UPWIND BASED NUMERICAL METHODS FOR TIME-DEPENDENT SINGULARLY PERTURBED PROBLEMS WITH BOUNDARY AND INTERIOR LAYERS

by

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DECLARATION

It is certified that the work contained in the thesis titled **"Upwind Based Numerical Methods for Time-Dependent Singularly Perturbed Problems with Boundary and Interior Layers"** has done by me under the guidance of Dr. Natesan Srinivasan, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is certified that the work contained in the thesis titled **"Upwind Based Numerical Methods for Time-Dependent Singularly Perturbed Problems with Boundary and Interior Layers"** by **Kaushik Mukherjee**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

January, 2010

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Associate Professor
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Dedicated

to

my Parents

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When the creation was new and all the stars
shone their first splendour, the gods
held their first splendour, the gods
held their assembly in the sky and sang
"Oh, the picture of perfection! The joy unalloyed!"

But one cried of a sudden—"It seems that
somewhere there is a break in the chain of light
and one of the stars has been lost."

The golden string of their harp snapped,
their song stopped, and they cried in dismay—
"Yes, that lost star was the best,
she was the glory of all heavens!"

From the day the search is unceasing for her,
and the cry goes from one to the other that
in her the world has lost its one joy!

Only the deepest silence of night
the stars smile and whisper among themselves—
"Vain is this seeking!
Unbroken perfection is over all!"

— **Rabindra Nath Tagore**

Abstract

This thesis provides some efficient numerical techniques for solving time-dependent singularly perturbed problems (SPPs) possessing boundary and interior layers. These types of problems are described by partial differential equations in which the highest spatial derivative is multiplied by an arbitrarily small parameter ε, known as "singular perturbation parameter". This leads to the occurrence of boundary (or interior) layers, which are basically thin regions in the neighbourhood of the boundary (or interior) of the domain, where the gradients of the solutions steepen as the perturbation parameter ε tends to zero. Due to this layer phenomena, it is a very difficult and challenging task to provide ε-uniform numerical methods for solving SPPs. The term "ε-uniform" is meant to identify those numerical methods in which the approximate solution converges (measured in the supremum norm) independently with respect to the parameter ε to the corresponding exact solution of SPP .

The purpose of this thesis is therefore to develop, analyze, improve and optimize the ε-uniform upwind based numerical methods for solving time-dependent singularly perturbed initial-boundary-value problems (IBVPs) with smooth and non-smooth data. This is accomplished by constructing spacial non-uniform meshes resolving boundary and interior layers.

At first, a uniformly convergent hybrid numerical scheme is proposed and analyzed on a layer resolving piecewise-uniform Shishkin mesh for singularly perturbed one-dimensional parabolic convection-diffusion IBVP with a regular boundary layer as well as a class of parabolic convection-diffusion IBVPs with strong interior layers. The scheme utilizes a proper combination of the midpoint upwind scheme and the classical central difference scheme for the spatial discretization and the backward-Euler scheme for discretizing the time derivative. The analogous study of a similar kind of hybrid scheme is also made for a class of singularly perturbed mixed parabolic-elliptic IBVPs exhibiting both boundary and interior layers. Further, the efficiency of the hybrid scheme (proposed for 1D parabolic IBVP with smooth data) is tested by extending it for solving two-dimensional singularly perturbed parabolic convection-diffusion IBVP on a spacial rectangular mesh, utilizing the Peaceman and Rachford method for the time discretization. In all the cases, the newly proposed hybrid schemes attain an almost second-order spatial accuracy. Moreover, a unified theory is derived to obtain an optimal order of convergence of the classical implicit upwind finite difference scheme on Shishkin-type meshes (including the piecewise-uniform Shishkin mesh and the Bakhalov-Shishkin mesh), for a class of singularly perturbed parabolic IBVPs exhibiting strong interior layers. Finally, a post-processing technique (Richardson extrapolation), which improves the accuracy of the standard upwind scheme, is analyzed on a piecewise-uniform Shishkin mesh for singularly perturbed parabolic convection-diffusion IBVP exhibiting a regular boundary layer.

Contents

List of Figures

List of Tables

Nomenclature

BVP	boundary-value problem
IBVP	initial-boundary-value problem
PDE	partial differential equation
SPP	singular perturbation problem
$\mathbb{R}$	set of all real numbers
ε	singular perturbation parameter
C(or subscripted), $\mathfrak{M}$	generic constants, independent of ε
$O(\cdot), o(\cdot)$	Landau symbols
N	number of mesh-intervals in spatial direction
M	number of mesh-intervals in time direction
x, x_i, y, y_j	spatial variables
t, t_n	temporal variables
$h, h_i, h_\ell, h_{(l)}, h_{(r)}, H_\ell, H, H_{(l)}, H_{(r)}$ $h_{1,i}, h_{2,i}, h_{\ell,k}$	mesh-sizes in spatial direction
Δt	mesh-size in time direction
D	bounded open subset in $\mathbb{R} \times [0,T]$
$\overline{D}$	closure of D
$\Vert \cdot \Vert_{\infty,D}$ or $\Vert \cdot \Vert_\infty$	standard supremum norm on D
$\mathcal{C}^k(D), \mathcal{C}^\lambda(D), \mathcal{C}^{k+\lambda}(D)$	function spaces
$\Omega, \overline{\Omega}, \Omega^*, \overline{\Omega^*}, \Omega^+, \overline{\Omega^+}, \Omega^-, \mathcal{D}, \overline{\mathcal{D}}$	continuous spatial domains
$G, \overline{G}, G^*, \overline{G^*}, G^+, \overline{G^+}, G^-, \overline{G^-}, \mathcal{G}$	domains for continuous non-stationary problems
S_t^M, S_t^{2M}	discrete time domains
$\overline{\Omega}_x^{N,\varepsilon}, \widetilde{\Omega}_x^{N,\varepsilon}, \widetilde{\Omega}_x^{2N,\varepsilon}, \overline{\Omega}_x^{N,\tau}, \overline{\Omega}_x^{2N,\tau}, I_{1,\varepsilon}^N, \overline{I}_{1,\varepsilon}^N,$ $I_{2,\varepsilon}^N, \overline{I}_{2,\varepsilon}^N, \overline{\mathcal{D}}_\varepsilon^N$	discrete spatial domains
$G_\varepsilon^{N,M}, \overline{G}_\varepsilon^{N,M}, \widetilde{G}_\varepsilon^{2N,2M}, G_\tau^{N,M}, \overline{G}_\tau^{N,M},$ $\overline{G}_\tau^{2N,2M}, \overline{\mathcal{G}}_\varepsilon^{N,M}$	domains for discrete non-stationary problems
$\partial\Omega, \partial D, \partial\mathcal{D}, \partial\mathcal{G}, \Gamma, \Gamma_\varepsilon^{N,M}, \Gamma_\tau^{N,M}$	boundary of given domins
$L_\varepsilon, L_\varepsilon^*, L_{x,\varepsilon}, L_{1,\varepsilon}, L_{2,\varepsilon}, \mathcal{L}_\varepsilon, \mathcal{L}_{1,\varepsilon}, \mathcal{L}_{2,\varepsilon}, \mathbf{L}_{x,\varepsilon}$	differential operators
$D_x^+, D_x^-, D_x^0, \delta_x^2, D_y^+, D_y^-, D_y^0, \delta_y^2, D_t^-,$ $D_x^F, D_x^B, L_\varepsilon^N, L_\varepsilon^{N,M}, L_H^{N,M}, L_{cen}^{N,M},$ $L_{mu}^{N,M,(+)}, L_{mu}^{N,M,(-)}, L_{1,cen}^{N,M}, L_{2,cen}^{N,M}, L_{2,mu}^{N,M},$ $\mathcal{L}_{1,cen}^N, \mathcal{L}_{1,mu}^N, \mathcal{L}_{2,cen}^N, \mathcal{L}_{2,mu}^N, \overline{\mathcal{L}}_{1,\varepsilon}^N, \overline{\mathcal{L}}_{2,\varepsilon}^N$	difference operators
$u_0(x), u_0(\mathbf{x}), s_0(x)$	initial data
$s_1(t), s_2(t), s(\mathbf{x}, t)$	boundary data
$e_\varepsilon^{N,\Delta t}, e_{N,\Delta t}, E_\varepsilon^{N,\Delta t}, E_{N,\Delta t}$	maximum point-wise errors
$p_\varepsilon^{N,\Delta t}, p_{N,\Delta t}, P_\varepsilon^{N,\Delta t}, P_{N,\Delta t}$	order of convergence

Chapter 1

Introduction

1.1 Brief Background

Time-dependent singularly perturbed problems (SPPs) with smooth and non-smooth data arise often in several branches of engineering and applied mathematics including fluid dynamics, hydrodynamics, heat transfer, semiconductor device modeling, financial modeling, mathematical biology, chemical-reactor theory etc. These types of problems are described by partial differential equations (PDEs) in which the highest spatial derivative is multiplied by an arbitrarily small parameter ε. The Navier-Stokes equation with a large *Reynolds number* is one of the most striking examples of SPPs. For instance, consider the unsteady incompressible viscous fluid flow problems governed by the following Navier-stokes equations:

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \dfrac{1}{Re} \nabla^2 \mathbf{v}, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{1.1}$$

where p is the pressure and $\mathbf{v} = (u, v)$ is the velocity field with the respective velocity components u, v along x and y directions. The parameter $Re = |\mathbf{v}|L/\nu$ is the *Reynolds number* with L being the length scale and ν the kinematic viscocity of the fluid. For sufficiently large Re ($\gg 1$), the equations given in (1.1) will be transformed to singularly perturbed PDEs. The drift-diffusion equation describing the flow of electrons through semi-conductor devices in electrical engineering is another significant example to be noted. Further details can be found in the book of Morton [56].

SPPs came into the picture at the Third International Congress of Mathematicians in Heidelberg in 1904; Prandtl's seven-page report published in the proceedings [75]. He pioneered the subject of boundary layer theory in his explanation of how a quantity as small as the viscocity of common fluids such as water and air could nevertheless play a crucial role in determining their flow. The term "singular perturbation" was first used by Friedrichs and Wasow in their paper [32]. In general, the solutions of SPPs possess boundary (or interior) layers which are basically thin regions in the neighbourhood of the boundary (or interior) of the domain, where the gradients of the solutions steepen as the perturbation parameter ε tends to zero. This characterizes the multi-scale nature of the solutions as they vary rapidly within the layer regions and behave regularly away from the layer regions. Away from any corner of the domain a boundary layer of either regular or parabolic type may occur. A boundary layer is said to be of parabolic type if the characteristics of the reduced equation, corresponding to $\varepsilon = 0$ are parallel to the boundary, and of regular type if these characteristics are not parallel to the boundary. Whereas, a boundary layer near a corner is said to be of corner type. The discussion along with the diagram is given in the book of Farrell *et al.* [27].

SPPs are of interest from theoretical as well as numerical points of view. Mathematicians and physicists study the behavior of the analytical solution as the parameter ε goes to zero, through an asymptotic expansion technique. This is the principle analytical approach to enter insight the qualitative properties of these multi-scale problems. A straightforward perturbation expansion using an *asymptotic sequence* in the small parameter ε leads to differential equations of lower order than the original governing equation. In consequence not all of the boundary or initial conditions can be satisfied by the perturbation expansion. The technique for overcoming this difficulty is to combine the straightforward expansion valid away from the boundary where a condition is not satisfied, with an expansion valid within a layer adjacent to this boundary. The straightforward expansion is called the *outer expansion*. The *inner expansion* associated within the boundary layer region is expressed in terns of a stretched variable, rather than the original independent variable, which takes account of the scale of certain derivative terms. The *inner* and *outer expansions* are matched over a region located at the 'edge of the boundary layer', using the method of *matched asymptotic expansions*. It is to be noted that before solving one has to know the width of the boundary layer which can be obtained from the *principle of least degeneracy* by Van Dyke [23]. For more details one can refer to the books of Bush [8], Eckhaus [24], Kevorkian and Cole [42], Lagerstrom [48], O'Malley [71] and the survey article of Lagerstrom [49].

Due to the presence of steep gradients in the analytical solution, inaccurate numerical solutions are obtained whenever the classical numerical methods (standard finite difference or finite element methods) are used to solve PDEs with boundary layers on uniform meshes. In this context, careful numerical experiments (see, e.g., the book of Roos *et al.* [77]) conducted even for simple two-point boundary-value problem (BVP) reveal that the classical methods usually fail to decrease the maximum point-wise error as the mesh is refined, until the mesh-size and the perturbation parameter ε have the same order of magnitude. This basically runs contrary to the natural expectation that the error of an acceptable numerical method should decrease while the mesh is refined. But from the practical point of view, this assumption is unsatisfactory when, for example, $\varepsilon = 10^{-5}$ and the mesh-size, say, $h < \varepsilon$. Consequently, to achieve accurate numerical solutions in two or three dimensional cases using the classical numerical methods, such a restriction would lead to unacceptably large (ε-dependent) number of mesh points and the dimension of the corresponding algebraic system generated would be too large for computation. Hence, this incorporates the massive computational cost. Even the use of computers with extremely large capacity will not guarantee acceptable accuracy in the answer. This drawback therefore arises the following relevant question:

- How to construct and analyze special numerical method for solving SPPs so that the error of the approximate solution obtained by such method, diminishes independent of the parameter ε and should be defined solely by the number of mesh points used?

Luckily, an optimistic answer to the above question is given by developing the concept of ε-uniform numerical methods; in which the order of convergence and the error constant, measured in the discrete supremum norm, are independent of the perturbation parameter ε. The choice of the discrete supremum norm (*i.e., maximum norm*) for the measurement of the error rather than an averaged norm, is due to the necessity to quantity the error inside the boundary or interior layer regions. Indeed, the averaged norm (root mean square) could not capture the rapid changes in the solutions and therefore may fail to detect the local behavior of the error in these layers. Further discussion about the choice of the norm can be found in the book of Miller *et al.* [54]. Certainly, providing such types of numerical methods are one of the most challenging job for numerical analysts to tackle multi-scale nature of the solutions of SPPs.

Over the last few decades, many ε-uniform numerical methods including finite difference, finite element, finite volume, spline collocations methods etc. have been developed by many researchers for singularly perturbed stationary and non-stationary problems; one may refer to the books of Doolan *et al.* [22], Farrell *et al.* [27], Miller *et al.* [54], Morton [56] and Roos *et al.* [77], Shishkin and Shishkina [84] and the survey articles of Ewing and Wang [26], Kadalbajoo and Patidar [38], Linβ [52], Stynes [86] and the references therein. In this regard, one can also look into a couple articles [62–67, 93, 94], in which Natesan and his collaborators made valuable contributions to SPPs, by developing several initial-value techniques and parallel boundary-value techniques to solve singularly perturbed BVPs. Two different approaches, which are generally used for the construction of ε-uniform numerical methods, are the followings:

The first one substitutes the standard finite difference operator by a difference operator which adapts the singularly perturbed nature of the differential operator. Such finite difference operators are called fitted finite difference operators and the numerical methods with a fitted finite difference operator defined on standard mesh (for example, on a uniform mesh) are referred to, in general, as *fitted operator methods* (first introduced by Allen *et al.* [19] for solving the problem of a viscous fluid flow past a cylinder). These methods are attractive since they allow the use of meshes with an arbitrary distribution of nodes, *i.e.*, uniform meshes. When regular boundary layers are present, it is often possible to obtain ε-uniform numerical method by constructing an appropriately fitted finite difference operator on a uniform mesh. However, this approach is not possible if a parabolic boundary layer is present. This negative result was first proved by Shishkin [78] (see also the book of Miller et al. [54] for a more detailed proof).

Numerical methods, that use the standard finite difference operator on a spacial mesh adapted to the boundary or interior layers, are known as *fitted mesh methods.* This is the second recognized technique for the construction of ε-uniform numerical methods to circumvent the limitations of the classical methods. The underlying idea of the adaptive mesh, particularly the well-known piecewise-uniform Shishkin mesh, is that it is a union of finite number of uniform meshes in which the mesh-sizes are different inside and outside the layer regions. These piecewise-uniform fitted meshes were first introduced by Shishkin [79]. Practically, the *fitted mesh methods* are recommended whenever possible, because of their simpler implementation than the *fitted operator methods.* Moreover, the *fitted mesh methods* have the extra advantage that they can be easily extended to the dimension higher than one and to the nonlinear problems. The recent development of ε-uniform numerical methods using newly developed adaptive grids, which are based on the equidistribution principle, can be found in Beckett [3].

As the model problems considered in the thesis are mainly parabolic PDEs, before proceeding to the next section, a concise literature survey of numerical methods concerning general parabolic PDEs are therefore presented here. Many applied mathematicians and engineers showed their interest to devise various numerical techniques for solving these parabolic PDEs; the information regrading these numerical methods can be obtained from the standard text books [37,57,85]. One may refer to the books of Ciarlet and Lions [11], Thomée [91] and the papers of Wade [43,95–97] to gain further theoretical knowledge of finite difference and finite element methods associated with parabolic PDEs.

1.2 Objective and Motivation

The main objective of the thesis revolves around developing, analyzing, improving and optimizing the ε-uniform upwind based *fitted mesh methods*, defined on the spacial non-uniform meshes (Shishkin meshes) resolving boundary and interior layers, for solving time-dependent singularly perturbed initial-boundary-value problems (IBVPs). A brief survey of the literature illustrating motivation behind the present work, carried out in the thesis, is presented below:

In recent years, for singularly perturbed parabolic convection-diffusion problems, a variety of uniformly convergent numerical methods on Shishkin meshes have been developed by several authors, mainly Cai and Liu ([9]), Clavero *et al.* ([16, 17]), O'Riordan *et al.* [25] in presence of boundary layers. In spit of these, Ng-Stynes *et al.* [69], Guo and Stynes [33], Stynes and O'Riordan [73] constructed numerous exponentially fitted uniformly convergent numerical methods for the same. But all of them are of first-order accurate in both spatial and temporal variables. A couple of articles (see [12, 14]) dealing with almost second-order uniformly convergent methods are available for singularly perturbed stationary convection-diffusion problems exhibiting boundary layers. Recently, in [2, 60], Natesan and Bawa applied cubic spline method to obtain second-order uniformly convergent hybrid numerical schemes for singularly perturbed BVPs. Later on, Natesan and Deb [61] showed that the spline based hybrid scheme can be used successfully to achieve second-order ε-uniform convergence in space, even for parabolic reaction-diffusion problems. Whereas, Clavero *et al.* [15], Hemker *et al.* [35], Kopteva [46] and Shishkin [81] constructed various uniformly convergent higher-order (two or more than two) time accurate numerical schemes for singularly perturbed parabolic convection-diffusion problems. However, most of these schemes are much more complex. Moreover, Natesan and Deb [21] currently proposed two ε-uniform numerical schemes of higher-order in time for singularly perturbed parabolic reaction-diffusion problems. On the other side, for singularly perturbed problems possessing interior layers, several authors, mainly Farrell *et al.* [28, 29], Shanthi *et al.* [92] developed ε-uniform numerical methods for stationary case and O'Riordan and Shishkin [72], Shishkin [80] for parabolic problems. These methods are based on piecewise-uniform Shishkin meshes but most of them are almost first-order spatial accurate, up to a logarithmic factor. In this context, we want to mention that there are only a handful of papers (see [5, 6]) dealing with singularly perturbed problems of mixed type exhibiting both boundary and interior layers. Here, the problems are discretized using an inverse-monotone finite volume method on Shishkin meshes to establish an almost second-order point-wise convergence in space, uniformly with respect to the perturbation parameter ε. However, its analysis is critical that is based on the discrete Green's function technique. Therefore, considering the above literature survey, one may still desire to construct efficient second-order uniformly convergent numerical methods in a simplest way.

It is a well-known fact that whenever the perturbation parameter ε is small, on a uniform mesh the classical central difference scheme leads to nonphysical oscillations in the discrete solution unless the mesh diameter is unsatisfactorily small. Again, the midpoint upwind scheme, which was first examined on an equidistant mesh by Abrahamsson *et al.* [1] and later on by Stynes and Roos [88] on an arbitrary mesh, is found to be second-order uniformly convergent outside the layer region. Besides, in the Shishkin mesh, the meshes will be divided equally inside and outside the layers, therefore one gets a fine mesh inside the layer region and a coarse mesh in the outer region. Utilizing this fact, Stynes and Roos [88] introduced a hybrid finite difference scheme (a proper combination of the midpoint upwind scheme in the outer region and the classical central difference scheme in the boundary layer region) on a piecewise-uniform Shishkin mesh for convection-diffusion BVP exhibiting boundary layer. The scheme converges uniformly with respect to the perturbation parameter ε, with almost second-order accuracy in the discrete supremum norm throughout the domain. Later, following the similar mechanism as described above, Cen [10] devised a hybrid scheme for convection-diffusion BVPs with interior layers. But still this is not the optimal order of convergence (up to a logarithmic factor), uniformly in ε, on a piecewise-uniform Shishkin mesh. The hybrid scheme has two advantages: firstly, its good stability property induced by the structure of the Shishkin mesh. Secondly, from computational point of view, the hybrid scheme utilizes coarser meshes to obtain accurate numerical solution, in comparison with the meshes required for lower-order schemes. Nevertheless, the scheme contains less complexity that permits one to easily handle it for doing computation. In this thesis, we propose and analyze such a uniformly convergent hybrid numerical scheme for singularly perturbed parabolic convection-diffusion IBVPs with boundary and interior layers, keeping the same accuracy with respect to the spatial variable. The proposed scheme comprises of the classical backward-Euler method for the time discretization. Note that a merely computationally study made in the recent paper [58] reveals that proposed hybrid scheme is superior to the implicit upwind scheme, although in both the schemes the backward-Euler method is used for approximating the time derivative and by this, the potential of the hybrid scheme is examined. A similar kind of hybrid scheme is also analyzed in this thesis for singularly perturbed problems of mixed parabolic-elliptic type. For the parabolic part of the mixed type problem, the classical central difference scheme is used for the spatial discretization and the classical backward-Euler method is used for discretizing the time derivative. While for the elliptic part, the hybrid finite difference scheme (a proper combination of the midpoint upwind scheme in the outer region and the classical central difference scheme in the interior layer region) is proposed. It is proved that the newly proposed scheme converges ε-uniformly with almost second-order spatial accuracy in the discrete supremum norm.

Next, designing low-cost second-order accurate uniformly convergent numerical methods for multidimensional evolutionary convection-diffusion problems is always a desirable task. Nevertheless, for multidimensional stationary singularly perturbed problems, the development of uniformly convergent method with order higher than one is also a very difficult task. A good number of uniformly convergent finite difference methods (see the books [27, 54], the papers of Linβ [50] and Kopteva [45]) have been developed and analyzed utilizing spacial non-uniform meshes, for two-dimensional stationary singularly perturbed problems. Most of these methods have the disadvantage of exhibiting lower-order accuracy. In recent past, for two-dimensional stationary convection-diffusion problem, Linβ and Stynes [87] proposed a hybrid scheme (a combination of upwinding and central difference schemes) on a spacial Shishkin mesh. This scheme is more accurate than the simple upwind scheme given in the book of Miller et al. [54], though it requires equal computational cost. However, the hybrid scheme is unable to produce the second-order convergence. On the other hand, Clavero *et al.* [18] introduced a new technique to obtain cheeper numerical scheme for multi-dimensional evolutionary convection-diffusion problems. This technique utilizes a first-order alternating direction method to discretize the time derivative that allows one to take the advantage of known-schemes applied for solving one-dimensional singularly perturbed problems. Further, a second-order numerical scheme has been developed by Bujanda *et al.* [7] for two-dimensional time-dependent reaction-diffusion problems and Clavero *et al.* [13] for time-dependent convection-diffusion problems, by using the Peaceman and Rachford method (see [74]) to discretize the time derivative, combined with a HODIE finite difference scheme (see [14]) for the spatial variable. The advantage of using these fractional step schemes is that it reduces the computational cost of the fully discrete scheme, because one has to solve only the tridiagonal linear systems instead of the block tridiagonal systems. In this thesis, taking into account this advantage and also considering the efficiency of the hybrid scheme [59] devised for one-dimensional time-dependent convection-diffusion problems, we develop and analyze a second-order (both in space and time) uniformly convergent hybrid numerical scheme on a Shishkin mesh (tensor-product of 1D piecewise-uniform Shishkin meshs) for two-dimensional singularly perturbed parabolic convection-diffusion problem, utilizing the Peaceman and Rachford method for the time discretization.

Recently, several researchers are interested in providing a unified theory for characterization of the layer resolving meshes which include the classical piecewise-uniform Shishkin mesh (see [54,77]) and the Bakhalov-Shishkin mesh (see [50,51])). These types of meshes are called as Shishkin-type meshes. Bakhalov-Shishkin meshes can be used whenever Shishkin meshes are applicable, but are easier to handle than pure Bakhalov meshes, in particular when curved layers or interior layers are present. In this context, the following natural question arises: Is it possible to characterize such meshes so that the characterization easily allows one to deduce the ε-uniform converges of the standard schemes on Shishkin-type meshes? This question has been firstly analyzed by Roos and Linβ [76] for simple upwinding and finite element method applied to singularly perturbed boundary value problem exhibiting boundary layer. Later on, Linβ *et al.* [53] extended this approach for studying nonstandard first-order scheme on Shishkin-type meshes applied to singularly perturbed quasi-linear BVP with an exponential boundary layer. Formerly, a similar kind of unified theory has been provided by Fröhner *et al.* [30] for Defect correction method and by Clavero and Gracia [12] for HODIE finite difference scheme. However, no theoretical framework for simple upwinding on Shishkin-type meshes resolving interior layers is available in the literature, as the boundary layers (left or right) counterpart. In this thesis, we consider a class of singularly perturbed parabolic convection-diffusion problems exhibiting strong interior layers and perform the analysis for the implicit upwind finite difference scheme (proposed by O'Riordan *et al.* [72]) on both the classical Shishkin mesh and the Bakhalov-Shishkin mesh. It has been shown that the bound obtained on the Bakhalov-Shishkin mesh is in fact more accurate than that on the piecewise-uniform Shishkin mesh.

Finally, in view of increasing the accuracy of numerical methods with less complexity, we turn our attention to construct and analyze a simple post-processing technique that can be applied to the standard upwind scheme defined on a Shishkin mesh. Unanimously, Richardson extrapolation, a well-known post-processing technique, provides a better approximation to the exact solution obtained by averaging the numerical solutions computed on two embedded meshes. This method has been studied in the literature [36, 82] to improve the accuracy of the numerical solution for singularly perturbed elliptic reaction-diffusion equation. But their analysis, based on the direct expansion of upwinding solution, is found to be complicated. This approach has been immutably followed by Shishkin and Shishkina [83] for quasilinear parabolic convection-diffusion problem. However, Natividad and Stynes [68] provide a comparatively easier and distinct analysis to deal with Richardson extrapolation for one-dimensional singularly perturbed convection-diffusion BVP. In this thesis, we study Richardson extrapolation by extending their technique for parabolic convection-diffusion IBVP. We show both theoretically and computationally that how the solution after extrapolation converges ε-uniformly in the discrete supremum norm with an accuracy of order almost two.

1.3 Preliminaries

This section introduces some basic definitions, notations and conventions which will be used throughout the thesis.

First of all, we define the following well-known function spaces. Let D be a bounded open subset in $\mathbb{R} \times [0, T]$. For non-negative integer k, we denote by $\mathcal{C}^k(D)$ the space of all functions whose derivatives upto order k are continuous on D. Suppose that $\lambda \in (0, 1)$. Then a function $g(x, t)$ defined on D is said to be Hölder continuous in D with exponent λ, if and only if

$$\sup_{(x,t),(x',t')\in D} \frac{\left|g(x,t)-g(x',t')\right|}{\left((x-x')^2+|t-t'|\right)^{\lambda/2}} < \infty.$$

This coincides with the definition of Friedman [31]. An equivalent definition is used by Ladyzenskaja *et al.* [47]. We denote the set of Hölder continuous functions in D with exponent λ by $\mathcal{C}^{\lambda}(D)$. For each integer $k \geq 1$, we define the parabolic Hölder space $\mathcal{C}^{k+\lambda}(D)$ as

$$\mathcal{C}^{k+\lambda}(D) = \left\{ g : \frac{\partial^{i+j} g}{\partial x^i \partial t^j} \in \mathcal{C}^{\lambda}(D) \text{ for all non-negative integers } i,j \text{ with } 0 \leq i+2j \leq k \right\}.$$

Note that for each integer $k \geq 0$, any function $g \in \mathcal{C}^{k+\lambda}(D)$ is uniformly continuous in D and admits a unique continuous extension on $\overline{D}$. This allows us to speak about values on $\partial D = \overline{D}\backslash D$ of a function $g \in \mathcal{C}^{k+\lambda}(D)$ and without ambiguity one can write $\mathcal{C}^{k+\lambda}(D) = \mathcal{C}^{k+\lambda}(\overline{D})$.

In the analysis, we use the standard supremum norm $\|\cdot\|_{\infty,D}$, which is defined by

$$\|g\|_{\infty,D} = \sup_{(x,t)\in D} |g(x,t)|.$$

It is a convention that when the domain is obvious, or of no particular significance, D is omitted.

Throughout the thesis, C (sometimes subscripted) denotes a generic positive constant that is independent of the perturbation parameter ε, N and M (number of mesh-intervals in the spatial and the temporal directions, respectively) and the mesh sizes. Note that an unsubscripted C may take different values in different places, but a subscripted C is a fixed constant that does not change throughout the thesis. For clarity of our presentation, we also use the notation $\mathfrak{M}$ to denote a generic positive constant as like C.

In the analysis, it is frequently assumed that $\varepsilon \leq N^{-1}$, which is the case of actual interest from the practical point of view. If $\varepsilon \geq N^{-1}$, then in practice the model problems considered in the thesis are not difficult to solve computationally. So the assumption is not restrictive. It can also be replaced by the hypothesis that $\varepsilon \leq C_0 N^{-1}$ for some fixed constant C_0 without altering the results obtained in the thesis.

We now define the following standard finite difference operators which will be used for describing the difference schemes (particularly, for the discretization of the one-dimensional parabolic IBVPs) in the subsequent chapters.

Let us consider the arbitrary meshes in the spatial and the temporal directions respectively, as $0 = x_0 < x_1 < \ldots < x_N = 1$ and $0 = t_0 < t_1 < \ldots < t_M = T$. Then, for a given mesh function $v(x_i,t_n) = v_i^n$, we define the forward, backward and central difference operators D_x^+, D_x^- and D_x^0 in space and the backward difference operator D_t^- in time by

$$\begin{cases} D_x^+ v_i^n = \dfrac{v_{i+1}^n - v_i^n}{x_{i+1}-x_i}, \; D_x^- v_i^n = \dfrac{v_i^n - v_{i-1}^n}{x_i - x_{i-1}}, \; D_x^0 v_i^n = \dfrac{v_{i+1}^n - v_{i-1}^n}{x_{i+1}-x_{i-1}} \\ \text{and} \quad D_t^- v_i^n = \dfrac{v_i^n - v_i^{n-1}}{t_n - t_{n-1}}, \end{cases}$$

respectively, and the second-order finite difference operator δ_x^2 in space is defined by

$$\delta_x^2 v_i^n = \frac{2\big(D_x^+ v_i^n - D_x^- v_i^n\big)}{x_{i+1} - x_{i-1}}.$$

In addition, we frequently use the notations $v_{i\mp 1/2}^n$, which are defined as $v_{i\mp 1/2}^n = \big(v_{i\mp 1}^n + v_i^n\big)/2$. Note that when $v(x_i) = v_i$, we similarly define $v_{i\mp 1/2} = \big(v_{i\mp 1} + v_i\big)/2$. Next, we insert the following definition of M-matrix, as given in the books ([27, 77]).

Definition 1.3.1. *A matrix* $\mathbf{A} = (a_{i,j}) \in \mathbb{R}^{k\times k}$ *is an* M*-matrix if* $\mathbf{A}$ *is nonsingular,* $\mathbf{A}^{-1} \geq 0$ *and* $a_{i,j} \leq 0$*, for all* $i \neq j,\ 1 \leq i, j \leq k$.

Finally, this section is concluded by introducing ***Landau's order symbols*** O (big-oh) and o (little-oh) to be used throughout the thesis. We refer the books ([42, 70]) for further discussion of the following definitions. Let $f(\varepsilon)$ and $g(\varepsilon)$ be two real valued functions, where $0 < \varepsilon \leq \varepsilon_0 \ll 1$.

Definition 1.3.2. *The expression* $f(\varepsilon) = O(g(\varepsilon))$ *as* $\varepsilon \to 0$ *means that there exist some constants* C, ε_o *such that in* $(0, \varepsilon_o]$,

$$\big|f(\varepsilon)\big| \leq C\big|g(\varepsilon)\big| \quad \text{as } \varepsilon \to 0.$$

Definition 1.3.3. *The expression* $f(\varepsilon) = o(g(\varepsilon))$ *as* $\varepsilon \to 0$ *means that*

$$\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0.$$

1.4 Model Problems

In this thesis, four types of model problems are considered and their concise descriptions are given below:

1.4.1 Singularly perturbed parabolic convection-diffusion problems with boundary layer

Here, we consider the following 1D singularly perturbed parabolic convection-diffusion IBVP posed on the domain $G = \Omega \times (0, T],\ \Omega = (0, 1)$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + L_{x,\varepsilon} u(x,t) = f(x,t), \quad (x,t) \in G, \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = u(1,t) = 0, \quad t \in (0,T], \end{cases} \tag{1.2}$$

where

$$L_{x,\varepsilon}u \equiv -\varepsilon\frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x)u,$$

$0 < \varepsilon \ll 1$ is a small parameter and the coefficients a, b are assumed to satisfy

$$a(x) > 0, \quad b(x) \geq 0 \quad \text{on } \overline{\Omega}. \tag{1.3}$$

In general, the solution $u(x,t)$ of the parabolic IBVP (1.2)-(1.3) exhibits a regular boundary layer of width $O(\varepsilon)$ at $x = 1$.

1.4.2 Singularly perturbed parabolic convection-diffusion problems with interior layers

Denote the domains for describing the model problem by

$$G^- = \Omega^- \times (0,T],\ G^+ = \Omega^+ \times (0,T],\ G = \Omega \times (0,T],\ \Omega^- = (0,\xi),\ \Omega^+ = (\xi,1),\ \Omega = (0,1).$$

Here, we consider the following class of singularly perturbed parabolic problems posed on the domain $G^- \cup G^+$:

$$\begin{cases} L_\varepsilon u(x,t) \equiv \left(\varepsilon\dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - b(x)u - \dfrac{\partial u}{\partial t}\right)(x,t) = f(x,t), \quad (x,t) \in G^- \cup G^+, \\ u(x,0) = s_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = s_1(t), \quad u(1,t) = s_2(t), \quad t \in (0,T], \end{cases} \tag{1.4}$$

where $0 < \varepsilon \ll 1$ is a small parameter, the functions a, b, f satisfy the assumptions

$$\begin{cases} b(x) \geq 0 \quad \text{on } \overline{\Omega}, \\ \big|[a]\big| \leq C, \quad \big|[f]\big| \leq C, \quad \text{at } x = \xi, \end{cases} \tag{1.5}$$

and the solution $u(x,t)$ satisfies the following interface conditions

$$[u] = 0, \quad \left[\frac{\partial u}{\partial x}\right] = 0, \quad \text{at } x = \xi. \tag{1.6}$$

The symbol $[u]$ denotes the jump of u across the point of discontinuity $x = \xi$ and is defined by $[u](\xi,t) = u(\xi^+,t) - u(\xi^-,t)$, where $u(\xi^\pm,t) = \lim_{x\to\xi\pm 0} u(x,t)$. We also assume that the convection coefficient $a(x)$ satisfy the following particular case:

$$\begin{cases} a(x) < 0, \quad x < \xi, \\ a(x) > 0, \quad x > \xi. \end{cases} \tag{1.7}$$

In general, the solution $u(x,t)$ of the parabolic IBVP (1.4)-(1.7) possesses strong interior layers of width $O(\varepsilon)$ in the neighborhood of the point $x = \xi$.

1.4.3 Singularly perturbed problems of mixed parabolic-elliptic type

Here, the following class of singularly perturbed mixed parabolic-elliptic problems are considered on the domain $G^- \cup G^+$:

$$\begin{cases} L_{1,\varepsilon}u(x,t) \equiv \left(\dfrac{\partial u}{\partial t} - \varepsilon\dfrac{\partial^2 u}{\partial x^2} + b(x,t)u\right)(x,t) \;=\; f(x,t), \quad (x,t) \in G^-, \\ L_{2,\varepsilon}u(x,t) \equiv \left(-\varepsilon\dfrac{\partial^2 u}{\partial x^2} - a(x,t)\dfrac{\partial u}{\partial x} + b(x,t)u\right)(x,t) \;=\; f(x,t), \quad (x,t) \in G^+, \\ u(x,0) = s_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = s_1(t), \quad u(1,t) = s_2(t), \quad t \in (0,T], \end{cases} \tag{1.8}$$

where $0 < \varepsilon \ll 1$ is a small parameter, the functions a, b, f satisfy the assumptions

$$\begin{cases} a(x,t) > 0, \; x > \xi, \quad b(x,t) \geq 0 \quad \text{on } \overline{G}, \\ \big|[f]\big| \leq C, \quad \text{at } x = \xi, \end{cases} \tag{1.9}$$

and the solution $u(x,t)$ satisfies the following interface conditions

$$[u] = 0, \quad \left[\frac{\partial u}{\partial x}\right] = 0, \quad \text{at } x = \xi. \tag{1.10}$$

In general, the solution $u(x,t)$ of the IBVP (1.8)-(1.10) exhibits a boundary layer at $x = 0$ and interior layers of different widths in the neighborhood of the point $x = \xi$.

1.4.4 2D singularly perturbed parabolic convection-diffusion problems

The following 2D singularly perturbed parabolic convection-diffusion IBVP is considered on the domain $\mathcal{G} = \mathcal{D} \times (0,T]$, $\mathcal{D} = (0,1)^2$, $\mathbf{x} = (x,y) \in \mathbb{R}^2$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(\mathbf{x},t) + \mathcal{L}_\varepsilon u(\mathbf{x},t) \;=\; f(\mathbf{x},t), \quad (\mathbf{x},t) \in \mathcal{G}, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\mathcal{D}}, \\ u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\mathcal{D} \times (0,T], \end{cases} \tag{1.11}$$

where

$$\mathcal{L}_\varepsilon u \;\equiv\; -\varepsilon \triangle u + \mathbf{a}(\mathbf{x}) \cdot \nabla u + b(\mathbf{x})u,$$

$0 < \varepsilon \ll 1$ is a small parameter and the coefficients $\mathbf{a} = (a_1, a_2)$, b are assumed to satisfy

$$a_i(\mathbf{x}) > 0, \; i = 1,2, \quad b(\mathbf{x}) \geq 0 \quad \text{on } \overline{\mathcal{D}}. \tag{1.12}$$

In general, the solution $u(x,t)$ of the parabolic IBVP (1.11)-(1.12) exhibits a regular boundary layer of width $O(\varepsilon)$ at the sides $x = 1$ and $y = 1$.

1.5 Organization of the Thesis

In this thesis, we first focus on the work related to the one-dimensional parabolic problems and then to the two-dimensional parabolic problem. The rest of the thesis consists of seven chapters and is organized as follows:

Chapter 2 presents the ε-uniform convergence of an almost second-order spatial accurate hybrid numerical scheme for one-dimensional singularly perturbed parabolic convection-diffusion problems of the form (1.2)-(1.3) on a piecewise-uniform Shishkin mesh resolving the boundary layer.

A post-processing technique is discussed in **Chapter 3**, which improves the accuracy of the standard upwind scheme on a piecewise-uniform Shishkin mesh, applied to one-dimensional singularly perturbed parabolic convection-diffusion problems of the form (1.2)-(1.3).

In **Chapter 4**, we analyze the ε-uniform convergence of an almost second-order spatial accurate hybrid numerical scheme for a class of singularly perturbed parabolic convection-diffusion problems of the form (1.4)-(1.7) with discontinuous convection coefficients on a piecewise-uniform Shishkin mesh resolving interior layers.

Chapter 5 provides a unified theory for analyzing the classical implicit upwind finite difference scheme on Shishkin-type meshes (including the classical piecewise-uniform Shishkin mesh and the Bakhalov-Shishkin mesh) for a class of singularly perturbed parabolic convection-diffusion problems of the form (1.4)-(1.7) exhibiting strong interior layers.

The analogous study of a hybrid numerical scheme for a class of singularly perturbed mixed parabolic-elliptic problems of the form (1.8)-(1.10) on a layer resolving piecewise-uniform Shishkin mesh, is presented in **Chapter 6**.

Chapter 7 is concerned with the construction of an ε-uniformly convergent efficient numerical scheme for solving two-dimensional singularly perturbed parabolic convection-diffusion problems of the form (1.11)-(1.12) by using a special rectangular mesh involving piecewise-uniform Shishkin meshes in the spatial directions.

Finally, **Chapter 8** addresses the summary of the results highlighting the contributions made by this thesis and also provides possible future scopes for further investigations of the present works.

Extensive numerical experiments are conducted to support the theoretical results and also to demonstrate the accuracy of the numerical methods. The corresponding numerical results are presented at the end of each chapter of the thesis. For clarity of the presentation, we have repeatedly described the model problems *i.e.,* (1.2)-(1.3) or (1.4)-(1.7) or (1.8)-(1.10) or (1.11)-(1.12) with suitable information on the given data at the beginning of the subsequent chapters.

Chapter 2

Parameter-Uniform Hybrid Numerical Scheme for Singularly Perturbed Parabolic Convection-Diffusion Problems exhibiting Boundary Layer

This chapter proposes a hybrid numerical scheme for one-dimensional singularly perturbed parabolic convection-diffusion problems exhibiting a regular boundary layer. The numerical scheme consists of the classical backward-Euler method to approximate the time derivative and a hybrid finite difference scheme (a proper combination of the midpoint upwind scheme in the outer region and the classical central difference scheme in the boundary layer region) for the spatial discretization. The scheme is analyzed on a piecewise-uniform Shishkin mesh to establish uniform convergence with respect to the perturbation parameter ε. It is proved that the method is almost second-order spatial accurate in the discrete supremum norm, provided that the perturbation parameter ε satisfies $\varepsilon \leq N^{-1}$. Here, N is the number of mesh-intervals in the spatial direction.

2.1 Introduction

Consider the following 1D singularly perturbed parabolic convection-diffusion IBVP posed on the domain $G = \Omega \times (0, T]$, $\Omega = (0, 1)$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) + L_{x,\varepsilon} u(x,t) \;=\; f(x,t), \quad (x,t) \in G, \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = u(1,t) = 0, \quad t \in (0,T], \end{cases} \tag{2.1}$$

where

$$L_{x,\varepsilon}u \equiv -\varepsilon\frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x)u,$$

$0 < \varepsilon \ll 1$ is a small parameter and the coefficients a, b are sufficiently smooth functions such that

$$a(x) \geq \alpha > 0, \quad b(x) \geq \beta \geq 0 \quad \text{on } \overline{\Omega}. \tag{2.2}$$

It is assumed that the functions u_0 and f are sufficiently smooth and satisfy the following compatibility conditions of order 1 [[47], §4.5] at the corner points $(0,0)$ and $(1,0)$:

$$u_0(0) = u_1(1) = 0, \tag{2.3}$$

and

$$\begin{cases} -\varepsilon\dfrac{\partial^2 u_0(0)}{\partial x^2} + a(0)\dfrac{\partial u_0(0)}{\partial x} + b(0)u_0(0) = f(0,0), \\ -\varepsilon\dfrac{\partial^2 u_0(1)}{\partial x^2} + a(1)\dfrac{\partial u_0(1)}{\partial x} + b(1)u_0(1) = f(1,0). \end{cases} \tag{2.4}$$

These conditions state that $u(x,0)$ and $\dfrac{\partial u}{\partial t}(x,0)$, which can be obtained from the differential equation and the initial condition given in (2.1), must agree at $x = 0$ and $x = 1$ with the values of u and $\dfrac{\partial u}{\partial t}$ determined from the boundary conditions given in (2.1). Then, it immediately follows from *Theorem 5.2* of [[47], §4.5] that the parabolic IBVP (2.1)-(2.2) has a unique solution $u \in \mathcal{C}^{2+\lambda}(\overline{G})$, which exhibits a regular boundary layer of width $O(\varepsilon)$ at $x = 1$ (see, e.g. [77]).

The outline of this chapter is as follows: Section 2.7 provides *a-priori* bounds on the analytical solution and its derivatives. Section 2.3 studies the uniform convergence of the semidiscrete scheme based on the backward-Euler method *via* an intermediate time semidiscretization step and state the asymptotic behavior of the solutions of the resulting semidiscrete problems and their spatial derivatives. Section 2.4 describes the piecewise-uniform Shishkin mesh and provides the detail construction of the newly proposed hybrid finite difference scheme for the spatial discretization of the semidiscrete problems resulting from the time discretization process. Here, the consistency, the stability and the uniform convergence of the proposed hybrid scheme are also studied. The fully discrete scheme is introduced in Section 2.5 and the main theoretical result, *i.e.*, the ε-uniform convergence of the fully discrete scheme, is proved. Finally, Section 2.6 provides the numerical experiments to verify the theoretical results.

2.2 Bounds on the Solution and its Derivatives

This section presents standard *a-priori* bounds on the analytical solution of the problem (2.1)-(2.2) and its derivatives. Let $\Gamma = \overline{G}\backslash G$ and we denote L_ε as the differential operator occurring in (2.1), which is defined by

$$L_\varepsilon u \equiv \left(\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x)u\right).$$

Then the operator L_ε satisfies the following minimum principle on $\overline{G}$, the proof of which follows from *Theorem 2.2* of [[77], Chapter II].

Lemma 2.2.1. (***Minimum Principle***) *Suppose that a function* $g \in \mathcal{C}^0(\overline{G}) \bigcap \mathcal{C}^2(G)$ *satisfies* $g(x,t) \geq 0,\ (x,t) \in \Gamma$ *and* $L_\varepsilon g(x,t) \geq 0,\ (x,t) \in G$. *Then we have* $g(x,t) \geq 0,\ \forall (x,t) \in \overline{G}$.

An immediate consequence of the above minimum principle is the following ε-uniform bound of the solution of the problem (2.1)-(2.2).

Lemma 2.2.2. *The solution u of the IBVP (2.1)-(2.2) satisfies*

$$\left|u(x,t)\right| \leq C, \quad (x,t) \in \overline{G}.$$

Proof. The proof follows from *Lemma 2.3* of [[77], Chapter II]. ■

The bounds on the derivatives of the analytical solution u are given in the following theorem.

