STUDY OF $N = 1$ AND $N = 2$ SUPERSYMMETRIC INTEGRABLE HIERARCHIES AND THEIR SOLITON SOLUTIONS

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Debojit Sarma

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Certificate

It is certified that the work contained in the thesis entitled "STUDY OF $N = 1$ AND $N = 2$ SUPERSYMMETRIC INTEGRABLE HIERARCHIES AND THEIR SOLITON SOLUTIONS", by Debojit Sarma, a student in the department of Physics, Indian Institute of Technology, Guwahati for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

Sasanka Ghosh

Dr. Sasanka Ghosh

Assoc. Professor

Physics Department

IIT Guwahati

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To my mother and father

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Preface

Integrable models have seen major advances in the the last four decades bringing together various branches of mathematics and having diverse physical applications. Of great interest is the realization that integrable systems give rise to nondispersive, localised travelling wave solutions called solitons. The task of constructing such solutions have resulted in the development of many different techniques to solve these systems, which are nonlinear, partial differential equations and difficult to solve analytically.

Another characteristic of integrable models is that they are Hamiltonian systems and their Hamiltonian structures are closely related to the conformal algebras. This was made clear when it was shown that the second Hamiltonian structure of the KdV hierarchy is isomorphic to nonlinear algebras known as the W_n algebras. Later on, the symmetry structures associated with the KP hierarchy, which includes the KdV hierarchy, have been investigated.

An important development in the study of integrable models came with the extension of these equations to include supersymmetric systems. It was found that many of the techniques in the study of ordinary (or bosonic) integrable systems could be applied to study supersymmetric integrable systems. In this context, the study of the KP hierarchy, specially its Hamiltonian structures and associated symmetry algebras has assumed great significance. Moreover, it is also important to ob-

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tain the soliton solutions of supersymmetric integrable systems as the existence of soliton solutions is essential from the point of view of integrability.

This work is concerned with the study of the $N = 2$ nonstandard flow KP system. The integrability of the system is studied via its biHamiltonian structures. The algebraic structure of the hierarchy is investigated and a representation in terms of free fields is found. In the latter part of the work soliton solutions are obtained.

The thesis consists of nine chapters. Chapter 1 is an overview of integrable systems in general, and their supersymmetric extensions with a discussion of the techniques to obtain soliton solutions with particular emphasis on the Hirota bilinear formalism. Chapter 2 is a review of the $N = 2$ nonstandard flow KP hierarchy. In chapters 3 to 5, the integrability as well as the algebraic and field theoretic aspects of the $N = 2$ nonstandard flow KP systems are discussed. The soliton solutions are obtained in chapters 6 to 8 for some $N = 1$ and $N = 2$ supersymmetric integrable systems using the Hirota bilinear formalism. Chapter 9 will be the concluding one.

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Chapter 1

Introduction

Integrable models have been an important field of study for over a period of thirty years. During this time it has developed into an extraordinarily rich field of study with applications ranging from condensed matter physics to unification of fundamental interactions. In fact, it would not be an overstatement to say that the integrable models encompass a wide range of fields in physics since some integrable systems such as the KdV are required to describe motion in areas as diverse as shallow water waves [1] to 2D quantum gravity [2, 3, 4, 5]. Integrable models can be defined briefly as nonlinear differential equations with localized solutions called solitons. Though solitons are essentially waves, they exhibit particle like behaviour as they propagate without dispersion and interact without change of form. There are, however, diverse criteria for integrability and various techniques for solving nonlinear differential equations have been developed. A necessary condition for integrability is the Liouville condition which states that an integrable field theoretical system possesses an infinite number of conserved quantities which are in involution. Integrable systems are also characterized by their biHamiltonian structure, that is the equations of motion can be generated by at least two different Hamiltonian structures. An-

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other condition for integrability for an nonlinear system is related to a method of finding solutions of such systems – the Lax method [6]. This method consists in determining a pair of differential operators for the system under question and these operators, which are linear, are used to determine the solutions of the system. It has been observed that only for nonlinear systems which are solvable can such pairs of operators be obtained and consequently the existence of Lax pairs is considered to be yet another criterion for integrability.

The connection of integrable models with supersymmetry was established with the supersymmetrization of bosonic integrable models such as the sine-Gordon equation [7], the KP hierarchy [8],the KdV equation [8, 9], the Boussinesq equation [10] and a few others. In addition, many of the methods developed in the study of integrable models were generalized to the supersymmetric regime among which are the Inverse Scattering Method [11, 12], Backlund transformation [11], Lax formalism [8], τ functions [13], Painleve analysis [14], Darboux transformation [15], to mention some of them. The study of supersymmetric systems has been in the forefront of theoretical physics since it is believed that a physical theory capable of unifying the four interactions will possess supersymmetry. This is a continuous spacetime symmetry relating bosonic fields with fermionic ones. Even though experimental evidence for supersymmetry has been absent, its advantages are numerous; one of them being that it is the only through supersymmetry can the Coleman-Mandula theorem [16], which limits the number of possible bosonic symmetries be circumvented. In the context of integrability, the most remarkable pioneering work was that of Manin and Radul [8], who formulated the supersymmetric version of the KP hierarchy in $N = 1$ superspace employing the supersymmetric extension of the Lax formalism. The KP system [17, 18, 19] has assumed tremendous importance as it incorporates the KdV system, which apart from

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possessing a rich symmetry structure has aquired additional significance as it has been observed that the equations of motion and symmetries of 2D quantum gravity [2, 3, 4, 5]can be formulated in terms of KdV-like equations . This indicates the possibility that the KP hierarchy could provide the underlying structure of 2D quantum gravity [20, 21]. Thus supersymmetric extensions of the KP equation are of undoubted physical relevance. On the other hand, although the bosonic KP hierarchy and their connection to matrix model have been studied extensively, very little is known about the higher spin extension of $N = 2$ superconformal symmetry. As conventional formulation of the supersymmetric matrix model failed to describe nonperturbative superstring theory – it gives nothing but the ordinary matrix model [22] – the super KP formalism may throw some light in this direction even if no supersymmetric extension of the matrix model can be formulated. Interest in the study of $N = 2$ and $N = 4$ supersymmetry in the context of quantum gravity and their connections with integrable systems has lately been of much importance and this has led to a series of studies relating to the symmetry structures of $N = 2$ and $N = 4$ supersymmetric integrable models [23, 24, 25, 26, 27, 28]. A major breakthrough in recent times is the nonperturbative solution of $N = 2$ super Yang-Mills equation and their connection with integrable systems [29, 30, 31]. Work has also been done in recent times to obtain τ functions for supersymmetric integrable hierarchies [32, 33, 35, 36, 37, 38].

A number of techniques of varying levels of applicability and difficulty have been developed for determining the exact solutions of nonlinear systems. An important method developed for solving such systems was the Inverse Scattering Method which is undoubtedly the most general one. This method was first applied by Gardner, Greene, Kruskal and Miura [39] to obtain the soliton solutions of the paradigmatic KdV equation. But it was only when Zakharov and Shabat [40] showed that

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this method could be used to solve another nonlinear differential equation, namely the nonlinear Schrodinger equation, it was realized that a powerful analytic method for the solution of nonlinear partial differential equations had been found. The fundamental idea behind the Inverse Scattering Method is: given the asymptotic form of the solution, to construct the potential. This method is a powerful one as it can handle very general initial conditions.

Over the years other techniques of solution of nonlinear systems have been developed and applied to a range of nonlinear partial differential equations. A method of solving certain classes of nonlinear partial differential equations is associated with the Backlund transformations which either connect two distinct solutions of the same equation or else the solutions of two distinct equations. Thus one can proceed from a simple solution such as one soliton solution to N soliton solutions by repeated application of the transformation. However, the most elegant and direct formalism of constructing soliton solutions is Hirota's method [41]. In this method, the nonlinear equation is cast into the so-called bilinear form employing the gauge invariant Hirota derivative. This procedure is highly nontrivial, there being no algorithm for choosing the correct transformation for obtaining the form of the equation which is amenable for bilinearization and relies heavily on intuition. The advantage of this technique is that once this goal, *i.e.* bilinearizing the equation, is achieved, the existence of one and two soliton solutions are guaranteed. The existence of three soliton solutions usually indicates the complete integrability [42] of the system and in this formalism it is possible to determine their existence by a simple algebraic procedure.

Integrable systems are Hamiltonian systems; that is, their equations of motion can be written in Hamiltonian form. The Hamiltonian structures of integrable systems are closely related to conformal al-

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gebras; more specifically the Hamiltonian structures of the integrable hierarchies have been found to be isomorphic to the various higher spin conformal algebras at the classical level. This was realized when it was shown that the W_n algebra, the higher spin extension of the Virasoro algebra, incorporates in its classical limit the Hamiltonian structure of the nonlinear integrable systems i.e. the generalized n^{th} KdV hierarchy [43, 44, 45]. Subsequently, the general linear deformation of the W_n in the large n limit, namely the $W_{1+\infty}$ [46, 47] and the W_∞ [48, 49, 50] were formulated. These algebras consist of an infinite number of bosonic fields with the lowest field having conformal spins 1 and 2 for $W_{1+\infty}$ and W_∞ algebras respectively and are isomorphic to the first Hamiltonian structure of the KP hierarchy [51, 52]. Being infinite dimensional symmetry algebras, these were candidates for a universal W-algebra which was conjectured to contain all W_n algebras by reduction. This expectation however turned out to be without basis, as no straightforward mechanism which could effectively truncate the spin content of these algebras and produces the nonlinear features of W_n could be found. An infinite dimensional nonlinear realization of the W algebra, namely the $\hat{W}_\infty$ algebra was obtained by the bootstrap approach and identified with the 2^{nd} Hamiltonian structure of the KP hierarchy [53, 54, 55]. It is a universal W algebra containing all W_n algebras and was obtained by associating the symmetry algebra of the $SL(2,R)_k/U(1)$ coset model with the $\hat{W}_\infty$ algebra characterized by the label k and then by showing that the symmetry algebra truncates to W_n algebra for $k = -n$ [55]. The extension of these theories to the supersymmetric framework has become significant in the light of superstring theories. Certain $N = 2$ superconformal field theories have been used in constructing the superstring vacuum, compactification, classification of conformal field theories and a host of other problems [56]. As a result, the task of establishing a firm foundation for the $N = 2$ superconformal

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symmetries has assumed importance. It is well known that Manin and Radul provided the supersymmetric extension of the KP hierarchy, and this was based on the odd parity superLax formulation. However, the construction of the Hamiltonian structure for odd parity Lax operator, following the Drinfeld Sokolov formalism, is not well understood yet[57]. Later on, an even parity Lax operator associated with supersymmetric KP hierarchy [58, 59] was obtained and a supersymmetric extension of the linear W algebra was realized [60]. The connection of the $N = 2$ super KdV hierarchies with affine Lie algebras was demonstrated by Inami and Kanno [61, 62] which is a step forward towards an $N = 2$ super analogue of the Drinfeld Sokolov formulation. This indicated that there ought to be consistent $N = 2$ superLax formulation of the super KP hierarchy which should be Hamiltonian with respect to one of the super Gelfand-Dikii bracket and should reduce to the Lax operators considered by Inami and Kanno under suitable reduction. Consequently, the existence of a nonlinear realization of the supersymmetric $\hat{W}_\infty$ algebra was shown through the super KP formulation [63]. Another aspect of conformal algebras are their free field representations, the significance of which cannot be overestimated since the underlying representation of a conformal field theory is essentially a free field representation. Moreover it plays a major role in the classification of various conformal algebras. Further, the quantization of classical symmetry algebras becomes straightforward in terms of free fields.

In the investigation of supersymmetric integrable systems, it is observed that one of the Hamiltonian structures happens to be nonlocal and may not be associated with a local conformal symmetry. The technique of obtaining classical conformal algebras through the integrable hierarchies is indeed a powerful one and its importance was recognized when the existence of a new higher spin conformal algebra was realized

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at the classical level [63]. It was seen that an even parity superLax operator in combination with the standard flow equation results in a KP hierarchy where the 2^{nd} Hamiltonian structure is nonlocal [59], while the 1^{st} is local [58]. This is a consequence of the fact that the standard flow produces trivial dynamics for the lowest spin supersymmetric field. The presence of the lowest superfield in the superLax operator is essential to obtain superLax operators for$N = 2$ generalized super KdV hierarchies by reduction. This difficulty is overcome by choosing a nonstandard flow equation. In this case, in contrast to the first, the 2^{nd} Hamiltonian structure turns out to be local [63]. The associated Poisson bracket algebra among the superfields is the nonlinear realization of the $N = 2$ super W_∞ algebra. Significantly, this algebra has the $N = 2$ superconformal algebra as a subalgebra and the bosonic and fermionic components carry the appropriate conformal weights [63].

Although bosonic integrable models have been thoroughly investigated, their supersymmetric extensions have not been widely studied. The methodologies of obtaining soliton solutions vis a vis their integrability are not developed or they have not been applied to many supersymmetric systems to show their integrability. For example, the biHamiltonian structures of a number supersymmetric integrable systems are known, but on the other hand, soliton solutions are not. This is more so for $N = 2$ supersymmetric integrable models, which although very important, the existence of soliton solutions for these systems are not known. Even for $N = 1$ supersymmetric integrable equations, this problem has not been addressed for a number of interesting systems due to the intricate nature of the systems. From the conformal symmetry point of view, in the case of bosonic algebras, the $\hat{W}_\infty$ is known to be a universal algebra as it contains all other algebras as reductions. Such type of classification for superalgebras is still unknown. In particular, in the supersymmetric framework, an equivalent superalgebra, which will have all bosonic and supersymmetric algebras as subalgebras has not been constructed. This thesis aims to study some of the aspects of supersymmetric integrable systems.

The thesis is concerned with various aspects of the $N = 2$ nonstandard KP hierarchy. In the first part of the thesis we show that nonstandard supersymmetric KP flows following the Gelfand Dikii method, are biHamiltonian [65]. We further show that one of the Hamiltonian structures is a candidate for a higher spin extension of $N = 2$ superconformal algebra and has the required number of spin fields and the bosonic sector of the algebra has the right structure with two commuting sets of bosonic generators. We will also obtain the free field representation of the generators which turns out to be nonlinear [65, 66]. The generators in the fermionic sector are exponential in the free fields. It will be apparent that in the bosonic sector the representation of generators are nontrivial generalizations of linear representation. This brings us closer to establishing that super $\hat{W}_\infty$ is a universal symmetry containing all finite dimensional bosonic and $N = 2$ supersymmetric W algebras. In the latter half of the thesis we will bilinearize [64] the $N = 2$ super KP equation associated with nonstandard flows, and its various reductions both with $N = 1$ as well as $N = 2$ supersymmetry. Specifically, these equations are the $N = 2$ KdV, the $N = 1$ mKdV, the $N = 1$ mKdV B and the $N = 1$ mKP [36, 67, 68]. The method will also be applied to generate N soliton solutions of these nonlinear supersymmetric integrable systems.

The chapterwise breakup of the thesis is given below. In *Chapter 2*, we will review the nonstandard flow $N = 2$ supersymmetric KP hierarchy, particularly earlier relevant work in this direction. The 2^{nd} Hamiltonian structure, and Gelfand-Dikii formalism through which it was obtained, of the above system will be discussed. In *Chapter 3*, the biHamiltonian property of the nonstandard flow $N = 2$ KP will

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be obtained by showing the existence of its 1^{st} Hamiltonian structure. It will be seen that the 1^{st} Hamiltonian structure is nonlocal and it reproduces the correct dynamical equations. The existence of a bi-Hamiltonian structure for the nonstandard flow $N = 2$ KP establishes its complete integrability. It will be established in *Chapter 4* that the second Hamiltonian structure of the $N = 2$ super KP has the appropriate symmetry structure of a supersymmetric $\hat{W}_\infty$ algebra. It will be shown that the generators of the algebra possess the correct conformal structure and the bosonic sector has the required $\hat{W}_{1+\infty} \oplus \hat{W}_\infty$ structure, which is essential in establishing its super $\hat{W}_\infty$ algebraic structure. *Chapter 5* will deal with the free field representation of the generators of the algebra discussed in the previous chapter. The method of construction of these nonlinear generators will be discussed. In *Chapter 6*, we will bilinearize the $N = 2$ KdV, $N = 1$ modified KP and the $N = 1$ modified KdV and $N = 1$ modified KdV B systems. The soliton solutions of the equations discussed in the previous chapter will be obtained in *Chapter 7* and the existence of N soliton solutions for these systems will be discussed. In *Chapter 8*, we will show that the N soliton solutions exist for the nonstandard flow $N = 2$ KP equation which are found through the Hirota formalism and thereby establish the integrability of the system from an alternate point of view. *Chapter 9* will be the concluding one.

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Chapter 2

Review of Non-standard $N = 2$ KP Flows

2.1 Introduction

A major development in the study of integrable models is their extension to include supersymmetric systems. This was done by extending the ordinary space variable x to a superspace variable (x, θ), where θ is an anticommuting Grassmann variable and hence $\theta^2 = 0$. This kind of supersymmetry with one Grassmann variable is referred to as $N = 1$ supersymmetry. For instance, the supersymmetrization of the KdV equation

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0 \tag{2.1}$$

is done by extending the u field to a fermionic superfield

$$u(x) \to \phi(x) = \xi(x) + \theta u(x) \tag{2.2}$$

where ξ and u are fermionic and bosonic fields respectively of spin $3/2$ and 2 respectively. Here the extension may be also to a bosonic superfield, but for the KdV the interesting extensions are provided by

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(2.2). This leads to the supersymmetric equation [9]

$$\partial_t \phi + 3\partial_x(\phi D\phi) + \partial_x^3 \phi = 0 \tag{2.3}$$

where D is the supersymmetric derivative defined by

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \tag{2.4}$$

with $D^2 = d/dx$. In component form the $N = 1$ KdV reads

$$\partial_t u + \partial_x^3 u + 6u\partial_x u - 3\xi\partial_x^2 \xi = 0 \tag{2.5}$$

and

$$\partial_t \xi + \partial_x^3 \xi + 3\partial_x(u\xi) = 0 \tag{2.6}$$

It is easily seen that the (2.5,2.6) are invariant under the supersymmetry transformation $\delta_\eta u = \eta\partial_x \xi$ and $\delta_\eta \xi = \eta\partial_x u$, where η is a constant Grassmann number. The integrability of the the $N = 1$ KdV is now well established on the basis of several different criterion.

The next extension of the supersymmetric KdV is by the addition of an extra supersymmetry *i.e.* by increasing the number of anti commuting variables. Thus the ordinary space variable x is extended to (x, θ_1, θ_2), with $\theta_1^2 = \theta_2^2 = 0$, $\theta_1\theta_2 = -\theta_2\theta_1$, and introducing the two superderivatives :

$$D_1 = \frac{\partial}{\partial \theta_1} + \theta_1 \frac{\partial}{\partial x} \qquad D_2 = \frac{\partial}{\partial \theta_2} + \theta_2 \frac{\partial}{\partial x} \tag{2.7}$$

with $D_1^2 = D_2^2 = \partial_x$ and $D_1 D_2 = -D_2 D_1$. The $N = 2$ KdV equation [69, 70, 71] can then be written as

$$\partial_t \Omega = -\partial_x^3 \Omega + 3\partial_x \left(\Omega D_1 D_2 \Omega\right) + \frac{\alpha - 1}{2}\partial_x \left(D_1 D_2 \Omega^2\right) + 3\alpha\Omega^2 \partial_x \Omega \tag{2.8}$$

In (2.8), the $N = 2$ KdV superfield Ω is a spin 1 bosonic superfield which in terms of its component fields is given by

$$\Omega(x, \theta_1, \theta_2) = \theta_2\theta_1 u(x) + \theta_1 \xi_1(x) + \theta_2 \xi_2(x) + v(x) \tag{2.9}$$

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ξ_1 and ξ_2 are two fermionic fields and v is a new bosonic field of spin 1. In the $N = 2$ super KdV equation (2.8), α is an arbitrary parameter and it was found that (2.8) is integrable for exactly three values of α, namely, $\alpha = -2, 1, 4$. Of special interest in the context of this work, is the $N = 2$ KdV with $\alpha = -2$. In $N = 1$ superspace the $N = 2$ KdV equation with $\alpha = -2$ is isomorphic to

$$\partial_t u_{-1} + u_{-1}^{[6]} + 3\left(u_{-1}^{[1]} u_0\right)^{[2]} - \frac{1}{2}\left(u_{-1}^3\right)^{[2]} = 0 \tag{2.10}$$

$$\partial_t u_0 + u_0^{[6]} - 3\left(u_0 u_0^{[1]}\right)^{[2]} - \frac{3}{2}\left(u_0 u_{-1}^2\right)^{[2]} + 3\left(u_0 u_{-1}^{[2]}\right)^{[2]} = 0 \tag{2.11}$$

In (2.10,2.11) u_{-1} and u_0 are superfields of conformal spin 1 and 3/2 respectively. (2.10) and (2.11) were obtained as the lowest equations of a $N = 2$ super KdV hierarchy by Inami and Kanno [61] from the nonstandard flow equation via the Lax operator

$$\tilde{L}_2 = D^4 + 2u_{-1}D^2 + 2u_0 D \tag{2.12}$$

and they have shown that this hierarchy arises in connection with affine Lie super algebras [61, 62]. The $N = 2$ KdV equations (2.10,2.11) were then extended to include another bosonic coordinate and thereby the $N = 2$ KP equation [63]

$$\begin{aligned} &\partial_t u_{-1} - \frac{1}{4}u_{-1}^{[6]} + \frac{1}{2}\left(u_{-1}^3\right)^{[2]} - \frac{3}{2}\left(u_0 u_{-1}^{[1]}\right)^{[2]} - \frac{3}{4}\partial_y^2 u_{-1}^{[-2]} \\ &-\frac{3}{2}u_{-1}^{[2]}\partial_y u_{-1}^{[-2]} + \frac{3}{2}u_{-1}^{[1]}\partial_y u_0^{[-2]} + \frac{3}{2}u_0\partial_y u_{-1}^{[-1]} - 3u_0\partial_y u_0^{[-2]} = 0 \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\partial_t u_0 - \frac{1}{4}u_0^{[6]} - \frac{3}{2}\left(u_0 u_0^{[1]}\right)^{[2]} + \frac{3}{2}\left(u_0 u_{-1}^{[2]}\right)^{[2]} + \frac{3}{2}\left(u_0 u_{-1}^2\right)^{[2]} \\ &-\frac{3}{4}\partial_y^2 u_0^{[-2]} - \frac{3}{2}\left(u_0\partial_y u_0^{[-2]}\right)^{[1]} - \frac{3}{2}u_0^{[2]}\partial_y u_{-1}^{[-2]} - \frac{3}{2}u_0\partial_y u_{-1} = 0 \end{aligned} \tag{2.14}$$

has been formulated. Note that in the limit y coordinate is set to zero, (2.13,2.14) become isomorphic to (2.10,2.11). (2.13,2.14) are the equations of motion for the lowest superfields in the $N = 2$ super KP hierarchy. In sections 2 and 3, we shall discuss the equations of motions for the $N = 2$ KP hierarchy and the associated superalgebraic structure.

2.2 The Lax Operator and the Nonstandard Flow

The first supersymmetric extension of the KP hierarchy was formulated by Manin and Radul [8] on the basis of a odd parity superLax operator. It was however not clear how to obtain the bosonic limit of the Hamiltonian structures associated with this Lax operator. Subsequently, the KP hierarchy was obtained in terms of an even parity superLax operator [58, 59]

$$L = D^2 + \sum_{i=0}^{\infty} u_i(X) D^{-i-1} \tag{2.15}$$

where D is the superderivative defined in (2.4). $u_i(X)$ are superfields in $X = (x, \theta)$ space. The grading of $u_i(X)$ is $|u_i| = i + 1$ so that u_{2i} and u_{2i+1} are fermionic and bosonic superfields respectively. Note that the superLax operator in (2.15) generates superfields with conformal spin $\frac{1}{2}(i+3)$. With the help of this operator the super KP hierarchy is obtained from the standard Lax equation

$$\frac{\partial L}{\partial t_n} = \left[L_+^n, L\right] = -\left[L_-^n, L\right] \qquad n = 1, 2, 3, \ldots \tag{2.16}$$

This produces a biHamiltonian structure in which the first Hamiltonian structure is local [58] while the second is nonlocal [59]. The nonlocality in the second Hamiltonian structure is due to the absence of the spin

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1 field in the superLax operator (2.15) which is essential to obtain the $N = 2$ superconformal algebra.

