DISTANCE PROBLEMS FOR HERMITIAN MATRIX POLYNOMIALS - AN ϵ-PSEUDOSPECTRA BASED APPROACH

A Thesis Submitted

in Partial Fulfillment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

by

Ravi Srivastava

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

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Contents

DECLARATION

It is certified that the work contained in the thesis titled **"Distance Problems For Hermitian Matrix Polynomials - An ϵ-Pseudospectra Based Approach"** has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under the guidance of **Dr. Shreemayee Bora** for the award of Doctor of Philosophy and that this work has not been submitted elsewhere for a degree.

August, 2012

Ravi Srivastava
Department of Mathematics
Indian Institute of Technology Guwahati

CERTIFICATE

It is certified that the work contained in the thesis titled **"Distance Problems For Hermitian Matrix Polynomials - An ϵ-Pseudospectra Based Approach"** by **Ravi Srivastava**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

August, 2012

Dr. Shreemayee Bora
Associate Professor
Department of Mathematics
Indian Institute of Technology Guwahati

Dedicated to my Mother.

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(Ravi Srivastava)

Abstract

Definite and definitizable pencils and hyperbolic, quasihyperbolic and definite polynomials are Hermitian matrix polynomials with real eigenvalues of definite type that arise in many applications in science and engineering. Given a member of any of these classes, it is therefore of practical importance to know the distance to a nearest Hermitian polynomial outside the class. These distance problems are analysed with respect to a specific norm in the setting of Hermitian ϵ-pseudospectra of these polynomials. Algorithms based on the bisection method are proposed for computing these distances and finding a nearest Hermitian polynomial outside the class. One of these algorithms computes the Crawford number which is the distance from a definite pencil to a nearest Hermitian pencil that is not definite. This algorithm also computes a nearest Hermitian pencil with a defective eigenvalue that attains the distance. The algorithms for computing the Crawford number and the solution of the distance problem for hyperbolic polynomials requires the computation of the smallest eigenvalue(s) of a positive definite matrix or the largest eigenvalue(s) of a negative definite matrix and corresponding eigenvectors at each step of the iteration.

A homogeneous definition of eigenvalue type for eigenvalues of Hermitian polynomials on the extended real line is proposed. Properties of Hermitian pencils based on their canonical form under congruence are investigated. Properties of the Hermitian ϵ-pseudospectra of regular Hermitian pencils and polynomials are also analysed with a view towards solving the distance problems.

Several bounds on the Crawford number are proposed. Some of these relate the Crawford number to the distribution of the eigenvalues of the definite pencil. A perturbation bound on the eigenvalues of the definite pencil in terms of the Crawford number is also derived in the setting of the Hermitian ϵ-pseudospectrum of the definite pencil.

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Chapter 1

Introduction

Given $n \times n$ matrices $A_k, k = 0, 1, \ldots, m$, the expression $P(z) = \Sigma_{k=0}^{m} z^k A_k$ is called a matrix polynomial [21] or a lambda-matrix [35] of size n. It has also been referred to as a polynomial matrix [44] as it may be also be viewed as a matrix whose entries are scalar polynomials. The associated polynomial eigenvalue problem (PEP) consists of finding a scalar λ and nonzero vectors x and y such that

$$P(\lambda)x = 0 \text{ and } y^* P(\lambda) = 0.$$

Such problems arise in solving a homogeneous system of linear differential equations [21]. Note that if $m = 1$, then the matrix polynomial becomes a matrix pencil $zA_1 + A_0$ and the polynomial eigenvalue problem reduces to the generalized eigenvalue problem (GEP) of finding a scalar λ and nonzero vectors x and y such that

$$(\lambda A_1 + A_0)x = 0 \text{ and } y^*(\lambda A_1 + A_0) = 0.$$

If A_0 is invertible, then this evidently reduces to the standard eigenvalue problem (SEP) for the matrix $A_0^{-1} A_1$. An important difference between the SEP and the PEP or GEP is that ∞ may be an eigenvalue in the latter two cases. This happens precisely when A_m is singular. Another crucial difference between the SEP and the other problems is that the set of eigenvalues of a matrix polynomial or pencil is either finite (with at most mn elements) or it is the entire complex plane. A matrix pencil or polynomials is said to be regular if its set of eigenvalues is not the entire complex plane.

It is well known that polynomial eigenvalue problems occur in a wide range of applications like vibration analysis of machines, buildings and vehicles, acoustics, thermodynamics, control theory, optimization, signal processing and a host of other applications [52, 18, 42]. In many instances, these applications give rise to matrix polynomials for which the coefficient matrices have additional properties like symmetry, skew-symmetry, Hermitian or skew-Hermitian structure to name a few [42]. This thesis is concerned with the case that all the coefficient matrices $A_k, k = 0, 1, \ldots, m$, of $P(z)$ are Hermitian. In such cases the eigenvalues of $P(z)$ are symmetrically placed with respect to the real line as they occur in pairs $(\lambda, \bar{\lambda})$. In particular, the real eigenvalues of $P(z)$ have the property that they are associated with a sign characteristic or eigenvalue type which is either positive, negative or mixed. The thesis work is mainly concerned with Hermitian matrix polynomials with real eigenvalues that are either of positive or negative type. These include the definite and definitizable pencils, hyperbolic, quasi-hyperbolic and definite polynomials and the overdamped quadratics. These classes of matrix polynomials are very important for a wide range of applications (for examples see [15, 33, 28, 29, 17, 35]). Therefore, it is important to know the distance from a given member of any of these classes to a nearest Hermitian pencil or polynomial outside the class with respect to a specified norm.

For the case of the definite pencils, the distance to a nearest Hermitian pencil that is not definite is called the Crawford number [38, 29]. This class deserves special mention as interest in the Crawford number goes back to more than 40 years since the time when Olga Taussky [50] first investigated pairs of $n \times n$ Hermitian matrices A and B with the property that the quadratic form $x^*(A+iB)x$ is never 0 for any nonzero vector x with n components. Definite pencils are defined to be Hermitian pencils with such coefficients and were first systematically studied by Crawford in his PhD thesis [13]. The Crawford number first appeared in [11] and came to be called so by Stewart [46]. Since then, there has been a lot of contribution to the theory of definite pencils. In particular the sensitivity analysis of eigenvalues of definite pencils have been investigated notably by Stewart and Sun ([47, 46, 48, 49] and Li [39, 40]. Relatively recent contributions to the theory of definite pencils and the Crawford number have been made in [38]. One of the important properties of definite pencils that emerge from these works is that they parallel the role of Hermitian matrices for the standard eigenvalue problems. In particular, if $L(z) = zA - B$ is a definite pencil where A is positive definite (which commonly happens in applications), then the eigenvalue problem is the same as the eigenvalue problem for the Hermitian matrix $zI - (G^*)^{-1}BG^{-1}$ where $*$ denotes complex conjugate transpose and G is the Cholesky factor of A. The problem of identifying a Hermitian pencil to be definite has been investigated in [12, 14, 29, 24]. There have also been several algorithms for computing the Crawford number [8, 10, 12, 14, 29, 24]. More recently, Uhlig [55] has proposed an algorithm that computes the generalized Crawford number of a Hermitian pencil $L(z) = zA - B$ which is the distance from 0 to the boundary of the field of values of $A+iB$ which is the set of all scalars of the form $x^*(A+iB)x$ for nonzero vectors x with n components. Since $L(z)$ is definite if and only if 0 is not in this set, the generalized Crawford number is nonpositive if $L(z)$ is not a definite pencil and equal to the Crawford number if $L(z)$ is a definite pencil.

The recent work of Ammari and Tisseur [4, 5] characterise definite pencils in terms of the distribution of their eigenvalues based on their type (Theorem 3.2 of [4]). Similar characterisations have also been made in [4] for the definitizable pencils (Theorem 4.2), hyperbolic polynomials (Theorem 3.4), quasihyperbolic polynomials (Theorem 4.6), definite polynomials (Theorem 3.8) and overdamped quadratics (Theorem 3.6). These characterisations have been crucial to the work done in this thesis.

Given a definite or definitizable pencil or a hyperbolic, quasihyperbolic, definite or quadratic overdamped polynomial, this thesis undertakes an analysis of its distance to a nearest Hermitian pencil/polynomial outside the class with respect to the norm $|||P||| = \sqrt{\Sigma_{k=0}^{m} \|A_k\|_2^2}$, where $\|\cdot\|_2$ denotes the induced 2-norm or spectral norm. These distance problems are analysed in the setting of the Hermitian ϵ-pseudospectrum of the pencil or polynomial which is the collection of all eigenvalues of perturbed Hermitian pencils or polynomials with perturbations strictly less than a given $\epsilon > 0$ with respect to the norm $|||\cdot|||$. In particular, we look into the relationship of these distances with the evolution of the components of the Hermitian ϵ-pseudospectrum and give a means of obtaining them from the ϵ-pseudospectra for all the classes except for certain types of definite polynomials. For such polynomials, we have an upper bound on the distance which we conjecture to be equal to the distance. In all cases where we find the distance from the pseudospectrum, we propose bisection type algorithms to compute it. For the definite pencils and hyperbolic polynomials of size n, this involves finding the smallest eigenvalue(s) of a positive definite matrix or the largest eigenvalue(s) of a negative definite matrix of size n and the corresponding eigenvectors at each step of the bisection. We also give the construction of a nearest Hermitian polynomial outside the class in all such cases. For the definite pencils, this construction results in a Hermitian pencil with an eigenvalue that is not semisimple. Numerical experiments for the definite pencils and quadratic hyperbolic polynomials suggest that a good approximation to the distance is obtained in the first few iterations of these algorithms.

The pseudospectra based analysis also leads to other interesting observations about Hermitian pencils and polynomials and their Hermitian ϵ-pseudospectra. In particular we have a definition of the eigenvalue type of real eigenvalues of such pencils based on the homogeneous form of the pencil which plays a crucial role in the analysis and computation of the distances. We also have estimates of the Crawford number of definite pencils in terms of upper and lower bounds. Some of these bounds link the Crawford number with the distribution of the eigenvalues of the pencil. One of the results which bound the distance between the eigenvalues of the definite pencil and those of a perturbed definite pencil in terms of the Crawford number is a slight modification of a similar result in [47] that is easily derived via the properties of the Hermitian ϵ-pseudospectrum of the pencil.

The thesis work is organized as follows. In Chapter 1 we introduce notations and give the preliminary results on which the work of the thesis is based. In Chapter 2, we undertake a linear perturbation of matrix pencils which is important for the work done in the following chapters. In Chapter 3, we analyse the distance problem for definite and definitizable pencils in terms of the Hermitian ϵ-pseudospectra of such pencils. In Chapter 4, we undertake a similar analysis for Hermitian matrix polynomials that relate the distance from a given hyperbolic, quasihyperbolic or definite polynomial or an overdamped quadratic polynomial to a nearest Hermitian polynomial outside the class to its Hermitian ϵ-pseudospectra. In Chapter 5, we propose algorithms based on the bisection method that compute these distances and find a nearest Hermitian polynomial outside the class in each case. It also contains upper and lower bounds on the Crawford number and results that relate the Crawford number with the distribution of its eigenvalues.

1.1 Notations

The following notations have been used throughout the thesis.

- $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers respectively.
- $\mathbb{R}^n$ and $\mathbb{C}^n$ are the set of real and complex vectors of length n respectively.
- $\mathbb{C}^{m\times n}$ is the set of complex matrices of size $m \times n$.
- A^* denotes the complex conjugate transpose of the matrix A.

- A Hermitian matrix is a matrix $A \in \mathbb{C}^{n\times n}$ such that $A^* = A$.
- $\operatorname{Ker} A$ denotes the kernel of the matrix A.
- $A > 0$ implies that $A \in \mathbb{C}^{n\times n}$ is a positive definite matrix, i.e., $A^* = A$ with $x^*Ax > 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$.
- $A < 0$ implies that $A \in \mathbb{C}^{n\times n}$ is negative definite, i.e., $-A > 0$.
- A is a definite matrix if $A \in \mathbb{C}^{n\times n}$ is positive or negative definite.
- A is an indefinite matrix if A is Hermitian but not positive or negative definite.
- Matrices A and B are of opposite *parity* if one of them is negative definite and the other is positive definite.
- A^{-1} denotes the inverse of A.
- I_n denotes the identity matrix of size n.
- 0_n denotes the zero matrix of size n.
- e_i denotes the i^{th} column of an identity matrix.
- $\det(A)$ denotes the determinant of $A \in \mathbb{C}^{n\times n}$.
- $\Lambda(A)$ denotes the set of eigenvalues of A.
- $\sigma_{\min}(A)$ denotes the smallest singular value of A.
- $\lambda_{\min}(A)$ denote the eigenvalue of A with the smallest modulus.
- $\|\cdot\|_2$ denotes the 2-norm of a vector or a matrix.
- $L(z)$ denotes the pencil $L(z) = zA - B$ of size n where $A, B \in \mathbb{C}^{n\times n}$.
- $\Delta L(z)$ denotes the pencil $\Delta L(z) = z\Delta A - \Delta B$ where $\Delta A, \Delta B \in \mathbb{C}^{n\times n}$.
- The homogeneous form of the pencil $L(z)$ is $L(\alpha, \beta) = \alpha A - \beta B$.
- $|||L|||$ denotes the norm $|||L||| = \sqrt{\|A\|_2^2 + \|B\|_2^2}$ of the pencil $L(z)$.
- A Hermitian pencil $L(z)$ is a pencil $L(z) = zA - B$ where A and B are Hermitian.

- A *regular* matrix pencil $L(z)$ is one for which $\det(L(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$.

- $\Lambda_\epsilon(L)$ denotes the ϵ-pseudospectrum of the pencil $L(z)$ with respect to the norm $|||\cdot|||$.

- $\Lambda_\epsilon^H(L)$ denotes the Hermitian ϵ-pseudospectrum of the Hermitian pencil $L(z)$ with respect to the norm $|||\cdot|||$.

- $P(z)$ denotes the matrix polynomial $P(z) = \Sigma_{k=0}^m z^k A_k$ of size n where $A_k \in \mathbb{C}^{n\times n}$, $k = 0, 1, \ldots, m$.

- $\Delta P(z)$ denotes the matrix polynomial $\Delta P(z) = \Sigma_{k=0}^m z^k \Delta A_k$ of size n where $\Delta A_k \in \mathbb{C}^{n\times n}$, $k = 0, 1, \ldots, m$.

- The degree of $P(z)$ is m if $A_m \neq 0$.

- A regular matrix polynomial $P(z)$ is one for which $\det P(\lambda) \neq 0$ for some $\lambda \in \mathbb{C}$.

- A Hermitian matrix polynomial $P(z)$ is a polynomial $P(z) = \Sigma_{k=0}^m z^k A_k$ where A_k is Hermitian for each $k = 0, 1, \ldots, m$.

- $\operatorname{rev} P(z)$ denotes the reversal of $P(z)$ defined by $\operatorname{rev} P(z) = z^m \Sigma_{k=0}^m \frac{1}{z^k} A_k$.

- The homogeneous form of the matrix polynomial $P(z)$ is $P(\alpha, \beta) = \Sigma_{k=0}^m \alpha^k \beta^{m-k} A_k$.

- $|||P|||$ denotes the norm $|||P||| = \sqrt{\Sigma_{k=0}^m \|A_k\|_2^2}$ of the polynomial $P(z)$.

- $\Lambda_\epsilon(P)$ denotes the ϵ-pseudospectrum of the matrix polynomial $P(z)$ with respect to the norm $|||\cdot|||$.

- $\Lambda_\epsilon^H(P)$ denotes the ϵ-pseudospectrum of the Hermitian matrix polynomial $P(z)$ with respect to the norm $|||\cdot|||$.

- Given two matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$, $A \otimes B$ denotes the Kronecker product of A and B defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}$$

where $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$. [30]

- Given two matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$, $A \oplus B$ denotes their direct sum defined by
$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}.$$
- Given a vector space V, $\dim V$ denotes the dimension of V.
- Given subspaces $\mathcal{S}_k, k = 1, 2, \ldots, m$ of a vector space V, $\oplus_{k=1}^{m}\mathcal{S}_k$ will denote the direct sum of $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$.
- $L'(\lambda_0)$ and $P'(\lambda_0)$ will denote the derivatives of $L(\lambda)$ and $P(\lambda)$ at $\lambda = \lambda_0$ respectively.
- $(zA - B)|\mathcal{S}$ will denote the restriction of $L(z)$ to the subspace $\mathcal{S}$.

1.2 Preliminaries

1.2.1 Regular matrix pencils

Given a regular matrix pencil $L(z) = zA - B$, its finite eigenvalue are the roots of the polynomial $\det(zA - B) = 0$. Given a finite eigenvalue λ of $L(z)$ any nonzero vector x in $\mathrm{Ker}\,\lambda A - B$ is a right eigenvector of $L(z)$ corresponding to λ. Any nonzero vector y in $\mathrm{Ker}\,(\lambda A - B)^*$ is a left eigenvector of $L(z)$ corresponding to λ. The pencil has an eigenvalue at ∞ if and only if 0 is an eigenvalue of $-\mathrm{rev}\,L(z)$, i.e., if and only if A is a singular matrix.

The algebraic multiplicity of a finite eigenvalue λ of $L(z)$ is its multiplicity as a root of $\det(zA - B) = 0$. The algebraic multiplicity of an infinite eigenvalue is defined to be that of 0 as an eigenvalue of $-\mathrm{rev}L(z)$. The eigenvalue λ is said to be a multiple eigenvalue of $L(z)$ if its algebraic multiplicity is more than 1. In fact an eigenvalue λ of $L(z)$ is a multiple eigenvalue if and only if there exists a pair of left and right eigenvectors x and y corresponding to λ such that $y^*Ax = 0$ [2, 3].

A set of vectors $\{x_1, x_2, \ldots, x_m\}$ is said to be a Jordan chain of $L(z)$ of length m corresponding to an eigenvalue λ if $x_1 \in \mathrm{Ker}\,(\lambda A - B) \setminus \{0\}$ and $L(\lambda)x_k + L'(\lambda)x_{k-1} = 0$ for $k = 2, 3, \ldots, m$. The vectors $x_2, x_3, \ldots, x_m$ are called generalised eigenvectors of $L(z)$ corresponding to λ. An eigenvalue λ of $L(z)$ is said to be defective if it has a Jordan chain of length greater than one. An eigenvalue of $L(z)$ is said to be semisimple if it is not defective.

A subspace $\mathcal{S}$ is said to be a deflating subspace for the pencil $L(z)$ if there exists a subspace $\mathcal{T}$ such that $\dim \mathcal{S} = \dim \mathcal{T}$ and $A\mathcal{S} \subset \mathcal{T}$, $B\mathcal{S} \subset \mathcal{T}$. If $\{x_1, x_2, \ldots, x_m\}$ be eigenvectors or a collection of Jordan chains of $L(z)$ then the subspace spanned by them is a deflating subspace. The spectral subspace corresponding to an infinite eigenvalue is the spectral subspace of 0 as an eigenvalue of $-\mathrm{rev}\, L(z)$. The following result due to Stewart [47] explains the significance of deflating subspaces and justifies the choice of their name.

Theorem 1.2.1. *Let $L(z)$ be a regular matrix pencil. If $\mathcal{S}$ be a deflating subspace of $L(z)$ of dimension r, then there exist nonsingular matrices X and Y such that*

$$YAX = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \; YBX = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}, \tag{1.1}$$

where $A_1, B_1 \in \mathbb{C}^{r\times r}$ and $\mathcal{S}$ is spanned by the first r columns of X. Conversely if (1.1) holds for some nonsingular matrices X and Y then the first r columns of X span a deflating subspace of $L(z)$.

If (1.1) holds, then the eigenvalues of $L(z)$ are clearly the union of the eigenvalues of the pencils $zA_1 - B_1$ and $zA_3 - B_3$ resulting from the deflation. The next result may be obtained as a corollary of Theorem 1.2.1.

Theorem 1.2.2. *Let $\mathcal{S}$ be a r dimensional deflating subspace of $L(z)$ such that $(\mathrm{Ker}\, A) \cap \mathcal{S} = \emptyset$. Then there exists $X_1 \in \mathbb{C}^{n\times r}$ of rank r and $K \in \mathbb{C}^{r\times r}$, such that $\mathrm{span}\, X_1 = \mathcal{S}$, and $AX_1K = BX_1$. Moreover, $K = A_1^{-1}B_1$ where the pencil $zA_1 - B_1 = (zA - B)|\mathcal{S}$.*

The spectral (deflating) subspace of $L(z)$ corresponding to a finite eigenvalue λ is the space spanned by all the eigenvectors and generalized eigenvectors of $L(z)$ corresponding to λ. The dimension of this subspace is the algebraic multiplicity of λ. The following result gives the Spectral Decomposition Theorem for regular matrix pencils.

Theorem 1.2.3. *[36] Let $L(z)$ be a regular pencil and $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the distinct eigenvalues of $L(z)$ including the eigenvalue at ∞ if A is singular. Then $\mathbb{C}^n$ admits the direct sum decomposition*

$$\mathbb{C}^n = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_m,$$

where $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$ are spectral subspaces of $L(z)$ corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively such that $(zA - B)|\mathcal{S}_j$ has the only eigenvalue λ_j, $j = 1, 2, \ldots, m$. If $\lambda_j \in$

$\mathbb{C}$, $1 \le j \le m$, *then,*

$$P_j = \frac{1}{2\pi i}\int_{\Gamma_j}(zA-B)^{-1}A dz$$

is the projection onto the subspace $\mathcal{S}_j$.

If $L_1(z) = X^*L(z)X$ where X is nonsingular, then $L_1(z)$ is said to be congruent to $L(z)$. If $L(z)$ is a Hermitian pencil then such congruences preserve the Hermitian form. We now state the canonical form for Hermitian pencils under congruence.

Theorem 1.2.4. *[37] Every Hermitian pencil* $L(z) = zA - B$ *is congruent to a Hermitian matrix pencil of the form*

$$\begin{aligned}
0_u \;&\oplus\; \left(z\begin{bmatrix} 0 & 0 & F_{\epsilon_1} \\ 0 & 0 & 0 \\ F_{\epsilon_1} & 0 & 0 \end{bmatrix} - G_{2\epsilon_1+1} \right) \oplus \cdots \oplus \left(z\begin{bmatrix} 0 & 0 & F_{\epsilon_p} \\ 0 & 0 & 0 \\ F_{\epsilon_p} & 0 & 0 \end{bmatrix} - G_{2\epsilon_p+1} \right) \\
&\oplus\; \delta_1(zG_{k_1} - F_{k_1}) \oplus \cdots \oplus \delta_r(zG_{k_r} - F_{k_r}) \\
&\oplus\; \eta_1((z-\alpha_1)F_{l_1} - G_{l_1}) \oplus \cdots \oplus \eta_q((z-\alpha_q)F_{l_q} - G_{l_q}) \\
&\oplus\; \left(\begin{bmatrix} 0 & (z-\beta_1)F_{m_1} \\ (z-\bar{\beta}_1)F_{m_1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & G_{m_1} \\ G_{m_1} & 0 \end{bmatrix} \right) \\
&\oplus\; \cdots \oplus \left(\begin{bmatrix} 0 & (z-\beta_s)F_{m_s} \\ (z-\bar{\beta}_s)F_{m_s} & 0 \end{bmatrix} - \begin{bmatrix} 0 & G_{m_s} \\ G_{m_s} & 0 \end{bmatrix} \right).
\end{aligned} \tag{1.2}$$

Here, $\epsilon_1 \le \cdots \le \epsilon_p$ *and* $k_1 \le \cdots \le k_r$ *are positive integers,* α_j *are real numbers,* β_j *are complex nonreal numbers,* $\delta_1, \delta_2, \ldots, \delta_r, \eta_1, \eta_2, \ldots, \eta_q$ *are signs each equal to* $+1$ *or* -1 *called the sign characteristics of* $L(z)$*, and* F_m *and* G_m *are the* $m \times m$ *matrices*

$$F_m = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad G_m = \begin{bmatrix} F_{m-1} & 0 \\ 0 & 0 \end{bmatrix}$$

with $F_1 = [1]$.

The form (1.2) is uniquely determined by $L(z)$ *up to a combination of the following permutations: a permutation of the blocks*

$$z\begin{bmatrix} 0 & 0 & F_{\epsilon_j} \\ 0 & 0 & 0 \\ F_{\epsilon_j} & 0 & 0 \end{bmatrix} - G_{2\epsilon_j+1}, \quad j = 1, 2, \ldots, p;$$

a permutation of the blocks

$$\delta_j(zG_{k_j} - F_{k_j}), \;\; j = 1, 2, \ldots, q;$$

a permutation of the blocks

$$\eta_j((z - \alpha_j)F_{l_j} - G_{l_j}), \;\; j = 1, 2, \ldots, r;$$

and a permutation of the blocks

$$\begin{bmatrix} 0 & (z - \beta_j)F_{m_j} \\ (z - \bar{\beta}_j)F_{m_j} & 0 \end{bmatrix} - \begin{bmatrix} 0 & G_{m_j} \\ G_{m_j} & 0 \end{bmatrix}, \;\; j = 1, 2, \ldots, s;$$

with possible replacement of β_j by $\bar{\beta}_j$ within each such block.

For proofs we refer to Trott [53], Turnbull [54], Ingraham and Wegner [31], [43], Williamson [58, 59], Dieudonné [16]. For an extensive list of references see [51].

1.2.2 Eigenvalue type and sign characteristic

Given a regular matrix polynomial $P(z)$, its finite eigenvalues are the roots of the equation $\det(P(z)) = 0$. Therefore, $P(z)$ has at most mn finite eigenvalues. Each such eigenvalue λ is associated with nonzero vectors x and y where $x \in \operatorname{Ker} P(\lambda)$ and $y \in \operatorname{Ker}(P(\lambda))^*$ and are respectively called the right and left eigenvectors of $P(z)$ corresponding to λ. The polynomial $P(z)$ has an eigenvalue at ∞ if and only if $\operatorname{rev} P(z)$ has an eigenvalue at 0. Equivalently, ∞ is an eigenvalue of $P(z)$ if and only if A_m is singular.

Given a finite eigenvalue λ of $P(z)$, its algebraic multiplicity is the multiplicity of λ as a root of $\det P(z) = 0$. The algebraic multiplicity of an eigenvalue at ∞ is that of 0 as an eigenvalue of $\operatorname{rev} P(z)$.

Given an eigenvalue λ of $P(z)$ with corresponding eigenvector x_0, the set of vectors $\{x_0, x_1 ...... x_m\}$ satisfying the relations

$$\sum_{l=0}^{j} \frac{1}{l!} \frac{d^l P}{dz^l}(\lambda) x_{j-l} = 0, \;\; j = 0, 1, ..., m$$

is called a Jordan chain of length $m + 1$ corresponding to λ. The vectors $x_1, \ldots, x_m$ are called generalized eigenvectors of $P(z)$ corresponding to λ. The eigenvalue λ is said to be defective if it has a Jordan chain of length at least 2. An eigenvalue of $P(z)$ is said to be semisimple if it is not defective.

If $P(z)$ is a Hermitian matrix polynomial, then a complex number λ is an eigenvalue of $P(z)$ if and only if $\bar{\lambda}$ is also an eigenvalue of $P(z)$. Also if x and y are respectively right and left eigenvectors corresponding to λ, then y and x are respectively right and left eigenvectors corresponding to $\bar{\lambda}$. In particular, if λ is real, then x is both a left and right eigenvector of $P(z)$. Real eigenvalues of $P(z)$ are associated with a sign characteristic or eigenvalue type.

Definition 1.2.5. *A finite real eigenvalue λ of a Hermitian polynomial $P(z)$ is of positive type*(negative type) *if $x^*P'(\lambda)x > 0(x^*P'(\lambda)x < 0)$ for all nonzero $x \in KerP(\lambda)$, respectively.*

Definition 1.2.6. *A real eigenvalue of a Hermitian matrix polynomial is of definite type if it is either of positive type or of negative type. It is said to be of mixed type if it is not of definite type.*

It may be easily established that if λ is a real eigenvalue of definite type, then $x^*P'(\lambda)x \neq 0$ for all nonzero $x \in KerP(\lambda)$ ([4], Lemma 2.3). Moreover in such a case λ is a semisimple eigenvalue of $P(z)$ ([7], Theorem 3.2).

If λ be a nonzero eigenvalue of a Hermitian polynomial $P(z)$ with corresponding right eigenvector x, we have

$$x^*P'(\lambda)x = -\lambda^{m-2}x^*(\mathrm{rev}P)'(1/\lambda)x. \tag{1.3}$$

In view of this identity, the type of an infinite eigenvalue of a Hermitian matrix polynomial is defined as follows.

Definition 1.2.7. *The type of an infinite eigenvalue $\lambda_0 = \infty$ of a Hermitian matrix polynomial $P(z)$ is defined to be the type of 0 as an eigenvalue of $-\mathrm{rev}P(z)$.*

For details on sign characteristic or eigenvalue types of real eigenvalues we refer to [22] and [21].

1.2.3 Linearizations of matrix polynomials

Given a $n \times n$ matrix polynomial $P(z) = \Sigma_{k=0}^{m} z^k A_k$ of degree m, the standard procedure for solving the associated polynomial eigenvalue problem is to convert it into an equivalent eigenvalue problem involving a matrix pencil. More specifically, a pencil $L(z)$ is a linearization of $P(z)$ if it satisfies

$$E(z)L(z)F(z) = \begin{bmatrix} P(z) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix}$$

for some matrix polynomials $E(z)$ and $F(z)$ with constant nonzero determinants. Clearly, $L(z)$ and $P(z)$ have the same eigenvalues.

The Jordan chains associated with the finite eigenvalues of $P(z)$ may be recovered from any linearization of $P(z)$ [21]. However to do the same for an infinite eigenvalue, the linearization needs to be a strong linearization [19, 34] which is defined to be a linearization of $P(z)$ with the additional property that $\operatorname{rev} L(z)$ is a linearization of $\operatorname{rev} P(z)$.

One of the simplest forms of linearizations are generalization of the first and second companion matrices for scalar polynomials. Indeed, setting $X_1 = X_2 = \begin{bmatrix} A_m & 0 \\ 0 & I_{(m-1)n} \end{bmatrix}$, and

$$Y_1 = \begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I_n & 0 \end{bmatrix}, \; Y_2 = \begin{bmatrix} A_{m-1} & -I_n & \cdots & 0 \\ A_{m-2} & 0 \ddots & & \vdots \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \cdots & 0 \end{bmatrix}$$

the pencils $L_1(z) = zX_1 + Y_1$ and $L_2(z) = zX_2 + Y_2$ are respectively the first and second companion linearizations of $P(z)$.

However, there exist many other linearizations of $P(z)$. Mackey et. al [41] have introduced the vector spaces $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ defined by

$$\mathbb{L}_1(P) = \{zX + Y : X, Y \in \mathbb{C}^{mn\times mn},\ L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{C}^m\},$$

$$\mathbb{L}_2(P) = \{zX + Y : X, Y \in \mathbb{C}^{mn\times mn},\ L(\lambda)(\Lambda^T \otimes I_n) = w^T \otimes P(\lambda),\ w \in \mathbb{C}^m\},$$

where $\Lambda = [\lambda^{m-1}, \lambda^{m-2}, ..., 1]^T \in \mathbb{C}^m$ which together with the space $\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$ provide a rich source of linearizations of $P(z)$. The vectors v and w are respectively called the right and left ansatz vectors. The linearizations of $P(z)$ in $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ generalize the first and second companion linearizations $L_1(z)$ and $L_2(z)$ respectively as $L_1(z) \in \mathbb{L}_1(P)$ and $L_2(z) \in \mathbb{L}_2(P)$ with the common ansatz vector e_1.

Mackey et. al ([41], Theorem 4.3, 4.7 and 6.8) have shown that if $P(z)$ is regular then all pencils belonging to these spaces are strong linearizations of $P(z)$ for all ansatz vectors $v \in \mathbb{C}^n$ except for those belonging to a closed nowhere dense subset of $\mathbb{C}^n$ of measure zero.

They have also shown that given $L(z) \in \mathbb{L}_1(P)$ with nonzero right ansatz vector v, then $x \in \mathbb{C}^n$ is a right eigenvector for $P(z)$ with finite eigenvalue λ, if and only if $\Lambda \otimes x$ is a right eigenvector for $L(z)$ with eigenvalue λ. Moreover, if $P(z)$ is regular and $L(z) \in \mathbb{L}_1(P)$ is a linearization for $P(z)$, then every eigenvector of $L(z)$ with finite eigenvalue λ is of the form $\Lambda \otimes x$ for some eigenvector x of $P(z)$ ([41], Theorem 3.8). The same holds for the left eigenvectors with respect to the linearizations in $\mathbb{L}_2(P)$ ([41], Theorem 3.14). These results extend to case $\lambda = \infty$ with Λ replaced by the vector e_1 ([41], Theorem 4.4).

The criterion for deciding whether a pencil in $\mathbb{DL}(P)$ is a linearization of $P(z)$ is given by the following theorem ([41], Theorem 6.7).

Theorem 1.2.8. *Suppose that $P(z)$ is a regular matrix polynomial and $L(z) \in \mathbb{DL}(P)$ with nonzero ansatz vector v. Then $L(z)$ is a linearization for $P(z)$ if and only if no root of the polynomial (including a root at ∞)*

$$p(\lambda; v) = \Lambda^T v = \lambda^{m-1} v_1 + \lambda^{m-2} v_2 + \cdots + \lambda v_{m-1} + v_m$$

is an eigenvalue of $P(z)$ with the convention that $p(\lambda; v)$ has a root at ∞ if $v_1 = 0$.

However, if $P(z)$ is Hermitian, we would like to seek Hermitian linearizations for $P(z)$. The work in [41] has been extended by Higham et. al ([26]) to show that in such a case, the following subspace of $\mathbb{L}_1(P)$

$$\mathbb{H}(P) := \{L(z) = zA - B \in \mathbb{L}_1(P) : A^* = A,\ B^* = B\}$$

is a rich source of Hermitian linearizations [26]. In fact it is shown that $\mathbb{H}(P)$ is exactly the set of linearizations in $\mathbb{DL}(P)$ corresponding to a real ansatz vector $v \in \mathbb{R}^n$. ([26], Theorem 6.1). Moreover, if $P(z)$ is regular, then every pencil in $\mathbb{H}(P)$ is a Hermitian linearization of $P(z)$ for all ansatz vectors belonging to $\mathbb{R}^n$ except for a closed nowhere dense subset of $\mathbb{R}^n$ of measure zero ([26], Theorem 7.2). The space $\mathbb{H}(P)$ may also be expressed as

$$\mathbb{H}(P) = \{\Sigma_{j=1}^m v_j \mathcal{L}_j(z) : v \in \mathbb{R}^m\}$$

where $\mathcal{L}_j(z) = z\mathcal{A}_j - \mathcal{B}_j$ are the pencils in $\mathbb{DL}(P)$ with corresponding ansatz vectors

e_j, $j = 1, 2, \ldots, m$ (for details see (3.6) - (3.8) of [26]). In particular,

$$\mathcal{L}_1(z) = \left[\begin{array}{ccccc} A_m & & & & \\ & -A_{m-2} & \cdots & -A_1 & -A_0 \\ & \vdots & \iddots & \iddots & \\ & -A_1 & \iddots & & \\ & -A_0 & & & \end{array}\right] - \left[\begin{array}{cccc} -A_{m-1} & \cdots & -A_1 & -A_0 \\ \vdots & \iddots & \iddots & \\ -A_1 & \iddots & & \\ -A_0 & & & \end{array}\right]$$

and

$$\mathcal{L}_m(z) = z \left[\begin{array}{ccccc} & & & & A_m \\ & & & \iddots & A_{m-1} \\ & & \iddots & \iddots & \vdots \\ & \iddots & \iddots & & A_2 \\ A_m & A_{m-1} & \cdots & A_2 & A_1 \end{array}\right] - \left[\begin{array}{ccccc} & & & A_m & \\ & & \iddots & A_{m-1} & \\ & \iddots & \iddots & \vdots & \\ A_m & A_{m-1} & \cdots & A_2 & \\ & & & & -A_0 \end{array}\right].$$

The following result is of great significance in relating the eigenvalue types of real eigenvalues of Hermitian polynomials and their linearizations.

Theorem 1.2.9. *Let λ be a eigenvalue of a Hermitian polynomial $P(z)$ with corresponding eigenvector x and let $\mathcal{L}(z) \in \mathbb{H}(P)$ with corresponding ansatz vector v be a linearization of $P(z)$. Then,*

$$x^* P'(\lambda) x = p(\lambda; v) w^* \mathcal{L}'(\lambda) w$$

where $w = [\lambda^{m-1} \lambda^{m-2} \cdots 1]^T \otimes x$.

For a proof, we refer to Section 3 of [27].

1.2.4 Homogeneous rotations

Consider the homogenous forms of the matrix polynomial $P(z) = \sum_{k=0}^{m} z^k A_k$, where $A_m \neq 0$, and pencil $L(z) = zA - B$, which are given by

$$P(\alpha, \beta) = \sum_{k=0}^{m} \alpha^k \beta^{m-k} A_k \ , \ L(\alpha, \beta) = \alpha A - \beta B.$$

An eigenvalue λ is identified with any pair $(\alpha, \beta) \neq (0, 0)$ for which $\lambda = \alpha/\beta$. The eigenvalue 0 is identified with the pair $(0, \beta)$ whereas an infinite eigenvalue may be identified with the pair $(\alpha, 0)$. If the eigenvalues of $P(z)$ are all on the extended real line, then $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{0, 0\}$. Without loss of generality we may assume that $\alpha^2 + \beta^2 = 1$ so that such eigenvalues correspond to points on the unit circle.

The matrix polynomial $\tilde{P}(\tilde{\alpha}, \tilde{\beta})$ is obtained from $P(\alpha, \beta)$ by homogenous rotation if

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = G \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \qquad c, s \in \mathbb{R},\ c^2 + s^2 = 1 \tag{1.4}$$

and

$$P(\alpha, \beta) = \sum_{j=0}^{m} \alpha^j \beta^{m-j} A_j =: \sum_{j=0}^{m} (c\tilde{\alpha} - s\tilde{\beta})^j (s\tilde{\alpha} + c\tilde{\beta})^{m-j} A_j =: \sum_{j=0}^{m} \tilde{\alpha}^j \tilde{\beta}^{m-j} \tilde{A}_j := \tilde{P}(\tilde{\alpha}, \tilde{\beta}). \tag{1.5}$$

The polynomials $P(\alpha, \beta)$ and $\tilde{P}(\tilde{\alpha}, \tilde{\beta})$ have the same eigenvectors but the corresponding eigenvalues are rotated. The relationship between the coefficient matrices A_k of $P(\alpha, \beta)$ and $\tilde{A}_k$ of the rotated polynomial $\tilde{P}(\tilde{\alpha}, \tilde{\beta})$ may be obtained from the equality $P(\alpha, \beta) = \tilde{P}(\tilde{\alpha}, \tilde{\beta})$, and the binomial expansion theorem. In particular,

$$\begin{aligned} \tilde{A}_m &= P(c, s), \\ \tilde{A}_{m-1} &= \sum_{j=0}^{m} (-jc^{j-1} s^{m+1-j} + (m-j) c^{j+1} s^{m-j-1}) A_j, \\ \tilde{A}_0 &= P(s, -c). \end{aligned}$$

The following result from [4] relates the types of the eigenvalue of a Hermitian pencil with the corresponding eigenvalues of its rotated counterpart and shows that although rotations do not preserve the eigenvalue type, they always preserve definite type.

Lemma 1.2.10. *Let $\tilde{P}(z)$ of degree m be obtained from $P(z)$ by a homogeneous rotation (1.4). Let the real numbers $\lambda_0 = \frac{\alpha_0}{\beta_0}$ and $\tilde{\lambda}_0 = \frac{\tilde{\alpha}_0}{\tilde{\beta}_0}$ with $\begin{bmatrix} \tilde{\alpha}_0 \\ \tilde{\beta}_0 \end{bmatrix} = G \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}$ be eigenvalues of $P(z)$ and $\tilde{P}(z)$ respectively, with corresponding eigenvector x.*

(i) If λ_0 and $\tilde{\lambda}_0$ are both real and finite then $c - \lambda_0 s \neq 0$ and

$$x^* P'_\lambda(\lambda_0) x = (c - \lambda_0 s)^{m-2} x^* \tilde{P}'_{\tilde{\lambda}}(\tilde{\lambda}_0) x.$$

(ii) If λ_0 is real and finite and $\tilde{\lambda}_0 = \infty$ then $s \neq 0$ and

$$x^* P'_\lambda(\lambda_0) x = s^{2-m} x^* \left(-(\mathrm{rev}\tilde{P})'_{\tilde{\lambda}}(0) \right) x.$$

(iii) If $\lambda_0 = \infty$ and $\tilde{\lambda}_0$ is real and finite then $s \neq 0$ and

$$x^* \left(-(\mathrm{rev}P)'_{\lambda}(0)\right) x = (-s)^{m-2} x^* \tilde{P}'_{\tilde{\lambda}}(\tilde{\lambda}_0) x.$$

1.2.5 Backward error and ϵ-pseudospectrum

Given a $n \times n$ matrix polynomial $P(z) = \Sigma_{k=0}^{m} z^k A_k$, let

$$|||P||| = \sqrt{\Sigma_{k=0}^{m} \|A_k\|_2^2}.$$

Then $||| \cdot |||$ is a norm on vector space of all $n \times n$ matrix polynomials $P(z)$ of degree less than or equal to m. Given a complex number $\tilde{\lambda} \in \mathbb{C}$ the normwise backward error of $\tilde{\lambda}$ with respect to the norm $||| \cdot |||$ considered as an approximate eigenvalue of $P(z)$ is defined as follows.

$$\eta(\tilde{\lambda}, P) \quad = \quad \min\{|||\Delta P||| : \Delta P(z) = \Sigma_{k=0}^{m} z^k \Delta A_k \text{ such that } \lambda \in \Lambda(P + \Delta P)\}. \quad (1.6)$$

It is given by the following formula [2].

$$\eta(\tilde{\lambda}, P) = \frac{\sigma_{\min}(P(\tilde{\lambda}))}{\sqrt{\Sigma_{k=0}^{m} |\tilde{\lambda}|^{2k}}}.$$

The backward error $\eta(\infty, P)$ of ∞ is defined to be the backward error of 0 considered as an approximate eigenvalue of $\mathrm{rev}P(z)$. The homogeneous form of the backward error of λ is often more convenient as it deals with the cases when $\lambda \in \mathbb{C}$ and $\lambda = \infty$ in the same framework. It is given by the formula

$$\eta(\alpha, \beta, P) = \frac{\sigma_{\min}(P(\alpha, \beta))}{\sqrt{\Sigma_{k=0}^{m} |\alpha|^{2(m-k)} |\beta|^{2k}}}. \quad (1.7)$$

where $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0, 0\}$ is any homogeneous representation of $\lambda \in \mathbb{C} \cup \{\infty\}$.

If $P(z)$ is Hermitian, then it is natural to consider the backward error with respect to perturbations that preserve the Hermitian structure of $P(z)$. Thus we have the following obvious definition of the Hermitian backward error of an approximate eigenvalue $\tilde{\lambda}$.

$$\begin{aligned} \eta^H(\tilde{\lambda}, P) \quad = \quad & \min\left\{|||\Delta P||| : \Delta P(z) = \Sigma_{k=0}^{m} z^k \Delta A_k, \Delta A_k^* = \Delta A_k, k = 0, 1, \ldots, m, \right. \\ & \left. \text{such that } \lambda \in \Lambda(P + \Delta P)\right\}. \end{aligned}$$

It is well known (for example, see [1]) that if $\tilde{\lambda} \in \mathbb{R} \cup \{\infty\}$, then $\eta(\tilde{\lambda}, P) = \eta^H(\tilde{\lambda}, P)$. In such a case, since $\tilde{\lambda}$ may be identified with a point $(\cos\tilde{\theta}, \sin\tilde{\theta})$ on the unit circle, setting $\tilde{c} = \cos\,\tilde{\theta}$, $\tilde{s} = \sin\,\tilde{\theta}$, and $P(\tilde{\theta}) = P(\tilde{c}, \tilde{s})$, the homogeneous form of $\eta^H(\tilde{\lambda}, P)$ may be written as

$$\eta(\tilde{\theta}, P) = \frac{\sigma_{\min}(P(\tilde{\theta}))}{\sqrt{\Sigma_{k=0}^{m} \tilde{c}^{2(m-k)} \tilde{s}^{2k}}}.$$

Note that $P(\tilde{\theta})$ is a Hermitian pencil. Hence, $\sigma_{\min}(P(\tilde{\theta}))$ may be replaced by $|\lambda_{\min}(P(\tilde{\theta}))|$ in the above expression where $\lambda_{\min}(P(\tilde{\theta}))$ is the eigenvalue of $P(\tilde{\theta})$ of smallest modulus. From the definition of the backward error, it follows that for $z \neq 0$, $\eta(z, P) = \eta(1/z, \mathrm{rev}\, P) = \eta(1/z, -\mathrm{rev}\, P)$.

Given a regular polynomial $P(z)$, the ϵ-pseudospectrum of $P(z)$ is defined as

$$\Lambda_\epsilon(P) = \bigcup \{\Lambda(P + \Delta P) : |||\Delta P||| < \epsilon\}.$$

Obviously this definition is equivalent to

$$\Lambda_\epsilon(P) = \{z \in \mathbb{C} \cup \{\infty\} : \eta(z, P) < \epsilon\}.$$

The homogeneous form of $\Lambda_\epsilon(P)$ is given by

$$\Lambda_\epsilon(P) = \{(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0, 0\} : \eta(\alpha, \beta, P) < \epsilon\}.$$

The Hermitian ϵ-pseudospectra of a regular Hermitian matrix polynomial $P(z)$ is defined as

$$\Lambda_\epsilon^H(P) := \bigcup \{\Lambda(P + \Delta P) : \Delta P(z) = \Sigma_{k=0}^{m} z^k \Delta A_k, \Delta A_k^* = \Delta A_k, k = 0, 1, \ldots, m, \ |||\Delta P||| < \epsilon\}.$$

Equivalently this may be written as

$$\Lambda_\epsilon^H(P) := \{z \in \mathbb{C} \cup \{\infty\} : \eta^H(z, P) < \epsilon\}$$

and in the homogeneous form as

$$\Lambda_\epsilon^H(P) = \{(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\} : \eta^H(\alpha, \beta, P) < \epsilon\}.$$

Evidently, $\Lambda_\epsilon^H(P) \cap (\mathbb{R} \cup \{\infty\}) = \Lambda_\epsilon(P) \cap (\mathbb{R} \cup \{\infty\})$ for a regular Hermitian polynomial $P(z)$. In particular, if all the eigenvalues of $P(z)$ are real and $\Lambda_\epsilon^H(P) \subset \mathbb{R} \cup \{\infty\}$, then $\Lambda_\epsilon^H(P) = \Lambda_\epsilon(P)$.