Theorem 2.2.3. *For all non-negative integers* l, m, *satisfying* $0 \leq l+m \leq 4$ *and* $0 \leq m \leq 3$, *the exact solution u of the IBVP (2.1)-(2.2) satisfies the estimate*

$$\left|\frac{\partial^{l+m} u}{\partial x^l \partial t^m}(x,t)\right| \leq C\bigg(1 + \varepsilon^{-l}\exp\big(-\alpha(1-x)/\varepsilon\big)\bigg), \quad (x,t) \in G. \tag{2.5}$$

Proof. This result was proved in [69] for $0 \leq l+m \leq 2$. Under necessary compatibility conditions and sufficient smoothness on the data, the proof of the estimate (2.5) for higher values of l, m follows similarly from [[15], *Lemma 2.1*]. ■

Remark 2.2.4. *In order to obtain the stronger bounds on the analytical solution in Chapter 3, one requires the estimate (2.5), for all non-negative integers* l, m, *satisfying* $0 \leq l+m \leq 5$.

2.3 The Time Semidiscretization

This section provides the time semidiscretization method which is essential for the convergence analysis of the fully discrete scheme, since it will contribute to the decomposition of the global error for the spatial and the temporal variables. Also, the asymptotic behavior of the solutions of the semidisctrete problems is given.

2.3.1 Discretization of the time domain

On the time domain $[0, T]$, the equidistant meshes are introduced with uniform time step Δt such that

$$\mathbb{S}_t^M = \{t_n = n\,\Delta t,\ n = 0, \ldots, M,\ t_0 = 0, t_M = T,\ \Delta t = T/M\},$$

where M denotes the number of mesh-intervals in the t-direction.

2.3.2 The semidiscrete scheme

Discretizing the problem (2.1)-(2.2) with respect to time by the backward-Euler method, we obtain the following semidiscrete scheme:

$$\begin{cases} u^0(x) = u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ \big(I + \Delta t\, L_{x,\varepsilon}\big) u^{n+1}(x) = u^n(x) + \Delta t\, f(x, t_{n+1}), \\ u^{n+1}(0) = u^{n+1}(1) = 0, \end{cases} \tag{2.6}$$

where $u^n(x)$ is the semidiscrete approximation to the exact solution $u(x,t)$ of the continuous problem (2.1)-(2.2) at time level $t_n = n\,\Delta t$.

2.3.3 Convergence analysis

In order to analyze the uniform convergence of the solution $u^n(x)$ of (2.6) to the exact solution $u(x, t_n)$, we shall do the stability analysis and also derive the consistency result of the scheme (2.6). It is clear that the operator $\big(I + \Delta t\, L_{x,\varepsilon}\big)$ satisfies a maximum principle so that

$$\big\|\big(I + \Delta t\, L_{x,\varepsilon}\big)^{-1}\big\|_\infty \leq \frac{1}{1 + \beta\,\Delta t}, \tag{2.7}$$

which ensures the stability of the scheme (2.6) (for more details see [16]). We define the local error e_{n+1} of the time semidiscretization scheme (2.6) by $e_{n+1} = u(x, t_{n+1}) - \widehat{u}^{n+1}(x)$, where $\widehat{u}^{n+1}(x)$ is the solution obtained after one step of the semidiscrete scheme (2.6) by taking the exact value $u(x, t_n)$, instead of $u^n(x)$ as the starting data. Consequently, we have the following system

$$\begin{cases} (I + \Delta t\, L_{x,\varepsilon})\widehat{u}^{n+1}(x) = u(x, t_n) + \Delta t\, f(x, t_{n+1}), \quad x \in \overline{\Omega}, \\ u^{n+1}(0) = u^{n+1}(1) = 0. \end{cases} \tag{2.8}$$

Now, from [17] one can obtain the following consistency result.

Lemma 2.3.1. *Assume that*

$$\left|\frac{\partial^i}{\partial t^i}u(x,t)\right| \leq C, \quad (x,t) \in \overline{\Omega} \times [0,T], \quad 0 \leq i \leq 2.$$

Then, the local error corresponding to the scheme (2.6) satisfies

$$\left\|e_{n+1}\right\|_\infty \leq C(\Delta t)^2. \tag{2.9}$$

Finally, combining the consistency result (2.9) with the stability result (2.7) of the scheme (2.6), we deduce the following convergence result.

Theorem 2.3.2. *Under the hypothesis of Lemma 2.3.1, we have*

$$\sup_{n \leq T/\Delta t} \left\|u(t_n) - u^n\right\|_\infty \leq C\Delta t. \tag{2.10}$$

Therefore, the time semidiscretization process is uniformly convergent of first-order in time.

2.3.4 Asymptotic behavior of the solution of semidiscrete problem

The following lemma shows that the asymptotic behavior of the exact solution $\widehat{u}^{n+1}(x)$ of the resulting semidiscrete problem (2.8) and their spatial derivatives, up to certain order, is essentially same as that of the exact solution u of the IBVP (2.1)-(2.2) and its spatial derivatives as shown in Theorem 2.2.3. To estimate the local truncation error associated with the hybrid finite difference scheme in the spatial discretization, we shall use the following result.

Lemma 2.3.3. *The exact solution of (2.8) can be decomposed as*

$$\widehat{u}^{n+1}(x) = \eta\widehat{w}^{n+1}(x) + \widehat{z}^{n+1}(x), \tag{2.11}$$

where the components of $\widehat{u}^{n+1}(x)$ satisfy the following

$$\begin{cases} \widehat{w}^{n+1}(x) = \exp\big(-a(1)(1-x)/\varepsilon\big), \quad \eta = \dfrac{\varepsilon}{a(1)}\dfrac{d\widehat{u}^{n+1}}{dx}(1), \\ \left|\dfrac{d^i\widehat{z}^{n+1}}{dx^i}\right| \leq C\left[1 + \varepsilon^{-i+1}\exp\big(-\alpha(1-x)/\varepsilon\big)\right], \quad 0 \leq i \leq 4. \end{cases} \tag{2.12}$$

Proof. The proof of this lemma is given in [15]. ■

2.4 The Spatial Discretization

This section describes the piecewise-uniform Shishkin mesh for the spatial discretization of the domain and studies the behavior of the difference scheme used to discretize the problem (2.8) with respect to the spatial variable.

2.4.1 The piecewise-uniform Shishkin mesh

Consider the spatial domain $\overline{\Omega} = [0, 1]$ and let $N \geq 4$ be an even positive integer. Since the problem (2.1)-(2.2) has only a regular layer at $x = 1$, to define the piecewise-uniform Shishkin mesh we divide the domain $\overline{\Omega}$ into two subintervals $[0, 1-\tau]$ and $[1-\tau, 1]$ and on each subinterval a uniform mesh with $N/2$ mesh-intervals is placed such that $\overline{\Omega}_x^{N,\varepsilon} = \{0 = x_0, x_1, \ldots, x_{N/2} = 1-\tau, \ldots, x_N = 1\}$. Here, the transition point $1-\tau$, which separates the coarse and fine portions of the mesh, is obtained by taking

$$\tau = \min\left\{\frac{1}{2}, \tau_0 \varepsilon \ln N\right\}, \tag{2.13}$$

where τ_0 is a positive constant will be chosen later on. In the analysis, we shall assume that $\tau = \tau_0 \varepsilon \ln N$, because otherwise N^{-1} is exponentially small relatively to ε, which is very unlikely in practice (and in this case the method can be analyzed in the classical way).

Let the mesh widths in space be denoted by

$$h_i = x_i - x_{i-1},\ i = 1, \ldots, N, \quad \widehat{h}_i = h_i + h_{i+1},\ i = 1, \ldots, N-1.$$

Let $\rho_i = h_i/\varepsilon$, $i = 1, \ldots, N$. Further, let $H = 2(1-\tau)/N$ and $h = 2\tau/N$ be the mesh widths in $[0, 1-\tau]$ and $[1-\tau, 1]$ respectively. Then, it is easy to see that

$$N^{-1} \leq H \leq 2N^{-1}, \quad h = 2\tau_0 \varepsilon N^{-1} \ln N.$$

2.4.2 The finite difference scheme

For spatial discretization of (2.8), we propose a hybrid scheme which is a proper combination of the midpoint upwind scheme in the outer region $[0, 1-\tau]$ and the classical central difference scheme in the boundary layer region $(1-\tau, 1]$. Then, the numerical scheme takes the following form:

$$\begin{cases} \widehat{U}_{i-1/2}^{n+1} + \Delta t\, L_{mu}^N \widehat{U}_i^{n+1} = \dfrac{1}{2}(u(x_{i-1}, t_n) + u(x_i, t_n)) + \Delta t\, f_{i-1/2}^{n+1}, & \text{for } 1 \leq i \leq N/2, \\ \widehat{U}_i^{n+1} + \Delta t\, L_{cen}^N \widehat{U}_i^{n+1} = u(x_i, t_n) + \Delta t\, f_i^{n+1}, & \text{for } N/2 < i \leq N-1, \end{cases} \tag{2.14}$$

where

$$\begin{cases} L_{mu}^N \widehat{U}_i^{n+1} \equiv -\varepsilon \delta_x^2 \widehat{U}_i^{n+1} + a_{i-1/2} D_x^- \widehat{U}_i^{n+1} + b_{i-1/2} \widehat{U}_{i-1/2}^{n+1}, \\ L_{cen}^N \widehat{U}_i^{n+1} \equiv -\varepsilon \delta_x^2 \widehat{U}_i^{n+1} + a_i D_x^0 \widehat{U}_i^{n+1} + b_i \widehat{U}_i^{n+1}. \end{cases} \tag{2.15}$$

After rearranging the terms in (2.14), we obtain the following form of the difference scheme:

$$\begin{cases} L_\varepsilon^N \widehat{U}_i^{n+1} \equiv r_i^- \widehat{U}_{i-1}^{n+1} + r_i^0 \widehat{U}_i^{n+1} + r_i^+ \widehat{U}_{i+1}^{n+1} \\ \qquad = \dfrac{1}{2}(u(x_{i-1}, t_n) + \Delta t\, f_{i-1}^{n+1}) + \dfrac{1}{2}(u(x_i, t_n) + \Delta t\, f_i^{n+1}), \qquad \text{for } 1 \le i \le N/2, \\ L_\varepsilon^N \widehat{U}_i^{n+1} \equiv r_i^- \widehat{U}_{i-1}^{n+1} + r_i^0 \widehat{U}_i^{n+1} + r_i^+ \widehat{U}_{i+1}^{n+1} \\ \qquad = u(x_i, t_n) + \Delta t\, f_i^{n+1}, \qquad \text{for } N/2 < i \le N-1, \end{cases} \tag{2.16}$$

where the coefficients are given by

$$r_i^- = \Delta t\, r_{mu,i}^- + \frac{1}{2}, \quad r_i^0 = \Delta t\, r_{mu,i}^0 + \frac{1}{2}, \quad r_i^+ = \Delta t\, r_{mu,i}^+, \quad \text{for } 1 \le i \le N/2, \tag{2.17}$$

and

$$r_i^- = \Delta t\, r_{cen,i}^-, \quad r_i^0 = \Delta t\, r_{cen,i}^0 + 1, \quad r_i^+ = \Delta t\, r_{cen,i}^+, \quad \text{for } N/2 < i \le N-1, \tag{2.18}$$

here

$$\begin{cases} r_{mu,i}^- = -\dfrac{2\varepsilon}{\widehat{h}_i h_i} - \dfrac{a_{i-1/2}}{h_i} + \dfrac{b_{i-1/2}}{2}, \\ r_{mu,i}^0 = \dfrac{2\varepsilon}{h_i h_{i+1}} + \dfrac{a_{i-1/2}}{h_i} + \dfrac{b_{i-1/2}}{2}, \\ r_{mu,i}^+ = -\dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}}, \end{cases} \quad \text{and} \quad \begin{cases} r_{cen,i}^- = -\dfrac{2\varepsilon}{\widehat{h}_i h_i} - \dfrac{a_i}{\widehat{h}_i}, \\ r_{cen,i}^0 = \dfrac{2\varepsilon}{h_i h_{i+1}} + b_i, \\ r_{cen,i}^+ = -\dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}} + \dfrac{a_i}{\widehat{h}_i}. \end{cases} \tag{2.19}$$

2.4.3 Error analysis

This section studies the consistency and stability of the proposed numerical scheme (2.16). Finally, the ε-uniform convergence is analyzed.

Lemma 2.4.1. *Assume that $N \ge N_0$, where*

$$\frac{N_0}{\ln N_0} \ge \tau_0 \|a\|_\infty \quad \textit{and} \tag{2.20}$$

$$\frac{(\|b\|_\infty + \Delta t^{-1})}{\alpha} \le N_0. \tag{2.21}$$

Then, we have

$$\begin{cases} r_i^- < 0,\ r_i^+ < 0, & \textit{for } 1 \le i \le N-1, \\ |r_1^0| - |r_1^+| \ge 0,\ |r_i^0| - |r_i^-| - |r_i^+| \ge 0, & \textit{for } 1 < i \le N/2, \\ |r_{N-1}^0| - |r_{N-1}^-| > 0,\ |r_i^0| - |r_i^-| - |r_i^+| > 0, & \textit{for } N/2 < i < N-1. \end{cases}$$

Proof. For $1 \le i \le N/2$, from (2.17) it is clear that $r_i^+ < 0$ and also using (2.21) we can obtain $r_i^- < 0$. Moreover, it is easy to prove that $|r_1^0| - |r_1^+| \ge 0$ and $|r_i^0| - |r_i^-| - |r_i^+| \ge 0$, for $1 < i \le N/2$.

Similarly, for $N/2 < i \le N-1$, from (2.18) it follows that $r_i^- < 0$ and also we have $r_i^+ < 0$ using (2.20). Now, using (2.20), from (2.18) we deduce that

$$|r_i^-| + |r_i^+| = \Delta t\left(\frac{2\varepsilon}{\widehat{h}_i h_i} + \frac{a_i}{\widehat{h}_i}\right) + \Delta t\left(\frac{2\varepsilon}{\widehat{h}_i h_{i+1}} - \frac{a_i}{\widehat{h}_i}\right) < \Delta t\left(\frac{2\varepsilon}{h_i h_{i+1}} + b_i\right) + 1 = |r_i^0|,$$

for $N/2 < i < N-1$ and

$$|r_{N-1}^-| = \Delta t\,\frac{2\varepsilon}{h_{N-1}h_N} - \Delta t\left(\frac{2\varepsilon}{\widehat{h}_{N-1}h_N} - \frac{a_{N-1}}{\widehat{h}_{N-1}}\right) < \Delta t\left(\frac{2\varepsilon}{h_{N-1}h_N} + b_{N-1}\right) + 1 = |r_{N-1}^0|.$$

This completes the proof. ■

Remark 2.4.2. *Lemma 2.4.1 shows that under the assumptions (2.20) and (2.21), the matrix associated with the difference operator L_ε^N defined in (2.16) is an M-matrix and therefore, the operator L_ε^N satisfies a* ***discrete maximum principle****. Hence, the method is uniformly stable in the supremum norm.*

Now, for the numerical scheme (2.16) the local truncation error is defined as

$$\begin{aligned}
\tau_{i,\widehat{u}^{n+1}} &= L_\varepsilon^N[\widehat{u}_i^{n+1} - \widehat{U}_i^{n+1}] \\
&= \begin{cases} r_i^-\widehat{u}_{i-1}^{n+1} + r_i^0\widehat{u}_i^{n+1} + r_i^+\widehat{u}_{i+1}^{n+1} - \dfrac{1}{2}\left(\widehat{u}_{i-1}^{n+1} + \Delta t\,(L_{x,\varepsilon}\widehat{u}^{n+1})(x_{i-1})\right) \\ -\dfrac{1}{2}\left(\widehat{u}_i^{n+1} + \Delta t\,(L_{x,\varepsilon}\widehat{u}^{n+1})(x_i)\right), \qquad \text{for } 1 \le i \le N/2, \\ r_i^-\widehat{u}_{i-1}^{n+1} + r_i^0\widehat{u}_i^{n+1} + r_i^+\widehat{u}_{i+1}^{n+1} \\ -\left(\widehat{u}_i^{n+1} + \Delta t\,(L_{x,\varepsilon}\widehat{u}^{n+1})(x_i)\right), \qquad \text{for } N/2 < i \le N-1, \end{cases} \\
&= \Delta t\,\tau_{i,\widehat{u}^{n+1}}^x,
\end{aligned} \tag{2.22}$$

where

$$\tau_{i,\widehat{u}}^x = \begin{cases} L_{mu}^N\widehat{u}_i - (L_{x,\varepsilon}\widehat{u})_{i-1/2}, & \text{for } 1 \le i \le N/2, \\ L_{cen}^N\widehat{u}_i - (L_{x,\varepsilon}\widehat{u})(x_i), & \text{for } N/2 < i \le N-1, \end{cases} \tag{2.23}$$

is the truncation error corresponding to the hybrid scheme used for the stationary 1D singularly perturbed convection-diffusion problem. The next lemma gives us useful estimates for the truncation error $\tau_{i,\widehat{u}}^x$.

Lemma 2.4.3. *Let $g(x)$ be a smooth function defined on $\overline{\Omega}$ and also let $g_i = g(x_i)$ on $\overline{\Omega}_x^{N,\varepsilon}$. Then the following estimates hold true:*

$$\left|L_{mu}^N(g_i) - (L_{x,\varepsilon}g)_{i-1/2}\right| \leq C\varepsilon \int_{x_{i-1}}^{x_{i+1}} \left|g'''(s)\right|ds + Ch_i \int_{x_{i-1}}^{x_i} \Big(\left|g'''(s)\right| + \left|g''(s)\right| + \left|g'(s)\right|\Big)ds, \tag{2.24}$$

for $1 \leq i \leq N/2$ and

$$\left|L_{cen}^N(g_i) - (L_{x,\varepsilon}g)(x_i)\right| \leq Ch \int_{x_{i-1}}^{x_{i+1}} \Big(\varepsilon\left|g^{(4)}(s)\right| + \left|g'''(s)\right|\Big)ds, \quad \textit{for } N/2 < i \leq N-1. \tag{2.25}$$

Proof. Use Taylor's formula with the integral form of remainder as in [[41], Lemma 3.3], where $\dfrac{1}{n!}\displaystyle\int_p^x (x-s)^n g(s)ds$ is the expression for the remainder obtained from the Taylor series expansions of the function $g(x)$ at a point $p \in \overline{\Omega}$. ∎

To obtain appropriate estimates for the local truncation error $\tau_{i,\hat{u}^{n+1}}$, the proper bounds on the truncation error $\tau_{i,\hat{u}}^x$ are derived in the following lemma.

Lemma 2.4.4. *Suppose that the assumption (2.20) of Lemma 2.4.1 holds true. Then, the truncation error given by (2.23) satisfies the following bounds:*

$$\left|\tau_{i,\hat{u}}^x\right| \leq \begin{cases} C\left[(\varepsilon + h_i)h_i + \dfrac{1}{\max\{\varepsilon, h_i\}}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)\right], & \textit{for } 1 \leq i < N/2, \\ C\left[(\varepsilon + h_i)h_i + \varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)\right], & \textit{for } i = N/2, \\ C\left[h^2 + h^2\varepsilon^{-3}\exp\big(-\alpha(1-x_i)/\varepsilon\big)\right], & \textit{for } N/2 < i \leq N-1. \end{cases}$$

Proof. In this proof we consider different cases depending on the location of the mesh point $x_i \in \overline{\Omega}_x^{N,\varepsilon}$ and for each case, we use the decomposition (2.11) denoted by $\tau_{i,\hat{z}}^x$ and $\tau_{i,\hat{w}}^x$, the truncation errors corresponding to $\hat{z}(x)$ and $\hat{w}(x)$ respectively.

Case 1. **(Outer region)** For $1 \leq i \leq N/2$. Here two subcases have been considered to find the appropriate estimates depending on the value of $\rho_i = h_i/\varepsilon$.

(*i*) When $\rho_i \leq 1$. Estimating the derivatives of $\hat{z}$ by using (2.12) and (2.24), we deduce that

$$\begin{aligned}\left|\tau_{i,\hat{z}}^x\right| &\leq C(\varepsilon + h_i)h_i + C\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big) + Ch_i\varepsilon^{-1}\exp\big(-\alpha(1-x_i)/\varepsilon\big) \\ &\leq C(\varepsilon + h_i)h_i + C\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big).\end{aligned}$$

Likewise,

$$\left|\tau_{i,\hat{w}}^x\right| \leq C\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big).$$

Hence, employing $\left|\tau^x_{i,\widehat{z}}\right|$ and $\left|\tau^x_{i,\widehat{w}}\right|$ in (2.11), we finally obtain

$$\left|\tau^x_{i,\widehat{u}}\right| \;\leq\; C(\varepsilon+h_i)h_i + C\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big), \quad \text{for } 1\leq i\leq N/2.$$

(ii) For $\rho_i \geq 1$. Arguing in the same way as in Case(i), it is easy to prove that

$$\left|\tau^x_{i,\widehat{z}}\right| \;\leq\; \begin{cases} C(\varepsilon+h_i)h_i + C\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big), & \text{for } 1\leq i< N/2,\\ C(\varepsilon+h_i)h_i + Ch_i\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big), & \text{for } i=N/2, \end{cases} \tag{2.26}$$

using $h_i = h_{i+1} = H$ and $s^k\exp(-s)\leq C$, for all $s\geq 0$, k a positive integer, in the first inequality. To find the estimates of $\left|\tau^x_{i,\widehat{w}}\right|$, we shall proceed in a distinct way as we know the function $\widehat{w}(x)$ and we shall calculate $\tau^x_{i,\widehat{w}} = L^N_{mu}\widehat{w}_i - (L_{x,\varepsilon}\widehat{w})_{i-1/2}$, exactly for $1\leq i\leq N/2$. Here,

$$\begin{aligned}\tau^x_{i,\widehat{w}} \;=\;& r^-_{mu,i}(\widehat{w}_{i-1}-\widehat{w}_i) + r^+_{mu,i}(\widehat{w}_{i+1}-\widehat{w}_i) + \frac{b_{i-1}}{2}(\widehat{w}_i-\widehat{w}_{i-1}) + \frac{1}{2}\varepsilon(\widehat{w}''_i+\widehat{w}''_{i-1})\\ &-\frac{1}{2}(a_i\widehat{w}'_i + a_{i-1}\widehat{w}'_{i-1}).\end{aligned} \tag{2.27}$$

Using (2.12), we obtain the followings

$$\begin{cases} \left|\frac{1}{2}\varepsilon(\widehat{w}''_i+\widehat{w}''_{i-1})\right| \;\leq\; C\varepsilon^{-1}\exp\big(-a(1)(1-x_i)/\varepsilon\big),\\ \left|\frac{1}{2}(a_i\widehat{w}'_i+a_{i-1}\widehat{w}'_{i-1})\right| \;\leq\; C\varepsilon^{-1}\exp\big(-a(1)(1-x_i)/\varepsilon\big),\\ \left|\frac{b_{i-1}}{2}(\widehat{w}_i-\widehat{w}_{i-1})\right| \;\leq\; C\exp\big(-a(1)(1-x_i)/\varepsilon\big). \end{cases} \tag{2.28}$$

Also, from (2.19), we have

$$\left|r^-_{mu,i}\right| \;\leq\; \frac{2\varepsilon}{\widehat{h}_i h_i} + \frac{a_{i-1/2}}{h_i} + \frac{b_{i-1/2}}{2} \;\leq\; \frac{C}{h_i}, \quad r^+_{mu,i} = -\frac{2\varepsilon}{\widehat{h}_i h_{i+1}}. \tag{2.29}$$

Now, for $1\leq i<N/2$, using (2.12) and (2.29), we deduce that

$$\left|r^-_{mu,i}(\widehat{w}_{i-1}-\widehat{w}_i) + r^+_{mu,i}(\widehat{w}_{i+1}-\widehat{w}_i)\right| \;\leq\; Ch_i^{-1}\exp\big(-a(1)(1-x_{i+1})/\varepsilon\big). \tag{2.30}$$

Therefore, by using (2.28) and (2.30) in equation (2.27), we obtain the following

$$\begin{aligned}\left|\tau^x_{i,\widehat{w}}\right| \;&\leq\; Ch_i^{-1}\left[1+\frac{h_{i+1}}{\varepsilon}\exp\big(-a(1)h_{i+1}/\varepsilon\big)\right]\exp\big(-a(1)(1-x_{i+1})/\varepsilon\big)\\ &\leq\; Ch_i^{-1}\exp\big(-a(1)(1-x_{i+1})/\varepsilon\big), \quad \text{for } 1\leq i<N/2.\end{aligned} \tag{2.31}$$

For $i = N/2$, using (2.12) and (2.29) we proceed as follows:

$$\begin{aligned}
&\Big|r^-_{mu,i}(\widehat{w}_{i-1}-\widehat{w}_i)+r^+_{mu,i}(\widehat{w}_{i+1}-\widehat{w}_i)\Big| \\
\le\;& \Big|r^-_{mu,i}\big(\exp\big(-a(1)H/\varepsilon\big)-1\big)+r^+_{mu,i}\big(\exp\big(a(1)h/\varepsilon\big)-1\big)\Big|\exp\big(-a(1)(1-x_i)/\varepsilon\big) \\
\le\;& \left[\frac{C}{H}\big(1-\exp\big(-a(1)H/\varepsilon\big)\big)+\frac{2\varepsilon}{h(H+h)}\big(\exp\big(a(1)h/\varepsilon\big)-1\big)\right]\exp\big(-a(1)(1-x_i)/\varepsilon\big) \\
\le\;& \left(\frac{C}{H}+\frac{C}{H+h}\right)\exp\big(-a(1)(1-x_i)/\varepsilon\big) \\
\le\;& CH^{-1}\exp\big(-a(1)(1-x_i)/\varepsilon\big), \qquad (2.32)
\end{aligned}$$

using $\exp(\psi) \ge 0$ and $\exp(\psi) \le 1 + C\psi$ in bounded intervals of ψ, in the above inequality. Hence, for $i = N/2$, using (2.28) and (2.32) in equation (2.27), we have

$$\left|\tau^x_{i,\widehat{w}}\right| \;\le\; C\varepsilon^{-1}\exp\big(-a(1)(1-x_i)/\varepsilon\big). \qquad (2.33)$$

Therefore, employing (2.26), (2.31) and (2.33) in equation (2.11), we finally obtain the following estimates

$$\left|\tau^x_{i,\widehat{u}}\right| \;\le\; \begin{cases} C(\varepsilon+h_i)h_i + Ch_i^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big), & \text{for } 1 \le i < N/2, \\ C(\varepsilon+h_i)h_i + C\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big), & \text{for } i = N/2. \end{cases}$$

Case 2. (**Inner region**) When $N/2 < i \le N-1$. In this case, using (2.12) and (2.25) it is straightforward to obtain

$$\begin{aligned}
\left|\tau^x_{i,\widehat{z}}\right| \;&\le\; Ch^2 + Ch\varepsilon^{-1}\Big[\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)-\exp\big(-\alpha(1-x_{i-1})/\varepsilon\big)\Big], \\
\left|\tau^x_{i,\widehat{w}}\right| \;&\le\; Ch\varepsilon^{-2}\Big[\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)-\exp\big(-\alpha(1-x_{i-1})/\varepsilon\big)\Big].
\end{aligned}$$

Finally, combining $\left|\tau^x_{i,\widehat{z}}\right|$ and $\left|\tau^x_{i,\widehat{w}}\right|$, we have

$$\begin{aligned}
\left|\tau^x_{i,\widehat{u}}\right| \;&\le\; Ch^2 + Ch\varepsilon^{-2}\Big[\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)-\exp\big(-\alpha(1-x_{i-1})/\varepsilon\big)\Big] \\
&=\; Ch^2 + Ch\varepsilon^{-2}\exp\big(-\alpha(1-x_i)/\varepsilon\big)\sinh(\alpha h/\varepsilon).
\end{aligned}$$

But according to our assumption, $N/\ln N \ge \tau_0\|a\|_\infty$, which is equivalent to $\alpha h/\varepsilon \le 2$ and since $\sinh\xi \le C\xi$ for $0 \le \xi \le 2$, so $\sinh(\alpha h/\varepsilon) \le C\alpha h/\varepsilon$. Thus, the required estimate for $N/2 < i \le N-1$ follows clearly. This completes the proof. ■

Lemma 2.4.5. *The local truncation error of (2.16) satisfies the following estimates*

$$\left|\tau_{i,\widehat{u}^{n+1}}\right| \;\le\; \begin{cases} C\Delta t\Big[(\varepsilon+h_i)h_i + \dfrac{1}{\max\{\varepsilon,h_i\}}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)\Big], & \text{for } 1\le i<N/2,\\ C\Delta t\Big[(\varepsilon+h_i)h_i + \varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)\Big], & \text{for } i=N/2,\\ C\Delta t\Big[h^2 + h^2\varepsilon^{-3}\exp\big(-\alpha(1-x_i)/\varepsilon\big)\Big], & \text{for } N/2<i\le N-1. \end{cases}$$

Proof. This result follows from Lemma 2.4.4. ■

Let us mention that we have just found estimates for the local truncation error $\tau_{i,\widehat{u}^{n+1}}$ that are not uniform in ε. Therefore, to obtain the uniform convergence of the method, we shall use the discrete maximum principle with the adequate barrier functions. To do this, we define the following mesh functions:

$$Z_i \;=\; 1+x_i, \quad i=0,\dots,N, \quad \phi_i(\gamma) \;=\; \prod_{j=i+1}^{N}\mu_j^{-1}(\gamma), \quad i=0,\dots,N-1,$$

where $\mu_i(\gamma) \;=\; 1+\gamma h_i/\varepsilon$, $1\le i\le N$, and we take $\phi_N(\gamma)=1$, where γ is a positive constant.

Lemma 2.4.6. *If $\gamma<\alpha/2$ and the assumption (2.20) of Lemma 2.4.1 holds true, then for some constant C, we have*

$$L_\varepsilon^N\phi_i(\gamma) \;\ge\; \begin{cases} \dfrac{C\Delta t}{\max\{\varepsilon,h_i\}}\phi_i(\gamma), & \text{for } 1\le i\le N/2,\\ \dfrac{C\Delta t}{\varepsilon}\phi_i(\gamma), & \text{for } N/2<i\le N-1. \end{cases} \tag{2.34}$$

Proof. Applying the operator L_ε^N to the discrete function ϕ_i, for $1\le i\le N/2$, we obtain

$$\begin{aligned} L_\varepsilon^N\phi_i(\gamma) &= r_i^-\phi_{i-1}(\gamma)+r_i^0\phi_i(\gamma)+r_i^+\phi_{i+1}(\gamma)\\ &= \Delta t\, L_{mu}^N\phi_i(\gamma)+\phi_i(\gamma)\big(1+\mu_i^{-1}(\gamma)\big)/2. \end{aligned}$$

Now, $\phi_i(\gamma)\big(1+\mu_i^{-1}(\gamma)\big)/2 \;\ge\; 0$ and if $\gamma<\alpha/2$ is satisfied, we can deduce that

$$L_{mu}^N\phi_i(\gamma) \ge \frac{C\gamma}{\varepsilon+\gamma h_i}\phi_i(\gamma).$$

Thus, under the condition $\gamma<\alpha/2$,

$$L_\varepsilon^N\phi_i(\gamma) \;\ge\; \frac{C\Delta t}{\max\{\varepsilon,h_i\}}\phi_i(\gamma), \quad \text{for } 1\le i\le N/2. \tag{2.35}$$

Similarly, for $N/2<i\le N-1$, we have

$$\begin{aligned} L_\varepsilon^N\phi_i(\gamma) &= r_i^-\phi_{i-1}(\gamma)+r_i^0\phi_i(\gamma)+r_i^+\phi_{i+1}(\gamma)\\ &= \Delta t\, L_{cen}^N\phi_i(\gamma)+\phi_i(\gamma). \end{aligned}$$

Again, $\phi_i(\gamma) \geq 0$ and if $\gamma < \alpha$ holds, we obtain

$$L^N_{cen}\phi_i(\gamma) \;\geq\; \frac{C\gamma}{\varepsilon(1+\frac{2\gamma}{\|a\|_\infty})}\phi_i(\gamma),$$

since the hypothesis (2.20) yields that

$$h/\varepsilon \;\leq\; 2/\big\|a\big\|_\infty. \tag{2.36}$$

Therefore, under the condition $\gamma < \alpha$,

$$L^N_\varepsilon\phi_i(\gamma) \geq \frac{C\Delta t}{\varepsilon}\phi_i(\gamma), \quad \text{for } N/2 < i \leq N-1. \tag{2.37}$$

Thus, both the results (2.35) and (2.37) will hold if $\gamma < \alpha/2$. Hence, this completes the proof. ■

To prove the uniform convergence of the hybrid scheme, the following results are used.

Lemma 2.4.7. *We have the following inequalities:*

$$(i) \quad \textit{If } \gamma < \alpha/2, \textit{ then } \exp\big(-\alpha(1-x_i)/\varepsilon\big) \;\leq\; \phi_i(\gamma), \quad i = 0,\ldots,N-1. \tag{2.38}$$

$$(ii) \quad \textit{There exists a constant } C \textit{ such that } \phi_{N/2}(\gamma) \;\leq\; CN^{-\gamma\tau_0}. \tag{2.39}$$

Proof. The proof of (i) follows from [[88], *Lemma 2.5*] and the proof of (ii) follows from [[89], *Lemma 3.1*]. ■

Theorem 2.4.8. *Let $\widehat{u}^{n+1}$ and $\{\widehat{U}^{n+1}\}$ be the exact and the discrete solutions of (2.8) and (2.16) respectively, and suppose that $N \geq N_0$ satisfies the conditions (2.20) and (2.21). Then, if $\gamma < \alpha/2$ and $\varepsilon \leq N^{-1}$, we have the following bounds*

$$\big|\widehat{u}^{n+1}_i - \widehat{U}^{n+1}_i\big| \;\leq\; \begin{cases} C\Big(N^{-2} + N^{-\gamma\tau_0}\Big), & \textit{for } 1 \leq i \leq N/2, \\ C\Big(\tau_0^2 N^{-2}\ln^2 N + N^{-\gamma\tau_0}\Big), & \textit{for } N/2 < i \leq N-1. \end{cases} \tag{2.40}$$

Proof. Define the following barrier function

$$\psi_i(\gamma) \;=\; C\Big[(\varepsilon + h_i)h_i Z_i + \mu_{i+1}(\gamma)\phi_i(\gamma)\Big], \quad \text{for } 0 \leq i \leq N.$$

Using (2.38) and Lemmas 2.4.5 and 2.4.6, we have

$$L^N_\varepsilon\psi_i(\gamma) \;\geq\; \big|\tau_{i,\widehat{u}^{n+1}}\big|, \quad \text{for } 1 \leq i \leq N-1.$$

Thus, by the discrete maximum principle for the operator L^N_ε, we obtain

$$\big|\widehat{u}^{n+1}_i - \widehat{U}^{n+1}_i\big| \;\leq\; \psi_i(\gamma), \quad \text{for } 1 \leq i \leq N-1.$$

Therefore, for $1 \le i \le N/2$,

$$\left|\widehat{u}_i^{n+1} - \widehat{U}_i^{n+1}\right| \le C\Big[(\varepsilon + H)H + \phi_{N/2+1}(\gamma)\Big] = C\Big[(\varepsilon + H)H + \Big(1 + \frac{\gamma h}{\varepsilon}\Big)\phi_{N/2}(\gamma)\Big].$$

Now, using $H \le 2N^{-1}$, $\varepsilon \le N^{-1}$ and invoking the inequalities (2.36), (2.39), finally we get

$$\left|\widehat{u}_i^{n+1} - \widehat{U}_i^{n+1}\right| \le C\Big(N^{-2} + N^{-\gamma\tau_0}\Big), \quad \text{for } 1 \le i \le N/2.$$

On the other hand, for $N/2 < i \le N-1$, consider the new barrier function

$$\chi_i(\gamma) = C\Big[(N^{-2} + N^{-\gamma\tau_0})Z_i + h^2\varepsilon^{-2}\phi_i(\gamma)\Big].$$

Then, it follows that

$$\begin{cases} L_\varepsilon^N \chi_i(\gamma) \ge \left|\tau_{i,\widehat{u}^{n+1}}\right|, & \text{for } N/2 < i \le N-1, \\ \chi_{N/2}(\gamma) \ge \left|\widehat{u}_{N/2}^{n+1} - \widehat{U}_{N/2}^{n+1}\right|, & \text{and} \quad \chi_N(\gamma) \ge \left|\widehat{u}_N^{n+1} - \widehat{U}_N^{n+1}\right|. \end{cases}$$

Therefore, it is clear that the operator L_ε^N satisfies the discrete maximum principle on $[1-\tau, 1]$ and hence we obtain

$$\left|\widehat{u}_i^{n+1} - \widehat{U}_i^{n+1}\right| \le \chi_i(\gamma) \le C\Big(\tau_0^2 N^{-2}\ln^2 N + N^{-\gamma\tau_0}\Big), \quad \text{for } N/2 < i \le N-1,$$

where we used $h = 2\tau_0\varepsilon N^{-1}\ln N$. ■

Corollary 2.4.9. *Theorem 2.4.8 implies that for fixed $\tau_0 \ge 2/\gamma$, there exists a constant C such that*

$$\left|\widehat{u}_i^{n+1} - \widehat{U}_i^{n+1}\right| \le \begin{cases} CN^{-2}, & \text{for } 1 \le i \le N/2, \\ CN^{-2}\ln^2 N, & \text{for } N/2 < i \le N-1. \end{cases} \tag{2.41}$$

Therefore, the method (2.16) is in practice uniformly convergent of order almost two with respect to the spatial variable but the error constant C depends only on the value of τ_0 (which is independent of ε).

Corollary 2.4.10. *If we take $N^{-q} \le C\Delta t$ with $0 < q < 1$, then from (2.41), we obtain*

$$\left|\widehat{u}_i^{n+1} - \widehat{U}_i^{n+1}\right| \le \begin{cases} C\Delta t\, N^{-2+q}, & \text{for } 1 \le i \le N/2, \\ C\Delta t\, N^{-2+q}\ln^2 N, & \text{for } N/2 < i \le N-1. \end{cases} \tag{2.42}$$

This bound is required to prove the uniform convergence of the fully discrete scheme.

2.5 Uniform Convergence of the Fully Discrete Scheme

Combining the time semidiscretization scheme (2.8) and the spatial discretization technique given in Section 2.4, the following fully discrete scheme is deduced on the mesh $\overline{G}_\varepsilon^{N,M} = \overline{\Omega}_x^{N,\varepsilon} \times \mathbb{S}_t^M$:

$$\begin{cases} U_i^0 = u_0(x_i), \quad i = 0, \dots, N, \\ \begin{cases} L_\varepsilon^N U_i^{n+1} & \equiv r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1} \\ & = \dfrac{1}{2}(U_{i-1}^n + \Delta t\, f_{i-1}^{n+1}) + \dfrac{1}{2}(U_i^n + \Delta t\, f_i^{n+1}), \qquad \text{for } 1 \le i \le N/2, \\ L_\varepsilon^N U_i^{n+1} & \equiv r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1} \\ & = U_i^n + \Delta t\, f_i^{n+1}, \qquad \text{for } N/2 < i \le N-1, \\ U_0^{n+1} = U_N^{n+1} = 0, \end{cases} \\ \text{for } n = 0, \dots, M-1, \end{cases} \tag{2.43}$$

where the coefficients r_i^-, r_i^0, r_i^+ are described in (2.17)-(2.19) and $U_i^n = U(x_i, t_n)$ is the fully discrete approximation to the exact solution $u(x,t)$ of (2.1)-(2.2) at the mesh point $(x_i, t_n) \in \overline{G}_\varepsilon^{N,M}$.

Theorem 2.5.1. ***(Global error)*** *Let $u(x, t_n)$ be the exact solution of (2.1)-(2.2) and $\{U^n\}$ be the discrete solution of the fully discrete scheme (2.43), at time level $t_n = n\,\Delta t$. Assume that $N \ge N_0$ satisfies the conditions (2.20), (2.21) and $\varepsilon \le N^{-1}$, $N^{-q} \le C\Delta t$ with $0 < q < 1$. Then, if $\gamma < \alpha/2$ and $\tau_0 \ge 2/\gamma$, the error associated with the total discretization of (2.1)-(2.2) by the fully discrete scheme (2.43) at time level t_n satisfies*

$$\big\|\{u(x_i, t_n)\}_i - \{U_i^n\}_i\big\|_\infty \le \begin{cases} C\Big(\Delta t + N^{-2+q}\Big), & \textit{for } 1 \le i \le N/2, \\ C\Big(\Delta t + N^{-2+q} \ln^2 N\Big), & \textit{for } N/2 < i \le N-1. \end{cases} \tag{2.44}$$

Proof. Let the global error at time level t_n be denoted by $E_i^n = u(x_i, t_n) - U_i^n$, for $0 \le i \le N$. Now, splitting the global error $\{E_i^n\}_i$, we have

$$\big\|\{E_i^n\}_i\big\|_\infty \le \big\|\{u(x_i, t_n)\}_i - \{\widehat{u}_i^n\}_i\big\|_\infty + \big\|\{\widehat{u}_i^n\}_i - \{\widehat{U}_i^n\}_i\big\|_\infty + \big\|\{\widehat{U}_i^n\}_i - \{U_i^n\}_i\big\|_\infty. \tag{2.45}$$

Next, using the estimates given by (2.9) and (2.42) in in the equation (2.45), we deduce that

$$\big\|\{E_i^n\}_i\big\|_\infty \le \begin{cases} C\Delta t\big(\Delta t + N^{-2+q}\big) + \big\|\{\widehat{U}_i^n\}_i - \{U_i^n\}_i\big\|_\infty, & \text{for } 1 \le i \le N/2, \\ C\Delta t\big(\Delta t + N^{-2+q} \ln^2 N\big) + \big\|\{\widehat{U}_i^n\}_i - \{U_i^n\}_i\big\|_\infty, & \text{for } N/2 < i \le N-1. \end{cases} \tag{2.46}$$

Then, applying the stability of the fully discrete scheme, it can be shown that

$$\left\|\{\widehat{U}_i^n\}_i - \{U_i^n\}_i\right\|_\infty \leq \left\|\{u(x_i, t_{n-1})\}_i - \{U_i^{n-1}\}_i\right\|_\infty. \tag{2.47}$$

Finally, from (2.46) and (2.47) we obtain the following recurrence relations

$$\left\|\{E_i^n\}_i\right\|_\infty \leq \begin{cases} C\Delta t\big(\Delta t + N^{-2+q}\big) + \left\|\{E_i^{n-1}\}_i\right\|_\infty, & \text{for } 1 \leq i \leq N/2, \\ C\Delta t\big(\Delta t + N^{-2+q}\ln^2 N\big) + \left\|\{E_i^{n-1}\}_i\right\|_\infty, & \text{for } N/2 < i \leq N-1, \end{cases}$$

and hence, the result (2.44) follows immediately from it. ∎

2.6 Numerical Results

This section presents the numerical results obtained by the fully discrete scheme (2.43) for the following test examples on the piecewise-uniform rectangular mesh $\overline{G}_\varepsilon^{N,M}$. In all the cases, the numerical experiments are performed by choosing the constant $\tau_0 = 4.2$ and $\Delta t = 0.8/N$, otherwise it is mentioned.

Example 2.6.1. *Consider the following parabolic IBVP:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + (1 + x(1-x))\dfrac{\partial u}{\partial x} = f(x,t), \quad (x,t) \in (0,1) \times (0,1], \\ u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \leq 1. \end{cases} \tag{2.48}$$

where the initial data $u_0(x)$ and the source term $f(x,t)$ have been chosen to fit

$$u(x,t) = \exp(-t)\big(m_1 + m_2 x - \exp(-(1-x)/\varepsilon)\big)$$

as the exact solution of the above problem, where $m_1 = \exp(-1/\varepsilon)$ and $m_2 = 1 - \exp(-1/\varepsilon)$. As the exact solution of the IBVP (2.48) is known, for each ε, the maximum point-wise error is calculated by

$$e_\varepsilon^{N,\Delta t} = \max_{(x_i,t_n)\in\overline{G}_\varepsilon^{N,M}} \left| u(x_i,t_n) - U^{N,\Delta t}(x_i,t_n) \right|,$$

where $u(x_i,t_n)$ and $U^{N,\Delta t}(x_i,t_n)$ denote the exact and the numerical solution obtained on the mesh $\overline{G}_\varepsilon^{N,M}$ with N mesh-intervals in the spatial direction and M mesh-intervals in the t-direction such that $\Delta t = T/M$ is the uniform time step. In addition, the corresponding order of convergence is determined by

$$p_\varepsilon^{N,\Delta t} = \log_2\left(\frac{e_\varepsilon^{N,\Delta t}}{e_\varepsilon^{2N,\Delta t/2}}\right).$$

Now, for each N and Δt, we define $e^{N,\Delta t} = \max_{\varepsilon} e_{\varepsilon}^{N,\Delta t}$ as the ε-uniform maximum point-wise error and the corresponding local ε-uniform order of convergence is defined by

$$p^{N,\Delta t} = \log_2\left(\frac{e^{N,\Delta t}}{e^{2N,\Delta t/2}}\right).$$

The calculated maximum point-wise errors $e_{\varepsilon}^{N,\Delta t}$ and the corresponding order of convergence $p_{\varepsilon}^{N,\Delta t}$ for Example 2.6.1 are presented in Table 2.1 for various values of ε and N.

