A STUDY OF SOME PARAMETERS IN SIGNED GRAPHS

by

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A STUDY OF SOME PARAMETERS IN SIGNED GRAPHS

A thesis submitted
in partial fulfillment of the requirements
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DOCTOR OF PHILOSOPHY

by

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Under the supervision of
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Dedicated

to

my family

and

my supervisor

Contents

DECLARATION

I do hereby declare that the work contained in this thesis entitled **"A STUDY OF SOME PARAMETERS IN SIGNED GRAPHS"** has been done by me under the supervision of **Dr. Bikash Bhattacharjya**, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy and that this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is certified that the work contained in this thesis entitled **"A STUDY OF SOME PARAMETERS IN SIGNED GRAPHS"** by **Deepak**, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

August, 2022

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Abstract

Graphs with positive or negative edges are called *signed graphs*. We denote a signed graph Σ by (G, ϕ), where G is called the underlying graph of Σ and ϕ is a function that assigns $+1$ or -1 to the edges of G. The set of negative edges in Σ is known as the *signature* of Σ. An unsigned graph can be realized as a signed graph in which all edges are positive. *Switching* Σ by a vertex v is to change the sign of each edge incident to v. Switching is a way of turning one signed graph into another. Two signed graphs are called *switching equivalent* if one can be obtained from the other by a sequence of switchings. Further, two signed graphs are said to be *switching isomorphic* to each other if one is isomorphic to a switching of the other. In Chapter 2 of the thesis, we classify the switching non-isomorphic signed graphs arising from K_6, $P_{3,1}$, $P_{5,1}$, $P_{7,1}$, and $B(m, n)$ for $m \geq 3$, $n \geq 1$, where K_6 is the complete graph on six vertices, $P_{n,k}$ denotes the generalized Petersen graph and $B(m, n)$ denotes the book graph consisting of n copies of the cycle $C_m$with exactly one common edge. We also count the switching non-isomorphic signatures of size two in $P_{2n+1,1}$ for $n \geq 1$. We prove that the size of a minimum signature of $P_{2n+1,1}$, up to switching, is at most $n + 1$.

A signed cycle is *positive* if the product of signs of the edges in the cycle is positive, and *negative*, otherwise. A signed graph is said to be *balanced* if every cycle in the graph is positive. A signed graph is called *unbalanced* if it is not balanced, that is, it has at least one negative cycle. The minimum number of edges required to delete from a signed graph to make it balanced is called the *frustration index* of the signed graph. The frustration index of a signed graph Σ is denoted by $l(\Sigma)$. The frustration index is a significant way to measure how unbalanced a signed graph is. V. Sivaraman [49] proved that if Σ is a signed graph whose underlying graph is simple, triangle-free and cubic, then $l(\Sigma) \leq \frac{3}{8}|V(\Sigma)|$. The *maximum frustration* of G, denoted $D(G)$, is the maximum frustration index over all possible signatures of G. Let $P_{n,k}$ be the generalized Petersen graph and $d := \gcd(n, k)$. In Chapter 3, we prove that $D(P_{n,k}) \leq 1 + \lfloor \frac{n}{2} \rfloor$ for $d = 1$, and $D(P_{n,k}) \leq 1 + d(1 + \lfloor \frac{n}{2d} \rfloor)$ for $d > 1$. These upper bounds on $D(P_{n,k})$ improve the Sivaraman's [49] bound for these graphs. We also determine the frustration indices of switching non-isomorphic signed complete graphs on six vertices. Finally, we show that if $B(m, n)$ is the book graph with $m \geq 3$ and $n \geq 1$, then $D(B(m, n)) = \lceil \frac{n}{2} \rceil$.

Ashraf and Germina [9] defined double domination in signed graphs as follows. Given a signed graph Σ with vertex set V, a subset D of V is said to be a *double dominating set* (DDS) of Σ if it satisfies the following conditions: (i) $|N[v] \cap D| \geq 2$ for each $v \in V$, and (ii) $\Sigma[D, D^c]$ is balanced, where $\Sigma[D, D^c]$ is the signed graph

induced by the edges of Σ with one end vertex in D and the other end vertex in D^c. The size of a minimum DDS of Σ is called the *double domination number* (DDN) of Σ, and it is denoted by $\gamma_{\times 2}(\Sigma)$. In Chapter 4, we prove that if D is a DDS of a cubic unsigned graph G such that $|D| = \frac{|V(G)|}{2}$, then $G[D, D^c]$ admits a cycle decomposition. We further give an example which shows that if D, with $|D| = \frac{|V(G)|}{2}$, is not a DDS of a cubic graph G, then it is not necessarily true that $G[D, D^c]$ admits a cycle decomposition. A lower bound on the DDN of signed cubic graphs are also obtained in Chapter 4. We also obtain some bounds on the DDN of signed generalized Petersen graphs and signed I-graphs. We prove that the DDN of a signed complete graph on n vertices is $n-1$ for $n \geq 5$. However, this number is known to be exactly 2 for unsigned complete graph on n vertices, when $n \geq 2$. Also, we compute the DDN of all switching non-isomorphic signed Petersen graphs and of all switching non-isomorphic signed complete graphs over K_6.

For $n \geq 1$, let $M_n = \{0, \pm 1, \ldots, \pm k\}$ if $n = 2k+1$ and $M_n = \{\pm 1, \ldots, \pm k\}$ if $n = 2k$. A *proper n-coloring* of a signed graph Σ is defined to be a mapping $c : V(G) \longrightarrow M_n$ that satisfies $c(x) \neq \phi(e)c(y)$ for each edge e, where $\Sigma = (G, \phi)$, $e = xy$ and $\phi(e)$ is the sign of e. Such a coloring is said to be *zero-free* if it never takes the value zero. The *chromatic number* $\chi(G, \phi)$ of (G, ϕ) is the smallest number n such that (G, ϕ) admits a proper n-coloring. The *chromatic polynomial* $\chi_\Sigma(2k+1)$ of Σ is defined to be the function whose value is equal to the number of proper colorings of Σ in $2k+1$ colors. The *zero-free chromatic polynomial* $\chi^b_\Sigma(\lambda)$ is the function such that $\chi^b_\Sigma(2k)$ counts the zero-free proper colorings in $2k$ colors. In Chapter 5, we first prove that the chromatic number of every signed book graph is either 2 or 3. We also obtain explicit formulas for the chromatic polynomials and the zero-free chromatic polynomials of switching non-isomorphic signed book graphs.

A signed graph Σ is said to be *parity signed* if there exists a bijective map $f : V(G) \longrightarrow \{1, \ldots, |V(G)|\}$ such that $\phi(uv) = +1$ if and only if $f(u)$ and $f(v)$ are of the same parity for each $uv \in E(G)$, where $\Sigma = (G, \phi)$. The rna number of a graph G, denoted $\sigma^-(G)$, is the least number of negative edges among all possible parity signed graphs over G. The rna number is also equal to the least size of a cut that has nearly equal sides. In Chapter 6, we prove that $3 \leq \sigma^-(P_{n,k}) \leq n$ and that these bounds are sharp. The values of $\sigma^-(P_{n,k})$ are also determined for $k \in \{1, 2\}$. The rna numbers of some famous generalized Petersen graphs are also determined. We show that the smallest order of $(4n-1)$-regular graphs having rna number 1 is bounded above by $12n-2$. The sharpness of this upper bound is established for $n = 1$ by showing that there is a unique cubic graph of order 10 whose rna number is 1. We also show that the smallest order of a $(4n+1)$-regular graph having rna number 1 is $8n+6$.

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Introduction

One of challenging problems in graph theory was the Four-Color Conjecture. K. Appel and W. Haken [5, 6] proposed a solution to this Conjecture in 1976, and the subject has exploded in popularity since then. Combinatorial optimization, computer science, topology and algebra all contribute to the development of graph theory in modern applied mathematics.

In the following sections, we briefly discuss existing literature and some basic definitions related to the problems considered in this thesis. In the first section, we define signed graphs and discuss one of the problems about signed graphs, namely, balanced signed graphs. In Section 1.2, we discuss about switching isomorphism classes of signed graphs. In Section 1.3, we define frustration in signed graphs and highlight its important existing results. In Section 1.4, we define double domination in signed graphs and discuss few existing results. In Section 1.5, we discuss vertex coloring of signed graphs and mention a few approaches of defining a chromatic number of signed graph. In Section 1.6, we discuss parity signed graphs and the rna number of a graph. In the last section, we describe the organization of the thesis.

1.1 Signed Graphs

A graph G is an ordered pair $(V(G), E(G))$ consisting of a non-empty set $V(G)$ and a multi-set $E(G)$ of 2-element subsets of $V(G)$. The elements of $V(G)$ and $E(G)$ are called the *vertices* and *edges* of G, respectively. A graph G is called *simple* if $E(G)$ is a set, that is, $E(G)$ does not contain repeated edges. Given a graph G, we denote the order and size of G by $|V(G)|$ and $|E(G)|$, respectively.

A graph can be used to describe many real-life situations. For example, suppose we have a group of people for whom we would like to learn about their likes, dislikes, and indifference among themselves. It can be represented as a simple graph, with vertices representing persons and edges labelled as positive (meaning two people like each other) or negative (meaning two people dislike each other), with no edge indicating indifference. A signed graph is a graph that is introduced in this way. Two-graphs, a topic established by J. J. Seidel, is closely related to signed graphs. See [48] for a survey on two-graphs.

Signed graphs have applications in social psychology [21], spin glasses [53], complex systems [24] etc. Various notions of graph theory are extended to the theory of signed graphs. For instance, the theory of minors, the theory of homomorphisms, the theory of vertex and edge coloring, the theory of domination etc. For details, see [2, 14, 28, 41, 59].

Formally, a *signed graph* Σ, where $\Sigma = (G, \phi)$, is a pair consisting of an ordinary graph G and a sign function $\phi : E(G) \to \{+1, -1\}$. The sign function labels each edge of G as positive or negative. We call G to be the *underlying graph* of Σ. The set of negative edges in Σ is known as the *signature* of Σ. We denote the signature of $\Sigma = (G, \phi)$ by σ, where $\sigma \coloneqq \{e \in E(G) : \phi(e) = -1\}$. Throughout the thesis, if we need to use the sign function of a signed graph in a particular discussion, then we denote the signed graph by (G, ϕ). Otherwise, a signed graph is denoted by (G, σ). A cycle is *positive* if the product of signs of its edges is positive, and *negative*, otherwise. A signed graph is said to be *balanced* if each cycle in the graph is positive. The terms signed graph and its balance appeared first in a paper of Frank Harary [31]. A signed graph is *all-positive* or *all-negative* according as all of its edges are positive or negative, respectively. An all-positive and an all-negative signed graph with underlying graph G are denoted by $+G$ and $-G$, respectively. An all-negative signed graph is also known as *antibalanced*. A signed graph is said to be *homogeneous* if it is either all-positive or all-negative, and *heterogeneous*, otherwise.

Throughout the thesis, solid lines of a graph in each figure represent positive edges, while dashed lines represent negative edges.

1.2 Switching Isomorphism in Signed Graphs

The set of all signed graphs with underlying graph G is denoted by $\mathcal{S}(G)$. *Switching* is a way of turning one signed graph into another signed graph with the same underlying graph. *Switching* a signed graph Σ by a vertex v is to change the sign of each edge incident to v. Consequently, switching Σ by a vertex set X is to switch Σ by each vertex in X. The signed graph obtained by switching Σ by a vertex set X is denoted by Σ^X. Two signed graphs are called *switching equivalent* (or simply *equivalent*) if one can be obtained by a sequence of switchings from the other. If two signed graphs are equivalent, then we say that their corresponding signatures are also equivalent. If the signed graphs (G, σ_1) and (G, σ_2) are switching equivalent, then we write $(G, \sigma_1) \sim (G, \sigma_2)$ or $\sigma_1 \sim \sigma_2$. Switching defines an equivalence relation on $\mathcal{S}(G)$ (also on the set of signatures). We write $[\Sigma]$ to denote the switching equivalence class of the signed graph Σ. If $\Sigma = (G, \sigma)$, then $[G, \sigma]$ stands for $[\Sigma]$. We use $\Omega_{\mathcal{S}}(G)$ to represent the set of all equivalence classes of signed graphs with the same underlying graph G. The notations $\mathcal{S}(G)$ and $\Omega_{\mathcal{S}}(G)$ are taken from [10].

For a signed graph (G, σ), a signature σ' is said to be a *minimum signature* if σ' is equivalent to σ and the number of negative edges in σ' is minimum among all equivalent signatures of σ. A signed graph (G, σ) is said to be *reduced* if σ is a minimum signature.

The following characterization for two signed graphs to be switching equivalent is given by Zaslavsky [58].

Lemma 1.2.1. *Two signed graphs* Σ_1 *and* Σ_2 *are switching equivalent if and only if they have the same set of negative cycles.*

We say that two signed graphs are *isomorphic* to each other if there exists a graph isomorphism between their underlying graphs preserving the edge signs. Two signed graphs are *switching isomorphic* if one is isomorphic to a switching of the other. Note that if two signed graphs are switching isomorphic to each other, then it may happen that they are neither switching equivalent nor isomorphic to each other. For example, consider the signed graphs shown in Figure 1.2.1. Observe that Σ_1 and Σ_2 are neither switching equivalent (due to Lemma 1.2.1) nor isomorphic (as they have different number of negative edges) to each other. However, the signed graphs Σ_1 and Σ_2 are switching isomorphic to each other, as Σ_2 is switching equivalent to Σ_3, and Σ_3 is isomorphic to Σ_1.

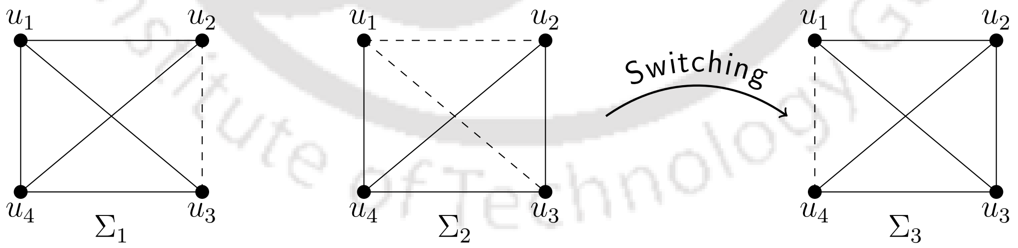


Figure 1.2.1. The signed graphs Σ_1 and Σ_2 are switching isomorphic but they are neither switching equivalent nor isomorphic

The classification of switching non-isomorphic signed graphs is very useful, especially when we deal with some switching invariant parameters in signed graphs. As far as we know, such classifications are known only for few graphs, namely Petersen graph [60], Heawood graph [49, Chapter 4] and complete graphs of order up to five. However, there are some general results on counting switching non-isomorphic graphs. For example, Y. Bagheri et al. [10] recently proved that the number of switching non-isomorphic signed graphs over a given graph G is equal to the number of orbits of its automorphism group acting on the set of all signed graphs over G. Also, Mallows and Sloane [39] proved that the number of switching non-isomorphic signed graphs over K_n is equal to the number of Euler graphs on n vertices.

Two signed graphs are *automorphic* if they are isomorphic to each other and their underlying graphs are identical. The group of all automorphisms of an unsigned graph G is denoted by $\mathrm{Aut}(G)$. Similarly, the group of all automorphisms of a signed graph Σ is denoted by $\mathrm{Aut}(\Sigma)$.

A *switching isomorphism class* of G is the collection of all signed graphs over a graph G in which any two signed graphs are either switching equivalent or switching isomorphic to each other. Clearly, the number of switching isomorphism classes of a graph G is equal to the number of pairwise switching non-isomorphic signed graphs over G. Describing all the switching isomorphism classes of a signed graph is one of the fundamental problems in signed graph theory. Note that for a given graph G with m edges, signed graphs over G can be constructed in 2^m ways. However, many of them may be switching equivalent to each other. In [41], the authors determined the number of switching non-equivalent signed graphs.

Lemma 1.2.2. *[41] There are 2^{m-n+c} switching non-equivalent signed graphs over a graph on n vertices with m edges and c components.*

1.3 Frustration in Signed Graphs

The study of structural balance in (social) networks can be traced back to Heider [35]. Cartwright and Harary [21] incorporated Heider's theory into the graph theoretical notion of signed graph balance. Signed graphs became an important topic of research after that. The interactive survey [56] presents an overview of the huge body of knowledge on signed graphs and related topics. Many different scientific disciplines use signed graphs. Balance in signed graphs received a lot of attention in a variety of applications, as evidenced in [7, 8]. The dissatisfaction of a signed graph, which shows how far away a signed graph is from being balanced, is very interesting.

A signed graph is called *unbalanced* if it is not balanced, that is, it has at least one negative cycle. The frustration index is a significant way to measure how unbalanced a signed graph is. An unbalanced signed graph can be made balanced by deleting some of its edges or vertices.

The least number of edges of a signed graph (G, σ) whose deletion results in a balanced signed graph is called the *frustration index* of (G, σ), and it is denoted by $l(G, \sigma)$. This number is implicated in certain questions of social psychology [1] and spin-glass physics [53]. The *maximum frustration*, denoted $D(G)$, of a graph G is the maximum frustration index over all possible signatures over the graph. The least number of vertices of a signed graph whose deletion results in a balanced signed graph is called the *frustration number* of the signed graph. The frustration number of a signed graph (G, σ) is denoted by $l_0(G, \sigma)$. The parameters frustration index and frustration number are invariant under switching, see [11, 60] for details.

In [60], the author determined the frustration indices of all signed Petersen graphs. In [49], the author determined the frustration indices of all signed Heawood graphs. Using coding theory, Zaslavsky and Sole [50] improved certain bounds on the frustration index of signed graphs and they treated several classes of graphs, including wheels, circular ladders and the rectangular grids etc. They also observed that the maximum frustration index of G is equal to the coverig radius of the cutset code (cocycle code) of G. For a detail study of the covering radius of the cutset code of a graph, we refer the reader to [50, 51]. Garry Bowlin [19] proved that the maximum frustration of a complete bipartite graph $K_{l,r}$ is bounded above by

$$\frac{lr}{2}\left(1 - \frac{1}{2^{l-1}}\binom{l-1}{\lfloor \frac{l-1}{2} \rfloor}\right).$$

The problem of computing the frustration index of a signed graph is equivalent to computing the ground state of an Ising spin glass model, which is known to be NP-hard problem [11]. However, the frustration index of signed planar graphs can be computed in polynomial time [37]. Samin Aref et al. [7] developed a method, which is efficient and exact for computing the frustration index of signed graphs.

1.3.1 Critical Signed Graphs

For a subset X of $V(G)$, we denote by $[X, X^c]$ the set of edges of G with one end vertex in X and the other end vertex in X^c, and by $|[X, X^c]|$ the number of edges in $[X, X^c]$. The set $[X, X^c]$ is called the *edge-cut* (or *cut*) of G associated with X. By $G[X, X^c]$, we denote the subgraph of G induced by the edges of $[X, X^c]$. Similarly, $\Sigma[X, X^c]$ denotes the signed subgraph induced by the edges of Σ with one end vertex in X and the other end vertex in X^c. The *weight* of $[X, X^c]$, denoted $\omega[X, X^c]$, is equal to the number of positive edges in $[X, X^c]$ minus the number of negative edges in $[X, X^c]$. Further, a cut $[X, X^c]$ is called *negative*, *positive* or *equilibrated* according as $\omega[X, X^c]$ is negative, positive or zero, respectively.

Lemma 1.3.1. *[19] A signed graph (G, σ) is reduced if and only if $\omega[X, X^c]$ is non-negative for each non-empty proper subset X of $V(G)$.*

Cappello and Steffen [20] recently studied signed graphs that are critical with respect to the frustration index. A signed graph (G,σ) with frustration index k is called *k-frustrated*. If the signatures π and σ of G are switching equivalent with $|\pi| = t$, then π is called a *t-signature* of (G,σ). Note that if π is a t-signature of (G,σ), then $t \geq l(G,\sigma)$. Let k be a positive integer. A k-frustrated signed graph (G,σ) is *k-critical* if $l(G-e,\sigma') < k$ for each edge e of G, where $\sigma' = \sigma \cap E(G-e)$. The following characterization of critical signed graphs is given in [20, Theorem 2.2].

Theorem 1.3.1. *[20] Let k be a positive integer. If (G,σ) is a k-frustrated signed graph, then the following statements are equivalent.*

1. *(G,σ) is k-critical.*
2. *Each edge of G is contained in a k-signature of (G,σ).*
3. *If π is a k-signature of (G,σ), then each positive edge is contained in an equilibrated edge-cut of (G,σ).*

The followings propositions give some examples of critical signed graphs.

Proposition 1.3.1. *[20, Prop. 2.4] Let G be a plane triangulation and $n \geq 3$. If G has n vertices, then $-G$ is $(n-2)$-critical.*

A *wheel* W_n, where $n \geq 3$, is the join of a vertex and a cycle C_n.

Proposition 1.3.2. *[20, Prop. 2.5] The antibalanced odd wheel $-W_{2k+1}$ is $(k+1)$-critical.*

The *hypercube* of dimension k, denoted H_k, is defined as follows. The vertex set of H_k consists of elements of $\mathbb{Z}_2^k$, and two vertices are adjacent if their Hamming distance is 1. In other words, if $\{e_1,...,e_k\}$ is the standard basis of $\mathbb{Z}_2^k$, then the vertices x and y of H_k are adjacent if $x-y \in \{e_1,...,e_k\}$. Two vertices of H_k are *antipodal* if their Hamming distance is k. For example, $(1,0,0,0)$ and $(0,1,1,1)$ are antipodal vertices in H_4. The *projective cube* of dimension k, denoted H_k^*, is a hypercube H_k along with a new edge between every pair of antipodal vertices. The *augmented hypercube* is the signed graph (H_k^*,σ), where $\sigma = E(H_k^*) \setminus E(H_k)$. In the study of signed graph homomorphism, augmented hypercubes are very useful, see [40, 41] for details.

Proposition 1.3.3. *[20, Prop. 2.6] If k is a positive integer, then the augmented hypercube (H_k^*,σ) of dimension k is 2^{k-1}-critical.*

1.4 Double Domination in Signed Graphs

Dominating set problem is one of the classical combinatorial optimization problems in graph theory. Double domination is a special kind of domination. The double domination in graph was studied by F. Harary and T. Haynes [32]. B. D. Acharya [2] later extended the domination theory to several forms of signed graphs. Domination theory is useful in a variety of theoretical and practical real-world problems such as optimization, communication and network challenges etc. See [16, 22, 29, 30] for more information on double domination number.

A subset D of vertices is said to be a *dominating set* (DS) of G if each vertex of G is either in D or is adjacent to at least one vertex in D. A DS of the least size of a graph is called a *minimum dominating set* of the graph. The size of a minimum DS is called the *domination number* (DN) of G. The literature on the study of various kind of domination numbers is available in the books [33] and [34].

The *open neighborhood* of a vertex v in a graph G, denoted $N(v)$, is the set $\{u \in V(G) : uv \in E(G)\}$. Similarly, the *closed neighborhood*, denoted $N[v]$, is $N(v) \cup \{v\}$. Harary and Haynes [32] defined a generalization of domination as follows: a subset D of $V(G)$ is a *k-tuple dominating* set of G if for every vertex $v \in V(G)$, either $v \in D$ and has at least $k-1$ neighbors of v in D or $v \in D^c$ and has at least k neighbors in D. Equivalently, a subset D of $V(G)$ is a *k-tuple dominating* set of G if $|N[v] \cap D| \geq k$ for each $v \in V(G)$. A 2-tuple dominating set of a graph G is called a *double dominating set* (DDS) of G. A DDS of the least size is called a *minimum DDS*. The size of a minimum DDS of G is called the *double domination number* (DDN) of G, and it is denoted by $\gamma_{\times 2}(G)$.

In 2013, Acharya [2] extended the theory of domination to signed graphs. Ashraf and Germina [9] defined double domination in signed graphs as follows.

Definition 1.4.1. For a signed graph Σ with vertex set V, a subset D of V is said to be a *double dominating set* (DDS) of Σ if it satisfies the following conditions: (i) $|N[v] \cap D| \geq 2$ for each $v \in V$, and (ii) $\Sigma[D, D^c]$ is balanced. The size of a minimum DDS of Σ is called the *double domination number* (DDN) of Σ, and it is denoted by $\gamma_{\times 2}(\Sigma)$.

Some major distinctions arise between the double domination in unsigned graphs and that of signed graphs. One of such distinctions is the following. Supersets of a double dominating set is a double dominating set in an unsigned graph. However, this fails for signed graphs. For example, see Figure 1.4.1.

Some existing results on DDN of signed graphs are presented below.

Theorem 1.4.1. *[9, Theorem 3.1] Every signed graph with no isolated vertex has a*

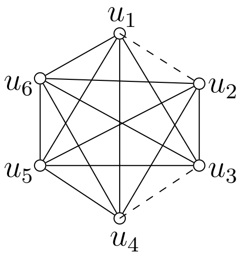


Figure 1.4.1. The set $\{u_1, u_2\}$ is a DDS but $\{u_1, u_2, u_3\}$ is not a DDS

double dominating set.

Theorem 1.4.2. *[9, Theorem 3.2] If Σ is a signed graph on n vertices without isolated vertices, then $2 \leq \gamma_{\times 2}(\Sigma) \leq n$. Moreover, these bounds are sharp.*

Theorem 1.4.3. *[9, Theorem 3.3] Let $l \geq 3$ and $r \geq 3$. If $(K_{l,r}, \sigma)$ is a signed complete bipartite graph, then $4 \leq \gamma_{\times 2}(K_{l,r}, \sigma) \leq l + r - 2$.*

The sharpness of the upper bound in Theorem 1.4.3 is not known.

1.5 Vertex Coloring of Signed Graphs

In [59], Zaslavsky initiated the study of vertex coloring of signed graphs. Zaslavsky used signed colors so that coloring remains compatible with switching operation. More precisely, a *coloring* of a signed graph (G, ϕ) in $2k + 1$ *signed colors* is a mapping $c : V(G) \longrightarrow \{0, \pm 1, \ldots, \pm k\}$. Similarly, a *zero-free coloring* of a signed graph (G, ϕ) in $2k$ *signed colors* is a mapping $c : V(G) \longrightarrow \{\pm 1, \ldots, \pm k\}$. A coloring c of (G, ϕ) is called *proper* if $c(x) \neq \phi(e)c(y)$ for each edge e of G, where $e = xy$ and $\phi(e)$ is the sign of the edge e.

Zaslavsky [57, 59] defined the *chromatic polynomial* $\chi_\Sigma(\lambda)$ of a signed graph Σ to be the function such that $\chi_\Sigma(2k + 1)$ equals the number of proper colorings of Σ in $2k + 1$ signed colors. The *zero-free chromatic polynomial* $\chi^b_\Sigma(\lambda)$ of a signed graph Σ is the function, where $\chi^b_\Sigma(2k)$ counts the zero-free proper colorings in $2k$ signed colors. Consequently, the *chromatic number* $\gamma(\Sigma)$ of a signed graph Σ is the smallest non-negative integer k for which $\chi_\Sigma(2k + 1) > 0$. Similarly, the *zero-free chromatic number* $\gamma^*(\Sigma)$ of a signed graph Σ is the smallest non-negative integer k for which $\chi^b_\Sigma(2k) > 0$. If two signed graphs are switching equivalent, then they have the same chromatic number and the same zero-free chromatic number. The interaction between colorings and zero-free colorings through the chromatic polynomials was a major focus of Zaslavsky's work on signed graph coloring.

The Petersen graph is used as a reference for many proposed results in graph theory. Zaslavsky [60] showed that there are only six distinct signed Petersen graphs up to switching isomorphism. These six signed Petersen graphs have distinct chromatic polynomials [60], and hence can be distinguished by this signed-graph invariant. Zaslavsky did not compute the chromatic polynomials of these six signed Petersen graphs. However, he demonstrated that at 3, the chromatic polynomials evaluate to distinct numbers [60, Table 9.2]. He even conjectured that the six distinct signed Petersen graphs have distinct zero-free chromatic polynomials, and that both types of chromatic polynomials have distinct evaluations at any positive integer [60, Conjecture 9.1]. Beck et al. [12, Theorem 1] confirmed this conjecture.

In 2016, Máčajová et al. [38] modified the definition of chromatic numbers given by Zaslavsky [57]. However, the definition of the chromatic number given in [38] also relies on the idea of Zaslavsky [59]. For $n \geq 1$, let $M_n = \{0, \pm 1, \ldots, \pm k\}$ if $n = 2k+1$, and $M_n = \{\pm 1, \ldots, \pm k\}$ if $n = 2k$. According to Máčajová et al. [38], an *n-coloring* of a signed graph (G, ϕ) is a mapping $c : V(G) \longrightarrow M_n$. An n-coloring c of (G, ϕ) is *proper* if $c(x) \neq \phi(e)c(y)$ for every edge e of G, where $e = xy$ and $\phi(e)$ is the sign of the edge e. The *chromatic number* $\chi(G, \phi)$ of (G, ϕ) is the smallest number n such that (G, ϕ) admits a proper n-coloring. Equivalently, the *chromatic number* $\chi(G, \phi)$ of (G, ϕ) is defined to be the least number of the set

$$\left\{2k+1 : \chi_{(G,\phi)}(2k+1) > 0,\ k \in \mathbb{N} \cup \{0\}\right\} \cup \left\{2r : \chi^{b}_{(G,\phi)}(2r) > 0,\ r \in \mathbb{N} \cup \{0\}\right\}.$$

Thus for every signed graph (G, ϕ), there is a direct relationship between the parameters χ, γ and γ^*. See [52] for details. This relationship is expressed as

$$\chi(G, \phi) = \gamma(G, \phi) + \gamma^*(G, \phi).$$

Furthermore, Máčajová et al. [38] presented bounds on the chromatic number of a signed graph in terms of the chromatic number, the acyclic chromatic number, and the arboricity of the underlying graph. They also proved a Brooks-type theorem, one of the central theorems in the theory of graph coloring, for signed graphs. It states that the chromatic number of every simple signed graph Σ is bounded above by $\Delta(\Sigma)$, the maximum degree of Σ, unless Σ is a balanced complete graph, a balanced odd cycle, or an unbalanced even cycle. See [38] for details.

The chromatic numbers and the chromatic polynomials of signed graphs, introduced by Zaslavsky [57, 59], are invariant under switching. The chromatic number $\chi(\Sigma)$ of a signed graph Σ is also switching invariant. The chromatic numbers of various signed graph families are studied in the literature. For instance, see [15, 41, 60]. However, chromatic polynomials of signed graphs are less studied. As far as we know, the chromatic polynomials are konwn only for a few classes of signed graphs, namely, signed Petersen graphs and signed complete graphs of order up to five. Therefore, it is worth seeing more of them. In [12], Mathias Beck and his team published a SAGE code that produces both kind of chromatic polynomials as output when a signed graph is given as input. Using that code, they presented explicit formulas for both kind of chromatic polynomials of the six switching non-isomorphic signed Petersen graphs. They also presented both kinds of polynomials of switching non-isomorphic signed complete graphs of order at most five.

1.5.1 Coloring via Homomorphism

Homomorphism on signed graphs is another approach to define a chromatic number of signed graphs. B. Guenin [28], in an unpublished manuscript, introduced the notion of signed graph homomorphism and showed how some well-known conjectures can be captured with the help of this notion. The authors in [41] used this concept to define a chromatic number for signed graphs. Homomorphisms on signed graphs are defined as homomorphisms on switching equivalence classes of signed graphs.

We say that there is a *homomorphism* from a switching equivalence class $[G, \sigma_1]$ to another class $[H, \sigma_2]$ if there is a representative $(G, \sigma_1^{'})$ of $[G, \sigma_1]$ and a representative $(H, \sigma_2^{'})$ of $[H, \sigma_2]$ together with a mapping $\psi : V(G) \longrightarrow V(H)$ that preserves adjacency and edge signs of $(G, \sigma_1^{'})$. If there is a homomorphism from $[G, \sigma_1]$ to $[H, \sigma_2]$, then we say that $[H, \sigma_2]$ bounds $[G, \sigma_1]$. The *h-chromatic number*, denoted $\chi_{Hom}(G, \sigma)$, of a signed graph (G, σ) is the smallest order of a signed graph (H, π) such that $[H, \pi]$ bounds $[G, \sigma]$. The chromatic number and the h-chromatic number of a signed graph can be distinguished by considering an unbalanced cycle of length 4. More precisely, if (C_4, σ) is unbalanced, then $\chi(C_4, \sigma) = 3$ while $\chi_{Hom}(C_4, \sigma) = 4$. Reza Naserasr et al. [41] also obtained bounds on the h-chromatic number of various classes of signed graphs. For example, the h-chromatic number of a signed graph, whose underlying graph is K_4-minor free, is bounded above by 5. For a survey on the theory of signed graph colorings, see Steffen and Vogel [52].

Vertex coloring of graphs and signed graphs can also be interpreted geometrically via hyper-plane arrangements. For details, we refer the reader to [13, 25].

1.6 Parity Signed Graphs

Recently, Acharya and Kureethara [3] introduced a special type of signed graph called the parity signed graph. In [4], Acharya et al. showed that parity signed graphs also have sociological aspects. This concept of parity signed labeling is based on the assignment of integers in $\{1, \ldots, |V(G)|\}$ to the vertices of a graph G. It is equivalent to a partition of the vertex set of a graph into two subsets A and B such that $||A| - |B|| \leq 1$. In [4], the authors characterized some families of parity signed graphs, namely, signed stars, bistars, cycles, paths and complete bipartite graphs.

A signed graph Σ is said to be *parity signed* if there exists a bijective map $f : V(G) \longrightarrow \{1, \dots, |V(G)|\}$ such that $\phi(uv) = +1$ if and only if $f(u)$ and $f(v)$ are of the same parity, where $\Sigma = (G, \phi)$ and uv is an edge of G. The rna number of a graph G, denoted $\sigma^{-}(G)$, is the least number of negative edges among all possible parity signed graphs over G. The rna number is also equal to the least size of a cut that has nearly equal sides. The rna number of some families of graphs such as stars, wheels, paths, cycles and complete graphs are known. For details, see [3, 4].

1.7 Organization of the Thesis

In this thesis, we discuss various signed graph parameters, mainly, for the class of generalized Petersen graphs, I-graphs and book graphs. The definitions of these classes are given below.

The family of generalized Petersen graphs was introduced by Coxeter [23] in 1950, and was given its name by Watkins [54] in 1969.

Throughout the thesis, the notation $[n]$ represents the set $\{0, 1, \dots, n-1\}$ for each $n \in \mathbb{N}$.

Definition 1.7.1. Let n and k be positive integers such that $2 \le 2k < n$. The *generalized Petersen graph* $P_{n,k}$ is defined by $V(P_{n,k}) = \{u_i, v_i : i \in [n]\}$ and $E(P_{n,k}) = \{u_i u_{i+1}, v_i v_{i+k}, u_i v_i : i \in [n]\}$, where the subscripts are read modulo n.

For example, the generalized Petersen graph $P_{6,1}$ is shown in Figure 1.7.1a. It is clear that $P_{n,k}$ is a cubic graph and $P_{5,2}$ is the well known Petersen graph. The vertices in $\{u_i : i \in [n]\}$ are called the *outer vertices* of $P_{n,k}$. Similarly, the vertices in $\{v_i : i \in [n]\}$ are called the *inner vertices* of $P_{n,k}$. Sometimes, we also call the outer vertices and the inner vertices to be the *u-vertices* and *v-vertices*, respectively. Further, the sets of all u-vertices and all v-vertices of $P_{n,k}$ are denoted by V_u and V_v, respectively. The edges $u_i v_i$ for $i \in [n]$ are called the *spokes* of $P_{n,k}$, and the set of all spokes is denoted by S_s. The cycle induced by the u-vertices is called the *outer cycle* of $P_{n,k}$, and it is denoted by C_o. The cycle(s) induced by the v-vertices is(are) called the *inner cycle(s)* of $P_{n,k}$. If $\gcd(n,k) = d$ and $d > 1$, then the subgraph induced by the v-vertices consists of d pairwise disjoint $\frac{n}{d}$-cycles. These inner cycles are denoted by $C_1, \dots, C_d$, where $v_i \in V(C_{i+1})$ for $i \in [d]$. If $d = 1$, then $P_{n,k}$ has only one inner cycle, and in this case the inner cycle is denoted by C_I.

Definition 1.7.2. An I-graph $I(n,j,k)$, for $1 \le j, k < n$, is a graph defined by

$$V(I(n,j,k)) = \{u_i, v_i : i \in [n]\} \text{ and}$$

$$E(I(n,j,k)) = \{u_i u_{i+j}, u_i v_i, v_i v_{i+k} : i \in [n]\},$$

where the subscripts are read modulo n.

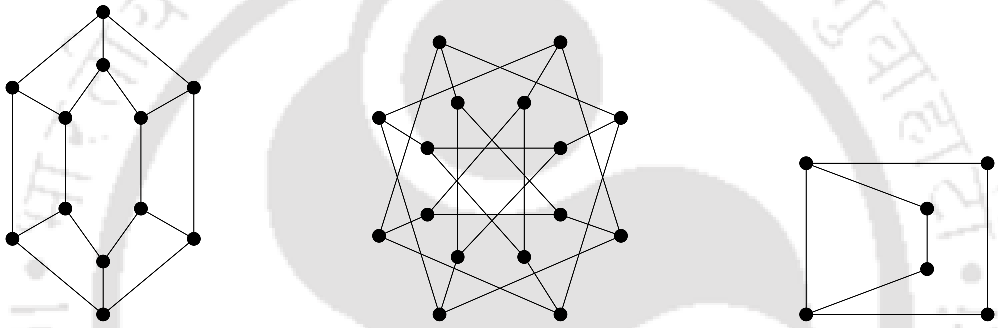
For example, the graph $I(8,2,3)$ is shown in Figure 1.7.1b. Note that the class of generalized Petersen graphs is a sub-class of the class of I-graphs.

Definition 1.7.3. For $m \geq 3$ and $n \geq 1$, the m-cycle *book graph* $B(m,n)$ consists of n copies of the cycle C_m with one common edge. Indeed

$$V(B(m,n)) = \{u,v\} \cup \{u_j^i : i \in \{1,\ldots,n\},\ j \in \{1,\ldots,m-2\}\},$$

where uv is the common edge to the cycles C_m^i, and $C_m^i = uu_1^iu_2^iu_3^i...u_{m-3}^iu_{m-2}^ivu$ for $1 \leq i \leq n$.

For example, the graph $B(4,2)$ is shown in Figure 1.7.1c.



(a) The graph $P_{6,1}$ (b) The graph $I(8,2,3)$ (c) The book graph $B(4,2)$

Figure 1.7.1. The graphs $P_{6,1}$, $I(8,2,3)$ and $B(4,2)$

The thesis contains seven chapters. Apart from the present chapter, the remaining chapters are organized as follows.

Chapter 2: Switching Isomorphism Classes. In this chapter, we prove that there are exactly 16 switching isomorphism classes of signed graphs arising from the complete graph K_6. We count the number of non-isomorphic signatures over $P_{2n+1,1}$ of size two for $n \geq 1$. Furthermore, we prove that every switching equivalence class of signatures of $P_{2n+1,1}$ contains a signature of size at most $n+1$. We also classify all the switching isomorphism classes of signed generalized Petersen graphs over $P_{3,1}, P_{5,1}$ and $P_{7,1}$. We prove that there are exactly $n+1$ switching non-isomorphic signed book graphs over $B(m,n)$.

Chapter 3: Frustration in Signed Graphs. In this chapter, we discuss frustrations in signed graphs and show that the frustration index is invariant under switching operation. We prove that the maximum frustration of the generalized Petersen graph $P_{n,k}$ is bounded above by $\left\lfloor \frac{n}{2} \right\rfloor + 1$ for $\gcd(n,k) = 1$, and this bound is proved to be sharp for $k \in \{1,2,3\}$. We also prove that the maximum frustration of $P_{n,k}$ is bounded above by $d\left\lfloor \frac{n}{2d} \right\rfloor + d + 1$, where $\gcd(n,k) = d \geq 2$. The frustration indices and the frustration numbers are also determined for all switching non-isomorphic signed complete graphs on six vertices. We also prove that the maximum frustration index of the book graph $B(m, n)$ is $\left\lceil \frac{n}{2} \right\rceil$.

Chapter 4: Double Domination in Signed Graphs. We prove in this chapter that if D is a DDS of a cubic graph G such that $|D| = \frac{|V(G)|}{2}$, then $G[D, D^c]$ admits a cycle decomposition. Also, we show that if D with $|D| = \frac{|V(G)|}{2}$ is not a DDS of a cubic graph G, then it is not necessarily true that $G[D, D^c]$ admits a cycle decomposition. We obtain a lower bound on the DDN of signed cubic graphs. Next, we obtain some bounds on the DDN of signed generalized Petersen graphs and signed I-graphs. We prove that the DDN of a signed complete graph over K_n can be $n - 1$ for $n \geq 5$. However, this number is known to be exactly 2 for unsigned complete graph K_n for $n \geq 2$. Also, we compute the DDN of all switching non-isomorphic signed Petersen graphs and of all switching non-isomorphic signed complete graphs over K_6.

Chapter 5: Chromatic Polynomials of Signed Book Graphs. In this chapter, we prove that the chromatic number for the class of signed book graphs is either 2 or 3. We also obtain explicit formulas for the chromatic polynomials and the zero-free chromatic polynomials of switching non-isomorphic signed book graphs.

Chapter 6: The rna Number of Generalized Petersen Graphs. In this chapter, we discuss the rna number of generalized Petersen graphs. We find an upper bound on the smallest order of $(4n - 1)$-regular graphs having rna number 1. We also construct a $(4n + 1)$-regular graph of smallest order having rna number 1.

