GOERITZ GROUPS OF GENUS TWO AND GENUS THREE HEEGARD SPLITTING OF THE THREE SPHERE

Swapnendu Panda

Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati, India- 781039

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GOERITZ GROUPS OF GENUS TWO AND GENUS THREE HEEGARD SPLITTING OF THE THREE SPHERE

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Swapnendu Panda
Roll Number: 126123003

to the
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati, India- 781039

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DECLARATION

I do hereby declare that this thesis titled **"Goeritz Groups of Genus Two and Genus Three Heegaard Splitting of the Three Sphere"** is a presentation of my original research work done under the supervision of **Dr. P.A.S. Sree Krishna**, Assistant Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

Swapnendu Panda

January, 2020

Swapnendu Panda
Roll No. 126123003
Department of Mathematics
Indian Institute of Technology

Guwahati

CERTIFICATE

It is to certify that the work contained in this thesis titled **“Goeritz Groups of Genus Two and Genus Three Heegaard Splitting of the Three Sphere”** has been carried out by **Swapnendu Panda,** a student at the Department of Mathematics, Indian Institute of Technology Guwahati, under my supervision for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

January, 2020

Dr. P.A.S. Sree Krishna
Assistant Professor
Department of Mathematics
Indian Institute of Technology
Guwahati

Indian Institute of Technology Guwahati

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January, 2020

Swapnendu Panda
Roll No. 126123003
Department of Mathematics
Indian Institute of Technology
Guwahati

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To My Parents

Sasadhar Panda

&

Malati Panda

My Brother

Swarnendu

and My wife

Sushmita


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Abstract

The study of Goeritz groups was initiated by Goeritz during the 1930s. He introduced the concept of Goeritz groups $\mathcal{H}_g$ of genus g Heegaard splitting of S^3 and showed that $\mathcal{H}_2$ is finitely generated. Powell enriched his work via an attempt towards generalization of Goeritz's work for higher genus cases. He introduced the Powell generators. But his work had a gap. Scharlemann pointed out this gap and updated Goeritz's proof in terms of modern findings. Akbas and Cho explored the presentation of $\mathcal{H}_2$. Later Freedman and Scharlemann established the finite generation of $\mathcal{H}_3$. But the proof of the Powell conjecture i.e. '$\mathcal{H}_g$ is finitely generated' remains wide open for $g \geq 4$.

We mainly consider the genus two and genus three Heegaard splitting of S^3 and the corresponding Goeritz groups. The essence of the third chapter of this thesis is to showcase an algorithm to determine a unique word for each element in $\mathcal{H}_2$. For that, we precisely determine a word w_Q in terms of our proposed generating set $\mathcal{G}_2$, which sends an arbitrary reducing sphere Q of the genus two Heegaard splitting of S^3 to the 'standard' reducing sphere P. In the process we also present a complexity measure on the reducing spheres of genus two Heegaard splitting.

The fourth chapter mainly presents an alternative generating set for $\mathcal{H}_3$.

In the final Chapter, we discuss a few directions for future research based on the work of this thesis.

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List of Symbols & Notations

S^3 : The three sphere typically represented by $\{x \in \mathbb{R}^4 : \|x\| = 1\}$.

Σ_g : The standard unknotted genus g connected closed compact orientable surface (connected sum of g tori)

V_g : The standard unknotted genus g handlebody bounded by Σ_g in S^3.

W_g : Complement handlebody of V_g in S^3.

$\mathcal{H}_g$: The Goeritz group of genus g Heegaard splitting of S^3.

$\mathcal{G}_g$: The generating set of $\mathcal{H}_g$ for $g = 2, 3$ proposed in this work.

$Aut(\Sigma_g)$: Set of all automorphisms of Σ_g.

B^n : Open n-ball in $\mathbb{R}^n$ typically represented by $\{x \in \mathbb{R}^n : \|x - a\| < r\}$.

D^n : Closed n-ball in $\mathbb{R}^n$ typically represented by $\{x \in \mathbb{R}^n : \|x - a\| \leq r\}$.

P : The standard reducing sphere of genus two Heegaard splitting of S^3.

$\Sigma_2^{\pm}$: The two genus one summand of Σ_2 separated by a standard reducing sphere P.

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Σ_2', Σ_2''	:	The pair of thrice boundered spheres in $\Sigma_2 - (A \cup B \cup C)$, where curves A, B, C are shown in figure 3.1
c_Q	:	The simple closed essential separating curve $Q \cap \Sigma_2$ (Q is a reducing sphere) on Σ_2. This curve is also referred to as the reducing curve of the reducing sphere Q.
$N(Q, \Sigma_2^{\pm}, a)$	:	Number of arcs of slope a on the genus one summands $\Sigma_2^{\pm}$ from c_Q.

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CHAPTER 1

Preliminaries

This chapter comprises of definitions, terminologies and established results from literature that are relevant to our work. Further illustration of this information is available in [3, 7] and [10].

1.1 Definitions & Results

Definition 1.1.1 (Handlebody [7])**.** Let $B_1, \ldots, B_n$ be a collection of closed 3-balls and let $D_1, \ldots, D_m, D'_1, \ldots, D'_m$ be a collection of pairwise disjoint disks in $\bigcup\limits_i \partial B_i$. For each $i \leq m$, let $\phi_i : D_i \rightarrow D'_i$ be a homeomorphism. Let H be the result of gluing along ϕ_1, then gluing along ϕ_2, and so on. After the final gluing, if H is connected then H is a handlebody.

Example 1.1.2. Figure 1.1 represents a handlebody made out of three 3-balls glued along four pairs of disks. The genus of the handlebody is same as the genus of the bounding surface. It may be observed that if m balls are glued together by n handles then the genus of the handlebody is $n - m + 1$. Moreover, the order in which these gluing are done does not affect the resultant handlebody. Two handlebodies are homeomorphic if and only if their boundaries have the same genus.

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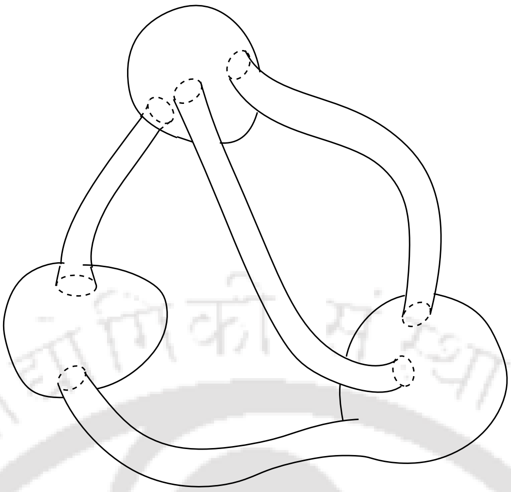

Figure 1.1: A handlebody is a collection of 3-balls, glued together along disks.

Definition 1.1.3 (Heegaard Splitting [7])**.** A Heegaard splitting for a 3-manifold M is an ordered triple (M, H_1, H_2) where H_1 and H_2 are handlebodies embedded in M such that $\partial H_1 = \Sigma = \partial H_2 = H_1 \cap H_2$ and $H_1 \cup H_2 = M$. The surface Σ is called a Heegaard surface. The Heegaard genus of M is the smallest possible genus of a Heegaard splitting of M.

Given a Heegaard surface Σ, the handlebodies are simply the closures of the components of the complement $M - \Sigma$. Thus the Heegaard surface determines the two handlebodies.

Example 1.1.4. Let $M = S^3$, and $V = V_g$ $(g = 2, 3, \cdots)$ be the standard unknotted genus g handlebody embedded in S^3 and $W = W_g$ is its closed complement in S^3. Then (S^3, V_g, W_g) denotes the genus g Heegaard splitting of S^3.

The following result is easy to prove. See [7] for example.

Theorem 1.1.5. *Every closed, connected, orientable* 3*-manifold allows a Heegaard splitting.*

Stabilization: [7, 10] Consider a genus g Heegaard splitting of a 3-manifold $M = H_1 \bigcup\limits_{\Sigma} H_2$. Consider an unknotted arc α embedded in H_2

with both ends on $\partial H_1 = \Sigma$. A connected arc α is said to be unknotted if there is a disk D embedded in H_2 such that ∂D consists of a single arc in ∂H_1 and the arc α in the interior of H_2. Let $N_1 \subset H_2$ be a closed regular neighborhood of α, as in Figure 1.2. Take $H_1' = \overline{H_1 \cup N_1}$ and $H_2' = H_2 \setminus N_1$.

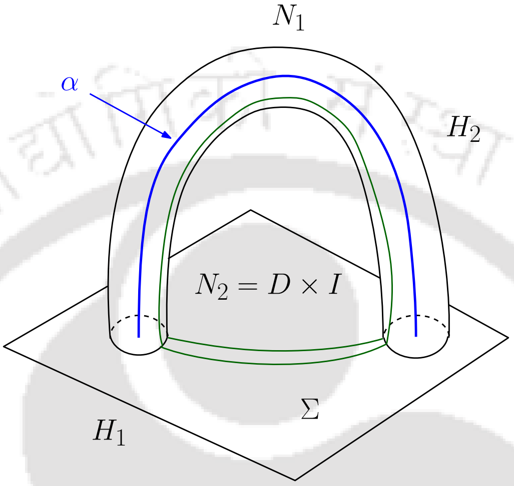

Figure 1.2: Stabilization of (M, H_1, H_2) defined by an unknotted arc α in H_1

Then $\partial H_1' = \partial H_2'$, call it Σ'. Thus $M = H_1' \cup_{\Sigma'} H_2'$ gives a genus $g + 1$ Heegaard splitting of M. Any Heegaard splitting constructed by iterating the above process one or more times, possibly reversing the roles of H_1 and H_2, is a stabilization of (M, H_1, H_2).

Definition 1.1.6 (Reducing Sphere[10])**.** Consider the Heegaard splitting (M, H_1, H_2) of a 3-manifold M. A 2-sphere Q embedded in M, that intersects $\partial H_1 = \Sigma$ transversally in a single essential circle and so intersects each handlebody in a single essential disk, is called a reducing sphere or an essential sphere.

The reducing spheres intersect the splitting surface in essential separating closed curves which bounds a disk in each handlebody, but not on the surface.

Definition 1.1.7 (Homotopy of functions)**.** Let X and Y be two topological spaces, and let f and g be continuous maps from X to Y. We shall say that f and g are homotopic if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$.

Definition 1.1.8 (Isotopy of simple closed curves on a surface)**.** Two simple closed curves α and β on a surface S are isotopic if there is a map

$$H : S \times [0,1] \to S$$

with the property that $H(S \times \{0\}) = identity, H(\alpha \times \{1\}) = \beta$ and$H(S \times \{t\})$ is a homeomorphism of S for each $t \in [0,1]$.

Definition 1.1.9 (Isotopy of surfaces in S^3)**.** Let S_1 and S_2 be two closed genus g surfaces embedded in S^3. We say S_1 and S_2 are isotopic in S^3 if there exists a continuous map $F : S^3 \times [0,1] \to S^3$ such that

$$F(S^3 \times \{0\}) = identity, F(S_1 \times \{1\}) = S_2$$

and $F(S^3 \times \{t\})$ is a homeomorphism of S^3 for all $t \in [0,1]$.

Definition 1.1.10 (Isotopy of homeomorphisms)**.** Let X be a topological space and $f, g : X \to X$ be two homeomorphisms. Then f and g are said to be isotopic if there exists a continuous map $H : X \times [0,1] \to X$ satisfying

$$H(x,0) = f(x), H(x,1) = g(x)$$

and $H(X \times \{t\})$ is a homeomorphism from X to X for all $t \in [0,1]$.

‘Is homotopic to’ and ‘is isotopic to’ are equivalence relations. In this work we have interchangably used curves, spheres or automorphisms, with their isotopy classes.

Definition 1.1.11 (Mapping Class Group[3])**.** Let S be a surface. Let us assume that S is the connected sum of $g \geq 0$ tori with $b \geq 0$ disjoint open disks removed and $n \geq 0$ points removed from the interior. Let $Homeo^+(S, \partial S)$ denote the group of orientation-preserving homeomorphisms of S that restrict to the identity on ∂S.
The mapping class group of S, denoted by $Mod(S)$, is the group of isotopy classes of elements of $Homeo^+(S, \partial S)$. Elements of $Mod(S)$ are called mapping classes.

Definition 1.1.12 (Goeritz Group[8])**.** Consider a genus g Heegaard splitting of S^3 given by

$$S^3 = V_g \bigcup_{\Sigma_g} W_g,$$

The group $\mathcal{H}_g$ of isotopy classes of orientation-preserving homeomorphisms of S^3 which leave the surface Σ_g and each handlebody invariant.

Definition 1.1.13 (Irreducible Manifold[9])**.** A compact 3-manifold M, possibly with boundary, is said to be irreducible if every embedded 2-sphere in M bounds a 3-ball in M.

Example 1.1.14. S^3 is an irreducible 3-manifold, i.e. every 2-sphere embedded in S^3, bounds a 3-ball in S^3.

Dehn-twist:[3] Consider the annulus $A = S^1 \times [0, 1]$. A is oriented by embedding it in the (θ, r)-plane via the map $(\theta, t) \to (\theta, t + 1)$ and the orientation induced by the standard orientation of the plane is taken. Consider the twist map $T : A \to A$ of A given by the formula

$$T(\theta, t) = (\theta + 2\pi t, t).$$

Clearly T is an orientation-preserving homeomorphism that fixes ∂A pointwise.

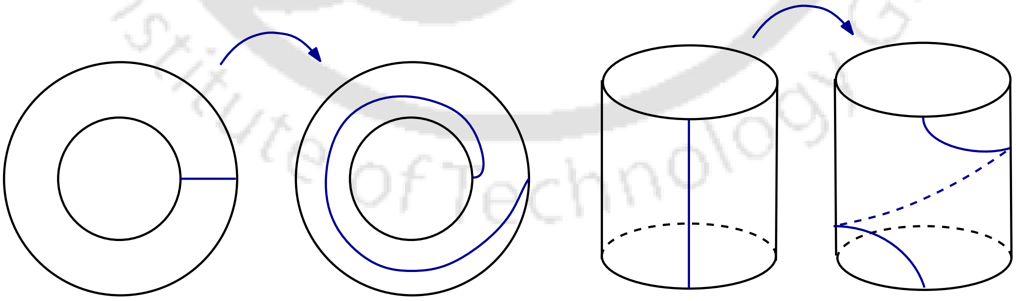

Figure 1.3: Dehn Twist

Now let S be an arbitrary (oriented) surface and let α be a simple closed curve in S. Let N be a regular neighborhood of α and choose an orientation

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preserving homeomorphism $\varphi : A \to N$. We obtain a homeomorphism $T_\alpha : S \to S$, called a Dehn twist about α, as follows:

$$T_\alpha(x) = \begin{cases} \varphi \circ T \circ \varphi^{-1}(x), & \text{if } x \in N \\ x, & \text{if } x \in S - N. \end{cases}$$

❦

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CHAPTER 2

Introduction

Heegaard splitting of the three sphere S^3 is a handlebody decomposition of S^3 expressed as (S^3, V_g, W_g) such that

$$S^3 = V_g \bigcup_{\Sigma_g} W_g,$$

where $\Sigma_g = \partial V_g = \partial W_g$ denotes a standard unknotted genus g surface embedded in S^3, V_g is a standard genus g handlebody and W_g is the closed complement of V_g in S^3. The surface Σ_g is called the Heegaard splitting surface or the Heegaard surface. The isotopy classes of automorphisms of S^3, that leaves the surface Σ_g and each handlebody V_g and W_g invariant, naturally forms a group, called the Goeritz group $\mathcal{H}_g$. Furthermore, elements in $\mathcal{H}_g$, when restricted to Σ_g give elements in $Mod(\Sigma_g)$ and these elements in fact forms a subgroup of $Mod(\Sigma_g)$.

Consequently, a natural question arises that which elements from $Mod(\Sigma_g)$ are restrictions of elements from $\mathcal{H}_g$?

2.1 Literature survey

The study of Goeritz group of the three sphere dates back to 1930s. Early work in this direction includes Goeritz's [5] result which said that the

Goeritz group of the genus two Heegaard splitting of S^3 is finitely generated. He also gave a set of four generators.

In 1980, Powell [8] attempted a generalization of Goeritz's result for higher genus cases. He introduced a set of generators for the Goeritz group $\mathcal{H}_g$. These automorphisms are termed as 'Powell generators'. But later on Scharlemann [10] identified a gap in Powell's proof. The statement made by Powell that '$\mathcal{H}_g$ is finitely generated' is well known in literature as Powell conjecture.

Scharlemann produced an updated proof for the finite generation of $\mathcal{H}_2$ in 2003 and he established that $\mathcal{H}_2$ is generated by the four automorphisms α, β, γ and δ described in Scharlemann [10].

In 2008, Akbas [1] extended Scharlemann's work by providing a finite presentation of this Goeritz group. Akbas showed that the graph constructed by Scharlemann is essentially a tree (i.e. connected, acyclic). If $N(Q, \Sigma_2^{\pm}, a)$ denote the number of arcs of slope a of c_Q in $\Sigma_2^{\pm}$, then Proposition 1 of Akbas [1] showed that for any non-standard reducing sphere Q, $N(Q, \Sigma_2^{-}, a) = N\left(Q, \Sigma_2^{+}, \frac{1}{a}\right)$, for $0 \leq a \leq \infty$. Lemma 1 of Akbas [1] proved that for any arbitrary non-standard reducing sphere Q, $N(Q, \Sigma_2^{+}, 0) \neq N(Q, \Sigma_2^{+}, \infty)$ i.e. the number of meridional arcs and the number of longitudinal arcs are not equal inside the same genus one summand. Finally Proposition 2 of Akbas [1] established the uniqueness of the third reducing sphere R from P and Q such that $P \cdot R = 4$ and $R \cdot Q < P \cdot Q$. Hence the acyclic nature of a certain graph Γ constructed in Scharlemann's [10] paper is established. Using this he showed that $\mathcal{H}_2$ is finitely presented and gave a presentation of $\mathcal{H}_2$.

Cho [2] produced an alternate proof of the fact that the graph $\tilde{\Gamma}$ in Scharlemann [10] and Akbas [1] is a tree. He used primitive disks and constructed a primitive disk complex $P(V)$. He finally constructed a graph T in the barycentric subdivision of $P(V)$ and showed that T is a tree. He also demonstrated that T and the tree in Akbas's [1] work can be reconciled.

In 2018 Freedman et al [4] proved the finite generation of the Goeritz group $\mathcal{H}_3$ of the genus three Heegaard splitting of the three sphere. They used the generators proposed by Powell [8]. They had further conjectured that the same set of generators will generate the Goeritz groups for the higher genus cases.

The latest work in this direction was by Zupan [14]. He constructed a curve complex by the reducing spheres on a standard genus g Heegaard splitting surface and studied some relations between the reducing sphere complex and the Powell Conjecture. He showed that Powell conjecture is true if and only if the said reducing sphere complex is connected. Recently Scharlemann [11] has announced that one of the Powell generators in Freedman and Scharlemann [4] is redundant.

2.2 Outline of the Thesis

In this work we make an alternate attempt to develop a general setup to address the problem on finite presentation of the Goeritz groups. Here we primarily deal with $\mathcal{H}_2$ and $\mathcal{H}_3$. We concentrate on these special subgroups of $Mod(\Sigma_2)$ and $Mod(\Sigma_3)$.

The first chapter contains the preliminaries relevant to this thesis and chapter 2 consists of a general introduction and a brief chronological survey of literature.

Chapter 3 contains the main work that address the genus two case. Here we present a generating set $\mathcal{G}_2$ of $\mathcal{H}_2$ which is algorithmically computable. We consider a special set of non-separating curves on the Heegaard surface Σ_2 and consider a special reducing sphere P as the standard. Now let Q be an arbitrary reducing sphere on Σ_2. The curve of intersection $c_Q = Q \cap \Sigma_2$ is referred to as the reducing curve corresponding to Q. The intersection number between two spheres Q and R is denoted by $Q \cdot R$ and defined as $Q \cdot R = |Q \cap \Sigma_2 \cap R|$. We produce an algorithm which monotonically decreases the intersection numbers of the reducing curve c_Q with the aforementioned special set of curves and brings it down to the 'standard' reducing curve. In the process we derive a complexity measure of a reducing sphere on Σ_2. Eventually we describe the elements in $\mathcal{H}_2$ uniquely in terms of words in the generators in $\mathcal{G}_2$.

In chapter 4, we present a similar generating set $\mathcal{G}_3$ of $\mathcal{H}_3$ and show that our proposed set is a subset of $\mathcal{H}_3$ and indeed generates $\mathcal{H}_3$.

In the final Chapter, we describe a few directions for future research based on the work of this thesis.

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CHAPTER 3

Genus two Heegaard splitting of the three sphere

3.1 Introduction

The genus two Heegaard splitting of the three sphere can be typically represented by

$$S^3 = V_2 \cup_{\Sigma_2} W_2$$

where V_2 denotes the unknotted genus two handlebody standardly embedded in S^3, W_2 is its closed complement in S^3 and Σ_2 is the closed, orientable, Heegaard splitting surface satisfying $\Sigma_2 = \partial V_2 = \partial W_2$. We are interested in the group of all isotopy classes of homeomorphisms of S^3 which leaves Σ_2, V_2 and W_2 invariant, termed as Goeritz group and denoted by $\mathcal{H}_2$.

3.1.1 Setup and Terminologies

Consider a standardly embedded genus two surface Σ_2 in S^3. Let $S^3 = V_2 \cup_{\Sigma_2} W_2$ be the corresponding Heegaard splitting of S^3. Consider the following curves on Σ_2 (refer to figure 3.1): A, B, C, X, Y, Z are non-separating curves on Σ_2. $A \cup B \cup C$ separates Σ_2 into two thrice boundered spheres, call them Σ_2' and Σ_2''. P is the reducing sphere whose reducing curve is c_P as shown in figure 3.1. We call P as the standard reducing sphere. P separates Σ_2 into two genus one surfaces with one boundary. We call these

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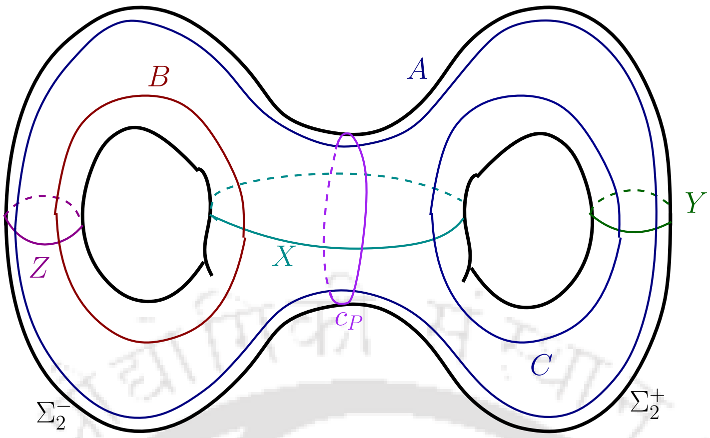

Figure 3.1: The standard set of curves on Σ_2

component surfaces as genus one summands and denote them by $\Sigma_2^\pm$ (see figure 3.1). Also c_P is disjoint from $B \cup Z$ and $C \cup Y$. We consider isotopy classes of simple closed curves on the surface Σ_2 and the action of elements in $\mathcal{H}_2$ on these curves. We assume all the basic terminology about such curves and such actions as in Farb and Margalit [3]. For terms and setup related to Heegaard splittings we follow Scharlemann [10] and Akbas [1].

Definition 3.1.1. (Akbas [1]) For any properly embedded arc $\nu \in \Sigma_2^\pm$ $\nu \in H_1\left(\Sigma_2^\pm, \partial\Sigma_2^\pm; \mathbb{Z}\right)$ can be written as $p\mu + q\lambda$ where $\mu = Z$ and $\lambda = B$ if $\nu \in \Sigma_2^-$ and $\mu = Y$ and $\lambda = C$ if $\nu \in \Sigma_2^+$ (figure 3.1). Then *slope* of ν is defined to be $|\frac{p}{q}| \in \mathbb{Q}^+ \cup \infty$.

Definition 3.1.2 (Meridional and longitudinal arcs)**.** In a genus one summand, an arc of slope 0 is referred to as a meridional arc and that of slope ∞ is termed as longitudinal arc.

By T_ω we denote the Dehn twist (Farb and Margalit [3]) about a simple closed curve ω on Σ_2. In this thesis, we follow the standard convention of function composition while writing the word for an automorphism in $\mathcal{H}_2$. For example $T_\omega T_\theta$ means we apply T_θ first and then T_ω.

If X and Y are isotopy classes of curves on Σ_2, then by $X \cdot Y$ we mean the geometric intersection of X and Y. For any reducing sphere Q, the

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essential separating circle $c_Q = Q \cap \Sigma_2$ on Σ_2 will be referred to here as the reducing curve corresponding to Q. We assume c_Q to be in minimal position with respect to arcs on Σ_2' and Σ_2'' i.e. c_Q intersects A, B and C minimally and transversally. We denote the arcs of c_Q on the thrice boundered spheres Σ_2' and Σ_2'' as elements in the cartesian product $\{a, b, c\} \times \{a, b, c\}$ where a, b, c represent points of intersection of c_Q with A, B and C respectively and thus (a, b) denotes an arc with ends on A and B. Therefore the arcs of c_Q can be listed as $\{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$.

Table 3.1 presents the restrictions of simultaneous existence of certain types of arcs listed above:

If exists	Ones that cannot exist
$\boxed{(a,a)}$	$\boxed{(b,b),\ (b,c),\ (c,c)}$
$\boxed{(b,b)}$	$\boxed{(a,a),(c,c)}, (a,c)$
$\boxed{(c,c)}$	$\boxed{(a,a),(b,b)}, (a,b)$
(a,b)	(c,c)
(a,c)	(b,b)
$\boxed{(b,c)}$	$\boxed{(a,a)}$

Table 3.1: Arcs with intersecting counterparts

$A \cup B \cup C$ separates c_Q into essential, proper, simple arcs with endpoints on A, B and C. The total number of arcs on both Σ_2' and Σ_2'' are equal. For an arbitrary reducing sphere Q, we denote $Q \cdot A = n_{AQ}, Q \cdot B = n_{BQ}$ and $Q \cdot C = n_{CQ}$.

3.1.2 Results

Scharlemann [10], Akbas[1] and Freedman et al [4], Zupan [14] respectively showed that $\mathcal{H}_2$ and $\mathcal{H}_3$ are finitely generated. Scharlemann [10] provided a recipe to construct the spheres in the concerned edge-path whereas Akbas

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[1] went on to establish the uniqueness of the said sphere constructed in the process demonstrated in Scharlemann [10].

In our present work we first propose a replacement of the automorphism δ in Scharlemann [10] by a new element φ of $\mathcal{MCG}(\Sigma_2)$ and express φ and β [10] as words in terms of Dehn twists about standard non-separating curves on (Σ_2). We also replace γ by ν. We show that $\varphi, \nu \in \mathcal{H}_2$. The main result then is a computationally simple algorithm to write any element of $\mathcal{H}_2$ as a word in terms of the automorphisms $\alpha, \beta, \nu, \varphi$ of $\mathcal{H}_2$. This proves that $\mathcal{G}_2 = \{\alpha, \beta, \nu, \varphi\}$ generates $\mathcal{H}_2$. The algorithm in this work is an alternative to the one given in Scharlemann [10] and Akbas [1]. In the process we derive a unique presentation for every element $f \in \mathcal{H}_2$ in the form

$$f = \alpha^a \nu^b \beta^c \prod \left(\varphi \nu^{s_i} \beta^{r_i}\right) = \alpha^a \nu^b \beta^c \left(\varphi \nu^{s_n} \beta^{r_n}\right) \circ \cdots \circ \left(\varphi \nu^{s_1} \beta^{r_1}\right).$$

Subsequently we derive a complexity measure of an arbitrary reducing sphere Q in terms of its intersection with the curves A, B and C (figure 3.1). The complexity measure is given by

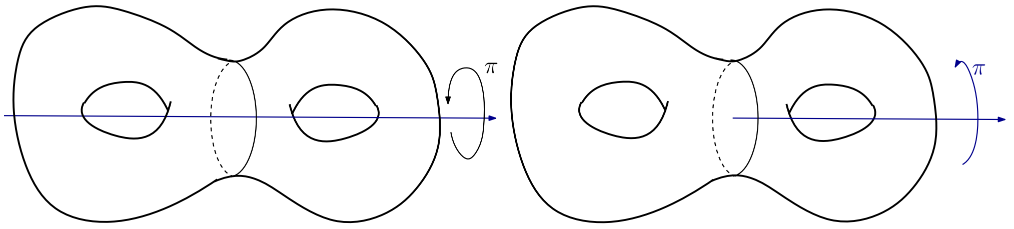
$$\mathcal{C}(Q) = \frac{1}{2} n_{AQ} + n_{BQ} + n_{CQ}.$$

3.2 The elements in $\mathcal{G}_2$

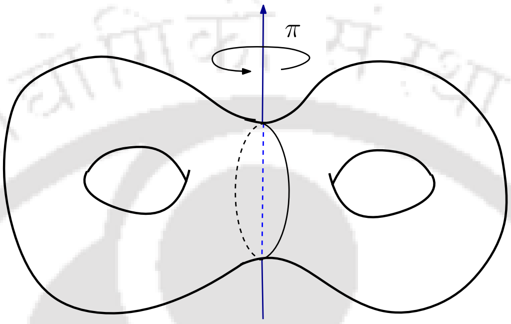
Consider the setup described in section 3.1.1 of the genus two Heegaard splitting of S^3.

Consider the set $S = \{\alpha, \beta, \gamma, \delta\}$ of generators of $\mathcal{H}_2$ as described in [10]. Here α represents the involution of Σ_2 (an order 2 rotation of Σ_2, see figure 3.2a), γ captures the rotational symmetry of Σ_2 (see figure 3.2c) and β represents the half-twists about the standard reducing curve c_P (figure 3.2b). It may be observed that these automorphisms constitute the stabilizer of the standard sphere.

We make a small modification to γ and introduce another order two rotation ν (figure 3.3b). γ swaps Σ_2' and Σ_2'' whereas ν leaves them invariant. Automorphisms α and ν are easy to implement in a computer



(a) Automorphism α (b) Automorphism β



(c) Automorphism γ

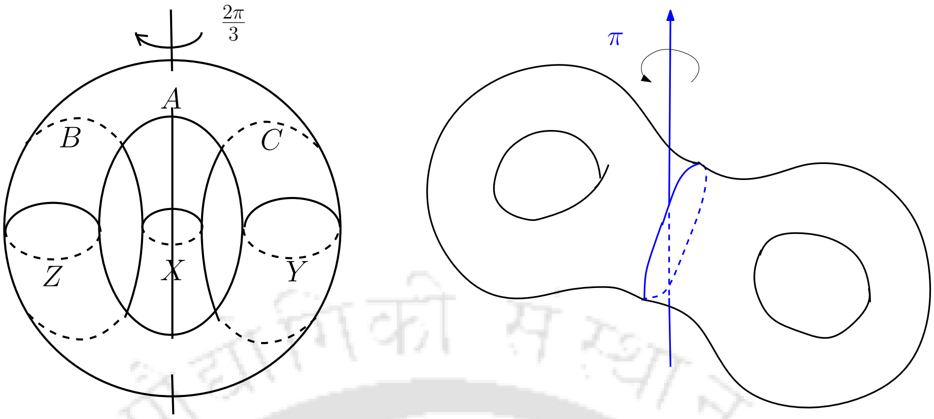
Figure 3.2: The generating set for the stabilizer of the standard sphere in $\mathcal{H}_2$ [10]

programme. Whereas the automorphism β i.e. half twist about the standard sphere and δ described in Scharlemann [10] are not easy to implement. Comparatively easier to implement are complete dehn-twists (twists about separating closed curves through an angle 2π). Since $\mathcal{MCG}(\Sigma_2)$ is generated by dehn twists about certain essential closed curves, we introduce an automorphism φ of $\mathcal{H}_2$ and describe both β and φ in terms of dehn-twists about certain separating closed curves on Σ_2 so that we have a computable set of automorphisms $\mathcal{G}_2 = \{\alpha, \beta, \nu, \varphi\}$. We then show that the automorphism γ and δ can be generated by elements from $\mathcal{G}_2$. This also justifies that our proposed set $\mathcal{G}_2$ indeed generates $\mathcal{H}_2$.

3.2.1 Automorphisms α and ν

Lemma 3.2.1 (Properties of α and ν). *(i)* $[\alpha]^2 = [\nu]^2 = 1$.

(ii) $\alpha\nu = \nu\alpha = \gamma$.



(a) Automorphism δ[10] (b) Automorphism ν

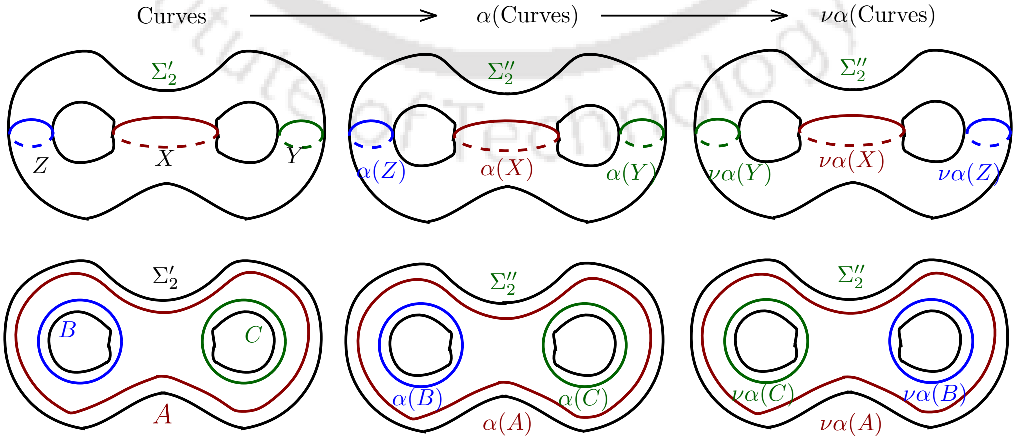
Figure 3.3: Automorphisms δ [10] and ν

Proof. The first part is immediate from the description of α and ν.

For the second part let us observe that

$$\alpha(A) = A, \alpha(B) = B, \alpha(C) = C, \alpha(X) = X, \alpha(Y) = Y, \alpha(Z) = Z,$$
$$\alpha(\Sigma_2') = \Sigma_2'' \text{ and } \alpha(\Sigma_2'') = \Sigma_2'.$$
$$\nu(A) = A, \nu(B) = C, \nu(C) = B, \nu(X) = X, \nu(Y) = Z, \nu(Z) = Y,$$
$$\nu(\Sigma_2') = \Sigma_2'' \text{ and } \nu(\Sigma_2'') = \Sigma_2'$$

Hence $\alpha\nu(*) = \nu\alpha(*)$ for all $* \in \{A, B, C, X, Y, Z, \Sigma_2', \Sigma_2''\}$ i.e. $\alpha\nu = \nu\alpha$. Now figure 3.4 shows that the action of $\gamma^{-1}\alpha\nu$ fixes $A, B, C, X, Y, Z, \Sigma_2'$ and



(a) $\alpha\nu(curves)$

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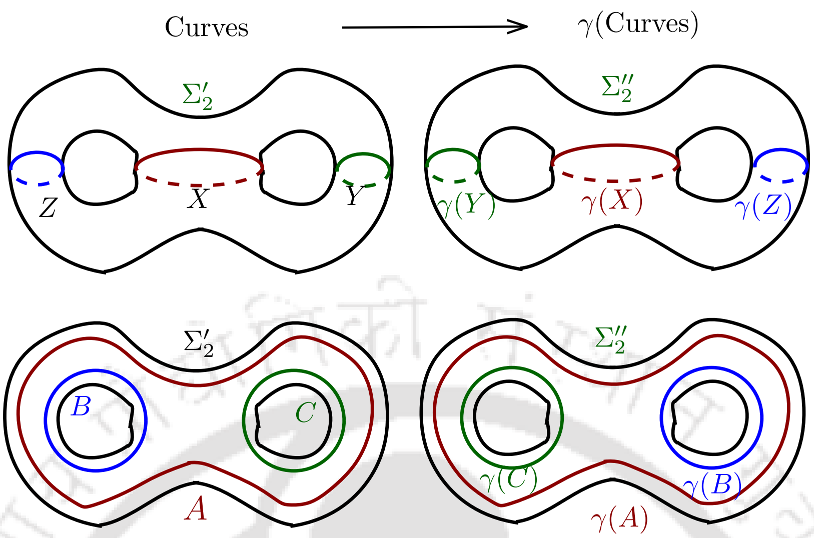

(b) $\gamma(curves)$

Figure 3.4: $\alpha\nu = \gamma$.