Example 2.6.2. *Consider the following parabolic IBVP:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon\dfrac{\partial^2 u}{\partial x^2} + (2-x^2)\dfrac{\partial u}{\partial x} + xu = 10t^2\exp(-t)x(1-x), \quad (x,t) \in (0,1)\times(0,3], \\ u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \le 3. \end{cases} \tag{2.49}$$

As the exact solution of the IBVP (2.49) is not known, to obtain the accuracy of the numerical solution and also to demonstrate the ε-uniform convergence of the proposed scheme, we use the double mesh principle as in [15] which is described as follows:

Let $\widetilde{U}^{2N,\Delta t/2}(x_i,t_n)$ be the numerical solution obtained on the fine mesh $\widetilde{G}_{\varepsilon}^{2N,2M} = \widetilde{\Omega}_x^{2N,\varepsilon} \times \mathbb{S}_t^{2M}$ with $2N$ mesh-intervals in the spatial direction and $2M$ mesh-intervals in the t-direction, where $\widetilde{\Omega}_x^{N,\varepsilon}$ is a piecewise-uniform Shishkin mesh as like $\overline{\Omega}_x^{N,\varepsilon}$ with the transition parameter $\widetilde{\tau}$ given by

$$\widetilde{\tau} = \min\left\{\frac{1}{2}, \tau_0\varepsilon\ln\left(\frac{N}{2}\right)\right\},$$

such that for $i = 0,1\ldots N$, the i^{th} point of the mesh $\overline{\Omega}_x^{N,\varepsilon}$ coincides with the $2i^{\text{th}}$ point of the mesh $\widetilde{\Omega}_x^{2N,\varepsilon}$. Then for each ε, we calculate the maximum point-wise error by

$$E_{\varepsilon}^{N,\Delta t} = \max_{(x_i,t_n)\in\overline{G}_{\varepsilon}^{N,M}} \left| U^{N,\Delta t}(x_i,t_n) - \widetilde{U}^{2N,\Delta t/2}(x_i,t_n)\right|,$$

and the corresponding order of convergence by

$$P_{\varepsilon}^{N,\Delta t} = \log_2\left(\frac{E_{\varepsilon}^{N,\Delta t}}{E_{\varepsilon}^{2N,\Delta t/2}}\right).$$

For each N and Δt, the quantities $E^{N,\Delta t}$ and $P^{N,\Delta t}$ are defined analogously to $e^{N,\Delta t}$ and $p^{N,\Delta t}$ based on the error $E_{\varepsilon}^{N,\Delta t}$ as in the previous example. In Table 2.2, we display the computed maximum point-wise errors $E_{\varepsilon}^{N,\Delta t}$ and the corresponding order of convergence $P_{\varepsilon}^{N,\Delta t}$ for Example 2.49.

From the results given in Tables 2.1 and 2.2 we see the monotonically decreasing behavior of the computed ε-uniform errors. This ensures that the proposed scheme (2.43) is ε-uniformly convergent. As a complement of this observation, we have plotted the maximum point-wise errors for Examples 2.48 and 2.49 in Figure 2.2. We have also included two surface plots for $N = 128$ in Figure 2.1 to clearly visualize the boundary layers of the corresponding numerical solutions of the IBVPs (2.48) and (2.49) for $\varepsilon = 10^{-4}$.

But the numerical results presented in Tables 2.1 and 2.2 do not clearly reflect the actual theoretical order of convergence of the proposed scheme (2.43) in space, as predicted by Theorem 2.5.1. We know that the error consists of two parts due to the spatial and the temporal discretization (briefly say, the spatial and the time error). Again, by the use of hybrid scheme we are improving only the accuracy in space, so we expect the spatial error will be essentially much smaller than the time error. Hence, the order of convergence observed in Tables 2.1 and 2.2 is due to the effect of the time error, where the contribution from the spatial error is relatively small.

Therefore, to justify the spatial order of convergence properly, we have carried out the numerical experiments for $M = N^2$ (as like in Hemker *et al.* [34]) and displayed the maximum point-wise errors as well as the corresponding order of convergence in Tables 2.3 and 2.4 respectively, for Examples 2.48 and 2.49. Further, we have highlighted these errors in Figure 2.3. This in fact reveals the second-order convergence in the outer region and the almost second-order convergence (reduced by the logarithmic factor) in the inner region of the proposed scheme (2.43), with respect to the spatial variable as derived in Corollary 2.4.9. Even this analysis also signifies the role of τ_0 to fulfill our claim in the same corollary.

Remark 2.6.3. *Looking towards the numerical experiments, we would like to point out that the restriction $N^{-q} \leq C\Delta t$, with $0 < q < 1$, seems not necessary to get the ε-uniform convergence and the corresponding uniform order of convergence of the proposed scheme (2.43).*

2.7 Conclusion

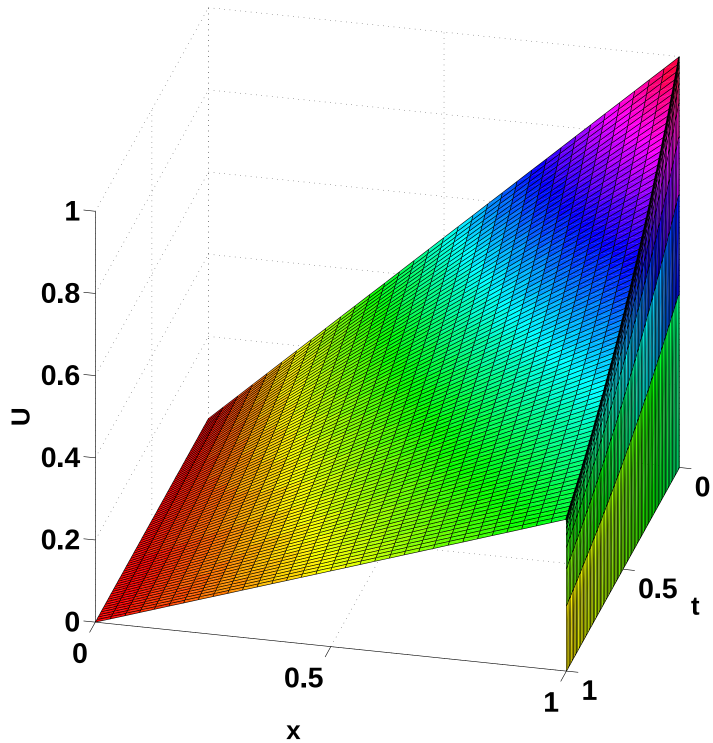
In this chapter, one-dimensional singularly perturbed parabolic convection-diffusion IBVP with regular boundary layer is solved by using an efficient hybrid numerical scheme on a piecewise-uniform Shishkin mesh. The hybrid scheme is constructed taking a proper combination of the midpoint upwind scheme and the classical central difference scheme for the spatial discretization and the backward-Euler scheme for discretizing the time derivative. It has been shown both theoretically and computationally that the newly prosed scheme is ε-uniform convergent with first-order accurate in time and almost second-order accurate in space. Moreover, we have computationally shown that the use of the backward-Euler method for time discretization does not reduce the order of convergence with respect to the spatial variable obtained by the hybrid numerical scheme. In the next chapter, we shall discuss a post processing technique based on the classical implicit upwind scheme to achieve better accuracy in both time and space.

Table 2.1: *Maximum point-wise errors and the corresponding order of convergence for Example 2.6.1.*

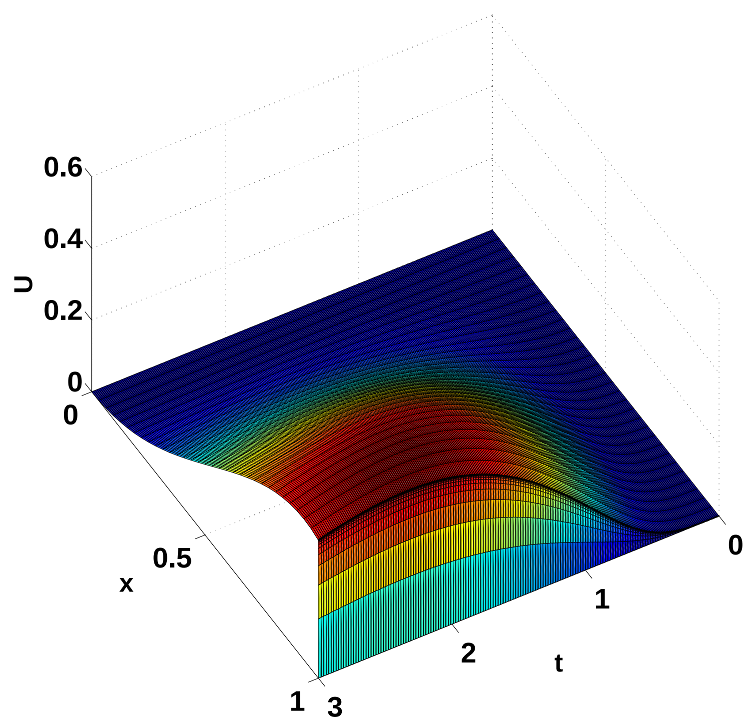
ε	Number of mesh-intervals N						
	16	32	64	128	256	512	1024
$1e-1$	7.7416e-3	2.3969e-3	8.6402e-4	3.5400e-4	1.5854e-4	7.4832e-5	3.6334e-5
	1.6914	1.4720	1.2873	1.1589	1.0831	1.0423	
$1e-2$	3.3344e-2	1.2246e-2	4.4419e-3	1.6249e-3	6.0951e-4	2.3759e-4	9.8234e-5
	1.4451	1.4631	1.4509	1.4146	1.3592	1.2741	
$1e-3$	3.2498e-2	1.1994e-2	4.3716e-3	1.6070e-3	6.0543e-4	2.3789e-4	9.9354e-5
	1.4380	1.4561	1.4438	1.4084	1.3477	1.2597	
$1e-4$	3.2413e-2	1.1969e-2	4.3644e-3	1.6051e-3	6.0497e-4	2.3791e-4	9.9460e-5
	1.4373	1.4554	1.4431	1.4078	1.3465	1.2582	
$1e-5$	3.2404e-2	1.1966e-2	4.3637e-3	1.6049e-3	6.0492e-4	2.3791e-4	9.9471e-5
	1.4372	1.4554	1.4430	1.4077	1.3464	1.2581	
$1e-6$	3.2404e-2	1.1966e-2	4.3637e-3	1.6049e-3	6.0492e-4	2.3791e-4	9.9472e-5
	1.4372	1.4553	1.4430	1.4077	1.3463	1.2580	
$1e-7$	3.2403e-2	1.1966e-2	4.3636e-3	1.6049e-3	6.0492e-4	2.3791e-4	9.9472e-5
to $1e-10$	1.4372	1.4553	1.4430	1.4077	1.3463	1.2580	
$e^{N,\Delta t}$	**3.3344e-2**	**1.2246e-2**	**4.4419e-3**	**1.6249e-3**	**6.0951e-4**	**2.3791e-4**	**9.9472e-5**
$p^{N,\Delta t}$	**1.4451**	**1.4631**	**1.4509**	**1.4146**	**1.3572**	**1.2580**	

Table 2.2: *Maximum point-wise errors and the corresponding order of convergence for Example 2.6.2.*

ε	Number of mesh-intervals N						
	16	32	64	128	256	512	1024
$1e-1$	4.1285e-3	9.6124e-4	4.6353e-4	2.2736e-4	1.1256e-4	5.6000e-5	2.7972e-5
	2.1027	1.0522	1.0277	1.0143	1.0072	1.0014	
$1e-2$	2.1294e-2	8.0080e-3	2.6024e-3	8.6121e-4	2.7054e-4	8.0862e-5	2.2722e-5
	1.4109	1.6216	1.5954	1.6705	1.7423	1.8314	
$1e-3$	2.0370e-2	7.7135e-3	2.5447e-3	8.4760e-4	2.6909e-4	8.1830e-5	2.3523e-5
	1.4010	1.5999	1.5860	1.6553	1.7174	1.7985	
$1e-4$	2.0330e-2	7.7067e-3	2.5523e-3	8.5269e-4	2.7219e-4	8.3605e-5	2.4462e-5
	1.3994	1.5943	1.5817	1.6474	1.7029	1.7730	
$1e-5$	2.0327e-2	7.7064e-3	2.5533e-3	8.5333e-4	2.7256e-4	8.3821e-5	2.4577e-5
	1.3992	1.5937	1.5812	1.6465	1.7012	1.7700	
$1e-6$	2.0326e-2	7.7063e-3	2.5534e-3	8.5339e-4	2.7260e-4	8.3843e-5	2.4588e-5
	1.3992	1.5936	1.5811	1.6464	1.7010	1.7697	
$1e-7$	2.0326e-2	7.7063e-3	2.5534e-3	8.5340e-4	2.7261e-4	8.3845e-5	2.4590e-5
to $1e-10$	1.3992	1.5936	1.5811	1.6464	1.7010	1.7697	
$E^{N,\Delta t}$	**2.1294e-2**	**8.0080e-3**	**2.6024e-3**	**8.6121e-4**	**2.7261e-4**	**8.3845e-5**	**2.7972e-5**
$P^{N,\Delta t}$	**1.4109**	**1.6216**	**1.5954**	**1.6595**	**1.7010**	**1.5837**	

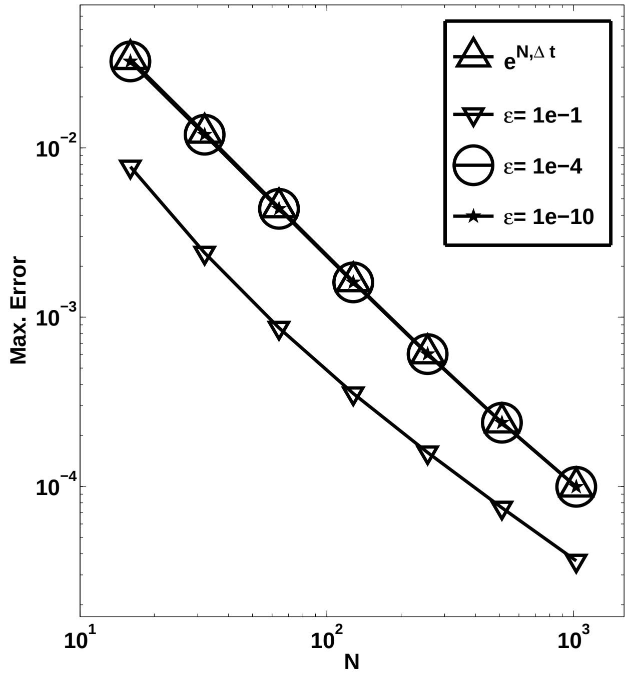


(a) Example 2.6.1.

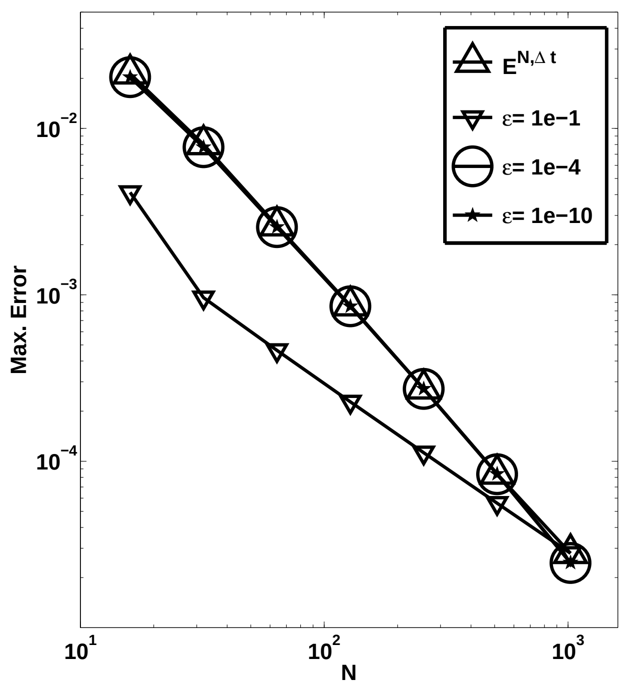


(b) Example 2.6.2.

Figure 2.1: *Surface plots of the Numerical solutions for* $\varepsilon = 1e-4, N = 64$.



(a) Example 2.6.1.



(b) Example 2.6.2.

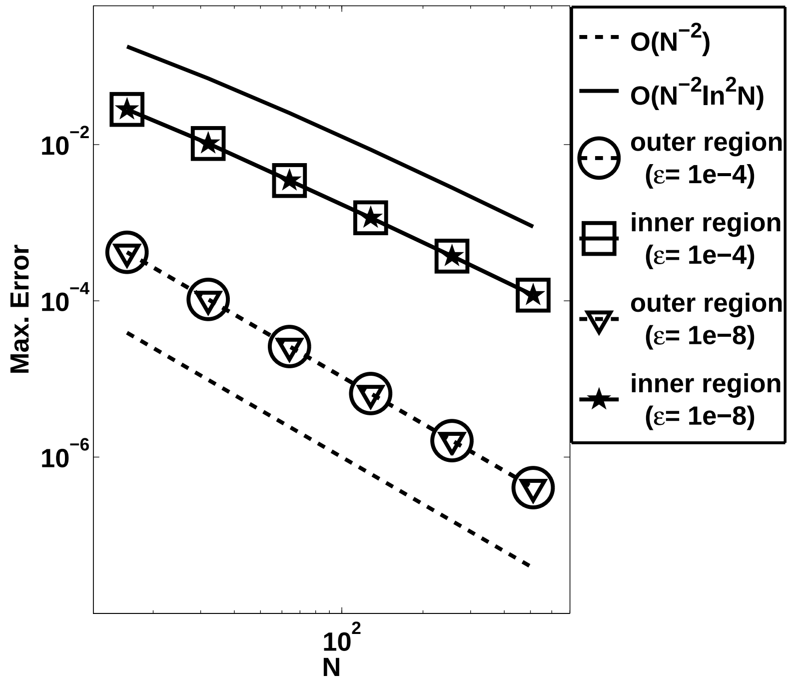
Figure 2.2: *Loglog plot of the maximum point-wise errors.*

Table 2.3: *Maximum point-wise errors and the corresponding order of convergence calculated for Example 2.6.1 by taking* $M = N^2$.

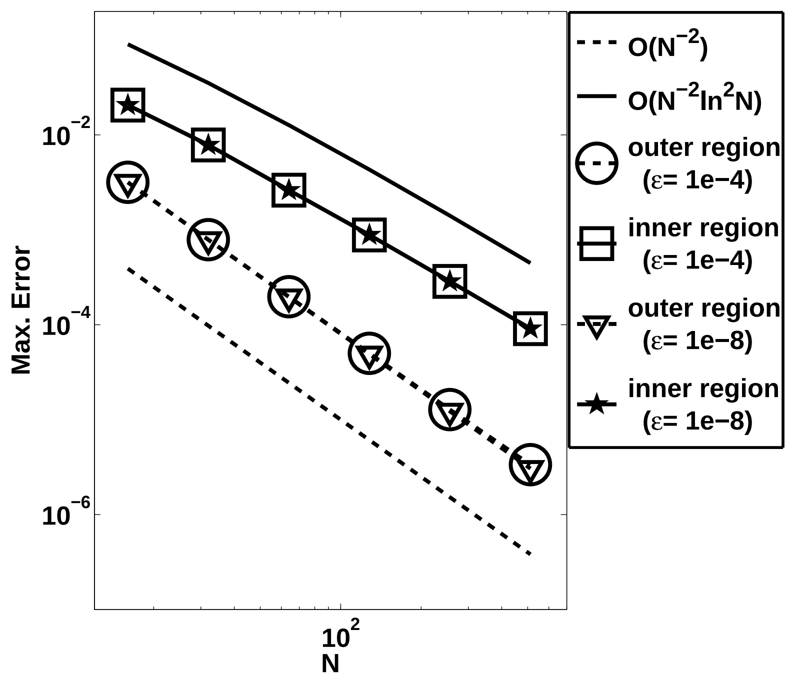
N	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-8}$	
	outer region $[0, 1-\tau]$	**inner region** $(1-\tau, 1]$	**outer region** $[0, 1-\tau]$	**inner region** $(1-\tau, 1]$
16	4.1927e-4 2.0057	2.8218e-2 1.4475	4.2047e-4 2.0046	2.8207e-2 1.4474
32	1.0440e-4 2.0017	1.0346e-2 1.5742	1.0478e-4 2.0006	2.8207e-2 1.5742
64	2.6069e-5 2.0013	3.4744e-3 1.5832	2.6184e-5 2.0001	3.4734e-3 1.5832
128	6.5116e-6 2.0012	1.1596e-3 1.6310	6.5455e-6 2.0000	1.1593e-3 1.6310
256	1.6265e-6 2.0012	3.7427e-4 1.6592	1.6363e-6 2.0000	3.7427e-4 1.6592

Table 2.4: *Maximum point-wise errors and the corresponding order of convergence calculated for Example 2.6.2 by taking* $M = N^2$.

N	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-8}$	
	outer region $[0, 1-\tau]$	**inner region** $(1-\tau, 1]$	**outer region** $[0, 1-\tau]$	**inner region** $(1-\tau, 1]$
16	3.1630e-3 2.0070	2.0601e-2 1.3981	3.1604e-3 2.0106	2.0598e-2 1.3979
32	7.8689e-4 1.9950	7.8168e-3 1.5807	7.8434e-4 2.0027	7.8164e-3 1.5800
64	1.9741e-4 1.9828	2.6134e-3 1.5683	1.9572e-4 2.0007	2.6145e-3 1.5678
128	4.9943e-5 1.9653	8.8124e-4 1.6229	4.8907e-5 2.0000	8.8195e-4 1.6219
256	1.2790e-5 1.9283	2.8613e-4 1.6558	1.2227e-5 2.0000	2.8655e-4 1.6538



(a) Example 2.6.1.



(b) Example 2.6.2.

Figure 2.3: *Loglog plot for the spatial order of convergence.*

Chapter 3

Richardson Extrapolation Technique for Singularly Perturbed Parabolic Convection-Diffusion Problems

This chapter deals with the study of a post-processing technique for one-dimensional singularly perturbed parabolic convection-diffusion problems exhibiting a regular boundary layer. For discretizing the time derivative, we use the classical backward-Euler method and for the spatial discretization the simple upwind scheme is used on a piecewise-uniform Shishkin mesh. We show that the use of Richardson extrapolation technique improves the ε-uniform accuracy of simple upwinding in the discrete supremum norm from $O\big(N^{-1}\ln N + \Delta t\big)$ to $O\big(N^{-2}\ln^2 N + \Delta t^2\big)$, where N is defined as before and Δt is the step size in the temporal direction.

3.1 Introduction

Consider the following 1D singularly perturbed parabolic convection-diffusion IBVP, as discussed in the previous chapter, posed on the domain $G = \Omega \times (0, T]$, $\Omega = (0, 1)$:

$$\begin{cases} L_\varepsilon u(x,t) \equiv \left(\dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} + b(x)u\right)(x,t) \;=\; f(x,t), \quad (x,t) \in G, \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = u(1,t) = 0, \quad t \in (0,T], \end{cases} \tag{3.1}$$

$0 < \varepsilon \ll 1$ is a small parameter and the coefficients a, b are sufficiently smooth functions such that

$$a(x) > 2\alpha > 0, \quad b(x) \geq \beta \geq 0 \quad \text{on } \overline{\Omega}. \tag{3.2}$$

Under sufficient smoothness and compatibility conditions (2.3) and (2.4) imposed on the functions u_0 and f, the parabolic IBVP (3.1)-(3.2), in general admits a unique solution $u(x,t) \in \mathcal{C}^{2+\lambda}(\overline{G})$ exhibiting a regular boundary layer of width $O(\varepsilon)$ at $x = 1$.

The outline of this chapter is as follows: Section 3.2 provides *a-priori* bounds on the derivatives of the analytical solution *via* decomposition. Section 3.3 describes the piecewise-uniform Shishkin mesh and the classical implicit upwind scheme for discretization of the continuous problem. The ε-uniform convergence result (the proof of which is given in Appendix A) for the standard upwind scheme is stated followed by some technical results that will be used later in this chapter. Afterwards, the Richardson extrapolation technique is introduced in Section 3.4 and the main theoretical result, *i.e.*, the extrapolated solution of the upwind scheme converges ε-uniformly with almost second-order accuracy to the exact solution of the continuous problem, is proved. Finally, Section 2.6 conducts the numerical experiments that confirm the theoretical results.

3.2 Bounds on the Solution Decomposition

In this section, stronger bounds on the derivatives of the analytical solution of the problem (3.1)-(3.2) are derived, by decomposing the solution into smooth and layer components. This will be used in the subsequent sections for the proof of ε-uniform error estimate.

We decompose the solution u of the IBVP (3.1)-(3.2) as $u = v + w$, where v and w are respectively the smooth component and the layer component. The smooth component is further decomposed into the sum

$$v = \sum_{i=0}^{4} \varepsilon^i v_i,$$

where the functions $v_i,\ i = 0,1,2,3$, are solutions to the following first-order problems

$$\begin{cases} \dfrac{\partial v_0}{\partial t} + a\dfrac{\partial v_0}{\partial x} + bv_0 \;=\; f \quad \text{in } G, \\ v_0(0,t) = u(0,t), \quad t \in (0,T], \quad v_0(x,0) = u(x,0), \quad x \in \overline{\Omega}; \\ \dfrac{\partial v_i}{\partial t} + a\dfrac{\partial v_i}{\partial x} + bv_i \;=\; \dfrac{\partial^2 v_{i-1}}{\partial x^2} \quad \text{in } G, \\ v_i(0,t) = 0, \quad t \in (0,T], \quad v_i(x,0) = 0, \quad x \in \overline{\Omega}, \quad i = 1,2,3, \end{cases} \tag{3.3}$$

and lastly, the function v_4 satisfies

$$\begin{cases} L_\varepsilon v_4 \;=\; \dfrac{\partial^2 v_4}{\partial x^2} \quad \text{in } G, \\ v_4(0,t) = v_4(1,t) = 0, \quad t \in (0,T], \quad v_4(x,0) = 0, \quad x \in \overline{\Omega}. \end{cases} \tag{3.4}$$

Hence, the smooth component v satisfies the following IBVP

$$\begin{cases} L_\varepsilon v = f \quad \text{in } G, \\ v(0,t) = 0, \quad v(1,t) = \sum_{i=0}^{4} \varepsilon^i v_i(1,t), \quad t \in (0,T], \\ v(x,0) = u(x,0), \quad x \in \overline{\Omega}, \end{cases} \tag{3.5}$$

and therefore, the layer component w must satisfy

$$\begin{cases} L_\varepsilon w = 0 \quad \text{in } G, \\ w(0,t) = 0, \quad w(1,t) = u(1,t) - v(1,t), \quad t \in (0,T], \\ w(x,0) = 0, \quad x \in \overline{\Omega}. \end{cases} \tag{3.6}$$

Theorem 3.2.1. *For all non-negative integers l, m, satisfying $0 \le l + m \le 5$, the smooth component v and the layer component w, defined in (3.5) and (3.6), respectively, satisfy the following bounds*

$$\left\| \frac{\partial^{l+m} v}{\partial x^l \partial t^m} \right\|_{\infty,G} \le C\big(1 + \varepsilon^{4-l}\big),$$

and

$$\left| \frac{\partial^{l+m} w}{\partial x^l \partial t^m}(x,t) \right| \le C\varepsilon^{-l} \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in G.$$

Proof. First, we shall obtain the stronger bounds on the smooth component v, defined in (3.5) and its derivatives. Since the functions $v_i,\ i = 0,1,2,3$, are solutions to the problems specified in (3.3), which are independent of the parameter ε, so they have the following ε-uniformly bounded derivatives

$$\left\| \frac{\partial^{l+m} v_i}{\partial x^l \partial t^m} \right\|_{\infty,G} \le C, \quad i = 0,1,2,3, \quad \text{for } 0 \le l + m \le 5. \tag{3.7}$$

Again, v_4 is the solution of the problem similar to (3.1) and hence, applying the estimate (2.5) analogously to v_4, we have the bounds

$$\left\| \frac{\partial^{l+m} v_4}{\partial x^l \partial t^m} \right\|_{\infty,G} \le C\Big(1 + \varepsilon^{-l} \exp\big(-\alpha(1-x)/\varepsilon\big)\Big), \quad \text{for } 0 \le l + m \le 5. \tag{3.8}$$

Thus, combining the estimates (3.7) and (3.8), for $0 \le l + m \le 5$, the required bounds on the smooth component v are obtained as follows

$$\left\| \frac{\partial^{l+m} v_i}{\partial x^l \partial t^m} \right\|_{\infty,G} \le \sum_{i=0}^{4} \varepsilon^i \left\| \frac{\partial^{l+m} v_i}{\partial x^l \partial t^m} \right\|_{\infty,G} \le C\big(1 + \varepsilon^{4-l}\big).$$

On the other hand, applying the minimum principle (Lemma 2.2.1) on $\overline{G}$ with the barrier function

$$\phi(x,t) = C \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in G,$$

for sufficiently large C, we obtain the required bound on w. Finally, the bounds on the derivatives of w follow from the arguments given in [25, 54] and hence, this completes the proof. ∎

3.3 Numerical Approximation

This section provides a suitable mesh for the discretization of the domain to obtain an ε-uniformly convergent difference scheme and also explicitly describes the difference scheme used to discretize the problem (3.1)-(3.2).

3.3.1 Discretization of the domain

Consider the domain $\overline{G} = \overline{\Omega} \times [0, T] = [0, 1] \times [0, T]$ and and let $N \geq 4$ be an even positive integer. Here, we will construct a rectangular mesh $\overline{G}_\tau^{N,M} = \overline{\Omega}_x^{N,\tau} \times \mathbb{S}_t^M$. Since the problem (3.1)-(3.2) has only a regular layer at $x = 1$, to define the piecewise-uniform Shishkin mesh we divide the domain $\overline{\Omega}$ into two subintervals $[0, 1-\tau]$ and $[1-\tau, 1]$ and on each subinterval a uniform mesh with $N/2$ mesh-intervals is placed such that $\overline{\Omega}_x^{N,\tau} = \{0 = x_0, x_1, \ldots, x_{N/2} = 1 - \tau, \ldots, x_N = 1\}$. Here, the transition point $1 - \tau$, which separates the coarse and fine portions of the mesh, is obtained by taking

$$\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon}{\alpha} \ln N\right\}. \tag{3.9}$$

In the analysis, we shall assume that $\tau = (2\varepsilon \ln N)/\alpha$, because otherwise N^{-1} is exponentially small relatively to ε, which is very unlikely in practice (and in this case the method can be analyzed in the classical way).

Let the mesh widths in space be denoted by

$$h_i = x_i - x_{i-1},\ i = 1, \ldots, N, \quad \widehat{h}_i = h_i + h_{i+1},\ i = 1, \ldots, N-1.$$

Further, let $H = 2(1-\tau)/N$ and $h = 2\tau/N$ be the mesh widths in $[0, 1-\tau]$ and $[1-\tau, 1]$ respectively. On the time domain $[0, T]$, the equidistant meshes are introduced in the temporal variable such that

$$\mathbb{S}_t^M = \{t_n = n\,\Delta t,\ n = 0, \ldots, M,\ t_0 = 0, t_M = T,\ \Delta t = T/M\},$$

where M denotes the number of mesh-intervals in the t-direction.

3.3.2 The classical implicit upwind scheme

Let $G_\tau^{N,M} = \overline{G}_\tau^{N,M} \bigcap G$ and $\Gamma_\tau^{N,M} = \overline{G}_\tau^{N,M} \backslash G_\tau^{N,M}$. For the discretization of the continuous problem (3.1), the following classical implicit upwind finite difference scheme is used on the mesh $\overline{G}_\tau^{N,M}$:

$$\begin{cases} L_\varepsilon^{N,M} U^{N,\Delta t} \equiv \Big(D_t^- - \varepsilon \delta_x^2 + a D_x^- + b \Big) U^{N,\Delta t} = f \quad \text{in } G_\tau^{N,M}, \\ U^{N,\Delta t} = u, \quad \text{on } \Gamma_\tau^{N,M}. \end{cases} \tag{3.10}$$

One can show that the finite difference operator $L_\varepsilon^{N,M}$ satisfies the following well-known discrete minimum principle, which lead to ε-uniform stability of the difference operator $L_\varepsilon^{N,M}$.

Lemma 3.3.1. ***(Discrete Minimum Principle)*** *Assume that a mesh function Z satisfies $Z \geq 0$ on $\Gamma_\tau^{N,M}$. Then $L_\varepsilon^{N,M} Z \geq 0$ in $G_\tau^{N,M}$ implies that $Z \geq 0$ at each point of $\overline{G}_\tau^{N,M}$.*

The following theorem states that the numerical scheme (3.10) converges ε-uniformly on the Shishkin mesh $\overline{G}_\tau^{N,M}$ with almost first-order accuracy.

Theorem 3.3.2. ***(Error due to Upwinding)*** *Let u be the solution of the continuous problem (3.1)-(3.2) and $U^{N,\Delta t}$ be the solution of the discrete problem (3.10). Then, the error associated with the discrete solution $U^{N,\Delta t}$ at time level t_n satisfies*

$$\Big| u(x_i, t_n) - U^{N,\Delta t}(x_i, t_n) \Big| \leq C \Big(N^{-1} \ln N + \Delta t \Big), \quad \text{for } 1 \leq i \leq N-1. \tag{3.11}$$

Proof. See Appendix A. ■

Our aim is to apply a post-processing technique on the discrete solution $U^{N,\Delta t}$ of the problem (3.10) so as to attain ε-uniform order of accuracy greater than one with respect to both the spatial and the temporal variables for the IBVP (3.1)-(3.2) and this will be obtained *via* Richardson extrapolation technique. Before introducing this technique, some technical lemmas which will be used in Section 3.4 are stated here.

Lemma 3.3.3. *On $\overline{\Omega}_x^{N,\tau} = \{x_i\}_0^N$, define the mesh function*

$$S_i = \prod_{j=1}^{i} \Big(1 + \frac{\alpha h_j}{\varepsilon} \Big),$$

(with the usual convention that for $i = 0$, $S_0 = 1$). Then for $1 \leq i \leq N-1$,

$$L_\varepsilon^{N,M} S_i \geq \Big(\frac{C_1}{\varepsilon + \alpha h_i} \Big) S_i, \tag{3.12}$$

for some constant C_1. Moreover, for $N/2 + 1 \leq i \leq N-1$, we have

$$L_\varepsilon^{N,M} S_i \geq C_2 \varepsilon^{-1} S_i, \tag{3.13}$$

for some constant C_2.

Proof. Firstly, $(S_i - S_{i-1}) = \dfrac{\alpha h_i}{\varepsilon} S_{i-1}$. So, we have

$$\begin{aligned} L_\varepsilon^{N,M} S_i &= -\frac{2\alpha}{(h_i + h_{i+1})}(S_i - S_{i-1}) + a_i \frac{\alpha}{\varepsilon} S_{i-1} + b_i S_i \\ &\geq \frac{\alpha}{\varepsilon} S_{i-1}\left[a_i - \frac{2\alpha h_i}{(h_i + h_{i+1})}\right] \geq \frac{C\alpha}{\varepsilon + \alpha h_i} S_i, \quad \text{for } 1 \leq i \leq N-1. \end{aligned}$$

Therefore, (3.12) is proved and (3.13) is an easy consequence of it, since $h/\varepsilon < 4/\alpha$. ■

Lemma 3.3.4. *For the Shishkin mesh $\overline{\Omega}_x^{N,\tau} = \{x_i\}_0^N$, the following inequalities hold true:*

$$(i) \quad \exp\big(-\alpha(1-x_i)/\varepsilon\big) \leq \prod_{j=i+1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}, \quad \textit{for all } i. \tag{3.14}$$

(ii) There exists a constant C such that

$$\prod_{j=i+1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \leq C N^{-4(1-i/N)}, \qquad \textit{for } N/2 \leq i \leq N-1. \tag{3.15}$$

Proof. The proof of (i) follows from [[88], *Lemma 2.5*] and the proof of (ii) follows from [[88], Lemma 3.1]. ■

3.4 Extrapolation of $U^{N,\Delta t}$

This section introduces the extrapolation technique and also provides the error analysis by decomposing the solution of the discrete problem (3.10). The ε-uniform error estimate associated with the extrapolated solution is established finally.

3.4.1 Extrapolation technique

To improve the accuracy of the solution $U^{N,\Delta t}$ by extrapolation, we shall solve the discrete problem (3.10) on the fine mesh $\overline{G}_\tau^{2N,2M} = \overline{\Omega}_x^{2N,\tau} \times \mathbb{S}_t^{2M}$ with $2N$ mesh intervals in the spatial direction and $2M$ mesh intervals in the t-direction, where $\overline{\Omega}_x^{2N,\tau}$ is a piecewise-uniform Shishkin mesh having the same transition point $1-\tau$ as $\overline{\Omega}_x^{N,\tau}$ and is obtained by bisecting each mesh interval of $\overline{\Omega}_x^{N,\tau}$. Clearly, $\overline{G}_\tau^{N,M} = \big\{(x_i, t_n)\big\} \subset \overline{G}_\tau^{2N,2M} = \big\{(\tilde{x}_i, \tilde{t}_n)\big\}$ and the following are the corresponding mesh widths in $\overline{G}_\tau^{2N,2M}$:

$$\tilde{x}_i - \tilde{x}_{i-1} = \begin{cases} H/2, & \text{for } \tilde{x}_i \in \overline{\Omega}_x^{2N,\tau} \cap [0, 1-\tau], \\ h/2, & \text{for } \tilde{x}_i \in \overline{\Omega}_x^{2N,\tau} \cap [1-\tau, 1], \end{cases} \quad \text{and} \quad \tilde{t}_n - \tilde{t}_{n-1} = \Delta t/2, \quad \text{for } \tilde{t}_n \in \mathbb{S}_t^{2M}.$$

Let $\widetilde{U}^{2N,\Delta t/2}$ be the solution of the discrete problem (3.10) on the mesh $\overline{G}_\tau^{2N,2M}$. Now, it follows from Theorem 3.3.2 that

$$\begin{aligned} U^{N,\Delta t}(x_i,t_n) - u(x_i,t_n) &= C_3\Big(N^{-1}\ln N + \Delta t\Big) + \mathcal{R}_{N,\Delta t}(x_i,t_n), \\ &= C_3\Big(N^{-1}(\alpha\tau/2\varepsilon) + \Delta t\Big) + \mathcal{R}_{N,\Delta t}(x_i,t_n), \\ &\qquad (x_i,t_n) \in \overline{G}_\tau^{N,M}, \end{aligned} \tag{3.16}$$

where C_3 is some fixed constant and the remainder term $\mathcal{R}_{N,\Delta t}$ is $o\Big(N^{-1}\ln N+\Delta t\Big)$. Keeping in mind that $\widetilde{U}^{2N,\Delta t/2}$ is obtained using the same transition point $1-\tau$, we have

$$\begin{aligned} \widetilde{U}^{2N,\Delta t/2}(\tilde{x}_i,\tilde{t}_n) - u(\tilde{x}_i,\tilde{t}_n) &= C_3\Big((2N)^{-1}(\alpha\tau/2\varepsilon) + (\Delta t/2)\Big) + \widetilde{\mathcal{R}}_{2N,\Delta t/2}(\tilde{x}_i,\tilde{t}_n), \\ &\qquad (\tilde{x}_i,\tilde{t}_n) \in \overline{G}_\tau^{2N,2M}, \end{aligned} \tag{3.17}$$

where the remainder term $\widetilde{\mathcal{R}}_{2N,\Delta t/2}$ is $o\Big(N^{-1}\ln N + \Delta t\Big)$. Now, elimination of the $O(N^{-1})$ and $O(\Delta t)$ terms from (3.16) and (3.17) lead to the following approximation

$$\begin{aligned} u(x_i,t_n) - \Big(2\widetilde{U}^{2N,\Delta t/2}(x_i,t_n) - U^{N,\Delta t}(x_i,t_n)\Big) &= 2\widetilde{\mathcal{R}}_{N,\Delta t}(x_i,t_n) - \mathcal{R}_{N,\Delta t}(x_i,t_n) \\ &= o\Big(N^{-1}\ln N + \Delta t\Big), \quad (x_i,t_n) \in \overline{G}_\tau^{N,M}. \end{aligned}$$

Therefore, we shall use the following simple extrapolation formula

$$U_{extp}^{N,\Delta t}(x_i,t_n) = 2\widetilde{U}^{2N,\Delta t/2}(x_i,t_n) - U^{N,\Delta t}(x_i,t_n), \quad (x_i,t_n) \in \overline{G}_\tau^{N,M}, \tag{3.18}$$

which is expected to yield an approximation of $u(x_i,t_n)$ more accurate than either $U^{N,\Delta t}(x_i,t_n)$ or $\widetilde{U}^{2N,\Delta t/2}(x_i,t_n)$.

3.4.2 Solution decomposition

To obtain the estimate of the nodal error $\Big|u(x_i,t_n) - U_{extp}^{N,\Delta t}(x_i,t_n)\Big|$ after extrapolation of $U^{N,\Delta t}$, we decompose the solution $U^{N,\Delta t}$ on the mesh $\overline{G}_\tau^{N,M}$ into the sum

$$U^{N,\Delta t} = V^{N,\Delta t} + W^{N,\Delta t},$$

where the smooth component $V^{N,\Delta t}$ and the layer component $W^{N,\Delta t}$ are respectively the solutions of the following discrete problems

$$\begin{cases} L_\varepsilon^{N,M} V^{N,\Delta t} = f \quad \text{in } G_\tau^{N,M}, \quad V^{N,\Delta t} = v \quad \text{on } \Gamma_\tau^{N,M}, \\ L_\varepsilon^{N,M} W^{N,\Delta t} = 0 \quad \text{in } G_\tau^{N,M}, \quad W^{N,\Delta t} = w \quad \text{on } \Gamma_\tau^{N,M}. \end{cases} \tag{3.19}$$

Similarly, we decompose the solution $\widetilde{U}^{2N,\Delta t/2}$ into the smooth component $V^{2N,\Delta t/2}$ and the layer component $W^{2N,\Delta t/2}$ on the fine mesh $\overline{G}_\tau^{2N,2M}$ as

$$\widetilde{U}^{2N,\Delta t/2} = \widetilde{V}^{2N,\Delta t/2} + \widetilde{W}^{2N,\Delta t/2}.$$

Then, the errors can be written in the following form

$$U^{N,\Delta t} - u = \big(V^{N,\Delta t} - v\big) + \big(W^{N,\Delta t} - w\big) \quad \text{and} \quad \widetilde{U}^{2N,\Delta t/2} - u = \big(\widetilde{V}^{2N,\Delta t/2} - v\big) + \big(\widetilde{W}^{2N,\Delta t/2} - w\big).$$

3.4.3 Extrapolation of $V^{N,\Delta t}$

Let

$$\zeta_1(x,t) = \frac{1}{2}a(x)\frac{\partial^2 v}{\partial x^2}(x,t) \quad \text{and} \quad \zeta_2(x,t) = \frac{1}{2}\frac{\partial^2 v}{\partial t^2}(x,t), \quad (x,t) \in G.$$

Lemma 3.4.1. *Assume that $\varepsilon \le N^{-1}$. Then, the local truncation error associated to the smooth component satisfies*

$$L_\varepsilon^{N,M}\big(V^{N,\Delta t} - v\big)(x_i, t_{n+1}) = h_i\,\zeta_1(x_i, t_{n+1}) + \Delta t\,\zeta_2(x_i, t_{n+1}) + O\Big(H^2 + \Delta t^2\Big), \quad \textit{for } 1 \le i \le N-1.$$

Proof. Employing the bounds on the derivatives of v given in Theorem 3.2.1 and $\varepsilon \le N^{-1} \le H$, it is straightforward to obtain the following result from the Taylor's expansion,

$$\begin{aligned} & L_\varepsilon^{N,M}\big(V^{N,\Delta t} - v\big)(x_i, t_{n+1}) \\ = \ & \frac{\varepsilon}{3\widehat{h}_i}\left[h_{i+1}^2\frac{\partial^3 v}{\partial x^3}(\xi_1, t_{n+1}) - h_i^2\frac{\partial^3 v}{\partial x^3}(\xi_2, t_{n+1})\right] + \frac{h_i}{2}a(x_i)\frac{\partial^2 v}{\partial x^2}(x_i, t_{n+1}) - \frac{h_i^2}{3!}a(x_i)\frac{\partial^3 v}{\partial x^3}(\xi_3, t_{n+1}) \\ & + \frac{\Delta t}{2}\frac{\partial^2 v}{\partial t^2}(x_i, t_{n+1}) - \frac{\Delta t^2}{3!}\frac{\partial^3 v}{\partial t^3}(x_i, \tilde{t}), \end{aligned}$$

for some $\xi_1 \in (x_i, x_{i+1})$, $\xi_2, \xi_3 \in (x_{i-1}, x_i)$ and $\tilde{t} \in (t_n, t_{n+1})$. ■

Now, according to the classical approach of Keller [39], define the functions E_k, $k = 1, 2$, to be the solutions of the following IBVPs

$$\begin{cases} L_\varepsilon E_k \ = \ \zeta_k \quad \text{in } G, \\ E_k(x,0) = 0, \quad x \in \overline{\Omega}, \\ E_k(0,t) = E_k(1,t) = 0, \quad t \in (0,T], \quad k = 1,2. \end{cases} \tag{3.20}$$

Next, decompose E_k into $E_k = \eta_k + \vartheta_k$, $k = 1, 2$, where the smooth component η_k and the layer component ϑ_k satisfy the following IBVPs

$$\begin{cases} L_\varepsilon \eta_k \ = \ \zeta_k, \quad L_\varepsilon \vartheta_k \ = \ 0 \quad \text{in } G, \\ \eta_k(x,0) = \vartheta_k(x,0) = 0, \quad x \in \overline{\Omega}, \\ \eta_k(0,t) = \vartheta_k(0,t) = 0, \quad \eta_k(1,t) = -\vartheta_k(1,t), \quad t \in (0,T], \quad k = 1,2. \end{cases} \tag{3.21}$$

Theorem 3.4.2. *For all non-negative integers l, m, satisfying $0 \le l + m \le 2$, the smooth components $\eta_k, k = 1, 2$, defined in (3.21) satisfy the following bounds*

$$\left\| \frac{\partial^{l+m}\eta_k}{\partial x^l \partial t^m} \right\|_{\infty,G} \le C, \qquad \left\| \frac{\partial^3 \eta_k}{\partial x^3} \right\|_{\infty,G} \le C\varepsilon^{-1}.$$

Proof. Firstly, following the approach of [69] and using the bounds on the derivatives of v given in Theorem 3.2.1, we obtain that the solution E_k of the IBVP (3.20) satisfies the estimate

$$\left| \frac{\partial^{l+m} E_k}{\partial x^l \partial t^m}(x,t) \right| \le C\Big(1 + \varepsilon^{-l} \exp\big(-\alpha(1-x)/\varepsilon\big)\Big), \quad (x,t) \in G, \quad \text{for } 0 \le l+m \le 2. \tag{3.22}$$

This shows that away from the side $x = 1$, the derivatives of E_k are bounded independent of ε.

