_f m$ to the slope of imm' at point m, we obtain

$$\frac{\mathbb{E}(R_m)-r_f}{\text{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,m};R_m)} = \frac{\Big[\mathbb{E}(R_i)-\mathbb{E}(R_m)\Big]\Big(\text{LPM}_\alpha^{1/\alpha}(\tau_{\lambda,m};R_m)\Big)^{\alpha-1}}{\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}\{(\tau_{\lambda,i}-R_i)-(\tau_{\lambda,m}-R_m)\}\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big]},$$

which can be rearranged as

$$\frac{\mathbb{E}(R_i)-\mathbb{E}(R_m)}{\mathbb{E}(R_m)-r_f} = \frac{\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}\{(\tau_{\lambda,i}-R_i)-(\tau_{\lambda,m}-R_m)\}\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big]}{\text{LPM}_\alpha(\tau_{\lambda,m};R_m)}. \tag{3.8}$$

Further simplifying, we have

$$\begin{aligned}&\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}\big((\tau_{\lambda,i}-R_i)-(\tau_{\lambda,m}-R_m)\big)\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big]\\ &=\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}(\tau_{\lambda,i}-R_i)\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big]-\text{LPM}_\alpha(\tau_{\lambda,m};R_m).\end{aligned}$$

Thus, equation (3.8) gives,

$$\frac{\mathbb{E}(R_i)-\mathbb{E}(R_m)}{\mathbb{E}(R_m)-r_f} = \frac{\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}\{(\tau_{\lambda,i}-R_i)\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big]}{\text{LPM}_\alpha(\tau_{\lambda,m};R_m)}-1$$

$$\implies \frac{\mathbb{E}(R_i)-r_f}{\mathbb{E}(R_m)-r_f} = \frac{\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}\{(\tau_{\lambda,i}-R_i)\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big]}{\text{LPM}_\alpha(\tau_{\lambda,m};R_m)} := \beta_{i;\lambda}^{\text{MLPM}_\alpha}.$$

We, therefore, finally get

$$\mathbb{E}(R_i)-r_f = \beta_{i;\lambda}^{\text{MLPM}_\alpha}\big[\mathbb{E}(R_m)-r_f\big],$$

where

$$\beta_{i;\lambda}^{\text{MLPM}_\alpha} = \frac{\text{CLPM}_\alpha(\tau_{\lambda,m},\tau_{\lambda,i};R_m,R_i)}{\text{LPM}_\alpha(\tau_{\lambda,m};R_m)}$$

and

$$\text{CLPM}_\alpha(\tau_{\lambda,m},\tau_{\lambda,i};R_m,R_i) = \mathbb{E}\Big[\big(\tau_{\lambda,m}-R_m\big)^{\alpha-1}\big(\tau_{\lambda,i}-R_i\big)\mathbb{1}_{\big\{\tau_{\lambda,m}>R_m\big\}}\Big].$$

■

The MLPM CAPM (3.5) is identical to the standard MV CAPM except that the measure of systematic risk, $\beta_i = \frac{\mathrm{Cov}(R_i,R_m)}{\mathrm{Var}(R_m)}$ is replaced by the generalized beta $\beta_{i;\lambda}^{\mathrm{MLPM}_\alpha}$. In the case of MLPM beta, a security contributes to the market risk only when market return falls below the target $\tau_{\lambda,m} = \lambda\mathbb{E}[R_m] + (1-\lambda)r_f$. Hence, as a measure of systematic risk the MLPM beta is theoretically superior to the usual beta.

3.3.1 Special cases

As we vary λ, special cases of equation (3.5) are obtained. For the case of $\lambda = 0$, the new CAPM is the same as Bawa-Lindenberg's (BL) CAPM because $\tau_{0,m} = r_f$ (see Bawa and Lindenberg (1977)). The mean−semivariance model of Hogan and Warren (1974) (HW) is obtained when $\lambda = 0$ and $\alpha = 2$. For $\lambda = 1$, the new CAPM reduces to the Lee-Rao's (LR) model (Lee and Rao, 1988). Besides these two, other values of λ produce infinitely many new models.

Table 3.1: MLPM betas

λ-beta ($\lambda \in \mathbb{R}$)	$\beta_{i;\lambda}^{\mathrm{MLPM}_\alpha}$	$\frac{\mathrm{CLPM}_\alpha(\tau_{\lambda,m},\tau_{\lambda,i};R_m,R_i)}{\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)}$
HW beta ($\lambda = 0$)	$\beta_{i;0}^{\mathrm{MLPM}_2}$	$\frac{\mathrm{CLPM}_2(r_f,r_f;R_m,R_i)}{\mathrm{LPM}_2(r_f;R_m)}$
BL beta ($\lambda = 0$)	$\beta_{i;0}^{\mathrm{MLPM}_\alpha}$	$\frac{\mathrm{CLPM}_\alpha(r_f,r_f;R_m,R_i)}{\mathrm{LPM}_\alpha(r_f;R_m)}$
LR beta ($\lambda = 1$)	$\beta_{i;1}^{\mathrm{MLPM}_\alpha}$	$\frac{\mathrm{CLPM}_\alpha(\mathbb{E}[R_m],\mathbb{E}[R_i];R_m,R_i)}{\mathrm{LPM}_\alpha(\mathbb{E}[R_m];R_m)}$

Moreover, the classical MV CAPM can also be seen as a special case of the new model for normally distributed returns. This result is presented below.

THEOREM 3.3. *If the joint density of R_i and R_m is bivariate normal, MLPM CAPM (3.8) reduces to MV CAPM for all $\alpha \geq 1$.*

PROOF. From the properties of bivariate normal distribution, we can write

$$\mathbb{E}\left[\tau_{\lambda,i} - R_i | R_m\right] = \tau_{\lambda,i} - \mathbb{E}[R_i] - \beta_i\left[R_m - \mathbb{E}[R_m]\right] \tag{3.9}$$

Using the relation (3.9), the CLPM can be expressed as follows:

$$\begin{aligned}
&\mathrm{CLPM}_\alpha\left(\tau_{\lambda,m},\tau_{\lambda,i};R_m,R_i\right)\\
&=\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}(\tau_{\lambda,i}-R_i)\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big]\\
&=\mathbb{E}\Big[\mathbb{E}\big[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}(\tau_{\lambda,i}-R_i)\big|R_m\big]\Big]\\
&=\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\mathbb{E}(\tau_{\lambda,i}-R_i|R_m)\Big]\\
&=\mathbb{E}\Big[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\{\tau_{\lambda,m}>R_m\}}\Big(\tau_{\lambda,i}-\mathbb{E}[R_i]-\beta_i(R_m-\mathbb{E}[R_m])\Big)\Big]\\
&=\beta_i\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)+\Big[\beta_i\big(\mathbb{E}[R_m]-\tau_{\lambda,m}\big)-\big(\mathbb{E}[R_i]-\tau_{\lambda,i}\big)\Big]\mathrm{LPM}_{\alpha-1}(\tau_{\lambda,m};R_m)\\
&=\beta_i\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)+(1-\lambda)\Big[\beta_i\big(\mathbb{E}[R_m]-r_f\big)-\big(\mathbb{E}[R_i]-r_f\big)\Big]\mathrm{LPM}_{\alpha-1}(\tau_{\lambda,m};R_m).
\end{aligned}$$

In the third and fourth lines, we use $\mathbb{E}[Z]=\mathbb{E}\big[\mathbb{E}[Z|Y]\big]$ and $\mathbb{E}[g(Y)X|Y]=g(Y)\mathbb{E}[X|Y]$, respectively. In the fifth line, equation (3.9) has been used.

Therefore the MLPM beta is now given by,

$$\begin{aligned}
\beta_{i;\lambda}^{\mathrm{MLPM}_\alpha}&=\frac{\mathrm{CLPM}_\alpha(\tau_{\lambda,m},\tau_{\lambda,i};R_m,R_i)}{\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)}\\
&=\beta_i+(1-\lambda)\Big[\beta_i\big(\mathbb{E}[R_m]-r_f\big)-\big(\mathbb{E}[R_i]-r_f\big)\Big]\frac{\mathrm{LPM}_{\alpha-1}(\tau_{\lambda,m};R_m)}{\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)}.
\end{aligned}\tag{3.10}$$

Using equation (3.5) and (3.10) and then simplifying, we obtain

$$\mathbb{E}[R_i]-r_f=\beta_i\big[\mathbb{E}[R_m]-r_f\big].$$

This is the traditional MV CAPM. ■

3.4 MLPM CAPM as a pricing formula

We now use the new MLPM CAPM to determine certainty equivalent pricing formula. Suppose P denotes the current price of a risky asset and Q represents its random payoff received after one time period. The expected value of Q is defined as $\overline{Q}=\mathbb{E}[Q]$. The random rate of return on this asset is given by $R_Q = \frac{Q-P}{P}$. Putting this into MLPM CAPM (3.5), we obtain

$$\frac{\overline{Q}-P}{P} = r_f + \beta_{Q;\lambda}^{\mathrm{MLPM}_\alpha}\left[\mathbb{E}[R_m]-r_f\right], \tag{3.11}$$

which implies

$$P = \frac{\overline{Q}}{1+r_f+\beta_{Q;\lambda}^{\mathrm{MLPM}_\alpha}\left[\mathbb{E}[R_m]-r_f\right]}, \tag{3.12}$$

where

$$\beta_{Q;\lambda}^{\mathrm{MLPM}_\alpha} = \frac{\mathrm{CLPM}_\alpha(\tau_{\lambda,m},\tau_{\lambda,Q};R_m,R_Q)}{\mathrm{LPM}_\alpha(\tau_{\lambda,m};R_m)}.$$

Further simplifying we have,

$$\begin{aligned}
&\mathrm{CLPM}_\alpha(\tau_{\lambda,m},\tau_{\lambda,Q};R_m,R_Q)\\
&= \mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}(\tau_{\lambda,Q}-R_Q)\right]\\
&= \mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}(\lambda\overline{Q}+(1-\lambda)r_f-R_Q)\right]\\
&= \mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}\left(\frac{\lambda\overline{Q}-\lambda P+P(1-\lambda)r_f-Q+P}{P}\right)\right]\\
&= \frac{1}{P}\,\mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}(\lambda\overline{Q}-Q)\right]+(1-\lambda)(1+r_f)\mathrm{LPM}_{\alpha-1}
\end{aligned} \tag{3.13}$$

Using (3.12) and (3.13), we get

$$P = \frac{1}{1+r_f}\left[\frac{\overline{Q}+\mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}(Q-\lambda\overline{Q})\right]\gamma}{1+(1-\lambda)\gamma\mathrm{LPM}_{\alpha-1}}\right] \tag{3.14}$$

We have now

$$
\begin{aligned}
&\overline{Q}+\mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}(Q-\lambda\overline{Q})\right]\gamma\\
&=\overline{Q}+\mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}(Q-\overline{Q}+\overline{Q}-\lambda\overline{Q})\right]\gamma\\
&=\overline{Q}+\mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}(Q-\overline{Q})\right]\gamma+(1-\lambda)\,\gamma\,\overline{Q}\,\mathrm{LPM}_{\alpha-1}(\tau_{\lambda,m};R_m)\\
&=\overline{Q}(1+(1-\lambda)\gamma\,\mathrm{LPM}_{\alpha-1})-\mathbb{E}\left[(\tau_{\lambda,m}-R_m)^{\alpha-1}\mathbb{1}_{\left\{\tau_{\lambda,m}>R_m\right\}}(\overline{Q}-Q)\right]\gamma\\
&=\overline{Q}(1+(1-\lambda)\gamma\,\mathrm{LPM}_{\alpha-1})-\gamma\,\mathrm{CLPM}_{\alpha}(\tau_{\lambda,m},\overline{Q};R_m,Q) \qquad (3.15)
\end{aligned}
$$

From (3.14) and (3.15), we obtain finally

$$P=\frac{1}{1+r_f}\left[\overline{Q}-\frac{\gamma\,\mathrm{CLPM}_{\alpha}(\tau_{\lambda,m},\overline{Q};R_m,Q)}{1+(1-\lambda)\gamma\mathrm{LPM}_{\alpha-1}}\right].$$

Thus. we can write

$$P=\frac{1}{1+r_f}\left[\overline{Q}-\Psi_{\lambda;Q}^{\mathrm{MLPM}_{\alpha}}\right], \qquad (3.16)$$

where

$$\Psi_{\lambda;Q}^{\mathrm{MLPM}_{\alpha}}=\frac{\gamma\,\mathrm{CLPM}_{\alpha}(\tau_{\lambda,m},\overline{Q};R_m,Q)}{1+(1-\lambda)\gamma\,\mathrm{LPM}_{\alpha-1}(\tau_{\lambda,m};R_m)},$$

the risk premium and

$$\gamma=\frac{\mathbb{E}[R_m]-r_f}{\mathrm{LPM}_{\alpha}(\tau_{\lambda,m};R_m)},$$

the market price of risk.

TThe risk premium $\Psi_{\lambda;Q}^{\mathrm{MLPM}_{\alpha}}$ critically depends on the co-lower partial moment between the risky cash flow Q and the market return R_m. It is also noted that P is linear function of Q as $\overline{Q}-\Psi_{\lambda;Q}^{\mathrm{MLPM}_{\alpha}}$ linearly depends on Q. Thus, the pricing formula is consistent with the principle of no arbitrage.

REMARK 3.1. If payoff Q is known in advance which essentially means if Q is non-random, i.e., $\overline{Q}=Q$, the risk premium $\Psi_{\lambda;Q}^{\mathrm{MLPM}_{\alpha}}=0$. Then, the pricing formula becomes $P=\frac{Q}{1+r_f}$, which is identical to the single period present value formula of a risk-free asset.

3.4.1 MLPM capital budgeting rule

Capital budgeting is a process used by firms to evaluate whether risky projects, long-term investments are worth funding through the firm's capitalization structure. The primary goal of capital budgeting is to increase the value of firm. Thus, a risky project is worth undertaken if, after the acceptance, the firm's share price increases. Using this concept and applying the pricing formula (4.2), we prove that a risky project is worth undertaking if and only if the risk-adjusted net present value (RNPV) is positive.

THEOREM 3.4 (MLPM NPV)**.** *Suppose all the assumptions of MLPM CAPM hold and a firm has opportunity to make an investment to a project i that costs C_i and generates future cash flow F_i over a period of time. Then, the project is accepted if and only if*

$$\mathrm{RNPV}_{i;\lambda} := -C_i + \frac{1}{1+r_f}\left[\overline{F}_i - \Psi^{\mathrm{MLPM}_\alpha}_{\lambda;F_i}\right] > 0.$$

PROOF. Suppose the firm is an all equity firm that has currently N outstanding shares. The equilibrium price of each share is P. Thus, the current market value V of the firm is

$$V = NP. \tag{3.17}$$

If F is the total return to the firm and thus the rate of return R is given by

$$R = \frac{F - V}{V}.$$

Using equation (3.16), we get

$$V = \frac{1}{1+r_f}\left[\overline{F} - \Psi^{\mathrm{MLPM}_\alpha}_{\lambda;F}\right]. \tag{3.18}$$

Suppose after acceptance of the project i, the firm's total return is

$$F^o = F + F_i.$$

Thus, the firm's new value can be determined as

$$
\begin{aligned}
V_l^o &= \frac{1}{1+r_f}\left[\overline{F^o} - \Psi_{\lambda;F^o}^{\mathrm{MLPM}_\alpha}\right] \\
&= \frac{1}{1+r_f}\left[\overline{F} - \Psi_{\lambda;F}^{\mathrm{MLPM}_\alpha}\right] + \frac{1}{1+r_f}\left[\overline{F_i} - \Psi_{\lambda;F_i}^{\mathrm{MLPM}_\alpha}\right]. \qquad (3.19)
\end{aligned}
$$

Suppose the firm issues N^o additional shares to finance the initial investment C_i. If the new equilibrium price of one share is P^o, then

$$C_i = N^o P^o.$$

Since the number of existing shares are N, the new market value of the firm is

$$V^o = NP^o + N^o P^o = NP^o + C_i. \qquad (3.20)$$

Subtracting (3.18) from (3.19) and using (3.20) and (3.17), we get

$$NP^o + C_i - NP = \frac{1}{1+r_f}\left[\overline{F_i} - \Psi_{\lambda;F_i}^{\mathrm{MLPM}_\alpha}\right],$$

which implies

$$N(P^o - P) = -C_i + \frac{1}{1+r_f}\left[\overline{F_i} - \Psi_{\lambda;F_i}^{\mathrm{MLPM}_\alpha}\right]$$

The project should be accepted only if and only if it will increase the firms price per share, i.e., $P^o - P > 0$. It can be easily seen that

$$P^o - P > 0 \iff \mathrm{RNPV}_{i;\lambda} > 0,$$

where

$$\mathrm{RNPV}_{i;\lambda} = -C_i + \frac{1}{1+r_f}\left[\overline{F_i} - \Psi_{\lambda;F_i}^{\mathrm{MLPM}_\alpha}\right] > 0.$$

■

EXAMPLE 3.1. The project cash flows and market return are given below:

Table 3.2: Project cash flows and market return

	Probability	Project 1	Project 2	Market return
State 1	0.20	101	90	0.10
State 2	0.20	150	100	0.08
State 3	0.20	113	103	-0.01
State 4	0.20	350	90	0.23
State 4	0.20	258	120	0.14

Assume that the initial costs of both the projects are same $C_1 = C_2 = 100$ and $\lambda = 0.5$. Now we have

$$\text{RNPV}_{1;0.5} = 41.82 \text{ and } \text{RNPV}_{2;0.5} = -1.94.$$

Therefore, project 1 is only acceptable as $\text{RNPV}_1 > 0$.

3.5 Empirical analysis

This section investigates the empirical behaviours of generalized MLPM beta and LPM based *alpha* in the context of asset ranking. We have collected adjusted (for both dividends and splits) monthly closing prices of 40 US mutual funds over a 10-year period starting from January 2009 to December 2018. The funds mainly invest in mega-cap, large-cap and mid-cap equities listed on NASDAQ and NYSE. The equity allocation of the funds are at least 80%. The monthly returns are calculated by using the standard formula $R_t = \frac{P(t+1)-P(t)}{P(t)}$. For computing the market index returns, we use adjusted monthly closing prices of S&P 500 for the same 10-year period.