If $P(z)$ is regular, both $\Lambda_\epsilon(P)$ and $\Lambda_\epsilon^H(P)$ are open sets with at most mn components and each component containing at least one eigenvalue of $P(z)$. In such cases, the homogeneous form of $\Lambda_\epsilon^H(P)$ shows that it varies continuously with respect to ϵ as a subset of $\mathbb{C}^2 \setminus \{0,0\}$ for values of ϵ strictly less than the distance to a nearest nonregular pencil. If any component C_ϵ of $\Lambda_\epsilon^H(P)$ is a subset of the extended real line $\mathbb{R} \cup \{\infty\}$, then it may be identified with a subset of the unit circle of the form

$$\{(\cos\,\theta, \sin\,\theta) : \theta \in [0,\pi),\, \eta(\theta, P) < \epsilon\}$$

where the interval $[0,\pi)$ may be replaced by any other interval I such that any point of $\mathbb{R} \cup \{\infty\}$ is uniquely represented by $(\cos\,\theta, \sin\,\theta)$, $\theta \in I$.

The boundary $\partial\Lambda_\epsilon(P)$ of $\Lambda_\epsilon(P)$ is a subset of $\{(\alpha,\beta) \in \mathbb{C}^2 \setminus \{0,0\} : \eta(\alpha,\beta,P) = \epsilon\}$. Similarly, $\partial\Lambda_\epsilon^H(P) \subseteq \{(\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\} : \eta^H(\alpha,\beta,P) = \epsilon\}$. In particular if a component C_ϵ of $\Lambda_\epsilon^H(P)$ is a subset of $\mathbb{R} \cup \{\infty\}$, then

$$\partial C_\epsilon \subseteq \{(\cos\,\theta, \sin\,\theta) : 0 \leq \theta \leq 2\pi,\, \sigma_{\min}(P(\theta)) = \epsilon\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}}\}.$$

Thus the boundary ∂C_ϵ is a subset of the set of all $(\cos\,\theta, \sin\,\theta)$ such that $\theta \in [0, 2\pi]$ and $\epsilon\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}}$ is a singular value of the Hermitian matrix $P(\theta) = \Sigma_{k=0}^m (\cos\,\theta)^{m-k}(\sin\,\theta)^k A_k$. These in turn are a subset of set of all pairs $(\cos\,\theta, \sin\,\theta)$ such that $\theta \in [0, 2\pi]$ and is a zero of $\det((P(\theta))^2 - \epsilon^2\Sigma_{k=0}^m (\sin\,\theta)^{2(m-k)}(\cos\,\theta)^{2k} I_n)$. For any $z \in \mathbb{C}$, $\det((P(z))^2 - \epsilon^2\Sigma_{k=0}^m (\sin\,z)^{2(m-k)}(\cos\,z)^{2k} I_n)$ is an analytic function of z. Therefore, its set of zeros are either isolated points of $\mathbb{C}$ or they form the entire complex plane and in the former case, there are only finitely many such zeros in the interval $[0, 2\pi]$. Therefore we have the following theorem.

Theorem 1.2.11. *If $P(z)$ is a regular Hermitian polynomial and C_ϵ is a component of $\Lambda_\epsilon^H(P)$ such that $C_\epsilon \subset \mathbb{R} \cup \{\infty\}$, and ϵ is less than the distance to a nearest pencil that is not regular with respect to the norm $|||\cdot|||$, then the boundary ∂C_ϵ of C_ϵ is a finite subset of the extended real line.*

Given a regular Hermitian matrix polynomial $P(z)$, we say that two components C_ϵ^1 and C_ϵ^2 of $\Lambda_\epsilon(P)$ or $\Lambda_\epsilon^H(P)$, have coalesced if $\partial C_\epsilon^1 \cap \partial C_\epsilon^2 \neq \emptyset$. Theorem 1.2.11 implies that if C_ϵ^1 and C_ϵ^2 are subsets of $\mathbb{R} \cup \{\infty\}$, then the intersection $\partial C_\epsilon^1 \cap \partial C_\epsilon^2$ form a finite subset of $\mathbb{R} \cup \{\infty\}$ if ϵ is less than the distance to a nearest pencil that is not regular with respect to the norm $|||\cdot|||$. We refer to points of these sets as the points of coalescence of these components.

1.2.6 Hermitian polynomials with real eigenvalues of definite type

In this section, we give definitions and important properties as given in [4] of some Hermitian pencils and polynomials with real eigenvalues of definite type that are considered in the subsequent chapters. The first of these are the definite pencils that have several important applications in science and engineering ([15, 33]).

A Hermitian pencil $L(z) = zA - B$ is said to be definite if its Crawford number $\gamma(A, B)$ defined by

$$\gamma(A, B) = \min_{\|x\|_2=1} \sqrt{|x^*Ax|^2 + |x^*Bx|^2}$$

is positive. Note that $\gamma(A, B)$ is the distance from 0 to the field of values

$$F(A + iB) = \{x^*(A + iB)x : x \in \mathbb{C}^n, \|x\|_2 = 1\}.$$

Therefore, Hermitian pencil is definite if and only if $0 \notin F(A+iB)$. The following result from [4] characterises definite pencils.

Theorem 1.2.12. *[4] For a Hermitian pencil $L(z) = zA - B$ of size n the following are equivalent.*

(P1) $\Lambda(L) \subset \mathbb{R} \cup \{\infty\}$ *with all eigenvalues of definite type and where the eigenvalues of positive type are separated from the eigenvalues of negative type.*

(P2) $L(\mu)$ *is a definite matrix for some* $\mu \in \mathbb{R} \cup \{\infty\}$*, or equivalently* $L(\alpha, \beta) > 0$ *for some* (α, β) *on the unit circle.*

(P3) *For every nonzero* $x \in \mathbb{C}^n$*, the scalar equation* $x^*L(\lambda)x = 0$ *has exactly one zero in* $\mathbb{R} \cup \{\infty\}$*.*

(P4) $(x^*Ax, x^*Bx) \neq 0$ *for all nonzero* $x \in \mathbb{C}^n$*.*

(D) *There exists a nonsingular* $X \in \mathbb{C}^{n\times n}$ *such that*

$$X^*L(\lambda)X = \begin{bmatrix} L_+(\lambda) & 0 \\ 0 & L_-(\lambda) \end{bmatrix}$$

where $L_+(\lambda) = \lambda D_+ - J_+$ *and* $L_-(\lambda) = \lambda D_- - J_-$ *are real diagonal pencils, such that* $[\lambda_{min}(L_+), \lambda_{max}(L_+)] \bigcap [\lambda_{min}(L_-), \lambda_{max}(L_-)] = \phi$, D_+ *has nonnegative entries,* D_- *has nonpositive entries and if* $(D_+)_{ii} = 0$ *then* $(J_+)_{ii} > 0$ *or if* $(D_-)_{ii} = 0$ *then* $(J_-)_{ii} < 0$*.*

A homogeneous rotation of a definite pencil $L(z)$ by an angle $\theta \in (0, 2\pi]$ results in a Hermitian pencil $L_\theta(z) = zA_\theta - B_\theta$ where $A_\theta = A\cos\theta - B\sin\theta$ and $B_\theta = A\sin\theta + B\cos\theta$. Evidently $0 \notin \gamma(A_\theta, B_\theta)$ with $\gamma(A, B) = \gamma(A_\theta, B_\theta)$. The following result which is established in [46] and ([47], Theorem 6.1.18) relates the Crawford number with the eigenvalues of A_θ as θ varies in $(0, 2\pi]$.

Theorem 1.2.13. *Given a definite pencil $L(z) = zA - B$ there exists $\theta \in [0, 2\pi)$ such that A_θ is positive definite and*

$$\gamma(A, B) = \max_{\theta \in [0, 2\pi]} \lambda_{\min}(A_\theta)$$

where $\lambda_{\min}(A_\theta)$ is the eigenvalues of A_θ with the smallest modulus.

Next we consider the hyperbolic polynomials. A Hermitian polynomial $P(z)$ is said to be hyperbolic if it satisfies any of the equivalent properties of Theorem 1.2.14.

Theorem 1.2.14. *[4] For a Hermitian matrix polynomial $P(z) = \Sigma_{k=0}^{m} z^k A_k$, of size n the following are equivalent.*

(P1) *All eigenvalues are real and finite, of definite type, and such that*

$$\underbrace{\lambda_{mn} \le \cdots \le \lambda_{(m-1)n+1}}_{(-1)^{m-1}type} < \cdots < \underbrace{\lambda_{2n} \le \cdots \le \lambda_{n+1}}_{negative\ type} < \underbrace{\lambda_n \le \cdots \le \lambda_1}_{positive\ type}$$

where "$(-1)^{(m-1)}$ type" denotes positive type for odd m and negative type for even m.

(P2) *There exists $\mu_j \in \mathbb{R} \cup \{\infty\}$ such that*

$$(-1)^j P(\mu_j) > 0, \quad j = 0, 1, \ldots, m-1, \quad \infty = \mu_0 > \mu_1 > \mu_2 > \cdots > \mu_{m-1}.$$

(P3) *$A_m > 0$ and for every nonzero $x \in \mathbb{C}^n$, the scalar equation $x^* P(z)x = 0$ has m distinct real and finite zeros.*

(L) *$P(z)$ has a definite linearization $L(z) \in \mathbb{H}(P)$ with vector $v \in \mathbb{R}^m$ such that $L(\infty) > 0$ if $v_1 > 0$ and $L(\infty) < 0$ if $v_1 < 0$.*

The class of overdamped quadratics form a subclass of the hyperbolic quadratic polynomials and arise in overdamped systems in structural mechanics. For examples see [17, 35]. They are defined to be Hermitian quadratic matrix polynomials that satisfy any one of the equivalent properties of Theorem 1.2.15.

Theorem 1.2.15. *[4] For a Hermitian quadratic matrix polynomial $Q(z) = z^2A_2 + zA_1 + A_0$, the following are equivalent.*

(P1) *All eigenvalues are real, finite, nonpositive and distributed in two disjoint closed intervals, the left-most containing n eigenvalues of negative type and the right most containing n eigenvalues of positive type.*

(P2) $A_2 > 0, A_1 > 0, A_0 \geq 0,$ *and* $Q(\mu) < 0$ *for some* $\mu < 0$.

(P3) $A_2 > 0$ *and for every nonzero* $x \in \mathbb{C}^n$, *the scalar equation* $x^*Q(\lambda)x = 0$ *has two distinct real and finite nonpositive zeros.*

(O) $A_2 > 0,\ A_1 > 0,\ A_0 \geq 0,$ *and*

$$(x^*A_1x)^2 > 4(x^*A_2x)(x^*A_0x) \text{ for all nonzero } x \in \mathbb{C}^n.$$

In [28], Higham et. al. introduced the class of definite matrix polynomials that generalize the hyperbolic matrix polynomials by relaxing the condition of positive definiteness of the leading coefficient matrix. Definite polynomials that are not hyperbolic arise in problems of acoustic-fluid structure interaction [28].

Definite polynomials are defined to be Hermitian matrix polynomials that satisfy any one of the equivalent properties of Theorem 1.2.16.

Theorem 1.2.16. *[4] Given a Hermitian matrix polynomial $P(z) = \Sigma_{k=0}^m z^k A_k$ of size n, the following are equivalent.*

(P1) *All eigenvalues are real, of definite type and such that*

$$\underbrace{\lambda_{mn} \leq \cdots \leq \lambda_{(m-1)n+p+1}}_{\substack{n-p \text{ eigenvalues} \\ \text{of } (-1)^{m-1}\epsilon \text{ type}}} < \cdots < \underbrace{\lambda_{jn+p} \leq \cdots \leq \lambda_{(j-1)n+p+1}}_{\substack{n \text{ eigenvalues of} \\ (-1)^{j-1}\epsilon \text{ type},\, 1\leq j\leq m-1}} < \cdots < \underbrace{\lambda_p \leq \cdots \leq \lambda_1}_{\substack{p \text{ eigenvalues} \\ \text{of } -\epsilon \text{ type}}}$$

with $0 \leq p < n$, where "$\alpha\epsilon$ type" denotes positive type when $\alpha\epsilon > 0$ and negative type otherwise, and λ_{p+1} is of ϵ type.

(P2) *There exists* $\mu_j \in \mathbb{R}\cup\{\infty\}$ *with* $\mu_0 > \mu_1 > \mu_2 > \cdots > \mu_{m-1}(\mu_0 = \infty$ being possible) *such that* $P(\mu_0), P(\mu_1), ..., P(\mu_{m-1})$ *are definite matrices with alternating parity.*

(P3) *There exists* $\mu \in \mathbb{R} \cup \{\infty\}$ *such that the matrix* $P(\mu)$ *is definite and for every nonzero* $x \in \mathbb{C}^n$ *the scalar equation* $x^*P(z)x = 0$ *has* m *distinct zeros in* $\mathbb{R}\cup\{\infty\}$.

(L) $P(z)$ *has a definite linearization* $L(z) \in \mathbb{H}(P)$.

If the leading coefficient A_m of $P(z)$ is definite then either $P(z)$ or $-P(z)$ is hyperbolic. In such a case, Theorem 1.2.14 implies that $p = 0$ in property (P1) of Theorem 1.2.16. Note that the definite polynomials include the definite pencils, hyperbolic polynomials and overdamped quadratics. The next result gives an important property of definite polynomials.

Theorem 1.2.17. *[5] For a definite matrix polynomial* $P(z)$ *of degree* m *with eigenvalues as in property (P1) of Theorem 1.2.16, let*

$$\mathcal{I}_j = (\lambda_{jn+p+1}, \lambda_{jn+p}), \;\; j = 1, 2, \ldots, m-1$$

and

$$\mathcal{I}_0 = \begin{cases} (\lambda_{p+1}, \lambda_p) & \text{if } p \neq 0, \\ (\lambda_1, +\infty) & \text{if } p = 0 \end{cases} \qquad \mathcal{I}_m = \begin{cases} \emptyset & \text{if } p \neq 0, \\ (-\infty, \lambda_{mn}) & \text{if } p = 0 \end{cases}$$

Then $P(\mu)$ *is definite for any* $\mu \in \mathcal{I}_j$, $j = 0, 1, \ldots, m$ *and if* $\mu_j \in \mathcal{I}_j$, $\mu_{j+1} \in \mathcal{I}_{j+1}$, *then* $P(\mu_j)$ *and* $P(\mu_{j+1})$ *have opposite parity.*

The class of definite pencils form a subset of a much wider class of pencils called definitizable pencils. These are Hermitian pencils that satisfy any one of the equivalent properties of Theorem 3.6.7.

Theorem 1.2.18. *[4] For a Hermitian pencil* $L(z) = zA - B$ *of size* n, *the following are equivalent.*

(P1) *All eigenvalues of* $L(z)$ *are real, finite and of definite type.*

(P2) A *is nonsingular and there exists a real polynomial* q *such that* $Aq(A^{-1}B) > 0$.

(P3) A *is nonsingular and the scalar equation* $x^*L(z)x = 0$ *has one zero in* $\mathbb{R}$ *for all eigenvectors* $x \in \mathbb{C}^n$ *of* L.

(P4) *A is nonsingular and $(x^*Ax, x^*Bx, ..., x^*A(A^{-1}B)^{n-1}x) \neq 0$ for all nonzero $x \in \mathbb{C}^n$.*

(D) *There exists a nonsingular $X \in \mathbb{C}^{n\times n}$ such that*

$$X^*AX = \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}, \quad X^*BX = \begin{bmatrix} J_+ & 0 \\ 0 & -J_- \end{bmatrix} \tag{1.8}$$

where $J_+ \in \mathbb{R}^{k\times k}$ and $J_- \in \mathbb{R}^{(n-k)\times(n-k)}$ are diagonal and $\Lambda(J_+) \cap \Lambda(J_-) = \Phi$.

The notion of definitizable pencils extend to matrix polynomials of higher degree to give the class of quasihyperbolic polynomials. These are Hermitian matrix polynomials that satisfy any of the equivalent properties of Theorem 1.2.19.

Theorem 1.2.19. *[4] For a Hermitian matrix polynomial $P(z) = \Sigma_{k=0}^{m} z^k A_k$ the following are equivalent.*

(P1) *All eigenvalues are real, finite and of definite type.*

(L) *Any linearization $L(z) \in \mathbb{H}(P)$ is definitizable.*

Chapter 2

Linear Perturbation of Matrix Pencils

Given an $n \times n$ pencil $L(z) := zA - B$, in this section we analyse the effect of linear perturbations of the form $L(z) + t\Delta L(z)$ on the deflating subspaces of $L(z)$ for a fixed pencil $\Delta L(z) := z\Delta A - \Delta B$ as t varies over $\mathbb{C}$. We also consider the effect of such perturbations with the restrictions that t varies over $\mathbb{R}$. This is useful for the case $L(z)$ is a Hermitian pencil and $\Delta L(z)$ is a fixed Hermitian pencil so that the perturbed pencils remain Hermitian as t varies over $\mathbb{R}$. Throughout this chapter we set $R(z) = (L(z))^{-1}$, $P_\Gamma := \frac{1}{2\pi i}\oint_\Gamma R(z)Adz$, $L_t(z) := (L + t\Delta L)(z)$, $R(t,z) := [(L + t\Delta L)(z)]^{-1}$ and $P_\Gamma(t) := \frac{1}{2\pi i}\oint_\Gamma R(t,z)(A + t\Delta A)dz$.

2.1 Effect of linear perturbation on spectral projections

Given a simple rectifiable closed curve Γ in $\mathbb{C}$ which does not contain any eigenvalues of $L(z)$, the spectral projection onto the deflating subspace of $L(z)$ associated with the eigenvalues in the interior of Γ has the integral representation [36]

$$P_\Gamma = \frac{1}{2\pi i}\oint_\Gamma R(z)Adz.$$

Thus if the pencil $L(z)$ has no eigenvalues in the interior of Γ, then $P_\Gamma = 0$ and if the interior of Γ contains all the eigenvalues of $L(z)$, then $P_\Gamma = I_n$. Consider the disc

$$\partial_\Gamma := \left\{ t \in \mathbb{C} : |t| < \frac{\inf_{z\in\Gamma} \eta(z, L)}{|||\Delta L|||} \right\}.$$

Recall that $\eta(z,L) := \frac{\sigma_{\min}(zA-B)}{\sqrt{1+|z|^2}}$ is the backward error of z as an approximate eigenvalue of $L(z)$ with respect to the norm $|||L||| := \sqrt{\|A\|_2^2 + \|B\|_2^2}$. We show that the projections $P_\Gamma(t)$ are analytic functions of $t \in \partial_\Gamma$. To this end we have the following result.

Lemma 2.1.1. *Let $t_0 \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \Lambda(L_{t_0})$ be fixed. For all $t \in D_0$, where*

$$D_0 := \left\{ t \in \mathbb{C} : |t - t_0| < \frac{\eta(z, L_{t_0})}{|||\Delta L|||} \right\}$$

we have $z \in \mathbb{C} \setminus \Lambda(L_t)$ with

$$R(t,z) = \Sigma_{n=0}^{\infty} (-1)^n R(t_0,z)\{\Delta L_{t_0}(z) R(t_0,z)\}^n (t-t_0)^n$$

where the series converges uniformly. It also converges absolutely for t belonging to any closed subset of D_0.

Proof: Whenever $t \in D_0$, we have $|||(t-t_0)\Delta L|||\sqrt{1+|z|^2} < \sigma_{\min}(L_{t_0}(z))$, which implies the following inequalities.

$$\|(t-t_0)\Delta L(z) R(t_0,z)\| < 1. \tag{2.1}$$

$$|t - t_0| < \frac{1}{\limsup_{n\to\infty} \|(\Delta L(z) R(t_0,z))^n\|^{\frac{1}{n}}}. \tag{2.2}$$

From (2.1) it follows that whenever $t \in D_0$, $I + (t-t_0)\Delta L(z) R(t_0,z)$ is invertible with

$$[I + (t-t_0)\Delta L(z) R(t_0,z)]^{-1} = \Sigma_{n=0}^{\infty} (-1)^n (\Delta L(z) R(t_0,z))^n (t-t_0)^n \tag{2.3}$$

Therefore $L_t(z) = [I + (t-t_0)\Delta L(z) R(t_0,z)] L_{t_0}(z)$ is also invertible for all $t \in D_0$. Multiplying the identity

$$L_t(z) - L_{t_0}(z) = (t-t_0)\Delta L(z)$$

on the right by $R(t_0,z)$ and on the left by $R(t,z)$, and using (2.3), we have

$$R(t,z) = R(t_0,z)[I+(t_0-t)\Delta L(z) R(t_0,z)]^{-1} = \Sigma_{n=0}^{\infty} (-1)^n R(t_0,z)\{\Delta L_{t_0}(z) R(t_0,z)\}^n (t-t_0)^n. \tag{2.4}$$

In view of (2.2), it follows that the series in (2.4) converges uniformly for all $t \in D_0$ and absolutely for all t in any closed subset of D_0. $\square$

Theorem 2.1.2. *Let Γ be a rectifiable simple closed curve in $\mathbb{C}$ which does not contain any eigenvalues of $L(z)$. The projection $P_\Gamma(t)$ is an analytic function of t for every $t \in \partial_\Gamma$.*

Proof: From the definition of ∂_Γ and that of the backward error $\eta(z, L)$, it follows that the curve Γ does not contain any eigenvalues of $L_t(z)$ for all $t \in \partial_\Gamma$. Thus $P_\Gamma(t) := \frac{1}{2\pi i}\oint_\Gamma R(t,z)(A + t\Delta A)dz$ is well defined for all $t \in \partial_\Gamma$. From Lemma 2.1.1, it follows that for each $t_0 \in \partial_\Gamma$, there exists a neighbourhood of t_0 such that $R(t,z) = \Sigma_{n=0}^{\infty} R(t_0,z)\{\Delta L_{t_0}(z)R(t_0,z)\}^n(t-t_0)^n$ where the series converges uniformly for all t in the neighbourhood and absolutely for all t in a closed subset of the neighbourhood. Therefore for all t in such a neighbourhood,

$$\begin{aligned} R(t,z)(A+t\Delta A) &= \Sigma_{n=0}^{\infty}(-1)^n R(t_0,z)\{\Delta L_{t_0}(z)R(t_0,z)\}^n(A+t_0\Delta A)(t-t_0)^n \\ &\quad +\Sigma_{n=0}^{\infty}(-1)^n R(t_0,z)\{\Delta L_{t_0}(z)R(t_0,z)\}^n \Delta A(t-t_0)^{n+1} \end{aligned} \tag{2.5}$$

where both the series on the right hand side converge uniformly in the neighbourhood and absolutely for all t in a closed subset of the neighbourhood. Therefore term by term integration of the series over Γ in (2.5) implies that $P_\Gamma(t)$ is analytic in a neighbourhood of t_0. Since $t_0 \in \partial_\Gamma$ is arbitrarily chosen, it follows that $P_\Gamma(t)$ is analytic for all $t \in \partial_\Gamma$. $\square$

Next we consider the perturbed pencils $L_t(z)$ with the restriction that $t \in \mathbb{R}$. In this case we have the following theorem. In view of Theorem 2.1.2, each fixed t belonging to an interval I on the real line has a neighbourhood in which $P_\Gamma(t)$ exists and is analytic.

Theorem 2.1.3. *Let Γ be a simple rectifiable closed curve and I an interval on the real line such that $P_\Gamma(t)$ exists for all $t \in I$. Then $P_\Gamma(t)$ is an analytic function of $t \in I$.*

Proof: In view of Theorem 2.1.2, each fixed $t \in I$ has a neighbourhood in which $P_\Gamma(t)$ exists and is analytic. $\square$

Note 2.1.4. *We may assume without loss of generality that the interval I in Theorem 2.1.3 is open. This is because even if I is closed at any end, then Theorem 2.1.2 implies that there exists a neighbourhood of the end point in which $P_\Gamma(t)$ is defined.*

Now consider the following result which is listed as Theorem I-4.10 in [32].

Theorem 2.1.5. *Let $P(t)$ be a projection depending continuously on a parameter t that varies in an open interval of the real line or a connected open set in the complex plane. Then the dimension of the range of $P(t)$ is constant for all values of t in such a set.*

In view of the above result and Theorems 2.1.2, (2.1.3) and Note 2.1.4, it follows that $\operatorname{rank} P_\Gamma(t) = \operatorname{rank} P_\Gamma(t_0)$ for all $t \in \partial_\Gamma$ if t varies over $\mathbb{C}$, and if t varies over $\mathbb{R}$ then $\operatorname{rank} P_\Gamma(t) = \operatorname{rank} P_\Gamma(t_0)$ for all $t \in I$ (where $I \doteq D_0 \cap \mathbb{R}$ with D_0 as defined in Lemma 2.1.1). Now consider the following result which is proved in Section 4.2, Chapter 2, of [32].

Theorem 2.1.6. *Let $P(t)$ be a projection which varies analytically in an open connected set say, D of the complex plane or is continuously differentiable in an open interval I of the real line. Then there exists $m \in \mathbb{N}$ and vectors $x_k(t)$, $k = 1, 2, \ldots, m$, which vary analytically with respect to t and form a basis of range of $P(t)$ for all $t \in \mathbb{D}$ or all $t \in I$.*

From the above theorem, and the preceding results we have the following theorem.

Theorem 2.1.7. *Let Λ be a subset of the set of eigenvalues of $L(z)$ which does not contain ∞ and P_Γ be the projection onto the direct sum of the spectral subspaces associated with the eigenvalues in Λ with $\operatorname{rank} P_\Gamma = m$. For all $t \in \partial_\Gamma$, there exist vectors $\{x_k(t) : 1 \le k \le m\}$ that vary analytically with respect to t and form a basis of $P_\Gamma(t)$.*

The above assertions also hold for all $t \in I$ where I is an interval on the real line containing 0, such that Γ does not contain any eigenvalues of $L_t(z)$ for all $t \in I$.

Consider a pencil $L(z)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If the eigenvalues are divided into k disjoint sets $\Lambda_1, \Lambda_2, \ldots, \Lambda_k$, and S_j be the deflating subspace associated with the eigenvalues in Λ_j, $j = 1, 2, \ldots, k$, then [36]

$$\mathbb{C}^n = S_1 \oplus S_2 \oplus \cdots \oplus S_k.$$

Let P_j be the projection onto the deflating subspace associated with the eigenvalues of $L(z)$ in Λ_j for $j = 1, 2, \ldots, k$. Suppose that all the eigenvalues of $L(z)$ in Λ_j are finite for $j = 1, 2, \ldots k-1$. Then, $P_j = \frac{1}{2\pi i}\oint_{\Gamma_j} R(z)A dz$ where Γ_j is a simple rectifiable closed curve in $\mathbb{C}$ such that $\Lambda(L) \bigcap \operatorname{Int}\Gamma_j = \Lambda_j$, $1 \le j \le k-1$. It follows from the previous results that for any $t \in \bigcap_{j=1}^{k-1} \partial_{\Gamma_j}$, and $j = 1, 2, \ldots, k-1$, the projections $P_j(t)$ are analytic functions of t. Hence, the perturbed pencils $L_t(z)$ have the same number of eigenvalues counting multiplicity as those of $L(z)$ in the interior of Γ_j for $1 \le j \le k-1$. In particular we have,

$$\mathbb{C}^{n\times n} = S_1(t) \oplus S_2(t) \oplus \cdots \oplus S_k(t) \text{ and } I = P_1(t) + P_2(t) + \cdots + P_k(t)$$

where $S_j(t)$ is the deflating subspace associated with the eigenvalues of $L_t(z)$ in the interior of Γ_j for $1 \leq j \leq k-1$ and $S_k(t)$ is the deflating subspace associated with the eigenvalues of $L_t(z)$ that lie on the exterior of each Γ_j, $1 \leq j \leq k-1$.

Since $\dim S_j = \dim S_j(t)$, $1 \leq j \leq k-1$, for all $t \in \bigcap_{j=1}^{k-1} \partial_{\Gamma_j}$, it follows that $\dim S_k(t) = \dim S_k$ for all $t \in \bigcap_{j=1}^{k-1} \partial_{\Gamma_j}$.

Also since $P_j(t)$ varies analytically for each $t \in \bigcap_{j=1}^{k-1} \partial_{\Gamma_j}$, it follows that $P_k(t) = I - \Sigma_{j=1}^{k-1} P_j(t)$ also varies analytically for $t \in \bigcap_{j=1}^{k-1} \partial_{\Gamma_j}$.

If t varies over $\mathbb{R}$ then the above statements continue to hold, for all $t \in I$ where I is an interval containing 0 such that each Γ_j, $1 \leq j \leq k-1$ does not contain any eigenvalues of the perturbed pencils $L_t(z)$. Now consider the following result which is proved in Section 4.5, Chapter 2 of [32].

Theorem 2.1.8. *Let $P_j(t), j = 1, 2, \ldots, k$ be mutually orthogonal projections associated with a vector space V of dimension n such that $\Sigma_{j=1}^{k} P_j(t) = I_V$ where I_V denotes the identity operator. Suppose that each $P_j(t)$, $1 \leq j \leq k$, varies analytically for all t in an open connected set D of the complex plane or is continuously differentiable for all $t \in I$ where I is an open interval of the real line. If $S_j(t)$ be the range of $P_j(t)$ for $1 \leq j \leq k$, then there exist numbers $m_j \in \mathbb{N}$, $j = 1, 2, \ldots, k$, such that $\dim S_j(t) = m_j$ for all $t \in D$ (or all $t \in I$). Also there exists a basis $\{x_{ji}(t) : i = 1 = 1, 2, \ldots, m_j,\ j = 1, 2, \ldots, k\}$ of V which varies analytically with respect to t for all $t \in D$ (or all $t \in I$) such that $\operatorname{span}\{x_{ji}(t) : i = 1 = 1, 2, \ldots, m_j\} = S_j(t)$ for $j = 1, 2, \ldots, k$.*

The following theorem is now an immediate consequence of the above result and the preceding analysis.

Theorem 2.1.9. *Let $L(z)$ be a pencil with $\Lambda(L) = \bigcup_{j=1}^{k} \Lambda_j$ where $\Lambda_j \cap \Lambda_i = \emptyset$ for $j \neq i$ and let P_j be the spectral projection onto the deflating subspace associated with the eigenvalues of $L(z)$ in Λ_j with $\operatorname{rank} P_j = m_j$ for $j = 1, 2, \ldots, k$. Also suppose that all the eigenvalues of $L(z)$ in $\Lambda_1, \Lambda_2, \ldots, \Lambda_{k-1}$ are finite. For $1 \leq j \leq k-1$, let Γ_j be rectifiable simple closed curves in $\mathbb{C}$ such that $\Lambda_j = \Lambda(L) \cap \operatorname{Int} \Gamma_j$. For all $t \in \bigcap_{j=1}^{k-1} \partial_{\Gamma_j}$, there exists a basis $\{x_{ji}(t) : i = 1 = 1, 2, \ldots, m_j,\ j = 1, 2, \ldots, k\}$ of $\mathbb{C}^n$ that varies analytically with respect to t such that $\operatorname{span}\{x_{ji}(t) : i = 1 = 1, 2, \ldots, m_j\} = S_j(t)$ for $j = 1, 2, \ldots, k$.*

In case t varies over $\mathbb{R}$, and there are no eigenvalues of $L_t(z)$ on each Γ_j, $j = 1, 2, \ldots, k-1$ for all t belonging to an interval I of the real line containing 0, then the above assertion holds for all $t \in I$.

2.2 Linear perturbation of definite pencils

Definite pencils have a close connection to Hermitian matrices. Apart from the fact that both definite pencils and Hermitian matrices have real eigenvalues, the eigenvalue problem for definite pencils may at times be equivalent to that for a Hermitian matrix. For instance an $n \times n$ Hermitian pencil $L(z) = zA - B$ with a definite coefficient A is a definite pencil such that $H := A^{-1/2}BA^{-1/2}$ is a Hermitian matrix and if U is a unitary matrix such that $U^*HU = D$ where D is diagonal, then $X^*BX = D$ and $X^*AX = I$ for $X = A^{-1/2}U$. Moreover, there is a variational characterisation of eigenangles of definite pencils similar to the variational characterisation of eigenvalues of Hermitian matrices (see for example, Section 3, Chapter 6, of [47]). In this section we observe that these similarities also extend to the effect of certain linear perturbations on definite pencils. Throughout this section we assume that $L(z) = zA - B$ is a definite pencil with A nonsingular. Note that, if A is singular, then it is possible to rotate $L(z)$ such that the resulting A is nonsingular. Consider the following well known theorem for Hermitian matrices, a proof of which may be found in Section 6.2, Chapter 2, of [32].

Theorem 2.2.1. *Let $H(t)$, $t \in \mathbb{R}$ be a family of $n \times n$ Hermitian matrices which depend analytically on t. Then there exists an orthonormal basis $\{x_k(t) : k = 1, 2, \ldots, n\}$ consisting of eigenvectors of $H(t)$ which varies analytically for all $t \in \mathbb{R}$.*

Given a definite pencil $L(z)$, consider the family of perturbed pencils $L_t(z) := zA - (B + t\Delta B)$ where ΔB is a Hermitian matrix and t varies over $\mathbb{R}$. For small enough t, the pencils $L_t(z)$ are definite and therefore there exist nonsingular matrices $X(t)$ such that $X(t)^*AX(t)$ and $X(t)^*BX(t)$ are diagonal. The columns of $X(t)$ in fact form the eigenvectors of $L_t(z)$ and the fact that $X(t)^*AX(t)$ is diagonal may be interpreted to mean that these columns span an A orthogonal subspace of $\mathbb{C}^n$. The following result states that $X(t)$ may be chosen to vary analytically with respect t for all t in an interval such that $L_t(z)$ is definite.

Theorem 2.2.2. *Let $L_t(z) = zA - (B + t\Delta B)$ be a family of definite pencils of size n as t varies in some interval I on the real line. There exists a nonsingular matrix $X(t)$ which varies analytically with respect to $t \in I$ such that $X(t)^*AX(t)$ and $X(t)^*(B + t\Delta B)X(t)$ are diagonal.*

Proof: Since $L_t(z)$ is definite for all $t \in I$, the proof considers two cases, the first case being the one in which the eigenvalues $\Lambda := \cup_{t\in I}\Lambda(L_t(z))$ form an interval on the real line and the second being the case in which this set is disconnected.

Case I: Let Λ form an interval on the real line. In this case we show that A must be definite. Suppose that A is indefinite. Since A is nonsingular, all eigenvalues of $L(z)$ are finite and of both positive and negative type. In view of Theorem 1.2.12, there exist two disjoint intervals say, I_1 and I_2 on the real line, such that I_1 contains all the eigenvalues of positive type and I_2 contains all the eigenvalues of negative type respectively. Therefore for $t_0 \in I$ close enough to 0,

$$\Lambda(t_0) := \cup_{t\in(-t_0,t_0)}\Lambda(L_t(z)) \subset I_1(t_0) \cup I_2(t_0)$$

where $I_1(t_0)$ and $I_2(t_0)$ are disjoint intervals on the real line with $I_1 \subset I_1(t_0)$ and $I_2 \subset I_2(t_0)$. For $t \in (-t_0, t_0)$, let the columns of $X_1(t)$ and $X_2(t)$ span the deflating subspaces of $L_t(z)$ associated with its eigenvalues lying in $I_1(t_0)$ and $I_2(t_0)$ respectively. Since A is nonsingular, $X(t)^*AX(t)$ is also nonsingular for $X(t) := [X_1(t)X_2(t)]$. This implies that both $X_1(t)^*AX_1(t)$ and $X_2(t)^*AX_2(t)$ are nonsingular as

$$X(t)^*AX(t) = \begin{bmatrix} X_1(t)^*AX_1(t) & \\ & X_2(t)^*AX_2(t) \end{bmatrix}.$$

Since $I_1(t_0)$ contains all the eigenvalues of $L(z)$ of positive type and $I_2(t_0)$ contains all the eigenvalues of $L(z)$ of negative type, $X_1(0)^*AX_1(0)$ is positive definite and $X_2(0)^*AX_2(0)$ is negative definite. This implies that $X_1(t)^*AX_1(t)$ is positive definite and $X_2(t)^*AX_2(t)$ is negative definite for all $t \in (-t_0, t_0)$. Therefore, for all $t \in (-t_0, t_0)$, $I_1(t_0)$ contains all the positive type eigenvalues of $L_t(z)$ while $I_2(t_0)$ contains all the negative type eigenvalues of $L_t(z)$. But since Λ is connected, there exists $t_1 \in I$ such that $\Lambda(t_1) := \cup_{t\in(-t_1,t_1)}\Lambda(L_t(z))$ is not a disjoint union of intervals such that one contains all the positive type eigenvalues of $L_t(z)$ and the other contains all the negative type eigenvalues of $L_t(z)$ as t varies over $(-t_1, t_1)$. Let $\hat{t}$ be such a point in I of minimum modulus. Then there exists an eigenvalue $\lambda(\hat{t})$ of $L_{\hat{t}}(z)$ and a sequence $\{t_n\}$ of points in I with $|t_n| < |\hat{t}|$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \lambda(t_n) = \lim_{n\to\infty} \tilde{\lambda}(t_n) = \lambda(\hat{t})$$

where $\lambda(t_n)$ is a positive type eigenvalue of $L_{t_n}(z)$ and $\tilde{\lambda}(t_n)$ is a negative type eigenvalue of $L_{t_n}(z)$. Let $\{x(t_n)\}$ and $\{\tilde{x}(t_n)\}$ be sequences of eigenvectors of $L_{t_n}(z)$ such that

$$\lambda(t_n)Ax(t_n) = (B + t_n\Delta B)x(t_n) \text{ and } \tilde{\lambda}(t_n)A\tilde{x}(t_n) = (B + t_n\Delta B)\tilde{x}(t_n) \text{ for all } n \in \mathbb{N}.$$

Since $L_{\hat{t}}(z)$ is a definite pencil, it has eigenvectors $x(\hat{t})$ and $\tilde{x}(\hat{t})$ corresponding to $\lambda(\hat{t})$ such that $x(\hat{t}) = \lim_{k\to\infty} x(t_{n_k})$ and $\tilde{x}(\hat{t}) = \lim_{k\to\infty} \tilde{x}(t_{n_k})$. Therefore,

$$x(\hat{t})^*Ax(\hat{t}) = \lim_{k\to\infty} x(t_{n_k})^*Ax(t_{n_k}) \geq 0 \text{ and } \tilde{x}(\hat{t})^*A\tilde{x}(\hat{t}) = \lim_{k\to\infty} \tilde{x}(t_{n_k})^*A\tilde{x}(t_{n_k}) \leq 0.$$

But this shows that $\lambda(\hat{t})$ is an eigenvalue of $L_{\hat{t}}(z)$ of mixed type which is impossible as $L_{\hat{t}}(z)$ is a definite pencil. Hence A is a definite matrix. Without loss of generality we assume that A is positive definite as if A is negative definite, we may replace A by $-A$ which will change only the signs of the eigenvalues leaving the eigenvectors intact. If A is positive definite, then $L_t(z)$ has the same eigenvalues as the Hermitian matrix $A^{-1/2}(B+t\Delta B)A^{-1/2}$ for all $t \in I$. Since the latter forms an analytic family of Hermitian matrices, by Theorem 2.2.1, there exists an orthogonal matrix $U(t)$ which varies analytically with respect to $t \in I$ such that $U(t)^*A^{-1/2}(B + t\Delta B)A^{-1/2}U(t) = D(t)$ where $D(t)$ is real and diagonal. Then $X(t) := A^{-1/2}U(t)$ is such that $X(t)^*(B + t\Delta B)X(t) = D(t)$ and $X(t)^*AX(t) = I_n$ and $X(t)$ varies analytically with respect to $t \in I$. This concludes the proof in this case.

Case II: Let $\Lambda = \Lambda_1 \cup \Lambda_2$ where $\Lambda_1 \cap \Lambda_2 = \emptyset$. Let Γ_1 and Γ_2 be two simple rectifiable closed curves in $\mathbb{C}$ such that $\Lambda \cap \operatorname{Int} \Gamma_j = \Lambda_j$, $j = 1, 2$. Then for any $t \in I$, $P_j(t) = \frac{1}{2\pi i} \oint_{\Gamma_j} R(t,z)Adz$, is the spectral projection onto the deflating subspace associated with the eigenvalues of $L_t(z)$ in Λ_j for $j = 1, 2$. Since $P_1(t) + P_2(t) = I_n$ and $P_j(t)$ is an analytic function of $t \in I$ for each $j = 1, 2$, by Theorem 2.1.9, for each $t \in I$, there exist bases for the deflating subspaces associated with the eigenvalues of $L_t(z)$ in Λ_1 and Λ_2 that vary analytically with respect to t. Since $L_t(z)$ is a definite pencil for all $t \in I$, the elements of the bases are eigenvectors of $L_t(z)$. The required matrix $X(t)$ is then obtained by setting the basis vectors of the deflating subspaces as its columns. $\square$

The proof of the above theorem will remain unchanged for the second case even if both A and B are perturbed such that the resulting pencil is definite. Hence we have the following corollary the proof of which is immediate from the proof of the above theorem.

Corollary 2.2.3. *Let $L_t(z)$ be an $n \times n$ family of definite pencils such that the union of the eigenvalues of $L_t(z)$ form a disconnected subset of the real line as t varies over an interval I on the real line. There exists a family of nonsingular matrices $X(t)$ that vary analytically with respect to $t \in I$ such that $X(t)^*(A+t\Delta A)X(t)$ and $X(t)^*(B+t\Delta B)X(t)$ are diagonal.*

Chapter 3

Distance Problems for Hermitian Matrix Pencils

3.1 Introduction

Given an $n \times n$ definite pencil $L(z) = zA - B$, its Crawford number

$$\gamma(A, B) := \min_{\|x\|_2=1} \sqrt{|x^*Ax|^2 + |x^*Bx|^2} \tag{3.1}$$

which is always positive is the distance of the field of values

$$F(A + iB) := \{x^*(A + iB)x : x \in \mathbb{C}^n, \|x\|_2 = 1\}$$

of $A+iB$ to the origin. The Crawford number is also the distance from $L(z)$ the nearest Hermitian pencil that is not definite [29]. In view of this, it is natural to expect that the Crawford number has some relationship with the Hermitian ϵ-pseudospectrum. In this chapter we analyse the relationship between the Crawford number of a definite pencil and the Hermitian ϵ-pseudospectrum of the pencil. The main result of this chapter is the fact that the Crawford number of a definite pencil may be obtained from its Hermitian ϵ-pseudospectrum. An interesting implication of this is the possibility of computing the Crawford number via an algorithm based on the bisection method. This also shows that the Crawford number is the distance with respect to the $|||\cdot|||$ to a nearest Hermitian pencil with a defective eigenvalue. One of the other significant results of the chapter is that if the pencil $L(z)$ is definitizable, then the distance to a nearest Hermitian pencil that is not definitizable with respect to the norm $|||\cdot|||$ may also be obtained from the Hermitian ϵ-pseudospectrum of the pencil. The methods used for the analysis is an extension of those in [6] for Hamiltonian matrices.

In the process we also obtain other interesting results that connect the Crawford number of a definite pencil $L(z)$ with its Hermitian ϵ-pseudospectrum. In particular we show that the Crawford number is the distance to a nearest Hermitian pencil with a complex eigenvalue.

We propose a new definition of eigenvalue type for a Hermitian polynomial $P(z)$ based on the homogeneous form of the polynomial which determines the type of both finite and infinite eigenvalues in the same framework. This definition, leads to a different proof of the fact that homogeneous rotations do not create eigenvalues of mixed type [4]) We also analyse certain properties of Hermitian pencils based on their canonical form under congruence as given in (1.2). Finally we analyse the components of the Hermitian ϵ-pseudospectrum of the pencil that contain eigenvalues of perturbed pencils of only one type, either positive or negative. In particular we investigate certain properties of the points of coalescence of such components. All these results are useful in proving the main results of the chapter.

3.2 The Crawford number and distance problems

The first result related to Crawford number is an alternative proof of the fact that the Crawford number is the distance to a nearest Hermitian pencil which is not definite. It also shows that a nearest Hermitian pencil which is not definite may be obtained via a rank one perturbation to $L(z)$.

Lemma 3.2.1. *Given an $n \times n$ definite pencil $L(z)$,*

$$\gamma(A,B) := \min\{|||\Delta L||| : (L+\Delta L)(z) \text{ is Hermitian but not definite } \}.$$

The above minimum is attained by a Hermitian pencil $\Delta L(z) := z\Delta A - \Delta B$ where ΔA and ΔB are rank one matrices.

Proof: Since the field of values $F(A+iB)$ is a closed convex set, it has a unique point z_0 such that $|\mathrm{Re} z_0|^2 + |\mathrm{Im} z_0|^2 = \gamma(A,B)$. Choose $x_0 \in \mathbb{C}^n$ with $\|x_0\|_2 = 1$ such that $x_0^* A x_0 = \mathrm{Re}\, z_0$ and $x_0^* B x_0 = \mathrm{Im}\, z_0$. Setting

$$\Delta A := -\mathrm{Re}\, z_0\, x_0 x_0^* \text{ and } \Delta B := -\mathrm{Im}\, z_0\, x_0 x_0^*$$

it follows $x_0^*[(A+\Delta A) + i(B+\Delta B)]x_0 = 0$ and

$$\sqrt{\|\Delta A\|_2^2 + \|\Delta B\|_2^2} = \sqrt{|\mathrm{Re}\, z_0|^2 + |\mathrm{Im}\, z_0|^2} = \gamma(A,B). \qquad \square$$

We show that the Crawford number $\gamma(A, B)$ is the distance from $L(z)$ to a nearest Hermitian pencil with an eigenvalue of mixed type. For this we need the following lemma.