The superLax operator (2.15) was modified to include the spin 1 field by the following choice of the even parity superLax operator [63], viz.

$$L = D^2 + \sum_{i=0}^{\infty} u_{i-1}(X) D^{-i} \tag{2.17}$$

The grading of $u_{i-1}(X)$ is $|u_{i-1}| = i$ so that u_{2i-1} are bosonic superfields, whereas u_{2i} are fermionic ones. As a result these superfields have the following component form

$$\begin{aligned} u_{2i-1}(X) &= u^b_{2i-1}(x) + \theta u^f_{2i-1}(x), \\ u_{2i}(X) &= u^f_{2i}(x) + \theta u^b_{2i}(x) \end{aligned} \tag{2.18}$$

An interesting consequence of this choice of the superLax operator is that under suitable reduction the superLax operator of the generalized $N = 2$ KdV hierarchies of Inami and Kanno [61] are obtained. To see this, the n_{th} reduction of L is defined as

$$\tilde{L}_n = L^n_{>0} \tag{2.19}$$

where > 0 implies the positive part of L^n without the D^0 term. Using definition (2.19) for the superLax operator (2.17), $\tilde{L}_n$ can be expressed in the following general form

$$\tilde{L}_n = L^n_{>0} = D^{2n} + \sum_{i=1}^{2n-2} \mathcal{U}^{(n)}_i D^i \tag{2.20}$$

where, the superfields $\mathcal{U}^{(n)}_i$ are functions of $u_{i-1}(X)$ and their superderivatives. $\tilde{L}_n$ in (2.20) is precisely the superLax operator considered by Inami and Kanno in [61]. For example, for $n = 2$ one obtains (2.12) which is the $N = 2$ super KdV Lax operator. Similarly, for $n = 3$

$$\tilde{L}_3 = D^6 + 3u_{-1}D^4 + 3u_0 D^3 + 3(u_1 + u^{[2]}_{-1} + u^2_{-1})D^2 + 3(u_2 + u^{[2]}_0 + 2u_{-1}u_0)D \tag{2.21}$$

is isomorphic to the Lax operator corresponding to $N = 2$ super Boussinesq hierarchy [61]. The importance of the presence of the field u_{-1} will be observed in the Poisson bracket structure. It makes the Gelfand Dikii bracket of second kind local [63] whereas the absence of u_{-1} makes the same nonlocal [59]. This is achieved by defining the nonstandard Lax equation

$$\frac{dL}{dt_n} = [L^n_{>0}, L] \tag{2.22}$$

The nonstandard flow equation (2.22) in combination with the superLax operator (2.17) allows one to obtain the dynamical equations for the fields. The evolution equations corresponding to the lowest three time-flows of the hierarchy are given below.

$$\frac{du_{i-1}}{dt_1} = u^{[2]}_{i-1} \tag{2.23}$$

$$\begin{aligned}
\frac{du_{i-1}}{dt_2} &= 2u^{[2]}_{i+1} + u^{[4]}_{i-1} + 2u_0 u^{[1]}_{i-1} + 2u_{-1}u^{[2]}_{i-1} - 2\left[\begin{array}{c} i+1 \\ 1 \end{array}\right] u_i u^{[1]}_{-1} \\
&-2(1+(-1)^i)u_0 u_i + 2\sum_{m=0}^{i-1}\left[\begin{array}{c} i \\ m+1 \end{array}\right](-1)^{i+1+[-m/2]} u_{i-m-1} u^{[m+1]}_0 \\
&+2\sum_{m=0}^{i-1}\left[\begin{array}{c} i+1 \\ m+2 \end{array}\right](-1)^{[m/2]} u_{i-m-1} u^{[m+2]}_{-1}
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
\frac{du_{i-1}}{dt_3} &= 3u^{[2]}_{i+3} + 3u^{[4]}_{i+1} + u^{[6]}_{i-1} + 6u_{-1}u^{[2]}_{i+1} + 3u_{-1}u^{[4]}_{i-1} \\
&-3\left[\begin{array}{c} i+3 \\ 1 \end{array}\right] u_{i+2}u^{[1]}_{-1} + 3\left[\begin{array}{c} i+3 \\ 2 \end{array}\right] u_{i+1}u^{[2]}_{-1} + 3\left[\begin{array}{c} i+3 \\ 3 \end{array}\right] u_i u^{[3]}_{-1} \\
&-3(1+(-1)^i)u_0 u_{i+2} + 3u_0 u^{[1]}_{i+1} - 3(-1)^i u_0 u^{[2]}_i + 3u_0 u^{[3]}_{i-1} \\
&+3\left[\begin{array}{c} i+2 \\ 1 \end{array}\right](-1)^i u_{i+1} u^{[1]}_0 - 3\left[\begin{array}{c} i+2 \\ 2 \end{array}\right](-1)^i u_i u^{[2]}_0 \\
&+3(u_1 + 2u^{[2]}_{-1} + u^2_{-1})u^{[2]}_{i-1} - 3\left[\begin{array}{c} i+1 \\ 1 \end{array}\right] u_i (u_1 + u^{[2]}_{-1} + u^2_{-1})^{[1]}
\end{aligned}$$

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$$
\begin{aligned}
&+3(u_2 + 2u_{-1}u_0 + u_0^{[2]})u_{i-1}^{[1]} - 3(1+(-1)^i)(u_2 + 2u_{-1}u_0 + u_0^{[2]})u_i \\
&-3\sum_{m=0}^{i-1} \left[\begin{array}{c} i+3 \\ m+4 \end{array} \right] (-1)^{[m/2]} u_{i-m-1} u_{-1}^{[m+4]} \\
&-3\sum_{m=0}^{i-1} \left[\begin{array}{c} i+2 \\ m+3 \end{array} \right] (-1)^{i+[-m/2]} u_{i-m-1} u_0^{[m+3]} \\
&+3\sum_{m=0}^{i-1} \left[\begin{array}{c} i+1 \\ m+2 \end{array} \right] (-1)^{[m/2]} u_{i-m-1} (u_1 + u_{-1}^{[2]} + u_{-1}^2)^{[m+2]} \\
&+3\sum_{m=0}^{i-1} \left[\begin{array}{c} i \\ m+1 \end{array} \right] (-1)^{i+[-m/2]} u_{i-m-1} (u_2 + 2u_{-1}u_0 + u_0^{[2]})^{[m+1]}
\end{aligned}
\tag{2.25}
$$

With the identification $t_1 = x$ and $t_2 = y$, (2.23) is the consistency condition, while (2.24) is the constraint equation. The time variables may be therefore identified as $t_3, t_4 \ldots$ and so on. The equations of motion for the lowest superfields u_{-1} and u_0 can be found from (2.25) by eliminating all the other fields using (2.24) and are precisely the equations (2.13,2.14).

2.3 Second Hamiltonian Structure

A significant consequence of the presence of the u_{-1} superfield is that in this case, unlike the standard flow super KP, the second Hamiltonian structure is local, thus making it a right candidate for a universal symmetry algebra containing all finite dimensional bosonic as well as $N = 2$ supersymmetric algebras. To obtain the local Poisson bracket algebra among the coefficient fields $u_{i-1}(X)$ the method of Gelfand and Dikii [72, 73, 74] is employed. The super Gelfand Dikii bracket of second kind is defined as

$$
\{F_P(L), F_Q(L)\}_2 = -Tr\left[\{L(PL)_-(LP)_-L\}\, Q\right] \tag{2.26}
$$

where, P, Q are auxiliary fields. P, Q are chosen in the form

$$P = \sum_{j=-2}^{\infty} D^j p_j \quad ; \quad Q = \sum_{j=-2}^{\infty} D^j q_j \tag{2.27}$$

with the grading $|p_j| = |q_j| = j$ so that the linear functional $F_P(L)$ (and similarly $F_Q(L)$) becomes

$$F_P(L) = Tr(LP) = \sum_{i=0}^{\infty} \int dX (-1)^{i+1} u_{i-1}(X) p_{i-1}(X) \tag{2.28}$$

Consequently the L.H.S. of (2.26) becomes

$$\{F_P(L), F_Q(L)\} = \sum_{i,j=0}^{\infty} \int dX \int dY (-1)^{i+j} p_{i-1}(X) \{u_{i-1}(X), u_{j-1}(Y)\} q_{j-1}(Y) \tag{2.29}$$

It is clear that (2.29) does not involve terms like p_{-2} and q_{-2} since the superfields in (2.17) starts from $u_{-1}(X)$. In order that this is consistent, the coefficients of p_{-2} and q_{-2} in the right hand side of (2.26) must be identically zero. This requires that the coefficient of the D term in $L(PL)_- - (LP)_- L$ vanishes. This leads to the constraint

$$p_{-1}(X) = -\sum_{r=0}^{\infty} \sum_{m=0}^{r-1} \left[\begin{array}{c} r-1 \\ m+1 \end{array} \right] (-1)^{m(r+1)} \left(p_{r-1}(X) u_{r-m-2}(X) \right)^{[m-1]} \tag{2.30}$$

It is emphasized that the vanishing of the coefficient of the D term ensures that the R.H.S. of (2.26) is independent of q_{-2}. Using the constraint (2.30) the coefficient of p_{-2} can be made zero. Thus the Poisson bracket algebra among the superfields can be obtained by using the relations (2.26)-(2.30). This has the explicit form

$$\{u_{j-1}(X), u_{k-1}(Y)\}_2 = \left[-\sum_{m=0}^{j+1} \left[\begin{array}{c} j+1 \\ m \end{array} \right] (-1)^{j(k+m+1)+[m/2]} u_{j+k-m} D^m \right.$$
$$+ \sum_{m=0}^{k+1} \left[\begin{array}{c} k+1 \\ m \end{array} \right] (-1)^{jm+(k+1)(m+1)} D^m u_{j+k-m}$$

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$$
\begin{aligned}
&+\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\left(\begin{bmatrix} j \\ m+1 \end{bmatrix}\begin{bmatrix} k \\ l+1 \end{bmatrix}-\begin{bmatrix} j-1 \\ m \end{bmatrix}\begin{bmatrix} k-1 \\ l \end{bmatrix}\right)(-1)^{j(m+1)+k+l+[m/2]} \\
&\times u_{j-m-2}D^{m+l+1}u_{k-l-2} \\
&+\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{bmatrix} k-n-1 \\ l \end{bmatrix}(-1)^{j(n+l)+(l+1)(n+k+1)}u_{n-1}D^{l}u_{j+k-n-l-2} \\
&-\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{bmatrix} j-1 \\ m \end{bmatrix}\begin{bmatrix} n+l-1 \\ l \end{bmatrix}(-1)^{j(m+n+l+k+1)+n(l+1)+[m/2]} \\
&\times u_{j+k-m-l-2}D^{m+l}u_{n-1}\Big]\,\Delta(X-Y)
\end{aligned}
\tag{2.31}
$$

The symmetry algebra (2.31) possesses the following features of interest.

1. The algebra is antisymmetric and satisfies the Jacobi identity.

2. The algebra is local. Thus it is possible to associate it with a superconformal algebra. To make contact with the $N = 2$ superconformal algebra consider the Poisson bracket algebra among the lowest superfields $u_{-1}(X)$ and $u_0(X)$. This is obtained from (2.31) and is given by

$$
\begin{aligned}
\{u_{-1}(X), u_{-1}(Y)\} &= (-2u_0(X) - u_{-1}(X)D_X + D_X u_{-1}(X))\Delta(X-Y) \\
\{u_0(X), u_{-1}(Y)\} &= (-D_X u_0(X) - u_{-1}(X)D_X^2)\Delta(X-Y) \\
\{u_{-1}(X), u_0(Y)\} &= (-u_0(X)D_X + D_X^2 u_{-1}(X))\Delta(X-Y) \\
\{u_0(X), u_0(Y)\} &= (u_0(X)D_X^2 + D_X^2 u_0(X))\Delta(X-Y)
\end{aligned}
\tag{2.32}
$$

This brackets are closed and hence is a subalgebra of (2.31). To show that (2.32) has the $N = 2$ superconformal structure it is convenient to write (2.32) in terms of the component fields in (2.18). Further redefining the fields as

$$
T = u_0^b - \frac{1}{2}(u_{-1}^b)', \qquad U = u_{-1}^b, \qquad G^+ = u_0^f, \qquad G^- = u_0^f - u_{-1}^f \tag{2.33}
$$

leads to the following classical analogue of the $N = 2$ superconformal

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algebra

$$
\begin{aligned}
\{T(x), T(y)\} &= (2T(y)\partial_y + T'(y))\delta(x-y), \\
\{T(x), U(y)\} &= (U(y)\partial_y + U'(y))\delta(x-y) \\
\{T(x), G^{\pm}(y)\} &= \pm(\frac{3}{2}G^{\pm}(y)\partial_y + (G^{\pm})'(y))\delta(x-y), \\
\{G^{\pm}(x), U(y)\} &= G^{\pm}(y))\delta(x-y) \\
\{G^{+}(x), G^{-}(y)\} &= (T(y) - U(y)\partial_y - \frac{1}{2}U'(y))\delta(x-y), \\
\{G^{\pm}(x), G^{\pm}(y)\} &= \{U(x), U(y)\} = 0 \qquad (2.34)
\end{aligned}
$$

From the algebra (2.31) the Poisson brackets of the component fields u^b_{2i-1}, u^b_{2i}, u^f_{2i-1} and u^f_{2i} with the energy momentum tensor T defined in (2.33) follow. They are

$$
\begin{aligned}
&\{T(x), u^b_{2i-1}(y)\} = \Big((i+1)u^b_{2i-1}(y)\partial_y + (u^b_{2i-1}(y))' \\
&- \sum_{m=0}^{i-2}(-1)^m \begin{pmatrix} i \\ m+2 \end{pmatrix} u^b_{2i-2m-3}(y)\partial_y^{m+2} \\
&-\frac{1}{2}\sum_{m=0}^{i-1}(-1)^m \begin{pmatrix} i \\ m+1 \end{pmatrix} u^b_{2i-2m-3}(y)\partial_y^{m+2}\Big)\,\delta(x-y) \quad (2.35)
\end{aligned}
$$

$$
\begin{aligned}
&\{T(x), u^b_{2i}(y)\} = \Big((i+2)u^b_{2i}(y)\partial_y + (u^b_{2i}(y))' \\
&- \sum_{m=0}^{i-2}(-1)^m \begin{pmatrix} i+1 \\ m+2 \end{pmatrix} u^b_{2i-2m-2}(y)\partial_y^{m+2} \\
&+\frac{1}{2}\sum_{m=0}^{i-1}(-1)^m \begin{pmatrix} i+1 \\ m+1 \end{pmatrix} u^b_{2i-2m-1}(y)\partial_y^{m+2}\Big)\,\delta(x-y) \quad (2.36)
\end{aligned}
$$

$$
\begin{aligned}
&\{T(x), u^f_{2i-1}(y)\} = \Big((i+\frac{3}{2})u^f_{2i-1}(y)\partial_y + (u^f_{2i-1}(y))' \\
&- \sum_{m=0}^{i-2}(-1)^m \begin{pmatrix} i \\ m+2 \end{pmatrix} u^f_{2i-2m-3}(y)\partial_y^{m+2} \\
&+\frac{1}{2}\sum_{m=0}^{i-1}(-1)^m \begin{pmatrix} i \\ m+1 \end{pmatrix} u^f_{2i-2m-3}(y)\partial_y^{m+2}\Big)\,\delta(x-y) \quad (2.37)
\end{aligned}
$$

$$
\begin{aligned}
\{T(x), u^f_{2i}(y)\} &= \Big((i+\frac{3}{2}) u^f_{2i}(y)\partial_y + (u^f_{2i}(y))' \\
&- \sum_{m=0}^{i-2} (-1)^m \binom{i+1}{m+2} u^f_{2i-2m-2}(y)\partial_y^{m+2} \Big) \delta(x-y) \quad (2.38)
\end{aligned}
$$

(2.35,2.36,2.37,2.38) clearly indicate that the component fields u^b_{2i-1}, u^b_{2i} and u^f_{2i-1}, u^f_{2i} are bosonic and fermionic conformal fields with respect to the energy momentum tensor T. The conformal spins of the odd bosonic fields u^b_{2i-1} and the even bosonic fields u^b_{2i} are $i+1$ and $i+2$ $(i = 0, 1, 2, \ldots)$ respectively, whereas the conformal spins of both even and odd fermionic fields, u^f_{2i-1} and u^f_{2i} become $i + \frac{3}{2}$ $(i = 0, 1, 2, \ldots)$.

2.4 Conclusion

The discussion above makes it clear that the $N = 2$ nonstandard flow KP hierarchy has a 2^{nd} Hamiltonian structure that is local with the $N = 2$ classical superconformal algebra as subalgebra. Moreover, the generators of the algebra are conformal fields with respect to the energy momentum tensor. This leads to the clear possibility that its second hamiltonian structure is a nonlinear realization of the super W_∞ algebra. In the chapter 4, we shall show the algebra possess the properties of such an algebra as well as that it has the necessary characteristics of a supersymmetric integrable system.

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Chapter 3

The BiHamiltonian Structure

3.1 Introduction

Integrable systems are essentially Hamiltonian systems. For the bosonic KdV equation

$$\partial_t u = u\partial_x u + \partial_x^3 u \tag{3.1}$$

Gardner [75] showed that it could be written in Hamiltonian form with respect to one of the conserved charges of the system that had been known for some time. As a matter of fact, relative to the Poisson bracket

$$\{u(x), u(y)\}_1 = \partial_x \delta(x - y) \tag{3.2}$$

and taking the conserved charge $H_3 = \int dx \quad \left(\frac{1}{3!}u^3 - \frac{1}{2}\left(\partial_x u\right)^2\right)$ as the Hamiltonian, the KdV equation can be written in the Hamiltonian form

$$\partial_t u = \{u(x), H_3\}_1 = \partial_x \left(\frac{1}{2}u^2 + \partial_x^2 u\right) = u\partial_x u + \partial_x^3 u = \partial_x \left(\frac{\delta H_3}{\delta u}\right) \tag{3.3}$$

Moreover, with respect to the bracket (3.2) all the conserved charges H_n are in involution $\{H_i, H_j\}_1 = 0$, which makes the KdV equation integrable in the formal sense. Interestingly, integrable models are bi-Hamiltonian, that is, the dynamical equations that reproduce the time

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evolution of such systems can be obtained from at least two distinct Poisson bracket relations. This was found first for the KdV equation when Magri [76] discovered that the KdV equation could be written in Hamiltonian form relative to a second Poisson bracket with a different Hamiltonian. In this case the Poisson bracket is

$$\{u(x), u(y)\}_2 = \left(\partial_x^3 + \frac{1}{3}\left(\partial_x u + u\partial_x\right)\right)\delta(x-y) \tag{3.4}$$

with the Hamiltonian given by $H_2 = \frac{1}{2}\int dx \;\; u^2$. Then one finds that

$$\partial_t u = \{u(x), H_2\}_2 = \left(\partial_x^3 + \frac{1}{3}\left(\partial_x u + u\partial_x\right)\right) u = u\partial_x u + \partial_x^3 u \tag{3.5}$$

In addition it was also found that that a linear combination of the two Poisson brackets is also a Poisson bracket – a nontrivial condition since the Poisson brackets have to satisfy the Jacobi identity. The biHamiltonian structure of the KdV equation thus implies that the conserved charges are related. In fact, for all $n > 1$

$$\partial_x \frac{\delta H_n}{\delta u} = \left(\partial_x^3 + \frac{1}{3}\left(\partial_x u + u\partial_x\right)\right) \frac{\delta H_{n+1}}{\delta u} \tag{3.6}$$

This relations can be used recursively to compute the conserved quantities of the KdV system. Another interesting fact that arises in this context is that the second Hamiltonian structure of the KdV (3.4) is isomorphic to the Virasoro algebra. For other integrable models the same connection was found with the higher spin extensions of the Virasoro algebra.

Another very interesting feature of integrable systems is that such nonlinear equations are associated with a linear equation which leads to a Lax [6] representation of the original nonlinear equation of the form

$$\frac{\partial L}{\partial t} = [L, A] \tag{3.7}$$

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where L and A are called the Lax pair and are pseudo differential operators associated with the linear system. This leads to a systematic procedure for determining the Hamiltonian structures starting from the Lax operator for a given problem via the Drinfeld-Sokolov formalism [72]. For a Lax operator

$$L = \sum a_n \partial_x^n \tag{3.8}$$

an auxiliary field Q with the product

$$Tr\,(LQ) = \int Res\,(LQ) = F_Q\,(L) \tag{3.9}$$

is defined with the residue indicating the coefficient of the ∂^{-1} term. The first and second hamiltonian structures of the system can be then obtained from the brackets

$$\{F_P(L), F_Q(L)\}_1 = -Tr\,([P, Q]\,L) = -Tr\,([L, P]\,Q) \tag{3.10}$$

and

$$\{F_P(L), F_Q(L)\}_2 = -Tr\,[\{L(PL)_-(LP)_-L\}\,Q] \tag{3.11}$$

which are called the Gelfand-Dikii brackets [73, 74] of the first and second kind respectively. The second Gelfand-Dikii bracket (3.11) was used to find the second Hamiltonian structure of the $N = 2$ KP which was discussed earlier in chapter 2. These brackets are antisymmetric and satisfy the Jacobi identity.

Although the use of the Gelfand-Dikii brackets is straightforward for the KdV hierarchy, it was seen that in case of the bosonic KP hierarchy the second Gelfand-Dikii bracket does not reproduce the equations of motion correctly [53]. This was essentially due to the fact that the Lax operator for the KP is not a pure differential operator but involves both positive and negative powers of ∂_x. This required a modification of the second Hamiltonian structure in order to reproduce the equations of motion correctly. In case of the $N = 2$ KP system, however, for

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the second Hamiltonian structure, modification is not required as was seen in chapter 2, but it is for the first Hamiltonian structure that modification becomes essential.

In this chapter, we shall show the existence of a first Hamiltonian structure for the $N = 2$ nonstandard flow KP. For this system also, the same problem arises as the first Hamiltonian structure following the definition (3.10) leads to equations of motion inconsistent with those obtained from the second Hamiltonian structure. Here we redefine the first Hamiltonian structure in order to get back the equations of motion correctly. As reviewed in chapter 2, the second Hamiltonian structure of the $N = 2$ KP hierarchy under consideration is local which allows us to associate it with the $N = 2$ superconformal algebra. It is generally observed for supersymmetric integrable models that one of the Hamiltonian structures are nonlocal. In this chapter, the 1^{st} Hamiltonian structure of the nonstandard flow $N = 2$ KP is explicitly derived and is found to be nonlocal as well as nonlinear. The fact that the $N = 2$ supersymmetric KP hierarchy is biHamiltonian demonstrates that dynamical equations associated with the non-standard flows are completely integrable.

3.2 The First Hamiltonian Structure

In the Gelfand Dikii bracket of the first kind (3.10) we define the auxiliary fields P and Q by

$$P = \sum_{j=-2}^{\infty} D^j p_j \qquad ; \qquad Q = \sum_{j=-2}^{\infty} D^j q_j \tag{3.12}$$

with the grading $|p_j| = |q_j| = j$ so that the linear functional $F_P(L)$ (and similarly $F_Q(L)$) becomes

$$F_P(L) = Tr(LP) = \sum_{i=0}^{\infty} \int dX (-1)^{i+1} u_{i-1}(X) p_{i-1}(X) \tag{3.13}$$

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Consequently the L.H.S. of (3.10) becomes

$$\{F_P(L), F_Q(L)\} = \sum_{i,j=0}^{\infty} \int dX \int dY (-1)^{i+j} p_{i-1}(X) \{u_{i-1}(X), u_{j-1}(Y)\}\, q_{j-1}(Y) \tag{3.14}$$

Notice that (3.14) does not involve terms like p_{-2} and q_{-2} since the superfields begin from $u_{-1}(X)$ in the Lax operator (2.17) defined in chapter 2, which has the form

$$L = D^2 + \sum_{i=0}^{\infty} u_{i-1}(X) D^{-i} \tag{3.15}$$

As in the case for the 2^{nd} Hamiltonian structure this consistency is ensured by setting the coefficient of the D term in the commutator $[L, P]$ to zero. This leads to the constraint

$$\begin{aligned} p_{-1} &= \sum_{j=-1}^{\infty} p_{2j+1}^{[2j+2]} + \sum_{i,m=0}^{\infty} (-1)^i u_{i-1} p_{i+2m+1}^{[2m]} \\ &+ \sum_{j=0}^{\infty} \sum_{m=0}^{j-1} \begin{bmatrix} j \\ m \end{bmatrix} (-1)^{j(m+1)} (p_j u_{j-m-2})^{[m]} \end{aligned} \tag{3.16}$$

Using the constraint (3.16) we obtain from (3.14) the following Poisson bracket among the superfields

$$\begin{aligned} \{u_{i-1}(X), u_{j-1}(Y)\}_1 = \Big[&-\delta_{i,0} D^{j+1} + (-1)^{[i/2]} \delta_{j,0} D^{i+1} \\ &- \sum_{m=0}^{i+j-1} (-1)^{i(j+m+1)+[m/2]} \begin{bmatrix} i-1 \\ m \end{bmatrix} u_{i+j-m-2} D^m \\ &- \sum_{m=0}^{i+j-1} (-1)^{m(i+1)+j(m+1)} \begin{bmatrix} j-1 \\ m \end{bmatrix} D^m u_{i+j-m-2} \Big] \Delta(X-Y) \end{aligned} \tag{3.17}$$

To check the consistency of the Poisson bracket structure (3.17), we consider one of the conserved charges [63], namely

$$H_2 = \int dX (u_2 + u_{-1} u_0) \tag{3.18}$$

which provides the consistent equation of motion (2.23) with respect to the second Hamiltonian structure and is in involution. However, with respect to the Poisson bracket structure (3.17) the equation of motion turns out to be

$$\frac{du_{i-1}}{dt_1} = \{u_{i-1}, H_2\}_1 = u_{i-1}^{[2]} - \sum_{m=0}^{i-1} (-1)^{m(i+1)+[m/2](u_{i-m-2}u_0)^{[m]}}$$
$$- \sum_{m=0}^{i-1} (-1)^{i(m+1)+[m/2]} \left[\begin{array}{c} i-1 \\ m \end{array} \right] u_{i-m-2}u_0^{[m]} - \delta_{i,0}u_{-1}^{[2]}$$
$$+ \sum_{m=0}^{i-1} (-1)^{i(m+1)+[m/2]} \left[\begin{array}{c} i-1 \\ m+1 \end{array} \right] u_{i-m}u_{-1}^{[m+1]} \qquad (3.19)$$

Clearly (3.19) does not reproduce the correct equation of motion *viz.* (2.23). This inconsistency arises because the superLax operator considered is not a pure differential operator and was observed also in other situations involving pseudo-differential operators [53]. This indicates, like in previous cases with the same inconsistency, one can either modify the Hamiltonians or alternately devise a different Hamiltonian structure. Since the Hamiltonians are the conserved quantities, we chose to go in for a modification of the Hamiltonian structure. If we modify Hamilton's equation of motion to the form given below,

$$\frac{du_{i-1}(X)}{dt_n} = \{u_{i-1}(X), H_{n+1}\}_1$$
$$-(-1)^i \int dY \, \{u_{i-1}(X), u_{-1}(Y)\}_1 \, sResL^n(Y)$$
$$-(-1)^i \int dY \, \{u_{i-1}(X), u_0(Y)\}_1 \, sRes\left(L^n D^{-1}(Y)\right) \qquad (3.20)$$

where

$$sResL^n(X) = \sum_{j=0}^{\infty} \delta_{j,0} \frac{\delta H_{n+1}}{\delta u_{j-1}(X)}$$
$$= \sum_{j=0}^{\infty} \sum_{l=0}^{j-1} \left[\begin{array}{c} j-1 \\ l+1 \end{array} \right] D^{l-1} u_{j-l-2}(X) \frac{\delta H_{n+1}}{\delta u_{j-1}(X)} \qquad (3.21)$$

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and

$$\begin{aligned} sResL^nD^{-1}(X) &= \sum_{j=0}^{\infty}(-1)^j D_X^j \frac{\delta H_{n+1}}{\delta u_j(X)} \\ &= \sum_{j=0}^{\infty}\sum_{l=0}^{j-1}\Big\{(-1)^j D^{-2}u_{j-l-2}(X)D_X^l + \begin{bmatrix} j-1 \\ l \end{bmatrix} \\ &\quad \times(-1)^{l(j+1)}D_X^{l-2}u_{j-l-2}(X)\Big\}\frac{\delta H_{n+1}}{\delta u_{j-1}(X)} \qquad (3.22) \end{aligned}$$

it reproduces the equations of motion correctly. Substituting (3.21) and (3.22) in (3.20), the equation of motion can be rewritten in the form

$$\frac{du_{i-1}(X)}{dt_n} = \{u_{i-1}(X), H_{n+1}\}_{\bar{1}} \qquad (3.23)$$

which eventually leads us to the correct form of the first Hamiltonian structure as

$$\begin{aligned} &\{u_{i-1}(X), u_{j-1}(Y)\}_{\bar{1}} = \\ &\Big[-\sum_{m=0}^{i+j-1}\Big\{\begin{bmatrix} i-1 \\ m \end{bmatrix}(-1)^{i(j+m+1)+[m/2]}u_{i+j-m-2}D^m \\ &+\begin{bmatrix} j-1 \\ m \end{bmatrix}(-1)^{m(i+1)+j(m+1)}D^m u_{i+j-m-2}\Big\} \\ &+\sum_{m=0}^{i-1}\sum_{l=0}^{j-1}\Big\{\begin{bmatrix} j-1 \\ l+1 \end{bmatrix}(-1)^{m(i+1)+j+[m/2]}D^m u_{i-m-2}D^{l-1}u_{j-l-2} \\ &-\begin{bmatrix} i-1 \\ m+1 \end{bmatrix}(-1)^{[-m/2]}u_{i-m-2}D^{m-1}u_{j-l-2}D^l \\ &+\begin{bmatrix} i-1 \\ m \end{bmatrix}\begin{bmatrix} j-1 \\ l+1 \end{bmatrix}(-1)^{j+i(m+1)+[m/2]}u_{i-m-2}D^{m+l-1}u_{j-l-2} \\ &+\begin{bmatrix} i-1 \\ m+1 \end{bmatrix}\begin{bmatrix} j-1 \\ l \end{bmatrix}(-1)^{(l+1)(j+1)+[-m/2]} \\ &\times u_{i-m-2}D^{m+l-1}u_{j-l-2}\Big\}\Big]\Delta(X-Y) \qquad (3.24) \end{aligned}$$

(3.24) correctly reproduces the dynamical equations of the hierarchy associated with the super Lax operator (2.17) and the nonstandard

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flows (2.22). It is evident that the above Hamiltonian structure is manifestly antisymmetric and satisfies the Jacobi identity.