The descriptive statistics (min, max, mean, std.dev, skewness and kurtosis) for the monthly returns of the funds and the market index are presented in Table 3.3. We conduct Jarque-Bera (JB) test to check whether the returns are normally distributed (see Table 3.3). We find that at 5%(10%) significance level, assumption of normality is rejected for 31 funds (33 funds).

Table 3.3: Descriptive statistics and Jarque-Bera test results

Funds	Descriptive Statistics						JB test results	
	Min	Max	Mean	Std.dev.	Skewness	Kurtosis	JB	*p*-value
S&P 500	-0.110	0.108	0.010	0.038	-0.386	3.684	5.279	0.071
AB Dis. Gr. A	-0.255	0.162	0.013	0.058	-0.777	6.134	60.666	0.000
AllianzGI Mid-Cap Inst.	-0.216	0.164	0.011	0.059	-0.325	4.506	13.336	0.001
Amana Gr. Inv	-0.107	0.172	0.011	0.045	0.220	5.141	23.696	0.000
American Beacon Lar Cap Value Inst	-0.194	0.167	0.010	0.051	-0.629	5.536	39.714	0.000
American Century Lar Com Value A	-0.139	0.114	0.010	0.042	-0.463	4.141	10.709	0.005
Ariel Fund Inv Class	-0.140	0.313	0.015	0.064	0.646	6.327	63.150	0.000
Ave Maria Gr	-0.144	0.156	0.012	0.050	-0.283	4.497	12.704	0.002
Baird MidCap Inst	-0.148	0.145	0.013	0.046	0.036	4.187	7.016	0.030
Baron Fifth Avenue Gr Retail	-0.110	0.147	0.013	0.047	-0.157	3.453	1.508	0.471
BlackRock Mid-Cap Gr Eq Inv A	-0.128	0.175	0.014	0.057	-0.149	3.676	2.707	0.258
BNY Mellon Lar Cap Stock M	-0.114	0.114	0.011	0.043	-0.371	3.451	3.737	0.154
Carillon Eagle Mid Cap Gr A	-0.127	0.155	0.013	0.051	-0.291	3.441	2.639	0.267
Champlain Mid Cap Adv	-0.146	0.126	0.012	0.046	-0.480	4.407	14.390	0.001
Columbia Disciplined Core A	-0.154	0.123	0.012	0.042	-0.549	4.761	21.352	0.000
Commerce Gr	-0.113	0.117	0.012	0.045	-0.302	3.346	2.406	0.300
Delaware Mid Cap Value A	-0.140	0.149	0.010	0.047	-0.326	4.264	10.032	0.007
DFA US Sustainability Core 1	-0.112	0.133	0.012	0.044	-0.264	3.852	4.974	0.083
Dodge & Cox Stock	-0.167	0.153	0.012	0.049	-0.467	4.536	16.034	0.000
Dreyfus Lar Cap Equity I	-0.193	0.115	0.011	0.046	-0.941	5.782	55.939	0.000
Federated Max-Cap Index R	-0.133	0.146	0.011	0.046	-0.405	3.991	8.126	0.017
First Eagle Fund of America C	-0.120	0.166	0.008	0.048	0.006	4.604	12.760	0.002
Gabelli Utilities AAA	-0.122	0.074	0.006	0.032	-0.671	4.230	16.424	0.000
GMO Quality IV	-0.210	0.152	0.011	0.048	-0.819	6.787	84.380	0.000
Goldman Sachs Large Cp Val Insghts Instl	-0.149	0.119	0.010	0.043	-0.724	4.695	24.644	0.000
Harbor Capital Appreciation Inv	-0.170	0.226	0.012	0.051	0.011	5.595	33.380	0.000
Hartford Core Equity A	-0.132	0.098	0.011	0.038	-0.527	4.176	12.358	0.002
Hartford MidCap A	-0.216	0.143	0.011	0.055	-0.833	5.053	34.673	0.000
JPMorgan Equity Index I	-0.173	0.127	0.012	0.046	-0.605	4.874	24.670	0.000
Kinetics Paradigm Adv A	-0.168	0.166	0.011	0.053	-0.235	4.351	10.151	0.006
Northern Lar Cap Value	-0.125	0.129	0.010	0.045	-0.231	3.882	4.910	0.086
Pax ESG Beta Quality Individual Inv	-0.148	0.135	0.011	0.044	-0.191	4.037	6.056	0.048
Pear Tree Quality Ordinary	-0.126	0.123	0.011	0.043	-0.471	4.001	9.362	0.009
Pioneer Global Equity Y	-0.102	0.117	0.009	0.043	-0.143	3.130	0.488	0.784
Principal MidCap A	-0.188	0.113	0.012	0.044	-0.818	5.787	51.800	0.000
Sound Shore Investor	-0.170	0.168	0.010	0.051	-0.491	4.559	16.845	0.000
USAA Aggressive Growth	-0.232	0.177	0.011	0.056	-0.467	5.603	37.917	0.000
Value Line Mid Cap Focused	-0.121	0.103	0.011	0.036	-0.597	4.083	12.875	0.002
Vanguard Total Stock Mkt Idx Inv	-0.105	0.120	0.012	0.040	-0.311	3.697	4.331	0.115
Wells Fargo Opportunity A	-0.195	0.195	0.011	0.055	-0.323	5.549	34.291	0.000
Wilshire Large Company Value Instl	-0.173	0.155	0.010	0.051	-0.402	4.436	13.420	0.001

3.5.1 Behaviour of generalized beta

We here explore the behaviour of G-beta given by (3.6). We first rank the funds according to the values of different G-beta and then find the corresponding rank correlation. Table 3.4 reports the rank correlations between seven G-betas. We observe that the rank correlation significantly deviates with the change of λ.

Table 3.4: Rank correlation between different G-betas

	β_{-10}^{MLPM}	β_{-7}^{MLPM}	β_{-3}^{MLPM}	β_{-1}^{MLPM}	β_{0}^{MLPM}	β_{1}^{MLPM}	β_{3}^{MLPM}	β_{7}^{MLPM}	β_{10}^{MLPM}
β_{-10}^{MLPM}	1.000	0.906	0.564	0.489	0.470	0.459	0.403	0.319	0.301
β_{-7}^{MLPM}	0.906	1.000	0.728	0.620	0.609	0.605	0.559	0.507	0.488
β_{-3}^{MLPM}	0.564	0.728	1.000	0.971	0.966	0.963	0.938	0.897	0.881
β_{-1}^{MLPM}	0.489	0.620	0.971	1.000	0.997	0.991	0.975	0.922	0.909
β_{0}^{MLPM}	0.470	0.609	0.966	0.997	1.000	0.996	0.983	0.933	0.922
β_{1}^{MLPM}	0.459	0.605	0.963	0.991	0.996	1.000	0.989	0.943	0.933
β_{3}^{MLPM}	0.403	0.559	0.938	0.975	0.983	0.989	1.000	0.973	0.966
β_{7}^{MLPM}	0.319	0.507	0.897	0.922	0.933	0.943	0.973	1.000	0.997
β_{10}^{MLPM}	0.301	0.488	0.881	0.909	0.922	0.933	0.966	0.997	1.000
Avg	0.546	0.669	0.879	0.875	0.875	0.875	0.865	0.832	0.822

Fig. 3.3 depicts the rank correlations of four particular G-betas $\beta_0^{MLPM}, \beta_1^{MLPM}, \beta_{-10}^{MLPM}, \beta_{10}^{MLPM}$ with other β_λ^{MLPM} where $\lambda \in [-15, 15]$. Here β_0^{MLPM} is the BL-beta and β_1^{MLPM} is the LR-beta. It is clearly seen that the rank correlation deviates significantly. Even the rank correlations between β_1^{MLPM} and β_{10}^{MLPM} fall below 0.

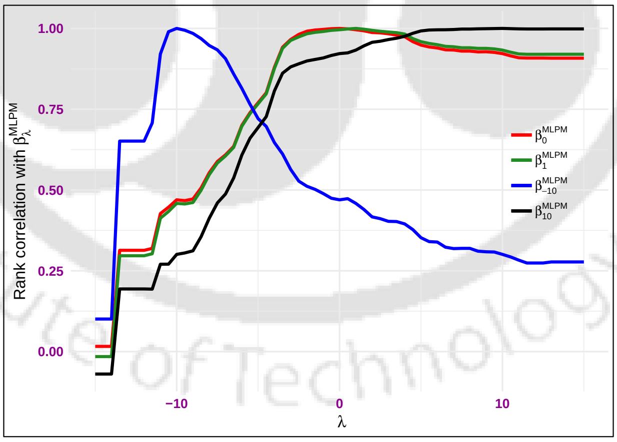


Figure 3.3: Ranking relation with

We also compare the rankings of standard beta β with the G-beta. We find that rank correlations increase as λ grows and decreases as λ falls. Table 3.5 and Fig. 3.4 clearly illustrate this behaviour.

Table 3.5: Rank correlation between MV beta and G-beta

	β_{-10}^{MLPM}	β_{-7}^{MLPM}	β_{-3}^{MLPM}	β_{-1}^{MLPM}	β_{0}^{MLPM}	β_{1}^{MLPM}	β_{3}^{MLPM}	β_{7}^{MLPM}	β_{10}^{MLPM}
β	0.277	0.472	0.868	0.894	0.908	0.920	0.956	0.994	0.998

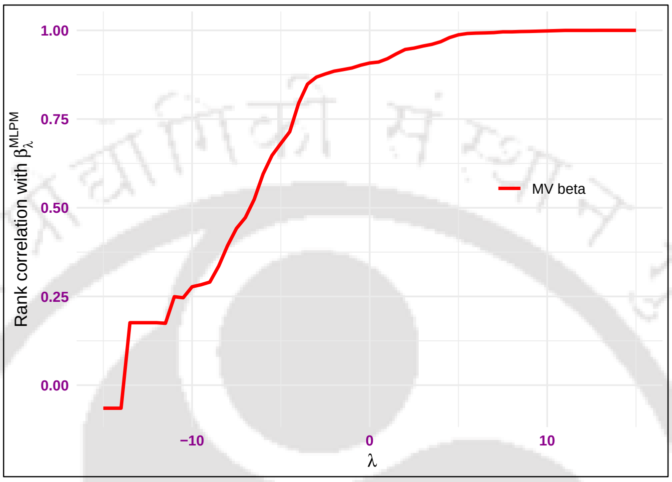


Figure 3.4: Ranking relation with standard beta

The empirical evidences reveal one interesting behaviour of β_λ^{MLPM}. When λ is very large, β_λ^{MLPM} behave similar to the Sharpe ratio and the usual beta β, respectively. Thus, the conclusion we can draw is that the MLPM selection rule gradually converges to the MV selection rule as λ tends to increase.

3.5.2 Behaviour of LPM based *alpha*

We now explore the behaviour of LPM based *alpha*, called LPM-*alpha*. The LPM-*alpha* is defined as

$$\alpha_{\lambda,i}^{LPM} = \left(\mathbb{E}[R_i] - r_f\right) - \beta_{\lambda,i}^{MLPM}\left(\mathbb{E}[R_m] - r_f\right).$$

Table 3.6 reports the rank correlations between seven α_λ^{LPM} with $\lambda = -10, -7, -3, -1, 0, 1, 3, 7, 10$. The α_λ^{LPM} does not display very high rank correlation among themselves. For example, the rank correlations between α_0^{LPM} and others varies from 0.625 to 0.930 (see column 2 of Table 3.6). The average correlation in this case is 0.725. In overall, the average correlations for α_λ^{LPM} (for β_λ^{MLPM}) lies between 0.725 to 0.894 (0.930 to 0.982).

Table 3.6: Rank correlations based on different α_{λ}^{LPM}

	α_{-10}^{LPM}	α_{-7}^{LPM}	α_{-3}^{LPM}	α_{-1}^{LPM}	α_{0}^{LPM}	α_{1}^{LPM}	α_{3}^{LPM}	α_{7}^{LPM}	α_{10}^{LPM}
α_{-10}^{LPM}	1.000	0.930	0.728	0.689	0.678	0.660	0.633	0.608	0.602
α_{-7}^{LPM}	0.930	1.000	0.796	0.730	0.716	0.698	0.656	0.591	0.560
α_{-3}^{LPM}	0.728	0.796	1.000	0.975	0.961	0.944	0.889	0.784	0.738
α_{-1}^{LPM}	0.689	0.730	0.975	1.000	0.994	0.986	0.952	0.862	0.820
α_{0}^{LPM}	0.678	0.716	0.961	0.994	1.000	0.996	0.969	0.885	0.844
α_{1}^{LPM}	0.660	0.698	0.944	0.986	0.996	1.000	0.984	0.909	0.871
α_{3}^{LPM}	0.633	0.656	0.889	0.952	0.969	0.984	1.000	0.959	0.931
α_{7}^{LPM}	0.608	0.591	0.784	0.862	0.885	0.909	0.959	1.000	0.994
α_{10}^{LPM}	0.602	0.560	0.738	0.820	0.844	0.871	0.931	0.994	1.000
Avg.	0.725	0.742	0.868	0.890	0.894	0.894	0.886	0.844	0.818

We also compare the rankings of Jensen's alpha or J-*alpha* and α_{λ}^{LPM}. We find that the rank correlations between J-*alpha* and α_{λ}^{LPM} is high for $\lambda \in [0,1]$ compare to the other values of λ. Table 3.7 and Fig. 3.5 clearly illustrate this behaviour. This indicates that the ranking differences should be low when $\lambda \in [0,1]$.

Table 3.7: Rank correlation between J-*alpha* and LPM *alpha*

	α_{-10}^{LPM}	α_{-7}^{LPM}	α_{-3}^{LPM}	α_{-1}^{LPM}	α_{0}^{LPM}	α_{1}^{LPM}	α_{3}^{LPM}	α_{7}^{LPM}	α_{10}^{LPM}
J-*alpha*	0.610	0.689	0.942	0.968	0.976	0.977	0.954	0.869	0.824

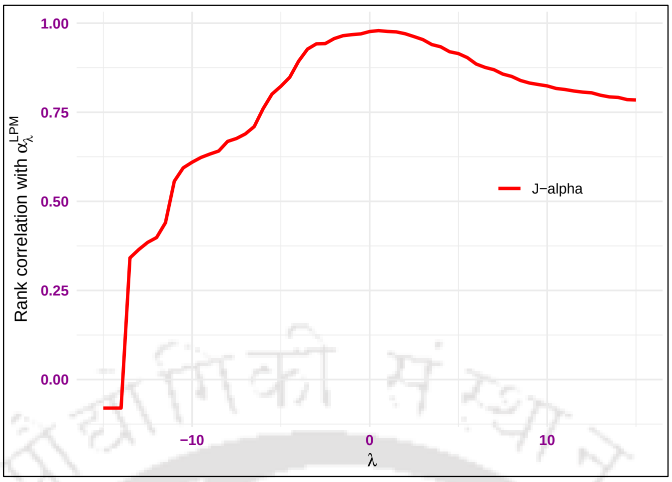


Figure 3.5: Ranking relation between J-*alpha* and α_λ^{LPM}

3.6 Concluding remarks

Using the generalized target for separation, we have developed a generalized version of CAPM in the MLPM framework. Our model contains other MLPM pricing models and the classical CAPM as special cases. As an application of the new pricing model, we have derived an acceptance criteria for capital investment projects. The criteria helps a firm to evaluate whether a risky project is worth funding. Finally, we presented some empirical experiments to study the behaviour of generalized MLPM beta β_λ^{MLPM} and LPM based *alpha*. Our findings reveal that the overall rankings of risky portfolios based on β_λ^{MLPM} converge to the rankings of usual beta as $\lambda \to \infty$. On the other hand, the ranking differences between LPM-*alpha* and J-*alpha* are very low when $\lambda \in [0,1]$. Moreover, the rankings of LPM-*alpha* converges to the rankings produced by J-*alpha* as $\lambda \to 0$.

Chapter 4

Asset Pricing without Risk-Free Asset

All the MLPM CAPMs discussed in the previous chapter are developed by adopting the standard assumptions of Sharpe-Lintner-Mossin (SLB) model (Sharpe, 1964; Lintner, 1965; Mossin, 1966). However, one of the assumptions is really unrealistic. The assumption is that the investors can lend and borrow any amount at a single risk-free rate. In real world, it never happens. Investors must borrow at a higher interest rate than the rate at which they lend. Otherwise, the banks or similar financial institutions are not going to survive. So, the borrowing rate (bank's loan rate) must be greater than the lending rate (bank's deposit rate). Hence, the concept of unique risk-free asset is truly impractical.

If the risk-free asset ceases to exist, the MV CAPM is invalid. In order to overcome this deficiency, Black (1972) proposed a new equilibrium model in the MV framework. This model is popularly known as the zero-beta CAPM. This CAPM is valid not only without the presence of risk-free asset but also when the lending and the borrowing rates are different. Hence, it is much more realistic than the traditional CAPM.

Similar to the MV CAPM, the MLPM CAPMs are also invalid without the assumption of unique risk-free asset. However, in the MLPM framework, no alternative model has been introduced yet. We analytically develop a new MLPM CAPM by relaxing the assumption of unique risk-free asset. Then, we discuss the validity of our CAPM under realistic situations. Finally, we show that the new CAPM is identical to Black's zero-beta CAPM when the underlying return distribution is normal.

The chapter is organized as follows. In section 4.1, issues of classical asset pricing model are discussed and a new MLPM CAPM is theoretically derived. In addition, we discuss the case of different borrowing and lending rates. Section 4.3 presents special cases of the new model. We show that our model specializes to Black's zero-beta CAPM, when the underlying distribution is normal. Section 4.4 provides some concluding comments.