Lemma 3.2.2. *Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a collection of finite eigenvalues of a regular Hermitian pencil $L(z)$ of size n of positive (negative) type that belong to an interval on the real line which does not contain any other eigenvalues of $L(z)$. Then for small enough $|||\Delta L|||$, eigenvalues of all perturbed Hermitian pencils $(L + \Delta L)(z)$ lying in neighbourhoods of these eigenvalues are also of positive (negative) type.*

If ∞ is an eigenvalue of $L(z)$ of positive (negative) type of multiplicity k, then there exists $z_1, z_2 \in \mathbb{R}$ such that for small enough $|||\Delta L|||$, the Hermitian polynomials $(L + \Delta L)(z)$ have k_1 eigenvalues of positive (negative) type counting multiplicity belonging to the set $(z_1, \infty) \cup \{\infty\}$ and k_2 eigenvalues of negative (positive) type counting multiplicity belonging to the set $(-\infty, z_2)$ such that $k_1 + k_2 = k$.

Proof: Without loss of generality we assume that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigenvalues of $L(z)$ of positive type. Since $L(z)$ is a regular pencil, let Γ be a simple rectifiable closed curve in $\mathbb{C}$ that does not contain any eigenvalues of $L(z)$ such that $\Lambda(L) \cap \operatorname{Int}\Gamma = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. Consider the spectral projection $P = \frac{1}{2\pi i}\oint_\Gamma R(z)A dz$ onto the spectral subspace associated with these eigenvalues. If $\operatorname{rank} P = m$, then as already argued in Chapter 2, for any Hermitian pencil $\Delta L(z)$ with $|||\Delta L||| < \min_{z\in\Gamma} \eta(z, L)$, the projections $P_t(\Gamma) = \frac{1}{2\pi i}\oint_\Gamma R(t, z)(A + t\Delta A)dz$ onto the spectral subspace associated with the eigenvalues of $(L + t\Delta L)(z)$ in the interior of Γ have rank m for $t \in [-1, 1]$. Also there exist full rank matrices $X(t)$ that vary analytically with respect to $t \in [-1, 1]$ such that the columns of $X(t)$ span the range of $P_\Gamma(t)$. Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are positive definite, they are all semisimple and therefore $X(0)^*AX(0)$ is positive definite. Therefore if $|||\Delta L|||$ is small enough, then $X(t)^*(A+t\Delta A)X(t)$ is positive definite for all $t \in [0, 1]$. In particular, $X(1)^*(A + \Delta A)X(1)$ is positive definite and hence all the m eigenvalues of $(L + \Delta L)(z)$ in the interior of Γ are of positive type.

If ∞ is an eigenvalue of $L(z)$ of positive (negative) type then 0 is an eigenvalue of $-\operatorname{rev} L(z)$ of positive (negative) type. By considering a rectifiable simple closed curve Γ in $\mathbb{C}$ which does not pass through any eigenvalues of $-\operatorname{rev} L(z)$ and satisfies $\Lambda(-\operatorname{rev} L(z)) \cap \operatorname{Int}\Gamma = \{0\}$, and replacing $L(z)$ by $-\operatorname{rev} L(z)$ in the above arguments, it follows that all Hermitian pencils $-\operatorname{rev}(L + \Delta L)(z)$ with $|||\Delta L|||$ small enough, have k eigenvalues of positive (negative) type in the interior of Γ counting multiplicity. Since

Γ is a closed curve enclosing 0, $\operatorname{Int}\Gamma \cap \mathbb{R} = (-\delta_1, \delta_2)$ where δ_1 and δ_2 are positive real numbers. Let $z_1 = 1/\delta_2$ and $z_2 = -1/\delta_1$. For any nonzero eigenvalue λ_0 of $(L+\Delta L)(z)$ with corresponding eigenvector x, the identity (1.3) implies that

$$x^*(L+\Delta L)'(\lambda_0)x = -(1/\lambda_0)\, x^*(\operatorname{rev}(L+\Delta L)'(1/\lambda_0))x. \tag{3.2}$$

This implies that the polynomials $(L+\Delta L)(z)$ have k_1 eigenvalues of negative (positive) type lying in $(z_1, \infty) \cup \{\infty\}$ and k_2 eigenvalues of positive (negative) type in $(-\infty, z_2)$ such that $k_1 + k_2 = k$. $\square$

From the above lemma it follows that for small enough values of ϵ, the Hermitian ϵ-pseudospectra $\Lambda_\epsilon^H(L)$ are such that if any component of $\Lambda_\epsilon^H(L)$ contains only positive type (respectively negative type) eigenvalues of $L(z)$, then all eigenvalues of perturbed Hermitian pencils $(L+\Delta L)(z)$ for $|||\Delta L||| < \epsilon$, lying in that component are of positive type (respectively negative type). This observation and the Theorem 1.2.4 on canonical forms for Hermitian pencils under congruence together give the following result.

Theorem 3.2.3. *Given a definite pencil $L(z)$,*

$$\begin{aligned}\gamma(A,B) \;=\; & \min\{|||\Delta L||| : (L+\Delta L)(z) \textit{ is Hermitian and has a complex eigenvalue} \\ & \textit{or a real or infinite eigenvalue of mixed type}\}\end{aligned} \tag{3.3}$$

If $\gamma(A,B)$ is less than the distance with respect to the $|||\cdot|||$ norm to a nearest pencil that is not regular, then any Hermitian pencil $(L+\Delta L)(z)$ that attains the minimum in (3.3) does not have any complex eigenvalue.

Proof: From Lemma 3.2.1 and Theorem 1.2.12, $\gamma(A,B)$ is the distance from $L(z)$ to a nearest Hermitian pencil which either has a complex eigenvalue or a real eigenvalue of mixed type or is such that all its eigenvalues are real and of definite type but the positive and negative type eigenvalues do not belong to two disjoint intervals on the real line. Let $\gamma(A,B)$ be less than the distance with respect to the $|||\cdot|||$ norm to a nearest pencil that is not regular.

Suppose that there exists a Hermitian pencil ΔL such that $|||\Delta L||| = \gamma(A,B)$ and $(L+\Delta L)(z)$ has a complex eigenvalue, say z_0. Since z_0 has a nonzero imaginary part, and $(L+\Delta L)(z)$ is a regular pencil, it is possible to choose a simple rectifiable closed curve Γ such that interior of Γ consists only of complex numbers and z_0 is the only eigenvalue of $(L+\Delta L)(z)$ in the interior. If the eigenvalue z_0 of $(L+\Delta L)(z)$ has algebraic multiplicity m_0, by Theorem 2.1.7 any Hermitian pencil close enough to $(L + \Delta L)(z)$ has m_0 eigenvalues in the interior of Γ counting multiplicity which are all complex. This applies in particular to pencils $(L + t\Delta L)(z)$ with $t \in (0, 1)$ close enough to 1. But then $|||t\Delta L||| < |||\Delta L||| < \gamma(A, B)$, which implies that $(L + t\Delta L)(z)$ must be definite. This contradicts the fact that $(L + t\Delta L)(z)$ has at least m_0 complex eigenvalues in a neighbourhood of z_0.

Next, suppose that all the eigenvalues of a Hermitian pencil nearest to $L(z)$ with respect to the norm $|||\cdot|||$ that is not definite are real or infinite and of definite type but the positive and negative type eigenvalues do not belong to disjoint intervals on the extended real line. If this pencil be $(L + \widehat{\Delta L})(z)$, then $|||\widehat{\Delta L}||| = \gamma(A, B)$, and it has eigenvalues λ_1, λ_2 and λ_3 with $\lambda_3 < \lambda_2 < \lambda_1$ such that λ_2 differs in type with that of both λ_1 and λ_3.

Suppose that $\lambda_1 < \infty$ and λ_2 is of negative type. By Lemma 3.2.2, Hermitian pencils $(L + \Delta L)(z)$ close enough to $(L + \widehat{\Delta L})(z)$ have eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2$ and $\tilde{\lambda}_3$ close to λ_1, λ_2 and λ_3 respectively such that $\tilde{\lambda}_3 < \tilde{\lambda}_2 < \tilde{\lambda}_1$ and $\tilde{\lambda}_1$ and $\tilde{\lambda}_3$ are of positive type whereas $\tilde{\lambda}_2$ is of negative type. This implies that such pencils are also not definite. In particular, this also holds for Hermitian pencils $(L+\Delta L)(z)$ close enough to $(L+\widehat{\Delta L})(z)$ such that $|||\Delta L||| < |||\widehat{\Delta L}|||$. But this is impossible as it implies $\gamma(A, B) < |||\widehat{\Delta L}|||$. The proof for the case when λ_2 is of positive type, follows analogously if $\lambda_1 < \infty$.

Next suppose that $\lambda_1 = \infty$. Suppose without loss of generality that λ_2 is of negative type as the proof for other alternative follows by similar arguments. Once again by Lemma 3.2.2, Hermitian pencils $(L+\Delta L)(z)$ close enough to $(L+\widehat{\Delta L})(z)$ have eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2$ and $\tilde{\lambda}_3$ close to λ_1, λ_2 and λ_3 respectively such that with $\tilde{\lambda}_3 < \tilde{\lambda}_2 < \tilde{\lambda}_1$ if $\tilde{\lambda}_1$ is positive or equal to ∞ and $\tilde{\lambda}_1 < \tilde{\lambda}_3 < \tilde{\lambda}_2$ otherwise. In each case, $\tilde{\lambda}_3$ is of positive type and $\tilde{\lambda}_2$ is of negative type. By Lemma 3.2.2, if $\tilde{\lambda}_1$ is positive or $\tilde{\lambda}_1 = \infty$, it is of positive type and otherwise of negative type. In each case $(L + \Delta L)(z)$ is not a definite pencil. In particular, choosing $(\Delta L)(z)$ such that $|||\Delta L||| < |||\widehat{\Delta L}|||$, we have a contradiction to the fact that $|||\widehat{\Delta L}||| = \gamma(A, B)$.

From the above arguments it follows that a Hermitian pencil nearest to $L(z)$ which is not definite has a real or infinite eigenvalue of mixed type and no complex eigenvalues.

To prove (3.3) we assume without loss of generality that $\gamma(A, B)$ is less than the distance to a nearest non regular pencil. In view of the preceding arguments, the proof follows by showing that any arbitrarily small Hermitian perturbation to a Hermitian pencil with an eigenvalue of mixed type results in a perturbed pencil with a complex eigenvalue. Let $L(z)$ be a Hermitian pencil with an eigenvalue λ_0 of mixed type. Then either λ_0 is a defective eigenvalue of $L(z)$ or it is a semisimple eigenvalue with eigenvectors x_1 and x_2 such that $x_1^*Ax_1 > 0$ and $x_2^*Ax_2 < 0$. In the former case if λ_0 has algebraic multiplicity m_0 then Theorem 1.2.4 implies that $L(z)$ is congruent to a block diagonal Hermitian pencil with the following block of size m_0 on its diagonal.

$$\begin{bmatrix} & & & -\eta & \eta(z-\lambda_0) \\ & & -\eta & \eta(z-\lambda_0) & \\ & \iddots & \iddots & & \\ -\eta & \eta(z-\lambda_0) & & & \\ \eta(z-\lambda_0) & & & & \end{bmatrix}$$

where η is either 1 or -1. Perturbing this block by placing small real number δ in the $(m_0, 2)$ and $(2, m_0)$ positions results in the block

$$\begin{bmatrix} & & & -\eta & \eta(z-\lambda_0) \\ & & -\eta & \eta(z-\lambda_0) & \delta \\ & \iddots & \iddots & & \\ -\eta & \eta(z-\lambda_0) & & & \\ \eta(z-\lambda_0) & \delta & & & \end{bmatrix}.$$

Setting δ to be a small positive number if $\eta = 1$ and a small negative number if $\eta = -1$, it follows that the perturbed diagonal block has a pair of eigenvalues $\lambda_0 + i\sqrt{2|\delta|}$ if $m_0 = 3$ and a pair of eigenvalues $\lambda_0 + i\sqrt{|\delta|}$ if $m_0 \neq 3$. If λ_0 is a semisimple eigenvalue of mixed type, then $L(z)$ is congruent to a block diagonal pencil with a 2×2 block on the diagonal of the form

$$\begin{bmatrix} -(z-\lambda_0) & \\ & (z-\lambda_0) \end{bmatrix}.$$

Perturbing this block by placing $i\delta$ and $-i\delta$ in the $(1, 2)$ and $(2, 1)$ positions, where $\delta > 0$, gives rise to the block

$$\begin{bmatrix} -(z-\lambda_0) & i\delta \\ -i\delta & (z-\lambda_0) \end{bmatrix}$$

which has the complex conjugate pair of eigenvalues $\lambda_0 \pm i\delta$.

Let X be the nonsingular matrix such that $X^*(zA - B)X$ produces the canonical form (1.2). Choosing $\Delta A = 0$ and ΔB such that $X^*\Delta BX$ is the appropriate Hermitian matrix that produces the perturbations as described above, in each case, there exists an arbitrarily small Hermitian perturbation such that the perturbed pencil has a complex conjugate pair of eigenvalues. This concludes the proof. □

Theorem 3.2.3 has the following immediate implication.

Corollary 3.2.4. *Given a definite pencil $L(z)$ if $\gamma(A, B)$ is less than the distance to a nearest Hermitian pencil that is not regular with respect to the norm $|||\cdot|||$, then for any complex number z_0, $\eta^H(z_0, L) > \gamma(A, B)$.*

3.3 Homogeneous definition of eigenvalue type

In this section we propose a homogeneous definition of the type of a real eigenvalue of a regular Hermitian pencil or polynomial that applies to both finite and infinite eigenvalues without having to deal with the reversal of the pencil or polynomial. Recall that a finite real eigenvalue λ of an $n \times n$ Hermitian polynomial $P(z) = \Sigma_{k=0}^m z^k A_k$ is said to be of positive (negative) type if $x^* P'(\lambda) x > 0$ (< 0) for every eigenvector x corresponding to λ. If λ is infinite, then its type is determined by the type of 0 as an eigenvalue of $-\mathrm{rev}\, P(z)$. We propose a homogeneous form of this definition which does not involve the reversal of $P(z)$ when determining the type of the infinite eigenvalue. For this purpose we consider the homogeneous form of the polynomial which is

$$P(\alpha, \beta) = \Sigma_{k=0}^m \alpha^k \beta^{m-k} A_k.$$

Definition 3.3.1. *A real eigenvalue λ_0 of a Hermitian polynomial $P(z)$ is said to be of positive (negative) type if the expression*

$$x^* \left(\beta_0 \frac{\partial P}{\partial \alpha}(\alpha_0, \beta_0) - \alpha_0 \frac{\partial P}{\partial \beta}(\alpha_0, \beta_0) \right) x$$

is positive (negative) for every eigenvector x corresponding to λ_0 and any representation (α_0, β_0) of λ_0 with $\beta_0 > 0$ if λ_0 is finite and $\alpha_0 > 0$ if $\lambda_0 = \infty$.

To see that the above definition coincides with the definition in [4] first consider the case when λ_0 is finite. Let (α_0, β_0) be a representation of λ_0 with $\beta_0 > 0$ and let

$s_0 = x^*\left(\beta_0 \frac{\partial P}{\partial \alpha}(\alpha_0, \beta_0) - \alpha_0 \frac{\partial P}{\partial \beta}(\alpha_0, \beta_0)\right) x$. Then,

$$\begin{aligned}
\beta_0 s_0 &= \beta_0 \left[\Sigma_{j=1}^{m} j\alpha_0^{j-1}\beta_0^{m-j+1} x^* A_j x\right] - \alpha_0 \left[\Sigma_{j=0}^{m-1}(m-j)\alpha_0^{j}\beta_0^{m-j} x^* A_j x\right] \\
&= \Sigma_{j=1}^{m-1}\left[j\alpha_0^{j-1}\beta_0^{m-j+2} - (m-j)\alpha_0^{j+1}\beta_0^{m-j}\right] x^* A_j x \\
&\quad + m\beta_0^2\alpha_0^{m-1} x^* A_m x - m\alpha_0\beta_0^m x^* A_0 x \\
&= \Sigma_{j=1}^{m-1}\left[j\alpha_0^{j-1}\beta_0^{m-j+2} - (m-j)\alpha_0^{j+1}\beta_0^{m-j} + \alpha_0^{j+1}\beta_0^{m-j}\right] x^* A_j x \\
&\quad + m\beta_0^2\alpha_0^{m-1} x^* A_m x + m\alpha_0^{m+1} x^* A_m x \\
&= \Sigma_{j=1}^{m-1}(\alpha_0^2+\beta_0^2) j\alpha_0^{j-1}\beta_0^{m-j} x^* A_j x + m(\alpha_0^2+\beta_0^2)\alpha_0^{m-1} x^* A_m x \\
&= (\alpha_0^2+\beta_0^2)\Sigma_{j=1}^{m} j\alpha_0^{j-1}\beta_0^{m-j} x^* A_j x \\
&= \beta_0^{m-1}(\alpha_0^2+\beta_0^2) x^* P'(\lambda_0) x
\end{aligned}$$

where we use the fact that $\beta_0^m x^* A_0 x = -\Sigma_{j=1}^{m}\alpha_0^j\beta_0^{m-j} x^* A_j x$ since x is an eigenvector corresponding to the eigenvalue (α_0, β_0). Thus

$$x^*\left(\beta_0 \frac{\partial P}{\partial \alpha}(\alpha_0, \beta_0) - \alpha_0 \frac{\partial P}{\partial \beta}(\alpha_0, \beta_0)\right) x = \beta_0^{m-2}(\alpha_0^2+\beta_0^2) x^* P'(\lambda_0) x. \tag{3.4}$$

Since $\beta_0 > 0$, the left hand side of (3.4) has the same sign as that of $x^* P'(\lambda_0) x$.

If $\lambda_0 = \infty$, then choosing a representation $(\alpha_0, 0)$ where $\alpha_0 > 0$,
$x^*\left(\beta_0 \frac{\partial P}{\partial \alpha}(\alpha_0, \beta_0) - \alpha_0 \frac{\partial P}{\partial \beta}(\alpha_0, \beta_0)\right) x = -\alpha_0^m x^* A_{m-1} x = \alpha_0^m x^*(-(\text{rev } P)'_\lambda(0)) x$.
Since $\alpha_0 > 0$, this shows that the type of λ_0 via the homogeneous definition coincides with the definition via the reversal of $P(z)$.

Next we present an alternative proof of Lemma 1.2.10 starting with the homogeneous definition of eigenvalue type. Consider a homogeneous rotation of the Hermitian polynomial $P(\alpha, \beta)$ by an angle θ which changes it to the polynomial $\tilde{P}(\tilde{\alpha}, \tilde{\beta})$. Recall that (α, β) and $(\tilde{\alpha}, \tilde{\beta})$ are related by

$$\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{3.5}$$

where $c = \cos\theta$ and $s = \sin\theta$. The following lemma is crucial towards relating the type of an eigenvalue with that of its rotated counterpart.

Lemma 3.3.2. *Let the Hermitian polynomial $P(\alpha, \beta) = \Sigma_{j=0}^{m}\alpha^j\beta^{m-j}A_j$ be rotated by an angle θ to the polynomial $\tilde{P}(\tilde{\alpha}, \tilde{\beta}) = \Sigma_{j=0}^{m}\tilde{\alpha}^j\tilde{\beta}^{m-j}\tilde{A}_j$ where (α, β) and $(\tilde{\alpha}, \tilde{\beta})$ are related by equation (3.5). Let λ_0 be an eigenvalue of $P(\alpha, \beta)$ represented by a pair (α_0, β_0) with* *corresponding rotated eigenvalue $\tilde{\lambda}_0$ represented by $(\tilde{\alpha}_0, \tilde{\beta}_0)$. Then for any eigenvector x corresponding to λ_0 and $\tilde{\lambda}_0$,*

$$x^* \left[\beta_0 \frac{\partial P}{\partial \alpha}(\alpha_0, \beta_0) - \alpha_0 \frac{\partial P}{\partial \beta}(\alpha_0, \beta_0)\right] x = x^* \left[\tilde{\beta}_0 \frac{\partial \tilde{P}}{\partial \tilde{\alpha}}(\tilde{\alpha}_0, \tilde{\beta}_0) - \tilde{\alpha}_0 \frac{\partial \tilde{P}}{\partial \tilde{\beta}}(\tilde{\alpha}_0, \tilde{\beta}_0)\right] x. \quad (3.6)$$

Proof: Using the equality $\tilde{P}(\tilde{\alpha}_0, \tilde{\beta}_0) = P(\alpha_0, \beta_0)$ and the relations (3.5) the chain rule for partial derivatives gives the following.

$$\begin{aligned}
x^* \left[\beta_0 \frac{\partial P}{\partial \alpha}(\alpha_0, \beta_0) - \alpha_0 \frac{\partial P}{\partial \beta}(\alpha_0, \beta_0)\right] x &= x^* \left[\beta_0 \frac{\partial \tilde{P}}{\partial \alpha}(\tilde{\alpha}_0, \tilde{\beta}_0) - \alpha_0 \frac{\partial \tilde{P}}{\partial \beta}(\tilde{\alpha}_0, \tilde{\beta}_0)\right] x \\
&= \beta_0 x^* \left[\frac{\partial \tilde{P}}{\partial \tilde{\alpha}}(\tilde{\alpha}_0, \tilde{\beta}_0)\frac{\partial \tilde{\alpha}_0}{\partial \alpha} + \frac{\partial \tilde{P}}{\partial \tilde{\beta}}(\tilde{\alpha}_0, \tilde{\beta}_0)\frac{\partial \tilde{\beta}_0}{\partial \alpha}\right] x \\
&\quad - \alpha_0 x^* \left[\frac{\partial \tilde{P}}{\partial \tilde{\alpha}}(\tilde{\alpha}_0, \tilde{\beta}_0)\frac{\partial \tilde{\alpha}}{\partial \beta} + \frac{\partial \tilde{P}}{\partial \tilde{\beta}}(\tilde{\alpha}_0, \tilde{\beta}_0)\frac{\partial \tilde{\beta}}{\partial \beta}\right] x \\
&= (s\tilde{\alpha}_0 + c\tilde{\beta}_0) x^* \left[c\frac{\partial \tilde{P}}{\partial \tilde{\alpha}}(\tilde{\alpha}_0, \tilde{\beta}_0) - s\frac{\partial \tilde{P}}{\partial \tilde{\beta}}(\tilde{\alpha}_0, \tilde{\beta}_0)\right] x \\
&\quad - (c\tilde{\alpha}_0 - s\tilde{\beta}_0) x^* \left[s\frac{\partial \tilde{P}}{\partial \tilde{\alpha}}(\tilde{\alpha}_0, \tilde{\beta}_0) + c\frac{\partial \tilde{P}}{\partial \tilde{\beta}}(\tilde{\alpha}_0, \tilde{\beta}_0)\right] x \\
&= x^* \left[\tilde{\beta}_0 \frac{\partial \tilde{P}}{\partial \tilde{\alpha}}(\tilde{\alpha}_0, \tilde{\beta}_0) - \tilde{\alpha}_0 \frac{\partial \tilde{P}}{\partial \tilde{\beta}}(\tilde{\alpha}_0, \tilde{\beta}_0)\right] x. \qquad \square
\end{aligned}$$

If $\lambda \in \mathbb{R} \cup \{\infty\}$ is an eigenvalue of $P(\alpha, \beta)$, then without loss of generality we may assume that it is represented by a pair $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha^2 + \beta^2 = 1$. This allows for unique representations of eigenvalues on the extended real line, with points both from the upper half and the lower half of the unit circle in $\mathbb{R}^2$. Note that the homogeneous definition of eigenvalue type restricts the representations to the upper half of the unit circle for finite eigenvalues while infinity corresponds to the point $(1, 0)$. With this convention for representation of eigenvalues on the extended real line, we have the following alternative proof of Lemma 1.2.10 which is restated for the sake of completeness.

Lemma 3.3.3. *Let the Hermitian polynomial $P(\alpha, \beta) = \Sigma_{j=0}^m \alpha^j \beta^{m-j} \tilde{A}_j$ be rotated by an angle θ to the polynomial $\tilde{P}(\tilde{\alpha}, \tilde{\beta}) = \Sigma_{j=0}^m \tilde{\alpha}^j \tilde{\beta}^{m-j} \tilde{A}_j$ where (α, β) and $(\tilde{\alpha}, \tilde{\beta})$ are related by equation (3.5). Let λ_0 be an eigenvalue of $P(\alpha, \beta)$ on the extended real line represented by a pair (α_0, β_0) that lies on the unit circle with $\beta_0 > 0$ if $\lambda_0 \in \mathbb{R}$ and $\alpha_0 = 1$ when*

$\lambda_0 = \infty$. *Also let the corresponding rotated eigenvalue* $\tilde{\lambda}_0$ *of* $\tilde{P}(\tilde{\alpha}, \tilde{\beta})$ *be represented by a point* $(\tilde{\alpha}_0, \tilde{\beta}_0)$ *on the unit circle and* x *be any eigenvector corresponding to* λ_0 *and* $\tilde{\lambda}_0$.

(a) *If* $\lambda_0, \tilde{\lambda}_0 \in \mathbb{R}$, *then* $c - \lambda_0 s \neq 0$ *and*

$$x^* P'_\lambda(\lambda_0)x = (c - \lambda_0 s)^{m-2} x^* \tilde{P}'_{\tilde{\lambda}}(\tilde{\lambda}_0)x.$$

(b) *If* $\lambda_0 \in \mathbb{R}$ *and* $\tilde{\lambda}_0 = \infty$, *then* $s \neq 0$ *and*

$$x^* P'_\lambda(\lambda_0)x = s^{2-m} x^*(-(\mathrm{rev}\, \tilde{P})'_{\tilde{\lambda}}(0))x.$$

(c) *If* $\lambda_0 = \infty$ *and* $\tilde{\lambda}_0 \in \mathbb{R}$, *then* $s \neq 0$ *and*

$$x^*(-(\mathrm{rev}\, P)'_\lambda(0))x = (-s)^{m-2} x^* \tilde{P}'_{\tilde{\lambda}}(\tilde{\lambda}_0)x.$$

Proof: [(a)] Since β_0 and $\tilde{\beta}_0$ are both nonzero, from (3.5) we have

$$c - \lambda_0 s = \frac{c\beta_0 - s\alpha_0}{\beta_0} = \frac{\tilde{\beta}_0}{\beta_0} \neq 0.$$

Equations (3.6) and (3.4) imply that

$$\beta_0^{m-2} x^* P'(\lambda_0)x = \tilde{\beta}_0^{m-2} x^* \tilde{P}'(\tilde{\lambda}_0)x.$$

The rest of the proof follows upon replacing $\tilde{\beta}_0$ by $c\beta_0 - s\alpha_0$ in the above identity and dividing both sides by β_0^{m-2}.

(b) In this case since $(\tilde{\alpha}_0, \tilde{\beta}_0) = (1, 0)$, and $\beta_0 \neq 0$, from (3.5) it follows that $s = \beta_0 \neq 0$. Putting $(\tilde{\alpha}_0, \tilde{\beta}_0) = (1, 0)$, in (3.6) we have

$$x^* \left[\beta_0 \frac{\partial P}{\partial \alpha}(\alpha_0, \beta_0) - \alpha_0 \frac{\partial P}{\partial \beta}(\alpha_0, \beta_0)\right] x = -x^* \tilde{A}_{m-1} x = x^*(-(\mathrm{rev}\, \tilde{P})'_{\tilde{\lambda}}(0))x.$$

Using (3.4) and the fact that $\beta_0 = s$, we have

$$s^{m-2} x^* P'_\lambda(\lambda_0)x = x^*(-(\mathrm{rev}\, \tilde{P})'_{\tilde{\lambda}}(0)x$$

from which the proof is immediate upon dividing both sides by s^{m-2}.

(c) In this case since $(\alpha_0, \beta_0) = (1, 0)$, (3.5) implies that $s = \tilde{\beta}_0 \neq 0$ as $\lambda_0 \neq 0$. Using these values of $\alpha_0, \beta_0, \tilde{\alpha}_0$ and $\tilde{\beta}_0$ in (3.6) and (3.4) we have,

$$(-s)^{m-2} x^* \tilde{P}'_{\tilde{\lambda}}(\tilde{\lambda}_0)x = -x^* A_{m-1} x = x^*(-(\mathrm{rev}\, P)'_\lambda(0))x.$$

The proof follows by dividing both sides by $(-s)^{m-2}$. $\square$

3.4 Properties of Hermitian pencils

In this section we investigate certain properties of regular Hermitian pencils via their canonical form (1.2) under congruence. Throughout the section we assume that $L(z) = zA - B$ is a regular Hermitian pencil of size n. We make the following definitions.

Definition 3.4.1. *Two vectors $x, y \in \mathbb{C}^n$ are said to be A-orthogonal if $x^*Ay = 0$.*

Definition 3.4.2. *Two subspaces $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{C}^n$, are said to be A-orthogonal if $x^*Ay = 0$ for all $x \in \mathcal{S}_1$ and $y \in \mathcal{S}_2$.*

Definition 3.4.3. *A subspace $\mathcal{S} \subset \mathbb{C}^n$ is said to be A-neutral if $x^*Ax = 0$ for all $x \in \mathcal{S}$.*

Definition 3.4.4. *A subspace $\mathcal{S} \subset \mathbb{C}^n$ is said to be A-nondegenerate if for every $x \in \mathcal{S} \setminus \{0\}$, there exists $y \in \mathcal{S}$ such that $x^*Ay \neq 0$.*

The following theorem analyses deflating subspaces of Hermitian pencil in terms of their A-orthogonality and A-nondegeneracy properties.

Theorem 3.4.5. *Let $L(z) = zA - B$ be a Hermitian pencil of size n. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be deflating subspaces of $L(z)$. Let Λ_1 and Λ_2 be the set of eigenvalues of $L(z)$ restricted to $\mathcal{S}_1$ and $\mathcal{S}_2$. If $\infty \notin \Lambda_1 \cup \Lambda_2$ and $\bar{\Lambda}_1 \cap \Lambda_2 = \emptyset$ where $\bar{\Lambda}_1$ denotes the set of the complex conjugates of the elements of Λ_1, then S_1 and S_2 are A-orthogonal.*

Each of the subspaces S_1 and S_2 are A-nondegenerate if they have the property that if they contain a vector belonging to a Jordan chain corresponding to some $\lambda \in \Lambda_i$, then it contains all the vectors of the chain.

Proof: Let the columns of $X_1 \in \mathbb{C}^{n \times k_1}$ and $X_2 \in \mathbb{C}^{n \times k_2}$ form the bases of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. The subspaces S_1 and S_2 will be A-orthogonal if it is shown that $X_1^*AX_2 = 0$. Since ∞ does not belong to any of Λ_1 and Λ_2, by Theorem 1.2.2, there exists matrices $K_1 \in \mathbb{C}^{k_1 \times k_1}$ and $K_2 \in \mathbb{C}^{k_2 \times k_2}$ such that $AX_1K_1 = BX_1$ and $AX_2K_2 = BX_2$. Also $\Lambda(K_1) = \Lambda_1$ and $\Lambda(K_2) = \Lambda_2$. The assumption $\bar{\Lambda_1} \cap \Lambda_2 = \emptyset$ implies that $0 \notin \Lambda(K_1^*) - \Lambda(K_2)$. Therefore the Sylvester operator $S(Y) = K_1^*Y - YK_2$ is nonsingular. The proof of $X_1^*AX_2 = 0$ now follows by observing that

$$\begin{aligned} S(X_1^*AX_2) &= K_1^*X_1^*AX_2 - X_1^*AX_2K_2 \\ &= X_1^*BX_2 - X_1^*BX_2 \\ &= 0. \end{aligned}$$

Let S_1 have the property that if it contains an eigenvector belonging to a Jordan chain of an eigenvalue $\lambda \in \Lambda_1$, then it contains all the vectors of that chain. Let $x \in \mathcal{S}_1 \setminus \{0\}$, and X be the nonsingular matrix such that $X^* L(z) X$ gives the canonical form (1.2) under congruence. Without loss of generality we may assume that some k_1 columns of X form a basis of $\mathcal{S}_1$. If these columns form the matrix $X_1 \in \mathbb{C}^{n \times k_1}$ then we may assume that $X_1 = [X_{1,1}, X_{1,2}, \ldots X_{1,m}, X_{1,m+1}, \ldots, X_{1,p}]$ where the columns of $X_{1,1}$ are eigenvectors corresponding to semisimple eigenvalues, the columns of each of $X_{1,2}, \ldots, X_{1,m}$ correspond to the blocks of the canonical form associated with real eigenvalues of the form

$$J(\lambda) := \eta[(z - \lambda)F_m - G_m],$$

and the columns of each of $X_{1,m+1}, \ldots, X_{1,p}$ correspond to blocks of the canonical form associated with complex eigenvalues of the form

$$J(\lambda) := \begin{bmatrix} 0 & (z-\lambda)F_m - G_m \\ (z-\bar{\lambda})F_m - G_m & 0 \end{bmatrix} \text{ or } J(\lambda) = (z-\lambda)F_m - G_m.$$

From the canonical form we observe that the spans of the columns of $X_{1,i}$ and $X_{1,j}$ are A-orthogonal for $i \neq j$ and each of them is individually A-nondegenerate. Since $x \in \oplus_{i=1}^{p} \operatorname{span} X_{i,i}$ it follows that $x^* A y$ is nonzero for at least some $y \in \mathcal{S}_1$. $\square$

It may be easily verified that any eigenvector corresponding to a complex eigenvalue of a Hermitian pencil is A-orthogonal. In [4] it is proved that any eigenvector of $L(z)$ corresponding to a defective eigenvalue is also A-orthogonal. The following proposition extends this result to that of vectors that form a Jordan chain for such an eigenvalue.

Proposition 3.4.6. *Let λ be a defective eigenvalue of an Hermitian pencil $L(z)$ of size n. If λ is complex, then there exists a Jordan chain corresponding to each Jordan block associated with λ such that the vectors of the chain are mutually A-orthogonal and they span an A-neutral subspace. In particular, the spectral subspace associated with λ is A-neutral.*

If λ is real or infinite and has a Jordan block of size m, then there exists a Jordan chain of length m such that the first $\lfloor \frac{m}{2} \rfloor$ vectors of the chain are A-orthogonal and span an A-neutral subspace.

Proof: The proof follows from the canonical form (1.2) for $L(z)$ under congruence. First consider the case when λ is complex. If λ has a Jordan block of size m, then the canonical form (1.2) has the following block on the diagonal.

$$J(\lambda) := \begin{bmatrix} 0 & (z-\lambda)F_m - G_m \\ (z-\bar{\lambda})F_m - G_m & 0 \end{bmatrix}.$$

Without loss of generality it may be assumed that this block is the first block of the block diagonal canonical form. Let X be the nonsingular matrix such that $X^*L(z)X$ gives this form. If $x_1, x_2, \ldots, x_{2m}$ be the first $2m$ columns of X, then $v_k = x_{2m-k+1}$ for $k = 1, 2, \ldots, m$ form a Jordan chain corresponding to λ. Let e_k denote the k^{th} column of the identity matrix I_{2m}. Since the (i, j) entries of $J(\lambda)$ are zero for $m+1 \leq i, j \leq 2m$, we have for $1 \leq i, j \leq m$,

$$v_i^* A v_j = x_{2m-i+1}^* A x_{2m-j+1} = e_{2m-i+1}^* J(\lambda) e_{2m-j+1} = 0.$$

This proves that the vectors $v_{m+1}, v_m, \ldots, v_{2m}$ are A-orthogonal and span an A-neutral subspace. This holds for all other Jordan chains that arise in a similar way from the corresponding blocks of the canonical form (1.2) and the spans of such mutually disjoint Jordan chains are mutually A-orthogonal. Since the union of these chains span the spectral subspace of λ, it must be an A-neutral subspace.

If $\lambda \in \mathbb{R}$, then $J(\lambda) := \eta[(z-\lambda)F_m - G_m]$ whereas $J(\lambda) = \delta[-F_m + zG_m]$ if $\lambda = \infty$ where η and δ are equal to either $+1$ or -1. In each case, the proof follows by observing that the (i, j) entries of $J(\lambda)$ for $1 \leq i, j \leq \lfloor \frac{m}{2} \rfloor$ are zero and applying the above arguments. $\square$

Given an eigenvalue λ of an $n \times n$ Hermitian pencil $L(z)$ of mixed type, there exists an eigenvector corresponding to such an eigenvalue such that $x^*Ax = 0$. If $X \in \mathbb{C}^{n \times k}$ is a matrix whose columns form a basis of the spectral subspace $\mathrm{Ker}\,(\lambda A - B)^n$ of $L(z)$ associated with λ, then x belongs to its column space so that there exist $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]^T \in \mathbb{C}^k$ such that $x = X\alpha$ and $\alpha^*(X^*AX)\alpha = 0$. On the other hand if λ is of definite type as an eigenvalue of $L(z)$, then the columns of X form a basis of $\mathrm{Ker}\,(\lambda A - B)$ as λ is necessarily semisimple. Moreover, λ is of positive (negative) type if X^*AX is positive (negative) definite. These observations are independent of the choice of X since if $\hat{X}$ is any other matrix whose columns form a basis of the spectral subspace $\mathrm{Ker}\,(\lambda A - B)^n$, then there exists a nonsingular matrix S such that $X = S\hat{X}$ so that X^*AX and $\hat{X}^*A\hat{X}$ are congruent. Therefore we have the following definitions of eigenvalue type for a Hermitian pencil.

Definition 3.4.7. *A finite eigenvalue λ of a Hermitian pencil $L(z)$ is said to be of positive (negative) type if for any matrix X whose columns form a basis of the spectral subspace of λ, we have X^*AX is positive (negative) definite.*

If $\lambda = \infty$, then it is of positive (negative) type if for any matrix X whose columns form a basis of the spectral subspace corresponding to 0 as an eigenvalue of $-\mathrm{rev}\, L(z)$, we have X^*BX is positive (negative) definite.

The homogeneous definition of eigenvalue type proposed in Section 3.3 may also be similarly modified.

Definition 3.4.8. *An eigenvalue λ of a Hermitian pencil $L(z)$ is said to be of positive (negative) type if $\bar{\beta}X^*AX + \bar{\alpha}X^*BX$ is positive (negative) definite for any matrix X whose columns form a basis of the spectral subspace of $L(z)$ corresponding to λ and any representation $(\alpha, \beta) \in \mathbb{R}^2$ of λ such that $\beta > 0$ if λ is finite and $\alpha > 0$ if λ is infinite.*

3.5 Properties of Hermitian ϵ-pseudospectra

In this section we investigate various properties of Hermitian ϵ-pseudospectra of regular Hermitian pencils. Throughout the section we assume that $L(z) = zA - B$ is a regular Hermitian pencil of size n. Given any non real eigenvalue λ of $L(z)$, with corresponding right eigenvector x, we have $x^*Ax = 0$ as otherwise we have $\lambda = \frac{x^*Bx}{x^*Ax} \in \mathbb{R}$ - a contradiction. Thus for such eigenvalues we have $x^*Ax = x^*Bx = 0$ for all $x \in \mathrm{Ker}\, L(\lambda)$. Due to this fact, whenever we say that λ is an eigenvalue of $L(z)$ of definite type, we will mean that λ is real. With this convention in mind we extend the definition of the type of a real eigenvalue of $L(z)$ to the components of it Hermitian ϵ-pseudospectrum. Let $\Lambda_\epsilon^H(L)$ be the Hermitian ϵ-pseudospectrum of a Hermitian pencil $L(z)$. The following definition specifies the type of a component of $\Lambda_\epsilon^H(L)$.

Definition 3.5.1. *A component C_ϵ of $\Lambda_\epsilon^H(L)$ is said to be of positive (negative) type if the finite eigenvalues of all perturbed Hermitian pencils $(L + \Delta L)(z)$ with $|||\Delta L||| < \epsilon$ belonging to C_ϵ are of positive (negative) type.*

The component C_ϵ is said to be of definite type if it is either of positive or negative type.

From the above definition and the preceding comments, it is clear that a definite type component of $\Lambda_\epsilon^H(L)$ is a subset of the extended real line. The following lemma states that if a component C_ϵ of $\Lambda_\epsilon^H(L)$ does not contain any eigenvalues of mixed type of perturbed Hermitian pencils $(L+\Delta L)(z)$ for $|||\Delta L||| < \epsilon$, then it is of positive or negative type if $\infty \notin C_\epsilon$. For the proof we recall the fact that two components $C_\epsilon^{(1)}$ and $C_\epsilon^{(2)}$ of $\Lambda_\epsilon^H(L)$ have coalesced if $\partial(C_\epsilon^{(1)}) \cap \partial(C_\epsilon^{(2)}) \neq \emptyset$.

Lemma 3.5.2. *Let C_{ϵ_0} be a component of $\Lambda_{\epsilon_0}^H(L)$ such that finite eigenvalues of all perturbed Hermitian pencils $(L+\Delta L)(z)$ with $|||\Delta L||| < \epsilon_0$ belonging to C_{ϵ_0} are not of mixed type. If C_{ϵ_0} is not of positive or negative type, then $\infty \in C_{\epsilon_0}$.*

Proof: If possible let $\infty \notin C_{\epsilon_0}$. Suppose that the eigenvalues of $L(z)$ in C_{ϵ_0} are finite and of positive type. From Lemma 3.2.2, for Hermitian perturbations $\Delta L(z)$ with $|||\Delta L|||$ small enough, the eigenvalues of $(L+\Delta L)(z)$ inside C_{ϵ_0} are of positive type. Let $\hat{\epsilon}$ be the minimal norm of all possible Hermitian perturbations $\Delta L(z)$ such that $(L+\Delta L)(z)$ has an eigenvalue of negative type in C_{ϵ_0}. Assuming this eigenvalue to be z_0 and the corresponding pencil to be $(L+\widehat{\Delta L})(z)$, by Lemma 3.2.2, there exists a neighbourhood of z_0 which has a nonempty intersection with C_{ϵ_0} such that all the points of the neighbourhood are eigenvalues of negative type of perturbed pencils $(L+\Delta L)(z)$ close enough to $(L+\widehat{\Delta L})(z)$. In particular, such a neighbourhood contains at least one eigenvalue of negative type of Hermitian pencils $(L+\Delta L)(z)$ for $|||\Delta L||| < |||\widehat{\Delta L}||| = \hat{\epsilon}$. But this contradicts the minimality of $\hat{\epsilon}$. Therefore perturbed Hermitian pencils $(L+\Delta L)(z)$ with $|||\Delta L||| < \epsilon_0$ do not have any eigenvalues of negative type in C_{ϵ_0}. Since such pencils also do not contain eigenvalues of mixed type in C_{ϵ_0}, it follows that C_{ϵ_0} is a component of positive type which leads to a contradiction. Similarly assuming that the eigenvalue of $L(z)$ inside C_{ϵ_0} are of negative type implies that C_{ϵ_0} is a component of negative type which also leads to a contradiction.

Suppose that $L(z)$ has eigenvalues of both positive and negative type in C_{ϵ_0}. Then there exists $\epsilon < \epsilon_0$ for which a component of $\Lambda_\epsilon^H(L)$ say $C_\epsilon^{(1)}$, containing the eigenvalues of $L(z)$ in C_{ϵ_0} of positive type coalesces with a component say $C_\epsilon^{(2)}$, which contains the eigenvalues of $L(z)$ in C_{ϵ_0} of negative type. The component of the closure of $\Lambda_\epsilon^H(L)$ formed as a result of this coalescence is a subset of C_{ϵ_0}. Therefore $\infty \notin C_\epsilon^{(1)} \cup C_\epsilon^{(2)}$. Also none of the components $C_\epsilon^{(1)}$ and $C_\epsilon^{(2)}$ contain any eigenvalues of perturbed Hermitian pencils $(L+\Delta L)(z)$ of mixed type. Therefore, by the preceding arguments $C_\epsilon^{(1)}$ and $C_\epsilon^{(2)}$ are respectively positive and negative type components. Any point of coalescence of these components is an eigenvalue of some Hermitian pencil $(L+\Delta L)(z)$ with $|||\Delta L||| = \epsilon$ and since it belongs to C_{ϵ_0}, it is a finite point. Let z_0 be any such coalescence point.

If z_0 is an eigenvalue of $(L+\Delta L)(z)$ of positive type, then by Lemma 3.2.2 there is a neighborhood of z_0 such that every point of the neighborhood is an eigenvalue of a positive type of Hermitian pencils $(L+\widehat{\Delta L})(z)$ close enough to $(L+\Delta L)(z)$. However since z_0 is a point of coalescence of $C_\epsilon^{(1)}$ and $C_\epsilon^{(2)}$ every neighborhood of z_0 also contains points of $C_\epsilon^{(2)}$ which is a component of negative type. This contradiction implies that z_0 is not an eigenvalue of $(L+\Delta L)(z)$ of positive type. By similar arguments it is also not an eigenvalue of $(L+\Delta L)(z)$ of negative type. Since z_0 is also not an eigenvalue of $(L+\Delta L)(z)$ of mixed type, we have a contradiction and the proof follows. $\square$

The following theorem characterizes positive or negative components of $\Lambda_\epsilon^H(L)$ and shows that if a component of $\Lambda_\epsilon^H(L)$ contains only positive (negative) type eigenvalues of $L(z)$, then it is a positive (negative) type component of $\Lambda_\epsilon^H(L)$ if it does not contain ∞ in its closure.

Theorem 3.5.3. *Let $L(z)$ be a Hermitian pencil of size n and C_ϵ be a component of $\Lambda_\epsilon^H(L)$ such that ∞ does not belong to its closure. Let $X(t)$ be a matrix with full column rank such that its columns form a basis of the direct sums of spectral subspaces associated with the eigenvalues of $(L+t\Delta L)(z)$ that lie in C_ϵ for a Hermitian pencil $\Delta L(z)$ with $|||\Delta L||| < \epsilon$ and $t \in [0,1]$. The following statements are equivalent.*

(i) The component C_ϵ is of positive (negative) type.

(ii) The eigenvalues of $L(z)$ in C_ϵ are of positive (negative) type.

(iii) The matrix $X(0)^*AX(0)$ is positive (negative) definite.