Chapter 7: Conclusion and Extension. In this chapter, we provide some concluding remarks based on the work presented in this thesis. We also identify few open problems for future research.

Some of the results in the thesis have already been published. The results of

- Sections 2.2 and 3.3 are published in

  [46] D. Sehrawat and B. Bhattacharjya. Signed complete graphs on six vertices and their frustration indices. *Advances and Appl. in Discrete Math.* 24 (2020), 129-142.

- Section 2.3 are published in

  [44] D. Sehrawat and B. Bhattacharjya. Non-isomorphic signatures on some generalized Petersen graph. *Elect. J. Graph Theory and Appl.* 9 (2) (2021), 235-255.

- Chapter 3, except Sections 3.3 and 3.4, are published in

[43] D. Sehrawat and B. Bhattacharjya. Maximum frustration in signed generalized Petersen graphs. *Indian J. Discrete Math.* 5(2) (2019), 77-93.

- Chapter 4, except Sections 4.5, are published in

  [47] D. Sehrawat and B. Bhattacharjya. Some bounds on the double domination of signed generalized Petersen graphs and signed I-graphs. *Indian J. Discrete Math.* 5(2) (2019), 63-75.

- Chapter 5 and Section 2.4 are published in

  [42] D. Sehrawat and B. Bhattacharjya. Chromatic polynomials of signed book graphs. *Theory and Appl. of Graphs* 9(1) (2022), #4.

All the graphs considered in this thesis are finite, undirected and simple, unless otherwise stated. For all the graph theoretic terms that are used in this thesis but not defined, we refer the reader to [18, 55].

2

Switching Isomorphism Classes of Signed Graphs

In this chapter, we classify switching isomorphism classes of signed graphs arising from K_6, $P_{3,1}$, $P_{5,1}$ and $P_{7,1}$, where $P_{n,k}$ denotes the generalized Petersen graph. We count the non-isomorphic signatures of size two in $P_{2n+1,1}$ for $n \geq 4$. We also show that each switching equivalence class of signatures of $P_{2n+1,1}$ contains a signature of size at most $n+1$. More precisely, the size of a minimum signature of $P_{2n+1,1}$ is at most $n+1$. Furthermore, the switching isomorphism classes of the book graph $B(m,n)$ are also determined, where $m \geq 3$ and $n \geq 1$.

2.1 Definitions and Preliminaries

A *signed graph* is a graph whose edges have $+1$ or -1 signs. More precisely, a signed graph Σ is a pair (G,ϕ), where G is the *underlying graph* of Σ and ϕ is a *function* from $E(G)$ into $\{+1,-1\}$. Clearly, such a function is determined if the set of edges mapped onto -1 is known. The function ϕ is called the *sign function* of (G,ϕ). Throughout the thesis, we assume that σ represents the set of negative edges of Σ, that is, $\sigma \coloneqq \phi^{-1}(-1) = \{e \in E(G) : \phi(e) = -1\}$. Therefore, a signed graph Σ is also denoted by (G,σ), where σ is called the *signature* of (G,ϕ). Similarly, Σ_i denotes the signed graph (G,σ_i). Sometimes, the underlying graph of a signed graph (G,σ) is clear from the context. In that case, we also denote the signed graph (G,σ) by σ itself.

A vertex v in G is said to have *degree* k if k number of edges are incident to v. Degree of a vertex v in G is denoted by $d_G(v)$. In a signed graph, the number of negative edges incident to a vertex v is known as the *negative degree* of v, while the number of positive edges incident to v is known as the *positive degree* of v. The positive degree and the negative degree of v in a signed graph Σ are denoted by $d^+_\Sigma(v)$ and $d^-_\Sigma(v)$, respectively. Clearly, $d_G(v) = d^+_\Sigma(v) + d^-_\Sigma(v)$ for every $v \in V(G)$, where $\Sigma = (G,\sigma)$. If the signed graph Σ and its underlying graph G are clear from the context, we simply write $d(v)$, $d^+(v)$ and $d^-(v)$ in place of $d_G(v)$, $d^+_\Sigma(v)$ and

$d^-_\Sigma(v)$, respectively.

Recall that a cycle is *positive* if the product of signs of the edges in the cycle is positive and *negative*, otherwise. A signed graph is said to be *balanced* if every cycle in the signed graph is positive. The sign $\phi(P)$ of a path P in a signed graph Σ is defined by $\phi(P) = \prod_{e\in E(P)} \phi(e)$. The following two characterizations for balanced signed graphs are given by Harary [31].

Theorem 2.1.1. *[31] A signed graph Σ is balanced if and only if its vertex set V is partitioned into two disjoint subsets V_1 and V_2, one of which may be empty, such that all edges having their both end vertices either in V_1 or in V_2 are positive and all edges in $[V_1, V_2]$ are negative.*

Theorem 2.1.2. *[31] A signed graph is balanced if and only if for each pair u, v of distinct vertices, all paths joining u and v have the same sign.*

Recall that a signed graph Σ_1 is switching equivalent to another signed graph Σ_2 if Σ_1 is obtained from Σ_2 by a sequence of switchings. Equivalently, we say that Σ_1 is switching equivalent to Σ_2 if there exists a function $f : V(G) \longrightarrow \{+1, -1\}$ such that $\phi_2(e) = f(u)\phi_1(e)f(v)$ for each edge e, where $\Sigma_1 = (G, \phi_1)$, $\Sigma_2 = (G, \phi_2)$ and $e = uv$. Switching defines an equivalence relation on the set of all signed graphs over G (also on the set of signatures). A switching equivalence class of a signed graph Σ is denoted by $[\Sigma]$.

We say that two signed graphs Σ_1 and Σ_2 are *isomorphic*, denoted $\Sigma_1 \cong \Sigma_2$, if there exists a graph isomorphism $\psi : V(G) \longrightarrow V(H)$ preserving the edge signs, where $\Sigma_1 = (G, \sigma)$ and $\Sigma_2 = (H, \pi)$. The signed graphs Σ_1 and Σ_2 are called *switching isomorphic* if Σ_1 is isomorphic to a switching of Σ_2. That is, there exists a signed graph Σ_2' which is equivalent to Σ_2 such that $\Sigma_1 \cong \Sigma_2'$. We use the notation $\Sigma_1 \simeq \Sigma_2$ to say that Σ_1 is switching isomorphic to Σ_2.

Recall that a *switching isomorphism class* of a graph G is the collection of all signed graphs over G in which any two signed graphs are either switching equivalent or switching isomorphic to each other. Zaslavsky [60] showed that there are only six switching isomorphism classes of signed graphs arising from the Petersen graph.

Remark 2.1.1. Let the signed graphs Σ_1 and Σ_2 have the same underlying graph and have different number of negative cycles. Due to Lemma 1.2.1, Σ_1 cannot be switching equivalent to Σ_2. Note that the image of a negative cycle under an isomorphism of a signed graph is also a negative cycle. Therefore, Σ_1 and Σ_2 cannot be switching isomorphic to each other.

The authors in [10] determined the number of switching non-isomorphic signed graphs over a graph G in terms of orbits of the action of $\mathrm{Aut}(G)$ on $\Omega_S(G)$.

Theorem 2.1.3. *[10] Let G be a simple graph and let $Aut(G)$ act on $\Omega_{\mathcal{S}}(G)$. Two signed graphs (G, σ_1) and (G, σ_2) are switching isomorphic if and only if (G, σ_1) and (G, σ_2) belong to the same orbit. In particular, the number of switching non-isomorphic signed graphs is equal to the number of orbits of this group action.*

We denote by $\psi(n, \Delta)$ the number of non-isomorphic graphs on n vertices with maximum degree at most Δ. A lower bound on the number of pairwise switching non-isomorphic signed complete graphs on n vertices is given in the following theorem.

Theorem 2.1.4. *[10] The number of pairwise switching non-isomorphic signed graphs over a complete graph on n vertices is at least $\psi(n, \lfloor \frac{n}{4} \rfloor - 1)$, where $n \geq 4$.*

Recall for a signed graph (G, σ) that a signature σ' is said to be a *minimum signature* if σ' is equivalent to σ and the number of negative edges in σ' is minimum among all equivalent signatures of σ. We denote the number of edges in σ' by $|\sigma'|$. We say that a signed graph is *reduced* if its signature is a minimum signature.

If (G, σ) is balanced, then it is equivalent to a reduced signed graph (G, σ'), where $\sigma' = \emptyset$. We observe that there may be more than one reduced signed graph equivalent to a given signed graph (G, σ). For example, for the signed graph (K_3, σ), where K_3 is the complete graph with vertex set $\{u_1, u_2, u_3\}$ and $\sigma = \{u_1u_2, u_2u_3, u_3u_1\}$, the equivalent signatures $\{u_1u_2\}$ and $\{u_2u_3\}$ are minimum. This shows that minimum signature (or reduced signed graph) of a signed graph is not unique.

We say that two signatures σ_1 and σ_2 of G are *automorphic* if there exists an automorphism f of G such that $uv \in \sigma_1$ if and only if $f(u)f(v) \in \sigma_2$. If two signatures are automorphic, then they are said to be *automorphic type* signatures. If two signatures σ_1 and σ_2 of G are not automorphic to each other, then we say that they are distinct automorphic type signatures. More precisely, saying that two signatures are automorphic means that their corresponding signed graphs are automorphic. For example, for the signed graphs $(K_6, \{u_1u_2\})$ and $(K_6, \{u_3u_5\})$, the signatures $\{u_1u_2\}$ and $\{u_3u_5\}$ are automorphic type signatures.

The *distance* between two vertices x and y in G, denoted $d(x, y)$, is the length of a shortest path connecting x and y. The *distance* between two edges e_1 and e_2 in a graph G, denoted $d(e_1, e_2)$, is $\min\{d(u_i, v_j) : i \in \{1, 2\}, j \in \{1, 2\}\}$, where $e_1 = u_1u_2$ and $e_2 = v_1v_2$.

For a signed graph Σ, we define the graph G_Σ such that $V(G_\Sigma) = V(G)$ and $E(G_\Sigma) = \sigma$, where $\Sigma = (G, \sigma)$. Basically, G_Σ is the spanning subgraph of G induced by the negative edges of Σ.

It is important to note that if $\Sigma = (G, \sigma)$ is equivalent to $\Sigma' = (G, \sigma')$ and σ' is minimum, then for each edge cut $[X, X^c]$ in Σ', the number of positive edges in

$[X, X^c]$ is at least as the number of negative edges. For each $v \in V(G)$, we apply this fact to the edge cut $[\{v\}, \{v\}^c]$, if needed. As a result, we obtain the following theorem easily.

Theorem 2.1.5. *Let* $\Sigma = (G, \sigma)$ *and* $\Sigma' = (G, \sigma')$, *where* G *is a graph on* n *vertices. If the signature* σ' *is minimum and equivalent to* σ, *then* $d_{G_{\Sigma'}}(v) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ *for each* $v \in V(G)$.

The following corollary is useful for the study of non-isomorphic signatures in sub-cubic graphs. Note that a graph is called *sub-cubic* if the degree of each vertex in the graph is at most 3.

Corollary 2.1.1. *[60] Every minimum signature of a sub-cubic graph is a matching.*

2.2 Switching Isomorphism Classes of K_6

The complete graph K_n is the graph whose vertex and edge sets are given by $V(K_n) = \{u_1, \ldots, u_n\}$ and $E(K_n) = \{u_iu_j : i, j \in \{1, \ldots, n\}$ and $i \neq j\}$, respectively. The complete graph K_6 is shown in Figure 2.2.1.

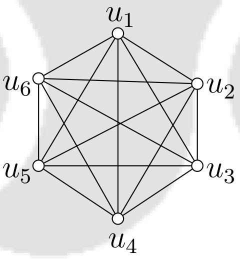


Figure 2.2.1. The complete graph K_6

In the literature, it is known that there are 1, 2, 3 and 7 switching isomorphism classes of K_2, K_3, K_4 and K_5, respectively. For details, see [39, Table 1].

As a particular case of Theorem 2.1.5, we get the following corollary.

Corollary 2.2.1. *Let* (K_6, σ) *be a signed graph. If* σ' *is a minimum signature equivalent to* σ, *then the negative degree of each vertex in* (K_6, σ') *is at most* 2, *and the size of* σ' *is at most six.*

It is clear that there are 2^{15} signed graphs with underlying graph K_6. We prove in Theorem 2.2.1 that only 16 of them are essentially different. Basically, our aim is to determine the distinct signatures of K_6 up to switching isomorphism. Due to Corollary 2.2.1, it is enough to examine the non-isomorphic signatures of K_6 of size at most six such that the negative degree of each vertex in their corresponding signed graphs is at most 2. Therefore, in the further discussion, we consider only those automorphic type signatures of K_6 whose edges form paths or cycles.

It is well known that $\mathrm{Aut}(K_n) = S_n$, where S_n denotes the symmetric group on n symbols. We use $\mathrm{Aut}(K_6)$ to find distinct automorphic type signatures of K_6 of various sizes in the following lemmas.

We denote a signature of size zero by σ_0. Also, we know that K_6 is edge-transitive. Therefore, all signatures of size one are automorphic to each other. One of such signatures is σ_1, where $\sigma_1 = \{u_1u_2\}$.

Lemma 2.2.1. *The number of distinct automorphic type signatures of K_6 of size two is* 2.

Proof. For an automorphic type signature of K_6 of size two, following are the only possibilities.

(i) Edges of the signature form a path of length 2. One of such signatures is σ_2, where $\sigma_2 = \{u_1u_2, u_2u_3\}$.

(ii) The distance between the edges of the signature is 1. One of such signatures is σ_3, where $\sigma_3 = \{u_1u_2, u_3u_4\}$.

It is easy to see that, any other signature of K_6 of size two is automorphic to σ_2 or σ_3. Further, σ_2 and σ_3 are non-automorphic to each other. This proves the lemma. ☐

Lemma 2.2.2. *The number of distinct automorphic type signatures of K_6 of size three is* 4.

Proof. For an automorphic type signature of size three in K_6, the following are the only possibilities.

(i) The edges of the signature form a path of length 3. One of such signatures is σ_4, where $\sigma_4 = \{u_1u_2, u_2u_3, u_3u_4\}$.

(ii) Two edges of the signature form a path and the third edge is at distance 1 from that path. One of such signatures is σ_5, where $\sigma_5 = \{u_1u_2, u_2u_3, u_4u_5\}$.

(iii) The distance between each pair of edges of the signature is 1. One of such signatures is σ_6, where $\sigma_6 = \{u_1u_2, u_6u_3, u_5u_4\}$.

(iv) The edges of the signature form a cycle. One of such signatures is σ_7, where $\sigma_7 = \{u_1u_2, u_2u_3, u_3u_1\}$.

Observe that any other signature of size three of K_6 is automorphic to one of $\sigma_4, \sigma_5, \sigma_6$ or σ_7. Further, these four signatures are pairwise non-automorphic. ☐

Lemma 2.2.3. *The number of distinct automorphic type signatures of size four in* K_6 *is* 5.

Proof. For an automorphic type signature of K_6 of size four, following are the only possibilities.

(i) All four edges of the signature form a path. One of such signatures is σ_8, where $\sigma_8 = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5\}$.

(ii) Three edges of the signature form a path and the remaining edge is at distance 1 from that path. One such signature is σ_9, where $\sigma_9 = \{u_1u_2, u_2u_3, u_3u_4, u_5u_6\}$.

(iii) The edges of the signature form two vertex-disjoint paths of length 2. One of such signatures is σ_{10}, where $\sigma_{10} = \{u_1u_2, u_6u_1, u_3u_4, u_4u_5\}$.

(iv) Three edges of the signature form a cycle and one edge is disjoint from that cycle. One of such signatures is given by $\sigma_{11} = \{u_1u_2, u_2u_3, u_3u_1, u_5u_6\}$.

(v) All four edges of the signature form a cycle. One of such signatures is σ_{12}, where $\sigma_{12} = \{u_1u_2, u_2u_3, u_3u_6, u_6u_1\}$.

Note that any other signature of size four in K_6 is automorphic to one of $\sigma_8, \sigma_9, \sigma_{10}$, σ_{11} or σ_{12}. Further, these five signatures are pairwise non-automorphic. This proves the lemma. □

Lemma 2.2.4. *The number of distinct automorphic type signatures of size five in* K_6 *is* 4.

Proof. It is easy to see that any subgraph of K_6 having five edges and having maximum degree 2 cannot have two disjoint paths of length 1 and 4 or of length 2 and 3. Therefore, for an automorphic type signature of size five in K_6, following are the only possibilities.

(i) All the edges of the signature form a path of length 5. One of such signatures is σ_{13}, where $\sigma_{13} = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6\}$.

(ii) Four edges of the signature form a cycle and one edge is disjoint from that cycle. One of such signatures is given by $\sigma_{14} = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_5u_6\}$.

(iii) Three edges of the signature form a cycle and two edges form a path of length 2. One of such signatures is given by $\sigma_{15} = \{u_1u_2, u_2u_3, u_3u_1, u_4u_6, u_5u_6\}$.

(iv) All the edges of the signature form a cycle. One of such signatures is σ_{16}, where $\sigma_{16} = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1\}$.

Clearly, any other signature of size five in K_6 is automorphic to one of $\sigma_{13}, \sigma_{14}, \sigma_{15}$ or σ_{16}. Also, these four signatures are pairwise non-automorphic. ☐

Lemma 2.2.5. *The number of distinct automorphic type signatures of size six in* K_6 *is* 2.

Proof. It is easy to see that a subgraph of K_6 having six edges and having maximum degree 2 is either a spanning cycle or union of two 3-cycles. One of the signatures whose edges form a cycle is σ_{17}, where $\sigma_{17} = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_1\}$. Again, one of the signatures whose edges form two disjoint 3-cycles is σ_{18}, where $\sigma_{18} = \{u_1u_2, u_2u_3, u_3u_1, u_4u_5, u_5u_6, u_6u_4\}$. Further, any other signature of K_6 of size six is automorphic to one of σ_{17} or σ_{18}. These two signatures are non-automorphic to each other. This completes the proof of the lemma. ☐

The signatures obtained in the previous five lemmas, along with σ_0 and σ_1, give us 19 distinct automorphic type signatures of K_6, namely, $\sigma_0, \sigma_1, \ldots, \sigma_{18}$. However, a few among these 19 signatures may be switching isomorphic to each other. We have the following observations.

- Switching σ_6 by $\{u_2, u_3, u_4\}$, we get a signature that is automorphic to σ_{17}. Thus σ_6 is switching isomorphic to σ_{17}, that is, $\sigma_6 \simeq \sigma_{17}$.
- Switching σ_{10} by $\{u_1, u_3, u_5\}$, we get a signature that is auotomprphic to σ_{14}. Thus $\sigma_{10} \simeq \sigma_{14}$.
- Switching σ_{13} by $\{u_2, u_4, u_6\}$, we get a signature that is automorphic to σ_9. Thus $\sigma_9 \simeq \sigma_{13}$.

Thus we are left with the signatures σ_0, σ_1, σ_2, σ_3, σ_4, σ_5, σ_6, σ_7, σ_8, σ_9, σ_{10}, σ_{11}, σ_{12}, σ_{15}, σ_{16} and σ_{18}. The signed graphs corresponding to these signatures are shown in Figure 2.2.2. We now proceed to show that no two of these 16 signatures are switching isomorphic.

Theorem 2.2.1. *There are exactly 16 distinct signed* K_6 *up to switching isomorphism.*

Proof. Observe that if two signed graphs with the same underlying graph have different number of negative cycles of a given order, then they cannot be switching isomorphic to each other. We use this fact to show that the 16 signed K_6, depicted in Figure 2.2.2, are pairwise switching non-isomorphic.

Let the number of negative 3-cycles, the number of negative 4-cycles and the number of negative 5-cycles of a signed graph be denoted by $|C_3^-|, |C_4^-|$ and $|C_5^-|$, respectively. These numbers for the signed graphs of Figure 2.2.2 are given in Table 2.2.1. Note that each pair of these 16 signed K_6 have different number of negative cycles of order 3 or 4 or 5. Hence all these 16 signed K_6 are pairwise switching non-isomorphic. ☐

	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{15}	Σ_{16}	Σ_{18}
$\lvert C_3^-\rvert$	0	4	6	8	8	10	12	10	10	12	12	14	8	16	10	20
$\lvert C_4^-\rvert$	0	12	18	20	24	22	24	18	26	24	20	18	24	12	30	0
$\lvert C_5^-\rvert$	0	24	24	32	40	36	24	36	36	32	40	36	48	48	36	72

Table 2.2.1. Number of negative k-cycles, for $3 \le k \le 5$, in distinct signed K_6

Corollary 2.2.2. *Each signed graph in Figure 2.2.2 is reduced in its switching isomorphism class.*

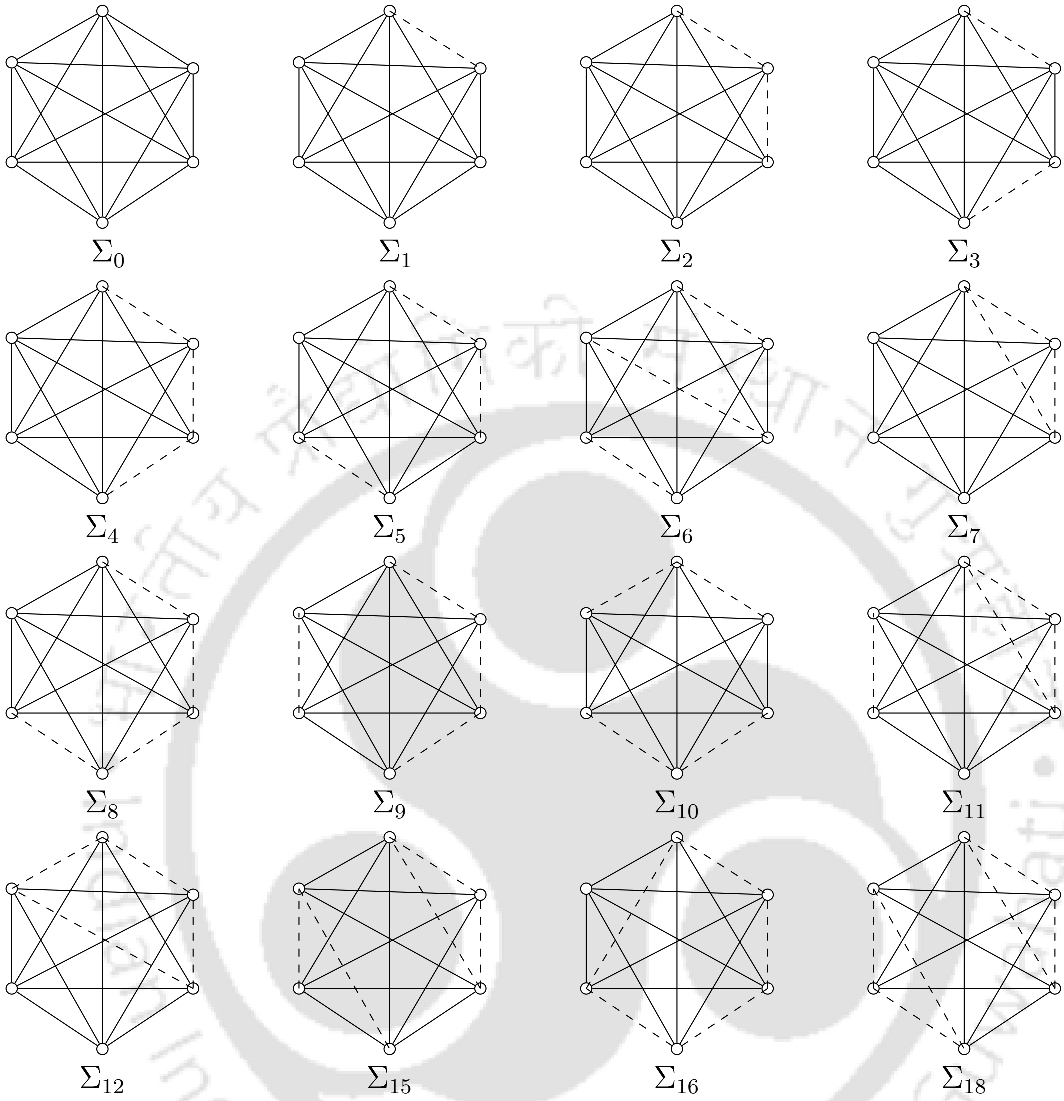
Proof. Let $A = [19] \setminus \{13, 14, 17\}$. We prove that for each $j \in A$, the signature σ_j is minimum in its switching isomorphism class. It is clear that σ_0 and σ_1 are minimum in their switching isomorphism classes. Now consider σ_j, where $j \in A \setminus \{0, 1\}$. If σ_j is not minimum, then σ_j is switching equivalent to a minimum signature τ such that $|\tau| < |\sigma_j|$. Further, the edges of τ must form paths or cycles. Considering the proofs of Lemmas 2.2.1, 2.2.2, 2.2.3, 2.2.4 and 2.2.5, we find that τ is isomorphic to some σ_k, where $k \in A$ and $k < j$. Therefore σ_j is switching isomorphic to σ_k with $|\sigma_k| < |\sigma_j|$, a contradiction to Theorem 2.2.1. Hence σ_j is minimum. ☐

As such, the proof technique by which we have computed the switching non-isomorphic signed K_6 can be used for higher values of n. However, the calculation would be very tedious even for $n = 7$. Indeed, the counting of these classes of K_n is given in [39, Table 1] for $n \le 21$. Nevertheless, classification of switching isomorphism classes of K_n is still unknown for $n \ge 7$.

2.3 Generalized Petersen Graphs

Let n and k be positive integers such that $k < \frac{n}{2}$. Recall that the vertex and edge sets of the generalized Petersen graph $P_{n,k}$ are given by $V(P_{n,k}) = \{u_i, v_i : i \in [n]\}$ and $E(P_{n,k}) = \{u_i u_{i+1}, v_i v_{i+k}, u_i v_i : i \in [n]\}$, respectively, where the subscripts are read modulo n. It is clear that $P_{2n+1,1}$ has $4n + 2$ vertices and $6n + 3$ edges. Now we discuss certain structural facts of $P_{2n+1,1}$ for $n \ge 1$.

Given a cycle C of $P_{2n+1,1}$, a path formed by consecutive u-vertices (respectively, v-vertices) in C is called a *u-segment* (respectively, *v-segment*) of C. For


Σ_0 Σ_1 Σ_2 Σ_3

Σ_4 Σ_5 Σ_6 Σ_7

Σ_8 Σ_9 Σ_{10} Σ_{11}

Σ_{12} Σ_{15} Σ_{16} Σ_{18}

Figure 2.2.2. The 16 switching non-isomorphic signed complete graphs over K_6

example, the cycles $u_iu_{i+1}v_{i+1}v_iu_i$, $u_iu_{i+1}v_{i+1}v_{i+2}v_{i+3}u_{i+3}u_{i+4}v_{i+4}...v_{i+2n}u_{i+2n}u_i$ and C_I have exactly one u-segment, exactly two u-segments and no u-segment, respectively. Further, if u_iu_{i+1} is an edge in C, then we say u_i *moves forward* along C and u_{i+1} *moves backward* along C. Similarly, if v_iv_{i+1} is an edge in C, then we say v_i *moves forward* along C and v_{i+1} *moves backward* along C.

Proposition 2.3.1. *If an odd cycle of $P_{2n+1,1}$ contains exactly one u-segment and exactly one v-segment, then the order of the cycle is $2n+3$.*

Proof. Let C be an odd cycle such that it contains exactly one u-segment and exactly one v-segment. Let the u-segment of C be $u_iu_{i+1}...u_{i+j}$ for some $i \in \{0, 1, \ldots, 2n\}$ and $j \in \{1, \ldots, 2n\}$. Since no u-vertex other than $u_i, u_{i+1}, \ldots, u_{i+j}$ belongs to $V(C)$, the edges u_iv_i and $u_{i+j}v_{i+j}$ must belong to $E(C)$. If v_i moves forward or v_{i+j} moves backward along C, then the cycle must be an even cycle of length $2j+2$. But this is a contradiction to the fact that C is an odd cycle. Hence v_i moves backward and v_{i+j} moves forward along C. In this case, the cycle C is $u_i u_{i+1} ... u_{i+j} v_{i+j} v_{i+j+1} ... v_{i+2n} v_i u_i$. Thus the order of C is $(j+1) + ((2n+1) - (j-1)) = 2n+3$, as desired. ☐

The following proposition is easy, so we omit the proof.

Proposition 2.3.2. *Any cycle other than* C_o *and* C_I *of* $P_{2n+1,1}$ *contains same number of* u*-segments and* v*-segments.*

Theorem 2.3.1. *The order of an odd cycle other than* C_o *and* C_I *of* $P_{2n+1,1}$ *is at least* $2n+3$.

Proof. Let C be an odd cycle in $P_{2n+1,1}$ of length $2r+1$ such that C contains u-vertices as well as v-vertices. We show that $r \geq n+1$.

Due to Proposition 2.3.2, C contains same number of u-segments and v-segments. If C contains exactly one u-segment and exactly one v-segment, then due to Proposition 2.3.1 we get $r = n+1$.

Now assume that the number of u-segments (as well as v-segments) of C be l, where $2 \leq l \leq n$. Let the number of u-vertices and v-vertices in these segments be $r_1, ..., r_l$ and $s_1, ..., s_l$, respectively. Clearly, $r_i \geq 2$ and $s_i \geq 2$ for each $i \in \{1, \ldots, l\}$, and $(r_1 + \cdots + r_l) + (s_1 + \cdots + s_l) = 2r+1$. It is important to note that corresponding to each u-segment and v-segment of order r_i and s_i, the number of v-vertices and u-vertices that are not in C are $r_i - 2$ and $s_i - 2$, respectively. Thus the total number of u-vertices and v-vertices that are not in C is $(r_1 + \cdots + r_l) + (s_1 + \cdots + s_l) - 4l$. Since the total number of vertices of P_{2n+1} is $4n+2$, we have

$$(2r+1) + (r_1 + \cdots + r_l) + (s_1 + \cdots + s_l) - 4l = 4n+2.$$

As $(r_1 + \cdots + r_l) + (s_1 + \cdots + s_l) = 2r+1$, the above equation gives $4r - 4l = 4n$. Therefore, $r \geq n+2$ since $l \geq 2$. This completes the proof. ☐

Theorem 2.3.2. *An even cycle of* $P_{2n+1,1}$ *must be of the form* $u_i u_{i+1} ... u_{i+j} v_{i+j} v_{i+j-1} ... v_{i+1} v_i u_i$ *for some* $i \in \{0, 1, \ldots, 2n\}$ *and* $j \in \{1, \ldots, 2n\}$. *Consequently, each even cycle of* $P_{2n+1,1}$ *has the same number of* u*-vertices and* v*-vertices.*

Proof. Let C be an even cycle of $P_{2n+1,1}$. First we assume that C contains exactly one u-segment and one v-segment. Let the u-segment of C be $u_i u_{i+1} ... u_{i+j}$ for some $i \in \{0, 1, \ldots, 2n\}$ and $j \in \{1, \ldots, 2n\}$. Since no u-vertex other than $u_i, u_{i+1}, \ldots, u_{i+j}$ belongs to $V(C)$, the edges $u_i v_i$ and $u_{i+j} v_{i+j}$ must belong to $E(C)$. If v_i moves backward and v_{i+j} moves forward along C, then the order of C must be $2n+3$, (see the proof of Proposition 2.3.1). Hence v_i must move forward and v_{i+j} must move backward along C. Accordingly, the cycle C is in the desired form.

Now assume that the number of u-segments (as well as v-segments) of C be l, where $2 \leq l \leq n$. Let the order of C be $2r$. Further, let the number of u-vertices and v-vertices in these segments be $r_1, ..., r_l$ and $s_1, ..., s_l$, respectively. Clearly, $r_i \geq 2$ and $s_i \geq 2$ for each $i \in \{1, \ldots, l\}$, and $(r_1 + \cdots + r_l) + (s_1 + \cdots + s_l) = 2r$. Corresponding to the v-segments of C, the total number of u-vertices not in C is $(s_1 + \cdots + s_l) - 2l$. Therefore, we have

$$(s_1 + \cdots + s_l) - 2l = 2n + 1 - (r_1 + \cdots + r_l).$$

Since $(r_1 + \cdots + r_l) + (s_1 + \cdots + s_l) = 2r$, we have $2r = 2(n + l) + 1$, which is absurd. Hence an even cycle of $P_{2n+1,1}$ cannot have more than one u-segment and one v-segment. Thus an even cycle of $P_{2n+1,1}$ must be of the form

$$u_i u_{i+1} ... u_{i+j} v_{i+j} v_{i+j-1} ... v_{i+1} v_i u_i,$$

and consequently each even cycle of $P_{2n+1,1}$ has the same number of u-vertices and v-vertices. □

Theorem 2.3.3. *For* $n \geq 1$ *and* $2 \leq l \leq 2n + 1$*, the number of* $2l$*-cycles and the number of* $(2n + 1)$*-cycles of* $P_{2n+1,1}$ *are* $2n + 1$ *and* 2*, respectively.*

Proof. Due to Theorem 2.3.1, the outer and inner cycles are the only cycles of length $2n + 1$. This proves the second part of the theorem.

We prove the first part of the theorem by counting the number of $2l$-cycles, where $2 \leq l \leq 2n + 1$. In view of Theorem 2.3.2, each even cycle of $P_{2n+1,1}$ contains equal number of inner and outer vertices. Again due to Theorem 2.3.2, for each $i \in \{0, 1, \ldots, 2n\}$, the cycle $u_i v_i v_{i+1} \ldots v_{i+(l-1)} u_{i+(l-1)} u_{i+(l-2)} \ldots u_{i+1} u_i$ is of length $2l$, and any even cycle of $P_{2n+1,1}$ is of this form. Hence there are $(2n + 1)$ cycles of $P_{2n+1,1}$ of length $2l$, where $2 \leq l \leq 2n + 1$. This proves the theorem. □

Theorem 2.3.4. *The distance between any two edges in* $P_{2n+1,1}$ *is at most* n*.*

Proof. It is clear that the distance between any two spokes of $P_{2n+1,1}$ is at most n. Further, the distance between any two edges of the outer as well as of the inner cycle, is at most n. Without loss of generality, if we pick the edge u_0u_1 from the outer cycle, then v_nv_{n+1} and $v_{n+1}v_{n+2}$ are the only edges of the inner cycle that are at maximum distance n from u_0u_1. Similarly, $u_{n+1}v_{n+1}$ is the only spoke that is at maximum distance n from u_0u_1. This completes the proof of the theorem. □

For each $k \in \{0, 1, \ldots, 2n\}$, define the permutations γ, ρ_k, and δ_k of $V(P_{2n+1,1})$ such that for all $i \in \{0, 1, \ldots, 2n\}$, we have

$$\gamma(u_i) = v_i, \gamma(v_i) = u_i; \;\; \rho_k(u_i) = u_{i+k}, \rho_k(v_i) = v_{i+k}; \;\text{ and}$$

$$\delta_k(u_i) = \begin{cases} u_i & \text{if } \; i = k, \\ u_l & \text{if } d(u_i, u_k) = d(u_l, u_k) \text{ and } i \neq k, i \neq l; \end{cases}$$

$$\delta_k(v_i) = \begin{cases} v_i & \text{if } \; i = k, \\ v_l & \text{if } d(v_i, v_k) = d(v_l, v_k) \text{ and } i \neq k, i \neq l. \end{cases}$$

Note that each ρ_k represents a clockwise rotation of $P_{2n+1,1}$. Also each δ_k represents a reflection of $P_{2n+1,1}$ about a line induced by the edge u_kv_k. Further, γ just swaps the inner and outer cycles of $P_{2n+1,1}$. From [26], we know that the automorphism group of $P_{2n+1,1}$ is given by

$$\text{Aut}(P_{2n+1,1}) = \langle \rho_k, \; \delta_k, \; \gamma : k \in \{0, 1, \ldots, 2n\} \rangle .$$

Thus, the automorphisms of $P_{2n+1,1}$ are some combination of rotations, reflections, and interchanges of the outer and inner vertices. Using this fact, if H_1 and H_2 are two given subgraphs of $P_{2n+1,1}$, then it is easy to decide whether there is an automorphism of $P_{2n+1,1}$ that maps H_1 onto H_2. The complete study of automorphism groups of the family of generalized Petersen graphs is given in [26].

Example 2.3.1. The graph $P_{3,1}$ is given in Figure 2.3.2. The automorphism ρ_1 rotates the graph $P_{3,1}$ clockwise through an angle $\frac{2\pi}{3}$, the automorphism δ_1 flips $P_{3,1}$ about the line containing the edge u_1v_1, and γ swaps the cycles $u_0u_1u_2$ and $v_0v_1v_2$ to each other.

From Corollary 2.1.1, it is easy to see that to find non-isomorphic signatures on $P_{n,k}$, it is enough to find the non-isomorphic matchings of $P_{n,k}$ of sizes up to n. Thus, we find non-isomorphic matchings of $P_{n,k}$ instead of non-isomorphic signatures over $P_{n,k}$. Two matchings of a graph are called *automorphic* or *automorphic type* if the corresponding signed graphs are automorphic to each other. Similarly, distinct automorphic type matchings are those whose corresponding signed graphs are not automorphic to each other. In what follows, we use the term matching in the sense of a signature. For $i \in \{0, 1, ..., n\}$, let M_i denote a matching of $P_{n,k}$ of size i.

2.3.1 Signatures on $P_{2n+1,1}$

In this section, we determine the number of switching non-isomorphic matchings of size two of $P_{2n+1,1}$ for $n \geq 4$. We also prove that any minimum signature of $P_{2n+1,1}$ is of size at most $n + 1$. The distance between two cycles C_1 and C_2 in a graph G is $d(C_1, C_2) := \min\{d(u, v) : u \in V(C_1), v \in V(C_2)\}$.

Lemma 2.3.1. *Let M_2' and M_2'' be two matchings of size two in $P_{2n+1,1}$. If the distance between the edges of M_2' is not equal to the distance between the edges of M_2'', then $(P_{2n+1,1}, M_2')$ and $(P_{2n+1,1}, M_2'')$ are switching non-isomorphic to each other.*

Proof. Let $M_2' = \{e_1, e_2\}$ and $M_2'' = \{e_3, e_4\}$ be two matchings in $P_{2n+1,1}$ satisfying $d(e_1, e_2) \neq d(e_3, e_4)$. Without loss of generality, assume that $d(e_3, e_4) \geq d(e_1, e_2) + 1$. Note that following are the only choices for the edges e and f of a matching of size two.

(1) $e, f \in E(C_o)$ (or $E(C_I)$),

(2) $e \in E(C_o)$ and $f \in E(C_I)$,

(3) $e \in E(C_o)$ (or $E(C_I)$) and $f \in S$,

(4) $e, f \in S$.

For $i \in \{1, 2, 3, 4\}$, if the edges of M_2' (resp. M_2'') are as in Case (i) of the preceding paragraph, then we write M_{2i}' (resp. M_{2i}'') in place of M_2' (resp. M_2''). To complete the proof it is enough to show that $(P_{2n+1,1}, \sigma) \nsim (P_{2n+1,1}, \pi)$ for any $\sigma \in \{M_{21}', M_{22}', M_{23}', M_{24}'\}$ and $\pi \in \{M_{21}'', M_{22}'', M_{23}'', M_{24}''\}$. Let Σ_σ and Σ_π denote the signed graphs $(P_{2n+1,1}, \sigma)$ and $(P_{2n+1,1}, \pi)$, respectively.

Note that switching does not change the sign of a cycle, and that isomorphism of a signed graph maps a negative k-cycle onto a negative k-cycle. Now we discuss the following cases.

Case 1. Let $\sigma = M_{21}'$ and $\pi = M_{21}''$. In this case, there are exactly two negative 4-cycles in each of Σ_σ and Σ_π. Due to the fact $d(e_1, e_2) \neq d(e_3, e_4)$, the distance between the negative 4-cycles in Σ_σ is not equal to the distance between the negative 4-cycles in Σ_π. Hence $\Sigma_\sigma \nsim \Sigma_\pi$.

Case 2. Let $\sigma = M_{22}'$ and $\pi = M_{22}''$. If $e_1 = u_i u_{i+1}$ and $e_2 = v_i v_{i+1}$ for some i, then Σ_σ has no negative 4-cycles, whereas Σ_π has two negative 4-cycles. Therefore $\Sigma_\sigma \nsim \Sigma_\pi$. Else, the argument is as in Case 1 to establish that $\Sigma_\sigma \nsim \Sigma_\pi$.

Case 3. Let $\sigma = M_{23}'$ and $\pi = M_{23}''$. Let $e_1, e_3 \in E(C_o)$ (or $E(C_I)$) and $e_2, e_4 \in S$. Let C_1', C_2' and C_3' be the negative 4-cycles in Σ_σ, where C_1' is obtained from e_1 and the consecutive 4-cycles C_2' and C_3' are obtained from e_2. Similarly, let C_1'', C_2'' and C_3'' be the negative 4-cycles in Σ_π, where C_1'' is obtained from e_3 and the consecutive 4-cycles C_2'' and C_3'' are obtained from e_4. Note that a switching isomorphism of Σ_σ onto Σ_π, if any, must map C_1' onto C_1''. Since $d(e_1, e_2) \neq d(e_3, e_4)$, in that case, the cycles in $\{C_2', C_3'\}$ cannot be mapped onto the cycles in $\{C_2'', C_3''\}$. Hence $\Sigma_\sigma \nsim \Sigma_\pi$.