3.3 Conclusion

The first Hamiltonian structure (3.24), unlike the second Hamiltonian structure (2.31), is nonlocal and hence cannot not be associated with a local conformal symmetry. As mentioned earlier this feature is noticed in other supersymmetric integrable hierarchies also that one of the two Hamiltonian structures becomes nonlocal [59]. The existence of two Hamiltonian structures, nonetheless, confirms the complete integrability of the even parity super KP hierarchy.

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Chapter 4

Super $\hat{W}_\infty$ Algebra

4.1 Introduction

The close relationship between the conformal algebras and the rich symmetry associated with integrable systems is well understood. The Hamiltonian structures of the integrable hierarchies have been found to be isomorphic to the various higher spin conformal algebras at the classical level. This was realized when it was shown that the W_n algebra incorporates in its classical limit the Hamiltonian structure of the nonlinear integrable systems i.e. the generalized n^{th} KdV hierarchy [43, 44, 45]. The technique of obtaining classical conformal algebras through the integrable hierarchies is indeed a powerful one and its importance was recognized when the existence of a new higher spin conformal algebra was realized at the classical level [63]. Prior to this all the known conformal algebras were obtained first in terms of free fields via the bootstrap approach and subsequently their relation to the symmetries of the integrable hierarchies was obtained.

It was initially believed that the large n limit of W_n algebra, namely $W_{1+\infty}$ [46, 47] and W_∞ [48, 49, 50] algebras would provide the universal algebras which would include all other algebras as subalgebras. The

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naive approach to the large n limit, however, gives rise to *linear* algebras which are truly infinite dimensional symmetry algebras containing all the conformal spins [49, 77, 51]. The infinite dimensional algebras have been interpreted as area preserving diffeomorphisms [77, 78] but the fact that they are linear prevents them from being the right candidates for universal algebras, there being no straightforward mechanism which effectively truncates the spin content of these algebras and produces the nonlinear features of W_n. A nonlinear realization of the W algebra namely $\hat{W}_\infty$ algebra was obtained by the bootstrap approach and identified with the 2^{nd} Hamiltonian structure of the KP hierarchy [55, 53, 54]. It is a universal W algebra containing all W_n algebras. This was obtained by associating the symmetry algebra of the $SL(2,R)_k/U(1)$ coset model with the $\hat{W}_\infty$ algebra characterized by the label k and then by showing that the symmetry algebra truncates to W_n algebra for $k=-n$ [55].

For supersymmetric systems, the supersymmetric KP hierarchy based on the odd parity superLax operator $L = D + \sum_{i=-1}^{\infty} u_i(X)D^{-i-1}$ was first obtained by Manin and Radul [8]. However, because of an inversion of the even-odd parity of the conserved supercharges, the super Hamiltonian formalism of the Manin-Radul KP cannot be reduced to the Hamiltonian structure of the bosonic KP. Meanwhile, the linear super W_∞ algebra was obtained through the bootstrap approach [60]. The bosonic sector of the super W_∞ algebra was the combination of the $W_{1+\infty}$ algebra, realized in terms of bilinears of the free fermion ψ and its conjugate ψ^*, and the W_∞ algebra, which was realized via bilinears of the free complex scalar boson ϕ and its conjugate $\bar{\phi}$ so that the bosonic sector possesses the $\hat{W}_{1+\infty} \oplus \hat{W}_\infty$ structure. On the other hand, its fermionic sector was constructed using the boson-fermion pairs $\psi\bar{\phi}$ and $\psi^*\phi$. Later on, the KP hierarchy, based on the even parity superLax operator $L = D^2 + \sum_{i=0}^{\infty} u_{i-2}(X)D^{-i+1}$ and associated with

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the standard flow was obtained and its first Hamiltonian structure was found to be isomorphic to the super W_∞ algebra [58]. Its 2^{nd} Hamiltonian structure [59], although nonlinear is nonlocal and hence cannot be associated with any symmetry algebra.

This nonlocality of the 2^{nd} Hamiltonian structure of the super KP hierarchy is overcome by the use of the nonstandard flow equation which results in a nonlinear but local 2^{nd} Hamiltonian structure [63] reviewed in chapter 2, where it was seen that the 2^{nd} Hamiltonian structure of the $N = 2$ nonstandard flow KP hierarchy includes the $N = 2$ superconformal algebra as a subalgebra. In this chapter we go further in demonstrating that the 2^{nd} Hamiltonian structure is a candidate for a higher spin extension of $N = 2$ superconformal algebra and has the required number of spin fields and the bosonic sector of the algebra has the desired $\hat{W}_{1+\infty} \oplus \hat{W}_\infty$ structure with two commuting sets of bosonic generators. Moreover it will be seen that while one of the bosonic sectors is represented by a linear basis the other is represented by a nonlinear one, the nonlinear set of bosonic generators being closely connected to the generators of the $\hat{W}_\infty$ algebra. It will also be shown that a consistent nonlinear basis for the fermionic sector also exists.

4.2 The Bosonic Sector

Since the superfields u_{2i-1} and u_{2i} are bosonic and fermionic superfields respectively, they can be expressed in component form as

$$\begin{aligned} u_{2i-1}(X) &= u^b_{2i-1}(x) + \theta u^f_{2i-1}(x), \\ u_{2i}(X) &= u^f_{2i}(x) + \theta u^b_{2i}(x) \end{aligned} \tag{4.1}$$

From the Poisson bracket of the component fields with the energy-momentum tensor (2.35,2.36,2.37,2.38) in chapter 2, we find that that the odd bosonic fields $u^b_{2i-1}(x)$ have conformal weights $i+1$, whereas the even bosonic fields $u^b_{2i}(x)$ have conformal weights $i+2$ $(i = 0, 1, 2, \ldots)$, with respect to the energy-momentum tensor

$$T(x) = u^b_0(x) - \frac{1}{2}\partial_x u^b_{-1}(x) \tag{4.2}$$

On the other hand, both odd fermionic fields $u^f_{2i-1}(x)$ and even fermionic fields $u^f_{2i}(x)$ have conformal weights $i + \frac{3}{2}$ $(i = 0, 1, 2, \ldots)$ with respect to the same stress tensor (4.2). The stress tensor $T(x)$ belongs to the $N = 2$ conformal algebra being a subalgebra of (2.31). This ensures the presence of a nonlinear supersymmetric conformal algebra in the Hamiltonian structure (2.31) of the $N = 2$ super KP hierarchy. The Poisson brackets among all the component fields are given in appendix A in a basis which will be defined later.

The $W_{1+\infty} \oplus W_\infty$ structure of the bosonic sector, however, is not apparent in our case from the Poisson bracket between two types of bosons, u^b_{2i-1} and u^b_{2i}. This is in contrast to the other supersymmetric algebra [58]. We shall establish that the bosonic sector of this algebra, indeed, has the required $W_{1+\infty} \oplus W_\infty$ structure [60]. This step is crucial in obtaining the free field representation of the generators. To carry out this program, a suitable basis is required. Notice that the odd bosons u_{2i-1} themselves form a closed algebra (A.5) and consequently for the odd bosons a suitable set of generators may be constructed from a linear combination of the fields as considered in [58], namely

$$\tilde{W}_{n+1} = \frac{2^{-n} n!}{(2n-1)!!} \sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} n+l \\ l \end{pmatrix} u^{b\,(n-l)}_{2l-1} \tag{4.3}$$

Since the odd bosons form a closed algebra among themselves, the algebra of the new set of generators are also closed and constitute the $\hat{W}_{1+\infty}$ algebra. Using the Poisson bracket amongst the component fields given in Appendix A (A.5), we obtain following Poisson brackets for the

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lower order odd boson generators.

$$
\begin{aligned}
\{\tilde{W}_1, \tilde{W}_1\} &= 0 \\
\{\tilde{W}_2, \tilde{W}_1\} &= \tilde{W}_1 \partial \delta(x-y) \\
\{\tilde{W}_2, \tilde{W}_2\} &= \left[2\tilde{W}_2\partial + \tilde{W}_2'\right] \delta(x-y) \\
\{\tilde{W}_3, \tilde{W}_1\} &= 2\tilde{W}_2 \partial \delta(x-y) \\
\{\tilde{W}_3, \tilde{W}_2\} &= \left[3\tilde{W}_3\partial + \tilde{W}_3' + \frac{1}{6}\tilde{W}_1\partial^3\right] \delta(x-y) \\
\{\tilde{W}_3, \tilde{W}_3\} &= \Big[4\tilde{W}_4\partial + 2\tilde{W}_4' + \frac{3}{5}\tilde{W}_2''\partial + 2\tilde{W}_3\tilde{W}_1\partial - \frac{1}{3}\tilde{W}_1''\tilde{W}_1\partial \\
&-2\tilde{W}_2^2\partial + \frac{1}{2}\tilde{W}_1^2\partial + \tilde{W}_2'\partial^2 + \frac{2}{3}\tilde{W}_3\partial^3 + \frac{2}{15}\tilde{W}_2''' \\
&+\tilde{W}_3'\tilde{W}_1 - \frac{1}{6}\tilde{W}_1'''\tilde{W}_1 - 2\tilde{W}_2'\tilde{W}_2 + \tilde{W}_3\tilde{W}_1' + \frac{1}{3}\tilde{W}_1''\tilde{W}_1'\Big] \delta(x-y) \\
\{\tilde{W}_4, \tilde{W}_1\} &= \left[3\tilde{W}_3\partial + \frac{1}{10}\tilde{W}_1''\partial - \frac{3}{10}\tilde{W}_1'\partial^2 + \frac{1}{10}\tilde{W}_1\partial^3\right] \delta(x-y) \\
\{\tilde{W}_4, \tilde{W}_2\} &= \left[4\tilde{W}_4\partial + \tilde{W}_4' + \frac{7}{10}\tilde{W}_2\partial^3\right] \delta(x-y) \qquad (4.4)
\end{aligned}
$$

In the even boson, u^b_{2i} sector, however, the Poisson bracket relation is complex and it appears that the generators neither form a closed algebra nor do they commute with the odd bosons. Apparently, therefore, the direct sum structure is not maintained as required for the supersymmetric W algebra. This problem can be circumvented and for the even bosons also a suitable basis with these desirable properties can be obtained. The first step in making them commute is to redefine the even bosons as a linear combination of odd and even bosons of equal spin as in [58].

$$v^b_{2j} = u^b_{2j+1} + u^b_{2j} \qquad (4.5)$$

and similarly we choose a linear combination of generators for the odd fermions

$$v^f_{2j-1} = u^f_{2j-1} - u^f_{2j} \qquad (4.6)$$

In the linear super W_∞ algebra, this is sufficient to ensure commutation between odd and even bosons, and thereby establish the $W_{1+\infty} \oplus W_\infty$ structure, but in this case we find from the Poisson bracket of u^b_{2j-1} with v^b_{2j}, *viz.*

$$\begin{aligned}
&\left\{u^b_{2j-1}(x), v^b_{2k}(y)\right\} = \\
&\Bigg[-\sum_{m=0}^{j-1}\sum_{l=0}^{k}\begin{pmatrix} j-1 \\ m \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix}(-1)^m u^f_{2j-2m-2}\partial^{m+l}v^f_{2k-2l-1} \\
&-\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\begin{pmatrix} j \\ m+1 \end{pmatrix}\begin{pmatrix} k \\ l+1 \end{pmatrix}(-1)^m u^b_{2j-2m-3}\partial^{m+l+1}v^b_{2k-2l-2} \\
&-\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} k-n-1 \\ l \end{pmatrix} u^f_{2n}\partial^l v^f_{2j+2k-2n-2l-3} \\
&+\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k}\sum_{l=0}^{k-n}\begin{pmatrix} j-1 \\ m \end{pmatrix}\begin{pmatrix} n+l-1 \\ l \end{pmatrix} \\
&\times(-1)^m u^f_{2j+2k-2m-2n-2l-2}\partial^{m+l}v^f_{2n-1}\Bigg]\,\delta(x-y)
\end{aligned} \tag{4.7}$$

that only the linear terms vanish. Consequently, the odd bosons commute with only the lowest spin even boson generator i.e.

$$\{u^b_{2j-1}, v^b_0\} = 0 \tag{4.8}$$

and the Poisson brackets with higher even bosons v^b_{2j}, $(j \neq 0)$ are nonzero. Another feature of the nonlinear superalgebra which distinguishes it from the linear one is the nature of the fermion brackets $\left\{u^f_{2j}(x), u^f_{2k}(y)\right\}$ and $\left\{v^f_{2j-1}(x), v^f_{2k-1}(y)\right\}$ given below

$$\begin{aligned}
&\left\{u^f_{2j}(x), u^f_{2k}(y)\right\} = \\
&\Bigg[-\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\begin{pmatrix} j \\ m+1 \end{pmatrix}\begin{pmatrix} k \\ l+1 \end{pmatrix}(-1)^m u^f_{2j-2m-2}\partial^{m+l+1}u^f_{2k-2l-2} \\
&-\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} k-n-1 \\ l \end{pmatrix} u^f_{2n}\partial^l u^f_{2j+2k-2n-2l-2}
\end{aligned}$$

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$$+\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{j}{m}\binom{n+l}{l}$$
$$\times(-1)^m u^f_{2j+2k-2m-2n-2l-2}\partial^{m+l}u^f_{2n}\Big]\,\delta(x-y) \qquad (4.9)$$

and

$$\left\{v^f_{2j-1}(x), v^f_{2k-1}(y)\right\} =$$
$$\left[\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\left\{\binom{j-1}{m}\binom{k-1}{l} - \binom{j}{m+1}\binom{k}{l+1}\right\}\right.$$
$$\times(-1)^m v^f_{2j-2m-3}\partial^{m+l+1}v^f_{2k-2l-3}$$
$$+\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{k-n-1}{l} v^f_{2n-1}\partial^l v^f_{2j+2k-2n-2l-3}$$
$$-\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{j-1}{m}\binom{n+l-1}{l}$$
$$\times(-1)^m v^f_{2j+2k-2m-2n-2l-3}\partial^{m+l}v^f_{2n-1}\Big]\,\delta(x-y) \qquad (4.10)$$

(4.9) and (4.10) clearly show that the fermion fields u^f_{2j} and v^f_{2j-1} are noncommuting for all $j > 0$. The noncommuting nature of the fermion brackets will be responsible for the necessity of a nonlinear free field representation of the generators of the nonlinear superalgebra.

Interestingly, nonlinear combinations of bosons and fermions exist which commute with all odd bosons. This may be achieved, as the second step, by taking the most general combinations of the fields of the appropriate conformal spin and the coefficient of the terms may be determined by allowing them to commute with the odd bosons. This procedure can be carried out for all even boson generators thereby yielding the mutually commuting set of generators. The explicit expressions of a few even boson generators are given below.

$$W_2 = v^b_0$$
$$W_3 = v^b_2 + \frac{1}{2}v^{b'}_0 + u^b_{-1}v^b_0 + u^f_0 v^f_{-1}$$

$$
\begin{aligned}
W_4 &= v_4^b + v_2^{b'} + \frac{1}{5}v_0^{b''} + 2v_2^b u_{-1}^b + u_2^f v_{-1}^f \\
&\quad + u_0^f v_1^f + v_0^{b'} u_{-1}^b + u_0^f v_{-1}^{f'} + v_0^b u_{-1}^{b^2} + u_{-1}^b u_0^f v_{-1}^f \qquad (4.11)
\end{aligned}
$$

and so on. The distinguishing character of this set is that spin 3 and higher generators are nonlinear. For the spin 3 generator it is a bilinear combination of bosonic as well as spin $\frac{3}{2}$ generators. For the spin 4 generator, this combination is more involved, having terms trilinear in the fields. This indicates that for higher spin generators, the basis becomes more and more nonlinear. But most importantly, these generators are such that the desired property is exhibited, namely

$$
\begin{aligned}
\{u_{2j-1}^b, W_2\} &= 0 \\
\{u_{2j-1}^b, W_3\} &= 0 \\
\{u_{2j-1}^b, W_4\} &= 0 \qquad (4.12)
\end{aligned}
$$

i.e. the new set of even boson generators commute with all the odd bosons. Moreover the W boson generators produce the requisite form of $\hat{W}_\infty$ algebra as can be observed from the following Poisson bracket relations.

$$
\begin{aligned}
\{W_2, W_2\} &= [-2W_2\partial - W_2']\,\delta(x-y) \\
\{W_3, W_2\} &= [-3W_3\partial - W_3']\,\delta(x-y) \\
\{W_3, W_3\} &= \Big[-4W_4\partial - 2W_2^2\partial - \frac{9}{20}W_2''\partial - 2W_4' - (W_2^2)' \\
&\quad - \frac{1}{10}W_2''' - \frac{3}{4}W_2'\partial^2 - \frac{1}{2}W_2\partial^3\Big]\,\delta(x-y) \\
\{W_4, W_2\} &= \Big[-4W_4\partial - W_4' - \frac{2}{5}W_2\partial^3\Big]\,\delta(x-y) \qquad (4.13)
\end{aligned}
$$

This algebra is isomorphic to classical analogue of the $\hat{W}_\infty$ algebra [55]. The procedure outlined above, although straightforward, becomes extremely difficult to use in obtaining the still higher spin generators and

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to show the closure of the algebra explicitly. To obtain the genenerators of higher spin we employ a different strategy. The W_4 generator, for example, may be obtained straightforwardly from the $\{W_3, W_3\}$ Poisson bracket algebra (4.13) by ensuring the closure of the algebra following the classical analogue of the $\hat{W}_\infty$ algebra [55]. Importantly the W_4 generator thus obtained matches with that of (4.11), which commutes with the odd bosons (4.12) and the form is unique. Similarly, the explicit form of the W_5 generator may be obtained from the Poisson bracket relation, $\{W_4, W_3\}$

$$\begin{aligned}\{W_4, W_3\} = \Big[&-5W_5\partial - 5W_5' - 4W_3W_2\partial - 4W_3W_2' \\ &-\frac{7}{5}W_3''\partial - \frac{2}{5}W_3''' - \frac{8}{5}W_3'\partial^2 - \frac{8}{5}W_3\partial^3\Big]\,\delta(x-y)\end{aligned} \tag{4.14}$$

by ensuring the closure of the algebra following the classical analogue of $\hat{W}_\infty$ algebra. It is found that the leading term of the $\{W_4, W_3\}$ algebra becomes v_6^b. This, indeed, ensures the presence of the W_5 generator in the algebra. The W_5 generator exhibits the explicit form

$$\begin{aligned}W_5 \;=\;& v_6^b + \frac{3}{2}v_4^{b'} + \frac{9}{14}v_2^{b''} + \frac{1}{14}v_0^{b'''} + 3u_{-1}^b v_4^b + 3u_{-1}^b v_2^{b'} + 3u_{-1}^{b^2} v_2^b \\ &+\frac{3}{2}u_{-1}^{b^2}v_0^{b'} + u_{-1}^{b^3}v_0^b - u_{-1}^{b'}v_0^{b'} + u_{-1}^b v_0^{b''} + \frac{1}{7}\left(u_{-1}^b v_0^b\right)'' \\ &+u_2^f v_1^f + u_4^f v_{-1}^f + u_0^f v_3^f + \frac{3}{2}u_0^f v_1^{f'} - \frac{1}{2}u_0^{f'}v_1^f + \frac{1}{2}u_2^{f'}v_{-1}^f \\ &+\frac{3}{2}u_2^f v_{-1}^{f'} - u_0^{f'}v_{-1}^{f'} + u_0^f v_1^{f''} + \frac{1}{7}\left(u_0^f v_{-1}^f\right)'' + \frac{1}{2}u_{-1}^b u_0^{f'} v_{-1}^f - \frac{1}{2}u_{-1}^{b'}u_0^f v_{-1}^f \\ &+\frac{3}{2}u_{-1}^b u_0^f v_{-1}^{f'} + u_{-1}^{b^2}u_0^f v_{-1}^f + 2u_{-1}^b u_2^f v_{-1}^f + 2u_{-1}^b u_0^f v_1^f + u_1^b u_0^f v_{-1}^f\end{aligned} \tag{4.15}$$

It is found by explicit calculation that the W_5 generator commutes with the odd bosons, u_{2j-1}^b, *i.e.*

$$\{u_{2j-1}^b, W_5\} = 0 \tag{4.16}$$

The spin six generator W_6, in a similar way, may be obtained from the Poisson bracket relation, $\{W_4, W_4\}$, whose leading term turns out

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to be v_8^b. Thus the first three generators may be obtained by the bootstrap approach and from spin four onwards all the generators may be found following the above procedure. This strategy of obtaining bosonic higher spin generators evidently guarantees the closure of the algebra being isomorphic to the classical $\hat{W}_\infty$ algebra. But to obtain the explicit forms of all higher generators becomes very difficult, although the strategy is quite clear. Moreover, this strategy also ensures that in the bosonic limit the super $\hat{W}_\infty$ reduces to the $\hat{W}_\infty$ algebra. In this way, we may establish the $\hat{W}_{1+\infty} \oplus \hat{W}_\infty$ structure in the bosonic sector of the super $\hat{W}_\infty$ algebra.

4.3 The Fermionic Sector

In the fermionic sector both the fermions, u_{2i}^f and v_{2i-1}^f form closed algebras (4.9,4.10) separately. This reveals that a linear representation such as the one in [58] always exists for the fermions. We shall, however, show the existence a nonlinear basis for both the fermions in conjunction with the odd bosons. This follows from the observation that the even fermions, u_{2i}^f as well as the odd fermions, v_{2i-1}^f and the odd bosons also satisfy separately a subalgebra (see (A.1,A.3) in appendix A). Nonetheless, it turns out that nonlinear basis has an interesting consequence. Both the fermions satisfy identical algebras in the nonlinear basis with an added advantage of generating the minimal algebra.

The new basis for the even fermion generators may be constructed as a suitable nonlinear combination of even fermions u_{2i}^f and odd bosons u_{2i-1}^b in the form

$$
\begin{aligned}
\tilde{J}_{3/2} &= -u_0^f \\
\tilde{J}_{5/2} &= -u_2^f - u_0^{f'} + u_{-1}^b u_0^f
\end{aligned}
$$

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$$\tilde{J}_{7/2} = -u_4^f - u_2^{f'} - \frac{5}{4}u_0^{f''} + u_{-1}^b u_2^f + u_1^b u_0^f + u_{-1}^{b'} u_0^f - u_{-1}^{b^2} u_0^f \tag{4.17}$$

Thus the new set of generators become more and more nonlinear as in the even boson case. But it is seen that the nonlinear basis for the even fermions (4.17) can be recast in terms of the bilinears of the generators only as

$$\begin{aligned} \tilde{J}_{3/2} &= -u_0^f \\ \tilde{J}_{5/2} &= -u_2^f + \tilde{J}'_{3/2} - \tilde{W}_1 \tilde{J}_{3/2} \\ \tilde{J}_{7/2} &= -u_4^f + \tilde{J}'_{5/2} + \frac{1}{4}\tilde{J}''_{3/2} - \tilde{W}_1 \tilde{J}_{5/2} + 2\tilde{W}_1 \tilde{J}'_{3/2} \\ &\quad -\tilde{W}_2 \tilde{J}_{3/2} + \frac{1}{2}\tilde{W}'_1 \tilde{J}_{3/2} \end{aligned} \tag{4.18}$$

This demonstrates that all the even fermions, in general, can be written as bilinears in $\tilde{J}$ and $\tilde{W}$ having the form

$$\begin{aligned} \tilde{J}_{n+3/2} &= -u_{2n}^f + \frac{2^{-n}(n-1)!}{(2n-1)!!}\sum_{l=0}^{n-1}\begin{pmatrix} n \\ l \end{pmatrix}\begin{pmatrix} n+l+1 \\ l+1 \end{pmatrix}\tilde{J}_{l+3/2}^{(n-l)} \\ &\quad -\sum_{m=0}^{n-1}\sum_{k=0}^{n-m-1}\sum_{l=0}^{m} B_{kl}^n C_{ml}^n (-1)^{k+l}\tilde{J}_{k+3/2}^{(n-m-l-1)}\tilde{W}_{l+1}^{m-l} \end{aligned} \tag{4.19}$$

($n = 0, 1, 2, \ldots$), where B_{kl}^n and C_{ml}^n are the c-number coefficients. While the values of C_{ml}^n can be easily extracted from (4.3) by determining u_{2i-1}^b in terms of $\tilde{W}_{i+1}$, it appears that there is no straightforward procedure to determine the explicit expressions of the B_{kl}^n for arbitary spins. The closed algebras among the even fermions and odd bosons strongly corroborates the existence of a nonlinear basis for all higher spin generators and thereby ensuring the coefficients B_{kl}^n can always be determined for all higher spin generators.

The even fermions of higher spins may be obtained from (A.1) and (A.9) through the Poisson brackets of the even fermions u_{2k}^f with the

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W_2 and $\tilde{W}_2$ generators,

$$\{W_2, u^f_{2k}\} = -u^f_{2k+2} - \sum_{m=0}^{k} \begin{pmatrix} k+1 \\ m+1 \end{pmatrix} \partial^{m+1} u^f_{2k-2m} \tag{4.20}$$

and

$$\begin{aligned}\{\tilde{W}_2, u^f_{2k}\} &= -u^f_{2k+2} - \frac{1}{2}u^f_{2k} + \frac{1}{2}u^f_{2k}\partial \\ &+ \sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} u^f_0 \partial^l u^b_{2k-2l-1} - \sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} u^b_{-1}\partial^l u^f_{2k-2l}\end{aligned} \tag{4.21}$$

The presence of the u^f_{2k+2} term in (4.20,4.21) clearly confirms that the next higher spin generator can be generated from the preceding one. Finally, (4.20) and (4.21) together with the closure of the Poisson brackets of the fermion generators determine the higher spin generators explicitly. As a consequence, the closure of the algebra among the even fermions is ensured even in the nonlinear basis. To explicitly show the closure property, the algebra among a few even fermion generators are given below.

$$\begin{aligned}
\{\tilde{J}_{3/2}, \tilde{J}_{3/2}\} &= 0 \\
\{\tilde{J}_{5/2}, \tilde{J}_{3/2}\} &= 0 \\
\{\tilde{J}_{5/2}, \tilde{J}_{5/2}\} &= 2\tilde{J}'_{3/2}\tilde{J}_{3/2}\delta(x-y) \\
\{\tilde{J}_{7/2}, \tilde{J}_{3/2}\} &= -\tilde{J}'_{3/2}\tilde{J}_{3/2}\delta(x-y) \\
\{\tilde{J}_{7/2}, \tilde{J}_{5/2}\} &= \left[-3\tilde{J}_{5/2}\tilde{J}'_{3/2} - 3\ \tilde{J}_{5/2}\tilde{J}_{3/2}\partial + 2\tilde{J}'_{3/2}\tilde{J}_{3/2}\partial\right]\delta(x-y) \\
\{\tilde{J}_{7/2}, \tilde{J}_{7/2}\} &= \Big[6\tilde{J}'_{5/2}\tilde{J}_{5/2} - 3\tilde{J}'_{5/2}\tilde{J}'_{3/2} + 3\tilde{J}_{5/2}\tilde{J}''_{3/2} + \frac{3}{4}\tilde{J}''_{3/2}\tilde{J}'_{3/2} \\
&-\frac{5}{4}\tilde{J}'''_{3/2}\tilde{J}_{3/2} - \frac{5}{2}\tilde{J}''_{3/2}\tilde{J}_{3/2}\partial - \frac{5}{2}\tilde{J}'_{3/2}\tilde{J}_{3/2}\partial^2\Big]\delta(x-y)
\end{aligned} \tag{4.22}$$

It is interesting to note that unlike the linear algebra, the fermionic generators do not commute among themselves. This is a significant departure from the linear representation and has a major effect on the

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free field representation: the free field representations of the generators of the algebra have to be nonlinear as will be observed in chapter 5. The difference between this algebra and the linear one becomes evident from the spin 5/2 generator onwards. For example, the Poisson bracket of the $\tilde{J}_{5/2}$ generator with itself becomes nonzero. In fact, the self brackets of the generators, in general, cannot be made to zero by any change of basis. Thus the nonlinear algebra cannot be reduced trivially to the linear one. A few Poisson bracket relations of $\tilde{W}$ and W bosons with even fermions are given in appendix B.