(iv) The matrix $X(t)^*(A+t\Delta A)X(t)$ is positive (negative) definite for any Hermitian pencil $\Delta L(z)$ with $|||\Delta L||| < \epsilon$ and $t \in [0,1]$.

Proof: The proof of (i) $\Rightarrow$ (ii) is evident. Indeed, if C_ϵ is a component of positive type, then all eigenvalues of perturbed Hermitian pencils $(L+\Delta L)(z)$ with $|||\Delta L||| < \epsilon$, in C_ϵ are of positive type so that in particular the eigenvalues of $L(z)$ are also of positive type. This proves (i) $\Rightarrow$ (ii).

If all the eigenvalues of $L(z)$ in C_ϵ are of positive type, then they are real and semisimple. Let $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ be the distinct eigenvalues of $L(z)$ in C_ϵ. Let $X = [X_1, X_2, \ldots, X_k]$ where the columns of X_j form a basis of Ker $(\lambda_j A - B)$ for $j = 1, 2, \ldots, k$. By Theorem 3.4.5, X^*AX is a block diagonal matrix with blocks $X_j^*AX_j$, $j = 1, 2, \ldots, k$, on the diagonal each of which is positive definite due to Definition (3.4.7). Therefore, X^*AX is positive definite. Since the columns of X and $X(0)$ form bases of the same space, there exists a non singular matrix S such that $X = SX(0)$ so that X^*AX and $X(0)^*AX(0)$ are congruent. Hence $X(0)^*AX(0)$ is positive definite. This proves (ii) $\Rightarrow$ (iii).

Conversely, if $X(0)^*AX(0)$ is positive definite, then so is X^*AX as the two matrices are congruent. Hence the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all of positive type. This proves (iii) $\Rightarrow$ (ii). Since the equivalence (iv) $\Leftrightarrow$ (i) is evident, the proof is completed by showing that (ii) $\Rightarrow$ (i).

Suppose that the eigenvalues of $L(z)$ in C_ϵ are of positive type. Let Γ be a simple rectifiable closed curve in $\mathbb{C}$ lying in the complement of $\Lambda_\epsilon^H(L)$ such that $\Lambda_\epsilon^H(L) \cap \operatorname{Int}\Gamma = C_\epsilon$. Therefore there are no eigenvalues of $(L + t\Delta L)(z)$ lying on Γ for any Hermitian pencil $\Delta L(z)$ with $|||\Delta L||| < \epsilon$ and $t \in [0,1]$. Hence the spectral projections $P(t) := \frac{1}{2\pi i}\oint_\Gamma R(t,z)(A + t\Delta A)dz$ exist for all $t \in [0,1]$. By Theorem 2.1.7, there exists a full column rank matrix $X(t)$ which varies continuously with respect to $t \in [0,1]$ such that its columns form a basis of the direct sum of the spectral subspaces corresponding to the eigenvalues of $(L + t\Delta L)(z)$ in C_ϵ. The space spanned by the columns of $X(t)$ is the direct sum of the spectral subspaces $S_{\lambda(t)}$ corresponding to the real eigenvalues $\lambda(t)$ of $(L + t\Delta L)(z)$ in C_ϵ and subspaces of the form $S_\mu(t) \oplus S_{\overline{\mu(t)}}$ where $S_{\mu(t)}$ and $S_{\overline{\mu(t)}}$ are respectively spectral subspaces corresponding to complex eigenvalues $\mu(t)$ and $\overline{\mu(t)}$ of $(L + t\Delta L)(z)$ in C_ϵ. Since C_ϵ does not contain any infinite eigenvalues of $(L + t\Delta L)(z)$, by Theorem 3.4.5 any two such subspaces are mutually $A + t\Delta A$-orthogonal and each of them is individually $A + t\Delta A$-nondegenerate. Therefore the span of $X(t)$ is $A + t\Delta A$ nondegenerate and hence $X(t)^*(A + t\Delta A)X(t)$ is nonsingular. Since the eigenvalues of $L(z)$ in C_ϵ are all of positive type, $X(0)^*AX(0)$ is positive definite. This implies that $X(t)^*(A + t\Delta A)X(t)$ is also positive definite for each $t \in [0,1]$ and hence all eigenvalues of $(L + t\Delta L)(z)$ in C_ϵ are of positive type for each $t \in [0,1]$. Since the Hermitian pencil $\Delta L(z)$ with $|||\Delta L||| < \epsilon$ is arbitrarily chosen, it follows that C_ϵ is a positive type component of $\Lambda_\epsilon^H(L)$. This proves (ii) $\Rightarrow$ (i) and the proof follows. $\square$

The next Proposition follows by arguments similar to those used for the proof of Theorem 3.5.3.

Proposition 3.5.4. *Let $L(z)$ be a Hermitian pencil and C_ϵ be a component of the closure of $\Lambda_\epsilon^H(L)$ which does not contain ∞ such that all eigenvalues of $L(z)$ in C_ϵ are* *of positive (negative) type. Then the eigenvalues of all Hermitian pencils $(L + \Delta L)(z)$ with $|||\Delta L||| \leq \epsilon$ in C_ϵ are of positive (negative) type.*

If Definition (3.5.1) be extended to components of the closure of $\Lambda_\epsilon^H(L)$, then Theorem 3.5.3 and Proposition 3.5.4 imply that if any component of the closure of $\Lambda_\epsilon^H(L)$ does not contain ∞ and is formed by the coalescence of components of $\Lambda_\epsilon^H(L)$ of positive (negative) type, then it is a positive (negative) type component of the closure. One of our aims is to prove that if components of $\Lambda_\epsilon^H(L)$ that are not of the same type coalesce at finite points, then this results in a component of the closure of $\Lambda_\epsilon^H(L)$ which has an eigenvalue of mixed type at the points of coalescence. To this end we prove certain other important results. We will need the following results.

Theorem 3.5.5. *[**Sun**, [23]] Let $s \in \mathbb{R}^n$ and $A(s) \in \mathbb{C}^{n\times n}$. Suppose that $\operatorname{Re} A(s)$ and $\operatorname{Im} A(s)$ are real analytic matrix valued functions of s in some neighbourhood $\mathcal{B}(0)$ of the origin. Let σ be a simple nonzero singular value of $A(0)$ with corresponding unit left and right singular vectors $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^n$ respectively. Then there exists a simple nonzero singular value $\sigma(s)$ of $A(s)$ which is a real analytic function of s in a neighbourhood $\mathcal{N}(0)$ of the origin such that $\sigma(0) = \sigma$ and*

$$\frac{\partial \sigma(0)}{\partial s_j} = \operatorname{Re}\left[u^* \left(\frac{\partial A(s)}{\partial s_j}\right)_{s=0} v\right].$$

Theorem 3.5.6. *[**Kato**, [32]] Let $H(t)$ be a family of Hermitian matrices of size n that depend analytically on a parameter $t \in \mathbb{R}$. Then there exists a decomposition $H(t) = X(t)^*D(t)X(t)$ where $D(t)$ is a diagonal matrix consisting of eigenvalues of $H(t)$ and $X(t)$ is a unitary matrix whose columns are the eigenvectors of $H(t)$ that varies analytically with respect to $t \in \mathbb{R}$.*

Note 3.5.7. *Note that Theorem3.5.6 is Theorem2.2.1 stated in a different form which is more relevant to the results that will be proved.*

Given an $n\times n$ Hermitian pencil $L(z) = zA - B$, as z varies in $\mathbb{R}$, $\operatorname{Re} L(z)$ and $\operatorname{Im} L(z)$ are real analytic matrix valued function of $z \in \mathbb{R}$. Theorem 3.5.5 therefore implies that if $\sigma_{\min}(L(z_0))$ is a simple singular value of $L(z_0)$ for some $z_0 \in \mathbb{R}$, then there exists a neighbourhood $\mathcal{B}(z_0)$ of z_0 in $\mathbb{R}$ such that $\sigma_{\min}(L(z))$ is an analytic function of z in $\mathcal{B}(z_0)$ with

$$\sigma'_{\min}(z_0) = \operatorname{Re}(\, u(z_0)^* A v(z_0)). \tag{3.7}$$

If we consider the homogeneous form $L(\alpha,\beta) = \alpha A - \beta B$, of the pencil, then this gives rise to the family of Hermitian matrices $L(\theta) = A\cos\theta - B\sin\theta$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Once again since $\mathrm{Re}\, L(\theta)$ and $\mathrm{Im}\, L(\theta)$ are real analytic functions of θ, defining $\sigma_{\min}(\theta) = \sigma_{\min}(L(\theta))$, it follows that if $\sigma_{\min}(\theta_0)$ is simple for some $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $\sigma_{\min}(\theta)$ is analytic in a neighbourhood of θ_0. Moreover if $u(\theta_0)$ and $v(\theta_0)$ be unit left and right singular vectors of $L(\theta_0)$ corresponding to $\sigma_{\min}(\theta_0)$ respectively, then by Theorem 3.5.5

$$\sigma'_{\min}(\theta_0) = -\mathrm{Re}\,(u(\theta_0)^*(A\sin\theta_0 + B\cos\theta_0)v(\theta_0))\,. \tag{3.8}$$

The above observations lead to the following result.

Proposition 3.5.8. *Let $L(\theta) = A\cos\theta - B\sin\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be a family of Hermitian matrices of size n. Let $\sigma_{\min}(\theta) = \sigma_{\min}(L(\theta))$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. There exist finitely many exceptional points $\theta_1, \theta_2, \ldots, \theta_m$ satisfying $-\frac{\pi}{2} < \theta_1 < \theta_2 < \cdots < \theta_m < \frac{\pi}{2}$ such that the following hold.*

(i) For $k = 0, 1, \ldots, m$, there exist functions $\lambda_{\min}(\theta)$ with the following properties. $L(\theta)v(\theta) = \lambda_{\min}(\theta)v(\theta)$, where $v(\theta) \in \mathbb{C}^n, \|v(\theta)\|_2 = 1$, such that

$$|\lambda_{\min}(\theta)| = \min_{\lambda \in \Lambda(L(\theta))} |\lambda|$$

and $\lambda_{\min}(\theta)$ varies analytically with respect to $\theta \in (\theta_k, \theta_{k+1})$, with $\theta_0 = -\frac{\pi}{2}$ and $\theta_{m+1} = \frac{\pi}{2}$. Also the one sided limits

$$\lim_{\theta \to \theta_k^{\pm}} v(\theta) \text{ and } \lim_{\theta \to \theta_k^{\pm}} \lambda_{\min}(\theta)$$

exist for each $k = 1, 2, \ldots, m$.

(ii) For each $\theta \in (\theta_k, \theta_{k+1})$, $k = 0, 1, \ldots, m$, either $\sigma_{\min}(\theta) = \lambda_{\min}(\theta)$ or $\sigma_{\min}(\theta) = -\lambda_{\min}(\theta)$.

(iii) For each $\theta \in (\theta_k, \theta_{k+1})$, $k = 0, 1, \ldots, m$, $\lambda'_{\min}(\theta) = -v(\theta)^*(A\sin\theta + B\cos\theta)v(\theta)$ where $v(\theta)$ is a unit right singular vector of $L(\theta)$ corresponding to $\lambda_{\min}(\theta)$ with properties as described in (i).

Proof: Since $L(\theta)$ is a Hermitian matrix for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, by Theorem 3.5.6 there exist $\lambda_1(\theta), \lambda_2(\theta), \ldots, \lambda_n(\theta) \in \mathbb{R}$ and an orthonormal basis $\{v_1(\theta), v_2(\theta), \ldots, v_n(\theta)\}$ that vary analytically with respect to $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $L(\theta)v_j(\theta) = \lambda_j(\theta)v_j(\theta)$ for $j = 1, 2, \ldots, n$. Also for each $p, q \in \{1, 2, \ldots, n\}$, if $|\lambda_p(\theta)| = |\lambda_q(\theta)|$ for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where $p \neq q$, then either $\lambda_p(\theta) = \lambda_q(\theta)$ or $\lambda_p(\theta) = -\lambda_q(\theta)$. In the former case, $\lambda_p(\theta)$ corresponds to a multiple zero of the algebraic curve $\{(\phi, \lambda) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} : \det(L(\phi) - \lambda I_n) = 0\}$ and in the latter case, $\lambda_p(\theta)$ is a common zero of the algebraic curves $\{(\phi, \lambda) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} : \det(L(\phi) - \lambda I_n) = 0\}$ and $\{(\phi, \lambda) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} : \det(L(\phi) + \lambda I_n) = 0\}$. In either case there are only finitely many such points θ. In particular, there are only finitely many points $\theta \in \mathbb{R}$ for which $|\lambda_p(\theta)| = |\lambda_q(\theta)| = \min\limits_{\lambda \in \Lambda(L(\theta))} |\lambda|$ with $p \neq q$. Let these be $\theta_1, \theta_2, \ldots, \theta_m$. Thus for each $k = 0, 1, \ldots, m$, there exists a unique $p(k) \in \{1, 2, \ldots, n\}$, such that either $\lambda_{p(k)}(\theta) = \min\limits_{j \in \{1,2,\ldots,n\}} |\lambda_j(\theta)|$ or $-\lambda_{p(k)}(\theta) = \min\limits_{j \in \{1,2,\ldots,n\}} |\lambda_j(\theta)|$ whenever $\theta \in (\theta_k, \theta_{k+1})$. Evidently either $\sigma_{\min}(\theta) = \lambda_{p(k)}(\theta)$ or $\sigma_{\min}(\theta) = -\lambda_{p(k)}(\theta)$ for $\theta \in (\theta_k, \theta_{k+1})$, $k = 0, 1, \ldots, m$, so that $\sigma_{\min}(\theta)$ is a simple singular value of $L(\theta)$ in each of these intervals with corresponding unit left and right singular vectors $u(\theta)$ and $v(\theta)$ satisfying $u(\theta) = v(\theta) = v_{p(k)}(\theta)$ in the former case and $u(\theta) = -v_{p(k)}(\theta)$ and $v(\theta) = v_{p(k)}(\theta)$ in the latter. Therefore from 3.8,

$$\lambda'_{p(k)}(\theta) = -(v_{p(k)}(\theta))^*(A \sin\theta + B\cos\theta)v_{p(k)}(\theta)$$

for each $\theta \in (\theta_k, \theta_{k+1})$, $k = 0, 1, \ldots, m$.

The proof now follows by defining $\lambda_{\min}(\theta) = \lambda_{p(k)}(\theta)$ and $v(\theta) = v_{p(k)}(\theta)$ for $\theta \in (\theta_k, \theta_{k+1}), k = 0, 1, \ldots, m$. $\square$

Note 3.5.9. *The choice of the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ in Theorem 3.5.8 is purely a matter of convention and may be replaced by any other interval for which every point on the extended real line may be uniquely identified with a point on the unit circle.*

Recall that the set $\Lambda^H_\epsilon(L) \cap (\mathbb{R} \cup \{\infty\})$ may be identified with the subset

$$\{(\cos\theta, \sin\theta),\, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] : \eta(\theta, L) < \epsilon\}$$

of the unit circle where $\eta(\theta, L) = \sigma_{\min}(\theta)$. In particular, if C_ϵ is a component of $\Lambda^H_\epsilon(L)$ of positive or negative type, then $C_\epsilon \subset \mathbb{R} \cup \{\infty\}$ so that it may be considered as a connected subset of the unit circle.

The next result analyses the nature of the point of coalescence of two components of $\Lambda^H_\epsilon(L)$ of definite type and shows that if the point of coalescence is finite and the types of the coalescing components are not the same, then it is an eigenvalue of mixed type of some Hermitian pencil $(L + \Delta L)(z)$ with $|||\Delta L||| = \epsilon$. It also shows that if the coalescing components are of the same type and the point of coalescence is ∞, then ∞ is an eigenvalue of mixed type of some Hermitian pencil $(L + \Delta L)(z)$ with $|||\Delta L||| = \epsilon$.

Theorem 3.5.10. *Let $L(z)$ be a Hermitian pencil and let $C_1(\epsilon)$ and $C_2(\epsilon)$ be components of $\Lambda_\epsilon^H(L)$ of definite type that coalesce at a point $(\cos\phi_0, \sin\phi_0)$. Also let $\sigma_{\min}(\theta)$ and $\lambda_{\min}(\theta)$ be as defined in Proposition 3.5.8. Then the following hold.*

(i) If $C_1(\epsilon)$ and $C_2(\epsilon)$ differ in type and $\phi_0 \neq 0$, then there exists a neighbourhood (ϕ_1, ϕ_2) of ϕ_0 such that either $\sigma_{\min}(\theta) = \lambda_{\min}(\theta)$ or $\sigma_{\min}(\theta) = -\lambda_{\min}(\theta)$ for all $\theta \in (\phi_1, \phi_2)$.

Moreover $(\cos\phi_0, \sin\phi_0)$ represents a defective eigenvalue of some Hermitian pencil $(L + \Delta L)(z)$ with $|||\Delta L||| = \epsilon$.

(ii) If $C_1(\epsilon)$ and $C_2(\epsilon)$ are of the same type and $\phi_0 \neq 0$, then there exists a neighbourhood (ϕ_1, ϕ_2) of ϕ_0 such that $\sigma_{\min}(\theta) = r_1\lambda_{\min}(\theta)$ for all $\theta \in (\phi_1, \phi_0)$ and $\sigma_{\min}(\theta) = r_2\lambda_{\min}(\theta)$ for all $\theta \in (\phi_0, \phi_2)$ where $r_1, r_2 \in \{-1, 1\}$ with $r_1 \neq r_2$.

(iii) If $C_1(\epsilon)$ and $C_2(\epsilon)$ are of same type and coalesce at $(1, 0)$, then there exists a neighbourhood (ϕ_1, ϕ_2) containing 0 such that $\sigma_{\min}(\theta) = \lambda_{\min}(\theta)$ or $\sigma_{\min}(\theta) = -\lambda_{\min}(\theta)$ for $\theta \in (\phi_1, \phi_2)$. In this case the point $(1, 0)$ represents a defective eigenvalue of a Hermitian pencil $(L + \Delta L)(c, s)$ with $|||\Delta L||| = \epsilon$.

If $C_1(\epsilon)$ and $C_2(\epsilon)$ differ in type and coalesce at $(1, 0)$, then $\sigma_{\min}(\theta) = r_1\lambda_{\min}(\theta)$ for $\theta \in (\phi_1, 0)$ and $\sigma_{\min}(\theta) = r_2\lambda_{\min}(\theta)$ for $\theta \in (0, \phi_2)$ where $r_1, r_2 \in \{-1, 1\}$ with $r_1 \neq r_2$.

Proof: Let $C_1(\epsilon)$ and $C_2(\epsilon)$ be of definite type and $(\cos\phi_0, \sin\phi_0)$, with $\phi_0 \neq 0$, be a point of coalescence. The proof of (i) will depend upon whether $\phi_0 > 0$ or $\phi_0 < 0$, the types of the components $C_1(\epsilon)$ and $C_2(\epsilon)$ and also on which of these components has points from the set $\{(\cos\theta, \sin\theta) : \theta < \phi_0\}$. To prove (i) we assume that $\phi_0 > 0$, and that the points of $C_1(\epsilon)$ satisfy $\{(\cos\theta, \sin\theta) : \theta < \phi_0\}$. The proofs for the other possible cases will follow with similar arguments. Since $(\cos\phi_0, \sin\phi_0)$ is a coalescence point, it is a boundary point of the coalescing components with the property that there exists a neighbourhood (ϕ_1, ϕ_2) of ϕ_0 with $\phi_1 > 0$ such that $\eta(\theta, L) = \sigma_{\min}(\theta)$ is monotone increasing for $\theta \in (\phi_1, \phi_0)$ and monotone decreasing for $\theta \in (\phi_0, \phi_2)$. In view of the Note 3.5.9, Proposition 3.5.8, also holds if the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ is replaced by the interval $(0, \pi)$. We assume without loss of generality that (ϕ_1, ϕ_2) does not contain any exceptional points except possibly ϕ_0 where the exceptional points are as defined in Proposition 3.5.8 with $(-\frac{\pi}{2}, \frac{\pi}{2})$ is replaced by the interval $(0, \pi)$. Considering any $\theta \in (\phi_1, \phi_0)$, we have $\sigma_{\min}(\theta) = r\lambda_{\min}(\theta)$ where $r = 1$ or $r = -1$. Therefore for all $\hat{\theta} \in (\phi_1, \phi_0)$,

$$\sigma'_{\min}(\hat{\theta}) = r\lambda'_{\min}(\hat{\theta}) = -rv(\hat{\theta})^*(A \sin \hat{\theta} + B \cos \hat{\theta})v(\hat{\theta}) \geq 0 \tag{3.9}$$

where $v(\hat{\theta})$ is a unit right eigenvector of $L(\hat{\theta})$ corresponding to $\lambda_{\min}(\hat{\theta})$ with properties as described in Proposition 3.5.8. Consider the Hermitian pencil $\Delta L(c, s) = c\Delta A - s\Delta B$ where

$$\Delta A = -\cos \hat{\theta}\lambda_{\min}(\hat{\theta})v(\hat{\theta})v(\hat{\theta})^* \text{ and } \Delta B = \sin \hat{\theta}\lambda_{\min}(\hat{\theta})v(\hat{\theta})v(\hat{\theta})^*.$$

We have $|||\Delta L||| = |\lambda_{\min}(\hat{\theta})| = \sigma_{\min}(\hat{\theta}) < \epsilon$ and $[(L+\Delta L)(\cos \hat{\theta}, \sin \hat{\theta})]v(\hat{\theta}) = L(\hat{\theta})v(\hat{\theta}) - \lambda_{\min}(\hat{\theta})v(\hat{\theta}) = 0$. Now suppose that $C_1(\epsilon)$ is of positive type. Then $(\cos \hat{\theta}, \sin \hat{\theta}) \in C_1(\epsilon)$ and hence it represents an eigenvalue of the pencil $(L + \Delta L)(\alpha, \beta)$ of positive type. Therefore, by the homogeneous definition of eigenvalue type,

$$0 < v(\hat{\theta})^*[(A + \Delta A) \sin \hat{\theta} + (B + \Delta B) \cos \hat{\theta}]v(\hat{\theta}) = v(\hat{\theta})^*[A \sin \hat{\theta} + B \cos \hat{\theta}]v(\hat{\theta}) \tag{3.10}$$

From (3.9) and (3.10) it follows that $r = -1$ so that $\sigma_{\min}(\theta) = -\lambda_{\min}(\theta)$ for all $\theta \in (\phi_1, \phi_0)$. Since $\sigma'_{\min}(\theta) \leq 0$ for all $\theta \in (\phi_0, \phi_2)$, if $C_2(\epsilon)$ is of negative type, by similar arguments it follows that $\sigma_{\min}(\theta) = -\lambda_{\min}(\theta)$ for all $\theta \in (\phi_0, \phi_2)$. Thus $\sigma_{\min}(\theta) = -\lambda_{\min}(\theta)$ for all $\theta \in (\phi_1, \phi_2)$. If $C_1(\epsilon)$ is of negative type and $C_2(\epsilon)$ is of positive type then $\sigma_{\min}(\theta) = \lambda_{\min}(\theta)$ for all $\theta \in (\phi_1, \phi_2)$.

From the proof of Proposition 3.5.8 we have $\lambda_{\min}(\theta) = \lambda_p(\theta)$ for $\theta \in (\phi_1, \phi_0)$ and $\lambda_{\min}(\theta) = \lambda_q(\theta)$ for all $\theta \in (\phi_0, \phi_2)$ where $\lambda_1(\theta), \lambda_2(\theta), \ldots, \lambda_n(\theta) \in \mathbb{R}$ are eigenvalues of $L(\theta)$ with corresponding eigenvectors $\{v_1(\theta), v_2(\theta), \ldots, v_n(\theta)\}$ that form an orthonormal basis of $\mathbb{C}^n$ such that all eigenvalues and eigenvectors vary analytically with respect to $\theta \in (\phi_1, \phi_2) \setminus \{\phi_0\}$. Assuming that $C_1(\epsilon)$ is of negative type as already observed above, we have $\sigma_{\min}(\theta) = \lambda_p(\theta)$ and $v_p(\theta)^*(A \sin \theta + B \cos \theta)v_p(\theta) > 0$ for all $\theta \in (\phi_1, \phi_0)$ and $\sigma_{\min}(\theta) = \lambda_q(\theta)$ and $v_q(\theta)^*(A \sin \theta + B \cos \theta)v_q(\theta) < 0$ for all $\theta \in (\phi_0, \phi_2)$. Therefore, $\sigma_{\min}(\phi_0) = \lambda_p(\phi_0) = \lambda_q(\phi_0)$ and

$$\begin{aligned} v_p(\phi_0)^*(A \sin \phi_0 + B \cos \phi_0)v_p(\phi_0) &= \lim_{\theta \to \phi_0^-} v_p(\theta)^*(A \sin \theta + B \cos \theta)v_p(\theta) \geq 0 \\ v_q(\phi_0)^*(A \sin \phi_0 + B \cos \phi_0)v_q(\phi_0) &= \lim_{\theta \to \phi_0^+} v_q(\theta)^*(A \sin \theta + B \cos \theta)v_q(\theta) \leq 0. \end{aligned}$$

If $p = q$, setting $v(\phi_0) = v_p(\phi_0)$, we have $v(\phi_0)^*(A \sin\, \phi_0 + B \cos\, \phi_0)v(\phi_0) = 0$. Consider the Hermitian pencil $\Delta L(\alpha, \beta) = \alpha\Delta A - \beta\Delta B$ where

$$\Delta A = -\cos\, \phi_0\, \sigma_{\min}(\phi_0)v(\phi_0)v(\phi_0)^* \text{ and } \Delta B = \sin\, \phi_0\sigma_{\min}(\phi_0)\, v(\phi_0)v(\phi_0)^*.$$

Therefore $|||\Delta L||| = \sigma_{\min}(\phi_0) = \epsilon$ and $(L + \Delta L)(\cos\, \phi_0, \sin\, \phi_0)v(\phi_0) = L(\phi_0)v(\phi_0) - \sigma_{\min}(\phi_0)v(\phi_0) = 0$. Also, $v(\phi_0)^*[(A+\Delta A) \sin\, \phi_0+(B+\Delta B) \cos,\, \phi_0]v(\phi_0) = v(\phi_0)^*(A \sin\, \phi_0+ B \cos\, \phi_0)v(\phi_0) = 0$. This proves that $(\cos\, \phi_0, \sin\, \phi_0)$ represents an eigenvalue of mixed type of the Hermitian pencil $(L + \Delta L)(\alpha, \beta)$ with $|||\Delta L||| = \epsilon$.

Next suppose that $p \neq q$. In this case if either $v_p(\phi_0)^*(A \sin\, \phi_0 + B \cos\, \phi_0)v_p(\phi_0) = 0$ or $v_q(\phi_0)^*(A \sin\, \phi_0 + B \cos\, \phi_0)v_q(\phi_0) = 0$ then arguing as above, it may be shown that $(\cos\, \phi_0, \sin\, \phi_0)$ is an eigenvalue of mixed type of some perturbed Hermitian pencil $(L + \Delta L)(\alpha, \beta)$ where $|||\Delta L||| = \epsilon$.

Suppose that

$$v_p(\phi_0)^*(A \sin\, \phi_0 + B \cos\, \phi_0)v_p(\phi_0) > 0 \text{ and } v_q(\phi_0)^*(A \sin\, \phi_0 + B \cos\, \phi_0)v_q(\phi_0) < 0.$$

This implies that the 2×2 Hermitian matrix $[v_p(\phi_0)\, v_q(\phi_0)]^*(A \sin\, \phi_0+B \cos\, \phi_0)[v_p(\phi_0)\, v_q(\phi_0)]$ is indefinite. Hence there exists $(x_1, x_2) \in \mathbb{C}^2$ with $|x_1|^2 + |x_2|^2 = 1$ such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^* [v_p(\phi_0)\, v_q(\phi_0)]^*(A \sin\, \phi_0 + B \cos\, \phi_0)[v_p(\phi_0)\, v_q(\phi_0)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

Setting $v(\phi_0) = x_1v_p(\phi_0) + x_2v_q(\phi_0)$, we have $v(\phi_0)^*(A \sin\, \phi_0 + B \cos\, \phi_0)v(\phi_0) = 0$. Since $\|v(\phi_0)\|_2 = 1$ and $L(\phi_0)v(\phi_0) = \sigma_{\min}(\phi_0)v(\phi_0)$, once again setting $\Delta L(\alpha, \beta) = \alpha\Delta A - \beta\Delta B$ where

$$\Delta A = -\cos\, \phi_0\sigma_{\min}(\phi_0)v(\phi_0)v(\phi_0)^* \text{ and } \Delta B = \sin\, \phi_0\sigma_{\min}(\phi_0)v(\phi_0)v(\phi_0)^*$$

it follows that $|||\Delta L||| = \epsilon$ and $(\cos\, \phi_0, \sin\, \phi_0)$ represents an eigenvalue of mixed type of $(L + \Delta L)(\alpha, \beta)$.

In each case, $(\cos\, \phi_0, \sin\, \phi_0)$ represents a multiple eigenvalue with only one eigenvector $v(\phi_0)$ up to linear dependence. Hence it is a defective eigenvalue of $(L + \Delta L)(\alpha, \beta)$. This completes the proof of (i).

The proof of (ii) - (iv) follow from similar arguments by using the monotonicity of $\sigma_{\min}(\theta)$ in (ϕ_1, ϕ_0) and (ϕ_0, ϕ_2) and the types of the coalescing components. In particular, for the proofs of (iii) and (iv) we use Proposition 3.5.8 with θ varying in $(-\frac{\pi}{2}, \frac{\pi}{2})$ so that the point of coalescence corresponds to $\phi_0 = 0$. In this case if $C_1(\epsilon)$ is of positive type, the inequality in (3.10) is reversed as the points $(\cos\theta, \sin\theta)$ of $C_1(\epsilon)$ are such that $\sin\theta < 0$. This implies that $\sigma_{\min}(\theta) = \lambda_{\min}(\theta)$ for all $\theta \in (\phi_1, 0)$ as $\sigma_{\min}(\theta)$ is monotone increasing in $(\phi_1, 0)$. Inequality (3.9) with its sign reversed also holds for $\theta \in (0, \phi_2)$ if $C_2(\epsilon)$ is of negative type since the points $(\cos\theta, \sin\theta)$ of $C_2(\epsilon)$ are such that $\sin\theta > 0$. Since $\sigma_{\min}(\theta)$ is monotone decreasing, in $(0, \phi_2)$, it follows that $\sigma_{\min}(\theta) = -\lambda_{\min}(\theta)$ for all $\theta \in (0, \phi_2)$ in this case. □

We have the following corollary of the above results.

Corollary 3.5.11. *Let $L(z)$ be a Hermitian pencil of size n with n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of definite type such that*

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 < \infty.$$

Let $C_1(\epsilon)$ and $C_2(\epsilon)$ be components of $\Lambda_\epsilon^H(L)$ of definite type that coalesce at a point $z_0 \in \mathbb{R} \cup \{\infty\}$. If $C_1(\epsilon)$ and $C_2(\epsilon)$ are of the same type and $z_0 \in (\lambda_{k+1}, \lambda_k)$, for $1 \leq k \leq n-1$, then $L(z)$ is indefinite for $z \in (\lambda_{k+1}, \lambda_k)$.

If $C_1(\epsilon)$ and $C_2(\epsilon)$ coalesce at ∞, and are not of the same type, then $L(z)$ is indefinite for $z \in (-\infty, \lambda_n) \cup (\lambda_1, \infty)$.

Proof: For $z \in \mathbb{R}$, let $f(z) = \sigma_{\min}(L(z))$ and $\lambda_{\min}(z) = \lambda_{\min}(\theta)$ where $(\cos\theta, \sin\theta)$, with $0 < \theta < \pi$, is a representation of z and $\lambda_{\min}(\theta)$ is as described in Proposition 3.5.8. Also let $C_1(\epsilon)$ and $C_2(\epsilon)$ be of positive type. Note that if $(\cos\phi_0, \sin\phi_0)$, $\phi_0 \in (0, \pi)$ represents z_0, then points $z_1 \in \mathbb{R}$ with $z_1 > z_0$ are represented by $(\cos\phi_1, \sin\phi_1)$ where $\phi_1 \in (0, \pi)$ with $\phi_1 < \phi_0$. Therefore if $z_0 \in \mathbb{R}$, it follows from part (ii) of Theorem 3.5.10 that there exists an interval (z_1, z_2) of z_0 with $z_1 \in C_1(\epsilon)$ such that $f(z) = -\lambda_{\min}(z)$ for $z \in (z_1, z_0)$ and $f(z) = \lambda_{\min}(z)$ for $z \in (z_0, z_2)$. Since $\lambda_{\min}(z)$ is an eigenvalue of $L(z)$ and $f(z) > 0$ for $z \in (z_1, z_2)$, it follows that $L(z)$ has a negative eigenvalue for each $z \in (z_1, z_0)$ and a positive eigenvalue for each $z \in (z_0, z_2)$. Since the pencil $L(z)$ has no eigenvalues in the interval $(\lambda_{k+1}, \lambda_k)$, if $L(z)$ is positive definite for any $z \in (\lambda_{k+1}, \lambda_k)$, by continuity of the eigenvalues, it must be positive definite for all $z \in (\lambda_{k+1}, \lambda_k)$. But this is impossible as $L(z)$ has a negative eigenvalue for $z \in (z_1, z_0)$. By the same arguments $L(z)$ cannot be negative definite for any $z \in (\lambda_{k+1}, \lambda_k)$. Therefore, $L(z)$ is indefinite for all $z \in (\lambda_{k+1}, \lambda_k)$.

The proof for the case when $z_0 = \infty$, follows from similar arguments by using part (iii) of Theorem 3.5.10. □

We now present some examples to illustrate some of the results of this section. To understand these examples it is important to note that if $\lambda_{\min}(\theta)$ and $v(\theta)$ be as in Theorem 3.5.8 and θ is not an exceptional point, then since

$$\lambda'_{\min}(\theta) = -v(\theta)^*(A \sin\,\theta + B \cos\,\theta)v(\theta),$$

from the proof of Theorem 3.5.10 it is clear that if $(\cos\,\theta, \sin\,\theta)$ represents an eigenvalue of positive type of the Hermitian pencil $z(A + \Delta A) - (B + \Delta B)$ where

$$\Delta A = -\cos\,\theta\lambda_{\min}(\theta)v(\theta)v(\theta)^* \text{ and } \Delta B = \sin\,\theta\lambda_{\min}(\theta)v(\theta)v(\theta)^*,$$

then the graph of $\theta \to \lambda_{\min}(\theta)$ has a negative (positive) slope for $\theta \in (0, \pi)$, $(\theta \in (-\pi, 0),)$ Thus if the graph of $\theta \to \lambda_{\min}(\theta)$ has both positive and negative slopes in a subinterval of $(0, \pi)$, then the points of this subinterval belong to a component of mixed type of $\Lambda^H_\epsilon(L)$.

The first example illustrates that the assumption $\infty \notin C(\epsilon)$ of $\Lambda^H_\epsilon(L)$ in Theorem 3.5.3 is necessary.

Example 3.5.12. *Let $L(z) = zA - B$ be a Hermitian pencil where*

$$A = \begin{bmatrix} 9.7147 & 2.9539 & 10.0045 & -0.1766 & 0.3242 \\ 2.9539 & 7.8157 & -2.7069 & -1.0477 & 2.3296 \\ 10.0045 & -2.7069 & 21.9120 & 3.1929 & -2.2609 \\ -0.1766 & -1.0477 & 3.1929 & 1.5282 & -0.9611 \\ 0.3242 & 2.3296 & -2.2609 & -0.9611 & 4.4911 \end{bmatrix} B = \begin{bmatrix} 4.6830 & -4.5779 & 14.5399 & 11.0972 & -19.1356 \\ -4.5779 & 28.0320 & -20.9328 & -6.6562 & 5.2371 \\ 14.5399 & -20.9328 & 45.9739 & 23.1175 & -41.6983 \\ 11.0972 & -6.6562 & 23.1175 & 12.6111 & -11.9098 \\ -19.1356 & 5.2371 & -41.6983 & -11.9098 & 31.6264 \end{bmatrix}.$$

$L(z)$ is in fact a definite pencil as A is a positive definite matrix. Therefore its eigenvalues $\lambda_1 = 23.423, \lambda_2 = 8.9338, \lambda_3 = 3.4394, \lambda_4 = 0.26$ and $\lambda_5 = -1.0371$ are all of positive type. They are respectively represented by pairs $(\cos\,\theta_k, \sin\,\theta_k), k = 1, 2, \ldots, 5$, where $\theta_1 = 0.043, \theta_2 = 0.115, \theta_3 = 0.283, \theta_4 = 1.3161$ and $\theta_5 = -0.767$ belong to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Figure 3.1 plots $\sigma_{\min}(\theta)$ and $\lambda_{\min}(\theta)$ with respect to $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Since $\lambda_{\min}(\theta)$ has both positive and negative slopes for $\theta \in [\theta_5, 0]$, the component of $\Lambda^H_\epsilon(L)$ for $\epsilon = 1.7049$ containing eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_5 is of mixed type although all these eigenvalues are of positive type. Note that this component contains ∞ as $0 \in (\theta_5, \theta_1)$.

The following example shows the coalescence of components of same type at ∞ and at a point on the real line.

Example 3.5.13. *Let $L(z) = zA - B$ be the pencil obtained by rotating the pencil in Example 3.5.12 by $\theta = -0.4775$. The matrix A of the rotated pencil continues to be positive*

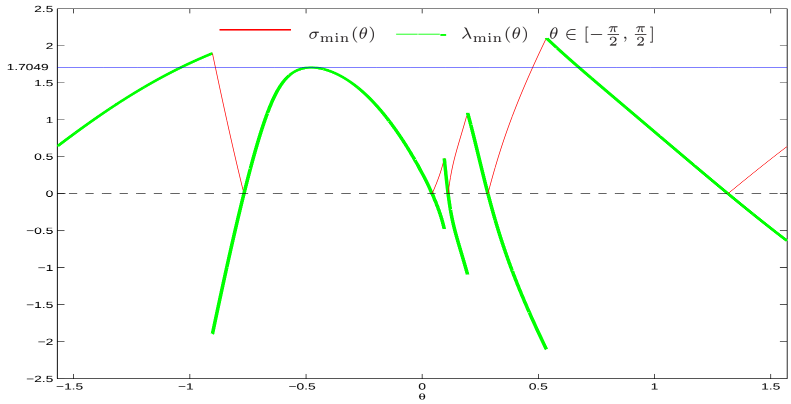

Figure 3.1: Eigenvalue and singular value curves for Example 3.5.12.

definite. Therefore the eigenvalues $\lambda_1 = 60.6668, \lambda_2 = 5.2575, \lambda_3 = 0.3646, \lambda_4 = -0.8571$ *and* $\lambda_5 = -19.095$ *of the rotated pencil are all of positive type. They are respectively represented by the pairs* $(\cos\,\theta_k, \sin\,\theta_k)$*,* $k = 1, 2, \ldots, 5$*, where* $\theta_1 = 0.5202, \theta_2 = 0.589, \theta_3 = 0.7605, \theta_4 = -0.2897$ *and* $\theta_5 = -1.348$ *belong to the interval* $[-\frac{\pi}{2}, \frac{\pi}{2}]$*. The graph on the top in Figure 3.2 shows a component of* $\Lambda^H_\epsilon(L)$ *of positive type containing the eigenvalue* λ_5 *coalescing at* ∞ *with another component of* $\Lambda^H_\epsilon(L)$ *of positive type containing the eigenvalues* $\lambda_1, \lambda_2,$ *and* λ_3 *for* $\epsilon = 1.7049$*.*

The graph at the bottom in Figure 3.2 plots $\lambda_{\min}(\theta)$ *and* $\sigma_{\min}(\theta)$ *for the same pencil for* $\theta \in [0, \pi]$*. In this case the eigenvalues* λ_k*,* $k = 1, 2, \ldots, 5$*, are represented by pairs* $(\cos\,\theta_k, \sin\,\theta_k)$*,* $k = 1, 2, \ldots, 5$*, respectively where* $\theta_1 = 0.5202, \theta_2 = 0.589, \theta_3 = 0.7605, \theta_4 = 1.792$ *and* $\theta_5 = 2.852$ *belong to the interval* $(0, \pi)$*. The graph shows the coalescence of a component of* $\Lambda^H_\epsilon(L)$ *for* $\epsilon = 1.093$ *of positive type containing eigenvalues* λ_4 *and* λ_5 *with another component of positive type containing eigenvalue* λ_3 *at a point represented by* $(\cos\,\theta, \sin\,\theta)$ *where* $\theta = 0.6729$*. The figure shows that* $L(z)$ *is indefinite for* z *in the interval* $[\lambda_4, \lambda_3]$*.*

The next example illustrates coalescence of components that differ in type at a point on the real line as well as at ∞.

Example 3.5.14. *Let* $L(z) = zA - B$ *be a Hermitian pencil where*

$$A = \begin{bmatrix} 43.0550 & 2.5057 & 18.7255 & 24.3296 \\ 2.5057 & 5.8110 & 3.0566 & 0 \\ 18.7255 & 3.0566 & 1.8170 & 9.7318 \\ 24.3296 & 0 & 9.7318 & 14.5977 \end{bmatrix} \quad B = \begin{bmatrix} -17.1573 & -5.6239 & -10.2372 & -6.9201 \\ -5.6239 & -19.1069 & -7.8711 & 0 \\ -10.2372 & -7.8711 & -12.0820 & -2.7680 \\ -6.9201 & 0 & -2.7680 & -4.1520 \end{bmatrix}.$$

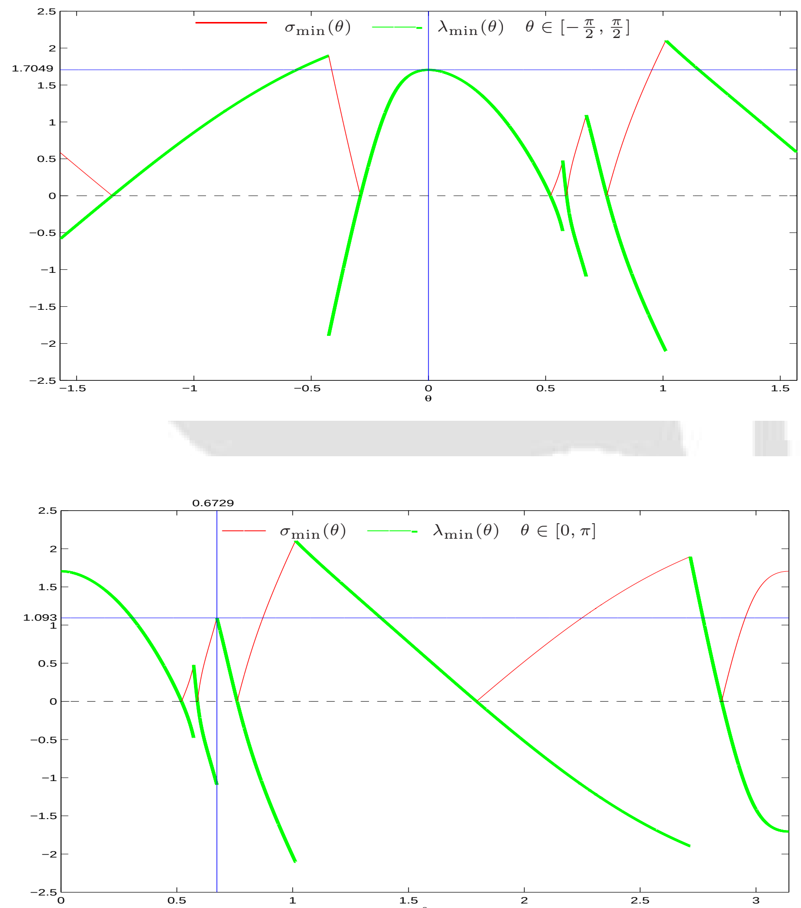

Figure 3.2: Eigenvalue and singular value curves for Example 3.5.13.

In this case also $L(z)$ is in fact a definite pencil with both A and B being indefinite matrices. The eigenvalues are $\lambda_1 = 0.58309, \lambda_2 = -.28443, \lambda_3 = -2.2444$ and $\lambda_4 = -4.0792$ of which λ_1 is of negative type while the rest are all of positive type. They are respectively represented by pairs $(\cos\,\theta_k, \sin\,\theta_k), k = 1, 2, \ldots, 4,$ where $\theta_1 = 1.0429, \theta_2 = 1.8479, \theta_3 = 2.7224$ and $\theta_4 = 2.9012$ belong to the interval $[0, \pi]$. The graph on the top of Figure 3.3 plots $\sigma_{\min}(\theta)$ and $\lambda_{\min}(\theta)$ with respect to $\theta \in [0, \pi]$. It shows the coalescence of a component of $\Lambda_\epsilon^H(L)$ of negative type containing λ_1 with a component of positive type containing λ_2 at the point represented by $(\cos\,\theta_0, \sin\,\theta_0)$ where $\theta_0 = 1.5277$ and $\epsilon = 0.7$.

The graph at the bottom of Figure 3.3 plots $\lambda_{\min}(\theta)$ and $\sigma_{\min}(\theta)$ for the same pencil rotated by an angle $\theta = 2.84$. The eigenvalues of the rotated pencil are $\lambda_1 = 13.8978, \lambda_2 = 0.2245, \lambda_3 = -0.6583$ and $\lambda_4 = -9.7603$ of which λ_1 and λ_2 are of positive type and λ_3 and λ_4 are of negative type. These are represented respectively by points $(\cos\,\theta_k, \sin\,\theta_k)$ where $\theta_1 = 0.07183, \theta_2 = 1.35, \theta_3 = -0.9859$ and $\theta_4 = -0.1181$ that belong to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The graph shows the coalescence of a component of $\Lambda_\epsilon^H(L)$ for $\epsilon = 0.41$ of negative type containing λ_3 with a component of positive type containing λ_2 at ∞. This example also illustrates the fact that ∞ is an exceptional boundary point of components of $\Lambda_\epsilon^H(L)$ of definite type since in such a case it can be an eigenvalue of positive (negative) type of of some Hermitian pencil $(L + \Delta L)(z)$ with $|||\Delta L||| = \epsilon$ even if the component itself is of negative (positive) type. Indeed in this example, it may be easily verified that ∞ is an eigenvalue of positive type of a Hermitian pencil $(L+\Delta L)(z)$ with $|||\Delta L||| = 0.41$ although it lies on the boundary of a component of negative type of $\Lambda_{0.41}^H(L)$.