We now introduce a new function η_k^* on the domain $\overline{G^*}$ which is a sufficiently large neighborhood of $\overline{G}$ beyond the point of discontinuity $x = 1$ so that $\overline{G} \subset \overline{G^*}$. For example, define an extended domain $G^* = \Omega^* \times (0, T],\ \Omega^* = (0, 1.5)$. Let the function η_k^* on $\overline{G^*}$ be defined to be the solution of the following IBVP

$$\begin{cases} L_\varepsilon^* \eta_k^* \, = \, \zeta_k^* \quad \text{in } G^*, \\ \eta_k^*(x,0) = 0, \quad x \in \overline{\Omega^*}, \\ \eta_k^*(0,t) = 0, \quad \eta_k^*(1.5,t) = 0, \quad t \in (0,T], \quad k = 1,2. \end{cases} \tag{3.23}$$

where

$$L_\varepsilon^* \eta_k^* \equiv \left(\frac{\partial \eta_k^*}{\partial t} - \varepsilon \frac{\partial^2 \eta_k^*}{\partial x^2} + a^*(x) \frac{\partial \eta_k^*}{\partial x} + b^*(x) \eta_k^* \right).$$

Here the coefficients a^* and b^* associated with the operator L_ε^* are respectively the smooth extensions of the functions a, b associated with the problem (3.20) from the domain $\overline{\Omega}$ onto the domain $\overline{\Omega^*}$. To deal with the compatibility easily at the corner $(1.5, 0)$, we define the function ζ_k^* as a smooth extension of the function ζ_k from the domain $\overline{G}$ onto the extended domain $\overline{G^*}$ so that $\zeta_k^* = 0$ in a neighborhood of the artificial corner $(1.5, 0)$. At the inflow corner $(0, 0)$, we impose extra compatibility conditions likewise [73].

Afterwards, the solution η_k^* of the problem (3.23) will analogously satisfy the estimate (3.22) on the extended domain $\overline{G^*}$ and hence we get

$$\left\| \frac{\partial^{l+m}\eta_k^*}{\partial x^l \partial t^m} \right\|_{\infty,G} \le C, \quad \text{for } 0 \le l+m \le 2. \tag{3.24}$$

Now, we define the smooth component η_k as a restriction of η_k^* onto the domain $\overline{G}$ and the function η_k is therefore the solution of the following problem

$$\begin{cases} L_\varepsilon \eta_k \, = \, \zeta_k \quad \text{in } G, \\ \eta_k(x,0) = 0, \quad x \in \overline{\Omega}, \\ \eta_k(0,t) = 0, \quad \eta_k(1,t) = \eta_k^*(1,t), \quad t \in (0,T], \quad k = 1,2. \end{cases} \tag{3.25}$$

Thus, the smooth component η_k satisfies the bound (3.24) and also, by differentiating the equation in (3.25) with respect to x, we obtain the required bound for $\dfrac{\partial^3\eta_k}{\partial x^3}$. Hereby, the proof is complete. ■

Remark 3.4.3. *It is to be noted that the bounds obtained in Theorem 3.2.1 on* $\dfrac{\partial^{l+m}v}{\partial x^l\partial t^m}$, *for* $0 \le l+m \le 4$, *are sufficient to derive the estimate (3.22) for the solution* E_k *of the IBVP (3.20).*

Lemma 3.4.4. *Assume that* $\varepsilon \le N^{-1}$. *Then, we have*

$$\big(V^{N,\Delta t}-v\big)(x_i,t_{n+1}) = h_i\,\eta_1(x_i,t_{n+1})+\Delta t\,\eta_2(x_i,t_{n+1})+O\Big(N^{-2}+\Delta t^2\Big), \quad \textit{for } 1\le i\le N-1.$$

Proof. Let $1 \le i \le N-1$. Then, using Lemma 3.4.1 and (3.21), we can easily obtain

$$L_\varepsilon^{N,M}\big(V^{N,\Delta t}-v\big)(x_i,t_{n+1}) = h_i\,L_\varepsilon\eta_1(x_i,t_{n+1})+\Delta t\,L_\varepsilon\eta_2(x_i,t_{n+1})+O\Big(H^2+\Delta t^2\Big). \tag{3.26}$$

Now, from the Taylor's expansion it follows that for $k=1,2$,

$$\Big|\big(L_\varepsilon-L_\varepsilon^{N,M}\big)\eta_k(x_i,t_{n+1})\Big| \le \Big[\frac{\varepsilon}{3}\big(h_i+h_{i+1}\big)\Big\|\frac{\partial^3\eta_k}{\partial x^3}\Big\|_\infty+\frac{h_i}{2}a(x_i)\Big\|\frac{\partial^2\eta_k}{\partial x^2}\Big\|_\infty+\frac{\Delta t}{2}\Big\|\frac{\partial^2\eta_k}{\partial t^2}\Big\|_\infty\Big],$$

and accordingly, using Theorem 3.4.2 and the inequality $h_i \le H$ for all i, we have

$$\begin{aligned}\Big|h_i\,\big(L_\varepsilon-L_\varepsilon^{N,M}\big)\eta_1(x_i,t_{n+1})+\Delta t\,\big(L_\varepsilon-L_\varepsilon^{N,M}\big)\eta_2(x_i,t_{n+1})\Big| &\le C\Big(H^2+H\Delta t+\Delta t^2\Big)\\ &\le C\Big(H^2+\Delta t^2\Big).\end{aligned} \tag{3.27}$$

Therefore, (3.26) and (3.27) together imply that

$$L_\varepsilon^{N,M}\big[V^{N,\Delta t}-v-h_i\,\eta_1-\Delta t\,\eta_2\big](x_i,t_{n+1}) = O\Big(H^2+\Delta t^2\Big), \quad \text{for } 1\le i\le N-1. \tag{3.28}$$

Define the following two discrete functions

$$\psi^{\pm}(x_i,t_n) = C_4\Big(N^{-2}+\Delta t^2\Big)(1+x_i)\pm\chi(x_i,t_n), \quad \text{for } 0\le i\le N \text{ and } n\ge 0,$$

where

$$\chi(x_i,t_n) = \begin{cases}\big[V^{N,\Delta t}-v-h_i\,\eta_1-\Delta t\,\eta_2\big](x_i,t_n), & \text{for } 1\le i\le N-1,\\ 0, & \text{for } i=0,N.\end{cases} \tag{3.29}$$

Now, for $1 \le i \le N-1$, using the inequality $H \le 2N^{-1}$ an easy calculation yields

$$L_\varepsilon^{N,M}\big((N^{-2}+\Delta t^2)(1+x_i)\big) \ge \alpha\Big(N^{-2}+\Delta t^2\Big) \ge \alpha\Big(H^2+\Delta t^2\Big)/4. \tag{3.30}$$

Therefore, from (3.28) and (3.30) it follows that one can choose the constant C_4 sufficiently large so that $L_\varepsilon^{N,M}\psi^{\pm} \geq 0$ in $G_\tau^{N,M}$ and also, by (3.19),(3.21) and (3.29), $\psi^{\pm} \geq 0$ on $\Gamma_\tau^{N,M}$. Thus, applying the discrete minimum principle (Lemma 3.3.1) over $\overline{G}_\tau^{N,M}$, we obtain

$$\left|\left[V^{N,\Delta t} - v - h_i\,\eta_1 - \Delta t\,\eta_2\right](x_i, t_{n+1})\right| \leq C\left(N^{-2} + \Delta t^2\right), \quad \text{for } 1 \leq i \leq N-1,$$

and hereby, the desired result follows. ∎

Remark 3.4.5. *An easier approach is provided in the proof of Lemma 3.4.4 compared to that in [68] and utilizing it, we deduce the estimate for the smooth part of the error after extrapolation, in the following lemma.*

Lemma 3.4.6. *Assume that $\varepsilon \leq N^{-1}$. Then, the error after extrapolation associated to the smooth component $V^{N,\Delta t}$ satisfies*

$$\left|v(x_i, t_{n+1}) - V_{extp}^{N,\Delta t}(x_i, t_{n+1})\right| \leq C\left(N^{-2} + \Delta t^2\right), \quad \textit{for } 1 \leq i \leq N-1.$$

Proof. Since the mesh widths of $\overline{G}_\tau^{2N,2M}$ are half of those of $\overline{G}_\tau^{N,M}$, applying Lemma 3.4.4 on the fine mesh $\overline{G}_\tau^{2N,2M}$, we have

$$\left(\widetilde{V}^{2N,\Delta t/2} - v\right)(x_i, t_{n+1}) = \begin{cases} (H/2)\,\eta_1(x_i, t_{n+1}) + (\Delta t/2)\,\eta_2(x_i, t_{n+1}) + O\left(N^{-2} + \Delta t^2\right), \\ \qquad\qquad \text{for } 1 \leq i \leq N/2, \\ (h/2)\,\eta_1(x_i, t_{n+1}) + (\Delta t/2)\,\eta_2(x_i, t_{n+1}) + O\left(N^{-2} + \Delta t^2\right), \\ \qquad\qquad \text{for } N/2+1 \leq i \leq N-1. \end{cases} \tag{3.31}$$

Therefore, according to the extrapolation formula (3.18), from Lemma 3.4.4 and (3.31) it immediately follows that

$$\begin{aligned} v(x_i, t_{n+1}) - V_{extp}^{N,\Delta t}(x_i, t_{n+1}) &= v(x_i, t_{n+1}) - \left(2\widetilde{V}^{2N,\Delta t/2}(x_i, t_{n+1}) - V^{N,\Delta t}(x_i, t_{n+1})\right) \\ &= -2\left(\widetilde{V}^{2N,\Delta t/2} - v\right)(x_i, t_{n+1}) + \left(V^{N,\Delta t} - v\right)(x_i, t_{n+1}) \\ &= O\left(N^{-2} + \Delta t^2\right), \quad \text{for } 1 \leq i \leq N-1. \end{aligned}$$ ∎

3.4.4 Extrapolation of $W^{N,\Delta t}$

Lemma 3.4.7. *The error after extrapolation associated to the layer component $W^{N,\Delta t}$ satisfies*

$$\left|w(x_i, t_{n+1}) - W_{extp}^{N,\Delta t}(x_i, t_{n+1})\right| \leq CN^{-2}, \quad \textit{for } 1 \leq i \leq N/2.$$

Proof. Let $1 \le i \le N/2$. Now, following the argument resulting to [[88], (2.7)] over $\overline{G}_\tau^{N,M}$ and using (3.15) , we can show that

$$|W^{N,\Delta t}(x_i, t_{n+1})| \le C \prod_{j=i+1}^{N} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1} \le CN^{-2}.$$

Next, from (3.14) and Theorem 3.2.1 we obtain $|w(x_i, t_{n+1})| \le CN^{-2}$. Hence, we have

$$\left|\big(W^{N,\Delta t} - w\big)(x_i, t_{n+1})\right| \le CN^{-2}.$$

Similarly, $\left|\big(\widetilde{W}^{2N,\Delta t/2} - w\big)(x_i, t_{n+1})\right| \le CN^{-2}$ and hereby, the required extrapolated error bound is obtained from the extrapolation formula (3.18). ■

Let

$$\rho_1(x,t) = \frac{2\varepsilon}{\alpha} a(x) \frac{\partial^2 w}{\partial x^2}(x,t) \quad \text{and} \quad \rho_2(x,t) = \frac{1}{2}\frac{\partial^2 w}{\partial t^2}(x,t), \quad (x,t) \in G.$$

Lemma 3.4.8. *For $N/2+1 \le i \le N-1$, the local truncation error associated to the layer component satisfies*

$$\begin{aligned} L_\varepsilon^{N,M}\big(W^{N,\Delta t} - w\big)(x_i, t_{n+1}) \;&=\; \big(N^{-1}\ln N\big)\,\rho_1(x_i, t_{n+1}) + \Delta t\,\rho_2(x_i, t_{n+1}) + \\ &O\Big(\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)N^{-2}\ln^2 N + \exp\big(-\alpha(1-x_i)/\varepsilon\big)\Delta t^2\Big). \end{aligned}$$

Proof. Let $N/2+1 \le i \le N-1$. Then, from the Taylor's expansion and using Theorem 3.2.1, for some $\xi_1 \in (x_i, x_{i+1})$, $\xi_2, \xi_3 \in (x_{i-1}, x_i)$ and $\tilde{t} \in (t_n, t_{n+1})$, we have

$$\begin{aligned} & L_\varepsilon^{N,M}\big(W^{N,\Delta t} - w\big)(x_i, t_{n+1}) \\ =\; & \frac{\varepsilon h^2}{4!}\left[\frac{\partial^4 w}{\partial x^4}(\xi_1, t_{n+1}) + \frac{\partial^4 w}{\partial x^4}(\xi_2, t_{n+1})\right] + \frac{h}{2}a(x_i)\frac{\partial^2 w}{\partial x^2}(x_i, t_{n+1}) - \frac{h^2}{3!}a(x_i)\frac{\partial^3 w}{\partial x^3}(\xi_3, t_{n+1}) + \\ & \frac{\Delta t}{2}\frac{\partial^2 w}{\partial t^2}(x_i, t_{n+1}) - \frac{\Delta t^2}{3!}\frac{\partial^3 w}{\partial t^3}(x_i, \tilde{t}) \\ =\; & \frac{2\varepsilon}{\alpha}\big(N^{-1}\ln N\big)a(x_i)\frac{\partial^2 w}{\partial x^2}(x_i, t_{n+1}) + \frac{\Delta t}{2}\frac{\partial^2 w}{\partial t^2}(x_i, t_{n+1}) + \\ & O\Big(\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)N^{-2}\ln^2 N + \exp\big(-\alpha(1-x_i)/\varepsilon\big)\Delta t^2\Big). \end{aligned}$$

Define the functions $\mathcal{F}_k$, $k = 1, 2$, to be the solutions of the following initial-boundary-value problems

$$\begin{cases} L_\varepsilon \mathcal{F}_k \;=\; \rho_k \quad \text{in } \mathfrak{D} = (1-\tau, 1) \times (0, T], \\ \mathcal{F}_k(x, 0) = 0, \quad x \in [1-\tau, 1], \\ \mathcal{F}_k(1-\tau, t) = \mathcal{F}_k(1, t) = 0, \quad t \in (0, T], \quad k = 1, 2. \end{cases} \tag{3.32}$$

We now obtain the bounds on the derivatives of $\mathcal{F}_k$, $k = 1, 2$, through Lemmas 3.4.9-3.4.12. Here, the detailed proof for the function $\mathcal{F}_1$ is proved and that of $\mathcal{F}_2$ follows similarly.

Lemma 3.4.9. *The temporal derivatives of $\mathcal{F}_1$ satisfy the following bounds*

$$\left|\frac{\partial^m \mathcal{F}_1}{\partial t^m}(x,t)\right| \le C \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}, \quad \text{for } 0 \le m \le 3.$$

Proof. From (3.32), we have

$$\mathcal{F}_1 \equiv 0 \quad \text{on } \Gamma_{\mathfrak{D}} = \overline{\mathfrak{D}} \backslash \mathfrak{D}.$$

Also, from Theorem 3.2.1, we obtain

$$\big|L_\varepsilon \mathcal{F}_1(x,t)\big| = \big|\rho_1\big| \le C\varepsilon^{-1} \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}.$$

Now, consider the barrier function

$$\phi(x,t) = C \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}, \tag{3.33}$$

where C is sufficiently large. It follows that $L_\varepsilon \phi(x,t) \ge \big|L_\varepsilon \mathcal{F}_1(x,t)\big|$ in $\mathfrak{D}$ and $\phi \ge \big|\mathcal{F}_1\big|$ on $\Gamma_{\mathfrak{D}}$. Since L_ε satisfies the minimum principle (Lemma 2.2.1) on $\overline{\mathfrak{D}}$, therefore we have $\big|\mathcal{F}_1\big| \le C \exp\big(-\alpha(1-x)/\varepsilon\big)$ in $\mathfrak{D}$.

Next, we shall obtain the bound on $\dfrac{\partial \mathcal{F}_1}{\partial t}$. On the sides $x = 1-\tau$ and $x = 1$ of $\overline{\mathfrak{D}}$, we have $\mathcal{F}_1 \equiv 0$ and so $\dfrac{\partial \mathcal{F}_1}{\partial t} \equiv 0$. On the side $t = 0$ we have $\mathcal{F}_1 \equiv 0$ and so $\dfrac{\partial \mathcal{F}_1}{\partial x} \equiv \dfrac{\partial^2 \mathcal{F}_1}{\partial x^2} \equiv 0$. Hence, from (3.32) we obtain

$$\frac{\partial \mathcal{F}_1}{\partial t}(x,0) = \frac{2\varepsilon}{\alpha} a(x) \frac{\partial^2 w}{\partial x^2}(x,0), \quad x \in [1-\tau, 1]. \tag{3.34}$$

Since on the side $t = 0$ we have $w \equiv 0$, which implies that $\dfrac{\partial^2 w}{\partial x^2} \equiv 0$ and hence from (3.34) we obtain $\dfrac{\partial \mathcal{F}_1}{\partial t}(x,0) = 0,\ x \in [1-\tau, 1]$. Therefore, it is clear that

$$\frac{\partial \mathcal{F}_1}{\partial t} \equiv 0 \quad \text{on } \Gamma_{\mathfrak{D}}.$$

Now, differentiating the equation given in (3.32) with respect to t, we get

$$\frac{\partial^2 \mathcal{F}_1}{\partial t^2} - \varepsilon \frac{\partial^3 \mathcal{F}_1}{\partial t \partial x^2} + a(x) \frac{\partial^2 \mathcal{F}_1}{\partial t \partial x} + b(x) \frac{\partial \mathcal{F}_1}{\partial t} = \frac{2\varepsilon}{\alpha} a(x) \frac{\partial^3 w}{\partial t \partial x^2} \quad \text{in } \mathfrak{D}, \tag{3.35}$$

and employing Theorem 3.2.1 in (3.35) yields that

$$\left|L_\varepsilon\left(\frac{\partial \mathcal{F}_1}{\partial t}\right)(x,t)\right| \le C\varepsilon^{-1} \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}.$$

Therefore, choosing the barrier function $\phi(x,t)$ given in (3.33), we obtain that $L_\varepsilon\phi(x,t) \geq \left|L_\varepsilon\Big(\frac{\partial \mathcal{F}_1}{\partial t}\Big)(x,t)\right|$ in $\mathfrak{D}$ and $\phi \geq \left|\frac{\partial \mathcal{F}_1}{\partial t}\right|$ on $\Gamma_{\mathfrak{D}}$. Thus, applying the minimum principle on $\overline{\mathfrak{D}}$, we have

$$\left|\frac{\partial \mathcal{F}_1}{\partial t}\right| \leq C\exp\big(-\alpha(1-x)/\varepsilon\big) \quad \text{in } \mathfrak{D}.$$

Next, we shall obtain the bound on $\frac{\partial^2 \mathcal{F}_1}{\partial t^2}$. On the sides $x = 1-\tau$ and $x = 1$ of $\overline{\mathfrak{D}}$, we have $\mathcal{F}_1 \equiv 0$ and so $\frac{\partial^2 \mathcal{F}_1}{\partial t^2} \equiv 0$. On the side $t = 0$ we obtain from (3.34), $\frac{\partial \mathcal{F}_1}{\partial t} = \Big(2\varepsilon a\frac{\partial^2 w}{\partial x^2}\Big)/\alpha$ and accordingly

$$\begin{cases} \dfrac{\partial^2 \mathcal{F}_1}{\partial t\partial x}(x,0) = \dfrac{2\varepsilon}{\alpha}a(x)\dfrac{\partial^3 w}{\partial x^3}(x,0), \quad \text{and} \\ \dfrac{\partial^3 \mathcal{F}_1}{\partial t\partial x^2}(x,0) = \dfrac{2\varepsilon}{\alpha}a(x)\dfrac{\partial^4 w}{\partial x^4}(x,0), \quad x \in [1-\tau,1]. \end{cases} \tag{3.36}$$

Again, on the side $t = 0$ we have $w \equiv 0$, which implies that $\frac{\partial^3 w}{\partial x^3} \equiv \frac{\partial^4 w}{\partial x^4} \equiv 0$ and hence from (3.36) we obtain $\frac{\partial^2 \mathcal{F}_1}{\partial t\partial x}(x,0) = \frac{\partial^3 \mathcal{F}_1}{\partial t\partial x^2}(x,0) = 0,\ x \in [1-\tau,1]$. Therefore, (3.35) implies that

$$\frac{\partial^2 \mathcal{F}_1}{\partial t^2}(x,0) = \frac{2\varepsilon}{\alpha}a(x)\frac{\partial^3 w}{\partial t\partial x^2}(x,0), \quad x \in [1-\tau,1]. \tag{3.37}$$

Since $w \equiv 0$ on the side $t = 0$ and $L_\varepsilon w = 0$ in G, we have $\frac{\partial w}{\partial t} \equiv 0$ and so $\frac{\partial^3 w}{\partial t\partial x^2} \equiv 0$. Therefore, from (3.37) we obtain $\frac{\partial^2 \mathcal{F}_1}{\partial t^2}(x,0) = 0,\ x \in [1-\tau,1]$. Thus, clearly

$$\frac{\partial^2 \mathcal{F}_1}{\partial t^2} \equiv 0 \quad \text{on } \Gamma_{\mathfrak{D}}.$$

Now, differentiating (3.35) with respect to t, we get

$$\frac{\partial^3 \mathcal{F}_1}{\partial t^3} - \varepsilon\frac{\partial^4 \mathcal{F}_1}{\partial t^2\partial x^2} + a(x)\frac{\partial^3 \mathcal{F}_1}{\partial t^2\partial x} + b(x)\frac{\partial^2 \mathcal{F}_1}{\partial t^2} = \frac{2\varepsilon}{\alpha}a(x)\frac{\partial^4 w}{\partial t^2\partial x^2} \quad \text{in } \mathfrak{D}. \tag{3.38}$$

and employing Theorem 3.2.1 in (3.38) yields that

$$\left|L_\varepsilon\Big(\frac{\partial^2 \mathcal{F}_1}{\partial t^2}\Big)(x,t)\right| \leq C\varepsilon^{-1}\exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}.$$

Henceforth, by choosing the barrier function $\phi(x,t)$, a similar argument as in the previous cases leads to the desired bound $\left|\frac{\partial^2 \mathcal{F}_1}{\partial t^2}\right| \leq C\exp\big(-\alpha(1-x)/\varepsilon\big)$ in $\mathfrak{D}$.

Finally, following the same approach as like $\frac{\partial^2 \mathcal{F}_1}{\partial t^2}$, one can obtain the bound on $\frac{\partial^3 \mathcal{F}_1}{\partial t^3}$ and this completes the proof. ■

Lemma 3.4.10. *The following mixed derivative of $\mathcal{F}_1$ satisfies the bound*

$$\left|\frac{\partial^3 \mathcal{F}_1}{\partial x \partial t^2}(x,t)\right| \le C\varepsilon^{-1} \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}.$$

Proof. Rewrite (3.38) in the following form

$$-\varepsilon \frac{\partial}{\partial x}\left(\frac{\partial^3 \mathcal{F}_1}{\partial x \partial t^2}\right) + a(x)\frac{\partial^3 \mathcal{F}_1}{\partial x \partial t^2} + b(x)\frac{\partial^2 \mathcal{F}_1}{\partial t^2} = \mathcal{H}_1 \quad \text{in } \mathfrak{D}. \tag{3.39}$$

where $\mathcal{H}_1(x,t) = -\dfrac{\partial^3 \mathcal{F}_1}{\partial t^3} + \dfrac{2\varepsilon}{\alpha} a(x) \dfrac{\partial^4 w}{\partial t^2 \partial x^2}$. From Lemma 3.4.9 and Theorem 3.2.1, we have

$$\big|\mathcal{H}_1(x,t)\big| \le C\varepsilon^{-1} \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}. \tag{3.40}$$

Now, fix $t \in \big[0,T\big]$. Then, the desired bound on $\dfrac{\partial^3 \mathcal{F}_1}{\partial x \partial t^2}$ is obtained by applying the argument of Kellogg and Tsan [41] on the line segment $\big\{(x,t), x \in [1-\tau, 1]\big\}$, as the requirements for this are equation (3.39), inequality (3.40) and the bound on $\dfrac{\partial^2 \mathcal{F}_1}{\partial t^2}$. ∎

Lemma 3.4.11. *The bounds on the derivatives of $\dfrac{\partial \mathcal{F}_1}{\partial t}$ are given by*

$$\left|\frac{\partial^l}{\partial x^l}\left(\frac{\partial \mathcal{F}_1}{\partial t}\right)(x,t)\right| \le C\varepsilon^{-l} \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}, \quad \textit{for } l = 1, 2.$$

Proof. Rewrite (3.35) in the following form

$$-\varepsilon \frac{\partial}{\partial x}\left(\frac{\partial^2 \mathcal{F}_1}{\partial x \partial t}\right) + a(x)\frac{\partial^2 \mathcal{F}_1}{\partial x \partial t} + b(x)\frac{\partial \mathcal{F}_1}{\partial t} = \mathcal{H}_2 \quad \text{in } \mathfrak{D}. \tag{3.41}$$

where $\mathcal{H}_2(x,t) = -\dfrac{\partial^2 \mathcal{F}_1}{\partial t^2} + \dfrac{2\varepsilon}{\alpha} a(x) \dfrac{\partial^3 w}{\partial t \partial x^2}$. From Lemmas 3.4.9, 3.4.10 and Theorem 3.2.1, we have

$$\left|\frac{\partial^l \mathcal{H}_2}{\partial x^l}(x,t)\right| \le C\varepsilon^{-l-1} \exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t) \in \mathfrak{D}, \quad \text{for } l = 0, 1. \tag{3.42}$$

Now, fix $t \in \big[0,T\big]$. Then, applying the argument of Kellogg and Tsan [41] on the line segment $\big\{(x,t), x \in [1-\tau, 1]\big\}$, by using equation (3.41), inequality (3.42) and the bound on $\dfrac{\partial \mathcal{F}_1}{\partial t}$, we obtain the desired bound on $\dfrac{\partial^2 \mathcal{F}_1}{\partial x \partial t}$.

Again, differentiating (3.41) with respect to x, we have

$$-\varepsilon \frac{\partial}{\partial x}\left(\frac{\partial^3 \mathcal{F}_1}{\partial x^2 \partial t}\right) + a(x)\frac{\partial^3 \mathcal{F}_1}{\partial x^2 \partial t} + \big(a'(x) + b(x)\big)\frac{\partial^2 \mathcal{F}_1}{\partial x \partial t} + b'(x)\frac{\partial \mathcal{F}_1}{\partial t} = \frac{\partial \mathcal{H}_2}{\partial x} \quad \text{in } \mathfrak{D}. \tag{3.43}$$

Likewise, fix $t \in \big[0,T\big]$ and a similar argument of Kellogg and Tsan [41] on the line segment $\big\{(x,t), x \in [1-\tau, 1]\big\}$, making use of equation (3.42), inequality (3.43) and the bounds on $\dfrac{\partial \mathcal{F}_1}{\partial t}, \dfrac{\partial^2 \mathcal{F}_1}{\partial x \partial t}$ leads to the desired bound on $\dfrac{\partial^3 \mathcal{F}_1}{\partial x^2 \partial t}$. ∎

Lemma 3.4.12. *The spatial derivatives of $\mathcal{F}_1$ satisfy the following bounds*

$$\left|\frac{\partial^l \mathcal{F}_1}{\partial x^l}(x,t)\right| \le C\varepsilon^{-l}\exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t)\in\mathfrak{D}, \quad \textit{for } 1\le l\le 3.$$

Proof. Rewrite (3.32) in the following form

$$-\varepsilon\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{F}_1}{\partial x}\right) + a(x)\frac{\partial \mathcal{F}_1}{\partial x} + b(x)\mathcal{F}_1 = \mathcal{H}_3 \quad \text{in } \mathfrak{D}. \tag{3.44}$$

where $\mathcal{H}_3(x,t) = -\dfrac{\partial \mathcal{F}_1}{\partial t} + \dfrac{2\varepsilon}{\alpha}a(x)\dfrac{\partial^2 w}{\partial x^2}$. From Lemmas 3.4.9, 3.4.11 and Theorem 3.2.1, we have

$$\left|\frac{\partial^l \mathcal{H}_3}{\partial x^l}(x,t)\right| \le C\varepsilon^{-l-1}\exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t)\in\mathfrak{D}, \quad \text{for } 0\le l\le 2. \tag{3.45}$$

Then, fix $t\in\big[0,T\big]$ and apply the argument of Kellogg and Tsan [41] on the line segment $\big\{(x,t), x\in[1-\tau,1]\big\}$, by using equation (3.44), inequality (3.45) and the bound on $\mathcal{F}_1$ to obtain the desired bound on $\dfrac{\partial \mathcal{F}_1}{\partial x}$.

Next, differentiate (3.44) with respect to x, we get

$$-\varepsilon\frac{\partial}{\partial x}\left(\frac{\partial^2 \mathcal{F}_1}{\partial x^2}\right) + a(x)\frac{\partial^2 \mathcal{F}_1}{\partial x^2} + \big(a'(x)+b(x)\big)\frac{\partial \mathcal{F}_1}{\partial x} + b'(x)\mathcal{F}_1 = \frac{\partial \mathcal{H}_3}{\partial x} \quad \text{in } \mathfrak{D}. \tag{3.46}$$

The bound on $\dfrac{\partial^2 \mathcal{F}_1}{\partial x^2}$ can be deduced similarly as in the previous cases, by using (3.46), (3.45) and the bounds on $\mathcal{F}_1$, $\dfrac{\partial \mathcal{F}_1}{\partial x}$. Finally, differentiating (3.46) with respect to x and using (3.44), one can obtain the bound on $\dfrac{\partial^3 \mathcal{F}_1}{\partial x^3}$ similarly. ■

Afterwards, we resume the previous lemmas in the following theorem.

Theorem 3.4.13. *For all non-negative integers l, m, satisfying $0\le l+m\le 3$, the solution $\mathcal{F}_k(x,t),\ k=1,2,$ of (3.32) satisfy the following bounds*

$$\left|\frac{\partial^{l+m} \mathcal{F}_k}{\partial x^l\partial t^m}(x,t)\right| \le C\varepsilon^{-l}\exp\big(-\alpha(1-x)/\varepsilon\big), \quad (x,t)\in\mathfrak{D}.$$

Lemma 3.4.14. *For $N/2+1\le i\le N-1$, we obtain*

$$\big(W^{N,\Delta t}-w\big)(x_i,t_{n+1}) = \big(N^{-1}\ln N\big)\,\mathcal{F}_1(x_i,t_{n+1}) + \Delta t\,\mathcal{F}_2(x_i,t_{n+1}) + O\Big(N^{-2}\ln^2 N + \Delta t^2\Big).$$

Proof. Let $N/2+1\le i\le N-1$. Then, Lemma 3.4.8 and (3.32) imply that

$$\begin{aligned} L_\varepsilon^{N,M}\big(W^{N,\Delta t}-w\big)(x_i,t_{n+1}) \;&=\; \big(N^{-1}\ln N\big)\,L_\varepsilon\mathcal{F}_1(x_i,t_{n+1}) + \Delta t\,L_\varepsilon\mathcal{F}_2(x_i,t_{n+1}) + \\ & O\Big(\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)N^{-2}\ln^2 N + \exp\big(-\alpha(1-x_i)/\varepsilon\big)\Delta t^2\Big). \end{aligned} \tag{3.47}$$

Again, from the Taylor's expansion and Theorem 3.4.13, it follows that for $k = 1, 2$,

$$(L_\varepsilon - L_\varepsilon^{N,M})\mathcal{F}_k(x_i, t_{n+1}) = O\Big(\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)N^{-1}\ln N + \exp\big(-\alpha(1-x_i)/\varepsilon\big)\Delta t\Big),$$

and so

$$\begin{aligned}
&\Big|\big(N^{-1}\ln N\big)\,\big(L_\varepsilon - L_\varepsilon^{N,M}\big)\mathcal{F}_1(x_i, t_{n+1}) + \Delta t\,\big(L_\varepsilon - L_\varepsilon^{N,M}\big)\mathcal{F}_2(x_i, t_{n+1})\Big| \\
\le\;& C\Big[\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)N^{-2}\ln^2 N + \exp\big(-\alpha(1-x_i)/\varepsilon\big)\Delta t^2 + \\
&\Big(\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big) + \exp\big(-\alpha(1-x_i)/\varepsilon\big)\Big)\big(N^{-1}\ln N\big)\,\Delta t\Big] \\
\le\;& C\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)\Big(N^{-2}\ln^2 N + \Delta t^2\Big). \qquad (3.48)
\end{aligned}$$

Therefore, combining (3.47) and (3.48), we have

$$\begin{aligned}
&\Big|L_\varepsilon^{N,M}\big[W^{N,\Delta t} - w - \big(N^{-1}\ln N\big)\mathcal{F}_1 - \Delta t\,\mathcal{F}_2\big](x_i, t_{n+1})\Big| \\
\le\;& C_5\Big(\varepsilon^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)\big(N^{-2}\ln^2 N + \Delta t^2\big)\Big), \quad \text{for } N/2+1 \le i \le N-1, \qquad (3.49)
\end{aligned}$$

for some constant C_5. Now, consider the barrier function

$$\varphi_i = C_6\Big[N^{-2}(1+x_i) + \Big(N^{-2}\ln^2 N + \Delta t^2\Big)S_i\prod_{j=1}^{N}\Big(1+\frac{\alpha h_j}{\varepsilon}\Big)^{-1}\Big], \quad \text{for } N/2 \le i \le N.$$

For $N/2+1 \le i \le N-1$, employing (3.13), we get

$$L_\varepsilon^{N,M}\varphi_i \ge C_2C_6\varepsilon^{-1}\Big(N^{-2}\ln^2 N + \Delta t^2\Big)\prod_{j=i+1}^{N}\Big(1+\frac{\alpha h_j}{\varepsilon}\Big)^{-1}, \qquad (3.50)$$

Also, for $i \ge N/2$, (3.14) yields

$$\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big) \le \exp\big(\gamma h/\varepsilon\big)\prod_{j=i+1}^{N}\Big(1+\frac{\alpha h_j}{\varepsilon}\Big)^{-1} \le \exp(4)\prod_{j=i+1}^{N}\Big(1+\frac{\alpha h_j}{\varepsilon}\Big)^{-1}. \qquad (3.51)$$

Hence, for $N/2+1 \le i \le N-1$, it is easy to show from (3.49)-(3.51) that

$$L_\varepsilon^{N,M}\varphi_i \ge \Big|L_\varepsilon^{N,M}\big[W^{N,\Delta t} - w - \big(N^{-1}\ln N\big)\mathcal{F}_1 - \Delta t\,\mathcal{F}_2\big](x_i, t_{n+1})\Big|, \qquad (3.52)$$

provided $C_6 \ge \exp(4)C_5/C_2$. Again, from the proof of Lemma 3.4.7, we have

$$\Big|\big(W^{N,\Delta t} - w\big)(1-\tau, t_{n+1})\Big| \le C_7 N^{-2},$$

for some constant C_7 and hence, by (3.32)

$$\varphi_{N/2} \geq C_6 N^2 \geq \left| \left[W^{N,\Delta t} - w - \big(N^{-1}\ln N\big)\mathcal{F}_1 - \Delta t\,\mathcal{F}_2\right](1-\tau, t_{n+1}) \right|, \tag{3.53}$$

provided $C_6 \geq C_7$. Also, recalling (3.19) and (3.32), we have

$$\varphi_N \geq 0 = \left| \left[W^{N,\Delta t} - w - \big(N^{-1}\ln N\big)\mathcal{F}_1 - \Delta t\,\mathcal{F}_2\right](1, t_{n+1}) \right|. \tag{3.54}$$

Therefore, it follows from (3.52)-(3.54) that one can choose $C_6 = \max\big\{\exp(4)C_5/C_2, C_7\big\}$ so that φ_i is a barrier function for $\pm\big[W^{N,\Delta t} - w - \big(N^{-1}\ln N\big)\mathcal{F}_1 - \Delta t\,\mathcal{F}_2\big](x_i, t_n)$. Thus, by the discrete minimum principle over $\big(\overline{G}_\tau^{N,M}\bigcap[1-\tau, 1]\big) \times [0, T]$, we obtain

$$\left| \left[W^{N,\Delta t} - w - \big(N^{-1}\ln N\big)\mathcal{F}_1 - \Delta t\,\mathcal{F}_2\right](x_i, t_{n+1}) \right| \leq \varphi_i \leq C\Big(N^{-2}\ln^2 N + \Delta t^2\Big),$$

for $N/2 + 1 \leq i \leq N-1$ and this completes the proof. ∎

Lemma 3.4.15. *The error after extrapolation associated to the layer component $W^{N,\Delta t}$ satisfies*

$$\left| w(x_i, t_{n+1}) - W_{extp}^{N,\Delta t}(x_i, t_{n+1}) \right| \leq C\Big(N^{-2}\ln^2 N + \Delta t^2\Big), \quad \textit{for } N/2 + 1 \leq i \leq N-1.$$

Proof. Let $N/2 + 1 \leq i \leq N-1$. Invoking $\tau = (2\varepsilon \ln N)/\alpha$, from Lemma 3.4.14, we have

$$\begin{aligned} \big(W^{N,\Delta t} - w\big)(x_i, t_{n+1}) &= N^{-1}\big(\alpha\tau/2\varepsilon\big)\mathcal{F}_1(x_i, t_{n+1}) + \Delta t\,\mathcal{F}_2(x_i, t_{n+1}) \\ &\quad + O\Big(N^{-2}\big(\alpha\tau/2\varepsilon\big)^2 + \Delta t^2\Big). \end{aligned} \tag{3.55}$$

Next, since the fine mesh $\overline{G}_\tau^{2N,2M}$ have the same transition point $1-\tau$ as that of $\overline{G}_\tau^{N,M}$, Lemma 3.4.14 implies that

$$\begin{aligned} \big(\widetilde{W}^{2N,\Delta t/2} - w\big)(x_i, t_{n+1}) &= (2N)^{-1}\big(\alpha\tau/2\varepsilon\big)\mathcal{F}_1(x_i, t_{n+1}) + (\Delta t/2)\,\mathcal{F}_2(x_i, t_{n+1}) \\ &\quad + O\Big(N^{-2}(\alpha\tau/2\varepsilon)^2 + \Delta t^2\Big). \end{aligned} \tag{3.56}$$

Henceforth, eliminating the $O(N^{-1})$ and $O(\Delta t)$ terms from (3.55) and (3.56), the required extrapolated error bound is obtained. ∎

3.4.5 Convergence result of the solution $U_{extp}^{N,\Delta t}$

The main result of this chapter is stated in the following theorem.

Theorem 3.4.16. *(**Error after extrapolation**) Assume that $\varepsilon \le N^{-1}$. Let u be the solution of the continuous problem (3.1)-(3.2) and $U_{extp}^{N,\Delta t}$ be the solution obtained via the Richardson extrapolation technique by solving the discrete problem (3.10) on two nested meshes $\overline{G}_\tau^{N,M}$ and $\overline{G}_\tau^{2N,2M}$. Then, the error associated with the solution $U_{extp}^{N,\Delta t}$ at time level t_n satisfies*

$$\left| u(x_i,t_n) - U_{extp}^{N,\Delta t}(x_i,t_n) \right| \le C\Big(N^{-2}\ln^2 N + \Delta t^2 \Big), \qquad \text{for } 1 \le i \le N-1. \tag{3.57}$$

Proof. For each $(x_i,t_n) \in \overline{G}_\tau^{N,M}$, we have

$$u(x_i,t_n) - U_{extp}^{N,\Delta t} = \Big(v(x_i,t_n) - V_{extp}^{N,\Delta t} \Big) + \Big(w(x_i,t_n) - W_{extp}^{N,\Delta t} \Big).$$

Therefore, the result (3.57) follows immediately after combining Lemma 3.4.6 for the smooth component and Lemmas 3.4.7 and 3.4.15 for the layer component. ■

3.5 Numerical Results

This section presents the numerical results obtained by the Richardson extrapolation technique to verify the theoretical results claimed in the previous section. To do this, the numerical experiments are profoundly carried out for the following test examples on the piecewise-uniform rectangular mesh $\overline{G}_\tau^{N,M}$. In all the cases, the numerical experiments are performed by choosing the transition parameter $\tau = 2$, $\alpha = 0.4672$ and $\Delta t = 1/N$.

Example 3.5.1. *Consider the following parabolic IBVP:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + (1 + x(1-x))\dfrac{\partial u}{\partial x} = f(x,t), \quad (x,t) \in (0,1) \times (0,1], \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \le 1. \end{cases} \tag{3.58}$$

where the initial data $u_0(x)$ and the source function $f(x,t)$ have been chosen to fit

$$u(x,t) = \exp(-t)\big(m_1 + m_2 x - \exp(-(1-x)/\varepsilon)\big)$$

as the exact solution of the above problem, where $m_1 = \exp(-1/\varepsilon)$ and $m_2 = 1 - \exp(-1/\varepsilon)$.

Example 3.5.2. *Consider the following parabolic IBVP:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + (2 - x^2)\dfrac{\partial u}{\partial x} + xu = 10t^2 \exp(-t)x(1-x), \quad (x,t) \in (0,1) \times (0,3], \\ u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \le 3. \end{cases} \tag{3.59}$$

For various values of ε and N, the calculated maximum point-wise errors $e_\varepsilon^{N,\Delta t}$ and the corresponding order of convergence $p_\varepsilon^{N,\Delta t}$ before and after extrapolation for Example 3.5.1 are presented in Table 3.1. On the other hand, the exact solution of the IBVP (3.59) is not known and the corresponding numerical results for Example 3.5.2 are demonstrated by using the double mesh principle as described in Chapter 2. In Table 3.2, we have presented the computed maximum point-wise errors $E_\varepsilon^{N,\Delta t}$ and the corresponding order of convergence $P_\varepsilon^{N,\Delta t}$ before and after extrapolation for Example 3.5.2. In addition, we have plotted the maximum point-wise errors for Examples 3.5.1 and 3.5.2 respectively, in Figures 3.1 and 3.2.

From the results given in Tables 3.1 and 3.2, we observe that the computed ε-uniform errors $e^{N,\Delta t}$ and $E^{N,\Delta t}$ respectively, for Examples 3.5.1 and 3.5.2 decrease monotonically as N increases. This ensures that the implicit upwind scheme (3.10) converges ε-uniformly before and after extrapolation. Moreover, Figures 3.1 and 3.2 clearly show that extrapolation increases the order of converges of the upwind scheme from $O(N^{-1} \ln N)$ to $O(N^{-2} \ln^2 N)$ for $\Delta t = 1/N$, which supports the theoretical bounds obtained in Theorems 3.3.2 and 3.4.16.

3.6 Conclusion

In this chapter, Richardson extrapolation technique is applied on the simple upwind scheme for solving one-dimensional singularly perturbed parabolic convection-diffusion IBVP with regular boundary layer. It has been shown both theoretically and computationally that the extrapolation technique improves the almost first-order convergence of the simple upwinding into almost second-order convergence in the discrete supremum norm.