Case 4. Let $\sigma = M'_{24}$ and $\pi = M''_{24}$. If $d(e_1, e_2) = 1$, then the number of negative 4-cycles in Σ_σ and Σ_π are 2 and 4, respectively. Therefore $\Sigma_\sigma \not\sim \Sigma_\pi$.

Else, let C'_1, C'_2, C'_3 and C'_4 be the negative 4-cycles in Σ_σ, where the consecutive 4-cycles C'_1 and C'_2 are obtained from e_1, and the consecutive 4-cycles C'_3 and C'_4 are obtained from e_2. Similarly, let C''_1, C''_2, C''_3 and C''_4 be the negative 4-cycles in Σ_π, where the consecutive 4-cycles C''_1 and C''_2 are obtained from e_3, and the consecutive 4-cycles C''_3 and C''_4 are obtained from e_4. Note that a switching isomorphism of Σ_σ onto Σ_π, if any, must map the cycles in $\{C'_1, C'_2\}$ onto the cycles in $\{C''_1, C''_2\}$ or onto the cycles in $\{C''_3, C''_4\}$. Without loss of generality, assume that the cycles in $\{C'_1, C'_2\}$ are mapped onto the cycles in $\{C''_1, C''_2\}$ by a switching isomorphism of Σ_σ. Since $d(e_1, e_2) \neq d(e_3, e_4)$, in that case, the cycles in $\{C'_3, C'_4\}$ cannot be mapped onto the cycles in $\{C''_3, C''_4\}$. Hence $\Sigma_\sigma \not\sim \Sigma_\pi$.

Case 5. Let $\sigma = M'_{21}$ and $\pi = M''_{22}$. In this case, both the cycles C_o and C_I are positive in Σ_σ, whereas both of these cycles are negative in Σ_π. Hence $\Sigma_\sigma \not\sim \Sigma_\pi$.

Case 6. Let $\sigma = M'_{21}$ and $\pi = M''_{23}$. In this case, both the cycles C_o and C_I are positive in Σ_σ, whereas exactly one of these cycles is negative in Σ_π. Hence $\Sigma_\sigma \not\sim \Sigma_\pi$.

Case 7. Let $\sigma = M'_{21}$ and $\pi = M''_{24}$. Since $d(e_3, e_4) \geq d(e_1, e_2) + 1 \geq 2$, the number of negative 4-cycles in Σ_σ and Σ_π are 2 and 4, respectively. Hence $\Sigma_\sigma \not\sim \Sigma_\pi$.

Case 8. Let $\sigma = M'_{22}$ and $\pi = M''_{21}$. This case is similar to Case 5.

Case 9. Let $\sigma = M'_{22}$ and $\pi = M''_{23}$. In this case, both the cycles C_o and C_I are negative in Σ_σ, whereas exactly one of these cycles is negative in Σ_π. Hence $\Sigma_\sigma \not\sim \Sigma_\pi$.

Case 10. Let $\sigma = M'_{22}$ and $\pi = M''_{24}$. In this case, both the cycles C_o and C_I are negative in Σ_σ, whereas both of these cycles are positive in Σ_π. Hence $\Sigma_\sigma \not\sim \Sigma_\pi$.

Case 11. Let $\sigma = M'_{23}$ and $\pi = M''_{21}$. This case is similar to Case 6.

Case 12. Let $\sigma = M'_{23}$ and $\pi = M''_{22}$. This case is similar to Case 9.

Case 13. Let $\sigma = M'_{23}$ and $\pi = M''_{24}$. In this case, exactly one of C_o and C_I is negative in Σ_σ, whereas both of these cycles are positive in Σ_π. Hence $\Sigma_\sigma \not\sim \Sigma_\pi$.

Case 14. Let $\sigma = M'_{24}$ and $\pi = M''_{21}$. If $d(e_1, e_2) = 1$, then there are exactly two negative 4-cycles in each of Σ_σ and Σ_π. Due to the fact $d(e_3, e_4) \geq d(e_1, e_2) + 1$, the distance between the negative 4-cycles in Σ_σ is not equal to the distance between the negative 4-cycles in Σ_π. If $d(e_1, e_2) \geq 2$, then the number of negative 4-cycles in Σ_σ and Σ_π are 4 and 2, respectively. Hence $\Sigma_\sigma \not\sim \Sigma_\pi$.

Case 15. Let $\sigma = M'_{24}$ and $\pi = M''_{22}$. This case is similar to Case 10.

Case 16. Let $\sigma = M'_{24}$ and $\pi = M''_{23}$. This case is similar to Case 13.

From Cases 1 to 16, the result follows. □

In the proof of Lemma 2.3.1, note that the hypothesis $d(e_1, e_2) \neq d(e_3, e_4)$ is needed only for the Cases 1 to 4, Case 7 and Case 14.

Theorem 2.3.5. *The number of switching non-isomorphic matchings of* $P_{2n+1,1}$ *of size two is* $4n-1$*, where* $n \geq 4$.

Proof. Let M_2 be a matching of $P_{2n+1,1}$ of size two. From Theorem 2.3.4, we know that edges of M_2 can be at maximum distance n from each other. If the edges of M_2 are at distance 1 from each other, then consider the matchings given by $M_2^{11} = \{u_0u_1, v_0v_1\}$, $M_2^{12} = \{u_0u_1, u_2v_2\}$, $M_2^{13} = \{u_0u_1, v_1v_2\}$ and $M_2^{14} = \{u_0u_1, u_2u_3\}$. Any other matching of $P_{2n+1,1}$ of size two, in which both the edges are at distance 1 from each other as well as both the edges lie either on the outer cycle or on the inner cycle, is automorphic to M_2^{14} via a rotation automorphism. Further, if a matching of $P_{2n+1,1}$ of size two contains only two consecutive spokes, then switching by the outer vertices of the spokes, we get a matching automorphic to M_2^{14}. Similarly, it can be shown that all other matchings of size two, whose edges are at distance 1 from each other, are automorphic to one of M_2^{11}, M_2^{12} or M_2^{13}.

Further, the number of negative 4-cycles in $M_2^{11}, M_2^{12}, M_2^{13}$ and M_2^{14} are $0, 3, 2$ and 2, respectively. The number of negative $(2n+1)$-cycles in $M_2^{11}, M_2^{12}, M_2^{13}$ and M_2^{14} are $2, 1, 2$ and 0, respectively. These numbers of negative cycles show that $M_2^{11}, M_2^{12}, M_2^{13}$ and M_2^{14} are pairwise switching non-isomorphic. Hence M_2^{11}, M_2^{12}, M_2^{13} and M_2^{14} are the only switching non-isomorphic matchings of size two whose edges are at distance 1 from each other.

For $i \in \{2, \ldots, n-1\}$, consider the matchings given by $M_2^{i1} = \{u_0u_1, u_{i+1}u_{i+2}\}$, $M_2^{i2} = \{u_0u_1, u_{i+1}v_{i+1}\}$, $M_2^{i3} = \{u_0u_1, v_iv_{i+1}\}$ and $M_2^{i4} = \{u_0v_0, u_iv_i\}$. Note that the distance between the edges of each M_2^{ik} is i for $k \in \{1, 2, 3, 4\}$. The number of negative 4-cycles in $M_2^{i1}, M_2^{i2}, M_2^{i3}$ and M_2^{i4} are $2, 3, 2$ and 4, respectively. Also, the number of negative $(2n+1)$-cycles in $M_2^{i1}, M_2^{i2}, M_2^{i3}$ and M_2^{i4} are $0, 1, 2$ and 0, respectively. These numbers of negative cycles show that these matchings are pairwise switching non-isomorphic. Now, we prove that any other matching of size two whose edges are at distance i from each other, is automorphic to one of $M_2^{i1}, M_2^{i2}, M_2^{i3}$ or M_2^{i4}. If the distance between two edges is i, where $2 \leq i \leq n-1$, then those two edges of a matching M_2 must be either from the outer cycle or from the inner cycle or one edge from the outer cycle and other from the inner cycle or one edge from the outer cycle and other from the spokes or both from the spokes. Note that any such matching is automorphic to one of the $M_2^{i1}, M_2^{i2}, M_2^{i3}$ or M_2^{i4}. Hence there are exactly four matchings of size two whose edges are at distance i from each other, where $2 \leq i \leq n-1$.

Consider the matchings of size two, whose edges are at distance n from each other, given by $M_2^{n1} = \{u_0u_1, u_{n+1}v_{n+1}\}$, $M_2^{n2} = \{u_0u_1, v_nv_{n+1}\}$ and $M_2^{n3} = \{u_0v_0, u_nv_n\}$. Observe that any other matching of size two, whose edges are at distance n from each other, is automorphic to one of M_2^{n1}, M_2^{n2} or M_2^{n3}. Further, the number of negative 4-cycles in M_2^{n1}, M_2^{n2} and M_2^{n3} are $3, 2$ and 4, respectively. Thus these three matchings are pairwise non-isomorphic.

Due to Lemma 2.3.1, it is clear that any two matchings of size two, whose edges are at different distances, are distinct up to switching isomorphism. Thus the total number of switching non-isomorphic matchings of $P_{2n+1,1}$ of size two is $4(n-1)+3 = 4n-1$, as desired. This completes the proof. $\square$

Definition 2.3.1. For $n \geq 3$, the graph G_n is defined by $V(G_n) = \{u_i, v_i : i \in [n]\}$ and $E(G_n) = \{u_iu_{i+1}, u_iv_i : i \in [n]\}$, where the subscripts are read modulo n.

For example, the graph G_4 is shown in Figure 2.3.1.

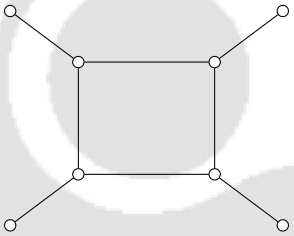


Figure 2.3.1. The graph G_4

We now show that every switching class of $P_{2n+1,1}$ contains a signature of size at most $n+1$. To do so, we make use of the following remark about minimum signatures of the graph G_{2n+1}.

Remark 2.3.1. It is well known that any signed cycle is equivalent to a signed cycle with at most one negative edge. Hence, a minimum signature of G_{2n+1} is always of size at most one. Further, if a minimum signature of G_{2n+1} has only one edge, then that edge must be one of the edges of C_{2n+1}.

Theorem 2.3.6. *Any minimum signature of $P_{2n+1,1}$ is of size at most $n+1$.*

Proof. We complete the proof by showing that if σ is a signature of $P_{2n+1,1}$, then σ is equivalent to a signature σ' such that $|\sigma'| \leq n+1$.

Let σ be a signature of $P_{2n+1,1}$. Note that $P_{2n+1,1} - E(C_o)$ is isomorphic to G_{2n+1}. Thus by Remark 2.3.1, we can switch $(P_{2n+1,1}, \sigma)$ to another signed graph $(P_{2n+1,1}, \sigma_1)$, where at most one of the edges of σ_1 is from $E(C_I)$ and the remaining edges are from C_o. That is, $\sigma_1 \subseteq \{v_iv_{i+1}, u_0u_1, u_1u_2, \ldots, u_{2n}u_0\}$ for some $i \in \{0, 1, \ldots, 2n\}$. Now we discuss the following cases.

Case 1. If $\sigma_1 \cap E(C_o) = \emptyset$, then we take $\sigma' = \sigma_1$.

Case 2. If $\sigma_1 \cap E(C_o) = E(C_o)$, then either $\sigma_1 = E(C_o)$ or $\sigma_1 = E(C_o) \cup \{v_iv_{i+1}\}$. For $\sigma_1 = E(C_o)$, switching by each vertex of the set $\{u_1, u_3, \ldots, u_{2n-1}\}$, we get a signature σ' such that $\sigma' = \{u_1v_1, u_3v_3, \ldots, u_{2n-1}v_{2n-1}, u_{2n}u_0\}$. Clearly, $|\sigma'| = n+1$.

On the other hand, if $\sigma_1 = E(C_o) \cup \{v_iv_{i+1}\}$, then switching by each vertex of the set $\{v_{i+1}, u_{i+1}, u_{i+3}, \dots, u_{i+(2n-1)}\}$, we get a signature σ' equivalent to σ_1 such that $\sigma' = \{v_{i+1}v_{i+2}, u_{i+3}v_{i+3}, u_{i+5}v_{i+5}, \dots, u_{i+2n-1}v_{i+2n-1}, u_{i-1}u_i\}$. It is easy to see that $|\sigma'| = n + 1$.

Case 3. If $\sigma_1 \cap E(C_o)$ is a non-empty proper subset of $E(C_o)$, then consider the subgraph H induced by $\sigma_1 \cap E(C_o)$. Note that each component of H is a path. If $P := u_{i_1}u_{i_1+1}\dots u_{i_1+2p-1}$ is a path component of H of even order at least 4, then switching by the vertices $u_{i_1+1}, u_{i_1+3}, \dots, u_{i_1+2p-3}$, will make all but one edge of P positive and also make $(p-1)$ spokes negative. Thus, this switching will replace the $(2p-1)$ negative edges of σ_1, corresponding to the path P, by p negative edges. Similar switching in a path component of order $(2q + 1)$ of H will replace the $2q$ negative edges of σ_1, corresponding to the path Q, by q negative edges (spokes). Apply this switching operation to each component of H to get the signature σ'.

Let the total number of vertices of all the even components of H be $2k$. Also, let there be s odd components of H, and that the total number of vertices of all the odd components of H be $2r + s$. It is clear that $|\sigma'| \le 1 + k + r$. Now $2k + 2r + s \le 2n + 1$ gives $k + r \le n$. Thus $|\sigma'| \le n + 1$.

From all these cases, the proof follows. □

In Chapter 3, we indeed prove for all positive integers n and k satisfying $2k < n$ and $\gcd(n, k) = 1$, that any minimum signature of $P_{n,k}$ is of size at most $n + 1$.

2.3.2 Switching Isomorphism Classes of $P_{3,1}$

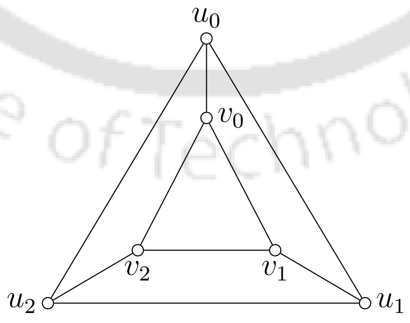


Figure 2.3.2. The graph $P_{3,1}$

Due to Theorem 2.3.6, we only need to determine the automorphic type matchings of $P_{3,1}$ of size at most two. We classify these matchings in the following lemmas. Let a matching of size zero be denoted by σ_0.

Lemma 2.3.2. *The number of distinct automorphic type matchings of size one in* $P_{3,1}$ *is* 2.

Proof. One of the matchings of size one that does not contain a spoke is given by $\sigma_1 = \{u_0u_1\}$. A matching of size one containing a spoke is given by $\sigma_2 = \{u_0v_0\}$. It is easy to see that any other matching of size one is automorphic to either σ_1 or σ_2, and that σ_1 is not automorphic to σ_2. This proves the lemma. □

Lemma 2.3.3. *The number of distinct automorphic type matchings of size two in* $P_{3,1}$ *is* 4.

Proof. We classify the automorphic type matchings M_2 of size two by considering the distance between their edges. Clearly, the distance between any two edges of $P_{3,1}$ is at most 1.

(i) Assume that M_2 does not contain a spoke. We may assume that one edge in M_2 is u_0u_1. There are two choices for such a matching of size two. Two such matchings are given by $\sigma_3 = \{u_0u_1, v_0v_1\}$ and $\sigma_4 = \{u_0u_1, v_1v_2\}$.

(ii) Assume that M_2 contains exactly one spoke, and let it be u_0v_0. One of such matchings is given by $\sigma_5 = \{u_0v_o, u_1u_2\}$.

(iii) Assume that M_2 contains two spokes. One of such matchings is given by $\sigma_6 = \{u_0v_0, u_1v_1\}$.

Any other matching of $P_{3,1}$ of size two is automorphic to $\sigma_3, \sigma_4, \sigma_5$ or σ_6. Further, no two of these matchings are automorphic to each other. This concludes the proof. □

The matchings obtained in the preceding lemmas, along with σ_0, give us seven automorphic type matchings of $P_{3,1}$, namely, $\sigma_0, \sigma_1, \ldots, \sigma_6$. However, some of these seven matchings may be switching isomorphic to each other. We have the following observation.

- By switching u_0, u_1 and u_2 in σ_6, we get a matching automorphic to σ_2. Thus $\sigma_6 \simeq \sigma_2$.

So we are left with the matchings $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ_5. The signed graphs corresponding to these matchings are depicted in Figure 2.3.3, where the label of the vertices correspond to that of Figure 2.3.2. In the following theorem, we show that these six matchings are not switching isomorphic to each other.

Theorem 2.3.7. *The number of switching non-isomorphic signed* $P_{3,1}$ *is* 6.

Proof. The number of negative 3-cycles and negative 4-cycles for the signed $P_{3,1}$, shown in Figure 2.3.3, are given in Table 2.3.1. We see that these number of negative cycles are different for all these six signatures. By Remark 2.1.1, we conclude that these six signatures are pairwise switching non-isomorphic. This completes the proof. □

	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5
$\lvert C_3^- \rvert$	0	1	0	2	2	1
$\lvert C_4^- \rvert$	0	1	2	0	2	3

Table 2.3.1. Number of negative 3-cycles and negative 4-cycles in signed $P_{3,1}$

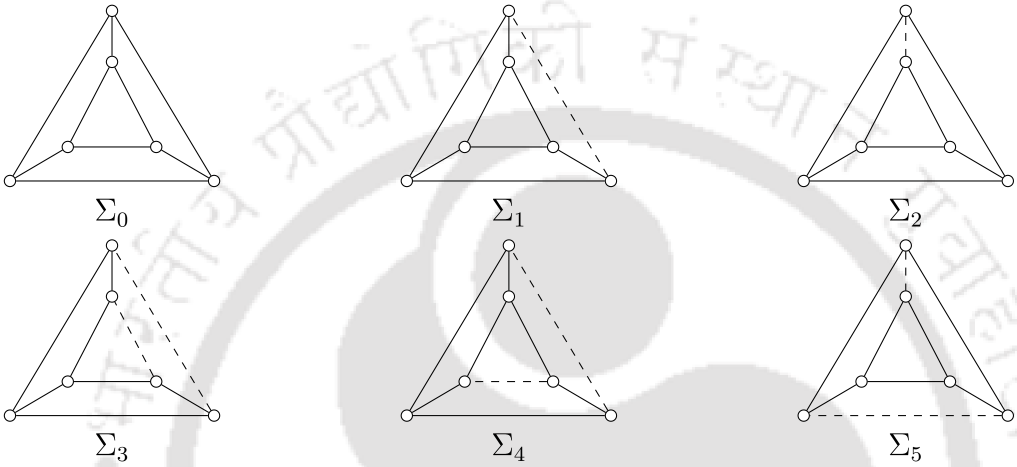


Figure 2.3.3. Six signed $P_{3,1}$

2.3.3 Switching Isomorphism Classes of $P_{5,1}$

The graph $P_{5,1}$ is shown in Figure 2.3.4. Due to Corollary 2.1.1 and Theorem 2.3.6, to find non-isomorphic signatures of $P_{5,1}$, it is enough to find matchings of $P_{5,1}$ of sizes $0, 1, 2$ and 3. We now classify all the automorphic type matchings of $P_{5,1}$ of size at most three. We denote a matching of size zero by σ_0. We observe that at most two edges of a matching can be from the outer cycle or from the inner cycle of $P_{5,1}$. We use this fact to determine the possible automorphic type matchings of different sizes.

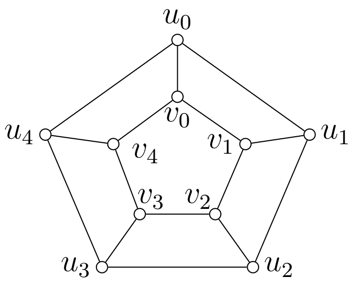


Figure 2.3.4. The graph $P_{5,1}$

Lemma 2.3.4. *The number of distinct automorphic type matchings of* $P_{5,1}$ *of size one is* 2.

Proof. We have the following two possibilities for a matching M_1 of size one.

(i) Assume that M_1 does not contain a spoke. There is only one such automorphic type matching of size one. One of such matchings is given by $\sigma_1 = \{u_0u_1\}$.

(ii) Assume that M_1 contains a spoke. There is only one such automorphic type matching of size one. One of such matchings is given by $\sigma_2 = \{u_0v_0\}$.

Any other matching of $P_{5,1}$ of size one is automorphic to σ_1 or σ_2, and that σ_1 is not automorphic to σ_2. This completes the proof. □

Lemma 2.3.5. *The number of distinct automorphic type matchings of* $P_{5,1}$ *of size two is* 8.

Proof. We classify the matchings M_2 of size two by considering the distance between the edges of the matching.

(i) Let the distance between the edges of the matching be 1. Consider the matchings of size two given by $\sigma_3 = \{u_0u_1, v_0v_1\}, \sigma_4 = \{u_0u_1, v_1v_2\}$, $\sigma_5 = \{u_0u_1, u_2u_3\}$, $\sigma_6 = \{u_0u_1, v_2u_2\}$ and $\sigma_7 = \{u_0v_0, u_1v_1\}$. Let M_2 be a matching of size two, other than $\sigma_3, \sigma_4, \sigma_5, \sigma_6$ and σ_7, where the distance between the edges of M_2 is 1. Note that M_2 must contain either two spokes, or two edges from the outer cycle, or two edges from the inner cycle, or one edge from the outer cycle and one from the inner cycle, or one edge from the outer/inner cycle and one spoke. In each of these cases, M_2 is automorphic to one of $\sigma_7, \sigma_5, \sigma_3, \sigma_4$ or σ_6. Thus $\sigma_3, \sigma_4, \sigma_5, \sigma_6$ and σ_7 are the only automorphic type matchings of size two, in which the distance between their edges is 1. It is easy to see that these matchings are pairwise non-automorphic.

(ii) Let the distance between the edges of matching M_2 be 2. Consider the matchings given by $\sigma_8 = \{u_0u_1, v_2v_3\}$, $\sigma_9 = \{u_0u_1, v_3u_3\}$ and $\sigma_{10} = \{u_0v_0, u_2v_2\}$. One can easily show that any other matching of size two, where the distance between their edges is 2, is automorphic to one of σ_8, σ_9 or σ_{10}. The matchings σ_8, σ_9 and σ_{10} are clearly pairwise non-automorphic.

This completes the proof of the lemma. □

Lemma 2.3.6. *The number of distinct automorphic type matchings of* $P_{5,1}$ *of size three is* 11.

Proof. We classify the automorphic type matchings M_3 of size three by considering the number of spokes contained in it.

(i) Assume that M_3 does not contain a spoke. Out of the three edges of M_3, two edges lie on the outer (inner) cycle and the remaining edge lies on the inner (outer) cycle. Because of the automorphism γ, we may assume that two edges of M_3 are lying on the outer cycle, and let they be u_0u_1 and u_2u_3. Therefore, a few possible matchings of size three for this case are given by $\sigma_{11} = \{u_0u_1, v_0v_1, u_2u_3\}$, $\sigma_{12} = \{u_0u_1, v_1v_2, u_2u_3\}$ and $\sigma_{13} = \{u_0u_1, v_3v_4, u_2u_3\}$. Any other matching of size three, that does not contain a spoke, is automorphic to one of σ_{11}, σ_{12} or σ_{13}. Further, these matchings are not automorphic to each other.

(ii) Assume that M_3 contains exactly one spoke, and let it be u_0v_0. If the other two edges of M_3 lie either on the outer cycle or on the inner cycle, then one of such matchings is given by $\sigma_{14} = \{u_0v_0, u_1u_2, u_3u_4\}$. If one edge of M_3 lies on the outer cycle and other lies on the inner cycle, then a few of the possible matchings are given by $\sigma_{15} = \{u_0v_0, u_1u_2, v_1v_2\}, \sigma_{16} = \{u_0v_0, u_1u_2, v_2v_3\}$ and $\sigma_{17} = \{u_0v_0, u_2u_3, v_2v_3\}$. Any other matching of size three containing only one spoke is automorphic to one of $\sigma_{14}, \sigma_{15}, \sigma_{16}$ or σ_{17}. Further, no two of these matchings are automorphic to each other.

(iii) Assume that M_3 contains exactly two spokes. If the spokes are consecutive, then one choice for such a matching is given by $\sigma_{18} = \{u_0v_0, u_1v_1, u_2u_3\}$. If the spokes are not consecutive, then one choice for such a matching is given by $\sigma_{19} = \{u_0v_0, v_2u_2, u_3u_4\}$. Any other matching of size three containing only two spokes is automorphic to σ_{18} or σ_{19}, and that σ_{18} is not automorphic to σ_{19}.

(iv) Assume that M_3 contains three spokes. In this case, there are only two automorphic type matchings of size three, and one of each such type of matchings are given by $\sigma_{20} = \{u_0v_0, u_1v_1, u_2v_2\}$ and $\sigma_{21} = \{u_0v_0, u_1v_1, u_3v_3\}$. Further, σ_{20} is not automorphic to σ_{21}.

This proves the lemma. □

The matchings obtained in the preceding lemmas, along with σ_0, give us 22 automorphic type matchings of $P_{5,1}$. However, some of these matchings are switching isomorphic to each other. We have the following observations.

- By switching u_0 and u_1 in σ_7, we get a matching automorphic to σ_5. Thus $\sigma_7 \simeq \sigma_5$.
- By switching u_1, u_2, v_1 and v_2 in σ_{11}, we get a matching automorphic to σ_1. Thus $\sigma_{11} \simeq \sigma_1$.

- By switching u_1, u_2 and v_1 in σ_{12}, we get a matching automorphic to σ_6. Thus $\sigma_{12} \simeq \sigma_6$.
- By switching u_1, v_1, u_2, v_2 and v_3 in σ_{13}, we get a matching automorphic to σ_9. Thus $\sigma_{13} \simeq \sigma_9$.
- By switching u_0, u_1 and u_4 in σ_{14}, we get a matching automorphic to σ_{10}. Thus $\sigma_{14} \simeq \sigma_{10}$.
- By switching u_0, u_1 and v_1 in σ_{15}, we get a matching automorphic to σ_4. Thus $\sigma_{15} \simeq \sigma_4$.
- By switching u_0, u_1, u_2, v_1 and v_2 in σ_{17}, we get a matching automorphic to σ_4. Thus $\sigma_{17} \simeq \sigma_4$.
- By switching u_1, u_2 and u_0 in σ_{18}, we get a matching automorphic to σ_9. Thus $\sigma_{18} \simeq \sigma_9$.
- By switching u_0, u_1 and u_2 in σ_{20}, we get a matching automorphic to σ_5. Thus $\sigma_{20} \simeq \sigma_5$.
- By switching u_0, u_1, u_2, u_3 and u_4 in σ_{21}, we get a matching automorphic to σ_{10}. Thus $\sigma_{21} \simeq \sigma_{10}$.

Thus, it is enough to consider the matchings $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_8, \sigma_9$, σ_{10}, σ_{16} and σ_{19}. The signed graphs corresponding to these 12 matchings are shown in Figure 2.3.5, where the label of the vertices correspond to that of Figure 2.3.4.

Theorem 2.3.8. *There are exactly 12 signed $P_{5,1}$ up to switching isomorphism.*

Proof. The number of negative cycles of length k, where $k \in \{4, 5, 6\}$, for the 12 signed $P_{5,1}$ shown in Figure 2.3.5 are given in Table 2.3.2.

Due to Remark 2.1.1 and Table 2.3.2, it follows that all the signed $P_{5,1}$, shown in Figure 2.3.5, are pairwise switching non-isomorphic. This concludes the proof of the theorem. ☐

2.3.4 Switching Isomorphism Classes of $P_{7,1}$

The graph $P_{7,1}$ is shown in Figure 2.3.6. By Theorem 2.3.3, we find that the number of 4-cycles, 6-cycles, 7-cycles and 8-cycles in $P_{7,1}$ are $7, 7, 2$ and 7, respectively. Due to Theorem 2.3.6, it is enough to determine the automorphic type matchings of $P_{7,1}$ of sizes at most 4. We now find the matchings of sizes $0, 1, 2, 3$ and 4 in $P_{7,1}$, up to switching isomorphism.

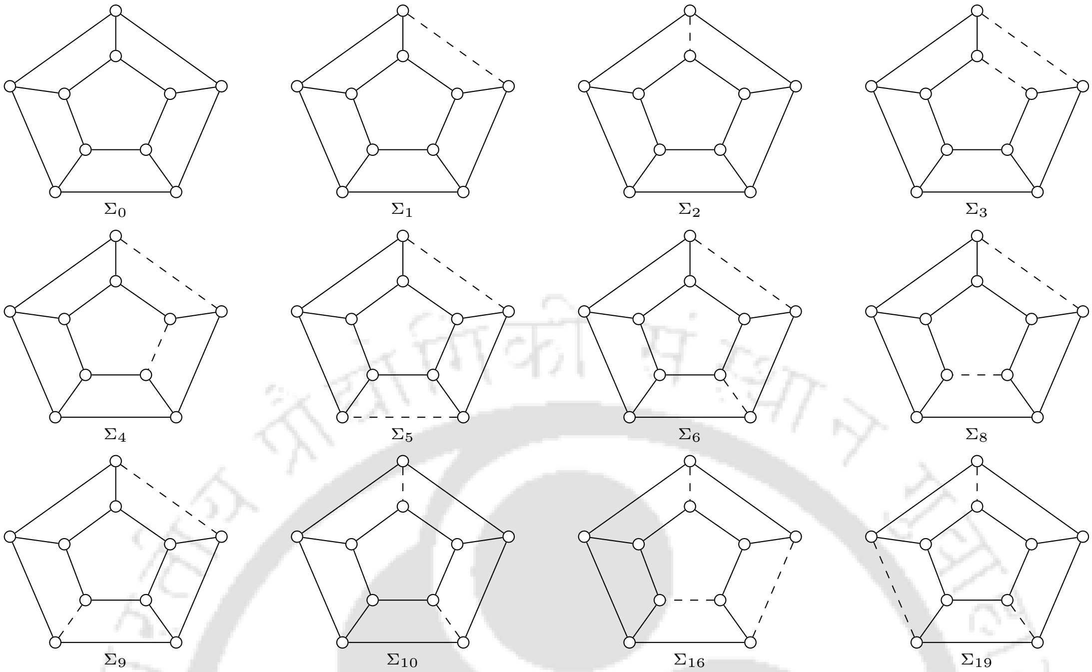


Figure 2.3.5. Twelve signed $P_{5,1}$

	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_8	Σ_9	Σ_{10}	Σ_{16}	Σ_{19}
$\lvert C_4^- \rvert$	0	1	2	0	2	2	3	2	3	4	4	5
$\lvert C_5^- \rvert$	0	1	0	2	2	0	1	2	1	0	2	1
$\lvert C_6^- \rvert$	0	2	2	0	2	4	2	4	4	2	2	0

Table 2.3.2. Number of negative k-cycles, where $4 \leq k \leq 6$, in some signed $P_{5,1}$

Consider the matchings $F_1 = \{u_0u_1, v_0v_1, v_2u_2\}$, $F_2 = \{u_0u_1, v_1v_2, u_2u_3\}$, $F_3 = \{u_0u_1, v_1v_2, v_4v_5\}$, $F_4 = \{u_0u_1, v_0v_1, u_3u_4\}$, $F_5 = \{u_0u_1, v_0v_6, u_3u_4\}$, $F_6 = \{u_0v_0, u_1v_1, v_2v_3\}$ and $F_7 = \{u_0v_0, u_1v_1, u_2v_2\}$ of $P_{7,1}$. The matchings F_i, where $1 \leq i \leq 7$, are said to be *forbidden matchings* of $P_{7,1}$.

Lemma 2.3.7. *If a forbidden matching is contained in a matching M_l of $P_{7,1}$, where $l \geq 3$, then M_l is switching equivalent to $M_{l'}$ for some l' with $l' \leq l - 1$.*

Proof. Let M_l^j be a matching of $P_{7,1}$ of size l that contain the set F_j, where $l \geq 3$ and $j \in \{1, \ldots, 7\}$. Consider the sets

$$S_1 = \{u_1, v_1, v_2\},\ S_2 = \{u_1, v_2, u_2\},\ S_3 = \{u_1, v_2, u_2, u_3, u_4, v_3, v_4\},$$

$$S_4 = \{u_1, v_1, u_2, v_2, v_3, u_3\},\quad S_5 = \{v_0, u_1, v_1, u_2, v_2, u_3, v_3\},$$

$$S_6 = \{v_1, v_2, v_0\}\ \text{ and }\ S_7 = \{u_0, u_1, u_2\}.$$

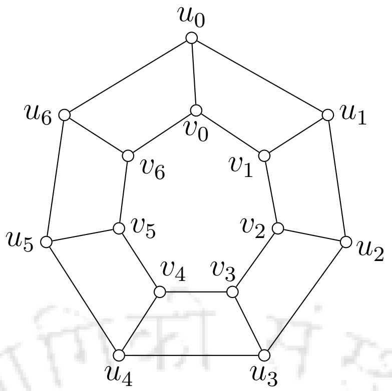


Figure 2.3.6. The graph $P_{7,1}$

If we switch by the vertices of S_j, then we get a matching of size at most $l-1$. Thus, M_l^j is switching equivalent to a matching of size l', where $l' \leq l-1$. This proves the lemma. □

In Lemma 2.3.7, the inequality $l' \leq (l-1)$ may be strict. For example, if $M_3 = F_4$, then switching by the vertices u_1, v_1, u_2, v_2, v_3 and u_3, we get $M_3 \simeq M_1$. Let M_0 denote the matching σ_0 of size zero, that is $M_0 = \varnothing$. Further, there are only two automorphic type matchings of size one given by $\sigma_1 = \{u_0u_1\}$ and $\sigma_2 = \{u_0v_0\}$. Any other matching of $P_{7,1}$ of size one is automorphic to either σ_1 or σ_2, and that σ_1 is not automorphic to σ_2.

Lemma 2.3.8. *The number of distinct automorphic type matchings of* $P_{7,1}$ *of size two is* 12.

Proof. We classify the matchings M_2 of size two by considering the distance between their edges. Note that the distance between any two edges of $P_{7,1}$ is at most 3.

(i) Let the edges of M_2 be at distance 1 from each other. Five such possible matchings are given by $\sigma_3 = \{u_0u_1, v_0v_1\}$, $\sigma_4 = \{u_0u_1, v_1v_2\}, \sigma_5 = \{u_0u_1, v_2u_2\}$, $\sigma_6 = \{u_0u_1, u_2u_3\}$ and $\sigma_7 = \{u_0v_0, u_1v_1\}$. Since the distance between the edges of M_2 is 1, the two edges of M_2 must be either two consecutive spokes, or one spoke and one edge from outer (inner) cycle, or two edges from the outer (inner) cycle, or one edge from the inner cycle and one from the outer cycle. Each such possible matching is automorphic to one of $\sigma_3, \sigma_4, \sigma_5, \sigma_6$ or σ_7. These five matchings are also pairwise non-automorphic.

(ii) Let the edges of M_2 be at distance 2 from each other. Four such matchings of size two are given by $\sigma_8 = \{u_0u_1, v_2v_3\}, \sigma_9 = \{u_0u_1, v_3u_3\}, \sigma_{10} = \{u_0v_0, v_2u_2\}$ and $\sigma_{11} = \{u_0u_1, u_3u_4\}$. Any other matching of size two, whose edges are at distance 2, is automorphic to one of $\sigma_8, \sigma_9, \sigma_{10}$ or σ_{11}. Further, these matchings are pairwise non-automorphic.

(iii) Let the edges of M_2 be at distance 3 from each other. Three such matchings of size two are given by $\sigma_{12} = \{u_0u_1, v_3v_4\}$, $\sigma_{13} = \{u_0u_1, v_4u_4\}$ and $\sigma_{14} = \{u_0v_0, v_3u_3\}$. Any other matching, whose edges are at distance 3, is automorphic to one of σ_{12}, σ_{13} or σ_{14}. Further, no two of these matchings are automorphic to each other.

This completes the proof. □

Lemma 2.3.9. *The number of distinct automorphic type matchings of $P_{7,1}$ of size three is* 23.

Proof. We classify the matchings M_3 of size three on the basis of the number of spokes contained in it. Since each forbidden matching is a matching of size three, and that they are switching equivalent to a matching of size at most two, we consider matchings other than the forbidden matchings.

(i) Assume that M_3 does not contain a spoke. Few such matchings are given by $\sigma_{15} = \{u_0u_1, u_2u_3, u_4u_5\}$, $\sigma_{16} = \{u_0u_1, u_2u_3, v_4v_5\}$ and $\sigma_{17} = \{u_0u_1, u_4u_3, v_5v_6\}$. Any other matching of size three with no spoke is either a forbidden matching or automorphic to one of σ_{15}, σ_{16} or σ_{17}. Further, it is easy to see that these matchings are pairwise non-automorphic.

(ii) Assume that M_3 contains exactly one spoke, say u_0v_0. A few of such matchings of $P_{7,1}$ are given by $\sigma_{18} = \{u_0v_0, u_1u_2, u_3u_4\}$, $\sigma_{19} = \{u_0v_0, u_1u_2, u_4u_5\}$, $\sigma_{20} = \{u_0v_0, u_1u_2, u_5u_6\}$, $\sigma_{21} = \{u_0v_0, u_2u_3, u_4u_5\}$, $\sigma_{22} = \{u_0v_0, u_1u_2, v_2v_3\}$, $\sigma_{23} = \{u_0v_0, u_1u_2, v_3v_4\}$, $\sigma_{24} = \{u_0v_0, u_1u_2, v_4v_5\}$, $\sigma_{25} = \{u_0v_0, u_1u_2, v_5v_6\}$, $\sigma_{26} = \{u_0v_0, u_2u_3, v_3v_4\}$, $\sigma_{27} = \{u_0v_0, u_2u_3, v_4v_5\}$, $\sigma_{28} = \{u_0v_0, u_2u_3, v_5v_6\}$ and $\sigma_{29} = \{u_0v_0, u_3u_4, v_5v_6\}$. Any other matching of size three containing only one spoke is automorphic to one of these 12 matchings. Further, any two of these matchings are non-automorphic.

(iii) Assume that M_3 contains exactly two spokes. If the distance between the spokes of M_3 is 1, let they be v_0u_0 and v_1u_1. One such matching is given by $\sigma_{30} = \{v_0u_0, v_1u_1, u_3u_4\}$. If the spokes are at distance 2 from each other, then let the spokes be v_0u_0 and v_2u_2. In this case, two such matchings are given by $\sigma_{31} = \{v_0u_0, v_2u_2, u_3u_4\}$ and $\sigma_{32} = \{v_0u_0, v_2u_2, u_4u_5\}$. If the spokes are at distance 3 from each other, then let they be v_0u_0 and v_3u_3. Two such possible matchings are given by $\sigma_{33} = \{v_0u_0, v_3u_3, u_1u_2\}$ and $\sigma_{34} = \{v_0u_0, v_3u_3, u_4u_5\}$.

Because of the forbidden matchings and the automorphisms of $P_{7,1}$, it is easy to see that any other matching of size three containing only two spokes is automorphic to one of $\sigma_{30}, \sigma_{31}, \sigma_{32}, \sigma_{33}$ or σ_{34}. Further, these matchings are pairwise non-automorphic.

(iv) Assume that M_3 contains three spokes. In this case, three such possible matchings are given by $\sigma_{35} = \{v_0u_0, v_1u_1, u_3v_3\}$, $\sigma_{36} = \{v_0u_0, v_1u_1, u_4v_4\}$ and $\sigma_{37} = \{v_0u_0, v_2u_2, u_4v_4\}$. Any matching of size three with three spokes is automorphic to σ_{35}, σ_{36} or σ_{37}. Also, these matchings are pairwise non-automorphic.

This completes the proof of the lemma. □

Lemma 2.3.10. *The number of switching isomorphism classes of* $P_{7,1}$ *such that each class has a minimum signature of size four is* 2.

Proof. It is enough to count the number of minimum matchings of size four, up to switching isomorphism. We classify such matchings on the basis of the number of spokes contained in it.

(i) Assume that M_4 does not contain a spoke. It is clear that any such M_4 has its three edges on the outer (inner) cycle and the remaining edge on the inner (outer) cycle, or two edges on the inner cycle and other two edges on the outer cycle. Therefore, any such M_4 must contain one of F_2, F_3 or F_4. Hence by Lemma 2.3.7, every matching of size four having no spoke is equivalent to a matching $M_{l'}$, where $l' \leq 3$. Thus a matching of size four containing no spoke cannot be minimum.

(ii) Assume that M_4 contains exactly one spoke, and let it be u_0v_0. It is clear that the remaining three edges of M_4 either lie on the outer (inner) cycle, or two edges lie on the outer (inner) cycle and other edge lies on the inner (outer) cycle. If three edges lie on the outer cycle, then one such matching is given by $\sigma_{38} = \{v_0u_0, u_1u_2, u_3u_4, u_5u_6\}$. If two edges lie on the outer cycle and one edge lies on the inner cycle, then two such possible matchings are given by $\sigma_{39} = \{v_0u_0, u_1u_2, u_3u_4, v_5v_6\}$ and $\sigma_{40} = \{v_0u_0, u_1u_2, v_3v_4, u_5u_6\}$. Any other matching of size four containing only one spoke either contains one of the forbidden matchings or automorphic to one of σ_{38}, σ_{39} or σ_{40}.