3.6 Crawford number and Hermitian ϵ-pseudospectra

In this section we analyse the relationship between the Crawford number of a definite pencil and its Hermitian ϵ-pseudospectrum and show that the Crawford number may be obtained from the pseudospectrum. We also show that if the Hermitian pencil is definitizable, then its Hermitian ϵ-pseudospectrum gives the distance to a nearest Hermitian pencil that is not definitizable.

Let $L(z)$ be a definite pencil of size n with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ where

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \leq \infty, \tag{3.11}$$

Since any $z \in \mathbb{R} \cup \{\infty\}$ may be uniquely represented by $(\cos\,\theta, \sin\,\theta)$ where $\theta \in [0, \pi)$,

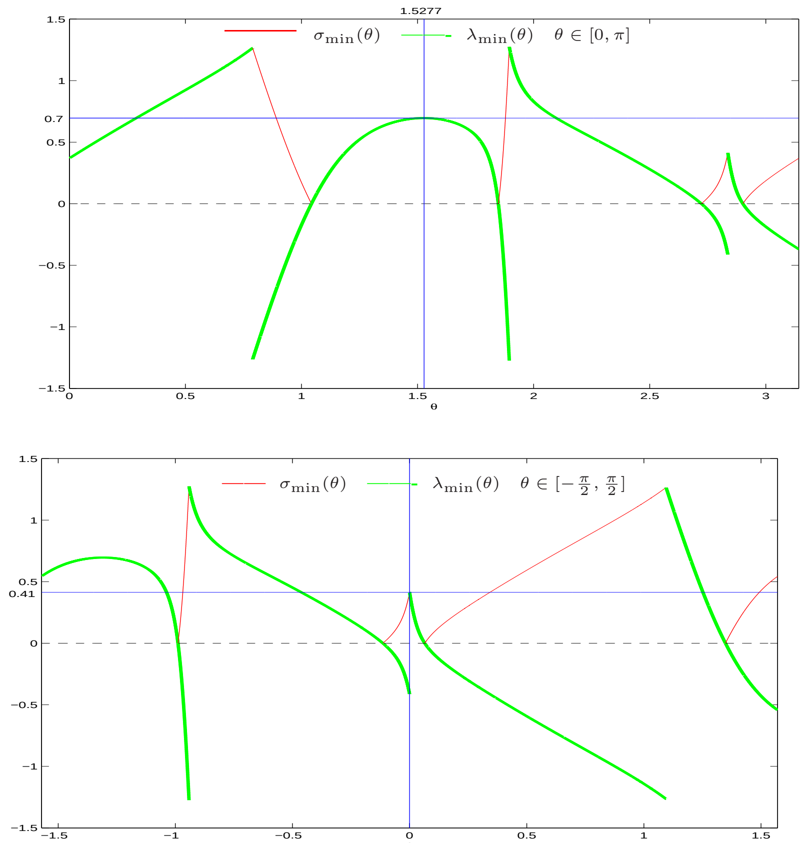

Figure 3.3: Eigenvalue and singular value curves for Example 3.5.14.

each eigenvalue λ_k has a unique representation $(\cos\,\theta_k, \sin\,\theta_k)$ where

$$0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_{n-1} \le \theta_n < \pi. \tag{3.12}$$

We will refer to θ_k, $k \in \{1, 2, \ldots, n\}$ as the k^{th} eigenangle of $L(z)$.

The finite eigenvalues of $L(z)$ are of positive type if A is a positive definite matrix, of negative type if A is a negative definite matrix and of both positive and negative type if A is an indefinite matrix. Also $L(z)$ has an eigenvalue at ∞ if A is singular. In such a case, choosing $\theta \in [0, \pi)$ which is not an eigenangle of $L(z)$, and homogeneously rotating $L(z)$ by the angle θ results in a Hermitian pencil $L_\theta(z) = zA_\theta - B_\theta$ where

$$A_\theta = A\cos\,\theta - B\sin\,\theta,\;\; B_\theta = A\sin\,\theta + B\cos\,\theta \tag{3.13}$$

which does not have any eigenvalues at ∞ as A_θ is nonsingular.

Theorem 1.2.17 implies that if A is singular and indefinite, then A_θ will be definite if $(\cos\,\theta, \sin\,\theta)$ represents a point of the interval $(\lambda_{k+1}, \lambda_k)$ where λ_{k+1} and λ_k are eigenvalues of $L(z)$ that differ in type. On the other hand, A_θ will be nonsingular and indefinite if $(\cos\,\theta, \sin\,\theta)$ represents a point z in any interval $(\lambda_{k+1}, \lambda_k)$ where λ_{k+1} and λ_k are eigenvalues of $L(z)$ of the same type. The same result also implies that if A is positive or negative semidefinite, then A_θ is definite if $(\cos\,\theta, \sin\,\theta)$ represents a point of $(-\infty, \lambda_n) \cup (\lambda_1, \infty)$ and indefinite but nonsingular if it represents an point of (λ_1, λ_n) other than the eigenvalues of $L(z)$.

Since $\gamma(A_\theta, B_\theta) = \gamma(A, B) > 0$, $L_\theta(z)$ is a definite pencil. As our aim is to relate the Crawford number $\gamma(A, B)$ with the Hermitian ϵ-pseudospectrum, we assume without loss of generality that the definite pencil $L(z)$ has no eigenvalues at ∞. We have the following result.

Theorem 3.6.1. *Let $L(z)$ be a definite pencil. Then*

$$\gamma(A, B) = \begin{cases} \max\limits_{\theta \in [\theta_n - \pi, \theta_1]} \sigma_{\min}(A_\theta) & \text{if } A \text{ is definite} \\ \max\limits_{\theta \in [\theta_k, \theta_{k+1}]} \sigma_{\min}(A_\theta) & \text{if } A \text{ is indefinite with} \\ & \lambda_{k+1} \text{ and } \lambda_k \text{ differing in type} \end{cases} \tag{3.14}$$

where A_θ is given by (3.13). Moreover, there exists a Hermitian pencil $(L + \Delta L)(z)$ with $|||\Delta L||| = \gamma(A, B)$ which has a defective eigenvalue at a point z_0 represented by $(\cos\,\theta_0, \sin\,\theta_0)$ where

$$\theta_0 \in \begin{cases} [\theta_n - \pi, \theta_1] & \text{if } A \text{ is definite} \\ [\theta_k, \theta_{k+1}] & \text{if } A \text{ is indefinite.} \end{cases}$$

is the point at which $\sigma_{\min}(A_\theta)$ attains the maximum in (3.14).

Proof: Let A be definite. By Theorem 3.5.3, $\epsilon < \sigma_{\min}(A) \Rightarrow \epsilon < \gamma(A,B)$. Therefore $\gamma(A,B) \geq \sigma_{\min}(A)$. The same obviously holds for all rotated pencils $L_\theta(z)$ for which A_θ is definite. Suppose that A is positive definite. Theorem 1.2.17 implies that A_θ is positive definite for all $\theta \in [0, \theta_n)$ and negative definite for all $\theta \in (\theta_1, \pi]$. Therefore,

$$\gamma(A,B) \geq \max_{\theta \in [\theta_n, \pi] \cup [0, \theta_1]} \sigma_{\min}(A_\theta)$$

The points of $\mathbb{R}$ represented by $(\cos\theta, \sin\theta)$ for $\theta \in [\theta_n, \pi)$ may also be uniquely represented by $(\cos\theta, \sin\theta)$ for $\theta \in [\theta_n - \pi, 0)$. Moreover since A_θ is negative definite for $\theta \in (\theta_n, \pi]$, it must be positive definite for $\theta \in (\theta_n - \pi, 0]$. Therefore the previous inequality may also be written as,

$$\gamma(A,B) \geq \max_{\theta \in [\theta_n - \pi, \theta_1]} \lambda_{\min}(A_\theta) \tag{3.15}$$

where $\lambda_{\min}(A_\theta)$ denotes a smallest eigenvalue of $L(z)$. Setting, $L(\theta) = A\cos\theta - B\sin\theta$, $\sigma_{\min}(\theta) = \sigma_{\min}(A_\theta)$ and $\lambda_{\min}(\theta) = \lambda_{\min}(A_\theta)$ as in Theorem 3.5.8, we see that

$$\sigma_{\min}(\theta) = \lambda_{\min}(\theta) = \eta(\theta, L) \text{ for all } \theta \in [\theta_n - \pi, \theta_1]$$

as $A_\theta \geq 0$ for all $\theta \in [\theta_n - \pi, \theta_1]$. Let $\epsilon_0 = \max_{\theta \in [\theta_n - \pi, \theta_1]} \sigma_{\min}(A_\theta)$. Then $\epsilon_0 = \max_{\theta \in [\theta_n - \pi, \theta_1]} \lambda_{\min}(\theta)$. Let $\phi_0 \in [\theta_n - \pi, \theta_1]$ such that $\lambda_{\min}(\phi_0) = \epsilon_0$. Note that since $\lambda_{\min}(\theta_n - \pi) = \lambda_{\min}(\theta_1) = 0$ and $\lambda_{\min}(\theta) > 0$ for all $\theta \in (\theta_n - \pi, \theta_1)$, it is clear that $\phi_0 \in (\theta_n - \pi, \theta_1)$.

As already discussed in Theorem 3.5.8, there exist eigenvectors $v(\theta)$ corresponding to $\lambda_{\min}(\theta)$ with $\|v(\theta)\|_2 = 1$ such that both $\lambda_{\min}(\theta)$ and $v(\theta)$ are analytic functions of $\theta \in [\theta_n - \pi, \theta_1]$ at all but finitely many exceptional points say $\psi_1, \psi_2, \ldots, \psi_m$. For each $j = 1, 2, \ldots, m$, $L(\psi_j)$ is positive definite and therefore it has more than one positive eigenvalue that attains $\lambda_{\min}(\psi_j)$. Moreover the limits $\lim_{\theta \to \psi_j^{\pm}} \lambda_{\min}(\theta)$ and $\lim_{\theta \to \psi_j^{\pm}} v(\theta)$ exist and give smallest eigenvalues and corresponding unit eigenvectors of $L(\psi_j)$ for each $j = 1, 2, \ldots, m$.

If $\phi_0 \notin \{\psi_1, \psi_2, \ldots, \psi_m\}$, then $\lambda'_{\min}(\phi_0) = -v(\phi_0)^*(A\sin\phi_0 + B\cos\phi_0)v(\phi_0) = 0$. If $\phi_0 = \psi_q$ for some $q \in \{1, 2, \ldots, m\}$, then there exists $\delta > 0$ small enough such that $\lambda'_{\min}(\theta)$ exists for $\theta \in (\phi_0 - \delta, \phi_0 + \delta) \setminus \{\phi_0\}$, $\lambda_{\min}(\theta)$ is increasing in $(\phi_0 - \delta, \phi_0)$ and decreasing in $(\phi_0, \phi_0 + \delta)$. Therefore we have,

$$\begin{aligned} -v(\theta)^*(A\sin\theta + B\cos\theta)v(\theta) &= \lambda'_{\min}(\theta) \geq 0 \text{ for all } \theta \in (\phi_0 - \delta, \phi_0) \\ -v(\theta)^*(A\sin\theta + B\cos\theta)v(\theta) &= \lambda'_{\min}(\theta) \leq 0 \text{ for all } \theta \in (\phi_0, \phi_0 + \delta). \end{aligned}$$

Proceeding as in the proof of part (i) of Theorem 3.5.10, there exists $v(\phi_0)$ with $\|v(\phi_0)\|_2 = 1$ such that $L(\phi_0)v(\phi_0) = \lambda_{\min}(\phi_0)v(\phi_0)$ and $v(\phi_0)^*(A\sin\,\phi_0 + B\cos\,\phi_0)v(\phi_0) = 0$.

In either case, setting

$$\Delta A = -\cos\,\phi_0\,\lambda_{\min}(\phi_0)v(\phi_0)v(\phi_0)^* \text{ and } \Delta B = \sin\,\phi_0\,\lambda_{\min}(\phi_0)v(\phi_0)v(\phi_0)^*,$$

the pencil $\Delta L(\alpha,\beta) = \alpha\Delta A - \beta\Delta B$ is Hermitian with $|||\Delta L||| = \lambda_{\min}(\phi_0) = \epsilon_0$ such that $(\cos\,\phi_0, \sin\,\phi_0)$ represents a defective eigenvalue of $(L+\Delta L)(\alpha,\beta)$. But this implies that

$$\gamma(A,B) \leq \max_{\theta\in[\theta_n-\pi,\theta_1]} \lambda_{\min}(A_\theta). \tag{3.16}$$

Hence the proof follows in this case from (3.15) and (3.16).

If A is negative definite, then Theorem 1.2.17 implies that A_θ is negative definite for $\theta \in [\theta_n - \pi, \theta_1]$. The proof follows in an identical manner by replacing $L(z)$ by the pencil $-L(z)$, which has the same Crawford number and same eigenvalues with types reversed.

If A is indefinite, then there exists $k \in \{1,2,\ldots,n-1\}$ such that λ_{k+1} and λ_k are eigenvalues of $L(z)$ of opposite types. Once again, Theorem 1.2.17, implies that $A\cos\theta - B\sin\theta$ is positive (negative) definite for all $\theta \in (\theta_k, \theta_{k+1})$ if λ_{k+1} is of positive (negative) type. If λ_{k+1} is of positive type, then $\sigma_{\min}(\theta) = \lambda_{\min}(\theta)$ for all $\theta \in (\theta_k, \theta_{k+1})$ and the proof follows by arguing as above. If λ_{k+1} is of negative type, then noting that $L(\alpha,\beta)$ and $-L(\alpha,\beta)$ have the same Crawford number and same eigenvalues as $L(\alpha,\beta)$ with their types changed, the proof follows in an identical way by replacing $L(z)$ by $-L(z)$. $\square$

Note 3.6.2. *Suppose that the definite pencil $L(z)$ in Theorem 3.6.1 is such that $A > 0$. Let $\phi \in (\theta_n - \pi, \theta_1)$ be such that $v(\phi)^*B_\phi v(\phi) = 0$ where $v(\phi) \in \mathbb{C}^n$ with $\|v(\phi)\|_2 = 1$ is an eigenvector of A_ϕ corresponding to the eigenvalue $\lambda_{\min}(\phi)$. Then by Theorem 3.6.1, $\lambda_{\min}(\phi) \leq \gamma(A,B)$. But by arguing as in the proof of Theorem 3.6.1, there exists a Hermitian pencil $\Delta L(\alpha,\beta) = \alpha\Delta A - \beta\Delta B$ with $|||\Delta L||| = \lambda_{\min}(\phi)$ such that $(\cos\,\phi, \sin\,\phi)$ represents an eigenvalue of $(L+\Delta L)(\alpha,\beta)$ of mixed type. Therefore, $\lambda_{\min}(\phi) = \gamma(A,B)$.*

Thus, if there exists $\phi \in (\theta_n - \pi, \theta_1)$, such that the matrix $V_\phi^* B_\phi V_\phi$ is indefinite where the columns of V_ϕ form an orthonormal basis of $\mathrm{Ker}(A_\phi - \lambda_{\min}(\phi)I_n)$, then $\lambda_{\min}(\phi) = \gamma(A,B)$. In particular, if ϕ is not an exceptional point of $\lambda_{\min}(\theta)$, then $V_\phi^* B_\phi V_\phi$ is indefinite if and only if $\lambda'_{\min}(\phi) = 0$. Thus $\gamma(A,B) = \lambda_{\min}(\phi)$ whenever ϕ is a zero of the equation $\lambda'_{\min}(\theta) = 0$ belonging to the interval $(\theta_n - \pi, \theta_1)$. Similar arguments also hold for the cases when A is negative definite or indefinite. This is the basis for the algorithm *proposed in Chapter 5 for computing $\gamma(A,B)$ and a nearest Hermitian pencil that is not definite.*

Note 3.6.3. *Let $L(z)$ be a definite pencil and $\gamma(A,B)$ be less than the distance from $L(z)$ to a nearest nonregular pencil with respect to the norm $|||\cdot|||$. Also let $\mathcal{I} = (\theta_k, \theta_{k+1})$ if A is indefinite and $\mathcal{I} = (\theta_n - \pi, \theta_1)$ if A is definite. If possible let $\phi_1, \phi_2 \in \mathcal{I}$ such that $\phi_1 \neq \phi_2$ and $\sigma_{\min}(\phi_1) = \sigma_{\min}(\phi_2) = \gamma(A,B)$. Assuming without loss of generality that $\phi_1 < \phi_2$, if $\sigma_{\min}(\phi) < \gamma(A,B)$ for all $\phi \in (\phi_1, \phi_2)$, then the set of points $\{(\cos\phi, \sin\phi) : \phi \in (\phi_1,\phi_2)\}$ represents a component of $\Lambda^H_{\gamma(A,B)}(L)$. But this is impossible as there are no eigenangles in the interval (ϕ_1, ϕ_2). Therefore, there exists at least one $\phi \in (\phi_1,\phi_2)$ such that $\sigma_{\min}(\phi) = \gamma(A,B)$. By repeating the above arguments, for the intervals (ϕ_1, ϕ) and (ϕ, ϕ_2), it follows that there exist points $\hat{\phi}$ in each of these intervals for which $\sigma_{\min}(\hat{\phi}) = \gamma(A,B)$. Continuing the process, it follows that $\sigma_{\min}(\theta) = \gamma(A,B)$ for all $\theta \in (\phi_1,\phi_2)$. But this is impossible in view of Theorem 1.2.11. Thus there exists a unique point θ_0 in $\mathcal{I}$ such that $\sigma_{\min}(\theta_0) = \gamma(A,B)$.*

Theorem 3.6.1 may be written in terms of the eigenvalue of A_θ of smallest modulus.

Theorem 3.6.4. *Let $L(z)$ be a definite pencil. If A is definite,*

$$\gamma(A,B) = \begin{cases} \max\limits_{\theta\in[\theta_n-\pi,\theta_1]} \lambda_{\min}(A_\theta) & \text{if } A \text{ is positive definite} \\ \max\limits_{\theta\in[\theta_n-\pi,\theta_1]} -\lambda_{\min}(A_\theta) & \text{if } A \text{ is negative definite.} \end{cases}$$

If A is indefinite,

$$\gamma(A,B) = \begin{cases} \max\limits_{\theta\in[\theta_k,\theta_{k+1}]} \lambda_{\min}(A_\theta) & \text{if } \lambda_{k+1} \text{ is of positive type and} \\ & \lambda_k \text{ is of negative type .} \\ \max\limits_{\theta\in[\theta_k,\theta_{k+1}]} -\lambda_{\min}(A_\theta) & \text{if } \lambda_{k+1} \text{ is of negative type and} \\ & \lambda_k \text{ is of positive type.} \end{cases}$$

Note that Theorem 1.2.13 is an immediate corollary of the above result.

The next result relates the Hermitian ϵ-pseudospectrum $\Lambda^H_\epsilon(L)$ with $\gamma(A,B)$.

Theorem 3.6.5. *Given a definite pencil $L(z)$, its Crawford number $\gamma(A,B)$ has the following relationship with $\Lambda^H_\epsilon(L)$.*
If A is definite then,

$$\gamma(A,B) = \min\left\{\epsilon : (-\infty, \lambda_n] \cup [\lambda_1, \infty] \subseteq \text{ closure of } \Lambda^H_\epsilon(L)\right\}. \tag{3.17}$$

In this case if $\gamma(A,B) = \sigma_{\min}(A)$ then either the closure of $\Lambda_\epsilon^H(L)$ is equal to $\mathbb{R} \cup \{\infty\}$ or $\Lambda_\epsilon^H(L)$ has components of same type coalescing at ∞. If $\gamma(A,B) > \sigma_{\min}(A)$ then the same holds for the rotated pencil $L_{\theta_0}(z) = zA_{\theta_0} - B_{\theta_0}$ where θ_0 satisfies

$$\lambda_{\min}(A_{\theta_0}) = \gamma(A,B).$$

In former case, ∞ is a defective eigenvalue of a nearest Hermitian pencil to $L(z)$ with respect to the norm $|||\cdot|||$ that is not definite. In the latter case, the same holds for a nearest Hermitian pencil to $L_{\theta_0}(z)$. If A is indefinite then,

$$\gamma(A,B) = \min\left\{\epsilon : [\lambda_{k+1}, \lambda_k] \subseteq \text{ closure of } \Lambda_\epsilon^H(L) \text{ where } \lambda_{k+1} \text{ and } \lambda_k \text{ differ in type}\right\}. \tag{3.18}$$

In this case, if $\gamma(A,B) < \sigma_{\min}(A)$, then it is the smallest value of ϵ for which a component of $\Lambda_{\gamma(A,B)}^H(L)$ of positive type coalesces with a component of negative type. If $\gamma(A,B) < \sigma_{\min}(B)$ and B is also indefinite, then the same holds for $-\mathrm{rev}\, L(z)$.

Proof: If A is definite, then the proof of Theorem 3.6.1 shows that the Hermitian backward error $\eta^H(\theta, L)$ is strictly less than $\gamma(A,B)$ for all $\theta \in [\theta_n - \pi, \theta_1] \setminus \{\theta_0\}$ where θ_0 is the unique point of $(\theta_n - \pi, \theta_1)$ such that $\eta^H(\theta_0, L) = \gamma(A,B)$. Since the set of points $\{(\cos\theta, \sin\theta) : \theta \in [\theta_n - \pi, 0]\}$ and $\{(\cos\theta, \sin\theta) : \theta \in [0, \theta_1]\}$ on the unit circle uniquely represent the points of the intervals $(-\infty, \lambda_n]$ and $[\lambda_1, \infty]$ respectively, evidently $\gamma(A,B)$ is the smallest value of ϵ for which the closure of $\Lambda_\epsilon^H(L)$ contains the intervals $(-\infty, \lambda_n]$ and $[\lambda_1, \infty]$ of the extended real line.

If A is definite and $\gamma(A,B) = \sigma_{\min}(A)$, then $\theta_0 = 0$ so that ∞ is a boundary point of $\Lambda_{\gamma(A,B)}^H(L)$. If $\Lambda_{\gamma(A,B)}^H(L)$ has only one component, then the closure of $\Lambda_{\gamma(A,B)}^H(L)$ is equal to $\mathbb{R} \cup \{\infty\}$. If $\Lambda_{\gamma(A,B)}^H(L)$ has more than one component, then ∞ is a point of coalescence of two such components. Evidently, in view of Theorem 3.5.3, they are of positive type if A is positive definite and of negative type if A is negative definite. By part (iv) of Theorem 3.5.10, ∞ is an eigenvalue of mixed type of a Hermitian pencil $(L + \Delta L)(z)$ such that $|||\Delta L||| = \gamma(A,B)$.

If $\gamma(A,B) > \sigma_{\min}(A)$, then defining $\tilde{\sigma}_{\min}(\theta) = \sigma_{\min}(A_{\theta_0}\cos\theta - B_{\theta_0}\sin\theta)$, we see that $\tilde{\sigma}_{\min}(\theta) = \sigma_{\min}(\theta_0 - \theta)$ where $\sigma_{\min}(\theta) = \sigma_{\min}(A_\theta)$. Since θ_0 is a local maximum of $\sigma_{\min}(\theta)$, $\tilde{\sigma}_{\min}(\theta)$ has a local maximum at $\theta = 0$. Thus ∞ is a point of coalescence of two components of $\Lambda_{\gamma(A,B)}^H(L_{\theta_0})$ and the proof follows from the preceding arguments as A_{θ_0} is also definite.

If A is indefinite, then once again, the arguments in the proof of Theorem 3.6.1 show that $\eta^H(\theta, L)$ is strictly less than $\gamma(A, B)$ for all $\theta \in [\theta_k, \theta_{k+1}]\backslash\{\theta_0\}$ where θ_0 is the unique point of (θ_k, θ_{k+1}) such that $\eta^H(\theta_0, L) = \gamma(A, B)$. Since the points $\{(\cos\theta, \sin\theta) : \theta \in [\theta_k, \theta_{k+1}]\}$ uniquely represent the points of the interval $[\lambda_{k+1}, \lambda_k]$, it is clear that $\gamma(A, B)$ is the smallest level of ϵ such that the closure of $\Lambda^H_\epsilon(L)$ contains the entire interval $[\lambda_{k+1}, \lambda_k]$.

If A is indefinite, and $\gamma(A, B) < \sigma_{\min}(A)$, then there are two components of $\Lambda^H_{\gamma(A,B)}(L)$ coalescing at a point, say, z_0 of $(\lambda_{k+1}, \lambda_k)$. Since λ_{k+1} and λ_k differ in type, we assume without loss of generality that λ_{k+1} is of positive type. Then the eigenvalues of $L(z)$ inside the component of $\Lambda^H_{\gamma(A,B)}(L)$ coalescing at z_0 and containing λ_k are all of positive type while those inside the other component are of negative type. Since $\gamma(A, B) < \sigma_{\min}(A)$, one of the coalescing components is of positive type and the other of negative type.

If B is indefinite, and $\gamma(A, B) < \sigma_{\min}(B)$, then the proof follows by applying the above arguments to $-\mathrm{rev}\, L(z)$ and noting that $\gamma(A, B) = \gamma(B, A)$. $\square$

Finally we have the following immediate corollary of Theorem 3.6.1.

Corollary 3.6.6. *Given a definite pencil $L(z)$ with eigenvalues of both positive and negative type,*

$$\gamma(A, B) = \min\{|||\Delta L||| : (L+\Delta L)(z) \text{ is a Hermitian pencil with a defective eigenvalue }\}.$$

Some of the Hermitian pencils considered in the examples in the previous section also illustrate the Crawford number computation from the corresponding Hermitian pseudospectrum.

The pencil $L(z) = zA-B$ considered in Example 3.5.12 is definite with all eigenvalues of positive type. The eigenangles are $\theta_1 = 0.043, \theta_2 = 0.115, \theta_3 = 0.283, \theta_4 = 1.3161$ and $\theta_5 = 2.3746$. The corresponding Figure 3.1 also illustrates that $L(\theta)$ is positive definite for $\theta \in (\theta_5 - \pi, \theta_1)$ and the Crawford number is 1.7049 as it is the maximum of $\lambda_{\min}(A_\theta)$ in the interval $[\theta_5-\pi, \theta_1]$. This is attained at $\theta_0 = -0.4775$ and the first pencil considered in Example 3.5.13 is a rotation of $L(z)$ by θ_0. The graph at the top of Figure 3.2 shows that the Crawford number of the rotated pencil (which remains unchanged) is attained at $\theta = 0$ as proved in Theorem 3.6.5.

The pencil $L(z)$ considered in Example 3.5.14 is also definite and has one eigenvalue - λ_1 - of negative type while all the other eigenvalues are of positive type. The graph at the top of Figure 3.3 illustrates that $L(\theta)$ is positive definite in the interval (θ_1, θ_2) and the Crawford number $\gamma(A, B) = 0.7$ as it is the maximum of $\lambda_{\min}(A_\theta)$ in that interval.

Next we consider a definitizable pencil $L(z)$ which is a Hermitian pencil with all eigenvalues real and of definite type. By arguing as in the proof of Theorem 3.2.3, a nearest Hermitian pencil which is not definitizable, is therefore one which either has an infinite eigenvalue or an eigenvalue of mixed type. Let

$$d_D(L) := \{|||\Delta L||| : (L + \Delta L)(z) \text{ is non definitizable and Hermitian }\}.$$

The following theorem relates $d_D(L)$ with the levels of the Hermitian ϵ-pseudospectrum of $L(z)$.

Theorem 3.6.7. *Given a definitizable pencil $L(z)$, $d_D(L) = \min\{\epsilon_0, \sigma_{\min}(A)\}$ where ϵ_0 is the smallest value of ϵ for which the closure of $\Lambda_\epsilon^H(L)$ has a component containing eigenvalues of $L(z)$ of both positive and negative type.*

If all the eigenvalues of $L(z)$ are either of positive type or of negative type, then $d_D(L) = \sigma_{\min}(A)$.

Proof: Suppose that $L(z)$ has eigenvalues of both positive and negative type and $d_D(L) < \sigma_{\min}(A)$. Then $d_D(L)$ is the distance to a nearest Hermitian pencil with an eigenvalue of mixed type. Consider the Hermitian ϵ-pseudospectrum $\Lambda_\epsilon^H(L)$ for $\epsilon < \sigma_{\min}(A)$. By Proposition 3.5.4, any component of the closure of $\Lambda_\epsilon^H(L)$ containing only positive (negative) type eigenvalues of $L(z)$ is of positive (negative) type. If all the components of the closure of $\Lambda_\epsilon^H(L)$ contain eigenvalues of $L(z)$ of only one type for every $\epsilon < \sigma_{\min}(A)$, then the perturbed Hermitian pencils $(L+\Delta L)(z)$ do not have any eigenvalues of mixed type for all $|||\Delta L||| < \sigma_{\min}(A)$. But this implies that $d_D(L) \geq \sigma_{\min}(A)$ leading to a contradiction.

So, suppose that $\epsilon_0 < \sigma_{\min}(A)$ is the smallest value of ϵ for which a component of the closure of $\Lambda_\epsilon^H(L)$ contains eigenvalues of $L(z)$ of both positive and negative type. Then, evidently it is formed from the coalescence of at least two components say $C_{\epsilon_0}^{(1)}$ and $C_{\epsilon_0}^{(2)}$ such that one of them, say, $C_{\epsilon_0}^{(1)}$ contains eigenvalues of $L(z)$ of positive type and $C_{\epsilon_0}^{(2)}$ contains eigenvalues of $L(z)$ of negative type. Since $\epsilon_0 < \sigma_{\min}(A)$, Theorem 3.5.3 implies that $C_{\epsilon_0}^{(1)}$ is a component of positive type and $C_{\epsilon_0}^{(2)}$ is a component of negative type. Part (i) of Theorem 3.5.10 shows that any point of coalescence of these components is an eigenvalue of mixed type of a Hermitian pencil $(L + \Delta L)(z)$ with $|||\Delta L||| = \epsilon_0$. Thus

$d_D(L) \leq \epsilon_0$. Also the choice of ϵ_0 implies that $\Lambda_\epsilon^H(L)$ has components of only positive or negative type for $\epsilon < \epsilon_0$. Thus, $\epsilon < \epsilon_0 \Rightarrow \epsilon < d_D(L)$ from which we have $\epsilon_0 \leq d_D(L)$. Hence the proof follows in this case.

If the eigenvalues of $L(z)$ are of only one type, say, positive, then once again by Theorem 3.5.3, all the components of $\Lambda_\epsilon^H(L)$ are of positive type for $\epsilon < \sigma_{\min}(A)$. Hence in this case any perturbed Hermitian pencil $(L + \Delta L)(z)$ with an eigenvalue of mixed type is such that $|||\Delta L||| \geq \sigma_{\min}(A)$ so that $d_D(L) = \sigma_{\min}(A)$. $\square$

Chapter 4

Distance problems for Hermitian Matrix Polynomials

4.1 Introduction

In this chapter we extend the results in Chapter 3 to the analysis of distance problems for regular Hermitian matrix polynomials of degree m of the form

$$P(z) = \Sigma_{k=0}^{m} z^k A_k \tag{4.1}$$

where $A_k \in \mathbb{C}^{n\times n}$, with $A_k^* = A_k$ for each $k = 0, 1, \ldots, m$. The first section analyses properties of the Hermitian ϵ-pseudospectrum $\Lambda_\epsilon^H(P)$ of these polynomials and shows that the results of Section 3.5 for the Hermitian pencils extend to these polynomials. These results are then used to show that if $P(z)$ is a hyperbolic polynomial, then the distance to a nearest Hermitian polynomial that is not hyperbolic with respect to the norm $|||\cdot|||$ may be obtained from the levels of the Hermitian ϵ-pseudospectrum $\Lambda_\epsilon^H(P)$. The same also holds for the case when $P(z)$ is a quasihyperbolic polynomial and certain classes of definite polynomials. We conjecture that given any definite polynomial, this distance may be read off from its Hermitian ϵ-pseudospectrum. An interesting observation about the class of Hermitian polynomials with real eigenvalues of definite type is that if $m > 1$ and it has no eigenvalue at ∞, then it must have eigenvalues of both positive and negative type. Throughout the chapter, whenever we refer to any Hermitian matrix polynomial $P(z)$ we will mean that it is a regular matrix polynomial. Also $\Delta P(z)$ will denote an $n\times n$ Hermitian matrix polynomial of degree m of the form

$$\Delta P(z) = \Sigma_{k=0}^{m} z^k \Delta A_k.$$

4.2 Properties of Hermitian ϵ-pseudospectra

Recall that a real eigenvalue λ_0 of a Hermitian matrix polynomial $P(z)$ is said to be of positive (negative) type if

$$x^* \left(\beta_0 \frac{\partial P}{\partial \alpha}(\alpha_0, \beta_0) - \alpha_0 \frac{\partial P}{\partial \beta}(\alpha_0, \beta_0) \right) x$$

is positive (negative) for all eigenvectors x corresponding to λ_0 and representations (α_0, β_0) of λ_0 with $\beta_0 > 0$ if λ_0 is finite.

Also, λ_0 is said to be of definite type if it is either of positive or negative type.

Given a Hermitian polynomial $P(z) = \Sigma_{k=0}^{m} z^k A_k$ such that A_m is nonsingular, Theorem 1.2.8 implies that it has a Hermitian linearization of the form

$$\mathcal{L}_m(z) := z \begin{bmatrix} & & & & A_m \\ & & & \iddots & A_{m-1} \\ & & \iddots & \iddots & \vdots \\ & \iddots & \iddots & & A_2 \\ A_m & A_{m-1} & \cdots & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} & & & A_m & \\ & & \iddots & A_{m-1} & \\ & \iddots & \iddots & \vdots & \\ A_m & A_{m-1} & \cdots & A_2 & \\ & & & & -A_0 \end{bmatrix}. \tag{4.2}$$

Also given any eigenvalue λ of $P(z)$, with corresponding eigenvector x, λ is an eigenvalue of $\mathcal{L}_m(z)$ with corresponding eigenvector $v = [\lambda^{m-1}, \lambda^{m-2}, \ldots, 1]^T \otimes x$ such that

$$x^* P'(\lambda) x = v^* \mathcal{L}_m'(\lambda) v. \tag{4.3}$$

On the other hand, if A_0 is not singular, then Theorem 1.2.8 implies that $P(z)$ has a linearization of the form

$$\mathcal{L}_1(z) = z \begin{bmatrix} A_m & & & & \\ & -A_{m-2} & \cdots & -A_1 & -A_0 \\ & \vdots & \iddots & \iddots & \\ & -A_1 & \iddots & & \\ & -A_0 & & & \end{bmatrix} - \begin{bmatrix} -A_{m-1} & \cdots & -A_1 & -A_0 \\ \vdots & \iddots & \iddots & \\ -A_1 & \iddots & & \\ -A_0 & & & \end{bmatrix}. \tag{4.4}$$

Moreover Theorem 1.2.9 implies that given any eigenvalue λ of $P(z)$ with corresponding eigenvector x, λ is an eigenvalue of $\mathcal{L}_1(z)$ with corresponding eigenvector $v = [\lambda^{m-1}, \lambda^{m-2}, \ldots, 1]^T \otimes x$ such that

$$x^* P'(\lambda) x = \lambda^{m-1} v^* \mathcal{L}_1'(\lambda) v. \tag{4.5}$$

Moreover, as $P(z)$ is regular, if $\mathcal{L}_m(z)$ or $\mathcal{L}_1(z)$ is a linearization of $P(z)$, then *any* eigenvector corresponding to a finite eigenvalue λ of $\mathcal{L}_m(z)$ or $\mathcal{L}_1(z)$, is of the form $[\lambda^{m-1}, \lambda^{m-2}, \ldots, 1]^T \otimes x$ where x is an eigenvector of $P(z)$ corresponding to λ. Thus Theorem 1.2.9 essentially establishes that the linearization $\mathcal{L}_m(z)$ of $P(z)$ preserves the types of the eigenvalues of $P(z)$. The same also holds for the finite eigenvalues of $P(z)$ with respect to the linearization $\mathcal{L}_1(z)$ if m is even. We use these to prove the first result of this chapter.

Proposition 4.2.1. *Let $P(z)$ be a Hermitian polynomial. If all the eigenvalues of $P(z)$ are real and of definite type, degree of $P(z)$ is at least 2 and ∞ is not an eigenvalue of $P(z)$, then its finite eigenvalues are of both positive and negative type.*

The same also holds for the finite eigenvalues of $P(z)$ even if ∞ is an eigenvalue, if it is of even degree and it does not have an eigenvalue at 0.

Proof: Suppose that all the eigenvalues of $P(z)$ are real and of definite type. If it has no eigenvalue at ∞, then it has a linearization $\mathcal{L}_m(z)$ of the form 4.2. Since A_m is nonsingular, the coefficient matrix

$$\mathcal{A}_m = \begin{bmatrix} & & & A_m \\ & & \iddots & A_{m-1} \\ & \iddots & \iddots & \vdots \\ \iddots & \iddots & & A_2 \\ A_m \quad A_{m-1} & \cdots & A_2 & A_1 \end{bmatrix}$$

is nonsingular and indefinite. Therefore $\mathcal{L}_m(z)$ has eigenvalues of both positive and negative type. In view of 4.3 the same also holds for $P(z)$.

Next suppose that $P(z)$ is of even degree and has no eigenvalue at 0. Then it has a linearization $\mathcal{L}_1(z)$ as given by 4.4. Suppose without loss of generality that all the finite real eigenvalues of $P(z)$ are of positive type. Then 4.5 implies that all the positive eigenvalues of $P(z)$ are positive type eigenvalues of $\mathcal{L}_1(z)$ and all the negative eigenvalues are negative type eigenvalues of $\mathcal{L}_1(z)$. But this implies that the coefficient matrix

$$\mathcal{B}_1 = \begin{bmatrix} -A_{m-1} & \cdots & -A_1 & -A_0 \\ \vdots & \iddots & \iddots & \\ -A_1 & \iddots & & \\ -A_0 & & & \end{bmatrix}$$

of $\mathcal{L}_1(z)$ is positive definite. But this is impossible from the form of $\mathcal{B}_1$ as A_0 is nonsingular. Hence the proof. $\square$

Note that conclusions similar to those made in Proposition 4.2.1, may also be drawn from Theorem 1.3 of [20]. However, the theorem assumes that A_m is nonsingular. Throughout the chapter, whenever we say that all the eigenvalues of a Hermitian polynomial are of positive or negative type, we will assume that the eigenvalues are all real. The next result analyses the effect of small perturbations on eigenvalues of positive or negative type.

Lemma 4.2.2. *Let $P(z)$ be a Hermitian polynomial and $\lambda_1, \lambda_2, \ldots, \lambda_k$ be k finite eigenvalues of $P(z)$ of positive (negative) type counting multiplicities in the interior of a rectifiable simple closed curve Γ in $\mathbb{C}$ which does not pass through any eigenvalues of $P(z)$. For small enough $|||\Delta P|||$, the eigenvalues of perturbed Hermitian polynomials $(P + \Delta P)(z)$ also have k eigenvalues of positive (negative) type counting multiplicities lying in the interior of Γ.*

If $P(z)$ is of even degree and ∞ is an eigenvalue of $P(z)$ of positive (negative) type of multiplicity k, then there exists $z_1, z_2 \in \mathbb{R}$ such that for small enough $|||\Delta P|||$, the Hermitian polynomials $(P + \Delta P)(z)$ have exactly k eigenvalues of positive (negative) type counting multiplicity belonging to the set $(z_1, \infty) \cup (-\infty, z_2) \cup \{\infty\}$.

If $P(z)$ is of odd degree and ∞ is an eigenvalue of $P(z)$ of positive (negative) type of multiplicity k, then there exists $z_1, z_2 \in \mathbb{R}$ such that for small enough $|||\Delta P|||$, the Hermitian polynomials $(P+\Delta P)(z)$ have k_1 eigenvalues of positive (negative) type counting multiplicity belonging to the set $(z_1, \infty) \cup \{\infty\}$ and k_2 eigenvalues of negative (positive) type counting multiplicity belonging to the set $(-\infty, z_2)$ such that $k_1 + k_2 = k$.

Proof: Suppose without loss of generality that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigenvalues of $P(z)$ of positive type. Let λ be a real number lying in the exterior of Γ such that $\lambda \notin \Lambda(P)$ and $\lambda < \min_{z \in \mathbb{R} \cap \operatorname{Int} \Gamma} z$ and consider $v = e_{m-1} - \lambda e_m$. Since $p(\hat{\lambda}; v) \neq 0$ for all $\hat{\lambda} \in \Lambda(P)$, $P(z)$ has a linearization, say, $\mathcal{L}_v(z) \in \mathbb{L}_1(P)$ with ansatz vector v. The form of v, implies that $\mathcal{L}_v(z) = \mathcal{L}_{m-1}(z) - \lambda \mathcal{L}_m(z) \in \mathbb{H}(P)$. Since $p(z; v) = z - \lambda > 0$ for all $z \in \operatorname{Int} \Gamma \cap \mathbb{R}$, therefore, by Theorem 1.2.9, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are also eigenvalues of $\mathcal{L}_v(z)$ of positive type.

The Hermitian polynomials $(P+\Delta P)(z)$ with $|||\Delta P||| < \eta(\lambda, P)$, also do not have λ as an eigenvalue and therefore they give rise to linearizations of the form $(\mathcal{L} + \Delta \mathcal{L})_v(z) \in \mathbb{H}(P)$. By arguing as in the proof of Lemma 3.2.2, such pencils also have exactly k

eigenvalues of positive type in the interior of Γ if $|||\Delta\mathcal{L}_v|||$ is small enough. So choosing $(\Delta P)(z)$ with $|||\Delta P|||$ small enough such that $|||\Delta\mathcal{L}_v|||$ is small, $(P + \Delta P)(z)$ also has exactly k eigenvalues counting multiplicity in the interior of Γ that are all of positive type.

If the degree of $P(z)$ is even and ∞ is an eigenvalue of $P(z)$ of positive (negative) type then 0 is an eigenvalue of $-\mathrm{rev}\, P(z)$ of positive (negative) type. By considering a rectifiable simple closed curve Γ in $\mathbb{C}$ which does not pass through any eigenvalues of $-\mathrm{rev}\, P(z)$ and satisfies $\Lambda(-\mathrm{rev}\, P(z)) \cap \mathrm{Int}\,\Gamma = \{0\}$, and replacing $P(z)$ by $-\mathrm{rev}\, P(z)$ in the above arguments, it follows that all Hermitian polynomials $-\mathrm{rev}\,(P + \Delta P)(z)$ with $|||\Delta P|||$ small enough, have k eigenvalues of positive (negative) type in the interior of Γ counting multiplicity. Since Γ is a closed curve enclosing 0, $\mathrm{Int}\,\Gamma \cap \mathbb{R} = (-\delta_1, \delta_2)$ where δ_1 and δ_2 are positive real numbers. Let $z_1 = 1/\delta_2$ and $z_2 = -1/\delta_1$. Since m is even, the identity

$$x^*(P+\Delta P)'(\lambda_0)x = -(\lambda_0)^{m-2}\, x^*(\mathrm{rev}\,(P+\Delta P)'(1/\lambda_0))x, \lambda_0 \neq 0 \text{ is an eigenvalue of } P(z) \tag{4.6}$$

implies that the polynomials $(P + \Delta P)(z)$ have k eigenvalues of positive (negative) type lying in $(z_1, \infty) \cup (-\infty, z_2) \cup \{\infty\}$.

If the degree of $P(z)$ is odd, then the proof follows by arguments similar to those used above. The only difference in this case is that since m is odd, 4.6 implies that the eigenvalues of close enough Hermitian polynomials lying in $(-\infty, z_2)$ are of positive (negative) type if ∞ is an eigenvalue of negative (positive) type. $\square$

The above result implies that for small enough $\epsilon > 0$, the components of $\Lambda_\epsilon^H(P)$ containing finite eigenvalues of $P(z)$ of only positive (negative) type are such that all eigenvalues of Hermitian polynomials $(P + \Delta P)(z)$ with $|||\Delta P||| < \epsilon$, belonging to them are of positive (negative type). This motivates an extension of definition of eigenvalue type to that of components of the Hermitian ϵ-pseudospectrum $\Lambda_\epsilon^H(P)$ of $P(z)$.

Definition 4.2.3. *A component C_ϵ of the Hermitian ϵ-pseudospectrum $\Lambda_\epsilon^H(P)$ of $P(z)$ is said to be of positive (negative) type if all the finite eigenvalues of perturbed Hermitian polynomials $(P+\Delta P)(z)$ with $|||\Delta P||| < \epsilon$ belonging to C_ϵ are of positive (negative) type.*

A component of $\Lambda_\epsilon^H(P)$ is said to be of definite type if it is either of positive or negative type.

In particular, the above definition implies that definite type components of $\Lambda_\epsilon^H(P)$ are subsets of the extended real line. The next result is an extension of Theorem 3.5.3

to the polynomials and is a characterisation of components of $\Lambda_\epsilon^H(P)$ of definite type which do not contain ∞ in its closure.

Theorem 4.2.4. *Let $P(z)$ be a Hermitian polynomial and let C_ϵ be a component of $\Lambda_\epsilon^H(P)$ such that ∞ does not belong to the closure of C_ϵ. It is a component of positive (negative) type if and only if all the eigenvalues of $P(z)$ belonging to C_ϵ are of positive (negative) type. Also if $P(z)$ has k eigenvalues of positive (negative) type in C_ϵ, counting multiplicity, then the same also holds for all perturbed Hermitian polynomials $(P + \Delta P)(z)$ such that $|||\Delta P||| < \epsilon$.*

Proof: All the eigenvalues of $P(z)$ in C_ϵ are naturally of positive (negative) type if C_ϵ is of positive (negative) type. Therefore, the proof is complete if it is established that if all the eigenvalues of $P(z)$ in C_ϵ are of positive (negative) type, then C_ϵ is of positive (negative) type.