Σ_2''. Therefore $\gamma^{-1}\alpha\nu$ is identity. Hence the equality holds. ☐

3.2.2 Automorphism β

β describes a half twist about the standard reducing curve c_P. We can express β as

$$\beta = (T_C T_Y)^3 = T_C T_Y T_C T_Y T_C T_Y,$$

where Y, C represents the meridional and longitudinal loops respectively in Σ_2^+ (figure 3.1); and T_Y, T_C denotes the dehn twists about the loops Y, C respectively. It is worth mentioning that this word-presentation is not unique. Other expression can be derived using the relations among the generators of $\mathcal{MCG}(\Sigma_g)$. For example, using the braid relation, i.e. $T_Y T_C T_Y = T_C T_Y T_C$ we can also express β as

$$\beta = (T_C T_Y T_C)^2 = (T_Y T_C T_Y)^2.$$

Figure 3.5 illustrates the computations of the application of β on A and X. Note that $(T_C T_Y T_C)$ exchanges Y and C. So they are invariant under β. Since β^{-1} composed with the half-twist in [10] fixes all the essential

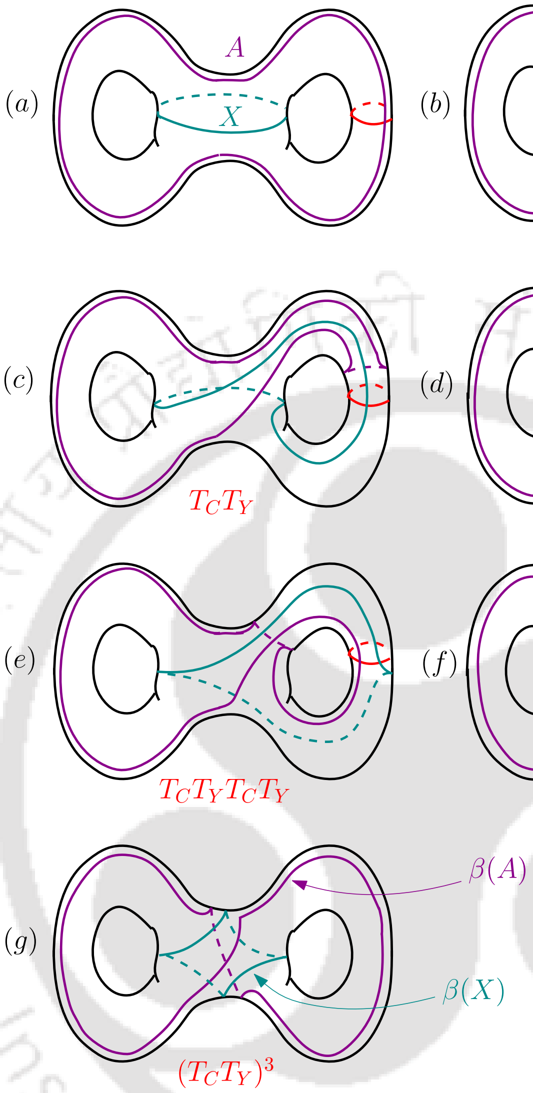

Figure 3.5: Computation of $\beta(X)$

non-separating loops A, B, C, X, Y and Z, the composition is identity on Σ_2. Therefore $\beta \in \mathcal{H}_2$.

Properties of β

Now from figure 3.6, one can observe that β leaves $\Sigma_2^{\pm}$ invariant and only increases or reduces the intersection of c_Q with A in a collar neighbourhood of c_P.

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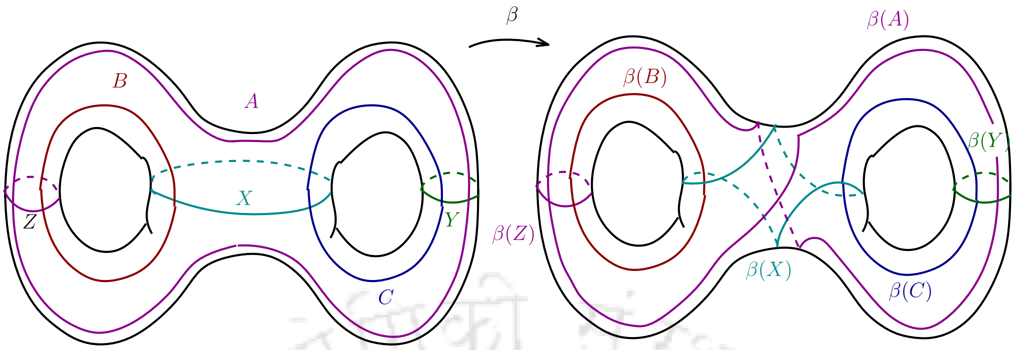

Figure 3.6: Action of β on Σ_2

Lemma 3.2.2. *β exhibits the following properties:*

(i) Order of β is infinite.

(ii) β commutes with α and ν.

Proof. The first part naturally follows from the fact that $\beta^{n+1}(X) \cdot A > \beta^n(X) \cdot A$, for all $n \in \mathbb{N}$.

For the second part, it is easy to verify that both $\nu\beta\nu\beta^{-1}$ and $\alpha\beta\alpha\beta^{-1}$ fixes all the standard loops on Σ_2. Therefore, both are identity in $\mathcal{MCG}(\Sigma_2)$. Also $[\alpha]^2 = [\gamma]^2 = 1$, therefore

$$\alpha\beta = \beta\alpha \qquad \text{and} \qquad \nu\beta = \beta\nu.$$

$\square$

3.2.3 Automorphism φ

An attempt to describe an automorphism that swaps the standard reducing sphere P with another member P' in the standard 2-simplex described in Scharlemann [10] led us to an automorphism φ. Figure 3.7a presents the action of φ on the three reducing spheres in the aforementioned 2-simplex.

φ can be described as

$$\varphi = T_Z^{-1} T_Y T_C T_Y T_X T_C T_Y.$$

The effect of φ on the standard loops on Σ_2 can be observed from figure 3.7b. It can be observed that φ exchanges the loops Y and X but leaves

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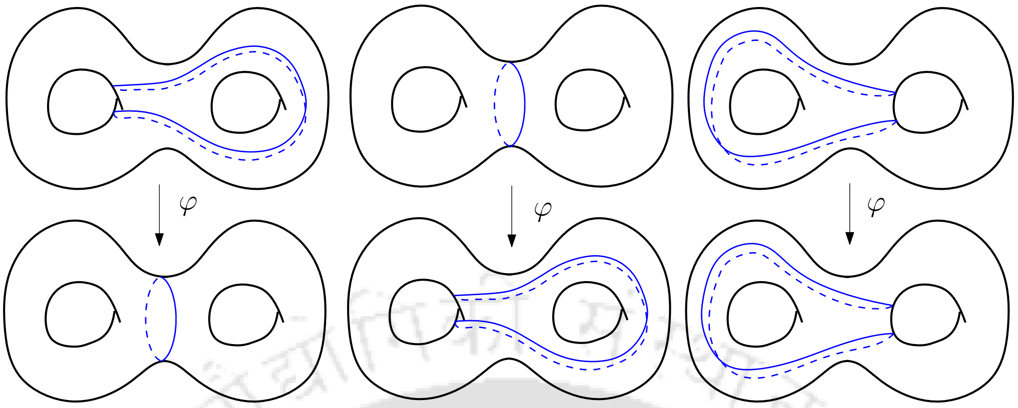
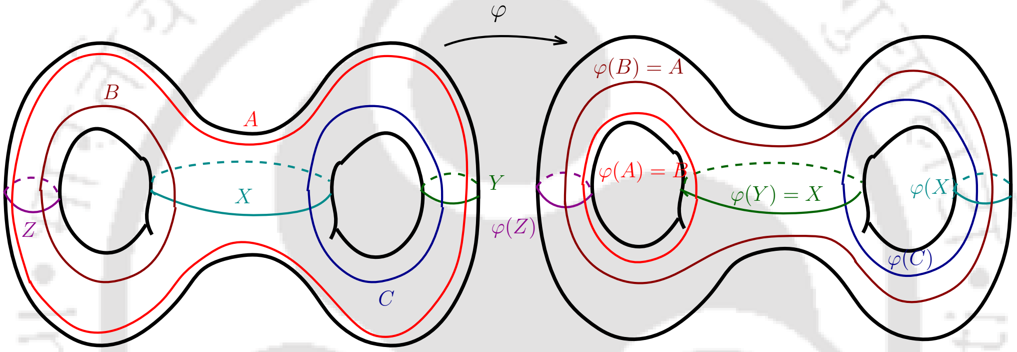


(a) φ acting on the standard 2-simplex described in [10]



(b) Action of φ on Σ_2

Figure 3.7: Action of φ

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C and Z invariant. From the action of φ on curves in figure 3.7b, we give the following lemma:

Lemma 3.2.3. *The automorphism φ satisfies the following:*

(i) $\varphi(A) = B$, $\varphi(B) = A$, $\varphi(C) = C$, $\varphi(X) = Y$, $\varphi(Y) = X$, $\varphi(Z) = Z$.

(ii) $\varphi \in \mathcal{H}_2$.

(iii) $\varphi^2 \simeq 1$ *(i.e. φ^2 is isotopic to identity).*

Proof. (i) Figure 3.8 demonstrates the verification of $\varphi(A) = B$. By computing in a similar manner, one can verify that the first result follows from figure 3.7b.

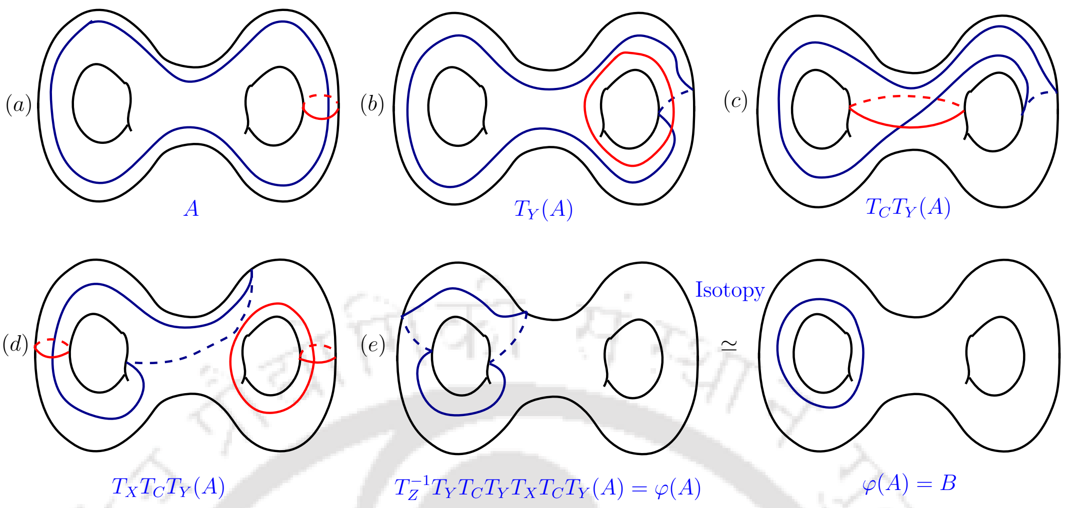

Figure 3.8: Computation of $\varphi(A)$

(ii) Now consider the eyeglass move φ_θ in [14] for Σ_2. Now figure 3.9 demonstrates that $\varphi^{-1}(\beta^{-1}\varphi_\theta)$ fixes all the standard non-separating curves (figure 3.1) on Σ_2. Moreover, $\varphi^{-1}(\beta^{-1}\varphi_\theta)$ leaves Σ_2' and Σ_2'' invariant. Therefore, $\varphi^{-1}(\beta^{-1}\varphi_\theta)$ is istopic to identity on Σ_2. This implies $\beta^{-1}\varphi_\theta \simeq \varphi$ on Σ_2. Hence $\varphi \in \mathcal{H}_2$.

(iii) From (i) it follows that φ^2 fixes $A, B, C, X, Y, Z, \Sigma_2'$ and Σ_2''. Therefore $\varphi^2 \simeq 1$ on Σ_2.

□

3.3 Complexity and reduction algorithm

In this section we propose an algorithm that will provide us a word in the elements from $\mathcal{G}_2$ that reduces an arbitrary reducing sphere to the standard one. We will first introduce a couple of results that will pave the foundation for our algorithm and will validate the same. For any reducing curve c_Q on Σ_2, we introduce a measure $\mathcal{C}(Q)$ defined as

$$\mathcal{C}(Q) = \frac{1}{2} n_{AQ} + n_{BQ} + n_{CQ}.$$

Lemma 3.3.1. *Every reducing curve c_Q on Σ_2 must intersect at least one of A, B and C.*

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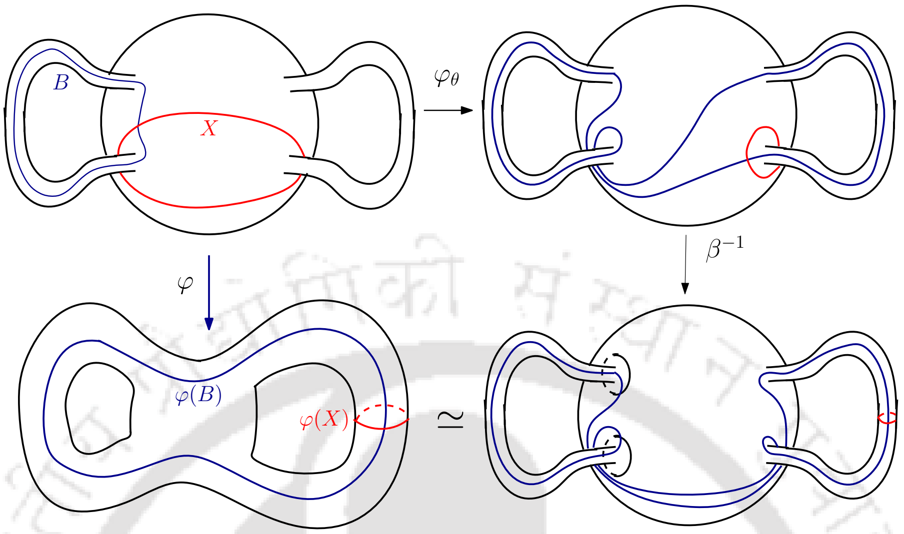
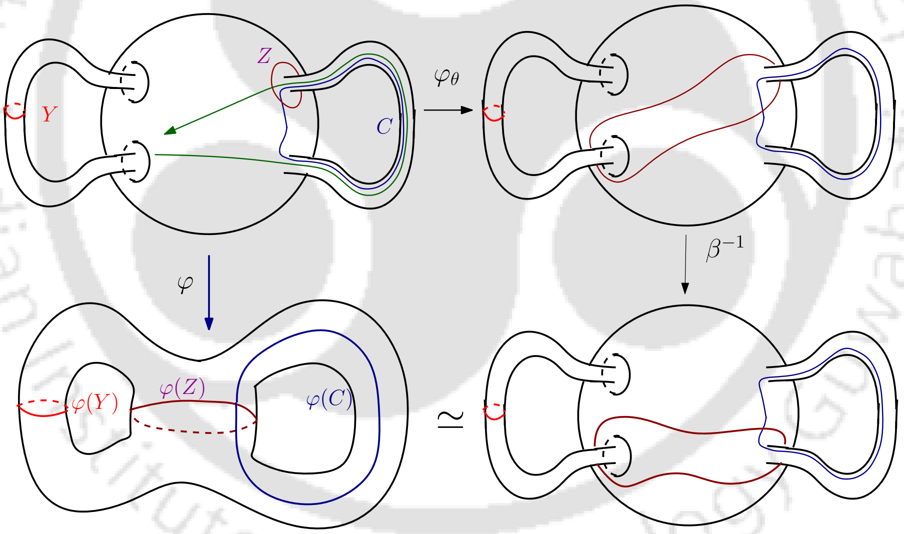


(a) Action of φ and $\beta^{-1}\varphi_\theta$ on B and X



(b) Action of φ and $\beta^{-1}\varphi_\theta$ on Y, Z and C

Figure 3.9: Automorphism φ comparision with eyeglass move [4, 14]

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Proof. Since Q is a reducing sphere, therefore c_Q must bound a disk in both V_2 and W_2. Now, if c_Q remains completely disjoint from $A \cup B \cup C$, then c_Q becomes a simple closed loop on a sphere with three boundaries (Σ_2' or Σ_2''). Then c_Q must either be isotopic to one of A, B or C or it must bound a disk on Σ_2. This leads to a contradiction as either c_Q fails to bound a disk inside V_2 or it is inessential. Therefore c_Q must intersect at least one of A, B and C. □

Lemma 3.3.2. *$n_{AQ} > n_{BQ} + n_{CQ}$ if and only if c_Q has an (a,a) arc on both sides of Σ_2, and $n_{AQ} < n_{BQ} + n_{CQ}$ if and only if c_Q has at least one of $(b,b),(c,c)$ or (b,c) arc.*

Proof. By virtue of lemma 3.3.1, c_Q must intersect $A \cup B \cup C$. Let us note that, each (a,b) arc contributes $+1$ to n_{AQ} and $+1$ to n_{BQ}. In a similar manner, we present the contribution of each type of arc to n_{AQ}, n_{BQ} and n_{CQ} in table 3.2. Let us assume that the number of $(a,a),(b,b),(c,c),(b,c),(a,b)$ and (a,c) arcs on Σ_2' are respectively $n'_{aa}, n'_{bb}, n'_{cc}, n'_{bc}, n'_{ab}$ and n'_{ac}. Also the number of $(a,a),(b,b),(c,c),(b,c),(a,b)$ and (a,c) arcs on Σ_2'' are respectively $n''_{aa}, n''_{bb}, n''_{cc}, n''_{bc}, n''_{ab}$ and n''_{ac}. Since only the $(a,a),(a,b)$ and (a,c) arcs contribute to n_{AQ}, only the $(b,b),(b,c)$ and (a,b) arcs contribute to n_{BQ} and only the $(c,c),(b,c)$ and (a,c) arcs contribute to n_{CQ}, hence we have

Type of arc	Each arc's contribution to		
	n_{AQ}	n_{BQ}	n_{CQ}
(a,a)	+2	0	0
(a,b)	+1	+1	0
(a,c)	+1	0	+1
(b,b)	0	+2	0
(b,c)	0	+1	+1
(c,c)	0	0	+2

Table 3.2: Contribution of each type of arc to n_{AQ}, n_{BQ} and n_{CQ}

$$\begin{aligned} n_{AQ} &= 2n'_{aa} + n'_{ab} + n'_{ac} = 2n''_{aa} + n''_{ab} + n''_{ac}, \\ n_{BQ} &= 2n'_{bb} + n'_{bc} + n'_{ab} = 2n''_{bb} + n''_{bc} + n''_{ab}, \\ \text{and } n_{CQ} &= 2n'_{cc} + n'_{bc} + n'_{ac} = 2n''_{cc} + n''_{bc} + n''_{ac}. \end{aligned} \tag{3.1}$$

Case 1: $n_{AQ} > n_{BQ} + n_{CQ}$**.** In this case, on $\Sigma_2^{'}$,

$$\begin{aligned} & 2n^{'}_{aa} + n^{'}_{ab} + n^{'}_{ac} > (2n^{'}_{bb} + n^{'}_{bc} + n^{'}_{ab}) + (2n^{'}_{cc} + n^{'}_{bc} + n^{'}_{ac}) \\ \iff & 2n^{'}_{aa} > 2(n^{'}_{bb} + n^{'}_{cc} + n^{'}_{bc}) \\ \iff & n^{'}_{aa} > n^{'}_{bb} + n^{'}_{cc} + n^{'}_{bc}. \end{aligned}$$

The strict inequality and the fact that $n^{'}_{bb}, n^{'}_{cc}, n^{'}_{bc} \geq 0$ together implies that $n^{'}_{aa} \geq 1$. Moreover, since n_{AQ}, n_{BQ} and n_{CQ} are same for both $\Sigma_2^{'}$ and $\Sigma_2^{''}$, a symmetric argument shows that, $n^{''}_{aa} \geq 1$ as well.
Thus $n_{AQ} > n_{BQ} + n_{CQ}$ implies c_Q has an (a, a) arc on both $\Sigma_2^{'}$ and $\Sigma_2^{''}$.

Conversely, Suppose c_Q has an (a, a) arc on $\Sigma_2^{'}$ (similarly on $\Sigma_2^{''}$). Then, $n^{'}_{aa} \neq 0$ and the fact that $(b, b), (c, c)$ and (b, c) cannot coexist with (a, a) implies $n^{'}_{bb} = 0$, $n^{'}_{cc} = 0$ and $n^{'}_{bc} = 0$. Then,

$$n_{BQ} + n_{CQ} = n^{'}_{ab} + n^{'}_{ac} < 2n^{'}_{aa} + n^{'}_{ab} + n^{'}_{ac} = n_{AQ} \quad (\because n^{'}_{aa} \geq 1).$$

Since, n_{AQ}, n_{BQ} and n_{CQ} are equal on $\Sigma_2^{'}$ and $\Sigma_2^{''}$, then c_Q must also have an (a, a) arc on $\Sigma_2^{''}$ i.e. $n^{''}_{aa} \neq 0$. Therefore, an (a, a) arc must exist on either sides of Σ_2 if and only if $n_{AQ} > n_{BQ} + n_{CQ}$.
Case 2: $n_{AQ} < n_{BQ} + n_{CQ}$**.** In this case we have,

$$\begin{aligned} & 2n^{'}_{aa} + n^{'}_{ab} + n^{'}_{ac} < (2n^{'}_{bb} + n^{'}_{bc} + n^{'}_{ab}) + (2n^{'}_{cc} + n^{'}_{bc} + n^{'}_{ac}) \\ \iff & 2n^{'}_{aa} < 2(n^{'}_{bb} + n^{'}_{cc} + n^{'}_{bc}) \\ \iff & n^{'}_{aa} < n^{'}_{bb} + n^{'}_{cc} + n^{'}_{bc}. \end{aligned}$$

By virtue of the strict inequality, at least one of $n^{'}_{bb}, n^{'}_{cc}$ or $n^{'}_{bc}$ must be non-zero. In other words, at least one of $(b, b), (c, c)$ or (b, c) must exist whenever $n_{AQ} < n_{BQ} + n_{CQ}$.
Conversely, if at least one of $(b, b), (c, c)$ or (b, c) arc exists on $\Sigma_2^{'}$, then (a, a) cannot exist on $\Sigma_2^{'}$, hence $n^{'}_{aa} = 0$ and $n^{'}_{bb} + n^{'}_{cc} + n^{'}_{bc} \geq 1$. Then

$$\begin{aligned} n_{BQ} + n_{CQ} &= (2n^{'}_{bb} + n^{'}_{bc} + n^{'}_{ab}) + (2n^{'}_{cc} + n^{'}_{bc} + n^{'}_{ac}) \\ &= 2(n^{'}_{bb} + n^{'}_{cc} + n^{'}_{bc}) + n^{'}_{ab} + n^{'}_{ac} > n^{'}_{ab} + n^{'}_{ac} = n_{AQ}. \end{aligned}$$

A symmetric argument shows that the above inequality holds even if at least one of $(b,b),(c,c)$ or (b,c) arc exists on Σ''_2.
Therefore, at least one of $(b,b),(c,c)$ or (b,c) exists on either sides of Σ_2 if and only if $n_{AQ} < n_{BQ} + n_{CQ}$. $\square$

Lemma 3.3.3. *For any reducing sphere Q, $n_{AQ} \neq n_{BQ} + n_{CQ}$.*

Proof. If $Q = P$, i.e. $c_Q = c_P$ then $n_{AQ} = 2$, $n_{BQ} + n_{CQ} = 0$ and the lemma holds.
Now suppose $c_Q \neq c_P$ be arbitrary. If possible suppose, $n_{AQ} = n_{BQ} + n_{CQ}$. In this case we have

$$\begin{aligned} & 2n'_{aa} + n'_{ab} + n'_{ac} = (2n'_{bb} + n'_{ab} + n_{bc})' + (2n'_{cc} + n'_{ac} + n'_{bc}) \\ \implies & 2n'_{aa} = 2(n'_{bb} + n'_{cc} + nbc') \implies n'_{aa} = n'_{bb} + n'_{cc} + n'_{bc} \end{aligned}$$

This eventually implies that, on both Σ'_2 as well as Σ''_2, existence of atleast one of $(b,b),(c,c)$ or (b,c) arcs enforce the existence of an (a,a)-arc and vice versa. But this is impossible by table 3.1. Therefore, this is the case, when c_Q can only have (a,b) and (a,c) arc on Σ'_2 and Σ''_2. Therefore, $n'_{aa} = n''_{aa} = 0$, $n'_{bb} = n''_{bb} = 0$, $n'_{cc} = n''_{cc} = 0$ and $n'_{bc} = n''_{bc} = 0$.

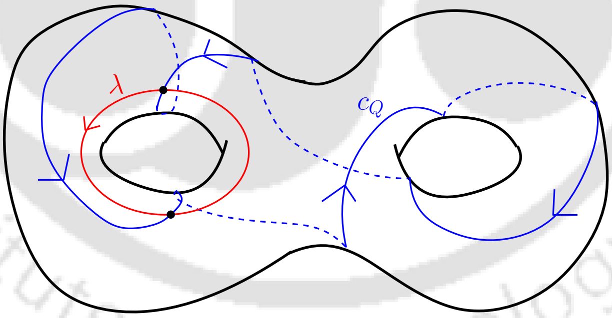

Figure 3.10: An arbitrary simple closed oriented curve on Σ_2 consisting of only (a,b) and (a,c)-arcs

Without loss of generality, let us assume that c_Q has an (a,b)-arc in both Σ'_2 and Σ''_2 (this argument works even with (a,c)-arcs). Orient c_Q such that one (a,b)-arc on Σ'_2 has an orientation from A to B. Now let us consider the (finite) sets of points

$$c_Q \cap A = \{a_1, a_2, \ldots, a_n\}$$

$$c_Q \cap B = \{b_1, b_2, \ldots, b_m\}$$

$$c_Q \cap C = \{c_1, c_2, \ldots, c_l\}.$$

When starting from a_1 we traverse along c_Q, then a sample of c_Q can be given by

$$\ldots a_{i_1} b_{j_1} a_{i_2} b_{j_2} a_{i_3} c_{k_1} \ldots, \text{ for } i_r \in \{1, \ldots, n\}, j_s \in \{1, \ldots, m\}, k_t \in \{1, \ldots, l\}$$

however in such a naming convention, every alternate letter should be from the set $c_Q \cap A$. This sequence tells us that if the $\overrightarrow{(a,b)}$-arc oriented from A to B is on Σ_2', then all $\overrightarrow{(a,b)}$-arcs in Σ_2' are oriented from A to B. Now if we orient B arbitrarily, then we note that the $\overrightarrow{(a,b)}$-arcs on Σ_2' intersect the curve B in such a manner that the absolute value of the algebraic intersection number of c_Q with B always increases. By our assumption that there exists one $\overrightarrow{(a,b)}$-arc, so this number is non-zero. This contradicts the assumption that c_Q is a separating curve on Σ_2 (since Q is a reducing sphere).

Thus c_Q cannot have only (a,b) and (a,c) arcs. Therefore, $n_{AQ} = n_{BQ} + n_{CQ}$ is not possible, i.e. either $n_{AQ} > n_{BQ} + n_{CQ}$ or $n_{AQ} < n_{BQ} + n_{CQ}$. □

Lemma 3.3.4. *The number of each of* $(a,a), (b,b), (c,c), (b,c), (a,b)$ *and* (a,c) *arcs of* c_Q *on* Σ_2' *and* Σ_2'' *are equal.*

Proof. Suppose the number of $(a,a), (b,b), (c,c), (b,c), (a,b)$ and (a,c) arcs on Σ_2' are respectively n'_{aa}, n'_{bb}, n'_{cc}, n'_{bc}, n'_{ab} and n'_{ac} and the number of $(a,a), (b,b), (c,c), (b,c), (a,b)$ and (a,c) arcs on Σ_2'' are respectively n''_{aa}, n''_{bb}, n''_{cc}, n''_{bc}, n''_{ab} and n''_{ac}.

Case 1: $n_{AQ} > n_{BQ} + n_{CQ}$. Then by lemma 3.3.2 both Σ_2' and Σ_2'' contain (a,a) arcs i.e. $n'_{aa}, n''_{aa} \neq 0$. As (a,a) arcs cannot coexist with $(b,b), (c,c)$ and (b,c) arcs, therefore $n'_{bb} = n''_{bb} = 0$, $n'_{cc} = n''_{cc} = 0$, and $n'_{bc} = n''_{bc} = 0$. In this case, $n'_{ab} = n_{BQ} = n''_{ab}$ and $n'_{ac} = n_{CQ} = n''_{ac}$, since each point of $c_Q \cap B$ (respectively $c_Q \cap C$) corresponds to exactly one (a,b) (respectively (a,c)) arc simultaneously on Σ_2' and Σ_2''.

Moreover, $n'_{aa} = n_{AQ} - n_{BQ} - n_{CQ} = n''_{aa}$.

So if $n_{AQ} > n_{BQ} + n_{CQ}$, then c_Q must have equal number of (a,a), (a,b) and (a,c) on both sides, while the other arcs cannot exist at all.

Case 2: $n_{AQ} < n_{BQ} + n_{CQ}$. In this case, by virtue of lemma 3.3.2, c_Q must have at least one of (b,b), (b,c) or (c,c) arc on either sides of Σ_2. Then neither side of Σ_2 can have an (a,a)-arc i.e. $n'_{aa} = 0 = n''_{aa}$.

It may be noted that, atleast one of n'_{bb} or n'_{cc} must be zero, since (b,b) and (c,c) cannot coexist. Similarly, atleast one of n''_{bb} or n''_{cc} must be zero. Therefore,

$$\begin{aligned} n_{AQ} &= n'_{ab} + n'_{ac} = n''_{ab} + n'_{ac}, \\ n_{BQ} &= 2n'_{bb} + n'_{bc} + n'_{ab} = 2n''_{bb} + n''_{bc} + n''_{ab}, \\ \text{and } n_{CQ} &= 2n'_{cc} + n'_{bc} + n'_{ac} = 2n''_{cc} + n''_{bc} + n''_{ac}. \end{aligned}$$

Now, let us consider the following subcases:

Subcase (i): $n'_{bb} \neq 0$. This implies, that there exists a (b,b) arc on Σ'_2. Therefore, (c,c) and (a,c) cannot exists on Σ'_2 i.e. $n'_{cc} = 0 = n'_{ac}$. So compairing n_{AQ}, n_{BQ} and n_{CQ} on Σ'_2 and Σ''_2, we have

$$\begin{aligned} n_{AQ} &= n'_{ab} = n''_{ab} + n''_{ac}, \\ n_{BQ} &= 2n'_{bb} + n'_{bc} + n'_{ab} = 2n''_{bb} + n''_{bc} + n''_{ab}, \\ \text{and } n_{CQ} &= n'_{bc} = 2n''_{cc} + n''_{bc} + n''_{ac}. \end{aligned}$$

Therefore, equating $n_{BQ} - (n_{AQ} + n_{CQ})$ on Σ''_2 and Σ'_2 we get,

$$\begin{aligned} (2n''_{bb} + n''_{bc} + n''_{ab}) - [(n''_{ab} + n''_{ac}) + (2n''_{cc} + n''_{bc} + n''_{ac})] &= (2n'_{bb} + n'_{bc} + n'_{ab}) - [n'_{ab} + n'_{bc}] \\ 2(n''_{bb} - n''_{cc} - n''_{ac}) &= 2n'_{bb} \\ n''_{bb} - (n''_{cc} + n''_{ac}) &= n'_{bb} > 0 \text{ (by assumption).} \end{aligned}$$

Therefore, $n''_{bb} > 0$ i.e. c_Q must have a (b,b) arc on Σ''_2. Therefore, $n''_{cc} = 0 = n''_{ac}$ (since (c,c) and (a,c) cannot coexist with (b,b)). From the above equations we now have

$$\begin{aligned} n_{AQ} &= n'_{ab} = n''_{ab}, \\ n_{BQ} &= 2n'_{bb} + n'_{bc} + n'_{ab} = 2n''_{bb} + n''_{bc} + n''_{ab}, \\ \text{and } n_{CQ} &= n'_{bc} = n''_{bc}. \end{aligned}$$

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Therefore combining all the above we have

$$n'_{aa}=n''_{aa}=0, n'_{bb}=n''_{bb}, n'_{cc}=n''_{cc}=0, n'_{bc}=n''_{bc}, n'_{ab}=n''_{ab}, n'_{ac}=n''_{ac}=0.$$

Subcase (ii): $n'_{cc}\neq 0$. By symmetric arguments as in the previous subcase, we have

$$n'_{aa}=n''_{aa}=0, n'_{bb}=n''_{bb}=0, n'_{cc}=n''_{cc}, n'_{bc}=n''_{bc}, n'_{ab}=n''_{ab}=0, n'_{ac}=n''_{ac}.$$

Subcase (iii): $n'_{bb}=n'_{cc}=0$. In this case, c_Q can only have (b,c), (a,b) and (a,c) arcs on Σ'_2. Also from previous cases, we have $n'_{aa}=0 \iff n''_{aa}=0$ and $n'_{bb}=0 \iff n''_{bb}=0$.

$$\begin{aligned} n_{AQ} &= n'_{ab}+n'_{ac}=n''_{ab}+n''_{ac},\\ n_{BQ} &= n'_{ab}+n'_{bc}=n''_{ab}+n''_{bc},\\ n_{CQ} &= n'_{bc}+n'_{ac}=n''_{bc}+n''_{ac}. \end{aligned}$$

Adding the above three equations we have,

$$n'_{bc}+n'_{ab}+n'_{ac}=n''_{bc}+n''_{ab}+n''_{ac}$$

From the above four equations it follows that

$$n'_{bc}=n''_{bc}, n'_{ab}=n''_{ab}, n'_{ac}=n''_{ac}.$$

Therefore, the number of (a,a) (similarly (b,b), (c,c), (b,c), (a,b) and (a,c)) arcs of c_Q are equal on both sides of Σ_2 i.e. Σ'_2 and Σ''_2. $\square$

Lemma 3.3.5. *Let $Q\neq P$ be any reducing sphere. Then $n_{AQ}<n_{BQ}+n_{CQ}$ if and only if c_Q has a (b,b) or (c,c) arc on either sides of Σ_2. Moreover,*

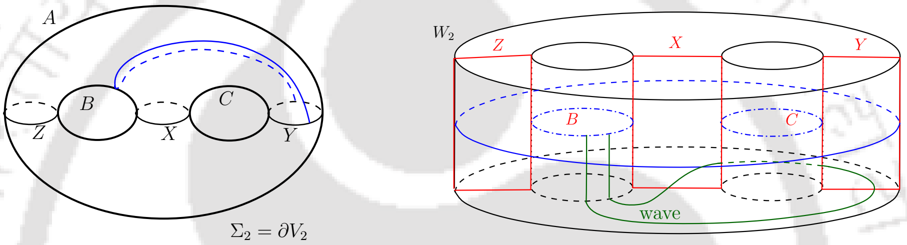
(i) $n_{AQ}<n_{BQ}+n_{CQ}$ and $n_{BQ}>n_{CQ}$ if and only if c_Q has a (b,b) arc,

(ii) $n_{AQ}<n_{BQ}+n_{CQ}$ and $n_{BQ}<n_{CQ}$ if and only if c_Q has a (c,c) arc,

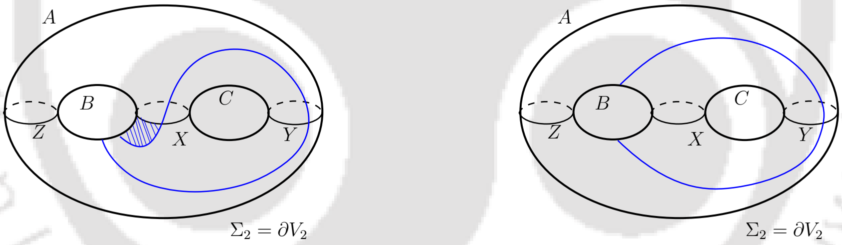
Proof. Suppose $c_Q\neq c_P$ such that $n_{AQ}<n_{BQ}+n_{CQ}$. By virtue of lemma 3.3.2, c_Q cannot have any (a,a) arc on either sides of Σ_2. From the work of Volodin et al[12] we have, $c_Q(\neq c_P)$ bounds a disk in V_2 implies c_Q must

have a 'wave' on ∂V_2 with respect to Y or Z. In other words, c_Q must either have a $y-b-y$ arc (see figure 3.11a) disjoint from $X \cup Z$ or a $z-c-z$ arc disjoint from $X \cup Y$.