(iii) Assume that M_4 contains exactly two spokes. If the spokes of M_4 are consecutive, then switching by the outer vertices of the spokes, we get a signature equivalent to M_4. Note that this resultant signature is either equivalent to a matching of size four of case (i) or it is not a matching. In both the cases, the resultant signature can be reduced to a matching $M_{l'}$, where $l' \leq 3$. Further, if the spokes of M_4 are at distance 2 or 3 from each other, then a few of such possible matchings are given by

$$\sigma_{41} = \{v_0u_0, v_2u_2, u_3u_4, u_5u_6\}, \quad \sigma_{42} = \{v_0u_0, v_2u_2, u_3u_4, v_5v_6\},$$

$$\sigma_{43} = \{v_0u_0, v_2u_2, u_4u_5, v_5v_6\}, \quad \sigma_{44} = \{v_0u_0, v_3u_3, u_1u_2, u_4u_5\},$$

$$\sigma_{45} = \{v_0u_0, v_3u_3, u_1u_2, v_4v_5\} \text{ and } \sigma_{46} = \{v_0u_0, v_3u_3, u_4u_5, v_5v_6\}.$$

Any other matching of size four containing only two spokes is automorphic to one of these six matchings.

(iv) Assume that M_4 contains exactly three spokes. It is clear that a matching of size four containing exactly three consecutive spokes is switching equivalent to a matching (signature) of size three. Also, if a matching contains exactly two consecutive spokes, then the matching is switching equivalent to a signature σ of size four. Observe that σ is either a matching M_4 having exactly one spoke or a signature of size four having two adjacent edges. The matching M_4 having exactly one spoke is already covered in case (ii). Also, a signature of size four having two adjacent edges is switching equivalent to a matching of size three. Thus all the three spokes of M_4 are non-consecutive. One such possible matching is given by $\sigma_{47} = \{v_0u_0, v_2u_2, v_4u_4, u_5u_6\}$. Any other such matching is automorphic to the matching σ_{47}.

(v) Assume that M_4 contains four spokes. Switching by all outer vertices, we get a matching of size three equivalent to M_4.

From the preceding cases, we see that the possible matchings M_4 of size four are given by $\sigma_{38}, \ldots, \sigma_{47}$. Observe that some of these matchings may be switching isomorphic to each other or switching isomorphic to a matching of size at most three. We have the following observations.

- By switching u_6, u_0 and u_1 in σ_{38}, we get a matching automorphic to σ_{32}. Thus $\sigma_{38} \simeq \sigma_{32}$.
- By switching $u_0, u_1, u_4, u_5, u_6, v_4$ and v_5 in σ_{39}, and using the automorphism γ, we get a matching automorphic to σ_{32}. Thus $\sigma_{39} \simeq \sigma_{32}$.
- By switching u_0, u_1 and u_6 in σ_{40}, and using the automorphism γ, we get a matching automorphic to σ_{32}. Thus $\sigma_{40} \simeq \sigma_{32}$.

- By switching u_0, u_1, u_2, u_3 and u_6 in σ_{41}, we get a matching automorphic to σ_{37}. Thus $\sigma_{41} \simeq \sigma_{37}$.

- By switching u_0, u_5, u_6 and v_5 in σ_{43}, and using the automorphism δ_4, we get a matching automorphic to σ_{45}. Thus $\sigma_{43} \simeq \sigma_{45}$.

- By switching u_0 and u_1 in σ_{44}, we get a matching automorphic to σ_{41}. Thus $\sigma_{44} \simeq \sigma_{41}$.

- By switching u_2 and u_3 in σ_{45}, and using the automorphism δ_5, we get a matching automorphic to σ_{42}. Thus $\sigma_{45} \simeq \sigma_{42}$.

- By switching u_4, v_5, v_4 and v_3 in σ_{46}, and using the automorphism γ, we get a matching automorphic to σ_{43}. Thus $\sigma_{46} \simeq \sigma_{43}$.

Thus the only left out matchings of size four are σ_{42} and σ_{47}. From Tables 2.3.3 and 2.3.4, it is clear that these two matchings are neither switching isomorphic to each other nor switching isomorphic to a matching of $P_{7,1}$ of size at most three. This completes the proof. ☐

The matchings obtained in Lemma 2.3.8 and Lemma 2.3.9, along with σ_0, σ_1 and σ_2, give us 38 distinct automorphic type matchings of $P_{7,1}$ of sizes at most 3. However, among these automorphic type matchings, few of them may be switching isomorphic to each other. We have the following observations.

- By switching u_0 and u_1 in σ_7, we get a matching automorphic to σ_6. Therefore $\sigma_7 \simeq \sigma_6$.

- By switching u_1, v_1, u_2 and v_2 in σ_{16}, and using the automorphism γ, we get a matching automorphic to σ_{15}. Thus $\sigma_{16} \simeq \sigma_{15}$.

- By switching u_0, v_0, u_6 and v_6 in σ_{17}, and using the automorphisms δ_3 followed by δ_5, we get a matching automorphic to σ_{16}. Thus $\sigma_{17} \simeq \sigma_{16}$.

- By switching u_1 and u_0 in σ_{19}, we get a matching automorphic to σ_{18}. Thus $\sigma_{19} \simeq \sigma_{18}$.

- By switching u_1, u_0 and u_6 in σ_{20}, we get a matching automorphic to σ_{10}. Thus $\sigma_{20} \simeq \sigma_{10}$.

- By switching u_0 and u_1 in σ_{24}, and using the automorphism δ_1, we get a matching automorphic to σ_{23}. Thus $\sigma_{24} \simeq \sigma_{23}$.

- By switching v_1, v_0, v_6 and u_1 in σ_{25}, we get a matching automorphic to σ_{22}. Thus $\sigma_{25} \simeq \sigma_{22}$.

- By switching v_6 and v_0 in σ_{28}, and using the automorphism γ, we get a matching automorphic to σ_{23}. Thus $\sigma_{28} \simeq \sigma_{23}$.

- By switching v_6 and v_0 in σ_{29}, and using the automorphism γ, we get a matching automorphic to σ_{24}. Thus $\sigma_{29} \simeq \sigma_{24}$.

- By switching u_0 and u_1 in σ_{30}, we get a matching automorphic to σ_{15}. Thus $\sigma_{30} \simeq \sigma_{15}$.

- By switching u_2 and u_3 in σ_{33}, we get a matching automorphic to σ_{31}. Thus $\sigma_{33} \simeq \sigma_{31}$.

- By switching u_0 and u_1 in σ_{35}, and using the automorphism δ_5, we get a matching automorphic to σ_{18}. Thus $\sigma_{35} \simeq \sigma_{18}$.

- By switching u_0 and u_1 in σ_{36}, we get a matching automorphic to σ_{21}. Thus $\sigma_{36} \simeq \sigma_{21}$.

Due to the preceding observations and Lemma 2.3.10, we only consider the following matchings: $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_8, \sigma_9,$ $\sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13},$ $\sigma_{14}, \sigma_{15}, \sigma_{18}, \sigma_{21}, \sigma_{22},$ σ_{23}, σ_{26}, $\sigma_{27}, \sigma_{31}, \sigma_{32}, \sigma_{34}, \sigma_{37}, \sigma_{42}$ and σ_{47}. The signed graphs corresponding to these matchings are shown in Figure 2.4.2, where the label of the vertices correspond to that of Figure 2.3.6.

Theorem 2.3.9. *There are exactly* 27 *distinct signed* $P_{7,1}$ *up to switching isomorphism.*

Proof. Let $|C_4^-|, |C_6^-|, |C_7^-|$ and $|C_8^-|$ denote the number of negative 4-cycles, negative 6-cycles, negative 7-cycles and negative 8-cycles, respectively, of a signed $P_{7,1}$. These numbers for the signed graphs, shown in Figure 2.4.2, are given in Table 2.3.3 and Table 2.3.4. Using Remark 2.1.1 in Table 2.3.3 and Table 2.3.4, we find that these 27 signed $P_{7,1}$ are switching non-isomorphic. This completes the proof. □

2.4 Switching Isomorphism Classes of $B(m, n)$

In this section, we obtain switching non-isomorphic signed book graphs. For $m \geq 3$ and $n \geq 1$, recall that the m-cycle *book graph* $B(m, n)$ consists of n copies

	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{13}	Σ_{14}
$\lvert C_4^-\rvert$	0	1	2	0	2	3	2	2	3	4	2	2	3	4
$\lvert C_6^-\rvert$	0	2	2	0	2	2	4	4	4	2	4	4	4	4
$\lvert C_7^-\rvert$	0	1	0	2	2	1	0	2	1	0	0	2	1	0
$\lvert C_8^-\rvert$	0	3	2	0	2	3	4	4	3	4	6	6	5	2

Table 2.3.3. Number of negative k-cycles, where $k \in \{4,6,7,8\}$, in some signed graphs of Figure 2.4.2

	Σ_{15}	Σ_{18}	Σ_{21}	Σ_{22}	Σ_{23}	Σ_{26}	Σ_{27}	Σ_{31}	Σ_{32}	Σ_{34}	Σ_{37}	Σ_{42}	Σ_{47}
$\lvert C_4^-\rvert$	3	4	4	4	4	4	4	5	5	5	6	6	7
$\lvert C_6^-\rvert$	6	4	6	2	4	4	6	2	4	4	2	2	0
$\lvert C_7^-\rvert$	1	0	0	2	2	2	2	1	1	1	0	2	1
$\lvert C_8^-\rvert$	5	4	2	4	4	2	2	5	3	1	4	4	6

Table 2.3.4. Number of negative k-cycles, where $k \in \{4,6,7,8\}$, in some signed graphs of Figure 2.4.2

of the cycle C_m with one common edge. The copies of the cycle C_m are called the *pages* of $B(m,n)$. Indeed

$$V(B(m,n)) = \{u,v\} \cup \{u_j^i : i \in \{1,\dots,n\},\ j \in \{1,\dots,m-2\}\},$$

where uv is the common edge to the cycles C_m^i, and $C_m^i = uu_1^i u_2^i u_3^i \dots u_{m-3}^i u_{m-2}^i vu$ for $1 \le i \le n$. For example, C_4^1 in $B(4,3)$ is the cycle $uu_1^1 u_2^1 vu$, where the graph $B(4,3)$ is shown in Figure 2.4.1.

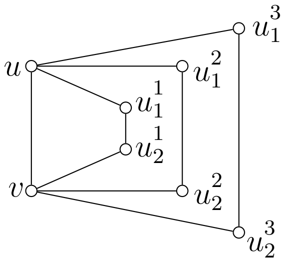


Figure 2.4.1. The book graph $B(4,3)$

Let $\mathrm{Aut}(G)$ denote the automorphism group of a graph G. It is clear that $B(m,1) = C_m$, and hence $\mathrm{Aut}(B(m,1)) \cong D_m$, the dihedral group of order $2m$.

Also, $B(m,n)-\{u,v\}$ is the disjoint union of n copies of the paths of order $m-2$. It is clear that any permutation of these n copies of P_{m-2} determines an automorphism of $B(m,n)$. Conversely, any non-trivial automorphism of $B(m,n)$ permutes the vertices u and v, and also permutes these n copies of P_{m-2}. Hence it is clear that $\text{Aut}(B(m,n)) \cong S_n \times S_2$, for $n \geq 2$. Thus every automorphism of $B(m,n)$ can only permute the vertices u and v, and permute the n pages of the graph.

Note that every signed cycle is switching equivalent to a signed cycle whose signature is either empty or of size one. We use this fact in the proof of the following theorem.

Theorem 2.4.1. *If $(B(m,n),\sigma)$ is a signed book graph, then $(B(m,n),\sigma)$ is equivalent to $(B(m,n),\tau)$, where $\tau \subseteq \{uu_1^1,\dots,uu_1^n\}$. Further, the number of switching non-isomorphic signed $B(m,n)$ is $n+1$.*

Proof. Let $(B(m,n),\sigma)$ be a signed book graph. By suitable switchings, if needed, we can make each negative edge of $B(m,n)$ incident to u. If the edge uv is negative, switching by u will make it positive. Thus we get a signature τ equivalent to σ such that $\tau \subseteq \{uu_1^1,\dots,uu_1^n\}$.

Note that if σ and τ are subsets of $\{uu_1^1,\dots,uu_1^n\}$ and $|\sigma| = |\tau|$, then an one-one correspondence of σ and τ determines an isomorphism between $(B(m,n),\sigma)$ and $(B(m,n),\tau)$. However, if $|\sigma| \neq |\tau|$, then $(B(m,n),\sigma)$ cannot be switching isomorphic to $(B(m,n),\tau)$ as the number of negative cycles of length m are different in $(B(m,n),\sigma)$ and $(B(m,n),\tau)$. Therefore, the number of switching non-isomorphic signed $B(m,n)$ is $n+1$. ☐

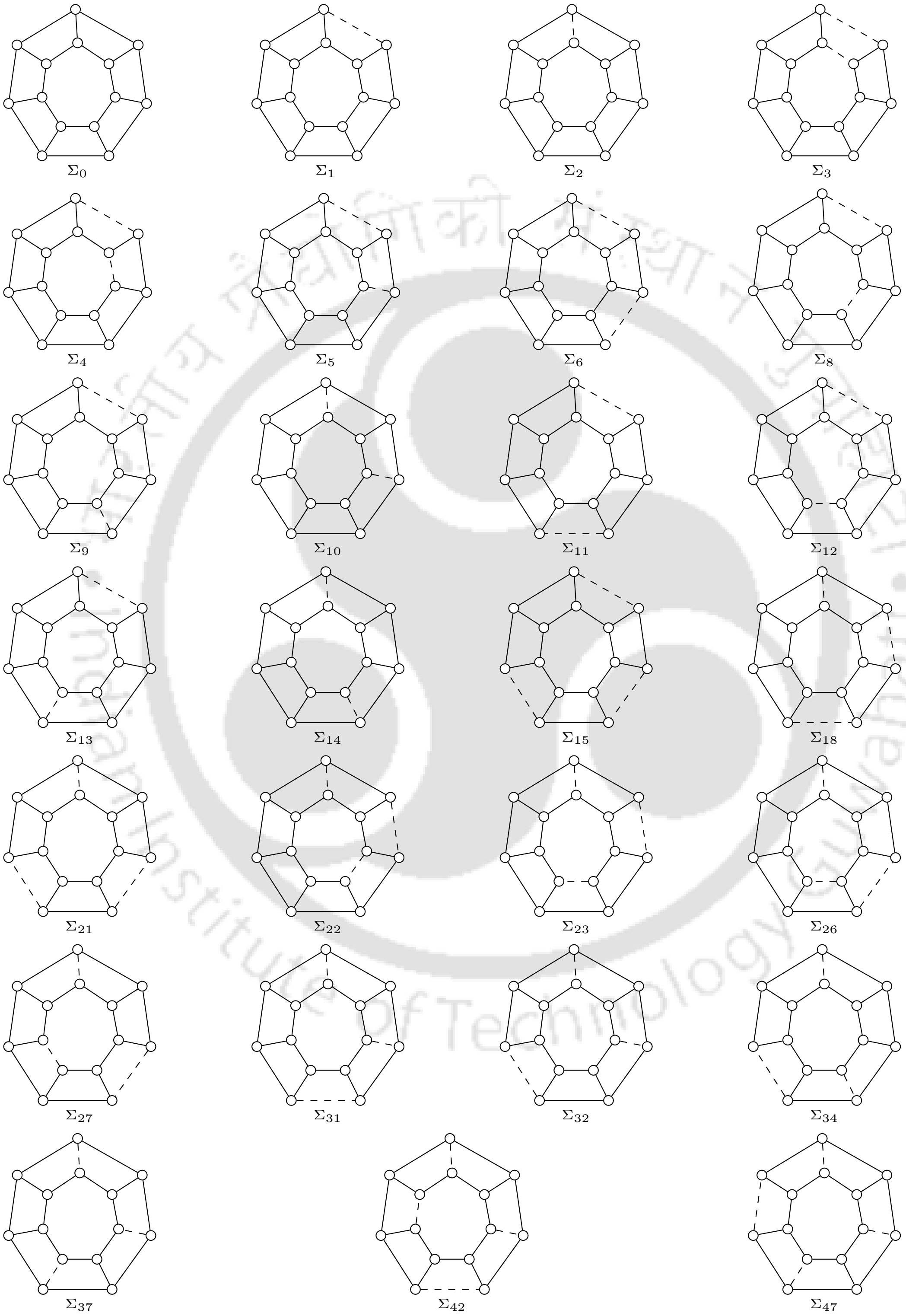


Figure 2.4.2. Twenty seven signed $P_{7,1}$

3

Frustration in Signed Graphs

Balance is the fundamental property of signed graphs. An unbalanced signed graph can be converted into a balanced signed graph by deleting some of its vertices or edges. In signed graph theory, the frustration index and the frustration number are among the most investigated invariants. Both of these parameters are invariant under switching. Zaslavsky [60] conjectured that these two numbers are equal for any signed cubic graph. Sivaraman [49] confirmed the conjecture of Zaslavsky and proved that $l(\Sigma) = l_0(\Sigma)$ for any signed sub-cubic graph. Sivaraman [49] also proved that if Σ is a signed graph whose underlying graph is simple, triangle-free and cubic, then $l(\Sigma) \le \frac{3}{8}|V(\Sigma)|$.

In this chapter, we prove that $D(P_{n,k}) \le 1 + \lfloor \frac{n}{2} \rfloor$ if $d := \gcd(n,k) = 1$, and $D(P_{n,k}) \le 1 + d(1 + \lfloor \frac{n}{2d} \rfloor)$ if $d > 1$. This improves the Sivaraman's bound for the class of generalized Petersen graphs. We also determine the frustration indices and the frustration numbers of switching non-isomorphic signed complete graphs over K_6. Finally, we show that $D(B(m,n)) = \lceil \frac{n}{2} \rceil$, where $m \ge 3$ and $n \ge 1$.

3.1 Preliminary Results

Throughout this chapter, we use the notation E_- in place of σ, for the set of negative edges of a signed graph (G, σ). That is $E_- = \sigma$. Let E_-^1 and E_-^2 be the negative edge sets of signed graphs Σ_1 and Σ_2, respectively. We say that E_-^1 is switching equivalent to E_-^2, denoted $E_-^1 \sim E_-^2$, to mean that Σ_1 is switching equivalent to Σ_2. We know that switching defines an equivalence relation on the set of all signed graphs over G (also on the set of negative edge sets). We write $[E_-]$ to denote the class of all negative edge sets equivalent to the negative edge set E_-.

Recall that the *frustration index* of a signed graph (G, E_-), denoted $l(G, E_-)$, is the least number of edges whose deletion results in a balanced signed graph. Implicit in [11], frustration index is switching invariant. More precisely, we have the following lemma.

Lemma 3.1.1. *The frustration index of a signed graph is invariant under switching.*

Proof. Let (G, E_-) be a signed graph. Suppose $(G, E_-) - D$ is a balanced signed graph, where $D \subseteq E(G)$. Since switching does not change the sign of a cycle, if we switch to $(G, E_-)^X$ and then delete D, there will still not be any negative cycles. Therefore $(G, E_-)^X - D$ is also balanced. Thus deletion of an edge set that makes one of the graphs in $[G, E_-]$ balanced, also make the other graphs in $[G, E_-]$ balanced. This completes the proof. ☐

Note that if (G, E_-) is balanced, then clearly $l(G, E_-) = 0$. Also, it is easy to see that

$$l(G, E_-) = \min_{E'_- \in [E_-]} |E'_-|. \tag{3.1.1}$$

Recall that the *maximum frustration* $D(G)$ of a graph G is the maximum frustration index over all possible signatures on G. That is,

$$D(G) = \max_{E_- \subseteq E(G)} l(G, E_-). \tag{3.1.2}$$

Further, the frustration number of a signed graph (G, E_-), denoted $l_0(G, E_-)$, is the least number of vertices whose deletion results in a balanced signed graph.

Lemma 3.1.2. *[60] Switching does not change the frustration number of a signed graph. Moreover,* $l_0(G, E_-) \leq l(G, E_-)$ *for every negative edge set* E_- *of a graph* G.

To emphasize that frustration index and frustration number are invariant under switching, we use the notations $l[G, E_-]$ and $l_0[G, E_-]$ in places of $l(G, E_-)$ and $l_0(G, E_-)$, respectively. In [49], the author proved that the frustration index and the frustration number are equal for signed sub-cubic graphs.

Theorem 3.1.1. *[49] If* Σ *is a signed sub-cubic graph, then* $l[\Sigma] = l_0[\Sigma]$.

From (3.1.1), it is clear that the frustration index of a signed graph is equal to the size of a minimum negative edge set. The following result, that easily follows from Theorem 2.1.1 of Chapter 2, says that the frustration index of a signed sub-cubic graph (G, E_-) is at most $\frac{|V(G)|}{2}$.

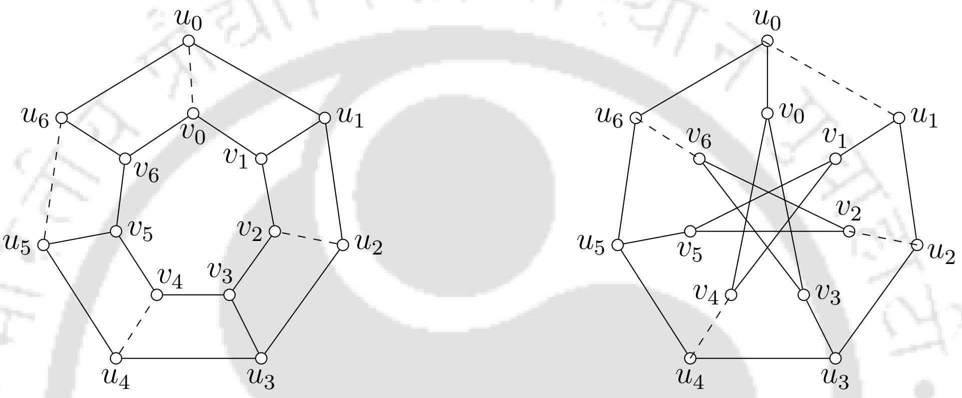
Theorem 3.1.2. *If* (G, E_-) *is a signed sub-cubic graph, then* $l[G, E_-] \leq \frac{|V(G)|}{2}$.

The bound obtained in Theorem 3.1.2 has an improvement for the cubic, triangle-free graphs.

Theorem 3.1.3. *[49] Let* (G, E_-) *be a signed graph. If* G *is simple, cubic and triangle-free, then* $l[G, E_-] \leq \frac{3}{8}|V(G)|$.

3.2 Maximum Frustration in $P_{n,k}$

Recall that $P_{n,k}$ denotes the generalized Petersen graph. The signed generalized Petersen graphs $P_{7,1}$ and $P_{7,3}$, with a minimum negative edge set of size four, are shown in Figure 3.2.1. The facts, that the negative edge sets of Figure 3.2.1a and Figure 3.2.1b are minimum, are proved in Lemma 3.2.2 and Example 3.2.1, respectively.



(a) A signed $P_{7,1}$ with $|E_-| = 4$ (b) A signed $P_{7,3}$ with $|E_-| = 4$

Figure 3.2.1. Reduced signed graphs over $P_{7,1}$ and $P_{7,3}$

3.2.1 Bound on $D(P_{n,k})$ when $\gcd(n,k) = 1$

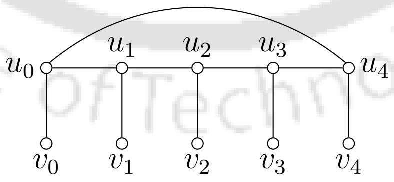


Figure 3.2.2. The graph G_5

Recall from Chapter 2 that the graph G_n is defined by $V(G_n) = \{u_i, v_i : i \in [n]\}$ and $E(G_n) = \{u_iu_{i+1}, u_iv_i : i \in [n]\}$, where $n \geq 3$ and the subscripts are read modulo n. For instance, the graph G_5 is shown in Figure 3.2.2. The vertices $u_0, \ldots, u_{n-1}$ are called the *u-vertices* of G_n. Similarly, the vertices $v_0, \ldots, v_{n-1}$ are called the *v-vertices* of G_n. Note that the graph induced by the edges of the outer cycle and the spokes of $P_{n,k}$ is isomorphic to G_n. Similarly, if $\gcd(n,k) = 1$, then the graph induced by the edges of the inner cycle and the spokes of $P_{n,k}$ is also isomorphic to

G_n. From the definition, it is also clear that the graph G_n is a sub-cubic graph, that is, each vertex of G_n has degree at most 3. Let C_n denote the cycle of G_n.

By Theorem 2.1.1, it is clear that the edges of any nonempty minimum negative edge set of G_n form a matching. Since the cycle C_n can be either positive or negative, and any negative edge of G_n incident to a v-vertex can be made positive by switching that v-vertex, a minimum negative edge set of G_n is either empty or a set of size one. This implies that $l[G_n, E_-] \leq 1$.

Clearly, (G_n, E_-) is balanced or unbalanced according as the cycle C_n is positive or negative, respectively. Further, if we do not allow switching operation by v-vertices, then the following lemma gives an upper bound on the least size of a negative edge set switching equivalent to E_-.

Lemma 3.2.1. *Let (G_n, E_-) be the signed graph defined in the preceding paragraph. If switching operation is not allowed by v-vertices, then the least size of a negative edge set equivalent to E_- is at most $\left\lfloor \frac{n}{2} \right\rfloor$ or $\left\lfloor \frac{n}{2} \right\rfloor + 1$ according as C_n is positive or negative in (G_n, E_-), respectively.*

Proof. Consider the signed graph (G_n, E_-). If the cycle C_n is positive in (G_n, E_-), then by switching u-vertices, if needed, we can make all the edges of C_n positive. Thus, the negative edges in the resulting signed graph are of the form $u_i v_i$ only. If the number of such negative edges are more than $\left\lfloor \frac{n}{2} \right\rfloor$, then we switch all u-vertices to get a signature of size at most $\left\lfloor \frac{n}{2} \right\rfloor$. We denote this new signature by E'_-. Thus $|E'_-| \leq \left\lfloor \frac{n}{2} \right\rfloor$.

If the cycle C_n is negative in (G_n, E_-), then by switching operation, if needed, we can make any given edge of C_n negative, and rest of the edges positive. Among all negative edge sets switching equivalent to E_-, obtained by switching only u-vertices, let E'_- be of least size. Note that E'_- is also a matching. We may assume that $E'_- \cap E(C_n) = \{u_0 u_1\}$. Therefore $E'_- \subseteq \{u_0 u_1\} \cup S$, where $S = \{u_2 v_2, \ldots, u_{n-1} v_{n-1}\}$. Take $E''_- = E'_- \setminus \{u_0 u_1\}$. To complete the proof, we show that $|E''_-| \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Suppose on the contrary that $|E''_-| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. We have

$$|E'_-| \geq \left\lfloor \frac{n}{2} \right\rfloor + 2. \tag{3.2.1}$$

Since $u_0 u_1 \in E'_-$ and $E''_- \subseteq S$, switching by each vertex of $\{u_1, \ldots, u_{n-1}\}$, we get a negative edge set E^*_- equivalent to E'_- such that $E^*_- = (S \setminus E''_-) \cup \{u_1 v_1, u_{n-1} u_0\}$. This gives

$$|E^*_-| = |S| - |E''_-| + 2 = n - |E''_-| \leq n - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, we have a negative edge set E^*_-, obtained by switching only some u-vertices in E'_-, such that $|E^*_-| \leq \left\lfloor \frac{n}{2} \right\rfloor$. This is a contradiction to the fact that $|E'_-| \geq \left\lfloor \frac{n}{2} \right\rfloor + 2$ and E'_- is of least size. Hence $|E''_-| \leq \left\lfloor \frac{n}{2} \right\rfloor$, and this completes the proof. ☐

Now we find an upper bound on the maximum frustration of $P_{n,k}$ in terms of n, where $\gcd(n,k) = 1$.

Theorem 3.2.1. *Let n and k be positive integers such that $2k < n$. If $\gcd(n,k) = 1$, then $D(P_{n,k}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Moreover, this bound is sharp.*

Proof. Since $\gcd(n,k) = 1$, the graph $P_{n,k}$ has exactly one inner cycle and this cycle is given by $C_I = v_0 v_k v_{2k} \dots v_{n-k} v_0$. Recall that the outer cycle is given by $C_o = u_0 u_1 \dots u_{n-1} u_0$. Switch by the vertices of C_o and C_I such that each of them has either 0 or 1 negative edge. Now consider the spokes. If more than half of them are negative, then switch by the vertices of C_o. The resulting signature will have at most $\frac{n}{2} + 2$ negative edges.

If either C_o or C_I has no negative edge, then the result follows. So, we consider the case that both the cycles C_o and C_I have exactly 1 negative edge. Without loss of generality, let $u_0 u_1$ be the negative edge in C_o and $v_r v_{r+k}$ be the negative edge in C_I for some $r \in \{0, 1, ..., n-1\}$. We complete the proof by showing that for any negative edge set E_- containing $u_0 u_1$ and $v_r v_{r+k}$, there is a minimum negative edge set, equivalent to E_-, of size at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$.

Let E'_- be a minimum negative edge set, equivalent to E_-, such that E'_- form a matching and $E'_- \subseteq \{u_0 u_1, v_r v_{r+k}\} \cup S$, where $S = S_s \setminus \{u_0 v_0, u_1 v_1, u_r v_r, u_{r+k} v_{r+k}\}$. Take $E''_- = E'_- \setminus \{u_0 u_1, v_r v_{r+k}\}$. Thus $|E'_-| = |E''_-| + 2$ and the edges of E''_- are some spokes only. Now we show that $|E''_-| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. To do so we consider the following sub-cases.

(a) Let neither v_r nor v_{r+k} be in $\{v_0, v_1\}$. In this case, we get $|S| = n - 4$. Suppose on the contrary that $|E''_-| \geq \left\lfloor \frac{n}{2} \right\rfloor$. Thus $|E'_-| \geq \left\lfloor \frac{n}{2} \right\rfloor + 2$.

By switching each vertex of $\{u_1, u_2, \dots, u_{n-1}, v_r\}$, we get an equivalent negative edge set E^*_-, where $E^*_- = (S \setminus E''_-) \cup \{u_1 v_1, v_{r-k} v_r, u_{r+k} v_{r+k}, u_{n-1} u_0\}$. Therefore,

$$|E^*_-| = |S| - |E''_-| + 4 = n - 4 - |E''_-| + 4 \leq n - \left\lfloor \frac{n}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

This is a contradiction to the fact that E'_- is minimum and $|E'_-| \geq \lfloor \frac{n}{2} \rfloor + 2$. Therefore $|E''_-| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, and this implies that $|E'_-| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

(b) Let $v_r = v_0$. If $u_{n-1} v_{n-1} \in E''_-$, then switching each vertex of $\{u_0, v_0, u_{n-1}\}$ gives us an equivalent negative edge set of size 1 less than the size of E'_-. This contradicts that E'_- is minimum. Hence $u_{n-1} v_{n-1} \notin E''_-$.

b(i) If $k > 1$ then $E''_- \subseteq T$, where $T = S_s \setminus \{u_0 v_0, u_1 v_1, u_k v_k, u_{n-1} v_{n-1}\}$. On the contrary, suppose that $|E''_-| \geq \left\lfloor \frac{n}{2} \right\rfloor$, giving $|E'_-| \geq \left\lfloor \frac{n}{2} \right\rfloor + 2$.

By switching each vertex of $\{u_1, \dots, u_{n-2}, v_k\}$, we get an equivalent negative edge set E^*_-, where $E^*_- = (T \setminus E''_-) \cup \{u_1v_1, v_kv_{2k}, u_{n-2}u_{n-1}\}$. Therefore,

$$|E^*_-| = |T| - |E''_-| + 3 = n - 4 - |E''_-| + 3 \le n - 1 - \left\lfloor \frac{n}{2} \right\rfloor \le \left\lfloor \frac{n}{2} \right\rfloor.$$

b(ii) If $k = 1$ then $E''_- \subseteq T$, where $T = S_s \setminus \{u_0v_0, u_1v_1, u_{n-1}v_{n-1}\}$. On the contrary, suppose that $|E''_-| \ge \left\lfloor \frac{n}{2} \right\rfloor$, giving $|E'_-| \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$.

By switching each vertex of $\{u_1, u_2, \ldots, u_{n-2}, v_1\}$, we get an equivalent negative edge set E^*_- such that $E^*_- = (T \setminus E''_-) \cup \{v_1v_2, u_{n-2}u_{n-1}\}$. Therefore,

$$|E^*_-| = |T| - |E''_-| + 2 = n - 3 - |E''_-| + 2 \le n - 1 - \left\lfloor \frac{n}{2} \right\rfloor \le \left\lfloor \frac{n}{2} \right\rfloor.$$

In both **b(i)** and **b(ii)**, we get a contradiction to the fact that E'_- is minimum and $|E'_-| \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$. Hence $|E''_-| \le \left\lfloor \frac{n}{2} \right\rfloor - 1$, and this implies that $|E'_-| \le \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Applying similar procedure, if $v_r = v_1$ or $v_{r+k} = v_0$ or $v_{r+k} = v_1$, then also we find that $|E'_-| \le \left\lfloor \frac{n}{2} \right\rfloor + 1$. For the sake of completeness, we provide the proofs of these cases as well.

(c) Let $v_r = v_1$. We consider two sub-cases.

c(i) Let $k > 1$. If $u_2v_2 \in E''_-$, then switching by each vertex of $\{u_1, v_1, u_2\}$ gives us an equivalent negative edge set of size 1 less than the size of E'_-. This contradicts the fact that E'_- is minimum. Hence we have $u_2v_2 \notin E''_-$. Thus $E''_- \subseteq T^*$, where $T^* = S_s \setminus \{u_0v_0, u_1v_1, u_2v_2, u_{k+1}v_{k+1}\}$. On contrary, suppose that $|E''_-| \ge \left\lfloor \frac{n}{2} \right\rfloor$, giving $|E'_-| \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$.

By switching each vertex of $\{u_1, \ldots, u_{n-1}, v_{k+1}\}$, we get an equivalent negative edge set E^*_-, where $E^*_- = (T^* \setminus E''_-) \cup \{u_1v_1, u_2v_2, v_{k+1}v_{2k+1}, u_{n-1}u_0\}$. Therefore,

$$|E^*_-| = |T^*| - |E''_-| + 4 = n - 4 - |E''_-| + 4 \le n - \left\lfloor \frac{n}{2} \right\rfloor \le \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

c(ii) Let $k = 1$. Thus $E''_- \subseteq T^*$, where $T^* = S_s \setminus \{u_0v_0, u_1v_1, u_2v_2\}$. On contrary, suppose that $|E''_-| \ge \left\lfloor \frac{n}{2} \right\rfloor$, giving $|E'_-| \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$.

By switching each vertex of $\{u_1, \ldots, u_{n-1}, v_1\}$, we get an equivalent negative edge set E^*_-, where $E^*_- = (T^* \setminus E''_-) \cup \{u_2v_2, v_0v_1, u_{n-1}u_0\}$. Therefore,

$$|E^*_-| = |T^*| - |E''_-| + 3 = n - 3 - |E''_-| + 3 \le n - \left\lfloor \frac{n}{2} \right\rfloor \le \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

In both **c(i)** and **c(ii)**, we get a contradiction as E'_- is minimum and $|E'_-| \ge \left\lfloor \frac{n}{2} \right\rfloor + 2$. Hence $|E''_-| \le \left\lfloor \frac{n}{2} \right\rfloor - 1$, and this implies that $|E'_-| \le \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Note that $v_{r+k} = v_0$ if and only if $v_r = v_{n-k}$. Similarly, $v_{r+k} = v_1$ if and only if $v_r = v_{n-k+1}$. Consider the reflection automorphism δ of $P_{n,k}$ determined by the interchanges of u_i and u_{n-i+1} for each $i \in [n]$. If $v_{r+k} = v_0$, then take $E^2_- = \delta(E'_-)$, the image of E'_- under δ. Observe that the negative outer and inner edges in E^2_-

are u_0u_1 and v_1v_{k+1} only. Further, $|E_-^2| = |E'_-|$, and hence E_-^2 is also minimum. By case **(c)**, we get, $|E_-^2| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$, that is, $|E'_-| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Similarly, if $v_{r+k} = v_1$, then also we get $|E'_-| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Thus the number of edges in a minimum negative edge set, equivalent to E_-, is at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$. Since E_- is arbitrary, we have

$$D(P_{n,k}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Sharpness of this bound follows from Theorem 3.2.2 and the proof is complete. □

Note that the graph $P_{n,1}$ is the n-gonal prism. Therefore, $P_{n,1}$ can be drawn as a ring of n quadrangles (cycles of length 4). For example, $P_{7,1}$ is the heptagonal prism and it is drawn as a ring of seven quadrangles in Figure 3.2.1a.

Observe that if (G, σ) is a planar signed graph with k negative faces, then $l(G, \sigma) \geq \frac{k}{2}$. This holds true because a negative edge can make at most two faces negative. We use this fact to prove the following lemma in which we show that there exists a signed $P_{n,1}$ whose frustration index is $\lfloor \frac{n}{2} \rfloor + 1$.

Lemma 3.2.2. *There exists an edge set E_- such that $l[P_{n,1}, E_-] = \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. The graph $P_{n,1}$ has exactly $n + 2$ faces. Thus, to get the desired result, we give a signature of size $\lfloor \frac{n}{2} \rfloor + 1$, where all but at most one of the faces are negative.

Let $n = 2k + 1$. Consider $E_-^1 = \{u_{2k-1}u_{2k}\} \cup \{u_{2j}v_{2j} : j \in \{0, 1, \ldots, k-1\}\}$. The edges of E_-^1 make all but one of the faces of $(P_{2k+1,1}, E_-^1)$ negative, and that $|E_-^1| = \lfloor \frac{n}{2} \rfloor + 1$. Hence $l[P_{2k+1,1}, E_-^1] = \lfloor \frac{n}{2} \rfloor + 1$.

Let $n = 2k$. Consider $E_-^2 = \{u_{2k-3}u_{2k-2}, v_{2k-2}v_{2k-1}\} \cup \{u_{2j}v_{2j} : j \in [k-1]\}$. The edges of E_-^2 make all the faces of $(P_{2k,1}, E_-^2)$ negative, and that $|E_-^2| = \lfloor \frac{n}{2} \rfloor + 1$. Hence $l[P_{2k,1}, E_-^2] = \lfloor \frac{n}{2} \rfloor + 1$. This completes the proof. □

From Theorem 3.2.1 and Lemma 3.2.2, we obtain the exact value of $D(P_{n,1})$.

Theorem 3.2.2. *If $n \geq 3$, then $D(P_{n,1}) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.*

In the following lemma, we show that there exists an edge set E_- such that $l[P_{2m+1,2}, E_-] = m + 1$, where $m \geq 2$.

Lemma 3.2.3. *For $m \geq 2$, there is an edge set E_- such that $l[P_{2m+1,2}, E_-] = m+1$.*

Proof. For $m = 2$, the graph $P_{5,2}$ is the Petersen graph. In [60], the author proved that there is a signed Petersen graph of frustration index 3. Thus the result is true for $m = 2$.

Let m, with $m \geq 3$, be odd. Consider the signed graph $(P_{2m+1,2}, E_-^1)$, where

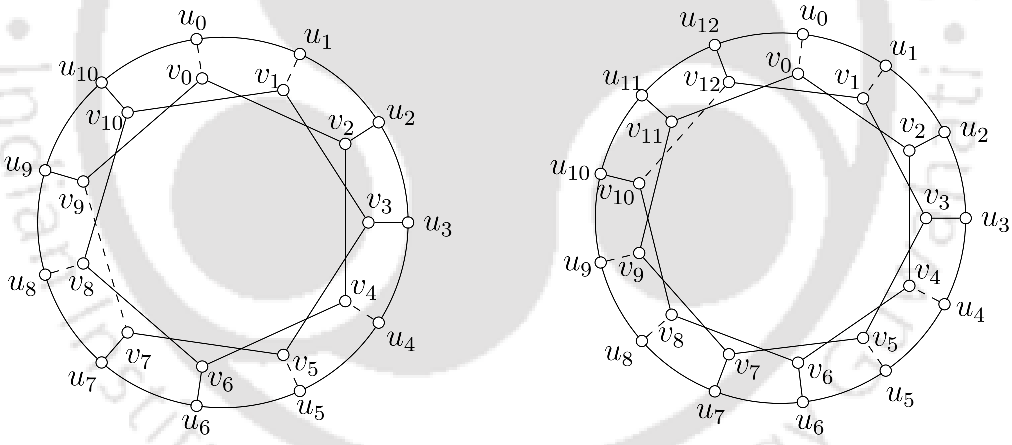
$$E_-^1 = \{u_{2m-2}v_{2m-2}, v_{2m-3}v_{2m-1}\} \cup \{u_{2j-2}v_{2j-2}, u_{(2j-2)+1}v_{(2j-2)+1} : j \in \{1, 3, \ldots, m-2\}\}$$

It is clear that $|E_-^1| = m + 1$. Further, $P_{2m+1,2}$ has exactly $(2m + 1)$ cycles of length 5. Note that with one negative edge, at most two 5-cycles can be made negative. Therefore, with at most m edges, at most $2m$ 5-cycles can be made negative. Consequently, any negative edge set equivalent to E_-^1 must have at least $m+1$ edges. Thus E_-^1 is minimum, and hence $l[P_{2m+1,2}, E_-^1] = m+1$. For instance, see Figure 3.2.3a.

Let m, with $m \geq 4$, be even. Consider the signed graph $(P_{2m+1,2}, E_-^2)$, where $E_-^2 = \{v_{2m-2}v_{2m}\} \cup \{u_{2j-2}v_{2j-2}, u_{(2j-2)+1}v_{(2j-2)+1} : j \in \{1, 3, \ldots, m-1\}\}$. It is clear that $|E_-^2| = m + 1$, and E_-^2 makes all the 5-cycles of $(P_{2m+1,2}, E_-^2)$ negative. By similar argument as in the preceding paragraph, it can be shown that E_-^2 is minimum. Hence $l[P_{2m+1,2}, E_-^2] = m + 1$. For instance, see Figure 3.2.3b. □

From Theorem 3.2.1 and Lemma 3.2.3, we obtain the exact value of $D(P_{2m+1,2})$.