Table 3.1: *Maximum point-wise errors and the corresponding order of convergence before and after extrapolation for Example 3.5.1.*

ε	Extrapolation	Number of mesh intervals N					
		32	64	128	256	512	1024
$1e-1$	**before**	1.9845e-2	1.0544e-2	5.4493e-3	2.7708e-3	1.3969e-3	7.0141e-4
		0.9123	0.9523	0.9757	0.9880	0.9939	
	after	1.3096e-3	3.7208e-4	9.9637e-5	2.5851e-5	6.5793e-6	1.6600e-6
		1.8154	1.9009	1.9465	1.9742	1.9867	
$1e-2$	**before**	4.3463e-2	2.9874e-2	1.8761e-2	1.1330e-2	6.5980e-3	3.7392e-3
		0.5408	0.6711	0.7276	0.7799	0.8192	
	after	6.7626e-3	2.9552e-3	1.2260e-3	4.4704e-4	1.5112e-4	4.8606e-5
		1.1943	1.2693	1.4555	1.5646	1.6365	
$1e-3$	**before**	4.2847e-2	2.9486e-2	1.8536e-2	1.1198e-2	6.5231e-3	3.6973e-3
		0.5391	0.6697	0.7270	0.7796	0.8190	
	after	6.7324e-3	2.9328e-3	1.2155e-3	4.4299e-4	1.4964e-4	4.8092e-5
		1.1988	1.2707	1.4562	1.5658	1.6376	
$1e-4$	**before**	4.2785e-2	2.9447e-2	1.8513e-2	1.1186e-2	6.5158e-3	3.6933e-3
		0.5389	0.6695	0.7269	0.7796	0.8190	
	after	6.7299e-3	2.9308e-3	1.2146e-3	4.4264e-4	1.4950e-4	4.8044e-5
		1.1993	1.2708	1.4563	1.5660	1.6377	
$1e-5$	**before**	4.2779e-2	2.9444e-2	1.8511e-2	1.1184e-2	6.5151e-3	3.6929e-3
		0.5389	0.6695	0.7269	0.7796	0.8190	
	after	6.7297e-3	2.9306e-3	1.2145e-3	4.4260e-4	1.4949e-4	4.8039e-5
		1.1993	1.2708	1.4563	1.5660	1.6378	
$1e-6$ to $1e-10$	**before**	4.2779e-2	2.9443e-2	1.8511e-2	1.1184e-2	6.5150e-3	3.6928e-3
		0.5389	0.6695	0.7269	0.7796	0.8190	
	after	6.7297e-3	2.9306e-3	1.2145e-3	4.4260e-4	1.4949e-4	4.8039e-5
		1.1993	1.2708	1.4563	1.5660	1.6378	
$e^{N,\Delta t}$	**before**	4.3463e-2	2.9874e-2	1.8761e-2	1.1330e-2	6.5980e-3	3.7392e-3
$p^{N,\Delta t}$		0.5408	0.6711	0.7276	0.7799	0.8192	
$e^{N,\Delta t}$	**after**	6.7626e-3	2.9552e-3	1.2260e-3	4.4704e-4	1.5112e-4	4.8606e-5
$p^{N,\Delta t}$		1.1943	1.2693	1.4555	1.5646	1.6365	

Table 3.2: *Maximum point-wise errors and the corresponding order of convergence before and after extrapolation for Example 3.5.2.*

ε	Extrapolation	Number of mesh intervals N					
		32	64	128	256	512	1024
	before	1.0354e-2	5.6933e-3	2.9937e-3	1.5377e-3	7.7932e-4	3.9234e-4
$1e-1$		0.8629	0.9273	0.9611	0.9804	0.9901	
	after	1.1110e-3	3.2392e-4	8.7925e-5	2.2940e-5	5.8610e-6	1.4814e-6
		1.7782	1.8813	1.9384	1.9686	1.9842	
	before	1.9623e-2	1.4118e-2	9.1132e-3	5.6409e-3	3.3388e-3	1.9136e-3
$1e-2$		0.4750	0.6315	0.6920	0.7565	0.8030	
	after	4.6274e-3	2.1270e-3	9.1367e-4	3.4022e-4	1.1660e-4	3.7847e-5
		1.1214	1.2191	1.4252	1.5449	1.6233	
	before	2.3816e-2	1.5889e-2	1.0079e-2	6.1108e-3	3.5550e-3	2.0150e-3
$1e-3$		0.5839	0.6566	0.7218	0.7815	0.8191	
	after	4.8759e-3	2.1920e-3	9.3640e-4	3.4756e-4	1.1862e-4	3.8365e-5
		1.1534	1.2271	1.4298	1.5510	1.6284	
	before	2.4652e-2	1.6265e-2	1.0334e-2	6.2456e-3	3.6295e-3	2.0559e-3
$1e-4$		0.5999	0.6544	0.7264	0.7830	0.8199	
	after	4.9395e-3	2.2134e-3	9.4539e-4	3.5073e-4	1.1947e-4	3.8549e-5
		1.1581	1.2273	1.4305	1.5537	1.6319	
	before	2.4740e-2	1.6306e-2	1.0362e-2	6.2605e-3	3.6383e-3	2.0608e-3
$1e-5$		0.6014	0.6541	0.7269	0.7830	0.8200	
	after	4.9464e-3	2.2158e-3	9.4644e-4	3.5114e-4	1.1961e-4	3.8588e-5
		1.1586	1.2272	1.4305	1.5537	1.6321	
	before	2.4750e-2	1.6310e-2	1.0365e-2	6.2622e-3	3.6393e-3	2.0613e-3
$1e-6$		0.6016	0.6540	0.7269	0.7830	0.8201	
to $1e-10$	**after**	4.9472e-3	2.2161e-3	9.4655e-4	3.5118e-4	1.1962e-4	3.8593e-5
		1.1586	1.2272	1.4305	1.5537	1.6321	
$E^{N,\Delta t}$	**before**	2.4750e-2	1.6310e-2	1.0365e-2	6.2622e-3	3.6393e-3	2.0613e-3
$P^{N,\Delta t}$		0.6016	0.6540	0.7269	0.7830	0.8201	
$E^{N,\Delta t}$	**after**	4.9472e-3	2.2161e-3	9.4656e-4	3.5118e-4	1.1962e-4	3.8593e-5
$P^{N,\Delta t}$		1.1586	1.2272	1.4305	1.5537	1.6321	

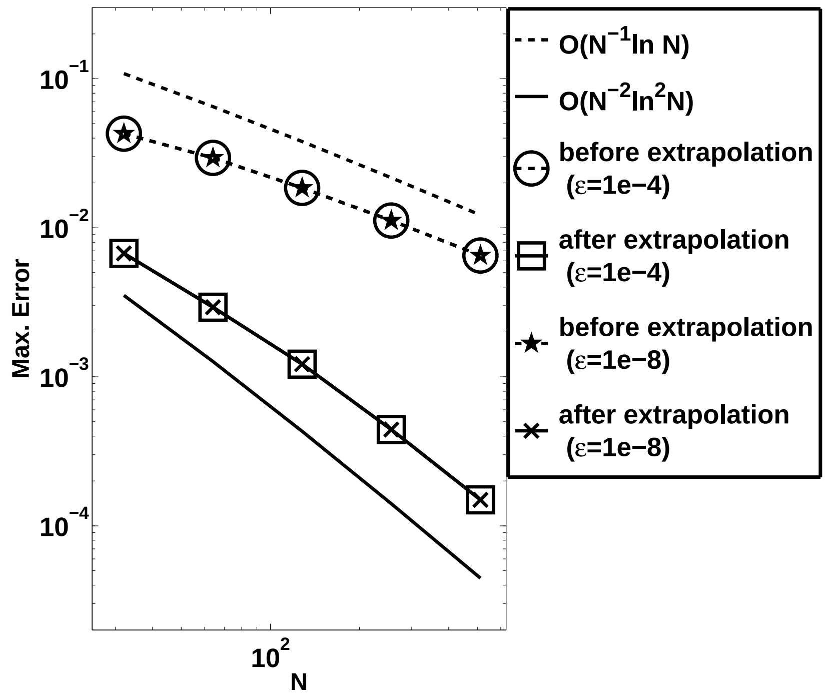


Figure 3.1: *Loglog plot of the maximum point-wise errors for Example 3.5.1.*

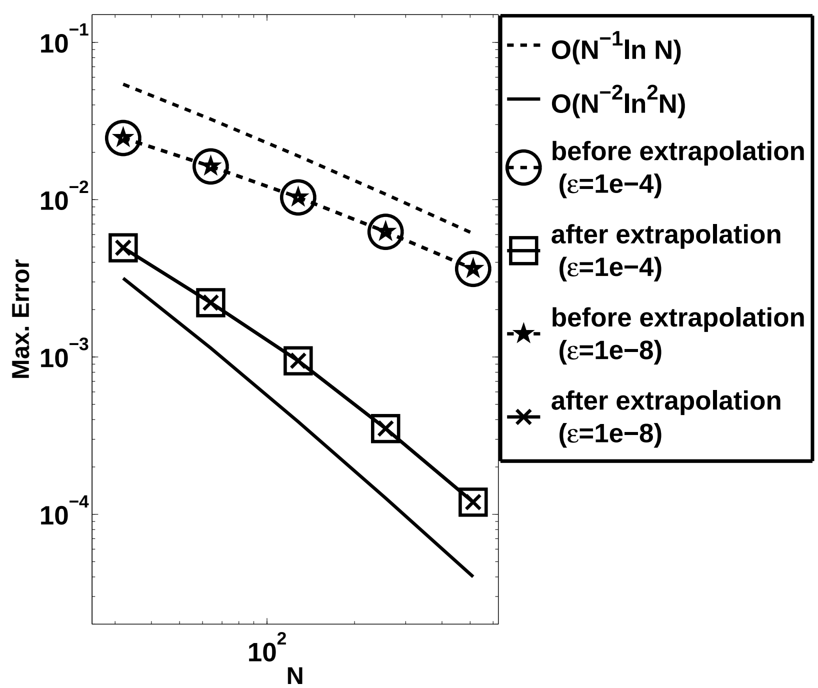


Figure 3.2: *Loglog plot of the maximum point-wise errors for Example 3.5.2.*

Chapter 4

Parameter-Uniform Hybrid Numerical Scheme for Singularly Perturbed Parabolic Problems with Interior Layers

The present chapter is devoted to the study of a hybrid numerical scheme for a class of singularly perturbed parabolic convection-diffusion problems with discontinuous convection coefficients. In general, the solutions of this class of problems possess strong interior layers. To solve these problems, we discretize the time derivative by the classical backward-Euler method and for the spatial discretization a hybrid finite difference scheme (a proper combination of the midpoint upwind scheme in the outer regions and the classical central difference scheme in the interior layer regions) is employed on a layer resolving piecewise-uniform Shishkin mesh. It is proved that the method converges uniformly in the discrete supremum norm with almost second-order spatial accuracy, provided that the perturbation parameter ε satisfies $\varepsilon \leq N^{-1}$. Moreover, an optimal order of convergence (up to a logarithmic factor) is obtained inside the layer regions.

4.1 Introduction

Let the domains for describing the model problem be denoted by $\Omega = (0,1),\ \Omega^- = (0,\xi),\ \Omega^+ = (\xi,1),\ G^- = \Omega^- \times (0,T],\ G^+ = \Omega^+ \times (0,T],\ G = \Omega \times (0,T]$. Consider the following class of singularly perturbed parabolic IBVPs posed on the domain $G^- \cup G^+$:

$$\begin{cases} L_\varepsilon u(x,t) \equiv \left(\varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - b(x)u - \dfrac{\partial u}{\partial t}\right)(x,t) \;=\; f(x,t), \quad (x,t) \in G^- \cup G^+, \\ u(x,0) = s_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = s_1(t), \quad u(1,t) = s_2(t), \quad t \in (0,T], \end{cases} \tag{4.1}$$

where $0 < \varepsilon \ll 1$ is a small parameter, the convection coefficient a is sufficiently smooth on $\Omega^- \cup \Omega^+$, the source term f is sufficiently smooth on $G^- \cup G^+$ and the coefficient b is sufficiently smooth on $\overline{\Omega}$ such that

$$\begin{cases} b(x) \geq \beta \geq 0 \quad \text{on } \overline{\Omega}, \\ \big|[a]\big| \leq C, \quad \big|[f]\big| \leq C, \quad \text{at } x = \xi, \end{cases} \tag{4.2}$$

and the solution $u(x,t)$ satisfies the following interface conditions

$$[u] = 0, \quad \left[\frac{\partial u}{\partial x}\right] = 0, \quad \text{at } x = \xi. \tag{4.3}$$

Here, the jump of u, denoted by $[u]$, across the point of discontinuity $x = \xi$ is defined by $[u](\xi,t) = u(\xi^+,t) - u(\xi^-,t)$, where $u(\xi^\pm,t) = \lim_{x \to \xi \pm 0} u(x,t)$. In general, due to the presence of discontinuity in the convection coefficient $a(x)$, the solution $u(x,t)$ of the problem (4.1)-(4.3) may possesses an interior layer of width $O(\varepsilon)$ in the neighborhood of the point $x = \xi$. But in reality, the nature of the interior layer depends on the sign of the convection coefficient a either side of the line of discontinuity. Therefore, to emphasize the occurrence of strong interior layers, the following particular case is considered:

$$\begin{cases} -\alpha_1^* < a(x) < -\alpha_1 < 0, \quad x < \xi, \\ \alpha_2^* > a(x) > \alpha_2 > 0, \qquad x > \xi. \end{cases} \tag{4.4}$$

Note that the coefficients in the differential operator are assumed to be independent of time and we confine here our assumptions to discontinuities in the spatial variable x only. Also, the data s_0, s_1 and s_2 are assumed to be sufficiently smooth on $\overline{G}$ and satisfy the following compatibility conditions at the two corners points $(0,0)$ and $(1,0)$:

$$\begin{cases} s_0(0) = s_1(0), \\ s_0(1) = s_2(1), \end{cases} \tag{4.5}$$

and

$$\begin{cases} \varepsilon \dfrac{\partial^2 s_0(0)}{\partial x^2} + a(0)\dfrac{\partial s_0(0)}{\partial x} - b(0)s_0(0) - f(0,0) = \dfrac{\partial s_1(0)}{\partial t}, \\ \varepsilon \dfrac{\partial^2 s_0(1)}{\partial x^2} + a(1)\dfrac{\partial s_0(1)}{\partial x} - b(1)s_0(1) - f(1,0) = \dfrac{\partial s_2(1)}{\partial t}. \end{cases} \tag{4.6}$$

The compatibility conditions at the transition point $(\xi,0)$ follows similarly. Under these hypotheses the parabolic IBVP (4.1)-(4.4) has a unique solution $u \in \mathcal{C}^{1+\lambda}(G) \bigcap \mathcal{C}^{2+\lambda}(G^- \cup G^+)$, which follows from *Theorem 13.1* of [[47], §3.13.1] and is also discussed in [72].

The outline of this chapter is as follows: Section 4.2 provides *a-priori* bound on the analytical solution and that on the derivatives of the solution *via* decomposition. Section 4.3

describes the piecewise-uniform Shishkin mesh and provides the detail construction of the newly proposed hybrid finite difference scheme. Next, the ε-uniform stability is studied in Section 4.4 and the main theoretical result, *i.e.*, the ε-uniform convergence of the proposed finite difference scheme, is proved. Finally, Section 4.5 carries out the numerical experiments to validate the theoretical results.

4.2 Bounds on the Solution and its Decomposition

This section presents the standard *a-priori* bound on the analytical solution and derive stronger bounds on the derivatives of the analytical solution of the problem (4.1)-(4.4), by decomposing the solution into smooth and layer components. This will be used later in the analysis for the proof of ε-uniform error estimate of the proposed difference scheme. Firstly, some preliminary results are stated below. Let $\Gamma = \overline{G}\backslash G$. Then the differential operator L_ε satisfies the following maximum principle on $\overline{G}$.

Lemma 4.2.1. (***Maximum Principle***) *Suppose that a function* $g \in \mathcal{C}^0(\overline{G}) \bigcap \mathcal{C}^2(G^- \cup G^+)$ *satisfies*

$$g(x,t) \leq 0,\ (x,t) \in \Gamma, \quad \left[\frac{\partial g}{\partial x}\right](\xi,t) \geq 0,\ t > 0, \quad L_\varepsilon g(x,t) \geq 0,\ (x,t) \in G^- \cup G^+,$$

then $g(x,t) \leq 0,\ \forall (x,t) \in \overline{G}$.

Proof. The proof follows from [72]. ■

An immediate consequence of the above maximum principle is the following stability result, which implies uniqueness of the solution of the IBVP (4.1)-(4.4).

Lemma 4.2.2. *Let u be the solution of the problem (4.1)-(4.4), then*

$$\|u\|_{\infty,\overline{G}} \leq \|u\|_{\infty,\Gamma} + \frac{1}{\mu}\|f\|_{\infty,\overline{G}},$$

where $\mu = \min\{\alpha_1/\xi, \alpha_2/(1-\xi)\}$.

Proof. The proof follows from [72]. ■

We now consider the following decomposition of the solution $u = v + w$, into a smooth component v and an interior layer component w. Here, the smooth component v is defined by the solution of the following IBVP

$$\begin{cases} L_\varepsilon v = f \quad \text{in } G^- \cup G^+, \\ v(0,t) = u(0,t), \quad v(\xi^-,t) = g_1(t), \\ v(\xi^+,t) = g_2(t), \quad v(1,t) = u(1,t), \quad t \in (0,T], \\ v(x,0) = u(x,0), \quad x \in \overline{\Omega}, \end{cases} \tag{4.7}$$

where the suitable choices of the functions $g_1(t),\ g_2(t)$ will be obtained from Theorem 4.2.4. With v thus defined, the layer component w of the decomposition must be defined as follows

$$\begin{cases} L_\varepsilon w \ = \ 0 \quad \text{in } G^- \cup G^+, \\ w(0,t) = w(1,t) = 0, \quad w(x,0) = 0, \quad x \in \overline{\Omega}, \\ [w](\xi,t) = -[v](\xi,t), \quad \left[\dfrac{\partial w}{\partial x}\right](\xi,t) = -\left[\dfrac{\partial v}{\partial x}\right](\xi,t), \quad t \in (0,T]. \end{cases} \tag{4.8}$$

Hence, $w(\xi^-,t) = u(\xi^-,t) - v(\xi^-,t)$ and $w(\xi^+,t) = u(\xi^+,t) - v(\xi^+,t)$. Note that since the problem (4.1)-(4.4) has a unique solution, the decomposition $u = v + w$ is valid. That's why, although both the components v and w are discontinuous at $x = \xi,\ t > 0$, but their sum is in $\mathcal{C}^{1+\lambda}(G)$. In Theorem 4.2.4, we shall obtain the bounds on the derivatives of the components v and w. Before proving this theorem, the following result which follows directly from [35] is stated below and this will be used in Theorem 4.2.4.

Theorem 4.2.3. *Assume in equation (4.1) that the data $a, b \in \mathcal{C}^2(\overline{\Omega})$, $f \in \mathcal{C}^{2+\lambda}(\overline{G})$ and the convection coefficient $a(x) \geq \alpha > 0,\ x \in \overline{\Omega}$. Also, let the initial-boundary data s_0, s_1 and s_2 be identically zero so that the compatibility conditions*

$$\frac{\partial^{l+m} f}{\partial x^l \partial t^m}(0,0) = \frac{\partial^{l+m} f}{\partial x^l \partial t^m}(1,0) = 0, \quad \text{for } 0 \leq l + 2m \leq 2,$$

be fulfilled. Then the solution $u(x,t)$ of the IBVP (4.1) lies in $\mathcal{C}^{4+\lambda}(\overline{G})$ and satisfies the estimate

$$\left\| \frac{\partial^{l+m} u}{\partial x^l \partial t^m} \right\|_{\infty, G} \leq C\varepsilon^{-l}, \quad \text{for } 0 \leq l + 2m \leq 4. \tag{4.9}$$

Theorem 4.2.4. *For all non-negative integers l, m, satisfying $0 \leq l \leq 3$ and $0 \leq l + 2m \leq 4$, there exist smooth functions $g_1(t),\ g_2(t)$ such that the smooth component v and the layer component w defined in (4.7) and (4.8), respectively, satisfy the following bounds*

$$\left\| \frac{\partial^{l+m} v}{\partial x^l \partial t^m} \right\|_{\infty, G^- \cup G^+} \leq C, \qquad \left\| \frac{\partial^4 v}{\partial x^4} \right\|_{\infty, G^- \cup G^+} \leq C\varepsilon^{-1},$$

and

$$\left| \frac{\partial^{l+m} w}{\partial x^l \partial t^m}(x,t) \right| \leq \begin{cases} C\Big(\varepsilon^{-l} \exp\big(-(\xi - x)\alpha_1/\varepsilon\big)\Big), & (x,t) \in G^-, \\ C\Big(\varepsilon^{-l} \exp\big(-(x - \xi)\alpha_2/\varepsilon\big)\Big), & (x,t) \in G^+, \end{cases}$$

$$\left| \frac{\partial^4 w}{\partial x^4}(x,t) \right| \leq \begin{cases} C\Big(\varepsilon^{-4} \exp\big(-(\xi - x)\alpha_1/\varepsilon\big)\Big), & (x,t) \in G^-, \\ C\Big(\varepsilon^{-4} \exp\big(-(x - \xi)\alpha_2/\varepsilon\big)\Big), & (x,t) \in G^+. \end{cases}$$

Proof. Here, the analysis is carried out separately to the subregions G^-, G^+ to obtain the stronger bounds on the smooth component v, defined in (4.7) and its derivatives. To do this, we first consider the subregion G^+ and introduce a new function v^* on the domain $\overline{G^*}$ which is a sufficiently large neighborhood of $\overline{G^+}$ beyond the point of discontinuity $x = \xi$ so that $\overline{G^+} \subset \overline{G^*}$. For example, define an extended domain $G^* = \Omega^* \times (0, T],\ \Omega^* = (-1, 1)$. Let the function v^* on $\overline{G^*}$ be defined to be the sum

$$v^* = v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^* + \varepsilon^3 v_3^*,$$

where the functions $v_i^*,\ i = 0, 1, 2$, are solutions to the following first-order problems

$$\begin{cases} a^* \dfrac{\partial v_0^*}{\partial x} - b^* v_0^* - \dfrac{\partial v_0^*}{\partial t} = f^* \quad \text{in } G^*, \\ v_0^*(1, t) = u(1, t), \quad t \in (0, T], \quad v_0^*(x, 0) = s_0^*(x), \quad x \in \overline{\Omega^*}; \\ a^* \dfrac{\partial v_i^*}{\partial x} - b^* v_i^* - \dfrac{\partial v_i^*}{\partial t} = -\dfrac{\partial^2 v_{i-1}^*}{\partial x^2} \quad \text{in } G^*, \\ v_i^*(1, t) = 0, \quad t \in (0, T], \quad v_i^*(x, 0) = 0, \quad x \in \overline{\Omega^*}, \quad i = 1, 2, \end{cases} \tag{4.10}$$

and lastly, the function v_3^* satisfies

$$\begin{cases} \varepsilon \dfrac{\partial^2 v_3^*}{\partial x^2} + a^* \dfrac{\partial v_3^*}{\partial x} - b^* v_3^* - \dfrac{\partial v_3^*}{\partial t} = -\dfrac{\partial^2 v_2^*}{\partial x^2} \quad \text{in } G^*, \\ v_3^*(-1, t) = v_3^*(1, t) = 0, \quad t \in (0, T], \quad v_3^*(x, 0) = 0, \quad x \in \overline{\Omega^*}. \end{cases} \tag{4.11}$$

Here, the coefficients a^*, b^* and the initial value s_0^* are respective smooth extensions of the functions a, b and s_0 associated with the problem (4.1)-(4.4) from the domain $\overline{\Omega^+}$ onto the domain $\overline{\Omega^*}$. Whereas the function f^* is the smooth extension of the function f from the domain $\overline{G^+}$ onto the extended domain $\overline{G^*}$. The extensions s_0^*, f^* are constructed so that $s_0^* = f^* = 0$ in a neighborhood of the artificial corner $(-1, 0)$. This freedom in the choice of extensions permits us to easily deal with the compatibility at the corner $(-1, 0)$.

Now, assume that the data a^*, b^*, f^* are sufficiently smooth on $\overline{G^*}$ and for simplicity, we assume all the initial-boundary data associated with (4.10) are identically zero. Afterwards, if we impose the following compatibility condition on f^* at the inflow corner $(1, 0)$

$$\frac{\partial^{l+m} f^*}{\partial x^l \partial t^m}(1, 0) = 0, \quad \text{for } 0 \leq l + m \leq 7,$$

then applying the results of Bobisud ([4], Appendix) to the first-order equations specified in (4.10), one can show that the inclusions $v_i^* \in \mathfrak{C}^{9-2i+\lambda}(\overline{G^*}) \bigcap \mathfrak{C}^{8-2i}(\overline{G^*}),\ i = 0, 1, 2,$ hold so that $\dfrac{\partial^2 v_2^*}{\partial x^2} \in \mathfrak{C}^{2+\lambda}(\overline{G^*})$ and hence $v_3^* \in \mathfrak{C}^{4+\lambda}(\overline{G^*})$. Moreover, since the functions $v_i^*,\ i = 0, 1, 2,$ are independent of ε and the solution v_3^* of (4.11) satisfies the estimate (4.9) analogously, so we obtain the following bounds

$$\left\| \frac{\partial^{l+m} v_i^*}{\partial x^l \partial t^m} \right\|_{\infty, G^*} \leq C,\ i = 0, 1, 2, \qquad \left\| \frac{\partial^{l+m} v_3^*}{\partial x^l \partial t^m} \right\|_{\infty, G^*} \leq C\varepsilon^{-l}, \quad \text{for } 0 \leq l + 2m \leq 4.$$

We now define the smooth component v as a restriction of v^* onto the domain $\overline{G^+}$ and also, define $g_2(t) = v^*(\xi, t)$. Hence, the function $v \in \mathfrak{C}^{4+\lambda}(\overline{G^+})$ is the solution of the following problem

$$\begin{cases} L_\varepsilon v \;=\; f \quad \text{in } G^+, \\ v(\xi,t) = v^*(\xi,t), \quad v(1,t) = u(1,t), \quad t \in (0,T], \\ v(x,0) = u(x,0), \quad x \in \overline{\Omega^+}. \end{cases}$$

Thus, the smooth component v satisfies

$$\left\| \frac{\partial^{l+m} v}{\partial x^l \partial t^m} \right\|_{\infty, G^+} \le C(1+\varepsilon^{3-l}), \quad \text{for } 0 \le l+2m \le 4.$$

Henceforth, applying the above arguments separately to the subregion G^-, one can define $g_1(t)$ similarly and thus, the required bounds on the smooth component v can be easily established.

Now, we shall proceed to find out the required bounds on the layer component w, defined in (4.8) and its derivatives on the domain $G^- \cup G^+$. To obtain the bound on w, define the following functions

$$\phi^{\pm}(x,t) = \begin{cases} \Big(\pm v(x,t) - \mathfrak{M}\Big) \exp\big(-(\xi - x)\alpha_1/\varepsilon\big) \pm w(x,t), & (x,t) \in G^-, \\ \Big(\pm v(x,t) - \mathfrak{M}\Big) \exp\big(-(x - \xi)\alpha_2/\varepsilon\big) \pm w(x,t), & (x,t) \in G^+, \end{cases} \tag{4.12}$$

where $\mathfrak{M}$ (a positive constant independent of ε) is chosen to be sufficiently large. Then, clearly

$$\phi^{\pm} \in \mathfrak{C}^0(\overline{G}), \quad \phi^{\pm}(x,t) \le 0,\ (x,t) \in \Gamma.$$

Now, from (4.8), (4.12) and employing the bound on v, we have for $t > 0$

$$[\phi^{\pm}](\xi,t) \;=\; \pm[w](\xi,t) \pm [v](\xi,t) \;=\; 0,$$

and

$$\begin{aligned} \left[\frac{\partial \phi^{\pm}}{\partial x}\right](\xi,t) &= \pm\left[\frac{\partial w}{\partial x}\right](\xi,t) \pm \left[\frac{\partial v}{\partial x}\right](\xi,t) - \left[\frac{\alpha_2}{\varepsilon}\Big(\pm v(\xi^+,t) - \mathfrak{M}\Big) + \frac{\alpha_1}{\varepsilon}\Big(\pm v(\xi^-,t) - \mathfrak{M}\Big)\right] \\ &= \left[\frac{\alpha_2}{\varepsilon}\Big(\mathfrak{M} \mp v(\xi^+,t)\Big) + \frac{\alpha_1}{\varepsilon}\Big(\mathfrak{M} \mp v(\xi^-,t)\Big)\right] \\ &\ge 0. \end{aligned}$$

Further, using the bounds on the derivatives of v, we obtain for $(x,t) \in G^-$,

$$\begin{aligned} L_\varepsilon \phi^{\pm}(x,t) &= \frac{\alpha_1}{\varepsilon}\left[(a+\alpha_1)\Big(\pm v(x,t) - \mathfrak{M}\Big) \pm \varepsilon\left(2\frac{\partial v}{\partial x} + \frac{f}{\alpha_1}\right) + \frac{\varepsilon b}{\alpha_1}\mathfrak{M}\right] \\ &\quad \times \exp\big(-(\xi - x)\alpha_1/\varepsilon\big) \\ &\ge 0, \end{aligned}$$

and for $(x,t) \in G^+$,

$$\begin{aligned} L_\varepsilon \phi^\pm(x,t) &= \frac{\alpha_2}{\varepsilon}\Big[(\alpha_2 - a)\Big(\pm v(x,t) - \mathfrak{M}\Big) \pm \varepsilon\Big(-2\frac{\partial v}{\partial x} + \frac{f}{\alpha_2}\Big) + \frac{\varepsilon b}{\alpha_2}\mathfrak{M}\Big] \\ &\quad \times \exp\big(-(x-\xi)\alpha_2/\varepsilon\big) \\ &\geq 0. \end{aligned}$$

Then, it follows from the maximum principle (Lemma 4.2.1) that $\big|\phi^\pm(x,t)\big| \leq 0,\ (x,t) \in \overline{G}$, which leads to the desired bound on w. The bounds on the derivatives of w follow from the arguments given in [55] and hence, the proof is finished. ■

Remark 4.2.5. *The bounds obtained in Theorem 4.2.4 for the smooth component v can also be obtained by following the arguments given in [72]. In that case, instead of the estimate (4.9) one has to establish the estimate*

$$\left|\frac{\partial^{l+m} u}{\partial x^l \partial t^m}(x,t)\right| \leq C\Big(1 + \varepsilon^{-l}\exp\big(-\alpha x/\varepsilon\big)\Big), \quad (x,t) \in G, \quad \text{for } 0 \leq l+m \leq 3, \tag{4.13}$$

for the same IBVP considered in Theorem 4.2.3, which shows that away from the side $x = 0$ all the partial derivatives up to order 3 are bounded independent of ε. One can derive this estimate by applying the technique described in [69], which is in fact a lengthy procedure compared to the proof of the estimate (4.9). In the proof of Theorem 4.2.3, the use of the estimate (4.13) is avoided and an alternative approach is provided to obtain the bounds on the smooth component v, defined in (4.7). Moreover, an explicit form of the barrier function is given by (4.12), which is very much crucial for getting the bounds on the layer component w.

4.3 Numerical Approximation

This section provides a suitable mesh for the discretization of the domain to obtain an ε-uniform convergent difference scheme and also explicitly describes the difference scheme used to discretize the problem (4.1)-(4.4).

4.3.1 Discretization of the domain

Consider the domain $\overline{G} = \overline{\Omega} \times [0,T] = [0,1] \times [0,T]$ and let $N \geq 8$ be an even positive integer. Here, we will construct a rectangular mesh $\overline{G}_\varepsilon^{N,M} = \overline{\Omega}_x^{N,\varepsilon} \times \mathbb{S}_t^M$, which is a combination of the piecewise-uniform Shishkin mesh condensed around the interior layers for the spatial variable and a uniform mesh for the temporal variable. Firstly, we define the piecewise-uniform Shishkin mesh by subdividing the spatial domain $\overline{\Omega}$ into four subintervals as

$$\overline{\Omega} = [0, \xi - \sigma_1] \cup [\xi - \sigma_1, \xi] \cup [\xi, \xi + \sigma_2] \cup [\xi + \sigma_2, 1],$$

for some σ_1, σ_2 that satisfy $0 < \sigma_1 \leq \xi/2$, $0 < \sigma_2 \leq (1-\xi)/2$. On each subinterval a uniform mesh with $N/4$ mesh-intervals is placed such that

$$\Omega_x^{N,\varepsilon} = \{x_i : 1 \leq i \leq N/2 - 1\} \bigcup \{x_i : N/2 + 1 \leq i \leq N - 1\}$$

denotes the set of interior points of the mesh. Clearly $x_{N/2} = \xi$ and $\overline{\Omega}_x^{N,\varepsilon} = \{x_i\}_0^N$. Note that this mesh is uniform when $\sigma_1 = \xi/2,\ \sigma_2 = (1-\xi)/2$. It is fitted to the problem (4.1)-(4.4) by choosing σ_1 and σ_2 to be the following functions of N and ε

$$\sigma_1 = \min\left\{\frac{\xi}{2}, \sigma_0 \varepsilon \ln N\right\}, \qquad \sigma_2 = \min\left\{\frac{1-\xi}{2}, \sigma_0 \varepsilon \ln N\right\},$$

where σ_0 is a positive constant will be chosen later on. On the time domain $[0, T]$, the equidistant meshes are introduced in the temporal variable such that

$$\mathbb{S}_t^M = \{t_n = n\,\Delta t,\ n = 0, \ldots, M,\ t_0 = 0, t_M = T,\ \Delta t = T/M\},$$

where M denotes the number of mesh-intervals in the t-direction. Let the step sizes in space be denoted by

$$h_i = x_i - x_{i-1},\ i = 1, \ldots, N, \qquad \widehat{h}_i = h_i + h_{i+1},\ i = 1, \ldots, N-1.$$

Further, denote the mesh width h_i in the spatial direction as follows:

$$h_i = \begin{cases} H_{(l)} = 4(\xi - \sigma_1)/N, & \text{for } i = 1, \ldots, N/4, \\ h_{(l)} = 4\sigma_1/N, & \text{for } i = N/4 + 1, \ldots, N/2, \\ h_{(r)} = 4\sigma_2/N, & \text{for } i = N/2 + 1, \ldots, 3N/4, \\ H_{(r)} = 4(1 - \xi - \sigma_2)/N, & \text{for } i = 3N/4 + 1, \ldots, N. \end{cases}$$

4.3.2 The backward-Euler hybrid finite difference scheme

For the discretization of the problem (4.1)-(4.4) in the spatial variable, we propose a hybrid scheme. The hybrid scheme consists of the midpoint upwind scheme in the outer regions $[0, \xi - \sigma_1], [\xi + \sigma_2, 1]$ and the classical central difference scheme in the interior layer regions $(\xi - \sigma_1, \xi), (\xi, \xi + \sigma_2)$. While at the point of discontinuity, a second-order one-sided difference approximations are used to keep the continuity of the spatial derivative and for the temporal discretization, we employ the backward-Euler method. Then, the proposed numerical scheme takes the following form:

$$\begin{cases} L_{mu}^{N,M,(-)}U_i^{n+1} = f_{i-1/2}^{n+1}, & \text{for } i = 1,\dots,N/4, \\ L_{cen}^{N,M}U_i^{n+1} = f_i^{n+1}, & \text{for } i = N/4+1,\dots,N/2-1, \\ & \qquad N/2+1,\dots,3N/4-1, \\ L_{mu}^{N,M,(+)}U_i^{n+1} = f_{i+1/2}^{n+1}, & \text{for } i = 3N/4,\dots,N-1, \\ D_x^F U_i^{n+1} - D_x^B U_i^{n+1} = 0, & \text{for } i = N/2, \end{cases} \tag{4.14}$$

where

$$\begin{cases} L_{mu}^{N,M,(-)}U_i^{n+1} = \varepsilon\delta_x^2 U_i^{n+1} + a_{i-1/2}D_x^- U_i^{n+1} - b_{i-1/2}U_{i-1/2}^{n+1} \\ \qquad\qquad -D_t^- U_{i-1/2}^{n+1}, \\ L_{cen}^{N,M}U_i^{n+1} = \varepsilon\delta_x^2 U_i^{n+1} + a_i D_x^0 U_i^{n+1} - b_i U_i^{n+1} - D_t^- U_i^{n+1}, \\ L_{mu}^{N,M,(+)}U_i^{n+1} = \varepsilon\delta_x^2 U_i^{n+1} + a_{i+1/2}D_x^+ U_i^{n+1} - b_{i+1/2}U_{i+1/2}^{n+1} \\ \qquad\qquad -D_t^- U_{i+1/2}^{n+1}, \end{cases} \tag{4.15}$$

and

$$D_x^F U_i^n = (-U_{i+2}^n + 4U_{i+1}^n - 3U_i^n)/2h_{(r)}, \quad D_x^B U_i^n = (U_{i-2}^n - 4U_{i-1}^n + 3U_i^n)/2h_{(l)}. \tag{4.16}$$

After rearranging the terms in (4.14), we obtain the following form of the difference scheme on the mesh $\overline{G}_\varepsilon^{N,M}$:

$$\begin{cases} U_i^0 = s_0(x_i), \quad \text{for } i = 0,\dots,N, \\ \begin{cases} L_\varepsilon^{N,M}U_i^{n+1} = \widetilde{f}_i^{n+1}, \quad \text{for } i = 1,\dots,N-1, \\ U_0^{n+1} = s_1(t_{n+1}), \quad U_N^{n+1} = s_2(t_{n+1}), \end{cases} \\ \text{for } n = 0,\dots,M-1, \end{cases} \tag{4.17}$$

where the finite difference operator $L_\varepsilon^{N,M}$ is defined as

$$L_\varepsilon^{N,M}U_i^{n+1} = \begin{cases} \left[r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1}\right] + \left[p_i^- U_{i-1}^n + p_i^0 U_i^n + p_i^+ U_{i+1}^n\right], \\ \qquad\qquad \text{for } i = 1,\dots,N/2-1, N/2+1,\dots,N-1, \\ q_i^{-,2}U_{i-2}^{n+1} + q_i^{-,1}U_{i-1}^{n+1} + q_i^0 U_i^{n+1} + q_i^{+,1}U_{i+1}^{n+1} + q_i^{+,2}U_{i+2}^{n+1}, \\ \qquad\qquad \text{for } i = N/2, \end{cases} \tag{4.18}$$

and the right hand side vector $\widetilde{f}^{n+1}$ as

$$\widetilde{f}_i^{n+1} = \begin{cases} \left[m_i^- f_{i-1}^{n+1} + m_i^0 f_i^{n+1} + m_i^+ f_{i+1}^{n+1}\right], & \text{for } i = 1, \dots, N/2-1, \\ & \qquad N/2+1, \dots, N-1, \\ 0, \quad \text{for } i = N/2. \end{cases} \tag{4.19}$$

Here, for $i = 1, \dots, N/4$, the coefficients are given by

$$\begin{cases} r_i^- = \left(\dfrac{2\varepsilon}{\widehat{h}_i h_i} - \dfrac{a_{i-1/2}}{h_i} - \dfrac{b_{i-1/2}}{2}\right) - \dfrac{1}{2\Delta t}, \\ r_i^0 = \left(-\dfrac{2\varepsilon}{h_i h_{i+1}} + \dfrac{a_{i-1/2}}{h_i} - \dfrac{b_{i-1/2}}{2}\right) - \dfrac{1}{2\Delta t}, \\ r_i^+ = \dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}}, \\ p_i^- = \dfrac{1}{2\Delta t}, \quad p_i^0 = \dfrac{1}{2\Delta t}, \quad p_i^+ = 0, \\ m_i^- = \dfrac{1}{2}, \quad m_i^0 = \dfrac{1}{2}, \quad m_i^+ = 0, \end{cases} \tag{4.20}$$

and for $i = N/4+1, \dots, N/2-1, N/2+1, \dots, 3N/4-1$,

$$\begin{cases} r_i^- = \left(\dfrac{2\varepsilon}{\widehat{h}_i h_i} - \dfrac{a_i}{\widehat{h}_i}\right), \\ r_i^0 = \left(-\dfrac{2\varepsilon}{h_i h_{i+1}} - b_i\right) - \dfrac{1}{\Delta t}, \\ r_i^+ = \left(\dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}} + \dfrac{a_i}{\widehat{h}_i}\right), \\ p_i^- = 0, \quad p_i^0 = \dfrac{1}{\Delta t}, \quad p_i^+ = 0, \\ m_i^- = 0, \quad m_i^0 = 1, \quad m_i^+ = 0, \end{cases} \tag{4.21}$$

and for $i = 3N/4, \dots, N-1$,

$$\begin{cases} r_i^- = \dfrac{2\varepsilon}{\widehat{h}_i h_i}, \\ r_i^0 = \left(-\dfrac{2\varepsilon}{h_i h_{i+1}} - \dfrac{a_{i+1/2}}{h_{i+1}} - \dfrac{b_{i+1/2}}{2}\right) - \dfrac{1}{2\Delta t}, \\ r_i^+ = \left(\dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}} + \dfrac{a_{i+1/2}}{h_{i+1}} - \dfrac{b_{i+1/2}}{2}\right) - \dfrac{1}{2\Delta t}, \\ p_i^- = 0, \quad p_i^0 = \dfrac{1}{2\Delta t}, \quad p_i^+ = \dfrac{1}{2\Delta t}, \\ m_i^- = 0, \quad m_i^0 = \dfrac{1}{2}, \quad m_i^+ = \dfrac{1}{2}, \end{cases} \tag{4.22}$$

and lastly, for $i = N/2$,

$$q_i^{-,2} = -\frac{1}{2h_{(l)}}, \quad q_i^{-,1} = \frac{2}{h_{(l)}}, \quad q_i^0 = -\frac{3}{2}\left(\frac{1}{h_{(r)}} + \frac{1}{h_{(l)}}\right), \quad q_i^{+,1} = \frac{2}{h_{(r)}}, \quad q_i^{+,2} = -\frac{1}{2h_{(r)}}. \tag{4.23}$$

4.4 Error Analysis

This section studies the stability of the proposed numerical scheme (4.17) and finally, the ε-uniform error estimate is obtained.

In the analysis, we shall assume that $\sigma_1 = \sigma_2 = \sigma = \sigma_0 \varepsilon \ln N$, because otherwise N^{-1} is exponentially small relatively to ε, which is very unlikely in practice (and in this case the method can be analyzed in the classical way). Then the spatial mesh widths are

$$h_i = \begin{cases} H_{(l)} = 4(\xi - \sigma)/N, & \text{for } i = 1, \ldots, N/4, \\ h = h_{(l)} = h_{(r)} = 4\sigma/N, & \text{for } i = N/4+1, \ldots, 3N/4, \\ H_{(r)} = 4(1 - \xi - \sigma)/N, & \text{for } i = 3N/4+1, \ldots, N. \end{cases} \tag{4.24}$$

Next, we see that the finite difference operator $L_\varepsilon^{N,M}$ defined in (4.18) does not satisfy the discrete maximum principle and accordingly, the present form (4.17) can not help us to find out the uniform stability criteria for the proposed numerical scheme (4.14). Due to this reason we need to transform the equation (4.17) for $i = N/2$, *i.e.*,

$$q_{N/2}^{-,2} U_{N/2-2}^{n+1} + q_{N/2}^{-,1} U_{N/2-1}^{n+1} + q_{N/2}^{0} U_{N/2}^{n+1} + q_{N/2}^{+,1} U_{N/2+1}^{n+1} + q_{N/2}^{+,2} U_{N/2+2}^{n+1} = 0, \tag{4.25}$$

so that the operator associated with the new system of equations will be able to reflect the stability property of the proposed scheme (4.14). Now, from (4.17) we have

$$\begin{cases} U_{N/2-2}^{n+1} = \dfrac{2h^2}{2\varepsilon - h a_{N/2-1}} \left[f_{N/2-1}^{n+1} - r_{N/2-1}^{0} U_{N/2-1}^{n+1} - r_{N/2-1}^{+} U_{N/2}^{n+1} - \dfrac{1}{\Delta t} U_{N/2-1}^{n} \right], \\ U_{N/2+2}^{n+1} = \dfrac{2h^2}{2\varepsilon + h a_{N/2+1}} \left[f_{N/2+1}^{n+1} - r_{N/2+1}^{0} U_{N/2+1}^{n+1} - r_{N/2+1}^{-} U_{N/2}^{n+1} - \dfrac{1}{\Delta t} U_{N/2+1}^{n} \right]. \end{cases} \tag{4.26}$$

Therefore, after substituting $U_{N/2-2}^{n+1}$ and $U_{N/2+2}^{n+1}$ from (4.25) using (4.26), the difference scheme (4.17) gets change into the following form on the mesh $\overline{G}_\varepsilon^{N,M}$:

$$\begin{cases} U_i^0 = s_0(x_i), \quad \text{for } i = 0, \ldots, N, \\ \begin{cases} L_H^{N,M} U_i^{n+1} = \widetilde{f}_{H,i}^{n+1}, \quad \text{for } i = 1, \ldots, N-1, \\ U_0^{n+1} = s_1(t_{n+1}), \quad U_N^{n+1} = s_2(t_{n+1}), \end{cases} \\ \text{for } n = 0, \ldots, M-1. \end{cases} \tag{4.27}$$

Here, the finite difference operator $L_H^{N,M}$ and the right hand side term $\widetilde{f}_{H,i}^{n+1}$ are respectively defined as

$$\begin{cases} L_H^{N,M} U_i^{n+1} = \left[r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1}\right] + \left[p_i^- U_{i-1}^n + p_i^0 U_i^n + p_i^+ U_{i+1}^n\right], \quad \text{and} \\ \widetilde{f}_{H,i}^{n+1} = \left[m_i^- f_{i-1}^{n+1} + m_i^0 f_i^{n+1} + m_i^+ f_{i+1}^{n+1}\right], \end{cases} \tag{4.28}$$

where for $i = 1, \ldots, N/2-1, N/2+1, \ldots, N-1$, the coefficients $r_i^-, r_i^0, r_i^+; p_i^-, p_i^0, p_i^+; m_i^-, m_i^0, m_i^+$ are described in (4.20)-(4.22) and for $i = N/2$,

$$\begin{cases} r_i^- = \dfrac{1}{2h}\left[4 - \dfrac{2(2\varepsilon + h^2 b_{i-1} + \frac{h^2}{\Delta t})}{2\varepsilon - h a_{i-1}}\right], \\ r_i^0 = \dfrac{1}{2h}\left[-6 + \dfrac{2\varepsilon + h a_{i-1}}{2\varepsilon - h a_{i-1}} + \dfrac{2\varepsilon - h a_{i+1}}{2\varepsilon + h a_{i+1}}\right], \\ r_i^+ = \dfrac{1}{2h}\left[4 - \dfrac{2(2\varepsilon + h^2 b_{i+1} + \frac{h^2}{\Delta t})}{2\varepsilon + h a_{i+1}}\right], \\ p_i^- = \dfrac{h}{(2\varepsilon - h a_{i-1})\Delta t}, \quad p_i^0 = 0, \quad p_i^+ = \dfrac{h}{(2\varepsilon + h a_{i+1})\Delta t}, \\ m_i^- = \dfrac{h}{(2\varepsilon - h a_{i-1})}, \quad m_i^0 = 0, \quad m_i^+ = \dfrac{h}{(2\varepsilon + h a_{i+1})}. \end{cases} \tag{4.29}$$

Let $G_\varepsilon^{N,M} = \overline{G}_\varepsilon^{N,M} \bigcap G$ and $\Gamma_\varepsilon^{N,M} = \overline{G}_\varepsilon^{N,M} \setminus G_\varepsilon^{N,M}$.

Lemma 4.4.1. *Assume that $N \geq N_0$, where*

$$\frac{N_0}{\ln N_0} \geq 2\sigma_0 \alpha^* \quad \textit{and} \tag{4.30}$$

$$\left(\|b\|_\infty + \Delta t^{-1}\right) \leq \frac{\alpha N_0}{2}, \tag{4.31}$$

*where $\alpha = \min\{\alpha_1, \alpha_2\}$ and $\alpha^* = \max\{\alpha_1^*, \alpha_2^*\}$. Then, the operator $L_H^{N,M}$ defined by (4.28) satisfies a **discrete maximum principle**, i.e., if the mesh function Z satisfies $Z \leq 0$ on $\Gamma_\varepsilon^{N,M}$, then $L_H^{N,M} Z \geq 0$ in $G_\varepsilon^{N,M}$ implies that $Z \leq 0$ at each point of $\overline{G}_\varepsilon^{N,M}$.*

Proof. On $G_\varepsilon^{N,M}$, set

$$-L_H^{N,M} Z_i^{n+1} = \left[A_{i,i-1} Z_{i-1}^{n+1} + A_{i,i} Z_i^{n+1} + A_{i,i+1} Z_{i+1}^{n+1}\right] - \left[B_{i,i-1} Z_{i-1}^n + B_{i,i} Z_i^n + B_{i,i+1} Z_{i+1}^n\right],$$

where the matrices $\mathbf{A} := (A_{i,j})$ and $\mathbf{B} := (B_{i,j})$ are respectively defined by

$$\begin{cases} A_{i,i-1} = -r_i^-, \ A_{i,i} = -r_i^0, \ A_{i,i+1} = -r_i^+, \quad \text{and} \\ B_{i,i-1} = p_i^-, \ B_{i,i} = p_i^0, \ B_{i,i+1} = p_i^+. \end{cases}$$

Under the assumptions (4.30) and (4.31), the matrix $\mathbf{A}$ is an M-matrix. Also, the matrix $\mathbf{B} \geq 0$. Therefore, the proof follows from Lemma 3.8 of the book of Roos *et al.* [77]. ■

An immediate consequence of the discrete maximum principle is the following stability result.

Lemma 4.4.2. *Let U be the solution of (4.27) and let the assumptions (4.30) and (4.31) of Lemma 4.4.1 hold true. Then*

$$\|U\|_{\infty,\overline{G}_\varepsilon^{N,M}} \le \|U\|_{\infty,\Gamma_\varepsilon^{N,M}} + \frac{1}{\mu}\|\widetilde{f_H}\|_{\infty,\overline{G}_\varepsilon^{N,M}},$$

where $\mu = \min\{\alpha_1/\xi, \alpha_2/(1-\xi)\}$.