Now, Q being a reducing sphere, c_Q must also bound a disk in W_2. Therefore, due to Volodin et al [12], c_Q must also have a 'wave' on ∂W_2 with respect to B or C (i.e. the dual curves to Z and Y respectively). The 'wave' on ∂W_2 with respect to B when viewed on ∂V_2, becomes a (b,b) arc on ∂V_2. The 'wave' on ∂W_2 with respect to B is shown in figure 3.11b, 3.11c and 3.11d respectively.



(a) Wave on ∂V_2 with respect to Y (b) Wave on ∂W_2 with respect to B



(c) Wave on ∂V_2 with respect to B (d) Wave w.r.t. B after isotopy

Figure 3.11: Wave of $c_Q (\neq c_P)$ w.r.t. Y and B

The isotopy from figure 3.11c to 3.11d may or may not decrease the intersection of c_Q with X, but this certainly does not increase the intersection.

Therefore, $n_{AQ} < n_{BQ} + n_{CQ}$ implies c_Q has a (b,b) or a (c,c) arc.

Moreover, if c_Q has a (b,b) arc, then by lemma 3.3.4, $n'_{bb} = n''_{bb} \geq 1$. Consequently, c_Q cannot have (c,c) or (a,c) arc i.e. $n'_{cc} = n''_{cc} = 0$ and $n'_{ac} = n''_{ac} = 0$. In this case,

$$n_{BQ} = 2n'_{bb} + n'_{bc} + n'_{ab} > n'_{bc} = n_{CQ}.$$

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Conversely, suppose $n_{BQ} > n_{CQ}$. By above, c_Q must have either (b, b) or (c, c) arc. If c_Q has a (c, c) arc then $n'_{cc} \geq 1$, there are no (b, b) or (a, b) arc, therefore $n'_{bb} = 0, n'_{ab} = 0$ and hence $n_{BQ} = n'_{bc} < 2n'_{cc} + n'_{bc} + n'_{ac} = n_{CQ}$. Therefore, $n_{CQ} > n_{BQ}$, a contradiction. Hence c_Q cannot have a (c, c) arc if $n_{BQ} > n_{CQ}$. Therefore, c_Q must have a (b, b) arc on either sides of Σ_2.

Thus $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{BQ} > n_{CQ}$ if and only if c_Q has a (b, b) arc.

By symmetric arguments, we conclude that $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{BQ} < n_{CQ}$ if and only if c_Q has a (c, c) arc. □

Corollary 3.3.6. *For $c_Q \neq c_P$, exactly one of the following three inequalities holds:*

(i) $n_{AQ} > n_{BQ} + n_{CQ}$

(ii) $n_{BQ} > n_{AQ} + n_{CQ}$

(iii) $n_{CQ} > n_{AQ} + n_{BQ}$

Proof. By lemma 3.3.2, $n_{AQ} > n_{BQ} + n_{CQ}$ if and only if c_Q has an (a, a) arc on either sides of Σ_2 and $n_{AQ} < n_{BQ} + n_{CQ}$ if and only if c_Q has a (b, b) or (c, c) arc on either sides of Σ_2. Moreover the above two cases are mutually exclusive.

Again by lemma 3.3.5, c_Q has (b, b) arc on either sides of Σ_2 if and only if $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{BQ} > n_{CQ}$. In such case c_Q cannot have any $(a, a), (c, c)$ or (a, c) arc. Therefore by equation 3.1 on Σ'_2 (similarly on Σ''_2), $n'_{bb} = n''_{bb} \geq 1$ and

$$n_{AQ} = n'_{ab}, n_{BQ} = 2n'_{bb} + n'_{ab} + n'_{bc}, \text{ and } n_{CQ} = n'_{bc}.$$

Therefore, $n_{BQ} = 2n'_{bb} + n'_{ab} + n'_{bc} > n_{AQ} + n_{CQ}$ if and only if c_Q has a (b, b) arc on either sides of Σ_2.

A similar argument shows that $n_{CQ} > n_{AQ} + n_{BQ}$ if and only if c_Q has a (c, c) arc on either sides of Σ_2.

Since, existence of $(a, a), (b, b)$ or (c, c) arc are mutually exclusive and c_Q must have atleast one of these three arcs, exactly one of the three inequalities must hold. □

3.3.1 β-reduction

In this section we first assume that the inequality $n_{AQ} > n_{BQ} + n_{CQ}$ holds and then using few results of our own we show that we can construct a curve A', an isotopic copy of either $\beta(A)$ or $\beta^{-1}(A)$, such that $c_Q \cdot A' < c_Q \cdot A$. Hence we show that β-reduction exists whenever the above inequality holds.

Lemma 3.3.7. *Let $Q \neq P$ be any reducing sphere. If $n_{AQ} > n_{BQ} + n_{CQ}$ then exactly one of the following occurs:*

(i) $\mathcal{C}(\beta(Q)) < \mathcal{C}(Q)$ *and* $\mathcal{C}(\beta^{-1}(Q)) > \mathcal{C}(Q)$*,*

(ii) $\mathcal{C}(\beta(Q)) > \mathcal{C}(Q)$ *and* $\mathcal{C}(\beta^{-1}(Q)) < \mathcal{C}(Q)$*.*

In any case if $n_{AQ} > n_{BQ} + n_{CQ}$, $\mathcal{C}(\varphi(Q)) > \mathcal{C}(Q)$ and $\mathcal{C}(\varphi\nu(Q)) > \mathcal{C}(Q)$.

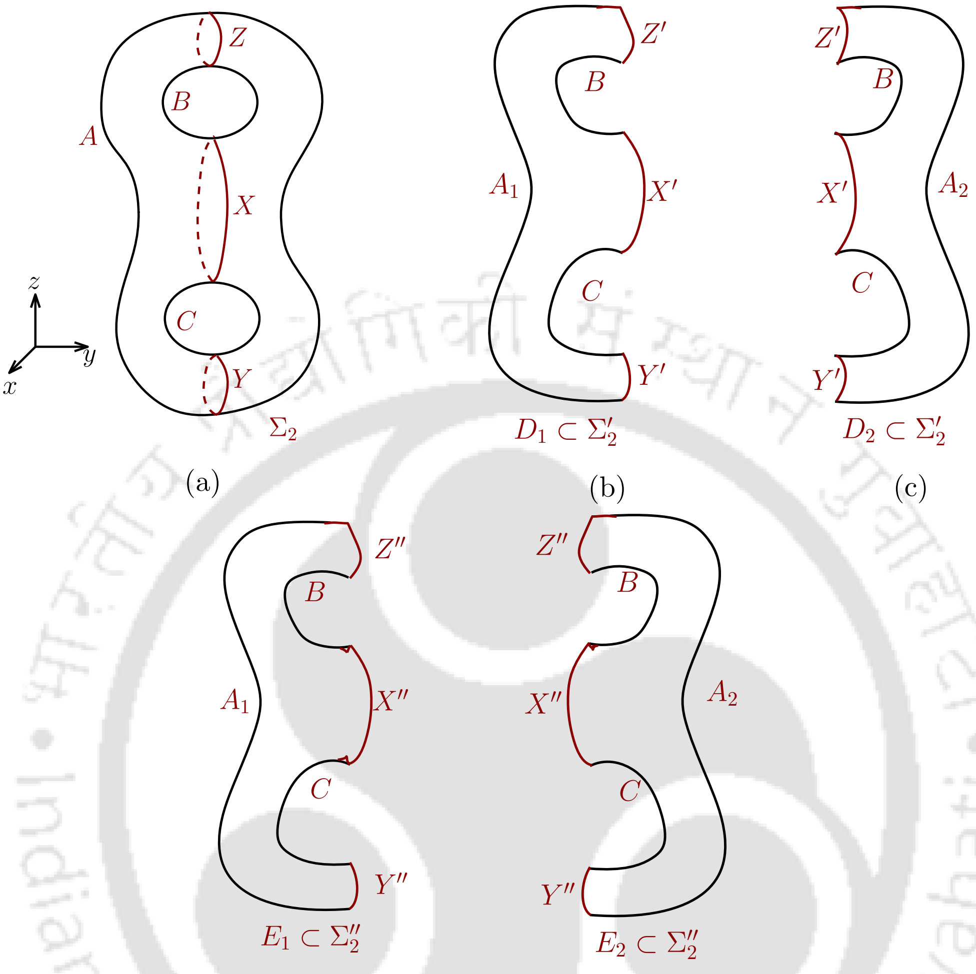
Proof of the beta reduction:

By our assumption that $n_{AQ} > n_{BQ} + n_{CQ}$, there exists an (a, a) arc on Σ_2' and an (a, a) arc on Σ_2''.

§ Setup and standard position of c_Q

We assume that c_Q is in minimal position with respect to A, B, C, X, Y, Z, and c_P. $A \cup B \cup C$ separate Σ_2 into two thrice-boundered spheres which we denote by Σ_2' and Σ_2''. We call Σ_2' and Σ_2'' the sides of Σ_2 throughout this proof. Denote $X \cap \Sigma_2'$ by X', $X \cap \Sigma_2''$ by X'', $Y \cap \Sigma_2'$ by Y', $Y \cap \Sigma_2''$ by Y'', $Z \cap \Sigma_2'$ by Z', and $Z \cap \Sigma_2''$ by Z''. We denote the two components of $A \setminus \{Y \cap A, Z \cap A\}$ by A_1 and A_2 as shown in figure 3.12. Further $X \cup Y \cup Z$ separate Σ_2' and Σ_2'' into disks. Such a disk on Σ_2' with one of the boundary as A_1 is denoted by D_1. The other disk on Σ_2' with one of the boundary arc as A_2 is denoted by D_2. Likewise the disk on Σ_2'' with one boundary arc as A_1 is denoted by E_1 and the disk on Σ_2'' with one boundary arc as A_2 is denoted by E_2.

We can arrange the standardly embedded genus-2 surface, ∂V_2 such that: (i) the points $Y \cap A$ and $Z \cap A$ lie along the z-axis with the z-value increasing from $Y \cap A$ to $Z \cap A$, (ii) no two points of A_1 have the same



(a) (b) (c)

(d) (e)

Figure 3.12: Sides of Σ_2 - $\Sigma_2^{'}$, $\Sigma_2^{''}$, and the arcs $X', Y', Z', X'', Y'', Z'', A_1, A_2$

z-coordinate and (iii) no two points of A_2 have the same z-coordinate. We will refer to this arrangement throughout this discussion. With this arrangement the z-coordinate can be thought of as a height function on A_1 and likewise on A_2. Owing to this observation, if $x_1, x_2 \in A_1$ and if the height of x_1 is greater than the height of x_2, then we say x_1 is above x_2 or that x_2 is below x_1. If $x_3 \in A_1$ such that the height of x_3 is between the heights of x_1 and x_2 then we say that x_3 is in between x_1 and x_2.

By an isotopy of c_Q, we can assume that it does not intersect the points $Z \cap A, Z \cap B, Y \cap A$ or $Y \cap C$. Since $|c_Q \cap A|$ is finite, there are only finitely many (a, a) arcs on $\Sigma_2^{'}$ or on $\Sigma_2^{''}$. These arcs are essential on $\Sigma_2^{'}$ or on $\Sigma_2^{''}$ as inessential arcs on a thrice boundered sphere are boundary reducible and a boundary reducing disk gives an isotopy of c_Q reducing the intersection number of c_Q with A. Also these (a, a) arcs do not intersect each other as c_Q is a simple curve. Since every essential arc on a thrice-boundered sphere from a boundary component to itself is separating, these (a, a) arcs are separating arcs of $\Sigma_2^{'}$ and $\Sigma_2^{''}$.

$A \cup B \cup C$ cuts c_Q into a sequence of arcs when we follow c_Q in a particular orientation. Starting at any one of these arcs and following the orientation of c_Q, we label these arcs as $\rho_1, \rho_2, \ldots, \rho_n$. We will follow a cyclic labelling for these arcs. For instance, ρ_{n+1} refers to ρ_1. A natural consequence of this ordering is that for each $k \in \mathbb{N}$, ρ_{2k-1} is on one side of Σ_2 and ρ_{2k} is on the other side of Σ_2. Without loss of generality we assume that for each $k \in \mathbb{N}$, ρ_{2k-1} is on $\Sigma_2^{'}$ and ρ_{2k} is on $\Sigma_2^{''}$.

Remark 3.3.8. *An (a, a) arc on $\Sigma_2^{'}$ intersects X' exactly once. Likewise, an (a, a) arc on $\Sigma_2^{''}$ intersects $\Sigma_2^{''}$ exactly once.*

Proof. Cut $\Sigma_2^{'}$ along X' to get an annulus S, which has two boundary components. One boundary component is the curve A and the other boundary component is a union of B and C with two copies of X' (see figure 3.13). For brevity, we denote these two boundary components of S by $\partial_A S$ and $\partial_X S$ respectively.

If an (a, a) arc does not intersect X', then it is an arc in the annulus S from a boundary component to itself and hence is boundary-reducible. The boundary-reducing disk is a bigon as the (a, a) arc intersects A only at its endpoints. Then by the bigon criterion, we have an isotopy of c_Q reducing its intersection with A contradicting the minimal intersection position of c_Q with A. So an (a, a) arc on $\Sigma_2^{'}$ has to intersect X' at least once.

Suppose now that an (a, a) arc, call it λ, intersects X' more than once. Orient λ from one end point on A to another and let $x_1, x_2, ..., x_n$ be the points of intersection of λ with X' listed in order when following the orientation of λ. S breaks λ into its component arcs on S. All such component

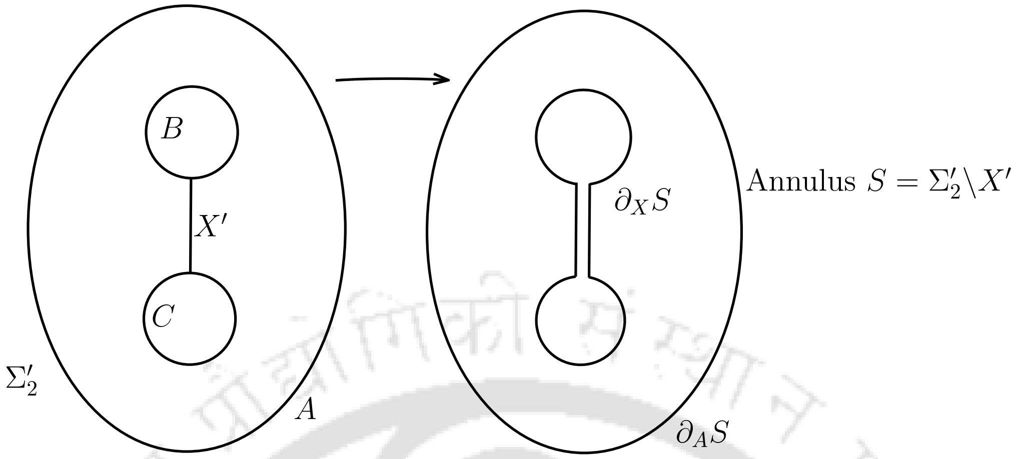


Figure 3.13: Annulus $S = \Sigma_2' \setminus X'$

arcs on the annulus S run from $\partial_X S$ to itself and hence are boundary reducible. Two boundary-reducing disks of such component arcs of λ are either disjoint or one is contained in the other, otherwise λ will have self-intersections. Hence by following a chain of containment of these boundary reducing disks, we get an innermost disk E which does not contain any other disk. We will now show that E is a bigon on the surface Σ_2 formed by λ and X'.

Let E be the boundary reducing disk for an arc λ_1 on S which is a component arc of λ on S. Let λ_1 considered on Σ_2' join x_i to x_j. E on S is bounded by λ_1 and a portion, call it ρ, of the boundary component $\partial_X S$ of S. If ρ includes B or C or both, then following the arc λ, in the orientation of λ or in the opposite orientation, beyond λ_1, we arrive at yet another component arc of λ, call it λ_2, on S which enters E (see figure 3.14).

Since λ_2 enters E, it is completely contained in E. This is because, λ_2 cannot intersect λ_1, B or C, so it has to intersect X' which then means that λ_2 is contained in E. Now, as remarked earlier, λ_2 is also boundary reducible and the boundary reduction disk of λ_2 to the boundary $\partial_X S$ is completely contained inside E, contradicting the assumption that E is the innermost disk. This proves that ρ cannot include B or C or both.

This implies that ρ contains only an arc from X'. So λ_1 along with ρ forms a bigon on the surface Σ_2. So by the bigon criterion there is an

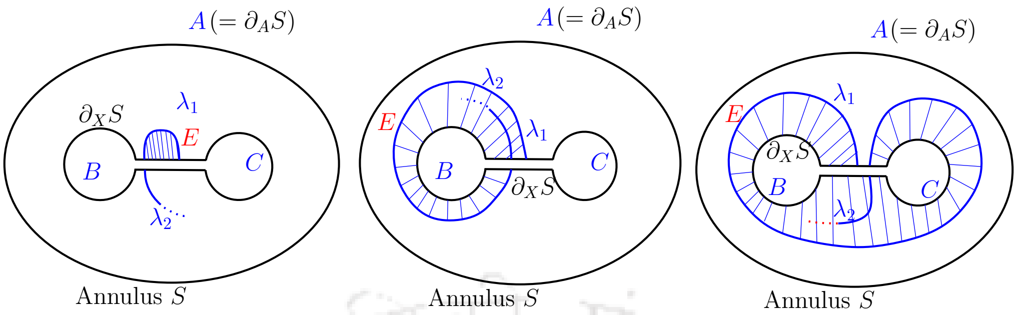


Figure 3.14: Any (a, a) arc intersects X exactly once

isotopy of c_Q reducing the intersection with X'. So if c_Q is in minimal intersection position with X, then an (a, a) arc on Σ_2' cannot intersect X' more than once.

Hence, we conclude that any (a, a) arc on Σ_2' intersects X' in exactly one point.

Likewise, we can repeat the same argument for an (a, a) arc on Σ_2'' and conclude that it intersects X'' in exactly one point. ☐

§ Fundamental Group of the handlebody V_2

Let D_X, D_Y and D_Z be the essential disks in V_2 bounded by the curves X, Y and Z respectively. Consider the core curves of the handlebody V_2, dual to the disks D_Z and D_Y, based at a point x_0 lying in the interior of D_X and denote them by Bx_0 and Cx_0 respectively. Note that Bx_0 and Cx_0 are freely isotopic in V_2 to B and C respectively. The element in $\pi_1(V_2, x_0)$ whose representative is a loop which traces Bx_0 once in the direction shown in figure 3.15 will be denoted by b. The element in $\pi_1(V_2, x_0)$ which traces Cx_0 once in the direction shown in this figure 3.15 will be denoted by c.

We consider the presentation $\langle b, c | - \rangle$ of the fundamental group of V_2 i.e. and henceforth refer to this presentation as $\pi_1(V_2)$. With a slight abuse of notation, we use the same letters to indicate the elements in $\pi_1(V_2)$ and the loops in V_2. Consider some orientation of c_Q. Give the $(a, a), (a, b)$ and (a, c) arcs of c_Q on Σ_2' and Σ_2'' the induced orientation. An (a, a) arc on Σ_2' or Σ_2'' has its end points on A_1 or A_2. One can draw an arc from any

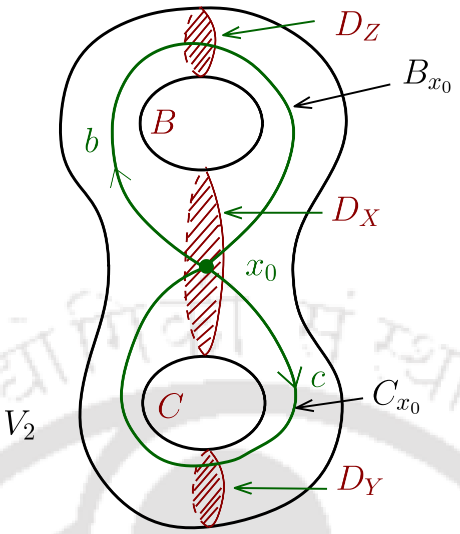


Figure 3.15: Fundamental group of handlebody V_2

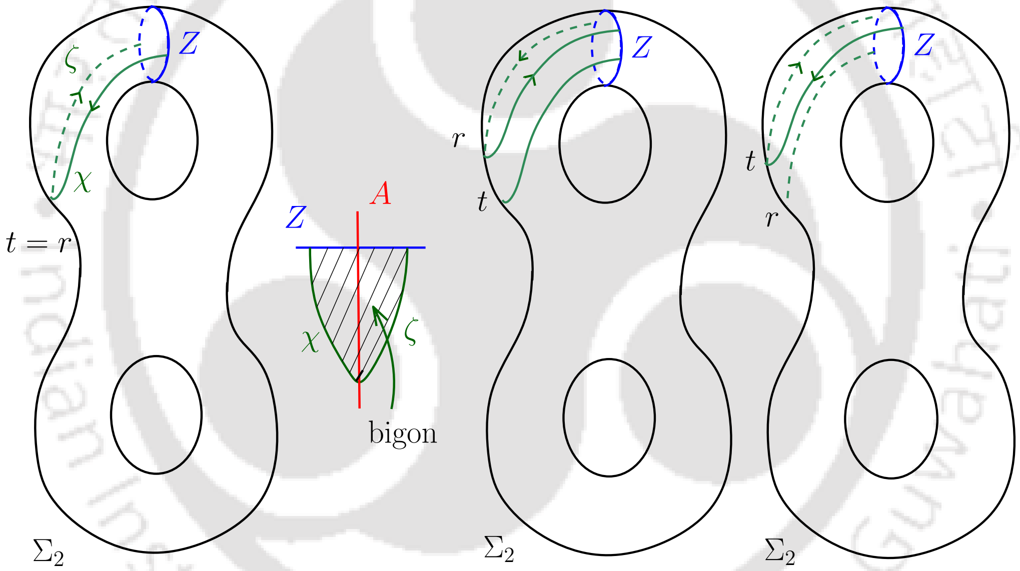
point on A_1 or from any point on A_2 to the point x_0 such that the arc is completely contained in V_2 and such that the arc does not intersect D_Y or D_Z. So by attaching such arcs from the endpoints of an (a, a) arc to x_0 and giving these arcs the orientation following the orientation of the (a, a) arc, we get a loop in V_2 and hence an element in $\pi_1(V_2)$. Likewise, an (a, b) or an (a, c) arc on Σ_2' or on Σ_2'' have end points on the curves A, B or C avoiding the points $A \cap Z, B \cap Z, A \cap Y$ and $C \cap Y$. One can draw an arc from the endpoints of these (a, b) or (a, c) arcs to x_0 lying entirely in V_2 such that the arc does not intersect D_Y or D_Z. We give these arcs the orientation induced by the (a, b) and (a, c) arcs. So for each (a, b) arc or (a, c) arc we get corresponding loops in V_2 and hence elements of $\pi_1(V_2)$. So every $(a, a), (a, b)$ and (a, c) arc defines a unique reduced word in $\pi_1(V_2)$ this way, which we call the word of that particular arc. Further, since c_Q is obtained by stitching its $(a, a), (a, b)$ and (a, c) arcs in the order c_Q describes them, the word in $\pi_1(V_2)$ described by c_Q is obtained by taking the product of the words described by these arcs in that order in $\pi_1(V_2)$. Since c_Q bounds a disk in V_2, it describes a word in $\pi_1(V_2)$ which reduces to the trivial word. We will use this fact to show that when $n_{AQ} > n_{BQ} + n_{CQ}$, the $(a, a), (a, b)$ and (a, c) arcs of c_Q cannot "wind around the genus holes too much" i.e.

their individual reduced words cannot have large word lengths.

§ Restriction on arcs of c_Q

Lemma 3.3.9. *Let F and G be any two of the four disks D_1, D_2, E_1, E_2 such that $F \cap G$ is either A_1 or A_2. Then, c_Q cannot have arcs on both F and G with one end-point on Z and the other on A. Likewise, c_Q cannot have arcs on both F and G with one end-point on Y and the other on A.*

Proof. We only need to prove the lemma in the case where $F = D_1, G = E_1$. The proof for all other cases follow by symmetry of Σ_2.



(a) t and r coincide (b) t is below r (c) t is above r

Figure 3.16: $(*, a) \vee (a, *)$ concatenation (i.e. $\chi \vee \eta$) bounding a ZAZ bigon

Let χ be an arc of c_Q in D_1 with one endpoint on Z' and the other on A_1. Let $t := \chi \cap A_1$. If possible let ζ be an arc on D_2 with end-points on Z'' and A_1 and $r := \zeta \cap A_1$. If $t = r$, then χ and ζ bound a bigon with Z on Σ_2 contradicting the minimal intersection position of c_Q with Z. If t is below r on A_1, then the continuation arc of ζ on D_1 is contained in the disk cut out of D_1 by χ, Z' and A_1 and hence is another arc ζ_1 with one endpoint as r and the other on Z'. But then, ζ and ζ_1 bound a bigon

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with Z on Σ_2, once again contradicting the minimal intersection position of c_Q with Z. If r is below t, then the continuation arc of χ on D_2 and χ together bound a bigon with Z leading us to a similar contradiction (figure 3.16).

So r cannot be above or below t on A_1 nor can it coincide with t and hence ζ does not exist. The proof for the case where χ is an arc of c_Q in D_1 with one endpoint on Y and the other on A_1 is similar. ☐

Corollary 3.3.10. *When the word of a* $(*, a)$ *arc of* c_Q *is concatenated with word of the following* $(a, *)$ *arc, the concatenated word cannot have trivial relators of* $\pi_1(V_2)$.

Proof. This is so because a trivial relator at the word interface contradicts Lemma 3.3.9.

Suppose that ρ_i is a $(*, a)$ arc on Σ_2' whose word ends with a letter b, and that ρ_{i+1} is an $(a, *)$ arc on Σ_2'' whose word starts with a letter b^{-1}. The terminal sub-arc, χ, of the arc ρ_i on Σ_2' which starts on Z and ends on A such that the interior of χ is disjoint from A, B, C, X, Y and Z and the initial sub-arc, ζ, of ρ_{i+1} on Σ_2'' starting on A and ending on Z such that the interior of ζ is disjoint from A, B, C, X, Y and Z are two arcs as in the proof of Lemma 3.3.9 whose endpoint on A coincide (figure 3.17). This is impossible by Lemma 3.3.9.

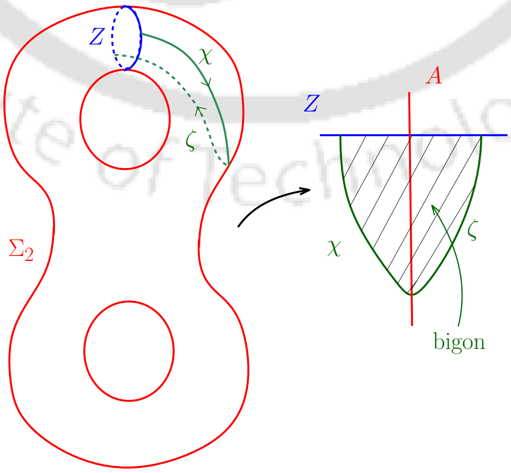


Figure 3.17: $(*, a) \vee (a, *)$ concatenation cannot have trivial relators

Similarly, if the word of ρ_i ends with any letter x, where x is one of b, b^{-1}, c or c^{-1} and the word of ρ_{i+1} begins with x^{-1} then c_Q forms a bigon with Y or Z on Σ_2 and this will lead to a contradiction. Since the roles of $\Sigma_2^{'}$ and $\Sigma_2^{''}$ can be reversed in the above argument, the corollary is proved. □

Corollary 3.3.11. *If the word of a $(*, b)$ arc of c_Q which ends in b or b^{-1} is concatenated with the word of the following $(b, *)$ arc then the concatenated word cannot have a trivial relator of $\pi_1(V_2)$. Similarly, if the word of a $(*, c)$ arc of c_Q which ends in c or c^{-1} is concatenated with the word of the following $(c, *)$ arc then the concatenated word cannot have a trivial relator of $\pi_1(V_2)$.*

Proof. As in the proof of Corollary 3.3.10 if ρ_i is an (a, b) arc on one side of Σ_2 whose word ends with a letter b or b^{-1}, then ρ_{i+1} will be a (b, a) arc on the other side of Σ_2, and its word cannot begin with b^{-1} or b respectively, because if it does, then there will be a bigon formed by a sub-arc of $\rho_i \cup \rho_{i+1}$ and Z leading to a contradiction with the assumption about c_Q (see figure 3.18).

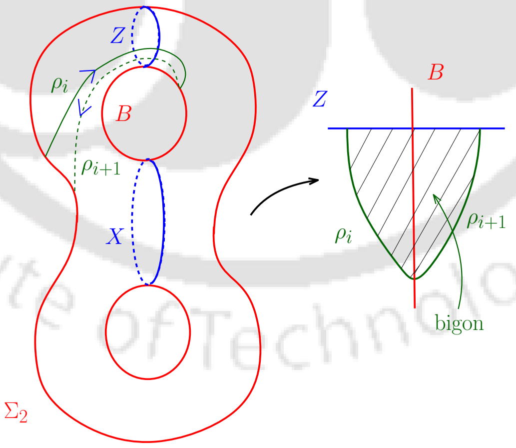


Figure 3.18: $(a, b) \vee (b, a)$ concatenation cannot have trivial relators

Analogously, if ρ_i is an (a, c) arc on one side of Σ_2 whose word ends with a letter c or c^{-1}, then ρ_{i+1} will be a (c, a) arc on the other side of Σ_2, and its word cannot begin with c^{-1} or c respectively, because if it does, then there will be a bigon formed by a sub-arc of $\rho_i \cup \rho_{i+1}$ and Y leading to a contradiction with the assumption about c_Q. □

Lemma 3.3.12. *A concatenated word of a sequence of* $(*, a), (a, b)$ *and* (b, a) *arcs of* c_Q *cannot contain the trivial relators* bb^{-1} *or* $b^{-1}b$*. Similarly, a concatenated word of a sequence of* $(*, a), (a, c)$ *and* (c, a) *arcs of* c_Q *cannot contain the trivial relators* cc^{-1} *or* $c^{-1}c$*.*

Proof. Corollary 3.3.11 implies that a trivial relator bb^{-1} or $b^{-1}b$ cannot occur when the word of a $(*, a)$ arc is concatenated with the word of a (a, b) arc. Corollary 3.3.10 implies that a trivial relator bb^{-1} or $b^{-1}b$ cannot occur when the word of a (a, b) arc is concatenated with the word of a (b, a) arc. So the only case remaining is when the word of the arc (a, b) is the empty word $\{\}$ and so the concatenated word of the $(*, a) - (a, b) - (b, a)$ arc sequence is of the form $w_1 b\{\}b^{-1}w_2$ or of the form $w_1 b^{-1}\{\}bw_2$ where w_1 and w_2 are some words. As in the proof of Corollary 3.3.11, this will imply that the terminal arc of the $(*, a)$ arc, the whole of (a, b) arc and the initial arc of the (b, a) arc form a bigon with Z as shown in figure 3.19. This

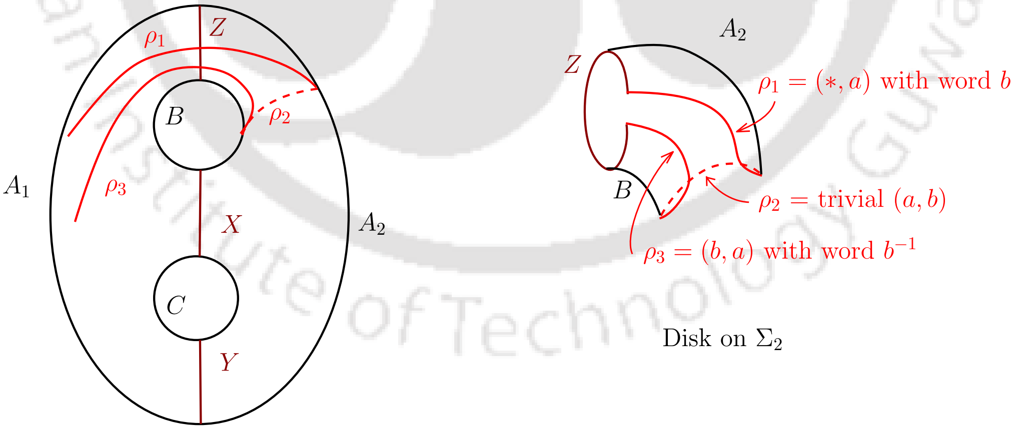


Figure 3.19: Concatenation of $(*, a), (a, b), (b, a)$ arcs bounds bigon on Σ_2 with Z

contradicts the assumption about c_Q. This proves the first statement of the lemma. The proof of the second statement of this lemma is similar. □

§ (a, a) arcs have word length atmost two

Theorem 3.3.13. *The word in $\pi_1(V)$ of an (a, a) arc of c_Q can have a length at most 2. Correspondingly, the word of an (a, b) or an (a, c) arc of c_Q can have a length at most 1.*

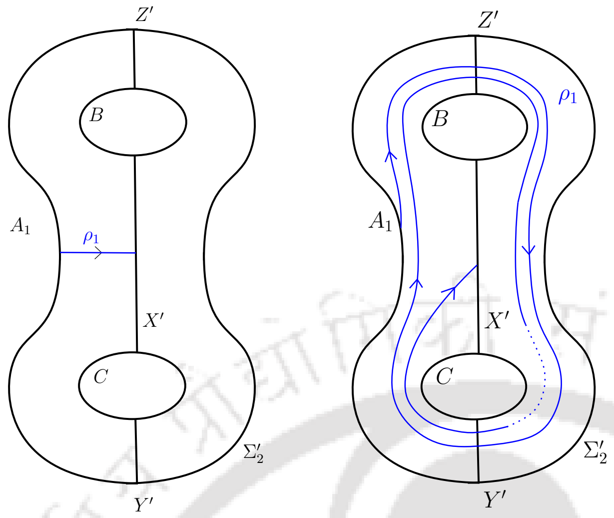
Proof. Let m be a non-negative integer, ϵ_1, ϵ_2 denote 0 or 1 and k be any integer for the following discussion. We will first describe the words of (a, a) arcs on Σ_2'. The same argument applies to words of (a, a) arcs on Σ_2''. Let S be the annulus $\Sigma_2' \backslash X'$. By Remark 3.3.8, an (a, a) arc of c_Q on Σ_2' intersects X' exactly once. Consider an (a, a) arc, ρ on Σ_2'. In the annulus S, ρ is a sequence of two sub-arcs, ρ_1 from A to X' and ρ_2 from X' to A. Since c_Q is in minimal intersection position with X, Y and Z, both ρ_1 and ρ_2 intersect X', Y' and Z' minimally.

As a first case, suppose that ρ_1 is an arc from A_1 to X'. If ρ_1 does not intersect Y' or Z', then then it contributes an empty word to the word of c_Q. If ρ_1 intersects Z' before intersecting Y', then we say it turns clockwise and it contributes a word of the form $b(cb)^m c^{\epsilon_1}$ to the word of c_Q (see figure 3.20). If ρ_1 intersects Y' before intersecting Z', then we say it turns anti-clockwise and it contributes a word of the form $c^{-1}(b^{-1}c^{-1})^m b^{-\epsilon_1}$ to the word of c_Q (see figure 3.21).

As a second case, suppose that ρ_1 is an arc from A_2 to X'. If ρ_1 does not intersect Y' or Z', then it contributes an empty word to the word of c_Q. If ρ_1 intersects Y' before intersecting Z', then we say it turns clockwise and it contributes a word of the form $c(bc)^m b^{\epsilon_1}$ to the word of c_Q (see Figure 3.22). If ρ_1 intersects Z' before intersecting Y', then we say it turns anti-clockwise and it contributes a word of the form $b^{-1}(c^{-1}b^{-1})^m c^{-\epsilon_1}$ to the word of c_Q (see figure 3.23).