The arguments regarding the existence of the nonlinear basis for the even fermion generators stated above hold good for the odd fermions as well since the odd fermions v^f_{2i-1} and the odd bosons u^b_{2i-1} form a closed algebra among themselves (A.3,A.4,A.5). We can therefore generate a nonlinear basis for the odd fermions v^f_{2i-1} as well in terms of odd fermions and the odd bosons u^b_{2i-1}. For example a consistent nonlinear basis for a few odd fermions may be identified as

$$\begin{aligned} J_{3/2} &= v^f_{-1} \\ J_{5/2} &= -v^f_1 + u^b_{-1}v^f_{-1} \\ J_{7/2} &= v^f_3 + v^{f'}_1 + \frac{5}{4}v^{f''}_{-1} - u^b_{-1}v^f_1 - u^b_1 v^f_{-1} - u^b_{-1}v^{f'}_{-1} + u^{b^2}_{-1}v^f_{-1} \end{aligned} \tag{4.23}$$

It is straightforward to observe that the generators in (4.23) can be rewritten as bilinear combinations of $\tilde{W}$ and J like the even fermions. This can also be argued from the fact that odd fermions and odd bosons form a subalgebra and thus can be generalised for other higher spin generators. The higher spin generators, however, may be obtained from the lower ones following the Poisson bracket relations resulting from (A.3,A.8)

$$\{W_2, v^f_{2k-1}\} = v^f_{2k+1} - v^f_{2k-1}\partial + u^b_{-1}v^f_{2k-1} - u^b_{2k-1}v^f_{-1} \tag{4.24}$$

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$$
\begin{aligned}
\{\tilde{W}_2, v^f_{2k-1}\} &= v^f_{2k+1} - \sum_{l=0}^{k-1} \begin{pmatrix} k-1 \\ l \end{pmatrix} u^b_{-1} \partial^l u^f_{2k-2l-3} \\
&+ \left[\sum_{m=0}^{k-1} \begin{pmatrix} k \\ m+1 \end{pmatrix} + \frac{1}{2} \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} \right] \partial^{m+1} v^f_{2k-2m-1} \qquad (4.25)
\end{aligned}
$$

and by demanding the closure of the algebra among the odd fermions. To demonstrate the identical algebra among the odd fermions and the even fermions in the nonlinear basis, we give below the algebra among a few odd fermion generators explicitly.

$$
\begin{aligned}
\{J_{3/2}, J_{3/2}\} &= 0 \\
\{J_{5/2}, J_{3/2}\} &= 0 \\
\{J_{5/2}, J_{5/2}\} &= 2J'_{3/2}J_{3/2}\delta(x-y) \\
\{J_{7/2}, J_{3/2}\} &= -J'_{3/2}J_{3/2}\delta(x-y) \\
\{J_{7/2}, J_{5/2}\} &= \left[-3J_{5/2}J'_{3/2} - 3J_{5/2}J_{3/2}\partial + 2J'_{3/2}J_{3/2}\partial\right]\delta(x-y) \\
\{J_{7/2}, J_{7/2}\} &= \Big[6J'_{5/2}J_{5/2} - 3J'_{5/2}J'_{3/2} + 3J_{5/2}J''_{3/2} + \frac{3}{4}J''_{3/2}J'_{3/2} \\
&- \frac{5}{4}J'''_{3/2}J_{3/2} - \frac{5}{2}J''_{3/2}J_{3/2}\partial - \frac{5}{2}J'_{3/2}J_{3/2}\partial^2\Big]\delta(x-y) \qquad (4.26)
\end{aligned}
$$

The algebra of odd fermions with even fermions as well as with the bosons are also given in appendix B.

4.4 Conclusion

The super $\hat{W}_\infty$ algebra, in its own right, deserves to be a candidate for a universal algebra, unifying all finite dimensional bosonic W algebras as well as supersymmetric W algebras. The presence of classical analogue of $\hat{W}_\infty$ algebra and a direct sum basis in the bosonic sector guarantees that all finite dimensional bosonic W algebras can be obtained under suitable truncation. Since the super $\hat{W}_\infty$ algebra is a higher spin extension of $N=2$ conformal algebra, it is expected to contain all finite

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dimensional $N = 2$ supersymmetric W algebras, like the bosonic univeral algebra. But a systematic analysis of the truncation of super $\hat{W}_\infty$ algebra through some non-compact coset model is yet to be studied and could provide the basis for a future study.

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Chapter 5

Free Field Representation

5.1 Introduction

It is well known that the underlying representation of a conformal field theory is essentially a free field representation. The free field representation in fact, is significant in a number of ways. Apart from playing a role in the classification of conformal algebras, the importance of the free field representation lies in the fact that the quantization of classical symmetry algebras becomes straightforward in terms of the free fields.

The simplest conformal algebra is the Virasoro algebra which can be viewed as the algebra of the spin 2 field $T(z)$ and can be realized in terms of a single real boson field as $T(z) = \partial\phi\partial\phi$. The Virasoro algebra, later on has been extended by the inclusion of generators of higher spin. The simplest higher spin extension of the Virasoro algebra is the W_3 algebra found by Zamolodchikov [43]. The W_3 comprises two generators, the energy momentum tensor $T(z)$ of spin 2 and a spin 3 generator $W(z)$. Subsequently, higher spin generalizations known as W_n algebras were constructed [44, 45]. These consist of the energy momentum tensor $T(z)$ together with higher spin generators each of spin $3 \leq s \leq n$. A characteristic feature of all the W_n algebras with

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$2 < n < \infty$ is that they are nonlinear. All W_n algebras are realizable in terms of $n-1$ free scalar fields. Extension of the W_n algebras to infinite number of spin fields *i.e.* $n = \infty$ have also been constructed. This is the W_∞ algebra [49, 50] generated by currents with spin $2 \leq s \leq \infty$. The W_∞ is a linear algebra and another related linear algebra, the $W_{1+\infty}$ [46, 47], with generators of all spins in the range $1 \leq s \leq \infty$ is also known. A feature of great significance in this connection is that the W_∞ algebra and its nonlinear deformation, namely the $\hat{W}_\infty$ algebra can realized in terms of a complex scalar boson $\partial\phi$ and its conjugate $\partial\bar{\phi}$ [48, 55]. On the other hand, the $W_{1+\infty}$ algebra has a representation in terms of a pair of free fermionic fields ψ and ψ^* [79]. While the representation of the generators of the nonlinear $\hat{W}_\infty$ algebra in terms of $\partial\phi$ and $\partial\bar{\phi}$ can be found only iteratively and cannot be written in a closed form, that for the linear W_∞ has the following closed form

$$V^i = \frac{2^{-i-1}(i+2)!}{(i+1)(2i+1)!!}\sum_{k=0}^{i}(-1)^k \begin{pmatrix} i+1 \\ k \end{pmatrix} \begin{pmatrix} i+l \\ k+1 \end{pmatrix} \partial^{i-k+1}\bar{\phi}\partial^{k+1}\phi \tag{5.1}$$

where the generators V^i have conformal spin $s = i+2$ $(i \geq 0)$. Another sequence of higher spin generators $\tilde{V}^i$, which constitute the $W_{1+\infty}$ algebra can be constructed from the fermion bilinears as

$$\tilde{V}^i = \frac{2^{-i-1}(i+1)!}{(2i+1)!!}\sum_{k=0}^{i+1}(-1)^k \begin{pmatrix} i+1 \\ k \end{pmatrix}^2 \partial^{i-k+1}\psi^*\partial^k\psi \tag{5.2}$$

The free complex bosons, $\phi(x,t)$ and $\bar{\phi}(x,t)$ and free fermions $\psi(x,t)$ and $\psi^*(x,t)$, satisfy the fundamental Poisson bracket algebras

$$\{\psi^*(x), \psi(y)\} = \delta(x-y) \tag{5.3}$$

and

$$\{\partial\phi(x), \partial\bar{\phi}(y)\} = \partial_x\delta(x-y) \tag{5.4}$$

The infinite dimensional linear algebras W_∞ and $W_{1+\infty}$, although possessing an infinite number of generators are not universal algebras as the nonlinear W_n algebras cannot be obtained from them in a natural way. However the nonlinear $\hat{W}_\infty$ algebra is a universal bosonic algebra containing all other algebras as subalgebras and is realizable by nonlinear combinations of the free complex scalar boson and its conjugate in contrast to the W_∞ which has a linear representation (5.1).

By combining the fermionic realization of the $W_{1+\infty}$ algebra (5.1) and the complex boson realization of the W_∞ algebra (5.2) the $N=2$ superalgebra was constructed whose bosonic component is $W_{1+\infty} \oplus W_\infty$ – a requirement which holds for all superalgebras. Its fermionic sector is constructed from bilinears of the complex scalar bosons and fermions and their conjugates [60]. The two types of fermionic generators $G^i(z)$ and $\tilde{G}^i(z)$ with conformal spin $i+\frac{3}{2}$ may be constructed as

$$G^i = \frac{2^{-i}(i+1)!}{(2i+1)!!} \sum_{k=0}^{i} (-1)^k \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} i+l \\ k \end{pmatrix} \partial^{i-k+1}\bar{\phi}\partial^k\psi \tag{5.5}$$

and

$$\tilde{G}^i = \frac{2^{-i}(i+1)!}{(2i+1)!!} \sum_{k=0}^{i} (-1)^k \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} i+l \\ k \end{pmatrix} \partial^{i-k+1}\phi\partial^k\psi^* \tag{5.6}$$

The $W_{1+\infty}$ generators $\tilde{V}^i$ $(i \geq -1)$, the W_∞ generators V^i $(i \geq 0)$ and the fermionic generators G^i and $\tilde{G}^i$ $(i \geq 0)$ close on a superalgebra which is called the $N=2$ super W_∞ algebra. Moreover, the fermionic generators are such that they commute, that is,

$$\left\{G^i, G^i\right\} = 0 = \left\{\tilde{G}^i, \tilde{G}^i\right\} \tag{5.7}$$

The super W_∞ algebra discussed above is obviously a linear algebra realizable with a basis bilinear in the free fields. The nonlinear superalgebra discussed in chapters 2 and 4 however is not realizable from

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a combination which is bilinear in free fields. In this chapter we construct a consistent nonlinear free field representation of the generators discussed in chapter 4. The nontrivial Poisson bracket algebras among the even fermions $\tilde{J}_{n+3/2}$ (4.22) as well as odd fermions $J_{n+3/2}$ (4.26) ensure a significant difference of the free field representations considered here from the linear ones (5.1,5.2,5.5,5.6). In fact, representations of the fermion generators become not only nonlinear, and in general exponential, but also a suitable combinations of both types of bosons and fermions. This makes the free field representation distinct and important. In the bosonic sector such a nontrivial change in the free field representation is not apparent from the Poisson bracket algebras of the generators. But it will be observed that the nontrivial Poisson bracket algebras of the fermions become responsible for an important change in the free field representation of the bosons over the linear ones. The bosonic sector is no longer such that one set of generators can be written in terms of bilinears of free fermionic fields while the other set in terms of free complex bosons. The free field represention is now such that free fermions and free bosons get mixed up in a nontrivial manner at the same time correctly reproducing the desired $W_\infty \oplus W_{1+\infty}$ structure.

5.2 Free Field Representation of Fermions

The representation of the lowest fermion generator $\tilde{J}_{3/2}$ is similar to the one for the linear super W_∞ algebra, except for a exponential term which makes the representation general. Explicitly,

$$\tilde{J}_{3/2} = \psi^* \partial\phi e^{i\epsilon\phi_2} \tag{5.8}$$

where $\phi_2 = \frac{1}{2i}(\phi - \bar{\phi})$ and ϵ is a real parameter. However, in order to reproduce the Poisson brackets for the even fermion generators of higher

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spin, the free field representation of all the generators turns out to be nonlinear. This is essentially because of the noncommuting nature of the even fermion generators of spin higher than 3/2. If we compare this with the linear superalgebra, we find that this results in a completely different representation. The representation for the $\tilde{J}_{5/2}$ generator is found by a simple procedure. Consider the Poisson bracket of the $\tilde{W}_2$ generator with the $\tilde{J}_{5/2}$ generator.

$$\{\tilde{W}_2, \tilde{J}_{3/2}\} = \left[-\tilde{J}_{5/2} + \frac{1}{2}\tilde{J}'_{3/2} + \frac{1}{2}\tilde{J}_{3/2}\partial - \tilde{W}_1\tilde{J}_{3/2}\right]\delta(x-y) \qquad (5.9)$$

Comparing (5.9) with the Poisson bracket algebra between the free field representations of $\tilde{W}_2$ (obtained in the next section) and $\tilde{J}_{3/2}$, namely

$$\begin{aligned}\{\tilde{W}_2, \tilde{J}_{3/2}\} = \Big[-&\left[\psi^{*'}\partial\phi + \psi^*(a\partial\phi + b\partial\bar{\phi})\partial\phi - b\psi\psi^*\psi^{*'}\right]e^{i\epsilon\phi_2}\Big]\\ \frac{1}{2}&\left[\psi^*\partial\phi e^{i\epsilon\phi_2}\right]' + \frac{1}{2}\psi^*\partial\phi e^{i\epsilon\phi_2}\partial\Big]\,\delta(x-y)\end{aligned} \qquad (5.10)$$

we find the free field representation of the $\tilde{J}_{5/2}$ generator,

$$\tilde{J}_{5/2} = [\psi^{*'}\partial\phi + \psi^*(a\partial\phi + b\partial\bar{\phi})\partial\phi - b\psi\psi^*\psi^{*'}]e^{i\epsilon\phi_2} \qquad (5.11)$$

where, a, b and ϵ are real parameters. The same result will be obtained by replacing $\tilde{W}_2$ by W_2.

Apart from the nonlinearity of the representation, a significant departure from the linear representation is the presence of terms with the fermionic field ψ and the bosonic field $\bar{\phi}$. In fact, the presence of both kinds of fermions, ψ and ψ^* is inevitable to reproduce the nonzero algebras consistently in the even fermion sector. This is in constrast to the linear super W_∞ algebra where the even fermion sector was constructed using bilinears of the fields ψ^* and ϕ [60]. The procedure outlined above can be used to find the next higher spin generator $\tilde{J}_{7/2}$ from the Poisson bracket between $\tilde{W}_2$ and $\tilde{J}_{5/2}$. This yields

$$\tilde{J}_{7/2} = \frac{5}{4}(\psi^*\partial\phi e^{i\epsilon\phi_2})'' - \psi^{*'}(\partial\phi e^{i\epsilon\phi_2})' + b\psi\psi^*\psi^{*'}(e^{i\epsilon\phi_2})'$$

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$$+\psi^{*'}(a\partial\phi + b\partial\bar{\phi})\partial\phi e^{i\epsilon\phi_2} - \psi^*(a\partial\phi + b\partial\bar{\phi})(\partial\phi e^{i\epsilon\phi_2})'$$
$$+\psi^*(a\partial\phi + b\partial\bar{\phi})^2\partial\phi e^{i\epsilon\phi_2} + b\psi'\psi^*\psi^{*'}e^{i\epsilon\phi_2}$$
$$+2b^2\psi\psi^*\psi^{*'}\partial\bar{\phi}e^{i\epsilon\phi_2} + 3ab\psi\psi^*\psi^{*'}\partial\phi e^{i\epsilon\phi_2} \qquad (5.12)$$

It is important to note that in order to reproduce the even fermion algebras (4.22), the consistency condition demands that all the parameters are not independent, but are related by

$$\frac{b\epsilon}{\sqrt{2}} - \frac{a\epsilon}{\sqrt{2}} - ab = 1 \qquad (5.13)$$

Evidently, two more parameters still remain arbitrary, which cannot be fixed at the classical level. The quantization of the algebra may fix the arbitrary parameters through its central charge. The relation (5.13) dictates that all the parameters a, b and ϵ cannot be set to be zero simultaneously. This implies the non-linear free field representation cannot be reduced to the linear one [60] trivially, which is a crucial observation and will also be seen in the odd fermion sector.

The similar Poisson bracket structures (4.22,4.26) of the odd fermions $J_{n+\frac{3}{2}}$ and the even fermions $\tilde{J}_{n+\frac{3}{2}}$, however, indicate the similar free field representations of the generators for the odd fermions. We give below the representations of the odd fermions upto spin $\frac{7}{2}$. The other higher spin generators can be constructed in a similar fashion as in the even fermion case. The representations of the odd fermions may given as

$$J_{3/2} = -\psi\partial\bar{\phi}e^{-i\epsilon\phi_2} \qquad (5.14)$$
$$J_{5/2} = [-\psi'\partial\bar{\phi} + \psi(a\partial\phi + b\partial\bar{\phi})\partial\bar{\phi} + a\psi\psi^*\psi']e^{-i\epsilon\phi_2} \qquad (5.15)$$
$$\begin{aligned} J_{7/2} = & -\frac{5}{4}(\psi\partial\bar{\phi}e^{-i\epsilon\phi_2})'' + \psi'(\partial\bar{\phi}e^{-i\epsilon\phi_2})' - a\psi\psi^*\psi'(e^{-i\epsilon\phi_2})' \\ & -\psi'(a\partial\phi + b\partial\bar{\phi})\partial\bar{\phi}e^{-i\epsilon\phi_2} - \psi(a\partial\phi + b\partial\bar{\phi})(\partial\bar{\phi}e^{-i\epsilon\phi_2})' \\ & -\psi(a\partial\phi + b\partial\bar{\phi})^2\partial\bar{\phi}e^{-i\epsilon\phi_2} - a\psi\psi^{*'}\psi'e^{-i\epsilon\phi_2} \\ & +2a^2\psi\psi^*\psi'\partial\phi e^{-i\epsilon\phi_2} - 3ab\psi\psi^*\psi'\partial\bar{\phi}e^{-i\epsilon\phi_2} \end{aligned} \qquad (5.16)$$

5.3 Freefield Representation of Bosons

In the bosonic sector, the free field representations become more involved. To be precise, in the linear representation [60], the $W_{1+\infty}$ algebra was realized in terms of the bilinears of a free fermion field and its conjugate whereas the W_∞ algebra was realized from the bilinears of a free complex scalar field and its conjugate. This had the advantage of automatically ensuring the direct sum basis of the generators of these algebras. But such a simple representation cannot be considered in the present case as it leads to an inconsistency in the fermionic sector. Thus a significant change in the free field representations in the bosonic sector is required so that the representation is consistent in all sectors.

For the odd bosons we have the following consistent representation. The linear part of the representation may be written in terms of the fermion bilinears. Thus the lowest one is identical to that of the linear representation, namely

$$\tilde{W}_1 = -\psi^* \psi \tag{5.17}$$

On the other hand in spin 2, we have trilinear terms,

$$\tilde{W}_2 = -\frac{1}{2}(\psi^{*'}\psi - \psi^*\psi') - \psi^*\psi(a\partial\phi + b\partial\bar{\phi}) \tag{5.18}$$

where a and b are the same parameters, already introduced in the fermionic sectors. Note that linear representation [60] consisted of only the first two terms which involved the free fermion field and its conjugate. The Poisson bracket of $\tilde{W}_2$ with itself is found employing the fundamental bracket (5.3,5.4) to be

$$\begin{aligned}\{\tilde{W}_2, \tilde{W}_2\} = &\left[2\left[-\frac{1}{2}(\psi^{*'}\psi - \psi^*\psi') - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})\right]\partial\right. \\ &\left.+\left[-\frac{1}{2}(\psi^{*'}\psi - \psi^*\psi') - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})\right]'\right]\delta(x-y)\end{aligned} \tag{5.19}$$

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thus correctly reproducing the Poisson bracket of the $\tilde{W}_2$ generator in (4.4). The term $\psi^*\psi(a\partial\phi + b\partial\bar{\phi})$ in (5.18) is the most general form of the spin 2 generator, but involving complex boson fields. From spin three onwards the representation becomes complicated having more and more nonlinear combinations of the free fields. We will therefore follow a different strategy from spin three onwards.

Following a similar procedure as in the odd boson case, we can obtain the spin 2 generator of the other sector. The most general form of the W_2 generator is

$$W_2 = -\partial\phi\partial\bar{\phi} - \psi^*\psi(a\partial\phi + b\partial\bar{\phi}) \tag{5.20}$$

For W_2 also we find that

$$\begin{aligned} \{W_2, W_2\} = \Big[&-2\left[-\partial\phi\partial\bar{\phi} - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})\right]\partial \\ &- \left[-\partial\phi\partial\bar{\phi} - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})\right]'\Big]\delta(x-y) \end{aligned} \tag{5.21}$$

i.e. the Poisson bracket structure of the W_2 given in (4.13) is correctly maintained and it also commutes with the odd boson generator $\tilde{W}_2$. Explicitly

$$\{\tilde{W}_2, W_2\} = 0 \tag{5.22}$$

Thus for both the spin 2 generators the last term turns out to be trilinear and more so this is the only possible term that exists at the spin 2 level being a mixture of bosonic and fermionic fields. We also emphasize that since the difference $W_2 - \tilde{W}_2$ is the the energy momentum tensor T, we find that the Virasoro algebra can be obtained from (5.19,5.21,5.22) and is therefore realizable in the basis (5.18,5.20).

For higher spin generators, however, the representations of both types of the bosonic generators may be obtained from the leading order terms of the Poisson brackets, $\{\tilde{J}_{n+3/2}, J_{3/2}\}$, the $\tilde{J}_{n+3/2}$ being given in

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(4.17). This will immediately follow from the Poisson bracket relation (A.7),

$$\{u^f_{2n}, v^f_{-1}\} = \left[(u^b_{2n+1} - v^b_{2n}) - \sum_{m=0}^{n} \begin{pmatrix} n+1 \\ m+1 \end{pmatrix} (-1)^m u^b_{2n-2m-1} \partial^{m+1}\right] \delta(x-y) \tag{5.23}$$

The leading order terms in (5.23) evidently will be a combination of $(\tilde{W}_{j+2} - W_{j+2})$ term and the suitable combinations of lower order spin terms. The representation of W_{j+2} generator and consequently the $\tilde{W}_{j+2}$ generator may be obtained explicitly by exploiting the commuting property of W and $\tilde{W}$ generators and from the Poisson bracket relation $\{W_{j+2}, W_2\}$. The consistency of these representations may be checked by comparing the algebra among the other bosonic generators. For example, the free field representations $\tilde{W}_3$ and W_3 generators may be obtained as follows. The Poisson bracket $\{\tilde{J}_{5/2}, J_{3/2}\}$ using the free field representations of the fermion generators $\tilde{J}_{5/2}$ and $J_{3/2}$ (5.11,5.14) is found to be

$$\begin{aligned}\{\tilde{J}_{5/2}, J_{3/2}\} \;=\; & \Big[-\partial\phi\partial^2\bar{\phi} + \frac{\epsilon}{\sqrt{2}}\partial\phi\partial\bar{\phi}(\partial\phi - \partial\bar{\phi}) - \partial\phi\partial\bar{\phi}(a\partial\phi + b\partial\bar{\phi}) \\ & -\psi^{*'}\psi' - 2a\psi^*\psi'\partial\phi - b\psi^*\psi'\partial\bar{\phi} + \psi^{*'}\psi\partial\bar{\phi} \\ & +a\frac{\epsilon}{\sqrt{2}}\psi^*\psi(\partial\phi)^2 - b\frac{\epsilon}{\sqrt{2}}\psi^*\psi(\partial\bar{\phi})^2 - b\psi^*\psi\partial^2\bar{\phi} \\ & - \left(\partial\phi\partial\bar{\phi} + \psi^{*'}\psi + 2\psi^*\psi(a\partial\phi + b\partial\bar{\phi})\right)\partial\Big]\,\delta(x-y)\end{aligned} \tag{5.24}$$

comparing the same with the Poisson bracket of generators $\tilde{J}_{5/2}$ and $J_{3/2}$

$$\begin{aligned}\{\tilde{J}_{5/2}, J_{3/2}\} = & \Big[-\tilde{W}_3 + W_3 + \frac{1}{2}W_2' + \tilde{W}_2\tilde{W}_1 - 2W_2\tilde{W}_1 \\ & -\frac{1}{2}\tilde{W}_1'\tilde{W}_1 + \frac{1}{6}\tilde{W}_1'' + \left(\tilde{W}_2 + W_2 + \frac{1}{2}\tilde{W}_1' - \tilde{W}_1^2\right)\partial\Big]\,\delta(x-y)\end{aligned} \tag{5.25}$$

allows us to determine the free field representation of $W_3 - \tilde{W}_3$ from the leading order term since the free field representation of the non leading order terms are already known. Explicitly,

$$\begin{aligned}
W_3 - \tilde{W}_3 = & \frac{1}{2}[\partial^2\phi\partial\bar{\phi} - \partial\phi\partial^2\bar{\phi}] + (-a + \frac{\epsilon}{\sqrt{2}})(\partial\phi)^2\partial\bar{\phi} - (b + \frac{\epsilon}{\sqrt{2}})\partial\phi(\partial\bar{\phi})^2 \\
& + \frac{1}{2}\psi^*\psi(a\partial^2\phi - b\partial^2\bar{\phi}) - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})^2 + 2\partial\phi\partial\bar{\phi}\psi^*\psi \\
& + \frac{\epsilon}{\sqrt{2}}\psi^*\psi[a(\partial\phi)^2 - b(\partial\bar{\phi})^2] - \frac{1}{2}(\psi^{*'}\psi + \psi^*\psi')(a\partial\phi - b\partial\bar{\phi}) \\
& + \frac{1}{6}(\psi^{*''}\psi - 4\psi^{*'}\psi' + \psi^*\psi'') + \psi^*\psi(a\partial\phi + b\partial\bar{\phi})^2 \\
& + (\psi^{*'}\psi - \psi^*\psi')(a\partial\phi + b\partial\bar{\phi})
\end{aligned} \tag{5.26}$$

Next, we determine the Poisson bracket between $W_3 - \tilde{W}_3$ and W_2. Since the odd and even boson generators commute (4.12), this operation allows one to find W_3 from the Poisson bracket

$$\left\{W_3 - \tilde{W}_3, W_2\right\} = \{W_3, W_2\} = [-3W_3\partial - W_3'] \, \delta(x-y) \tag{5.27}$$

and hence $\tilde{W}_3$. This procedure eventually leads to the following explicit form of the free field representation of the spin 3 generator W_3

$$\begin{aligned}
W_3 = & \frac{1}{2}[\partial^2\phi\partial\bar{\phi} - \partial\phi\partial^2\bar{\phi}] + (-a + \frac{\epsilon}{\sqrt{2}})(\partial\phi)^2\partial\bar{\phi} - (b + \frac{\epsilon}{\sqrt{2}})\partial\phi(\partial\bar{\phi})^2 \\
& + \frac{1}{2}\psi^*\psi(a\partial^2\phi - b\partial^2\bar{\phi}) - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})^2 + 2\partial\phi\partial\bar{\phi}\psi^*\psi \\
& + \frac{\epsilon}{\sqrt{2}}\psi^*\psi[a(\partial\phi)^2 - b(\partial\bar{\phi})^2] - \frac{1}{2}(\psi^{*'}\psi + \psi^*\psi')(a\partial\phi - b\partial\bar{\phi})
\end{aligned} \tag{5.28}$$

Cosequently the representation for the spin 3 generator in the odd boson sector, namely $\tilde{W}_3$ is obtained from (5.26,5.28) to be

$$\begin{aligned}
\tilde{W}_3 = & -\frac{1}{6}(\psi^{*''}\psi - 4\psi^{*'}\psi' + \psi^*\psi'') - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})^2 \\
& -(\psi^{*'}\psi - \psi^*\psi')(a\partial\phi + b\partial\bar{\phi})
\end{aligned} \tag{5.29}$$

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Notice that the spin 3 generators acquire a complex structure and possess terms quadrilinear in free fields. In a similar manner both the spin four generators may be obtained from the Poisson bracket $\{\tilde{J}_{7/2}, J_{3/2}\}$. This constitutes an algorithmic procedure by means of which free field representations of the higher spin generators may be constructed. An interesting consequence of the free field representations considered here is that in the bosonic limit the generators in the even boson sector reduce to that of the bosonic super $\hat{W}_\infty$ algebra thus lending weight to the possibility that the second Hamiltonian structure of the $N = 2$ nonstandard flow KP is truly a supersymmetric universal algebra.