4.1 The pricing model without risk-free asset

Let us consider a situation where all the standard assumptions of Sharpe's CAPM are employed except one. We drop the assumption that there is a single risk-free rate at which unlimited borrowing and lending can take place. As a result, the concept of unique risk-free asset is no more valid. Hence, the usual capital market line (CML) does not exist and thereby the standard CAPM cannot be obtained. Even if we assume the existence of a single risk-free asset, the linear separation cannot be achieved in the MLPM framework when the target rate is not equal to the affine combination of risk-free rate and expected return of portfolio whose risk is being measured. Hence, the MLPM CML cannot be obtained for arbitrary target levels. Thus, the usual equilibrium relationship in the MLPM framework cannot be guaranteed. In order to overcome this drawback, we develop a new MLPM CAPM.

We assume that all investors find their optimal portfolios solely on the basis of mean and LPM of return. Neither risk-free lending nor risk-free borrowing is permitted. We further assume that the market portfolio m (with expected return $\mu_m = \mathbb{E}(R_m)$) is efficient.

A tangent to the efficient frontier at m can be drawn (see Fig. 4.1). We assume that the tangent touches Y-axis at $(0, \tau_1)$. Since the minimum $\mathrm{LPM}_{\alpha}^{1/\alpha}$ boundary is convex, we can uniquely find a minimum risky portfolio z such that $\tau_1 = \mathbb{E}(R_z) = \mu_z$.

Suppose that a portfolio p contains an arbitrary asset i and the market portfolio m in the proportions of w_i and $(1 - w_i)$, respectively. The expected return and the risk of the portfolio p are

$$\mu_p = w_i \mu_i + (1 - w_i)\mu_m$$

and

$$\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; R_p) = \left(\mathbb{E}\left[\left(\tau - R_p\right)^{\alpha} \mathbb{1}_{\{\tau > R_p\}}\right]\right)^{\frac{1}{\alpha}},$$

respectively. As w_i varies, these values trace out curve imm' in Fig. 4.1.

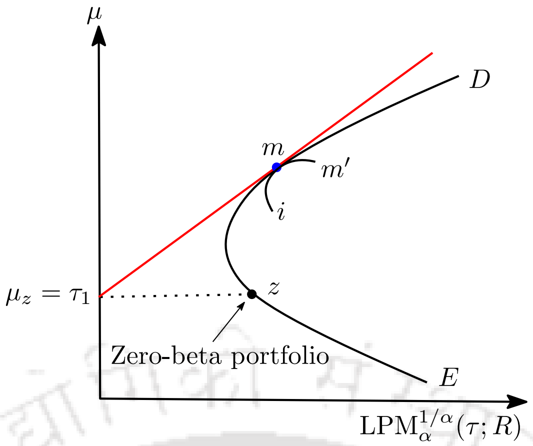


Figure 4.1: Efficient frontier, market portfolio and zero-beta portfolio

Since

$$\frac{\partial \mathbb{E}(R_p)}{\partial w_i} = \mu_i - \mu_m$$

and

$$\frac{\partial \mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_p)}{\partial w_i} = \frac{\mathbb{E}\Big[(\tau - R_p)^{\alpha-1}(R_m - R_i)\mathbb{1}_{\{\tau>R_p\}}\Big]}{\Big(\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_p)\Big)^{\alpha-1}},$$

for all $\alpha \geq 1$, the slope of imm' at m is given by

$$\frac{\partial \mathbb{E}(R_p)}{\partial \mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_p)}\bigg|_{w_i=0} = \frac{(\mu_i - \mu_m)\Big(\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_m)\Big)^{\alpha-1}}{\mathbb{E}\Big[(\tau - R_m)^{\alpha-1}(R_m - R_i)\mathbb{1}_{\{\tau>R_m\}}\Big]}.$$

Since the slope of line $\tau_1 m$ must be equal to the slope of imm' at m,

$$\frac{\mu_m - \mu_z}{\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_m)} = \frac{(\mu_i - \mu_m)\Big(\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_m)\Big)^{\alpha-1}}{\mathbb{E}\Big[(\tau - R_m)^{\alpha-1}(R_m - R_i)\mathbb{1}_{\{\tau>R_m\}}\Big]}$$

which can be rearranged as

$$\frac{\mu_i - \mu_m}{\mu_m - \mu_z} = \frac{\mathbb{E}\Big[(\tau - R_m)^{\alpha-1}(R_m - R_i)\mathbb{1}_{\{\tau>R_m\}}\Big]}{\mathrm{LPM}_\alpha(\tau;R_m)}. \tag{4.1}$$

We can write

$$\begin{aligned}
&\mathbb{E}\left[(\tau-R_m)^{\alpha-1}(R_m-R_i)\mathbb{1}_{\{\tau>R_m\}}\right]\\
&=\mathbb{E}\left[(\tau-R_m)^{\alpha-1}\{(\tau-R_i)-(\tau-R_m)\}\mathbb{1}_{\{\tau>R_m\}}\right]\\
&=\mathbb{E}\left[(\tau-R_m)^{\alpha-1}(\tau-R_i)\mathbb{1}_{\{\tau>R_m\}}\right]-\mathrm{LPM}_{\alpha}(\tau;R_m). \qquad (4.2)
\end{aligned}$$

Equations (4.1) and (4.2) yield

$$\frac{\mu_i-\mu_m}{\mu_m-\mu_z}=\frac{\mathbb{E}\left[(\tau-R_m)^{\alpha-1}\{(\tau-R_i)\mathbb{1}_{\{\tau>R_m\}}\right]}{\mathrm{LPM}_{\alpha}(\tau;R_m)}-1,$$

which implies

$$\frac{\mu_i-\mu_z}{\mu_m-\mu_z}=\frac{\mathbb{E}\left[(\tau-R_m)^{\alpha-1}(\tau-R_i)\mathbb{1}_{\{\tau>R_m\}}\right]}{\mathrm{LPM}_{\alpha}(\tau;R_m)}.$$

Therefore, the MLPM CAPM without the risk-free asset is given by

$$\mu_i-\mu_z=\beta_i^{\mathrm{MLPM}_{\alpha}(\tau)}(\mu_m-\mu_z), \qquad (4.3)$$

where

$$\beta_i^{\mathrm{MLPM}_{\alpha}(\tau)}=\frac{\mathrm{CLPM}_{\alpha}(\tau,\tau;R_m,R_i)}{\mathrm{LPM}_{\alpha}(\tau;R_m)},$$

the MLPM beta, and

$$\mathrm{CLPM}_{\alpha}(\tau,\tau;R_m,R_i)=\mathbb{E}\left[(\tau-R_m)^{\alpha-1}(\tau-R_i)\mathbb{1}_{\left\{\tau>R_m\right\}}\right], \qquad (4.4)$$

the co-lower partial moment of order α between the market portfolio m and the asset i.

There are several differences between this beta and the MV beta. In the case of MLPM beta, a security adds to the risk of a portfolio only when the market's return fall below the target return. When the market return exceed the target level, by definition it is not risky and thus individual security returns contribute nothing to the market risk. Hence, unlike the MV beta, the MLPM beta punishes only the downside outcomes. Thus, in bearish markets, the MLPM beta must perform better than the usual MV beta.

REMARK 4.1. Since the linear separation is not needed to obtain the MLPM zero-beta CAPM, it is valid for any target level τ. The only requirement is that the market portfolio should be efficient.

4.1.1 Zero-MLPM beta portfolio

For $i = z$, the relation (4.3) becomes

$$\mu_z - \mu_z = \beta_z^{\mathrm{MLPM}_\alpha(\tau)}(\mu_m - \mu_z).$$

Since $\mu_m - \mu_z \neq 0$,

$$\beta_z^{\mathrm{MLPM}_\alpha(\tau)} = 0.$$

So, z is a zero-beta portfolio (see Fig. 4.1). Moreover, the portfolios which have expected return equal to μ_z are all zero-beta portfolios. Among those, portfolio z has minimum risk. The new CAPM given by (4.3) is similar to the Black's zero-beta CAPM except that the measure of systematic risk, MV beta is replaced by MLPM beta. Hence, we call it zero-MLPM beta CAPM.

4.2 Validation under realistic scenarios

Let us consider the situation of real world where the risk-free lending and borrowing rates are different. Suppose the lending and the borrowing rates are r_{fl} and r_{fb}, respectively. Since the lending rate must be less than the borrowing rate, we consider $r_{fl} < r_{fb}$. We also assume that $r_{fl} < \mu_z < r_{fb}$.

Now investors face a new efficient frontier. In order to identify the new frontier, we first focus on the shape of an investment curve, which is formed by the affine combinations of a risk-free asset and a risky portfolio. In Chapter 2, three properties of investment curve have been described. The properties are convexity, monotonicity and convergent (see Theorem 2.9, 2.10 and 2.11).

The convergent property states that slopes of an investment curve always converge. For $\alpha = 1$, piecewise linearity of the curves reduces to perfect linearity, and for $\alpha \geq 2$, non-linearity converges to perfect linearity. Thus, second derivative must converge to zero for $\alpha \geq 2$.

THEOREM 4.1. *Convexity of investment curve approaches perfect linearity in* μ_p-$\mathrm{LPM}_\alpha^{1/\alpha}$ *space as* $\mu_p \to \infty$ *for all* τ *and* $\alpha \geq 2$.

PROOF. It is sufficient to show that $\frac{\partial^2 \mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_p)}{\partial \mu_p^2} \to 0$ as $\mu_p \to \infty$. Rearranging equation (2.14) (see Chapter 2), we obtain

$$\begin{aligned}\frac{\partial^2 \mathrm{LPM}_\alpha^{1/\alpha}(\tau;r_p)}{\partial \mu_p^2} &= \frac{\mu_p^{2\alpha-2}(AB-C^2)}{\mu_p^{2\alpha-1}B^{\frac{2\alpha-1}{\alpha}}} \\ &= \frac{1}{\mu_p}\frac{AB-C^2}{B^{\frac{2\alpha-1}{\alpha}}},\end{aligned} \tag{4.5}$$

where

$$A = \mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha-2} \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}} X^2\right],$$

$$B = \mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha} \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}}\right] \text{ and}$$

$$C = \mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha-1} \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}} X\right].$$

In Theorem 2.10, we have shown that

$$\lim_{\mu_p\to\infty} \mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha} \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}}\right] = \mathbb{E}[X^\alpha \mathbb{1}_{\{r_f>R_q\}}].$$

and

$$\lim_{\mu_p\to\infty} \mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha-1} \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}} X\right] = \mathbb{E}[X^\alpha \mathbb{1}_{\{r_f>R_q\}}].$$

Similarly

$$\lim_{\mu_p\to\infty} \mathbb{E}\left[\left(\frac{\tau-R_p}{\mu_p}\right)^{\alpha-2} \mathbb{1}_{\left\{\frac{\tau-R_p}{\mu_p}>0\right\}} X^2\right] = \mathbb{E}[X^\alpha \mathbb{1}_{\{r_f>R_q\}}].$$

Taking limit as $\mu_p \to \infty$ in equation (4.5), we finally get

$$\lim_{\mu_p\to\infty} \frac{\partial^2 \mathrm{LPM}_\alpha^{1/\alpha}(\tau;R_p)}{\partial \mu_p^2} = 0.$$

■

The new efficient frontier in the presence of a risk-free asset is depicted in Fig 4.2. For $\tau = r_f$, the efficient frontier fC will be linear and it will originate from the point $(0, r_f)$.

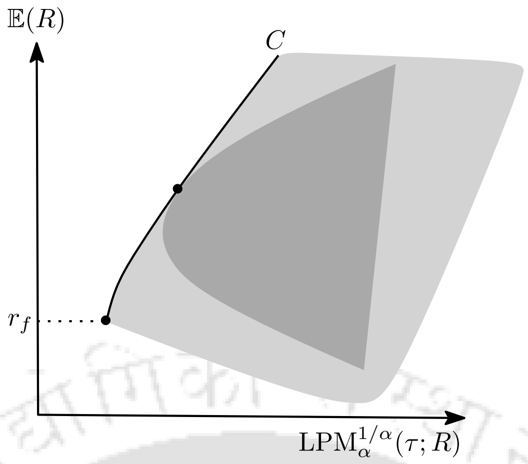


Figure 4.2: Efficient frontier with a single risk-free asset

4.2.1 Restricted borrowing

Suppose that borrowing is not permitted and only lending (investing) is allowed at rate r_{fl}. Then, the new efficient frontier lT_lmD (see Fig. 4.3) originates from l and goes upto the tangency point T_l and beyond that it coincides with the efficient frontier of risky assets. It is seen that the market portfolio is still efficient. Hence, the zero-beta CAPM is still valid.

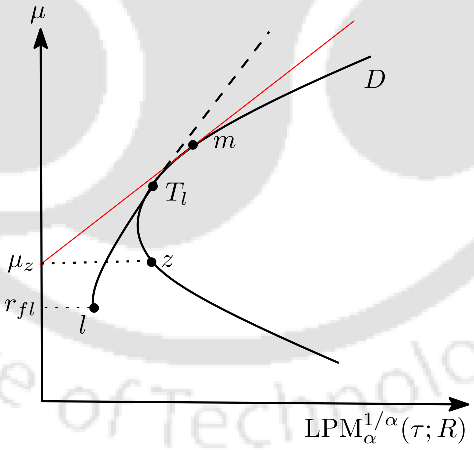


Figure 4.3: Efficient frontier with restricted borrowing

If $\tau = r_{fl}$, then we will have a linear efficient frontier from $(0, r_{fl})$ to the tangency point T_l and then the usual curvy frontier T_lmD.

4.2.2 Different lending and borrowing rates

Let us assume that besides the lending at r_{fl}, the risk-free borrowing is allowed at the rate r_{fb}. In this case, the efficient frontier will be lT_lmT_bD' (see Fig. 4.4). However, the market portfolio remains efficient and hence, the zero-beta CAPM holds. Thus, as long as the market portfolio is efficient, the MLPM zero-beta CAPM will always hold.

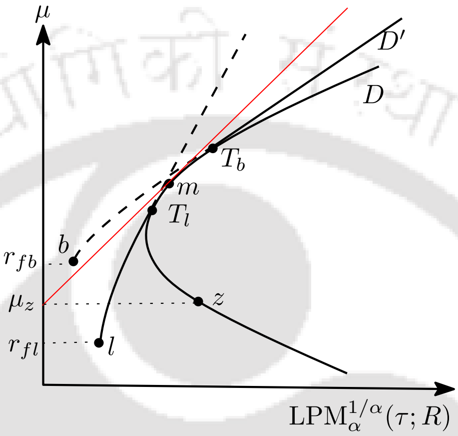


Figure 4.4: Efficient frontier with different borrowing and lending rates

If target rate $\tau = r_{fl}$, the efficient frontier will be same lT_lmT_bD' but the portion lT_l will be linear. Similarly, for $\tau = r_{bl}$, the portion T_bD' of the whole frontier lT_lmT_bD' will be linear.

REMARK 4.2. In the above figures, it is considered that the target rate τ is strictly greater than r_{fl} and r_{fb}. When $\tau < r_{fl}$, the corresponding investment curves (lT_l in Fig. 4.3 and Fig. 4.4) originates from $(0, r_{fl})$ as $\mathrm{LPM}_{\alpha}^{1/\alpha}(\tau; r_f) = 0$ for $\tau < r_{fl}$. Similarly, for $\tau < r_{fb}$, the curve bT_b in Fig. 4.4 intersects at $(0, r_{fb})$. For the case of equality ($\tau = r_{fl}$ or $\tau = r_{fb}$), the corresponding curves (lT_l or bT_b) become straight lines.

4.3 Special cases

If we vary the parameters of LPM, α and τ, several types of risk measures and the corresponding pricing models can be obtained. For example, when $\alpha = 2$ and τ is expected return, equation (3.5) explains the equilibrium pricing relationship in mean-semivariance framework. For $\alpha = 1$, mean-semideviation model can be achieved. Similarly, we can obtain various other models by changing the parameters. Moreover, the MV zero-beta CAPM can also be seen as a special case of the new model. The following theorem proves this result.

THEOREM 4.2. *If the joint density of R_i and R_m is bivariate normal and $\tau = \mu_z$, the MLPM zero-beta CAPM (4.3) reduces to MV zero-beta CAPM:*

$$\mu_i - \mu_z = \beta_i(\mu_m - \mu_z)$$

PROOF. From the properties of bivariate normal distributions, we can write

$$\mathbb{E}\big(\mu_z - R_i|R_m\big) = \mu_z - \mu_i - \beta_i(R_m - \mu_m). \tag{4.6}$$

Using relation (4.6), the CLPM given by (4.4) can be expressed as follows:

$$\begin{aligned}
&\mathrm{CLPM}_\alpha\big(\mu_z, \mu_z; R_m, R_i\big)\\
&= \mathbb{E}\Big[(\mu_z - R_m)^{\alpha-1}(\mu_z - R_i)\mathbb{1}_{\{\mu_z > R_m\}}\Big]\\
&= \mathbb{E}\Big[\mathbb{E}\Big((\mu_z - R_m)^{\alpha-1}\mathbb{1}_{\{\mu_z > R_m\}}(\mu_z - R_i)\big|R_m\Big)\Big]\\
&= \mathbb{E}\Big[(\mu_z - R_m)^{\alpha-1}\mathbb{1}_{\{\mu_z > R_m\}}\mathbb{E}(\mu_z - R_i|R_m)\Big]\\
&= \mathbb{E}\Big[(\mu_z - R_m)^{\alpha-1}\mathbb{1}_{\{\mu_z > R_m\}}\big(\mu_z - \mu_i - \beta_i[R_m - \mu_m]\big)\Big]\\
&= \beta_i\,\mathrm{LPM}_\alpha(\mu_z; R_m) + \Big[\beta_i(\mu_m - \mu_z) - (\mu_i - \mu_z)\Big]\mathrm{LPM}_{\alpha-1}(\mu_z; R_m).
\end{aligned}$$

Therefore, the MLPM beta becomes

$$\begin{aligned}
\beta_i^{\mathrm{MLPM}_\alpha(\mu_z)} &= \frac{\mathrm{CLPM}_\alpha(\mu_z, \mu_z; R_m, R_i)}{\mathrm{LPM}_\alpha(\mu_z; R_m)}\\
&= \beta_i + \Big[\beta_i(\mu_m - \mu_z) - (\mu_i - \mu_z)\Big]\frac{\mathrm{LPM}_{\alpha-1}(\mu_z; R_m)}{\mathrm{LPM}_\alpha(\mu_z; R_m)}.
\end{aligned} \tag{4.7}$$

Using equations (3.5) and (4.7), we obtain

$$\begin{aligned}(\mu_i-\mu_z)&\left[1+(\mu_m-\mu_z)\frac{\mathrm{LPM}_{\alpha-1}(\mu_z;R_m)}{\mathrm{LPM}_{\alpha}(\mu_z;R_m)}\right]\\ &=\beta_i(\mu_m-\mu_z)\left[1+(\mu_m-\mu_z)\frac{\mathrm{LPM}_{\alpha-1}(\mu_z;R_m)}{\mathrm{LPM}_{\alpha}(\mu_z;R_m)}\right].\end{aligned}\tag{4.8}$$

Since $(\mu_m-\mu_z)>0$ and $\frac{\mathrm{LPM}_{\alpha-1}(\mu_z;R_m)}{\mathrm{LPM}_{\alpha}(\mu_z;R_m)}>0$,

$$1+(\mu_m-\mu_z)\frac{\mathrm{LPM}_{\alpha-1}(\mu_z;R_m)}{\mathrm{LPM}_{\alpha}(\mu_z;R_m)}\neq 0.$$

Therefore, equation (4.8) gives

$$\mu_i-\mu_z=\beta_i(\mu_m-\mu_z).$$

■

4.4 Concluding remarks

In this chapter, we have analytically developed a new MLPM asset pricing model which is valid not only without the assumption of unique risk-free asset but also when the risk-free lending and borrowing rates are different. The model is analogues to the Black's MV zero-beta CAPM. It is developed for arbitrary probability distribution of asset returns and reduces to the MV zero-beta CAPM when the underlying distribution is normal. Hence, the new model should perform better than the MV zero-beta CAPM under any situation.