Theorem 3.2.3. *If $m \geq 2$, then $D(P_{2m+1,2}) = m + 1$.*



(a) $(P_{11,2}, E_-)$ with $|E_-| = 6$ (b) $(P_{13,2}, E_-)$ with $|E_-| = 7$

Figure 3.2.3. Reduced signed graphs over $P_{11,2}$ and $P_{13,2}$

Lemma 3.2.4. *Let E_- be a negative edge set (matching) of $P_{7,3}$. If the outer cycle and each cycle of length 6 are negative in $(P_{7,3}, E_-)$, then $|E_-| \geq 4$.*

Proof. Clearly, the graph $P_{7,3}$ has exactly seven cycles of length 6. Note that each edge of the form u_iu_{i+1} appears in exactly three 6-cycles. Similarly, each edge of the form v_iv_{i+3} appears in exactly one 6-cycle. Thus, by one edge of C_o we can make at most three 6-cycles negative. Similarly, by one edge of C_I we can make at most one 6-cycle negative and by one spoke, at most two 6-cycles can be made negative. These facts imply that the number of negative edges in a signed $(P_{7,3}, E_-)$, in which each 6-cycle as well as the outer cycle are negative, must be at least three. Thus, to complete the proof, it is enough to show that there is no matching (negative edge set) of size three such that the outer cycle and the 6-cycles are negative in its corresponding signed graph over $P_{7,3}$.

Let, if possible, E_- be a matching of size three such that the outer cycle and all the 6-cycles are negative in $(P_{7,3}, E_-)$. Since C_o is negative, E_- must contain odd number of edges of C_o. Thus, E_- contains either one edge or three edges of C_o. We discuss these two cases separately.

Case 1. Let E_- have one edge of C_o. Without loss of generality, let this edge be u_0u_1. If $E_- \setminus \{u_0u_1\}$ contains exactly two inner edges, then the number of negative 6-cycles in $(P_{7,3}, E_-)$ is at most 5. Similarly, if $E_- \setminus \{u_0u_1\}$ contains one inner edge and one spoke, then also the number of negative 6-cycles in $(P_{7,3}, E_-)$ is at most 6. Thus, the two edges of $E_- \setminus \{u_0u_1\}$ must be spokes. Accordingly, $E_- \setminus \{u_0u_1\} \subset \{u_2v_2, u_3v_3, u_4v_4, u_5v_5, u_6v_6\}$. Consequently, one of the two spokes of E_- is u_iv_i for some $i \in \{2,3,5,6\}$. So, the 6-cycle containing u_0u_1 and u_iv_i is positive in $(P_{7,3}, E_-)$. Therefore, such a matching cannot make all the 6-cycles of $P_{7,3}$ negative.

Case 2. Let E_- have three edges of C_o. Since E_- is a matching of $P_{7,3}$, the edges of E_- are u_0u_1, u_2u_3 and u_4u_5, up to rotations. Clearly, the 6-cycle $u_0u_1u_2u_3v_3v_0u_0$ is positive in $(P_{7,3}, E_-)$. Hence such a matching cannot make all the 6-cycles of $P_{7,3}$ negative.

From cases 1 and 2, we conclude that $|E_-| \geq 4$. This completes the proof. □

Example 3.2.1. Consider the signed graph $(P_{7,3}, E_-)$ shown in Figure 3.2.1b. Note that each 6-cycle as well as the outer cycle in $(P_{7,3}, E_-)$ are negative. Since $|E_-| = 4$, we conclude by Lemma 3.2.4 that E_- is minimum. Thus $l[P_{7,3}, E_-] = 4$, and therefore by Theorem 3.2.1, $D(P_{7,3}) = 4$.

Theorem 3.2.3 and Example 3.2.1 establish the sharpness of the bound in Theorem 3.2.1 for $k \in \{2,3\}$ as well.

3.2.2 Bound on $D(P_{n,k})$ when $\gcd(n,k) \geq 2$

In this section, we find an upper bound for the maximum frustration of $P_{n,k}$, where $\gcd(n,k) = d \geq 2$. Recall that if $\gcd(n,k) = d$, then the graph $P_{n,k}$ has d pairwise disjoint inner cycles of length $\frac{n}{d}$. Let these d cycles be denoted by $C_{I_1}, C_{I_2}, \ldots, C_{I_d}$, where $C_{I_i} = v_{i-1}v_{i+k-1}v_{i+2k-1}\ldots v_{n-k+i-1}v_{i-1}$ for $1 \leq i \leq d$.

For each $i \in \{1, \ldots, d\}$, consider the subgraph G_{I_i} of $P_{n,k}$, where

$$V(G_{I_i}) = V(C_{I_i}) \cup \{u_{i+j-1} : j \in \{0, k, 2k, \ldots, n-k\}\} \text{ and}$$

$$E(G_{I_i}) = E(C_{I_i}) \cup \{u_{i+j-1}v_{i+j-1} : j \in \{0, k, 2k, \ldots, n-k\}\}.$$

Note that (G_{I_i}, E_-) is balanced or unbalanced according as C_{I_i} is positive or negative, respectively. Further, the graph G_{I_i} is isomorphic to G_r with $r = \frac{n}{d}$, where G_r is given in Definition 2.3.1. If switching by u-vertices are not allowed in G_{I_i}, then by Lemma 3.2.1, the least size of a negative edge set, switching equivalent to E_-, is at most $\left\lfloor \frac{n}{2d} \right\rfloor + 1$ or $\left\lfloor \frac{n}{2d} \right\rfloor$ according as (G_{I_i}, E_-) is unbalanced or balanced, respectively. Now we proceed to determine an upper bound on the maximum frustration of $P_{n,k}$ in terms of n and d.

Theorem 3.2.4. *If* $\gcd(n,k) = d \geq 2$*, then* $D(P_{n,k}) \leq d\left\lfloor \frac{n}{2d} \right\rfloor + d + 1$.

Proof. To complete the proof, we show that each signature of $P_{n,k}$ is switching equivalent to a signature having at most $d\lfloor \frac{n}{2d} \rfloor + d + 1$ edges. Given a signed graph $(P_{n,k}, E_-)$, first switch by the vertices of each inner and outer cycle in $(P_{n,k}, E_-)$ to have at most 1 negative edge in each of those cycles. This gives a total of $d+1$ negative edges. Note that the set of spokes connecting an inner cycle to the outer cycle consists of $\frac{n}{d}$ spokes, and for each such set at most $\lfloor \frac{n}{2d} \rfloor$ edges can be negative. Consequently the total number of negative spokes are at most $d\lfloor \frac{n}{2d} \rfloor$. Therefore, at most $d\lfloor \frac{n}{2d} \rfloor + d + 1$ edges are negative in the resulting signature, and hence the proof follows. ☐

Sharpness of the bound in Theorem 3.2.4 is still open to explore. However, we show that the bound of $d\left\lfloor \frac{n}{2d} \right\rfloor + d + 1$ on $D(P_{n,k})$ can be improved to $\left\lfloor \frac{n}{2} \right\rfloor + 2$ for the case $n = 3k$, where $k \geq 2$.

Consider $P_{3k,k}$, where $k \geq 2$. Clearly, the inner cycles of $P_{3k,k}$ are triangles and there are k such pairwise disjoint triangles induced by the v-vertices. Let G_{I_i} be the subgraph of $P_{n,k}$ defined in the beginning of this section. For example, the subgraph G_{I_1} of $P_{3k,k}$ is shown in Figure 3.2.4.

For $1 \leq i \leq k$, let (G_{I_i}, E_-) be a signed graph. If switching by the u-vertices are not allowed, then by Lemma 3.2.1 we get a negative edge set E'_-, switching equivalent to E_-, of size at most two or one according as (G_{I_i}, E_-) is unbalanced or balanced, respectively. Further, if (G_{I_i}, E'_-) is unbalanced, where E'_- be a matching of size two, then E'_- can be chosen to be $\{v_{i-1}v_{i+k-1}, u_{i+2k-1}v_{i+2k-1}\}$ up to switching. In this case, we call the edges $v_{i-1}v_{i+k-1}$ and $u_{i+2k-1}v_{i+2k-1}$ the *primary edge* and *secondary edge* of G_{I_i}, respectively. Note that the secondary edge of an unbalanced (G_{I_i}, E'_-), if any, belongs to the set S, where $S = \{u_{2k}v_{2k}, u_{2k+1}v_{2k+1}, \ldots, u_{3k-1}v_{3k-1}\}$.

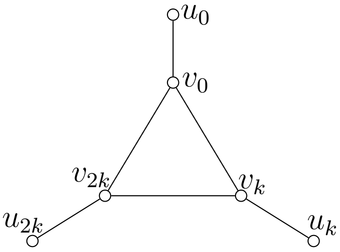


Figure 3.2.4. The graph G_{I_1}

Theorem 3.2.5. *If* $k \geq 2$, *then* $D(P_{3k,k}) \leq \left\lfloor \frac{3k}{2} \right\rfloor + 2$.

Proof. To prove the theorem it is enough to show that any negative edge set E_- of $P_{3k,k}$ is switching equivalent to E'_- such that $|E'_-| \leq \left\lfloor \frac{3k}{2} \right\rfloor + 2$. We consider $(P_{3k,k}, E_-)$ and discuss the following cases.

Case 1. For each $i \in \{1, \ldots, k\}$, let C_{I_i} be positive in $(P_{3k,k}, E_-)$. Without loss of generality, assume that all the edges of the inner triangles are positive. Therefore, all the edges of E_- lie in the graph G_o. By Lemma 3.2.1, there exists a negative edge set E'_-, equivalent to E_-, such that $|E'_-| \leq \left\lfloor \frac{3k}{2} \right\rfloor + 1$.

Case 2. Let C_o be positive and let r number of inner cycles be negative in $(P_{3k,k}, E_-)$, where $1 \leq r \leq k$. Without loss of generality, assume that all the edges of C_o are positive. Thus $E_- \subseteq \cup_{i=1}^{k} E(G_{I_i})$. Switching by v-vertices of each G_{I_i}, if needed, we get a signature E'_- of least size. Note that each G_{I_i} contributes at most one or two edges in E'_- according as G_{I_i} is balanced or unbalanced in $(P_{3k,k}, E_-)$, respectively. If the r unbalanced subgraphs of the form G_{I_i} have at most $\left\lfloor \frac{k}{2} \right\rfloor + 1$ secondary edges, then the total number of edges in E'_- is at most $(r + \left\lfloor \frac{k}{2} \right\rfloor + 1) + (k - r)$. Thus $|E'_-| \leq k + \left\lfloor \frac{k}{2} \right\rfloor + 1 = \left\lfloor \frac{3k}{2} \right\rfloor + 1$.

On the other hand, let the r unbalanced subgraphs of the form G_{I_i} have l secondary edges, where $l \geq \left\lfloor \frac{k}{2} \right\rfloor + 2$. Let the set of these l negative secondary edges be given by $A = \{u_{j_1}v_{j_1}, u_{j_2}v_{j_2}, \ldots, u_{j_l}v_{j_l}\}$, where $j_t \in \{2k, 2k+1, \ldots, 3k-1\}$ for $1 \leq t \leq l$. Note that the number of edges in $E'_- \setminus A$ is at most $r + (k - r) = k$. Let E''_- be the signature obtained by switching each vertex of $\{u_{2k}, u_{2k+1}, \ldots, u_{3k-1}\}$ in $(P_{3k,k}, E'_-)$. We see that $E''_- \subseteq \left(E'_- \setminus A\right) \cup (S \setminus A) \cup \{u_{2k-1}u_{2k}, u_{3k-1}u_0\}$. Therefore,

$$|E''_-| \leq k + (k - l) + 2 = 2k + 2 - l \leq 2k + 2 - \left(\left\lfloor \frac{k}{2} \right\rfloor + 2\right) = 2k - \left\lfloor \frac{k}{2} \right\rfloor \leq \left\lfloor \frac{3k}{2} \right\rfloor + 1.$$

Case 3. Let C_o as well as r number of inner cycles be negative in $(P_{3k,k}, E_-)$, where $1 \leq r \leq k$. Switching by u-vertices, if needed, we make the edge u_1u_2 negative and the other edges of C_o positive. Let the negative edge set obtained by this switching be denoted by E'_-. Note that all the edges of E'_-, except u_1u_2, lie in $\cup_{i=1}^{k} E(G_{I_i})$.

Now applying the same procedure as in Case 2, we get E''_- such that E''_- is switching equivalent to E'_- and $|E''_-| \leq \left\lfloor \frac{3k}{2} \right\rfloor + 2$.

In all the preceding cases, we observe that any negative edge set E_- of $P_{3k,k}$ is switching equivalent to E'_- such that $|E'_-| \leq \left\lfloor \frac{3k}{2} \right\rfloor + 2$. Since E_- is arbitrary, we have $D(P_{3k,k}) \leq \left\lfloor \frac{3k}{2} \right\rfloor + 2$. □

3.3 Frustrations in K_6

We proved in Chapter 2 that there are exactly 16 different signed K_6 up to switching isomorphism. To compute the frustration indices and the frustration numbers of all signed K_6, due to Lemma 3.1.1 and Lemma 3.1.2, it is enough to determine these invariants for the 16 switching non-isomorphic signed K_6.

Theorem 3.3.1. *The frustration indices of the* 16 *signed* K_6 *in Figure 2.2.2 are given in Table 3.3.1.*

Proof. Recall from Corollary 2.2.2 that each signed graph in Figure 2.2.2 of Chapter 2 is reduced in its switching isomorphism class. Therefore, from (3.1.1), the frustration indices of the 16 signed K_6 are the size of their corresponding signatures. Using this, we calculate the frustration indices of these 16 signed K_6 and tabulate in Table 3.3.1. This completes the proof. □

Σ	$l(\Sigma)$	Σ	$l(\Sigma)$	Σ	$l(\Sigma)$	Σ	$l(\Sigma)$
Σ_0	0	Σ_4	3	Σ_8	4	Σ_{12}	4
Σ_1	1	Σ_5	3	Σ_9	4	Σ_{15}	5
Σ_2	2	Σ_6	3	Σ_{10}	4	Σ_{16}	5
Σ_3	2	Σ_7	3	Σ_{11}	4	Σ_{18}	6

Table 3.3.1. Frustration indices of switching non-isomorphic signed K_6

Theorem 3.3.2. *The frustration numbers of the signed* K_6 *in Figure 2.2.2 are given in Table 3.3.2.*

Proof. Observe that to make a reduced (minimum) signed complete graph balanced by deleting some vertices, all of its negative edges must get deleted. In the second column of Table 3.3.2, we give a set of vertices (Ref. the labeling of Figure 2.2.1) whose deletion also deletes all the negative edges of the corresponding signed K_6. Further, it is easy to see that no set of vertices, whose size is less than the size of the given set in the second column of Table 3.3.2, can delete all the negative edges of the corresponding signature. Using this, we calculate the frustration numbers of these 16 signed K_6 and present in the third column of Table 3.3.2. This completes the proof. $\square$

Σ	Deleted set	$l_0(\Sigma)$
Σ_0	$\{\}$	0
Σ_1	$\{u_2\}$	1
Σ_2	$\{u_2\}$	1
Σ_3	$\{u_2, u_3\}$	2
Σ_4	$\{u_2, u_3\}$	2
Σ_5	$\{u_2, u_4\}$	2
Σ_6	$\{u_2, u_3, u_4\}$	3
Σ_7	$\{u_2, u_3\}$	2
Σ_8	$\{u_2, u_4\}$	2
Σ_9	$\{u_2, u_3, u_5\}$	3
Σ_{10}	$\{u_1, u_4\}$	2
Σ_{11}	$\{u_2, u_3, u_5\}$	3
Σ_{12}	$\{u_1, u_3\}$	2
Σ_{15}	$\{u_2, u_3, u_6\}$	3
Σ_{16}	$\{u_2, u_3, u_5\}$	3
Σ_{18}	$\{u_2, u_3, u_4, u_5\}$	4

Table 3.3.2. Frustration numbers of switching non-isomorphic signed K_6

3.4 Maximum Frustration in Signed Book Graphs

In Theorem 2.4.1 of Chapter 2, we proved that for $m \geq 3$ and $n \geq 1$, the signed book graph $(B(m,n), E_-)$ is equivalent to $(B(m,n), E'_-)$, where $E'_- \subseteq \{uu_1^1, \dots, uu_1^n\}$. Furthermore, we proved that the number of switching non-isomorphic signed $B(m,n)$ is $n+1$. Now we determine the maximum frustration in the signed book graphs.

Theorem 3.4.1. *If $m \geq 3$ and $n \geq 1$, then $D(B(m,n)) = \left\lceil \frac{n}{2} \right\rceil$.*

Proof. It is easy to check that any signed book $(B(m,n), E_-)$ is switching equivalent to some $(B(m,n), E'_-)$, where $|E'_-| \leq \left\lceil \frac{n}{2} \right\rceil$. Therefore $D(B(m,n)) \leq \left\lceil \frac{n}{2} \right\rceil$.

If n is even and $E_- = \{uu_1^1, uu_1^2, \dots, uu_1^{\frac{n}{2}}\}$, then $l(B(m,n), E_-) = \frac{n}{2}$. Further, if n is odd and $E_- = \{uu_1^1, uu_1^2, \dots, uu_1^{\frac{n+1}{2}}\}$, then $l(B(m,n), E_-) = \frac{n+1}{2}$. This implies that $D(B(m,n)) = \left\lceil \frac{n}{2} \right\rceil$. $\square$

Let G^* be the dual graph of a planar graph G. In [50], it is proved that if G^* is Hamiltonian and p^* is the number of faces of a planar embedding of G, then $D(G) = \left\lfloor \frac{p^*}{2} \right\rfloor$. Now we give an alternative proof of $D(B(m,n)) = \left\lceil \frac{n}{2} \right\rceil$ using this planarity argument, in the following paragraph.

It is known that if G is a connected planar graph of order p and size q, then $p^* = q-p+2$. Since $B(m,n)$ is a planar graph of order $mn-2n+2$ and size $mn-n+1$, the number of faces of a planar embedding of $B(m,n)$ equals $n+1$. If we embed $B(m,n)$ in the plane, similar to the embedding of $B(4,3)$ in Figure 2.4.1, then $B^*(m,n)$ contains a spanning cycle of order $n+1$ and so $B^*(m,n)$ is Hamiltonian. Hence we have $D(B(m,n)) = \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$, as desired.

4

Double Domination in Signed Graphs

In this chapter, we prove that if D is a double dominating set (DDS) of a cubic unsigned graph G such that $|D| = \frac{|V(G)|}{2}$, then the subgraph $G[D, D^c]$ admits a cycle decomposition. We further give an example to show that if D, with $|D| = \frac{|V(G)|}{2}$, is not a DDS of a cubic graph G, then it is not necessarily true that $G[D, D^c]$ admits a cycle decomposition. A lower bound on the DDN of signed cubic graphs is also obtained. We also obtain some bounds on the DDN of signed generalized Petersen graphs and signed I-graphs. We prove that the DDN of a signed complete graph over K_n can be $n-1$ for $n \geq 5$. However, this number is known to be exactly 2 for unsigned complete graph K_n for $n \geq 2$. Also, we compute the DDN of all switching non-isomorphic signed Petersen graphs and of all switching non-isomorphic signed complete graphs over K_6.

4.1 Preliminaries

Recall that a subset D of $V(G)$ is said to be a *double dominating set* (DDS) of G if $|N[v] \cap D| \geq 2$ for each $v \in V(G)$. Further, a subset D of $V(G)$ is said to be a *double dominating set* of a signed graph Σ with underlying graph G, if it satisfies the following conditions:

(i) $|N[v] \cap D| \geq 2$ for each $v \in V(G)$, and

(ii) $\Sigma[D, D^c]$ is balanced.

The size of a minimum DDS of Σ is called the *double dominating number* (DDN) of Σ, and it is denoted by $\gamma_{\times 2}(\Sigma)$.

Note that condition (ii) of the preceding definition is always satisfied for every subset D of vertices of an all-positive signed graph $+G$. Recall that an all-positive signed graph can be considered as an unsigned graph. Thus, the notion of double domination in signed graphs is a generalization of the notion of double domination of unsigned graphs.

Any property or parameter of a signed graph that depends only on the signs of its cycles remains invariant under switching. It is shown in [9] that the double domination number is an example of such a parameter. More precisely, we have the following theorem.

Theorem 4.1.1. *[9] Double domination number of a signed graph is invariant under switching.*

Remark 4.1.1. Observe that for two switching non-equivalent signed graphs having the same underlying graph, a DDS of one signed graph need not be a DDS of the other signed graph. For instance, let K_4 be the complete graph with the vertex set $\{u_1, u_2, u_3, u_4\}$. Further, suppose that Σ_1 and Σ_2 are the signed graphs over K_4 with signatures $\{u_1u_2\}$ and $\{u_1u_3\}$, respectively. By Lemma 1.2.1, it is clear that Σ_1 and Σ_2 are non-equivalent. Take $D = \{u_1, u_2\}$. Note that D is a DDS of Σ_1, but D is not a DDS of Σ_2.

Throughout the chapter, the set of all filled vertices enclosed by a circle forms a DDS of the signed graph in each figure.

4.2 Double Domination in Signed Cubic Graphs

A *decomposition* of a graph G is a family $\mathcal{F}$ of edge-disjoint subgraphs of G such that $\cup_{H\in\mathcal{F}} E(H) = E(G)$. If all the members of the family $\mathcal{F}$ are cycles, we call $\mathcal{F}$ a *cycle decomposition* of G. We show that if D is a DDS of G of size $\frac{|V(G)|}{2}$, then $G[D, D^{\mathsf{c}}]$ admits a vertex-disjoint cycle decomposition.

Theorem 4.2.1. *For $n \geq 2$, let G be a cubic graph on $2n$ vertices. If D is a DDS of G such that $|D| = n$, then $G[D, D^{\mathsf{c}}]$ admits a cycle decomposition.*

Proof. To complete the proof it is sufficient to show that each vertex in D is adjacent to exactly two vertices of D^{c}, and vice versa.

Since D is a DDS of G, a vertex in D cannot be adjacent to three vertices of D^{c}. Let, if possible, there be l vertices in D each of which is adjacent to at most one vertex of D^{c}, where $1 \leq l \leq n$. Thus, each of the remaining $n - l$ vertices in D is adjacent to exactly two vertices of D^{c}. Therefore,

$$|[D, D^{\mathsf{c}}]| \leq 2(n - l) + l = 2n - l. \tag{4.2.1}$$

Since D is a DDS of G, each vertex in D^{c} must be adjacent to at least two vertices of D. Therefore,

$$|[D^{\mathsf{c}}, D]| \geq 2n. \tag{4.2.2}$$

As $|[D, D^{\mathsf{c}}]| = |[D^{\mathsf{c}}, D]|$, inequalities (4.2.1) and (4.2.2) cannot hold simultaneously. Hence, every vertex in D is adjacent to exactly two vertices of D^{c}. Similarly, every vertex in D^{c} is adjacent to exactly two vertices of D. This shows that the degree of each vertex of the subgraph $G[D, D^{c}]$ is 2. Hence $G[D, D^{c}]$ is a 2-regular subgraph of G, and the proof is complete. □

Remark 4.2.1. In Theorem 4.2.1, the condition that D is a DDS of G is essential. That is, if the set D of size $\frac{|V(G)|}{2}$ is not a DDS of a cubic graph G, then it is not necessary that $G[D, D^{c}]$ is the union of vertex disjoint cycles. For instance, let $G = P_{4,1}$ and take $D = \{u_0, u_1, u_2, u_3\}$. Clearly, $|D| = \frac{|V(G)|}{2}$ and D is not a DDS of G. It is easy to see that $G[D, D^{c}]$ is not the union of vertex disjoint cycles.

For a signed graph Σ with underlying graph G, it is clear that $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(\Sigma)$. Using this inequality, Ashraf and Germina [9] proved for $l \geq 3$ and $r \geq 3$ that $4 \leq \gamma_{\times 2}(K_{l,r}, \sigma) \leq l + r - 2$, where $(K_{l,r}, \sigma)$ denotes a signed complete bipartite graph. They also obtained the following lower and upper bound on the DDN of a signed graph.

Theorem 4.2.2. *[9] If Σ is a signed graph of order n without isolated vertices, then $2 \leq \gamma_{\times 2}(\Sigma) \leq n$. Moreover, these bounds are sharp.*

For the class of signed cubic graphs, we improve the lower bound in Theorem 4.2.2 in the next result.

Theorem 4.2.3. *If Σ is a signed cubic graph on $2n$ vertices, then*

$$n \leq \gamma_{\times 2}(\Sigma) \leq 2n.$$

Proof. The upper bound follows from Theorem 4.2.2.

Let G be the underlying graph of Σ. Since $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(\Sigma)$, to establish the lower bound, it is enough to show that $n \leq \gamma_{\times 2}(G)$. For this, we claim that G cannot have a DDS of size at most $n-1$. On the contrary, let D be a DDS of G such that $|D| \leq n-1$. Since D is a DDS, each vertex in D is adjacent to at most two vertices of D^{c}. Thus

$$|[D, D^{c}]| \leq 2n - 2. \tag{4.2.3}$$

Similarly, since D is a DDS, each vertex in D^{c} is adjacent to at least two vertices of D. Thus

$$|[D^{c}, D]| \geq 2(2n - |D|) \geq 2n + 2. \tag{4.2.4}$$

Clearly, the inequalities (4.2.3) and (4.2.4) cannot hold simultaneously. This implies that any DDS of G must be of size at least n. Therefore $\gamma_{\times 2}(G) \geq n$, and this completes the proof. □

There is an example for which the lower bound in Theorem 4.2.3 is achieved. To see this, let G be the disjoint union of n copies of K_4. Take $\Sigma = (G, \sigma)$, where $\sigma = \emptyset$. Note that $\gamma_{\times 2}(K_4) = 2$. Thus Σ is a signed cubic graph on $4n$ vertices and $\gamma_{\times 2}(\Sigma) = 2n$.

4.3 Bounds on $\gamma_{\times 2}(P_{n,k}, \sigma)$

Observe that the class of generalized Petersen graphs can be partitioned into the classes $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, where $\mathcal{S}_1 = \{P_{n,k} : k = 1\}$, $\mathcal{S}_2 = \{P_{n,k} : k \geq 2 \text{ and } \gcd(n,k) = 1\}$, and $\mathcal{S}_3 = \{P_{n,k} : \gcd(n,k) \geq 2\}$. Since $|V(P_{n,k})| = 2n$, it follows from Theorem 4.2.3 that $n \leq \gamma_{\times 2}(P_{n,k}, \sigma) \leq 2n$. An improved upper bound on the DDN of any signed $P_{n,k}$, where $P_{n,k} \in \mathcal{S}_1$, is obtained in Theorem 4.3.1. An improved upper bound on the DDN of any signed $P_{n,k}$, where $P_{n,k} \in \mathcal{S}_2$, is obtained in Theorem 4.3.2. An improved upper bound on the DDN of any signed $P_{n,k}$, where $P_{n,k} \in \mathcal{S}_3$, is obtained in Theorem 4.3.3.

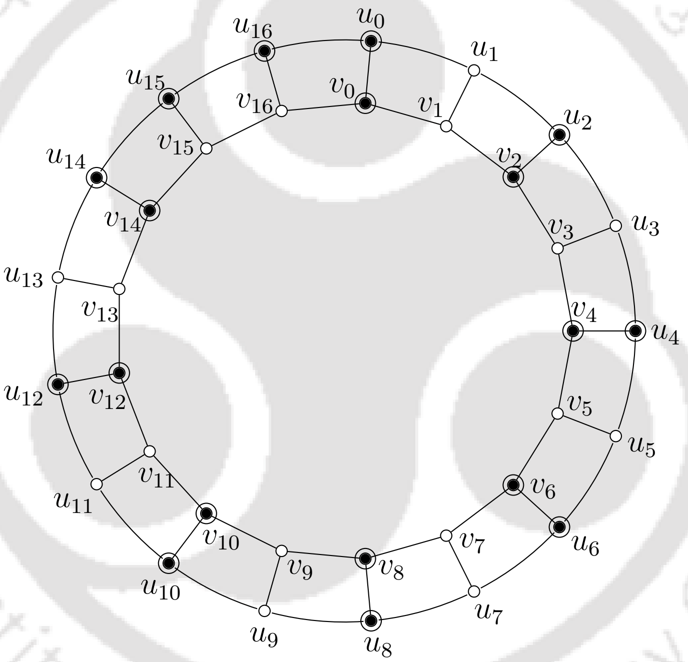


Figure 4.3.1. The graph $P_{17,1}$ with a DDS (Ref. Lemma 4.3.1)

The next two lemmas together determine bounds on the DDN of signed $P_{n,k}$, where $P_{n,k} \in \mathcal{S}_1$.

Lemma 4.3.1. *If Σ is a signed graph with underlying graph $P_{2n+1,1}$, then*

$$2n + 1 \leq \gamma_{\times 2}(\Sigma) \leq 2n + 2.$$

Moreover, the upper bound is sharp.

Proof. The lower bound follows from Theorem 4.2.3.

To get the upper bound, we construct a DDS of Σ having $2n+2$ vertices. Consider $D = \{u_{2i}, v_{2i} : i = 0, \ldots, n-1\} \cup \{u_{2n-1}, u_{2n}\}$. For instance, see Figure 4.3.1. Clearly, $|D| = 2n + 2$ and that D is a DDS of the unsigned $P_{2n+1,1}$.

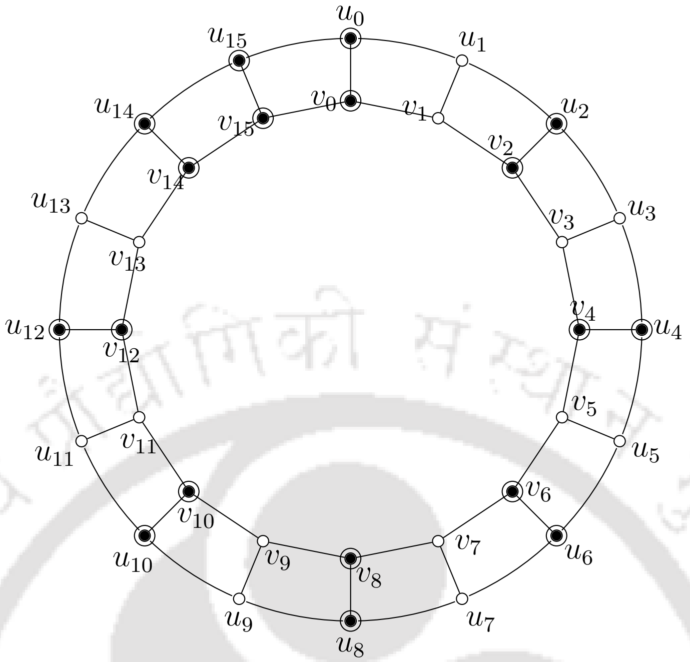


Figure 4.3.2. The graph $P_{16,1}$ with a DDS (Ref. Lemma 4.3.2)

It remains to show that $\Sigma[D, D^c]$ is balanced. Note that the subgraph $\Sigma[D, D^c]$ is the union of two vertex disjoint paths P_1 and P_2, where $P_1 = u_0u_1 \dots u_{2n-3}u_{2n-2}$ and $P_2 = u_{2n}v_{2n}v_0v_1 \dots v_{2n-1}u_{2n-1}$. Therefore $\Sigma[D, D^c]$ is acyclic, and hence $\Sigma[D, D^c]$ is balanced. Thus D is a DDS of Σ.

To prove the sharpness of the upper bound, consider the graph $P_{3,1}$. The upper bound in the theorem gives $\gamma_{\times 2}(P_{3,1}) \leq 4$. Let $D \subseteq V(P_{3,1})$ and $|D| = 3$. Clearly, D contains one, two or three outer vertices. In all these cases, it is easy to verify that D cannot be a DDS of $P_{3,1}$. Therefore $\gamma_{\times 2}(P_{3,1}) \geq 4$, and hence $\gamma_{\times 2}(P_{3,1}) = 4$. This completes the proof. □

Lemma 4.3.2. *If Σ is a signed graph with underlying graph $P_{2n,1}$, then*

$$2n \leq \gamma_{\times 2}(\Sigma) \leq 2n + 2.$$

Moreover, the lower bound is sharp.

Proof. The lower bound follows from Theorem 4.2.3.

To get the upper bound, we produce a DDS of Σ having $2n+2$ vertices. Consider $D = \{u_{2i}, v_{2i} : i = 0, \dots, m-1\} \cup \{u_{2n-1}, v_{2n-1}\}$. For example, see Figure 4.3.2. It is clear that each vertex of $P_{2n,1}$ is dominated at least twice by D, and that $|D| = 2n + 2$.

We now show that $\Sigma[D, D^c]$ is balanced. Notice that $[D, D^c] = E(P_1) \cup E(P_2)$, where $P_1 = u_0u_1 \dots u_{2n-2}$ and $P_2 = v_0v_1 \dots v_{2n-2}$. Thus $\Sigma[D, D^c]$ is acyclic, and hence $\Sigma[D, D^c]$ is balanced. Thus D is a DDS of Σ, and hence $\gamma_{\times 2}(\Sigma) \leq 2n + 2$.

To prove the second part, let $\Sigma = (P_{2n,1}, \sigma)$, where σ is a signature such that both C_o and C_I are positive in Σ. Consider $D = \{u_{2i}, v_{2i} : i = 0, \dots, n-1\}$. Clearly, D is a DDS of the unsigned $P_{2n,1}$, and that $|D| = 2n$. It is easy to see that $\Sigma[D, D^c] = C_0 \cup C_I$. Thus $\Sigma[D, D^c]$ is balanced, as C_o and C_I are positive in Σ. Hence $\gamma_{\times 2}(\Sigma) = 2n$. $\square$

Lemma 4.3.1 and Lemma 4.3.2 together give the following theorem.

Theorem 4.3.1. *If Σ is a signed graph with underlying graph $P_{n,1}$, then*

$$n \leq \gamma_{\times 2}(\Sigma) \leq 2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right).$$

Moreover, the bounds are sharp.

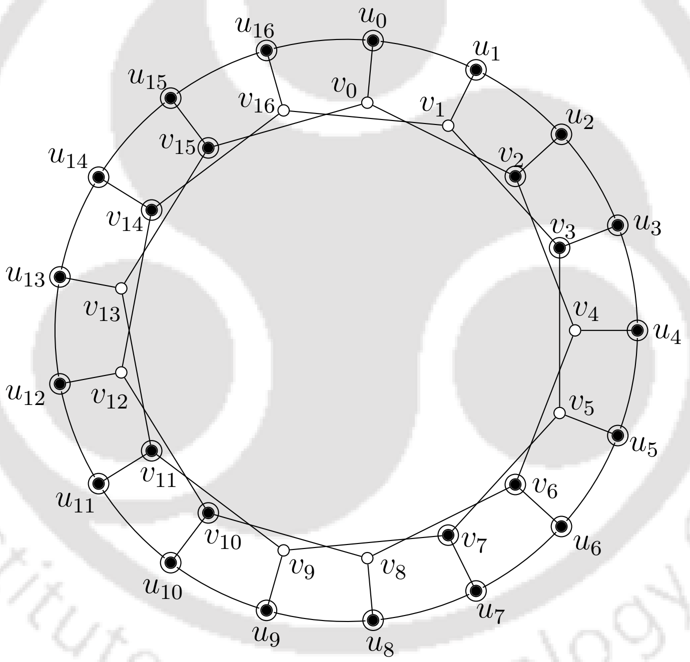


Figure 4.3.3. The graph $P_{17,2}$ with a DDS (Ref. Lemma 4.3.3)

The next two lemmas together determine bounds on the DDN of signed $P_{n,k}$, where $P_{n,k} \in \mathcal{S}_2$. Recall that the subscripts used in writing the vertices of a generalized Petersen graph $P_{n,k}$ are always read modulo n.

Lemma 4.3.3. *Let $P_{n,k} \in \mathcal{S}_2$ be such that $\left\lceil \frac{n}{k} \right\rceil = 2q + 1$ for some positive integer q. If Σ is a signed graph with underlying graph $P_{n,k}$, then $n \leq \gamma_{\times 2}(\Sigma) \leq n + qk$.*

Proof. The lower bound follows from Theorem 4.2.3.

Recall that V_u denotes the set of u-vertices and V_v denotes the set of v-vertices of $P_{n,k}$. Let $\{V_1, \dots, V_{2q+1}\}$ be the partition of V_v, where $V_i = \{v_{(i-1)k+j} : j \in [k]\}$

for $1 \le i \le 2q$ and $V_{2q+1} = V_v - \cup_{i=1}^{2q} V_i$. For $1 \le i \le 2q$, it is clear that $|V_i| = k$, and $|V_{2q+1}| = n - 2qk < k$.

To get the upper bound, take $D = V_u \cup \left(\cup_{i=1}^{q} V_{2i}\right)$. For example, see Figure 4.3.3. It is clear that $|D| = n + kq$. Clearly, each u-vertex is dominated at least twice by D. Further, each v-vertex is either in D and has a neighbour in D, or in D^{c} and is adjacent to at least two vertices of D. This shows that each v-vertex is also dominated at least twice by D. Hence D is a DDS of the unsigned $P_{n,k}$.

To show that $\Sigma[D, D^{\mathsf{c}}]$ is balanced, it is enough to show that $\Sigma[D, D^{\mathsf{c}}]$ is acyclic. Since C_o lies in $\Sigma[D]$, each u-vertex is adjacent to at most one vertex of D^{c}. Therefore, no u-vertex appears in a cycle of $\Sigma[D, D^{\mathsf{c}}]$. Also, $P_{n,k}$ has only one inner cycle induced by the v-vertices, as $\gcd(n, k) = 1$. Therefore, if $\Sigma[D, D^{\mathsf{c}}]$ contains a cycle, then that cycle must be the inner cycle itself. Note that the vertices v_{k-1} and v_{n-1} are adjacent, and both belong to the set D^{c}. That is, v_{k-1} and v_{n-1} do not have two v-vertices as neighbours in the set D. Thus $\Sigma[D, D^{\mathsf{c}}]$ cannot contain a cycle. This implies that $\Sigma[D, D^{\mathsf{c}}]$ is balanced. Hence $\gamma_{\times 2}(\Sigma) \le n + qk$. □

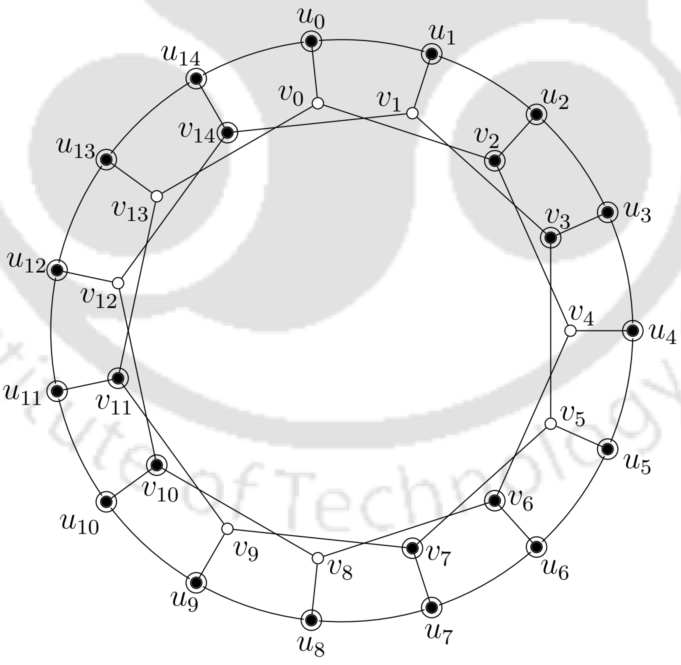


Figure 4.3.4. The graph $P_{15,2}$ with a DDS (Ref. Lemma 4.3.4)

Lemma 4.3.4. *Let $P_{n,k} \in \mathcal{S}_2$ be such that $\left\lceil \frac{n}{k} \right\rceil = 2q$ for some positive integer q. If Σ is a signed graph with underlying graph $P_{n,k}$, then $n \le \gamma_{\times 2}(\Sigma) \le 2n - qk$.*

Proof. The lower bound follows from Theorem 4.2.3.

Let $\{V_1, \ldots, V_{2q}\}$ be the partition of V_v, where $V_i = \{v_{(i-1)k+j} : j \in [k]\}$ for $1 \le i \le 2q-1$ and $V_{2q} = V_v - \cup_{i=1}^{2q-1} V_i$. Clearly, $|V_i| = k$ for $1 \le i \le 2q-1$ and $|V_{2q}| = n - k(2q-1) \le k-1$.

Now consider $D = V_u \cup \{\cup_{i=1}^{q} V_{2i}\}$. For example, see Figure 4.3.4. Note that $|D| = n + k(q-1) + n - k(2q-1) = 2n - kq$. We prove that D is a DDS of Σ, and this will give us the required upper bound.

Clearly, every u-vertex is dominated at least twice by D. Note that each vertex of V_{2i-1} is adjacent to a vertex of V_{2i} for each $i \in \{1, \ldots, q-1\}$, and that $V_{2i} \subseteq D$. Further, each vertex of V_{2q-1} is adjacent to a vertex of V_{2q-2}. Also for each v-vertex, the corresponding u-vertex belongs to D. This shows that each vertex of V_v is either in D and has one neighbour in D, or in D^{c} and adjacent to at least two vertices of D. Hence D is a DDS of the unsigned $P_{n,k}$.