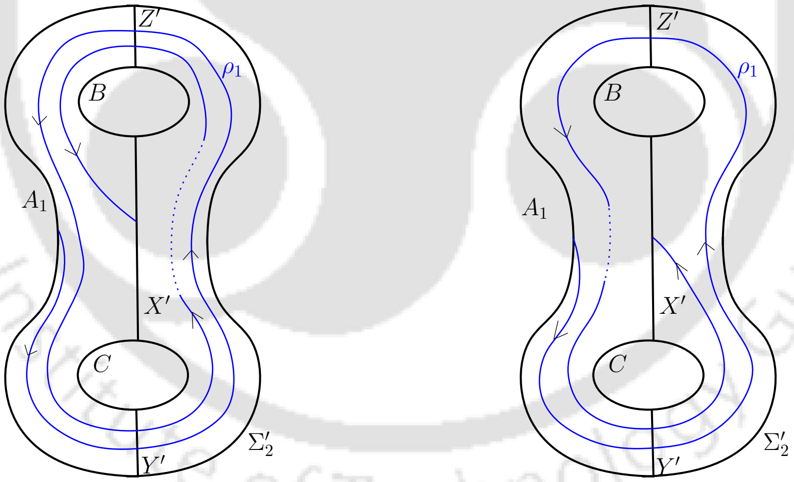
The word of ρ_2 should be almost the same as the inverse word of ρ_1 as they both are essential arcs in the same annulus S, however ρ_2 has a different initial letter and a freedom of a half or one turn in the annulus S at the terminal end which could potentially change the end letters.

Figure 3.24 shows the three different possibilities of words for ρ_2 when the word for ρ_1 is bcb.



(a) $\rho_1 \to$ empty word (b) $\rho_1 \to b(cb)^m c$ (c) $\rho_1 \to b(cb)^m c^0$

Figure 3.20: ρ_1 starts from A_1 and turns clockwise describing either the empty word or $b(cb)^m c^\epsilon$, $\epsilon = 0, 1$



(a) $\rho_1 \to c^{-1}(b^{-1}c^{-1})^m b^{-1}$ (b) $\rho_1 \to c^{-1}(b^{-1}c^{-1})^m b^{-0}$

Figure 3.21: ρ_1 starts from A_1 and turns anticlockwise describing the word $c^{-1}(b^{-1}c^{-1})^m b^\epsilon$, $\epsilon = 0, 1$

We conclude that the possibility of words for an (a, a) arc are as follows. The word of an (a, a) arc which starts on A_1 and ends on A_2 could be an empty word, or a word of the form $b(cb)^m c^{\epsilon_1} b^{-\epsilon_1}(c^{-1}b^{-1})^m c^{-1}$ or of

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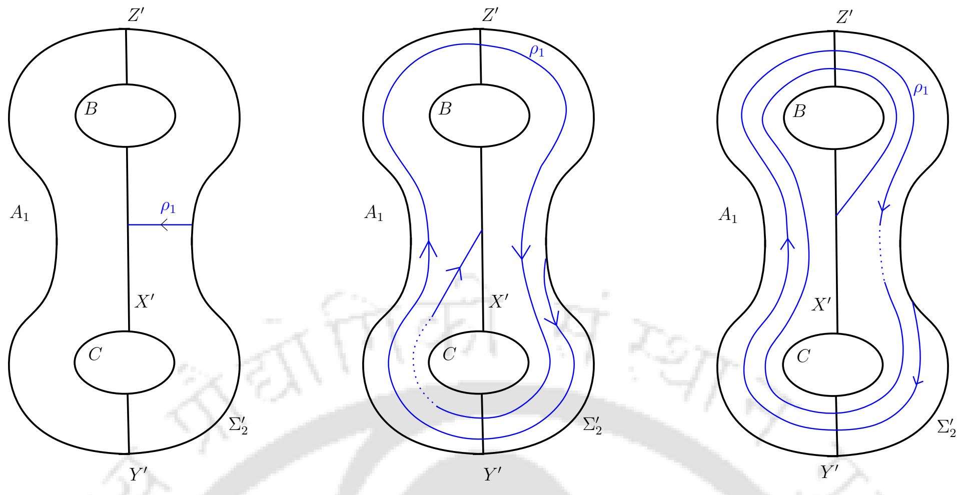
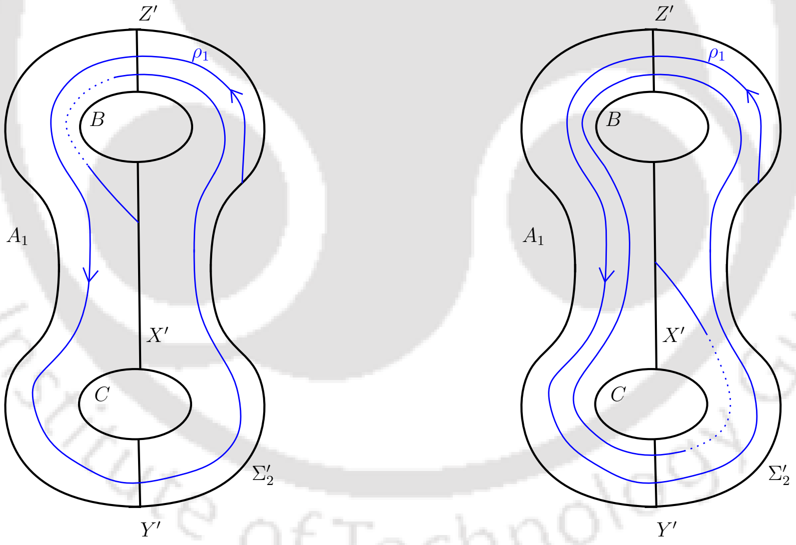


(a) $\rho_1 \to$ empty word (b) $\rho_1 \to c(bc)^m b^0$ (c) $\rho_1 \to c(bc)^m b^1$

Figure 3.22: ρ_1 starting from A_2 turns clockwise describing either the empty word or $c^{-1}(b^{-1}c^{-1})^m b^{\epsilon}$, $\epsilon = 0, 1$



(a) $\rho_1 \to b^{-1}(c^{-1}b^{-1})^m c^0$ (b) $\rho_1 \to b^{-1}(c^{-1}b^{-1})^m c^{-1}$

Figure 3.23: ρ_1 starts from A_2 and turns anticlockwise describing the word $b^{-1}(c^{-1}b^{-1})^m c^{-\epsilon}$, $\epsilon = 0, 1$

the form $c^{-1}(b^{-1}c^{-1})^m b^{-\epsilon_1} c^{\epsilon_1}(bc)^m b$. An (a, a) arc which starts on A_2 and ends on A_1 describes a word which is the inverse word of the above word forms from A_1 to A_2. So an (a, a) arc from A_2 to A_1 could describe an

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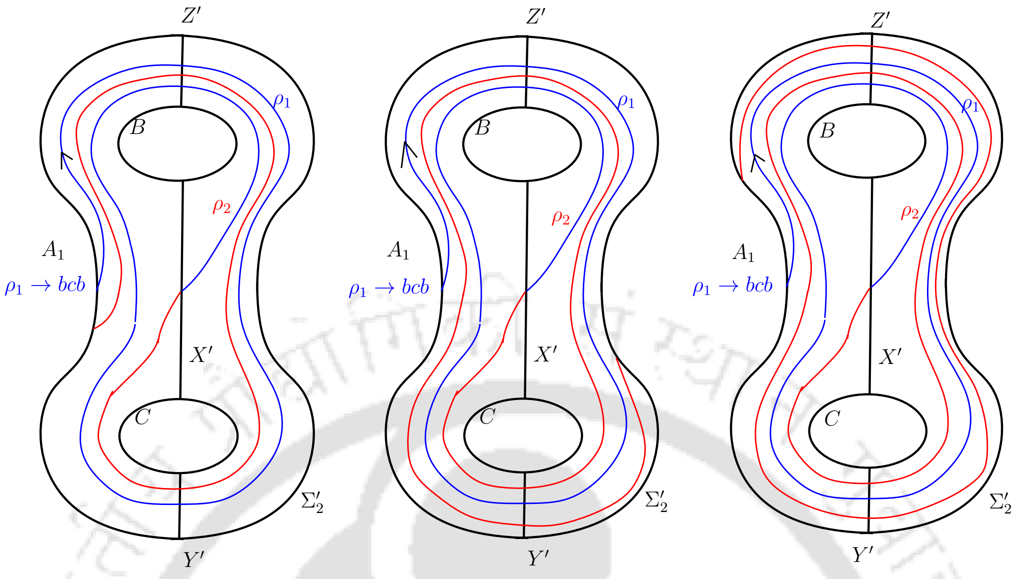


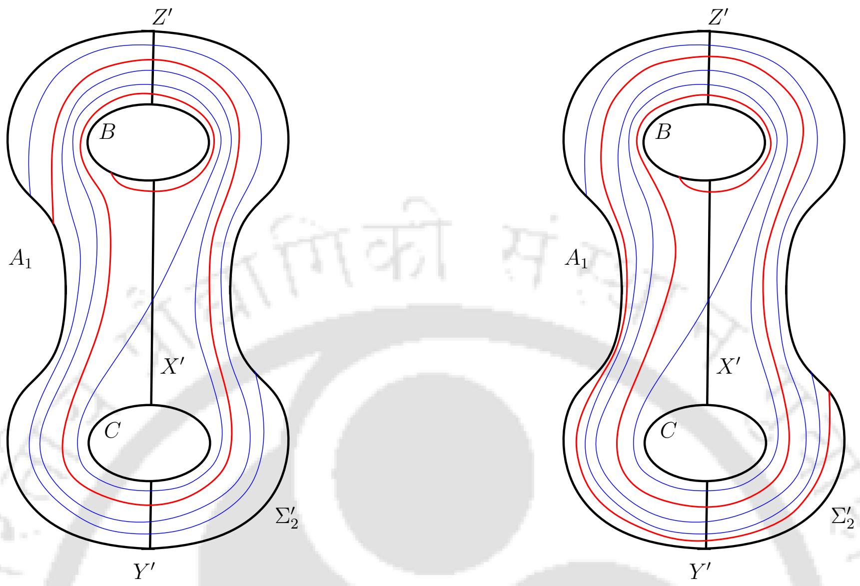
(a) $\rho_2 \to c^{-1}b^{-1}$ (b) $\rho_2 \to c^{-1}(b^{-1}c^{-1})$ (c) $\rho_2 \to c^{-1}(b^{-1}c^{-1})b^{-1}$

Figure 3.24: Possible cases of ρ_2 when word of ρ_1 is bcb

empty word or a word of the form $c(bc)^m b^{\epsilon_1} c^{-\epsilon_1}(b^{-1}c^{-1})^m b^{-1}$ or of the form $b^{-1}(c^{-1}b^{-1})^m c^{-\epsilon_1} b^{\epsilon_1}(cb)^m c$.

An (a, a) arc from A_1 to A_1 can be seen as an (a, a) arc from A_1 to A_2 augmented or truncated by a half turn in the annulus S in a direction so that there is no intersection-number-reducing isotopy with A, B, C, X, Y and Z. Hence, the word of an (a, a) arc from A_1 to A_1 is b, b^{-1}, c or c^{-1}, or of the form $b(cb)^m c^{\epsilon_1} b^{-\epsilon_1}(c^{-1}b^{-1})^{m+\epsilon_2}$ or $c^{-1}(b^{-1}c^{-1})^m b^{-\epsilon_1} c^{\epsilon_1}(bc)^{m+\epsilon_2}$. Likewise, an (a, a) arc from A_2 to A_2 can be seen as an (a, a) arc from A_2 to A_1 augmented or truncated by a half turn in a direction so that there is no intersection-number-reducing isotopy with A, B, C, X, Y and Z. Hence, the word of an (a, a) arc from A_2 to A_2 is b, b^{-1}, c or c^{-1} or of the form $c(bc)^m b^{\epsilon_1} c^{-\epsilon_1}(b^{-1}c^{-1})^{m+\epsilon_2}$ or $b^{-1}(c^{-1}b^{-1})^m c^{-\epsilon_1} b^{\epsilon_1}(cb)^{m+\epsilon_2}$.

The words of (a, b), (b, a), (a, c) and the (c, a) arcs on Σ_2' , if any, of c_Q are constrained by the (a, a) arc present on Σ_2'. In particular, if Σ_2' contains an (a, a) arc whose word contains more than two letters, then none of the (a, b), (b, a), (a, c) or the (c, a) arcs on Σ_2' describe a trivial word in $\pi_1(V_2)$. This is evident from the way an (a, a) arc describing a word in $\pi_1(V_2)$ with more than two letters cuts out an annulus from Σ_2' (see figure 3.25).



(a) $(a,a) \to b(cb)(c^{-1}b^{-1})c^{-1}$, and $(a,b) \to bcb^k$

(b) $(a,a) \to b(cb)(c^{-1}b^{-1})c^{-1}$, and $(a,b) \to c(bc)b^k$

Figure 3.25: Words of (a,b) arcs depend on the words of (a,a) arcs

Table 3.3 gives the various possibilities for (a,a), (a,b), (b,a), (a,c) and the (c,a) arcs on Σ_2'.

Case 1: A_1 **to** A_2, $\epsilon_1 = 0$

Arc	Start	End	Word	Remark
(a,a)	A_1	A_2	$b(cb)^m[c^0b^{-0}](c^{-1}b^{-1})^mc^{-1}$	$\epsilon_1 = 0$, $m \in \mathbb{Z}^+$
(a,a)	A_2	A_1	inverse of above	
(a,b)	A_1	B	$b(cb)^{m-1}cb^k$, $k \in \mathbb{Z}$	
(a,b)	A_2	B	$cb(cb)^{m-1}cb^k$, $k \in \mathbb{Z}$	
(a,c)	A_1	C	$b(cb)^mc^k$, $k \in \mathbb{Z}$	
(a,c)	A_2	C	$(cb)^mc^k$, $k \in \mathbb{Z}$	(Clockwise from start)

Case 2: A_1 **to** A_2, $\epsilon_1 = 1$

Arc	Start	End	Word	Remark
(a,a)	A_1	A_2	$b(cb)^m[c^1b^{-1}](c^{-1}b^{-1})^mc^{-1}$	$\epsilon_1 = 1$, $m \in \mathbb{Z}^+$

(a,a)	A_2	A_1	inverse of above	
(a,b)	A_1	B	$b(cb)^m cb^k,\ k \in \mathbb{Z}$	
(a,b)	A_2	B	$(cb)^m cb^k,\ k \in \mathbb{Z}$	
(a,c)	A_1	C	$b(cb)^m c^k,\ k \in \mathbb{Z}$	
(a,c)	A_2	C	$cb(cb)^m c^k,\ k \in \mathbb{Z}$	(Clockwise from start)

Case 3: A_1 to A_1, $\epsilon_1 = \epsilon_2 = 0$

Arc	Start	End	Word	Remark
(a,a)	A_1	A_1	$b(cb)^m[c^0b^{-0}](c^{-1}b^{-1})^{m+0}$	$\epsilon_1 = \epsilon_2 = 0$, $m \in \mathbb{Z}^+$
(a,a)	A_1	A_1	inverse of above	
(a,b)	A_1	B	$b(cb)^{m-1}cb^k,\ k \in \mathbb{Z}$	
(a,b)	A_2	B	not possible	
(a,c)	A_1	C	$b(cb)^m c^k$ or $b(cb)^{m-1}c^k,\ k \in \mathbb{Z}$	depends on starting points relative to starting points of (a,a)
(a,c)	A_2	C	$(cb)^m c^k,\ k \in \mathbb{Z}$	(Clockwise from start)

Case 4: A_1 to A_1, $\epsilon_1 = \epsilon_2 = 1$

Arc	Start	End	Word	Remark
(a,a)	A_1	A_1	$b(cb)^m[c^1b^{-1}](c^{-1}b^{-1})^{m+1}$	$\epsilon_1 = \epsilon_2 = 1$, $m \in \mathbb{Z}^+$
(a,a)	A_1	A_1	inverse of above	
(a,b)	A_1	B	$b(cb)^m cb^k,\ k \in \mathbb{Z}$	
(a,b)	A_2	B	not possible	
(a,c)	A_1	C	$b(cb)^m cbc^k$ or $b(cb)^m c^k,\ k \in \mathbb{Z}$	depends on starting points relative to starting points of (a,a)
(a,c)	A_2	C	$cb(cb)^m c^k,\ k \in \mathbb{Z}$	(Clockwise from start)

Case 5: A_1 to A_1, $\epsilon_1 = 0, \epsilon_2 = 1$

Arc	Start	End	Word	Remark

(a,a)	A_1	A_1	$b(cb)^m[c^0b^{-0}](c^{-1}b^{-1})^{m+1}$	$\epsilon_1=0,\epsilon_2=1,\ m\in\mathbb{Z}^+$
(a,a)	A_1	A_1	inverse of above	
(a,b)	A_1	B	$b(cb)^{m-1}cb^k,\ k\in\mathbb{Z}$	
(a,b)	A_2	B	$cb(cb)^{m-1}cb^k,\ k\in\mathbb{Z}$	
(a,c)	A_1	C	$b(cb)^mc^k,\ k\in\mathbb{Z}$	(Clockwise from start)
(a,c)	A_2	C	not possible	

Case 6: A_1 to A_1, $\epsilon_1=1,\epsilon_2=0$

Arc	Start	End	Word	Remark
(a,a)	A_1	A_1	$b(cb)^m[c^1b^{-1}](c^{-1}b^{-1})^{m+0}$	$\epsilon_1=1,\epsilon_2=0,\ m\in\mathbb{Z}^+$
(a,a)	A_1	A_1	inverse of above	
(a,b)	A_1	B	$b(cb)^mcb^k$ or $b(cb)^{m-1}cb^k,\ k\in\mathbb{Z}$	
(a,b)	A_2	B	$(cb)^mcb^k,\ k\in\mathbb{Z}$	
(a,c)	A_1	C	$b(cb)^mc^k,\ k\in\mathbb{Z}$	(Clockwise from start)
(a,c)	A_2	C	not possible	

Case 7: A_2 to A_2, $\epsilon_1=\epsilon_2=0$

Arc	Start	End	Word	Remark
(a,a)	A_2	A_2	$c(bc)^m[b^0c^{-0}](b^{-1}c^{-1})^{m+0}$	$\epsilon_1=\epsilon_2=0,\ m\in\mathbb{Z}^+$
(a,a)	A_2	A_2	inverse of above	
(a,b)	A_1	B	$bc(bc)^{m-1}b^k=(bc)^mb^k,\ k\in\mathbb{Z}$	
(a,b)	A_2	B	$c(bc)^{m-1}b^k$ or $c(bc)^mb^k,\ k\in\mathbb{Z}$	
(a,c)	A_1	C	not possible	
(a,c)	A_2	C	$c(bc)^{m-1}bc^k,\ k\in\mathbb{Z}$	(Clockwise from start)

Case 8: A_2 to A_2, $\epsilon_1=\epsilon_2=1$

Arc	Start	End	Word	Remark
(a,a)	A_2	A_2	$c(bc)^m[b^1c^{-1}](b^{-1}c^{-1})^{m+1}$	$\epsilon_1=\epsilon_2=1,\ m\in\mathbb{Z}^+$
(a,a)	A_2	A_2	inverse of above	

(a,b)	A_1	B	$bc(bc)^m b^k,\ k\in\mathbb{Z}$	
(a,b)	A_2	B	$c(bc)^m b^k$ or $c(bc)^{m+1}b^k,\ k\in\mathbb{Z}$	
(a,c)	A_1	C	not possible	
(a,c)	A_2	C	$c(bc)^m bc^k,\ k\in\mathbb{Z}$	(Clockwise from start)

Case 9: A_2 to A_2, $\epsilon_1=0,\epsilon_2=1$

Arc	Start	End	Word	Remark
(a,a)	A_2	A_2	$c(bc)^m[b^0c^{-0}](b^{-1}c^{-1})^{m+1}$	$\epsilon_1=0,\epsilon_2=1,\ m\in\mathbb{Z}^+$
(a,a)	A_2	A_2	inverse of above	
(a,b)	A_1	B	$bc(bc)^{m-1}b^k,\ k\in\mathbb{Z}$	
(a,b)	A_2	B	$c(bc)^{m-1}b^k$ or $c(bc)^m b^k,\ k\in\mathbb{Z}$	
(a,c)	A_1	C	not possible	
(a,c)	A_2	C	$c(bc)^{m-1}bc^k,\ k\in\mathbb{Z}$	(Clockwise from start)

Case 10: A_2 to A_2, $\epsilon_1=1,\epsilon_2=0$

Arc	Start	End	Word	Remark
(a,a)	A_2	A_2	$c(bc)^m[b^1c^{-1}](b^{-1}c^{-1})^{m+0}$	$\epsilon_1=1,\epsilon_2=0,\ m\in\mathbb{Z}^+$
(a,a)	A_2	A_2	inverse of above	
(a,b)	A_1	B	not possible	
(a,b)	A_2	B	$c(bc)^m b^k,\ k\in\mathbb{Z}$	
(a,c)	A_1	C	$c(bc)^{m-1}bc^k$ or $c(bc)^m bc^k$	
(a,c)	A_2	C	$bc(bc)^{m-1}bc^k=(bc)^m bc^k,\ k\in\mathbb{Z}$	(Clockwise from start)

Case 11: A_1 to A_2, $\epsilon_1=0$ (compare with case 1)

Arc	Start	End	Word	Remark
(a,a)	A_1	A_2	$c^{-1}(b^{-1}c^{-1})^m[b^{-0}c^0](bc)^m b$	$\epsilon_1=0,\ m\in\mathbb{Z}^+$
(a,a)	A_2	A_1	inverse of above	
(a,b)	A_1	B	$c^{-1}(b^{-1}c^{-1})^m b^k,\ k\in\mathbb{Z}$	
(a,b)	A_2	B	$(b^{-1}c^{-1})^m b^k,\ k\in\mathbb{Z}$	
(a,c)	A_1	C	$c^{-1}(b^{-1}c^{-1})^{m-1}b^{-1}c^k,\ k\in\mathbb{Z}$	

(a,c)	A_2	C	$b^{-1}c^{-1}(b^{-1}c^{-1})^{m-1}b^{-1}c^k$ $=$ $(b^{-1}c^{-1})^m b^{-1}c^k$	(Anticlockwise from start)

Case 12: A_1 to A_2, $\epsilon_1 = 1$ (compare with case 2)

Arc	Start	End	Word	Remark
(a,a)	A_1	A_2	$c^{-1}(b^{-1}c^{-1})^m[b^{-1}c^1](bc)^m b$	$\epsilon_1 = 1$, $m \in \mathbb{Z}^+$
(a,a)	A_2	A_1	inverse of above	
(a,b)	A_1	B	$c^{-1}(b^{-1}c^{-1})^m b^k$, $k \in \mathbb{Z}$	
(a,b)	A_2	B	$b^{-1}c^{-1}(b^{-1}c^{-1})^m b^k$, $k \in \mathbb{Z}$	
(a,c)	A_1	C	$c^{-1}(b^{-1}c^{-1})^m b^{-1}c^k$, $k \in \mathbb{Z}$	
(a,c)	A_2	C	$(b^{-1}c^{-1})^m b^{-1}c^k$, $k \in \mathbb{Z}$	(Anticlockwise from start)

Case 13: A_1 to A_1, $\epsilon_1 = \epsilon_2 = 0$ (compare with case 7)

Arc	Start	End	Word	Remark
(a,a)	A_1	A_1	$c^{-1}(b^{-1}c^{-1})^m[b^{-0}c^0](bc)^{m+0}$	$\epsilon_1 = \epsilon_2 = 0$, $m \in \mathbb{Z}^+$
(a,a)	A_1	A_1	inverse of above	
(a,b)	A_1	B	$c^{-1}(b^{-1}c^{-1})^{m-1}b^k$ or $c^{-1}(b^{-1}c^{-1})^m b^k$, $k \in \mathbb{Z}$	
(a,b)	A_2	B	$(b^{-1}c^{-1})^m b^k$, $k \in \mathbb{Z}$	
(a,c)	A_1	C	$c^{-1}(b^{-1}c^{-1})^{m-1}b^{-1}c^k$, $k \in \mathbb{Z}$	
(a,c)	A_2	C	not possible	(Anticlockwise from start)

Case 14: A_1 to A_1, $\epsilon_1 = \epsilon_2 = 1$ (compare with case 8)

Arc	Start	End	Word	Remark
(a,a)	A_1	A_1	$c^{-1}(b^{-1}c^{-1})^m[b^{-1}c^1](bc)^{m+1}$	$\epsilon_1 = \epsilon_2 = 1$, $m \in \mathbb{Z}^+$
(a,a)	A_1	A_1	inverse of above	
(a,b)	A_1	B	$c^{-1}(b^{-1}c^{-1})^m b^k$ or $c^{-1}(b^{-1}c^{-1})^{m+1}b^k$, $k \in \mathbb{Z}$	
(a,b)	A_2	B	$b^{-1}c^{-1}(b^{-1}c^{-1})^m b^k$ $=$ $(b^{-1}c^{-1})^{m+1}b^k$, $k \in \mathbb{Z}$	
(a,c)	A_1	C	$c^{-1}(b^{-1}c^{-1})^m b^{-1}c^k$, $k \in \mathbb{Z}$	
(a,c)	A_2	C	not possible	(Anticlockwise from start)

Case 15: A_1 to A_1, $\epsilon_1 = 0, \epsilon_2 = 1$ (compare with case 9)

Arc	Start	End	Word	Remark
(a,a)	A_1	A_1	$c^{-1}(b^{-1}c^{-1})^m[b^{-0}c^0](bc)^{m+1}$	$\epsilon_1 = 0, \epsilon_2 = 1$, $m \in \mathbb{Z}^+$
(a,a)	A_1	A_1	inverse of above	
(a,b)	A_1	B	$c^{-1}(b^{-1}c^{-1})^{m-1}b^k$ or $c^{-1}(b^{-1}c^{-1})^m b^k$, $k \in \mathbb{Z}$	
(a,b)	A_2	B	$b^{-1}c^{-1}(b^{-1}c^{-1})^{m-1}b^k = (b^{-1}c^{-1})^m b^k$, $k \in \mathbb{Z}$	
(a,c)	A_1	C	$c^{-1}(b^{-1}c^{-1})^{m-1}b^{-1}c^k$, $k \in \mathbb{Z}$	
(a,c)	A_2	C	not possible	(Anticlockwise from start)

Case 16: A_1 to A_1, $\epsilon_1 = 1, \epsilon_2 = 0$ (compare with case 10)

Arc	Start	End	Word	Remark
(a,a)	A_1	A_1	$c^{-1}(b^{-1}c^{-1})^m[b^{-1}c^1](bc)^{m+0}$	$\epsilon_1 = 1, \epsilon_2 = 0$, $m \in \mathbb{Z}^+$
(a,a)	A_1	A_1	inverse of above	
(a,b)	A_1	B	$c^{-1}(b^{-1}c^{-1})^m b^k$, $k \in \mathbb{Z}$	
(a,b)	A_2	B	not possible	
(a,c)	A_1	C	$c^{-1}(b^{-1}c^{-1})^{m-1}b^{-1}c^k$, or $c^{-1}(b^{-1}c^{-1})^m b^{-1}c^k$	
(a,c)	A_2	C	$(b^{-1}c^{-1})^m b^{-1}c^k$, $k \in \mathbb{Z}$	(Anticlockwise from start)

Case 17: A_2 to A_2, $\epsilon_1 = \epsilon_2 = 0$ (compare with case 3)

Arc	Start	End	Word	Remark
(a,a)	A_2	A_2	$b^{-1}(c^{-1}b^{-1})^m[c^{-0}b^0](cb)^{m+0}$	$\epsilon_1 = \epsilon_2 = 0$, $m \in \mathbb{Z}^+$
(a,a)	A_2	A_2	inverse of above	
(a,b)	A_1	B	not possible	
(a,b)	A_2	B	$b^{-1}(c^{-1}b^{-1})^{m-1}c^{-1}b^k$	
(a,c)	A_1	C	$c^{-1}b^{-1}(c^{-1}b^{-1})^{m-1}c^k = (c^{-1}b^{-1})^m c^k$	

(a,c)	A_2	C	$b^{-1}(c^{-1}b^{-1})^{m-1}c^k$, or $b^{-1}(c^{-1}b^{-1})^m c^k$	(Anticlockwise from start)

Case 18: A_2 to A_2, $\epsilon_1 = \epsilon_2 = 1$ (compare with case 4)

Arc	Start	End	Word	Remark
(a,a)	A_2	A_2	$b^{-1}(c^{-1}b^{-1})^m[c^{-1}b^1](cb)^{m+1}$	$\epsilon_1 = \epsilon_2 = 1, m \in \mathbb{Z}^+$
(a,a)	A_2	A_2	inverse of above	
(a,b)	A_1	B	not possible	
(a,b)	A_2	B	$b^{-1}(c^{-1}b^{-1})^m c^{-1}b^k$	
(a,c)	A_1	C	$c^{-1}b^{-1}(c^{-1}b^{-1})^m c^k =$ $(c^{-1}b^{-1})^{m+1}c^k$	
(a,c)	A_2	C	$b^{-1}(c^{-1}b^{-1})^m c^k$, or $b^{-1}(c^{-1}b^{-1})^{m+1}c^k$	(Anticlockwise from start)

Case 19: A_2 to A_2, $\epsilon_1 = 0, \epsilon_2 = 1$ (compare with case 5)

Arc	Start	End	Word	Remark
(a,a)	A_2	A_2	$b^{-1}(c^{-1}b^{-1})^m[c^{-0}b^0](cb)^{m+1}$	$\epsilon_1 = 0, \epsilon_2 = 1, m \in \mathbb{Z}^+$
(a,a)	A_2	A_2	inverse of above	
(a,b)	A_1	B	$(c^{-1}b^{-1})^m c^{-1}b^k$	
(a,b)	A_2	B	$b^{-1}(c^{-1}b^{-1})^{m-1}c^{-1}b^k$ or $b^{-1}(c^{-1}b^{-1})^m c^{-1}b^k$	
(a,c)	A_1	C	not possible	
(a,c)	A_2	C	$b^{-1}(c^{-1}b^{-1})^m c^k$	(Anticlockwise from start)

Case 20: A_2 to A_2, $\epsilon_1 = 1, \epsilon_2 = 0$ (compare with case 6)

Arc	Start	End	Word	Remark
(a,a)	A_2	A_2	$b^{-1}(c^{-1}b^{-1})^m[c^{-1}b^1](cb)^{m+0}$	$\epsilon_1 = 1, \epsilon_2 = 0, m \in \mathbb{Z}^+$
(a,a)	A_2	A_2	inverse of above	
(a,b)	A_1	B	$(c^{-1}b^{-1})^m c^{-1}b^k$	
(a,b)	A_2	B	$b^{-1}(c^{-1}b^{-1})^{m-1}c^{-1}b^k$ or $b^{-1}(c^{-1}b^{-1})^m c^{-1}b^k$	
(a,c)	A_1	C	not possible	

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(a, c)	A_2	C	$b^{-1}(c^{-1}b^{-1})^m c^k$	(Anticlockwise from start)

Table 3.3: Possibilities of arcs on Σ_2'

We summarize of the words of various arcs in $\pi_1(V_2)$ as follows. We will use the orientation of c_Q for giving a meaning to starting and ending of arcs. Let w represent some reduced word in $\pi_1(V_2)$. An (a, a) arc starting on A_1 and ending on A_2 has words with three possible reduced forms, viz: empty, bwc^{-1}, $c^{-1}wb$. An (a, a) arc starting on A_2 and ending on A_1 has words with three possible reduced forms, viz: empty, cwb^{-1}, $b^{-1}wc$. An (a, a) arc starting on A_1 and ending on A_1 has words with the following possible reduced forms: b, b^{-1}, c, c^{-1}, bwb^{-1}, $c^{-1}wc$. An (a, a) arc starting on A_2 and ending on A_2 has words with following possible reduced forms: b, b^{-1}, c, c^{-1}, cwc^{-1}, $b^{-1}wb$. Depending on the beginning letter of the word described by an (a, a) arc, an (a, b) arc starting on A_1 describes a reduced word of the following forms: b^k, $bwcb^k$, $c^{-1}b^k$ or $c^{-1}wc^{-1}b^k$. Depending on the beginning letter of the word described by an (a, a) arc, an (a, b) arc starting on A_2 describes a reduced word of the following forms: b^k, cb^k, $cwcb^k$, $b^{-1}wc^{-1}b^k$. Likewise, an (a, c) arc starting on A_1 describes a reduced word of the following forms: c^k, bc^k, $bwbc^k$, $c^{-1}wb^{-1}c^k$. An (a, c) arc starting on A_2 describes a reduced word of the following forms: c^k, $cwbc^k$, $b^{-1}c^k$, $b^{-1}wb^{-1}c^k$. The word of a (b, a) arc is the inverse of the word of an (a, b) arc and the word of a (c, a) arc is the inverse of the word of an (a, c) arc of the appropriate kind.

Now, when we concatenate the words described by the various arcs of c_Q on Σ_2' and Σ_2'', in order described by c_Q, then the words should reduce to an empty word. Since the words described by various arcs, as summarized above are already in their reduced form, at least one trivial relator, a relator of the form xx^{-1}, where x could be any of b, c, b^{-1}, c^{-1}, has to occur at the join of two appropriate arcs as dictated by c_Q. We will show that when the word w in the summary of arcs is non-trivial a trivial relator does not arise when words are concatenated, leaving us with a conclusion that c_Q cannot have arcs with non-trivial words w. The word w is precisely empty when $m = \epsilon_1 = \epsilon_2 = 0$.

Now consider an arc ρ_i of c_Q. Suppose that ρ_i is a $(*, a)$ arc, whose word is of any of the types described above where w is not the empty word.

Case 1: ρ_{i+1} and ρ_{i+2} is an $(a,b)(b,a)$ or an $(a,c)(c,a)$ arc combination. Suppose first that ρ_{i+1} is an (a,b) arc and ρ_{i+2} is a (b,a) arc. In this case, by Corollary 3.3.10 and Lemma 3.3.12, for any trivial relator to arise, if at all it does, when words in $\pi_1(V)$ are concatenated, the words of both ρ_{i+1} and ρ_{i+2} must be empty. But this is impossible as ρ_{i+2} is on the same side of Σ_2 as ρ_i and ρ_{i+2} is a (b,a) arc which cannot describe an empty word when w is not the empty word. So no trivial relator arises when concatenating the words of ρ_i, ρ_{i+1} and ρ_{i+2}. A similar argument shows that no trivial relator arises when concatenating the words of ρ_i, ρ_{i+1} and ρ_{i+2} when $\rho_{i+1}\rho_{i+2}$ is an $(a,c)(c,a)$ arc combination.

Case 2: ρ_{i+1} is an (a,a) arc. Even in this case, by Corollary 3.3.10, there cannot be a trivial relator when the words of ρ_i and ρ_{i+1} are concatenated. However, the word of ρ_{i+1} could be empty, in which case it should be an arc from A_1 to A_2 or an arc from A_2 to A_1.

Subcase 1: ρ_i ends on A_2: In this sub-case, the word of ρ_i ends with b or c^{-1}. Here we refer to the summary of words of arcs. So if the word of ρ_{i+1} is empty it must be an arc from A_2 to A_1, and so ρ_{i+2} has to start on A_1 and hence has a word starting with either b or c^{-1}. As a result there is no trivial relator while concatenating these three words.

Subcase 2: ρ_i ends on A_1: In this sub-case, the word of ρ_i ends with b^{-1} or c. So if the word of ρ_{i+1} is empty it must be an arc from A_1 to A_2, and so ρ_{i+2} has to start on A_2 and hence has a word starting with either b^{-1} or c. As a result there is no trivial relator while concatenating these three words.

If ρ_i is an (a,b) or an (a,c) arc whose word is of any of the types described above where w is not the empty word, then the corresponding $(a,b)(b,a)$ arc combination or an $(a,c)(c,a)$ arc combination can never be the empty word by Corollary 3.3.11.