5.4 Conclusion

Thus we see that free field representation of the nonlinear super algebra is not a trivial extension of the linear one [60]. This is essentially because the relation (5.13) dictates the both the parameters a and b cannot be set to be zero simultaneously, making the representation essentially nonlinear. The presence of an admixture of the bosonic and fermionic terms, on the other hand, makes the representation of bosonic sector nontrivial, unlike the linear case. Importantly, even with the presence of the fermionic and bosonic fields together, we observe that the odd bosons commute with the even bosons. All the higher spin generators may be constructed from the algebra amongst the even and odd fermionic generators (5.23) and consistency of these representations may be checked by comparing the algebra among the bosonic generators. But the explicit forms of the higher spin generators in terms of the free fields become more and more complicated as observed from the spin 3 generators (5.29,5.28). However, the strategy is quite obvious.

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Chapter 6

Bilinearization

6.1 Introduction to the Hirota Formalism

The integrability of the $N = 2$ nonstandard flow KP equation has been established from various viewpoints among them being the existence of the biHamiltonian structures which was shown in chapter 3 and also from the existence of the Lax pair. Another aspect of integrability of a nonlinear system is the existence of nondispersing wavelike solutions called solitons. In fact, the existence of soliton solutions is an indispensible criterion for determining the integrability of a nonlinear evolution equation. Soliton solutions can be generated by various methods such as the inverse scattering transform, the Backlund transformation, the trace method, and other methods of varying applicability. These techniques have also been extended to include supersymmetric equations in their purview [11, 13, 14, 15]. However, a powerful and general method of generating soliton solutions in bosonic nonlinear differential equations, namely the Hirota bilinear method [41] has been applied to supersymmetric extensions of only a few nonlinear systems; these have been mainly $N = 1$ supersymmetric equations [32, 33, 35, 37, 38]. But the bilinear method for $N = 2$ supersymmetric equations are yet to be

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studied.

The direct method for constructing multisoliton solutions of nonlinear evolution equations was introduced by Hirota in 1971 [41]. This is based on a transformation of the dependent variable(s) so that in these new variables the multisoliton solutions appear in a simple form. While the inverse scattering method is more powerful as it can handle general initial condition, the direct method is easier to use as it is algebraic, rather than analytic. The effectiveness of this method, also called the Hirota method, was demonstrated when soliton solutions of a number of nonlinear differential equations were obtained using this technique (see for example [80, 81, 82, 83, 84, 85, 86, 87, 88, 89]). It was later shown that the transformed variables, called τ functions [90, 91, 92] possess nice properties as they play the role of the effective action in field theories. The Hirota method essentially consists of casting the equation in a form that involves the bilinears of the τ functions. This is called the bilinear form of the equation and procedure is given the name bilinearization and requires the introduction of a novel differential operator. The main difficulty in the Hirota method is that the bilinearization process is not algorithmic and this problem is made more difficult as one cannot predict beforehand how many dependent variables will be required for bilinearization. It is important to note that bilinearization is intimately related to the notion of integrability.

As an example of the bilinearization process in the Hirota formalism we consider the KdV equation

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0 \tag{6.1}$$

In order to bilinearize (6.1) the required transformation is

$$u = 2\partial_x^2 \log \tau \tag{6.2}$$

Substituting (6.2) in (6.1), and after one integration over x, we find

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that

$$2\partial_x\partial_t \log\tau + 2\partial_x^4 \log\tau + 12\left(\partial_x^2 \log\tau\right)^2 = 0 \qquad (6.3)$$

Using the Hirota operator $\mathbf{D}$ [41] defined by

$$\mathbf{D}_x^n f.g = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_1=x_2=x} \qquad (6.4)$$

the transformed KdV equation (6.3) can be cast in the bilinear form

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tau.\tau) = 0 \qquad (6.5)$$

The Hirota operator acts on a product of two functions like the Leibnitz rule, except for a crucial sign difference. From (6.4) the Leibnitz rule for the Hirota operator follows as

$$\mathbf{D}_x^n f.g = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (-1)^i \partial_x^{n-i} f \partial_x^i g \qquad (6.6)$$

Note that the choice of the transformation (6.2) is a non algorithmic process and is a mixture of guesswork and intuition.

An important property of equations in Hirota's bilinear form is their gauge invariance. It can be shown that for quadratic expression homogeneous in the derivatives, *i.e.* of the form $\sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} c_i \partial_x^{n-i} f \partial_x^i g$, the requirement of gauge invariance under $f \to e^{kx} f$, $g \to e^{kx} g$ implies that $c_i = (-1)^i$ [93]. Any equation that can be written in the bilinear form possesses two soliton solutions. The situation for three and higher soliton solutions is more complicated. From a study of the integrability of nonlinear evolution equations via the Painleve analysis, it appears that the existence of three soliton solution goes hand in hand with the Painleve property thereby indicating that the three soliton solution implies complete integrability [42]. This is also shown by the example of the bilinear equation given in [85], namely

$$(k\mathbf{D}_t^2 + \mathbf{D}_x^3\mathbf{D}_t + \mathbf{D}_x^6)\tau.\tau = 0 \qquad (6.7)$$

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which passes the Painleve test for only $k = -\frac{1}{5}$ and possesses three soliton solution only for that value.

The extension of the Hirota method to supersymmetric systems was provided by McArthur and Yung [32] who bilinearized the $N = 1$ KdV equation of Manin-Radul-Mathieu [8, 9] and thus extended the Hirota method to the supersymmetric regime. Later on, N soliton solutions of this equation were also constructed by the super Hirota method [33]. Several other equations with $N = 1$ supersymmetry, such as supersymmetric Sawada-Kotera-Ramani, the supersymmetric Burgers equation have been bilinearized with the super Hirota method [34]. The bilinearization is carried out in manner similar to the bosonic case, but by using the supersymmetric extension of the Hirota operator [32] defined by

$$\mathbf{S}\mathrm{D}_x^n f.g = (D_{\theta_1} - D_{\theta_2})(\partial_{x_1} - \partial_{x_2})^n f(x_1, \theta_1) g(x_2, \theta_2)|_{\substack{x_1 = x_2 = x \\ \theta_1 = \theta_2 = \theta}} \tag{6.8}$$

where D is the supersymmetric derivative (2.4) and θ is the Grassmann odd coordinate. The super Hirota operator $\mathbf{S}$ is super gauge invariant so that

$$\mathbf{S}\mathrm{D}_x^n (e^{\Theta} f.e^{\Theta} g) = e^{2\Theta} \mathbf{S}\mathrm{D}_x^n f.g \tag{6.9}$$

where $\Theta = kx + \zeta\theta$, ζ being a Grassmann odd parameter. Following McArthur and Yung [32], we will demonstrate the bilinearization of the $N = 1$ supersymmetric KdV equation

$$\partial_t \phi + D^6 \phi + 3D^2(\phi D\phi) = 0 \tag{6.10}$$

where ϕ is a spin 3/2 superfield. The superfield can be written in terms of a spin 0 super τ function through the transformation

$$\phi = 2D^3 \log \tau \tag{6.11}$$

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Substituting (6.11) in (6.10) we obtain the $N = 1$ super KdV in the form

$$2D\partial_t \log\tau + 3[(2D^3\log\tau)(2\partial_x^2\log\tau)] + 2D^7\log\tau = 0 \qquad (6.12)$$

Using the super Hirota operator defined in (6.8), (6.12) can be cast in the bilinear form

$$(\mathbf{SD}_t + \mathbf{SD}_x^3)\tau.\tau = 0 \qquad (6.13)$$

In this chapter, the extension of the Hirota formalism in $N = 1$ superspace will be utilised to bilinearize several supersymmetric equations. We will first consider the $N = 2$ KdV equation in $N = 1$ superspace [36]. In addition, several supersymmetric equations with $N = 1$ supersymmetry, namely the $N = 1$ mKdV equation, $N = 1$ mKdV B equation and the $N = 1$ mKP, which have not been studied earlier, will also be bilinearized [67, 68].

6.2 The $N = 2$ KdV Equation

A supersymmetric system of special interest is the $N = 2$ KdV equation obtained from the non-standard superLax representation by Inami and Kanno [61]. This system is essentially the $N = 2$ superspace KdV equation proposed by Labelle, Laberge, Mathieu and Walton [69, 70, 71],

$$\partial_t\Omega = -\partial_x^3\Omega + 3\partial_x\left(\Omega D_1 D_2\Omega\right) + \frac{\alpha - 1}{2}\partial_x\left(D_1 D_2\Omega^2\right) + 3\alpha\Omega^2\partial_x\Omega \quad (6.14)$$

with the value of the free parameter $\alpha = -2$, and Ω is spin 1 superfield in $N = 2$ superspace. It was later shown that this system could be obtained from the reduction of a $N = 2$ superKP hierarchy [63] arising out of the non-standard flow equation. In $N = 1$ superspace the $N = 2$

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KdV equation [69, 70, 71, 61] has the explicit form

$$\partial_t u_{-1} + u_{-1}^{[6]} + 3\left(u_{-1}^{[1]} u_0\right)^{[2]} - \frac{1}{2}\left(u_{-1}^3\right)^{[2]} = 0 \tag{6.15}$$

$$\partial_t u_0 + u_0^{[6]} - 3\left(u_0 u_0^{[1]}\right)^{[2]} - \frac{3}{2}\left(u_0 u_{-1}^2\right)^{[2]} + 3\left(u_0 u_{-1}^{[2]}\right)^{[2]} = 0 \tag{6.16}$$

In (6.15,6.16) u_{-1} and u_0 are superfields of conformal spin 1 and 3/2 respectively. It is clear from the form of the equations that they are two coupled $N = 1$ supersymmetric equations which in two different limits, namely $u_{-1} = 0$ and $u_0 = 0$ reduce to the $N = 1$ KdV equation [8, 9] and the $N = 1$ mKdV B equation respectively. Interestingly, in yet another limit, namely $u_0 = Du_{-1}$ both the equations (6.15,6.16) acquire an identical form and ultimately reduce to the $N = 1$ mKdV B equation. All these reductions are bilinearizable and N soliton solutions exist for each of them. On the other hand, the bosonic limit of (6.15,6.16) gives rise to a set of coupled equations of the form

$$\partial_t u_{-1}^b + \partial^3 u_{-1}^b - \frac{1}{2}\partial u_{-1}^{b^3} = 0 \tag{6.17}$$

and

$$\partial_t u_0^b + \partial^3 u_0^b - 3\partial u_0^{b^2} - \frac{3}{2}\partial\left(u_0^b u_{-1}^{b^2}\right) + 3\partial\left[u_0^b(\partial u_{-1}^b)\right] = 0 \tag{6.18}$$

where the super fields are written in the component forms as

$$u_{-1} = u_{-1}^b + \theta u_{-1}^f \tag{6.19}$$

and

$$u_0 = u_0^f + \theta u_0^b \tag{6.20}$$

The first equation, (6.17) is nothing but the mKdV equation, whereas the second one, (6.18) is a coupled one and reduces to the KdV equation if u_{-1}^b is set to zero. These bosonic equations (6.17,6.18) can be shown to be bilinearizable and possess N soliton solutions.

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In order to cast the supersymmetric equations (6.15,6.16) in the bilinear form, we transform the superfields as

$$u_{-1} = 2D^2 \log \frac{\tau_1}{\tau_2} \tag{6.21}$$

for u_{-1} and

$$u_0 = 2D^3 \log \tau_1 \tag{6.22}$$

for u_0. τ_1 and τ_2 are independent bosonic superfields. Notice that in (6.22), u_0 depends only the superfield τ_1, whereas the other superfield u_{-1} is a function of both τ_1 and τ_2. Substituting for u_{-1} and u_0 in (6.15), we have

$$\begin{aligned} &2\partial_t D^2 \log \frac{\tau_1}{\tau_2} + 2D^8 \log \frac{\tau_1}{\tau_2} + 3D^2 \left[(2D^2 \log \frac{\tau_1}{\tau_2})(2D^2 \log \tau_1)\right] \\ &-\frac{1}{2}D^2 \left[(D^2 \log \frac{\tau_1}{\tau_2})^3\right] = 0 \end{aligned} \tag{6.23}$$

After operating on (6.23) by D^{-1}, we find that (6.23) can be rewritten in the form

$$\begin{aligned} &2\partial_t D \log \tau_1 + 2D^7 \log \tau_1 + 12D^4 \log \tau_1 D^3 \log \tau_1 \\ &- \left(2\partial_t D \log \tau_2 + 2D^7 \log \tau_2 + 12D^4 \log \tau_2 D^3 \log \tau_2\right) \\ &12 \left[D^4 \log(\tau_1\tau_2) + \left(D^2 \log \frac{\tau_1}{\tau_2}\right)\right] \left[D^3 \log \tau_1 - D^3 \log \tau_2\right] = 0 \end{aligned} \tag{6.24}$$

(6.24) can now be cast in a form involving only bilinears. Explicitly,

$$\begin{aligned} &\frac{(\mathrm{SD}_t + \mathrm{SD}_x^3)(\tau_1.\tau_1)}{\tau_1^2} - \frac{(\mathrm{SD}_t + \mathrm{SD}_x^3)(\tau_2.\tau_2)}{\tau_2^2} \\ &-6\frac{\mathrm{D}_x^2(\tau_1.\tau_2)}{\tau_1\tau_2}\left[\frac{\mathrm{SD}_x(\tau_1.\tau_1)}{\tau_1^2} - \frac{\mathrm{SD}_x(\tau_2.\tau_2)}{\tau_2^2}\right] = 0 \end{aligned} \tag{6.25}$$

A similar procedure applied to the (6.16) yields

$$\frac{(\mathrm{SD}_t + \mathrm{SD}_x^3)(\tau_1.\tau_1)}{\tau_1^2} - 6\frac{\mathrm{D}_x^2(\tau_1.\tau_2)}{\tau_1\tau_2}\frac{\mathrm{SD}_x(\tau_1.\tau_1)}{\tau_1^2} = 0 \tag{6.26}$$

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Thus in terms of the supersymmetric Hirota derivative S (6.8), the $N = 2$ super KdV equations can be cast in the following bilinear form:

$$(\mathbf{SD}_t + \mathbf{SD}_x^3)(\tau_1.\tau_1) = 0 \qquad (6.27)$$

$$(\mathbf{SD}_t + \mathbf{SD}_x^3)(\tau_2.\tau_2) = 0 \qquad (6.28)$$

$$\mathbf{D}_x^2(\tau_1.\tau_2) = 0 \qquad (6.29)$$

6.3 The $N = 1$ mKdV Equations

In this section, we consider the $N = 1$ mKdV equation [9, 94, 95, 96] involving spin 1/2 super field $\psi(x, \theta, t)$,

$$\partial_t\psi + D^6\psi - 3\psi D^3\psi D\psi - 3(D\psi)^2 D^2\psi = 0 \qquad (6.30)$$

and $N = 1$ mKdV B equation having spin 1 super field $\Psi(x, \theta, t)$

$$\partial_t\Psi + D^6\Psi - 2D^2\Psi^3 = 0 \qquad (6.31)$$

and show that these equations can be bilinearised following the supersymmetric extension of the Hirota method. It will be seen that these two equations have different bilinear forms.

The $N = 1$ mKdV Equation

(6.30) is related to the $N = 1$ KdV (6.10) equation through the super Miura transformation [9, 94]

$$\phi = D^2\psi + \psi D\psi \qquad (6.32)$$

The superfield ψ is fermionic having spin 1/2. The $N = 1$ mKdV also follows as an $N = 1$ reduction of the $N = 2$ mKdV system [61]

$$\partial_t\psi_1 = -D\left[D^5\psi_1 + 3\psi_1 D^2\psi_2 D\psi_2 - \frac{1}{2}(D\psi_1)^3 - \frac{3}{2}D\psi_1(D\psi_2)^2\right] \qquad (6.33)$$

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$$\partial_t \psi_2 = -D\left[D^5\psi_2 + 3\psi_2 D^2\psi_1 D\psi_1 - \frac{1}{2}(D\psi_2)^3 - \frac{3}{2}D\psi_2(D\psi_1)^2\right] \tag{6.34}$$

which is connected to the $N = 2$ KdV (6.15,6.16) through the Miura transformation

$$u_{-1} = D(\psi_1 + \psi_2) - \psi_1\psi_2 \qquad u_0 = D^2\psi_2 - \psi_2 D\psi_1 \tag{6.35}$$

By imposing the constraint $\psi_1 = -\psi_2 = \psi$, (6.33,6.34) immediately reduce to (6.30).

In order to cast the $N = 1$ mKdV equation in the bilinear form the superfield ψ is written in terms of the τ functions as

$$\psi = D\log\frac{\tau_1}{\tau_2} \tag{6.36}$$

In terms of the supersymmetric Hirota derivative $\mathbf{S}$ (6.8) the $N = 1$ mKdV equations can be cast in the following bilinear forms:

$$(\mathbf{SD}_t + \mathbf{SD}_x^3)(\tau_1.\tau_1) = 0 \tag{6.37}$$

$$(\mathbf{SD}_t + \mathbf{SD}_x^3)(\tau_2.\tau_2) = 0 \tag{6.38}$$

$$\mathbf{D}_x^2(\tau_1.\tau_2) = 0 \tag{6.39}$$

$$\mathbf{SD}_x(\tau_1.\tau_2) = 0 \tag{6.40}$$

In obtaining the bilinear equations above, use has been made of the identities

$$D^4\log(\tau_1\tau_2) + \left(D^2\log\frac{\tau_1}{\tau_2}\right)^2 = \frac{\mathbf{D}^2(\tau_1.\tau_2)}{\tau_1\tau_2} \tag{6.41}$$

and

$$D^3\log(\tau_1\tau_2) + D^2\log\frac{\tau_1}{\tau_2}D\log\frac{\tau_1}{\tau_2} = \frac{\mathbf{SD}(\tau_1.\tau_2)}{\tau_1\tau_2} \tag{6.42}$$

It is interesting to note that in spite of earlier claims that the $N = 1$ mKdV is not super bilinearizable [33, 34], the contrary seems to be the case as we have seen above.

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The $N = 1$ mKdV B equation

Just as the $N = 1$ mKdV equation (6.30) is related to the $N = 1$ KdV equation of Manin-Radul-Mathieu [8, 9] through super Miura transformation (6.32), the $N = 1$ mKdV B equation (6.31) is related to the $N = 1$ KdV B equation [97, 98]

$$\partial_t \Phi + D^6 \Phi + 6 D^2 \Phi D \Phi = 0 \tag{6.43}$$

$\Phi(x, \theta)$ being a spin 3/2 superfield, through a Miura tranformation,

$$D\Phi = D^2 \Psi - \Psi^2 \tag{6.44}$$

Note that the Miura transformation (6.44) is nonlocal. Interestingly, in the bosonic limit (6.31) also reduces to the modified KdV equation and is invariant under the supersymmetric transformation

$$\delta \Psi^b = \eta \Psi^f \quad ; \quad \delta \Psi^f = \eta \partial_x \Psi^b \tag{6.45}$$

As mentioned earlier, the equation (6.31) directly results from $N = 2$ KdV equations [61] in the limit $u_0 = 0$ by identifying $u_{-1} = \Psi$. Interestingly, in yet another limit, namely $u_0 = Du_{-1}$ both the equations (6.15,6.16) acquire the same form which is identical to the $N = 1$ mKdV B equation. One can also obtain the $N = 1$ mKdV B equation form the $N = 2$ mKdV B proposed by Popowicz [99], namely

$$\partial_t U + \partial_x^3 U + 2 (\partial_x U)^3 - 6 (D_1 D_2 U)^2 \partial_x U = 0 \tag{6.46}$$

which reduces to the $N = 1$ mKdV B in the limit $U^f = 0$, where $U = U^b + \theta_2 U^f$, U^f and U^b being $N = 1$ superfields and under the redefinition $\Psi = \partial_x U^b$.

With the following dependent variable transformation

$$\Psi = D^2 \log \frac{\tau_1}{\tau_2} \tag{6.47}$$

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the $N = 1$ mKdV B can be cast in the bilinear equations

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tau_1.\tau_1) = 0 \qquad (6.48)$$

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tau_2.\tau_2) = 0 \qquad (6.49)$$

$$\mathbf{D}_x^2(\tau_1.\tau_2) = 0 \qquad (6.50)$$

The bilinearization for $N = 1$ mKdV B, however, is not unique. It possesses an alternate, but important set of bilinear forms

$$(\mathbf{D}_t + \mathbf{D}_x^3)(\tau_1.\tau_2) = 0 \qquad (6.51)$$

$$\mathbf{D}_x^2(\tau_1.\tau_2) = 0 \qquad (6.52)$$

The equations (6.51,6.52) become useful to show its connection with bosonic mKdV equation. Both the equivalent bilinear equations of the $N = 1$ mKdV B equation do not involve the supersymmetric Hirota operator leading different types of solitons solutions than those of $N = 1$ mKdV equation. This is essentially because the field involved, though supersymmetric, is a bosonic superfield.

6.4 The $N = 1$ mKP Equation

The evolution equations for the lowest spin fields u_{-1} and u_0 of the $N = 2$ KP hierarchy constitute the $N = 2$ KP equations (2.13,2.14) where the superfield u_{-1} is a spin 1 bosonic superfield while u_0 is a fermionic superfield of spin 3/2. Imposing the constraint $u_0 = Du_{-1}$ in the $N = 2$ KP equation both these equations reduce to a single equation, the $N = 1$ mKP equation and ultimately becomes

$$\partial_t u_{-1} + u_{-1}^{[6]} - \frac{1}{2}\left(u_{-1}^3\right)^{[2]} + 12\partial_y^2 u_{-1}^{[-2]} + 6u_{-1}^{[2]}\partial_y u_{-1}^{[-2]} = 0 \qquad (6.53)$$

with the scaling of $t = -4t$, $y = -\frac{1}{2}y$ and $u_{-1} = -\frac{1}{2}u_{-1}$. Consider the dependent variable transformation

$$u_{-1} = 2D^2 \log \frac{\tau_1}{\tau_2} \tag{6.54}$$

Substituting (6.54) in (6.53) leads to the following Hirota form for the $N = 1$ mKP equation

$$(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + 12\mathrm{D}_y^2)(\tau_1.\tau_1) = 0 \tag{6.55}$$

$$(\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + 12\mathrm{D}_y^2)(\tau_2.\tau_2) = 0 \tag{6.56}$$

$$(\mathrm{D}_x^2 - 2\mathrm{D}_y)(\tau_1.\tau_2) = 0 \tag{6.57}$$

In the case of the $N = 1$ mKP, as with the $N = 1$ mKdV B, the Hirota forms do not involve the super Hirota operator S and reduces to those for the $N = 1$ mKdV B equation if the fields become y independent.

6.5 Conclusion

The $N = 2$ supersymmetric KdV equation is thus bilinearizable in the Hirota formalism in $N = 1$ superspace. This system is intimately related to the $N = 2$ nonstandard flow KP hierarchy. Moreover three different $N = 1$ supersymmetric evolution equations, to which the Hirota formalism was not applied earlier, have also been bilinearized. Bilinearization automatically implies the existence of one and two soliton solutions. In the next chapter *i.e.* 7, we shall show the existence of one, two and higher soliton solutions of these supersymmetric nonlinear evolution equations.

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Chapter 7

Soliton Solutions

7.1 Introduction

The Hirota method can be used to construct multisoliton solutions by an algebraic procedure in contrast to other techniques which are are analytic. Most partial differential equations have travelling wave solutions but only in exceptional cases do they have solutions that where these travelling waves scatter elastically *i.e.* are true solitons. An important advantage of the Hirota method is that if the equation is bilinearizable, it will always have at least two soliton solutions. As an example, consider once again the KdV equation (6.1) and its form in terms of τ functions (6.3). Notice that $\tau = 1$ is trivially a solution; it corresponds to $u = 0$. If this solution is altered by adding the exponential of a linear term: $\tau = 1 + e^{\eta}$ with $\eta = kx + \omega t$, then we find that this is a solution of (6.3) provided $\omega = -k^3$ which is the KdV dispersion relation. A two soliton solution can be attempted by adding another term so that $\tau = 1 + e^{\eta_1} + e^{\eta_2}$ with $\eta_i = k_i x - k_i^3 t$ $(i = 1, 2)$. We find that this is not a solution as the terms with $e^{2\eta_i}$ cancel but the mixed terms $e^{\eta_1+\eta_2}$ do not. This term can be cancelled by adding a term $A_{12}e^{\eta_1+\eta_2}$ to the τ

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function with some constant A_{12}. Remarkably, the modified τ function

$$\tau_1 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} \tag{7.1}$$

is a solution of (6.3) provided

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \tag{7.2}$$

This procedure can be carried further so that N soliton solutions can be constructed. However, the existence of three and higher soliton solutions is not guaranteed as these give rise to certain nontrivial conditions which have to be satisfied by the one and two soliton parameters.