Proof. Consider the following discrete functions

$$\varphi_i^{\pm,n} = -\|U\|_{\infty,\Gamma_\varepsilon^{N,M}} - \begin{cases} \dfrac{x_i}{\mu\xi}\|\widetilde{f_H}\|_\infty \mp U_i^n, & \text{for } 0 \le i \le N/2, \\ \dfrac{(1-x_i)}{\mu(1-\xi)}\|\widetilde{f_H}\|_\infty \mp U_i^n, & \text{for } N/2 < i \le N. \end{cases}$$

Then, clearly $\varphi_0^{\pm,n+1} \le 0$, $\varphi_N^{\pm,n+1} \le 0$, $\varphi_i^{\pm,0} \le 0$ and $L_H^{N,M}\varphi_i^{\pm,n+1} \ge 0$, for $i = 1,\dots,N/2-1, N/2+1,\dots,N-1$. Further,

$$L_H^{N,M}\varphi_{N/2}^{\pm,n+1} \ge \left(D_x^F - D_x^B\right)\varphi_{N/2}^{\pm,n+1} = \left[\frac{1}{\mu(1-\xi)} + \frac{1}{\mu\xi}\right]\|\widetilde{f_H}\|_\infty \ge 0.$$

Therefore, applying Lemma 4.4.1 we obtain the required stability bound. ■

To estimate the nodal error $\left|U_i^{n+1} - u(x_i,t_{n+1})\right|$ separately outside and inside the layers, the discrete solution U is decomposed by applying the similar technique as that of [72]. First, define the mesh functions V_L and V_R, which approximate v respectively to the left and the right of the point of discontinuity $x = \xi$. Then, the mesh functions W_L and W_R are constructed (to approximate w on either side of $x = \xi$) so that the amplitude of the jump $W_R(\xi) - W_L(\xi)$ is determined by the size of the jump $\left|[v](\xi)\right|$.

Here, the mesh functions V_L and V_R are defined to be the solutions of the following discrete problems

$$\begin{cases} L_H^{N,M}V_{L,i}^{n+1} = \widetilde{f_{H,i}}^{n+1}, & \text{for } i = 1,\dots,N/2-1, \\ V_{L,0}^{n+1} = v(0,t_{n+1}), \quad V_{L,N/2}^{n+1} = v(\xi^-,t_{n+1}), & n \ge 0, \\ V_{L,i}^0 = v(x_i,0), \quad i \le N/2, \end{cases}$$

and

$$\begin{cases} L_H^{N,M}V_{R,i}^{n+1} = \widetilde{f_{H,i}}^{n+1}, & \text{for } i = N/2+1,\dots,N-1, \\ V_{R,N/2}^{n+1} = v(\xi^+,t_{n+1}), \quad V_{R,N}^{n+1} = v(1,t_{n+1}), & n \ge 0, \\ V_{R,i}^0 = v(x_i,0), \quad i \ge N/2. \end{cases}$$

Also, define the mesh functions W_L and W_R to be the solutions of the following system of finite difference equations

$$\begin{cases} L_H^{N,M} W_{L,i}^{n+1} = 0, \quad \text{for } i = 1, \ldots, N/2 - 1, \\ L_H^{N,M} W_{R,i}^{n+1} = 0, \quad \text{for } i = N/2 + 1, \ldots, N - 1, \\ W_{L,0}^{n+1} = 0, \quad W_{R,N}^{n+1} = 0, \\ W_{L,i}^{0} = 0, \quad i \le N/2, \quad W_{R,i}^{0} = 0, \quad i \ge N/2, \\ W_{R,N/2}^{n+1} + V_{R,N/2}^{n+1} = W_{L,N/2}^{n+1} + V_{L,N/2}^{n+1}, \\ D_x^F W_{R,N/2}^{n+1} + D_x^F V_{R,N/2}^{n+1} = D_x^B W_{L,N/2}^{n+1} + D_x^B V_{L,N/2}^{n+1}, \quad n \ge 0. \end{cases}$$

So, one can define U to satisfy the following decomposition

$$U_i^{n+1} = \begin{cases} V_{L,i}^{n+1} + W_{L,i}^{n+1}, & \text{for } i = 1, \ldots, N/2 - 1, \\ V_{L,i}^{n+1} + W_{L,i}^{n+1} = V_{R,i}^{n+1} + W_{R,i}^{n+1}, & \text{for } i = N/2, \\ V_{R,i}^{n+1} + W_{R,i}^{n+1}, & \text{for } i = N/2 + 1, \ldots, N - 1. \end{cases}$$

First, the errors are analyzed separately for the smooth components V_L and V_R of the solution U in the region $G^- \cup G^+$.

Lemma 4.4.3. *Assume that $\varepsilon \le N^{-1}$. Then under the assumptions (4.30) and (4.31) of Lemma 4.4.1, the errors associated to the smooth components satisfy the following estimates*

$$\begin{cases} \left| V_{L,i}^{n+1} - v(x_i, t_{n+1}) \right| \le C\Big(N^{-2} + \Delta t\Big) x_i, & \textit{for } 1 \le i \le N/2 - 1, \\ \left| V_{R,i}^{n+1} - v(x_i, t_{n+1}) \right| \le C\Big(N^{-2} + \Delta t\Big)(1 - x_i), & \textit{for } N/2 + 1 \le i \le N - 1. \end{cases}$$

Proof. Define the following barrier function

$$\Psi_{L,i} = -C\Big(N^{-2} + \Delta t\Big) x_i, \quad \text{for } 0 \le i \le N/2.$$

Now, the truncation error corresponding to the smooth component on the left side of the discontinuity satisfies the following estimate

$$\left| L_H^{N,M} \big(V_{L,i}^{n+1} - v(x_i, t_{n+1})\big) \right|$$

$$\le \begin{cases} C\Big[(\varepsilon + h_i)(h_i + h_{i+1}) \left\| \frac{\partial^3 v}{\partial x^3} \right\|_\infty + h_i^2 \Big(\left\| \frac{\partial^2 v}{\partial x^2} \right\|_\infty + \left\| \frac{\partial v}{\partial x} \right\|_\infty \Big) + \Delta t \left\| \frac{\partial^2 v}{\partial t^2} \right\|_\infty \Big], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \le i \le N/4, \\ C\Big[h^2 \Big(\varepsilon \left\| \frac{\partial^4 v}{\partial x^4} \right\|_\infty + \left\| \frac{\partial^3 v}{\partial x^3} \right\|_\infty \Big) + \Delta t \left\| \frac{\partial^2 v}{\partial t^2} \right\|_\infty \Big], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } N/4 + 1 \le i \le N/2 - 1. \end{cases}$$

Then, using $h_i \leq CN^{-1}$ (see (4.24)), the assumption $\varepsilon \leq N^{-1}$ and the bounds on the derivatives of v given in Theorem 4.2.4, we have

$$\left| L_H^{N,M}\left(V_{L,i}^{n+1} - v(x_i, t_{n+1})\right) \right| \leq C\left(N^{-2} + \Delta t\right) \leq L_H^{N,M}\Psi_{L,i}, \quad \text{for } 1 \leq i \leq N/2 - 1.$$

Thus, applying the discrete maximum principle to $\Psi_{L,i} \pm \left(V_{L,i}^n - v(x_i, t_n)\right)$ over $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^-}$, we have

$$\left| V_{L,i}^{n+1} - v(x_i, t_{n+1}) \right| \leq C\left(N^{-2} + \Delta t\right)x_i, \quad \text{for } 1 \leq i \leq N/2 - 1.$$

On the other hand, the truncation error corresponding to the smooth component on the right hand side of the discontinuity satisfies the following estimate

$$\left| L_H^{N,M}\left(V_{R,i}^{n+1} - v(x_i, t_{n+1})\right) \right|$$
$$\leq \begin{cases} C\left[h^2\left(\varepsilon\left\|\dfrac{\partial^4 v}{\partial x^4}\right\|_\infty + \left\|\dfrac{\partial^3 v}{\partial x^3}\right\|_\infty\right) + \Delta t\left\|\dfrac{\partial^2 v}{\partial t^2}\right\|_\infty\right], & \text{for } N/2 + 1 \leq i \leq 3N/4 - 1, \\ C\left[\left(\varepsilon + h_{i+1}\right)\left(h_i + h_{i+1}\right)\left\|\dfrac{\partial^3 v}{\partial x^3}\right\|_\infty + h_{i+1}^2\left(\left\|\dfrac{\partial^2 v}{\partial x^2}\right\|_\infty + \left\|\dfrac{\partial v}{\partial x}\right\|_\infty\right) + \Delta t\left\|\dfrac{\partial^2 v}{\partial t^2}\right\|_\infty\right], & \\ & \text{for } 3N/4 \leq i \leq N - 1. \end{cases}$$

Therefore, using a new barrier function

$$\Psi_{R,i} = -C\left(N^{-2} + \Delta t\right)(1 - x_i), \quad \text{for } N/2 \leq i \leq N,$$

and arguing similarly for $(V_R - v)$ on the right side of the discontinuity, we can easily get the following error bound

$$\left| V_{R,i}^{n+1} - v(x_i, t_{n+1}) \right| \leq C\left(N^{-2} + \Delta t\right)(1 - x_i), \quad \text{for } N/2 + 1 \leq i \leq N - 1.$$

Hence the proof. ■

Now, on $\overline{\Omega}_x^{N,\varepsilon} = \{x_i\}_0^N$, define the following two mesh functions

$$S_i = \prod_{j=1}^{i}\left(1 + \frac{\gamma h_j}{\varepsilon}\right), \quad \text{for } 1 \leq i \leq N/2, \quad Q_i = \prod_{j=1}^{N-i}\left(1 + \frac{\gamma \hbar_j}{\varepsilon}\right), \quad \text{for } N/2 \leq i \leq N - 1,$$

(with the usual convention that if $i = 0$, then $S_0 = 1$ and if $i = N$, then $Q_N = 1$) where γ is a positive constant and

$$\hbar_j = \begin{cases} H_{(r)}, & \text{for } 1 \leq j \leq N/4, \\ h, & \text{for } N/4 + 1 \leq j \leq N/2. \end{cases}$$

Lemma 4.4.4. *If $\gamma \leq \alpha/2$, then for some constant C, we have the followings inequalities:*

$$-L_H^{N,M} S_i \geq \begin{cases} \dfrac{C}{\varepsilon + \gamma h_i} S_i, & \text{for } 1 \leq i \leq N/4, \\ \dfrac{C}{\varepsilon + \gamma h} S_i, & \text{for } N/4+1 \leq i \leq N/2-1, \end{cases}$$

and

$$-L_H^{N,M} Q_i \geq \begin{cases} \dfrac{C}{\varepsilon + \gamma h} Q_i, & \text{for } N/2+1 \leq i \leq 3N/4-1, \\ \dfrac{C}{\varepsilon + \gamma \hbar_{N-i}} Q_i, & \text{for } 3N/4 \leq i \leq N-1. \end{cases}$$

Proof. Firstly, $(S_i - S_{i-1}) = \dfrac{\gamma h_i}{\varepsilon} S_{i-1}$ and $a_i < -\alpha_1 \leq -2\gamma$, for $i < N/2$. Hence, a simple calculation yields

$$\begin{aligned} -L_H^{N,M} S_i &= -\frac{2\gamma}{(h_i + h_{i+1})}(S_i - S_{i-1}) - a_{i-1/2}\frac{\gamma}{\varepsilon} S_{i-1} + b_{i-1/2} S_{i-1/2} \\ &\geq -\frac{\gamma}{\varepsilon} S_{i-1}\left[\frac{2\gamma h_i}{(h_i + h_{i+1})} + a_{i-1/2}\right] \geq \frac{C}{\varepsilon + \gamma h_i} S_i, \quad \text{for } 1 \leq i \leq N/4, \end{aligned}$$

and

$$\begin{aligned} -L_H^{N,M} S_i &= -\frac{\gamma}{h}(S_i - S_{i-1}) - a_i\left[\frac{\gamma}{\varepsilon} + \frac{\gamma^2 h}{2\varepsilon^2}\right] S_{i-1} + b_i S_i \\ &\geq -\frac{\gamma}{\varepsilon} S_{i-1}(\gamma + a_i) - a_i \frac{\gamma^2 h}{2\varepsilon^2} S_{i-1} \geq \frac{C}{\varepsilon + \gamma h} S_i, \quad \text{for } N/4+1 \leq i \leq N/2-1. \end{aligned}$$

Next, $(Q_{i+1} - Q_i) = -\dfrac{\gamma \hbar_{N-i}}{\varepsilon} Q_{i+1}$ and $a_i > \alpha_2 \geq 2\gamma$, for $i > N/2$. Hence, it follows that

$$\begin{aligned} -L_H^{N,M} Q_i &= \frac{\gamma}{h}(Q_{i+1} - Q_i) + a_i\left[\frac{\gamma}{\varepsilon} + \frac{\gamma^2 h}{2\varepsilon^2}\right] Q_{i+1} + b_i Q_i \\ &\geq \frac{\gamma}{\varepsilon} Q_{i+1}(a_i - \gamma) + a_i \frac{\gamma^2 h}{2\varepsilon^2} Q_{i+1} \geq \frac{C}{\varepsilon + \gamma h} Q_i, \quad \text{for } N/2+1 \leq i \leq 3N/4-1, \end{aligned}$$

and

$$\begin{aligned} -L_H^{N,M} Q_i &= \frac{2\gamma}{(h_i + h_{i+1})}(Q_{i+1} - Q_i) + a_{i+1/2}\frac{\gamma}{\varepsilon} Q_{i+1} + b_{i+1/2} Q_{i+1/2} \\ &\geq \frac{\gamma}{\varepsilon} Q_{i+1}\left[a_{i+1/2} - \frac{2\gamma \hbar_{N-i}}{(h_i + h_{i+1})}\right] \geq \frac{C}{\varepsilon + \gamma \hbar_{N-i}} Q_i, \quad \text{for } 3N/4 \leq i \leq N-1. \quad \blacksquare \end{aligned}$$

Lemma 4.4.5. *Let $\sigma_0 \geq 2/\gamma$. Then for the Shishkin mesh $\overline{\Omega}_x^{N,\varepsilon} = \{x_i\}_0^N$, there exists a constant C such that*

$$\prod_{j=i+1}^{N/2}\left(1+\frac{\gamma h_j}{\varepsilon}\right)^{-1} \leq CN^{-4(1-2i/N)}, \quad \textit{for } N/4 \leq i \leq N/2-1. \tag{4.32}$$

and

$$\prod_{j=N-i+1}^{N/2}\left(1+\frac{\gamma \hbar_j}{\varepsilon}\right)^{-1} \leq CN^{-4(2i/N-1)}, \quad \textit{for } N/2+1 \leq i \leq 3N/4. \tag{4.33}$$

Proof. Here, the arguments are used as in the proof of [[88], Lemma 3.1]. For $N/4 \leq i \leq N/2-1$,

$$\begin{aligned}
\prod_{j=i+1}^{N/2}\left(1+\frac{\gamma h_j}{\varepsilon}\right)^{-1} &= \prod_{j=i+1}^{N/2}\left(1+\frac{\gamma h}{\varepsilon}\right)^{-1} \leq \exp\big(-\gamma(\xi - x_i)/(\varepsilon+\gamma h)\big)\\
&= N^{-2\gamma\sigma_0(1-2i/N)/(1+4\gamma\sigma_0 N^{-1}\ln N)}\\
&= N^{-2\gamma\sigma_0(1-2i/N)}N^{[8(\gamma\sigma_0)^2(1-2i/N)(N^{-1}\ln N)/(1+4\gamma\sigma_0 N^{-1}\ln N)]}\\
&\leq CN^{-2\gamma\sigma_0(1-2i/N)},
\end{aligned}$$

because the sequence $N^{[8(\gamma\sigma_0)^2(1-2i/N)(N^{-1}\ln N)/(1+4\gamma\sigma_0 N^{-1}\ln N)]}$ is bounded. Henceforth, the result (4.32) is obtained from the assumption $\sigma_0 \geq 2/\gamma$.

Next, for $N/2+1 \leq i \leq 3N/4$, one can get

$$\begin{aligned}
\prod_{j=N-i+1}^{N/2}\left(1+\frac{\gamma \hbar_j}{\varepsilon}\right)^{-1} &= \prod_{j=N-i+1}^{N/2}\left(1+\frac{\gamma h}{\varepsilon}\right)^{-1} \leq \exp\big(-\gamma(x_i-\xi)/(\varepsilon+\gamma h)\big)\\
&= N^{-2\gamma\sigma_0(2i/N-1)/(1+4\gamma\sigma_0 N^{-1}\ln N)},
\end{aligned}$$

and hereby, the result (4.33) follows similarly. ■

Now, we shall proceed to analyze the errors for the layer components W_L and W_R of the solution U in the region $\big((0,\xi-\sigma_1]\cup[\xi+\sigma_2,1)\big)\times(0,T]$. Here, we shall use Lemmas 4.4.4 and 4.4.5 instead of the usual truncation error argument.

Lemma 4.4.6. *Let $\gamma \leq \alpha/2$ and $\sigma_0 \geq 2/\gamma$. Then under the assumptions (4.30) and (4.31) of Lemma 4.4.1, the errors associated to the singular components satisfy the following estimates*

$$\begin{cases}
\left|W_{L,i}^{n+1} - w(x_i,t_{n+1})\right| \leq CN^{-2}, & \textit{for } 1 \leq i \leq N/4,\\[2mm]
\left|W_{R,i}^{n+1} - w(x_i,t_{n+1})\right| \leq CN^{-2}, & \textit{for } 3N/4 \leq i \leq N-1.
\end{cases}$$

Proof. Firstly, by definition W_L satisfies the following homogeneous difference equation

$$\begin{cases} L_H^{N,M} W_{L,i}^{n+1} = 0, \quad \text{for } 1 \le i \le N/2-1, \\ W_{L,0}^{n+1} = 0, \quad W_{L,i}^0 = 0, \quad i \le N/2. \end{cases} \tag{4.34}$$

Also, using Lemma 4.4.2 and Theorem 4.2.4, we can easily obtain

$$|W_{L,N/2}^{n+1}| \le C. \tag{4.35}$$

Therefore, it follows from (4.34), (4.35) and Lemma 4.4.4 that

$$\Phi_{L,i} = -C\left[\prod_{j=1}^{N/2}\left(1+\frac{\gamma h_j}{\varepsilon}\right)^{-1}\right] S_i, \quad \text{for } 0 \le i \le N/2,$$

is a barrier function for $\{W_{L,i}^n\}$. Thus, by the discrete maximum principle over $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^-}$, we have

$$|W_{L,i}^{n+1}| \le C \prod_{j=i+1}^{N/2}\left(1+\frac{\gamma h_j}{\varepsilon}\right)^{-1}, \quad \text{for } 1 \le i \le N/2-1. \tag{4.36}$$

Now, for $1 \le i \le N/4$, inequality (4.36) and Lemma 4.4.5 yield

$$|W_{L,i}^{n+1}| \le C \prod_{j=N/4+1}^{N/2}\left(1+\frac{\gamma h_j}{\varepsilon}\right)^{-1} \le CN^{-2}. \tag{4.37}$$

Next, since $\gamma \le \alpha/2 < \alpha$ and $\sigma_0 \ge 2/\gamma$, from Theorem 4.2.4, we obtain

$$|w(x_i, t_{n+1})| \le C\exp\big(-\alpha\sigma_1/\varepsilon\big) \le C\exp\big(-\gamma\sigma/\varepsilon\big) \le CN^{-2}, \quad \text{for } 1 \le i \le N/4. \tag{4.38}$$

Therefore, using (4.37) and (4.38), for $1 \le i \le N/4$, we have

$$\left|W_{L,i}^{n+1} - w(x_i, t_{n+1})\right| \le \Big(|W_{L,i}^{n+1}| + |w(x_i, t_{n+1})|\Big) \le CN^{-2}.$$

On the other hand, with a new barrier function

$$\Phi_{R,i} = -C\left[\prod_{j=1}^{N/2}\left(1+\frac{\gamma \hbar_j}{\varepsilon}\right)^{-1}\right] Q_i, \quad \text{for } N/2 \le i \le N,$$

arguing in the same way for the mesh function W_R over $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^+}$, we have

$$|W_{R,i}^{n+1}| \le C \prod_{j=N-i+1}^{N/2}\left(1+\frac{\gamma \hbar_j}{\varepsilon}\right)^{-1}, \quad \text{for } N/2+1 \le i \le N-1. \tag{4.39}$$

Hence, for $3N/4 \le i \le N-1$, inequality (4.39) and Lemma 4.4.5 imply that

$$|W_{R,i}^{n+1}| \le C \prod_{j=N/4+1}^{N/2}\left(1+\frac{\gamma \hbar_j}{\varepsilon}\right)^{-1} \le CN^{-2}. \tag{4.40}$$

Therefore, using Theorem 4.2.4 and (4.40), for $3N/4 \le i \le N-1$, we have

$$\left| W_{R,i}^{n+1} - w(x_i, t_{n+1}) \right| \le CN^{-2}.$$

Thus, the proof is complete. ■

Remark 4.4.7. *The similar construction of two new barrier functions $\Phi_{L,i}$ and $\Phi_{R,i}$ by introducing mesh functions S_i and Q_i respectively, for $i \le N/2$ and $i \ge N/2$ are very much essential to obtain the following bounds*

$$|W_{L,i}^N| \le CN^{-2}, \quad \textit{for } i \le N/4 \quad \textit{and} \quad |W_{R,i}^N| \le CN^{-2}, \quad \textit{for } i \ge 3N/4,$$

as mentioned in [10].

Some technical lemmas which will be used later to obtain ε-uniform error bounds of the proposed numerical scheme are stated here.

Lemma 4.4.8. *The following inequalities hold true:*

$$\exp\big(-\gamma(\xi - x_i)/\varepsilon\big) \le \prod_{j=i+1}^{N/2} \left(1 + \frac{\gamma h_j}{\varepsilon}\right)^{-1}, \quad \textit{for } 1 \le i \le N/2 - 1. \tag{4.41}$$

and

$$\exp\big(-\gamma(x_i - \xi)/\varepsilon\big) \le \prod_{j=N-i+1}^{N/2} \left(1 + \frac{\gamma \hbar_j}{\varepsilon}\right)^{-1}, \quad \textit{for } N/2 + 1 \le i \le N - 1. \tag{4.42}$$

Proof. For each j, we have

$$\exp\big(-\gamma h_j/\varepsilon\big) = \Big(\exp\big(\gamma h_j/\varepsilon\big)\Big)^{-1} \le \left(1 + \frac{\gamma h_j}{\varepsilon}\right)^{-1}, \tag{4.43}$$

and similarly,

$$\exp\big(-\gamma \hbar_j/\varepsilon\big) \le \left(1 + \frac{\gamma \hbar_j}{\varepsilon}\right)^{-1}. \tag{4.44}$$

Hereby, we deduce the result (4.41) by multiplying the inequalities obtained from (4.43), for $j = i+1, \ldots, N/2$ and the result (4.42) follows from multiplication of (4.44), for $j = N-i+1, \ldots, N/2$. ■

Lemma 4.4.9. *For $i = N/2$, we have the following results*

$$D_x^B S_i \ge \frac{C}{\varepsilon + \gamma h} S_i, \quad \textit{and} \quad -D_x^F Q_i \ge \frac{C}{\varepsilon + \gamma h} Q_i.$$

Proof. Using $(S_i - S_{i-1}) = \dfrac{\gamma h_i}{\varepsilon} S_{i-1}$ and $(Q_{i+1} - Q_i) = -\dfrac{\gamma \hbar_{N-i}}{\varepsilon} Q_{i+1}$, it is straightforward to derive the result from (4.16), for $i = N/2$. ■

4.4.1 The main convergence result

Theorem 4.4.10. *Assume that $N \geq N_0$ satisfies the conditions given in (4.30), (4.31) and $\varepsilon \leq N^{-1}$. Then, if $\gamma \leq \alpha/2$ and $\sigma_0 \geq 2/\gamma$, the respective solutions u and U of (4.1)-(4.4) and (4.27) satisfy the following error bounds at time level t_n:*

$$\Big|U_i^n - u(x_i, t_n)\Big| \leq \begin{cases} C\Big(N^{-2} + \Delta t\Big), & \text{for } 1 \leq i \leq N/4 \bigcup 3N/4 \leq i \leq N-1, \\ C\Big(N^{-2}\ln^2 N + \Delta t\Big), & \text{for } N/4 < i < 3N/4. \end{cases} \tag{4.45}$$

Proof. The proof is splitted up into two different cases depending on the location of the mesh point $x_i \in \overline{\Omega}_x^{N,\varepsilon}$.

Case 1. $\big($**Outer region**$\big)$ For $1 \leq i \leq N/4$ and $3N/4 \leq i \leq N-1$. Here, the estimate of $\big|U_i^n - u(x_i, t_n)\big|$ follows easily from Lemmas 4.4.3 and 4.4.6, by invoking the triangle inequality to the error

$$U - u = (V_L - v) + (W_L - w).$$

Case 2. $\big($**Interior layer region**$\big)$ Here, we need to find out the estimate of $\big|U_i^n - u(x_i, t_n)\big|$ for $N/4 < i < 3N/4$. From **Case 1**, clearly we have

$$\Big|U_i^{n+1} - u(x_i, t_{n+1})\Big| \leq C\Big(N^{-2} + \Delta t\Big), \quad \text{for } i = N/4, 3N/4. \tag{4.46}$$

Now, a straightforward calculation using the Taylor's formula with the integral form of remainder as in [[41], Lemma 3.3] and the bounds of the derivatives stated in Theorem 4.2.4 shows that for $N/4 + 1 \leq i \leq N/2 - 1$,

$$\begin{aligned} &\Big|L_H^{N,M}\big(U_i^{n+1} - u(x_i, t_{n+1})\big)\Big| \\ \leq\;& h\int_{x_{i-1}}^{x_{i+1}} \left(\varepsilon\left|\frac{\partial^4 u}{\partial x^4}(x, t_{n+1})\right| + \left|\frac{\partial^3 u}{\partial x^3}(x, t_{n+1})\right|\right) dx + C\Delta t\left\|\frac{\partial^2 u}{\partial t^2}\right\|_\infty \\ \leq\;& \left(Ch^2 + C\frac{h}{\varepsilon^2}\Big[\exp\big(-(\xi - x_{i+1})\alpha_1/\varepsilon\big) - \exp\big(-(\xi - x_{i-1})\alpha_1/\varepsilon\big)\Big]\right) + C\Delta t \\ \leq\;& C\left[\left(h^2 + \frac{h}{\varepsilon^2}\exp\big(-(\xi - x_i)\alpha_1/\varepsilon\big)\sinh(\alpha_1 h/\varepsilon)\right) + \Delta t\right]. \end{aligned}$$

Clearly, the assumption (4.30) implies $\alpha_1 h/\varepsilon \leq 2$ and since $\sinh\zeta \leq C\zeta$ for $0 \leq \zeta \leq 2$, so $\sinh(\alpha_1 h/\varepsilon) \leq C\alpha_1 h/\varepsilon$. Thus, for $N/4 + 1 \leq i \leq N/2 - 1$,

$$\Big|L_H^{N,M}\big(U_i^{n+1} - u(x_i, t_{n+1})\big)\Big| \leq C\left[\left(h^2 + \frac{h^2}{\varepsilon^3}\exp\big(-(\xi - x_i)\alpha_1/\varepsilon\big)\right) + \Delta t\right]. \tag{4.47}$$

Similarly, for $N/2+1 \le i \le 3N/4-1$, we get

$$\left|L_H^{N,M}\big(U_i^{n+1}-u(x_i,t_{n+1})\big)\right| \le C\Bigg[\bigg(h^2+\frac{h^2}{\varepsilon^3}\exp\big(-(x_i-\xi)\alpha_2/\varepsilon\big)\bigg)+\Delta t\Bigg]. \tag{4.48}$$

Now, at the mesh point $x_{N/2}=\xi$, using (4.47) and (4.48), we obtain the following estimate:

$$\begin{aligned}
&\left|m_{N/2}^{-}f_{N/2-1}^{n+1}+m_{N/2}^{+}f_{N/2+1}^{n+1}-L_H^{N,M}u_{N/2}^{n+1}\right| \\
\le\;& m_{N/2}^{-}\left|f_{N/2-1}^{n+1}-L_H^{N,M}u_{N/2-1}^{n+1}\right|+m_{N/2}^{+}\left|f_{N/2+1}^{n+1}-L_H^{N,M}u_{N/2+1}^{n+1}\right|+\left|\left(D_x^F-D_x^B\right)u_{N/2}^{n+1}\right| \\
\le\;& C\left|L_H^{N,M}\big(U_{N/2-1}^{n+1}-u(x_{N/2-1},t_{n+1})\big)\right|+C\left|L_H^{N,M}\big(U_{N/2+1}^{n+1}-u(x_{N/2+1},t_{n+1})\big)\right|+ \\
&\left|\left(D_x^F-D_x^B\right)u_{N/2}^{n+1}-\left[\frac{\partial u}{\partial x}\right](x_{N/2},t_{n+1})\right| \\
\le\;& C\Big(h^2+\Delta t\Big)+C\frac{h^2}{\varepsilon^3}\Big(\exp\big(-(\xi-x_{N/2-1})\alpha_1/\varepsilon\big)+\exp\big(-(x_{N/2+1}-\xi)\alpha_2/\varepsilon\big)\Big)+ \\
&\left|D_x^F u_{N/2}^{n+1}-\frac{\partial u}{\partial x}(x_{N/2},t_{n+1})\right|+\left|D_x^B u_{N/2}^{n+1}-\frac{\partial u}{\partial x}(x_{N/2},t_{n+1})\right| \\
\le\;& C\bigg(\frac{h^2}{\varepsilon^3}+\Delta t\bigg).
\end{aligned} \tag{4.49}$$

Define the discrete function

$$\Theta_i=\begin{cases}
-C\Big(N^{-2}+\Delta t\Big)\Big(1+\Big[x_i-(\xi-\sigma)\Big]\Big)-C\dfrac{h^2}{\varepsilon^2}\left[\displaystyle\prod_{j=1}^{N/2}\left(1+\frac{\gamma h_j}{\varepsilon}\right)^{-1}\right]S_i, \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for } N/4+1\le i\le N/2-1, \\
-C\Big(N^{-2}+\Delta t\Big)\Big(1+\Big[(\xi+\sigma)-x_i\Big]\Big)-C\dfrac{h^2}{\varepsilon^2}\left[\displaystyle\prod_{j=1}^{N/2}\left(1+\frac{\gamma \hbar_j}{\varepsilon}\right)^{-1}\right]Q_i, \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for } N/2+1\le i\le 3N/4-1,
\end{cases}$$

where C is chosen sufficiently large. Next, a direct calculation yields

$$L_H^{N,M}\Theta_i \ge \begin{cases}
C\Big(N^{-2}+\Delta t\Big)\big(-a_i\big)-C\dfrac{h^2}{\varepsilon^2}\left[\displaystyle\prod_{j=1}^{N/2}\left(1+\frac{\gamma h_j}{\varepsilon}\right)^{-1}\right]L_H^{N,M}S_i, \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for } N/4+1\le i\le N/2-1, \\
C\Big(N^{-2}+\Delta t\Big)\big(a_i\big)-C\dfrac{h^2}{\varepsilon^2}\left[\displaystyle\prod_{j=1}^{N/2}\left(1+\frac{\gamma \hbar_j}{\varepsilon}\right)^{-1}\right]L_H^{N,M}Q_i, \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for } N/2+1\le i\le 3N/4-1.
\end{cases}$$

Consequently, under the condition $\gamma \le \alpha/2$, this inequality together with Lemmas 4.4.4 and 4.4.8 yield

$$L_H^{N,M}\Theta_i \ge \begin{cases} C\alpha_1\Big(N^{-2}+\Delta t\Big) + C\dfrac{h^2}{\varepsilon^3}\exp\big(-(\xi-x_i)\alpha_1/\varepsilon\big), \\ \qquad\qquad\qquad\qquad \text{for } N/4+1\le i\le N/2-1, \\ C\alpha_2\Big(N^{-2}+\Delta t\Big) + C\dfrac{h^2}{\varepsilon^3}\exp\big(-(x_i-\xi)\alpha_2/\varepsilon\big), \\ \qquad\qquad\qquad\qquad \text{for } N/2+1\le i\le 3N/4-1, \end{cases} \tag{4.50}$$

since the assumption (4.30) yields $h/\varepsilon \le 2/\alpha_*$. Now, using this argument and Lemma 4.4.9, we get

$$\begin{aligned} L_H^{N,M}\Theta_{N/2} &\ge (D_x^F - D_x^B)\Theta_{N/2} \\ &= 2C\Big(N^{-2}+\Delta t\Big) + C\frac{h^2}{\varepsilon^2}\Bigg[\prod_{j=1}^{N/2}\Big(1+\frac{\gamma h_j}{\varepsilon}\Big)^{-1} D_x^B S_{N/2} - \prod_{j=1}^{N/2}\Big(1+\frac{\gamma\hbar_j}{\varepsilon}\Big)^{-1} D_x^F Q_{N/2}\Bigg] \\ &\ge 2C\Big(N^{-2}+\Delta t\Big) + 2C\frac{h^2}{\varepsilon^3}. \end{aligned} \tag{4.51}$$

Therefore, it follows from (4.46)-(4.51) that Θ_i is a barrier function for $(U-u)$. Now, applying the discrete maximum principle to $\Theta_i \pm \big(U_i^n - u(x_i,t_n)\big)$ over the domain $\big(\overline{G}_\varepsilon^{N,M}\bigcap[\xi-\sigma,\xi+\sigma]\big)\times[0,T]$, we obtain for $N/4<i<3N/4$,

$$\Big|U_i^n - u(x_i,t_n)\Big| \le C\Big(\frac{h^2}{\varepsilon^2}+\Delta t\Big) \le C\Big(\frac{N^{-2}\sigma^2}{\varepsilon^2}+\Delta t\Big) \le C\Big(N^{-2}\ln^2 N+\Delta t\Big).$$

Hence, this completes the proof. ■

Remark 4.4.11. *The spatial error estimate provided by Theorem 4.4.10 is optimal over the hybrid difference scheme [10], whose nodal error e_i satisfies the weaker estimate*

$$\big|e_i\big| \le CN^{-2}ln^3N \quad \textit{for all } i. \tag{4.52}$$

Furthermore, if we use our approach to analyze the hybrid difference scheme for the stationary case, we can sharpen (4.52) to $O(N^{-2}ln^2N)$. The key step is to construct the barrier function Θ_i adequately on the interval $[\xi-\sigma,\xi+\sigma]$, which can be of the following form:

$$\Theta_i = \begin{cases} -CN^{-2}\Big(1+\Big[x_i-(\xi-\sigma)\Big]\Big) - C\dfrac{h^2}{\varepsilon^2}\Bigg[\displaystyle\prod_{j=1}^{N/2}\Big(1+\frac{\gamma h_j}{\varepsilon}\Big)^{-1}\Bigg]S_i, \\ \qquad\qquad\qquad\qquad \textit{for } N/4+1\le i\le N/2-1, \\ -CN^{-2}\Big(1+\Big[(\xi+\sigma)-x_i\Big]\Big) - C\dfrac{h^2}{\varepsilon^2}\Bigg[\displaystyle\prod_{j=1}^{N/2}\Big(1+\frac{\gamma\hbar_j}{\varepsilon}\Big)^{-1}\Bigg]Q_i, \\ \qquad\qquad\qquad\qquad \textit{for } N/2+1\le i\le 3N/4-1. \end{cases}$$

4.5 Numerical Results

This section presents the numerical results obtained by the proposed scheme (4.14) to verify the theoretical results claimed in the previous section. To do this, we have profoundly carried out the numerical experiments for the following test examples on the piecewise-uniform rectangular mesh $\overline{G}_{\varepsilon}^{N,M}$. In all the cases, the numerical experiments are performed by choosing the constant $\sigma_0 = 4.2$ and $\Delta t = 0.8/N$, otherwise it is mentioned.

Example 4.5.1. *Consider the following parabolic IBVP:*

$$\begin{cases} \varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - x(1-x)u - \dfrac{\partial u}{\partial t} = f(x,t), \quad (x,t) \in (0,1) \times (0,1], \\ u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = t^2, \quad u(1,t) = 0, \quad 0 < t \le 1, \end{cases} \tag{4.53}$$

where the convective coefficient is

$$a(x) = \begin{cases} -\big(1 + x(1-x)\big), & 0 \le x \le 0.5, \\ \big(1 + x(1-x)\big), & 0.5 < x \le 1, \end{cases}$$

and the source term is given by

$$f(x,t) = \begin{cases} 2(1+x^2)t^2, & 0 \le x \le 0.5, \\ 3(1+x^2)t^2, & 0.5 < x \le 1. \end{cases}$$

Example 4.5.2. *Consider the following parabolic IBVP:*

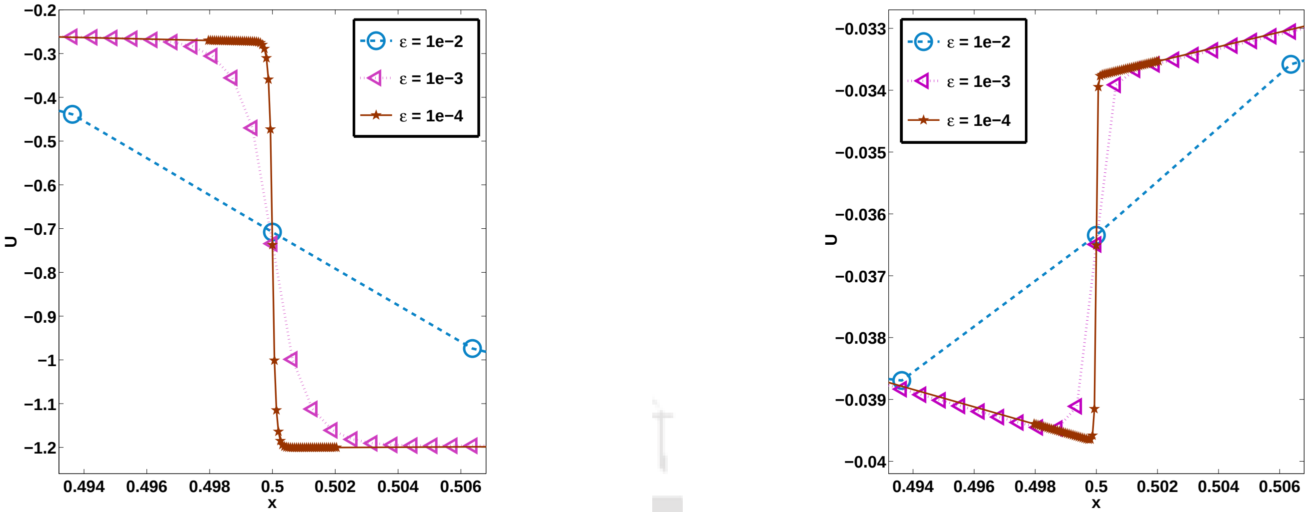
$$\begin{cases} \varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial t} = f(x,t), \quad (x,t) \in (0,1) \times (0,1], \\ u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \le 1, \end{cases} \tag{4.54}$$

where the convective coefficient

$$a(x) = \begin{cases} -\big(2 + x^2\big), & 0 \le x \le 0.5, \\ 3 - x^2, & 0.5 < x \le 1, \end{cases}$$

and the source term is given by

$$f(x,t) = \begin{cases} 2x\exp(-t)t^2, & 0 \le x \le 0.5, \\ 2(1-x)\exp(-t)t^2, & 0.5 < x \le 1. \end{cases}$$



(a) Example 4.5.1. *(b) Example 4.5.2.*

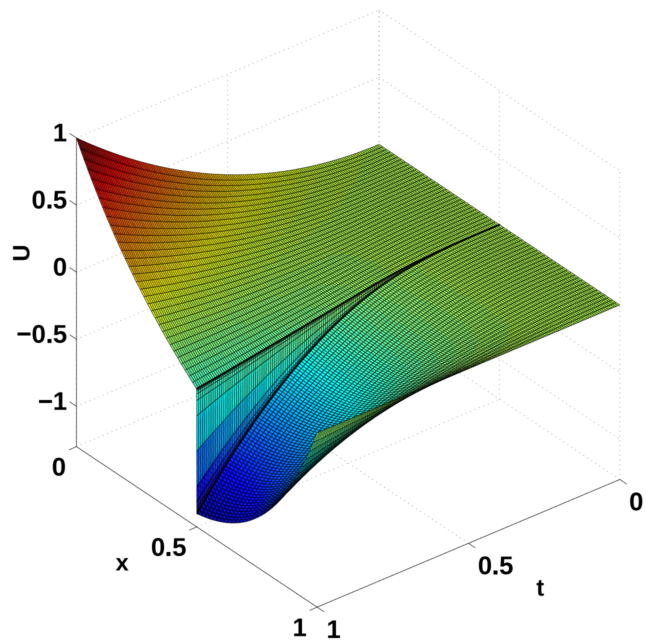
Figure 4.1: *Numerical solutions at time $t = 1$ for $N = 128$ (The zoomed view near $x = 0.5$).*

Note that the source term of Example 4.5.1 has a finite jump discontinuity at $(0.5, t)$, $t > 0$ and that of Example 4.5.2 is continuous on the full domain. Whereas both the IBVPs (4.5.1) and (4.5.2) contain discontinuous convective coefficients of type (4.4). In Figure 4.1, we have displayed the presence of interior layers in the numerical solutions of the IBVPs (4.53) and (4.54) near the point of discontinuity $x = 0.5$, where the gradient of the solutions steepens as the parameter ε decreases. Also to visualize the complete numerical solutions of the IBVPs (4.53) and (4.54) for $\varepsilon = 10^{-4}$, we have included two surface plots for $N = 128$ in Figure 4.2.

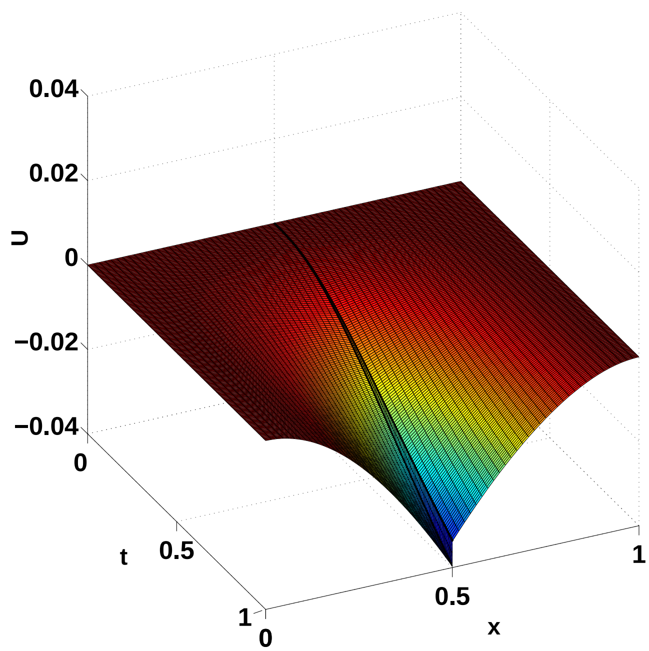
As the exact solutions of the IBVPs (4.53) and (4.54) are not known, to show the ε-uniform convergence of the proposed scheme (4.14) and also to obtain the accuracy of the numerical solutions, the double mesh principle is used as described in Chapter 2.

In Tables 4.1 and 4.2, we have presented the computed maximum point-wise errors $E_\varepsilon^{N,\Delta t}$ and the corresponding order of convergence $P_\varepsilon^{N,\Delta t}$ respectively, for Examples 4.5.1 and 4.5.2. Clearly, from the results given in Tables 4.1 and 4.2 we see that the computed ε-uniform errors $E^{N,\Delta t}$ decrease monotonically as N increases. This ensures that the proposed scheme (4.14) is ε-uniformly convergent. As a complement of this observation, we have plotted the maximum point-wise errors for Examples 4.5.1 and 4.5.2 in Figure 4.3.

Next, we see that the numerical results presented in Tables 4.1 and 4.2 do not clearly reflect the actual theoretical order of convergence of the proposed scheme (4.14) for the spatial variable, as predicted by Theorem 4.4.10. Firstly, it is known to us that the error consists of two parts due to the spatial and the temporal discretization (briefly say, the



(a) Example 4.5.1.



(b) Example 4.5.2.

Figure 4.2: *Surface plots of the Numerical solutions for $\varepsilon = 1e-4, N = 128$.*

spatial and the time error). Again, by the use of hybrid scheme we are improving only the accuracy in space and that's why we expect the spatial error will be essentially smaller than the temporal error. Hence, the errors computed in Tables 4.1 and 4.2 are combination of the temporal and the spatial errors.

Therefore, to justify the spatial order of convergence properly, the numerical experiments are carried out by taking $M = N^2$ (as like Hemker *et al.* [34]) and displayed the maximum point-wise errors as well as the corresponding order of convergence in Tables 4.3 and 4.4 respectively, for Examples 4.5.1 and 4.5.2. Further, we have highlighted these errors in Figures 4.4 and 4.5. This in fact shows that the proposed scheme (4.14) is second-order spatial accurate outside the interior layers. Whereas the spatial order of convergence of the proposed scheme (4.14) inside the layer regions is reduced by the presence of logarithmic factor as derived in Theorem 4.4.10. Even this analysis also signifies the role of σ_0 to fulfill our claim in the same Theorem.