When c_Q is viewed as a sequence of (a,a) arcs, $(a,b)(b,a)$ arc combinations and $(a,c)(c,a)$ arc combinations and the corresponding words in $\pi_1(V_2)$ are concatenated by following c_Q, at-least one of these pieces gives a non-trivial reduced word when w is not the empty word and none of the concatenation results in a trivial relator. This shows that the word described by c_Q cannot be the empty word resulting in a contradiction to the assumption that c_Q bounds a disk. This shows that the word of any (a,a) arc on Σ_2' or Σ_2'' can only be of the form xwy with an empty w and where $x,y \in \{b,c,b^{-1},c^{-1},\{\}\}$. Hence the words of (a,a) arcs on Σ_2' or Σ_2'' can contain at most two letters. Correspondingly, we infer from the summary of arcs above that the word of an (a,b) or of an (a,c) arc of c_Q is of the form wb^k and wc^k respectively, for some integer k, where w can have a length at most 1. ☐

Recall that an (a,a) arc with empty word and an (a,a) arc with two-letter word cannot occur on the same side of Σ_2. By Theorem 3.3.13 if $n_{AQ} > n_{BQ} + n_{CQ}$, then exactly one of the following cases is possible.

Case I c_Q has an (a,a)-arc on Σ_2' or on Σ_2'' with a single letter word.

Case II c_Q does not have an (a,a) arc with a single letter word on any side of Σ_2 and c_Q has an (a,a)-arc on Σ_2' or on Σ_2'' with an empty word while it has an (a,a) arc with a two-letter word on the other side.

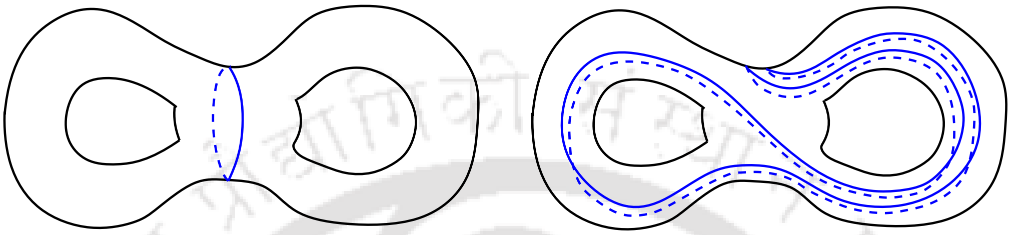
Case III c_Q does not have an (a,a) arc with a single letter word on any side of Σ_2 and c_Q has an (a,a) arc with a two-letter word on both sides of Σ_2

Case IV n_{AQ} is greater than 2; c_Q does not have an (a,a) arc with a single letter word on any side of Σ_2 and c_Q has an (a,a) arc with the empty word on both sides of Σ_2 .

Case V $Q = P$, $n_{AQ} = 2$ and $n_{BQ} = n_{CQ} = 0$.

Proposition 3.3.14. *$\mathcal{C}(Q) = 1$ if and only if $n_{AQ} = 2$ and $n_{BQ} = n_{CQ} = 0$.*

Proof. If $Q = P$, then $n_{AQ} = 2$ and $n_{BQ} = n_{CQ} = 0$. Conversely, if $n_{AQ} = 2$ and $n_{BQ} = n_{CQ} = 0$, then c_Q must have an (a, a) arc, say ρ_1, on Σ_2' and an (a, a) arc, say ρ_2, on Σ_2'' and no other arcs. By corollary 3.3.10, if one of these arcs has a non-trivial word, then the other arc cannot have a word such that a trivial relator arises in the concatenation and hence c_Q will not have an empty reduced word, contradicting the fact that c_Q bounds a disk in V. So, the (a, a) arc on Σ_2' or on Σ_2'' must have the empty word. Hence these (a, a) arcs are disjoint from Y and Z in addition to B and C. If $\mathcal{N}(T)$ denotes the tubular neighborhood of a simple closed curve T on Σ_2, then consider an annulus $\mathcal{A}$ defined as $\Sigma_2 \setminus (\mathcal{N}(B) \cup \mathcal{N}(Z) \cup \mathcal{N}(C) \cup \mathcal{N}(Y))$, where the tubular neighborhoods are chosen small enough so that they do not intersect a tubular neighborhood of c_P. Since ρ_1 and ρ_2 are the only arcs of c_Q, by an isotopy if necessary, c_Q can be assumed to be contained in the annulus $\mathcal{A}$. But every closed essential curve in an annulus is isotopic to the core curve of the annulus, and so in this case c_Q is isotopic to c_P and so $Q = P$. ☐



$\mathcal{C}(Q) = 1$ and $Q = P$ $\quad$ $\mathcal{C}(Q) = 1$ and c_Q is isotopic to c_P

Figure 3.26: Possible c_Q on Σ_2 with $\mathcal{C}(Q) = 1$

Remark 3.3.15. *If $Q = P$, then β or β^{-1} fix Q and hence there is no reduction or increase in complexity by applying β or β^{-1}. Case I, Case II and Case III are mutually exclusive with Case V because P does not have an (a, a) arc with one or two letter words on any side of Σ_2. Case IV is exclusive with Case V because of the condition $n_{AQ} > 2$ in Case IV, for if*

$n_{AQ} = 2$, the inequality $n_{AQ} > n_{BQ} + n_{CQ}$ forces $n_{BQ} = n_{CQ} = 0$ which means $Q = P$.

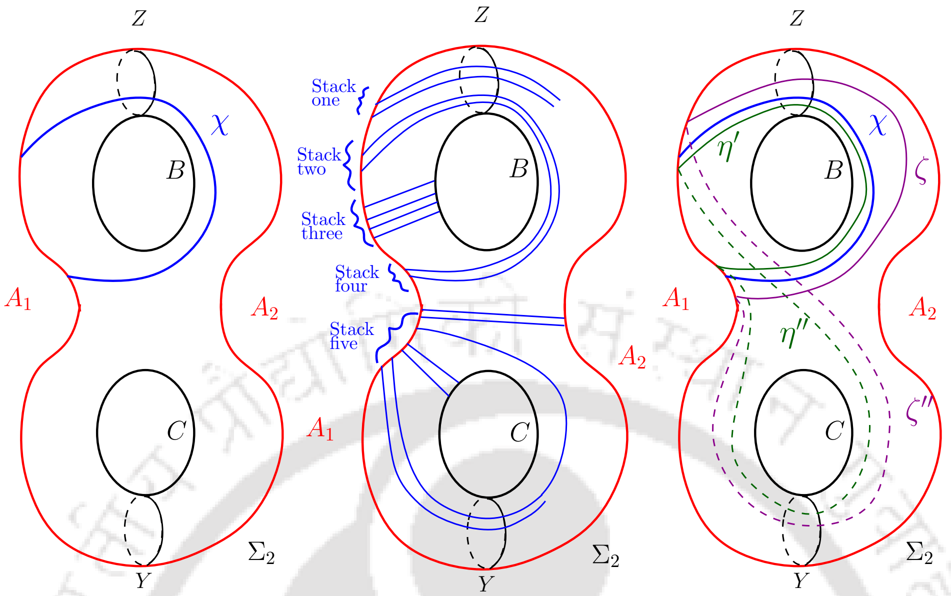
We now consider each of the cases I, II, III and IV separately and show that Case III is impossible whereas in the other cases, β or β^{-1} can be applied to reduce the complexity, $\mathcal{C}(Q)$, as long as $\mathcal{C}(Q) > 1$. Recall that β or β^{-1} do so by reducing n_{AQ} and not altering n_{BQ} and n_{CQ}.

Case I: (a, a) arc with one letter word.

Suppose that c_Q has an (a, a) arc on Σ_2' or Σ_2'' whose word is exactly one letter. In this case, we will prove that $n_{AQ} > n_{BQ} + n_{CQ}$ implies that we can apply the automorphism β or β^{-1} to reduce the complexity described earlier.

Let us assume that c_Q has an (a, a) arc on Σ_2', call it χ, which has both its endpoints on A_1 and its word is b or b^{-1} based on its orientation, the schematic for which is as shown in figure 3.27a. Every (a, a) arc on Σ_2' which satisfies the description of χ, *i.e.* which starts and ends on A_1 and has the word b or b^{-1} will be called an (a, a) arc *parallel to* χ.

Since χ cuts an annulus S_χ out of Σ_2' containing the circle B, all the a-ends of the (a, b) arcs of c_Q on Σ_2' have to lie on A_1 and between the endpoints of χ. With the z-coordinate as height we classify the points of $c_Q \cap A_1$ into five stacks. The *first stack of points* with the largest z-coordinates consists of a-ends of arc segments of c_Q which lie outside S_χ, if any, which connect Z' to A_1 and which are not end points of (a, a) arcs on Σ_2' parallel to χ. Every (a, a) arc on Σ_2' which is parallel to χ has two ends, one with a higher z-coordinate and one with a lower z-coordinate. The *second stack of points* consists of those a-ends of (a, a) arcs on Σ_2' parallel to χ, which have higher z-coordinate than their counterparts. The *fourth stack of points* consists of those a-ends of (a, a) arcs on Σ_2' parallel to χ which have lower z-coordinate than their counterparts. The *third stack of points* consists of the a-ends of (a, b) arcs of c_Q, if any. The *fifth stack of points* consists of the a-ends of the arc segments of c_Q on Σ_2', if any, which



(a) Schematic of χ (b) Stacks of points on A_1 (c) Schematic of $\eta' \cup \eta''$

Figure 3.27: On the arc-configurations of c_Q on Σ_2' and Σ_2''

have a z-coordinate lower than that of any point in the fourth stack of points (see figure 3.27b).

Now without loss of generality, we can assume that χ is the innermost among all these parallel (a, a)-arcs on Σ_2' having word b or b^{-1}.

Let ω_1 and ω_2 be distinct (a, a) arcs of c_Q on Σ_2' parallel to χ. Let the z-coordinates of the endpoints of ω_1 be z_{11} and z_{12} with $z_{11} < z_{12}$ and the z-coordinates of the endpoints of ω_2 be z_{21} and z_{22} with $z_{21} < z_{22}$. Then either $z_{11} < z_{21} < z_{22} < z_{12}$ or $z_{21} < z_{11} < z_{12} < z_{22}$. This is because, if the endpoints of ω_1 and ω_2 alternate on A_1, then ω_1 would intersect ω_2 on Σ_2'. If $z_{11} < z_{21} < z_{22} < z_{12}$ then we say that ω_2 is nested inside ω_1. Or if $z_{21} < z_{11} < z_{12} < z_{22}$, then we say that ω_1 is nested inside ω_2.

§ **Construction of A':** We now define an (a, a)-arcs, η' and ζ' on Σ_2' which are not arcs of c_Q and are disjoint from arcs of c_Q on Σ_2' and are parallel to χ as follows. Define η' to be an arc whose ends on A_1 are nested inside the ends of χ so that a-ends of all (a, b)-arcs on Σ_2', if any, are between the ends of η'. Likewise, define ζ' to be an arc whose ends on

A_1 are such that (i) every arc of c_Q on Σ_2' parallel to χ is nested inside ζ', (ii) the a-ends of every (a, b)-arc on Σ_2' is also between the ends of ζ' and (iii) the a-ends of no other arc of c_Q on Σ_2' is between the ends of ζ'.

Define η'' and ζ'' to be (a, a) arcs (also not of c_Q) on Σ_2'' such that: (i) $\partial\eta'' = \partial\eta'$, $\partial\zeta'' = \partial\zeta'$, (ii) the word of η'' or ζ'' is c^{-1} or c and (iii) η'' and ζ'' intersect c_Q minimally on Σ_2'' where the end points of all arcs on Σ_2'' are rigid *i.e.*cannot be moved while considering their intersection numbers. The simple closed curves $\eta' \cup \eta''$ or $\zeta' \cup \zeta''$ are isotopic to each other on Σ_2 and in turn are isotopic to $A' := \beta^{-1}(A)$. In order to show that β reduces complexity we show that $|c_Q \cap A'| < |c_Q \cap A|$. By construction, $|c_Q \cap \eta'| = |c_Q \cap \zeta'| = 0$. So it is enough to show that either $|c_Q \cap A| > |c_Q \cap \eta''|$ or $|c_Q \cap A| > |c_Q \cap \zeta''|$. We will first show that $|c_Q \cap A| \geq |c_Q \cap \eta''|$ and $|c_Q \cap A| \geq |c_Q \cap \zeta''|$. Figure 3.28 shows η and ζ. In this context, we note the following restrictions on (a, a), (a, b) and (a, c) arcs of c_Q on Σ_2''.

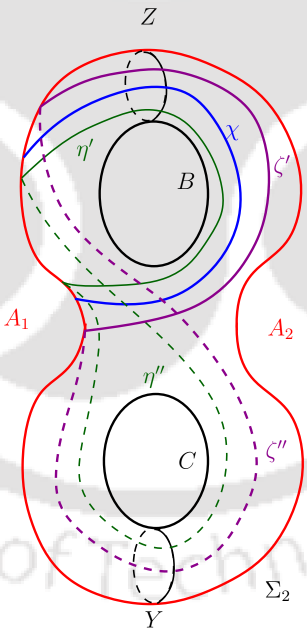


Figure 3.28: The construction of the arcs η', ζ', η'' and ζ''

§ Restrictions on (a, a)-arcs

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Proposition 3.3.16. *c_Q cannot have an (a,a)-arc on Σ_2'' with both ends on A_1, whose word is either b or b^{-1}. Also c_Q cannot have an (a,a)-arc on Σ_2'' with word bc^{-1} or cb^{-1}.*

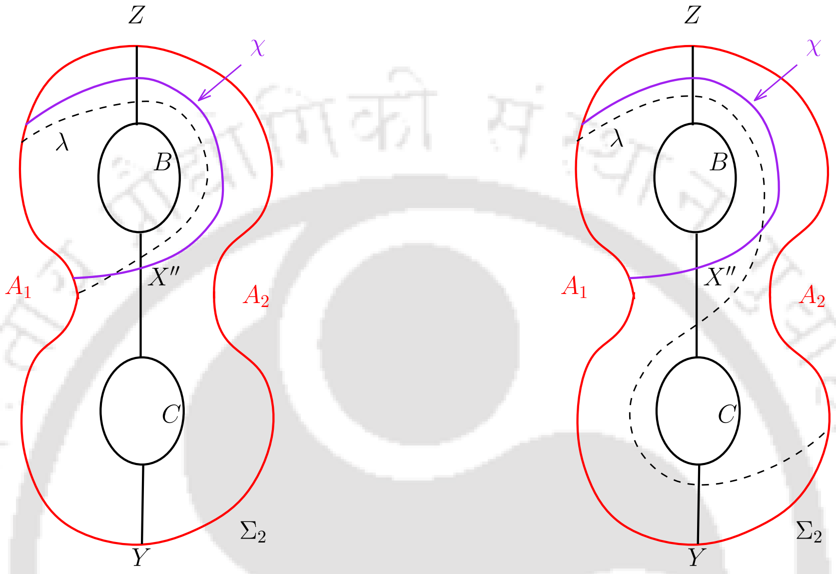

Figure 3.29: (a,a) arcs on Σ_2'' describing words b, b^{-1}, bc^{-1} or cb^{-1} with one end on A_1 are impossible

Proof. The presence of such an arc (see figure 3.29) on Σ_2'' along with χ contradicts Lemma 3.3.9. □

So the possibilities for (a,a)-arc of c_Q on Σ_2'' are: (i) an (a,a) arc with the empty word with two endpoints, one each on A_1 and A_2 (ii) an (a,a) arc with the word c or c^{-1} with both ends on A_1 (iii) an (a,a) arc with the word c or c^{-1} with both ends on A_2 (iv) an (a,a) arc with the word b or b^{-1} with both ends on A_2 (v) an (a,a) arc with the word $b^{-1}c$ or $c^{-1}b$ with endpoints on A_1 and A_2.

As a result, the intersection of these possible (a,a) arcs of c_Q on Σ_2'' with η'' and ζ'' are as shown in Figure 3.30 (a-d).

§ Restrictions on (a,b)-arcs

In the presence of the above-mentioned (a,a)-arcs the possible words of

(a, b) arcs of c_Q on Σ_2'' are b^k (with a-ends on A_1 or A_2), $c^{-1}b^k$ (with a-ends on A_1) and cb^k (with a-end on A_2), $k \in \mathbb{Z}$ (see figure 3.31).

We have the following restrictions or limitations for these arcs.

Proposition 3.3.17. *c_Q cannot have a (a, b)-arc on Σ_2'' with word $b^k, k \in \mathbb{Z}$ with an end on stack four or five of A_1.*

Proof. If possible, suppose c_Q has such an (a, b) arc λ with one end on stack four or five. Then all arcs on Σ_2'' starting from stacks one, two and three must end on B (see figure 3.32a). But each point on B is connected to a point in stack three by a (b, a)-arc on Σ_2'. Thus if $|B \cap c_Q| = n_B$, then stack three must have n_B points. Further, λ starts from a point outside stack three. And all the points in stack one, two and three must connect to a distinct point on B. Therefore, points of $c_Q \cap B = n_B =$ points in stack one, two and three $\geq n_B + 1$ ($+1$ for λ), a contradiction.
Hence, such a λ cannot exist. ☐

Proposition 3.3.18. *If Σ_2'' contains an (a, b) arc of c_Q with the word $c^{-1}b^k$ with its a-end on A_1, then such an a-end should be in stack three,*

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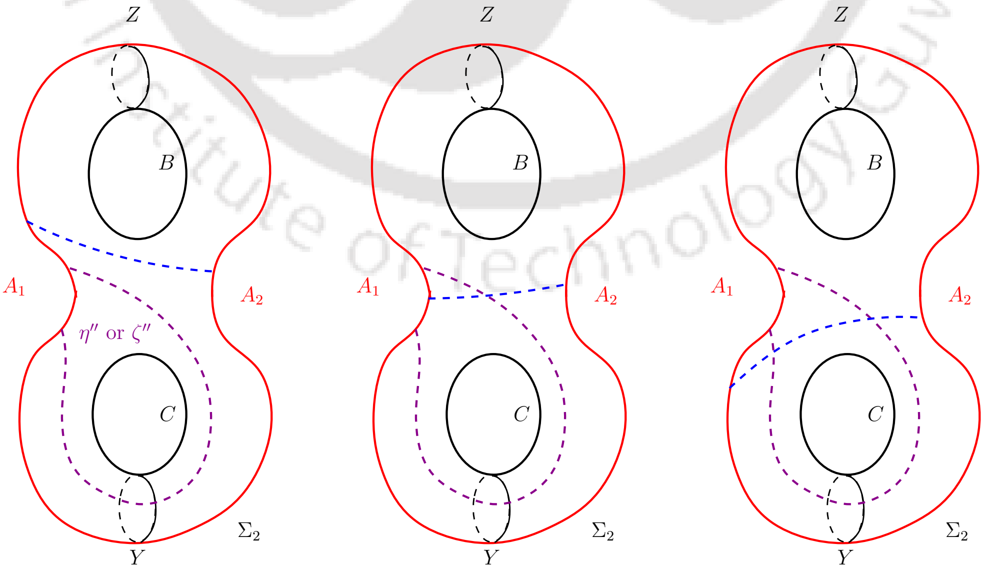
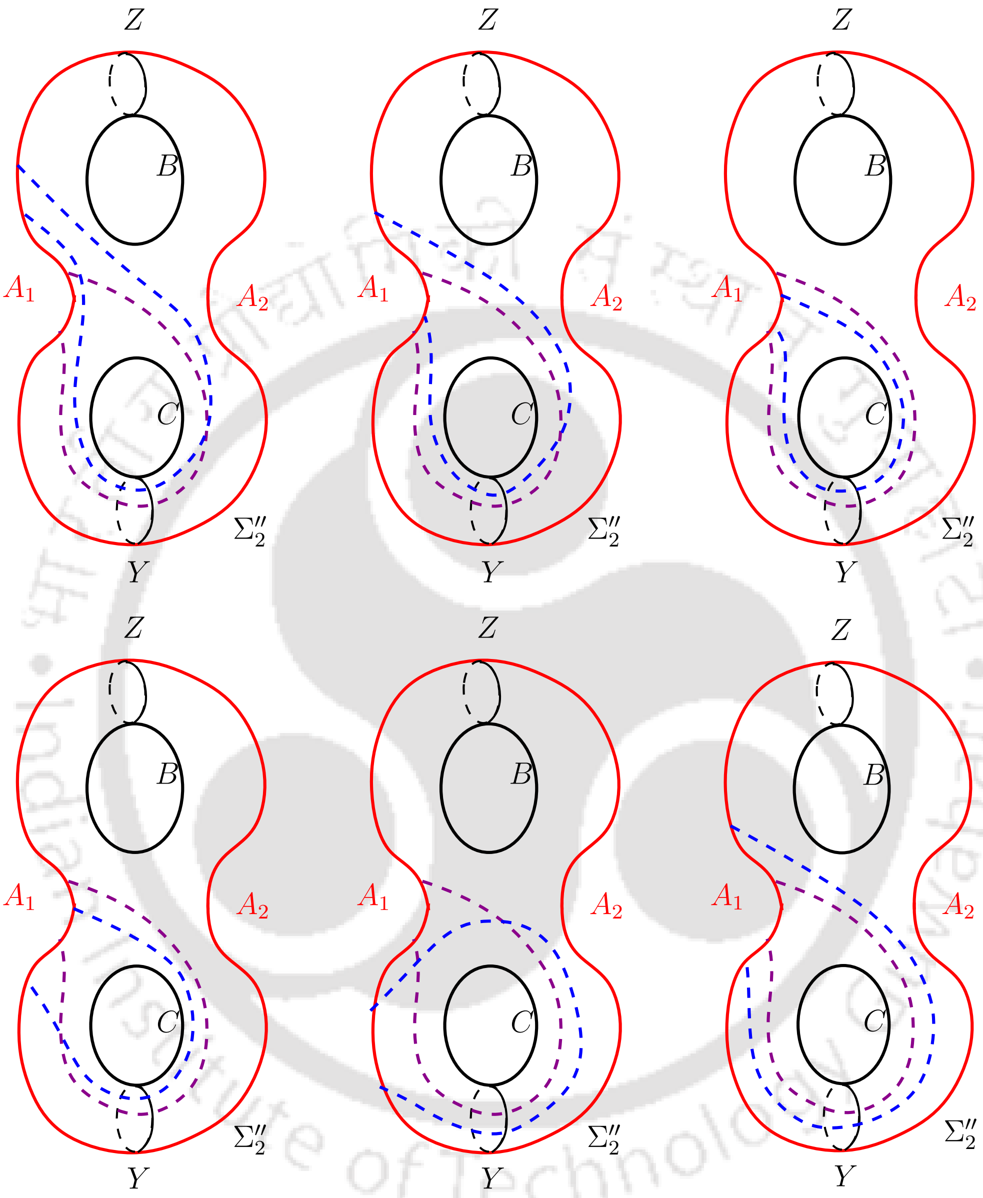
Figure 3.30: Various intersection pattern of (a, a) arcs with η''



(a) Intersection between η'' and (a, a) arcs describing empty word

Figure 3.30: Various intersection pattern of (a, a) arcs with η''

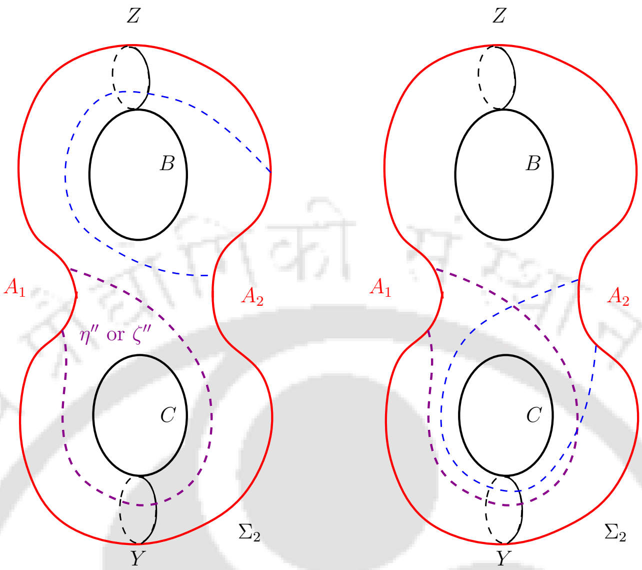


(b) All possible intersection patterns between η'' and (a, a) arcs describing c or c^{-1} with both ends on A_1

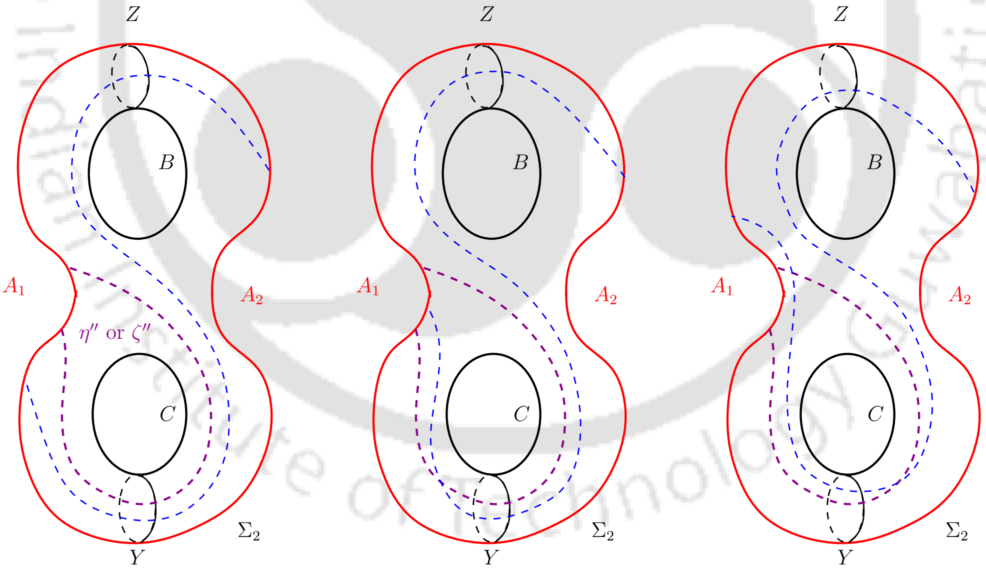
four or five and it cannot be the topmost (w.r.t. the height function on A_1*) point in the third stack.*

Proof. Suppose to the contrary that λ is such an (a, b)-arc on Σ_2'' with an endpoint on A_1 with word $c^{-1}b^k$ and with its endpoint in stack one or two

Figure 3.30: Various intersection pattern of (a, a) arcs with η''



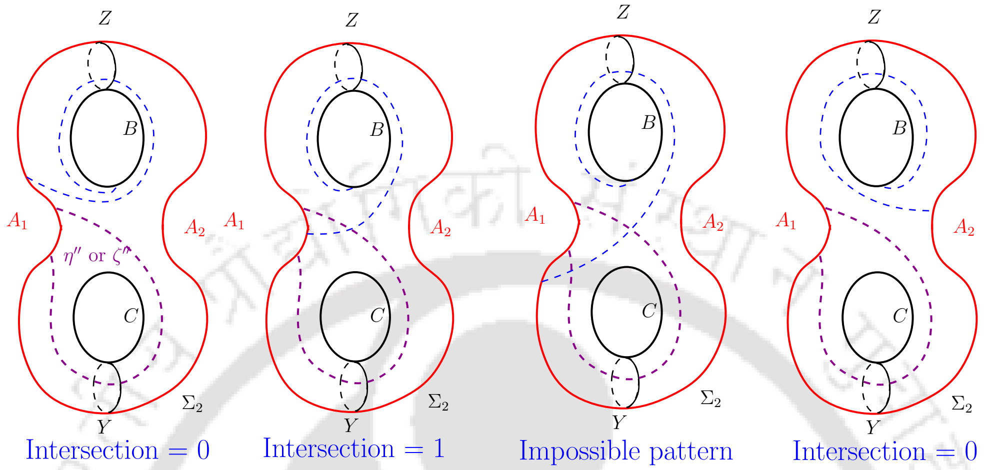
(c) Intersection between η'' and (a, a) arcs describing b, b^{-1}, c or c^{-1} with both ends on A_2



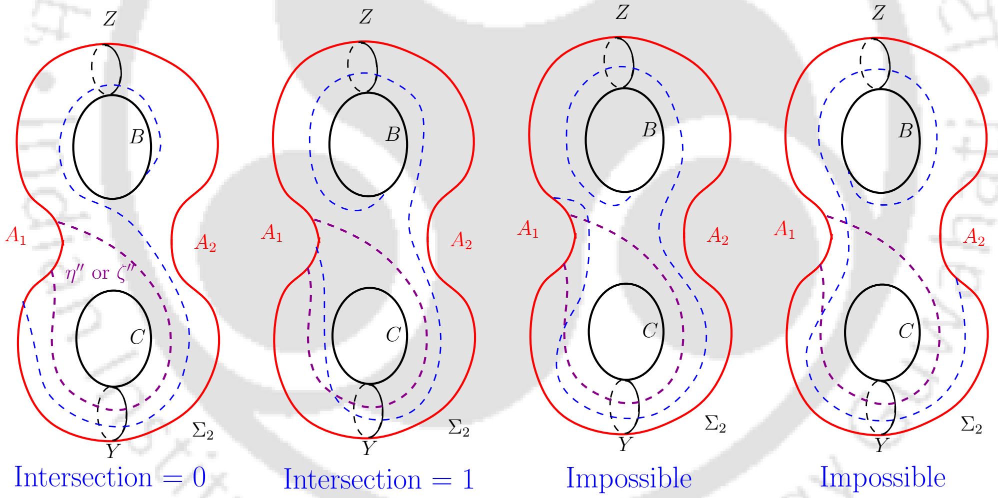
(d) Intersection between η'' and (a, a) arcs describing $b^{-1}c$ or $c^{-1}b$

of A_1 or the topmost point in stack three on A_1 (see figure 3.32b). Then λ will force every arc of c_Q on Σ_2'' starting from stacks three, four and five to end on B. But existence of χ implies, stack four is non-empty and stack three shares equal number of points, say n_B, with B. In other words, at

Figure 3.31: Intersction between various types of (a, b)-arcs on Σ_2'' with η'' or ζ''



(a) Intersection pattern of (a, b) arcs on Σ_2'' having word b^k with η'' and ζ''

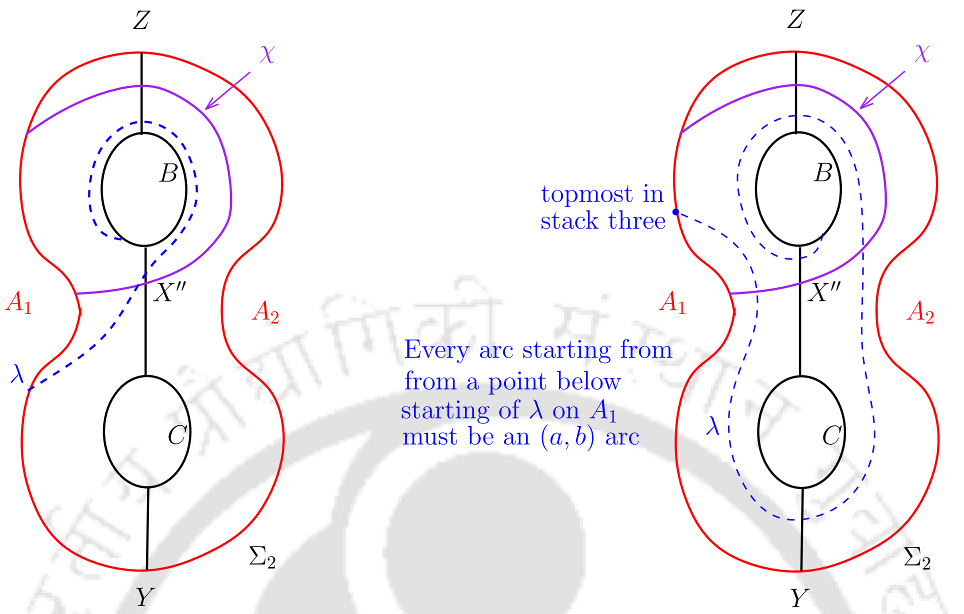


(b) Intersection pattern of (a, b) arc on Σ_2'' having word $c^{-1}b^k$ with η'' or ζ''

least $n_B + 1$ (a, b)-arcs must end on distinct points of B, i.e. $n_B \geq n_B + 1$. But this is impossible and hence the lemma is true. □

Proposition 3.3.19. *Σ_2'' cannot contain an (a, b) arc of c_Q with its a-end on A_2 and with the word cb^k.*

Proof. If possible suppose λ is an (a, b)-arc on Σ_2'' starting from A_2 with word cb^k (see figure 3.33). But such a λ will force all arcs from A_1 to end on



(a) (a, b) describing word b^k from stack 4 or 5 of A_1

(b) (a, b) describing $c^{-1}b$ from topmost point in stack three

Figure 3.32: Impossible (a, b)-arcs in propositions 3.3.17 and 3.3.18

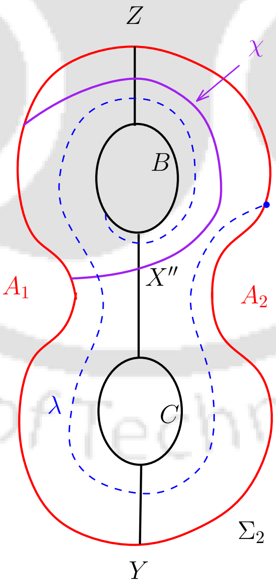


Figure 3.33: (a, b) from A_2 describing word cb^k

B. Existence of χ, and the fact that stack three and B shares equal number of points (say, n_B) implies $n_B \geq n_B + 2$, which is impossible. Therefore,

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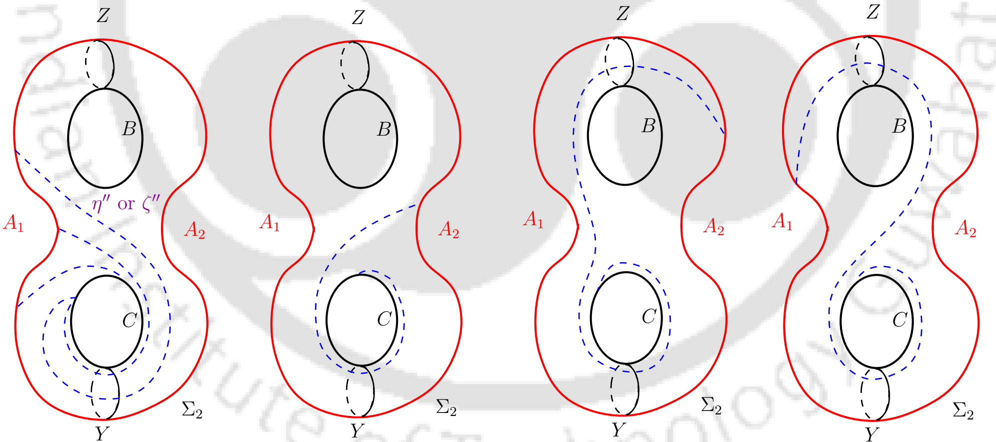
such a λ cannot exist. □

So the possibilities for (a, b)-arc of c_Q on Σ_2'' are: (i) an (a, b) arc with the empty word or the word of the form b^k, $k \in \mathbb{Z}$, with the a-end on A_1 (ii) an (a, b) arc with the empty word or the word of the form b^k, $k \in \mathbb{Z}$, with the a-end A_2 (iii) an (a, b) arc with the word $c^{-1}b^k$, $k \in \mathbb{Z}$, with its a-end on A_1.

As a result, the intersection of these possible (a, b) arcs of c_Q on Σ_2'' with η'' and ζ'' are as shown in Figure 3.31.

§ Restrictions on (a, c)-arcs

In the presence of the above-mentioned (a, a)-arcs the possibilities for the words of an (a, c)-arc of c_Q on Σ_2'' are c^k (a-end either on A_1 or on A_2), bc^k (with a-end on A_1) or $b^{-1}c^k$ (with a-end on A_2), where $k \in \mathbb{Z}$ (see figure 3.34).