Suppose without loss of generality that $P(z)$ has k eigenvalues of positive type, say, $\lambda_1, \lambda_2, \ldots, \lambda_k$ in C_ϵ counting multiplicity. Let $\Delta P(z)$ be a Hermitian perturbation to $P(z)$ with $|||\Delta P\ || < \epsilon$ and Γ be a simple rectifiable closed curve in $\mathbb{C}$ lying in the complement of $\Lambda_\epsilon^H(P)$ such that $\Lambda_\epsilon^H(P) \cap \operatorname{Int}\Gamma = C_\epsilon$. Then Γ does not contain any eigenvalues of $(P + t\Delta P)(z)$ for all $t \in [0,1]$. Let λ be a real number in the exterior of Γ and the complement of $\Lambda_\epsilon^H(P)$ such that $\lambda < \min\limits_{z \in \mathbb{R}\cap \operatorname{Int}\Gamma} z$. Therefore, λ is not an eigenvalue of the Hermitian polynomials $(P + t\Delta P)(z)$ for all $t \in [0,1]$ and hence as already shown in the proof of Lemma 4.2.2, each of them admit a linearization of the form $(\mathcal{L} + t\Delta\mathcal{L})_v(z) \in \mathbb{H}(P)$ corresponding to the ansatz vector $v = e_{m-1} - \lambda e_m$. Let $(\mathcal{L} + t\Delta\mathcal{L})_v = z(\mathcal{A} + t\Delta\mathcal{A})_v - (\mathcal{B} + t\Delta\mathcal{B})_v$, $t \in [0,1]$.

Since none of these pencils have any eigenvalues on Γ, the spectral projections $P_\Gamma(t) = \frac{1}{2\pi i}\oint_\Gamma((\mathcal{L}+t\Delta\mathcal{L})_v(z))^{-1}(\mathcal{A}+t\Delta\mathcal{A})_v dz$ exist and vary analytically for all t belonging to an open interval containing $[0,1]$. Theorem2.1.7 implies that there exists a full rank matrix $X(t)$ whose columns form a basis of the direct sum of the spectral subspaces associated with the eigenvalues of $(\mathcal{L} + t\Delta\mathcal{L})_v(z)$ lying in C_ϵ which varies analytically with respect to t for all $t \in [0,1]$. Since $P(z)$ has k eigenvalues counting multiplicity in C_ϵ, the same is also the case for each of the linearizations $(\mathcal{L} + t\Delta\mathcal{L})_v(z)$, so that $X(t) \in \mathbb{C}^{mn\times k}$, $t \in [0,1]$.

For each $t \in [0,1]$, the span of the columns of $X(t)$ is a direct sum of spectral subspaces $S_{\lambda(t)}$ corresponding to the real eigenvalues $\lambda(t)$ of $(\mathcal{L} + t\Delta\mathcal{L})_v(z)$ in C_ϵ and subspaces of the form $S_{\mu(t)} \oplus S_{\overline{\mu(t)}}$ corresponding to complex eigenvalues $\mu(t)$ and $\overline{\mu(t)}$

of $(\mathcal{L}+t\Delta\mathcal{L})_v(z)$ in C_ϵ. Since $\infty \notin C_\epsilon$, all the eigenvalues of $(\mathcal{L}+t\Delta\mathcal{L})_v(z)$ in C_ϵ are finite. Therefore, by Theorem 3.4.5, these subspaces are mutually $(\mathcal{A}+t\Delta\mathcal{A})_v$-orthogonal. Also each of the subspaces $\mathrm{Ker}\,(\lambda(t)(\mathcal{A}+t\Delta\mathcal{A})_v-(\mathcal{B}+t\Delta\mathcal{B})_v)^{mn}$ and $\mathrm{Ker}\,(\mu(t)(\mathcal{A}+t\Delta\mathcal{A})_v-(\mathcal{B}_v+t\Delta\mathcal{B}_v))^{mn}\oplus\mathrm{Ker}\,(\overline{\mu(t)}(\mathcal{A}+t\Delta\mathcal{A})_v-(\mathcal{B}+t\Delta\mathcal{B})_v)^{mn}$ are individually $(\mathcal{A}+t\Delta\mathcal{A})_v$-nondegenerate. This implies that for all $t\in[0,1]$, the span of $X(t)$ is $(\mathcal{A}+t\Delta\mathcal{A})_v$-nondegenerate so that $X(t)^*(\mathcal{A}+t\Delta\mathcal{A})_vX(t)$ is nonsingular.

Since all the eigenvalues of $P(z)$ in C_ϵ are of positive type, the same also holds for all the eigenvalues of $\mathcal{L}_v(z)$ in C_ϵ as $p(\lambda_j;v)=\lambda_j-\lambda>0$ for all $j=1,2,\ldots,k$. Hence $X(0)^*\mathcal{A}_vX(0)$ is positive definite. Thus by continuity, $X(t)^*(\mathcal{A}+t\Delta\mathcal{A})_vX(t)$ is also positive definite for all $t\in[0,1]$. But this implies that all the k eigenvalues of $(\mathcal{L}+t\Delta\mathcal{L})_v(z)$ in C_ϵ are of positive type for all $t\in[0,1]$. This ensures that $(P+\Delta P)(z)$ has k eigenvalues counting multiplicity in C_ϵ. If these be $\hat{\lambda}_1,\hat{\lambda}_2,\ldots,\hat{\lambda}_k$, then the choice of λ ensures that $p(\hat{\lambda}_j;\lambda)=\hat{\lambda}_j-\lambda>0$ for all $j=1,2,\ldots,k$. Hence they are all eigenvalues of $(P+\Delta P)(z)$ of positive type. Since $\Delta P(z)$ is an arbitrarily chosen Hermitian polynomial with $|||\Delta P|||<\epsilon$, it follows that C_ϵ is a component of $\Lambda_\epsilon^H(P)$ of positive type. $\square$

The next result, is an extension of Theorem 4.2.4 to a component of the closure of $\Lambda_\epsilon^H(P)$.

Proposition 4.2.5. *Let $P(z)$ be a Hermitian polynomial without any eigenvalues at ∞ and C_ϵ be a component of the closure of $\Lambda_\epsilon^H(P)$ which does not contain ∞ such that $P(z)$ has k eigenvalues of positive (negative) type in C_ϵ. Then eigenvalues of all Hermitian polynomials $(P+\Delta P)(z)$ with $|||\Delta P|||\le\epsilon$ in C_ϵ are of positive (negative) type.*

Thus if a component C_ϵ of the closure of $\Lambda_\epsilon^H(P)$ be defined to be of positive (negative) type if all the finite eigenvalues of Hermitian polynomials $(P+\Delta P)(z)$ for $|||\Delta P|||\le\epsilon$ lying in C_ϵ are of positive (negative) type, then Theorem 4.2.4 and Proposition 4.2.5 imply that components of the closure of $\Lambda_\epsilon^H(P)$ formed from the coalescence of components of $\Lambda_\epsilon^H(P)$ of positive (negative) type are of positive (negative) type if $\epsilon<\sigma_{\min}(A_m)$.

We now extend the analysis of the points of coalescence of components of $\Lambda_\epsilon^H(L)$ of definite type done in Chapter 3 to that of Hermitian polynomials. The homogeneous form $P(\alpha,\beta)=\Sigma_{k=0}^m\alpha^k\beta^{m-k}A_k$, of the Hermitian polynomial $P(z)$ gives rise to the Hermitian family of matrices $P(\theta)=\Sigma_{k=0}^m(\cos\theta)^k(\sin\theta)^{m-k}A_k$, where $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$. Then the real part $\mathrm{Re}\,P(\theta)$ and imaginary part $\mathrm{Im}\,P(\theta)$ are clearly analytic functions of $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$. Let $\sigma_{\min}(\theta)=\sigma_{\min}(P(\theta))$ be the smallest singular value of $P(\theta)$. If $\sigma_{\min}(\theta_0)$

is simple for some $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then by Theorem 3.5.5, there exists a neighbourhood of θ_0 in which $\sigma_{\min}(\theta)$ is an analytic function of θ. Let $u(\theta_0)$ and $v(\theta_0)$ be unit left and right singular vectors of $P(\theta_0)$ corresponding to $\sigma_{\min}(\theta_0)$. Setting $c_0 = \cos\theta_0$, and $s_0 = \sin\theta_0$, by Theorem 3.5.5 we have

$$\sigma'_{\min}(\theta_0) = -\mathrm{Re}\left[u(\theta_0)^*\left(s_0\frac{\partial P}{\partial c}(c_0, s_0) - c_0\frac{\partial P}{\partial s}(c_0, s_0)\right)v(\theta_0)\right]. \tag{4.7}$$

The next result is an extension of Proposition 3.5.8 to the case of the Hermitian matrices $P(\theta)$.

Proposition 4.2.6. *Let $P(\theta) = \Sigma_{k=0}^{m}(\cos\theta)^k(\sin\theta)^{m-k}A_k$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be a family of Hermitian matrices of size n. Let $\sigma_{\min}(\theta) = \sigma_{\min}(P(\theta))$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. There exist finitely many points $\theta_1, \theta_2, \ldots, \theta_m$ satisfying $-\frac{\pi}{2} < \theta_1 < \theta_2 < \cdots < \theta_m < \frac{\pi}{2}$ such that the following hold.*

(i) *For $k = 0, 1, \ldots, m$, there exist functions $\lambda_{\min}(\theta)$ with the following properties. $P(\theta)v(\theta) = \lambda_{\min}(\theta)v(\theta)$, where $v(\theta) \in \mathbb{C}^n, \|v(\theta)\|_2 = 1$, such that*

$$|\lambda_{\min}(\theta)| = \min_{\lambda \in \Lambda(P(\theta))} |\lambda|$$

and $\lambda_{\min}(\theta)$ varies analytically with respect to $\theta \in (\theta_k, \theta_{k+1})$, with $\theta_0 = -\frac{\pi}{2}$ and $\theta_{m+1} = \frac{\pi}{2}$. Also the one sided limits

$$\lim_{\theta \to \theta_k^{\pm}} v(\theta) \text{ and } \lim_{\theta \to \theta_k^{\pm}} \lambda_{\min}(\theta)$$

exist for each $k = 1, 2, \ldots, m$.

(ii) *For each $\theta \in (\theta_k, \theta_{k+1})$, $k = 0, 1, \ldots, m$, either $\sigma_{\min}(\theta) = \lambda_{\min}(\theta)$ or $\sigma_{\min}(\theta) = -\lambda_{\min}(\theta)$.*

(iii) *For each $\theta \in (\theta_k, \theta_{k+1})$, $k = 0, 1, \ldots, m$,*

$$\lambda'_{\min}(\theta) = -v(\theta)^*\left(\frac{\partial P}{\partial c}(c, s)\sin\theta + \frac{\partial P}{\partial s}(c, s)\cos\theta\right)v(\theta)$$

where $c = \cos\theta$, $s = \sin\theta$ and $v(\theta)$ is a unit right eigenvector of $P(\theta)$ corresponding to $\lambda_{\min}(\theta)$ with properties as described in (i).

Proof: The proof follows from Theorem 3.5.6 and Theorem 3.5.5 by using arguments identical to those used to establish Proposition 3.5.8. □

The positive and negative components of $\Lambda_\epsilon^H(P)$ are subsets of the extended real line $\mathbb{R}\cup\{\infty\}$. Considering the homogeneous form of $\Lambda_\epsilon^H(P)$, the points of these components form connected subsets of the form

$$\{(\cos\,\theta, \sin\,\theta) : \theta \in (\theta_1, \theta_2), \eta(\theta, P) < \epsilon\}$$

where $(\theta_1, \theta_2) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\eta(\theta, P) = \dfrac{\sigma_{\min}(\theta)}{\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}}}$ is the backward error of a point represented by $(\cos\,\theta, \sin\,\theta)$ on the unit circle as an approximate eigenvalue of $P(\alpha, \beta)$ with respect to both Hermitian and arbitrary perturbations. We need the following lemma to analyse points of coalescence of components of $\Lambda_\epsilon^H(P)$ of positive or negative type.

Lemma 4.2.7. *Let* $g(\theta) = \frac{\lambda_{\min}(\theta)}{\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}}}$ *where* $\lambda_{\min}(\theta)$ *is an eigenvalue of* $P(\theta) = \Sigma_{k=0}^m (\cos\,\theta)^k (\sin\,\theta)^{m-k} A_k$ *such that* $|\lambda_{\min}(\theta)| = \min\limits_{\lambda\in\Lambda(P(\theta_0))} |\lambda|$. *Suppose that* $g(\theta)$ *has a derivative at* $\theta = \theta_0$, *and let* $v(\theta_0)$ *be a unit right eigenvector of* $P(\theta_0)$ *corresponding to* $\lambda_{\min}(\theta_0)$. *Setting* $c = \cos\,\theta, s = \sin\,\theta, c_0 = \cos\,\theta_0$ *and* $s_0 = \sin\,\theta_0$, *there exists a Hermitian polynomial* $\Delta P(\alpha, \beta)$ *with* $|||\Delta P||| = |g(\theta_0)|$ *such that* $(P+\Delta P)(c_0, s_0)v(\theta_0) = 0$ *and*

$$g'(\theta_0) = -\frac{v(\theta_0)^* \left(s_0 \frac{\partial (P+\Delta P)}{\partial c}(c_0, s_0) - c_0 \frac{\partial (P+\Delta P)}{\partial s}(c_0, s_0)\right) v(\theta_0)}{\sqrt{\Sigma_{k=0}^m c_0^{2(m-k)} s_0^{2k}}}. \tag{4.8}$$

Moreover, if $s_0 > 0$, *then* (c_0, s_0) *represents an eigenvalue of* $(P + \Delta P)(\alpha, \beta)$ *of positive type if* $g'(\theta_0) < 0$, *of negative type if* $g'(\theta_0) > 0$ *and of mixed type if* $g'(\theta_0) = 0$. *If* $s_0 = 0$, *and* $c_0 > 0$, *then* $(P + \Delta P)(\alpha, \beta)$ *has an infinite eigenvalue of positive type if* $g'(\theta_0) < 0$, *of negative type if* $g'(\theta_0) > 0$ *and of mixed type if* $g'(\theta_0) = 0$.

Proof: Let $X(\theta) = \Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}$. Since $g'(\theta_0)$ exists, $\lambda_{\min}(\theta_0)$ is a simple eigenvalue of $P(\theta_0)$. Therefore by part (iii) of Proposition 4.2.6,

$$\begin{aligned} g'(\theta_0) &= \frac{\lambda'_{\min}(\theta_0)}{\sqrt{X(\theta_0)}} - \frac{\lambda_{\min}(\theta_0)\, X'(\theta_0)}{2(X(\theta_0))^{\frac{3}{2}}} \\ &= \frac{1}{\sqrt{X(\theta_0)}} \left[-v(\theta_0)^* \left(s_0 \frac{\partial P}{\partial c}(c_0, s_0) - c_0 \frac{\partial P}{\partial s}(c_0, s_0) \right) v(\theta_0) - \frac{\lambda_{\min}(\theta_0)\, X'(\theta_0)}{2X(\theta_0)} \right]. \end{aligned} \tag{4.9}$$

Now,

$$\begin{aligned} X'(\theta_0) &= -\Sigma_{k=0}^{m-1} 2(m-k) c_0^{2m-2k-1} s_0^{2k+1} + \Sigma_{k=1}^{m} 2k c_0^{2m-2k+1} s_0^{2k-1} \\ &= \Sigma_{k=1}^{m-1} \left(2k s_0^{2k-1} c_0^{2m-2k+1} - 2(m-k) s_0^{2k+1} c_0^{2m-2k-1} \right) \\ & \quad -2m c_0^{2m-1} s_0 + 2m s_0^{2m-1} c_0. \end{aligned} \tag{4.10}$$

Let $\Delta P(\alpha, \beta) = \Sigma_{k=0}^{m} \alpha^k \beta^{m-k} \Delta A_k$ where

$$\Delta A_k = -\frac{c_0^k s_0^{m-k} \lambda_{\min}(\theta_0)}{X(\theta_0)} v(\theta_0) v(\theta_0)^*, \quad k = 0, 1, \ldots, m. \tag{4.11}$$

Then clearly $\Delta P(\alpha, \beta)$ is a Hermitian polynomial with $|||\Delta P||| = |g(\theta_0)|$ and

$$(P + \Delta P)(c_0, s_0) v(\theta_0) = P(\theta_0) v(\theta_0) - \lambda_{\min}(\theta_0) v(\theta_0) = 0$$

which shows that $(P + \Delta P)(\alpha, \beta)$ has an eigenvalue at the point represented by (c_0, s_0). We also have,

$$\begin{aligned} & -v(\theta_0)^* \left[s_0 \frac{\partial \Delta P}{\partial c}(c_0, s_0) - c_0 \frac{\partial \Delta P}{\partial s}(c_0, s_0) \right] v(\theta_0) \\ = & \left[s_0 \Sigma_{k=0}^{m-1} \left\{ (m-k) c_0^{m-k-1} s_0^k \right\} c_0^{m-k} s_0^k - c_0 \Sigma_{k=1}^{m} \left\{ k c_0^{m-k} s_0^{k-1} \right\} c_0^{m-k} s_0^k \right] \frac{\lambda_{\min}(\theta_0)}{X(\theta_0)} \\ = & \left[\Sigma_{k=1}^{m-1} \left\{ (m-k) c_0^{2m-2k-1} s_0^{2k+1} - k c_0^{2m-2k-1} s_0^{2k-1} \right\} \right. \\ & \left. + m c_0^{2m-1} s_0 - m s_0^{2m-1} c_0 \right] \frac{\lambda_{\min}(\theta_0)}{X(\theta_0).} \end{aligned} \tag{4.12}$$

From (4.10) and (4.12) we have,

$$v(\theta_0)^* \left[s_0 \frac{\partial \Delta P}{\partial c}(c_0, s_0) - c_0 \frac{\partial \Delta P}{\partial s}(c_0, s_0) \right] v(\theta_0) = \frac{X'(\theta_0)}{2X(\theta_0)} \lambda_{\min}(\theta_0).$$

The proof of (4.8) now follows by using the above relation in (4.9).

From (4.11) it is clear that the coefficient matrices of $\Delta P(\alpha, \beta)$ are all of rank one. Thus the polynomial $(P+\Delta P)(\alpha, \beta)$ has only one eigenvector $v(\theta_0)$ corresponding to the eigenvalue represented by (c_0, s_0) up to linear independence. Thus the numerator of the right hand side of (4.8) gives the type of the eigenvalue of $(P + \Delta P)(\alpha, \beta)$ represented by $(\cos\theta_0, \sin\theta_0)$. The rest of the proof now follows from the homogeneous definition of eigenvalue type by noting that this numerator is positive if $g'(\theta_0) < 0$, negative when $g'(\theta_0) > 0$ and equal to 0 if $g'(\theta_0) = 0$. $\square$

The following result analyses coalescence points of components of $\Lambda_\epsilon^H(P)$ of definite type on the lines of Theorem 3.5.10 for Hermitian pencils.

Theorem 4.2.8. *Let $P(z)$ be a Hermitian polynomial and $C_1(\epsilon)$ and $C_2(\epsilon)$ be components of $\Lambda_\epsilon^H(P)$ of definite type that coalesce at a point $(\cos\theta_0, \sin\theta_0)$. Let $g(\theta)$ be as defined in Lemma 4.2.7 and $r, r_1, r_2 \in \{1, -1\}$. Then the following hold.*

(i) If $\theta_0 \neq 0$ and $C_1(\epsilon)$ and $C_2(\epsilon)$ differ in type then there exists a neighbourhood (θ_1, θ_2) of θ_0 such that either $\eta(\theta, P) = g(\theta)$ or $\eta(\theta, P) = -g(\theta)$ for all $\theta \in (\theta_1, \theta_2)$.

Moreover $(\cos\theta_0, \sin\theta_0)$ represents an eigenvalue of mixed type of some Hermitian polynomial $(P + \Delta P)(z)$ with $|||\Delta P||| = \epsilon$.

(ii) If $\theta_0 \neq 0$ and $C_1(\epsilon)$ and $C_2(\epsilon)$ are of the same type then there exists a neighbourhood (θ_1, θ_2) of θ_0 such that $\eta(\theta, P) = r_1 g(\theta)$ for all $\theta \in (\theta_1, \theta_0)$ and $\eta(\theta, P) = r_2 g(\theta)$ for all $\theta \in (\theta_0, \theta_2)$ where $r_1 \neq r_2$.

(iii) Let $\theta_0 = 0$ and $C_1(\epsilon)$ and $C_2(\epsilon)$ be of the same type. Then there exists a neighbourhood (θ_1, θ_2) of 0 such that $\eta(\theta, P) = r g(\theta)$ for $\theta \in (\theta_1, \theta_2)$ if the degree of $P(z)$ is odd.

In this case ∞ is an eigenvalue of of mixed type of some Hermitian polynomial $(P + \Delta P)(z)$ with $|||\Delta P||| = \epsilon$.

If $P(z)$ is of even degree, then $\eta(\theta, P) = r_1 g(\theta)$ for $\theta \in (\theta_1, 0)$ and $\eta(\theta, P) = r_2 g(\theta)$ for $\theta \in (0, \theta_2)$ where $r_1 \neq r_2$.

(iv) Let $\theta_0 = 0$ and $C_1(\epsilon)$ and $C_2(\epsilon)$ differ in type. Then there exists a neighbourhood (θ_1, θ_2) of 0 such that $\eta(\theta, P) = r g(\theta)$ for $\theta \in (\theta_1, \theta_2)$ if the degree of $P(z)$ is even.

In this case ∞ is an eigenvalue of of mixed type of some Hermitian polynomial $(P + \Delta P)(z)$ with $|||\Delta P||| = \epsilon$.

If $P(z)$ is of odd degree, then $\eta(\theta, P) = r_1 g(\theta)$ for $\theta \in (\theta_1, 0)$ and $\eta(\theta, P) = r_2 g(\theta)$ for $\theta \in (0, \theta_2)$ where $r_1 \neq r_2$.

Proof: The proof of (i) will depend upon the types of $C_1(\epsilon)$ and $C_2(\epsilon)$, the sign of θ_0, the degree of $P(z)$ and also on which of the components $C_1(\epsilon)$ and $C_2(\epsilon)$ has points corresponding to $\{(\cos\theta, \sin\theta) : \theta < \theta_0\}$. We assume that $\theta_0 > 0$, $C_1(\epsilon)$ is of positive type and $C_2(\epsilon)$ is of negative type and $\theta < \theta_0$ for the points of $C_1(\epsilon)$. Therefore the point of coalescence is a point on the real line, say, z_0. Since, $C_1(\epsilon)$ and $C_2(\epsilon)$ are definite components of $\Lambda_\epsilon^H(L)$, it follows that $C_1(\epsilon) \cup C_2(\epsilon) \cup \{z_0\} \subset \mathbb{R}$ and the backward error

function $\eta(\theta, P)$ has a local maximum at $\theta = \theta_0$. Hence there exists an interval $(\theta_1, \theta_2) \subset (0, \pi)$ containing θ_0 that does not contain any exceptional points of the function $\lambda_{\min}(\theta)$ as described in Theorem 4.2.6, except possibly at θ_0, such that $\eta(\theta, P)$ is monotone increasing in (θ_1, θ_0) and monotone decreasing in (θ_0, θ_2). Also for $\theta \in (\theta_1, \theta_2)$, $\eta(\theta, P) = rg(\theta)$, where $g(\theta)$ is as in Lemma 4.2.7 and $r \in \{-1, 1\}$. Since, there are no exceptional points in (θ_1, θ_2) except possibly at θ_0, r is constant in each of the intervals (θ_1, θ_0) and (θ_0, θ_2). Also $g(\theta)$ and hence $\eta(\theta, P)$ is differentiable for all $\theta \in (\theta_1, \theta_0) \cup (\theta_0, \theta_2)$.
Since $\eta'(\theta, P) \geq 0$ for all $\hat{\theta} \in (\theta_1, \theta_0)$, setting $\hat{c} = \cos\hat{\theta}$ and $\hat{s} = \sin\hat{\theta}$, from (4.8), we have

$$0 \leq -r \frac{v(\hat{\theta})^* \left(\hat{s}\frac{\partial(P+\Delta P)}{\partial c}(\hat{c}, \hat{s}) - \hat{c}\frac{\partial(P+\Delta P)}{\partial s}(\hat{c}, \hat{s})\right) v(\hat{\theta})}{\sqrt{\Sigma_{k=0}^{m} (\hat{c})^{2(m-k)} (\hat{s})^{2k}}} \tag{4.13}$$

where $\Delta P(\alpha, \beta)$ is a Hermitian polynomial with coefficients ΔA_k, $k = 0, 1, \ldots, m$ specified by (4.11) such that $|||\Delta P||| = \eta(\hat{\theta}, P)$ and $(P + \Delta P)(\alpha, \beta)$ has an eigenvalue at a point represented by $(\cos\hat{\theta}, \sin\hat{\theta})$. Assuming that $(\cos\theta, \sin\theta) \in C_1(\epsilon)$ for $\theta \in (\theta_1, \theta_0)$, since $\theta_0 > 0$, it follows that

$$v(\hat{\theta})^* \left(\hat{s}\frac{\partial(P+\Delta P)}{\partial c}(\hat{c}, \hat{s}) - \hat{c}\frac{\partial(P+\Delta P)}{\partial s}(\hat{c}, \hat{s})\right) v(\theta_0) > 0$$

for all $\theta \in (\theta_1, \theta_0)$ irrespective of the degree of $P(z)$. Hence from (4.13) we have $r = -1$ for all $\theta \in (\theta_1, \theta_0)$. Similarly, it may be established that $r = -1$ for all $\theta \in (\theta_0, \theta_1)$. Thus we have $\eta(\theta, P) = -g(\theta)$ and in particular $\lambda_{\min}(\theta) = -\sigma_{\min}(\theta) < 0$ for all $\theta \in (\theta_1, \theta_2)$.

By Theorem 3.5.6, the Hermitian family of matrices $P(\theta)$, $\theta \in (\theta_1, \theta_2)$, has eigenvalues $\lambda_1(\theta), \lambda_2(\theta), \ldots, \lambda_n(\theta)$ an corresponding eigenvectors $v_1(\theta), v_2(\theta), \ldots, v_n(\theta)$ that form an orthonormal basis of $\mathbb{C}^n$ such that all the eigenvalues and eigenvectors vary analytically with respect to $\theta \in (\theta_1, \theta_2) \setminus \{\theta_0\}$.

Thus there exist $p, q \in \{1, 2, \ldots, n\}$, such that $\lambda_{\min}(\theta) = \lambda_p(\theta)$, $v(\theta) = v_p(\theta)$ for all $\theta \in (\theta_1, \theta_0]$ and $\lambda_{\min}(\theta) = \lambda_q(\theta)$, $v(\theta) = v_q(\theta)$ for all $\theta \in [\theta_0, \theta_2)$.

Suppose that $p = q$. Then $\eta'(\theta_0, P) = -g'(\theta_0) = 0$. Thus by Lemma 4.2.7, $(\cos\theta_0, \sin\theta_0)$ represents an eigenvalue of mixed type of the Hermitian polynomial $(P+\Delta P)(\alpha, \beta)$ with $|||\Delta P||| = \eta(\theta_0, P)$ where the coefficient matrices of $(\Delta P)(\alpha, \beta)$ are given by (4.11).

Next suppose that $p \neq q$. Since $g'(\theta) = -\eta'(\theta, P) \leq 0$ for all $\theta \in (\theta_1, \theta_0)$ and $g'(\theta) = -\eta'(\theta, P) \geq 0$ for all $\theta \in (\theta_0, \theta_2)$, setting $c_0 = \cos\theta_0$ and $s_0 = \sin\theta_0$ we have

$$v_p(\theta_0)^* \left(s_0\frac{\partial(P+\Delta P^{(p)})}{\partial c}(c_0, s_0) - c_0\frac{\partial(P+\Delta P^{(p)})}{\partial s}(c_0, s_0)\right) v_p(\theta_0) \leq 0$$

where $\Delta P^{(p)}(\alpha,\beta) = \Sigma_{k=0}^{m}\alpha^k\beta^{m-k}\Delta A_k^{(p)}$ is obtained by replacing $\lambda_{\min}(\theta_0)$ and $v(\theta_0)$ by $\lambda_p(\theta_0)$ and $v_p(\theta_0)$ respectively in (4.11). Similarly,

$$v_q(\theta_0)^*\left(s_0\frac{\partial(P+\Delta P^{(q)})}{\partial c}(c_0,s_0) - c_0\frac{\partial(P+\Delta P^{(q)})}{\partial s}(c_0,s_0)\right)v_q(\theta_0) \geq 0$$

where $\Delta P^{(q)}(\alpha,\beta) = \Sigma_{k=0}^{m}\alpha^k\beta^{m-k}\Delta A_k^{(q)}$ is obtained by replacing $\lambda_{\min}(\theta_0)$ and $v(\theta_0)$ by $\lambda_q(\theta_0)$ and $v_q(\theta_0)$ respectively in (4.11). If equality holds in any of the above relations, say, the first one, then replacing $v(\theta_0)$ and $\lambda_{\min}(\theta)$ by $v_p(\theta_0)$ and $\lambda_p(\theta_0)$ respectively in (4.11), $(\cos\theta_0, \sin\theta_0)$ represents an eigenvalue of mixed type of the Hermitian polynomial $(P+\Delta P)(\alpha,\beta)$ with $|||\Delta P||| = \eta(\theta_0, P)$.

So suppose that strict inequality holds for each case. Observing that $\lambda_p(\theta_0) = \lambda_q(\theta_0) = -\sigma_{\min}(\theta_0)$, let $\Delta P(\alpha,\beta)) = \Sigma_{k=0}^{m}\alpha^k\beta^{m-k}\Delta A_k$ where

$$\Delta A_k = -\frac{c_0^k s_0^{m-k}(-\sigma_{\min}(\theta_0))}{X(\theta_0)}[v_p(\theta_0)\,v_q(\theta_0)][v_p(\theta_0)\,v_q(\theta_0)]^*,\quad k = 0,1,\ldots,m,$$

where $X(\theta_0) = \Sigma_{k=0}^{m}(c_0)^{2(m-k)}s_0^{2k}$. Since $v_p(\theta_0)$ and $v_q(\theta_0)$ are mutually orthogonal, $|||\Delta P||| = \eta(\theta_0,P)$. Also, $v_p(\theta_0)$ and $v_q(\theta_0)$ are eigenvectors of $(P+\Delta P)(\alpha,\beta)$ corresponding to the eigenvalue represented by (c_0,s_0). Since they are mutually orthogonal, we have

$$\begin{aligned} & v_p(\theta_0)^*\left(s_0\frac{\partial(\Delta P)}{\partial c}(c_0,s_0) - c_0\frac{\partial(\Delta P)}{\partial s}(c_0,s_0)\right)v_p(\theta_0) \\ = \;& v_p(\theta_0)^*\left(s_0\frac{\partial(\Delta P^{(p)})}{\partial c}(c_0,s_0) - c_0\frac{\partial(\Delta P^{(p)})}{\partial s}(c_0,s_0)\right)v_p(\theta_0) \end{aligned}$$

and

$$\begin{aligned} & v_q(\theta_0)^*\left(s_0\frac{\partial(\Delta P)}{\partial c}(c_0,s_0) - c_0\frac{\partial(\Delta P)}{\partial s}(c_0,s_0)\right)v_q(\theta_0) \\ = \;& v_q(\theta_0)^*\left(s_0\frac{\partial(\Delta P^{(q)})}{\partial c}(c_0,s_0) - c_0\frac{\partial(\Delta P^{(q)})}{\partial s}(c_0,s_0)\right)v_q(\theta_0), \end{aligned}$$

the common value in each case being $\sigma_{\min}(\theta_0)\dfrac{X'(\theta_0)}{2X(\theta_0)}$. Thus we have,

$$v_p(\theta_0)^*\left(s_0\frac{\partial(P+\Delta P)}{\partial c}(c_0,s_0) - c_0\frac{\partial(P+\Delta P)}{\partial s}(c_0,s_0)\right)v_p(\theta_0) < 0$$

$$v_q(\theta_0)^*\left(s_0\frac{\partial(P+\Delta P)}{\partial c}(c_0,s_0) - c_0\frac{\partial(P+\Delta P)}{\partial s}(c_0,s_0)\right)v_q(\theta_0) > 0$$

which shows that (c_0, s_0) represents an eigenvalue of mixed type of $(P + \Delta P)(\alpha, \beta)$.

The proof of (i) for the other cases as well as those for the statements $(ii)-(iv)$ follow with similar arguments and the observation that given an eigenvalue λ_0 of a Hermitian polynomial $P(z)$ represented by a point (c_0, s_0) on the unit circle, with corresponding eigenvector x_0,

$$x_0^* \left(c_0 \frac{\partial P}{\partial c}(c_0, s_0) - s_0 \frac{\partial P}{\partial s}(c_0, s_0) \right) x_0 = \begin{cases} x_0^* \left((-c_0) \frac{\partial P}{\partial c}(-c_0, -s_0) - (-s_0) \frac{\partial P}{\partial s}(-c_0, -s_0) \right) x_0 & \text{if } m \text{ is even} \\ -x_0^* \left((-c_0) \frac{\partial P}{\partial c}(-c_0, -s_0) - (-s_0) \frac{\partial P}{\partial s}(-c_0, -s_0) \right) x_0 & \text{if } m \text{ is odd.} \end{cases}$$

We have the following immediate corollary of Theorem 4.2.8.

Corollary 4.2.9. *Let $P(z)$ be a Hermitian polynomial with mn real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{mn}$ of definite type such that*

$$-\infty < \lambda_{mn} \le \lambda_{mn-1} \le \cdots \le \lambda_1 < \infty.$$

Let $C_1(\epsilon)$ and $C_2(\epsilon)$ be components of $\Lambda_\epsilon^H(P)$ of definite type that coalesce at a point $z_0 \in \mathbb{R} \cup \{\infty\}$. If $C_1(\epsilon)$ and $C_2(\epsilon)$ are of the same type and $z_0 \in (\lambda_{k+1}, \lambda_k)$, for $1 \le k \le mn - 1$, then $P(z)$ is indefinite for all $z \in (\lambda_{k+1}, \lambda_k)$.

If $C_1(\epsilon)$ and $C_2(\epsilon)$ coalesce at ∞, and are not of the same type, then $P(z)$ is indefinite for $z \in (-\infty, \lambda_{mn}) \cup (\lambda_1, \infty)$.

Proof: The proof follows on the lines of the proof of Corollary 3.5.11. □

We now illustrate some of the above results with the aid of numerical examples. To understand these examples we note that in view of the relation (4.8) if m is even and $g(\theta)$ has a negative (positive) slope at a point θ which is not an exceptional point, then $(\cos\,\theta, \sin\,\theta)$ represents an eigenvalue of positive(negative) type of perturbed Hermitian pencils $(L + \Delta L)(z)$ where $\Delta L(z) = \Sigma_{k=0}^m z^k \Delta A_k$ with coefficient $\Delta A_k, k = 0, 1, \ldots, m$, given by (4.11). Thus if the curve of $g(\theta)$ has both positive and negative slopes for θ belonging to a subinterval of $(0, \pi)$, then the corresponding points $(\cos\,\theta, \sin\,\theta)$ belong to a component of $\Lambda_\epsilon^H(P)$ of mixed type. All the examples are those of quadratic Hermitian polynomials with real eigenvalues of definite type and have been generated by using the NLEVP software [9].

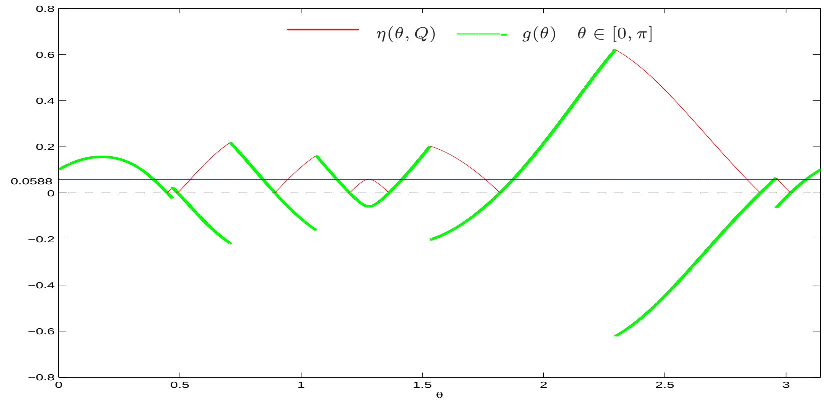

Figure 4.1: Plots of $\eta(\theta, Q)$ and $g(\theta)$ for Example 4.2.10.

Example 4.2.10. *Consider the hyperbolic polynomial* $Q(z) = z^2A + zB + C$ *where*

$$A = \begin{bmatrix} 0.7874 & 0.5476 & 0.1132 & -0.1082 \\ 0.5476 & 0.6416 & -0.0393 & -0.0427 \\ 0.1132 & -0.0393 & 0.5133 & 0.0487 \\ -0.1082 & -0.0427 & 0.0486 & 0.1980 \end{bmatrix} B = \begin{bmatrix} 0.0012 & -0.6858 & -0.0636 & -0.1315 \\ -0.6858 & -0.5291 & 0.5926 & -0.0681 \\ -0.0636 & 0.5926 & -0.7845 & -0.0268 \\ -0.1315 & -0.0681 & -0.0268 & 0.8318 \end{bmatrix}$$

and

$$C = \begin{bmatrix} -0.6881 & 0.6261 & 0.1728 & -0.2741 \\ 0.6261 & -0.5350 & -0.4698 & 0.2481 \\ 0.1728 & -0.4698 & -0.1114 & 0.0984 \\ -0.2741 & 0.2481 & 0.0984 & -0.5679 \end{bmatrix}.$$

Its eigenvalues are $\lambda_1 = 2.085, \lambda_2 = 1.87, \lambda_3 = 0.8068, \lambda_4 = 0.3887, \lambda_5 = 0.2119, \lambda_6 = -0.2542, \lambda_7 = -3.951$ *and* $\lambda_8 = -8.0870$. *These are represented respectively by the pairs* $(\cos\,\theta_k, \sin\,\theta_k)$, $k = 1, 2, \ldots, 8$, *where* $\theta_1 = 0.4472, \theta_2 = 0.4911, \theta_3 = 0.8919, \theta_4 = 1.2000, \theta_5 = 1.3620, \theta_6 = 1.8200, \theta_7 = 2.8940$ *and* $\theta_8 = 3.0190$ *belong to the interval* $(0, \pi)$. *Figure 4.1 plots the backward error function* $\eta(\theta, Q)$ *and* $g(\theta)$ *for* $\theta \in [0, \pi]$ *and shows the coalescence of a component of* $\Lambda_\epsilon^H(Q)$ *containing the eigenvalue* λ_4 *of positive type with a component containing the eigenvalue* λ_5 *of negative type for* $\epsilon = 0.0588$. *These components are respectively of positive type and negative type as they do not contain* ∞ *in their closure. Thus the point of coalescence represented by* $(\cos\,\theta_0, \sin\,\theta_0)$ *where* $\theta_0 = 1.278$ *is an eigenvalue of mixed type of a Hermitian polynomial* $(Q + \Delta Q)(z)$ *where* $|||\Delta Q||| = 0.0588$.

Example 4.2.11. *Let* $Q_1(z)$ *be the polynomial obtained by homogeneously rotating the polynomial* $Q(z)$ *considered in Example 4.2.10 by an angle* $\theta = 1.279$. *The eigenvalues of* $Q_1(z)$ *are* $\lambda_1 = 12.09, \lambda_2 = 1.667, \lambda_3 = -0.0434, \lambda_4 = -0.1699, \lambda_5 = -0.9104, \lambda_6 = -0.9941, \lambda_7 = -2.45$ *and* $\lambda_8 = -12.58$. *They are respectively represented by* $(\cos\,\theta_k, \sin\,\theta_k)$,

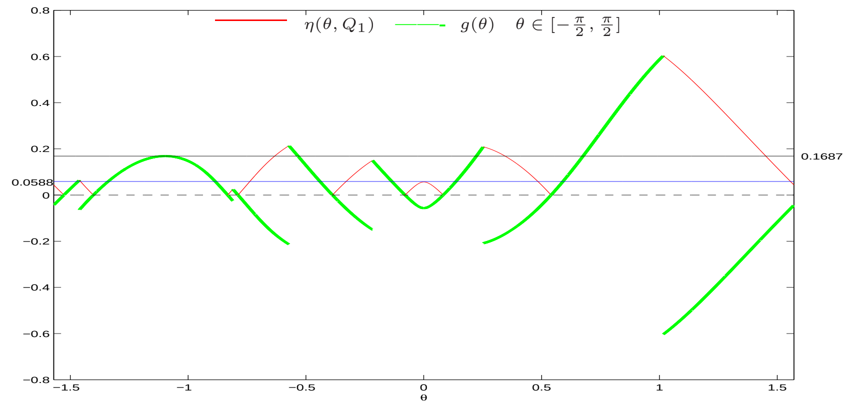

Figure 4.2: Plots of $\eta(\theta, Q)$ and $g(\theta)$ for Example 4.2.11.

$k = 1, 2, \ldots, 8$, *where* $\theta_1 = 0.0825, \theta_2 = 0.5403, \theta_3 = -1.527, \theta_4 = -1.402, \theta_5 = -0.8323, \theta_6 = -0.7883, \theta_7 = -0.3875$, *and* $\theta_8 = -0.0793$ *belong to the interval* $[-\frac{\pi}{2}, \frac{\pi}{2}]$. *Figure 4.2 plots* $\eta(\theta, Q_1)$ *and* $g(\theta)$ *for* $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *and shows that the eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *are of negative type while* $\lambda_5, \lambda_6, \lambda_7$ *and* λ_8 *are of positive type. Thus* $Q_1(z)$ *is quasihyperbolic. It also shows that for* $\epsilon = 0.0588$ *two components of* $\Lambda_\epsilon^H(Q_1)$ *each of negative type one containing* λ_1 *and the other containing* λ_8 *have coalesced at* ∞. *It may be verified that* ∞ *is an eigenvalue of mixed type of a Hermitian polynomial* $(Q_1 + \Delta Q_1)(z)$ *with* $|||\Delta Q_1||| = 0.0588$ *where the coefficients of* $\Delta Q_1(z)$ *are given by (4.11).*

Example 4.2.12. *Consider the Hermitian polynomial* $Q_2(z)$ *obtained by rotating the polynomial* $Q(z)$ *by the angle* $\theta = 2.296$. *The eigenvalues of* $Q_2(z)$ *are* $\lambda_1 = 1.469, \lambda_2 = 1.134, \lambda_3 = 0.2853, \lambda_4 = 0.2384, \lambda_5 = -0.1684, \lambda_6 = -0.5143, \lambda_7 = -0.7397$ *and* $\lambda_8 = -1.939$. *They are represented by the pairs* $(\cos\,\theta_k, \sin\,\theta_k)$, $k = 1, 2, \ldots, 8$, *where* $\theta_1 = 0.5978, \theta_2 = 0.7227, \theta_3 = 1.2930, \theta_4 = 1.3370, \theta_5 = -1.404, \theta_6 = -1.096, \theta_7 = -0.9339$ *and* $\theta_8 = -0.4762$ *belong to the interval* $[-\frac{\pi}{2}, \frac{\pi}{2}]$. *Figure 4.3 plots* $\eta(\theta, Q_2)$ *and* $g(\theta)$ *with respect to* $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. *From the figure we see that* $\lambda_1, \lambda_2, \lambda_7$ *and* λ_8 *are of negative type while the other eigenvalues are of positive type. Hence* $Q_2(z)$ *is a definite polynomial with* $p = 2$. *Figure 4.3 also shows that for* $\epsilon = 0.063$ *a component of* $\Lambda_\epsilon^H(Q_2)$ *of containing* λ_7 *coalesces with a component containing* λ_6 *at the point represented by* $(\cos\,\theta_0, \sin\,\theta_0)$ *for* $\theta_0 = -1.012$. *Since* ∞ *is evidently not an element of the closure of* $\Lambda_\epsilon^H(Q_2)$, *these components are respectively of negative and positive type with the point of*

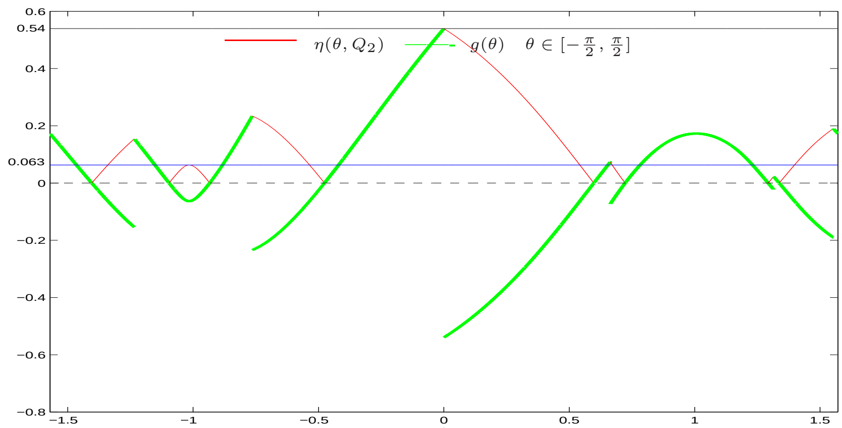

Figure 4.3: Plots of $\eta(\theta, Q)$ and $\overset{o}{g}(\theta)$ for Example 4.2.12.

coalescence being an eigenvalue of mixed type of a Hermitian polynomial $(Q_2 + \Delta Q_2)(z)$ *with* $|||\Delta Q_2||| = 0.063$.

4.3 Distance problems via Hermitian ϵ-pseudospectra.

In this section we consider the distance from a hyperbolic or quasihyperbolic polynomial to a nearest Hermitian polynomial outside the class with respect to the $|||\cdot|||$ norm. We show that these distance problems may be solved by the pseudospectra approach for the case of the hyperbolic and quasihyperbolic polynomials. The same is also true for the definite polynomials with certain assumptions.

4.3.1 Hyperbolic and quasihyperbolic polynomials

Let $P(z)$ be a hyperbolic polynomial of degree m. If $m = 1$, then $P(z)$ is a definite pencil with all eigenvalues of positive type - a case which has already been considered in Chapter 3. Therefore we assume that $m \geq 1$. A nearest Hermitian polynomial with respect to the norm $|||\cdot|||$ is one which violates property (P1) of Theorem 1.2.14. We prove that such a polynomial is one which either has a finite real eigenvalue of mixed type or an eigenvalue at ∞. Lemma 4.2.2 implies that for small enough $\epsilon > 0$, the perturbed Hermitian polynomials $(P + \Delta P)(z)$ for $|||\Delta P||| < \epsilon$ are also hyperbolic. The following result, states that the distance to a nearest Hermitian polynomial to $P(z)$ which is not hyperbolic is the distance to a nearest Hermitian polynomial which has a real eigenvalue of mixed type or an infinite eigenvalue.