In this chapter, we shall construct the three soliton solutions of the equations bilinearized in chapter 6 and discuss the existence of N soliton solutions for these equations. It will be observed that for the $N = 1$ mKdV B and the $N = 1$ mKP equations the fermionic parameters play a trivial role at all levels of soliton solutions. On the other hand, for the $N = 2$ KdV and the $N = 1$ mKdV equations, constraints imposed on the fermionic parameters from the two soliton level onwards are crucial in demonstrating the the existence of higher soliton solutions.

7.2 One Soliton Solution

For the one soliton solution of the $N = 2$ KdV, the $N = 1$ mKdV and the $N = 1$ mKdV B equations, we use the the following forms of the τ functions:

$$\tau_1 = 1 + \alpha e^{\eta} \tag{7.3}$$

and

$$\tau_2 = 1 + \beta e^{\eta} \tag{7.4}$$

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α and β in (7.3,7.4) are nonzero bosonic parameters. The parameters α and β can be determined from the initial value conditions. In (7.3,7.4),

$$\eta = kx + \omega t + \zeta \theta \tag{7.5}$$

where k and ω are the bosonic parameters and ζ is the Grassmann odd parameter. We shall now consider each equation separately below.

The $N = 2$ KdV Equation

Substituting (7.3) and (7.4) back into third equation of the Hirota form of the $N = 2$ KdV equation (6.29), we find the non-trivial solutions exist provided

$$\beta = -\alpha. \tag{7.6}$$

The dispersion relation, however, follows from (6.27,6.28) as

$$\omega + k^3 = 0 \tag{7.7}$$

which is identical to that for the bosonic KdV or mKdV equations. In obtaining the dispersion relation the following property of the super Hirota operator has been used.

$$\mathbf{SD}^n \left(e^{\eta_1}.e^{\eta_2}\right) = (k_1 - k_2)^n \left[-(\zeta_1 - \zeta_2) + \theta\,(k_1 - k_2)\right] e^{\eta_1+\eta_2} \tag{7.8}$$

The fermionic parameter, ζ, however, remains arbitrary at the one soliton level.

The explicit forms of the the soliton solutions for the superfields u_{-1} and u_0 may be found by substituting the τ functions in (6.21) and (6.22) respectively. This yields

$$u_{-1} = -(2k)\mathrm{cosech}(\phi + \gamma_0) - \theta(2k\zeta)\cosh(\phi + \gamma_0)\mathrm{cosech}^2(\phi + \gamma_0) \tag{7.9}$$

and

$$u_0 = -\frac{1}{2}k\zeta\mathrm{sech}^2\frac{1}{2}(\phi + \gamma_0) + \theta\frac{k^2}{2}\mathrm{sech}^2\frac{1}{2}(\phi + \gamma_0) \tag{7.10}$$

where we have chosen $\phi = k\dot{x} - k^3t$ and $\alpha = -\beta = e^{\gamma_0}$, γ_0 being nonzero, real parameter. The bosonic and fermionic components of the superfields are easily found by comparing (7.9) and (7.10) with the component form of the superfields u_{-1} and u_0, (6.19) and (6.20). It is observed that the soliton solution of u_{-1}^b becomes isomorphic to the mKdV soliton and u_0^b to the KdV soliton as expected.

If we choose α to be negative in (7.3), the solutions for u_{-1} and u_0 become

$$u_{-1} = (2k)\text{cosech}(\phi + \gamma_0) + \theta(2k\zeta)\cosh(\phi + \gamma_0)\text{cosech}^2(\phi + \gamma_0) \quad (7.11)$$

and

$$u_0 = \frac{1}{2}k\zeta\text{cosech}^2\frac{1}{2}(\phi + \gamma_0) - \theta\frac{k^2}{2}\text{cosech}^2\frac{1}{2}(\phi + \gamma_0) \quad (7.12)$$

Interestingly, the solutions (7.9,7.10) and (7.11,7.12) are connected through the transformations

$$u_{-1} = -u_{-1} \quad (7.13)$$

and

$$u_0 = u_0 - Du_{-1} \quad (7.14)$$

On the other hand, at the equation level, the $N = 2$ KdV equation remain invariant with respect to the transformations (7.13,7.14). Evidently, $\alpha \to -\alpha$ is a symmetry of the equations of motion. At the τ functions level (6.21,6.22), in fact, this symmetry is manifested in the tranformation $\tau_1 \to \tau_2$ and vice versa and thus will be observed at the N soliton solutions.

The $N = 1$ mKdV and mKdV B Equations

For the $N = 1$ mKdV and the $N = 1$ mKdV B, we choose the same form for the τ functions *viz.* (7.3,7.4,7.5). Substituting (7.3) and (7.4)

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in the Hirota forms of the $N = 1$ mKdV equation (6.39) and (6.40) and that of the $N = 1$ mKdV B equation (6.50), we find that for nontrivial solutions of these two equations the constraint (7.6) has to be satisfied. Moreover, from (6.37) and (6.38) for the $N = 1$ mKdV and (6.48) and (6.49) for the $N = 1$ mKdV B, we find that the dispersion relation is identical to that of the $N = 2$ KdV *i.e.*, $\omega = -k^3$. Though the one soliton τ functions for the $N = 2$ KdV, the $N = 1$ mKdV and the $N = 1$ mKdV B are identical, the explicit one soliton solutions themselves are not so. This is clear from the one soliton solutions of the $N = 1$ mKdV and the $N = 1$ mKdV B, which are obtained by substituting (7.3) and (7.4) in (6.36) and (6.47) respectively. Thus for the $N = 1$ mKdV

$$\psi = \zeta \text{cosech}(\phi + \gamma_0) - \theta k \text{cosech}(\phi + \gamma_0) \tag{7.15}$$

and for the $N = 1$ mKdV B

$$\Psi = -k \text{cosech}(\phi + \gamma_0) - \theta(k\zeta)\cosh(\phi + \gamma_0)\text{cosech}^2(\phi + \gamma_0) \tag{7.16}$$

are the explicit one soliton solutions where, $\phi = kx - k^3 t$ and $\alpha = -\beta = e^{\gamma_0}$, γ_0 is a nonzero, real parameter. Notice that the bosonic component in (7.16) becomes identical with that of (7.15), as expected, since both the equations reduce to the mKdV equation in the bosonic limit. But the fermion components are quite different.

The $N = 1$ mKP Equation

For the $N = 1$ mKP, we choose (7.3) and (7.4) as the τ functions, but as it involves an additional space dimension, namely y, the phase η is now

$$\eta = k_x x + k_y y + \omega t + \zeta\theta \tag{7.17}$$

It is easily shown that plugging these back in first two equations, namely (6.55) and (6.56) of the Hirota form of the $N = 1$ mKP, the dispersion

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relation may be obtained as

$$\omega k_x + k_x^4 + 12k_y^2 = 0 \tag{7.18}$$

From the third equation of this set (6.57), k_y is related to k_x through

$$k_y = \frac{1}{2}\frac{(\alpha+\beta)}{(\alpha-\beta)}k_x^2 \tag{7.19}$$

This imposes the following constraint between α and β parameters like

$$\beta \neq \alpha \tag{7.20}$$

Notice that for the $N = 1$ mKP equation the condition on α and β parameters are less restrictive. However, if the further restriction $\beta = -\alpha$ is imposed on the parameters, evidently k_y in (7.19) becomes zero and as a consequence the one soliton solution τ function reduces to that for the $N = 1$ KdV B equation. This reduction, in fact, will be observed for all the N soliton solutions. The components of the one soliton solution of the $N = 1$ mKP equation are obtained by putting back (7.3) and (7.4) in (6.54) to get

$$u^b_{-1} = k_x \sinh\frac{1}{2}(\gamma_0 - \delta_0)\,\mathrm{sech}\frac{1}{2}(\phi + \gamma_0)\,\mathrm{sech}\frac{1}{2}(\phi+\delta_0) \tag{7.21}$$

$$\begin{aligned} u^f_{-1} = -\frac{1}{2}k_x^2\xi \sinh\frac{1}{2}(\gamma_0 - \delta_0)\,\mathrm{sech}\frac{1}{2}(\phi+\gamma_0)\,\mathrm{sech}\frac{1}{2}(\phi+\delta_0) \\ \times\left[\tanh(\phi+\gamma_0) + \tanh(\phi+\delta_0)\right] \end{aligned} \tag{7.22}$$

Here $\alpha = e^{\gamma_0}$ and $\beta = e^{\delta_0}$, γ_0 and δ_0 being nonzero, real parameters $\phi = k_x x + k_y y + \omega t$.

It is clear from the derivation of the one soliton solutions above that the fermionic parameter ζ is not significant; but in the next section, it will be seen that for the two soliton solution of the $N = 2$ KdV and the $N = 1$ mKdV equations the fermionic parameters play an essential role in proving their existence.

7.3 Two Soliton Solution

The two soliton solution of the $N = 2$ KdV, the $N = 1$ mKdV and the $N = 1$ mKdV B equations may be obtained by choosing the τ functions in the form

$$\tau_1 = 1 + \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_1 \alpha_2 A_{12} e^{\eta_1+\eta_2} \tag{7.23}$$

and

$$\tau_2 = 1 + \beta_1 e^{\eta_1} + \beta_2 e^{\eta_2} + \beta_1 \beta_2 B_{12} e^{\eta_1+\eta_2} \tag{7.24}$$

where

$$\eta_1 = k_1 x + \omega_1 t + \zeta_1 \theta \tag{7.25}$$

$$\eta_2 = k_2 x + \omega_2 t + \zeta_2 \theta \tag{7.26}$$

In (7.25,7.26) the parameters k_1, k_2, ω_1 and ω_2 are bosonic, while ζ_1 and ζ_2 are fermionic ones, as before.

<u>The $N = 2$ KdV Equation</u>

For the two soliton solutions of the $N = 2$ KdV equation, the third bilinear form (6.29) yields $\alpha_1 = -\beta_1$ and $\alpha_2 = -\beta_2$ in order that the solutions to be nontrivial. These are obtained by setting the coefficient of the first order terms e^{η_i} $(i = 1, 2)$ to zero. On the other hand from the coefficient of $e^{\eta_1+\eta_2}$ we find that the interaction term to be

$$A_{12} = B_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \tag{7.27}$$

which is the same as that for the bosonic KdV equation (7.2). Substituting the two soliton solutions (7.23) and (7.24) in the bilinear equations (6.27) and (6.28), on setting the coefficient of the linear terms to zero, we obtain the dispersion relations

$$\omega_1 + k_1^3 = 0 \tag{7.28}$$

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and

$$\omega_2 + k_2^3 = 0 \tag{7.29}$$

However in order that the coefficient of the second order term $e^{\eta_1+\eta_2}$ consistently vanish, in addition to (7.27) it is required that the following constraint be imposed on the fermionic parameters ζ_1 and ζ_2

$$k_1\zeta_2 = k_2\zeta_1 \tag{7.30}$$

The $N = 1$ mKdV and mKdV B Equations

For the $N = 1$ mKdV equation the choice of (7.23) and (7.24) as the two soliton τ functions leads to the same results as in the case of the $N = 2$ KdV, that is we obtain identical interaction term (7.27) from (6.39) and from (6.40) provided the condition (7.30) is satisfied. Moreover, the dispersion relations are the same as (7.29,7.30). For the $N = 1$ mKdV B equation the bilinear equations (6.48,6.49,6.50) ensure the nontrivial two soliton solutions invoking the set of identical results as in $N = 1$ mKdV equation but for the last condition (7.30). In particular, the conditions on the parameter α_i and β_i, the dispersion relations (7.28,7.29) and the interaction terms A_{12} and B_{12} (7.27) become identical in both the cases. However, in contrast to the $N = 1$ mKdV, the relation among the fermionic and bosonic parameters does not arise for the $N = 1$ mKdV B equation. This fact is reflected in the structure of the bilinear equations, which are expressed only in terms of the bosonic Hirota operator $\mathbf{D}$ and will be observed for all higher soliton solutions also.

The $N = 1$ mKP Equation

For the two soliton solution for the $N = 1$ mKP equation, the τ functions may be written once again as (7.23) and (7.24), with η_1 and

η_2 being now dependent on the y coordinate. The explicit forms of η_1 and η_2 are given by

$$\eta_1 = k_{1x}x + k_{1y}y + \omega_1 t + \zeta_1\theta \tag{7.31}$$

$$\eta_2 = k_{2x}x + k_{2y}y + \omega_2 t + \zeta_2\theta \tag{7.32}$$

For nontrivial soliton solution we find as for the one soliton case

$$\beta_i \neq \alpha_i \qquad i = 1, 2 \tag{7.33}$$

We will obtain two dispersion relations corresponding to the the parameters η_1 and η_2. The dispersion relations resulting from (6.55) and (6.56) are

$$\omega_i k_{ix} + k_{ix}^4 + 12k_{iy}^2 = 0, \qquad i = 1, 2 \tag{7.34}$$

The expressions for A_{12} and B_{12} may be obtained from (6.57) by equating the coefficient of $e^{\eta_1+\eta_2}$ to zero. It turns out that

$$A_{12} = B_{12} = -\frac{(\alpha_1\beta_2 + \alpha_2\beta_1)(k_{1x} - k_{2x})^2 - 2(\alpha_1\beta_2 - \alpha_2\beta_1)(k_{1y} - k_{2y})}{(\alpha_1\alpha_2 + \beta_1\beta_2)(k_{1x} + k_{2x})^2 - 2(\alpha_1\alpha_2 - \beta_1\beta_2)(k_{1y} + k_{2y})} \tag{7.35}$$

Quite clearly, (7.35) reduces to the analogous expression for KdV in the limit $\beta_i = -\alpha_i$ $(i = 1, 2)$.

7.4 N Soliton Solution

It is well known that the existence of one and two soliton solutions do not ensure the exact integrability by Hirota method. In fact, the method of obtaining one and two soliton solutions considered in the previous sections always leads to the nontrivial solutions if the equations of motion can be cast in the bilinear forms. This method determines the unknown parameters involved in the trial solutions upto two soliton solutions. It is thus necessary to show the existence of atleast three

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soliton solution, which checks the consistency of the solution. In this section we discuss the N soliton solutions and obtain the three soliton solutions in particular. The τ function for the N soliton for all four equations discussed earlier may be written as

$$\tau_1 = \sum_{\mu_i=0,1} exp\left(\sum_{i,j=1}^{N} \phi(i,j)\mu_i\mu_j + \sum_{i=1}^{N} \mu_i(\eta_i + \log \alpha_i)\right) \qquad (i < j) \tag{7.36}$$

and $A_{ij} = e^{\phi(i,j)}$. For τ_2 we replace α_i by β_i and A_{ij} by B_{ij}. It is important to note that in a recent work [37] a more general solution for $N = 1$ KdV is proposed by dressing the fermionic part of the soliton solutions through the interactions. But the presence of the bilinear forms (6.29) for the $N = 2$ KdV and (6.39) for the $N = 1$ mKdV equations with only bosonic Hirota derivatives ensures that the N soliton solution will be of the form (7.36). Consequently the dressing of the fermionic part will *not* be effective in the solutions for $N = 1$ mKdV and $N = 2$ KdV equations.

Thus the explicit forms of τ_1 and τ_2 for the three soliton solution become

$$\begin{aligned}\tau_1 &= 1 + \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_3 e^{\eta_3} + \alpha_1\alpha_2 A_{12} e^{\eta_1+\eta_2} + \alpha_1\alpha_3 A_{13} e^{\eta_1+\eta_3} \\ &+ \alpha_2\alpha_3 A_{23} e^{\eta_2+\eta_3} + \alpha_1\alpha_2\alpha_3 A_{12}A_{13}A_{23} e^{\eta_1+\eta_2+\eta_3}\end{aligned} \tag{7.37}$$

and

$$\begin{aligned}\tau_2 &= 1 + \beta_1 e^{\eta_1} + \beta_2 e^{\eta_2} + \beta_3 e^{\eta_3} + \beta_1\beta_2 B_{12} e^{\eta_1+\eta_2} + \beta_1\beta_3 B_{13} e^{\eta_1+\eta_3} \\ &+ \beta_2\beta_3 B_{23} e^{\eta_2+\eta_3} + \beta_1\beta_2\beta_3 B_{12}B_{13}B_{23} e^{\eta_1+\eta_2+\eta_3}\end{aligned} \tag{7.38}$$

where

$$\eta_i = k_{ix}x + \omega_i t + \zeta_i\theta \quad (i = 1, 2, 3) \tag{7.39}$$

for the $N = 2$ KdV, $N = 1$ mKdV and the $N = 1$ mKdV B equations, but for the $N = 1$ mKP, it is also a function of the y coordinate so that

$$\eta_i = k_{ix}x + k_{iy}y + \omega_i t + \zeta_i\theta \quad (i = 1, 2, 3) \tag{7.40}$$

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Notice that (7.37,7.38) do not contain any new unknown parameter; it is expressed in terms of the parameters of the two soliton solutions only. Three soliton solutions thus verifies the consistency of the parameters determined by one and two soliton solutions. Substitution of the three soliton solutions in the bilinear equations gives rise to a set of nontrivial relations among the parameters, which determine the consistency of the solutions. In the three soliton solution we find the conditions $\beta_i = -\alpha_i$ and the dispersion relations $\omega_i + k_i^3 = 0$ $(i = 1, 2, 3)$ hold good for the $N = 2$ KdV, the $N = 1$ mKdV and the $N = 1$ mKdV B equations. The interaction terms for the three equations become

$$A_{ij} = B_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \quad (i, j = 1, 2, 3; \quad i \neq j) \tag{7.41}$$

The $N = 2$ KdV and $N = 1$ mKdV equation however admits additional constraint as in the two soliton solution on the fermionic parameters:

$$k_i \zeta_j = k_j \zeta_i \quad (i, j = 1, 2, 3; \quad i \neq j) \tag{7.42}$$

which are not found for the $N = 1$ mKdV B equation. The constraints (7.42) are essential to ensure the coefficients of the terms $e^{\eta_1+\eta_2}$, $e^{\eta_1+\eta_3}$, $e^{\eta_2+\eta_3}$, $e^{2\eta_1+\eta_2+\eta_3}$, $e^{\eta_1+2\eta_2+\eta_3}$ and $e^{\eta_1+\eta_2+2\eta_3}$ to vanish. Apart from determing the unknown parameters, three soliton solutions provide a nontrivial relation among the parameters. This identity follows from the coefficient of the term $e^{\eta_1+\eta_2+\eta_3}$ in (6.27,6.28) or (6.37,6.38) to be zero and is given by

$$\begin{aligned}
&A_{12}A_{13}A_{23}[(\omega_1 + \omega_2 + \omega_3) + (k_1 + k_2 + k_3)^3] \\
&\times[-(\zeta_1 + \zeta_2 + \zeta_3) + \theta(k_1 + k_2 + k_3)] \\
&+A_{23}[(\omega_1 - \omega_2 - \omega_3) + (k_1 - k_2 - k_3)^3] \\
&\times[-(\zeta_1 - \zeta_2 - \zeta_3) + \theta(k_1 - k_2 - k_3)] \\
&+A_{13}[(\omega_2 - \omega_1 - \omega_3) + (k_2 - k_1 - k_3)^3]
\end{aligned}$$

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$$\begin{aligned}
&\times[-(\zeta_2 - \zeta_1 - \zeta_3) + 0(k_2 - k_1 - k_3)] \\
&+A_{12}[(\omega_3 - \omega_1 - \omega_2) + (k_3 - k_1 - k_2)^3] \\
&\times[-(\zeta_3 - \zeta_1 - \zeta_2) + 0(k_3 - k_1 - k_2)] = 0
\end{aligned} \tag{7.43}$$

and interestingly is satisfied by using the dispersion relations and the parameters in (7.41,7.42). In fact, (7.43) makes the three soliton solution nontrivial. For the $N = 1$ mKdV B, however the nontrivial coefficient *i.e.* the coefficient of $e^{\eta_1+\eta_2+\eta_3}$ which follows from the bilinear equations (6.48) and (6.49) is the same as that for the bosonic KdV or mKdV equation, that is

$$\begin{aligned}
A_{12}A_{13}A_{23}&[(\omega_1 + \omega_2 + \omega_3) + (k_1 + k_2 + k_3)^3](k_1 + k_2 + k_3) \\
&+A_{23}[(\omega_1 - \omega_2 - \omega_3) + (k_1 - k_2 - k_3)^3](k_1 - k_2 - k_3) \\
&+A_{13}[(\omega_2 - \omega_1 - \omega_3) + (k_2 - k_1 - k_3)^3](k_2 - k_1 - k_3) \\
+A_{12}&[(\omega_3 - \omega_1 - \omega_2) + (k_3 - k_1 - k_2)^3](k_3 - k_1 - k_2) = 0
\end{aligned} \tag{7.44}$$

Note that the constraint (7.42) implies that $\zeta_2 = \frac{k_2}{k_1}\zeta_1$ and $\zeta_3 = \frac{k_3}{k_1}\zeta_1$ so that (7.43) and (7.44) are identical upto a multiplicative factor which makes it easy to realize that the constraint on the fermionic parameters are essential in establishing the integrability of the equations. From the bilinear form $\mathbf{D}_{\mathbf{x}}^2(\tau_1.\tau_2)$, which is common to the Hirota forms of all the three equations discussed above, the coefficient of $e^{\eta_1+\eta_2+\eta_3}$ is identically zero.

For $N = 1$ mKP equation, with the τ functions for the three soliton solution given by (7.37,7.38), and η_i from (7.40), we find that if we substitute the form of A_{ij} for $i < j$ corresponding to (7.35) as

$$A_{ij} = B_{ij} = -\frac{(\alpha_i\beta_j + \alpha_j\beta_i)(k_{ix} - k_{jx})^2 - 2(\alpha_i\beta_j - \alpha_j\beta_i)(k_{iy} - k_{jy})}{(\alpha_i\alpha_j + \beta_i\beta_j)(k_{ix} + k_{jx})^2 - 2(\alpha_i\alpha_j - \beta_i\beta_j)(k_{iy} + k_{jy})} \tag{7.45}$$

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and the dispersion relation corresponding to each η_i from (7.34) as

$$\omega_i k_{ix} + k_{ix}^4 + 12k_{iy}^2 = 0 \tag{7.46}$$

in the equations (7.37) and (7.38), τ_1 and τ_2 eventually satisfy the equations (6.55,6.56,6.57), provided k_{ix} and k_{iy} are related as earlier like

$$k_{iy} = \frac{1}{2}\frac{(\alpha_i + \beta_i)}{(\alpha_i - \beta_i)}k_{ix}^2 \tag{7.47}$$

This, indeed, ensures the existence of three soliton solution for the $N = 1$ KP equation and is seen from the vanishing of the coefficient of $e^{\eta_1+\eta_2+\eta_3}$ in the Hirota equations (6.55,6.56,6.57). The explicit form of the coefficient in first two equations (6.55,6.56) is

$$\begin{aligned}
&A_{12}A_{13}A_{23}[(\omega_1 + \omega_2 + \omega_3)(k_{1x} + k_{2x} + k_{3x}) + (k_{1x} + k_{2x} + k_{3x})^4 \\
&+(k_{1y} + k_{2y} + k_{3y})^2] \\
&+A_{23}[(\omega_1 - \omega_2 - \omega_3)(k_{1x} - k_{2x} - k_{3x}) + (k_{1x} - k_{2x} - k_{3x})^4 \\
&+(k_{1y} - k_{2y} - k_{3y})^2] \\
&+A_{13}[(\omega_2 - \omega_1 - \omega_3)(k_{2x} - k_{1x} - k_{3x}) + (k_{2x} - k_{1x} - k_{3x})^4 \\
&+(k_{2y} - k_{1y} - k_{3y})^2] \\
&+A_{12}[(\omega_3 - \omega_1 - \omega_2)(k_{3x} - k_{1x} - k_{2x}) + (k_{3x} - k_{1x} - k_{2x})^4 \\
&+(k_{3y} - k_{1y} - k_{2y})^2]
\end{aligned} \tag{7.48}$$

while from the third Hirota equation (6.57) we get for the coefficient of $e^{\eta_1+\eta_2+\eta_3}$ the following expression

$$\begin{aligned}
&A_{12}A_{13}A_{23}[(\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3)(k_{1x} + k_{2x} + k_{3x})^2 \\
&-2(\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3)(k_{1y} + k_{2y} + k_{3y})] \\
&+A_{23}[(\alpha_1\beta_2\beta_3 + \alpha_2\alpha_3\beta_1)(k_{1x} - k_{2x} - k_{3x})^2 \\
&-2(\alpha_1\beta_2\beta_3 - \alpha_2\alpha_3\beta_1)(k_{1y} - k_{2y} - k_{3y})] \\
&+A_{13}[(\alpha_2\beta_1\beta_3 + \alpha_1\alpha_3\beta_2)(k_{2x} - k_{1x} - k_{3x})^2
\end{aligned}$$

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$$-2(\alpha_2\beta_1\beta_3 - \alpha_1\alpha_3\beta_2)(k_{2y} - k_{1y} - k_{3y})]$$
$$+A_{12}[(\alpha_3\beta_1\beta_2 + \alpha_1\alpha_2\beta_3)(k_{3x} - k_{1x} - k_{2x})^2$$
$$-2(\alpha_3\beta_1\beta_2 - \alpha_1\alpha_2\beta_3)(k_{3y} - k_{1y} - k_{2y})] \tag{7.49}$$

It follows from (7.47) straightforwardly that $\beta_i \neq \alpha_i$ for a consistent solution. The three soliton solution for the $N = 1$ mKdV B equation may be obtained from (7.45-7.47) by imposing the conditions $\alpha_i = -\beta_i$, which make the solutions independent of the y coordinate. This procedure of obtaining three soliton solutions can be extended to N soliton solutions.

7.5 Conclusion

In this chapter we extracted soliton solutions of the $N = 2$ KdV and three $N = 1$ supersymmetric equations, the mKdV, mKdV B and mKP. All these systems can be shown to possess N soliton solutions. It was also found that for the $N = 2$ KdV and $N = 1$ mKdV equations, the Grassmann odd parameters in the τ functions play a major role in the existence of two and higher soliton solutions, while for the $N = 1$ mKdV B and the $N = 1$ mKP equations, they do not. This is essentially due to presence of the super Hirota operator in the bilinear forms for the $N = 2$ KdV and the $N = 1$ mKdV equations. This was also a first attempt to solve a $N = 2$ supersymmetric system, namely the $N = 2$ KdV through the Hirota method and we believe it is worth mentioning that earlier attempts to bilinearize the $N = 1$ mKdV equation were unsuccessful [33, 34]. The integrability of both these systems were known via the Lax method; the N soliton solutions considered in this chapter establishes their integrability from an alternate point of view.