Word = c^k, from A_1 Word c^k from A_2 Word $b^{-1}c^k$ from A_2 Impossible pattern

Figure 3.34: Intersection of (a, c) arcs on Σ_2'' describing c^k or $b^{-1}c^k$ with η'' and ζ''

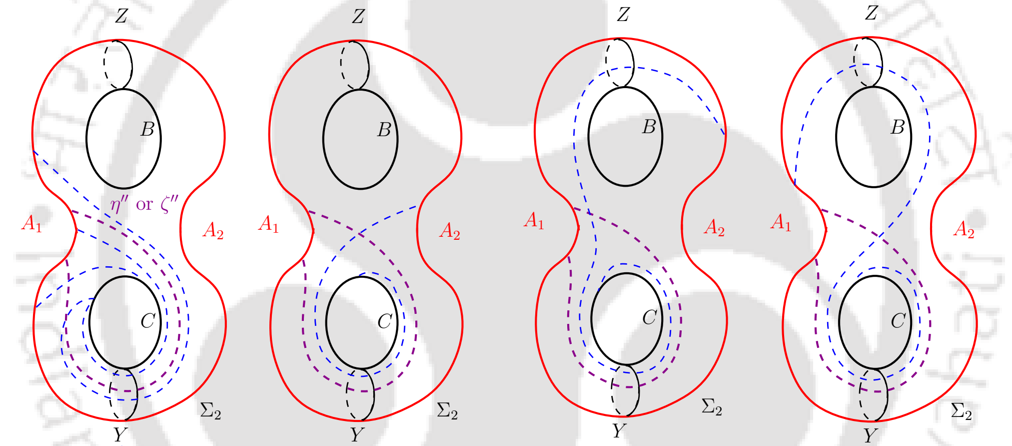
We have the following restriction for these arcs.

Proposition 3.3.20. *c_Q cannot have an (a, c)-arc on Σ_2'' with word bc^k, where $k \in \mathbb{Z}$*

Proof. The presence of such an arc on Σ_2'' along with χ contradicts Lemma 3.3.9. ☐

So the possibilities for (a, c)-arc of c_Q on Σ_2'' are: (i) an (a, c) arc with the empty word or the word of the form c^k, $k \in \mathbb{Z}$, with the a-end on A_1 or (ii) an (a, c) arc with the empty word or the word of the form c^k, $k \in \mathbb{Z}$, with the a-end on A_2 (iii) an (a, c) arc with the word $b^{-1}c^k$ with its a-end on A_2.

As a result, the intersection of these possible (a, c) arcs of c_Q on Σ_2'' with η'' and ζ'' are as shown in Figure 3.35.



Word $= c^k$, from A_1 Word c^k from A_2 Word $b^{-1}c^k$ from A_2 Impossible pattern

Figure 3.35: Intersection of η'' with various types of (a, c)-arcs on Σ_2''

From the above discussion and figures in figure 3.30, 3.31 and 3.35, we conclude the following.

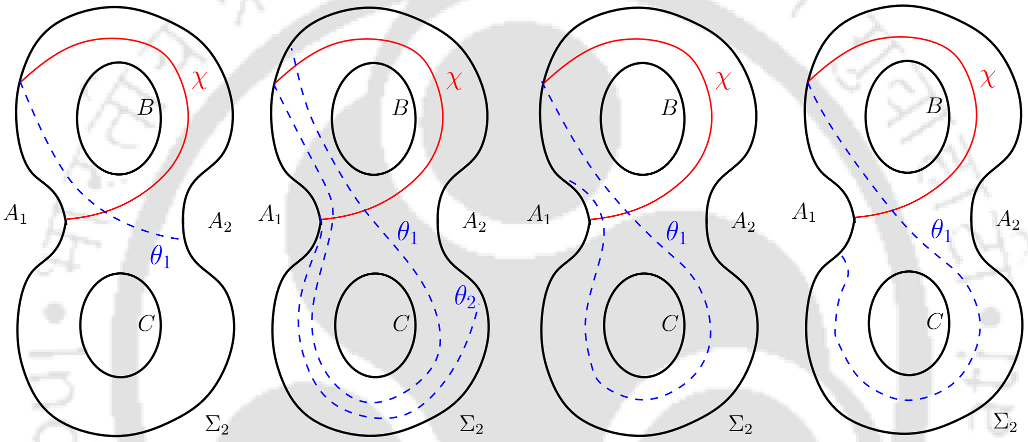
Proposition 3.3.21. *Every arc κ of c_Q on Σ_2'' satisfies $|\kappa \cap \eta''| \leq |\kappa \cap A|$ and $|\kappa \cap \zeta''| \leq |\kappa \cap A|$.*

Next we show that at least one of the two strict inequalities: $|c_Q \cap A| > |c_Q \cap \eta''|$ or $|c_Q \cap A| > |c_Q \cap \zeta''|$ holds, by showing the following.

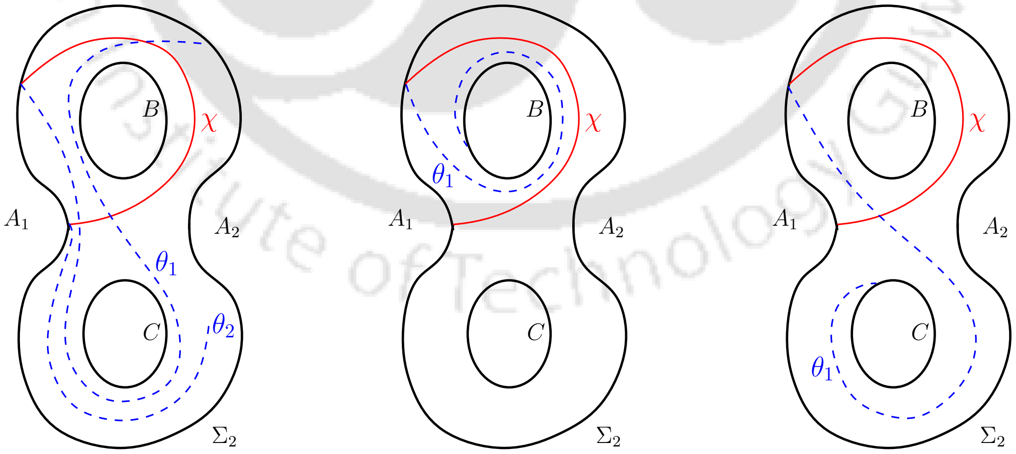
Lemma 3.3.22. *There exists an arc λ of c_Q on Σ_2'' such that $|\lambda \cap \eta''| < |\lambda \cap A|$ or $|\lambda \cap \zeta''| < |\lambda \cap A|$.*

Proof. Consider the points $\chi \cap A_1$. Let the arc of c_Q on Σ_2'' with the point in stack two of $\chi \cap A_1$ as an endpoint be θ_1 and the arc with the point in stack four of $\chi \cap A_1$ as an endpoint be θ_2. By proposition 3.3.18, θ_1 cannot be an (a, b) arc with the word $c^{-1}b^k$. We have the following cases.

Case 1 Suppose that θ_1 is an (a, a)-arc with endpoints on A_1 and A_2 with the empty word. In this case, θ_1 is disjoint from η'' and is the required λ (figure 3.36).



(a) Case 1: $\lambda = \theta_1$ (b) Case 2: $\lambda = \theta_2$ (resp. θ_1), if θ_1 ends in stack 1,2 (resp. 3,4 or 5)



(c) Case 3: $\lambda = \theta_2$ (d) Case 4: $\lambda = \theta_1$ (e) Case 5: $\lambda = \theta_1$

Figure 3.36: The arc λ leading to complexity reduction

Case 2 Suppose θ_1 is an (a, a)-arc with both its endpoints on A_1 with the word c or c^{-1}. If the other end point of θ_1 is in stack one or two, then θ_1 cuts off an annulus from Σ_2'' containing C as one boundary component and so θ_2, cannot be an (a, c) arc. So, if the other end point of θ_1 is in stack one or two then θ_2 is an (a, a) arc with the word $c^{-1}b$, c or c^{-1} or an (a, b) arc with the word $c^{-1}b^k$, $k \in \mathbb{Z}$, and in any case, θ_2 is disjoint from η'' and is the required λ. If the other end point of θ_1 is in stack three then it intersects η'' once, whereas θ_1 intersects A twice. So θ_1 is the required λ. Finally, if the other end point of θ_1 is in stack four or five then it is disjoint from η'' and θ_1 is the required λ.

Case 3 If θ_1 is an (a, a) arc with endpoints on A_1 and A_2 with the word $c^{-1}b$, then θ_2 is either an (a, a) arc with the word $c^{-1}b$ or is an (a, b) arc with the word $c^{-1}b^k$. In any case, θ_2 is disjoint from η'' and is the required λ.

Case 4 If θ_1 is an (a, b) arc with the word b^k, $k \in \mathbb{Z}$, then it is disjoint from η'' and is the required λ.

Case 5 Suppose that θ_1 is an (a, c)-arc with the word c^k, $k \in \mathbb{Z}$. Then, θ_1 is disjoint from ζ'' and is the required λ.

□

Owing to the symmetry of Σ_2, all of the above discussion holds even if we replace χ by any (a, a) arc on Σ_2 with both endpoints on A_1 or both endpoints on A_2 describing a single letter word in $\pi_1(V)$. For comprehensiveness, figure 3.37 shows the eight possibilities for such an (a, a) arc. We note that, in some cases we might have to use $A'' = \beta(A)$ instead of $A' = \beta^{-1}(A)$ throughout the above discussion for the inequalities to hold.

So the above discussion proves the following.

Theorem 3.3.23. *If $n_{AQ} > n_{BQ} + n_{CQ}$ and if c_Q has an (a, a)-arc on Σ_2' or on Σ_2'' describing a single letter word b, b^{-1}, c or c^{-1} in $\pi_1(V)$ then either $|\beta(c_Q) \cap A| < |c_Q \cap A|$ or $|\beta^{-1}(c_Q) \cap A| < |c_Q \cap A|$.*

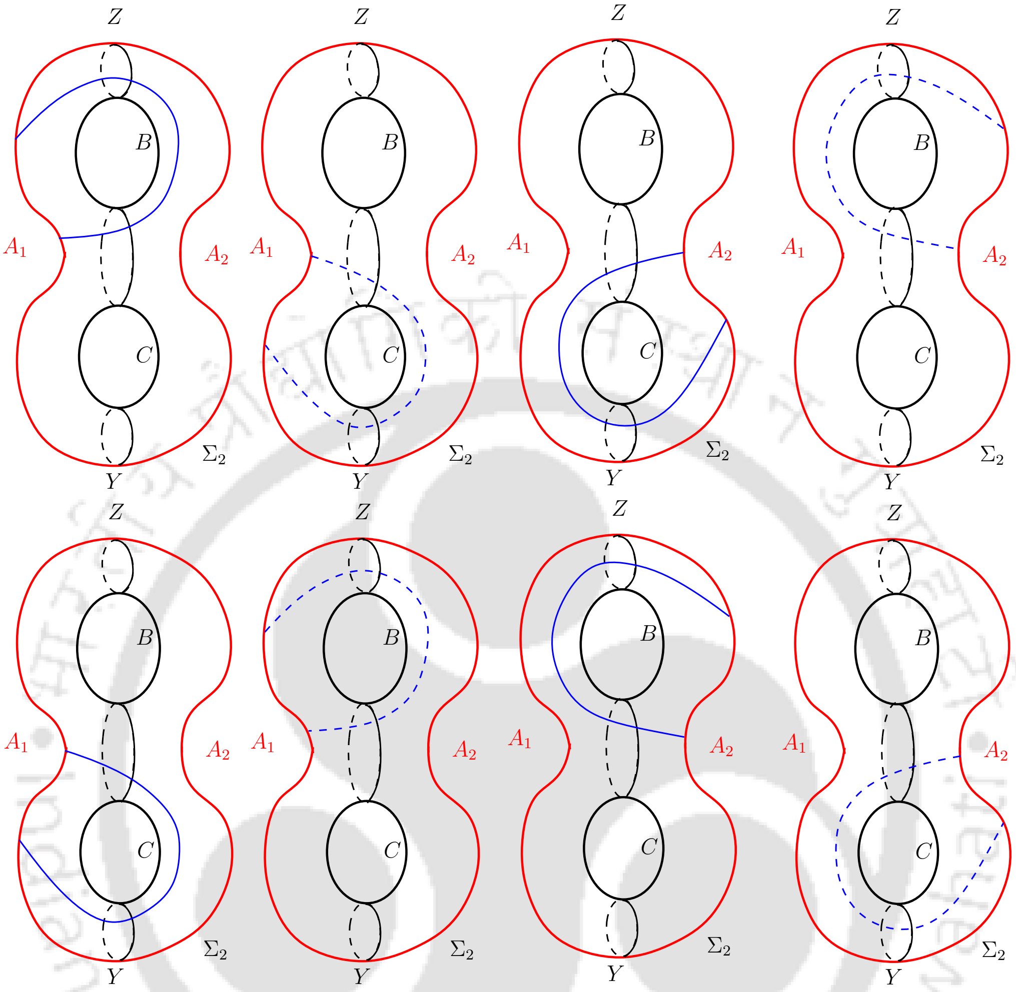


Figure 3.37: The eight possibilities for the (a, a)-arc λ

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Case II: (a, a) arc with two letter word on one side and empty word on the other.

Theorem 3.3.24. *Suppose that $n_{AQ} > n_{BQ} + n_{CQ}$ and that c_Q has no (a, a)-arc on Σ_2' or on Σ_2'' describing a single letter word. Now if c_Q contains an (a, a) arc with a two letter word on one side of Σ_2 and an (a, a) arc with the empty word on the other side of Σ_2, then either $|\beta(c_Q) \cap A| < |c_Q \cap A|$ or $|\beta^{-1}(c_Q) \cap A| < |c_Q \cap A|$.*

Proof. Let us first assume that c_Q has an (a, a) arc on Σ_2', call it ρ, with the word bc^{-1} and has an (a, a) arc on Σ_2'' with the empty word. Let the orientation of the two-letter (a,a) arc on $\Sigma_2^{'}$ be such that the arc starts on A_1 and ends on A_2. As before, let us classify the points in $c_Q \cap A_1$ into five stacks. Notice that since there are no (a,a) arcs with one letter words on $\Sigma_2^{'}$, the second and the fourth stacks are empty. Also the first stack is non-empty because of the presence of ρ. Further, with the presence of ρ, there cannot be any points in the fifth stack as well. So there are points only in the first and the third stacks.

Construct an (a,a) arc, η', on $\Sigma_2^{'}$ with both endpoints on A_1 such that all the points on A_1 in the third stack, if any, are nested between the endpoints of η' and such that it describes a word b or b^{-1}, depending on its orientation, in $\pi_1(V)$. Similarly construct an (a,a) arc, η'' on $\Sigma_2^{''}$ such that $\partial\eta' = \partial\eta''$ and that η'' describes a word c or c^{-1}, depending on its orientation, in $\pi_1(V)$. Define $A' = \eta' \cup \eta''$. Note that A' is isotopic to $\beta^{-1}(A)$. So in order to show that the automorphism β^{-1} reduces complexity, it is enough to show that $|c_Q \cap A'| < |c_Q \cap A|$.

Since there is an (a,a) arc, call it χ_1, on $\Sigma_2^{''}$ with the empty word, that arc should be continuation arc from some point in the first or the third stack. We also know that the first stack is non-empty.

First, ρ cuts out an annulus from $\Sigma_2^{'}$ containing the circle B. So the possible arcs of c_Q on this annulus are (a,a) arcs parallel to ρ, (a,b) (or (b,a)) arcs with their a-ends on A_1, (a,b) (or (b,a)) arcs with their a-ends on A_2. Of these, the (a,b) (or (b,a)) arcs which end on A_2 intersect both η' and A exactly once. So such arcs of c_Q, if any, contribute exactly 1 to each of $|c_Q \cap A'|$ and $|c_Q \cap A|$. No other arcs of c_Q intersect η' as can be seen in the following schematic Figure 3.38.

Case 1: χ_1 is a continuation arc on $\Sigma_2^{''}$ from some point in the third stack. In this case the continuation arcs on $\Sigma_2^{''}$ from all the points in the first stack on A_1 either have to be (a,a) arcs on $\Sigma_2^{''}$ describing the empty word or have to be (a,b) arcs on $\Sigma_2^{''}$ describing the word b^k, for some $k \in \mathbb{Z}$. In any case, each such arc contributes at least 1 to $|c_Q \cap A|$, whereas it contributes 0 to $|c_Q \cap A'|$. If the continuation arc on $\Sigma_2^{''}$ from a point in the third stack on A_1 is an (a,b) arc describing the word b^k, for some $k \in \mathbb{Z}, k < 0$ then

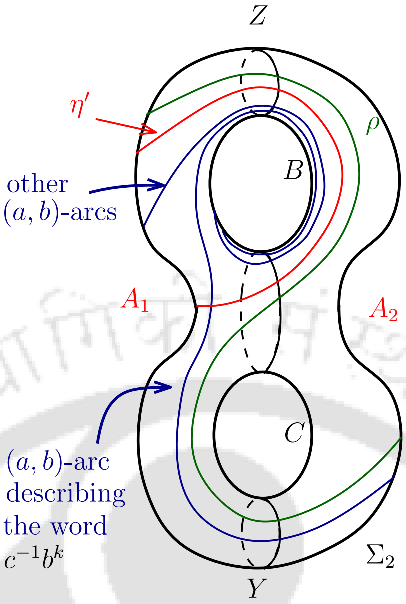


Figure 3.38: (a,b)-arc with a-end on A_2 and describing word $c^{-1}b^k$

the initial arc segment of this (a,b) arc on Σ_2'' with endpoints on A_1 and Z'' along with ρ violate Lemma 3.3.9. If the continuation arc on Σ_2'' from a point in the third stack on A_1 is an (a,b) arc describing the word b^k, for some $k \in \mathbb{Z}, k \geq 0$ then such an arc intersects both A and η'' once. If the continuation arc on Σ_2'' from a point in the third stack on A_1 is an (a,a) arc describing the empty word then such an arc intersects A twice whereas it intersects η'' once. If the continuation arc on Σ_2'' from a point in the third stack on A_1 is an (a,c) arc describing a word of the form $c^k, k \in \mathbb{Z}$ then such an arc intersects A once, whereas it is disjoint from η''. In any case the continuation arcs on Σ_2'' from points in the third stack on A_1 contribute at least as much to $|c_Q \cap A|$ as they contribute to $|c_Q \cap A'|$. Finally, (a,c) arcs on Σ_2'' with their a ends on A_2 contribute one each to $|c_Q \cap A|$ and $|c_Q \cap A'|$ and (a,b) arcs on Σ_2'' with their a ends on A_2 contribute one to $|c_Q \cap A|$ and 0 to $|c_Q \cap A'|$. Since the first stack of points on A_1 is non-empty, we conclude that the theorem holds in this case.

Case 2: χ_1 is a continuation arc on Σ_2'' of some point in the first stack. In this case, if the continuation arc on Σ_2'' from a point in the first stack on A_1 is an (a,b) arc describing the word of the form $b^k, k \in \mathbb{Z}$, then it contributes 1 to $|c_Q \cap A|$, whereas it contributes 0 to $|c_Q \cap A'|$. If the continuation arc on Σ_2'' from a point in the first stack on A_1 is an (a, a) arc describing the empty word, and we know that one such arc exists, then it contributes 2 to $|c_Q \cap A|$, whereas it contributes 0 to $|c_Q \cap A'|$. If the continuation arc on Σ_2'' from a point in the first stack on A_1 is an (a, c) arc describing a word $c^k, k \in \mathbb{Z}$ then it contributes 1 to each of $|c_Q \cap A|$ and $|c_Q \cap A'|$. The continuation arc from a point in the third stack on A_1 in this case cannot be an (a, b) arc. If the continuation arc on Σ_2'' from a point in the third stack on A_1 is an (a, a) arc describing the empty word then it contributes 2 to $|c_Q \cap A|$, whereas it contributes 1 to $|c_Q \cap A'|$. If the continuation arc on Σ_2'' from a point in the third stack on A_1 is an (a, c) arc with a word $c^k, k \in \mathbb{Z}$ then it contributes 1 to $|c_Q \cap A|$, whereas it contributes 0 to $|c_Q \cap A'|$. Finally, (a, c) arcs on Σ_2'' with their a ends on A_2 contribute one each to $|c_Q \cap A|$ and $|c_Q \cap A'|$ and (a, b) arcs on Σ_2'' with their a ends on A_2 contribute one to $|c_Q \cap A|$ and 0 to $|c_Q \cap A'|$. So the theorem holds in this case as well.

Since χ_1 should continue from some point in the first or the third stack *i.e.* since the above two cases are exhaustive, the theorem is proved. ☐

Case III: (a, a) arc with two letter words on both sides.

Next we consider the case where both the sides of Σ_2, Σ_2' and Σ_2'', contain an (a, a) arc with two letter words and none of the sides contain an (a, a) arc with one letter word. We will show that this cannot occur when c_Q is in minimal intersection position with all the curves X, Y, Z, A, B and C.

Theorem 3.3.25. *Suppose that* $n_{AQ} > n_{BQ} + n_{CQ}$ *and that* c_Q *has no* (a, a)*-arc on* Σ_2' *or on* Σ_2'' *describing a single letter word. Then both the sides of* Σ_2*,* Σ_2' *and* Σ_2''*, cannot contain an* (a, a) *arc with two letter words.*

Proof. Without loss of generality assume that Σ_2' contains an (a, a) arc with end points on A_1 and A_2 which describes the word bc^{-1}. In this case Σ_2'' cannot contain an (a, a) arc describing the same word bc^{-1} or the inverse word cb^{-1} owing to Lemma 3.3.9. So, the (a, a) arc on Σ_2'' will have to describe the word $b^{-1}c$ or the word $c^{-1}b$ depending on whether it starts on A_2 or A_1. We will now show that even this cannot occur. For the purpose of this argument, we note that an (a, b) arc on one side of Σ_2 has to continue on the other side as a (b, a) arc. So we tie up such (a, b) and (b, a) arc combinations and call them $a - b - a$ arcs. We concatenate the words described by the (a, b) and the (b, a) arcs involved and call the concatenated and reduced word in $\pi_1(V)$ as the word of the $a - b - a$ arc. Likewise we tie up (a, c) and (c, a) arc combinations and call them $a-c-a$ arcs. We likewise define the words of $a-c-a$ arcs. Now, c_Q can be seen as a sequence of (a, a), $a-b-a$ and $a-c-a$ arcs. Further, the concatenation of the words described by these arcs should reduce to the empty word. Now, we gather the possible words described by these arcs when they are traced from A_i to A_j for $i, j \in \{1, 2\}$ in Table 3.4.

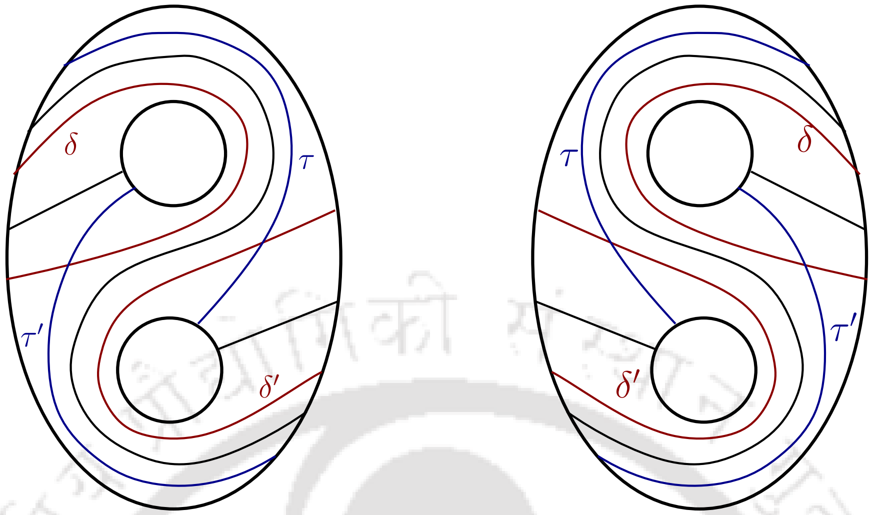
In this table 3.4 an arc starts on A_1' means that an arc starts on A_1 and on Σ_2', whereas an arc starts on A_1'' means that it starts on A_1 and on Σ_2''. An arc ends on A_1' means that the arc ends on A_1 and on Σ_2' whereas it ends on A_1'' means that the arc ends on A_1 and on Σ_2''.

Ends on $\longrightarrow$ Starts on $\downarrow$	A_1'	A_1''	A_2'	A_2''
A_1'		c, bc, b, bc	bc^{-1}	ϕ, b
A_1''	$c^{-1}, b^{-1}, c^{-1}b^{-1}$		ϕ, c^{-1}	$c^{-1}b$
A_2'	cb^{-1}	ϕ, c		c, b, cb
A_2''	ϕ, b^{-1}	$b^{-1}c$	$c^{-1}, b^{-1}c^{-1}, b^{-1}$	

Table 3.4: Words described by the $(a, a), a-b-a$, and $a-c-a$ arcs traced from A_i to A_j, $i, j \in \{1, 2\}$

Figure 3.39a shows the possible arcs on Σ_2' along with the (a, a) arc whose word is bc^{-1}. Figure 3.39b shows the possible arcs on Σ_2'' along with the (a, a) arc whose word is $b^{-1}c$ or $c^{-1}b$ depending on how it is traced.

We orient c_Q such that ρ is traced to have the word bc^{-1}. Starting at ρ as σ_1 we number the (a, a), $a - b - a$ and $a - c - a$ arcs following c_Q in the order they occur as $\sigma_i's$ and number them cyclically. We first define w_1 to



(a) Possible arcs with (a, a) describing bc^{-1} (b) Possible arcs with (a, a) describing $b^{-1}c$

Figure 3.39: Arcs with two lettered (a, a)

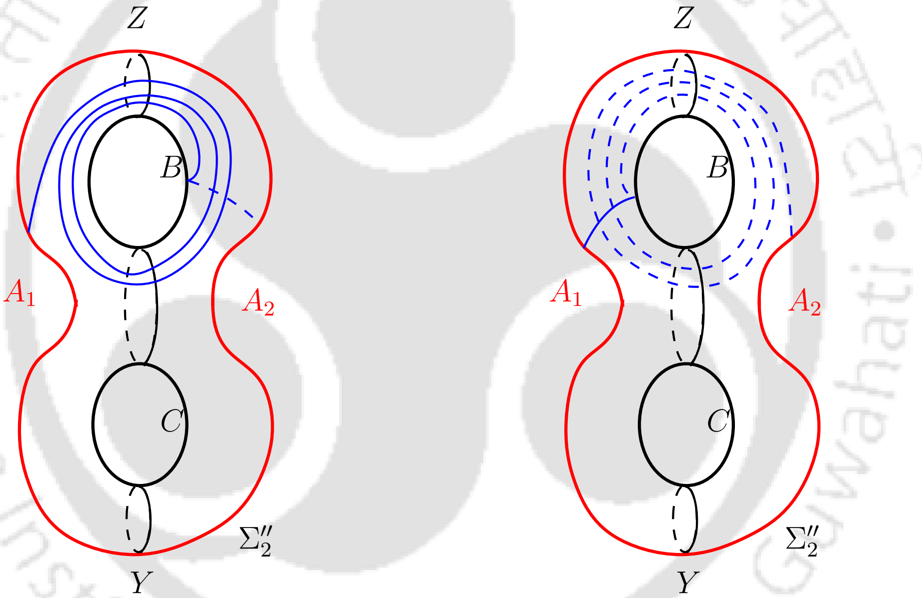
be the word of σ_1. At the i^{th} step we concatenate the word w_i to the word of σ_{i+1} and reduce this word to form word w_{i+1}. We show that the length of w_i, written $|w_i|$ is such that $w_i > 1$ for all $i \in \mathbb{N}$, by using induction. Since w_i represents the word of c_Q for some n, and since the word of c_Q is trivial, we arrive at a contradiction. Initially by the existence of ρ, $|w_1| \geq 2$. ρ ends on A_2. So its continuation arc on Σ_2'' has the possible words from the row against A_2'' in the above table. Note that none of the words start with c. So the continuation arc of ρ on Σ_2'' contributes a word to c_Q which when concatenated to bc^{-1} still gives a word which is of length ≥ 2. This can be taken as the base case. Now suppose that, for some $k \in \mathbb{N}$, k consecutive continuation arcs to ρ contribute words to c_Q which when concatenated to bc^{-1} still give a reduced word which is of length ≥ 2. This k^{th} continuation arc of ρ ends on any of A_1', A_1'', A_2' or A_2'' and so the $k + 1^{th}$ continuation arc of ρ will start on A_1'', A_1', A_2'' or A_2' respectively. In the above table, for any of these combinations, we notice that there is no reduction upon concatenation of the words, except possibly when the ending word of the k^{th} continuation arc of ρ is empty. So, except when such a word is empty, the concatenated word has a length ≥ 2 by mathematical induction. Now suppose that the k^{th} continuation arc of ρ contributes an empty word to the word of c_Q. The $k+1^{th}$ continuation arc has four possibilities *viz.* it can start on A_1', A_2', A_1'' or A_2''. Let us suppose that it starts on A_1'. Then the k^{th} continuation arc of ρ should have ended on A_1''. From the table 3.4, there is only one possibility for an arc describing the empty word and ending on A_1'', namely an $a-c-a$ arc which starts on A_2', which in turn means that the $k-1^{th}$ continuation arc must have ended on A_2''. For all arcs ending on A_2'' we notice that the ending letter of such arcs is b or c, unless it is the empty word. So such a letter cannot be reduced with the word of any arc starting on A_1' and in particular with the word of the $k+1^{th}$ continuation arc of ρ. So the property holds unless the $k-1^{th}$ arc described an empty word. If the $k-1^{th}$ arc described an empty word, from the table 3.4, we infer that it must have been an $a-b-a$ arc starting on A_1'. Continuing thus, we see that the $k+1^{th}$ continuation arc of ρ could be preceded by a finite sequence of $a-c-a$ and $a-b-a$ arcs which end on A_1'' and A_2'' respectively which contribute empty words to the word of c_Q. The arc immediately preceding this sequence of arcs will contribute a word which ends with b or c and so there is no reduction on concatenation with the word of the $k+1^{th}$ continuation arc of ρ. Hence the concatenated word so far has a length ≥ 2. The argument when the $k+1^{th}$ continuation arc starts on A_2', A_1'' or A_2'' is similar. So, even when the word of the k^{th} continuation arc of ρ is empty, we see that the word of c_Q has a word length ≥ 2. Since c_Q bounds a disk its word should be trivial when reduced and this is a contradiction. This completes the proof of this theorem. $\square$

Case IV: (a, a) arc with empty word on both sides.

Now, assuming $n_{AQ} > n_{BQ} + n_{CQ}$, suppose that c_Q has an (a, a) arc with an empty word on both sides and does not contain an (a, a) arc with one letter word on any side of Σ_2. We show that even in this case, β or β^{-1} can be used to reduce the complexity, $\mathcal{C}(Q)$.

Theorem 3.3.26. *Suppose that* $n_{AQ} > n_{BQ} + n_{CQ}$ *and that* c_Q *has no* (a,a)*-arc on* Σ_2' *or on* Σ_2'' *describing a single letter word. Further assume that* $n_{AQ} > 2$*. Now if* c_Q *contains* (a,a) *arc with empty word on both sides of* Σ_2 *then either* $|\beta(c_Q) \cap A| < |c_Q \cap A|$ *or* $|\beta^{-1}(c_Q) \cap A| < |c_Q \cap A|$*.*

Proof. Note that the (a,b) and (a,c) arcs in this case can only have the words of the form b^k and c^k respectively. By a first isotopy, we arrange c_Q so that all intersections of (a,b) arcs with X occur on X'' and all the intersections of (a,c) arcs with X occur on X'. Figure 3.40 shows how this can be done for an (a,b) arc describing the word b^3 as an example.



(a) An (a,b) arc describing b^3 followed by a (b,a) arc on Σ_2'' describing empty word. This arc intersects X'

(b) After isotopy this arc intersects only X'' and not X'

Figure 3.40: First isotopy for (a,b) describing b^3

An (a,b) or an (a,c) arc is necessarily followed by a (b,a) or a (c,a) arc respectively and the concatenation of the corresponding words in $\pi_1(V)$ cannot contain any trivial relator by corollary 3.3.11. So the illustration in Figure 3.40 is generic. If $\mathcal{N}(T)$ denotes the tubular neighborhood of a simple closed curve T on Σ_2, then an annulus $\mathcal{A}$ can be obtained as

$\Sigma_2 \setminus (\mathcal{N}(B) \cup \mathcal{N}(Z) \cup \mathcal{N}(C) \cup \mathcal{N}(Y))$, where the tubular neighborhoods are chosen small enough so that they do not intersect a tubular neighborhood of c_P. See Figure 3.41.

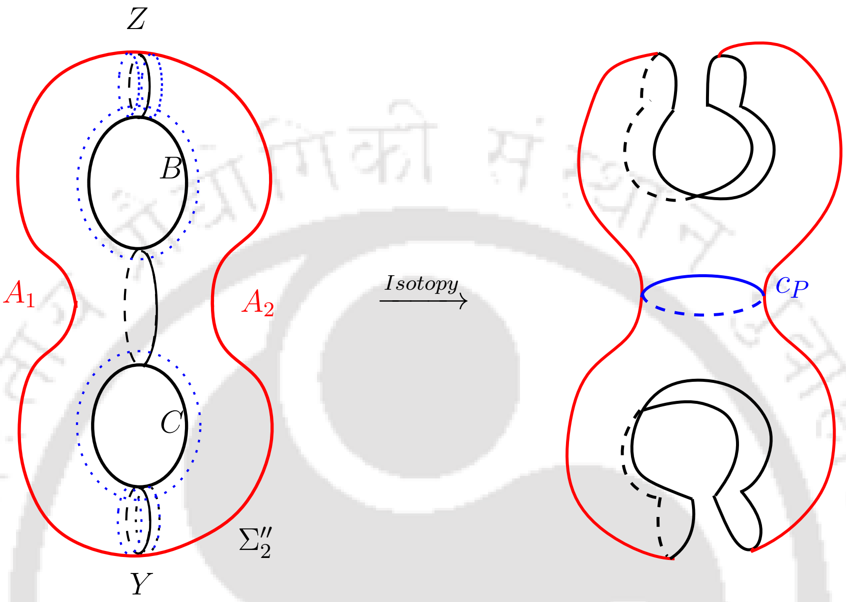


Figure 3.41: The annulus $\mathcal{A} = \Sigma_2 \setminus (\mathcal{N}(B) \cup \mathcal{N}(Z) \cup \mathcal{N}(C) \cup \mathcal{N}(Y))$

This annulus has two boundary components, one of them is the boundary of the closure of $\mathcal{N}(B) \cup \mathcal{N}(Z)$, which we denote as $\partial_B\mathcal{A}$ and the other boundary is the boundary of the closure of $\mathcal{N}(C) \cup \mathcal{N}(Y))$, which we denote as $\partial_C\mathcal{A}$. Since all the (a, a) arcs of c_Q on any side of Σ_2, in this case, describe the empty word, they do not intersect Y, Z, B or C. So, by a second isotopy, we assume that all the (a, a) arcs of c_Q are contained inside $\mathcal{A}$.