4.6 Conclusion

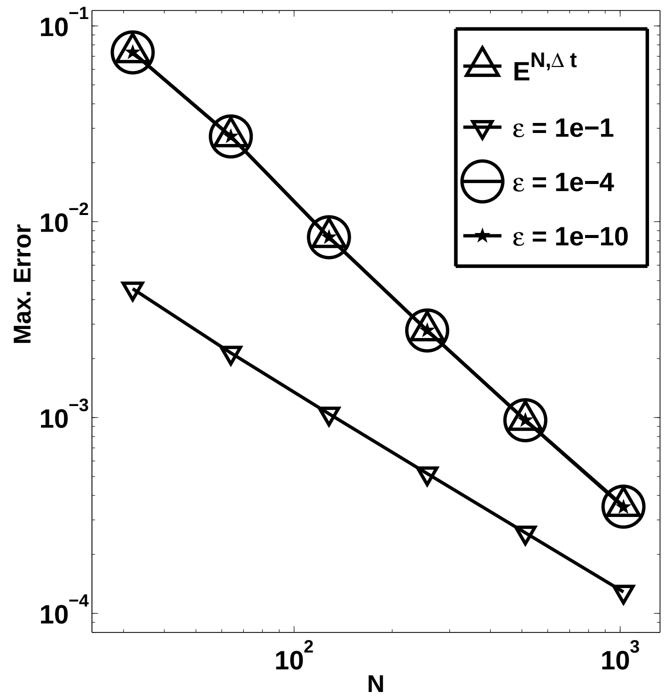
In this chapter, a class of singularly perturbed parabolic IBVPs with discontinuous convection coefficients is solved by using an efficient hybrid numerical scheme on a piecewise-uniform Shishkin mesh resolving strong interior layers. For the spatial discretization, the hybrid scheme utilizes a proper combination of the midpoint upwind scheme and the classical central difference scheme and for discretizing the time derivative, the backward-Euler scheme is used. It has been shown both theoretically and computationally that the newly prosed scheme is ε-uniform convergent with first-order accurate in time and almost second-order accurate in space. Moreover, we have computationally shown that the use of the backward-Euler method for time discretization does not reduce the order of convergence with respect to the spatial variable obtained by the hybrid numerical scheme. In the next chapter, we derive an optimal error estimate of the implicit upwind scheme on both the classical Shsiskin mesh and Bakhalov-Shsiskin mesh for the same class of problems possessing strong interior layers.

Table 4.1: *Maximum point-wise errors and the corresponding order of convergence for Example 4.5.1.*

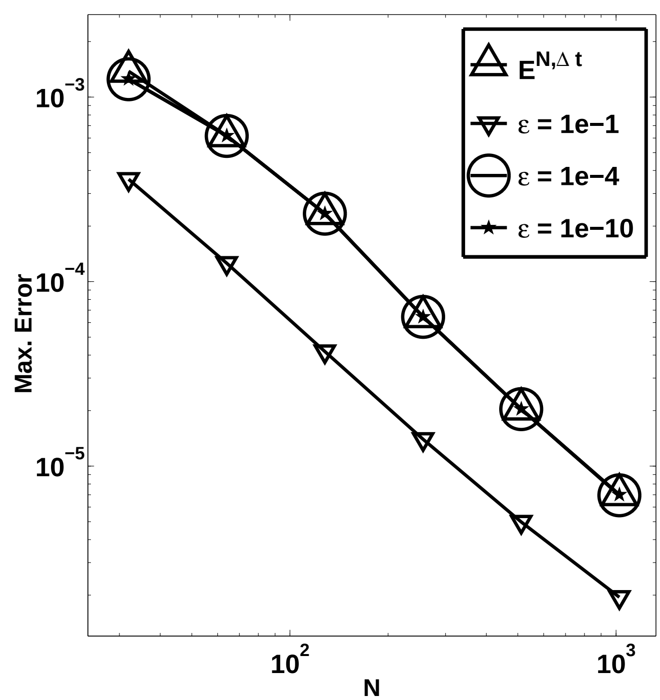
ε	Number of mesh-intervals N					
	32	64	128	256	512	1024
$1e-1$	4.5469e-3	2.1384e-3	1.0448e-3	5.1766e-4	2.5780e-4	1.2867e-4
	1.0883	1.0333	1.0131	1.0057	1.0026	
$1e-2$	6.9956e-2	2.6389e-2	8.0409e-3	2.6831e-3	8.8442e-4	2.9357e-4
	1.4065	1.7145	1.5834	1.6011	1.5910	
$1e-3$	7.2931e-2	2.7208e-2	8.3226e-3	2.7834e-3	9.7122e-4	3.5273e-4
	1.4225	1.7089	1.5802	1.5190	1.4612	
$1e-4$	7.3215e-2	2.7267e-2	8.3339e-3	2.7829e-3	9.6863e-4	3.5052e-4
	1.4250	1.7101	1.5824	1.5226	1.4664	
$1e-5$	7.3243e-2	2.7272e-2	8.3348e-3	2.7828e-3	9.6829e-4	3.5026e-4
	1.4253	1.7102	1.5826	1.5230	1.4670	
$1e-6$	7.3246e-2	2.7273e-2	8.3349e-3	2.7828e-3	9.6826e-4	3.5023e-4
	1.4253	1.7102	1.5826	1.5231	1.4671	
$1e-7$	7.3246e-2	2.7273e-2	8.3349e-3	2.7828e-3	9.6825e-4	3.5023e-4
	1.4253	1.7102	1.5826	1.5231	1.4671	
$1e-8$	7.3246e-2	2.7273e-2	8.3349e-3	2.7828e-3	9.6826e-4	3.5024e-4
	1.4253	1.7102	1.5826	1.5231	1.4671	
$1e-9$	7.3246e-2	2.7273e-2	8.3349e-3	2.7828e-3	9.6836e-4	3.5013e-4
	1.4253	1.7102	1.5826	1.5229	1.4676	
$1e-10$	7.3246e-2	2.7273e-2	8.3349e-3	2.7827e-3	9.6925e-4	3.4857e-4
	1.4253	1.7102	1.5827	1.5215	1.4754	
$E^{N,\Delta t}$	**7.3246e-2**	**2.7273e-2**	**8.3349e-3**	**2.7834e-3**	**9.7122e-4**	**3.5273e-4**
$P^{N,\Delta t}$	**1.4253**	**1.7102**	**1.5823**	**1.5190**	**1.4612**	

Table 4.2: *Maximum point-wise errors and the corresponding order of convergence for Example 4.5.2.*

ε	Number of mesh-intervals N					
	32	64	128	256	512	1024
$1e-1$	3.5895e-4	1.2585e-4	4.1801e-5	1.3981e-5	4.9816e-6	1.9473e-6
	1.5121	1.5901	1.5801	1.4888	1.3551	
$1e-2$	1.3812e-3	6.1068e-4	2.1006e-4	6.4531e-5	1.6906e-5	4.2732e-6
	1.1774	1.5396	1.7028	1.9325	1.9841	
$1e-3$	1.2118e-3	6.0182e-4	2.3009e-4	6.4495e-5	2.0006e-5	6.6592e-6
	1.0097	1.3871	1.8349	1.6888	1.5870	
$1e-4$	1.2496e-3	6.1578e-4	2.3344e-4	6.4413e-5	2.0363e-5	6.9445e-6
	1.0209	1.3994	1.8576	1.6614	1.5520	
$1e-5$	1.2533e-3	6.1718e-4	2.3378e-4	6.4404e-5	2.0399e-5	6.9734e-6
	1.0220	1.4006	1.8599	1.6586	1.5486	
$1e-6$	1.2537e-3	6.1732e-4	2.3381e-4	6.4403e-5	2.0403e-5	6.9763e-6
	1.0221	1.4007	1.8601	1.6583	1.5482	
$1e-7$	1.2537e-3	6.1733e-4	2.3381e-4	6.4403e-5	2.0403e-5	6.9766e-6
	1.0221	1.4007	1.8602	1.6583	1.5482	
$1e-8$	1.2537e-3	6.1733e-4	2.3381e-4	6.4403e-5	2.0403e-5	6.9763e-6
	1.0221	1.4007	1.8602	1.6583	1.5483	
$1e-9$	1.2537e-3	6.1733e-4	2.3381e-4	6.4403e-5	2.0402e-5	6.9747e-6
	1.0221	1.4007	1.8602	1.6584	1.5485	
$1e-10$	1.2537e-3	6.1733e-4	2.3381e-4	6.4402e-5	2.0397e-5	7.0005e-6
	1.0221	1.4007	1.8602	1.6588	1.5428	
$E^{N,\Delta t}$	**1.3812e-3**	**6.1733e-4**	**2.3381e-4**	**6.4531e-5**	**2.0403e-5**	**7.0005e-6**
$P^{N,\Delta t}$	**1.1618**	**1.4007**	**1.8573**	**1.6612**	**1.5433**	



(a) Example 4.5.1.



(b) Example 4.5.2.

Figure 4.3: *Loglog plot of the maximum point-wise errors.*

Table 4.3: *Maximum point-wise errors and the corresponding order of convergence calculated for Example 4.5.1 by taking $M = N^2$.*

N	$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$		
	left outer region $[0, \xi - \sigma_1]$	**interior layer region** $(\xi - \sigma_1, \xi + \sigma_2)$	**right outer region** $[\xi + \sigma_2, 1]$	**left outer region** $[0, \xi - \sigma_1]$	**interior layer region** $(\xi - \sigma_1, \xi + \sigma_2)$	**right outer region** $[\xi + \sigma_2, 1]$
32	7.7095e-4	6.6961e-2	2.3345e-3	7.7143e-4	6.6990e-2	2.3364e-3
	1.9913	1.4343	1.9926	2.0024	1.4347	2.0015
64	1.9391e-4	2.4776e-2	5.8665e-4	1.9254e-4	2.4782e-2	5.8348e-4
	1.9764	1.7501	1.9807	2.0006	1.7502	2.0004
128	4.9278e-5	7.3658e-3	1.4863e-4	4.8115e-5	7.3665e-3	1.4583e-4
	1.9507	1.6892	1.9598	2.0001	1.6895	2.0001
256	1.2747e-5	2.2842e-3	3.8208e-5	1.2028e-5	2.2839e-3	3.6456e-5
	1.9041	1.6871	1.9210	2.0000	1.6878	2.0000
512	3.4059e-6	7.0935e-4	1.0089e-5	3.0068e-6	7.0891e-4	9.1139e-6

Table 4.4: *Maximum point-wise errors and the corresponding order of convergence calculated for Example 4.5.2 by taking $M = N^2$.*

N	$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$		
	left outer region $[0, \xi - \sigma_1]$	**interior layer region** $(\xi - \sigma_1, \xi + \sigma_2)$	**right outer region** $[\xi + \sigma_2, 1]$	**left outer region** $[0, \xi - \sigma_1]$	**interior layer region** $(\xi - \sigma_1, \xi + \sigma_2)$	**right outer region** $[\xi + \sigma_2, 1]$
32	2.4319e-5	1.2390e-3	6.0953e-6	2.4419e-5	1.2432e-3	6.2366e-6
	1.9279	1.0227	2.0259	1.9331	1.0239	1.9965
64	6.3913e-6	6.0983e-4	1.4968e-6	6.3945e-6	6.1138e-4	1.5629e-6
	1.9870	1.4055	2.0564	1.9996	1.4069	1.9982
128	1.6123e-6	2.3020e-4	3.5985e-7	1.5990e-6	2.3057e-4	3.9123e-7
	1.9726	1.8216	2.1211	1.9999	1.8222	1.9998
256	4.1079e-7	6.5127e-5	8.2722e-8	3.9979e-7	6.5201e-5	9.7818e-8
	1.9447	1.6046	2.2724	2.0000	1.6035	2.0000
512	1.0671e-7	2.1416e-5	1.7122e-8	9.9949e-8	2.1456e-5	2.4455e-8

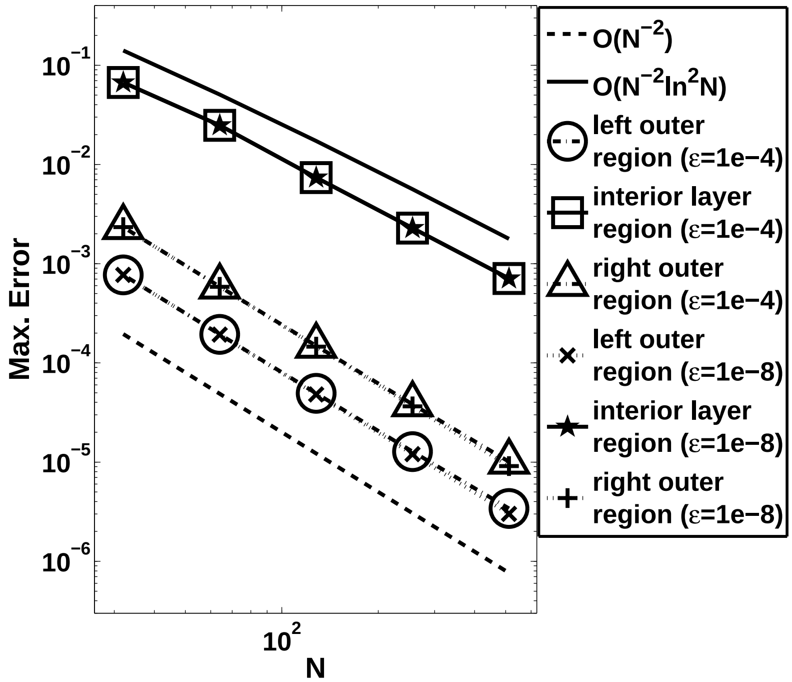


Figure 4.4: *Loglog plot of the spatial order of convergence for Example 4.5.1.*

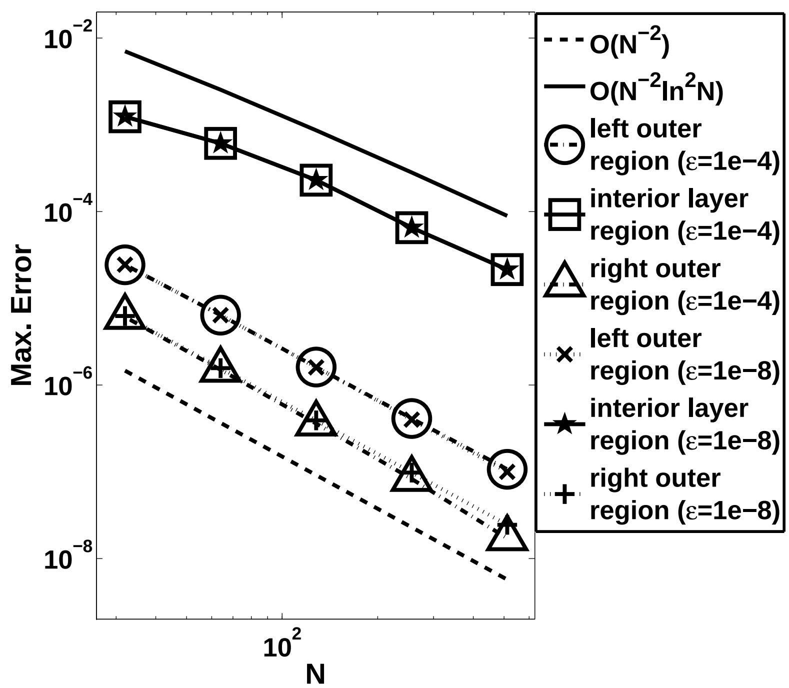


Figure 4.5: *Loglog plot of the spatial order of convergence for Example 4.5.2.*

Chapter 5

Optimal Error Estimate of Upwind Scheme on Shishkin-type meshes for Singularly Perturbed Parabolic Problems with Discontinuous Convection Coefficients

This chapter analyzes the implicit upwind finite difference scheme on Shishkin-type meshes (including the classical piecewise-uniform Shishkin mesh and the Bakhalov-Shishkin mesh) for a class of singularly perturbed parabolic convection-diffusion problems exhibiting strong interior layers. Suitable conditions on the mesh-generating functions are derived that are sufficient for the convergence of the method, uniformly with respect to the perturbation parameter ε. Utilizing these conditions, it is shown that the method converges uniformly in the discrete supremum norm with an optimal error bound of $O\big(N^{-1} + \Delta t\big)$ on the Bakhalov-Shishkin mesh, while on the piecewise-uniform Shishkin mesh it is of $O\big(N^{-1}\ln N + \Delta t\big)$.

5.1 Introduction

Let us denote the domains for describing the model problem by $\Omega = (0,1)$, $\Omega^- = (0,\xi)$, $\Omega^+ = (\xi, 1)$, $G^- = \Omega^- \times (0,T]$, $G^+ = \Omega^+ \times (0,T]$, $G = \Omega \times (0,T]$. We once again consider the following class of singularly perturbed parabolic IBVPs posed on the domain $G^- \cup G^+$:

$$\begin{cases} L_\varepsilon u(x,t) \equiv \left(\varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - b(x)u - \dfrac{\partial u}{\partial t}\right)(x,t) \;=\; f(x,t), \quad (x,t) \in G^- \cup G^+, \\ u(x,0) = s_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = s_1(t), \quad u(1,t) = s_2(t), \quad t \in (0,T], \end{cases} \tag{5.1}$$

where $0 < \varepsilon \ll 1$ is a small parameter, the convection coefficient a is sufficiently smooth on $\Omega^- \cup \Omega^+$, the source term f is sufficiently smooth on $G^- \cup G^+$ and the coefficient b is sufficiently smooth on $\overline{\Omega}$ such that

$$\begin{cases} b(x) \geq \beta \geq 0 \quad \text{on } \overline{\Omega}, \\ \big|[a]\big| \leq C, \quad \big|[f]\big| \leq C, \quad \text{at } x = \xi, \end{cases} \tag{5.2}$$

and the solution $u(x,t)$ satisfies the following interface conditions

$$[u] = 0, \quad \left[\frac{\partial u}{\partial x}\right] = 0, \quad \text{at } x = \xi. \tag{5.3}$$

As like the previous chapter, we assume that the convection coefficient $a(x)$ satisfies the following particular case:

$$\begin{cases} -\alpha_1^* < a(x) < -\alpha_1 < 0, \quad x < \xi, \\ \alpha_2^* > a(x) > \alpha_2 > 0, \qquad x > \xi. \end{cases} \tag{5.4}$$

Afterwards, assuming that the data s_0, s_1 and s_2 are sufficiently smooth on $\overline{G}$ and satisfy the compatibility conditions (4.5) and (4.6) at the two corner points $(0,0), (1,0)$ together with suitable compatibility condition at the transition point $(\xi, 0)$, the parabolic IBVP (5.1)-(5.4), in general admits a unique solution $u \in \mathcal{C}^{1+\lambda}(G) \bigcap \mathcal{C}^{2+\lambda}(G^- \cup G^+)$, which possesses strong interior layers of width $O(\varepsilon)$ in the neighborhood of the point $x = \xi$.

The outline of this chapter is as follows: Section 5.2 describes the Shishkin-type meshes along with some properties of these meshes and also introduce the implicit upwind finite difference scheme. This section ends up by stating some technical lemmas which will be used in the sequel. Afterwards, the error analysis is carried out for the upwind scheme in Section 5.3 and the main theoretical result, *i.e.*, the ε-uniform optimal error bounds of the implicit upwind scheme on the Shishkin-type meshes, is proved. This chapter concludes in Section 5.4 by conducting the numerical experiments for verification of the theoretical results.

5.2 Numerical Approximation

This section describes the nonuniform Shishkin-type meshes resolving the interior layers and these meshes are constructed with the help of the mesh generating functions following the idea introduced in [53]. Further, the difference scheme used to solve the problem (5.1)-(5.4) is described. Some auxiliary results are provided for these meshes as the prerequisites for performing error analysis in the subsequent section.

5.2.1 Shishkin-type meshes

Consider the domain $\overline{G} = \overline{\Omega} \times [0,T] = [0,1] \times [0,T]$ and let $N \geq 8$ be an even positive integer. Here, we will construct a rectangular mesh $\overline{G}_{\varepsilon}^{N,M} = \overline{\Omega}_{x}^{N,\varepsilon} \times \mathbb{S}_{t}^{M}$, which is a combination of the Shishkin-type mesh condensed around the interior layers for the spatial variable and the uniform mesh for the temporal variable.

On the time domain $[0,T]$, the equidistant meshes are introduced in the temporal variable such that

$$\mathbb{S}_{t}^{M} = \{t_n = n\,\Delta t,\; n = 0,\ldots,M,\; t_0 = 0, t_M = T,\; \Delta t = T/M\},$$

where M denotes the number of mesh-intervals in the t-direction.

Next, fix the transitions points as in the Shishkin mesh, by choosing the transition parameters σ_1 and σ_2 to be the following functions of N and ε

$$\sigma_1 = \min\left\{\frac{\xi}{2}, \sigma_0\varepsilon\ln N\right\}, \quad \sigma_2 = \min\left\{\frac{1-\xi}{2}, \sigma_0\varepsilon\ln N\right\},$$

where $\sigma_0 = 2/\gamma$ and γ is a positive constant. Note that the mesh will be uniform when $\sigma_1 = \xi/2,\; \sigma_2 = (1-\xi)/2$.

Assumption 5.2.1. *A very mild assumption is made such that $\sigma_1 = \sigma_2 = \sigma = \sigma_0\varepsilon\ln N$, as otherwise N^{-1} is exponentially small relatively to ε (and in this case the method can be analyzed in the classical way).*

Now, the spatial domain $\overline{\Omega}$ is subdivide into four subintervals as

$$\overline{\Omega} = [0, \xi-\sigma_1] \cup [\xi-\sigma_1, \xi] \cup [\xi, \xi+\sigma_2] \cup [\xi+\sigma_2, 1],$$

and on each of the subintervals $[0, \xi-\sigma_1], [\xi+\sigma_2, 1]$, a uniform mesh with $N/4$ mesh-intervals is placed such that $x_0 = 0,\; x_{N/4} = \xi - \sigma_1$ and $x_{3N/4} = \xi + \sigma_2,\; x_N = 1$; while the subintervals $[\xi-\sigma_1, \xi]$ and $[\xi, \xi+\sigma_2]$ are respectively partitioned into the same number of mesh-intervals by using continuous, monotonically increasing, piecewise continuously differentiable mesh-generating functions $\varphi_l(\tau), \tau \in [1/4, 1/2]$ and $\varphi_r(\tau), \tau \in [1/2, 3/4]$ such that

$$\varphi_l(1/4) = -\ln N,\; \varphi_l(1/2) = 0 \quad \text{and} \quad \varphi_r(1/2) = 0,\; \varphi_r(3/4) = \ln N.$$

Then, clearly $x_{N/2} = \xi$ and $\overline{\Omega}_{x}^{N,\varepsilon} = \{x_i\}_0^N$, where the mesh points are given by

$$x_i = \begin{cases} 4(\xi-\sigma_1)i/N, & \text{for } i = 0,\ldots,N/4-1, \\ \xi + \sigma_0\varepsilon\varphi_l(\tau_i), & \text{for } \tau_i = i/N,\; i = N/4,\ldots,N/2, \\ \xi + \sigma_0\varepsilon\varphi_r(\tau_i), & \text{for } \tau_i = i/N,\; i = N/2,\ldots,3N/4, \\ (\xi+\sigma_2) + 4(i-3N/4)(1-\xi-\sigma_2)/N, & \text{for } i = 3N/4+1,\ldots,N. \end{cases}$$

Let the mesh widths in space be denoted by

$$h_i = x_i - x_{i-1},\ i = 1, \ldots, N, \qquad \widehat{h}_i = h_i + h_{i+1},\ i = 1, \ldots, N-1.$$

From the above definition, we have on the coarse mesh

$$h_i = \begin{cases} H_{(l)} = 4(\xi - \sigma_1)/N, & \text{for } i = 1, \ldots, N/4, \\ H_{(r)} = 4(1 - \xi - \sigma_2)/N, & \text{for } i = 3N/4 + 1, \ldots, N. \end{cases} \tag{5.5}$$

and on the fine mesh

$$\begin{cases} h_i \geq h_{i+1}, & \text{for } i = N/4 + 1, \ldots, N/2 - 1, \\ h_i \leq h_{i+1}, & \text{for } i = N/2 + 1, \ldots, 3N/4 - 1. \end{cases}$$

Now, define two new functions ψ_l and ψ_r that are closely related to φ_l and φ_r, respectively by

$$\varphi_l = \ln \psi_l, \quad \varphi_r = -\ln \psi_r,$$

where the function ψ_l is monotonically increasing with $\psi_l(1/4) = N^{-1}$, $\psi_l(1/2) = 1$ and the function ψ_r is monotonically decreasing with $\psi_r(1/2) = 1$, $\psi_r(3/4) = N^{-1}$. Different examples of *mesh characterizing functions* ψ_l and ψ_r are:

- *Standard Shishkin mesh* (**S** mesh) [54, 77]:

$$\psi_l(\tau) = \exp(-4(1/2 - \tau) \ln N), \quad \psi_r(\tau) = \exp(-4(\tau - 1/2) \ln N). \tag{5.6}$$

- *Bakhalov-Shishkin mesh* (**B** $-$ **S** mesh) [50, 51]:

$$\psi_l(\tau) = 1 - 4(1 - N^{-1})(1/2 - \tau), \quad \psi_r(\tau) = 1 - 4(1 - N^{-1})(\tau - 1/2). \tag{5.7}$$

It is acknowledged in [30, 53, 76] that the properties of ψ admit an easy characterization of the uniform convergence behavior of certain numerical methods for linear convection-diffusion problems having smooth data.

Here, we show that this can also be applied to the difference scheme (5.13) in case of the problem (5.1)-(5.4) having discontinuous convection-coefficient. In the following, we state some properties of the Shishkin-type meshes that will be used later in the analysis.

Assumption 5.2.2. *Assume that the mesh-generating functions φ_l, φ_r satisfy the following conditions*

$$\max_{\tau\in[1/4,1/2]} \left|\varphi_l'(\tau)\right| \le CN, \qquad \max_{\tau\in[1/2,3/4]} \left|\varphi_r'(\tau)\right| \le CN, \tag{5.8}$$

and

$$\int_{1/4}^{1/2} \left\{\varphi_l'(\tau)\right\}^2 d\tau \le CN, \quad \int_{1/2}^{3/4} \left\{\varphi_r'(\tau)\right\}^2 d\tau \le CN. \tag{5.9}$$

It is easy to check that both the **S** mesh and the **B** $-$ **S** mesh satisfy the conditions given in (5.8) and (5.9).

Lemma 5.2.3. *Assume that $\varepsilon \le N^{-1}$ and let (5.8) be satisfied. Then, the step size h_i on the layer regions satisfies*

$$h_i \le CN^{-1} \quad \textit{and} \tag{5.10}$$

$$\frac{h_i}{\varepsilon} \le C, \quad \textit{for } i = N/4+1,\ldots,3N/4. \tag{5.11}$$

Proof. Let $i = N/4+1,\ldots,N/2$. From the proof of [[76], *Theorem 1*], we have

$$h_i \le \sigma_0 \varepsilon N^{-1} \max_{\tau\in[\tau_{i-1},\tau_i]} \left|\varphi_l'(\tau)\right|. \tag{5.12}$$

Then using (5.8) and $\varepsilon \le N^{-1}$, we obtain inequality (5.10), whereas (5.12) and the assumption $\varepsilon \le N^{-1}$ yield inequality (5.11). The proof of (5.10) and (5.11) follows similarly for $i = N/2+1,\ldots,3N/4$. ■

5.2.2 The implicit upwind finite difference scheme

In this section, we apply the implicit upwind finite difference scheme for the discretization of the problem (5.1)-(5.4) on the piecewise-uniform rectangular mesh $\overline{G}_\varepsilon^{N,M}$. O'Riordan et al. [72] used this scheme to solve the IBVP (5.1)-(5.4). While at the point of discontinuity, a first-order one-sided difference approximations are used to keep the continuity of the spatial derivative. Then, the numerical scheme takes the following form:

$$\begin{cases} U_i^0 = s_0(x_i), \quad \text{for } i = 0,\ldots,N, \\ \begin{cases} L_\varepsilon^{N,M} U_i^{n+1} \equiv \varepsilon\delta_x^2 U_i^{n+1} + a_i D_x^* U_i^{n+1} - b_i U_i^{n+1} - D_t^- U_i^{n+1} \\ \qquad\qquad = f_i^{n+1}, \quad \text{for } i = 1,\ldots,N/2-1, N/2+1,\ldots,N-1, \\ D_x^+ U_i^{n+1} - D_x^- U_i^{n+1} = 0, \quad \text{for } i = N/2, \\ U_0^{n+1} = s_1(t_{n+1}), \quad U_N^{n+1} = s_2(t_{n+1}), \end{cases} \\ \text{for } n = 0,\ldots,M-1, \end{cases} \tag{5.13}$$

where

$$D_x^* U_i^n = \begin{cases} D_x^- U_i^n, & i < N/2, \\ D_x^+ U_i^n, & i > N/2. \end{cases}$$

From [72], one can show that the finite difference operator $L_\varepsilon^{N,M}$ satisfies the following well-known discrete maximum principle, which lead to ε-uniform stability of the difference operator $L_\varepsilon^{N,M}$. Let $G_\varepsilon^{N,M} = \overline{G}_\varepsilon^{N,M} \bigcap G$ and $\Gamma_\varepsilon^{N,M} = \overline{G}_\varepsilon^{N,M} \backslash G_\varepsilon^{N,M}$.

Lemma 5.2.4. ***(Discrete Maximum Principle)*** *Suppose that a mesh function Z satisfies*

$$Z \leq 0 \textit{ on } \Gamma_\varepsilon^{N,M}, \quad L_\varepsilon^{N,M} Z \geq 0 \textit{ in } G_\varepsilon^{N,M} \bigcap (G^- \cup G^+) \quad \textit{and} \quad D_x^+ Z_{N/2}^{n+1} - D_x^- Z_{N/2}^{n+1} \geq 0,$$

for $n = 0, \ldots, M-1$, then $Z \leq 0$ at each point of $\overline{G}_\varepsilon^{N,M}$.

Before closing this section, some technical lemmas which will be used in the next section are stated here. Now, on $\overline{\Omega}_x^{N,\varepsilon} = \{x_i\}_0^N$, define the following two mesh functions

$$\mathcal{S}_i = \prod_{j=1}^{i} \left(1 + \frac{\gamma h_j}{2\varepsilon}\right), \quad \text{for } 1 \leq i \leq N/2, \quad \mathcal{Q}_i = \prod_{j=i+1}^{N} \left(1 + \frac{\gamma h_j}{2\varepsilon}\right), \quad \text{for } N/2 \leq i \leq N-1,$$

with the usual convention that for $i = 0$, $\mathcal{S}_0 = 1$ and for $i = N$, $\mathcal{Q}_N = 1$.

Lemma 5.2.5. *If $\gamma \leq \alpha$, then for some constant C, we have the following inequalities*

$$\begin{cases} -L_\varepsilon^{N,M} \mathcal{S}_i \geq \dfrac{C}{2\varepsilon + \gamma h_i} \mathcal{S}_i, & \textit{for } 1 \leq i \leq N/2 - 1, \quad \textit{and} \\ -L_\varepsilon^{N,M} \mathcal{Q}_i \geq \dfrac{C}{2\varepsilon + \gamma h_{i+1}} \mathcal{Q}_i, & \textit{for } N/2 + 1 \leq i \leq N-1. \end{cases}$$

Proof. Firstly, $(\mathcal{S}_i - \mathcal{S}_{i-1}) = \dfrac{\gamma h_i}{2\varepsilon} \mathcal{S}_{i-1}$ and $a_i < -\alpha_1 \leq -\gamma$, for $i < N/2$. Hence, it is straightforward that

$$\begin{aligned} -L_\varepsilon^{N,M} \mathcal{S}_i &= -\frac{\gamma}{(h_i + h_{i+1})} (\mathcal{S}_i - \mathcal{S}_{i-1}) - a_i \frac{\gamma}{2\varepsilon} \mathcal{S}_{i-1} + b_i \mathcal{S}_i \\ &\geq -\frac{\gamma}{2\varepsilon} \mathcal{S}_{i-1} \left[\frac{\gamma h_i}{(h_i + h_{i+1})} + a_i \right] \geq \frac{C}{2\varepsilon + \gamma h_i} \mathcal{S}_i, \quad \text{for } 1 \leq i \leq N/2 - 1, \end{aligned}$$

Next, $(\mathcal{Q}_{i+1} - \mathcal{Q}_i) = -\dfrac{\gamma h_{i+1}}{2\varepsilon} \mathcal{Q}_{i+1}$ and $a_i > \alpha_2 \geq \gamma$, for $i > N/2$. Hence, it follows that

$$\begin{aligned} -L_\varepsilon^{N,M} \mathcal{Q}_i &= \frac{\gamma}{(h_i + h_{i+1})} (\mathcal{Q}_{i+1} - \mathcal{Q}_i) + a_i \frac{\gamma}{2\varepsilon} \mathcal{Q}_{i+1} + b_i \mathcal{Q}_i \\ &\geq \frac{\gamma}{2\varepsilon} \mathcal{Q}_{i+1} \left[a_i - \frac{\gamma h_{i+1}}{(h_i + h_{i+1})} \right] \geq \frac{C}{2\varepsilon + \gamma h_{i+1}} \mathcal{Q}_i, \quad \text{for } N/2 + 1 \leq i \leq N-1. \blacksquare \end{aligned}$$

Lemma 5.2.6. *For the Shishkin-type mesh* $\overline{\Omega}_x^{N,\varepsilon} = \{x_i\}_0^N$*, the following inequalities relating the mesh-size and the perturbation parameter with the mesh-generating functions hold true:*

$$\sum_{j=N/4+1}^{N/2} \left(\frac{h_j}{\varepsilon}\right)^2 \leq \sigma_0^2 N^{-1} \int_{1/4}^{1/2} \left\{\varphi_l'(\tau)\right\}^2 d\tau, \tag{5.14}$$

and

$$\sum_{j=N/2+1}^{3N/4} \left(\frac{h_j}{\varepsilon}\right)^2 \leq \sigma_0^2 N^{-1} \int_{1/2}^{3/4} \left\{\varphi_r'(\tau)\right\}^2 d\tau. \tag{5.15}$$

Proof. Employing the following representations

$$\frac{h_j}{\varepsilon} = \begin{cases} \sigma_0 \displaystyle\int_{\tau_{j-1}}^{\tau_j} \varphi_l'(\tau) d\tau, & \text{for } j = N/4+1, \ldots, N/2, \\ \sigma_0 \displaystyle\int_{\tau_{j-1}}^{\tau_j} \varphi_r'(\tau) d\tau, & \text{for } j = N/2+1, \ldots, 3N/4, \end{cases}$$

the inequalities (5.14) and (5.15) respectively follow from

$$\sum_{j=N/4+1}^{N/2} \left(\frac{h_j}{\varepsilon}\right)^2 \leq \sigma_0^2 \sum_{j=N/4+1}^{N/2} (\tau_j - \tau_{j-1}) \int_{\tau_{j-1}}^{\tau_j} \left\{\varphi_l'(\tau)\right\}^2 d\tau = \sigma_0^2 N^{-1} \int_{\tau_{N/4}}^{\tau_{N/2}} \left\{\varphi_l'(\tau)\right\}^2 d\tau$$

and

$$\sum_{j=N/2+1}^{3N/4} \left(\frac{h_j}{\varepsilon}\right)^2 \leq \sigma_0^2 \sum_{j=N/2+1}^{3N/4} (\tau_j - \tau_{j-1}) \int_{\tau_{j-1}}^{\tau_j} \left\{\varphi_r'(\tau)\right\}^2 d\tau = \sigma_0^2 N^{-1} \int_{\tau_{N/2}}^{\tau_{3N/4}} \left\{\varphi_r'(\tau)\right\}^2 d\tau. \quad \blacksquare$$

Lemma 5.2.7. *Suppose that (5.9) holds true. Then for the Shishkin-type mesh* $\overline{\Omega}_x^{N,\varepsilon} = \{x_i\}_0^N$*, there exists a constant C such that*

$$\prod_{j=N/4+1}^{N/2} \left(1 + \frac{\gamma h_j}{2\varepsilon}\right)^{-1} \leq CN^{-1}, \tag{5.16}$$

and

$$\prod_{j=N/2+1}^{3N/4} \left(1 + \frac{\gamma h_j}{2\varepsilon}\right)^{-1} \leq CN^{-1}. \tag{5.17}$$

Proof. Since for $x \geq 0$, $\ln(1+x) \geq x - x^2/2$, so we have

$$\begin{aligned} \ln\Big(\prod_{j=N/4+1}^{N/2} \Big(1 + \frac{\gamma h_j}{2\varepsilon}\Big)\Big) &\geq \sum_{j=N/4+1}^{N/2} \left[\frac{\gamma h_j}{2\varepsilon} - \frac{1}{2}\left(\frac{\gamma h_j}{2\varepsilon}\right)^2\right] \\ &\geq \left[\frac{\gamma}{2\varepsilon}(\xi - x_{N/4}) - \frac{1}{2} \sum_{j=N/4+1}^{N/2} \left(\frac{\gamma h_j}{2\varepsilon}\right)^2\right] \\ &= \left[\frac{\gamma\sigma_0}{2} \ln N - \frac{\gamma^2}{8} \sum_{j=N/4+1}^{N/2} \left(\frac{h_j}{\varepsilon}\right)^2\right]. \end{aligned}$$

Therefore,

$$\prod_{j=N/4+1}^{N/2}\left(1+\frac{\gamma h_j}{2\varepsilon}\right)^{-1} \leq CN^{-\gamma\sigma_0/2}\exp\left(\frac{\gamma^2}{8}\sum_{j=N/4+1}^{N/2}\left(\frac{h_j}{\varepsilon}\right)^2\right). \tag{5.18}$$

Henceforth, using (5.9) and (5.14) in (5.18), we obtain the result (5.16).

On the other hand, following the above argument, we get

$$\begin{aligned}\ln\left(\prod_{j=N/2+1}^{3N/4}\left(1+\frac{\gamma h_j}{2\varepsilon}\right)\right) &\geq \left[\frac{\gamma}{2\varepsilon}(x_{3N/4}-\xi)-\frac{1}{2}\sum_{j=N/2+1}^{3N/4}\left(\frac{\gamma h_j}{2\varepsilon}\right)^2\right]\\ &= \left[\frac{\gamma\sigma_0}{2}\ln N-\frac{\gamma^2}{8}\sum_{j=N/4+1}^{N/2}\left(\frac{h_j}{\varepsilon}\right)^2\right].\end{aligned}$$

and hereby, from the above inequality with (5.9) and (5.15), the result (5.17) follows. ■

Lemma 5.2.8. *The following inequalities hold true:*

$$\exp\big(-\gamma(\xi-x_i)/2\varepsilon\big) \leq \prod_{j=i+1}^{N/2}\left(1+\frac{\gamma h_j}{2\varepsilon}\right)^{-1}, \quad \textit{for } 1\leq i\leq N/2-1.$$

and

$$\exp\big(-\gamma(x_i-\xi)/2\varepsilon\big) \leq \prod_{j=N/2+1}^{i}\left(1+\frac{\gamma h_j}{2\varepsilon}\right)^{-1}, \quad \textit{for } N/2+1\leq i\leq N-1.$$

Proof. The proof of this lemma is analogous to that of Lemma 4.4.8. ■

Lemma 5.2.9. *Let $\gamma \leq \alpha$. Also, let the integers i, j be such that $i < j$. Then, for $i, j \in \{N/4, \dots, N/2\}$, we have*

$$\int_{x_i}^{x_j}\exp\big(-(\xi-x)\alpha_1/\varepsilon\big)dx \leq C\varepsilon N^{-1}\exp\big(-\gamma(\xi-x_j)/2\varepsilon\big)\max_{\tau\in[\tau_i,\tau_j]}\big|\psi_l'(\tau)\big|, \tag{5.19}$$

and for $i, j \in \{N/2, \dots, 3N/4\}$, we have

$$\int_{x_i}^{x_j}\exp\big(-(x-\xi)\alpha_2/\varepsilon\big)dx \leq C\varepsilon N^{-1}\exp\big(-\gamma(x_i-\xi)/2\varepsilon\big)\max_{\tau\in[\tau_i,\tau_j]}\big|\psi_r'(\tau)\big|. \tag{5.20}$$

Proof. Let $i, j \in \{N/4, \dots, N/2\}$. Then, by substituting $x = \xi + \sigma_0\varepsilon\varphi_l(\tau)$ in the left hand side of (5.19), we have

$$\begin{aligned}\int_{x_i}^{x_j}\exp\big(-(\xi-x)\alpha_1/\varepsilon\big)dx &= \sigma_0\varepsilon\int_{\tau_i}^{\tau_j}\exp\Big(\alpha_1\sigma_0\varphi_l(\tau)\Big)\varphi_l'(\tau)d\tau\\ &\leq \sigma_0\varepsilon\int_{\tau_i}^{\tau_j}\exp\Big((\alpha_1\sigma_0-1)\varphi_l(\tau)\Big)\big|\psi_l'(\tau)\big|d\tau\\ &\leq \sigma_0\varepsilon\int_{\tau_i}^{\tau_j}\exp\big(\varphi_l(\tau)\big)\big|\psi_l'(\tau)\big|d\tau,\end{aligned}$$

since by the assumptions $\big(\alpha_1\sigma_0 - 1\big) \geq 1$. Now,

$$\max_{\tau\in[\tau_i,\tau_j]} \exp\big(\varphi_l(\tau)\big) = \max_{x\in[x_i,x_j]} \exp\big(-(\xi - x)/\sigma_0\varepsilon\big) = \exp\big(-(\xi - x_j)/\sigma_0\varepsilon\big),$$

which yields the result (5.19).

Next, let $i, j \in \big\{N/2, \ldots, 3N/4\big\}$ and substitute $x = \xi + \sigma_0\varepsilon\varphi_r(\tau)$ in the left hand side of (5.20) to get

$$\begin{aligned} \int_{x_i}^{x_j} \exp\big(-(x-\xi)\alpha_2/\varepsilon\big)dx &\leq \sigma_0\varepsilon \int_{\tau_i}^{\tau_j} \exp\Big(\big(1 - \alpha_2\sigma_0\big)\varphi_r(\tau)\Big)\big|\psi_r'(\tau)\big|d\tau \\ &\leq \sigma_0\varepsilon \int_{\tau_i}^{\tau_j} \exp\big(-\varphi_r(\tau)\big)\big|\psi_r'(\tau)\big|d\tau, \end{aligned}$$

since by the assumptions $\big(\alpha_2\sigma_0 - 1\big) \geq 1$. Again,

$$\max_{\tau\in[\tau_i,\tau_j]} \exp\big(-\varphi_r(\tau)\big) = \max_{x\in[x_i,x_j]} \exp\big(-(x-\xi)/\sigma_0\varepsilon\big) = \exp\big(-(x_i-\xi)/\sigma_0\varepsilon\big),$$

and hence the result (5.20) follows. ∎

5.3 Error Analysis

This section deals with the ε-uniform error estimate of the numerical scheme (5.13). To estimate the nodal error $\big|U_i^{n+1} - u(x_i, t_{n+1})\big|$ separately outside and inside the layers, the discrete solution U is decomposed by applying the similar technique as that of [72].

First, define the mesh functions V_L and V_R (which approximate v respectively to the left and to the right of the point of discontinuity $x = \xi$) as the solutions of the following discrete problems

$$\begin{cases} L_\varepsilon^{N,M} V_{L,i}^{n+1} = f_i^{n+1}, \quad \text{for } i = 1, \ldots, N/2 - 1, \\ V_{L,0}^{n+1} = v(0, t_{n+1}), \quad V_{L,N/2}^{n+1} = v(\xi^-, t_{n+1}), \quad n \geq 0, \\ V_{L,i}^0 = v(x_i, 0), \quad i \leq N/2, \end{cases}$$

and

$$\begin{cases} L_\varepsilon^{N,M} V_{R,i}^{n+1} = f_i^{n+1}, \quad \text{for } i = N/2 + 1, \ldots, N - 1, \\ V_{R,N/2}^{n+1} = v(\xi^+, t_{n+1}), \quad V_{R,N}^{n+1} = v(1, t_{n+1}), \quad n \geq 0, \\ V_{R,i}^0 = v(x_i, 0), \quad i \geq N/2. \end{cases}$$

Hence, the mesh functions W_L and W_R (which approximate w on either side of $x = \xi$) must satisfy the following system of finite difference equations

$$\begin{cases} L_\varepsilon^{N,M} W_{L,i}^{n+1} = 0, \quad \text{for } i = 1, \dots, N/2 - 1, \\ L_\varepsilon^{N,M} W_{R,i}^{n+1} = 0, \quad \text{for } i = N/2 + 1, \dots, N - 1, \\ W_{L,0}^{n+1} = 0, \quad W_{R,N}^{n+1} = 0, \\ W_{L,i}^{0} = 0, \quad i \le N/2, \quad W_{R,i}^{0} = 0, \quad i \ge N/2, \\ W_{R,N/2}^{n+1} + V_{R,N/2}^{n+1} = W_{L,N/2}^{n+1} + V_{L,N/2}^{n+1}, \\ D_x^+ W_{R,N/2}^{n+1} + D_x^+ V_{R,N/2}^{n+1} = D_x^- W_{L,N/2}^{n+1} + D_x^- V_{L,N/2}^{n+1}, \quad n \ge 0, \end{cases}$$

so that U can satisfy the following decomposition

$$U_i^{n+1} = \begin{cases} V_{L,i}^{n+1} + W_{L,i}^{n+1}, & \text{for } i = 1, \dots, N/2 - 1, \\ V_{L,i}^{n+1} + W_{L,i}^{n+1} = V_{R,i}^{n+1} + W_{R,i}^{n+1}, & \text{for } i = N/2, \\ V_{R,i}^{n+1} + W_{R,i}^{n+1}, & \text{for } i = N/2 + 1, \dots, N - 1. \end{cases}$$

Afterwards, the errors are analyzed separately for the smooth components V_L and V_R of the solution U in the region $G^- \cup G^+$.