To prove that $\Sigma[D, D^{\mathsf{c}}]$ is balanced, it is enough to show that $\Sigma[D, D^{\mathsf{c}}]$ is acyclic. Clearly, every u-vertex is adjacent to at most one vertex of D^{c}. Therefore, no u-vertex appears in a cycle of $\Sigma[D, D^{\mathsf{c}}]$. Thus if $\Sigma[D, D^{\mathsf{c}}]$ contains a cycle, then that cycle must be the inner cycle itself. If $|V_{2q}| = r$, where $1 \le r \le k-1$, then we see that $2qk - 1 \equiv k - r - 1 \pmod{n}$ and that $k - r - 1 \in [k]$. Therefore $v_{2qk-k-1}$ is adjacent to v_{k-r-1}. As $v_{2qk-k-1}, v_{k-r-1} \in D^{\mathsf{c}}$, we find that $v_{2qk-k-1}v_{k-r-1} \notin [D, D^{\mathsf{c}}]$. Thus $\Sigma[D, D^{\mathsf{c}}]$ is acyclic, and so $\Sigma[D, D^{\mathsf{c}}]$ is balanced. Hence D is a DDS of Σ. □

Theorem 4.3.2. *If $P_{n,k} \in \mathcal{S}_2$ and Σ is a signed graph with underlying graph $P_{n,k}$, then $n \le \gamma_{\times 2}(\Sigma) \le \frac{3n}{2}$.*

Proof. For any positive integers n and k, we know that $\left\lfloor \frac{n}{k} \right\rfloor \le \frac{n}{k} \le \left\lceil \frac{n}{k} \right\rceil$. Thus if $\gcd(n,k) = 1$ and $\left\lceil \frac{n}{k} \right\rceil = 2q+1$, then $2q = \left\lfloor \frac{n}{k} \right\rfloor < \frac{n}{k}$, and so $qk \le \frac{n}{2}$. Thus $n + qk \le \frac{3n}{2}$.

Again for $\left\lceil \frac{n}{k} \right\rceil = 2q$, we get $\frac{n}{2} \le qk$. Therefore $2n - qk \le \frac{3n}{2}$. Hence the theorem follows from Lemma 4.3.3 and Lemma 4.3.4. □

Finally, we give a lower bound and an upper bound on the DDN of $(P_{n,k}, \sigma)$, where $P_{n,k} \in \mathcal{S}_3$.

Theorem 4.3.3. *If $P_{n,k} \in \mathcal{S}_3$ and Σ is a signed graph with underlying graph $P_{n,k}$, then $n \le \gamma_{\times 2}(\Sigma) \le n + d\left\lceil \frac{n}{3d} \right\rceil$.*

Proof. The lower bound follows from Theorem 4.2.3.

Since $\gcd(n,k) = d \ge 2$, the graph $P_{n,k}$ has exactly d disjoint inner cycles of length $\frac{n}{d}$. Let the inner cycles be given by $C_r := v_{r-1}v_{(r-1)+k} \cdots v_{(r-1)+(\frac{n}{d}-1)k}v_{r-1}$ for $r \in \{1, \ldots, d\}$. Consider $V_r = \{v_{(r-1)+3(j-1)k} : j \in \{1, \ldots, \left\lceil \frac{n}{3d} \right\rceil\}\}$ for $1 \le r \le d$. It is clear that $|V_r| = \left\lceil \frac{n}{3d} \right\rceil$ for $1 \le r \le d$. Also, every vertex of the set $V(C_r) \setminus V_r$ is adjacent to at least one vertex of V_r for $1 \le r \le d$.

To get the upper bound, consider $D = V_u \cup \left(\cup_{r=1}^{d} V_r\right)$. For example, see Figure 4.3.5. It is clear that $|D| = n + d\left\lceil \frac{n}{3d} \right\rceil$, and that D dominates every vertex of

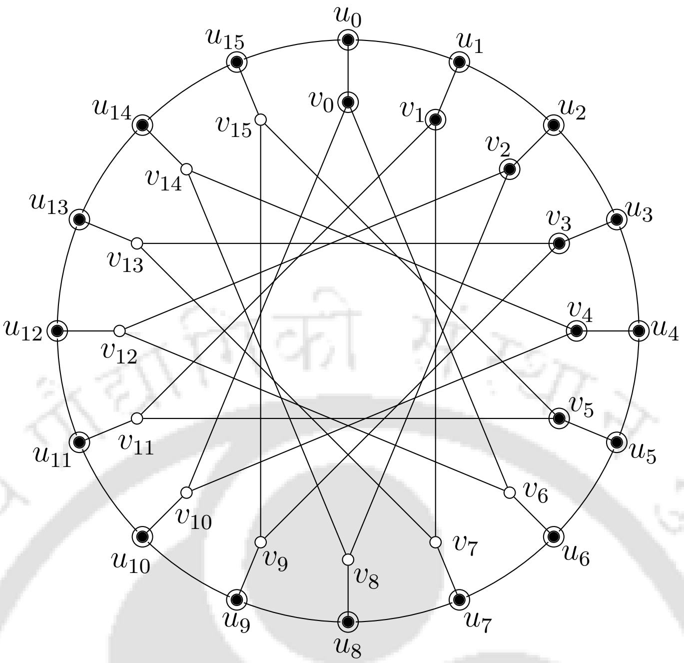


Figure 4.3.5. The graph $P_{16,6}$ with a DDS (Ref. Theorem 4.3.3)

V_u at least twice. Further, every vertex of V_v is either in D having one neighbour in D, or in D^c having two neighbors in D. Therefore D is a DDS of the unsigned $P_{n,k}$.

We now prove that $\Sigma[D, D^c]$ is balanced. Note that the outer cycle C_o lies in $\Sigma[D]$. Therefore, if $\Sigma[D, D^c]$ contains a cycle, then that cycle must be of the form C_r, where $1 \leq r \leq d$. However, the consecutive vertices $v_{(r-1)+k}$ and $v_{(r-1)+2k}$ of C_r are contained in D^c for $1 \leq r \leq d$. Thus, the vertices $v_{(r-1)+k}$ and $v_{(r-1)+2k}$ are not adjacent in $\Sigma[D, D^c]$ for $1 \leq r \leq d$. Accordingly, the cycle C_r does not appear in $\Sigma[D, D^c]$ for $1 \leq r \leq d$. Therefore $\Sigma[D, D^c]$ is acyclic, and hence $\Sigma[D, D^c]$ is balanced. Thus we have $\gamma_{\times 2}(\Sigma) \leq n + d\left\lceil \frac{n}{3d} \right\rceil$. □

4.3.1 The DDN of Signed Petersen Graphs

Recall that $P_{5,2}$ is the well known Petersen graph. In what follows, we denote the graph $P_{5,2}$ by P. The graph P is depicted in Figure 4.3.6. In Section 4.3, we obtained bounds on the DDN of signed generalized Petersen graphs. In this section, we calculate the DDN of all signed Petersen graphs. By Theorem 4.3.2, we have $5 \leq \gamma_{\times 2}(P, \sigma) \leq 7$. If D is a DDS of P of size five, then due to Theorem 4.2.1, $P[D, D^c]$ is a 2-regular subgraph of P. Since $P[D, D^c]$ is bipartite and the girth of P is 5, the component(s) of $P[D, D^c]$ must be even cycle(s) of length at least 6. Thus $P[D, D^c]$ is either a 10-cycle or vertex disjoint union of a 6-cycle and a

4-cycle. Since P has neither a 10-cycle nor a 4-cycle, D cannot be a DDS of P. Hence $\gamma_{\times 2}(P) \geq 6$. Thus the preceding inequality reduces to

$$6 \leq \gamma_{\times 2}(P,\sigma) \leq 7. \tag{4.3.1}$$

Our aim is to show that $\gamma_{\times 2}(P,\sigma) = 6$. There are 2^{15} ways of assigning $+1$ or -1 to the edges of P. Zaslavsky [60] proved that, up to switching isomorphism, only six of them are essentially distinct. The six switching non-isomorphic signed Petersen graphs are shown in Figure 4.3.7.

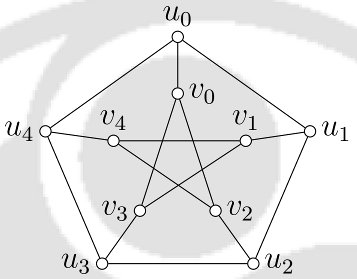
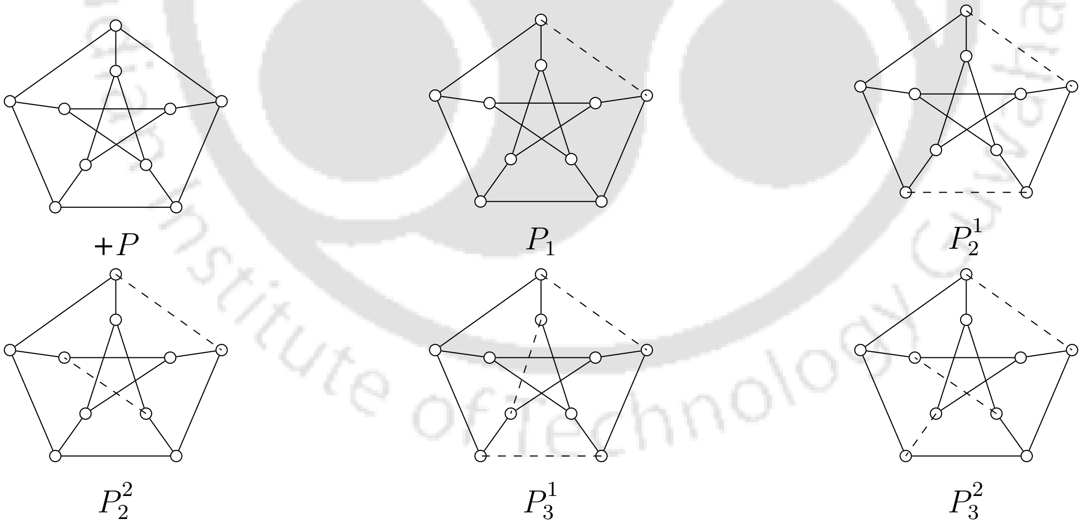


Figure 4.3.6. The Petersen graph


$+P$ P_1 P_2^1

P_2^2 P_3^1 P_3^2

Figure 4.3.7. The six switching non-isomorphic signed Petersen graphs

In Figure 4.3.7, $+P$ represents the all-positive Petersen graph, and P_1 represents a signed Petersen graph that has only one negative edge. Further, for each $r \in \{2,3\}$ and $s \in \{1,2\}$, the graph P_r^s represents a signed Petersen graph having r negative edges, in which the distance between each pair of the negative edges is s.

Theorem 4.3.4. *The DDN of each of the six switching non-isomorphic signed Petersen graphs is* 6.

Proof. By (4.3.1), the DDN of each of $+P$, P_1, P_2^1, P_2^2, P_3^1 and P_3^2 is at least 6. It is easy to verify that $\{u_0, u_1, u_3, v_2, v_3, v_4\}$ is a DDS for each of $+P$, P_1, P_2^2 and P_3^2. Further, $\{u_0, u_1, u_2, u_3, v_1, v_2\}$ is a DDS of both P_2^1 and P_3^1. Hence the result follows. ☐

Every signed Petersen graph is either switching equivalent or switching isomorphic to one of the six signed Petersen graphs in Figure 4.3.7. Also double domination number is switching invariant. Therefore, using Theorem 4.3.4, we conclude that every signed Petersen graph has double domination number 6.

4.4 Double Domination in Signed I-Graphs

Recall that the I-graph $I(n, j, k)$ is a graph defined by

$$V(I(n,j,k)) = \{u_i, v_i : i \in [n]\} \quad \text{and}$$
$$E(I(n,j,k)) = \{u_iu_{i+j}, u_iv_i, v_iv_{i+k} : i \in [n]\},$$

where the subscripts are read modulo n.

In [17], Boben, Pisanski and Zitnik studied various properties of I-graphs. They also determined the automorphism groups of I-graphs. Note that $I(n, 1, k) = P_{n,k}$ and $I(n, j, k) = I(n, k, j)$. Therefore, we assume that $2 \le j \le k$. We now obtain a lower bound and an upper bound on $\gamma_{\times 2}(I(n, j, k), \sigma)$ for $\gcd(n, k) = 1$.

Theorem 4.4.1. *Let* $\Sigma = (I(n, j, k), \sigma)$*. If* $\gcd(n, k) = 1$ *and* $j \ge 2$*, then*

$$n \le \gamma_{\times 2}(\Sigma) \le \frac{3n}{2}.$$

Proof. The lower bound follows from Theorem 4.2.3.

If $\left\lceil \frac{n}{k} \right\rceil = 2q + 1$, then the set D, considered in the proof of Lemma 4.3.3, is a DDS of Σ. Therefore $\gamma_{\times 2}(\Sigma) \le n + qk$. If $\left\lceil \frac{n}{k} \right\rceil = 2q$, then the set D, considered in the proof of Lemma 4.3.4, is a DDS of Σ. Therefore $\gamma_{\times 2}(\Sigma) \le 2n - qk$. For both the cases $\left\lceil \frac{n}{k} \right\rceil = 2q$ and $\left\lceil \frac{n}{k} \right\rceil = 2q + 1$, as in the proof of Theorem 4.3.2, we obtain the required upper bound. This completes the proof. ☐

Theorem 4.4.2. *If* $\Sigma = (I(n, j, k), \sigma)$ *and* $\gcd(n, k) = d \ge 2$*, then*

$$n \le \gamma_{\times 2}(\Sigma) \le n + d\left\lceil \frac{n}{3d} \right\rceil.$$

Proof. The lower bound follows from Theorem 4.2.3.

Let $V_v = \{v_0, \ldots, v_{n-1}\}$ and $V_u = \{u_0, \ldots, u_{n-1}\}$. Note that the structure of cycles induced by the vertices of V_v in $I(n, j, k)$ is same as the structure of cycles induced by the vertices of V_v in $P_{n,k}$. Observe that the set D considered in the proof of Theorem 4.3.3 contains the whole set V_u. Therefore, that set D is a DDS of Σ as well. Hence

$$n \leq \gamma_{\times 2}(\Sigma) \leq n + d\left\lceil \frac{n}{3d} \right\rceil.$$

This completes the proof. □

4.5 Double Domination Number of Signed K_n

In this section, we prove that if $n \geq 5$, then the DDN of a signed complete graph on n vertices can be $n-1$. However, this number is known to be exactly 2 for unsigned complete graphs K_n for $n \geq 2$. Also, we compute the DDN of all switching non-isomorphic signed complete graphs over K_6. We define our desired signed complete graph, for which the DDN is $n-1$, as follows.

Definition 4.5.1. Let K_n be the complete graph on the vertex set $\{u_1, \ldots, u_n\}$. For $k \geq 3$, the signed graph (K_n, σ^*) is defined by

$$\sigma^* = \begin{cases} \{u_1u_2, u_2u_3, \ldots, u_{n-2}u_{n-1}, u_{n-1}u_1\} & \text{if } n = 2k \\ \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_nu_1\} & \text{if } n = 2k-1. \end{cases}$$

For example, the signed graphs (K_6, σ^*) and (K_7, σ^*) are shown in Figure 4.5.1.

(K_6, σ^*) (K_7, σ^*)

Figure 4.5.1. The signed complete graphs (K_6, σ^*) and (K_7, σ^*)

Let $K_{l,r}$ denote the complete bipartite graph with $V(K_{l,r}) = A \cup B$, where $A = \{x_1, \ldots, x_l\}$ and $B = \{y_1, \ldots, y_r\}$. The vertices of A and B are called the *left* and *right vertices* of $K_{l,r}$, respectively. Throughout this section, we assume that $2 \leq l \leq r$ and $r \geq 3$. A signed complete bipartite graph is a signed graph whose underlying graph is a complete bipartite graph. In the next four lemmas, we introduce some unbalanced signed complete bipartite graphs. These unbalanced signed complete bipartite graphs are used to show that the DDN of (K_n, σ^*) is $n-1$.

Lemma 4.5.1. *Let* $\sigma_1 = \{x_1y_1, x_1y_2\}$, $\sigma_2 = \{x_1y_1, x_2y_2\}$, $\sigma_3 = \{x_1y_1, x_1y_2, x_2y_1, x_2y_3\}$ *and* $\sigma_4 = \{x_1y_1, x_1y_2, x_2y_3, x_2y_4\}$. *Let* $(K_{2,r}, \sigma)$ *be a signed complete bipartite graph, where* $r \geq 3$. *If* $\sigma \in \{\sigma_1, \sigma_2, \sigma_3\}$ *for* $r \geq 3$ *and* $\sigma = \sigma_4$ *for* $r \geq 5$, *then* $(K_{2,r}, \sigma)$ *is unbalanced.*

Proof. The following are the only possible cases.

1. If $\sigma \in \{\sigma_1, \sigma_2, \sigma_3\}$, then the cycle $x_1y_1x_2y_3x_1$ is negative in $(K_{2,r}, \sigma)$.
2. If $\sigma = \sigma_4$, then the cycle $x_1y_1x_2y_5x_1$ is negative in $(K_{2,r}, \sigma)$.

Thus, it follows that $(K_{2,r}, \sigma)$ is unbalanced. □

Lemma 4.5.1 is used to show that the DDN of (K_n, σ^*) cannot be 2.

Lemma 4.5.2. *Let the positive integers* l, r *and* n *be such that* $3 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\left\lceil \frac{n}{2} \right\rceil \leq r \leq n-3$ *and* $l + r = n \geq 6$. *Let* $(K_{l,r}, \sigma)$ *be a signed complete bipartite graph, where* $\sigma \neq \emptyset$. *If every vertex has negative degree at most 2 and at least one left vertex has negative degree* 0, *then* $(K_{l,r}, \sigma)$ *is unbalanced.*

Proof. Given that at least one left vertex of $(K_{l,r}, \sigma)$ has negative degree 0. Let the left vertex x_j be such that $d^-(x_j) = 0$. Since $\sigma^{-1}(-1) \neq \emptyset$, there exists at least one vertex x_k having a negative neighbor, say, y_i. As $d^-(x_j) = 0$, it is clear that y_i is adjacent to x_j by a positive edge. Since $d^+(x_k) \geq 1$, there must be a vertex y_s, where $s \neq i$, such that y_s is adjacent to x_k and x_j by positive edges. Hence the cycle $x_ky_ix_jy_sx_k$ is negative in $(K_{l,r}, \sigma)$. This implies that $(K_{l,r}, \sigma)$ is unbalanced. □

Lemma 4.5.3. *Let the positive integers* l, r *and* n *be such that* $3 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\left\lceil \frac{n}{2} \right\rceil \leq r \leq n-3$ *and* $l + r = n \geq 7$. *Let* $(K_{l,r}, \sigma)$ *be a signed complete bipartite graph, where* $\sigma \neq \emptyset$. *If every vertex has negative degree at most* 2 *and at least one left vertex has negative degree exactly* 1, *then* $(K_{l,r}, \sigma)$ *is unbalanced.*

Proof. Given that at least one left vertex of $(K_{l,r}, \sigma)$ has negative degree 1. Let the left vertex x_j be such that $d^-(x_j) = 1$. Without loss of generality, assume that the negative neighbor of x_j is y_1. Note that every left vertex has positive degree at least 2, while every right vertex has positive degree at least 1. Since $d^+(y_1) \geq 1$, there exists a vertex x_k such that y_1x_k is a positive edge. Further, the inequalities $d^+(x_k) \geq 2$ and $d^+(x_j) = r - 1$ allow us to choose a vertex y_i, where $i \neq 1$, such that y_i is a positive neighbor of both x_j and x_k. Hence the cycle $x_jy_1x_ky_ix_j$ is negative in $(K_{l,r}, \sigma)$. This implies that $(K_{l,r}, \sigma)$ is unbalanced. □

Lemma 4.5.4. *Let the positive integers* l, r *and* n *be such that* $3 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\left\lceil \frac{n}{2} \right\rceil \leq r \leq n-3$ *and* $l + r = n \geq 7$. *Let* $(K_{l,r}, \sigma)$ *be a signed complete bipartite* *graph, where $\sigma \neq \varnothing$, such that every left vertex has negative degree 2. If the edges of σ induce a path of order $n-1$ or n according as n is even or odd, then $(K_{l,r}, \sigma)$ is unbalanced.*

Proof. Given that every left vertex has negative degree 2. Also, if n is even, then the edges of σ induce a path of order $n-1$. Therefore, $l = \frac{n}{2} - 1$ and $r = \frac{n}{2} + 1$. Without loss of generality, assume that the path induced by the edges of σ to be $y_1 x_1 y_2 x_2 \dots y_{\frac{n}{2}-1} x_{\frac{n}{2}-1} y_{\frac{n}{2}}$. Here the cycle $x_1 y_1 x_2 y_4 x_1$ is negative in $(K_{l,r}, \sigma)$. Hence $(K_{l,r}, \sigma)$ is unbalanced.

If n is odd, then the edges of σ induce a path of order n. Therefore, $l = \left\lfloor \frac{n}{2} \right\rfloor$ and $r = \left\lceil \frac{n}{2} \right\rceil$. Again assume that the path induced by the edges of σ to be $y_1 x_1 y_2 x_2 \dots y_{\lceil \frac{n}{2} \rceil - 1} x_{\lceil \frac{n}{2} \rceil - 1} y_{\lceil \frac{n}{2} \rceil}$. Now the cycle $x_1 y_1 x_2 y_4 x_1$ is negative in $(K_{l,r}, \sigma)$. Hence $(K_{l,r}, \sigma)$ is unbalanced. $\square$

Lemma 4.5.5. *Let $n \geq 5$. If D is a DDS of (K_n, σ^*), then $|D| > \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let D be a subset of $V(K_n)$ such that $2 \leq |D| \leq \left\lfloor \frac{n}{2} \right\rfloor$. It is clear that the cut $[D, D^c]$ in (K_n, σ^*) induces a signed complete bipartite graph $(K_{l,r}, \sigma)$, where $l = |D|, r = |D^c|$ and $l + r = n$. Note that $2 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lceil \frac{n}{2} \right\rceil \leq r \leq n-2$.

If $l = 2$, then σ takes the form of one of the signatures mentioned in Lemma 4.5.1. If $3 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor$, then $(K_{l,r}, \sigma)$ satisfies the hypothesis of either Lemma 4.5.2, Lemma 4.5.3 or Lemma 4.5.4. Consequently, $(K_{l,r}, \sigma)$ is unbalanced. Thus the graph induced by $[D, D^c]$ is always unbalanced in (K_n, σ^*). Therefore, D cannot be a DDS of (K_n, σ^*). This completes the proof. $\square$

Theorem 4.5.1. *If $n \geq 5$, then $\gamma_{\times 2}(K_n, \sigma^*) = n - 1$.*

Proof. Let $D \subseteq V(K_n)$. If $2 \leq |D| \leq \left\lfloor \frac{n}{2} \right\rfloor$, then by Lemma 4.5.5, D is not a DDS of (K_n, σ^*). If $\left\lfloor \frac{n}{2} \right\rfloor < |D| \leq n-2$, then $2 \leq |D^c| \leq \left\lfloor \frac{n}{2} \right\rfloor$. Therefore, as in the proof of Lemma 4.5.5, the cut $[D^c, D]$ is unbalanced in (K_n, σ^*). As $[D, D^c] = [D^c, D]$, we find that D is not a DDS of (K_n, σ^*). Therefore, we have $\gamma_{\times 2}(K_n, \sigma^*) \geq n-1$. It is easy to verify that any subset of $V(K_n)$ of size $n-1$ is a DDS of (K_n, σ^*). Hence $\gamma_{\times 2}(K_n, \sigma^*) = n - 1$. $\square$

4.5.1 The DDN of Signed K_6

In Section 2.2, we found that there are 16 signed complete graphs over K_6 up to switching isomorphism. Those 16 signed graphs are shown in Figure 2.2.2 of Chapter 2. Since double domination number in signed graphs is invariant under switching operation, it is enough to determine the DDN of these 16 switching non-isomorphic signed complete graphs.

Lemma 4.5.6. *If* $V(K_6) = \{u_1, \ldots, u_6\}$ *and* $\Sigma = (K_6, \{u_1u_2, u_2u_3, u_3u_4, u_4u_5\})$, *then* $\gamma_{\times 2}(\Sigma) = 5$.

Proof. Let $D \subseteq V(K_6)$. If $|D| = 2$, then the subgraph $\Sigma[D, D^c]$ induces a $(K_{2,4}, \sigma)$, where σ takes the form of one of the signatures described in Lemma 4.5.1 or takes the form of one of $\{x_1y_1\}$, $\{x_1y_1, x_2y_2, x_2y_3\}$ or $\{x_1y_1, x_2y_1, x_2y_2\}$.

If σ takes the form of one of the signatures described in Lemma 4.5.1, then $(K_{2,4}, \sigma)$ is indeed unbalanced. If σ is either $\{x_1y_1\}$ or $\{x_1y_1, x_2y_2, x_2y_3\}$, then the cycle $x_1y_1x_2y_4x_1$ is negative in $(K_{2,4}, \sigma)$. Finally, if $\sigma = \{x_1y_1, x_2y_1, x_2y_2\}$, then the cycle $x_1y_1x_2y_2x_1$ is negative in $(K_{2,4}, \sigma)$. Thus in all the cases, $(K_{2,4}, \sigma)$ is unbalanced. Hence D is not a DDS of Σ.

Now let $D \subseteq V(K_6)$ such that $|D| = 3$. Since $[D, D^c] = [D^c, D]$, without loss of generality, assume that $u_6 \in D$. Also the subgraph $\Sigma[D, D^c]$ induces a $(K_{3,3}, \sigma)$. Note that this $(K_{3,3}, \sigma)$ satisfies the hypothesis of Lemma 4.5.2. Thus $(K_{3,3}, \sigma)$ is unbalanced. Hence D is not a DDS of Σ.

From the preceding paragraphs, we conclude that $\gamma_{\times 2}(\Sigma) \geq 5$. However, any subset of $V(K_6)$ of size five is a DDS of Σ. This completes the proof. ☐

Theorem 4.5.2. *The DDN of the* 16 *switching non-isomorphic signed complete graphs over* K_6 *are given in Table 4.5.1 and Table 4.5.2.*

Proof. A DDS of each of $\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7, \Sigma_9, \Sigma_{10}, \Sigma_{11}, \Sigma_{12}, \Sigma_{15}$ and Σ_{18} is given in the second rows of Table 4.5.1 and Table 4.5.2. Consequently, the DDN of these graphs is 2. Note that Σ_{16} is nothing but (K_6, σ^*), and therefore the DDN of Σ_{16} is 5 due to Theorem 4.5.1. The DDN of Σ_8 is 5 due to Lemma 4.5.6. This completes the proof. ☐

	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7
DDS	$\{5,6\}$	$\{5,6\}$	$\{5,6\}$	$\{5,6\}$	$\{5,6\}$	$\{4,5\}$	$\{4,5\}$	$\{5,6\}$
DDN	2	2	2	2	2	2	2	2

Table 4.5.1. DDS and DDN of some signed K_6

	Σ_8	Σ_9	Σ_{10}	Σ_{11}	Σ_{12}	Σ_{15}	Σ_{16}	Σ_{18}
DDS	$\{1,2,3,4,5\}$	$\{5,6\}$	$\{1,4\}$	$\{5,6\}$	$\{4,5\}$	$\{1,3\}$	$\{1,2,3,4,5\}$	$\{1,3\}$
DDN	5	2	2	2	2	2	5	2

Table 4.5.2. DDS and DDN of some (that are not listed in Table 4.5.1) signed K_6

5

Chromatic Polynomials of Signed Book Graphs

Unlike the chromatic polynomial of an unsigned graph, the chromatic polynomial of a signed graph comes with a companion, namely, the zero-free chromatic polynomial. In this chapter, we determine the chromatic number and both kind of chromatic polynomials of switching non-isomorphic signed book graphs.

5.1 Preliminaries

For $n \geq 1$, let $M_n = \{0, \pm 1, ..., \pm k\}$ if $n = 2k + 1$, and $M_n = \{\pm 1, ..., \pm k\}$ if $n = 2k$. According to Máčajová et al. [38], an *n-coloring* of a signed graph (G, ϕ) is a mapping $c : V(G) \longrightarrow M_n$. An n-coloring c is *proper* if $c(x) \neq \phi(e)c(y)$ for every edge e of G, where $e = xy$ and $\phi(e)$ is the sign of the edge e. The *chromatic number* $\chi(G, \sigma)$ of (G, σ) is the smallest number n such that (G, ϕ) admits a proper n-coloring, where $\phi^{-1}(-1) = \sigma$. If n is even, then an n-coloring is called a *zero-free coloring*.

Recall that the *chromatic polynomial* $\chi_\Sigma(\lambda)$ of a signed graph Σ is the function such that $\chi_\Sigma(2k+1)$ is equal to the number of proper colorings of Σ in $2k+1$ signed colors. The *zero-free chromatic polynomial* of a signed graph Σ is the function $\chi^b_\Sigma(\lambda)$, where $\chi^b_\Sigma(2k)$ counts the zero-free proper colorings in $2k$ signed colors. The chromatic polynomial of an ordinary graph G is denoted by $\chi_G(\lambda)$.

Let the signed graphs Σ and Σ' be switching equivalent. It is clear that a coloring c of Σ is proper if and only if the coloring c' of Σ' is proper, where c' is obtained from c after negating the colors of the vertices by which Σ is switched. Thus the chromatic number of a signed graph remains invariant under switching operation. Consequently, both kind of chromatic polynomials of signed graphs are also switching invariant. There is a strong conclusion if Σ is balanced. More precisely, we have the following lemma.

Lemma 5.1.1 ([58]). *Let Σ be a signed graph with underlying graph G. If Σ is* *balanced, then* $\chi_G(\lambda) = \chi_\Sigma(\lambda) = \chi^b_\Sigma(\lambda)$.

In perfect analogy to ordinary graph coloring theory, we have the following theorem.

Theorem 5.1.1 ([58])**.** *If* Σ *is a signed graph on* n *vertices, then* $\chi_\Sigma(\lambda)$ *and* $\chi^b_\Sigma(\lambda)$ *are monic polynomial functions of* λ *of degree* n.

Let $\Sigma = (G, \sigma)$. The *contraction* of Σ by a positive edge e, denoted Σ/e, is obtained by identifying the end vertices of e and deleting e. See [59] for details.

We also have the signed analogue of edge deletion-contraction formula for the chromatic polynomial of simple graph.

Theorem 5.1.2 ([58])**.** *If* e *is a positive edge in a signed graph* Σ*, then*

$$\chi_\Sigma(\lambda) = \chi_{\Sigma\setminus e}(\lambda) - \chi_{\Sigma/e}(\lambda) \text{ and } \chi^b_\Sigma(\lambda) = \chi^b_{\Sigma\setminus e}(\lambda) - \chi^b_{\Sigma/e}(\lambda).$$

5.2 Chromatic Number of Signed Book Graphs

Recall that, for $m \geq 3$ and $n \geq 1$, the m-cycle *book graph* $B(m, n)$ consists of n copies of the cycle C_m with one common edge. Indeed

$$V(B(m,n)) = \{u, v\} \cup \{u^i_j : i \in \{1, \ldots, n\}, \ j \in \{1, \ldots, m-2\}\},$$

where uv is the common edge to the cycles C^i_m, and $C^i_m = uu^i_1u^i_2u^i_3 \ldots u^i_{m-3}u^i_{m-2}vu$ for $1 \leq i \leq n$. For example, the graph $B(4, 3)$ is shown in Figure 2.4.1. We also recall Theorem 2.4.1 from Chapter 2, and present once again in the next result.

Theorem 5.2.1. *If* $(B(m, n), \sigma)$ *is a signed book graph, then* $(B(m, n), \sigma)$ *is equivalent to* $(B(m, n), \tau)$*, where* $\tau \subseteq \{uu^1_1, \ldots, uu^n_1\}$*. Further, the number of switching non-isomorphic signed* $B(m, n)$ *is* $n + 1$.

Let $\sigma_0 = \emptyset$. For $1 \leq l \leq n$, let $\sigma_l = \{uu^1_1, \ldots, uu^l_1\}$. For $i, j \in \{0, 1, \ldots, n\}$ with $i \neq j$, it is clear that $(B(m, n), \sigma_i)$ and $(B(m, n), \sigma_j)$ have different number of negative cycles. Hence σ_i and σ_j are switching non-isomorphic to each other for $i, j \in \{0, 1, \ldots, n\}$ with $i \neq j$. Thus (B, σ_l) is a representative of one of the $n + 1$ switching isomorphism classes of $B(m, n)$, where $0 \leq l \leq n$.

It is proved in [38, Proposition 2.4] that $\chi_{(C_m,\sigma)} \leq 3$, where (C_m, σ) is a signed cycle. As $B(m, 1) = C_m$, we have $\chi_{(B(m,1),\sigma)} \leq 3$. More precisely, if $(B(m, 1), \sigma)$ is balanced and $(B(m, 1), \tau)$ is unbalanced, then

$$\chi(B(m,1),\sigma) = \begin{cases} 2 & \text{if } m \text{ is even} \\ 3 & \text{if } m \text{ is odd} \end{cases}$$

and

$$\chi(B(m,1),\tau) = \begin{cases} 3 & \text{if } m \text{ is even} \\ 2 & \text{if } m \text{ is odd.} \end{cases}$$

In the following theorem, we consider the case $n \geq 2$.

Theorem 5.2.2. *If* $m \geq 3, n \geq 2$ *and* $0 \leq l \leq n$, *then*

$$\chi(B(m,n),\sigma_l) = \begin{cases} 2 & \textit{if } m \textit{ is odd and } l = n \\ 2 & \textit{if } m \textit{ is even and } l = 0 \\ 3 & \textit{otherwise.} \end{cases}$$

Proof. Let k be a positive integer. In $(B(2k+1,n),\sigma_n)$, assign colors 1 and -1 to u and v, respectively. For $1 \leq r \leq n$ and $1 \leq i \leq 2k-1$, assign the color 1 or -1 to u_i^r according as i is odd or even, respectively. This gives a proper 2-coloring of $(B(2k+1,n),\sigma_n)$.

In $(B(2k,n),\sigma_0)$, assign the colors 1 and -1 to the vertices u and v, respectively. For $1 \leq r \leq n$ and $1 \leq i \leq 2k-2$, the vertex u_i^r is colored by -1 or 1 according as i is odd or even, respectively. This gives a proper 2-coloring of $(B(2k,n),\sigma_0)$.

Note that the odd cycle $uu_1^n u_2^n \dots u_{2k-2}^n u_{2k-1}^n vu$ is positive in $(B(2k+1,n),\sigma_l)$, where $0 \leq l \leq n-1$. It is well known that at least three colors are needed to properly color the vertices of a positive odd cycle. Therefore, $\chi(B(2k+1,n),\sigma_l) \geq 3$. We now give a proper 3-coloring of $(B(2k+1,n),\sigma_l)$. Assign the colors 0 and -1 to u and v, respectively. For $1 \leq r \leq n$ and $1 \leq i \leq 2k-1$, color the vertex u_i^r by 1 or -1 according as i is odd or even, respectively. This assignment is a proper 3-coloring of $(B(2k+1,n),\sigma_l)$, where $0 \leq l \leq n-1$.

Finally, $(B(2k,n),\sigma_l)$ contains a negative $2k$-cycle for $1 \leq l \leq n$. The fact that an unbalanced even cycle needs at least 3 colors to have a proper coloring implies that $\chi(B(2k,n),\sigma_l) \geq 3$. To complete the proof, it is sufficient to give a proper 3-coloring of $(B(2k,n),\sigma_l)$. Assign the colors 0 and 1 to u and v, respectively. For $1 \leq r \leq n$ and $1 \leq i \leq 2k-2$, color the vertex u_i^r by 1 or -1 according as i is odd or even, respectively. This assignment is a proper 3-coloring of $(B(2k,n),\sigma_l)$. □

As the chromatic number of a signed graph is invariant under switching operation, we conclude from Theorem 5.2.2 that the chromatic number of a signed book graph is either 2 or 3.

5.3 Chromatic Polynomials of Signed $B(m,n)$

For convenience, we write $B_l(m,n)$ to denote the signed book graph $(B(m,n),\sigma_l)$. Also, we write $B^{uv}(m,n)$ to denote the signed graph $(B(m,n),\{uv\})$. It is clear that $B_n(m,n) \sim B^{uv}(m,n)$. Since the chromatic polynomials of a signed graph are switching invariant, it is sufficient to determine the chromatic polynomials of $B^{uv}(m,n)$ and $B_l(m,n)$ for $0 \le l \le n-1$.

Let an unbalanced cycle on n vertices be denoted by C_n^-. For instance, C_2^- is shown in Figure 5.3.1. In the following example, we count all possible assignments of colors from the set $\{0, \pm 1, \dots, \pm k\}$ to the vertices of C_2^- to calculate $\chi_{C_2^-}(2k+1)$.

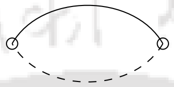


Figure 5.3.1. An unbalanced cycle of length 2

Example 5.3.1. Let the colors $0, \pm 1, \dots, \pm k$ be available. The number of proper colorings of C_2^- with these $2k+1$ colors, in which one of the vertices is colored with 0, is $4k$. Else, the number of proper colorings of C_2^- is $2k(2k-2)$. Thus $\chi_{C_2^-}(2k+1) = 4k + 2k(2k-2) = (2k)^2$.

From Example 5.3.1, we find that $\chi_{C_2^-}(\lambda) = (\lambda-1)^2$. In [12], it is proved that $\chi_{C_3^-}(\lambda) = (\lambda-1)^3$. Now we give the formula of $\chi_{C_n^-}(\lambda)$ for all n.

Lemma 5.3.1. *Let $n \ge 2$. If the cycle C_n^- is unbalanced, then $\chi_{C_n^-}(\lambda) = (\lambda-1)^n$.*

Proof. We prove this lemma by induction on n. If $n = 2$, the result is true by Example 5.3.1. Let us assume that the result holds for $n = r-1$, where $r \ge 3$. We prove that the result is also true for $n = r$. If e is a positive edge of C_r^-, then by edge deletion-contraction formula, we get

$$\chi_{C_r^-}(\lambda) = \chi_{P_r}(\lambda) - \chi_{C_{r-1}^-}(\lambda). \tag{5.3.1}$$

It is well known that $\chi_{P_r}(\lambda) = \lambda(\lambda-1)^{r-1}$, and by induction hypothesis we have $\chi_{C_{r-1}^-}(\lambda) = (\lambda-1)^{r-1}$. Therefore, Equation (5.3.1) gives

$$\chi_{C_r^-}(\lambda) = \lambda(\lambda-1)^{r-1} - (\lambda-1)^{r-1} = (\lambda-1)^r.$$

Thus the proof follows by induction. □

It is well known that $\chi_{C_m}(\lambda) = (\lambda-1)^m + (-1)^m(\lambda-1)$. Using the function $\chi_{C_m}(\lambda)$, the polynomial function γ_m of degree $m-2$ is defined by

$$\gamma_m = \frac{\chi_{C_m}(\lambda)}{\lambda(\lambda-1)} = \frac{(\lambda-1)^{m-1} - (-1)^{m-1}}{\lambda}. \tag{5.3.2}$$

The function γ_m can also be expressed as a finite geometric series. More precisely, we have

$$\gamma_m = \frac{(\lambda-1)^{m-1}+(-1)^m}{\lambda} = \sum_{i=0}^{m-2}(-1)^i(\lambda-1)^{(m-2)-i}. \tag{5.3.3}$$

The function γ_m is useful for obtaining the explicit formulas of both kind of chromatic polynomials of signed book graphs.

Let Σ be a given signed graph. Construct the signed graph Σ_{t+1} by attaching the all-positive path P_{t+1} to a vertex u of Σ, where $P_{t+1} := uu_1u_2\dots u_t$. If the chromatic polynomial of Σ is known, then we compute the chromatic polynomial of Σ_{t+1} using the following lemma.

Lemma 5.3.2. *Let Σ be a signed graph. Then $\chi_{\Sigma_{t+1}}(\lambda) = (\lambda-1)^t\chi_\Sigma(\lambda)$.*

Proof. We prove the lemma by induction on t. Note that $\chi_{P_{t+1}}(\lambda) = \lambda(\lambda-1)^t$. Let $t = 1$, and let the all-positive path P_2 be attached to a vertex u of Σ, where $P_2 := uu_1$. Using edge deletion-contraction formula on uu_1, we get

$$\chi_{\Sigma_2}(\lambda) = \lambda\chi_\Sigma(\lambda) - \chi_\Sigma(\lambda) = (\lambda-1)\chi_\Sigma(\lambda).$$

Now assume that the result holds for $t = r-1$, that is, $\chi_{\Sigma_r}(\lambda) = (\lambda-1)^{r-1}\chi_\Sigma(\lambda)$, where $r \geq 3$. Now let $t = r$ and consider the edge uu_1 of Σ_{r+1}. Using edge deletion-contraction formula on the edge uu_1 of Σ_{r+1}, we get $\chi_{\Sigma_{r+1}}(\lambda) = \chi_{P_r\cup\Sigma}(\lambda) - \chi_{\Sigma_r}(\lambda)$, where P_r and Σ are disjoint. We have

$$\begin{aligned}\chi_{\Sigma_{r+1}}(\lambda) &= \lambda(\lambda-1)^{r-1}\chi_\Sigma(\lambda) - \chi_{\Sigma_r}(\lambda)\\ &= \lambda(\lambda-1)^{r-1}\chi_\Sigma(\lambda) - (\lambda-1)^{r-1}\chi_\Sigma(\lambda)\\ &= (\lambda-1)^r\chi_\Sigma(\lambda).\end{aligned}$$

Hence the proof follows by induction. □

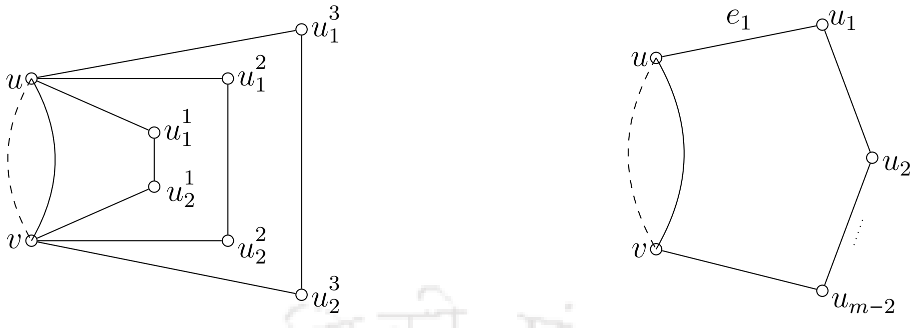
Replace the edge uv of $B(m,n)$ by an unbalanced cycle of length 2, and denote the graph so obtained by $\mathbf{B}_m^n$. For example, $\mathbf{B}_4^3$ and $\mathbf{B}_m^1$ are shown in Figure 5.3.2. As a convention, write $\mathbf{B}_2^1 = C_2^-$. Thus $\chi_{\mathbf{B}_2^1}(\lambda) = (\lambda-1)^2$.