The only essential closed curve on Σ_2 contained in the annulus $\mathcal{A}$ is isotopic to c_P. Since, we are assuming that $n_{AQ} > 2$, c_Q is not completely contained in $\mathcal{A}$. So, $c_Q \cap \mathcal{A}$ consists of a finite collection of essential and possibly boundary-reducible arcs on $\mathcal{A}$. The boundary-reducible arcs occur if a (b, a) or a (c, a) arc is followed by a (a, b) or a (a, c) arc respectively. See Figure 3.42

By a third isotopy, we remove the boundary-reducible arcs from $\mathcal{A}$ by performing the boundary reduction. This isotopy removes the corresponding points of $c_Q \cap A$ from $\mathcal{A}$. So, after this isotopy, $\mathcal{A} \cap c_Q$ consists only of

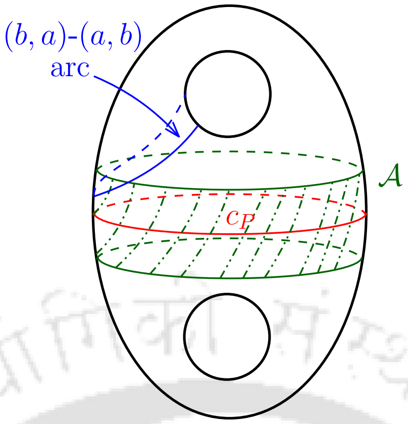
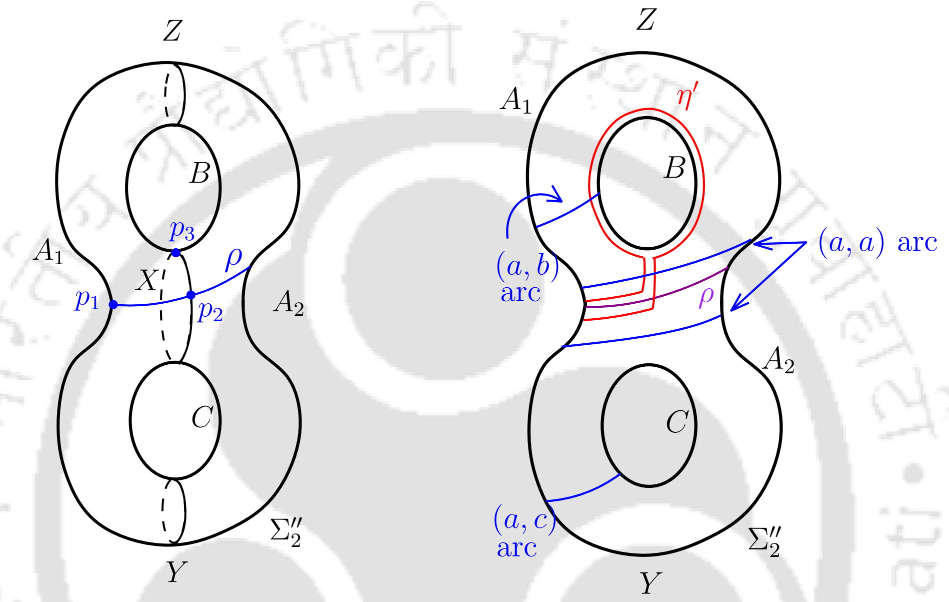


Figure 3.42: A (b,a)-(a,b) arc resulting in a boundary reducible disk in $\mathcal{A}$

essential arcs on the annulus $\mathcal{A}$. By a fourth isotopy, we can arrange the essential arcs in $\mathcal{A}$ such that the height (z-coordinate, using the Euclidean model) of an essential arc is strictly increasing. For the foregoing argument, we ignore the orientation induced by c_Q on these essential arcs and orient them from $\partial_C\mathcal{A}$ to $\partial_B\mathcal{A}$ *i.e.* in the direction of increasing height.

Note that the above four isotopies do not change the minimal intersection position of c_Q with any of A, B, C, X, Y and Z.

By assumption, there is at-least one (a,a) arc, call it ρ, on Σ_2'. ρ could be an arc from A_1 to A_2 or A_2 to A_1 when considering it in the direction of its increasing height. Suppose that it is an arc from A_1 to A_2. By a simple property of the annulus, all the (a,a) arcs of c_Q on Σ_2' will then have to increase from A_1 to A_2. Let $p_1 := \rho \cap A_1, p_2 := \rho \cap X'$ and $p_3 := X' \cap B$. Consider an arc ρ_1 of Σ_2' which is the join of the arc segment of ρ from p_1 to p_2 with the arc segment of X' from p_2 to p_3. Let $\mathcal{N}(\rho_1)$ be a tubular neighborhood of ρ_1 on Σ_2' such that no point of $c_Q \cap A$ other than p_1 is contained in this neighborhood. Consider the boundary of the closure in Σ_2' of the union, $\mathcal{N}(\rho_1) \cup \mathcal{N}(B)$. It consists of two components, one of which is the curve B itself. We name the other curve of this boundary as η'. Every (a,a) arc of c_Q on Σ_2' whose point of intersection with A_1 lies above p_1 will intersect both A and η' exactly twice. Every (a,a) arc of c_Q on Σ_2' whose point of intersection with A_1 lies below p_1 will intersect A twice but will be disjoint from η'. Every (a,b) arc on Σ_2' intersects η' exactly once, near its intersection point with B and also intersects A once. Every (a,c) arc on Σ_2' will be disjoint from η' whereas it intersects A once. ρ itself intersects η' once and A twice. So, considering all these intersections, $|c_Q \cap \Sigma_2' \cap \eta'| < |c_Q \cap \Sigma_2' \cap A|$. See Figure 3.43.



(a) The arc ρ and points p_1, p_2, p_3 shown on Σ_2

(b) η' shown on Σ_2' with various arcs intersecting it

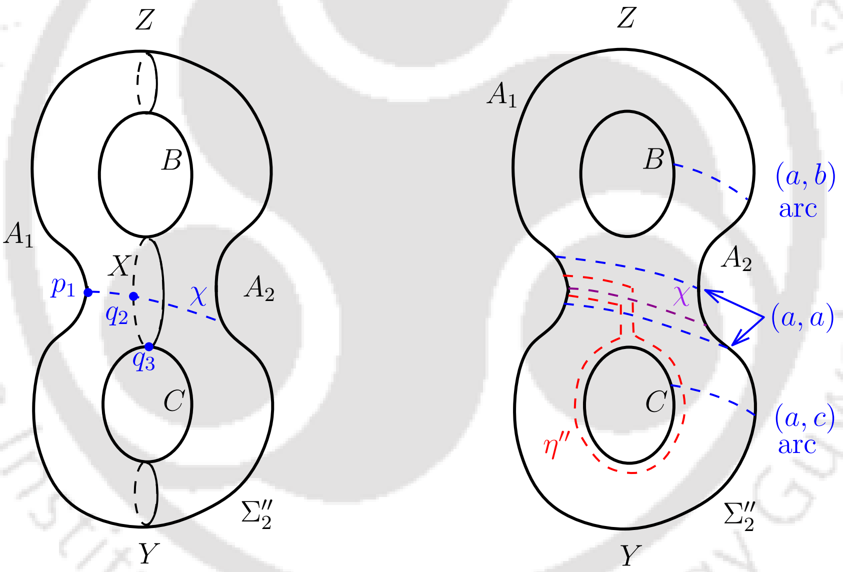
Figure 3.43: Construction of η'

We let χ be the preceding arc on Σ_2'' of ρ which joins ρ at p_1. χ cannot be an (a,b) arc as that would contradict the arrangement that the height of every essential arc on $\mathcal{A}$ is strictly increasing. So, it can be an (a,a) arc or an (a,c) arc.

Case (i):χ is an (a,a) arc: In this case, as above we let $q_2 := \chi \cap X''$ and $q_3 := X'' \cap C$, χ_1 be the join of the arc of χ from p_1 to q_2 and the arc of X'' from q_2 to q_3 and $\mathcal{N}(\chi_1)$ be a tubular neighborhood of χ_1 on Σ_2'' such that no point of $c_Q \cap A$ other than p_1 is contained in this neighborhood. As above, we consider the boundary of the closure in Σ_2'' of the union, $\mathcal{N}(\chi_1) \cup \mathcal{N}(C)$. It consists of two components, one of which is the curve C itself. We name the other curve of this boundary as η''. We adjust η''

so that its endpoints coincide exactly with the endpoints of η' and so that $\eta' \cup \eta''$ is a simple closed curve isotopic to A'.

Every (a, a) arc of c_Q on Σ_2'' whose point of intersection with A_1 lies below p_1 will intersect both A and η'' exactly twice. Every (a, a) arc of c_Q on Σ_2'' whose point of intersection with A_1 lies above p_1 will intersect A twice but will be disjoint from η'. Every (a, c) arc on Σ_2'' intersects η' exactly once, near its intersection point with C and also intersects A once. Every (a, b) arc on Σ_2'' will be disjoint from η'' whereas it intersects A once. χ intersects both η'' and A once, without counting its intersection with A at p_1 twice. So, considering all these intersections, $|c_Q \cap \Sigma_2'' \cap \eta''| \leq |c_Q \cap \Sigma_2'' \cap A|$. See Figure 3.44.



(a) The arc χ and points p_1, q_2, q_3 shown on Σ_2

(b) η'' shown on Σ_2' with various arcs intersecting it

Figure 3.44: Construction of η''

Case (ii):χ is an (a, c) arc: In this case, let $\mathcal{N}(\chi_1)$ be a tubular neighborhood of χ on Σ_2''. As above, we consider the boundary of the closure in Σ_2'' of the union, $\mathcal{N}(\chi) \cup \mathcal{N}(C)$. It consists of two components, one of which is the curve C itself. We name the other curve of this boundary as η''. We adjust η'' so that its endpoints coincide exactly with the endpoints of η' and so that $\eta' \cup \eta''$ is a simple closed curve isotopic to $A' = \beta^{-1}(A)$.

Every (a, a) arc of c_Q on Σ_2'' whose point of intersection with A_1 lies below p_1 will intersect both A and η'' exactly twice. Every (a, a) arc of c_Q on Σ_2'' whose point of intersection with A_1 lies above p_1 will intersect A twice but will be disjoint from η'. Every (a, c) arc on Σ_2'' intersects η' exactly once, near its intersection point with C and also intersects A once. Every (a, b) arc on Σ_2'' will be disjoint from η'' whereas it intersects A once. χ is disjoint from η'' and does not intersect A any more, without counting its intersection with A at p_1 twice. So, considering all these intersections, $|c_Q \cap \Sigma_2'' \cap eta''| \leq |c_Q \cap \Sigma_2'' \cap A|$. See Figure 3.45.

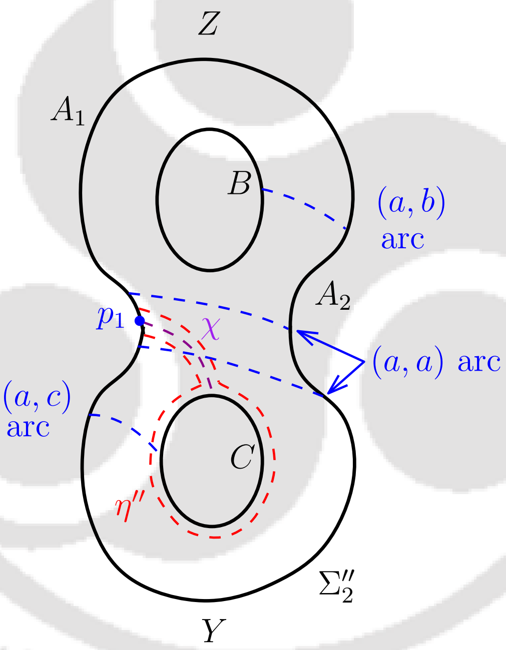


Figure 3.45: The case when χ is an (a, c) arc, η'' is shown with various possible arcs intersection it

In any of the above cases, we get $|c_Q \cap (\eta' \cup \eta'')| < |c_Q \cap A|$.

If ρ were an (a, a) arc increasing from A_2 to A_1 then we would do a similar construction with p_1 being on A_2 and then we would get $\eta' \cup \eta''$ to be a simple closed curve on Σ_2 isotopic to $A'' = \beta(A)$. We still would get the inequality $|c_Q \cap (\eta' \cup \eta'')| < |c_Q \cap A|$ in this case.

So, in summary, even in this Case IV, if $n_{AQ} > n_{BQ} + n_{CQ}$ and $n_{AQ} > 2$,

$|\beta(c_Q) \cap A| < |c_Q \cap A|$ or $|\beta^{-1}(c_Q) \cap A| < |c_Q \cap A|$. ☐

Case V: $Q = P$, $n_{AQ} = 2$ and $n_{BQ} = n_{CQ} = 0$.

Lemma 3.3.27. *For a reducing sphere Q, if $n_{BQ} = n_{CQ} = 0$ then $n_{AQ} = 2$ and $Q = P$.*

Proof. Since c_Q has to intersect at least one of A, B, C, $n_{BQ} = n_{CQ} = 0$ implies $n_{AQ} > 0$. Further since c_Q is separating on Σ_2, n_{AQ} has to be even. If possible, suppose that $n_{AQ} > 2$. Since $n_{AQ} > n_{BQ} + n_{CQ}$ and $n_{AQ} > 2$, $\mathcal{C}(Q) > 1$ and hence one of Case I, II or IV for the complexity reduction using β or β^{-1} is applicable. This reduction only reduces n_{AQ} without changing n_{BQ} or n_{CQ}. So by repeated application of this reduction n_{AQ} keeps decreasing until it reaches 2. So, after $|k|$ steps, for some integer k, we obtain $\beta^k(Q) = P$. But then $Q = \beta^{-k}(P)$ and P is stabilized by β. So $Q = P$ and $n_{AQ} = 2$, a contradiction to the assumption that $n_{AQ} > 2$. So n_{AQ} has to equal 2. ☐

We now prove the final part that whenever $n_{AQ} > n_{BQ} + n_{CQ}$ holds, φ and $\varphi\nu$ increases the complexity.

φ (or $\varphi\nu$) increases $\mathcal{C}(Q)$. φ (or $\varphi\nu$) only exchanges endpoints of arcs between A and B (or C) keeping C (or B) unchanged. This implies that if $n_{AQ} > n_{BQ} + n_{CQ}$, then an application of φ (similarly $\varphi\nu$) will give

$$\begin{aligned}
\mathcal{C}(\varphi(Q)) - \mathcal{C}(Q) &= \mathcal{C}(R) - \mathcal{C}(Q) \qquad (\text{taking} \quad R = \varphi(Q)) \\
&= \frac{1}{2}n_{AR} + n_{BR} + n_{CR} - \frac{1}{2}n_{AQ} - n_{BQ} - n_{CQ} \\
&= \frac{1}{2}n_{BQ} + n_{AQ} + n_{CQ} - \frac{1}{2}n_{AQ} - n_{BQ} - n_{CQ} \\
&= \frac{1}{2}n_{AQ} - \frac{1}{2}n_{BQ} > 0 \qquad (\because n_{AQ} > n_{BQ}).
\end{aligned}$$

(Similar argument works for $\varphi\nu$ since $n_{AQ} > n_{CQ}$)
Thus application of φ (or $\varphi\nu$) too increases $\mathcal{C}(Q)$. Therefore, the reduction is done only by a unique choice between β or β^{-1}. Hence the result. ☐

3.3.2 φ-reduction

This section shows that the inequality $n_{AQ} < n_{BQ} + n_{CQ}$ implies only the automorphism φ reduces the complexity measure of c_Q. We recall that by Lemma 3.3.5, $Q \neq P$ implies $n_{BQ} \neq n_{CQ}$.

Lemma 3.3.28. *Let* $Q \neq P$ *be any reducing sphere and* $\varphi(Q) = R$ *and* $\varphi\nu(Q) = S$. *Then*

(i) if $n_{AQ} < n_{BQ} + n_{CQ}$ *and* $n_{BQ} > n_{CQ}$ *then* $\mathcal{C}(R) < \mathcal{C}(Q)$ *and* $n_{AR} > n_{BR} + n_{CR}$*;*

(ii) if $n_{AQ} < n_{BQ} + n_{CQ}$ *and* $n_{BQ} < n_{CQ}$ *then* $\mathcal{C}(S) < \mathcal{C}(Q)$ *and* $n_{AS} > n_{BS} + n_{CS}$.

In either case, $\mathcal{C}(\beta(Q)) > \mathcal{C}(Q)$ *and* $\mathcal{C}(\beta^{-1}(Q)) > \mathcal{C}(Q)$.

Proof. We see that, $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{BQ} > n_{CQ}$ implies c_Q must contain a (b, b)-arc in both Σ_2' and Σ_2''. Therefore, on Σ_2' and Σ_2'', c_Q can only have $(a, b), (b, b)$ or (b, c)-arcs with distinct end-points. Therefore, n_{AQ} and n_{CQ} denote the numbers of (a, b)-arcs and (c, b)-arcs respectively. All such arcs contibutes to n_{BQ} as well. Also there exists at least one $(b.b)$ arc. Therefore, $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{BQ} > n_{CQ}$ implies $n_{BQ} > n_{AQ} + n_{CQ}$. By symmetric arguments, we also conclude that, $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{BQ} < n_{CQ}$ implies $n_{CQ} > n_{AQ} + n_{BQ}$.

(i) If $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{BQ} > n_{CQ}$ then

$$\begin{aligned}
\mathcal{C}(R) - \mathcal{C}(Q) &= \frac{1}{2}n_{AR} + n_{BR} + n_{CR} - \left(\frac{1}{2}n_{AQ} + n_{BQ} + n_{CQ}\right) \\
&= \frac{1}{2}n_{BQ} + n_{AQ} + n_{CQ} - \left(\frac{1}{2}n_{AQ} + n_{BQ} + n_{CQ}\right) \\
&= \frac{1}{2}n_{AQ} - \frac{1}{2}n_{BQ} < 0,
\end{aligned}$$

and $n_{AR} = n_{BQ} > n_{AQ} + n_{CQ} = n_{BR} + n_{CR}$.

(ii) Similarly, if $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{BQ} < n_{CQ}$ then

$$\begin{aligned}\mathcal{C}(S) - \mathcal{C}(Q) &= \frac{1}{2}n_{AS} + n_{BS} + n_{CS} - \left(\frac{1}{2}n_{AQ} + n_{BQ} + n_{CQ}\right)\\ &= \frac{1}{2}n_{CQ} + n_{BQ} + n_{AQ} - \left(\frac{1}{2}n_{AQ} + n_{BQ} + n_{CQ}\right)\\ &= \frac{1}{2}n_{AQ} - \frac{1}{2}n_{CQ} < 0,\end{aligned}$$

and $n_{AS} = n_{CQ} > n_{BQ} + n_{AQ} = n_{BS} + n_{CS}$. This also proves that reduction of $\mathcal{C}(Q)$ by φ or $\varphi\nu$ is by $\frac{1}{2}(n_{BQ} - n_{AQ})$ or $\frac{1}{2}(n_{CQ} - n_{AQ})$ respectively.

Now we will show that, if $n_{AQ} < n_{BQ} + n_{CQ}$ then both β as well as β^{-1} increases the complexity $\mathcal{C}(Q)$. Now, recall that, since β is an automorphism,

$$\beta^{\pm1}(Q){\cdot}\beta^{\pm1}(A) = Q{\cdot}A, \beta^{\pm1}(Q){\cdot}\beta^{\pm1}(B) = Q{\cdot}B, \text{ and } \beta^{\pm1}(Q){\cdot}\beta^{\pm1}(C) = Q{\cdot}C.$$

Also, $\beta^{\pm1}(B) = B$ and $\beta^{\pm1}(C) = C$ implies $\beta^{\pm1}(Q) \cdot B = Q \cdot B$ and $\beta^{\pm1}(Q) \cdot C = Q \cdot C$. Therefore by taking $\beta^{\pm1}(Q) = T$ we have,

$$\begin{aligned}\mathcal{C}(T) - \mathcal{C}(Q) &= \frac{1}{2}n_{AT} + n_{BT} + n_{CT} - \left(\frac{1}{2}n_{AQ} + n_{BQ} + n_{CQ}\right)\\ &= \frac{1}{2}n_{AT} + n_{BQ} + n_{CQ} - \frac{1}{2}n_{AQ} - n_{BQ} - n_{CQ}\\ &= \frac{1}{2}(n_{AT} - n_{AQ})\end{aligned}$$

Therefore, it is enough to show that, $n_{A\beta(Q)} > n_{AQ}$ and $n_{A\beta^{-1}(Q)} > n_{AQ}$ whenever $n_{AQ} < n_{BQ} + n_{CQ}$.

Now, $n_{AQ} < n_{BQ} + n_{CQ}$ implies c_Q cannot have any (a, a) arc. Again by lemma 3.3.5, c_Q must have atleast one of (b, b) or (c, c) arcs. Since c_Q does not have any (a, a) arc, so each point on A must correspond to an (a, b) or (a, c) arc on both Σ_2' and Σ_2''. Furthermore, an (a, b) arc from Σ_2' and an (a, c) arc on Σ_2'' cannot share an end-point on A and conversely. This is because, otherwise, existence of such an arc will ensure that c_Q cannot have a (b, b) or (b, c) arc, a contradiction. So we can cluster the arc segments of c_Q as $(b, b), (b, c), (c, c), (b, a, b)$ and (c, a, c).

Type of arc	Each arc's contribution to		
	$c_Q \cdot A$	$c_Q \cdot A'$	$c_Q \cdot A''$
(b, b)	0	+2	+2
(c, c)	0	+2	+2
(b, c)	0	+1	+1
(b, a, b)	+1	+1	+1
(c, a, c)	+1	+1	+1

Table 3.5: Contribution of each type of arc-combination to $c_Q \cdot A, c_Q \cdot A'$ and $c_Q \cdot A''$, where $A' = \beta(A), A'' = \beta^{-1}(A)$

From table 3.5 it can be noted that each one of $(b, b), (c, c)$ and (b, c) arc of c_Q contributes more to $c_Q \cdot A'$ and $c_Q \cdot A''$ than to $c_Q \cdot A$. Whereas, the contribution is equal for the $b - a - b$ and $c - a - c$ concatenations. Since atleast one (b, b) or (c, c) exists by lemma 3.3.5, therefore $c_Q \cdot A' > c_Q \cdot A$ and $c_Q \cdot A'' > c_Q \cdot A$.

Consequently, we can conclude that both β as well as β^{-1} increases complexity whenever $n_{AQ} < n_{BQ} + n_{CQ}$. $\square$

The corollary 3.3.6 also shows that the conditions for β and φ reduction are mutually exclusive.
The above results lead us to the reduction algorithm described in the following section.

3.3.3 The algorithm

Based on the discussion presented above we can present the reduction of an arbitrary reducing sphere of the genus two Heegaard splitting of S^3 via a finite step algorithm. Let Q be a reducing sphere. Determine n_{AQ}, n_{BQ} and n_{CQ}.

Step-1 If $n_{BQ} = n_{CQ} = 0$ then Q is standard (lemma 3.3.27). Return.

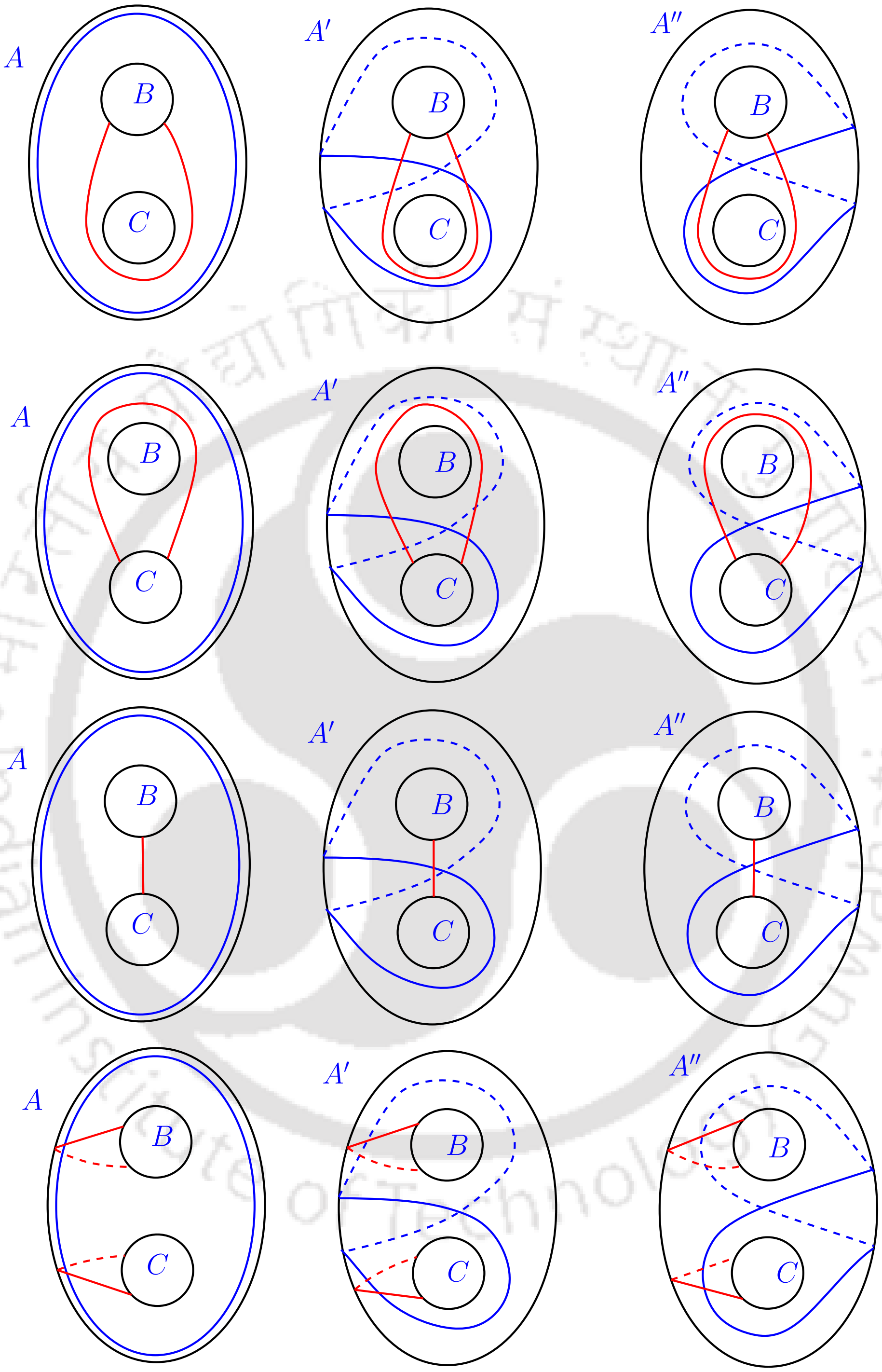


Figure 3.46: Arcs of c_Q intersecting curves A, A' and A''

Else goto step-2.

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Step-2 While $n_{AQ} > n_{BQ} + n_{CQ}$, do the following:

(i) apply β or β^{-1} so that $\mathcal{C}(Q)$ decreases,

(ii) update n_{AQ}, n_{BQ} and n_{CQ}.

Goto Step-3.

Step-3 If $n_{AQ} < n_{BQ}+n_{CQ}$ and $n_{BQ} > n_{CQ}$ apply φ and update n_{AQ}, n_{BQ} and n_{CQ}. Else if $n_{AQ} < n_{BQ} + n_{CQ}$ and $n_{CQ} > n_{BQ}$ apply $\varphi\nu$ and update n_{AQ}, n_{BQ} and n_{CQ}. Goto Step-1.

As $\mathcal{C}(Q)$ decreases strictly at each step until $\mathcal{C}(Q) = 1$, and n_{AQ}, n_{BQ}, n_{CQ} are finite, therefore the above algorithm terminates in finitely many steps.

Now since for any arbitrarily chosen reducing sphere Q, this algorithm provides an automorphism f such that $f(Q) = P$ using only elements from $\mathcal{G}_2$, therefore we summarize our findings in the following proposition:

Proposition 3.3.29. *$\mathcal{G}_2$ generates $\mathcal{H}_2$.*

Theorem 3.3.30. *Every element f of $\mathcal{H}_2$ can be uniquely expressed in the form*

$$f = \alpha^a \nu^b \beta^c \prod \left(\varphi\nu^{s_i}\beta^{r_i}\right) = \alpha^a \nu^b \beta^c \left(\varphi\nu^{s_n}\beta^{r_n}\right) \circ \cdots \circ \left(\varphi\nu^{s_1}\beta^{r_1}\right),$$

where $a, b, s_i = 0, 1$ and $c.r_i \in \mathbb{Z}$.

Proof. Let $f \in \mathcal{H}_2$. Then $f(P) = Q$ is a reducing sphere. Now if $Q = P$, then f is of the form $\alpha^a\nu^b\beta^c$, for some a, b, c as described in the algorithm. If $Q \neq P$, then the algorithm starts at step 2 with an integral power (can be zero) of β. Once $n_{AQ} < n_{BQ} + n_{CQ}$ is obtained, the algorithm moves to step 3. We apply φ or $\varphi\nu$ and both of which can be written as $\varphi\nu^{s_1}$. This again changes the inequality. So either we get the standard sphere and the algorithm exits, or we repeat the algorithm from step 2. Once the algorithm exits, f can have a prefix of the form $\alpha^a\nu^b\beta^c$, for some a, b, c. So f is written in the form described in the theorem.

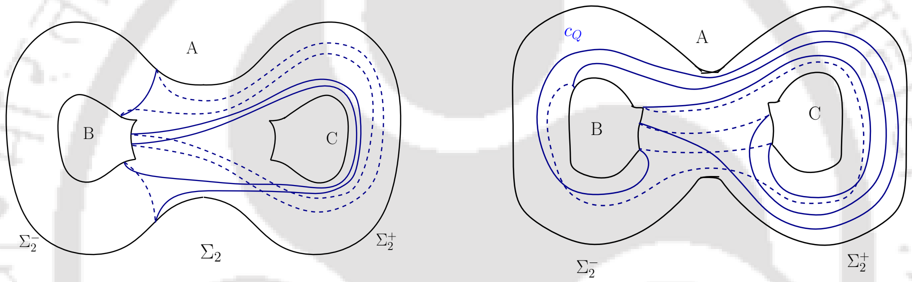
For the uniqueness of the expression, except for the prefix term, for any reducing sphere Q we have

$$\mathcal{C}(\varphi\nu^{s_i}\beta^{r_i}(Q)) > \mathcal{C}(Q).$$

So, $\varphi\nu^{s_i}\beta^{r_i}(Q) \neq Q$. Hence f cannot have two different expressions in the given form. $\square$

3.3.4 Illustration of the algorithm

Here we present a couple of examples of two reducing spheres and observe the application of the above algorithm. Consider the following examples:



(a) Example 1

(b) Example 2

Figure 3.47: Application of the algorithm

In figure 3.47a, $n_{AQ} < n_{BQ} + n_{CQ}$, and $n_{BQ} > n_{CQ}$. So we apply φ. In figure 3.47b we have, $n_{BQ} + n_{CQ} = 3 > 0 = n_{AQ}$ also $n_{BQ} > n_{CQ}$. Here too we apply φ. On application of φ we get the spheres in figure 3.48a and 3.49a respectively.

Let $\varphi(Q) = R$. Then from both figure 3.48a and figure 3.48b, we have $n_{AR} = 4 > 2 = n_{BR} + n_{CR}$. So now we apply β or β^{-1} suitably. For instance here we apply β^{-1} in both cases. The result is presented in figure 3.49a and figure 3.49b respectively. Now if $\beta^{-1}(R) = S$, in both cases we have $n_{AS} = 0$. In the first case, we have $n_{BS} = 2, n_{CS} = 0$ whereas in the other we have $n_{BS} = 0, n_{CS} = 2$. So in both cases, $n_{AS} < n_{BS} + n_{CS}$. In first case $n_{BS} > n_{CS}$ and we apply φ again whereas in the second one $n_{CS} > n_{BS}$ and so we apply $\varphi\nu$. It can be easily calculated that in both cases we are left with the standard curve c_P.

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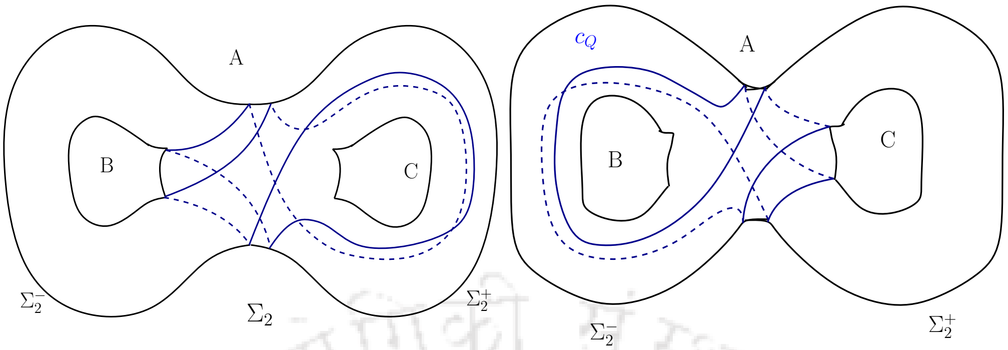
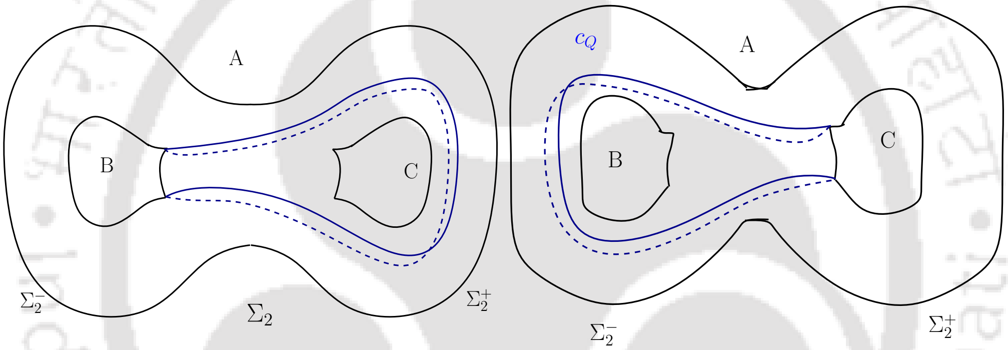


(a) Example 1

(b) Example 2

Figure 3.48: Application of φ in algorithm



(a) Example 1

(b) Example 2

Figure 3.49: Application of β in algorithm

Therefore the automorphism that takes the first one to the standard is given by $f = \varphi\beta^{-1}\varphi$ and the one that takes the second one to standard is given by $g = \varphi\nu\beta^{-1}\varphi$.

3.3.5 The automorphism δ is in $\langle\mathcal{G}_2\rangle$

From the description of α and β, they are already identical with the corresponding generators in [10]. Also we have seen that $\gamma = \alpha\nu$. Consider the subset of $\mathcal{MCG}(\Sigma_2)$ given by

$$\mathcal{G}_2 = \{\nu, \alpha, \beta, \varphi\}.$$

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Want to show that the automorphism δ described in Scharlemann [10] is in $\langle \mathcal{G}_2 \rangle$.

Proposition 3.3.31. *The automorphism δ from* [10] *is generated by ν and φ and we can express δ as*

$$\delta = \nu^{-1}\varphi\nu\varphi = (\nu\varphi)^2,$$

as $[\nu]^2 = 1$.

Proof. From the earlier discussion, we have $\varphi(B) = A$. We also have

$$\varphi(A) = B, \varphi(C) = C, \varphi(X) = Y, \varphi(Y) = X, \text{ and } \varphi(Z) = Z.$$

Now since ν exchanges the two genus one summands and leaves X and A invariant therefore

$$\nu\varphi(A) = C, \nu\varphi(B) = A, \nu\varphi(C) = B, \nu\varphi(X) = Z, \nu\varphi(Y) = X \text{ and } \nu\varphi(Z) = Y.$$

Now if $\psi = \nu\varphi\nu\varphi$, then

$$\psi(A) = B, \psi(B) = C, \psi(C) = A, \psi(X) = Y, \psi(Y) = Z, \text{ and } \psi(Z) = X.$$

Now from the description of δ we have

$$\delta(A) = B, \delta(B) = C, \delta(C) = A, \delta(X) = Y, \delta(Y) = Z \text{ and } \delta(Z) = X.$$

Therefore, $\psi^{-1}\delta$ fixes all the above mentioned loops on Σ_2. But that implies $[\psi^{-1}\delta] = 1$ i.e. $\delta = \psi = (\nu\varphi)^2$.
This completes the proof. □

Therefore, elements in $\mathcal{G}_2$ generate the generators of $\mathcal{H}_2$ proposed by Scharlemann [10]. Thus this gives another proof of the fact that $\mathcal{G}_2$ generates $\mathcal{H}_2$.