Chapter 5

Ideal Performance Measure

Generally, a risk-adjusted performance measure is a ratio where the denominator is a risk measure and the numerator is a reward measure. As we vary the reward and the risk measures, various performance measures are obtained. Now a major question is which of these measures are appropriate in the context of portfolio management. Cherny and Madan (2008) proposed a set of axioms to characterise performance measures for cash flows generated by writing options. Measures satisfying these axioms are termed as indexes of acceptability. On the other hand, by defining desirable properties of risk and reward measures, Rachev et al. (2008) axiomatically defined coherent ratios to measure portfolio's performance. These axioms mainly emphasize on the characteristics of reward and risk measures rather than the performance measures. However, our main focus is to observe only those properties of performance measures which have realistic economical interpretations as well as proper relevance to the practical investment management.

In this chapter, we present and justify three minimal and desirable properties that a performance measure should satisfy. The properties are quasi-concavity, monotonicity and size invariance. A performance measure satisfying these properties is called "*ideal*". Then, various popular performance measures are examined whether they are ideal or not. However, we mainly review and discuss the LPM based performance measures in light of the above mentioned properties. In the literature, there exist mainly two types of LPM based ratios: Kappa and FT ratio. All the Other LPM based ratios can be seen as a special case of these ratios. We show that the Kappa ratio is an ideal performance measure. On the other hand, the FT ratio is not an ideal measure as it fails to satisfy the very first criteria, the quasi-concavity

The chapter is organized as follows. Section 5.1 presents and justifies three axioms of an ideal performance measure. Section 5.2 discusses the properties of two LPM based performance measures (Sotino and FT). In addition, various properties of UPM are described. In Section 5.3, we examine various other performance ratios. Finally, we conclude the chapter in Section 5.4.

5.1 Ideal performance measure

Let $R_i, i = 1, 2, \ldots, n$, indicate random return of portfolio (or fund) i and X_i denote corresponding excess return over some benchmark level. The measure of reward and risk are defined as $\pi(X_i)$ and $\rho(X_i)$, respectively, where $\rho(X_i) > 0$. The performance measure $\psi(X_i)$ as a reward to risk ratio of portfolio i is defined as

$$\psi(X_i) = \frac{\pi(X_i)}{\rho(X_i)}. \tag{5.1}$$

The higher the values of $\psi(X_i)$, the better the rank of X_i. Hence, portfolio i is preferable to portfolio j if $\psi(X_i) > \psi(X_j)$.

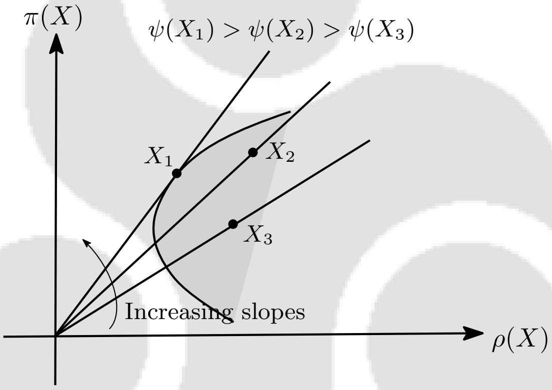


Figure 5.1: Increasing performance ratio

Before identifying a good investment opportunity by using performance measures, we need to see how good the performance measure itself. It is really difficult for any performance measure to reflect all features of an investor's preferences. However, a good performance measure should have some properties which can at least capture the basic notions of portfolio theory such as investment diversification, preferring (disliking) higher return (higher risk), no dependency on wealth sizes. These are the minimal properties that every investor would like to see in any performance measure used in the context of portfolio management. We now present and justify three desirable properties of a performance measure and these are given in the form of axioms.

AXIOM 1 (Quasi-concavity)**.** *For any* $X_i, X_j \in \mathscr{G}$ *and* $w \in [0,1]$, $\psi(wX_i + (1-w)X_j) \geq \min\{\psi(X_i), \psi(X_j)\}$.

This axiom guarantees that diversification does not decrease performance. It means that the overall performance of a portfolio is not worse than the lowest performing asset of the portfolio. As a result, the exposure of whole portfolio to any particular asset is reduced. The diversification is a minimal requirement of portfolio management. This property can be explained by quasi-concavity of ψ. A sufficient condition for the quasi-concavity is presented below.

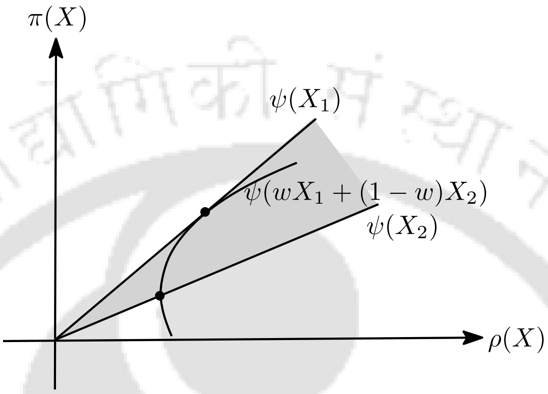


Figure 5.2: Diversification

THEOREM 5.1. *If $\rho(X)$ is convex and $\pi(X)$ is concave, then $\psi(X)$ is quasi-concave.*

PROOF. Suppose that $\min\{\psi(X_i),\psi(X_j)\}=\delta$, where $\delta>0$. Hence, $\psi(X_i)\geq\delta$ and $\psi(X_j)\geq\delta$. This implies $\pi(X_i)\geq\delta\rho(X_i)$ and $\pi(X_j)\geq\delta\rho(X_j)$. We get

$$\begin{aligned}\psi(wX_i+(1-w)X_j) &= \frac{\pi(wX_i+(1-w)X_j)}{\rho(wX_i+(1-w)X_j)}\\ &\geq \frac{w\pi(X_i)+(1-w)\pi(X_j)}{w\rho(X_i)+(1-w)\rho(X_j)}\\ &\geq \frac{w\delta\rho(X_i)+(1-w)\delta\rho(X_j)}{w\rho(X_i)+(1-w)\rho(X_j)}\\ &= \delta.\end{aligned}$$

Hence, for any X_i and X_j,

$$\psi(wX_i+(1-w)X_j)\geq\min\{\psi(X_i),\psi(X_j)\}.$$

■

A mean to risk ratio, where the measure of risk is convex, always satisfies this axiom. Thus, ratios of mean to coherent risk measures (Artzner et al., 1999) and ratios of mean to deviation risk measures (Rockafellar et al., 2006) are quasi-concave.

AXIOM 2 (Monotonicity)**.** *For any* $X_i, X_j \in \mathscr{G}$, *if* $X_i \geq X_j$ *almost surely (a.s.), then* $\psi(X_i) \geq \psi(X_j)$.

Monotonicity states that if X_i generates higher return than X_j in all states (in the *almost surely* sense), then X_i should have better rank than X_j. It is really illogical to think why any investor would invest in a portfolio that always provides lower return than another potential investment. Hence, the performance measures should follow this notion. It is also noticed that ψ satisfies this axiom if π is non-decreasing and ρ is non-increasing:

$$X_i \geq X_j \ a.s. \implies \pi(X_i) \geq \pi(X_j) \text{ and } \rho(X_i) \leq \rho(X_j).$$

AXIOM 3 (Size invariance)**.** *For any* X_i, $\psi(\lambda X_i) = \psi(X_i)$ *for any* $\lambda > 0$.

The size invariance axiom encapsulates the fact that a performance measure should not be influenced by the size of portfolios. Here, the size means the amount of invested wealth. Suppose λ is the invested money ($) to a portfolio with rate of return X. Then, the dollar return of this portfolio is λX. Through this axiom, we mean to say that a performance measure should not depend on λ.

The main aim of a performance ratio is not to measure how much wealth a portfolio generates but to quantify the reward it provides at the cost of risk. According to Ingersoll et al. (2007), the value of performance measure should not depend on the investment amount. The reason is that the returns (X) should be sufficient statistics rather than dollar gains or loses. If a performance ratio does not satisfy the size invariance property, ranking of funds based on the ratio can be manipulated by varying the size of one or more funds. The following example clarify this situation.

EXAMPLE 5.1. Suppose that there are two efficient portfolios with random returns X_1 and X_2. To measure the performance, we use the following ratio:

$$\psi(X) = \frac{\pi(X)}{\rho(X)}.$$

We assume that

$$\rho(X_1) = 10, \pi(X_1) = 15; \ \rho(X_2) = 10, \pi(X_2) = 5.$$

It is obvious that every investor chooses fund 1 over fund 2 as, at the same level of risk, fund 1 provides higher return. A performance measure should reflect this behaviour. This means, we should get $\psi(X_1) > \psi(X_2)$. We obtain $\psi(X_1) = 1.5$ and $\psi(X_2) = 0.5$.

The performance of dollar return should also reflect this attitude. We mean to say that for any λ_1 and λ_2 , $\psi(\lambda_1 X_1) > \psi(\lambda_2 X_2)$, where λ_1 and λ_2 denote investment amounts ($). It always holds if ψ is size invariant.

Let us assume that a performance ratio ψ^* is not size invariance. In particular, we consider $\psi^*(\lambda X) = \lambda\psi^*(X)$. Suppose that $\lambda_1 = 6, \lambda_2 = 22$. Hence, $\psi^*(\lambda_1 X_1) = 9$ and $\psi^*(\lambda_2 X_2) = 11$. According to this ratio, fund 2 is more profitable for investment. Due to the higher investment amount, ψ^* ranks fund 2 higher than fund 1. Since the ratio ψ^* is influenced by the investment size, the actual return and risk can not fully depict the performance. This distortion will hamper the actual rankings of financial assets. Furthermore, it may cause huge losses as a bad investment fund (high risk with low return) can get higher rank. Thus, a performance ratio should be size invariant.

Furthermore, the size invariance property is connected to a desirable property of risk and reward measures. It can be easily observed that ψ is size invariant if π and ρ both satisfy positive homogeneity, i.e.,

$$\pi(\lambda X) = \lambda\pi(X), \rho(\lambda X) = \lambda\rho(X) \ \forall \ \lambda > 0.$$

Artzner et al. (1999) described that a good risk measure should be positively homogenous. On the other hand, according to Rachev et al. (2008), a good reward measure should also be positively homogenous. Therefore, if an investor doubles ($\lambda = 2$) the size of a portfolio, risk and reward of the portfolio will be doubled but performance remains same. As a result, rankings of funds can not be altered by changing portfolio size.

DEFINITION 5.1. *A performance measure that satisfies the axioms of quasi-concavity, monotonicity and size invariance is called ideal.*

THEOREM 5.2 (Sufficient condition)**.** *A a performance ratio* $\psi(X_i) = \frac{\pi(X_i)}{\rho(X_i)}$ *is ideal if it satisfies the following conditions:*

(1) The reward measure $\pi(X_i)$ *is concave, monotonically non-decreasing, positive homogeneous and*

(2) The risk measure $\rho(X_i)$ is convex, monotonically non-increasing, positive homogenous.

Since coherent risk measures are convex, positively homogenous and monotonous, ratios of mean to coherent risk measures are ideal. For example, mean-CVaR ratio and mini-max ratio are ideal measures. Ratios of mean to deviation risk measures are not ideal because they fail to satisfy Axiom 2. For example, Sharpe ratio, mean-semi deviation ratio, mean-mean absolute deviation (MAD) ratio are not ideal. For the same reason, drawdown based ratios are not ideal. Mean-VaR ratio and Rachev ratio are not ideal because they are not quasi-concave. However, our main focus is to study the LPM based performance ratios. The next section discusses this topic in detail.

5.2 Partial moment based performance measures

5.2.1 Kappa ratio

In order to use LPM in asset ranking, Sortino and Price (1994) proposed the following ratio known as Sortino ratio:

$$S_i(\tau;\alpha) = \frac{\mathbb{E}(R_i)-\tau}{\mathrm{LPM}_2^{1/2}(\tau;R_i)} = \frac{\mathbb{E}(X_i)}{\mathrm{LPM}_2^{1/2}(0;X_i)}.$$

Kaplan and Knowles (2004) generalized the Sortino ratio to Kappa ratio given by

$$\kappa_i(\tau;\alpha) = \frac{\mathbb{E}(X_i)}{\mathrm{LPM}_\alpha^{1/\alpha}(0;X_i)},\ \alpha \geq 1.$$

In Chapter 1, various properties of $\mathrm{LPM}_\alpha^{1/\alpha}(\tau;R)$ are presented. Thus, the following properties of $\mathrm{LPM}_\alpha^{1/\alpha}(0;X)$ are easily obtained.

(i). (Positive homogeneity) For any $\lambda > 0$, $\mathrm{LPM}_\alpha^{1/\alpha}(0;\lambda X_i) = \lambda \mathrm{LPM}_\alpha^{1/\alpha}(0;X_i)$

(ii). (Convexity[1]) For any X_i, X_j and $\lambda \in [0,1]$,

$$\mathrm{LPM}_\alpha^{1/\alpha}(0;\lambda X_i + (1-\lambda)X_j) \leq \lambda \mathrm{LPM}_\alpha^{1/\alpha}(0;X_i) + (1-\lambda)\mathrm{LPM}_\alpha^{1/\alpha}(0;X_j)$$

(iii). (Monotonicity) For any X_i and X_j, $X_i \geq X_j$ a.s. $\implies \mathrm{LPM}_\alpha^{1/\alpha}(0;X_i) \leq \mathrm{LPM}_\alpha^{1/\alpha}(0;X_j)$

It is evident that the Kappa ratio satisfies all the axioms of ideal performance measure.

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[1] Convexity follows from the fact that $\mathrm{LPM}_\alpha^{1/\alpha}(X;\tau)$ is sub-additive for any τ and positively homogenous for $\tau = 0$.

5.2.2 Upper partial moment

Although, the Sortino or Kappa ratio penalizes the downside deviations, it does not properly utilize the upside deviations or the potential returns. The upside deviation is measured by the upper partial moment, i.e.,

$$\mathrm{UPM}_\gamma(\tau;R) = \mathbb{E}\left[(R-\tau)^\gamma \mathbb{1}_{\{R>\tau\}}\right]$$

and the corresponding root-UPM is defined as

$$\mathrm{UPM}_\gamma^{1/\gamma}(\tau;R) = \left(\mathbb{E}\left[(R-\tau)^\gamma \mathbb{1}_{\{R>\tau\}}\right]\right)^{\frac{1}{\gamma}}.$$

Here, γ captures investor's behaviour above the target level τ. For $\gamma = 1$, investors are indifferent above the target return and for $\gamma > 1$ (< 1), they are potential-seeking (potential-averse). Some good references related to UPM are Holthausen (1981), Viole and Nawrocki (2011), Cumova and Nawrocki (2014) and Caporin et al. (2014).

5.2.3 Zero reward

The reward is positive when investment portfolio produces return higher than the desired target level. If a portfolio always generates lower return than the target rate, then there is no reward and thus upside upside deviation of the portfolio should be zero.

THEOREM 5.3. $R \leq \tau$ *a.s.* $\iff \mathrm{UPM}_\gamma^{1/\gamma}(\tau;R) = 0.$

5.2.4 Asset monotonicity

If portfolio R_1 provides higher return than the return of portfolio R_2 in all scenarios, then for same target return R_1 have higher UPD than R_2.