Theorem 4.3.1. *Let $P(z)$ be a hyperbolic polynomial and*

$$d_H(P) = \min\{|||\Delta P||| : (P + \Delta P)(z) \text{ is Hermitian but not hyperbolic .}\} \tag{4.14}$$

Then $d_H(P)$ is the distance to a nearest Hermitian polynomial with respect to the norm $|||\cdot|||$ which either has an eigenvalue at ∞, or a finite real eigenvalue of mixed type.

Proof: Suppose that $d_H(P) < \sigma_{\min}(A_m)$. Then $d_H(P)$ is the distance with respect to the norm $|||\cdot|||$ to a nearest Hermitian polynomial which either has a complex eigenvalue, a real eigenvalue of mixed type or its eigenvalues are real and of definite type but they do not conform to the separation as described in property (P1).

Suppose that there exists a Hermitian polynomial $(P+\Delta P)(z)$ which has a complex eigenvalue, say, z_0, such that $|||\Delta P||| = d_H(P)$. Since $d_H(P) < \sigma_{\min}(A_m)$, $(P + \Delta P)(z)$ is regular. Consider a simple rectifiable closed curve Γ in $\mathbb{C}$ which does not pass through any eigenvalues of $P(z)$ such that $(\Gamma \cup \operatorname{Int}\Gamma) \cap \mathbb{R} = \emptyset$, $\Gamma \cap \Lambda(P) = \emptyset$, and $\operatorname{Int}\Gamma \cap \Lambda(P) = \{z_0\}$. Consider any Hermitian linearization $(\mathcal{L} + \Delta\mathcal{L})(z)$ say, from the space $\mathbb{H}(P + \Delta P)$ associated with $(P+\Delta P)(z)$. The only eigenvalue of $(\mathcal{L}+\Delta\mathcal{L})(z)$ in $\operatorname{Int}\Gamma$ is z_0. Therefore, for each $t \in (0, 1)$, close enough to 1, the pencils $(\mathcal{L}+t\Delta\mathcal{L})(z)$ have at least one eigenvalue in $\operatorname{Int}\Gamma$. This implies that $(P + t\Delta P)(z)$ has a complex eigenvalue for $t \in (0, 1)$ close enough to 1. But since $|||t\Delta P||| < d_H(P)$, for all such t, it follows that all the eigenvalues of $(P+t\Delta P)(z)$ must be real leading to a contradiction. Therefore, a nearest Hermitian polynomial to $P(z)$ which is not hyperbolic does not have a complex eigenvalue.

Next suppose that a nearest Hermitian polynomial to $P(z)$ with respect to the norm $|||\cdot|||$ which is not hyperbolic is one for which all the eigenvalues are real and finite but they are not grouped into disjoint intervals as described in (P1) of Theorem 1.2.14. By Lemma 4.2.2, for small enough ϵ, all Hermitian polynomial $(P+\Delta P)(z)$ with $|||\Delta P||| < \epsilon$ are hyperbolic. Therefore there exists $\epsilon_0 = d_H(P)$ such that eigenvalues of all perturbed Hermitian polynomials $(P + \Delta P)(z)$ for $|||\Delta P||| < \epsilon_0$, satisfy the eigenvalue distribution (P1) of Theorem 1.2.14 but there is a Hermitian polynomial $(P+\widehat{\Delta P})(z)$ with $|||\widehat{\Delta P}||| = \epsilon_0$ such that all its eigenvalues are real and of definite type but it does not satisfy the distribution (P1). This implies that $(P+\widehat{\Delta P})(z)$ has a real eigenvalue say λ_0 of definite type belonging to an interval $\mathcal{I}$ on the real line such that any eigenvalue of Hermitian polynomial $(P+\Delta P)(z)$ with $|||\Delta P||| < |||\widehat{\Delta P}|||$ belonging to $\mathcal{I}$ is also definite but differs in type with λ_0 as an eigenvalue of $(P + \widehat{\Delta P})(z)$. However this is not possible in view of Lemma 4.2.2 as Hermitian polynomials $(P+\Delta P)(z)$ with $|||\Delta P||| < |||\widehat{\Delta P}|||$ that are close enough to $(P+\widehat{\Delta P})(z)$ will have eigenvalues in $\mathcal{I}$ of the same type as that of λ_0.

Thus, a Hermitian polynomial nearest to $P(z)$ with respect to the norm $|||\cdot|||$ which is not hyperbolic either has an infinite eigenvalue, or a real eigenvalue of mixed type. $\square$

Since quasihyperbolic polynomials are Hermitian polynomials with all eigenvalues real, finite and of definite type, we have the following corollary of Theorem 4.3.1.

Corollary 4.3.2. *Given a quasihyperbolic polynomial $P(z)$, if*

$$d_Q(P) = \min\{|||\Delta P||| : (P+\Delta P)(z) \text{ is Hermitan but not quasihyperbolic }\}, \qquad (4.15)$$

then $d_Q(P)$ is the distance with respect to the $|||\cdot|||$ norm to a nearest Hermitian polynomial with an infinite eigenvalue or a finite real eigenvalue of mixed type.

The following result now establishes that the distance problems for hyperbolic and quasihyperbolic polynomials may be solved by observing the levels of the Hermitian ϵ-pseudospectrum of these polynomials.

Theorem 4.3.3. *Given a hyperbolic polynomial $P(z)$,*

$$d_H(P) = \min\{\sigma_{\min}(A_m), \epsilon_0\} \qquad (4.16)$$

where ϵ_0 is the smallest value of ϵ such that a component of the closure of $\Lambda_\epsilon^H(P)$ contains eigenvalues of $P(z)$ of both positive and negative type.

Similarly, if $P(z)$ is quasihyperbolic, then $d_Q(P) = \sigma_{\min}(A_m)$ if it has eigenvalues of only positive or only negative type. If the degree of $P(z)$ is 1 and it has eigenvalues of both positive and negative type or the degree is 2 or more, then (4.16) also holds for $d_Q(P)$.

Proof: Let $P(z)$ be a hyperbolic polynomial and $d_H(P) < \sigma_{\min}(A_m)$. If all the components of the closure $\Lambda_\epsilon^H(P)$ contain eigenvalues of $P(z)$ of only one type, either positive or negative, for all $\epsilon < \sigma_{\min}(A)$, then in view of Proposition 4.2.5, $(P+\Delta P)(z)$ will be a hyperbolic polynomial for all $\epsilon < \sigma_{\min}(A)$ implying that $d_H(P) \geq \sigma_{\min}(A_m)$. Therefore, there exists $0 < \epsilon < \sigma_{\min}(A_m)$ such that the closure of $\Lambda_\epsilon^H(P)$ has at least one component that contains eigenvalues of $P(z)$ of both positive and negative type. Let ϵ_0 be the smallest such ϵ and C_{ϵ_0} be a component of the closure of $\Lambda_{\epsilon_0}^H(P)$ containing eigenvalues of $P(z)$ of both positive and negative type. Therefore C_{ϵ_0} must be formed from the coalescence of components of $\Lambda^H_{\epsilon_0}(P)$ such that one of them contains eigenvalues of $P(z)$ of only positive type and the other contains eigenvalues of $P(z)$ of only negative type. Since $\epsilon_0 < \sigma_{\min}(A_m)$, these components are respectively, positive and negative type components of $\Lambda^H_{\epsilon_0}(P)$. By part (i) of Theorem 4.2.8, their point of coalescence is an eigenvalue of mixed type of a Hermitian polynomial $(P + \Delta P)(z)$ with $|||\Delta P||| = \epsilon_0$. Also, in view of Theorem 4.2.4, the eigenvalues of Hermitian polynomials $(P + \Delta P)(z)$ with $|||\Delta P||| < \epsilon_0$, are such that they are all of definite type and satisfy the distribution specified by property (P1) of Theorem 1.2.14. Therefore, $d_H(P) = \epsilon_0$ and the proof follows.

When $P(z)$ is quasihyperbolic with $m = 1$, it is a definitizable pencil and the proof follows from Theorem 3.6.7. If $P(z)$ has degree at least 2, then it has eigenvalues of both positive and negative type in view of Proposition 4.2.1 and the rest of the proof follows by arguing as in the case of the hyperbolic polynomials. □

Note 4.3.4. *Note that in view of Proposition 4.2.1, if a quasihyperbolic polynomial has eigenvalues of only one type then it is a definite pencil.*

Figure 4.1 shows that $\epsilon_0 = 0.0588$ for the hyperbolic polynomial $Q(z)$ considered in Example 4.2.10. Since this is less than $\sigma_{\min}(A) = 0.1005$, $d_H(Q) = 0.0588$. For the rotated polynomial $Q_2(z)$ considered in Example 4.2.12 which is quasihyperbolic, Figure 4.3 shows that $\epsilon_0 = 0.063$ while $\sigma_{\min}(A) = 0.54$. In this the case $d_Q(Q_1) = 0.063$.

4.3.2 Overdamped quadratics

As already defined in Theorem 1.2.15, the overdamped quadratics of size n form a subclass of the class of hyperbolic polynomials such that all the eigenvalues are real and nonpositive and divided into two disjoint groups of n eigenvalues each where the group on the left contains all the eigenvalues of negative type and the group on the right contains all the eigenvalues of positive type. The following result gives the distance to a nearest quadratic Hermitian polynomial that is not overdamped.

Theorem 4.3.5. *Given an overdamped quadratic polynomial $Q(z) = z^2A_2 + zA_1 + A_0$ of size n, if*

$$d_{OQ}(Q) = \min\{|||\Delta Q||| : (Q+\Delta Q)(z) \textit{ is a Hermitian polynomial but not overdamped }\},$$

then $d_{OQ}(Q) = \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0), \epsilon_0\}$ *where* $\lambda_{\min}(A_2)$ *and* $\lambda_{\min}(A_0)$ *denote the smallest eigenvalues of* A_2 *and* A_0 *respectively and* ϵ_0 *is the smallest value of* ϵ *for which a component of the closure of* $\Lambda_\epsilon(Q)$ *contains eigenvalues of* $Q(z)$ *of both positive and negative type.*

Proof: We observe that $d_{OQ}(Q) \leq \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0)\}$. First consider the case when $Q(z)$ has a zero eigenvalue. In such a case $\eta(\frac{\pi}{2}, Q) = \lambda_{\min}(A_0) = 0$ and given any $\epsilon > 0$, there exists $\theta < \frac{\pi}{2}$ close enough to $\frac{\pi}{2}$ such that $\eta(\theta, Q) < \epsilon$. Since $(\cos(\theta), \sin(\theta))$ represents a positive real number, it follows that $d_{OQ}(Q) < \epsilon$. Since $\epsilon > 0$ is arbitrarily chosen, we have that $d_{OQ}(Q) = 0$ in this case.

Next consider the case that all the eigenvalues of $Q(z)$ are strictly negative. Suppose that $d_{OQ}(Q) < \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0)\}$. If $\lambda_{\min}(A_2) \leq \epsilon_0 \leq \lambda_{\min}(A_0)$, then for any $\epsilon \in (0, \lambda_{\min}(A_2))$, no component of the closure of $\Lambda_\epsilon(Q)$ contains ∞. Also the eigenvalues of $Q(z)$ in any component are all of positive type or negative type. Therefore, by Theorem 4.2.5, $(Q + \Delta Q)(z)$ is an overdamped quadratic for all quadratic Hermitian polynomials $\Delta Q(z)$ such that $|||\Delta Q||| \leq \epsilon$. Since, $\epsilon \in (0, \lambda_{\min}(A_2))$, is arbitrary, it follows that $d_{OQ}(Q) = \lambda_{\min}(A_2) = \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0)\}$, which gives a contradiction. Similarly, the case $\lambda_{\min}(A_0) \leq \epsilon_0 \leq \lambda_{\min}(A_2)$ also leads to a contradiction. Thus we have, $\epsilon_0 < \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0)\}$ in this case. The definition of ϵ_0 implies that the closure of $\Lambda^H_{\epsilon_0}(Q)$ has a component say C_{ϵ_0}, that contains eigenvalues of $Q(z)$ of both positive and negative type and is formed from the coalescence of components of $\Lambda^H_\epsilon(Q)$ such that the eigenvalues of $Q(z)$ belonging to them are either of positive type or of negative type. Such a component also does not contain any infinite eigenvalues in its closure as $\epsilon_0 < \lambda_{\min}(A_2)$. Therefore, by Theorem 4.2.4, C_{ϵ_0} is formed by the coalescence of components of $\Lambda_{\epsilon_0}(Q)$ of positive and negative type and by Theorem 4.2.8, any point of coalescence of these components is an eigenvalue of mixed type of a Hermitian polynomial $(Q+\Delta Q)(z)$ where $|||\Delta Q||| = \epsilon_0$. The minimality of ϵ_0 and the fact that $\epsilon_0 < \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0)\}$ implies that for all Hermitian quadratic polynomials $(\Delta Q)(z)$ such that $|||\Delta Q||| < \epsilon$, $(Q + \Delta Q)(z)$ is an overdamped quadratic polynomial in view of Theorem 4.2.4. Thus we have $d_{OQ}(Q) = \epsilon_0 = \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0), \epsilon_0\}$.

If $d_{OQ}(Q) = \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0)\}$, then we claim that $\epsilon_0 \geq \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0)\}$. Indeed if this inequality does not hold and $\lambda_{\min}(A_0) \leq \lambda_{\min}(A_2)$, then $\epsilon_0 < \lambda_{\min}(A_0)$, and by arguing as above, the closure of $\Lambda^H_{\epsilon_0}(-\mathrm{rev}\, Q)$ has a component which does not contain ∞ and is formed from the coalescence of components of positive and negative type of $\Lambda^H_{\epsilon_0}(-\mathrm{rev}\, Q)$. Also, the minimality of ϵ_0 implies that the closure of $\Lambda^H_{\epsilon_0}(-\mathrm{rev}\, Q)$ does not have such a component for any $\epsilon < \epsilon_0$. Thus there exists a quadratic Hermitian polynomial $\Delta Q(z)$ with $|||\Delta Q||| = \epsilon_0$, such that $-\mathrm{rev}(Q + \Delta Q)(z)$ has a finite eigenvalue, say μ of mixed type. But this implies that $1/\mu$ is an eigenvalue of mixed type of $(Q + \Delta Q)(z)$. This implies that $d_{OQ}(Q) = \epsilon_0 < \lambda_{\min}(A_2) = \min\{\lambda_{\min}(A_2), \lambda_{\min}(A_0)\}$ leading to a contradiction. By similar arguments the case when $\lambda_{\min}(A_2) \leq \lambda_{\min}(A_0)$ also leads to a contradiction. This establishes our claim and completes the proof. □

4.3.3 Definite polynomials

Given a definite polynomial $P(z)$, a nearest Hermitian polynomial that is not definite is one that does not satisfy property (P1) of Theorem 1.2.16. Assuming the $P(z)$ is a definite polynomial, Theorem 4.2.2 implies that for small enough $\epsilon > 0$, Hermitian polynomials $(P + \Delta P)(z)$ satisfying $|||\Delta P||| < \epsilon$ are also definite. The following result shows that given a definite polynomial $P(z)$, a nearest Hermitian polynomial which is not definite is the distance to a nearest definite polynomial with an eigenvalue of mixed type.

Theorem 4.3.6. *Let $P(z)$ be a definite polynomial. Let*

$$d_D(P) = \min\{|||\Delta P||| : (P + \Delta P)(z) \text{ is Hermitian but not definite}\}. \tag{4.17}$$

Then $d_D(P)$ is the distance with respect to the $|||\cdot|||$ norm, to a Hermitian polynomial that has a complex eigenvalue or an eigenvalue of mixed type on the extended real line.

Proof: If the degree of $P(z)$ is one, then $P(z)$ is a definite pencil - a case which has already been dealt with in Chapter 3. So we assume without loss of generality that $m > 1$.

Statement (P1) of Theorem 1.2.16 implies that a nearest Hermitian polynomial that is not definite either has a complex eigenvalue, a real or infinite eigenvalue of mixed type or is such that all its eigenvalues are on the extended real line but they do not satisfy the eigenvalue distribution specified by (P1). Suppose that the last alternative holds and let $(P + \widehat{\Delta P})(z)$ be a Hermitian polynomial with $|||\widehat{\Delta P}||| = d_D(P)$ such that all its eigenvalues are on the extended real line and of definite type but they do not satisfy the eigenvalue distribution as described in (P1) of Theorem 1.2.16. In view of Lemma 4.2.2 for small enough $|||\Delta P|||$, all Hermitian polynomials $(P + \Delta P)(z)$ are definite with an eigenvalue distribution that is identical to that of $P(z)$. If the violation of this distribution by the eigenvalues of $(P+\widehat{\Delta P})(z)$ occurs due to a finite eigenvalue say, z_0, then all Hermitian polynomials $(P+\Delta P)(z)$ with $|||\Delta P||| < d_D(P)$ have eigenvalues in a neighbourhood of z_0 that differ in type from z_0. By arguing as in the proof of Theorem 4.3.1, it may be shown that this is not possible. If the eigenvalue distribution of $(P+\widehat{\Delta P})(z)$ does not satisfy (P1) on account of an eigenvalue at ∞ and $p \neq 0$ for $P(z)$, then its eigenvalue distribution is identical to that of $P(z)$ except for the eigenvalue at ∞. Thus it satisfies the following distribution:

$$\text{(a)}\quad \underbrace{\lambda_{mn} \leq \cdots \leq \lambda_{(m-1)n+p+1}}_{\substack{n-p \text{ eigenvalues} \\ \text{of } (-1)^{m-1}\epsilon \text{ type}}} < \cdots < \underbrace{\lambda_{jn+p} \leq \cdots \leq \lambda_{(j-1)n+p+1}}_{\substack{n \text{ eigenvalues of} \\ (-1)^{j-1}\epsilon \text{ type}, 1\leq j\leq m-1}} < \cdots < \underbrace{\lambda_{p} \leq \cdots \leq \lambda_{k+1}}_{\substack{p-k \text{ eigenvalues} \\ \text{of } -\epsilon \text{ type}}} < \underbrace{\infty}_{\epsilon \text{ type}}$$

where $1 \leq k < p$. If $p = 0$ for $P(z)$, then the eigenvalues of $(P+\widehat{\Delta P})(z)$ satisfy a distribution of the following type.

$$\text{(b)}\quad \underbrace{\lambda_{mn} \leq \cdots \leq \lambda_{(m-1)n+1}}_{\substack{n \text{ eigenvalues} \\ \text{of } (-1)^{m-1}\epsilon \text{ type}}} < \cdots < \underbrace{\lambda_{jn} \leq \cdots \leq \lambda_{(j-1)n+1}}_{\substack{n \text{ eigenvalues of} \\ (-1)^{j-1}\epsilon \text{ type}, 2\leq j\leq m-1}} < \cdots < \underbrace{\lambda_{n} \leq \cdots \leq \lambda_{k+1}}_{\substack{n-k \text{ eigenvalues} \\ \text{of } -\epsilon \text{ type}}} < \underbrace{\infty}_{\epsilon \text{ type}}$$

where $1 \leq k < n$. In the above distributions "$\alpha\epsilon$ type" denotes positive type when $\alpha\epsilon > 0$ and negative type otherwise.

Suppose that (a) holds and m is odd. Also suppose that the p largest eigenvalues of $P(z)$ are of positive type. As m is odd, the $mn-p$ smallest eigenvalues of $P(z)$ are of negative type and the same holds for all Hermitian polynomials $(P+\Delta P)(z)$ with $|||\Delta P||| < |||\widehat{\Delta P}|||$. Also, (a) implies that ∞ is an eigenvalue of $(P+\widehat{\Delta P})(z)$ of negative type of multiplicity k. Since m is odd, Lemma 4.2.2 implies that Hermitian polynomials $(P+\Delta P)(z)$ close enough to $(P+\widehat{\Delta P})(z)$ have k_1 positive eigenvalues of negative type and k_2 negative eigenvalues of positive type close to ∞ where $k_1+k_2 = k$. In particular, the same also holds for all such polynomials with $|||\Delta P||| < |||\widehat{\Delta P}|||$. But this violates the fact that any Hermitian polynomial $(P+\Delta P)(z)$ with $|||\Delta P||| < |||\widehat{\Delta P}|||$ is definite and has the same eigenvalue distribution type as $P(z)$. Identical arguments lead to the same contradiction for the case when the p largest eigenvalues are of negative type.

Next suppose that the eigenvalues of $(P+\widehat{\Delta P})(z)$ satisfy distribution (a) and m is even. Also suppose that the p largest and $mn-p$ smallest eigenvalues of $P(z)$ are of positive type. Then the same holds for the p largest and $mn-p$ smallest eigenvalues of Hermitian polynomials $(P+\Delta P)(z)$ with $|||\Delta P||| < |||\widehat{\Delta P}|||$ whereas ∞ is an eigenvalue of $(P+\widehat{\Delta P})(z)$ of negative type. By Lemma 4.2.2, all Hermitian polynomials close enough to $(P+\Delta P)(z)$ have k_1 positive and k_2 negative eigenvalues all of negative type close to ∞ with $k_1+k_2=k$. In particular, choosing Hermitian perturbations $(\Delta P)(z)$ such that $(P+\Delta P)(z)$ is close enough to $(P+\widehat{\Delta P})(z)$ and $|||\Delta P||| < |||\widehat{\Delta P}|||$, this violates the assumptions that the p largest and $mn-p$ smallest eigenvalues of such polynomials are of positive type. Once again, this contradicts the assumption that $|||\Delta P||| < |||\widehat{\Delta P}||| = d_D(P)$.

If $p=0$ for $P(z)$ then the eigenvalues distribution of $(P+\widehat{\Delta P})(z)$ satisfies (b). In this case also, using Lemma 4.2.2 and arguing as in the case when $p \neq 0$ leads to a contradiction.

Thus, it follows that a Hermitian polynomial nearest to $P(z)$ that is not definite either has a complex eigenvalue or an eigenvalue of mixed type on the extended real line. Hence the proof. □

We show that the distance $d_D(P)$ may be read off from the Hermitian ϵ-pseudospectrum of $P(z)$ under certain conditions. For this we restate Theorem 1.2.17 in terms of eigenangles. Recall that for each $k=1,2,\ldots,mn$, the eigenvalue λ_k of $P(z)$ may be uniquely represented by the pair $(\cos\theta_k, \sin\theta_k)$ where $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{mn} < \pi$. Therefore, Theorem 1.2.17 takes the following form.

Theorem 4.3.7. *Let $P(z)$ be a definite polynomial with eigenvalues satisfying property (P1) of Theorem 1.2.16. Let*

$$\mathcal{I}_j = (\theta_{jn+p}, \theta_{jn+p+1}),\ j=1,2,\ldots,m-1,$$

and

$$\mathcal{I}_0 = \begin{cases} (\theta_p, \theta_{p+1}) & \text{if } p \neq 0 \\ (\theta_{mn}-\pi, \theta_1) & \text{if } p = 0 \end{cases}.$$

Then $P(\theta) = \Sigma_{k=0}^{m} (\cos\theta)^k (\sin\theta)^{m-k} A_k$ is definite for all $\theta \in \mathcal{I}_j$, $j=0,1,\ldots,m-1$. Also if $\phi_j \in \mathcal{I}_j$, and $\phi_{j+1} \in \mathcal{I}_{j+1}$, then $P(\phi_j)$ and $P(\phi_{j+1})$ are of opposite parity.

Proof: Observe that $P(z) = (\sin\theta)^{-m} P(\theta)$ for all $\theta \in (0,\pi)$. The proof for the case when $p \neq 0$ is obvious as $\mathcal{I}_j \subset (0,\pi)$ for all $j=0,1,\ldots,m-1$. Suppose that $p=0$. Also assume without loss of generality that $P(z)$ is positive definite for $z \in (\lambda_1, \infty)$. If m is odd then $P(z)$ is negative definite for $z \in (-\infty, \lambda_{mn})$. Therefore $P(\theta)$ is positive definite for $(0,\theta_1)$. Since $\sin\theta < 0$ for $\theta \in (\theta_{mn}-\pi, 0)$ and m is odd, $P(\theta)$ is also positive definite for $\theta \in (\theta_{mn}-\pi, 0)$. Since $p=0$, ∞ is not an eigenvalue of $P(z)$ and hence $P(\theta)$ is positive definite for $\theta \in (\theta_{mn}-\pi, \theta_1)$.

If m is even, and $P(z)$ is positive definite for $z \in (\lambda_1, \infty)$, then $P(z)$ is also positive definite for $z \in (-\infty, \lambda_{mn})$. Hence $P(\theta)$ is also positive definite for $\theta \in (\theta_{mn} - \pi, \theta_1)$. Hence the proof. $\square$

Theorem 4.3.8. *Let $P(z)$ be a definite polynomial of degree $m \geq 2$ such that any one of the following conditions hold.*

(1) All the eigenvalues of $P(z)$ are finite and $d_D(P) < \sigma_{\min}(A_m)$.

(2) $P(z)$ is such that $p \neq 0$ and $0 \in [\lambda_{(m-1)n+p-1}, \lambda_{p+1}]$.

Then for $S_j = \{(\cos\theta, \sin\theta) : \theta \in \mathcal{I}_j\}$, $j = 0, 1, ..., m-1$, where intervals $\mathcal{I}_j$, $j = 0, 1, ..., m-1$, are as defined in Theorem 4.3.7,

$d_D(P) = \min\{\epsilon : S_j$ *belongs to the closure of* $\Lambda_\epsilon^H(P)$ *for some* j *satisfying* $0 \leq j \leq m-1\}$

and is given by

$$d_D(P) = \min_{0 \leq j \leq m-1} \max_{\theta \in \mathcal{I}_j} \frac{\sigma_{\min}(P(\theta))}{\sqrt{\Sigma_{k=0}^m (\cos\theta)^{2(m-k)} (\sin\theta)^{2k}}}.$$

where the intervals I_j, $j = 0, 1, \ldots, m-1$ are given by Theorem 4.3.7.

Proof: If $P(z)$ satisfies (1) then since it has no eigenvalues at ∞ and $d_D(P) < \sigma_{\min}(A_m)$, by arguments identical to those used in the proof of Theorem 4.3.3 it follows that $d_D(P)$ is the smallest value of ϵ such that the closure of $\Lambda_\epsilon^H(P)$ contains an eigenvalue of $P(z)$ of both positive and negative type. Since ∞ does not belong to such a component, this implies that such a component is formed from the coalescence of components of $\Lambda_\epsilon^H(P)$ such that one of them contains eigenvalues from the group $\{\lambda_{j_0 n+p}, \lambda_{j_0 n+p+1}, \ldots, \lambda_{(j_0-1)n+p+1}\}$ and the other from an adjacent group where $2 \leq j_0 \leq m-1$, if $p = 0$ and $1 \leq j_0 \leq m-1$ if $p \neq 0$. The gap between these two groups of eigenvalues which is one of the sets S_{j_0} or S_{j_0-1} is then a subset of the component. The minimality of ϵ, ensures that no other set S_j belongs to $\Lambda_\epsilon^H(P)$ for any smaller value of ϵ. Therefore, in this case,

$$d_D(P) = \min\{\epsilon : S_j \text{ is a subset of the closure of } \Lambda_\epsilon^H(P), 0 \leq j \leq m-1\}$$

with the corresponding j being nonzero for the minimal value of ϵ in the case when $p = 0$.

Suppose that $P(z)$ satisfies statement (2) and is of even degree. We first establish that if C_ϵ is a component of $\Lambda_\epsilon^H(P)$ containing k eigenvalues of $P(z)$ of positive (negative) type counting multiplicities, then the component is of positive (negative) type and each Hermitian polynomial $(P+\Delta P)(z)$ for $|||\Delta P||| < \epsilon$ has k eigenvalues of positive (negative) type in C_ϵ counting multiplicities. If ∞ does not belong to the closure of C_ϵ, then the proof follows from Theorem 4.2.4. Suppose that ∞ belongs to the closure of C_ϵ and assume without loss of generality that $P(z)$ has k eigenvalues in C_ϵ of positive type counting multiplicities. Since $p \neq 0$, these eigenvalues are a subset of $\{\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{(m-1)n+p+1}, \lambda_{(m-1)n+p+2}, \ldots, \lambda_{mn}\}$. Also since $0 \in [\lambda_{(m-1)n+p-1}, \lambda_{p+1}]$, therefore 0 is not in the closure of C_ϵ. This implies that the corresponding component $\hat{C}_\epsilon$ of $\Lambda_\epsilon^H(-\text{rev}\, P)$ which contains the reciprocals of the k eigenvalues of $P(z)$ in C_ϵ does not contain ∞ in its closure. These reciprocals are the k eigenvalues of $-\text{rev}\, P(z)$ in $\hat{C}_\epsilon$ and since m is even, the relation (1.3) implies that they are eigenvalues $-\text{rev}\, P(z)$ of positive type. Therefore applying Theorem 4.2.4, it follows that $\hat{C}_\epsilon$ is a component of positive type of $\Lambda_\epsilon^H(-\text{rev}\, P)$. Therefore the Hermitian polynomials $-\text{rev}\,(P+\Delta P)(z)$ have k eigenvalues of positive type in $\hat{C}_\epsilon$ counting multiplicity for $|||\Delta P||| < \epsilon$. Since m is even, once again by (4.6), $(P+\Delta P)(z)$ has k eigenvalues of positive type in C_ϵ for $|||\Delta P||| < \epsilon$ so that C_ϵ is a component of $\Lambda_\epsilon^H(P)$ positive type in particular.

From the above arguments it follows that if C_ϵ is a component of $\Lambda_\epsilon^H(P)$ or its closure such that all the eigenvalues of $P(z)$ belonging to C_ϵ are of positive (negative) type, then it is a component of positive (negative) type. Moreover, in view of Theorem 4.2.4, the sum of the multiplicities of the eigenvalues of $(P+\Delta P)(z)$ in C_ϵ is constant for $|||\Delta P||| < \epsilon$. Thus any Hermitian polynomial $(P+\Delta P)(z)$ with $|||\Delta P||| \leq \epsilon$ is definite unless the closure of $\Lambda_\epsilon^H(P)$ has a component containing eigenvalues of $P(z)$ of both positive and negative type. If ϵ_0 be the smallest value of ϵ for which the closure of $\Lambda_\epsilon^H(P)$ has such a component, then it is formed from the coalescence of components say $C_{\epsilon_0}^1$ and $C_{\epsilon_0}^2$ of $\Lambda_\epsilon^H(P)$ such that $C_{\epsilon_0}^1$ contains only positive type eigenvalues of $P(z)$ and $C_{\epsilon_0}^2$ contains only negative type eigenvalues of $P(z)$. But then $C_{\epsilon_0}^1$ and $C_{\epsilon_0}^2$ are components of $\Lambda_{\epsilon_0}^H(P)$ of positive and negative type respectively and by Theorem 4.2.8 any point of coalescence is an eigenvalue of mixed type of a certain Hermitian polynomial $(P+\Delta P)(z)$ with $|||\Delta P||| = \epsilon_0$. Since $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ and $\{\lambda_{(m-1)n+p+1}, \lambda_{(m-1)n+p+2}, \ldots, \lambda_{mn}\}$ are of the same type, one of the components among $C_{\epsilon_0}^1$ and $C_{\epsilon_0}^2$ contain eigenvalues from the group $\{\lambda_{j_0 n+p}, \lambda_{j_0 n+p+1}, \ldots, \lambda_{(j_0-1)n+p+1}\}$ $1 \leq j_0 \leq m-1$, and the other from any group adjacent to it. Thus the corresponding gap which is either S_{j_0} or S_{j_0-1} is evidently a subset of the component formed from the coalescence of $C^1_{\epsilon_0}$ and $C^2_{\epsilon_0}$. Also, the minimality of ϵ_0 ensures that no other interval S_j belongs to any component of the closure of $\Lambda^H_\epsilon(P)$ for smaller values of ϵ. Thus we have

$$d_D(P) = \min\{\epsilon : S_j \subset \text{ closer of } \Lambda^H_\epsilon(P), 0 \leq j \leq m-1\}$$

in this case also.

Next suppose that $P(z)$ satisfies statement (2) and is of odd degree. In this case property (P1) of Theorem 1.2.16 implies that the eigenvalues of $P(z)$ are grouped into $m+1$ sets $E_j, j = 0, 1, \ldots, m$, where $E_0 = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$, $E_j = \{\lambda_{jn+p}, \lambda_{jn+p+1}, \ldots, \lambda_{(j-1)n+p+1}\}$, $1 \leq j \leq m-1$ and $E_m = \{\lambda_{(m-1)n+p+1}, \lambda_{(m-1)n+p+2}, \ldots, \lambda_{mn}\}$. We show that $d_D(P)$ is the smallest value of ϵ for which the closure of $\Lambda^H_\epsilon(P)$ contains at least one eigenvalue each from E_j and E_{j-1} for $j \in \{1, 2, \ldots, m\}$. Let this value be ϵ_0. If each of the components of the closure of $\Lambda^H_\epsilon(P)$ contain eigenvalues from only one of these groups and $\epsilon < \sigma_{\min}(A_m)$, then Theorem 4.2.4 implies that $\epsilon < d_D(P)$. Let $\sigma_{\min}(A_m) \leq \epsilon < \epsilon_0$. Then there exists a component of $\Lambda^H_\epsilon(P)$ say C_ϵ such that ∞ belongs to the closure of C_ϵ and C_ϵ contains eigenvalues from at least one or both of E_0 and E_m but none from $E_j, 1 \leq j \leq m-1$. Since m is odd and $0 \in [\lambda_{(m-1)n+p-1}, \lambda_{p+1}]$, the eigenvalue groups E_0 and E_m differ in sign as well as in type. Therefore, (4.6) implies that the groups $\{1/\lambda_1, \ldots, 1/\lambda_p\}$ and $\{1/\lambda_{(m-1)n+p+1}, 1/\lambda_{(m-1)n+p+2}, \ldots, 1/\lambda_{mn}\}$ are eigenvalues of $-\text{rev}\, P(z)$ of the same type. Without loss of generality suppose that they are all of positive type. If $\hat{C}_\epsilon$ be the component of $\Lambda^H_\epsilon(-\text{rev}\, P)$ corresponding to C_ϵ, then all the eigenvalues of $-\text{rev}\, P(z)$ in $\hat{C}_\epsilon$ are of positive type. Let these be k in number, counting multiplicities. Also ∞ does not belong to the closure of $\hat{C}_\epsilon$, as $0 \in [\lambda_{(m-1)n+p-1}, \lambda_{p+1}]$. Therefore all perturbed Hermitian polynomials $(P + \Delta P)(z)$ with $|||\Delta P||| < \epsilon$ are such that $-\text{rev}\,(P + \Delta P)(z)$ has k eigenvalues of positive type in $\hat{C}_\epsilon$. In particular if 0 is an eigenvalue, then it is of positive type. If $\hat{C}_\epsilon$ contains both positive and negative numbers, then the same is the case with C_ϵ leading to the possibility that $(P + \Delta P)(z)$ has k_1 eigenvalues of positive type and k_2 eigenvalues of negative type in C_ϵ where $k_1 + k_2 = k$. In particular, C_ϵ does not contain any eigenvalues of mixed type of Hermitian polynomials $(P + \Delta P)(z)$ with $|||\Delta P||| \leq \epsilon$. Since $\epsilon < \epsilon_0$, the other components of the closure of $\Lambda^H_\epsilon(P)$ contain eigenvalues of $P(z)$ of only positive or only negative type and Theorem 4.2.4 implies that all Hermitian pencils $(P + \Delta P)(z)$ with $|||\Delta P||| < \epsilon$ have a distribution that satisfies (P1) with the possibility that $p \neq 0$ for some of them. This proves that $\epsilon_0 \leq d_D(P)$.

To prove the equality let $\hat{C}_{\epsilon_0}$ be a component of the closure of $\Lambda^H_{\epsilon_0}(P)$ containing an eigenvalue from one of the groups E_j and E_{j-1} for $j \in \{1, 2, \ldots, m\}$. If $\infty \notin \hat{C}_{\epsilon_0}$ then the components of $\Lambda^H_{\epsilon_0}(P)$ that coalesce to form $\hat{C}_{\epsilon_0}$ are of positive and negative type so that any point of coalescence is an eigenvalue of mixed type of some Hermitian polynomial $(P + \Delta P)(z)$ with $|||\Delta P||| = \epsilon_0$. Otherwise, $\hat{C}_{\epsilon_0}$ is formed from the coalescence of components of $\Lambda^H_{\epsilon_0}(P)$ such that one of them (which contains ∞ in its closure) is possibly not of positive or negative type. However, as shown above, it does not contain any eigenvalues of mixed type of perturbed Hermitian pencils $(P + \Delta P)(z)$ with $|||\Delta P||| < \epsilon_0$. Therefore, $\hat{C}_{\epsilon_0}$ is a subset of the extended real line. Thus if $(\cos\theta_0, \sin\theta_0)$ represents a corresponding point of coalescence of components of $\Lambda^H_{\epsilon_0}(P)$, that result in $\hat{C}_{\epsilon_0}$, then θ_0 is a point at which the backward error $\eta(\theta, P)$ has a maximum. Note that $\theta_0 \in S_j$ where $j \in \{0, m-1\}$. Theorem 4.3.7 implies that $\eta(\theta, P) = g(\theta)$ if $P(\theta)$ is positive definite and $\eta(\theta, P) = -g(\theta)$ if $P(\theta)$ is negative definite for $\theta \in S_j, j \in \{0, m-1\}$, where $g(\theta) = \frac{\lambda_{\min}(\theta)}{\sqrt{\Sigma_{k=0}^m (\cos\theta)^{2(m-k)}(\sin\theta)^{2k}}}$. Therefore, $g(\theta)$ has a local extremum at θ_0. If $g'(\theta_0)$ exists then it is zero and by Lemma 4.2.7, there exists a Hermitian polynomial $\Delta P(z)$ with $|||\Delta P||| = \eta(\theta_0, P)$ such that $(P + \Delta P)(z)$ has an eigenvalue of mixed type. If $g'(\theta_0)$ does not exist, then there exists $\delta > 0$, such that $g'(\theta)$ exists and differs in sign for $\theta \in (\theta_0 - \delta, \theta_0)$ and $\theta \in (\theta_0, \theta_0 + \delta)$. By arguing as in the proof of Theorem 4.2.8, in this case also there exists a Hermitian polynomial $\Delta P(z)$ with $|||\Delta P||| = \eta(\theta_0, P)$ such that $(P + \Delta P)(z)$ has an eigenvalue of mixed type. The minimality of ϵ_0 ensures that $d_D(P) = \epsilon_0$. Also, clearly ϵ_0 is the smallest value of ϵ such that the closure of $\Lambda^H_\epsilon(P)$ contains at least one of the sets S_j, $0 \le j \le m-1$.

In all the cases, there is a point of coalescence of components of $\Lambda^H_\epsilon(P)$ represented by $(\cos\theta_0, \sin\theta_0)$ where $\theta_0 \in \mathcal{I}_{j_0}$, $0 \le j_0 \le m-1$ which results in an eigenvalue of mixed type for a nearest Hermitian polynomial that is not definite. Also, the coalescing components are subsets of the extended real line and θ_0 is a point at which $\eta(\theta, P)$ attains its maximum for $\theta \in \mathcal{I}_{j_0}$. The minimality of the coalescing level, implies that

$$\eta(\theta_0, P) = \min_{0 \le j \le m-1} \max_{\theta \in \mathcal{I}_j} \eta(\theta, P)$$

where the outer minimization is attained by some $j \in \{1, 2, \ldots, m-1\}$ if $P(z)$ satisfies (1) and $p = 0$. The proof now follows from the fact that

$$\eta(\theta, P) = \frac{\sigma_{\min}(\theta)}{\sqrt{\Sigma_{k=0}^m (\cos\theta)^{2(m-k)}(\sin\theta)^{2k}}}. \qquad \square$$

We note that the definite polynomial $Q_2(z)$ considered in Example 4.2.12 satisfies condition (2) of Theorem 4.3.8. As shown by Figure 4.3, the level $\epsilon = 0.063$ gives the smallest value of ϵ for which the closure of $\Lambda_\epsilon^H(Q_2)$ contains an eigenvalue of both positive and negative type. Hence $d_D(Q_2) = 0.063$.

We give an upper bound on $d_D(P)$ for the case when $P(z)$ is an arbitrary definite polynomial.

Theorem 4.3.9. *Let $P(z)$ be a definite polynomial and intervals $\mathcal{I}_j$, $j = 0, 1, \ldots, m$, be as in Theorem 4.3.7. Then*

$$d_D(P) \leq \min_{0\leq j\leq m-1} \max_{\theta\in\mathcal{I}_j} \frac{\sigma_{\min}(\theta)}{\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}}}. \tag{4.18}$$

Proof: Since $\eta(\theta, P) = \frac{\sigma_{\min}(\theta)}{\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}}}$, we have either $\eta(\theta, P) = g(\theta)$ or $\eta(\theta, P) = -g(\theta)$ according as whether $P(\theta)$ is positive definite or negative definite for each $\theta \in \mathcal{I}_j$, $j = 0, 1, \ldots, m-1$ where $g(\theta) = \frac{\lambda_{\min}(\theta)}{\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}}}$. Therefore $g(\theta)$ has a local extremum at any point $\theta_0 \in \mathcal{I}_j$ where $\eta(\theta_0, P) = \max_{\theta\in\mathcal{I}_j} \eta(\theta, P)$. Since $\eta(\theta, P) = 0$ at each boundary point of $\mathcal{I}_j$, it is clear that there exists at least one such $\theta_0 \in \mathcal{I}_j$ for each $j = 0, 1, \ldots, m-1$. If $g'(\theta_0)$ exists then it is zero and by Lemma 4.2.7, there exists a Hermitian polynomial $\Delta P(z)$ with $|||\Delta P||| = \eta(\theta_0, P)$ such that $(P+\Delta P)(z)$ has an eigenvalue of mixed type. If $g'(\theta_0)$ does not exist, then there exists $\delta > 0$, such that $g'(\theta)$ exists and differs in sign for $\theta \in (\theta_0 - \delta, \theta_0)$ and $\theta \in (\theta_0, \theta_0 + \delta)$. By arguing as in the proof of Theorem 4.2.8, in this case also there exists a Hermitian polynomial $\Delta P(z)$ with $|||\Delta P||| = \eta(\theta_0, P)$ such that $(P + \Delta P)(z)$ has an eigenvalue of mixed type. Hence we have,

$$d_D(P) \leq \min_{0\leq j\leq m-1} \max_{\theta\in\mathcal{I}_j} \eta(\theta, P) \tag{4.19}$$

and the proof follows. $\square$

Theorem 4.3.8 shows that equality holds in (4.19) for definite polynomials that satisfy the conditions (1) and (2). As already proved in Chapter 3, the inequality (4.19) is also an equality for $m = 1$ in which case $P(z)$ is a definite pencil. We conjecture that the same also holds for all definite polynomials.

Conjecture 4.3.10. *Let $P(z)$ be a definite polynomial and intervals $\mathcal{I}_j$, $j = 0, 1, \ldots, m$, be as in Theorem 4.3.7. Then*

$$d_D(P) = \min_{0 \le j \le m-1} \max_{\theta \in \mathcal{I}_j} \frac{\sigma_{\min}(\theta)}{\sqrt{\Sigma_{k=0}^{m} (\cos\,\theta)^{2(m-k)} (\sin\,\theta)^{2k}}}.$$

Chapter 5

Computing and estimating solutions of distance problems

5.1 Introduction

In this chapter we use the results of the previous chapters to design algorithms to compute the Crawford number of a definite pencil and the distance from a given hyperbolic polynomial to a nearest Hermitian polynomial that is not hyperbolic. Each bisection step of these algorithms finds the smallest eigenvalue(s) of a positive definite matrix or the largest eigenvalue(s) of a negative definite matrix of size n and the corresponding eigenvectors. These algorithms also produce a nearest Hermitian pencil or polynomial outside the respective class. Several numerical experiments are also performed to illustrate these algorithms for the definite pencils and quadratic hyperbolic polynomials. For the case of the Crawford number these include important classes of definite pencils previously considered in the literature and comparisons are made with algorithms proposed in [29, 57] and the latest algorithm proposed in [55, 56]. In particular, these experiments show that the proposed algorithm produces a very good lower bound in a few iterations.

Similar algorithms are also suggested for solving other distance problems discussed in the previous chapters. It may be mentioned here, that there are algorithms [29, 25] that computes the distance from a given quadratic hyperbolic polynomial to a nearest Hermitian matrix polynomial that is non hyperbolic. This distance is computed with respect to the norm $|||Q|||_F = \sqrt{4\|A\|_F^2 + 2\|B\|_F^2 + 4\|C\|_F^2}$ where $Q(z) = z^2A + zB + C$ and $\|\cdot\|_F$ denotes the Frobenius norm of a matrix in [29]. A slight variation of this set up is considered in [25].

Finally we also obtain bounds on the Crawford number. Some of these bounds relate the Crawford number of the distribution of the eigenvalues of the definite pencil. We also give a pseudospectra based derivation of an eigenvalue perturbation bound for definite pencils via the Crawford number on the lines of a similar result in [47]. As in the previous chapters, $L(z) = zA - B$ will denote a Hermitian pencil of size n and $P(z) = \Sigma_{k=0}^{m} z^k A_k$ will denote a Hermitian polynomial of size n and degree m. We will also use the notation $Q(z) = z^2 A_2 + zA_1 + A_0$ to denote a quadratic Hermitian polynomial of size n.

5.2 Computing the Crawford number and other distances

Let $L(z)$ be a definite pencil with its n real eigenvalues ordered as follows.