3.4 Conclusion

The proposed generating set $\mathcal{G}_2 = \{\nu, \alpha, \beta, \varphi\}$ indeed generates $\mathcal{H}_2$. Also for an arbitrary reducing sphere Q, the reduction algorithm provides a recipe to retrieve the automorphism $f \in \mathcal{H}_2$ uniquely as words in $\mathcal{G}_2$ such that $f(Q)$ is standard. If $f_Q \in \mathcal{H}_2, f_Q(Q) = P$ implies

$$f_Q = \alpha^a \nu^b \beta^c \prod \left(\varphi \nu^{s_i} \beta^{r_i}\right) = \alpha^a \nu^b \beta^c \left(\varphi \nu^{s_n} \beta^{r_n}\right) \circ \cdots \circ \left(\varphi \nu^{s_1} \beta^{r_1}\right),$$

where $a, b, s_i = 0, 1$ and $c, r_i \in \mathbb{Z}$. (Here $\prod$ represents composition.)
Also against each application of φ or $\varphi\nu$ with a possible β in the above algorithm, $\mathcal{C}(Q)$ decreases strictly until $\mathcal{C}(Q) = 1$, i.e. when Q becomes standard. Whereas for α and ν, $\mathcal{C}(Q)$ remains invariant. Therefore, $\mathcal{C}(Q)$ is indeed a complexity measure of the reducing sphere Q. This complexity measure is a significant finding of this work. The combined work of Scharlemann[10], Akbas [1] and Cho [2] showed that, for any reducing sphere Q with $P \cdot Q = 4$, a unique reducing sphere R can be constructed so that

$$P \cdot R = 4 \text{ and } R \cdot Q < P \cdot Q.$$

Then an application of δ or δ^{-1}, brings R to P and Q to some Q' such that $P \cdot Q' < P \cdot Q$. This process terminates when $P \cdot Q \leq 4$. But once Q' is obtained, the process does not specify whether one needs to apply β in the immediate step or not. It also does not specify $P \cdot Q'$. Finally the factor $\alpha^a \nu^b \beta^c$ takes care of capturing all the automorphisms that stabilize P.

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CHAPTER 4

Genus three Heegaard splitting of S^3

4.1 Introduction

We describe the genus three heegaard splitting of the three sphere as

$$S^3 = V_3 \bigcup_{\Sigma_3} W_3$$

or (S^3, V_3, W_3), where V_3 denote the standard unknotted genus three handlebody embedded in S^3, W_3 is its closed complement in S^3 and Σ_3 represents the heegaard splitting surface homeomorphic to ∂V_3 as well as ∂W_3. Let $\mathcal{H}_3$ denote the Goeritz group of genus three heegaard splitting of S^3. Freedman and Scharlemann[4] proved that $\mathcal{H}_3$ is finiely generated. They proposed a generating set containing the Powell generators $\varphi_\omega, \varphi_\eta, \varphi_{\eta_{12}}, \varphi_v, \varphi_\theta$.

In this work we first propose a generating set of four elements in $\mathcal{H}_3$ where each elements are described as either rotation of Σ_3 about the standard axes or a composition of dehn-twists about essential simple non-separating closed curves on Σ_3. The motivation behind this presentation is the fact that $Mod(\Sigma_3)$ is generated by dehn twists about such non-separating curves (for details we refer to Farb and Margalit [3]) and each element in $\mathcal{H}_3$ corresponds to a unique element $Mod(\Sigma_3)$ upto isotopy. Next

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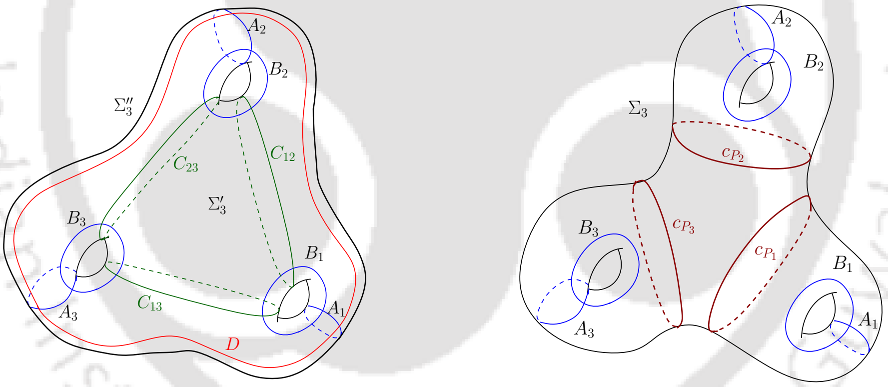
we show that our proposed set actually generates the Powell generators and hence all of $\mathcal{H}_3$.

4.2 Terminologies

Hereafter for any reducing sphere Q, we will call the essential separating simple closed curve $c_Q = Q \cap \Sigma_3$ as reducing curve of Q. Each genus one summand upto isotopy contains a pair of loops A_i and B_i (for $i = 1, 2, 3$) that can be isotoped to intersect at single points such that A_i bounds a disk D_{A_i} in V_3 and B_i bounds a disk D_{B_i} in W_3 (see figure 4.1a). We will call such pairs (of both loops and disks) as standard orthogonal pairs. Also we will refer to each of the loops A_i, B_i, C_{ij} and D in figure 4.1a as standard loops on Σ_3. There are exactly three non-isotopic reducing curves on Σ_3



(a) The standard curves on Σ_3 (b) The standard spheres on Σ_3

Figure 4.1: Standard curves and standard triple on Σ_3

which are disjoint from all wedges of loops $(A_i \wedge B_i)$, for $i = 1, 2, 3$. We call these curves standard and denote them as c_{P_1}, c_{P_2} and c_{P_3} (see figure 4.1b). $B_1 \cup B_2 \cup B_3 \cup D$ separates Σ_3 into two components Σ_3' and Σ_3'', both of which are spheres with four boundaries. Moreover, corresponding to each A_i, upto isotopy there is a unique loop A_i' which differs from A_i by a hyperelliptic involution of Σ_3. We will refer to these loops as variants of standard.

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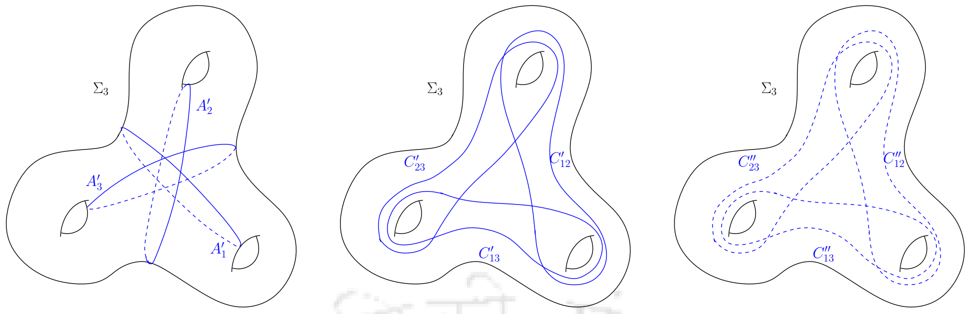

(c) Involution of A_i (d) Dual of C_{ij} on Σ'_3 (e) Dual of C_{ij} on Σ''_3

Figure 4.1: Standard curves and standard triple on Σ_3

Each curve c_{P_i} in the standard triple bounds a genus one summand of Σ_3, call it Σ_3^i for $i = 1, 2, 3$. Also we use $\Sigma_3^{i\prime}$ and $\Sigma_3^{i\prime\prime}$ to refer to $\Sigma_3^i \cap \Sigma'_3$ and $\Sigma_3^i \cap \Sigma''_3$ respectively for $i = 1, 2, 3$. We use Σ_3^0 to refer to the pair of pants component of Σ_3 bounded by the standard triple. Figures 4.1c,4.1d and 4.1e shows other curves such as - A'_i, C'_{ij} and C''_{ij} for $i, j = 1, 2, 3 (i < j)$ on Σ_3.

Basic terminologies used for surfaces, curves and automorphisms are aligned with Farb et al [3] and those for Heegaard splitting are in parity with Scharlemann [10], Akbas[1] and Zupan [14]. We also refer to the preliminary part for the frequently used terminologies.

4.3 The subset $\mathcal{G}_3$ of $Mod(\Sigma_3)$

It may be observed that there can be atmost three disjoint non-isotopic reducing curves placed simultaneously on Σ_3. Such a triple separates Σ_3 into genus one summands and a pair of pants component. Any essential simple separating closed curve on any of these components must be boundary parallel. Therefore, we can consider the set of all unordered triples of reducing curves on Σ_3. The aforementioned three standard reducing curves also forms such a triple. We will refer to this triple $P = (c_{P_1}, c_{P_2}, c_{P_3})$ as the standard triple (see figure 4.1b). Now, if we consider the set $\mathcal{H}_3^P$ of all automorphisms in $\mathcal{H}_3$ that leaves the standard triple invariant, then it is

easy to observe that $\mathcal{H}_3^P$ is a subgroup of $\mathcal{H}_3$. We call this subgroup the stabilizer of the standard triple. Since each element in $\mathcal{H}_3^P$ fixes the standard triple, therefore we can isotope any element in the stabilizer to fix all the three wedge of loops $A_i \wedge B_i$ for $i = 1, 2, 3$ pointwise. Therefore any element in the stabilizer is uniquely determined upto isotopy, dehn twists about the standard reducing curves and permutations of the reducing curves. Consequently, the stabilizer of a triple is generated by the dehn-twists about reducing curves and automorphisms of Σ_3 permuting them.
Consider the set

$$\mathcal{G}_3 = \{\alpha, \beta, \nu, \varphi\}$$

where

(i) The automorphism ν is the first one from the generating set of the stabilizer $\mathcal{H}_3^p$ that captures the rotational symmetry of the genus one summands in Σ_3. This is an element of order 3 and thus $\nu^3 = 1$. Figure 4.2 illustrates the action of ν on Σ_3. From the description of ν it follows that

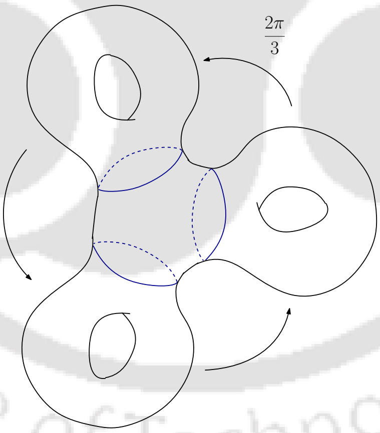

Figure 4.2: Rotation ν on Σ_3

$\nu(A_i) = A_{i+1}, \nu(B_i) = B_{i+1}$ for $i = 1, 2$ and $\nu(A_3) = A_1, \nu(B_3) = B_1$.

Also $\nu(D) = D, \nu(c_{P_1}) = c_{P_2}, \nu(c_{P_2}) = c_{P_3}$ and $\nu(c_{P_3}) = c_{P_1}$

$$\nu(\Sigma_3') = \Sigma_3' \text{ and } \nu(\Sigma_3'') = \Sigma_3''.$$

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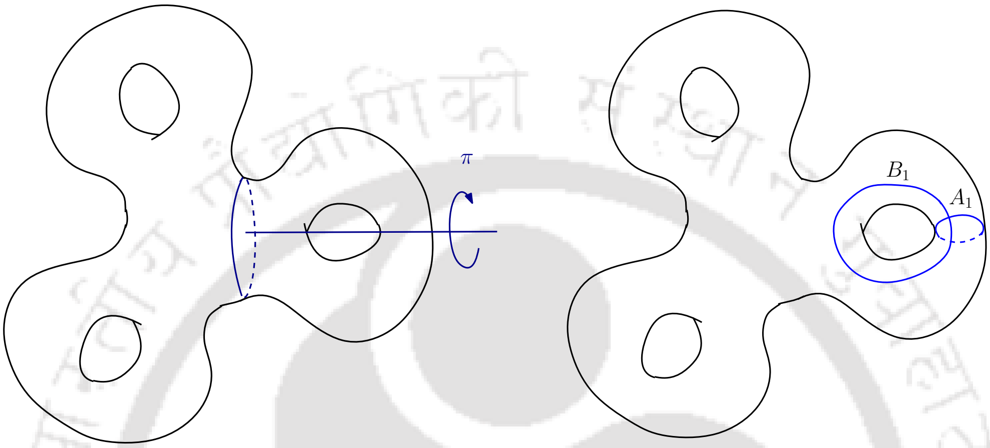
(ii) The automorphism β in $\mathcal{H}_3^p$ is an element of infinite order which can be interpreted as involution of a genus one summand while fixing the rest. This is just half-a-twist about the standard reducing curve bounding the concerned genus of summand. Figure 4.3a describes β.



(a) Half-twist β on Σ_3 (b) β in terms of dehn-twists

Figure 4.3: The automorphism β in $\mathcal{H}_3$

Figure 4.3b illustrates β by expressing it as a word in the generators of $Mod(\Sigma_3)$. We can express β as

$$\beta = (T_{B_1}T_{A_1})^3 = T_{B_1}T_{A_1}T_{B_1}T_{A_1}T_{B_1}T_{A_1},$$

where A_1, B_1 represents the meridianal and longitudinal loops respectively in the concerned genus one summand; and T_λ denotes the dehn twist about the loop λ. It may be observed that the action ob β is restricted completely inside a collar neighbourhood of the concerned standard reducing curve on Σ_3. From the description of β. Therefore we have

$$\beta(A_i) = A_i, \beta(B_i) = B_i \text{ and } \beta(c_{P_i}) = c_{P_i} \text{ for } i = 1, 2, 3.$$
$$\beta(\Sigma_3^{1\prime}) = \Sigma_3^{1\prime\prime}, \beta(\Sigma_3^{1\prime\prime}) = \Sigma_3^{1\prime}, \beta(\Sigma_3^{i\prime}) = \Sigma_3^{i\prime} \ \&\ \beta(\Sigma_3^{i\prime\prime}) = \Sigma_3^{i\prime\prime} \text{ for } i = 0, 2, 3.$$

(iii) The automorphism α can be described as the following composition of dehn-twists about essential, simple non-separating curves shown in

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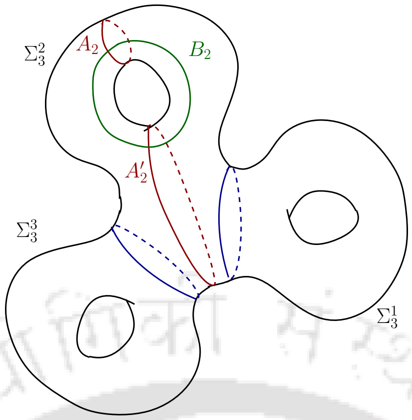

Figure 4.4: The automorphism α in $\mathcal{H}_3$

figure 4.4. We express α as $\alpha = T_{A_2'}T_{B_2}T_{A_2'}T_{A_2}T_{B_2}T_{A_2'}$, where T_{A_2}, T_{B_2} and $T_{A_2'}$ represent dehn twists about the non-separating curves A_2, B_2 and A_2' respectively. The automorphism α can also be interpreted as the composition of a half-twist of the genus two summand $\Sigma_3 - \Sigma_3^3$ about its boundary and a inverse half-twist of the genus one summand Σ_3^1 about its boundary. In the process α swaps two genus one summands keeping the third (i.e. Σ_3^3) fixed. But it also changes the standard reducing curve bounding Σ_3^2. Therefore

$$\alpha(A_1) = A_2, \alpha(A_2) = A_1, \alpha(A_3) = A_3 \text{ and}$$
$$\alpha(B_1) = B_2, \alpha(B_2) = B_1, \alpha(B_3) = B_3,$$
$$\alpha(c_{P_1}) = c_{P_2}, \alpha(c_{P_2}) = c_{P_1} \text{ and } \alpha(c_{P_3}) = c_{P_3}.$$
$$\alpha(\Sigma_3^{1\prime}) = \Sigma_3^{2\prime}, \alpha(\Sigma_3^{1\prime\prime}) = \Sigma_3^{2\prime\prime}, \alpha(\Sigma_3^{2\prime}) = \Sigma_3^{1\prime\prime}, \alpha(\Sigma_3^{2\prime\prime}) = \Sigma_3^{1\prime},$$
$$\alpha(\Sigma_3^{3\prime}) = \Sigma_3^{3\prime}, \alpha(\Sigma_3^{3\prime\prime}) = \Sigma_3^{3\prime\prime}, \alpha\Sigma_3^{0\prime} = \Sigma_3^{0\prime\prime} \text{ and } \alpha\Sigma_3^{0\prime\prime} = \Sigma_3^{0\prime}.$$

(iv) The final element in our proposed set of generators of $\mathcal{H}_3$ is the automorphism φ. The description of φ in terms of composition of dehn twists about some specific non-separating curves is as follows:

$$\varphi = T_{A_2}^{-1}T_{A_1}T_{B_1}T_{A_1}T_{C_{12}}T_{B_1}T_{A_1},$$

where the curves A_1, B_1, A_2, C_{12} are as shown in figure 4.5a.

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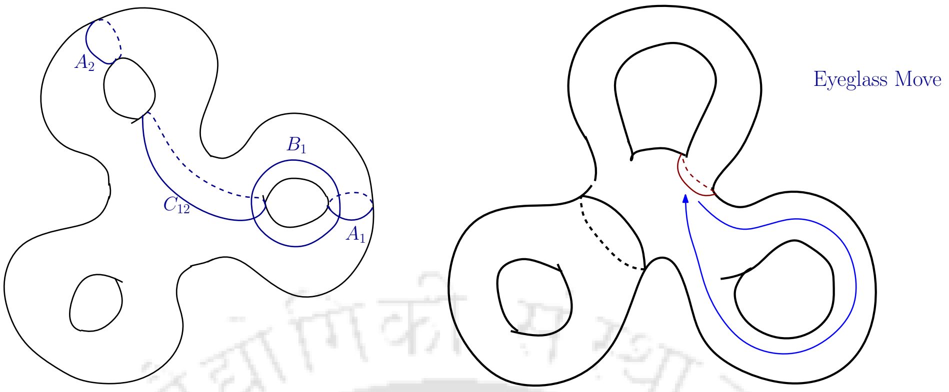
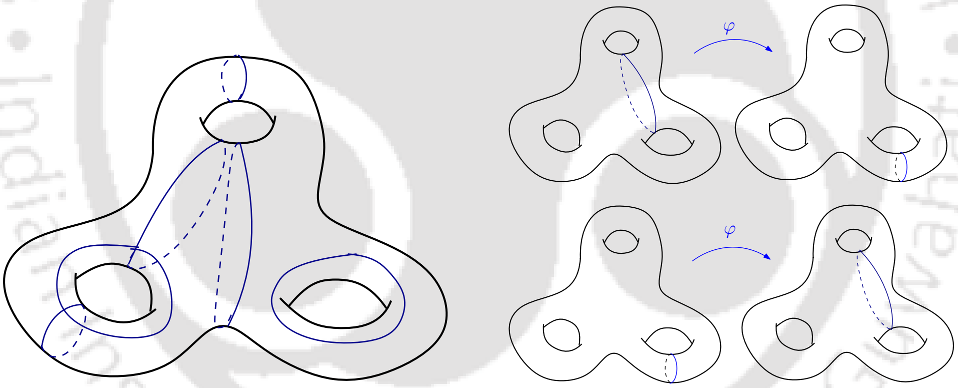


(a) The automorphism φ (b) The automorphism φ_θ

Figure 4.5: The automorphism φ in $\mathcal{H}_3$ and the eyeglass move

The action of φ on the standard set of loops on Σ_3 is described in figure 4.6 and 4.7. From figure 4.6 and 4.7 it follows that,



(a) Loops invariant under φ (b) Loops having order 2 transformations under φ

Figure 4.6: Loops invariant under φ^2

$$\varphi(A_1) = C_{12}, \varphi(A_2) = A_2, \varphi(A_3) = A_3,$$
$$\varphi(B_1) = B_1, \varphi(B_2) = C''_{12}, \varphi(B_3) = B_3,$$
$$\varphi(C_{12}) = A_1, \varphi(C_{23}) = C_{23}.$$

Also $\varphi(C_{13})$ is the curve same as $T_{A'_2}(C_{13})$ (see figure 4.7, second row).

It may further be observed that the action of φ is restricted to a genus

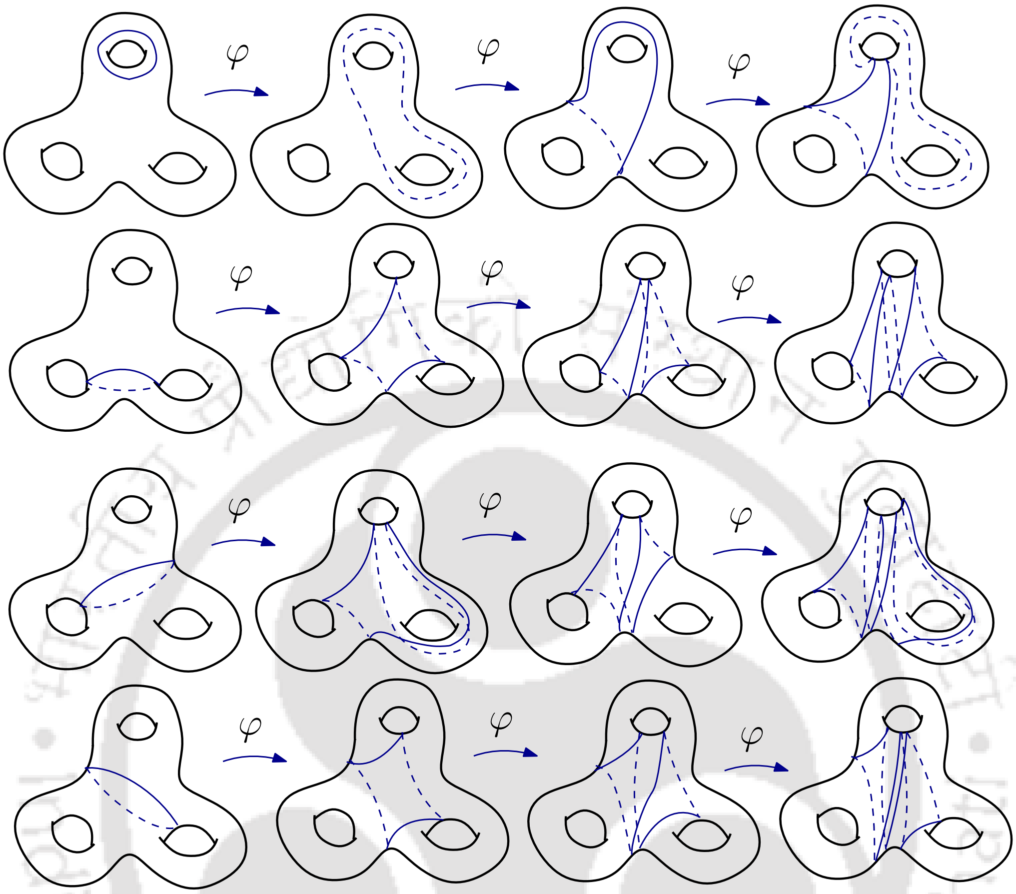


Figure 4.7: Loops having infinite orbit under φ

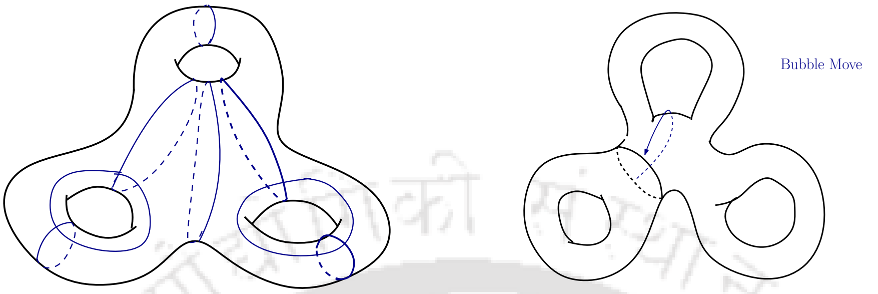
two summand of Σ_g for $g \geq 3$. More precisely, according to our description φ leaves the genus one summand Σ_3^3 fixed pointwise as none of the generating loops intersect Σ_3^3.

4.4 $\mathcal{G}_3$ generates all of $\mathcal{H}_3$

One can observe that φ^2 replicates a variant of the bubble move φ_v in [4, 14]. Taking $h = \nu^{-1}\beta\nu\beta^{-1}\alpha$, we can observe that $\varphi_v^{-1}\beta^{-1}(\nu^{-1}h\nu)\varphi^{-2}$ leaves all the standard loops as well as Σ_3' and Σ_3'' invariant. Therefore $\varphi_v^{-1}\beta^{-1}(\nu^{-1}h\nu)\varphi^{-2}$ is identity on Σ_3. So the action of φ_v in [14] can be reproduced as

$$\varphi_v = \beta^{-1}(\nu^{-1}h\nu)\varphi^{-2}$$

From figure 4.8 and 4.9 it can be observed that the action of φ^2 on the standard loops on Σ_3 matches with that of bubble move.



(a) Loops fixed under φ^2 (b) Bubble move corresponding to φ^2

Figure 4.8: φ^2 versus Bubble move

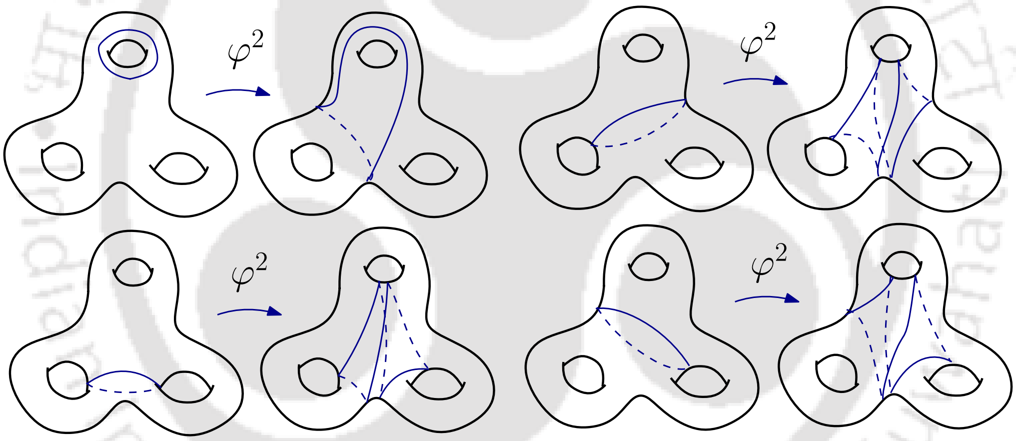


Figure 4.9: Action of φ^2: Other loops

Anyway ν, capturing the rotational symmetry of Σ_3, remains same as the corresponding Powell move φ_η described in [4, 14].

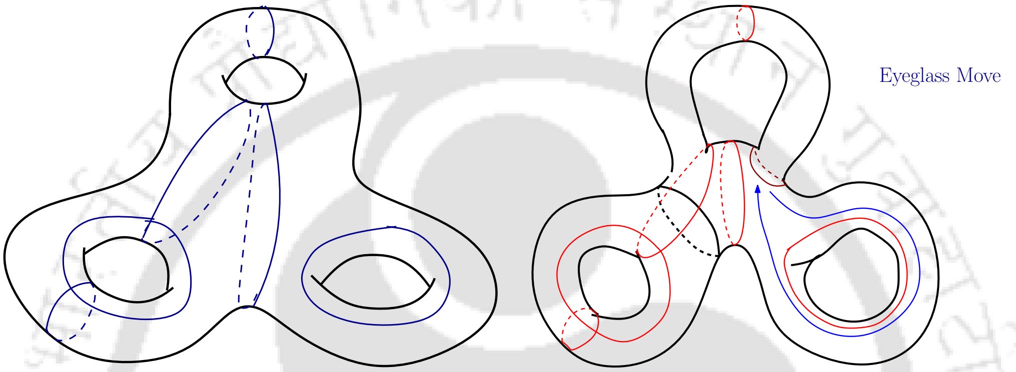
Also β being described as dehn twists about the curve of intersection of a standard sphere matches exactly with the corresponding element φ_ω in the set of Powell generators.

The automorphism $\varphi_{\eta_{12}}$ in [4, 14] which moves two genus one summand around each other is a composition of α, β, ν and β^{-1}. In fact we can express that automorphism as

$$\varphi_{\eta_{12}} = \nu\beta\nu^{-1}\beta^{-1}\alpha.$$

One can actually verify that $\varphi_{\eta_{12}}^{-1} \circ (\nu\beta\nu^{-1}\beta^{-1}\alpha)$ fixes all the standard loops on Σ_3. Further, this composition also fixes Σ_3' and Σ_3''. Therefore, this composition is identity on Σ_3 and hence the equality holds.

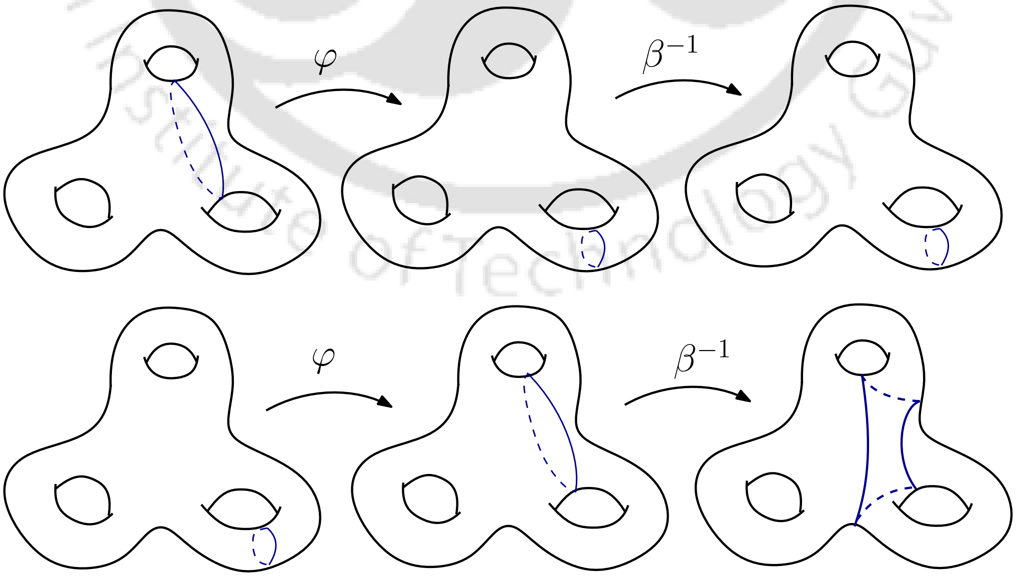
Moreover $\beta^{-1}\varphi$ reproduces the actions of a variant of the eyeglass move φ_θ mentioned in [4, 14]. Figures 4.10 and 4.11 show a comparision of the action of $\beta^{-1}\varphi$ with the corresponding eyeglass move. Thus the set of



(a) Loops fixed under $\beta^{-1}\varphi$: $B_1, A_2, A_2', A_{23}, A_3, B_3$

(b) Loops fixed under φ_θ corresponding to $\beta^{-1}\varphi$

Figure 4.10: $\beta^{-1}\varphi$ versus Eyeglass move on curves $B_1, A_2, A_2', A_{23}, A_3, B_3$



(a) Action of $\beta^{-1}\varphi$ on C_{12}, A_1

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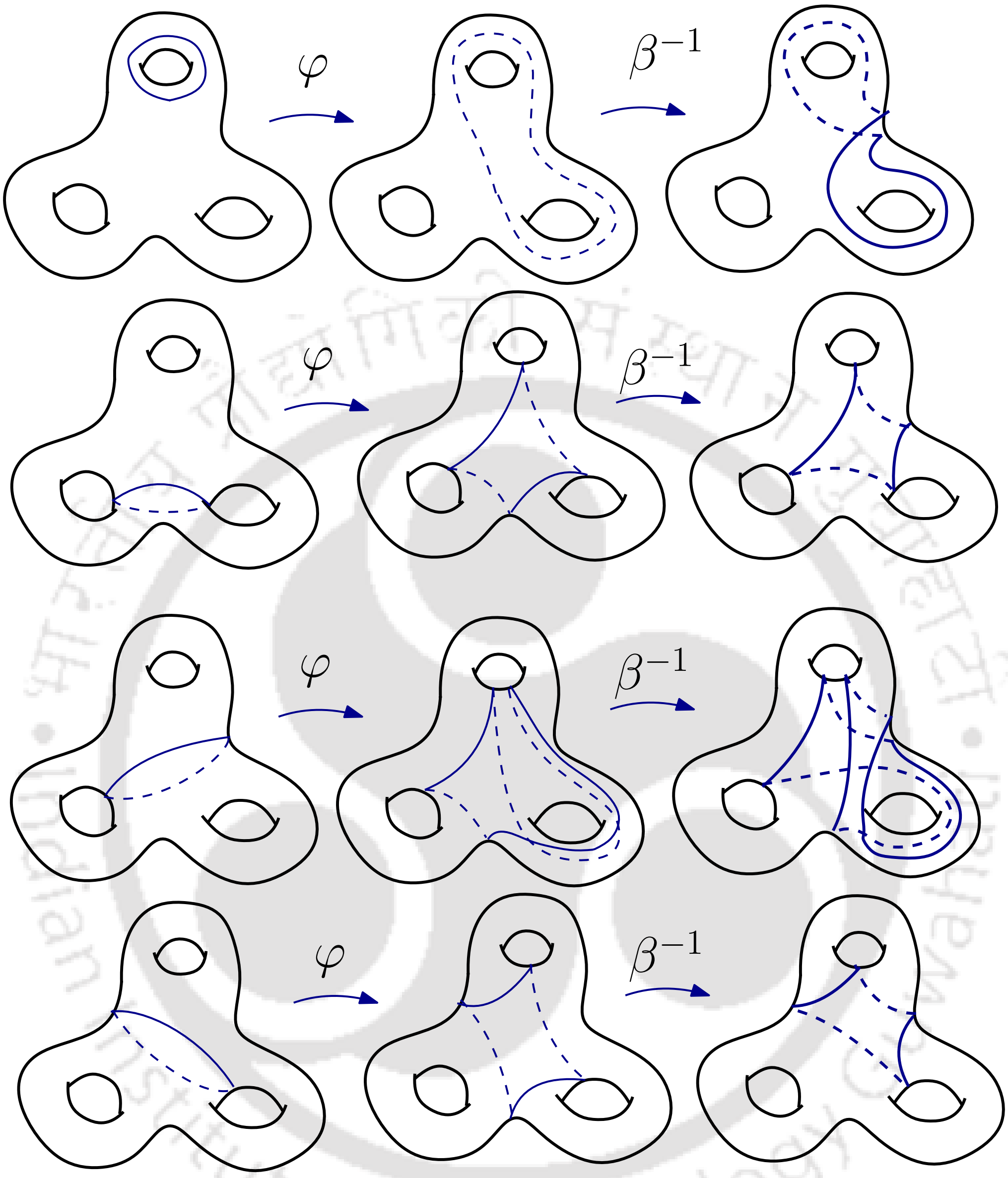

(b) Action of $\beta^{-1}\varphi$ on B_2, C_{23}, A'_3 and A'_1

Figure 4.11: Action of $\beta^{-1}\varphi$ on other loops

generators proposed in this work generates each element in the generating set of Freedman et al [4] and Zupan [14]. Thus $\mathcal{G}_3$ generate the desired subgroup of the mapping class group of Σ_3.

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4.5 Conclusion

The main outcome of this work is that it proposes an alternative generating set of $\mathcal{H}_3$, where the elements are algorithmically computable.

Now, we have already observed that $\beta^{-1}\varphi$ recreates an eyeglass move φ_θ. Therefore φ can be written as $\beta\varphi_\theta$. Again φ^2 describes a variant of bubble move upto a symmetry of Σ_3. This implies $(\beta\varphi_\theta)^2$ reconciles with φ_v upto a symmetry of Σ_3.

The generating set proposed in [4, 14], contains five elements, whereas $\mathcal{G}_3$ has just four. In [4], the symmetries are captured by φ_η and $\varphi_{\eta_{12}}$. β is also in that generating set. So this result reconciles with the one given by Scharlemann [11] where he has shown that one of the Powell generators is redundant i.e. the revised generating set will have four elements.

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CHAPTER 5

Future plan

Some of the questions that immediately follows from this work can be presented as follows:

1. What can be a possible reduction algorithm for the genus three case?
2. Similar to $\mathcal{H}_2$, can we have such unique presentations for elements in $\mathcal{H}_3$? Can we have a complexity measure for reducing spheres of (Σ_3, V_3, W_3) of S^3?
3. What is the presentation of $\mathcal{H}_3$?
4. What can we say about such an algorithm and complexity measure for higher genus ($g \geq 4$) cases?
5. Is there a significant connection between Cayley graphs and the presentation of $\mathcal{H}_g$ in general?

❦

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Publications:

Based on the work in this thesis, the following research articles are communicated.

List of Papers Communicated/Under Preparation.

1. Sreekrishna Palaparthi and Swapnendu Panda, *Describing elements of the genus-2 Goeritz group of* S^3, arXiv:1912.08894 [math.GT] (2019).

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