Lemma 5.3.3. *For $m \geq 2$, the chromatic polynomial of $\mathbf{B}_m^1$ is given by*

$$\chi_{\mathbf{B}_m^1}(\lambda) = (\lambda-1)^2\gamma_m.$$

Proof. We prove the lemma by induction on m. Consider $\mathbf{B}_m^1$ as given in Figure 5.3.2(b). Since $\gamma_2 = 1$, the result is true by Example 5.3.1 for $m = 2$. Assume that the result is true for $m = r-1$, where $r \geq 3$. That is,

$$\chi_{\mathbf{B}_{r-1}^1}(\lambda) = (\lambda-1)^2\gamma_{r-1}. \tag{5.3.4}$$



(a) The signed graph $\mathbf{B}_4^3$ (b) The signed graph $\mathbf{B}_m^1$

Figure 5.3.2. The signed graphs $\mathbf{B}_4^3$ and $\mathbf{B}_m^1$

An application of edge deletion-contraction on the edge uu_1 of $\mathbf{B}_r^1$ is shown in Figure 5.3.3. Since $\chi_{C_2^-}(\lambda)$ is known, the chromatic polynomial of the graph in the middle of Figure 5.3.3 is computed using Lemma 5.3.2. The chromatic polynomial of the third graph of Figure 5.3.3 is given in Equation (5.3.4). Therefore,

$$\begin{aligned}\chi_{\mathbf{B}_r^1}(\lambda) &= (\lambda-1)^{r-2}\chi_{C_2^-}(\lambda) - (\lambda-1)^2\gamma_{r-1}\\ &= (\lambda-1)^r - (\lambda-1)^2\frac{(\lambda-1)^{r-2} - (-1)^{r-2}}{\lambda}\\ &= (\lambda-1)^2\frac{(\lambda-1)^{r-1} - (-1)^{r-1}}{\lambda}\\ &= (\lambda-1)^2\gamma_r.\end{aligned}$$

This completes the proof. □

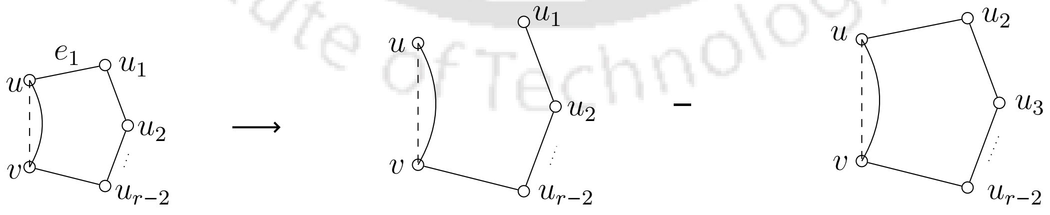


Figure 5.3.3. An application of edge deletion-contraction on $\mathbf{B}_r^1$

Now we give the formula for the chromatic polynomial of the signed graph $\mathbf{B}_m^n$, where $n \geq 2$. To do so, we use Theorem 5.1.2, Lemma 5.3.2 and Lemma 5.3.3.

Theorem 5.3.1. *For $m \geq 3$ and $n \geq 2$, the chromatic polynomial of the signed graph $\mathbf{B}_m^n$ is given by*

$$\chi_{\mathbf{B}_m^n}(\lambda) = (\lambda-1)^2\gamma_m^n.$$

Proof. We sequentially use edge deletion-contraction formula on the edges uu_1^n, uu_2^n, uu_3^n, and so on. This process gives

$$\begin{aligned}\chi_{\mathbf{B}_m^n}(\lambda) &= \left[\sum_{i=0}^{m-2}(-1)^i(\lambda-1)^{(m-2)-i}\right]\chi_{\mathbf{B}_m^{n-1}}(\lambda)\\ &= \gamma_m\chi_{\mathbf{B}_m^{n-1}}(\lambda).\end{aligned}$$

Since $\chi_{\mathbf{B}_m^1}(\lambda)$ is known from Lemma 5.3.3, we find after $(n-2)$ recursions that

$$\chi_{\mathbf{B}_m^n}(\lambda) = (\lambda-1)^2\gamma_m^n.$$

This completes the proof. ☐

It is well known that if G and H are two simple graphs such that $G \cap H$ is a complete graph, then the chromatic polynomial of $G \cup H$ is given by

$$\chi_{G\cup H}(\lambda) = \frac{\chi_G(\lambda)\cdot\chi_H(\lambda)}{\chi_{G\cap H}(\lambda)}. \qquad (5.3.5)$$

Using Equation (5.3.5), we now determine the chromatic polynomial of an unsigned book graph.

Theorem 5.3.2. *For $m \geq 3$ and $n \geq 1$, the chromatic polynomial of $B(m,n)$ is given by*

$$\chi_{B(m,n)}(\lambda) = \lambda(\lambda-1)\gamma_m^n.$$

Proof. We prove the theorem by induction on n. Clearly, $B(m,1) = C_m$. By the definition of γ_m, we have

$$\chi_{B(m,1)}(\lambda) = \chi_{C_m}(\lambda) = \lambda(\lambda-1)\gamma_m. \qquad (5.3.6)$$

Further, the graph $B(m,2)$ is the union of two m-cycles whose intersection is K_2. Thus by Equation (5.3.5), we get

$$\chi_{B(m,2)}(\lambda) = \lambda(\lambda-1)\gamma_m^2.$$

This shows that the result is true for $n = 1$ and $n = 2$. Let us assume that the result is true for $n = r-1$, where $r \geq 3$. That is,

$$\chi_{B(m,r-1)}(\lambda) = \lambda(\lambda-1)\gamma_m^{r-1}. \qquad (5.3.7)$$

Now we prove that the result holds for $n = r$. The graph $B(m,r)$ can be considered as the union of the graphs $B(m,r-1)$ and C_m, whose intersection is K_2. Therefore by Equation (5.3.5), we have

$$\chi_{B(m,r)}(\lambda) = \frac{\chi_{B(m,r-1)}(\lambda)\cdot\chi_{C_m}(\lambda)}{\lambda(\lambda-1)}. \qquad (5.3.8)$$

Using Equations (5.3.6) and (5.3.7) in Equation (5.3.8), we get

$$\chi_{B(m,r)}(\lambda) = \lambda(\lambda-1)\gamma_m^r.$$

Hence the proof follows by induction. □

From Lemma 5.1.1 and Theorem 5.3.2, it follows that

$$\chi_{B_0(m,n)}(\lambda) = \chi_{B(m,n)}(\lambda) = \lambda(\lambda-1)\gamma_m^n.$$

Clearly, $B_1(m,1) \cong C_m^-$. Thus by Lemma 5.3.1, we have $\chi_{B_1(m,1)}(\lambda) = (\lambda-1)^m$. We now compute the chromatic polynomials of the signed book graphs $B_1(m,n)$ for $m \geq 3$ and $n \geq 1$.

Theorem 5.3.3. *For $m \geq 3$ and $n \geq 1$, the chromatic polynomial of $B_1(m,n)$ is given by*

$$\chi_{B_1(m,n)}(\lambda) = (\lambda-1)^m\gamma_m^{n-1}.$$

Proof. Recall that $B_1(m,n)$ is the signed book graph with the signature $\{uu_1^1\}$. For $n = 1$, the result holds true due to Lemma 5.3.1. For $n \geq 2$, consider $e_1 = uu_1^n$. Using edge deletion-contraction formula on e_1, we get

$$\chi_{B_1(m,n)}(\lambda) = \chi_{B_1'(m,n-1)}(\lambda) - \chi_{B_1''(m,n)}(\lambda),$$

where $B_1'(m,n-1)$ denotes the graph obtained from $B_1(m,n-1)$ by attaching the all-positive path P_{m-1} to the vertex v, and $B_1''(m,n)$ denotes the graph that is almost same as $B_1(m,n)$ but one of its pages is a cycle of length $m-1$ and the vertex obtained by contraction of e_1 is denoted by u again.
Using Lemma 5.3.2, the chromatic polynomial of the signed graph $B_1'(m,n-1)$ is obtained in terms of $\chi_{B_1(m,n-1)}(\lambda)$. For $B_1''(m,n)$, we apply the edge deletion-contraction formula again on the edge uu_2^n. Repeated use of edge deletion-contraction formula and Lemma 5.3.2 give the chromatic polynomial of $B_1(m,n)$ as

$$\begin{aligned}\chi_{B_1(m,n)}(\lambda) =& (\lambda-1)^{m-2}\chi_{B_1(m,n-1)}(\lambda) - (\lambda-1)^{m-3}\chi_{B_1(m,n-1)}(\lambda) + \cdots \\ &+ (-1)^{m-2}(\lambda-1)^{(m-2)-(m-2)}\chi_{B_1(m,n-1)}(\lambda) \\ =& \left[\sum_{i=0}^{m-2}(-1)^i(\lambda-1)^{(m-2)-i}\right]\chi_{B_1(m,n-1)}(\lambda) \\ =& \gamma_m\chi_{B_1(m,n-1)}(\lambda).\end{aligned}$$

Since $\chi_{B_1(m,1)}(\lambda) = (\lambda-1)^m$, we find after $(n-2)$ recursions that

$$\chi_{B_1(m,n)}(\lambda) = (\lambda-1)^m\gamma_m^{n-1}.$$

This completes the proof. □

Theorem 5.3.4. *For $m \geq 3$ and $n \geq 2$, the chromatic polynomial of $B^{uv}(m,n)$ is given by*

$$\begin{aligned}\chi_{B^{uv}(m,n)}(\lambda) &= (\lambda-1)\gamma_{m-1}\chi_{B^{uv}(m,n-1)}(\lambda) + (-1)^{m-2}(\lambda-1)^2\gamma_m^{n-1}\\ &= (\lambda-1)^{m+n-1}\gamma_{m-1}^{n-1} + (-1)^{m-2}\gamma_m(\lambda-1)^2\frac{(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}-\gamma_m^{n-1}}{(\lambda-1)\gamma_{m-1}-\gamma_m}.\end{aligned}$$

Proof. Recall that $B^{uv}(m,n)$ is the signed book graph with the signature $\{uv\}$. Consider $e_1 = uu_1^n$. Using edge deletion-contraction formula on e_1, we have

$$\chi_{B^{uv}(m,n)}(\lambda) = \chi_{\widetilde{B^{uv}}(m,n-1)}(\lambda) - \chi_{\widetilde{\widetilde{B^{uv}}}(m,n)}(\lambda),$$

where $\widetilde{B^{uv}}(m,n-1)$ denotes the graph obtained from $B^{uv}(m,n-1)$ by attaching the all-positive path P_{m-1} to the vertex v, and $\widetilde{\widetilde{B^{uv}}}(m,n)$ denotes the graph that is almost same as $B^{uv}(m,n)$ but one of its pages is a cycle of length $m-1$.

Using Lemma 5.3.2, we obtain the chromatic polynomial of the signed graph $\widetilde{B^{uv}}(m,n-1)$ in terms of $\chi_{B^{uv}(m,n-1)}(\lambda)$. For the graph $\widetilde{\widetilde{B^{uv}}}(m,n)$, we apply the edge deletion-contraction formula again on the edge uu_2^n.

Applying edge deletion-contraction formula repeatedly and using Lemma 5.3.2, we obtain the chromatic polynomial of $B^{uv}(m,n)$ as

$$\chi_{B^{uv}(m,n)}(\lambda) = \left[\sum_{i=0}^{m-3}(-1)^i(\lambda-1)^{(m-2)-i}\right]\chi_{B^{uv}(m,n-1)}(\lambda) + (-1)^{m-2}\chi_{\mathbf{B}_m^{n-1}}(\lambda).$$

In the last step of edge deletion-contraction formula of the preceding paragraph, the resulting graph is nothing but the signed graph $\mathbf{B}_m^{n-1}$. We know by Theorem 5.3.1 that $\chi_{\mathbf{B}_m^{n-1}}(\lambda) = (\lambda-1)^2\gamma_m^{n-1}$. Thus we have,

$$\chi_{B^{uv}(m,n)}(\lambda) = (\lambda-1)\gamma_{m-1}\chi_{B^{uv}(m,n-1)}(\lambda) + (-1)^{m-2}(\lambda-1)^2\gamma_m^{n-1}.$$

After $(n-2)$ recursions, we have

$$\begin{aligned}\chi_{B^{uv}(m,n)}(\lambda) &= (\lambda-1)^{m+n-1}\gamma_{m-1}^{n-1} + (-1)^{m-2}\frac{\gamma_m^{n+1}}{\gamma_{m-1}^2}\sum_{i=2}^{n}\left(\frac{(\lambda-1)\gamma_{m-1}}{\gamma_m}\right)^i\\ &= (\lambda-1)^{m+n-1}\gamma_{m-1}^{n-1} + (-1)^{m-2}\gamma_m(\lambda-1)^2\frac{(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}-\gamma_m^{n-1}}{(\lambda-1)\gamma_{m-1}-\gamma_m}.\end{aligned}$$

This completes the proof. □

We now give a formula for the chromatic polynomial of $B_l(m,n)$ for $2 \leq l \leq n-1$.

Theorem 5.3.5. *Let $m \geq 3$, $n \geq 3$ and $2 \leq l \leq n-1$. The chromatic polynomial of $B_l(m,n)$ is given by*

$$\chi_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\chi_{B^{uv}(m,l)}(\lambda).$$

Proof. By repeated use of edge deletion-contraction formula and Lemma 5.3.2 on the graph $B_l(m,n)$, we get

$$\chi_{B_l(m,n)}(\lambda) = \frac{(\lambda-1)^{m-1} + (-1)^m}{\lambda}\chi_{B_l(m,n-1)}(\lambda) = \gamma_m \chi_{B_l(m,n-1)}(\lambda). \qquad (5.3.9)$$

Applying the recursion of Equation (5.3.9) $(n-l-1)$ times, we have

$$\chi_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\chi_{B_l(m,l)}(\lambda).$$

Note that $B_l(m,l)$ is switching equivalent to $B^{uv}(m,l)$. Therefore we have

$$\chi_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\chi_{B^{uv}(m,l)}(\lambda).$$

This completes the proof. □

5.4 Zero-free Chromatic Polynomials of Signed Book Graphs

In [58], the author explained that the chromatic polynomial and the zero-free chromatic polynomial of a signed graph Σ are different unless Σ is balanced. In this section, we determine the zero-free chromatic polynomials of signed book graphs.

We present all the results of Section 5.3 in terms of the zero-free chromatic polynomials. The proofs of these results are similar to the corresponding proofs presented for chromatic polynomials in Section 5.3. For the sake of completeness, we provide the proofs.

Lemma 5.4.1. *If Σ is a signed graph, then*

$$\chi^b_{\Sigma_{t+1}}(\lambda) = (\lambda-1)^t\chi^b_{\Sigma}(\lambda).$$

Proof. We prove the lemma by induction on t. Note that $\chi^b_{P_{t+1}}(\lambda) = \lambda(\lambda-1)^t$. Let $t = 1$, and let the all-positive path P_2 be attached to a vertex u of Σ, where $P_2 := uu_1$. Using edge deletion-contraction formula on uu_1, we get

$$\chi^b_{\Sigma_2}(\lambda) = \lambda\chi^b_{\Sigma}(\lambda) - \chi^b_{\Sigma}(\lambda) = (\lambda-1)\chi^b_{\Sigma}(\lambda).$$

Now assume that the result holds for $t = r-1$, that is, $\chi^b_{\Sigma_r}(\lambda) = (\lambda-1)^{r-1}\chi^b_{\Sigma}(\lambda)$, where $r \geq 3$. Let $t = r$ and consider the edge uu_1 of Σ_{r+1}. Using edge deletion-contraction formula on the edge uu_1 of Σ_{r+1}, we get $\chi^b_{\Sigma_{r+1}}(\lambda) = \chi^b_{P_r \cup \Sigma}(\lambda) - \chi^b_{\Sigma_r}(\lambda)$, where P_r and Σ are disjoint. Thus we have

$$\begin{aligned}\chi^b_{\Sigma_{r+1}}(\lambda) &= \lambda(\lambda-1)^{r-1}\chi^b_{\Sigma}(\lambda) - \chi^b_{\Sigma_r}(\lambda)\\ &= \lambda(\lambda-1)^{r-1}\chi^b_{\Sigma}(\lambda) - (\lambda-1)^{r-1}\chi^b_{\Sigma}(\lambda)\\ &= (\lambda-1)^r\chi^b_{\Sigma}(\lambda).\end{aligned}$$

Hence the proof follows by induction. □

Theorem 5.4.1. *For $n \geq 2$, the zero-free chromatic polynomial of C_n^- is given by*

$$\chi^b_{C_n^-}(\lambda) = (\lambda - 1)^n - (-1)^n = \lambda\gamma_{n+1}.$$

Proof. The proof easily follows by induction on n. □

Lemma 5.4.2. *For $m \geq 2$, the zero-free chromatic polynomial of $\mathbf{B}_m^1$ is given by*

$$\chi^b_{\mathbf{B}_m^1}(\lambda) = \lambda(\lambda - 2)\gamma_m.$$

Proof. We prove the lemma by induction on m. Consider $\mathbf{B}_m^1$ as given in Figure 5.3.2(b). Since $\mathbf{B}_2^1 = C_2^-$, the result is true by Theorem 5.4.1 for $m = 2$. Assume that the result is true for $m = r - 1$, where $r \geq 3$. That is,

$$\chi^b_{\mathbf{B}_{r-1}^1}(\lambda) = \lambda(\lambda - 2)\gamma_{r-1}. \tag{5.4.1}$$

An application of edge deletion-contraction on the edge uu_1 of $\mathbf{B}_r^1$ is shown in Figure 5.3.3. Since $\chi^b_{C_2^-}(\lambda)$ is known, the zero-free chromatic polynomial of the graph in the middle of Figure 5.3.3 is computed using Lemma 5.4.1. The zero-free chromatic polynomial of the third graph of Figure 5.3.3 is given in Equation (5.4.1). Therefore

$$\begin{aligned}\chi^b_{\mathbf{B}_r^1}(\lambda) &= (\lambda - 1)^{r-2}\chi^b_{C_2^-}(\lambda) - \lambda(\lambda - 2)\gamma_{r-1} \\ &= (\lambda - 1)^{r-2}\lambda(\lambda - 2) - \lambda(\lambda - 2)\frac{(\lambda - 1)^{r-2} - (-1)^{r-2}}{\lambda} \\ &= \lambda(\lambda - 2)\frac{(\lambda - 1)^{r-1} - (-1)^{r-1}}{\lambda} \\ &= \lambda(\lambda - 2)\gamma_r.\end{aligned}$$

This completes the proof. □

Theorem 5.4.2. *For $m \geq 3$ and $n \geq 2$, the zero-free chromatic polynomial of $\mathbf{B}_m^n$ is given by*

$$\chi^b_{\mathbf{B}_m^n}(\lambda) = \lambda(\lambda - 2)\gamma_m^n.$$

Proof. We sequentially use edge deletion-contraction formula on the edges uu_1^n, uu_2^n, uu_3^n and so on. This process gives

$$\begin{aligned}\chi^b_{\mathbf{B}_m^n}(\lambda) &= \left[\sum_{i=0}^{m-2}(-1)^i(\lambda - 1)^{(m-2)-i}\right]\chi^b_{\mathbf{B}_m^{n-1}}(\lambda) \\ &= \gamma_m\chi^b_{\mathbf{B}_m^{n-1}}(\lambda).\end{aligned}$$

Since $\chi^b_{\mathbf{B}_m^1}(\lambda)$ is known from Lemma 5.4.2, we find after $(n-2)$ recursions that

$$\chi^b_{\mathbf{B}_m^n}(\lambda) = \lambda(\lambda - 2)\gamma_m^n.$$

This completes the proof. □

From Lemma 5.1.1 and Theorem 5.3.2, it follows that

$$\overset{b}{\chi}_{B_0(m,n)}(\lambda) = \lambda(\lambda-1)\gamma_m^n.$$

Theorem 5.4.3. *For $m \geq 3$ and $n \geq 1$, the zero-free chromatic polynomial of $B_1(m,n)$ is given by*

$$\overset{b}{\chi}_{B_1(m,n)}(\lambda) = \lambda\gamma_m^{n-1}\gamma_{m+1}.$$

Proof. Recall that $B_1(m,n)$ is the signed book graph with the signature $\{uu_1^1\}$. For $n = 1$, result holds true due to Theorem 5.4.1. Repeated use of edge deletion-contraction formula and Lemma 5.4.1 give the zero-free chromatic polynomial of $B_1(m,n)$ as

$$\begin{aligned}\overset{b}{\chi}_{B_1(m,n)}(\lambda) &= \left[\sum_{i=0}^{m-2}(-1)^i(\lambda-1)^{(m-2)-i}\right]\overset{b}{\chi}_{B_1(m,n-1)}(\lambda)\\ &= \gamma_m\overset{b}{\chi}_{B_1(m,n-1)}(\lambda).\end{aligned}$$

Since $\overset{b}{\chi}_{B_1(m,1)}(\lambda) = \lambda\gamma_{m+1}$, we find after $(n-2)$ recursions that

$$\overset{b}{\chi}_{B_1(m,n)}(\lambda) = \lambda\gamma_{m+1}\gamma_m^{n-1}.$$

This completes the proof. ☐

Theorem 5.4.4. *For $m \geq 3$ and $n \geq 2$, the zero-free chromatic polynomial of $B^{uv}(m,n)$ is given by*

$$\begin{aligned}\overset{b}{\chi}_{B^{uv}(m,n)}(\lambda) &= (\lambda-1)\gamma_{m-1}\overset{b}{\chi}_{B^{uv}(m,n-1)}(\lambda) + (-1)^{m-2}\lambda(\lambda-2)\gamma_m^{n-1}\\ &= \lambda(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}\gamma_{m+1} + (-1)^{m-2}\lambda(\lambda-2)\gamma_m\frac{(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}-\gamma_m^{n-1}}{(\lambda-1)\gamma_{m-1}-\gamma_m}.\end{aligned}$$

Proof. Recall that $B^{uv}(m,n)$ is the signed book graph with the signature $\{uv\}$. Applying edge deletion-contraction formula repeatedly and using Lemma 5.4.1, we obtain the zero-free chromatic polynomial of $B^{uv}(m,n)$ as

$$\overset{b}{\chi}_{B^{uv}(m,n)}(\lambda) = \left[\sum_{i=0}^{m-3}(-1)^i(\lambda-1)^{(m-2)-i}\right]\overset{b}{\chi}_{B^{uv}(m,n-1)}(\lambda) + (-1)^{m-2}\overset{b}{\chi}_{\mathbf{B}_m^{n-1}}(\lambda).$$

In the last step of edge deletion-contraction formula of the preceding paragraph, the resulting graph is nothing but the signed graph $\mathbf{B}_m^{n-1}$. We know by Theorem 5.4.2 that $\overset{b}{\chi}_{\mathbf{B}_m^{n-1}}(\lambda) = \lambda(\lambda-2)\gamma_m^{n-1}$. Thus we have

$$\overset{b}{\chi}_{B^{uv}(m,n)}(\lambda) = (\lambda-1)\gamma_{m-1}\overset{b}{\chi}_{B^{uv}(m,n-1)}(\lambda) + (-1)^{m-2}\lambda(\lambda-2)\gamma_m^{n-1}.$$

After $(n-2)$ recursions, we have

$$\begin{aligned}\overset{b}{\chi}_{B^{uv}(m,n)}(\lambda) &= \lambda(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}\gamma_{m+1} + (-1)^{m-2}\lambda(\lambda-2)\sum_{i=0}^{n-2}\left((\lambda-1)\gamma_{m-1}\gamma_{m}^{n-1-i}\right)^{i} \\ &= \lambda(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}\gamma_{m+1} + (-1)^{m-2}\lambda(\lambda-2)\gamma_m\frac{(\lambda-1)^{n-1}\gamma_{m-1}^{n-1}-\gamma_m^{n-1}}{(\lambda-1)\gamma_{m-1}-\gamma_m}.\end{aligned}$$

This completes the proof. ☐

Theorem 5.4.5. *For $m \geq 3$, $n \geq 2$ and $2 \leq l \leq n-1$, the zero-free chromatic polynomial of $B_l(m,n)$ is given by*

$$\overset{b}{\chi}_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\overset{b}{\chi}_{B^{uv}(m,l)}(\lambda).$$

Proof. Repeated use of edge deletion-contraction formula and Lemma 5.4.1 give that

$$\overset{b}{\chi}_{B_l(m,n)}(\lambda) = \frac{(\lambda-1)^{m-1}+(-1)^m}{\lambda}\overset{b}{\chi}_{B_l(m,n-1)}(\lambda) = \gamma_m\overset{b}{\chi}_{B_l(m,n-1)}(\lambda). \tag{5.4.2}$$

Applying the recursion of Equation (5.4.2) $(n-l-1)$ times, we have

$$\overset{b}{\chi}_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\overset{b}{\chi}_{B_l(m,l)}(\lambda).$$

Note that $B_l(m,l)$ is switching equivalent to $B^{uv}(m,l)$. Therefore we have

$$\overset{b}{\chi}_{B_l(m,n)}(\lambda) = \gamma_m^{n-l}\overset{b}{\chi}_{B^{uv}(m,l)}(\lambda).$$

This completes the proof. ☐

6

The rna Number of Generalized Petersen Graphs

Recall that $\sigma^{-}(G)$ denotes the rna number of a graph G. In this chapter, we prove that $3 \le \sigma^{-}(P_{n,k}) \le n$, and that these bounds are sharp. The value of $\sigma^{-}(P_{n,k})$ is also determined for $k \in \{1,2\}$. The number $\sigma^{-}(P_{n,k})$ is also calculated for some famous generalized Petersen graphs, namely, the Petersen graph $P_{5,2}$, Dürer graph $P_{6,2}$, Möbius-Kantor graph $P_{8,3}$, Dodecahedron $P_{10,2}$, Desargues graph $P_{10,3}$ and the Nauru graph $P_{12,5}$.

We show that for each positive integer n, the smallest order of $(4n-1)$-regular graphs having rna number 1 is bounded above by $12n-2$. The sharpness of this upper bound is established for $n = 1$ by showing that there is a unique cubic graph of order 10 whose rna number is 1. We also show that the smallest order of a $(4n+1)$-regular graph, whose rna number is 1, is $8n+6$.

6.1 Preliminaries

Throughout this chapter, all graphs are considered to be connected. Recall that the length of a shortest path joining the vertices x and y is called the *distance*, denoted $d(x,y)$, between x and y. The *k-th power* of a simple graph G is the graph G^k whose vertex set is $V(G)$ and two distinct vertices are adjacent in G^k if their distance in G is at most k.

An edge e of a graph G is said to be a *cut-edge* of G if deletion of e results in a disconnected graph. Recall that an *edge-cut* (or simply a *cut*) of G is a set of edges whose deletion results in a disconnected graph. If exactly one end vertex of each of the edges of a cut belongs to a subset A of vertices of G, then the cut is denoted by $[A, A^c]$. The *size* of the cut $[A, A^c]$ is the number of edges in $[A, A^c]$. A cut is said to be an *even cut* or an *odd cut* according as its size is even or odd, respectively. The numbers $|A|$ and $|A^c|$ are called the sides of the cut $[A, A^c]$. The *edge-connectivity* $\kappa'(G)$ of a graph G is the least size of its cuts. For a connected graph G with minimum degree δ, it is well known that $1 \le \kappa'(G) \le \delta$.

Now we give some necessary definitions and results.

Definition 6.1.1. *[4]* For a given graph G of order n and a bijective mapping $f : V(G) \longrightarrow \{1, \ldots, n\}$, define $\phi_f : E(G) \longrightarrow \{+1, -1\}$ such that $\phi_f(uv) = +1$ if $f(u)$ and $f(v)$ are of the same parity and $\phi_f(uv) = -1$ if $f(u)$ and $f(v)$ are of the different parity, where $uv \in E(G)$. We define Σ_f to be the signed graph (G, ϕ_f).

Definition 6.1.2. *[3]* A signed graph (G, ϕ) of order n is called a *parity signed graph* if there exists a bijective map $f : V(G) \longrightarrow \{1, \ldots, n\}$ such that $\phi = \phi_f$.

In [3, Theorem 6], the authors proved that a parity signed cycle is always balanced. Consequently, every parity signed graph is balanced, see [4, Theorem 1].

Definition 6.1.3. *[3]* The rna number of a graph G, denoted $\sigma^-(G)$, is the least number of negative edges among all possible parity signed graphs over G.

Note that finding the least number of negative edges among all parity signed graphs over a graph G is equivalent to finding the size of a minimum cut of G with nearly equal sides [3, 4]. More precisely, if G is of even order, then $\sigma^-(G)$ is the size of a minimum cut whose sides are equal. If G is of odd order, then $\sigma^-(G)$ is the size of a minimum cut whose sides differ by exactly 1.

Now we mention the rna number of some well known graphs.

Theorem 6.1.1. *[3] If P_n is the path of order n, then $\sigma^-(P_n) = 1$.*

Theorem 6.1.2. *[3] If C_n is the cycle on n vertices, then $\sigma^-(C_n) = 2$.*

Theorem 6.1.3. *[3] If $K_{1,n}$ is the star on $n+1$ vertices, then $\sigma^-(K_{1,n}) = \left\lceil \frac{n}{2} \right\rceil$.*

Theorem 6.1.4. *[4] If W_n is the wheel on $n+1$ vertices, then $\sigma^-(W_n) = \left\lceil \frac{n+4}{2} \right\rceil$.*

Theorem 6.1.5. *[3] If $n \geq 2$, then $\sigma^-(K_n) = \left\lfloor \frac{n^2}{4} \right\rfloor$.*

By Theorem 6.1.5, we have $\sigma^-(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$, where G is any graph on n vertices and $n \geq 4$. Note that this bound is a trivial upper bound for $\sigma^-(G)$. Recently, Kang et al. [36] proved that $\sigma^-(G) \leq \left\lfloor \frac{m}{2} + \frac{n}{4} \right\rfloor$, where G is any graph on m edges and n vertices.

6.2 Forbidden Cuts in $P_{n,k}$

Let $P_{n,k}$ be a generalized Petersen graph. The outer vertex u_i and the inner vertex v_i of $P_{n,k}$ are called the *partner* of each other for each $i \in \{0, 1, \ldots, n-1\}$. If $\gcd(n, k) = d > 1$, then $P_{n,k}$ has d inner cycles. These inner cycles are denoted by $C_1, \ldots, C_d$, where $v_i \in V(C_{i+1})$ for $i \in \{0, 1, \ldots, d-1\}$. For $d = 1$, $P_{n,k}$ has only one inner cycle, and in this case the inner cycle is denoted by C_I.

Recall that for a subset A of vertices of G, the *induced subgraph* $G[A]$ is the subgraph of G whose vertex set is A and edge set consists of all edges of G having both end vertices in A.

Lemma 6.2.1. *If* $n \geq 4$ *and* $k \geq 1$*, then* $P_{n,k}$ *cannot have a cut of size three of equal sides.*

Proof. We analyse two cases depending on whether n is odd or even.
Case 1. Let $n = 2l$ for $l \geq 2$. Let there exist a subset A of $V(P_{2l,k})$ such that $|A| = 2l$ and $|[A, A^c]| = 3$. Denote the degree of a vertex a in $P_{2l,k}[A]$ by $d_A(a)$. We have

$$\sum_{a \in A} d_A(a) = 3(2l) - 3, \text{ an odd integer.}$$

This shows that $P_{2l,k}[A]$ does not satisfy the handshaking lemma. Hence no such A is possible.
Case 2. Let $n = 2l + 1$ for $l \geq 2$. Let there exist a subset A of $V(P_{2l+1,k})$ such that $|A| = 2l + 1$ and $|[A, A^c]| = 3$. If A contains either all u-vertices or all v-vertices, then all the spokes are in $[A, A^c]$. This contradicts the fact that $|[A, A^c]| = 3$. Therefore, A must contain u-vertices as well as v-vertices. Consequently $[A, A^c]$ contains at least two edges of C_o, since u-vertices lie in both A and A^c.

Now we consider two sub-cases.

Sub-case 2(i). Let $\gcd(2l + 1, k) = 1$. In this sub-case, $P_{2l+1,k}$ has exactly one inner cycle C_I. The condition that the v-vertices lie in both A and A^c forces $[A, A^c]$ to contain at least two edges of C_I. Thus we have $|[A, A^c]| \geq 4$, a contradiction to the fact that $|[A, A^c]| = 3$.

Sub-case 2(ii). Let $\gcd(2l + 1, k) = d \geq 2$. In this sub-case, $P_{2l+1,k}$ has the inner cycles $C_1, \ldots, C_d$. Note that all vertices of C_i lie entirely in A or in A^c. Otherwise, we get a contradiction on the size of $[A, A^c]$. Therefore, the vertices of at least one inner cycle do not lie in A, and the vertices of at least one inner cycle do not lie in A^c. Hence both A and A^c contain at least three u-vertices.

It is easy to see that if $|[A, A^c]| = 3$, then exactly two edges of $[A, A^c]$ must be edges of C_o, and the third edge must be a spoke. Let this spoke be $u_j v_j$ for some $j \in \{0, 1, \ldots, 2l\}$. Without loss of generality, let $u_j \in A$ and $v_j \in A^c$. Since exactly one spoke lies in $[A, A^c]$, the remaining u-vertices of A must have their partners in A. Thus the number of u-vertices and v-vertices in A are $l + 1$ and l, respectively. Also, $[A, A^c]$ has exactly two edges of C_o. Therefore, there exists a path of length l induced by the u-vertices of A. Let the end vertices of this path be u_r and u_{r+l} for some $r \in \{0, 1, \ldots, 2l\}$. Consequently, the set of v-vertices in A is $\{v_r, v_{r+1}, \ldots, v_{r+l}\} \setminus \{v_j\}$ and the subgraph induced by these v-vertices of A must be union of some inner cycle(s).

The condition $2k < 2l+1$, together with $\gcd(2l+1,k) = d \geq 2$, imply that $3 \leq k \leq l$. For $v_{r+l} \neq v_j$, consider the inner cycle $v_{r+l}v_{r+l+k}\dots v_{r+l-k}v_{r+l}$ containing v_{r+l}. We see that $v_{r+l+k} \in \{v_r, v_{r+1}, \dots, v_{r+l}\}$ only if $k \geq l+1$. As $k \leq l$, we conclude that all the vertices of the inner cycle containing v_{r+l} do not lie in A. This is a contradiction to the fact that all vertices of C_i lie entirely in A or in A^c. If $v_{r+l} = v_j$, then we consider the inner cycle containing v_{r+l-1} and get a similar contradiction.

This completes this proof. ☐

Lemma 6.2.2. *If n is even and $n \geq 4$, then $P_{n,k}$ cannot have an odd cut of equal sides.*

Proof. Let $n = 2l$ for some $l \geq 2$. On the contrary, let $P_{n,k}$ have an odd cut of equal sides. Therefore, there exists a subset A of $V(P_{2l,k})$ such that $|A| = 2l$ and $|[A, A^c]| = 2r+1$ for some positive integer r.

If $d_A(a)$ is the degree of the vertex a in $P_{2l,k}[A]$, then

$$\sum_{a\in A} d_A(a) = 3(2l) - (2r+1), \text{ an odd integer.}$$

This is a contradiction to the handshaking lemma. Hence no odd cut of equal sides is possible in $P_{2l,k}$. This completes the proof. ☐

Lemma 6.2.3. *If n is odd and $n \geq 5$, then $P_{n,k}$ cannot have an even cut of equal sides.*

Proof. Let $n = 2l+1$ for some $l \geq 2$. Let, if possible, $P_{2l+1,k}$ have an even cut of equal sides. Therefore, there exists a subset A of $V(P_{2l+1,k})$ such that $|A| = 2l+1$ and $|[A, A^c]| = 2r$ for some positive integer r.

If $d_A(a)$ is the degree of the vertex a in $P_{2l+1,k}[A]$, then

$$\sum_{a\in A} d_A(a) = 3(2l+1) - 2r, \text{ an odd integer.}$$

This is a contradiction to the handshaking lemma. Hence no even cut of equal sides is possible in $P_{2l+1,k}$. This completes the proof. ☐

6.3 Main Results

According to Kang et al. [36], $\sigma^-(P_{n,k}) \leq 2n$. In this section, we prove that the rna number of $P_{n,k}$ is bounded above by n, improving the upper bound of Kang et al. [36] for the class of generalized Petersen graphs. Further, we compute $\sigma^-(P_{n,1})$ for $n \geq 3$, and $\sigma^-(P_{n,2})$ for $n \geq 6$. A simple but important result is the following.

Theorem 6.3.1. *If G is a graph of edge-connectivity k, then $\sigma^-(G) \geq k$.*

Proof. Since $\kappa'(G) = k$, no cut of G with nearly equal sides can have less than k edges. Hence $\sigma^-(G) \geq k$. $\square$

Theorem 6.3.2. *If* $n \geq 3$ *and* $k \geq 1$*, then* $3 \leq \sigma^-(P_{n,k}) \leq n$.

Proof. Since $\kappa'(P_{n,k}) = 3$, the lower bound follows from Theorem 6.3.1.

Define $f : V(P_{n,k}) \longrightarrow \{1, \ldots, 2n\}$ such that $f(u_i) = 2i + 1$ and $f(v_i) = 2i + 2$ for $0 \leq i \leq n - 1$. The labeling f induces the parity signed graph $(P_{n,k}, \phi_f)$. Note that all the spokes are negative in $(P_{n,k}, \phi_f)$. Thus, the number of negative edges in $(P_{n,k}, \phi_f)$ is n. Hence $\sigma^-(P_{n,k}) \leq n$. $\square$

Sharpness of the bounds in Theorem 6.3.2 follow from Theorem 6.3.5 and Example 6.4.1, in which we show that $\sigma^-(P_{3,1}) = 3$ and $\sigma^-(P_{5,2}) = 5$.

Theorem 6.3.3. *Let* $n \geq 5$ *and* $k \geq 2$*. If* $\gcd(n, k) = 1$*, then* $5 \leq \sigma^-(P_{n,k}) \leq n$.

Proof. The upper bound follows from Theorem 6.3.2.

For odd n, the lower bound follows from Lemma 6.2.1 and Lemma 6.2.3. Let n be an even integer. By Lemma 6.2.1, $\sigma^-(P_{n,k}) \geq 4$. Now we show that the rna number of $P_{n,k}$ cannot be 4.

Let, if possible, $P_{n,k}$ have a cut of size four of equal sides. Therefore, there exists a subset A of $V(P_{n,k})$ such that $|A| = n$ and $|[A, A^c]| = 4$. If A contains either all u-vertices or all v-vertices, then $[A, A^c]$ contains precisely n spokes of $P_{n,k}$, that is, $|[A, A^c]| = n \geq 5$, a contradiction to the fact that $|[A, A^c]| = 4$. Therefore, A must contain u-vertices as well as v-vertices.

Since $\gcd(n, k) = 1$, there is only one inner cycle in $P_{n,k}$. Note that $[A, A^c]$ must contain exactly two edges of C_o and two edges of C_I, because A (and also A^c) contains vertices of both C_o and C_I. Also, A contains as many u-vertices as v-vertices. Otherwise, $[A, A^c]$ will contain at least one spoke, giving $|[A, A^c]| \geq 5$. Further, the condition that $[A, A^c]$ contains exactly two edges of C_I enforces the v-vertices of A to induce a path of order $\frac{n}{2}$. Let this path be given by $P := v_r v_{r+k} \ldots v_{r+(\frac{n}{2}-1)k}$ for some $r \in \{0, \ldots, n - 1\}$.

Similarly, all u-vertices of A induce a path of order $\frac{n}{2}$. Let this path be given by $Q := u_j u_{j+1} \ldots u_{j+(\frac{n}{2}-1)}$ for some $j \in \{0, \ldots, n - 1\}$. Since $k \geq 2$, at least one vertex of Q cannot have its partner among the vertices of P. Hence at least two spokes belong to $[A, A^c]$. Therefore, $|[A, A^c]| \geq 6$, a contradiction. This completes the proof. $\square$

The sharpness of the lower bound in Theorem 6.3.3 is followed by Example 6.4.1.

Theorem 6.3.4. *Let* n *be an even integer and* k *be an odd integer such that* $n \geq 8$ *and* $k \geq 3$*. If* $\gcd(n, k) = 1$*, then* $6 \leq \sigma^-(P_{n,k}) \leq n$.