THEOREM 5.4. *For any two random returns* $R_1, R_2 \in \mathscr{G}$, $R_1(\omega) \geq R_2(\omega)\ \forall\, \omega \in \Omega \implies \mathrm{UPM}_\gamma^{1/\gamma}(\tau;R_1) \geq \mathrm{UPM}_\gamma^{1/\gamma}(\tau;R_2).$

Proof. We can see that

$$
\begin{aligned}
R_1 \geq R_2 &\implies (R_1-\tau) \geq (r_2-\tau)\\
&\implies (R_1-\tau)\mathbb{1}_{\{R_2\geq\tau\}} \geq (R_2-\tau)\mathbb{1}_{\{R_1\geq\tau\}} \geq (R_2-\tau)\mathbb{1}_{\{R_2\geq\tau\}}\\
&\implies (R_1-\tau)^{\gamma}\mathbb{1}_{\{R_1\geq\tau\}} \leq (R_2-\tau)^{\gamma}\mathbb{1}_{\{R_2\geq\tau\}}\\
&\implies \mathbb{E}\left[(R_1-\tau)^{\gamma}\mathbb{1}_{\{R_1\geq\tau\}}\right] \geq \mathbb{E}\left[(R_2-\tau)^{\gamma}\mathbb{1}_{\{R_2\geq\tau\}}\right]\\
&\implies \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R_1) \geq \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R_2).
\end{aligned}
$$

■

5.2.5 SSD consistency

Corollary 5.1. *For any* $R_1, R_2 \in \mathscr{G}$*,* $R_1 \succcurlyeq_2 R_2 \implies \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R_1) \geq \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R_2)\ \forall\, 0 \leq \gamma \leq 1.$

Proof. Holthausen (1981) showed that $R_1 \succcurlyeq_2 R_2 \implies \mathrm{UPM}_{\gamma}(\tau;R_1) \geq \mathrm{UPM}_{\gamma}(\tau;R_2)$ for all $0 \leq \gamma \leq 1$. Now it is easy to see that $\mathrm{UPM}_{\gamma}(\tau;R_1) \geq \mathrm{UPM}_{\gamma}(\tau;R_2) \implies \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R_1) \geq \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R_2)$. Hence, $R_1 \succcurlyeq_2 R_2 \implies \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R_1) \geq \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R_2)$. ■

Remark 5.1. We know that investor's upside-seeking behaviour is captured by $\gamma \geq 1$. However, for these values, the SSD consistency results may not be obtained. For $\gamma \geq 1$, the UPD is only consistent with the FSD.

5.2.6 Target monotonicity

It indicates that UPD of a portfolio decreases as target return increases. The proof is presented below.

Theorem 5.5. *For any* τ_1, τ_2 *and* $R_1 \in \mathscr{G}$*,* $\tau_1 \geq \tau_2 \implies \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau_1;R_1) \leq \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau_2;R_1).$

PROOF. We have

$$\begin{aligned}
\tau_1 \geq \tau_2 &\implies (R_1-\tau_1) \leq (R_1-\tau_2)\\
&\implies (R_1-\tau_1)\mathbb{1}_{\{R_1\geq\tau_1\}} \leq (R_1-\tau_2)\mathbb{1}_{\{R_1\geq\tau_1\}} \leq (R_1-\tau_2)\mathbb{1}_{\{R_1\geq\tau_2\}}\\
&\implies (R_1-\tau_1)^{\gamma}\mathbb{1}_{\{R_1\geq\tau_1\}} \leq (R_1-\tau_2)^{\gamma}\mathbb{1}_{\{R_1\geq\tau_2\}}\\
&\implies \mathbb{E}\left[(R_1-\tau_1)^{\gamma}\mathbb{1}_{\{R_1\geq\tau_1\}}\right] \leq \mathbb{E}\left[(R_1-\tau_2)^{\gamma}\mathbb{1}_{\{R_1\geq\tau_2\}}\right]\\
&\implies \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau_1;R_1) \leq \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau_2;R_1).
\end{aligned}$$

■

5.2.7 Shift invariance and positive homogeneity

It is easy to see that the UPD satisfies both the shift invariance and positive homogeneity. The implications are similar to the case of LPD: the reward remains same after the adding (or subtracting) constant amount to (from) both the asset return and target return.

$$\mathrm{UPM}_{\gamma}^{1/\gamma}(\tau+\lambda;R+\lambda) = \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R)\ \forall\ \lambda \in \mathbb{R}$$

5.2.8 Positive homogeneity

Positive homogeneity implies that reward should be proportionally increases in both the asset and target returns.

$$\mathrm{UPM}_{\gamma}^{1/\gamma}(\lambda\tau;\lambda R) = \lambda\ \mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R)\ \forall\ \lambda > 0$$

Although the UPD has several desirable properties, it lacks a very important characteristics. It does not encourage the diversification. $\mathrm{UPM}_{\gamma}^{1/\gamma}(\tau;R)$ is neither not concave. It is not even quasi-concave. To see these violations, let us take a look at the following example.

EXAMPLE 5.2. Let us consider two portfolios C and D with random returns R_C and R_D, respectively. The random returns and their joint probabilities are given below:

$$(R_c, R_d) = \begin{cases} (-2\%, 7\%), & \text{Prob} = 1/2 \\ (5\%, -5\%), & \text{Prob} = 1/2 \\ (8\%, -4\%), & \text{Prob} = 1/2 \\ (5\%, 5\%), & \text{Prob} = 1/2 \\ (3\%, 1\%), & \text{Prob} = 1/2 \end{cases}$$

We choose a target return $\tau = 6\%$. We obtain

$$\text{UPM}_\gamma^{1/\gamma}(R_C; 6\%) > 0, \text{UPM}_\gamma^{1/\gamma}(R_D; 6\%) > 0$$

for all $\gamma \geq 1$ and

$$\text{UPM}_\gamma^{1/\gamma}(wR_C + (1-w)R_D; 6\%) = 0$$

for $w = \frac{1}{2}$. Hence,

$$\text{UPM}_\gamma^{1/\gamma}(wR_C + (1-w)R_D; 6\%) = 0 < \text{Min}\{\text{UPM}_\gamma^{1/\gamma}(R_C; 6\%), \text{UPM}_\gamma^{1/\gamma}(R_D; 6\%)\}.$$

So, the UPM is not quasi-concave and thus it is not concave.

5.2.9 Farinelli-Tibiletti ratio

In order to reflect both the upside and downside behaviours in asset ranking, Farinelli and Tibiletti (2008) use UPM and LPM as measures of reward and risk, respectively and then propose the following ratio, known as FT ratio:

$$\text{FT}_i(\tau; \alpha, \gamma) = \frac{\text{UPM}_\gamma^{1/\gamma}(\tau; R_i)}{\text{LPM}_\alpha^{1/\alpha}(\tau; R_i)} = \frac{\text{UPM}_\gamma^{1/\gamma}(0; X_i)}{\text{LPM}_\alpha^{1/\alpha}(0; X_i)}.$$

For $\alpha = 1$ and $\gamma = 1$, the FT ratio becomes Omega ratio. This ratio can be applied when investors does not worry about the losses as well as about the potential returns. The upside potential ratio

(UPR) is obtained when $\alpha = 2$ and $\gamma = 1$. The UPR should be used when investors are downside risk-averse and upside potential-neutral. The most common behaviour, upside potential-seeking and downside risk-averse, can be achieved by setting $\gamma > 1$ and $\alpha > 1$. In this way, several types of investment decision can be accomplished just by changing the values of α and γ. However, the FT ratio has a serious drawback. It does not satisfy Axiom 1. The main culprit for this phenomenon is the reward measure, $\text{UPM}_\gamma^{1/\gamma}$. According to the definition of coherent reward measure proposed by Rachev et al. (2008), a reward measure should be concave. We have seen that $\text{UPM}_\gamma^{1/\gamma}(0;X)$ is not concave (see Example 5.2). As result, the FT ratio is not quasi-concave. Let us see the following example.

Example 5.3. Let us choose two portfolios with their return given in the Example 5.2. We choose a target return $\tau = 6\%$. Setting $X_C = R_C - 6\%$ and $X_D = R_D - 6\%$ yields

$$\frac{\text{UPM}_\gamma^{1/\gamma}(wX_C + (1-w)X_D;0)}{\text{LPM}_\alpha^{1/\alpha}(wX_C + (1-w)X_D;0)} = 0 < \text{Min}\left\{\frac{\text{UPM}_\gamma^{1/\gamma}(X_C;0)}{\text{LPM}_\alpha^{1/\alpha}(X_C;0)}, \frac{\text{UPM}_\gamma^{1/\gamma}(X_D;0)}{\text{LPM}_\alpha^{1/\alpha}(X_D;0)}\right\}$$

for $w = \frac{1}{2}$. Hence, the FT ratio is not quasi-concave and thereby it is not ideal.

5.3 Some popular performance measures

5.3.1 MVaR ratio

VaR is the maximum expected loss on an investment that cannot be exceeded over a given time horizon at a predefined confidence level. See 1.6.4 for the mathematical explanation. It is well-known that the VaR is neither sub-additive nor convex. Thus, the mean-VaR (MVaR) ratio does not follow the axiom of quasi-concavity.

$$\text{MVaR}_i = \frac{\mathbb{E}[X_i]}{VaR_\varepsilon(X_i)}.$$

Example 5.4. X and Y are random variables with the joint distribution given below

$$(X,Y) = \begin{cases} (-1,0), & \text{Prob} = 1/4 \\ (0,-1), & \text{Prob} = 1/4 \\ (1,1), & \text{Prob} = 1/4 \\ (2,2), & \text{Prob} = 1/4. \end{cases}$$

For $\varepsilon = 0.25$, $\mathrm{VaR}_{0.25}(X) = \mathrm{VaR}_{0.25}(Y) = -1$ and $\mathrm{VaR}_{0.25}(X+Y/2) = -1/2$. Also, $\mathbb{E}(X) = \mathbb{E}(Y) = 1/2$ and $\mathbb{E}(X+Y/2) = 1/2$. Thus,

$$\mathrm{MVaR}(\tfrac{X+Y}{2}) = -1 < \min\{\mathrm{MVaR}(X), \mathrm{MVaR}(Y)\} = -1/2.$$

Thus, MVaR violates the quasi-concavity property.

5.3.2 Coherent risk based performance measure

Artzner et al. (1999) developed four axioms that a risk measure should satisfy. Risk measures satisfying those axioms are called coherent. The formal definition of coherent risk measure is given below.

Definition 5.2 (Coherent risk measure)**.** *A functional $\rho : \mathscr{G} \to \mathbb{R}$ is called coherent risk measure if the following holds:*

1. *(Monotonicity) $\rho(X) \leq \rho(Y)$ when $X \geq Y$,*
2. *(Sub-additivity) $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathscr{G}$,*
3. *(Positive homogeneity) $\rho(0) = 0$, and $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in \mathscr{G}$ and $\lambda > 0 (\in \mathbb{R})$,*
4. *(Translation invariance) $\rho(X+c) = \rho(X) - c$ for all $X \in \mathscr{G}$ and $c \in \mathbb{R}$.*

Since the coherent risk measures are convex, positively homogenous and monotonous, ratios of mean to coherent risk measures are ideal. For example, mean-CVaR (MCVaR) ratio and gain-loss (GL) ratio are ideal. Furthermore, as spectral risk measures (Acerbi, 2002) are coherent, ratios of mean to spectral risk measures are also ideal.

$$\begin{aligned} \text{MCVaR}_i &= \frac{\mathbb{E}(X_i)}{\text{CVaR}_\varepsilon(X_i)} \\ \text{GL}_i &= \frac{\mathbb{E}(X_i)}{\mathbb{E}(-X_i \mathbb{1}_{\{X_i<0\}})} \end{aligned}$$

There are several coherent risk based performance measures which are not ideal. For example, Rachev ratio (RR), Generalized Rachev ratio (GRR) are not ideal.

$$\begin{aligned} \text{RR} &= \frac{\text{CVaR}_{\varepsilon_1}(-X_i)}{\text{CVaR}_{\varepsilon_2}(X_i)} \\ \text{GRR} &= \frac{\text{CVaR}_{\gamma,\varepsilon_1}(-X_i)}{\text{CVaR}_{\alpha,\varepsilon_2}(X_i)} \end{aligned}$$

Here, $\text{CVaR}_\varepsilon(X) = \mathbb{E}[-X \mid -X > \text{VaR}_\varepsilon(X)]$ and $\text{CVaR}_{\alpha,\varepsilon_1}(X) = \mathbb{E}[(\max\{-X,0\})^\alpha \mid -X > \text{VaR}_{\varepsilon_1}(X)]$. For more explanations see Biglova et al. (2004). The RR satisfy all the axioms expect the quasi-concavity. On the other hand, GRR satisfies only the monotonicity. Thus, both the measures are not quasi-concave.

5.3.3 Deviation risk based performance measure

Rockafellar et al. (2006) proposed some axioms for deviation type risk measures. The deviation risk measure is a generalized concept of standard deviation.

Definition 5.3 (Deviation risk measure)**.** *A functional $\mathfrak{D} : \mathscr{G} \to [0,\infty]$ is called deviation risk measure if the following holds:*

1. *(Non-negativity) $\mathfrak{D}(X) \geq 0$ for all X and $\mathfrak{D}(X) > 0$ for nonconstant X.*
2. *(Sub-additivity) $\mathfrak{D}(X+Y) \leq \mathfrak{D}(X) + \mathfrak{D}(Y)$ for all $X, Y \in \mathscr{G}$,*
3. *(Positive homogeneity) $\mathfrak{D}(0) = 0$, and $\mathfrak{D}(\lambda X) = \lambda \mathfrak{D}(X)$ for all $X \in \mathscr{G}$ and $\lambda > 0 (\in \mathbb{R})$,*
4. *(Translation invariance) $\mathfrak{D}(X+c) = \mathfrak{D}(X)$ for all $X \in \mathscr{G}$ and $c \in \mathbb{R}$.*

Ratios of mean to deviation risk measures are not ideal because they fail to satisfy Axiom 2. For example, Sharpe ratio, mean-semi deviation ratio, mean-MAD (mean absolute deviation) ratio are not ideal.

$$\text{Sharpe ratio} = \frac{\mathbb{E}(X_i)}{\sqrt{\mathbb{E}[(X_i - \mathbb{E}(X_i))^2]}}$$

$$\text{Mean-MAD ratio } = \frac{\mathbb{E}(X_i)}{\mathbb{E}|X_i - \mathbb{E}(X_i)|}$$

$$\text{Mean-semi deviation ratio } = \frac{\mathbb{E}(X_i) - r_f}{\sqrt{\mathbb{E}[(X_i - \mathbb{E}(X_i))^2 \mathbb{1}_{\{X_i < \mathbb{E}(X_i)\}}]}}$$

5.3.4 Drawdown based performance measure

Drawdown measures downturn of an investment from a peak to trough over a specific time period. There are several determinants of drawdown risk measures. For example, average drawdown, maximum drawdown, conditional expected drawdown (Zabarankin et al., 2014; Goldberg and Mahmoud, 2017).

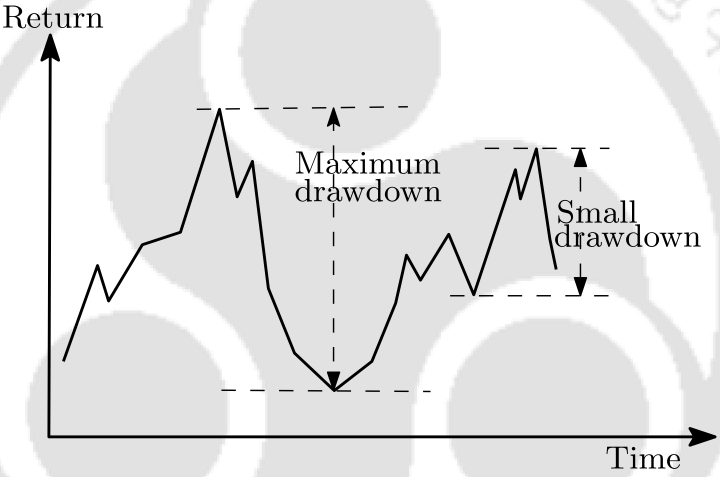


Figure 5.3: Drawdown measure

$$DD_T(X) = \max_{s \in [0,T]} X(s) - X(T)$$

Drawdown risk measures are generally not monotonous, i.e., for any $X, Y \in \mathscr{G}$, $X \geq Y$ a.s. does not always imply $DD(X) \leq DD(Y)$. Thus, mean to drawdown risk ratio fails to satisfy Axiom 2.

$$\text{Clamar ratio } = \frac{\mathbb{E}(X_i)}{DD_T(X_i)}$$

5.3.5 Rachev ratio

There are several coherent risk based performance measures which are not ideal. For example, Rachev ratio (RR), Generalized Rachev ratio (GRR) (Biglova et al., 2004) are not ideal.

$$\begin{aligned} \text{RR} &= \frac{CVaR_{\alpha}(-X_i)}{CVaR_{\beta}(X_i)} \\ \text{GRR} &= \frac{CVaR_{\gamma,\alpha}(-X_i)}{CVaR_{\delta,\beta}(X_i)} \end{aligned}$$

The RR satisfy all the axioms expect the quasi-concavity. On the other hand, GRR satisfies only the monotonicity. Thus, both the measures violates quasi-concavity.

5.3.6 Treynor ratio

The Treynor ratio is given by

$$T_i = \frac{\mathbb{E}(X_i)}{\beta_i},$$

where $\beta_i = \frac{Cov(X_i,X_m)}{\sigma_m^2}$. Since β violates monotonicity property of risk measure, this ratio does not satisfy the Axiom 2. Hence, it is not ideal. However, the Treynor ratio always satisfies Axiom 1 and Axiom 3.

EXAMPLE 5.5. Let X_1, X_2 and M be three random variables with the joint distribution given as:

$$(X_1,X_2,M) = \begin{cases} (2,4,0), & \text{Prob} = 1/3 \\ (3,8,3), & \text{Prob} = 1/3 \\ (3,12,-1), & \text{Prob} = 1/3. \end{cases}$$

It is easy to observe that: $X_2 \geq X_1$ a.s.; $\mathbb{E}(X_1) = 2.67$ and $\beta_{X_1} = \frac{\text{Cov}(X_1,M)}{\sigma_M^2} = 0.08$; $\mathbb{E}(Y_1) = 8$ and $\beta_{X_2} = \frac{\text{Cov}(X_2,M)}{\sigma_M^2} = -0.46$. The ratios T_{X_1} and T_{X_2} are given by

$$T_1 = 34.67 > T_2 = -17.33.$$

Hence, $X_2 \geq X_1$ a.s. does not imply $T_1 \geq T_2$.