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Chapter 8

The $N = 2$ super KP Equation

8.1 Introduction

The dynamical equations for the lowest spin superfields u_{-1} and u_0 of the $N = 2$ KP hierarchy (2.13,2.14) constitute the $N = 2$ supersymmetric KP equations. In this chapter, the $N = 2$ super KP equation is bilinearized following the Hirota method and the existence of N soliton solutions is shown. Unlike the $N = 2$ KdV equation, we are unable to bilinearize the $N = 2$ super KP equation using the super analogue of the Hirota operator as the $N = 2$ superKP equation, in contrast to the $N = 2$ super KdV equation, involves the space-time dependence in a complicated way. However, in terms of the component fields, the $N = 2$ super KP equation is in fact bilinearizable and is found to possess N soliton solutions. In the next section, *i.e.* section 2, the bilinearization of the $N = 2$ super KP equation is discussed. In section 3, we obtain its one soliton solution. In section 4, the existence of higher soliton solutions is discussed. Section 5 is the concluding one.

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8.2 Bilinear Forms of $N = 2$ KP Equation

The $N = 2$ KP equations can be rewritten in a convenient form after a rescaling of $t = -4t$, $y = -\frac{1}{2}y$, $u_{-1} = -\frac{1}{2}u_{-1}$ and $u_0 = -\frac{1}{2}u_0$ so that from (2.13) and (2.14), we obtain

$$\begin{aligned}&\partial_t u_{-1} + u_{-1}^{[6]} - \frac{1}{2}\left(u_{-1}^3\right)^{[2]} + 3\left(u_{-1}^{[1]}u_0\right)^{[2]} + 12\partial_y^2 u_{-1}^{[-2]}\\ &+6u_{-1}^{[2]}\partial_y u_{-1}^{[-2]} - 6u_{-1}^{[1]}\partial_y u_0^{[-2]} - 6u_0\partial_y u_{-1}^{[-1]} + 12u_0\partial_y u_0^{[-2]} = 0\end{aligned} \tag{8.1}$$

$$\begin{aligned}&\partial_t u_0 + u_0^{[6]} - 3\left(u_0 u_0^{[1]}\right)^{[2]} + 3\left(u_0 u_{-1}^{[2]}\right)^{[2]} - \frac{3}{2}\left(u_0 u_{-1}^2\right)^{[2]}\\ &+12\partial_y^2 u_0^{[-2]} + 6\left(u_0\partial_y u_0^{[-2]}\right)^{[1]} + 6u_0^{[2]}\partial_y u_{-1}^{[-2]} + 6u_0\partial_y u_{-1} = 0\end{aligned} \tag{8.2}$$

Here the superfields u_{-1} is a bosonic superfield, while u_0 is a fermionic one, having conformal spins 1 and 3/2 respectively.

It is interesting to note that if the superfields are made independent of the y coordinate in (8.1,8.2), the $N = 2$ superKP equations reduce to the $N = 2$ supersymmetric KdV equation [69, 70, 71, 61]. In chapter 6, it was shown that $N = 2$ supersymmetric KdV equation is bilinearizable in $N = 1$ superspace and possesses N soliton solutions. The bilinear forms of the $N = 2$ KdV equation were obtained by transforming the superfields as

$$u_{-1} = 2D^2 \log\frac{\tau_1}{\tau_2} \tag{8.3}$$

for u_{-1} and

$$u_0 = 2D^3 \log\tau_1 \tag{8.4}$$

for u_0. In (8.3,8.4) τ_1 and τ_2 are bosonic superfields which can be written in the component forms as

$$\tau_i = \tau_i^b + \theta\tau_i^f \qquad (i = 1, 2) \tag{8.5}$$

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τ_i^b and τ_i^b being the bosonic and fermionic component fields respectively. We find however that the $N = 2$ super KP equation cannot be bilinearized in terms of the τ functions defined in (8.3,8.4), although identical τ functions (without y dependence) allow us to bilinearize the $N = 2$ KdV equation. This difficulty arises due to the nonlocality as well as the y coordinate dependence present in the equations of motion (8.1,8.2) in an intricate way. On the other hand, it is important to observe that if the evolution equations of its component fields are considered, the system is amenable to bilinearization through the Hirota method.

The superfields u_{-1} and u_0 have the following component form

$$u_{-1} = u_{-1}^b + \theta u_{-1}^f \tag{8.6}$$

and

$$u_0 = u_0^f + \theta u_0^b \tag{8.7}$$

The time evolution of the component fields u_{-1}^b, u_{-1}^f, u_0^f and u_0^b consequently can be obtained from the dynamical equations for the superfields (8.1, 8.2) to give

$$\begin{aligned}
&\partial_t u_{-1}^b + \partial_x^3 u_{-1}^b - \frac{1}{2}\partial_x \left(u_{-1}^b\right)^3 + 3\partial_x \left(u_{-1}^f u_0^f\right) + 12\partial_y^2\partial_x^{-1} u_{-1}^b - 6u_{-1}^f \partial_y \partial_x^{-1} u_0^f \\
&+6\left(\partial_x u_{-1}^b\right)\partial_y\partial_x^{-1}u_{-1}^b - 6u_0^f\partial_y\partial_x^{-1}u_{-1}^f + 12u_0^f\partial_y\partial_x^{-1}u_0^f = 0
\end{aligned} \tag{8.8}$$

$$\begin{aligned}
&\partial_t u_{-1}^f + \partial_x^3 u_{-1}^f - \frac{3}{2}\partial_x \left(u_{-1}^{b^2} u_{-1}^f\right) + 3\partial_x\left[(\partial_x u_{-1}^b) u_0^f\right] - 3\partial_x\left(u_0^b u_{-1}^f\right) \\
&+12\partial_y^2\partial_x^{-1}u_{-1}^f + 6(\partial_x u_{-1}^b)\partial_y\partial_x^{-1}u_{-1}^f + 6(\partial_x u_{-1}^f)\partial_y\partial_x^{-1}u_{-1}^b + 6u_0^f\partial_y u_{-1}^b \\
&-6u_{-1}^f\partial_y\partial_x^{-1}u_0^b - 6u_0^b\partial_y\partial_x^{-1}u_{-1}^f - 6(\partial_x u_{-1}^b)\partial_y\partial_x^{-1}u_0^f + 12u_0^b\partial_y\partial_x^{-1}u_0^f \\
&-12u_0^f\partial_y\partial_x^{-1}u_0^b = 0
\end{aligned} \tag{8.9}$$

$$\partial_t u_0^f + \partial_x^3 u_0^f - 3\partial_x\left(u_0^b u_0^f\right) + 3\partial_x\left(u_0^f\partial_x u_{-1}^b\right) - \frac{3}{2}\partial_x\left(u_{-1}^{b^2} u_0^f\right)$$

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$$12\partial_y^2\partial_x^{-1}u_0^f + 6u_0^b\partial_y\partial_x^{-1}u_0^f - 6u_0^f\partial_y\partial_x^{-1}u_0^b + 6\left(\partial_x u_0^f\right)\partial_y\partial_x^{-1}u_{-1}^b$$
$$+6u_0^f\partial_y u_{-1}^b = 0 \qquad (8.10)$$

$$\partial_t u_0^b + \partial_x^3 u_0^b - 3\partial_x u_0^{b^2} + 3\partial_x\left(u_0^f\partial_x u_0^f\right) + 3\partial_x\left(u_0^b\partial_x u_{-1}^b\right)$$
$$-3\partial_x\left(u_0^f\partial_x u_{-1}^f\right) - \frac{3}{2}\partial_x\left(u_0^b u_{-1}^{b^2}\right) - 3\partial_x\left(u_{-1}^b u_{-1}^f u_0^f\right) + 12\partial_y^2\partial_x^{-1}u_0^b$$
$$+6(\partial_x u_0^f)\partial_y\partial_x^{-1}u_0^f + 6u_0^f\partial_y u_0^f + 6(\partial_x u_0^b)\partial_y\partial_x^{-1}u_{-1}^b - 6(\partial_x u_0^f)\partial_y\partial_x^{-1}u_{-1}^f$$
$$+6u_0^b\partial_y u_{-1}^b - 6u_0^f\partial_y u_{-1}^f = 0 \qquad (8.11)$$

To bilinearize the above equations, consider the following transformations of the component fields

$$u_{-1}^b = 2\partial_x \log\frac{\tilde{\tau}_1}{\tilde{\tau}_2} \qquad u_{-1}^f = 2\xi\partial_x^2\log\frac{\tilde{\tau}_1}{\tilde{\tau}_2} \qquad (8.12)$$

$$u_0^f = 2\xi\partial_x^2\log\tilde{\tau}_1 \qquad u_0^b = 2\partial_x^2\log\tilde{\tau}_1 \qquad (8.13)$$

where ξ is a spin $-\frac{1}{2}$ Grassmann odd parameter. Note that the Grassmann parameter is required in order that the fields have the correct conformal dimension. The τ functions in (8.12,8.13) are bosonic fields in contrast to those in (8.3,8.4) where the τ functions are bosonic superfields. Substituting (8.12,8.13) in (8.8,8.9,8.10,8.11) leads to the following bilinear forms

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4 + 12\mathbf{D}_y^2)(\tilde{\tau}_1.\tilde{\tau}_1) = 0 \qquad (8.14)$$

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4 + 12\mathbf{D}_y^2)(\tilde{\tau}_2.\tilde{\tau}_2) = 0 \qquad (8.15)$$

$$(\mathbf{D}_x^2 - 2\mathbf{D}_y)(\tilde{\tau}_1.\tilde{\tau}_2) = 0 \qquad (8.16)$$

To compare the bilinear forms for $N = 2$ KdV equation we observe that if the above equations become y independent, bilinear forms reduce to

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tilde{\tau}_1.\tilde{\tau}_1) = 0 \qquad (8.17)$$

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$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tilde{\tau}_2.\tilde{\tau}_2) = 0 \tag{8.18}$$

$$\mathbf{D}_x^2(\tilde{\tau}_1.\tilde{\tau}_2) = 0 \tag{8.19}$$

Notice that the above bilinear equations for $N = 2$ KdV equation do not involve super Hirota operator (6.8) in contrast to the forms (6.27,6.28, 6.28). But both the Hirota forms for KdV equation give rise to identical soliton solutions. This allows us to identify (8.12) and (8.13) with (8.3) and (8.4) provided

$$\tau_1^f = \xi\partial_x\tau_1^b \qquad \tau_2^f = \xi\partial_x\tau_2^b \tag{8.20}$$

where τ_i are given in (8.5). The relation between the fermionic and bosonic components (8.20), however, become evident from the soliton solutions.

8.3 One Soliton Solution

For the one soliton solution of the $N = 2$ superKP equation we consider the following forms for the τ functions

$$\tilde{\tau}_1 = 1 + \alpha e^{\eta} \tag{8.21}$$

$$\tilde{\tau}_2 = 1 + \beta e^{\eta} \tag{8.22}$$

with

$$\eta = k_x x + k_y y + \omega t \tag{8.23}$$

It is easily shown that substituting (8.21,8.22) back in first two equations, namely (8.14) and (8.15) of the Hirota form of the $N = 2$ KP equation, the dispersion relation may be obtained as

$$\omega k_x + k_x^4 + 12k_y^2 = 0 \tag{8.24}$$

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The dispersion relation is identical to that of the bosonic KP equation. From the third equation of this set (8.16), k_y is related to k_x through

$$k_y = \frac{1}{2}\frac{(\alpha+\beta)}{(\alpha-\beta)}k_x^2 \tag{8.25}$$

This imposes the following constraint between α and β parameters like

$$\beta \neq \alpha \tag{8.26}$$

and consequently reduces to the soliton solution of $N = 2$ KdV equation (7.3-7.6), if $\alpha = -\beta$. This is obvious from (8.25) as under this condition k_y becomes zero.

The explicit forms of the one soliton solution for the $N = 2$ KP equation are easily found by substituting the τ functions from (8.21,8.22) in the transformation equations (8.12,8.13). For the components of the bosonic superfield u_{-1}, we have

$$u_{-1}^b = k_x \sinh\frac{1}{2}(\gamma_0 - \delta_0)\,\mathrm{sech}\frac{1}{2}(\phi+\gamma_0)\,\mathrm{sech}\frac{1}{2}(\phi+\delta_0) \tag{8.27}$$

$$\begin{aligned} u_{-1}^f = -\frac{1}{2}k_x^2\xi \sinh\frac{1}{2}(\gamma_0 - \delta_0)\,\mathrm{sech}\frac{1}{2}(\phi+\gamma_0)\,\mathrm{sech}\frac{1}{2}(\phi+\delta_0) \\ \times\left[\tanh(\phi+\gamma_0) + \tanh(\phi+\delta_0)\right] \end{aligned} \tag{8.28}$$

Similarly for the fermionic superfield u_0, the components are

$$u_0^f = \frac{1}{2}k_x^2\xi\,\mathrm{sech}^2(\phi+\gamma_0) \tag{8.29}$$

$$u_0^b = \frac{1}{2}k_x^2\,\mathrm{sech}^2(\phi+\gamma_0) \tag{8.30}$$

where we have chosen $\phi = k_x x + k_y y + \omega t$, $\alpha = e^{\gamma_0}$ and $\beta = e^{\delta_0}$. γ_0 and δ_0 are nonzero, real parameters.

It is straightforward that (8.27,8.28,8.29,8.30) reduce to (7.9,7.10) provided $\beta = -\alpha$ and if we identify

$$\xi = -\frac{\zeta}{k_x} \tag{8.31}$$

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In fact, both sets of bilinear forms of $N = 2$ KdV equation, namely (6.27,6.28,6.29) and (8.17,8.18,8.19), will give rise to identical higher soliton solutions. As observed in chapter 7, the fermionic parameter ζ does not play a nontrivial role at the one soliton solution level but the fermionic parameters obey a set of relations among themselves for two soliton solutions onwards. As a result the identification of soliton solutions for higher solitons obtained from both the bilinear forms is not straightforward for $N = 2$ KdV equation.

8.4 N Soliton Solution

For the $N = 2$ KP equation, we choose the same set of trial functions as in chapter 7, namely (7.36). Then for the τ functions for the two soliton solutions, in particular, we have

$$\tilde{\tau}_1 = 1 + \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_1 \alpha_2 A_{12} e^{\eta_1 + \eta_2} \tag{8.32}$$

and

$$\tilde{\tau}_2 = 1 + \beta_1 e^{\eta_1} + \beta_2 e^{\eta_2} + \beta_1 \beta_2 B_{12} e^{\eta_1 + \eta_2} \tag{8.33}$$

where

$$\eta_1 = k_{1x} x + k_{1y} y + \omega_1 t \tag{8.34}$$

and

$$\eta_2 = k_{2x} x + k_{2y} y + \omega_2 t \tag{8.35}$$

and α_i, β_i $(i = 1, 2)$ and A_{12}, B_{12} are unknown parameters. The parameters α_i and β_i follow the same relation as in one soliton solution, but A_{12}, B_{12} are additional parameters to be determined at the two soliton level. As for the one soliton solution, we find that for nontrivial solutions we must have

$$\beta_i \neq \alpha_i \quad (i = 1, 2) \tag{8.36}$$

We will obtain two dispersion relations corresponding to the space time dependent parameters η_1 and η_2. The dispersion relations resulting from (8.14) and (8.15) are given by

$$\omega_i k_{ix} + k_{ix}^4 + 12k_{iy}^2 = 0, \qquad i = 1, 2 \tag{8.37}$$

The two soliton interaction terms A_{12} and B_{12} are obtained by substituting the τ functions (8.32,8.33) in the bilinear equation (8.16) and we find that

$$A_{12} = B_{12} \tag{8.38}$$

and A_{12} is given by

$$A_{12} = -\frac{(\alpha_1\beta_2 + \alpha_2\beta_1)(k_{1x} - k_{2x})^2 - 2(\alpha_1\beta_2 - \alpha_2\beta_1)(k_{1y} - k_{2y})}{(\alpha_1\alpha_2 + \beta_1\beta_2)(k_{1x} + k_{2x})^2 - 2(\alpha_1\alpha_2 - \beta_1\beta_2)(k_{1y} + k_{2y})} \tag{8.39}$$

Also one finds from (8.16) that k_{ix} and k_{iy} $(i = 1, 2)$ are connected through the relation

$$k_{iy} = \frac{1}{2}\frac{(\alpha_i + \beta_i)}{(\alpha_i - \beta_i)}k_{ix}^2 \tag{8.40}$$

The two soliton solution of $N = 2$ KdV equation obtained from the bilinear forms (8.17,8.18,8.19) immediately follows from above solutions if $\beta_i = -\alpha_i$ with $(i = 1, 2)$ and as a result dispersion relations become

$$\omega_i + k_{ix}^3 = 0, \qquad i = 1, 2 \tag{8.41}$$

and the interaction terms reduce to

$$A_{12} = B_{12} = \frac{(k_{1x} - k_{2x})^2}{(k_{1x} + k_{2x})^2} \tag{8.42}$$

To compare the above result (8.41,8.42) with those obtained in chapter 7, we note the bilinear forms (6.27,6.28,6.29) result an additional condition on the fermionic parameters (7.30); namely

$$k_i\zeta_j = k_j\zeta_i \qquad (i, j = 1, 2; i \neq j) \tag{8.43}$$

where the spin 1/2 parameter ζ is introduced in (7.5). The condition (8.43) makes the two soliton solutions of $N=2$ KdV equation apparently inconsistent. But notice that (8.43) ensures the ratio $\frac{\zeta_i}{k_{ix}}$ to be same for both the solitons and explicit form of one soliton dictates that the ratio will be same as $-\xi$ (8.31) *i.e.*

$$\xi = -\frac{\zeta_1}{k_1} = -\frac{\zeta_2}{k_2} \tag{8.44}$$

This condition also follows from the expressions of u_{-1} and u_0 when written in terms of two soliton solutions. Importantly, with this identification two soliton solutions of $N=2$ KdV equation obtained from both the bilinear forms become consistent, although a condition such as (8.43) does not arise separately when the equations of motions are written in the component fields. This strongly indicates that the bilinearization of the $N=2$ KP equation in terms of the bosonic Hirota operator by using the τ functions defined in (8.12,8.13) is equivalent to bilinearization in terms of the super Hirota operator in $N=1$ superspace.

Let us now consider three soliton solution to check the consistency of the parameters determined in one and two soliton solutions. The explicit forms of the τ functions for three soliton solution of the $N=2$ KP are obtained from (7.36) to be

$$\begin{aligned}\tilde{\tau}_1 &= 1+\alpha_1 e^{\eta_1}+\alpha_2 e^{\eta_2}+\alpha_3 e^{\eta_3}+\alpha_1\alpha_2 A_{12} e^{\eta_1+\eta_2}+\alpha_1\alpha_3 A_{13} e^{\eta_1+\eta_3}\\ &+\alpha_2\alpha_3 A_{23} e^{\eta_2+\eta_3}+\alpha_1\alpha_2\alpha_3 A_{12}A_{13}A_{23} e^{\eta_1+\eta_2+\eta_3}\end{aligned} \tag{8.45}$$

and

$$\begin{aligned}\tilde{\tau}_2 &= 1+\beta_1 e^{\eta_1}+\beta_2 e^{\eta_2}+\beta_3 e^{\eta_3}+\beta_1\beta_2 B_{12} e^{\eta_1+\eta_2}+\beta_1\beta_3 B_{13} e^{\eta_1+\eta_3}\\ &+\beta_2\beta_3 B_{23} e^{\eta_2+\eta_3}+\beta_1\beta_2\beta_3 B_{12}B_{13}B_{23} e^{\eta_1+\eta_2+\eta_3}\end{aligned} \tag{8.46}$$

with

$$\eta_i = k_{ix}x + k_{iy}y + \omega_i t \qquad i=1,2,3 \tag{8.47}$$

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If the τ functions are substituted in the bilinear forms (8.14,8.15, 8.16), then one would get conditions which were already obtained from the one soliton and two soliton solutions as well as some additional relations which should be satisfied. In fact as before from the bilinear form (8.16) we obtain the condition

$$\beta_i \neq \alpha_i \qquad i = 1, 2, 3 \tag{8.48}$$

by setting the coefficient of e^{η_i} and $e^{2\eta_i+\eta_j}$, $(i = 1, 2, 3)$ to zero and the interaction terms from the vanishing of the coefficients of $e^{\eta_i+\eta_j}$ $(i \neq j)$

$$A_{ij} = B_{ij} = -\frac{(\alpha_i\beta_j + \alpha_j\beta_i)(k_{ix} - k_{jx})^2 - 2(\alpha_i\beta_j - \alpha_j\beta_i)(k_{iy} - k_{jy})}{(\alpha_i\alpha_j + \beta_i\beta_j)(k_{ix} + k_{jx})^2 - 2(\alpha_i\alpha_j - \beta_i\beta_j)(k_{iy} + k_{jy})} \tag{8.49}$$

The parameters obtained from (8.49) are those already found in the two soliton solution. The only nontrivial condition at three soliton solution arising from (8.16) is the coefficient of $e^{\eta_1+\eta_2+\eta_3}$, namely

$$\begin{aligned}
&A_{12}A_{13}A_{23}[(\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3)(k_{1x} + k_{2x} + k_{3x})^2 - 2(\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3) \\
&\times(k_{1y} + k_{2y} + k_{3y})] \\
&+A_{23}[(\alpha_1\beta_2\beta_3 + \alpha_2\alpha_3\beta_1)(k_{1x} - k_{2x} - k_{3x})^2 - 2(\alpha_1\beta_2\beta_3 - \alpha_2\alpha_3\beta_1) \\
&\times(k_{1y} - k_{2y} - k_{3y})] \\
&+A_{13}[(\alpha_2\beta_1\beta_3 + \alpha_1\alpha_3\beta_2)(k_{2x} - k_{1x} - k_{3x})^2 - 2(\alpha_2\beta_1\beta_3 - \alpha_1\alpha_3\beta_2) \\
&\times(k_{2y} - k_{1y} - k_{3y})] \\
&+A_{12}[(\alpha_3\beta_1\beta_2 + \alpha_1\alpha_2\beta_3)(k_{3x} - k_{1x} - k_{2x})^2 - 2(\alpha_3\beta_1\beta_2 - \alpha_1\alpha_2\beta_3) \\
&\times(k_{3y} - k_{1y} - k_{2y})]
\end{aligned} \tag{8.50}$$

and is found to be zero with the choice A_{ij} in (8.49).

Similarly from the first two bilinear forms (8.14,8.15), we obtain the the dispersion relation

$$\omega_i k_{ix} + k_{ix}^4 + 12k_{iy}^2 = 0 \qquad i = 1, 2, 3 \tag{8.51}$$

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by equating the coefficient of e^{η_i} and $e^{2\eta_i+\eta_j}$, $(i = 1,2,3)$ to zero. With the dispersion relation (8.51) and (8.49) we find that the only nontrivial term arising from the bilinear forms (8.14,8.15), namely the coefficient of $e^{\eta_1+\eta_2+\eta_3}$ which is given by

$$
\begin{aligned}
&A_{12}A_{13}A_{23}[(\omega_1+\omega_2+\omega_3)(k_{1x}+k_{2x}+k_{3x})+(k_{1x}+k_{2x}+k_{3x})^4\\
&+(k_{1y}+k_{2y}+k_{3y})^2]\\
&+A_{23}[(\omega_1-\omega_2-\omega_3)(k_{1x}-k_{2x}-k_{3x})+(k_{1x}-k_{2x}-k_{3x})^4\\
&+(k_{1y}-k_{2y}-k_{3y})^2]\\
&+A_{13}[(\omega_2-\omega_1-\omega_3)(k_{2x}-k_{1x}-k_{3x})+(k_{2x}-k_{1x}-k_{3x})^4\\
&+(k_{2y}-k_{1y}-k_{3y})^2]\\
&+A_{12}[(\omega_3-\omega_1-\omega_2)(k_{3x}-k_{1x}-k_{2x})+(k_{3x}-k_{1x}-k_{2x})^4\\
&+(k_{3y}-k_{1y}-k_{2y})^2]
\end{aligned}
\tag{8.52}
$$

vanishes and thereby ensures the existence of three soliton solutions for the $N = 2$ KP equation.

It is seen that upto three soliton solution, the only condition that is not trivially satisfied comes from the coefficient of $e^{\sum_{i=1}^{N}\eta_i}$. The coefficients of all other exponents will be zero identically by ensuring the existence of soliton solutions of lower order. Consequently, for N soliton solution, the vanishing of the coefficient of $e^{\sum_{i=1}^{N}\eta_i}$ indeed confirms the existence of N soliton solution. From the τ function for the N soliton solution (7.36), we find that in the product $\tau_1.\tau_2$, the terms that will generate only $e^{\sum_{i=1}^{N}\eta_i}$ after being acted on by the Hirota operator $\mathbf{D}$ can be written as

$$
1.\prod_{\substack{a,b,c=1\\ a<b}}^{N}\beta_c A_{ab} + \sum_{i_1=1}^{N}\alpha_{i_1}e^{\eta_{i_1}}.\prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1}}^{N}\beta_c A_{ab}e^{\eta_c}
$$

$$+\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{N}\alpha_{i_1}\alpha_{i_2}A_{i_1i_2}e^{\eta_{i_1}+\eta_{i_2}}.\prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1,i_2}}^{N}\beta_cA_{ab}e^{\eta_c}$$

$$+\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}A_{i_1i_2}A_{i_1i_3}A_{i_2i_3}e^{\eta_{i_1}+\eta_{i_2}+\eta_{i_3}}.\prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1,i_2,i_3}}^{N}\beta_cA_{ab}e^{\eta_c}$$

$$+\ldots+\sum_{\substack{i_1,i_2,i_3\ldots i_N=1\\ i_1<i_2<i_3<\ldots<i_N}}^{N}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\ldots\alpha_{i_N}A_{i_1i_2}A_{i_1i_3}A_{i_2i_3}\ldots A_{i_{N-1}i_N}$$
$$\times e^{\eta_{i_1}+\eta_{i_2}+\eta_{i_3}+\ldots+\eta_{i_N}}.1 \tag{8.53}$$

Using (8.53), we obtain the nontrivial coefficient of the term $e^{\sum_{i=1}^{N}\eta_i}$ in the Hirota equation (8.16) to be of the form

$$\prod_{\substack{a,b,c=1\\ a<b}}^{N}\beta_cA_{ab}[X^2-2Y]+\sum_{i_1=1}^{N}\prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1}}^{N}[\{2k_{i_1}x-X\}^2-\{2k_{i_1}y-Y\}]$$
$$\times\alpha_{i_1}A_{ab}\beta_c+\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{N}\prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1,i_2}}^{N}[\{2(k_{i_1}x+k_{i_2}x)-X\}^2$$
$$-2\{2(k_{i_1}y+k_{i_2}y)-Y\}]\alpha_{i_1}\alpha_{i_2}A_{i_1i_2}A_{ab}\beta_c$$
$$+\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N}\prod_{\substack{a,b,c=1\\ a<b\\ a,b,c\neq i_1,i_2,i_3}}^{N}[\{2(k_{i_1}x+k_{i_2}x+k_{i_3}x)-X\}^2$$
$$-2\{k_{i_1}y+k_{i_2}y+k_{i_3}y)-Y\}]\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}A_{i_1i_2}A_{i_1i_3}A_{i_2i_3}A_{ab}\beta_c$$
$$+\ldots+\sum_{\substack{i_1,i_2,i_3\ldots i_N=1\\ i_1<i_2<i_3<\ldots<i_N}}^{N}[(k_{i_1}x+k_{i_2}x+k_{i_3}x+\ldots+k_{i_N}x)^2$$
$$-2(k_{i_1}y+k_{i_2}y+k_{i_3}y+\ldots+k_{i_N}y)]\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\ldots\alpha_{i_N}$$
$$\times A_{i_1i_2}A_{i_1i_3}A_{i_2i_3}\ldots A_{i_{N-1}i_N} \tag{8.54}$$

where $X=\sum_{m=1}^{N}k_{mx}$ and $Y=\sum_{n=1}^{N}k_{ny}$. It is easily seen that (8.54) reproduce the nontrivial condition coming from the Hirota equation

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(8.16) at all orders of soliton solutions. For caxample, at the one soliton level with N=1, we get (8.25) and with N=2, we get (8.39). At the three soliton level, with the substitution N=3, the condition (8.50) is obtained.