Proof. The result follows from Theorem 6.3.3 and Lemma 6.2.2. ☐

The sharpness of the lower bound in Theorem 6.3.4 is followed by Example 6.4.3.

6.3.1 The rna Number of $P_{n,1}$

In this section, we determine $\sigma^{-}(P_{n,1})$ for $n \geq 3$.

Theorem 6.3.5. *If $n \geq 3$, then*

$$\sigma^{-}(P_{n,1}) = \begin{cases} 3 & \text{if } n = 3 \\ 4 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd and } n \geq 5. \end{cases}$$

Proof. Clearly, $\sigma^{-}(P_{3,1}) \geq 3$. Define the mapping $f : V(P_{3,1}) \longrightarrow \{1, \ldots, 6\}$ such that $f(u_i) = 2i + 1$ and $f(v_i) = 2i + 2$ for $0 \leq i \leq 2$. Clearly, all the spokes of $P_{3,1}$ are negative and all the edges of the C_o and C_I are positive in $(P_{3,1}, \phi_f)$. Thus $\sigma^{-}(P_{3,1}) = 3$.

For $n \geq 4$, we analyze two cases depending on whether n is even or odd.

Case 1. Let $n = 2l$ for $l \geq 2$. By Lemma 6.2.1, we have $\sigma^{-}(P_{2l,1}) \geq 4$. To show that $\sigma^{-}(P_{2l,1}) = 4$, we produce a parity signed $P_{2l,1}$ that contains exactly four negative edges. Let $f : V(P_{2l,1}) \longrightarrow \{1, \ldots, 4l\}$ be defined by

$$f(u_i) = \begin{cases} 2i + 2 & \text{for } 0 \leq i \leq l - 1 \\ 2i - 2l + 1 & \text{for } l \leq i \leq 2l - 1 \end{cases}$$

and

$$f(v_i) = \begin{cases} 2i + 2l + 2 & \text{for } 0 \leq i \leq l - 1 \\ 2i + 1 & \text{for } l \leq i \leq 2l - 1. \end{cases}$$

Let $A = \{u_l, \ldots, u_{2l-1}, v_l, v_{l+1}, \ldots, v_{2l-1}\}$ and $B = \{u_0, \ldots, u_{l-1}, v_0, \ldots, v_{l-1}\}$. Hence, every edge of $P_{2l,1}$ having both end vertices in A (or in B) is positive in $(P_{2l,1}, \phi_f)$. Consequently, all edges of $P_{2l,1}$ belonging to $[A, B]$ get a negative sign in $(P_{2l,1}, \phi_f)$. Clearly, $[A, B] = \{u_{2l-1}u_0, u_{l-1}u_l, v_{2l-1}v_0, v_{l-1}v_l\}$. Hence $\sigma^{-}(P_{2l,1}) = 4$.

Case 2. Let $n = 2l + 1$ for $l \geq 2$. By Lemma 6.2.1 and Lemma 6.2.3, we have $\sigma^{-}(P_{2l+1,1}) \geq 5$. Now we produce a parity signed graph over $P_{2l+1,1}$ having exactly five negative edges.

Let $f : V(P_{2l+1,1}) \longrightarrow \{1, \ldots, 4l + 2\}$ be defined by

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } 0 \leq i \leq l \\ 2i - 2l & \text{for } l + 1 \leq i \leq 2l \end{cases}$$

and

$$f(v_i) = \begin{cases} 2i + (2l + 3) & \text{for } 0 \leq i \leq l - 1 \\ 2i + 2 & \text{for } l \leq i \leq 2l. \end{cases}$$

Let $A = \{u_0, \ldots, u_l, v_0, \ldots, v_{l-1}\}$ and $B = \{u_{l+1}, \ldots, u_{2l}, v_l, \ldots, v_{2l}\}$. Note that $[A, B] = \{u_l u_{l+1}, u_{2l} u_0, u_l v_l, v_{l-1} v_l, v_{2l} v_0\}$. Each edge of $P_{2l+1,1}$, except these five edges of $[A, B]$, is positive in the parity signed graph $(P_{2l+1,1}, \phi_f)$. Consequently, the number of negative edges in $(P_{2l+1,1}, \phi_f)$ is five. Hence $\sigma^-(P_{2l+1,1}) = 5$. This completes the proof. □

6.3.2 The rna Number of $P_{n,2}$

In this section, our aim is to prove the following two results.

Theorem 6.3.6. *If $l \geq 3$, then $\sigma^-(P_{2l+1,2}) = 7$.*

Theorem 6.3.7. *If $l \geq 4$, then $\sigma^-(P_{2l,2}) = 6$.*

In the light of Lemma 6.2.3, it is clear that the rna number of $P_{2l+1,2}$ cannot be 4 or 6 for $l \geq 3$.

Lemma 6.3.1. *If $l \geq 3$, then the rna number of $P_{2l+1,2}$ cannot be 5.*

Proof. We prove that no cut of size five of equal sides is possible in $P_{2l+1,2}$.

On the contrary, let $P_{2l+1,2}$ have a cut of size five of equal sides. Therefore, there exists a subset A of $V(P_{2l+1,2})$ such that $|A| = 2l + 1$ and $|[A, A^c]| = 5$. If A contains either all u-vertices or all v-vertices, then the set of all spokes of $P_{2l+1,2}$ constitute $[A, A^c]$. This gives $|[A, A^c]| \geq 7$, a contradiction. Therefore, A contains some u-vertices as well as v-vertices.

Since $\gcd(2l+1, 2) = 1$, the graph $P_{2l+1,2}$ has only one inner cycle induced by the v-vertices. Also, A contains both u-vertices and v-vertices. Therefore, $[A, A^c]$ must consist of two edges of C_o, two edges of C_I and one spoke. Let this spoke be $u_j v_j$ for some $j \in \{0, \ldots, 2l\}$. Without loss of generality, let $u_j \in A$ and $v_j \in A^c$. The conditions that $|A| = 2l + 1$, $u_j \in A$ and that $[A, A^c]$ contains exactly one spoke, together imply that the number of u-vertices and v-vertices in A are $l + 1$ and l, respectively.

Consequently, there exists a path P of order $l + 1$ induced by the u-vertices of A. Similarly, there exists a path Q of order l induced by the v-vertices of A. Let the paths P and Q be $u_r u_{r+1} \ldots u_{r+l-1} u_{r+l}$ and $v_s v_{s+2} \ldots v_{s+(2l-2)}$, respectively, for some $r, s \in \{0, 1, \ldots, 2l\}$.

It is easy to check that at least one vertex of Q cannot have its partner among the vertices of P. This means at least one v-vertex of A has its partner in A^c, and consequently at least two spokes lie in $[A, A^c]$. This gives $|[A, A^c]| \geq 6$, a contradiction to the assumption that $|[A, A^c]| = 5$. This establishes the lemma. $\square$

Proof of Theorem 6.3.6. By Theorem 6.3.3, Lemma 6.2.3 and Lemma 6.3.1, we have $\sigma^-(P_{2l+1,2}) \geq 7$. To complete the proof, we produce a parity signed graph over $P_{2l+1,2}$ with exactly seven negative edges.

Define $f : V(P_{2l+1,2}) \longrightarrow \{1, \ldots, 4l+2\}$ by

$$f(u_i) = \begin{cases} 2i+1 & \text{for } i = 0, \ldots, l \\ 2i-2l & \text{for } i = l+1, \ldots, 2l \end{cases}$$

and

$$f(v_i) = \begin{cases} 4l+2 & \text{for } i = 0 \\ 2l+(2i+1) & \text{for } i = 1, \ldots, l \\ 2i & \text{for } i = l+1, \ldots, 2l. \end{cases}$$

Let A and B be the sets of vertices of $P_{2l+1,2}$ labeled with odd and even integers, respectively. Thus $A = \{u_0, \ldots, u_l, v_1, \ldots, v_l\}$, $B = \{u_{l+1}, \ldots, u_{2l}, v_0, v_{l+1}, \ldots, v_{2l}\}$ and $|A| = |B| = 2l+1$. Observe that the set of negative edges in $(P_{2l+1,2}, \phi_f)$ is $[A, B]$ and $[A, B] = \{u_0v_0, u_0u_{2l}, u_lu_{l+1}, v_0v_2, v_{2l}v_1, v_{l-1}v_{l+1}, v_lv_{l+2}\}$. Thus the number of negative edges in $(P_{2l+1,2}, \phi_f)$ is seven. Hence $\sigma^-(P_{2l+1,2}) = 7$. $\square$

Lemma 6.3.2. *If $l \geq 4$, then $P_{2l,2}$ cannot have a cut of size four of equal sides.*

Proof. On the contrary, let $P_{2l,2}$ have a cut of size four with equal sides. Therefore, there exists a subset A of $V(P_{2l,2})$ such that $|A| = 2l$ and $|[A, A^c]| = 4$. Clearly, A must contain some u-vertices as well as some v-vertices.

Note that the graph $P_{2l,2}$ has only two inner cycles given by $C_1 = v_0v_2 \ldots v_{2l-2}v_0$ and $C_2 = v_1v_3 \ldots v_{2l-1}v_1$. Since the u-vertices lie in both A and A^c, the cut $[A, A^c]$ must contain at least two edges of C_o. Thus the following are the only possible choices for the edges of $[A, A^c]$.

1. All four edges of $[A, A^c]$ are in $E(C_o)$.
2. Two edges of $[A, A^c]$ are in $E(C_o)$ and the remaining two edges are spokes.
3. Two edges of $[A, A^c]$ are in $E(C_o)$ and the remaining two edges are in one of the inner cycles.

Case 1. Let $[A, A^c]$ have four edges of C_o. Thus $V(C_1) \subseteq A$, $V(C_2) \subseteq A^c$ or $V(C_2) \subseteq A$, $V(C_1) \subseteq A^c$. Without loss of generality, assume that $V(C_1) \subseteq A$ and

$V(C_2) \subseteq A^c$. Since $[A, A^c]$ contains no spoke, we have

$$A = \{u_0, u_2, \ldots, u_{2l-2}\} \cup \{v_0, v_2, \ldots, v_{2l-2}\} \text{ and}$$
$$A^c = \{u_1, u_3, \ldots, u_{2l-1}\} \cup \{v_1, v_3, \ldots, v_{2l-1}\}.$$

Thus, $[A, A^c]$ contains all the edges of C_o. That is, $|[A, A^c]| = 2l \geq 8$, a contradiction.

Case 2. Let $[A, A^c]$ have two edges of C_o and two spokes. Since $[A, A^c]$ contains no edge of the inner cycles, without loss of generality, assume that $\{v_0, v_2, \ldots, v_{2l-2}\} \subseteq A$ and $\{v_1, v_3, \ldots, v_{2l-1}\} \subseteq A^c$.

Recall that $[A, A^c]$ contains exactly two spokes. Consequently, the set of u-vertices in A is $\{u_r\} \cup \{u_0, u_2, \ldots, u_{2l-2}\} \setminus \{u_j\}$ for some $r \in \{1, 3, \ldots, 2l-1\}$ and $j \in \{0, 2, \ldots, 2l-2\}$. Since $l \geq 4$, it is easy to check that $[A, A^c]$ will contain at least four edges of C_o, a contradiction.

Case 3. Let $[A, A^c]$ have two edges of C_o and two edges of one of the inner cycles. Without loss of generality, assume that the two edges of C_1 lie in $[A, A^c]$. Thus both A and A^c contain at least one vertex of C_1. Clearly, all the vertices of C_2 lie entirely in A or in A^c. Hence either A or A^c contains at least $(l+1)$ v-vertices. If A contains at least $(l+1)$ v-vertices, then the number of u-vertices in A is at most $l-1$. This observation shows that at most $(l-1)$ u-vertices of A can have their partners in A. Hence at least two spokes belong to $[A, A^c]$, contradicting our assumption that $[A, A^c]$ contains no spoke. Similarly, we arrive at a contradiction if A^c contain at least $(l+1)$ v-vertices.

The contradictions in the preceding cases establish the lemma. □

Proof of Theorem 6.3.7. By Lemma 6.2.2 and Lemma 6.3.2, it is clear that $\sigma^-(P_{2l,2}) \geq 6$. Define $f : V(P_{2l,2}) \longrightarrow \{1, \ldots, 4l\}$ by

$$f(u_i) = \begin{cases} 2i+1 & \text{for } i \in \{0, \ldots, l-1\} \\ 2i-2l+2 & \text{for } i \in \{l, \ldots, 2l-1\} \end{cases}$$

and

$$f(v_i) = \begin{cases} 2l+2i+1 & \text{for } i \in \{0, \ldots, l-1\} \\ 2i+2 & \text{for } i \in \{l, \ldots, 2l-1\}. \end{cases}$$

Let $A = \{u_0, u_1, \ldots, u_{l-1}, v_0, v_1, \ldots, v_{l-1}\}$ and $B = \{u_l, u_{l+1}, \ldots, u_{2l-1}, v_l, \ldots, v_{2l-1}\}$. Clearly, the edges of $P_{2l,2}$ joining the vertices of A and B are the only negative edges in $(P_{2l,2}, \phi_f)$. Note that $[A, B] = \{u_0u_{2l-1}, u_{l-1}u_l, v_{2l-2}v_0, v_{2l-1}v_1, v_{l-2}v_l, v_{l-1}v_{l+1}\}$, and so $|[A, B]| = 6$. This proves that $\sigma^-(P_{2l,2}) = 6$. □

6.4 The rna Number of Some Well Known Generalized Petersen Graphs

In this section, we compute the rna number of some well known generalized Petersen graphs, namely, the Petersen graph, Dürer graph, Möbius-Kantor graph, Dodecahedron, Desargues graph and the Nauru graph.

Example 6.4.1. The generalized Petersen graph $P_{5,2}$ is known as the *Petersen* graph. Since $\gcd(5,2) = 1$, Theorem 6.3.3 gives $\sigma^{-}(P_{5,2}) \geq 5$. Label the vertices of $P_{5,2}$ by the map $f : V(P_{5,2}) \longrightarrow \{1,\dots,10\}$ such that $f(u_i) = 2i + 1$ and $f(v_i) = 2i + 2$ for $0 \leq i \leq 4$. Thus, all the spokes of $P_{5,2}$ are negative while other edges of $P_{5,2}$ are positive in $(P_{5,2}, \phi_f)$. Hence the rna number of the Petersen graph is 5. That is, $\sigma^{-}(P_{5,2}) = 5$.

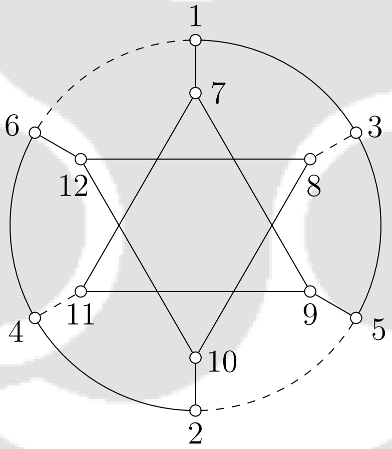


Figure 6.4.1. A parity signed Dürer graph with 4 negative edges

Example 6.4.2. The generalized Petersen graph $P_{6,2}$ is known as the Dürer graph. It is depicted in Figure 6.4.1. By Lemma 6.2.1, we have $\sigma^{-}(P_{6,2}) \geq 4$. Let the map $f : V(P_{6,2}) \longrightarrow \{1,\dots,12\}$ be defined by $f(v_i) = i + 7$ for $i \in \{0,\dots,5\}$ and

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i \in \{0,1,2\} \\ 2i - 4 & \text{for } i \in \{3,4,5\}. \end{cases}$$

This vertex labeling of $P_{6,2}$ is described in Figure 6.4.1. Clearly, $(P_{6,2}, \phi_f)$ have exactly four negative edges. Hence the rna number of the Dürer graph is 4.

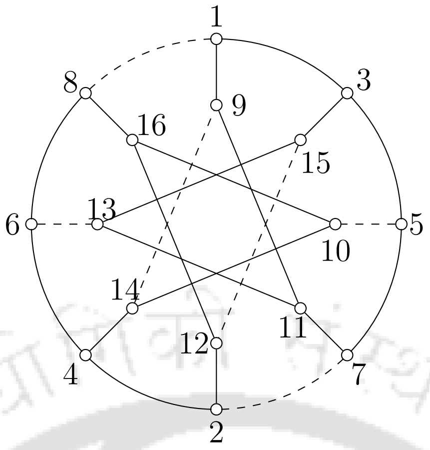


Figure 6.4.2. A parity signed Möbius-Kantor graph with 6 negative edges

Example 6.4.3. The generalized Petersen graph $P_{8,3}$ is known as the *Möbius-Kantor* graph. A parity signed Möbius-Kantor is depicted in Figure 6.4.2. By Theorem 6.3.4, we have $\sigma^{-}(P_{8,3}) \geq 6$. Let $f : V(P_{8,3}) \longrightarrow \{1, \ldots, 16\}$ be defined by $f(v_0) = 9$, $f(v_1) = 15$, $f(v_2) = 10$, $f(v_3) = 11$, $f(v_4) = 12$, $f(v_5) = 14$, $f(v_6) = 13$, $f(v_7) = 16$, and

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i \in \{0, 1, 2, 3\} \\ 2i - 6 & \text{for } i \in \{4, 5, 6, 7\}. \end{cases}$$

The vertex labeling f is shown in Figure 6.4.2. We see that $(P_{8,3}, \phi_f)$ has exactly six negative edges. Hence the rna number of Möbius-Kantor graph is 6.

Example 6.4.4. The generalized Petersen graph $P_{10,2}$ is known as the *Dodecahedron*. By Theorem 6.3.7, we get $\sigma^{-}(P_{10,2}) = 6$. The parity signed labeling of $P_{10,2}$ described in Theorem 6.3.7 is depicted in Figure 6.4.3.

Example 6.4.5. The generalized Petersen graph $P_{10,3}$ is known as the *Desargues* graph. Since $\gcd(10, 3) = 1$, Theorem 6.3.4 gives $\sigma^{-}(P_{10,3}) \geq 6$. Let the mapping $f : V(P_{10,3}) \longrightarrow \{1, \ldots, 20\}$ be defined by $f(v_0) = 11$, $f(v_1) = 13$, $f(v_2) = 12$, $f(v_3) = 15$, $f(v_4) = 17$, $f(v_5) = 14$, $f(v_6) = 16$, $f(v_7) = 19$, $f(v_8) = 18$, $f(v_9) = 20$, and

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i \in \{0, 1, 2, 3, 4\} \\ 2i - 8 & \text{for } i \in \{5, 6, 7, 8, 9\}. \end{cases}$$

This vertex labeling is shown in Figure 6.4.4. It is clear that $(P_{10,3}, \phi_f)$ has exactly six negative edges. Hence the rna number of the Desargues graph is 6.

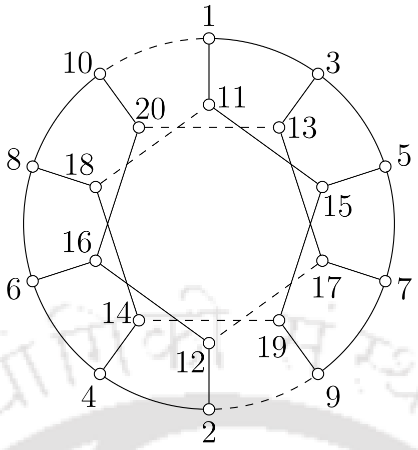


Figure 6.4.3. A parity signed Dodecahedron graph with 6 negative edges

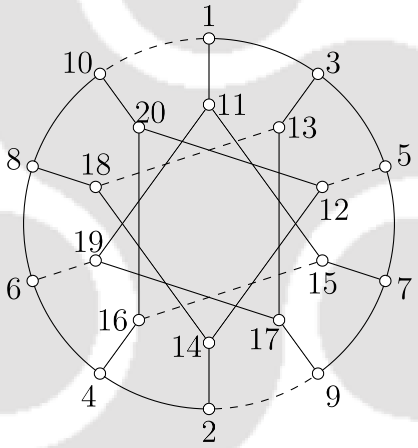


Figure 6.4.4. A parity signed Desargues graph with 6 negative edges

The generalized Petersen graph $P_{12,5}$ is known as the *Nauru* graph. The Nauru graph is depicted in Figure 6.4.5.

Lemma 6.4.1. *The* rna *number of Nauru graph is at least 8.*

Proof. Due to Theorem 6.3.4, we have $\sigma^{-}(P_{12,5}) \geq 6$. Also, Lemma 6.2.2 shows that the rna number of $P_{12,5}$ cannot be 7. Thus it remains to prove that $\sigma^{-}(P_{12,5}) \neq 6$.

Let, if possible, $P_{12,5}$ have a cut of size six of equal sides. Therefore, there exists a subset A of $V(P_{12,5})$ such that $|A| = 12$ and $|[A, A^c]| = 6$. Note that A must contain u-vertices as well as v-vertices. Since $\gcd(12, 5) = 1$, there is only one inner cycle in $P_{12,5}$. Thus the cut $[A, A^c]$ will contain even number of edges from each of C_o and C_I. We analyze three cases.

Case 1. Let $[A, A^c]$ contain four edges of C_o and two edges of C_I. Since $[A, A^c]$

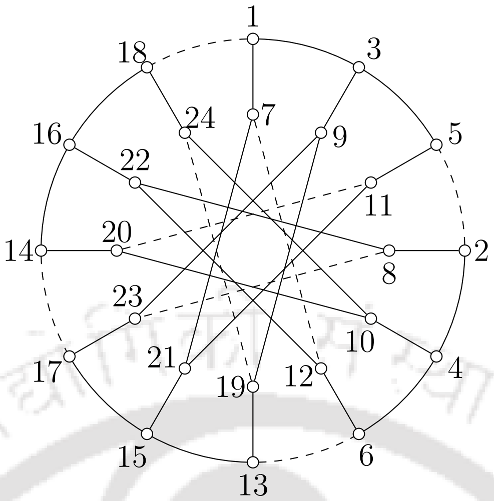


Figure 6.4.5. A parity signed Nauru graph with 8 negative egdes

contains no spoke, A must have six u-vertices and six v-vertices. Again, since exactly two edges of C_I are in $[A, A^c]$, all v-vertices of A induce a path of order six. Thus, the set of v-vertices of A is $\{v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}, v_{r+1}\}$ for some $r \in \{0, 1, \ldots, 11\}$. Hence the set of u-vertices of A is $\{u_r, u_{r+5}, u_{r+10}, u_{r+3}, u_{r+8}, u_{r+1}\}$. Thus $[A, A^c]$ contains the edges $u_{r+1}u_{r+2}, u_{r+2}u_{r+3}, u_{r+3}u_{r+4}, u_{r+4}u_{r+5}$, $u_{r+5}u_{r+6}$, $u_{r+7}u_{r+8}$, $u_{r+8}u_{r+9}$, $u_{r+9}u_{r+10}$, $u_{r+10}u_{r+11}$ and $u_{r+11}u_r$ for some $r \in \{0, \ldots, 11\}$. Hence $|[A, A^c]| \geq 12$, a contradiction.

Case 2. Let $[A, A^c]$ contain two edges of C_0 and four edges of C_I. In this case also, we proceed as in Case 1. We find that the u-vertices of A must be $u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}$ and u_{j+5} for some $j \in \{0, 1, \ldots, 11\}$. Accordingly, the v-vertices of A are $v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}$ and v_{j+5}. This gives at least 12 edges in $[A, A^c]$, a contradiction.

Case 3. Let $[A, A^c]$ contain two edges of C_o, two edges of C_I and two spokes. In this case, A (or A^c) cannot have more than seven u-vertices or seven v-vertices. Otherwise, the number of spokes in $[A, A^c]$ will exceed 2 and a contradiction will occur. Note that if A has less than five u-vertices, then A^c will have more than seven u-vertices. Further, $[A^c, A] = [A, A^c]$. Therefore, if A has less than five u-vertices, then the cut $[A^c, A]$ will have more than 2 spokes, a contradiction. Thus the possible number of u-vertices of A are five, six and seven only. Also, if A has five u-vertices, then A^c has seven u-vertices. Due to $[A^c, A] = [A, A^c]$ again, the cases that A has five u-vertices and A has seven u-vertices are similar. Thus we consider two sub-cases depending on whether A has seven u-vertices or six u-vertices.

Sub-case 3(i). Assume that A has seven u-vertices and five v-vertices. Since the cut $[A, A^c]$ contain only two edges from each of C_o and C_I, these seven u-vertices and five v-vertices must induce paths of order 7 and 5, respectively. Hence the set A must be $\{u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}, u_{j+5}, u_{j+6}, v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}\}$ for some $j, r \in \{0, 1, \ldots, 11\}$. Observe that these five v-vertices of A cannot be adjacent to five u-vertices of A, for any j and r. That is, at most four v-vertices of A have their partners in A. Therefore, at least one v-vertex and at least three u-vertices of A must have their partners in A^c. Consequently, $[A, A^c]$ will contain at least four spokes, a contradiction.

Sub-case 3(ii). Let A have six u-vertices and six v-vertices. Since $[A, A^c]$ contains only two edges from each of C_o and C_I, the u-vertices and v-vertices of A form two paths of order six. Thus there are some $j, r \in \{0, 1, \ldots, 11\}$ such that $A = \{u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}, u_{j+5}, v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}, v_{r+1}\}$. For any j and r, observe that at most four v-vertices of A can have their partners in A. Therefore, $[A, A^c]$ will contain at least four spokes, a contradiction.

From these cases, we conclude that $\sigma^-(P_{12,5}) \geq 8$. □

Example 6.4.6. By Lemma 6.4.1, we have $\sigma^-(P_{12,5}) \geq 8$. Define the mapping $f : V(P_{12,5}) \longrightarrow \{1, \ldots, 24\}$ by

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i \in \{0, 1, 2, 6, 7, 8\} \\ 2i - 4 & \text{for } i \in \{3, 4, 5, 9, 10, 11\} \end{cases}$$

and

$$f(v_i) = \begin{cases} 2i + 7 & \text{for } i \in \{0, 1, 2, 6, 7, 8\} \\ 2i + 2 & \text{for } i \in \{3, 4, 5, 9, 10, 11\}. \end{cases}$$

This vertex labeling is shown in Figure 6.4.5. Clearly, $(P_{12,5}, \phi_f)$ has exactly eight negative edges. Hence the rna number of Nauru graph is 8.

6.5 Regular Graphs with rna Number 1

Obvious lower and upper bounds on the rna number of a graph G are 1 and m, respectively, where m is the size of G. It is shown in [3, Proposition 4] that the rna number of a path of order n is 1, where $n \geq 2$. Acharya et al. [4] characterized the structure of those graphs whose rna number is 1. More precisely, we have the following theorem.

Theorem 6.5.1. *[4, Theorem 3.5] For any connected graph G, $\sigma^-(G) = 1$ if and only if G has a cut-edge joining two graphs whose orders differ by at most 1.*

An *even regular* graph is a regular graph in which every vertex has even degree. Similarly, an *odd regular* graph is a regular graph in which every vertex has odd degree. Since an even regular graph cannot have a cut-edge, in light of Theorem 6.5.1, the rna number of an even graph is at least 2. Therefore, the following problem is worth exploring.

Problem 6.5.1. *If k is odd and $k \geq 3$, then what is the smallest order of a k-regular graph whose rna number is 1?*

In this section, we find a solution to this problem. Note that an odd positive integer can be written as $4n-1$ or $4n+1$ for some n. We consider these two cases separately. First, we construct a $(4n-1)$-regular graph on $12n-2$ vertices with a cut-edge joining two graphs of order $6n-1$ each. Recall that the *k-th power* of G is the graph G^k whose vertex set is $V(G)$, and two distinct vertices are adjacent in G^k if their distance in G is at most k.

Lemma 6.5.1. *For each positive integer n, there exists a $(4n-1)$-regular graph on $12n-2$ vertices with a cut-edge joining two graphs of order $6n-1$ each.*

Proof. Consider the cycle C_{6n-1} such that $V(C_{6n-1}) = \{v_i : i \in [6n-3]\}$ and $E(C_{6n-1}) = \{v_iv_{i+1} : i \in [6n-3]\}$, where the subscripts are read modulo $6n-1$. Construct the power graph C_{6n-1}^{2n-1} from C_{6n-1}. Note that the degree of each vertex of C_{6n-1}^{2n-1} is $4n-2$. Now for each $i \in \{1, \ldots, 3n-1\}$, insert an edge between v_i and $v_{i+(3n-1)}$ in C_{6n-1}^{2n-1}, and denote this new graph by G_r. Clearly the order of G_r is $6n-1$, the degree of v_0 in G_r is $4n-2$, and the degree of all other vertices in G_r is $4n-1$. Now take two disjoint copies of G_r and join the vertices corresponding to v_0 by an edge. This resulting graph is the required graph. ☐

Lemma 6.5.2. *A cubic graph of order four cannot have rna number 1.*

Proof. The only cubic graph on four vertices is K_4, which does not have a cut-edge. Hence the result follows by Theorem 6.5.1. ☐

Lemma 6.5.3. *A cubic graph of order six cannot have rna number 1.*

Proof. The only two non-isomorphic cubic graphs of order six are shown in Figure 6.5.1. Clearly, none of these graphs contain a cut-edge. Thus, in light of Theorem 6.5.1, the result follows. ☐

Lemma 6.5.4. *A cubic graph of order eight cannot have rna number 1.*

Proof. There are five non-isomorphic cubic graphs of order eight that are depicted in Figure 6.5.2. It is clear that none of these graphs contain a cut-edge. Hence by Theorem 6.5.1, the result follows. ☐

Figure 6.5.1. Non-isomorphic cubic graphs of order 6

Lemma 6.5.5. *The smallest order of a parity signed cubic graph having* rna *number* 1 *is* 10.

Proof. By Lemmas 6.5.2, 6.5.3 and 6.5.4, we know that the order of a cubic graph having rna number 1 is at least 10. Let Σ be the parity signed cubic graph as shown in Figure 6.5.3. Clearly, it is a cubic graph of order 10, and it has a cut-edge joining two graphs of the same order. Thus by Theorem 6.5.1, we have $\sigma^{-1}(\Sigma) = 1$. This completes the proof. □

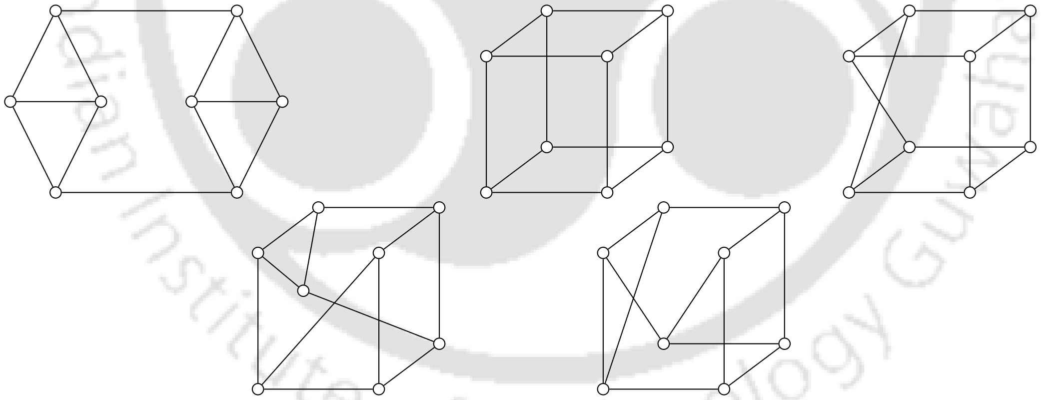


Figure 6.5.2. Non-isomorphic cubic graphs of order 8

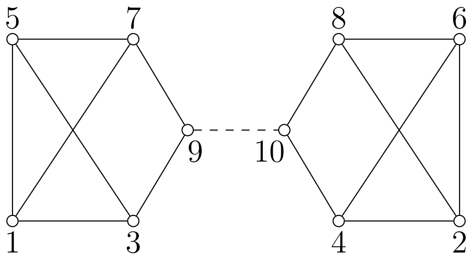


Figure 6.5.3. A parity signed cubic graph of order 10 with exactly 1 negative edge

Theorem 6.5.2. *For each positive integer n, the smallest order of $(4n-1)$-regular graphs having* rna *number 1 is bounded above by $12n-2$. Moreover, this bound is sharp for $n=1$.*

Proof. The proof follows from Lemma 6.5.1 and Lemma 6.5.5. □

Now for each positive integer n, we construct a $(4n+1)$-regular graph on $8n+6$ vertices with a cut-edge joining two graphs of order $4n+3$ each.

Lemma 6.5.6. *For each positive integer n, there exists a $(4n+1)$-regular graph on $8n+6$ vertices with a cut-edge joining two graphs of order $4n+3$ each.*

Proof. Consider the cycle C_{4n+3} such that $V(C_{4n+3}) = \{v_i : i \in [4n+3]\}$ and $E(C_{4n+3}) = \{v_i v_{i+1} : i \in [4n+3]\}$, where the subscripts are read modulo $4n+3$. Construct the power graph C_{4n+3}^{2n} from C_{4n+3}. Note that the degree of each vertex of C_{4n+3}^{2n} is $4n$. Now for each $i \in \{1, \ldots, 2n+1\}$, insert an edge between v_i and $v_{i+(2n+1)}$ in C_{4n+3}^{2n}, and denote this new graph by G_s. Clearly, the order of G_s is $4n+3$ and the degree of v_0 is $4n$, while the degree of all other vertices of G_s is $4n+1$. Now take two disjoint copies of G_s and join the vertices corresponding to v_0 by an edge. This resulting graph is the required graph. □

Theorem 6.5.3. *For each positive integer n, the smallest order of a $(4n+1)$-regular graph having* rna *number 1 is $8n+6$.*

Proof. Let G be a $(4n+1)$-regular graph having rna number 1. By Theorem 6.5.1, G must be obtained by joining two graphs H_1 and H_2 of equal order by a cut-edge. Therefore, exactly one vertex of H_1 (and also of H_2) must have degree $4n$ in H_1 (and in H_2), while the degree of the remaining vertices of H_1 (and also of H_2) must be $4n+1$. Thus the order of H_1 (and of H_2) must be at least $4n+3$. Note that the graph constructed in Lemma 6.5.6 satisfies these properties. Hence the smallest order of a $(4n+1)$-regular graph having rna number 1 is $8n+6$. □

7

Conclusion and Extension

In Chapter 2, we classified all switching isomorphism classes of signed graphs arising from the complete graph K_6, the generalized Petersen graphs $P_{3,1}$, $P_{5,1}$ and $P_{7,1}$, and the book graph $B(m,n)$ for $m \geq 3, n \geq 1$.

We also proved that every switching equivalence class of $P_{2n+1,1}$ contains a minimum signature of size at most $n+1$. We obtained the number of switching non-isomorphic signatures (matchings) of size two in $P_{2n+1,1}$ for each $n \in \mathbb{N}$. Hence the following problem is interesting.

Problem 7.0.1. *Can one find the number of non-isomorphic matchings of size* k *in* $P_{2n+1,1}$*, where* $n \geq 4$ *and* $3 \leq k \leq n+1$.

In Chapter 3, we studied frustration index of signed graphs. Particularly, our focus was on the class of signed generalized Petersen graphs. We obtained the value of $D(P_{n,k})$, where $k \in \{1,2\}$ and $\gcd(n,k) = 1$. Among the few questions raised from the discussion in Chapter 3, the following can be of particular interest.

Problem 7.0.2. *What is the value of* $D(P_{n,k})$*, where* $k \geq 3$ *and* $\gcd(n,k) = 1$*?*

Problem 7.0.3. *Can a new bound on* $D(P_{n,k})$ *be obtained in terms of* n *and* d*, where* $d = \gcd(n,k)$*, such that the bound gives* $\left\lfloor \frac{n}{2} \right\rfloor + 1$ *when evaluated at* $d = 1$.

Theorem 3.1.2 gives $l[G, E_-] \leq \frac{n}{2}$, where (G, E_-) is any signed cubic graph on n vertices. Can this bound be improved? In [49], the author gives a family of signed cubic graphs to show that without any additional restrictions on the underlying graph, the bound cannot be improved. For example, consider the disjoint union of k copies of K_4 with two disjoint negative edges in each component. However, if we restrict to connected graphs, then the following natural questions can be asked.

Problem 7.0.4. *Can the bound of* $\frac{n}{2}$ *on* $l[G, E_-]$ *be improved for cubic connected signed graphs* (G, E_-) *on* n *vertices?*

Problem 7.0.5. *Can the bound of* $\frac{n}{2}$ *on* $l[G, E_-]$ *be improved for cubic connected signed bipartite graphs* (G, E_-) *on* n *vertices?*

Problem 7.0.6. *Can the bound of* $\frac{n}{2}$ *on* $l[G, E_-]$ *be improved for cubic planar connected signed graphs* (G, E_-) *on* n *vertices?*

The study of double domination of signed generalized Petersen graphs occupied the bulk of Chapter 4. In Chapter 4, we also computed the DDN of different signed Petersen graphs and of different signed complete graphs on 6 vertices. It was also proved that the difference between the DDN of a complete graph and the DDN of a signed complete graph on n vertices can be $n - 3$. On the basis of the study of double domination in Chapter 4, we pose the following problems.

Problem 7.0.7. *Does there exist a connected signed graph on* n *vertices with double domination number* k*, where* $2 \leq k \leq n$*?*

If we restrict Problem 7.0.7 to only those signed graphs whose underlying graph is complete, then the following problem is worth exploring.

Problem 7.0.8. *Does there exist a signed complete graph on* n *vertices with double domination number* k*, where* $3 \leq k \leq n - 2$*?*

The upper bound of Theorem 4.2.3 is a trivial upper bound that does not use a sign function. It would be better if one constructs a minimal DDS set of signed cubic graphs giving a better non-trivial upper bound. So, we pose the following open problem.

Problem 7.0.9. *Can we get a better non-trivial upper bound for the double domination number of signed cubic graphs?*

In Chapter 5, we obtained explicit formulas for the chromatic polynomials and the zero-free chromatic polynomials of switching non-isomorphic signed book graphs.

In ordinary graph theory, it is well known that the chromatic polynomial of the union $G \cup H$ of two graphs G and H, where $G \cap H$ is a complete graph, is given by

$$\chi_{G \cup H}(k) = \frac{\chi_G(k) \cdot \chi_H(k)}{\chi_{G \cap H}(k)}.$$

In the context of signed graph theory, we pose the following problem.

Problem 7.0.10. *Let* Σ_1 *and* Σ_2 *be two signed graphs such that* $\Sigma_1 \cap \Sigma_2$ *is a signed complete graph, and that* $\chi_{\Sigma_1}(\lambda), \chi_{\Sigma_2}(\lambda)$ *and* $\chi_{\Sigma_1 \cap \Sigma_2}(\lambda)$ *be known. What is the formula for* $\chi_{\Sigma_1 \cup \Sigma_2}(\lambda)$*?*

We pose the same problem for the zero-free chromatic polynomial of the union of two signed graphs.

Problem 7.0.11. *Let Σ_1 and Σ_2 be two signed graphs such that $\Sigma_1 \cap \Sigma_2$ is a signed complete graph, and that $\chi^b_{\Sigma_1}(\lambda), \chi^b_{\Sigma_2}(\lambda)$ and $\chi^b_{\Sigma_1 \cap \Sigma_2}(\lambda)$ be known. What is the formula for $\chi^b_{\Sigma_1 \cup \Sigma_2}(\lambda)$?*

In Problem 7.0.9 and Problem 7.0.10, it is implicitly assumed that the signatures of Σ_1 and Σ_2 agree on their intersection part so that $\Sigma_1 \cup \Sigma_2$ is well defined.

Note that a signed book graph $(B(m,2),\sigma)$ is the union of two m-cycles having exactly one edge, namely uv, in common. The edge uv, being the intersection of two signed cycles, is indeed a signed complete graph on two vertices. Thus if we find answers of Problem 7.0.9 and Problem 7.0.10, then the calculations of $\chi_{(B(m,2),\sigma)}(\lambda)$ and $\chi^b_{(B(m,2),\sigma)}(\lambda)$ are straightforward.

Note that $(B(m,n),\sigma)$ is the union of a signed $B(m,n-1)$ and a signed m-cycle, where their intersection is a signed complete graph on two vertices. Therefore, if the answers of Problem 7.0.9 and Problem 7.0.10 are known for $n \geq 3$, then both kind of chromatic polynomials of $(B(m,n),\sigma)$ can be obtained easily.

In Chapter 6, we discussed the rna number of generalized Petersen graphs. We determined the rna number of $P_{n,k}$ for $k \in \{1,2\}$. For $k \geq 3$, the distribution of odd and even integers to the vertices of $P_{n,k}$ to obtain the value of $\sigma^-(P_{n,k})$ seems to be difficult. Thus the following problem is worth exploring.

Problem 7.0.12. *What is the value of $\sigma^-(P_{n,k})$, where $n \geq 7$ and $k \geq 3$?*

In Chapter 6, we proved that the smallest order of $(4n-1)$-regular graphs having rna number 1 is bounded above by $12n-2$. We also proved that the smallest order of a $(4n+1)$-regular graph with rna number 1 is $8n+6$. This implies that best possible lower bound on the rna number of regular graphs is 1. To the best of our knowledge, a best possible upper bound on the rna number of regular graphs is not known. Hence the following problem is also interesting.

Problem 7.0.13. *Determine a best possible upper bound on the rna number of regular graphs?*

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4. D. Sehrawat and B. Bhattacharjya. Non-isomorphic signatures on some generalised Petersen graph. *Elect. J. Graph Theory and Appl.* 9 (2) (2021), 235-255.
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6. D. Sehrawat and B. Bhattacharjya. **rna** number of some parity signed generalized Petersen graphs. arXiv:2110.03264, 2021.

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