5.4 Concluding remarks

In this chapter, we have demonstrated that a performance measure should have three desirable properties– quasi-concavity, monotonicity and size invariance. The practical relevance of the axioms are also explained. A measure satisfying these properties is called ideal. We observed that the FT ratio does not follow the most important property, quasi-concavity. It means that the FT ratio does not encourage diversification. This is a serious drawback for any performance measure. Various other measures fail to satisfy all the properties of ideal performance measure. Sharpe ratio, drawdown based ratios, deviation measures based ratios and Treynor ratio violate monotonicity. On the other hand, Mean-VAR ratio, Rachev ratio do not satisfy the quasi-concavity property. However, there are some ratios that are ideal. For example, Kappa ratio, mean-CVaR ratio and mini-max ratio are ideal performance measure.

Table 5.1: Formulas of performance measures

Performance ratio	Reward measure	Risk measure	Axiom 1	Axiom 2	Axiom 3
Sharpe	Mean	Std.dev	✓	×	✓
Sortino	Mean	LPM_2	✓	✓	✓
UPR	UPM_1	LPM_2	×	✓	✓
FT	UPM	LPM	×	✓	✓
Mean-CVaR	Mean	CVaR	✓	✓	✓
Mean-VaR	Mean	VaR	×	✓	✓
Treynor	Mean	Beta	✓	×	✓
Gini	Mean	Gini	×	✓	✓
Rachev	$\text{CVaR}_{\alpha}(-X)$	$\text{CVaR}_{\beta}(X)$	×	✓	✓
Generalized Rachev	$\text{CVaR}_{\gamma,\alpha}(-X)$	$\text{CVaR}_{\delta,\beta}(X)$	×	✓	×
Mean-MAD	Mean	MAD	✓	×	✓
Calmar	Mean	MDD	✓	×	✓

Chapter 6

Upside Beta Ratio

In the previous Chapter, it is shown that an ideal performance measure should satisfy three desirable properties, quasi-concavity, monotonicity and size invariance. The Kappa ratio is an ideal performance measure, but the FT ratio is not an ideal measure as it fails to satisfy the very first criteria, the quasi-concavity. The FT ratio is suitable for the investors who seek upside potential rather than the expected return. The primary goal of this ratio is not just to earn threshold return but to beat the threshold by good margin. Without sacrificing this essence, we propose a new performance measure, upside beta ratio (UBR), which is ideal (in the sense of the defined properties). We also analyse its performance through an empirical investigation. In the empirical work, we use monthly returns of 40 US mutual funds and rank them for UBR and other four performance measures (Sharpe, Sortino, FT and Jensen's *alpha*). We use Jensen's *alpha* in the empirical analysis because it is widely used in industry for observing high potential financial assets. We present empirical comparison between the rankings of UBR and other measures. Moreover, in order to see the actual performance, the top-ranked funds are compared through in-sample and out-of-sample analysis. The empirical results suggest that funds chosen by UBR perform better than the other top-ranked funds in most scenarios. It is also important to see how different choices of parameter affect the rankings produced by the new performance measure. Thus, a sensitivity analysis of UBR is also conducted.

The chapter is structured as follows. Section 6.1 presents the drawbacks of beta as a measure of risk. In section 6.2, we develop a new ideal performance measure (UBR) and describe its analytical properties. Section 6.3 comprises an empirical study, where descriptive statistics for 40 mutual funds data are reported and then rankings of funds for five different performance measures are presented. We also discuss the ranking differences and then compare the performance of top-ranked funds. In addition, we present a parameter sensitivity analysis of the UBR. Section 6.4 concludes this chapter.

6.1 The beta

Beta of an asset i is defined as

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}. \tag{6.1}$$

The beta measures volatility of a stock or a portfolio in comparison to the overall market return. It depicts how much a portfolio grows or falls with the market movements. According to the theoretical definition, portfolios with $\beta_i > 1$ are more volatile and hence tend to grow or fall faster than the market movements. On the other hand, portfolios with $\beta_i < 1$ are less volatile and hence they might move slower than the market. Thus, the assets with a higher beta are considered more risky than the assets with a lower beta.

The main issue with the beta is that it treats both the upside and the downside sensitivities identically. According to the beta, the magnitude of gain during market upswing will be same as the magnitude of loss during market downturn. However, it rarely happens in reality. Thus, the beta does not give any additional value to upside potential of asset returns.

The beta can be expressed as

$$\beta_i = \lambda_1 \beta_i^- + \lambda_2 \beta_i^+, \tag{6.2}$$

where

$$\text{downside beta: } \beta_i^- = \frac{\mathbb{E}[(\mu_i - R_i)(\mu_m - R_m)\mathbb{1}_{\{R_m<\mu_m\}}]}{\mathbb{E}[(\mu_m - R_m)^2\mathbb{1}_{\{R_m<\mu_m\}}]},$$

$$\text{upside beta: } \beta_i^+ = \frac{\mathbb{E}[(R_i - \mu_i)(R_m - \mu_m)\mathbb{1}_{\{R_m>\mu_m\}}]}{\mathbb{E}[(R_m - \mu_m)^2\mathbb{1}_{\{R_m>\mu_m\}}]},$$

$$\lambda_1 = \frac{\mathbb{E}[(\mu_m - R_m)^2\mathbb{1}_{\{R_m<\mu_m\}}]}{\sigma_m^2} > 0$$

and

$$\lambda_2 = \frac{\mathbb{E}[(R_m - \mu_m)^2\mathbb{1}_{\{R_m>\mu_m\}}]}{\sigma_m^2} > 0.$$

The upside beta measures the potential of an asset when market grows and the downside beta captures the downside sensitivity during market downturn. Therefore, the higher upside beta implies higher reward and the lower downside beta indicates lower risk. Suppose, portfolio A with high upside volatility (β_A^+) and portfolio B with high downside volatility (β_B^-) produce their respective betas as

β_A and β_B such that $\beta_A > \beta_B$. According to the traditional theory, asset A is riskier than B. However, in reality, asset A has less risk because investors are mainly concerned about the downside movements rather than the upside deviations. Thus, to know the real picture, investors should look into both the upside and downside betas. In order to illustrate this point, we present the following example.

EXAMPLE 6.1. Random returns of portfolio A, portfolio B and market portfolio m, and their joint probabilities are given as follows:

Table 6.1: Returns and joint distribution of three portfolios

R_A	R_B	R_m	$\mathbb{P}_{R_A,R_B,R_m}$
-4%	-10%	-4%	0.2
$+7\%$	$+3\%$	$+8\%$	0.2
-3%	$+2\%$	$+5\%$	0.2
$+2\%$	$+1\%$	$+7\%$	0.2
$+15\%$	$+1\%$	$+12\%$	0.2

Beta, upside beta and downside beta of A and B are given below:

$$\beta_A = 1.1, \beta_B = 0.77;$$

$$\beta_A^+ = 1.6, \beta_B^+ = 0.43;$$

$$\beta_A^- = 0.8, \beta_B^- = 0.95.$$

Although $\beta_A > \beta_B$, portfolio A is less risky than portfolio B. This is because $\beta_A^+ > \beta_B^+$ and $\beta_A^- < \beta_B^-$.

Therefore, high beta does not necessarily stipulate higher risk. Similarly, it can be also shown that low beta does not imply lower risk. Thus, the traditional concept, that high beta indicates higher risk with higher reward and low beta displays lower risk with lower reward, is not purely correct. The higher reward can be achieved from a low beta asset if upside beta of the asset is high. Hence, investors seeking high upside potential should use upside beta instead of usual beta.

6.2 Upside beta ratio

As most investors prefer return above their own threshold level or the MAR (say τ), we use τ instead of μ as a benchmark to determine the upside beta. The benchmark portfolio (BP) can be any well diversified portfolio. It is not necessary to use market index as a benchmark. Now the γ-th degree target upside beta (hereafter, upside beta) of R_i can be defined as

$$\beta^+(\gamma,\tau;R_i) = \frac{\mathbb{E}[(R_i-\tau)(R_b-\tau)^{\gamma-1}\mathbb{1}_{\{R_b>\tau\}}]}{\mathrm{UPM}_\gamma(\tau;R_b)}, \tag{6.3}$$

where R_b denotes the returns of BP.

The upside beta describes how a stock behaves when BP grows beyond τ (or when excess market return is positive). Suppose that BP is the market portfolio and $\beta^+(\gamma,\tau;R_A) = 1.5$. This means every 1% growth in the market return beyond τ, portfolio A grows 1.5%. An asset with higher upside beta grows faster during market upswing and generates larger returns. Thus, the potential-seeking investors should allocate funds in high upside beta portfolios to maximize their gain. Figure 6.1 shows the monthly return of Starbucks Corp. and S&P 500. The upside beta of Starbucks Corp. is very high (1.53). Thus, it generates high returns (upto 30% per month) with average return of 2.6% per month. Therefore, it beats both the threshold return 0.5% (per month) and the market's mean return 1% (per month) by high margin.

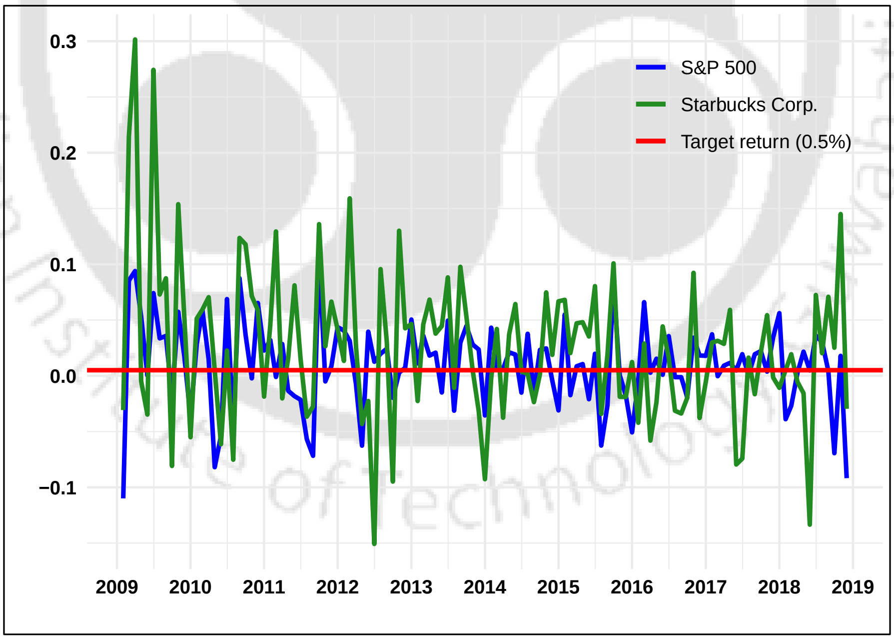


Figure 6.1: Monthly returns of S&P 500 and Starbucks Corp.

Table 6.2: Betas, upside betas and mean returns of S&P 500 and Starbucks Corp.

Asset	Beta	Upside beta	Mean return
S&P 500	1	1	1%
Starbucks Corp.	0.94	1.53	2.6%

To see the actual growth of the assets over the same time period, initial value adjusted monthly prices[1] are plotted in Figure 6.2. As we can see, price of Starbucks Corp. grows substantially faster than the price of market given that their initial prices are same (1 USD).

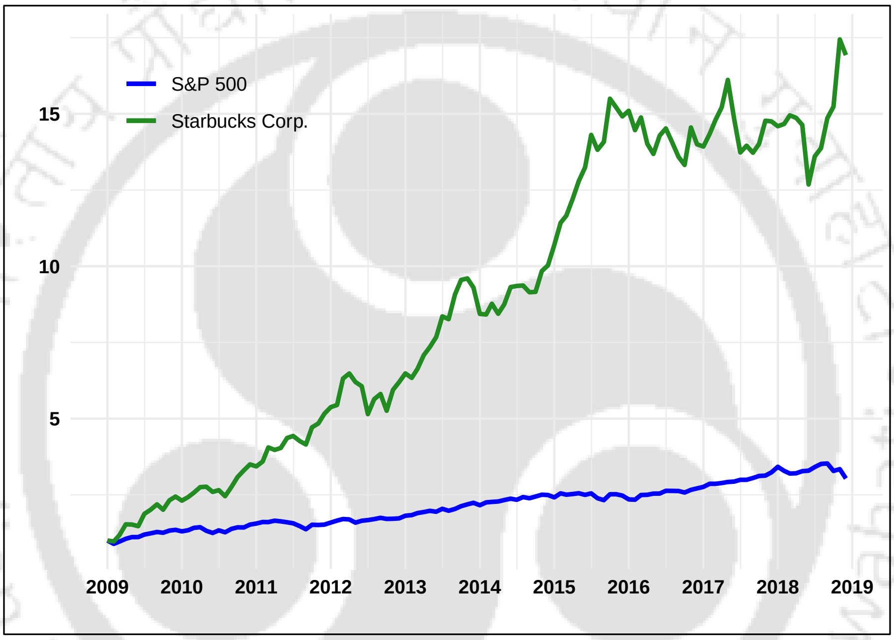


Figure 6.2: Initial value adjusted monthly prices S&P 500 and Starbucks Corp.

The upside beta of X_i $(= R_i - \tau)$ can be written as

$$\beta^+(\gamma,0,X_i) = \frac{\mathbb{E}[X_i(X_b)^{\gamma-1}\mathbb{1}_{\{X_b>0\}}]}{\mathrm{UPM}_\gamma(0;X_b)}. \tag{6.4}$$

As a measure of reward, $\beta^+(\gamma,0;X)$ satisfies the following properties:

(i). (Linearity) For any X_i and X_j, $\beta^+(\gamma;0,X_i+X_j) = \beta^+(\gamma,0;X_i)+\beta^+(\gamma,0;X_j)$;

(ii). (Positive homogeneity) For any X_i and $\lambda > 0$, $\beta^+(\gamma,0;\lambda X_i) = \lambda\beta^+(\gamma,0;X_i)$;

---

[1] Initial value adjusted price of an asset i is defined as the price $P_i(t)$ divided by its initial price $P_i(t_0)$, i.e., $\frac{P_i(t)}{P_i(t_0)}$ for all $t \in [t_0,t_T]$. Hence, prices of all the assets are same at initial time point. It gives better picture to compare the growths of multiple assets.

(iii). (Monotonicity) For any X_i and X_j, $X_i \geq X_j$ a.s. $\implies \beta^+(\gamma,0;X_i) \geq \beta^+(\gamma,0;X_j)$.

Since upside beta is a linear functional of X, it is more robust to the outliers of X than the UPM, especially when $\gamma > 1$.

Upside beta ratio: A risk-averse and potential-seeking investor should choose optimal portfolio by maximizing β^+ and minimize $\text{LPM}_\alpha^{1/\alpha}$. Ranking of funds based on this behaviour can be done by using the upside beta ratio (UBR):

$$\text{UBR}_i(\alpha,\gamma,\tau) = \frac{\beta^+(\gamma,0,X_i)}{\text{LPM}_\alpha^{1/\alpha}(0;X_i)}. \tag{6.5}$$

Using the properties of $\beta^+(\gamma,0,X_i)$ and $\text{LPM}_\alpha^{1/\alpha}(0;X_i)$, we can easily show that the UBR satisfies the axioms of quasi-concavity, monotonicity and size invariance, as described in Chapter 5. Hence, the UBR is an ideal performance measure. Therefore, the UBR is theoretically superior to the FT ratio. Now we need to see how the UBR performs in the practical world of investment. In the sequel, we examine the performance of UBR in comparison to other performance measures.

6.3 Empirical analysis

We have collected adjusted (for both dividends and splits) monthly closing prices of 40 US mutual funds over a 10-year period starting from January 2009 to December 2018. The descriptive statistics and other information are provided in Table 3.3 of Chapter 4.

6.3.1 Ranking of funds

We rank all the funds for comparison of five different performance measures: Sharpe, Sortino, FT, UBR and Jensen's alpha (J-*alpha*). Using the historical returns $R_{i1}, R_{i2}, \ldots, R_{iT}$ of an asset i, the ex-post formulas of all the measures are presented below:

Table 6.3: Ex-post formulas of performance measures

Performance measure	Ex-post formula
Sharpe	$\frac{\overline{R_i}-r_f}{\sigma_i}$
Sortino	$\frac{\overline{R_i}-\tau}{\left(\frac{1}{T}\sum_{t=1}^{T}(max[R_{it}-\tau,0])^{\alpha}\right)^{\frac{1}{\alpha}}}$
FT	$\frac{\left(\frac{1}{T}\sum_{t=1}^{T}(max[R_{it}-\tau,0])^{\gamma}\right)^{\frac{1}{\gamma}}}{\left(\frac{1}{T}\sum_{t=1}^{T}(max[\tau-R_{it},0])^{\alpha}\right)^{\frac{1}{\alpha}}},$
UBR	$\frac{\frac{\sum_{t=1}^{T}[(R_{it}-\tau)(max[R_{mt}-\tau,0])^{\gamma-1}]}{\sum_{t=1}^{T}(max[R_{mt}-\tau,0])^{\gamma}}}{\left(\frac{1}{T}\sum_{t=1}^{T}(max[\tau-R_{it},0])^{\alpha}\right)^{\frac{1}{\alpha}}}$
J-*alpha*	$(\overline{R_i}-r_f)-\beta_i(\overline{R_m}-r_f)$

$\overline{R_i}=\frac{1}{T}\sum_{t=1}^{T}R_{it}$, r_f = risk-free rate

The rankings based on all the measures are displayed in Table 6.4. For determining the Sharpe ratio and J-*alpha*, we set the risk-free monthly rate as 0.2%. This corresponds to the 10-year US treasury rate 2.67% (per year) at 31st December 2018. Since investors allocate their funds in risky portfolio to earn higher return than the risk-free rate, we fix MAR as $\tau = 0.5\%$ (per month). To determine the upside beta and usual beta, we choose S&P 500 as benchmark portfolio. We also assume $\alpha = 2$ and $\gamma = 2$ as these values capture the rational behaviour, downside risk-averse and upside potential-seeking. It is also be noted that the Kappa ratio is identical to Sortino ratio for $\alpha = 2$.