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1.$$

As already discussed in Section 3.6, for the purpose of computing the Crawford number, we may assume without loss of generality that all the eigenvalues of $L(z)$ are finite as otherwise $L(z)$ may be homogeneously rotated so that this is the case. Therefore, each of these eigenvalues is associated with a unique angle $\theta \in [0, \pi]$ which we refer to as an eigenangle of the pencil $L(z)$. If θ_k be the eigenangle associated with the eigenvalue λ_k for $k = 1, 2, \ldots, n$, then

$$0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n < \pi.$$

Recall that by Theorem 3.6.1, the Crawford number is the maximum of $\sigma_{\min}(A_\theta)$ where $A_\theta = A\cos\theta - B\sin\theta$ in an interval $\mathcal{I}$ where

$$\mathcal{I} = \begin{cases} [\theta_k, \theta_{k+1}], \ 1 \leq k \leq n-1 & \text{if } A \text{ is indefinite} \\ [\theta_n - \pi, \theta_1] & \text{if } A \text{ is definite} \end{cases}.$$

Also since A_θ is either positive or negative definite in each case, the Crawford number is a local extremum of the eigenvalue A_θ with the smallest modulus which we denote by $\lambda_{\min}(A_\theta)$. We propose the following algorithm for computing the Crawford number $\gamma(A, B)$ and a nearest Hermitian pencil that is not definite in view of Theorem 3.5.8 and Notes 3.6.2 and 3.6.3.

Algorithm 1:

Step 1. Compute the eigenvalues of $L(z)$ and make a homogeneous rotation if necessary so that all the eigenvalues are finite. Determine the types of the eigenvalues of the rotated pencil and the corresponding eigenangles.

Step 2. If there exists $k \in \{1, 2, \ldots, n-1\}$ such that λ_k and λ_{k+1} are not of the same type set $\theta_l = \theta_k$ and $\theta_r = \theta_{k+1}$. Else set $\theta_l = \theta_n - \pi$ and $\theta_r = \theta_1$.

Step 3. Compute $\theta_m = (\theta_l + \theta_r)/2$.

Step 4. Compute $\lambda_{\min}(A_\theta)$ and the matrix V_θ whose columns span an orthonormal basis of $\mathrm{Ker}\,(A_\theta - \lambda_{\min}(A_\theta)I_n)$ for $\theta = \theta_l$, $\theta = \theta_r$ and $\theta = \theta_m$.

Step 5. Compute $S_\star = V_{\theta_\star}^*(A \sin\,\theta_\star + B \cos\,\theta_\star)V_{\theta_\star}$ for each $\star \in \{l, r, m\}$ and set

$$s_\star = \begin{cases} 1 & \text{if } S_\star \text{ is positive definite} \\ -1 & \text{if } S_\star \text{ is negative definite} \\ 0 & \text{otherwise.} \end{cases}$$

Step 6. If $s_m = 0$ or $\theta_r - \theta_l < tol_1$ then set $\gamma(A, B) = |\lambda_{\min}(A_{\theta_m})|$. Else go to **Step 9.**

Step 7. If $s_m = 0$, compute $x \in \mathbb{C}^m$ with $\|x\|_2 = 1$ and such that

$$x^* V_{\theta_m}^*(A \sin\,\theta_m + B \sin\,\theta_m)V_{\theta_m}x < tol_2$$

where $m = \dim\,\mathrm{Ker}(A_{\theta_m} - \lambda_{\min}(A_{\theta_m})I_n)$ and set $v_m = V_{\theta_m}x$. Else compute $x \in \mathbb{C}^2$ with $\|x\|_2 = 1$ such that

$$x^*[v_l\, v_r]^*(A \sin\,\theta_m + B \cos\,\theta_m)[v_l\, v_r]x < tol_2$$

where $A_{\theta_l}v_l = \lambda_{\min}(A_{\theta_l})v_l$, $A_{\theta_r}v_r = \lambda_{\min}(A_{\theta_r})v_r$, with $\|v_l\|_2 = \|v_r\|_2 = 1$ and set $v_m = [v_l\, v_r]x$. If such x does not exist, lower tol_1 and repeat Steps 1-6.

Step 8. Set $(L + \Delta L)(\alpha, \beta) = \alpha(A + \Delta A) - \beta(B + \Delta B)$ where

$$\Delta A = -\cos\,\theta_m\, \lambda_{\min}(A_{\theta_m})v_m v_m^* \text{ and } \Delta B = \sin\,\theta_m\, \lambda_{\min}(A_{\theta_m})v_m v_m^*$$

and exit.

Step 9. If $s_l s_m < 0$, then set $\theta_r = \theta_m$. Else, set $\theta_l = \theta_m$. Repeat **Step 3-7.**

In the above algorithm, tol_1 and tol_2 are tolerances set by the user. Steps 7 and 8 may be skipped if only $\gamma(A, B)$ is desired. Also, in most cases, the matrices $V_{\theta_l}, V_{\theta_m}$ and V_{θ_r} will be a single vector making it easy to calculate the quantities s_l, s_r and s_m in **Step 5** and the vector x in **Step 7.**

If $L(z)$ be a definitizable pencil, with all eigenvalues of either positive or negative type, then as already shown in Theorem 3.6.7, $d_D(L) = \sigma_{\min}(A)$. If $L(z)$ has eigenvalues of both positive and negative types, then computing the distance to a nearest Hermitian pencil that is not definitizable with respect to the norm $|||\cdot|||$, will involve finding the maximum of $\sigma_{\min}(A_\theta)$ in a finite number of intervals of the real line of the form $[\theta_k, \theta_{k+1}]$, $1 \leq k \leq n-1$, where the corresponding eigenvalues λ_k and λ_{k+1} are not of the same type. Let $I_1, I_2, \ldots, I_p$ be these intervals. Each interval gives a positive real number ϵ_j, $j = 1, 2, \ldots, p$, where $\epsilon_j = \max_{\theta \in I_j} \sigma_{\min}(A_\theta)$. Let $\epsilon_0 = \min_{1 \leq j \leq p} \epsilon_j$. If $d_D(L) < \sigma_{\min}(A)$, this gives the smallest value of ϵ such that a positive type component of $\Lambda_\epsilon^H(L)$ coalesces with a negative type component of $\Lambda_\epsilon^H(L)$ and gives the distance to a nearest Hermitian pencil with a finite eigenvalue of mixed type. Thus, the procedure for calculating $d_D(L)$ in this case, involves computing ϵ_0 and finding the smaller of ϵ_0 and $\sigma_{\min}(A)$.

A similar procedure leads to the distance with respect to the $|||\cdot|||$ from a hyperbolic polynomial $P(z)$ to a nearest Hermitian polynomial that is not hyperbolic. In this case we assume that the mn real eigenvalues of $P(z)$ satisfy the ordering

$$\lambda_{mn} \leq \lambda_{mn-1} \leq \cdots \leq \lambda_2 \leq \lambda_1 < \infty.$$

and the corresponding eigenangles satisfy the ordering

$$0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{mn-1} \leq \theta_{mn} < \pi.$$

Setting $I_j = [\theta_{jn}, \theta_{jn+1}]$, $j = 1, 2, \ldots, m-1$, if $d_H(P) < \sigma_{\min}(A_0)$, then $d_H(P) = \epsilon_0$ where

$$\epsilon_0 = \min_{1 \leq j \leq m-1} \max_{\theta \in I_j} \frac{\sigma_{\min}(P(\theta))}{\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)} (\sin\,\theta)^{2k}}},$$

with $P(\theta) = \Sigma_{k=0}^m A_k (\cos\,\theta)^k (\sin\,\theta)^{m-k}$. Since $P(\theta)$ is either positive definite or negative definite in each interval I_j, the inner maximum is obtained by finding either a maximum of $g(\theta) = \dfrac{\lambda_{\min}(P(\theta))}{\sqrt{\Sigma_{k=0}^m (\cos\,\theta)^{2(m-k)} (\sin\,\theta)^{2k}}}$ if $P(\theta)$ is positive definite or a minimum of $g(\theta)$ and taking its absolute value if $P(\theta)$ is negative definite. The extremum of $g(\theta)$ in each of the intervals I_j, $j = 1, 2, \ldots, m-1$ may be obtained by applying a modification of Algorithm 1 to each of these intervals. We propose the following algorithm in view of the proof of part (i) of Theorem 4.2.8.

Algorithm 2:

Step 1. Calculate the eigenvalues of $P(z)$ and their types. Also find the corresponding eigenangles.

Step 2. For $j = 1, 2, \ldots, m-1$, repeat the following steps:

(i) Set $\theta_l^{(j)} = \theta_{jn}, \theta_r^{(j)} = \theta_{jn+1}$ and $\theta_{mid}^{(j)} = \frac{\theta_r^{(j)} + \theta_l^{(j)}}{2}$.

(ii) Calculate the eigenvalue $\lambda_{\min}(P(\theta))$ of $P(\theta)$ of smallest modulus and a matrix V_θ whose columns span an orthonormal basis of $\mathrm{Ker}\,(P(\theta) - \lambda_{\min}(P(\theta))I_n)$ for $\theta = \theta_l^{(j)}, \theta_r^{(j)}, \theta_{mid}^{(j)}$.

(ii) Form the polynomial $\Delta P^\star(\alpha, \beta) = \Sigma_{k=0}^m \alpha^k \beta^{m-k} \Delta A_k^\star$ where

$$\Delta A_k^\star = \frac{c_\star^k s_\star^{m-k} \lambda_{\min}(P(\theta_\star))}{\Sigma_{k=0}^m c_\star^{2(m-k)} s_\star^{2k}} V_{\theta_\star} V_{\theta_\star}^*,\ k = 0, 1, \ldots, m-1$$

and the matrices

$$S_\star = V_{\theta_\star}^* \left[s_\star \frac{\partial(P + \Delta P^\star)}{\partial c_\star}(c_\star, s_\star) - c_\star \frac{\partial(P + \Delta P^\star)}{\partial s_\star}(c_\star, s_\star) \right] V_{\theta_\star}$$

where $c_\star = \cos\, \theta_\star^{(j)}, s_\star = \sin\, \theta_\star^{(j)}$ for each $\star \in \{l, r, mid\}$.

(iv) Compute

$$s_\star = \begin{cases} 1 & \text{if } S_\star \text{ is positive definite} \\ -1 & \text{if } S_\star \text{ is negative definite} \\ 0 & \text{otherwise} \end{cases}$$

for each $\star \in \{l, r, mid\}$.

(v) If $s_{mid} = 0$ or $\theta_r^{(j)} - \theta_l^{(j)} < tol_1$ set

$$\epsilon_j = |g(\theta_{mid}^{(j)})|$$

and exit. Else set

$$\begin{aligned} \theta_r^{(j)} &= \theta_{mid}^{(j)} \text{ if } s_l s_{mid} < 0, \\ \theta_l^{(j)} &= \theta_{mid}^{(j)} \text{ if } s_{mid} s_r < 0 \end{aligned}$$

$$\theta_{mid}^{(j)} = \frac{\theta_l^{(j)} + \theta_r^{(j)}}{2}$$

and repeat **Steps (ii) - (v)**.

Step 3. Set $\epsilon_0 = \min_{1 \leq j \leq m-1} \epsilon_j$ and $d_H(P) = \min\{\epsilon_0, \sigma_{\min}(A_0)\}$.

Step 4. If $d_H(P) = \epsilon_0$, identify $j_0 \in \{1, 2, \ldots, m-1\}$ such that $\epsilon_0 = \epsilon_{j_0}$ and form the polynomial $(P + \Delta P)(\alpha, \beta) = \Sigma_{k=0}^{m} \alpha^k \beta^{m-k}(A_k + \Delta A_k)$ where

$$\Delta A_k = \frac{\left(\cos\, \theta_{mid}^{(j_0)}\right)^k \left(\sin\, \theta_{mid}^{(j_0)}\right)^{m-k} \lambda_{\min}\left(P(\theta_{mid}^{(j_0)})\right)}{\Sigma_{k=0}^{m}(\cos\, \theta_{mid}^{(j_0)})^{2(m-k)}(\sin\, \theta_{mid}^{(j_0)})^{2k}} V_{\theta_{mid}}^{(j_0)} \left(V_{\theta_{mid}}^{(j_0)}\right)^*, \; k = 0, 1, \ldots, m$$

where $\epsilon_0 = |g(\theta_{mid}^{(j_0)})|$.

The following points are to be noted for the above algorithm.

Note 5.2.1. *The quantity tol_1 is a tolerance level set by the user.*

Note 5.2.2. *In most cases, $V_{\theta_\star}$ will be a single vector so that determining $s_\star$ is not difficult.*

Note 5.2.3. *Since it cannot be guranteed that any component of $\Lambda_\epsilon^H(P)$ which contains eigenvalues of $P(z)$ of only positive (negative) type will be of positive (negative) type if it contains ∞, it is possible that in such a case a perturbed Hermitian polynomial $(P + \Delta P)(z)$ with $|||\Delta P||| < \epsilon$ has an eigenvalue of mixed type in a component which contains only positive or only negative type eigenvalues of $P(z)$. This leads to the possibility that $s_{mid} = 0$ in* **Step (v)** *for one of the intervals I_j, $1 \leq j \leq m-1$ even though the point $\theta_{mid}^{(j)}$ is not a local extremum of $g(\theta)$. However, this is not a problem for the computation of $d_H(P)$ as the corresponding $|g(\theta_{mid}^{(j)})|$ will then be greater than or equal to $\sigma_{\min}(A_m)$ by Theorem 4.2.4.*

If $P(z)$ is a definite polynomial that satisfies the hypothesis of Theorem 4.3.8, then a modification of Algorithm 2 applied to the intervals

$$I_j = (\theta_{jn+p}, \theta_{jn+p+1}), \; j = 1, 2, \ldots, m-1,$$

and

$$I_0 = \begin{cases} (\theta_p, \theta_{p+1}) & \text{if } p \neq 0 \\ (\theta_{mn} - \pi, \theta_1) & \text{if } p = 0 \end{cases}$$

gives

$$\epsilon_0 = \min_{0 \leq j \leq m-1} \max_{\theta \in I_j} \frac{\sigma_{min}(P(\theta))}{\sqrt{\Sigma_{k=0}^{m}(\cos\, \theta)^{2(m-k)}(\sin\, \theta)^{2k}}}$$

which is the distance $d_D(P)$ to a nearest Hermitian polynomial which is not definite. We have conjectured that $\epsilon_0 = d_D(P)$ for all definite polynomials.

If $P(z)$ is a quasihyperbolic polynomial of degree at least 2, and $d_H(P) < \sigma_{\min}(A_m)$, then the smallest level of ϵ such that a positive type component of $\Lambda_\epsilon^H(P)$ coalesces with a negative type component of $\Lambda_\epsilon^H(P)$ is obtained by taking the minimum of all the maximas of the backward error function $\frac{\sigma_{\min}(P(\theta))}{\sqrt{\Sigma_{k=0}^{m}(\cos\,\theta)^{2(m-k)}(\sin\,\theta)^{2k}}}$ in every interval of the form $[\theta_k, \theta_{k+1}]$ where θ_k and θ_{k+1} are eigenangles corresponding to eigenvalues λ_k and λ_{k+1} respectively that differ in type. Thus $d_Q(P)$ is the lesser of this minimum and $\sigma_{\min}(A_m)$.

The Table 5.1 gives some numerical examples that illustrate Algorithm 1 for the Crawford number. Method 1 in the table obtains the Crawford number by laying a fine grid on the interval $[0, 2\pi]$ and finding the maximum of $\lambda_{\min}(A_\theta)$ on the grid. Method 2 calculates the Crawford number by laying a grid on the interval $\mathcal{I}$ where

$$\mathcal{I} = \begin{cases} [\theta_k, \theta_{k+1}] & \text{if } A \text{ is indefinite} \\ [\theta_n - \pi, \theta_1] & \text{if } A \text{ is definite} \end{cases}$$

and finding the maximum of the backward error $\eta(\theta, L) = \sigma_{\min}(A_\theta)$ on the grid. The figures in the brackets against the results obtained from Algorithm 1 give the number of iterations required to produce the output. Note that in most cases we observe a very good lower bound of the Crawford number within a few iterations. The second column of the table gives the point z_0 represented by $(\cos\,\theta_0, \sin\,\theta_0)$ where θ_0 is the point at which the maximum of $\sigma_{\min}(A_\theta)$ (which gives the Crawford number) occurs.

Table 5.1: *Crawford Number.*

Size	z_0	Method 1	Method 2	Algorithm 1	
6	1.4712	5.71635 $\times 10^{-1}$	5.71635 $\times 10^{-1}$	5.716359595168258 $\times 10^{-1}$ (18)	5.71635 $\times 10^{-1}$ (12)
3	-0.34572	1.4583 $\times 10^{-3}$	1.4583 $\times 10^{-3}$	1.458295799903395 $\times 10^{-3}$ (22)	1.45829 $\times 10^{-3}$ (9)
4	-0.31707	5.1117 $\times 10^{-1}$	5.1117 $\times 10^{-1}$	5.111723097662407 $\times 10^{-1}$ (22)	5.1117 $\times 10^{-1}$ (12)
5	0.19595	5.6663 $\times 10^{-1}$	5.6663 $\times 10^{-1}$	5.666419715164018 $\times 10^{-1}$ (19)	5.6664 $\times 10^{-1}$ (9)
7	-4.965	1.07139 $\times 10^{-1}$	1.07139 $\times 10^{-1}$	1.071390231916839 $\times 10^{-1}$ (19)	1.07139 $\times 10^{-1}$ (9)
9	2.07667	7.05105 $\times 10^{-1}$	7.05105 $\times 10^{-1}$	7.051053626850798 $\times 10^{-1}$ (20)	7.05105 $\times 10^{-1}$ (10)
10	-3.54269	1.0828594 $\times 10^{-1}$	1.0828594 $\times 10^{-1}$	1.082859401693458 $\times 10^{-1}$ (20)	1.0828594 $\times 10^{-1}$ (10)

Table 5.2 gives the Crawford number for various sizes of the definite pencil $A-zB$ where

$$\begin{aligned} A &= |i-j| \text{ (Fiedler matrix)} \\ B &= U^T U \; U \text{ is an unit upper triangular matrix} \\ & \qquad \text{with } u_{ij} = -1,\ j > i. \text{ (Moler matrix).} \end{aligned}$$

The first column gives the Crawford number by Algorithm 1 with the figure in the bracket indicating the number of iterations required to obtain it. For sizes up to 40, these outputs were produced with $tol_1 = |||L||| \times 10^{-8}$. For sizes 80, 160 and 320 they were produced with tol_1 values of $|||L||| \times 10^{-10}$, $|||L||| \times 10^{-12}$, and $|||L||| \times 10^{-13}$ respectively. For sizes 640 and 1280 the outputs were produced with $tol_1 = |||L|| \times 10^{-14}$. The second column gives the point z_0 represented by $(\cos\theta_0, \sin\theta_0)$ where θ_0 is the point of $\mathcal{I}$ at which the the maximum of $\sigma_{\min}(A_\theta)$ (which gives the Crawford number) is attained. For size 5, Algorithm 1 produces the Crawford number as 0.460512945276378 in just 6 iterations with $tol_1 = 10^{-3}$. This improves to 0.460516199240031 in 13 iterations with $tol_1 = 10^{-5}$. Note that both these results are good approximations of the value of the Crawford given in Table 5.2 obtained with $tol_1 = 10^{-8}$.

This pencil was considered in [29] for $n = 10$. It has the property that its Crawford number decreases with increase in size. The same example has also been recently considered by Uhlig [55]. In fact the data in columns 4, 5 and 6 have been taken from [55]. These record the number of iterations required to compute the Crawford number via the MATLAB algorithm `crawfno.m` proposed by Uhlig [55, 56], a Chebfun based algorithm [57] and Algorithm 2.3 of [29] respectively. It may be noted here that for size 10, Uhlig [55] reports that the Crawford number obtained by `crawfno.m` is 0.186778387832906 while the same obtained by Chebfun and Algorithm 2.3 of [29] are 0.186778387832912 and 0.186778387832918 respectively.

Algorithm 1 was also used to compute the Crawford number for a definite pencil $A - zB$ of size 25 generated by the following MATLAB code [24]:

```
V = gallery('triw',n,1,2);
theta = zeros(n,1);
for i = 2:n, theta(i) = theta(i-1) + pi/2^(i-1); end
A = V*diag(sin(theta))*V;
B = V*diag(cos(theta))*V;
```

The coefficient A is positive semidefinite while B is indefinite and singular for these

Table 5.2: *Crawford number for the Fiedler/Moler Pencil*

Size	Algorithm 1	z_0	Uhlig (2011)	Chebfun	HTvD
5	0.460516199322903 (23)	-3.87894	12	202	397
10	0.186778387832905 (19)	-10.7219	14	172	712
20	0.077907742140276 (16)	-25.826	15	201	1215
40	0.035906797923880 (13)	-55.8903	15	219	1982
80	0.017284338656944 (16)	-1.159214×10^2	15	262	2990
160	0.008484748278810 (20)	-2.7610×10^2	17	247	5555
320	0.004204156531129 (20)	-4.759384×10^2	20	252	19885
640	0.002092661042589 (20)	-9.559415×10^2	20	254	—
1280	0.001043993112866 (17)	9.3947×10^3	22	252	—

pencils. These pencils have been considered by Moon (as Example 3.0, p. 80 of [45]), Guo et. al. (Experiment 2, p. 1144 of [24]) and more recently by Uhlig (Example 2, p. 9 of [55]). The pencil was homogeneously rotated to remove the infinite eigenvalue before applying Algorithm 1. This produced the Crawford number 7.46322×10^{-9} in 12 iterations with $tol_1 = |||L||| \times 10^{-8}$. As per Example 2 of [55], the same answer was produced in 25 iterations by the algorithm `crawfno.m` [55] while the Chebfun based algorithm [57] computed the Crawford number as 7.4632×10^{-9} in 965 iterations and Algorithm 2.3 of [29] computed it to be 1.7293×10^{-8} in 1873 iterations.

The next table lists the distance from a quadratic hyperbolic polynomial to a nearest Hermitian polynomial that is not hyperbolic. This is computed by finding the maximum of the backward error $\eta(\theta, P)$ on a fine grid on the interval $[\theta_n, \theta_{n+1}]$ (Method 1) and by Algorithm 2. The figures in the brackets against the value of ϵ_0 given by Algorithm 2, indicate the number of iterations required to obtain ϵ_0 with $tol_1 = |||P||| \times 10^{-8}$. The third column gives the point $z_0 \in \mathbb{R}$ at which the maximum ϵ_0 is achieved in each case. The examples considered in these tables have been generated by using the NLEVP software [9].

5.3 Bounds for the Crawford number

In this section we present some bounds on the Crawford number of a definite pencil $L(z)$. Let $D(z) = zD_A - D_B$ be a real diagonal pencil such that $X^*AX = D_A$ and $X^*BX = D_B$ for a nonsingular pencil $X \in \mathbb{C}^{n \times n}$. The first bound on $\gamma(A, B)$ is in terms of $\|X\|_2$ and $\gamma(D_A, D_B)$ and is a consequence of the definition of $\gamma(A, B)$.

Table 5.3: *The distance to a nearest nonhyperbolic Hermitian polynomial*

Size	$\sigma_{\min}(A_2)$	z_0	Method 1 (ϵ_0)	Algorithm 2 (ϵ_0)	$d_h(P)$ $= \min(\epsilon_0, \sigma_{\min}(A_2))$
3	1.1416×10^{-1}	2.54407×10^{-1}	2.8759×10^{-1}	2.8759×10^{-1} (26)	1.1416×10^{-1}
3	1.2655×10^{-1}	-3.73056	3.8843×10^{-2}	3.8843×10^{-2} (25)	3.8843×10^{-2}
3	1.5879×10^{-1}	2.40699×10^{-1}	9.1123×10^{-1}	9.1123×10^{-1} (28)	1.5879×10^{-1}
4	1.2127×10^{-1}	2.76	6.4819×10^{-3}	6.4819×10^{-3} (21)	6.4819×10^{-1}
4	1.0382×10^{-1}	2.59884	6.4346×10^{-2}	6.4346×10^{-2} (26)	6.4346×10^{-2}
5	6.0847×10^{-2}	-2.14065	4.7109×10^{-2}	4.7109×10^{-2} (25)	4.7109×10^{-2}
5	7.5107×10^{-2}	7.83358×10^{-1}	5.8685×10^{-1}	5.8685×10^{-1}(28)	7.5107×10^{-2}
6	6.6095×10^{-2}	-3.06100×10^{-1}	6.4154×10^{-1}	6.4154×10^{-1} (26)	6.6095×10^{-2}

Proposition 5.3.1. *Let $L(z) = zA - B$ be a definite pencil and X a nonsingular matrix such that $D_A = X^*AX$ and $D_B = X^*BX$ are real diagonal matrices. If $\gamma(D_A, D_B)$ denotes the Crawford number of $D(z) = zD_A - D_B$, then*

$$\frac{\gamma(D_A, D_B)}{\|X\|_2^2} \leq \gamma(A, B) \leq \gamma(D_A, D_B)\|X^{-1}\|_2^2. \tag{5.1}$$

Proof: From the definition of $\gamma(A, B)$ we have,

$$\begin{aligned}
\gamma(D_A, D_B) &= \min_{\|y\|_2=1} \sqrt{|y^*D_Ay|^2 + |y^*D_By|^2} \\
&= \min_{y\neq 0} \frac{\sqrt{|y^*X^*AXy|^2 + |y^*X^*BXy|^2}}{\|y\|_2^2} \\
&= \min_{z\neq 0} \frac{\sqrt{|z^*Az|^2 + |z^*Bz|^2}}{\|X^{-1}z\|_2^2} \\
&\geq \frac{\gamma(A, B)}{\|X^{-1}\|_2^2}.
\end{aligned} \tag{5.2}$$

Interchanging the roles of (A, B) and (D_A, D_B) in (5.2), we have

$$\gamma(A, B) \geq \frac{\gamma(D_A, D_B)}{\|X\|_2^2}. \tag{5.3}$$

Combining the inequalities (5.2) and (5.3) we have,

$$\frac{\gamma(D_A, D_B)}{\|X\|_2^2} \leq \gamma(A, B) \leq \gamma(D_A, D_B)\|X^{-1}\|_2^2. \qquad \square$$

Given two pencils $L_1(z) = zA_1 - B_1$ and $L_2(z) = zA_2 - B_2$, we define a notion of separation between $L_1(z)$ and $L_2(z)$ in terms of their ϵ-pseudospectra as

$$\text{sep}_\lambda(L_1, L_2) = \inf\{\epsilon : \Lambda_\epsilon(L_1) \cap \Lambda_\epsilon(L_2) \neq \emptyset\}.$$

If $L(z)$ be a definite pencil in the block diagonal form

$$L(z) = \begin{bmatrix} L_1(z) & \\ & L_2(z) \end{bmatrix},$$

then $\Lambda(L) = \Lambda(L_1) \cup \Lambda(L_2)$. Also, we have $\Lambda_\epsilon(L) = \Lambda_\epsilon(L_1) \cup \Lambda_\epsilon(L_2)$ as

$$\begin{aligned} z \in \Lambda_\epsilon(L) &\Leftrightarrow \eta(z, L) < \epsilon \\ &\Leftrightarrow \min\{\eta(z, L_1), \eta(z, L_2)\} < \epsilon \\ &\Leftrightarrow \text{ either } \eta(z, L_1) < \epsilon \text{ or } \eta(z, L_2) < \epsilon \\ &\Leftrightarrow z \in \Lambda_\epsilon(L_1) \cup \Lambda_\epsilon(L_2). \end{aligned}$$

In particular if the eigenvalues of $L_1(z)$ and $L_2(z)$ are respectively all the positive type and all the negative type eigenvalues of $L(z)$, then

$$\text{sep}_\lambda(L_1, L_2) = \min\left\{\epsilon : (\Lambda_\epsilon(L_1) \cap \Lambda_\epsilon(L_2)) \cap \{[\lambda_{k+1}, \lambda_k] \cup [-\infty, \lambda_n] \cup [\lambda_1, \infty]\} \neq \emptyset\right\}.$$

If this is written in terms of the homogeneous form of the backward error, we have

$$\text{sep}_\lambda(L_1, L_2) = \min\left\{\epsilon : \Lambda_\epsilon(L_1) \cap \Lambda_\epsilon(L_2) \cap \{[\theta_k, \theta_{k+1}] \cup [\theta_n - \pi, \theta_1]\} \neq \emptyset\right\}.$$

In view of the equality (3.18), $\text{sep}_\lambda(L_1, L_2) = \gamma(A, B)$ if $\gamma(A, B) < \sigma_{\min}(A) = \eta(\infty, L)$ and a lower bound of $\gamma(A, B)$ otherwise. In the former case it is the minimum value of ϵ such that $\Lambda_\epsilon(L_1) \cap \Lambda_\epsilon(L_2) \cap [\theta_k, \theta_{k+1}] \neq \emptyset$. In particular, if $L_1(z) = L_+(z)$ and $L_2(z) = L_-(z)$ where $L_+(z)$ and $L_-(z)$ are diagonal pencil as mentioned in part (D) of Theorem 1.2.12, then

$$\text{sep}_\lambda(L_-, L_+) = \eta(z, L) = \eta(z, L_+) = \eta(z, L_-)$$

for some $z \in \mathbb{R}$.

Since $L(z)$ is a diagonal pencil, $\eta(z, L) = \min_{1 \le i \le n} \frac{|\alpha_i z - \beta_i|}{\sqrt{1 + |z|^2}}$ where α_i and β_i are the i^{th} diagonal entries of A and B respectively. If these diagonal entries also satisfy $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, 2, \ldots, n$, and $(\alpha_i, \beta_i) = (\sin\theta_i, \cos\theta_i), \theta_i \in (0, \pi), i = 1, 2, \ldots, n$, then for each $z \in \mathbb{R}$,

$$\eta(z, L) = \min_{1 \le i \le n} \chi(z, \lambda_i) = \min_{1 \le i \le n} |\sin(\theta - \theta_i)|$$

where $(\cos\theta, \sin\theta)$, $0 \leq \theta \leq \pi$, represents the point z and $\chi(\mu, \hat{\mu}) = \frac{|\alpha\hat{\beta}-\hat{\alpha}\beta|}{\sqrt{\alpha^2+\beta^2}\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}$ is the distance between μ and $\hat{\mu} \in \mathbb{R}$ represented by the pairs (α, β) and $(\hat{\alpha}, \hat{\beta})$ in $\mathbb{R}^2$ with respect to the chordal metric. Therefore,

$$\gamma(A, B) = \min_{1\leq i\leq k} |\sin(\theta - \theta_i)| = \min_{k+1\leq i\leq n} |\sin(\theta - \theta_i)|$$

for some $\theta \in (\theta_k, \theta_{k+1})$ where as before, θ_k and θ_{k+1} are eigenangles corresponding to eigenvalues λ_k and λ_{k+1} that differ in type. The continuity of sine function implies that

$$\gamma(A, B) = \inf_{\theta\in(\theta_k,\theta_{k+1})} \max\left\{\min_{1\leq i\leq k} |\sin(\theta - \theta_i)|, \min_{k+1\leq i\leq n} |\sin(\theta - \theta_i)|\right\}$$

where the right hand side is equal to $\mathrm{sep}_\lambda(L_1, L_2)$ if $\gamma(A, B) < \sigma_{\min}(A)$ and an upper bound of $\mathrm{sep}_\lambda(L_1, L_2)$ otherwise.

Given a definite pencil $L(z)$ with eigenvalues λ_k and λ_{k+1}, $1 \leq k \leq n-1$, differing in type, following Stewart and Sun (pp. 319, [47]), we define

$$\delta = \min_{\substack{1\leq i\leq k \\ k+1\leq j\leq n}} \chi(\lambda_i, \lambda_j). \tag{5.4}$$

Then δ is a measure of the separation between the sets of eigenvalues of positive and negative type with respect to the chordal metric. This leads to the following bounds on $\gamma(A, B)$ in terms of δ.

Proposition 5.3.2. *Let $L(z)$ be a definite pencil with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of positive type and $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n$ of negative type such that*

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_{k+1} \leq \lambda_k \leq \cdots \leq \lambda_1$$

and corresponding eigenangles $\theta_1, \theta_2, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_n$ that satisfy

$$\theta_1 \leq \theta_2 \leq \cdots \theta_k \leq \theta_{k+1} \leq \cdots \leq \theta_n.$$

Let X be the nonsingular matrix such that

$$X^* L(z) X = \begin{bmatrix} L_-(z) & 0 \\ 0 & L_+(z) \end{bmatrix}$$

where $L_+(z) = zD_A^+ - D_B^+$ and $L_-(z) = zD_A^- - D_B^-$ are diagonal pencils with $\Lambda(L_+) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ and $\Lambda(L_-) = \{\lambda_{k+1}, \lambda_{k+2} \ldots, \lambda_n\}$. Also suppose that the diagonal entries *of D_A^+ and D_A^- are of the form* $\sin\theta_i$ *and those of D_B^+ and D_B^- of the form* $\cos\theta_i$ *for* $1 \leq i \leq n$. *Then for δ defined by (5.4),*

$$\frac{\delta}{2\|X\|_2^2} \leq \gamma(A, B). \tag{5.5}$$

Further, if $\max\{\theta_{k+1} - \theta_1, \theta_n - \theta_k\} \leq \frac{\pi}{2}$, *then*

$$\gamma(A, B) \leq |\sin(\theta_k - \theta_{k+1})| \|X^{-1}\|_2^2. \tag{5.6}$$

Proof: Let $D_A = \begin{bmatrix} D_A^+ & 0 \\ 0 & D_A^- \end{bmatrix}$ and $D_B = \begin{bmatrix} D_B^+ & 0 \\ 0 & D_B^- \end{bmatrix}$. In view of the bounds given by (5.1), the proof is complete if we show that $\frac{\delta}{2} \leq \gamma(D_A, D_B)$ and $\gamma(D_A, D_B) \leq |\sin(\theta_k - \theta_{k+1})|$ if the eigenangles satisfy $\max\{\theta_{k+1} - \theta_1, \theta_n - \theta_k\} \leq \frac{\pi}{2}$.

There exist $i_0 \in \{1, 2, \ldots, k\}$, $j_0 \in \{k+1, k+2, \ldots, n\}$ and $\theta_0 \in (\theta_k, \theta_{k+1})$ be such that

$$\gamma(D_A, D_B) = |\sin(\theta_0 - \theta_{i_0})| = |\sin(\theta_0 - \theta_{j_0})|. \tag{5.7}$$

From definition 5.4 of δ,

$$\delta = \chi(\lambda_{i_0}, \lambda_{j_0}) \leq \chi(\lambda_{i_0}, \lambda_0) \leq \chi(\lambda_0, \lambda_{j_0}) = |\sin(\theta_0 - \theta_{i_0})| + |\sin(\theta_0 - \theta_{j_0})|.$$

Therefore from (5.7) we have $\delta \leq 2\gamma(D_A, D_B)$. If the eigenangles satisfy $\max\{\theta_{k+1} - \theta_1, \theta_n - \theta_k\} \leq \frac{\pi}{2}$, then $i_0 = k$ and $j_0 = k+1$ in (5.7). Also for any $\theta \in (\theta_k, \theta_{k+1})$, $\max\{|\sin(\theta - \theta_k)|, |\sin(\theta - \theta_{k+1})|\} \leq |\sin(\theta_k - \theta_{k+1})|$. Hence we have the upper bound

$$\gamma(D_A, D_B) \leq |\sin(\theta_k - \theta_{k+1})|. \qquad \square$$

The lower bound (5.5) shows that if the separation δ in terms of the chordal metric is large then $L(z)$ may not be close to a pencil that is not definite. On the other hand, if the conditions for the upper bound to hold are satisfied, then a small chordal distance between the eigenvalues λ_k and λ_{k+1} that differ in type indicates the possibility of a proximity to a Hermitian pencil that is not definite.

A tighter bound than the ones obtained from (5.5) and (5.6) may be obtained by using the fact that if the definite pencil $L(z) = zD_A - D_B$ is in diagonal form with the i^{th} diagonal entries of D_A and D_B satisfying $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, 2, \ldots, n$, then the field of values $\{(x^*D_Ax, x^*D_Bx) : x \in \mathbb{C}^n, \|x\|_2 = 1\}$ is a convex combination of the pairs (α_i, β_i), $i = 1, 2, \ldots, n$. An implication of this is the fact that $\gamma(D_A, D_B)$ is then the length of the perpendicular from the origin to the line joining the points $(\cos\,\theta_1, \sin\,\theta_1)$ and $(\cos\,\theta_n, \sin\,\theta_n)$ on the unit circle. If this be denoted by $\ell(A, B)$, we have the following result.

Proposition 5.3.3. *Let $L(z)$ be a definite pencil with eigenvalues, eigenangles and a corresponding nonsingular matrix X with properties as described in Proposition 5.3.2. If $\ell(A, B)$ be the length of the perpendicular from the origin to the line joining the points $(\cos\,\theta_1, \sin\,\theta_1)$ and $(\cos\,\theta_n, \sin\,\theta_n)$, then*

$$\frac{\ell(A,B)}{\|X\|_2^2} \leq \gamma(A,B) \leq \ell(A,B)\|X^{-1}\|_2^2. \tag{5.8}$$

Table 5.4 illustrates upper and lower bounds on the Crawford number as given by (5.5), (5.6) and (5.8). All the definite pencils $L(z) = zA - B$ considered in this table are such that A is indefinite.

Table 5.5 illustrates bounds given by (5.8) for definite pencils $L(z) = zA - B$ where A is a positive definite matrix.

Table 5.4: *Bounds for Crawford number for pencil with leading coefficient indefinite*

S.N.	Size	Lower bound (5.5)	Lower bound (5.8)	$\gamma(A,B)$	Upper bound (5.8)	Upper bound (5.6)
1	6	6.8699×10^{-2}	8.8834×10^{-2}	9.4221×10^{-2}	3.7160×10	5.7474×10
2	3	8.3407×10^{-1}	8.5721×10^{-1}	1.2757	1.1354×10	2.2095×10
3	5	3.7668×10^{-2}	3.8733×10^{-2}	5.4509×10^{-2}	1.1081	2.1552
4	3	7.7867×10^{-1}	8.0263×10^{-1}	9.4275×10^{-1}	1.1508	2.2330

Table 5.5: *Bounds for Crawford number for pencils with definite leading coefficient.*

Size	Lower bound (5.8)	$\gamma(A,B)$	Upper bound (5.8)
4	0.5373	0.6209	35.0300
5	0.0962	0.1517	32.2400
6	0.0016	0.0030	26.7199
4	0.01096537	0.01267113	0.71480899
6	1.51169478×10^{-4}	3.45478315×10^{-4}	2.27851188×10^{-1}
7	2.23070879×10^{-3}	4.94459841×10^{-3}	2.98258546×10^{-1}

In [47] it is shown that given a definite pencil $L(z) = zA - B$, a perturbed Hermitian pencil $(L + \Delta L)(z) = z(A + \Delta A) - (B + \Delta B)$ is also definite if

$$\max_{\|x\|_2=1} \sqrt{\frac{(x^*\Delta Ax)^2 + (x^*\Delta Bx)^2}{(x^*Ax)^2 + (x^*Bx)^2}} < 1.$$

In such a case if $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ and $\theta_1 \leq \theta_2 \leq \cdots \leq \lambda_n$ be the eigenvalues and corresponding eigenangles of $L(z)$ and $\tilde{\lambda}_n, \tilde{\lambda}_{n-1} \leq \cdots \leq \tilde{\lambda}_1$ and $\tilde{\theta}_1 \leq \tilde{\theta}_2 \leq \cdots \leq \tilde{\theta}_n$ be those of $(L + \Delta L)(z)$, then $|\theta_i - \tilde{\theta}_i| < \frac{\pi}{2}$ and

$$\xi(\lambda_i, \tilde{\lambda}_i) \leq \frac{|||\Delta L|||}{\gamma(A, B)}, \; i = 1, 2, \ldots, n.$$

We prove an analogous result by using the ϵ-pseudospectrum based approach. First of all we have the following lemma that relates the backward errors $\eta(z, L)$ and $\eta(z, X^{-1}LX)$ which is a direct consequence of the definition of backward error. A similar proof may be found in ([2]) but we reproduce it here for the sake of completeness.

Lemma 5.3.4. *Given a pencil $L(z)$ and a nonsingular matrix X both of size n, for any $z \in \mathbb{C}$, the backward errors $\eta(z, L)$ and $\eta(z, X^*LX)$ where $X^*LX = X^*L(z)X$ are related as follows.*

$$\frac{\eta(z, L)}{\|X^{-1}\|_2^2} \leq \eta(z, X^*LX) \leq \eta(z, L)\|X\|_2^2.$$

Proof: The proof follows by observing that for any $z \in \mathbb{C}$,

$$\begin{aligned}
\eta(z, X^*LX) &= \inf\{|||\Delta L||| : z \in \Lambda(X^*LX + \Delta L)\} \\
&= \inf\{|||\Delta L||| : z \in \Lambda(L + (X^*)^{-1}\Delta LX^{-1})\} \\
&= \inf\{|||X^*(X^*)^{-1}\Delta LX^{-1}X||| : z \in \Lambda(L + (X^*)^{-1}\Delta LX^{-1})\} \\
&\leq \eta(z, L)\|X\|_2^2.
\end{aligned}$$

and,

$$\begin{aligned}
\eta(z, L) &= \inf\{|||\Delta L||| : z \in \Lambda(L + \Delta L)\} \\
&= \inf\{|||\Delta L||| : z \in \Lambda(X^*LX + X^*\Delta LX)\} \\
&= \inf\{|||(X^*)^{-1}X^*\Delta LXX^{-1}||| : z \in \Lambda(X^*LX + X^*\Delta LX)\} \\
&\leq \eta(z, X^*LX)\|X^{-1}\|_2^2.
\end{aligned}$$

□

Theorem 5.3.5. *Let $L(z)$ be a definite pencil and $\Delta L(z)$ be a Hermitian pencil such that $(L+\Delta L)(z)$ is also definite. Given an eigenvalue λ of $L(z)$ there exists an eigenvalue $\tilde{\lambda}$ of $(L+\Delta L)(z)$ such that*

$$\chi(\lambda, \tilde{\lambda}) \leq \frac{|||\Delta L|||}{\gamma(A,B)}.$$

Proof: By the definition of Hermitian backward error, $\eta^H(\tilde{\lambda}, L) \leq |||\Delta L|||$. Let X be a nonsingular matrix such that $\tilde{L}(z) = X^*L(z)X$ is a diagonal pencil with i^{th} diagonal entries α_i of X^*AX and and β_i of X^*BX satisfying $\alpha_i^2 + \beta_i^2 = 1$ for $i = 1, 2, \ldots, n$. Since $(L+\Delta L)(z)$ is also a definite pencil, $\tilde{\lambda}$ is real and hence $\eta^H(\tilde{\lambda}, L) = \eta(\tilde{\lambda}, L)$. By Lemma 5.3.4,

$$\frac{\eta(\tilde{\lambda}, \tilde{L})}{\|X\|_2^2} \leq |||\Delta L|||. \tag{5.9}$$

If x_i be the i^{th} column of X, for $i = 1, 2, \ldots, n$, we have

$$\sqrt{|x_i^*Ax_i|^2 + |x_i^*Bx_i|^2} = \sqrt{\alpha_i^2 + \beta_i^2} = 1.$$

Therefore, for $i = 1, 2, \ldots, n$,

$$\|x_i\|_2^2 = \frac{\|x_i\|_2^2}{\sqrt{|x_i^*Ax_i|^2 + |x_i^*Bx_i|^2}} \leq \frac{1}{\gamma(A,B)}$$

where the last inequality follows from the definition of $\gamma(A,B)$. Let $v \in \mathbb{C}^n$ be such that $\|v\|_2 = 1$ and $\|X\|_2 = \|Xv\|_2$. This implies that

$$\|X\|_2^2 \leq \max_{i=1,2,\ldots,n} \|x_i\|_2^2 \|v\|_2^2 \leq \frac{1}{\gamma(A,B)}.$$

Using the above inequality in (5.9), we have

$$\eta(\tilde{\lambda}, \tilde{L}) \leq \frac{|||\Delta L|||}{\gamma(A,B)}.$$

Since $\tilde{L}(z)$ is a diagonal pencil, there exists an eigenvalue λ of $L(z)$ such that $\eta(\tilde{\lambda}, \tilde{L}) = \chi(\tilde{\lambda}, \lambda)$. This gives,

$$\chi(\tilde{\lambda}, \lambda) \leq \frac{|||\Delta L|||}{\gamma(A,B)}.$$

Hence the proof. □

Summary and conclusion

We have undertaken a Hermitian ϵ-pseudospectra based analysis of the distance from certain classes of Hermitian matrix polynomials with real eigenvalues of definite type to a nearest Hermitian polynomial outside the class with respect to a prespecified norm. These include the definite and definitizable pencils as well as hyperbolic, quasihyperbolic and definite polynomials which have many applications in science and engineering. This has resulted in bisection type algorithms that compute these distances and produce a nearest Hermitian pencil or polynomial outside the respective class for all but certain classes of definite polynomials. In particular, we have an algorithm for computing the Crawford number which also produces a nearest Hermitian pencil with a defective eigenvalue.

In the process other interesting results that connect the Crawford number of a definite pencil with its Hermitian ϵ-pseudospectrum have been obtained. In particular it has been shown that the Crawford number is the distance with respect to the chosen norm to a nearest Hermitian pencil with an eigenvalue of mixed type on the extended real line as well as to a Hermitian pencil with a complex eigenvalue. We have proposed a definition of eigenvalue type for a Hermitian polynomial $P(z)$ based on the homogeneous form of the polynomial which determines the type of both finite and infinite eigenvalues in the same framework and is crucial to the proofs of the main results of the thesis. We have also analysed certain properties of Hermitian pencils based on their canonical form under congruence. Finally we have analysed the components of the Hermitian ϵ-pseudospectrum of the pencil that contain eigenvalues of perturbed pencils of only one type, either positive or negative. This analysis has also been extended to that of Hermitian ϵ-pseudospectra of regular Hermitian matrix polynomials.

Several numerical experiments have been carried out to illustrate the proposed algorithms to compute the Crawford number and the distance from a hyperbolic polynomial to a nearest Hermitian polynomial that is not hyperbolic. In particular, these experiments have shown that the proposed algorithm for computing the Crawford number produces a very good lower bound in a few iterations. Finally certain bounds have also been obtained on the Crawford number. Some of these bounds relate the Crawford number to the distribution of the eigenvalues of the definite pencil. A pseudospectra based derivation of an eigenvalue perturbation bound for definite pencils via the Crawford number has also been obtained.

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