For the other two bilinear forms for the $N = 2$ KP equation, namely (8.14,8.15), the expression equivalent to (8.54) *i.e.* the coefficient of the term $e^{\sum_{i=1}^{N}\eta_i}$ is found to have the form

$$\begin{aligned}
&\prod_{\substack{a,b=1\\a<b}}^{N} A_{ab}[X\Omega + X^4 + 12Y^2] \\
&+\sum_{i_1=1}^{N} \prod_{\substack{a,b=1\\a<b\\a,b\neq i_1}}^{N} [\{2k_{i_1}x - X\}\{2\omega_{i_1} - \Omega\} + \{2k_{i_1}x - X\}^4 + 12\{2k_{i_1}y - Y\}^2]A_{ab} \\
&+ \sum_{\substack{i_1,i_2=1\\i_1<i_2}}^{N} \prod_{\substack{a,b=1\\a<b\\a,b\neq i_1,i_2}}^{N} [\{2(k_{i_1}x + k_{i_2}x) - X\}\{2(\omega_{i_1} + \omega_{i_2}) - \Omega\} \\
&+\{2(k_{i_1}x + k_{i_2}x) - X\}^4 + 12\{2(k_{i_1}y + k_{i_2}y) - Y\}^2]A_{i_1i_2}A_{ab} \\
&+ \sum_{\substack{i_1,i_2,i_3=1\\i_1<i_2<i_3}}^{N} \prod_{\substack{a,b=1\\a<b\\a,b\neq i_1,i_2,i_3}}^{N} [\{2(k_{i_1}x + k_{i_2}x + k_{i_3}x) - X\}\{2(\omega_{i_1} + \omega_{i_2} + \omega_{i_3}) - \Omega\} \\
&+\{2(k_{i_1}x + k_{i_2}x + k_{i_3}x) - X\}^4 + 12\{k_{i_1}y + k_{i_2}y + k_{i_3}y) - Y\}^2] \\
&\times A_{i_1i_2}A_{i_1i_3}A_{i_2i_3}A_{ab} \\
&+\ldots+ \sum_{\substack{i_1,i_2,i_3\ldots i_N=1\\i_1<i_2<i_3<\ldots<i_N}}^{N} [(k_{i_1}x + k_{i_2}x + k_{i_3}x + \ldots + k_{i_N}x) \\
&\times(\omega_{i_1} + \omega_{i_2} + \omega_{i_3} + \ldots + \omega_{i_N}) + (k_{i_1}x + k_{i_2}x + k_{i_3}x + \ldots + k_{i_N}x)^4 \\
&+12(k_{i_1}y + k_{i_2}y + k_{i_3}y + \ldots + k_{i_N}y)^2]A_{i_1i_2}A_{i_1i_3}A_{i_2i_3}\ldots A_{i_{N-1}i_N}
\end{aligned} \tag{8.55}$$

where $\Omega = \sum_{\lambda=1}^{N}\omega_\lambda$. It is easily checked that (8.55) also consistently gives the the nontrivial coefficient at each level of soliton solution.

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8.5 Conclusion

In this chapter, we have shown that the $N = 2$ nonstandard flow KP equation is bilinearizable in component form, although we were unable to bilinearize it in $N = 1$ superspace. By comparing with the $N = 2$ KdV equation, it is seen that the right choice of τ functions ensures that this is equivalent to bilinearization via the super Hirota formalism in $N = 1$ superspace. Though this is not evident from the Hirota equations themselves, it becomes obvious from the soliton solutions that in fact the methods are equivalent and it is expected that this will also be valid for the $N = 2$ KP equation, although the exact comparison may be made only if it is bilinearized in $N = 1$ superspace.

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Chapter 9

Conclusion

This work has been mainly concerned with the study of the $N = 2$ supersymmetric KP hierarchy – its Hamiltonian structures and soliton solutions. We have shown that the $N = 2$ KP obtained from the nonstandard flow equation has a second Hamiltonian structure which is local and the superalgebra comprises an infinite number of generators with the appropriate conformal spins. In a suitable nonlinear basis, it was found that bosonic sector consists of two infinite towers of bosonic generators which mutually commute so that this sector has the $W_{1+\infty} \oplus W_\infty$ structure similar to that found in the linear super W_∞ algebra of Bergshoeff *et al.* However due to the highly nonlinear nature of the W_∞ sector, the generators cannot be written in closed form but an algorithmic procedure for finding the all higher spin generators have been indicated. Moreover in the bosonic limit, the algebra of the nonlinear generators is isomorphic to the classical bosonic $\hat{W}_\infty$ algebra of Bakas and Kiritsis. We therefore believe that the super $\hat{W}_\infty$ algebra is a candidate for a universal algebra, unifying all bosonic and supersymmetric W algebras. Since the classical $\hat{W}_\infty$ algebra is present in the algebra it is guaranteed that all finite dimensional bosonic algebras can be obtained from it under suitable truncation. As the super $\hat{W}_\infty$

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algebra contains the $N = 2$ superconformal algebra as a subalgebra, it is reasonable to expect that it contains all finite dimensional $N = 2$ supersymmetric W algebras as was the case for the bosonic universal algebra. This may be done by analysing the truncation of the super $\hat{W}_\infty$ algebra through some noncompact coset model and could prove to be the basis of future work.

The fermionic sector of this algebra also posseses some unique properties. Although it can be written in terms of a linear basis, we find that a nonlinear basis for the fermionic sector also exists. The difference of the nonlinear super $\hat{W}_\infty$ algebra with the linear super W_∞ algebra lies also in the fermionic sector as the fermionic generators in nonlinear algebra do not commute among themselves in any basis. This results in a drastic change in the freefield representation of the nonlinear super $\hat{W}_\infty$ algebra. The consistent representation turns out to be nonlinear in free complex bosons and free fermions and the nonlinear representation cannot be reduced to the linear one trivially. Moreover, the representations of the bosonic generators possess a structure much more complex than the linear super W_∞ algebra as they are now an admixture of complex bosons and fermions. At the same time, the two types of bosonic generators mutually commute with each other maintaining the $W_{1+\infty} \oplus W_\infty$ structure. This is a nontrivial generalization when compared to the linear representation of the $N = 2$ super W_∞ algebra. In the fermionic sector the most general representation becomes exponential in terms of the free fields.

The second part of the work has been involved with finding the soliton solutions of $N = 1$ and $N = 2$ supersymmetric systems through the supersymmetric extension of the Hirota method. In this regard, the method adopted was the generalization of the Hirota method to $N = 1$ superspace by MacArthur and Yung. Using this technique N soliton solutions of an $N = 2$ KdV equation, the $N = 1$ mKdV,

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$N = 1$ mKdV B, and $N = 1$ mKP equations were obtained. Howevever, for the $N = 2$ KP we were unable to bilinearize the equation using the above mentioned formalism, but we could obtain the N soliton solutions for it after applying the Hirota method to the component equations of this supersymmetric system. Bilinearization of the $N = 2$ KP using the Hirota formalism in $N = 1$ superspace constitutes another problem to be addressed in future. But of greater significance, in this connection, is the lack of the Hirota formalism in $N = 2$ superspace. We believe that generalization of this elegant and direct method to $N = 2$ superspace would result in a major advance in the study of supersymmetric integrable models.

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Appendix A

Poisson Brackets of Component Fields

Poisson bracket algebra amongst the component fields v^b_{2i}, u^b_{2i-1}, u^f_{2i} and v^f_{2i-1}.

$$
\begin{aligned}
\left\{u^b_{2j-1}(x), u^f_{2k}(y)\right\} = & \Bigg[-\sum_{m=0}^{j}\begin{pmatrix} j \\ m \end{pmatrix}(-1)^m u^f_{2j+2k-2m}\partial^m \\
& + \sum_{m=0}^{j-1}\sum_{l=0}^{k}\begin{pmatrix} j-1 \\ m \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix}(-1)^m u^f_{2j-2m-2}\partial^{m+l}u^b_{2k-2l-1} \\
& - \sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\begin{pmatrix} j \\ m+1 \end{pmatrix}\begin{pmatrix} k \\ l+1 \end{pmatrix}(-1)^m u^b_{2j-2m-3}\partial^{m+l+1}u^f_{2k-2l-2} \\
& + \sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} k-n-1 \\ l \end{pmatrix}u^f_{2n}\partial^l u^b_{2j+2k-2n-2l-3} \\
& - \sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k}\sum_{l=0}^{k-n}\begin{pmatrix} j-1 \\ m \end{pmatrix}\begin{pmatrix} n+l-1 \\ l \end{pmatrix}(-1)^m \\
& \times u^f_{2j+2k-2m-2n-2l-2}\partial^{m+l}u^b_{2n-1}\Bigg]\delta(x-y)
\end{aligned}
\tag{A.1}
$$

$$\left\{u^f_{2j}(x), u^f_{2k}(y)\right\} =$$

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$$\Bigg[-\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\binom{j}{m+1}\binom{k}{l+1}(-1)^m u^f_{2j-2m-2}\partial^{m+l+1}u^f_{2k-2l-2}$$
$$-\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{k-n-1}{l}u^f_{2n}\partial^l u^f_{2j+2k-2n-2l-2}$$
$$+\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{j}{m}\binom{n+l}{l}$$
$$\times(-1)^m u^f_{2j+2k-2m-2n-2l-2}\partial^{m+l}u^f_{2n}\Bigg]\delta(x-y) \quad \text{(A.2)}$$

$$\left\{u^b_{2j-1}(x), v^f_{2k-1}(y)\right\} = \Bigg[\sum_{m=0}^{k}\binom{k}{m}\partial^m v^f_{2j+2k-2m-1}$$
$$+\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\left\{\binom{j-1}{m}\binom{k-1}{l}-\binom{j}{m+1}\binom{k}{l+1}\right\}$$
$$\times(-1)^m u^b_{2j-2m-3}\partial^{m+l+1}v^f_{2k-2l-3}$$
$$+\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{k-n-1}{l}u^b_{2n-1}\partial^l v^f_{2j+2k-2n-2l-3}$$
$$-\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{j-1}{m}\binom{n+l-1}{l}$$
$$\times(-1)^m u^b_{2j+2k-2m-2n-2l-3}\partial^{m+l}v^f_{2n-1}\Bigg]\delta(x-y) \quad \text{(A.3)}$$

$$\left\{v^f_{2j-1}(x), v^f_{2k-1}(y)\right\} =$$
$$\Bigg[\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\left\{\binom{j-1}{m}\binom{k-1}{l}-\binom{j}{m+1}\binom{k}{l+1}\right\}$$
$$\times(-1)^m v^f_{2j-2m-3}\partial^{m+l+1}v^f_{2k-2l-3}$$
$$+\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{k-n-1}{l}v^f_{2n-1}\partial^l v^f_{2j+2k-2n-2l-3}$$
$$-\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{j-1}{m}\binom{n+l-1}{l}$$

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$$\times(-1)^m v^f_{2j+2k-2m-2n-2l-3}\partial^{m+l}v^f_{2n-1}\Big]\,\delta(x-y) \tag{A.4}$$

$$\left\{u^b_{2j-1}(x),u^b_{2k-1}(y)\right\}=\Bigg[-\sum_{m=0}^{j}\binom{j}{m}(-1)^m u^b_{2j+2k-2m-1}\partial^m$$
$$+\sum_{m=0}^{k}\binom{k}{m}\partial^m u^b_{2j+2k-2m-1}$$
$$+\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\left\{\binom{j-1}{m}\binom{k-1}{l}-\binom{j}{m+1}\binom{k}{l+1}\right\}$$
$$\times(-1)^m u^b_{2j-2m-3}\partial^{m+l+1}u^b_{2k-2l-3}$$
$$+\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{k-n-1}{l}u^b_{2n-1}\partial^l u^b_{2j+2k-2n-2l-3}$$
$$-\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{j-1}{m}\binom{n+l-1}{l}$$
$$\times(-1)^m u^b_{2j+2k-2m-2n-2l-3}\partial^{m+l}u^b_{2n-1}\Big]\,\delta(x-y) \tag{A.5}$$

$$\left\{v^b_{2j}(x),v^b_{2k}(y)\right\}=$$
$$\Bigg[\sum_{m=0}^{j+1}\binom{j+1}{m}(-1)^m v^b_{2j+2k-2m+2}\partial^m-\sum_{m=0}^{k+1}\binom{k+1}{m}\partial^m v^b_{2j+2k-2m+2}$$
$$-\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\binom{j}{m+1}\binom{k}{l+1}(-1)^m v^b_{2j-2m-2}\partial^{m+l+1}v^b_{2k-2l-2}$$
$$-\sum_{l=0}^{k}\binom{k}{l}u^b_{-1}\partial^l v^b_{2j+2k-2l}-\sum_{l=0}^{k}\binom{k}{l}u^f_{2j}\partial^l v^f_{2k-2l-1}$$
$$+\sum_{m=0}^{j}\binom{j}{m}(-1)^m v^b_{2j+2k-2m}\partial^m u^b_{-1}-\sum_{m=0}^{j}\binom{j}{m}(-1)^m v^f_{2j-2m-1}\partial^m u^f_{2k}$$
$$-\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\binom{k-n-1}{l}v^b_{2n}\partial^l v^b_{2j+2k-2n-2l-2}$$

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$$+\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} j \\ m \end{pmatrix}\begin{pmatrix} n+l \\ l \end{pmatrix}$$
$$\times(-1)^m v^b_{2j+2k-2m-2n-2l-2}\partial^{m+l}v^b_{2n}\Big]\,\delta(x-y) \tag{A.6}$$

$$\left\{u^f_{2j}(x), v^f_{2k-1}(y)\right\} =$$
$$\left[\sum_{m=0}^{j+1}\begin{pmatrix} j+1 \\ m \end{pmatrix}(-1)^m u^b_{2j+2k-2m+1}\partial^m - \sum_{m=0}^{k}\begin{pmatrix} k \\ m \end{pmatrix}\partial^m v^b_{2j+2k-2m}\right.$$
$$-\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\begin{pmatrix} j \\ m+1 \end{pmatrix}\begin{pmatrix} k \\ l+1 \end{pmatrix}(-1)^m u^f_{2j-2m-2}\partial^{m+l+1}v^f_{2k-2l-3}$$
$$-\sum_{m=0}^{j}\begin{pmatrix} j \\ m \end{pmatrix}(-1)^m u^b_{2j-2m-1}\partial^m u^b_{2k-1} + \sum_{m=0}^{j}\begin{pmatrix} j \\ m \end{pmatrix}(-1)^m u^b_{2j+2k-2m-1}\partial^m u^b_{-1}$$
$$-\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} k-n-1 \\ l \end{pmatrix}u^f_{2n-1}\partial^l u^b_{2j+2k-2n-2l-2}$$
$$+\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} j \\ m \end{pmatrix}\begin{pmatrix} n+l \\ l \end{pmatrix}$$
$$\times(-1)^m u^b_{2j+2k-2m-2n-2l-3}\partial^{m+l}v^b_{2n}\Big]\,\delta(x-y) \tag{A.7}$$

$$\left\{v^b_{2j}(x), v^f_{2k-1}(y)\right\} = \left[\sum_{m=0}^{j+1}\begin{pmatrix} j+1 \\ m \end{pmatrix}(-1)^m v^f_{2j+2k-2m+1}\partial^m\right.$$
$$-\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\begin{pmatrix} j \\ m+1 \end{pmatrix}\begin{pmatrix} k \\ l+1 \end{pmatrix}(-1)^m v^b_{2j-2m-2}\partial^{m+l+1}v^f_{2k-2l-3}$$
$$+\sum_{m=0}^{j}\begin{pmatrix} j \\ m \end{pmatrix}(-1)^m v^f_{2j+2k-2m-1}\partial^m u^b_{-1}$$
$$-\sum_{m=0}^{j}\begin{pmatrix} J \\ m \end{pmatrix}(-1)^m v^f_{2j-2m-1}\partial^m u^b_{2k-1}$$
$$-\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} k-n-1 \\ l \end{pmatrix}v^f_{2n-1}\partial^l v^b_{2j+2k-2n-2l-2}$$

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$$+\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} j \\ m \end{pmatrix}\begin{pmatrix} n+1 \\ l \end{pmatrix}$$
$$\times(-1)^m v^f_{2j+2k-2m-2n-2l-3}\partial^{m+l}v^b_{2n}\Big]\,\delta(x-y) \tag{A.8}$$

$$\left\{v^b_{2j}(x), u^f_{2k}(y)\right\} = \Big[-\sum_{m=0}^{k+1}\begin{pmatrix} k+1 \\ m \end{pmatrix}\partial^m u^f_{2j+2k-2m+2}$$
$$-\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\begin{pmatrix} j \\ m+1 \end{pmatrix}\begin{pmatrix} k \\ l+1 \end{pmatrix}(-1)^m v^b_{2j-2m-2}\partial^{m+l+1}u^f_{2k-2l-2}$$
$$+\sum_{l=0}^{k}\begin{pmatrix} k \\ l \end{pmatrix}u^f_{2j}\partial^l u^b_{2k-2l-1}-\sum_{l=0}^{k}\begin{pmatrix} k \\ l \end{pmatrix}u^b_{-1}\partial^l u^f_{2j+2k-2l}$$
$$-\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} k-n-1 \\ l \end{pmatrix}v^b_{2n}\partial^l u^f_{2j+2k-2n-2l-2}$$
$$+\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} j \\ m \end{pmatrix}\begin{pmatrix} n+l \\ l \end{pmatrix}$$
$$\times(-1)^m v^b_{2j+2k-2m-2n-2l-2}\partial^{m+l}u^f_{2n}\Big]\,\delta(x-y) \tag{A.9}$$

$$\left\{u^b_{2j-1}(x), v^b_{2k}(y)\right\} =$$
$$\Big[-\sum_{m=0}^{j-1}\sum_{l=0}^{k}\begin{pmatrix} j-1 \\ m \end{pmatrix}.\begin{pmatrix} k \\ l \end{pmatrix}(-1)^m u^f_{2j-2m-2}\partial^{m+l}v^f_{2k-2l-1}$$
$$-\sum_{m=0}^{j-1}\sum_{l=0}^{k-1}\begin{pmatrix} j \\ m+1 \end{pmatrix}\begin{pmatrix} k \\ l+1 \end{pmatrix}(-1)^m u^b_{2j-2m-3}\partial^{m+l+1}v^b_{2k-2l-2}$$
$$-\sum_{n=0}^{k-1}\sum_{l=0}^{k-n-1}\begin{pmatrix} k-n-1 \\ l \end{pmatrix}u^f_{2n}\partial^l v^f_{2j+2k-2n-2l-3}$$
$$+\sum_{m=0}^{j+k-n-l-1}\sum_{n=0}^{k}\sum_{l=0}^{k-n}\begin{pmatrix} j-1 \\ m \end{pmatrix}\begin{pmatrix} n+l-1 \\ l \end{pmatrix}$$
$$\times(-1)^m u^f_{2j+2k-2m-2n-2l-2}\partial^{m+l}v^f_{2n-1}\Big]\,\delta(x-y) \tag{A.10}$$

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Appendix B

Poisson Brackets of Generators

The Poisson brackets between odd bosons and even fermions are

$$
\begin{aligned}
\{\tilde{W}_1, \tilde{J}_{3/2}\} &= -\tilde{J}_{3/2}\delta(x-y) && \text{(B.1)}\\
\{\tilde{W}_2, \tilde{J}_{3/2}\} &= \left[-\tilde{J}_{5/2} + \frac{1}{2}\tilde{J}'_{3/2} + \frac{1}{2}\tilde{J}_{3/2}\partial - \tilde{W}_1\tilde{J}_{3/2}\right]\delta(x-y) && \text{(B.2)}\\
\{\tilde{W}_1, \tilde{J}_{5/2}\} &= \left[-\tilde{J}_{5/2} + \tilde{J}'_{3/2} + \tilde{J}_{3/2}\partial\right]\delta(x-y) && \text{(B.3)}\\
\{\tilde{W}_2, \tilde{J}_{5/2}\} &= \left[-\tilde{J}_{7/2} + \frac{1}{2}\tilde{J}'_{5/2} + \frac{3}{4}\tilde{J}''_{3/2} - \tilde{W}_1\tilde{J}_{5/2}\right.\\
&\quad \left. + \frac{3}{2}\tilde{J}_{5/2}\partial - \tilde{J}'_{3/2}\partial - \frac{1}{2}\tilde{J}_{3/2}\partial^2\right]\delta(x-y) && \text{(B.4)}
\end{aligned}
$$

The Poisson brackets between odd bosons and odd fermions are:

$$
\begin{aligned}
\{\tilde{W}_1, J_{3/2}\} &= J_{3/2}\delta(x-y) && \text{(B.5)}\\
\{\tilde{W}_2, J_{3/2}\} &= \left[-J_{5/2} + \frac{1}{2}J'_{3/2} + \frac{1}{2}J_{3/2}\partial + \tilde{W}_1 J_{3/2}\right]\delta(x-y) && \text{(B.6)}\\
\{\tilde{W}_1, J_{5/2}\} &= \left[J_{5/2} - J'_{3/2} - J_{3/2}\partial\right]\delta(x-y) && \text{(B.7)}\\
\{\tilde{W}_2, J_{5/2}\} &= \left[-J_{7/2} + \frac{1}{2}J'_{5/2} + \frac{3}{4}J''_{3/2} + \tilde{W}_1 J_{5/2}\right.\\
&\quad \left. + \frac{3}{2}J_{5/2}\partial - J'_{3/2}\partial - \frac{1}{2}J_{3/2}\partial^2\right]\delta(x-y) && \text{(B.8)}
\end{aligned}
$$

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The Poisson brackets between even bosons and even fermions are:

$$\{W_2, \tilde{J}_{3/2}\} = \left[-\tilde{J}_{5/2} - \tilde{J}_{3/2}\partial - \tilde{W}_1\tilde{J}_{3/2}\right]\delta(x-y) \tag{B.9}$$

$$\{W_2, \tilde{J}_{5/2}\} = \left[-\tilde{J}_{7/2} - \tilde{J}'_{5/2} + \frac{5}{4}\tilde{J}''_{3/2} - \tilde{W}_1\tilde{J}_{5/2} - \tilde{J}_{5/2}\partial\right]\delta(x-y) \tag{B.10}$$

The Poisson brackets between even bosons and odd fermions are:

$$\{W_2, J_{3/2}\} = \left[-J_{5/2} - J_{3/2}\partial + \tilde{W}_1 J_{3/2}\right]\delta(x-y) \tag{B.11}$$

$$\{W_2, J_{5/2}\} = \left[-J_{7/2} - J'_{5/2} + \frac{5}{4}J''_{3/2} + \tilde{W}_1 J_{5/2} - J_{5/2}\partial\right]\delta(x-y) \tag{B.12}$$

The Poisson bracket between even fermions and odd fermions are:

$$\{\tilde{J}_{3/2}, J_{3/2}\} = \left[-\tilde{W}_2 + W_2 + \frac{1}{2}\tilde{W}'_1 + \tilde{W}_1\partial\right]\delta(x-y) \tag{B.13}$$

$$\begin{aligned}\{\tilde{J}_{3/2}, J_{5/2}\} = \Big[&\tilde{W}_3 - W_3 - \tilde{W}'_2 - \frac{1}{2}W'_2 - \tilde{W}_2\tilde{W}_1 + 2W_2\tilde{W}_1\\ &+\frac{3}{2}\tilde{W}'_1\tilde{W}_1 + \frac{1}{3}\tilde{W}''_1 - \left(\tilde{W}_2 + W_2 - \frac{1}{2}\tilde{W}'_1 - \tilde{W}_1^2\right)\partial\Big]\delta(x-y)\end{aligned} \tag{B.14}$$

$$\begin{aligned}\{\tilde{J}_{5/2}, J_{3/2}\} = \Big[&-\tilde{W}_3 + W_3 + \frac{1}{2}W'_2 + \tilde{W}_2\tilde{W}_1 - 2W_2\tilde{W}_1\\ &-\frac{1}{2}\tilde{W}'_1\tilde{W}_1 + \frac{1}{6}\tilde{W}''_1 + \left(\tilde{W}_2 + W_2 + \frac{1}{2}\tilde{W}'_1 - \tilde{W}_1^2\right)\partial\Big]\delta(x-y)\end{aligned} \tag{B.15}$$

$$\begin{aligned}\{\tilde{J}_{5/2}, J_{5/2}\} = \Big[&\tilde{W}_4 - W_4 - \frac{1}{2}\tilde{W}'_3 - W'_3 - \tilde{W}_3\tilde{W}_1 + 3W_2\tilde{W}_1\\ &+W_2\tilde{W}_2 + \frac{3}{2}W'_2\tilde{W}_1 + \frac{1}{6}\tilde{W}'_2\tilde{W}_1 + \frac{3}{2}W_2\tilde{W}'_1 + \tilde{W}_2\tilde{W}'_1 - 3W_2\tilde{W}_1^2\\ &+\tilde{W}_2\tilde{W}_1^2 - \frac{3}{10}W''_2 - \frac{1}{10}\tilde{W}''_2 - \tilde{W}_2^2 + \frac{1}{12}\tilde{W}''_1\tilde{W}_1 + \frac{3}{4}\tilde{W}_1^2\\ &+\frac{1}{12}\tilde{W}'''_1 - \frac{3}{2}\tilde{W}'_1\tilde{W}_1^2 - J'_{3/2}\tilde{J}_{3/2} - \Big(\tilde{W}_3 + 2W_3 - 3W_2\tilde{W}_1 - 2\tilde{W}_2\tilde{W}_1\\ &-\frac{1}{6}\tilde{W}''_1 + W'_2 - \tilde{W}_1^3 + J_{3/2}\tilde{J}_{3/2}\Big)\partial - W_2\partial^2\Big]\delta(x-y)\end{aligned} \tag{B.16}$$

and so on.

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2. "Integrability and Symmetry Algebra Associated with $N = 2$ KP Flows", Sasanka Ghosh and Debojit Sarma, Nucl. Phys. B 616 (2001) 549

3. "Soliton Solution for the $N = 2$ Supersymetric KdV Equation", Sasanka Ghosh and Debojit Sarma, Phys. Lett. B 522 (2001) 189

4. "Bilinearization of Supersymmetric KP Hierarchies Associated with Non-Trivial Flows", Sasanka Ghosh and Debojit Sarma, nlin.SI/0107019

5. "Bilinearization of $N = 1$ Supersymmetric Modified KdV Equations", Sasanka Ghosh and Debojit Sarma, nlin.SI/0202031

6. "Soliton Solutions for the $N = 2$ nonstandard flow KP Equation", Sasanka Ghosh and Debojit Sarma, under preparation.

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