Table 6.4: Rankings of funds

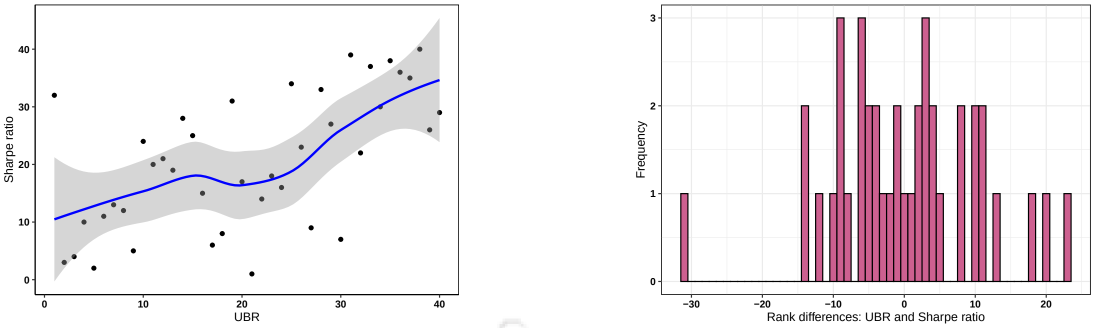
Funds	Sharpe	Sortino	FT	UBR	J-*alpha*
AB Dis. Gr. A	30	20	32	34	18
AllianzGI Mid-Cap Inst.	39	29	23	31	29
Amana Gr. Inv	16	16	4	24	9
American Beacon Lar Cap Value Instl	33	36	35	28	35
American Century Lar Com Value A	19	33	30	13	33
Ariel Fund Inv Class	32	3	1	1	22
Ave Maria Gr	22	23	15	32	13
Baird MidCap Inst	10	1	2	4	6
Baron Fifth Avenue Gr Retail	11	4	3	6	10
BlackRock Mid-Cap Gr Eq Inv A	27	11	10	29	2
BNY Mellon Lar Cap Stock M	12	22	14	8	24
Carillon Eagle Mid Cap Gr A	21	9	9	12	12
Champlain Mid Cap Adv	9	10	17	27	3
Columbia Disciplined Core A	5	13	13	9	20
Commerce Gr	8	7	11	18	1
Delaware Mid Cap Value A	25	27	27	15	32
DFA US Sustainability Core 1	4	6	6	3	19
Dodge & Cox Stock	20	19	18	11	25
Dreyfus Lar Cap Equity I	18	24	38	23	28
Federated Max-Cap Index R	17	21	24	20	14
First Eagle Fund of America C	40	39	33	38	40
Gabelli Utilities AAA	29	40	40	40	39
GMO Quality IV	26	31	36	39	8
Goldman Sachs Large Cp Val Insghts Instl	15	25	34	16	30
Harbor Capital Appreciation Inv	23	17	8	26	16
Hartford Core Equity A	2	5	12	5	15
Hartford MidCap A	35	30	39	37	23
JPMorgan Equity Index I	14	14	25	22	5
Kinetics Paradigm Adv A	31	26	16	19	31
Northern Lar Cap Value	24	34	19	10	37
Pax ESG Beta Quality Individual Inv	13	15	5	7	21
Pear Tree Quality Ordinary	7	18	22	30	4
Pioneer Global Equity Y	28	38	21	14	38
Principal MidCap A	6	12	28	17	11
Sound Shore Investor	36	37	37	36	36
USAA Aggressive Growth	38	28	29	35	26
Value Line Mid Cap Focused	1	8	20	21	7
Vanguard Total Stock Mkt Idx Inv	3	2	7	2	17
Wells Fargo Opportunity A	37	32	26	33	27
Wilshire Large Company Value Instl	34	35	31	25	34

In order to compare the rankings of UBR with the other ratios, we present Spearman's rank correlation coefficients in Table 6.5. It is seen that rankings based on UBR exhibit moderately high correlation with the rankings of Sharpe, Sortino and FT ratios, and low correlation with the rankings of J-*alpha*. Hence, the rankings of UBR is significantly different from the rankings of J-*alpha*. To depict the ranking differences more clearly, descriptive statistics (min, max and mean absolute deviation (MAD)) of ranking differences are determined (see Table 6.5). As we can see, one fund moves down 18 (31, 24 and 27) places and another fund moves up 20 (23, 18 and 31) places while using the UBR in place of FT ratio (Sharpe, Sortino and J-*alpha*). On average one fund moves 7 (8, 8 and 13) places in the rankings while using UBR instead of FT ratio (Sharpe, Sortino and J-*alpha*).

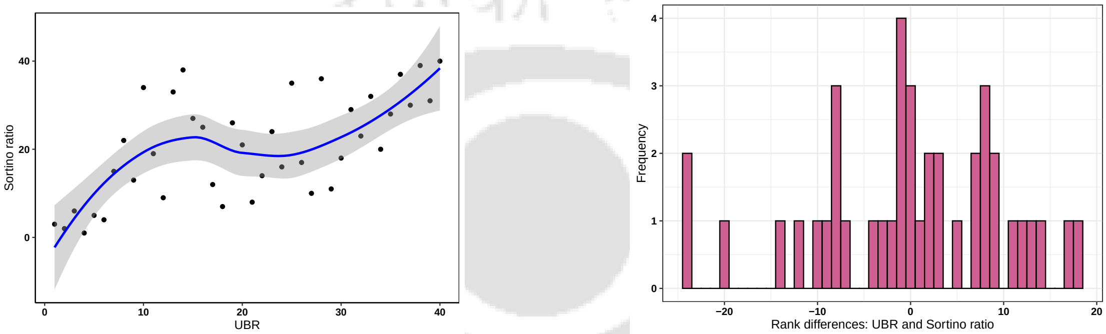
Table 6.5: Spearman rank correlations; descriptive statistics for ranking differences

Pairs of performance measure	Rank correlations	Ranking differences		
		Min	Max	MAD
UBR−Sharpe	0.607	-31	23	8
UBR−Sortino	0.616	-24	18	8
UBR−FT	0.686	-18	20	7
UBR−J-*alpha*	0.123	-27	31	13

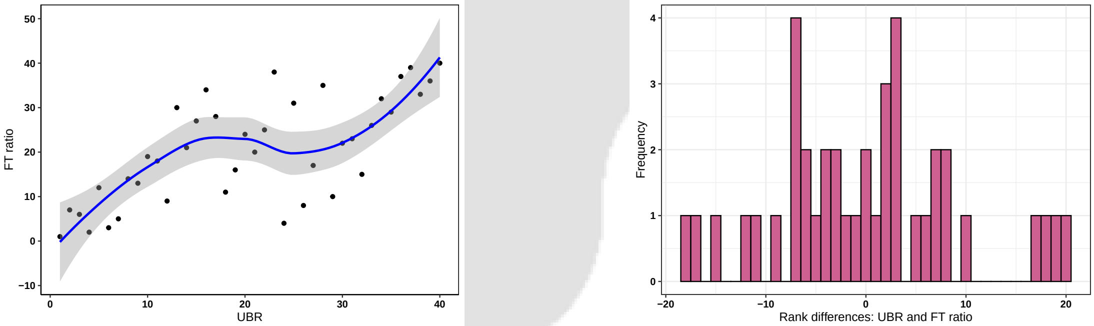
Figure 6.3 illustrates a more detailed picture about the ranking differences. We present scatter plots of the rankings of UBR and other ratios. The histograms of corresponding ranking differences are also displayed. The less the differences, the more the pairs lie on straight lines and the taller the zero-peak of corresponding histograms. It is clearly seen that the UBR−J-*alpha* scatter plot depict lower correlations. Thus, unlike the other histograms, the UBR−J-*alpha* histogram has short zero-peak.



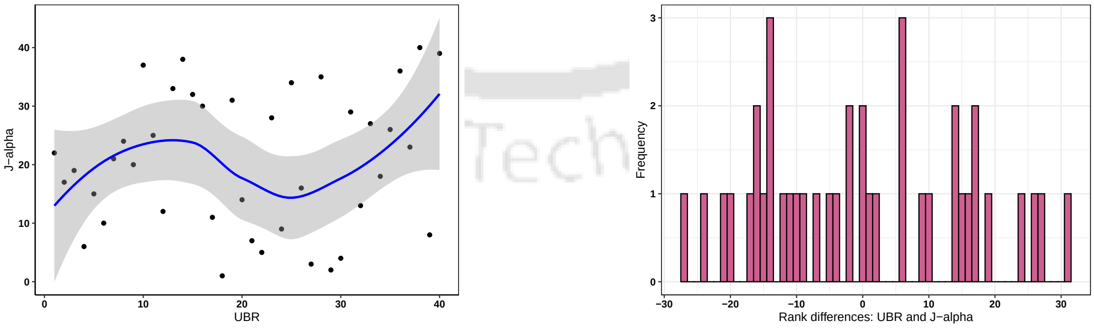
(a) UBR and Sharpe (b) UBR and Sharpe



(c) UBR and Sortino (d) UBR and Sortino



(e) UBR and FT (f) UBR and FT



(g) UBR and J-*alpha* (h) UBR and J-*alpha*

Figure 6.3: Scatter plots of the rankings based on UBR and other ratios; histograms of corresponding ranking differences

6.3.2 In-sample performance analysis

Investors are mainly interested in identifying best portfolios to allocate their wealth. We thus need to compare historical performance of the best funds identified by the performance measures. According to UBR and FT ratio, ARIEL[2] is the best fund. On other side, the Sharpe, Sortino, and J-*alpha* choose VALUE LINE, BAIRD MID, COMMERCE, respectively. Initial value adjusted monthly prices ($) of the top ranked funds and the market portfolio (S&P 500) are depicted in Figure 6.4. Since, upside beta of ARIEL is the highest ($\beta^+ = 1.61$), it grows faster compare to the others. Due to less upside beta ($\beta^+ = 0.81$), growth of VALUE LINE is low.

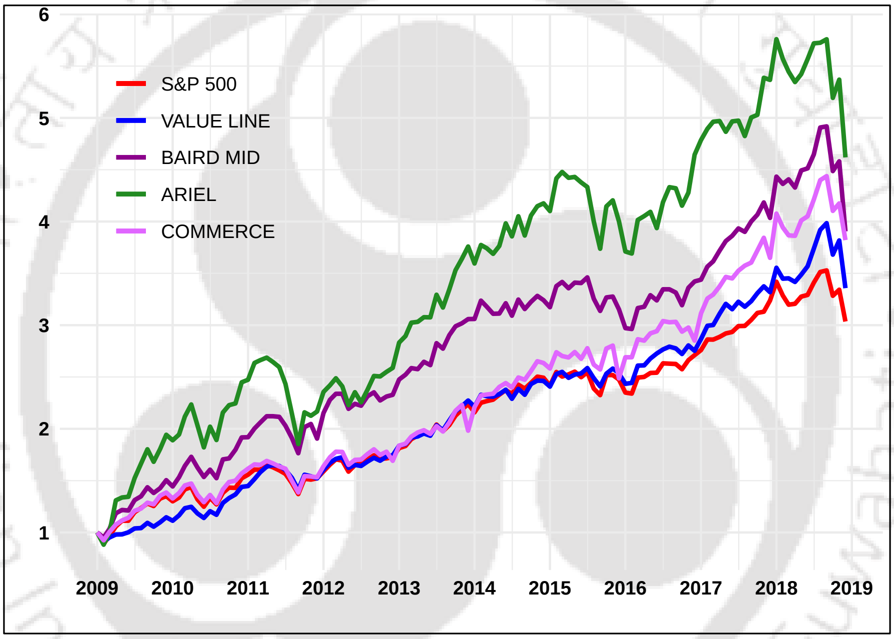


Figure 6.4: Initial value adjusted monthly prices of S&P 500 and top funds

Moreover, ARIEL has highest monthly expected return (see Table 6.6), which is also greater than the market expected return. As the ARIEL has highest upside beta as well as highest UPM, both FT and UBR select it as best fund. On the other side, the monthly expected return of the top fund chosen by Sharpe ratio is lowest.

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[2] ARIEL = Ariel Fund Inv Class, BAIRD MID = Baird Mid Cap Inst, COMMERCE = Commerce Gr, VALUE LINE = Value line Mid Cap Focused

Table 6.6: Mean return, LPM, UPM and upside beta of S&P 500 and top funds

Funds	$\mathbb{E}(R)$	$\mathrm{LPM}_2^{1/2}(0.5\%;R)$	$\mathrm{UPM}_2^{1/2}(0.5\%;R)$	$\beta^+(0.5\%,2;R)$
S&P 500	0.010	0.026	0.029	1.00
VALUE LINE	0.011	0.025	0.027	0.81
BAIRD MID	0.013	0.029	0.037	1.13
ARIEL	0.015	0.037	0.052	1.61
COMMERCE	0.012	0.032	0.034	0.99

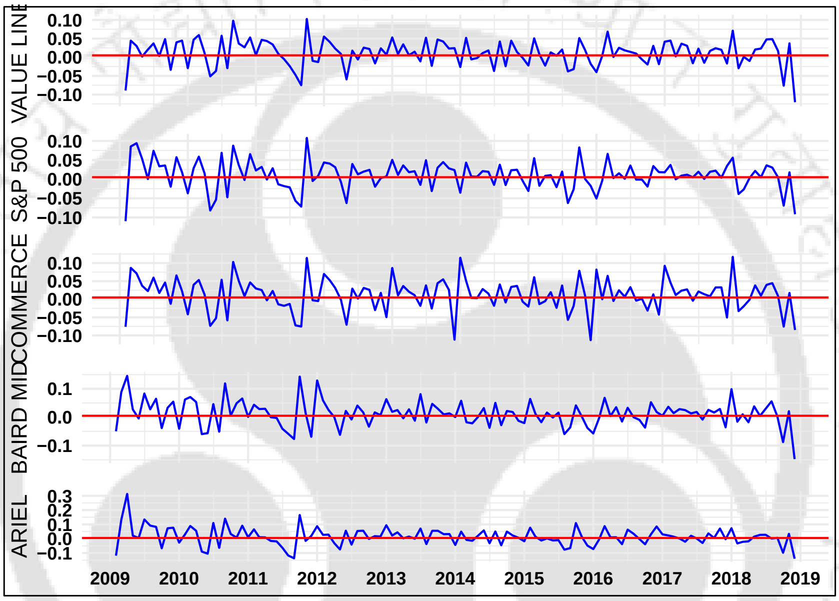

Figure 6.5: Monthly returns of S&P 500 and top funds

In order to observe whether any outliers affect the ratio estimations, monthly returns of the top funds and the corresponding box plots are displayed in Figure 6.5 and Figure 6.6, respectively. As we can notice, there are no high magnitude outliers that can badly affect the estimations.

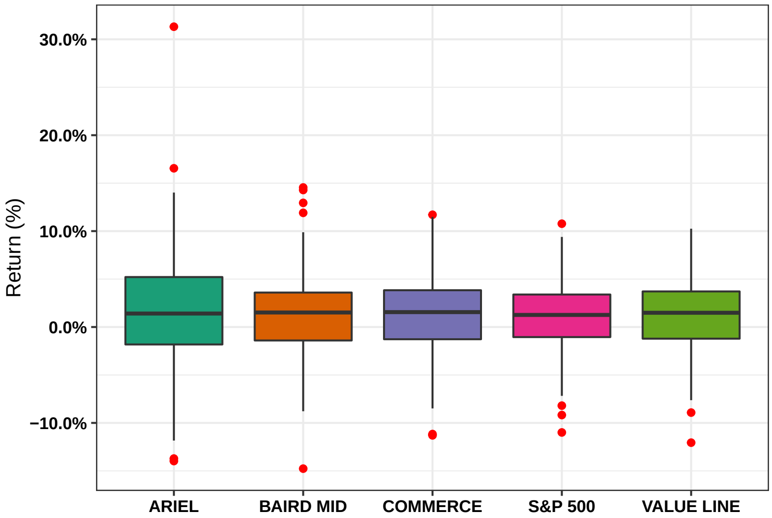

Figure 6.6: Box plots for monthly returns of S&P 500 and top funds

To know the actual potential of an asset, we need to see the upside tail areas of density function of the asset returns (see Figure 6.7). The higher the area, the fatter the tail is. The fat upside tail indicates high potential of generating larger returns. Table 6.7 summarizes the tail areas beyond three different target levels (5%, 8%, 10%). We see that ARIEL has the highest tail areas in all the three levels. It indicates that ARIEL has greater potential than the others to generate larger returns.

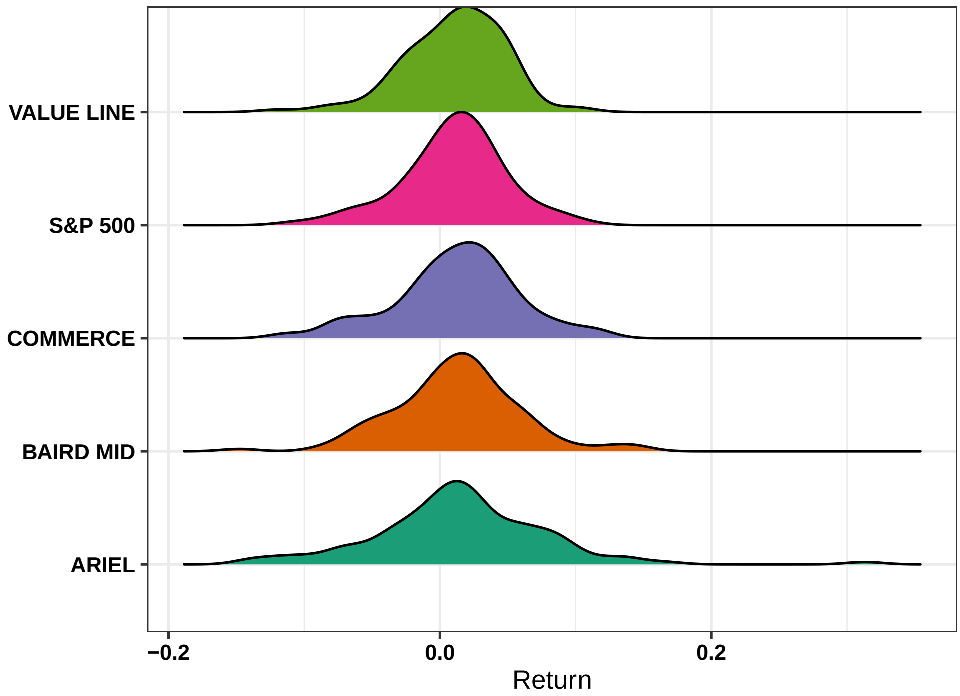

Figure 6.7: Probability density functions of S&P 500 and top funds

Table 6.7: Upside tail areas of S&P 500 and top funds

Funds	$\mathbb{P}(R > 5\%)$	$\mathbb{P}(R > 8\%)$	$\mathbb{P}(R > 10\%)$
S&P 500	0.126	0.042	0.008
VALUE LINE	0.109	0.017	0.008
BAIRD MID	0.185	0.067	0.034
ARIEL	0.261	0.126	0.059
COMMERCE	0.176	0.067	0.034

Table 6.8 reports 3-month, 6-month, 1-year, 3-year, 5-year expected rolling returns of the top ranked funds and the market portfolio. It is seen that ARIEL has outperformed all the other funds as well as market portfolio in all scenarios. Thus, all the above empirical evidences reveal that the best fund according to UBR and FT has outperformed all the other top funds.

Table 6.8: Expected rolling returns (%) of S&P 500 and top funds

Funds	3-month	6-month	1-year	3-year	5-year
S&P 500	3.35	6.76	13.00	11.86	12.32
VALUE LINE	3.62	7.59	15.40	14.02	14.16
BAIRD MID	4.09	8.25	16.10	13.46	13.46
ARIEL	4.91	9.72	18.16	14.41	15.32
COMMERCE	3.91	7.94	15.38	14.14	14.62

Note: Returns upto 1 year are absolute and returns above 1 year are annualized.

6.3.3 Out-of-sample performance analysis

This section presents out-of-sample analysis to compare performance of all the performance measures. In order to do that, we use a rolling sample methodology. We set a period of 6 years (for example, 2009-2014), in which monthly data of first 3 years (2009-2011) will be used to estimate the ratios for selecting best funds. Then, we compare the performance of all the top ranked funds in subsequent 3-year period (2012-2014). We repeat this procedure for next 6-year period (2010-2015) by dropping the data of earliest year (2009). This method will be continued until the end of the data set is reached. Thus, based on the monthly returns from 2009 to 2018, we will have 5 in-sample windows (2009-2011, 2010-2012, 2011-2013, 2012-2014, 2013-2015) for the selection of top funds. Next, we will have corresponding 5 out-of-sample windows (2012-2014, 2013-2015, 2014-2016, 2015-2017, 2016-2018) for measuring the performance. This analysis is undoubtedly very useful and realistic.

Table 6.9 displays the list of top ranked funds in each period with respect to each performance measure. The corresponding out-of-sample expected returns and risks are presented in Table 6.10 and Table 6.11, respectively. The expected returns (risks) of top funds chosen by a ratio in all the periods are given in the column of that ratio. The market returns (risks) are provided in the last column.

Table 6.9: Names of top ranked funds in different in-sample time periods

Time window	Sharpe	Sortino	FT	UBR	J-*alpha*
2009-2011	GMO	ABD	ARIEL	ARIEL	ABD
2010-2012	GABELI	ABD	ABD	ABD	ABD
2011-2013	PEAR TREE	PRINCIPLE	BAIRD MID	PRINCIPLE	PRINCIPLE
2012-2014	DODGE	DODGE	HARTFORD C	HARTFORD C	COMMERCE
2013-2015	HARTFORD C	HARTFORD C	HARTFORD C	HARTFORD C	BLACK ROCK

ABD = AB Dis. Gr. A, ARIEL = Ariel Fund Inv Class, DELWARE = Delaware Mid Cap Value A, DODGE = Dodge & Cox Stock, GABELI = Gabelli Utilities AAA, GMO = GMO Quality IV, HARTFORD C = Hartford Core Equity A, PRINCIPLE = Principal MidCap A, PEAR TREE = Pear Tree Quality Ordinary

The average return in the case of UBR over all the time periods is the second highest (0.975%). It outperforms the market index (0.925%) and all the ratios except J-*alpha*. The average return in the case of FT is the third highest.

Table 6.10: Monthly expected returns of S&P 500 and top funds in out-of-sample time windows

Time window	Sharpe	Sortino	FT	UBR	J-*alpha*	S&P 500
2012-14	1.094	1.035	1.723	1.723	1.035	1.328
2013-15	0.322	0.763	0.763	0.763	0.763	0.936
2014-16	0.801	0.743	0.376	0.743	0.743	0.698
2015-17	1.021	1.021	0.897	0.897	1.077	0.880
2016-18	0.709	0.709	0.709	0.709	1.242	0.781
Average	0.789	0.854	0.894	0.967	0.972	0.925

We also notice that the average risks of Sharpe ratio and J-*alpha* are relatively higher than the others. On the other hand, the top funds according to UBR and FT exhibit lower average risks (2.501% and 2.483%). Although the funds based on Sharpe, Sortino and J-*alpha* have higher average risks than the UBR, their average returns are lower than the average return of UBR. This evidence suggests that the UBR not only provides good return but also protects investor from incurring huge losses.

Table 6.11: Monthly risk ($\text{LPM}_2^{1/2}(0.5\%;R)$) of S&P 500 and top funds in out-of-sample time windows

Time window	Sharpe	Sortino	FT	UBR	J-*alpha*	S&P 500
2012-14	3.583	2.780	2.310	2.310	2.780	1.658
2013-15	2.357	3.384	3.384	3.384	3.384	1.915
2014-16	3.395	2.473	2.377	2.473	2.473	1.953
2015-17	2.361	2.361	1.541	1.541	2.752	1.789
2016-18	2.799	2.799	2.799	2.799	3.104	2.311
Average	2.899	2.759	2.483	2.501	2.897	1.925

In Table 6.12, we report the long term (3-month, 6-month, 1-year) expected rolling returns of all the top ranked funds in the above described time horizons. The average rolling return for all the periods can also be found in the last column of the table.

Table 6.12: Expected rolling returns (%) of S&P 500 and top funds in out-of-sample time windows

Rolling Period	PM / Market	Time window					Average
		2012-14	2013-15	2014-16	2015-17	2016-18	
3-month	Sharpe	3.58	1.00	1.94	2.95	2.89	2.472
	Sortino	3.35	2.62	1.99	2.95	2.89	2.76
	FT	5.13	2.62	0.57	2.52	2.89	2.746
	UBR	5.13	2.62	1.99	2.52	2.89	3.03
	J-*alpha*	3.35	2.62	1.99	3.22	4.63	3.162
	S&P 500	3.84	2.86	1.68	2.29	2.84	2.702
6-month	Sharpe	7.23	1.96	3.98	5.72	6.66	5.11
	Sortino	7.61	5.67	4.22	5.72	6.66	5.976
	FT	11.39	5.67	1.29	4.78	6.66	5.958
	UBR	11.39	5.67	4.22	4.78	6.66	6.544
	J-*alpha*	7.61	5.67	4.22	6.55	10.81	6.972
	S&P 500	8.18	5.44	3.12	4.39	6.46	5.518
1-year	Sharpe	16.80	5.39	7.28	14.00	15.19	11.732
	Sortino	19.49	11.21	6.57	14.00	15.19	13.292
	FT	27.19	11.21	0.97	9.73	15.19	12.858
	UBR	27.19	11.21	6.57	9.73	15.19	13.978
	J-*alpha*	19.49	11.21	6.57	14.31	24.40	15.196
	S&P 500	18.50	11.79	4.23	9.79	14.21	11.704

PM = Performance measure

As we observe, the top funds according to UBR generate second highest average returns in all the time periods. On the other side, average returns generated by the top funds of Sharpe ratio are lowest in all the scenarios. The average returns of these funds are even less than the market's average returns in 3-month and 6-month periods. Although, the average out-sample returns for J-*alpha* are the highest, the corresponding average risks are much higher than the average risk of UBR based funds. Hence, the return per unit of risk for UBR is much higher. Thus, from the outcomes of out-of-sample analysis, we can infer that the UBR has consistently performed well in all the scenarios.

6.3.4 Sensitivity analysis

In the above sections, we have presented the empirical analysis by choosing $\alpha = 2$ and $\gamma = 2$. However, investors can select desired values for the parameters of UBR to select best funds. The choice of parameters purely depends on the risk-averse and potential-seeking attitude of the investors. Investors who do not prefer downside deviations of return, should choose $\alpha > 1$. On the other side, since most investor would like to have higher potential or higher upside deviation, the appropriate choice should be $\gamma > 1$. Furthermore, in order to reflect higher degree of risk-aversion (potential-seeking) in asset ranking, the higher values of α (γ) can be set. The higher the values of parameters, the more the investors are downside risk-averse and upside potential-seeking.

We now present sensitivity analysis of UBR with respect to the parameter γ to see how different choices of γ affect the rankings. We fix $\alpha = 2$, $\tau = 0.005$ and then define $\mathrm{U}_\gamma := \mathrm{UBR}(2,\gamma;0.005)$. We mainly investigate the variation in asset rankings produced by $\mathrm{UBR}(\alpha,\gamma;\tau)$ with respect to the variation of γ. For that, all the funds are ranked based on different UBR by varying the values of γ and then the corresponding rank correlations are computed. Table 6.13 presents the rank correlations for different UBR_γ, where $\gamma = 2,5,8,12,15,20,25,30$.

Table 6.13: Rank correlations between different UBRs

	U_2	U_5	U_8	U_{12}	U_{15}	U_{20}	U_{25}	U_{30}
U_2	1.000	0.881	0.821	0.790	0.769	0.740	0.725	0.724
U_5	0.881	1.000	0.982	0.945	0.908	0.870	0.839	0.832
U_8	0.821	0.982	1.000	0.982	0.952	0.917	0.883	0.873
U_{12}	0.790	0.945	0.982	1.000	0.991	0.970	0.945	0.937
U_{15}	0.769	0.908	0.952	0.991	1.000	0.990	0.974	0.969
U_{20}	0.740	0.870	0.917	0.970	0.990	1.000	0.992	0.989
U_{25}	0.725	0.839	0.883	0.945	0.974	0.992	1.000	0.999
U_{30}	0.724	0.832	0.873	0.937	0.969	0.989	0.999	1.000
Avg.	0.779	0.894	0.916	0.937	0.936	0.924	0.908	0.903

It is seen that all the UBRs exhibit moderate to high rank correlations amongst them. For example, the rank correlation between U_2 and others varies from 0.724 to 0.881. On average, the correlations for U_2 in relation to the other UBRs is 0.779. For the case of U_{15}, the correlation varies between 0.769 to 0.991, and the average correlation (0.936) with the other UBRs is very high. It is also noted that the average correlations for all the other UBRs are greater than 0.90. Thus, the rankings have not differed vastly with the small change of the parameter γ. Furthermore, as the values of γ differ from each other, the corresponding rank correlation decreases. For example, see the rank correlations between U_2 and others (first column of Table 6.13). As γ deviates from 2, the rank correlation with U_2 decreases.

In order to investigate the sensitivity in more detail, we compute the rank correlations between U_2 and numerous U_γ. The values of γ are now varied from 2 to 150. Fig. 6.8 illustrates the rank correlations.

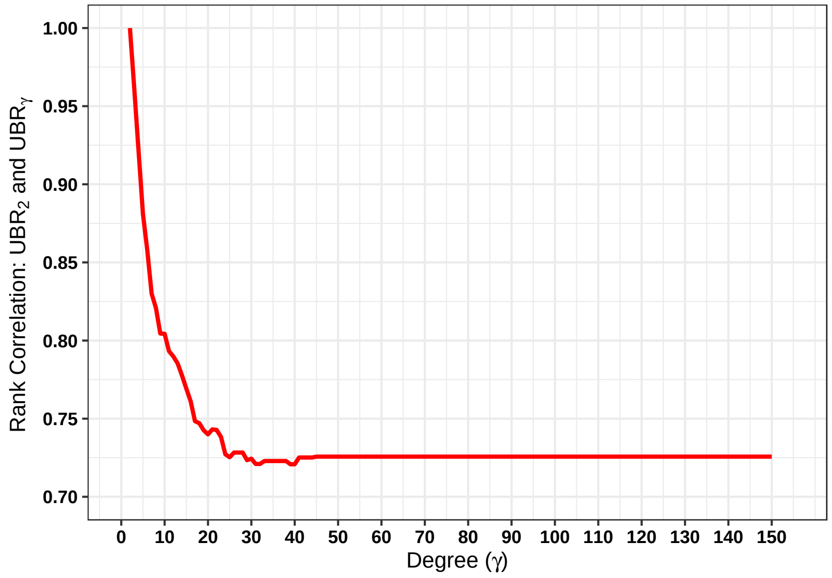

Figure 6.8: Rank correlations between U_2 and U_γ

As mentioned earlier, the correlation gradually decreases as the degree increases beyond 2. However, the correlations do not fall below 0.70. It indicates that small to moderate variation in γ has not significantly changed the rankings of UBR. Therefore, the new performance measure, UBR, is not highly sensitive to the degree of its reward measure.

6.4 Concluding remarks

In the previous chapter, we showed that FT ratio violates the quasi-concavity that resembles the idea of diversification. This is a serious drawback for any performance measure. To overcome this issue, we have developed a new performance measure, upside beta ratio (UBR). The UBR is consistent with all the proposed axioms, and hence it is an ideal performance measure. Our work is not just limited to the development of a new ideal performance measure. In order to see practical applications of the new ratio, we have conducted empirical experiments using monthly returns of 40 US mutual funds. We first examined the normality of return distribution through Jarque-Bera test. The test revealed that most of the fund's returns are different from normally distributed. We then ranked the funds based on UBR, FT, Sortino, Sharpe ratio and J-*alpha*. To compare the rankings of UBR with the rankings of other four measures, we computed rank correlations. We found that the rank correlations between UBR and J-*alpha* are lower than the other pairs. To get more clarity, minimum, maximum and MAD of corresponding ranking differences are determined. We observed that the ranking difference between UBR and J-*alpha* is highest. In order to examine the actual performances, we compared the performance of top ranked funds using back-testing and out-of-sample methodology. In the historical performances, we noticed that the best fund according to UBR has outperformed not only the market portfolio but also all the other top funds. On the other hand, the out-of-sample performance of UBR is also better than others in most scenarios. Lastly, we conduct sensitivity analysis of UBR with respect to the degree of upside beta. It is seen that the UBR is not highly sensitive to its parameter.

From the theoretical justifications and from the empirical evidences, we conclude that the UBR is a suitable performance measure for ranking risky funds. Hence, it is recommended that the investors who seek upside potential to gain larger returns should use UBR.

Chapter 7

Conclusion and Future Direction

In this thesis, we have addressed various important problems of portfolio theory with LPM. We have not only solved some interesting open problems but also developed new tools and technique that can be used in real investment world for the purpose of asset allocation. Each and every development of this thesis has its own characteristic and practical relevance. The development of generalized target helps investors to find their suitable target returns than allow the tow-fund separation. The uniqueness of this target make investors more informed so that unknowingly they will not face excess financial risk. Thus, unlike the traditional MV model, the MLPM model protects investor from incurring huge investment losses. In addition to that, two new pricing models have been developed in the MLPM framework. Both the models reduce to their MV counterparts when the probability distributions of asset's returns follow multivariate normal. One of these MLPM model is developed by abandoning an unrealistic assumption of classical CAPM, and hence this model valid under the practical scenarios. One of the most important contributions of the thesis is the characterization of performance measure. We have observed three desirable properties that a good performance measure should satisfy. A performance measure that satisfies these properties is called "ideal". This theory help individual investors, portfolio managers and others financial advisors in selecting appropriate performance measure to find good investment portfolios. We also propose a new ideal performance measure for potential-seeking investors. It is called UBR. Several empirical experiments have been conducted to depict the efficiency of the new measure. The new measure can provide investors higher return along with downside risk protection. Hopefully, our developments will not only strengthen the theoretical foundation of MLPM portfolio theory but also encourage investors and researcher to use LPM more efficiently.

The main challenge of the LPM based portfolio selection model is the analytical solution of MLPM Optimization / LPM based ratio optimization. The complexity increases as the degree of

LPM increases. It is even more difficult to solve when target does not allow TFMS. Another problem is the choice of appropriate target. It is not easy to find appropriate target return. There is also the problem of empirical validation of pricing models. The appropriate techniques are not readily available to conduct robust validation. However, a number of numerical and statistical techniques have been introduced in recent years to tackle the issues of computational complexity.

In future research, we would like to extend the usages of LPM in active portfolio management. This thesis restricts the target rate or the threshold level to a real number, more specifically, a non-random quantity. If we allow the target rate to be stochastic quantity, we can accurately measure risk that arises in active portfolio selection. An active portfolio manager forms portfolios with the goal of outperforming a benchmark portfolio, which could be a market index, or any other broadly diversified portfolio. The risk that occurs for attempting to beat the benchmark return is called active risk. The active risk is mainly measured by tracking error, the standard deviation of the difference between return on the portfolio and the return on the benchmark. Some variants of tracking error has also been used for this purpose. For example, non-central second moment return deviation and mean absolute return deviation. However, none of them are appropriate measures of active risk. The reason is that they treat both under-performance and out-performance in a same way. Thus, they fail to reflect the actual purpose of measuring active risk− penalizing only the under-performance. Also, the other classes of portfolio risk measures (for example deviation measure, coherent measure) are not suitable in this framework because these measures do not take into account the return of benchmark portfolio against which the managed portfolio will be compared. In spite of the serious drawback, no appropriate metric has been developed for measuring the active risk. Since the LPM penalizes only the downside returns, it can be utilized to measure active risk by choosing target rate as the return of benchmark portfolio. This will have a large number of applications in the area of active risk management, benchmark portfolio tracking optimization and performance analysis.

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   - Paper title: Asset pricing through capital market curve

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