Lemma 5.3.1. *Assume that $\varepsilon \le N^{-1}$. Then, if (5.8) holds true, the errors associated to the smooth components satisfy the following estimates*

$$\begin{cases} \left| V_{L,i}^{n+1} - v(x_i, t_{n+1}) \right| \le C\Big(N^{-1} + \Delta t \Big) x_i, & \textit{for } 1 \le i \le N/2 - 1, \\ \left| V_{R,i}^{n+1} - v(x_i, t_{n+1}) \right| \le C\Big(N^{-1} + \Delta t \Big) (1 - x_i), & \textit{for } N/2 + 1 \le i \le N - 1. \end{cases}$$

Proof. First, define the following barrier function

$$\Psi_{L,i} = -C\Big(N^{-1} + \Delta t \Big) x_i, \quad \text{for } 0 \le i \le N/2.$$

Next, corresponding to the smooth component on the left side of the discontinuity, we have the following truncation error bound

$$\left| L_\varepsilon^{N,M} \big(V_{L,i}^{n+1} - v(x_i, t_{n+1}) \big) \right|$$

$$\le \begin{cases} C\left[\dfrac{\varepsilon}{12} h_i^2 \left\| \dfrac{\partial^4 v}{\partial x^4} \right\|_\infty + \dfrac{h_i}{2} a(x_i) \left\| \dfrac{\partial^2 v}{\partial x^2} \right\|_\infty + \dfrac{\Delta t}{2} \left\| \dfrac{\partial^2 v}{\partial t^2} \right\|_\infty \right], \\ \qquad\qquad\qquad\qquad \text{for } 1 \le i \le N/4, \\ C\left[\dfrac{\varepsilon}{3} \big(h_i + h_{i+1} \big) \left\| \dfrac{\partial^3 v}{\partial x^3} \right\|_\infty + \dfrac{h_i}{2} a(x_i) \left\| \dfrac{\partial^2 v}{\partial x^2} \right\|_\infty + \dfrac{\Delta t}{2} \left\| \dfrac{\partial^2 v}{\partial t^2} \right\|_\infty \right], \\ \qquad\qquad\qquad\qquad \text{for } N/4 + 1 \le i \le N/2 - 1. \end{cases}$$

Now, applying the similar argument as given in Lemma 4.4.3 over $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^-}$, and invoking the inequality $h_i \leq CN^{-1}$ (see (5.5), (5.10)), the assumption $\varepsilon \leq N^{-1}$ and the bounds on the derivatives of v given in Theorem 4.2.4, we obtain the following estimate

$$\left|V_{L,i}^{n+1} - v(x_i, t_{n+1})\right| \leq C\Big(N^{-1} + \Delta t\Big)x_i, \quad \text{for } 1 \leq i \leq N/2 - 1.$$

Likewise, employing a new barrier function

$$\Psi_{R,i} = -C\Big(N^{-1} + \Delta t\Big)(1 - x_i), \quad \text{for } N/2 \leq i \leq N,$$

with the corresponding truncation error bound to the right side of the discontinuity, it is easy to obtain the following error bound

$$\left|V_{R,i}^{n+1} - v(x_i, t_{n+1})\right| \leq C\Big(N^{-1} + \Delta t\Big)(1 - x_i), \quad \text{for } N/2 + 1 \leq i \leq N - 1.$$

Hence the proof. ■

Now, the errors are derived for the layer components W_L and W_R of the solution U in the region $\big((0, \xi - \sigma_1] \cup [\xi + \sigma_2, 1)\big) \times (0, T]$.

Lemma 5.3.2. *Let $\gamma \leq \alpha$. Then, if (5.9) holds true, the errors associated to the layer components satisfy the following estimates*

$$\begin{cases} \left|W_{L,i}^{n+1} - w(x_i, t_{n+1})\right| \leq CN^{-1}, & \textit{for } 1 \leq i \leq N/4, \\ \left|W_{R,i}^{n+1} - w(x_i, t_{n+1})\right| \leq CN^{-1}, & \textit{for } 3N/4 \leq i \leq N - 1. \end{cases}$$

Proof. Using the barrier functions

$$\begin{cases} \Phi_{L,i} = -C\left[\displaystyle\prod_{j=1}^{N/2}\left(1 + \frac{\gamma h_j}{2\varepsilon}\right)^{-1}\right]\mathcal{S}_i, & \text{for } 0 \leq i \leq N/2, \\ \Phi_{R,i} = -C\left[\displaystyle\prod_{j=N/2+1}^{N}\left(1 + \frac{\gamma h_j}{2\varepsilon}\right)^{-1}\right]\mathcal{Q}_i, & \text{for } N/2 \leq i \leq N, \end{cases}$$

and applying the similar arguments given in Lemma 4.4.6 respectively, for $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^-}$ and $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^+}$, we can easily deduce from Lemma 5.2.7 that

$$\begin{cases} |W_{L,i}^{n+1}| \leq C \displaystyle\prod_{j=N/4+1}^{N/2}\left(1 + \frac{\gamma h_j}{2\varepsilon}\right)^{-1} \leq CN^{-1}, & \text{for } 1 \leq i \leq N/4, \\ |W_{R,i}^{n+1}| \leq C \displaystyle\prod_{j=N/2+1}^{3N/4}\left(1 + \frac{\gamma h_j}{2\varepsilon}\right)^{-1} \leq CN^{-1}, & \text{for } 3N/4 \leq i \leq N - 1. \end{cases}$$

Therefore, for $1 \le i \le N/4$, from Theorem 4.2.4 we have

$$\Big|W_{L,i}^{n+1} - w(x_i, t_{n+1})\Big| \le \Big(|W_{L,i}^{n+1}| + |w(x_i, t_{n+1})|\Big) \le CN^{-1} + C\exp\big(-\alpha\sigma_1/\varepsilon\big) \le CN^{-1}.$$

Similarly, for $3N/4 \le i \le N-1$, we obtain

$$\Big|W_{R,i}^{n+1} - w(x_i, t_{n+1})\Big| \le CN^{-1}.$$

and the proof is complete. ■

The ε-uniform convergence of the numerical scheme (5.13) on the Shishkin-type mesh $\overline{G}_{\varepsilon}^{N,M}$ is proved in the following theorem.

5.3.1 The main convergence result

Theorem 5.3.3. *Assume that $\varepsilon \le N^{-1}$ and let the conditions (5.8) and (5.9) be satisfied. Then, if $\gamma \le \alpha$, the respective solutions u and U of (5.1)-(5.4) and (5.13) satisfy the following error bounds at time level t_n:*

$$\Big|U_i^n - u(x_i, t_n)\Big| \le \begin{cases} C\Big[N^{-1} + \Delta t\Big], & \textit{for } 1 \le i \le N/4 \bigcup 3N/4 \le i \le N-1, \\ C\Big[N^{-1}\Big(\max\limits_{\tau\in[1/4,1/2]} \big|\psi_l'(\tau)\big| + \max\limits_{\tau\in[1/2,3/4]} \big|\psi_r'(\tau)\big|\Big) + \Delta t\Big], & \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for } N/4 < i < 3N/4. & \end{cases} \tag{5.21}$$

Proof. The proof is splitted up into two different cases depending on the location of the mesh point $x_i \in \overline{\Omega}_x^{N,\varepsilon}$.

Case 1. $\big($**Outer region**$\big)$ For $1 \le i \le N/4$ and $3N/4 \le i \le N-1$. Here, the estimate of $\big|U_i^n - u(x_i, t_n)\big|$ follows easily from Lemmas 5.3.1 and 5.3.2, by invoking the triangle inequality to the error

$$U - u = (V_L - v) + (W_L - w).$$

Case 2. $\big($**Interior layer region**$\big)$ Here, we need to find out the estimate of $\big|U_i^n - u(x_i, t_n)\big|$ for $N/4 < i < 3N/4$. From **Case 1**, clearly we have

$$\Big|U_i^{n+1} - u(x_i, t_{n+1})\Big| \le C\Big(N^{-1} + \Delta t\Big), \quad \text{for } i = N/4, 3N/4. \tag{5.22}$$

Now, using the Taylor's formula with the integral form of remainder as in [[41], *Lemma 3.3*], the bounds of the derivatives stated in Theorem 4.2.4 and the inequalities (5.10), (5.19), we obtain that for $N/4+1 \leq i \leq N/2-1$,

$$\begin{aligned}
&\left|L_\varepsilon^{N,M}(U_i^{n+1}-u(x_i,t_{n+1}))\right| \\
\leq\ & C\Big(\varepsilon\int_{x_{i-1}}^{x_{i+1}}\left|\frac{\partial^3 u}{\partial x^3}(x,t_{n+1})\right|dx+\int_{x_i}^{x_{i+1}}\left|\frac{\partial^2 u}{\partial x^2}(x,t_{n+1})\right|dx\Big)+C\Delta t\left\|\frac{\partial^2 u}{\partial t^2}\right\|_\infty \\
\leq\ & C\Big(h_i+\varepsilon^{-2}\int_{x_{i-1}}^{x_{i+1}}\exp\big(-(\xi-x)\alpha_1/\varepsilon\big)dx\Big)+C\Delta t \\
\leq\ & C\Big[\Big(N^{-1}+\varepsilon^{-1}N^{-1}\exp\big(-\gamma(\xi-x_{i+1})/2\varepsilon\big)\max_{\tau\in[\tau_i,\tau_j]}\left|\psi_l'(\tau)\right|\Big)+\Delta t\Big].
\end{aligned}$$

Thus, for $N/4+1 \leq i \leq N/2-1$,

$$\begin{aligned}
&\left|L_\varepsilon^{N,M}(U_i^{n+1}-u(x_i,t_{n+1}))\right| \\
\leq\ & C\Big[\Big(N^{-1}+\varepsilon^{-1}N^{-1}\exp\big(-\gamma(\xi-x_i)/2\varepsilon\big)\max_{\tau\in[\tau_i,\tau_j]}\left|\psi_l'(\tau)\right|\Big)+\Delta t\Big],
\end{aligned}\tag{5.23}$$

since inequality (5.11) implies that

$$\exp\big(-\gamma(\xi-x_{i+1})/2\varepsilon\big)\leq\exp\big(-\gamma(\xi-x_i)/2\varepsilon\big)\times\exp\big(\gamma h_{i+1}/2\varepsilon\big)\leq C\exp\big(-\gamma(\xi-x_i)/2\varepsilon\big).$$

Similarly, for $N/2+1 \leq i \leq 3N/4-1$, we get

$$\begin{aligned}
&\left|L_\varepsilon^{N,M}(U_i^{n+1}-u(x_i,t_{n+1}))\right| \\
\leq\ & C\Big[\Big(N^{-1}+\varepsilon^{-1}N^{-1}\exp\big(-\gamma(x_i-\xi)/2\varepsilon\big)\max_{\tau\in[\tau_i,\tau_j]}\left|\psi_r'(\tau)\right|\Big)+\Delta t\Big].
\end{aligned}\tag{5.24}$$

Now, at the mesh point $x_{N/2}=\xi$, using (5.19) and (5.20), we obtain the following estimate:

$$\begin{aligned}
&\left|(D_x^+-D_x^-)(U_{N/2}^{n+1}-u(x_{N/2},t_{n+1}))\right| \\
=\ & \left|(D_x^- - D_x^+)\,u_{N/2}^{n+1}+\left[\frac{\partial u}{\partial x}\right](x_{N/2},t_{n+1})\right| \\
\leq\ & \left|D_x^- u_{N/2}^{n+1}-\frac{\partial u}{\partial x}(x_{N/2},t_{n+1})\right|+\left|D_x^+ u_{N/2}^{n+1}-\frac{\partial u}{\partial x}(x_{N/2},t_{n+1})\right| \\
\leq\ & C\Big(h_{N/2}+h_{N/2+1}\Big) \\
& +C\varepsilon^{-2}\Big(\int_{x_{N/2-1}}^{x_{N/2}}\exp\big(-(\xi-x)\alpha_1/\varepsilon\big)dx+\int_{x_{N/2}}^{x_{N/2+1}}\exp\big(-(x-\xi)\alpha_2/\varepsilon\big)dx\Big) \\
\leq\ & 2CN^{-1}+C\varepsilon^{-1}N^{-1}\mathfrak{F},
\end{aligned}\tag{5.25}$$

where

$$\mathfrak{F} = \left(\max_{\tau\in[1/4,1/2]} \left|\psi_l'(\tau)\right| + \max_{\tau\in[1/2,3/4]} \left|\psi_r'(\tau)\right| \right).$$

Define the discrete function

$$\Theta_i = \begin{cases} -C\Big(N^{-1}+\Delta t\Big)\Big(1+\Big[x_i-(\xi-\sigma)\Big]\Big) - CN^{-1}\mathfrak{F}\Big[\prod_{j=1}^{N/2}\Big(1+\frac{\gamma h_j}{2\varepsilon}\Big)^{-1}\Big]\mathcal{S}_i, \\ \qquad\qquad \text{for } N/4+1\le i\le N/2-1, \\ -C\Big(N^{-1}+\Delta t\Big)\Big(1+\Big[(\xi+\sigma)-x_i\Big]\Big) - CN^{-1}\mathfrak{F}\Big[\prod_{j=N/2+1}^{N}\Big(1+\frac{\gamma h_j}{2\varepsilon}\Big)^{-1}\Big]\mathcal{Q}_i, \\ \qquad\qquad \text{for } N/2+1\le i\le 3N/4-1, \end{cases}$$

where C is chosen sufficiently large. A direct calculation yields

$$L_\varepsilon^{N,M}\Theta_i \ge \begin{cases} C\Big(N^{-1}+\Delta t\Big)\big(-a_i\big) - CN^{-1}\mathfrak{F}\Big[\prod_{j=1}^{N/2}\Big(1+\frac{\gamma h_j}{\varepsilon}\Big)^{-1}\Big]L_\varepsilon^{N,M}\mathcal{S}_i, \\ \qquad\qquad \text{for } N/4+1\le i\le N/2-1, \\ C\Big(N^{-1}+\Delta t\Big)\big(a_i\big) - CN^{-1}\mathfrak{F}\Big[\prod_{j=N/2+1}^{N}\Big(1+\frac{\gamma h_j}{2\varepsilon}\Big)^{-1}\Big]L_\varepsilon^{N,M}\mathcal{Q}_i, \\ \qquad\qquad \text{for } N/2+1\le i\le 3N/4-1. \end{cases}$$

Consequently, under the condition $\gamma\le\alpha$, this inequality together with Lemmas 5.2.5, 5.2.8 and the inequality (5.11) yield

$$L_\varepsilon^{N,M}\Theta_i \ge \begin{cases} C\alpha_1\Big(N^{-1}+\Delta t\Big) + C\varepsilon^{-1}N^{-1}\mathfrak{F}\exp\big(-(\xi-x_i)\alpha_1/\varepsilon\big), \\ \qquad\qquad \text{for } N/4+1\le i\le N/2-1, \\ C\alpha_2\Big(N^{-1}+\Delta t\Big) + C\varepsilon^{-1}N^{-1}\mathfrak{F}\exp\big(-(x_i-\xi)\alpha_2/\varepsilon\big), \\ \qquad\qquad \text{for } N/2+1\le i\le 3N/4-1. \end{cases} \tag{5.26}$$

Again, using the inequality (5.11), we get

$$\begin{aligned} \big(D_x^+-D_x^-\big)\Theta_{N/2} &= 2C\Big(N^{-1}+\Delta t\Big) \\ &\quad +CN^{-1}\mathfrak{F}\Big[\prod_{j=1}^{N/2}\Big(1+\frac{\gamma h_j}{2\varepsilon}\Big)^{-1}D_x^-\mathcal{S}_{N/2} - \prod_{j=N/2+1}^{N}\Big(1+\frac{\gamma h_j}{2\varepsilon}\Big)^{-1}D_x^+\mathcal{Q}_{N/2}\Big] \\ &\ge 2C\Big(N^{-1}+\Delta t\Big) + 2C\varepsilon^{-1}N^{-1}\mathfrak{F}. \end{aligned} \tag{5.27}$$

Therefore, it follows from (5.22)-(5.27) that Θ_i is a barrier function for $(U-u)$. Now, applying the discrete maximum principle (Lemma 5.2.4) to $\Theta_i \pm \big(U_i^n - u(x_i,t_n)\big)$ over the domain $\big(\overline{G}_\varepsilon^{N,M} \bigcap [\xi-\sigma,\xi+\sigma]\big) \times [0,T]$, we obtain for $N/4 < i < 3N/4$,

$$\Big|U_i^n - u(x_i,t_n)\Big| \le C\Big[N^{-1}\Big(\max_{\tau\in[1/4,1/2]}\big|\psi_l'(\tau)\big| + \max_{\tau\in[1/2,3/4]}\big|\psi_r'(\tau)\big|\Big) + \Delta t\Big].$$

Hence, this completes the proof. ■

Table 5.1:

		$\max\|\psi_l'\|(\text{or}\max\|\psi_r'\|)$
S mesh	(5.6)	$C\ln N$
B – **S** mesh	(5.2.1)	C

Remark 5.3.4. *Following the above approach for the hybrid finite difference scheme (4.14), proposed in Chapter 4, we would get following error estimate:*

$$\Big|U_i^n - u(x_i,t_n)\Big| \le \begin{cases} C\Big[N^{-2}+\Delta t\Big], & \text{for } 1\le i\le N/4 \bigcup 3N/4 \le i \le N-1,\\ C\Big[N^{-2}\Big(\Big\{\max\limits_{\tau\in[1/4,1/2]}\big|\psi_l'(\tau)\big|\Big\}^2 + \Big\{\max\limits_{\tau\in[1/2,3/4]}\big|\psi_r'(\tau)\big|\Big\}^2\Big) + \Delta t\Big], & \\ \qquad\qquad \text{for } N/4 < i < 3N/4, & \end{cases} \tag{5.28}$$

provided the conditions (5.8) and (5.9) are satisfied.

Nevertheless, we see that Table 5.1 contains the maximum value of $|\psi'|$ for the Shiskin-type meshes and henceforth, the estimate (5.21) in Theorem 5.3.3 reduces to the following form:

$$\big\|U-u\big\|_{\infty,\overline{G}_\varepsilon^{N,M}} \le \begin{cases} C\Big[N^{-1}\ln N + \Delta t\Big], & (\mathbf{S}\ \textit{mesh}),\\ C\Big[N^{-1}+\Delta t\Big], & (\mathbf{B}-\mathbf{S}\ \textit{mesh}). \end{cases} \tag{5.29}$$

5.4 Numerical Results

This section computationally verifies the theoretical results obtained in the previous section. For this, the performance of the implicit upwind scheme (5.13) is studied by applying to the following test examples on the Shishkin-type meshes and displayed the numerical results for $\varepsilon = 10^{-4}, 10^{-8}$. In all the cases, the numerical experiments are executed by choosing the constant $\sigma_0 = 2.2$ and $\Delta t = 1/N$.

Example 5.4.1. *Consider the following parabolic IBVP:*

$$\begin{cases} \varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - x(1-x)u - \dfrac{\partial u}{\partial t} = f(x,t), & (x,t) \in (0,1) \times (0,1], \\ u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = t^2, \quad u(1,t) = 0, \quad 0 < t \le 1, \end{cases} \tag{5.30}$$

where the convective coefficient is

$$a(x) = \begin{cases} -\big(1 + x(1-x)\big), & 0 \le x \le 0.5, \\ \big(1 + x(1-x)\big), & 0.5 < x \le 1, \end{cases}$$

and the source term is given by

$$f(x,t) = \begin{cases} 2(1+x^2)t^2, & 0 \le x \le 0.5, \\ 3(1+x^2)t^2, & 0.5 < x \le 1. \end{cases}$$

Example 5.4.2. *Consider the following parabolic IBVP:*

$$\begin{cases} \varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial t} = f(x,t), & (x,t) \in (0,1) \times (0,1], \\ u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \le 1, \end{cases} \tag{5.31}$$

where the convective coefficient is

$$a(x) = \begin{cases} -\big(2 + x^2\big), & 0 \le x \le 0.5, \\ 3 - x^2, & 0.5 < x \le 1, \end{cases}$$

and the source term is given by

$$f(x,t) = \begin{cases} 2x\exp(-t)t^2, & 0 \le x \le 0.5, \\ 2(1-x)\exp(-t)t^2, & 0.5 < x \le 1. \end{cases}$$

As the exact solutions of the IBVPs (5.30) and (5.31) are not known, the numerical results of the upwind scheme (5.13) are illustrated by using the double mesh principle as described in Chapter 2.

In Tables 5.2 and 5.3 for both the **S** mesh and the **B** − **S** mesh, we have presented the computed maximum point-wise errors $E_\varepsilon^{N,\Delta t}$ and the corresponding order of convergence $P_\varepsilon^{N,\Delta t}$ respectively, for Examples 5.4.1 and 5.4.2. Besides, we have displayed the maximum point-wise errors over the full domain for Examples 5.4.1 and 5.4.2 respectively, in Figures 5.1 and 5.2.

Clearly, one can see that the numerical results displayed in all the tables and figures do agree well with the theoretical bounds obtained in Theorem 5.3.3 and (5.29). Indeed, we can conclude that the implicit upwind finite difference scheme (5.13) on the $\mathbf{B}-\mathbf{S}$ mesh outperforms the **S** mesh, irrespective of the perturbation parameter ε.

5.5 Conclusion

In this chapter, for a class of singularly perturbed parabolic IBVPs with discontinuous convection coefficient, a unified theory for characterization of Shishkin-type meshes (including the piecewise-uniform Shishkin mesh and the Bakhalov-Shishkin mesh) resolving strong interior layers is provided. Moreover, we have shown both theoretically and computationally that the error bound obtained for the implicit upwind scheme on the Bakhalov-Shishkin mesh is optimal and more accurate than that obtained on the piecewise-uniform Shishkin mesh.

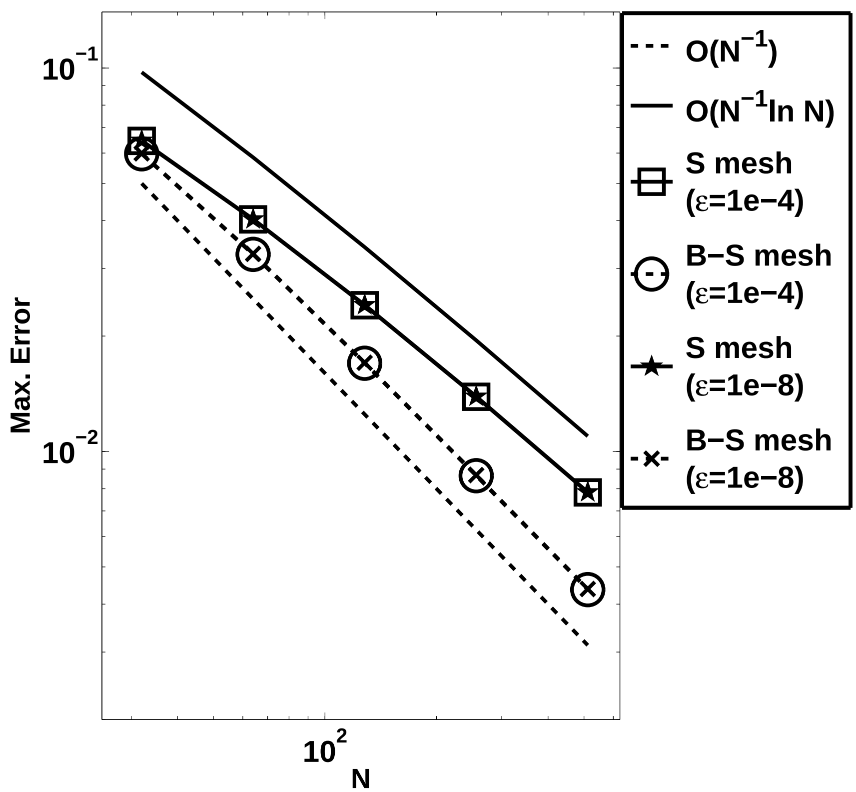


Figure 5.1: *Loglog plot of the maximum point-wise errors for Example 5.4.1 over the full domain.*

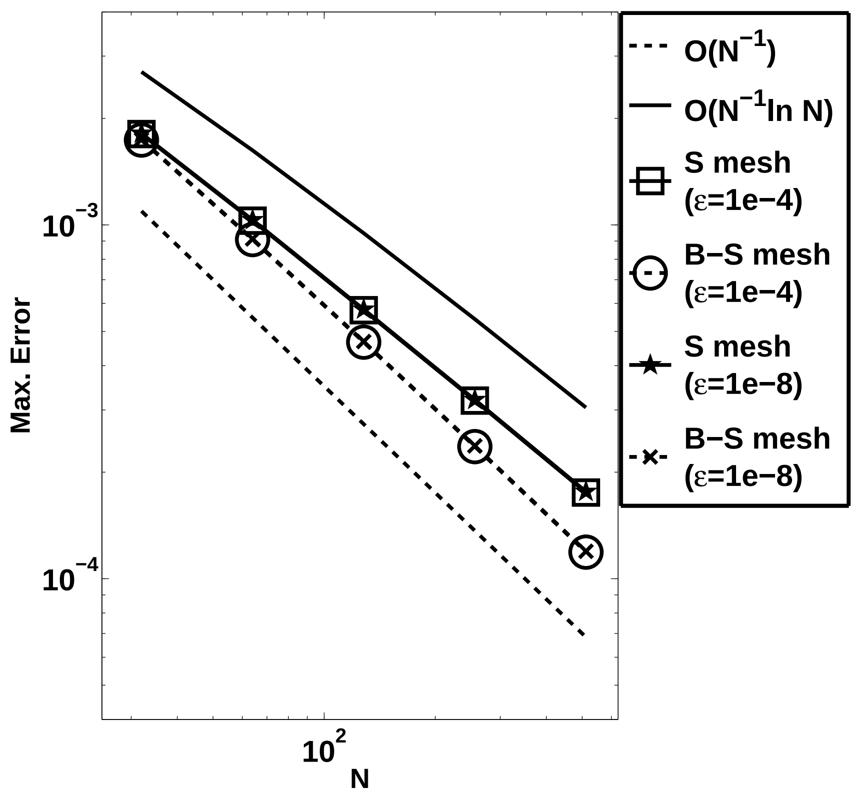


Figure 5.2: *Loglog plot of the maximum point-wise errors for Example 5.4.2 over the full domain.*

Table 5.2: *Maximum point-wise errors and the corresponding order of convergence calculated for Example 5.4.1.*

N	$\varepsilon = 10^{-4}$					
	left outer region $[0, \xi - \sigma_1]$		**interior layer region** $(\xi - \sigma_1, \xi + \sigma_2)$		**right outer region** $[\xi + \sigma_2, 1]$	
	S mesh	**B-S mesh**	**S mesh**	**B-S mesh**	**S mesh**	**B-S mesh**
32	2.7026e-2	2.7021e-2	6.4582e-2	5.9750e-2	4.2758e-2	4.2758e-2
	0.9116	0.9116	0.6805	0.8713	0.8901	0.8901
64	1.4366e-2	1.4363e-2	4.0293e-2	3.2661e-2	2.3070e-2	2.3070e-2
	0.9593	0.9596	0.7436	0.9432	0.9475	0.9475
128	7.3882e-3	7.3854e-3	2.4064e-2	1.6985e-2	1.1962e-2	1.1962e-2
	0.9805	0.9806	0.7924	0.9734	0.9739	0.9739
256	3.7442e-3	3.7427e-3	1.3893e-2	8.6504e-3	6.0897e-3	6.0897e-3
	0.9909	0.9906	0.8282	0.9869	0.9872	0.9872
512	1.8838e-3	1.8835e-3	7.8247e-3	4.3644e-3	3.0720e-3	3.0720e-3
	0.9957	0.9955	0.8592	0.9935	0.9938	0.9938
	$\varepsilon = 10^{-8}$					
	left outer region $[0, \xi - \sigma_1]$		**interior layer region** $(\xi - \sigma_1, \xi + \sigma_2)$		**right outer region** $[\xi + \sigma_2, 1]$	
	S mesh	**B-S mesh**	**S mesh**	**B-S mesh**	**S mesh**	**B-S mesh**
32	2.7084e-2	2.7084e-2	6.4673e-2	5.9900e-2	4.2865e-2	4.2865e-2
	0.9113	0.9113	0.6800	0.8707	0.8896	0.8896
64	1.4401e-2	1.4401e-2	4.0364e-2	3.2757e-2	2.3136e-2	2.3136e-2
	0.9589	0.9593	0.7435	0.9428	0.9469	0.9469
128	7.4083e-3	7.4063e-3	2.4108e-2	1.7040e-2	1.2001e-2	1.2001e-2
	0.9801	0.9802	0.7924	0.9730	0.9734	0.9734
256	3.7556e-3	3.7541e-3	1.3919e-2	8.6811e-3	6.1119e-3	6.1119e-3
	0.9905	0.9902	0.8283	0.9866	0.9867	0.9867
512	1.8901e-3	1.8897e-3	7.8391e-3	4.3810e-3	3.0843e-3	3.0843e-3
	0.9953	0.9951	0.8592	0.9932	0.9933	0.9933

Table 5.3: *Maximum point-wise errors and the corresponding order of convergence calculated for Example 5.4.2.*

N	$\varepsilon = 10^{-4}$					
	left outer region $[0, \xi - \sigma_1]$		**interior layer region** $(\xi - \sigma_1, \xi + \sigma_2)$		**right outer region** $[\xi + \sigma_2, 1]$	
	S mesh	**B-S mesh**	**S mesh**	**B-S mesh**	**S mesh**	**B-S mesh**
32	1.7249e-3	1.7207e-3	1.8059e-3	1.7375e-3	1.5053e-3	1.5069e-3
	0.9537	0.9507	0.8129	0.9346	0.9584	0.9598
64	8.9057e-4	8.9024e-4	1.0279e-3	9.0899e-4	7.7463e-4	7.7471e-4
	0.9772	0.9767	0.8406	0.9637	0.9793	0.9795
128	4.5237e-4	4.5235e-4	5.7398e-4	4.6607e-4	3.9289e-4	3.9289e-4
	0.9889	0.9888	0.8481	0.9808	0.9900	0.9900
256	2.2793e-4	2.2793e-4	3.1884e-4	2.3615e-4	1.9781e-4	1.9781e-4
	0.9947	0.9947	0.8635	0.9901	0.9953	0.9953
512	1.1438e-4	1.1438e-4	1.7524e-4	1.1889e-4	9.9223e-5	9.9224e-5
	0.9976	0.9976	0.8839	0.9952	0.9980	0.9980
	$\varepsilon = 10^{-8}$					
	left outer region $[0, \xi - \sigma_1]$		**interior layer region** $(\xi - \sigma_1, \xi + \sigma_2)$		**right outer region** $[\xi + \sigma_2, 1]$	
	S mesh	**B-S mesh**	**S mesh**	**B-S mesh**	**S mesh**	**B-S mesh**
32	1.7290e-3	1.7248e-3	1.8131e-3	1.7438e-3	1.5092e-3	1.5108e-3
	0.9530	0.9500	0.8118	0.9341	0.9577	0.9590
64	8.9309e-4	8.9276e-4	1.0329e-3	9.1263e-4	7.7705e-4	7.7712e-4
	0.9765	0.9760	0.8395	0.9631	0.9786	0.9787
128	4.5386e-4	4.5384e-4	5.7719e-4	4.6812e-4	3.9431e-4	3.9431e-4
	0.9882	0.9882	0.8483	0.9802	0.9893	0.9893
256	2.2878e-4	2.2878e-4	3.2057e-4	2.3729e-4	1.9862e-4	1.9862e-4
	0.9941	0.9941	0.8631	0.9894	0.9946	0.9946
512	1.1486e-4	1.1486e-4	1.7624e-4	1.1951e-4	9.9678e-5	9.9678e-5
	0.9970	0.9970	0.8837	0.9946	0.9973	0.9973

Chapter 6

Parameter-Uniform Hybrid Numerical Scheme for Singularly Perturbed Problems of Mixed Parabolic-Elliptic type

The present chapter is dealt with the study of a hybrid numerical scheme for a class of singularly perturbed mixed parabolic-elliptic problems possessing both boundary and interior layers. The domain under consideration is partitioned into two subdomains. In the first subdomain, the given problem takes the form of parabolic reaction-diffusion type, whereas in the second subdomain elliptic convection-diffusion-reaction types of problems are posed. To solve these problems, the time derivative is discretized by the classical backward-Euler method. While for the spatial discretization of the problem, the classical central difference scheme is used on the first subdomain and a hybrid finite difference scheme (a proper combination of the midpoint upwind scheme in the outer regions and the classical central difference scheme in the interior layer regions) is proposed on the second subdomain. The proposed method is analyzed on a layer resolving piecewise-uniform Shishkin mesh and is shown to be ε-uniformly convergent with almost second-order spatial accuracy in the discrete supremum norm, provided that the perturbation parameter ε satisfies $\varepsilon \leq N^{-1}$.

6.1 Introduction

Let us denote the domains for describing the model problem by $\Omega = (0, 1)$, $\Omega^- = (0, \xi)$, $\Omega^+ = (\xi, 1)$, $G^- = \Omega^- \times (0, T]$, $G^+ = \Omega^+ \times (0, T]$, $G = \Omega \times (0, T]$. Consider the following class of singularly perturbed mixed parabolic-elliptic problems posed on the domain $G^- \cup G^+$:

$$\begin{cases} L_{1,\varepsilon} u(x,t) \equiv \left(\dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + b(x,t)u \right)(x,t) \; = \; f(x,t), \quad (x,t) \in G^-, \\ L_{2,\varepsilon} u(x,t) \equiv \left(-\varepsilon \dfrac{\partial^2 u}{\partial x^2} - a(x,t) \dfrac{\partial u}{\partial x} + b(x,t)u \right)(x,t) \; = \; f(x,t), \quad (x,t) \in G^+, \\ u(x,0) = s_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = s_1(t), \quad u(1,t) = s_2(t), \quad t \in (0,T], \end{cases} \tag{6.1}$$

where $0 < \varepsilon \ll 1$ is a small parameter and

$$\begin{cases} b(x,t) \geq \beta \geq 0 \quad \text{on } \overline{G}, \\ \alpha^* > a(x,t) > \alpha > 0, \qquad x > \xi, \end{cases} \tag{6.2}$$

and the solution $u(x,t)$ satisfies the following interface conditions

$$[u] = 0, \qquad \left[\frac{\partial u}{\partial x} \right] = 0, \quad \text{at } x = \xi. \tag{6.3}$$

Here, the jump of u, denoted by $[u]$, across the point of discontinuity $x = \xi$ is defined by $[u](\xi,t) = u(\xi^+,t) - u(\xi^-,t)$, where $u(\xi^\pm,t) = \lim_{x \to \xi \pm 0} u(x,t)$. Under sufficient smoothness and necessary compatibility conditions imposed on the data s_0, s_1 and s_2, the solution $u(x,t) \in \mathcal{C}^{1+\lambda}(G) \bigcap \mathcal{C}^{2+\lambda}(G^- \cup G^+)$ of the IBVP (6.1)-(6.3), exhibits a boundary layer at $x = 0$ and interior layers of different widths in the neighborhood of the point $x = \xi$. Such kind of problems describe, for example, an electromagnetic field arising in the motion a train on an air-pillow (see [44]). An asymptotic expansion of this problem is constructed in [90]. Since the convection coefficient is positive in Ω^+, an interior layer of width $O(\varepsilon)$ appears on the left side of the boundary of G^+ . Whereas the differential equation in (6.1) is of parabolic reaction-diffusion type in G^-, which leads to occurrence of parabolic boundary layers of width $O(\sqrt{\varepsilon})$ on both the boundaries of G^- (see, e.g., [5]).

The outline of this chapter is as follows: Section 6.2 provides *a-priori* bound on the analytical solution and that on the derivatives of the solution *via* decomposition. Section 6.3 describes the piecewise-uniform Shishkin mesh and provides the detail construction of the newly proposed hybrid finite difference scheme. Next, the ε-uniform stability is studied in Section 6.4 and the main theoretical result, *i.e.*, the ε-uniform convergence of the proposed finite difference scheme, is proved. Finally, Section 6.5 performs the numerical experiments to support the theoretical results.

6.2 Bounds on the Solution and its Decomposition

Here, *a-priori* bound on the analytical solution is derived and also stronger bounds are obtained on the derivatives of the analytical solution of the problem (6.1)-(6.3), which is being decomposed into smooth and layer components. This will be used later in the analysis for the proof of ε-uniform error estimate of the proposed difference scheme.

Firstly, some preliminary results are stated here. Let $\Gamma = \overline{G}\backslash G$. Then we obtain the following maximum principle on $\overline{G}$.

Lemma 6.2.1. (***Maximum Principle***) *Suppose that a function* $g \in \mathcal{C}^0(\overline{G}) \bigcap \mathcal{C}^2(G^- \cup G^+)$ *satisfies*

$$g(x,t) \le 0,\ (x,t) \in \Gamma, \quad \left[\frac{\partial g}{\partial x}\right](\xi, t) \ge 0,\ t > 0 \quad \text{and} \quad \begin{cases} L_{1,\varepsilon} g(x,t) \le 0,\ (x,t) \in G^-, \\ L_{2,\varepsilon} g(x,t) \le 0,\ (x,t) \in G^+, \end{cases}$$

then $g(x,t) \le 0,\ \forall (x,t) \in \overline{G}$.

Proof. The proof follows from [5]. ■

An immediate consequence of the above maximum principle is the following stability result, which implies the uniqueness of the solution of the IBVP (6.1)-(6.3).

Lemma 6.2.2. *Let u be the solution of the problem (6.1)-(6.3), then*

$$\|u\|_{\infty,\overline{G}} \le \|u\|_{\infty,\Gamma} + \frac{(1+T)}{\mu}\|f\|_{\infty,\overline{G}},$$

where $\mu = \min\{1, \alpha/(1-\xi)\}$.

Proof. Consider the following functions

$$\psi^{\pm}(x,t) \;=\; -\|u\|_{\infty,\Gamma} - \begin{cases} \dfrac{(1+t)}{\mu}\|f\|_\infty \mp u(x,t), & (x,t) \in G^-, \\ \dfrac{(1+t)(1-x)}{\mu(1-\xi)}\|f\|_\infty \mp u(x,t), & (x,t) \in G^+. \end{cases}$$

Then, clearly $\psi^{\pm} \in \mathcal{C}^0(\overline{G})$ and the required stability bound will be obtained by applying Lemma 6.2.1 to $\psi^{\pm}$. ■

Now, consider the following decomposition of the solution $u = v + w$, into a smooth component v and a layer component w. Here, the smooth component v is defined to be the sum

$$v = \sum_{i=0}^{3} \varepsilon^i v_i + \varepsilon^4 \mathfrak{R}, \quad v_i = \begin{cases} v_i^- & \text{in } G^-, \\ v_i^+ & \text{in } G^+, \end{cases} \tag{6.4}$$

where the functions $v_i^-,\ i = 0,1,2,3$ are solutions to the following first-order problems

$$\begin{cases} \dfrac{\partial v_0^-}{\partial t} + b v_0^- = f \quad \text{in } G^-, \quad v_0^-(x,0) = u(x,0), \quad x \in \overline{\Omega^-}; \\ \dfrac{\partial v_i^-}{\partial t} + b v_i^- = \dfrac{\partial^2 v_{i-1}^-}{\partial x^2} \quad \text{in } G^-, \quad v_i^-(x,0) = 0, \quad x \in \overline{\Omega^-}, \quad i = 1,2,3, \end{cases} \tag{6.5}$$

while the functions v_i^-, $i = 0, 1, 2, 3$ satisfy the following first-order problems

$$\begin{cases} -a\dfrac{\partial v_0^+}{\partial x} + bv_0^+ \;=\; f \quad \text{in } G^+, \\ v_0^+(1,t) = u(1,t), \quad t \in (0,T], \quad v_0^+(x,0) = u(x,0), \quad x \in \overline{\Omega^+}; \\ -a\dfrac{\partial v_i^+}{\partial x} + bv_i^+ \;=\; \dfrac{\partial^2 v_{i-1}^+}{\partial x^2} \quad \text{in } G^+, \\ v_i^+(1,t) = 0, \quad t \in (0,T], \quad v_i^+(x,0) = 0, \quad x \in \overline{\Omega^+}, \quad i = 1, 2, 3, \end{cases} \tag{6.6}$$

and lastly, the remainder $\mathfrak{R}$ satisfies

$$\begin{cases} L_{1,\varepsilon}\mathfrak{R} \;=\; \dfrac{\partial^2 v_3^-}{\partial x^2} \quad \text{in } G^-, \quad L_{2,\varepsilon}\mathfrak{R} \;=\; \dfrac{\partial^2 v_3^+}{\partial x^2} \quad \text{in } G^+, \\ \mathfrak{R}(0,t) = \mathfrak{R}(1,t) = 0, \quad \mathfrak{R}(x,0) = 0, \quad x \in \overline{\Omega}, \\ [\mathfrak{R}](\xi,t) = 0, \quad \left[\dfrac{\partial \mathfrak{R}}{\partial x}\right](\xi,t) = 0, \quad t \in (0,T]. \end{cases} \tag{6.7}$$

With v thus defined, the layer component w is further decomposed into the sum

$$w = \begin{cases} w^1 + w^2 \quad \text{in } G^-, \\ w^2 \quad \text{in } G^+, \end{cases} \tag{6.8}$$

where w^1 and w^2 respectively satisfy the following IBVPs

$$\begin{cases} L_{1,\varepsilon}w^1 \;=\; 0 \quad \text{in } G^-, \\ w^1(0,t) = u(0,t) - v(0,t), \quad w^1(\xi,t) = 0, \quad t \in (0,T], \\ w^1(x,0) = 0, \quad x \in \overline{\Omega^-}, \end{cases} \tag{6.9}$$

and

$$\begin{cases} L_{1,\varepsilon}w^2 \;=\; 0 \quad \text{in } G^-, \quad L_{2,\varepsilon}w^2 \;=\; 0 \quad \text{in } G^+, \\ w^2(0,t) = w^2(1,t) = 0, \quad w^2(x,0) = 0, \quad x \in \overline{\Omega}, \\ [w^2](\xi,t) = -[v](\xi,t), \quad \left[\dfrac{\partial w^2}{\partial x}\right](\xi,t) = -\left[\dfrac{\partial v}{\partial x}\right](\xi,t) + \dfrac{\partial w^1}{\partial x}(\xi,t), \quad t \in (0,T]. \end{cases} \tag{6.10}$$

Hence, $w(\xi^-,t) = u(\xi^-,t) - v(\xi^-,t)$ and $w(\xi^+,t) = u(\xi^+,t) - v(\xi^+,t)$. Note that since the problem (6.1)-(6.3) has a unique solution, the decomposition $u = v + w$ is valid. That is why, although both the components v and w are discontinuous at $x = \xi$, $t > 0$, but their sum is in $\mathcal{C}^{1+\lambda}(G)$. In Theorem 6.2.4, the bounds on the derivatives of the components v, w_1 and w_2 are obtained. Before proving this theorem, the following result which will be used in Theorem 6.2.4 is stated below.

Theorem 6.2.3. *Assume that the data* $a, b \in \mathcal{C}^{2+\lambda}(\overline{G})$, $f \in \mathcal{C}^{2+\lambda}(G^- \cup G^+)$ *and the initial-boundary data* s_0, s_1 *and* s_2 *are assumed to be satisfy sufficient smoothness and compatibility conditions. Then the solution* $u(x,t)$ *of the IBVP (6.1)-(6.3) lies in* $\mathcal{C}^{1+\lambda}(G) \bigcap \mathcal{C}^{4+\lambda}(G^- \cup G^+)$ *and satisfies the estimates*

$$\left\| \frac{\partial^{l+m} u}{\partial x^l \partial t^m} \right\|_{\infty, G^-} \leq C\varepsilon^{-l/2} \quad \text{and} \quad \left\| \frac{\partial^{l+m} u}{\partial x^l \partial t^m} \right\|_{\infty, G^+} \leq C\varepsilon^{-l}, \tag{6.11}$$

for all non-negative integers l, m, *satisfying* $0 \leq l + 2m \leq 4$.

Proof. Here, the similar arguments as in the proof of [[55], *Theorem 3*] are used. First, the variable x is transformed to the stretched variable

$$\widetilde{x} = \begin{cases} x/\sqrt{\varepsilon}, & x \in \Omega^-, \\ x/\varepsilon, & x \in \Omega^+, \end{cases}$$

and using the variables $(\widetilde{x}, t)$, the differential equations in (6.3) are independent of the parameter ε. Now, the estimate (10.5) given in [[47], §4.10] is appropriate for the function $\widetilde{u}(\widetilde{x}, t) = u(x,t)$ and applying this result separately to the subregions G^- and G^+, we can obtain the required estimates. ■

Theorem 6.2.4. *For all non-negative integers* l, m, *satisfying* $0 \leq l + 2m \leq 4$, *the smooth component* v *defined in (6.4) satisfies the bounds*

$$\left\| \frac{\partial^{l+m} v}{\partial x^l \partial t^m} \right\|_{\infty, G^- \cup G^+} \leq C,$$

and the layer components w^1 *and* w^2 *defined in (6.9) and (6.10), respectively, satisfy the bounds*

$$\left| \frac{\partial^{l+m} w^1}{\partial x^l \partial t^m}(x,t) \right| \leq C\Big(\varepsilon^{-l/2} \exp\big(-x/\sqrt{\varepsilon}\big)\Big), \quad (x,t) \in G^-,$$

$$\left| \frac{\partial^{l+m} w^2}{\partial x^l \partial t^m}(x,t) \right| \leq \begin{cases} C\Big(\varepsilon^{-l/2} \exp\big(-(\xi - x)/\sqrt{\varepsilon}\big)\Big), & (x,t) \in G^-, \\ C\Big(\varepsilon^{-l} \exp\big(-(x - \xi)\alpha/\varepsilon\big)\Big), & (x,t) \in G^+. \end{cases}$$

Proof. First, the analysis is carried out to obtain the stronger bounds on the smooth component v, defined in (6.4) and its derivatives. Since the functions v_i^- and v_i^+ are solutions to the problems (6.5) and (6.6) respectively, which are independent of the parameter ε, so they have the following ε-uniformly bounded derivatives

$$\left\| \frac{\partial^{l+m} v_i^{\pm}}{\partial x^l \partial t^m} \right\|_{\infty, G^{\pm}} \leq C, \quad i = 0, 1, 2, 3, \quad \text{for } 0 \leq l + 2m \leq 4. \tag{6.12}$$

Further, for $0 \leq l + 2m \leq 4$, applying the estimates given in (6.11) analogously to the remainder $\mathfrak{R}$, defined in (6.7), we obtain that

$$\left\| \varepsilon^4 \frac{\partial^{l+m}\mathfrak{R}}{\partial x^l \partial t^m} \right\|_{\infty, G^-} \leq C\varepsilon^{4-l/2} \leq C, \tag{6.13}$$

and

$$\left\| \varepsilon^4 \frac{\partial^{l+m}\mathfrak{R}}{\partial x^l \partial t^m} \right\|_{\infty, G^+} \leq C\varepsilon^{4-l} \leq C. \tag{6.14}$$

Henceforth, the required bounds on the smooth component v and its derivatives follow from (6.4) and the estimates (6.12)-(6.14).

Next, the required bounds on the layer component w^1, defined in (6.9) and its derivatives can be obtained similarly as given in [[55], *Theorem 4*]. Now, we shall proceed to find out the required bounds on the layer component w^2, defined in (6.10) and its derivatives on the domain $G^- \cup G^+$. To obtain the bound on w^2, define the following functions

$$\phi^{\pm}(x,t) = \begin{cases} \Big(\pm v(x,t) - \mathfrak{M}\exp(2t) \Big) \exp\big(-(\xi - x)/\sqrt{\varepsilon} \big) \pm w^2(x,t), & (x,t) \in G^-, \\ \Big(\pm v(x,t) - \mathfrak{M}\exp(2t) \Big) \exp\big(-(x - \xi)\alpha/\varepsilon \big) \pm w^2(x,t), & (x,t) \in G^+, \end{cases} \tag{6.15}$$

where $\mathfrak{M}$ (a positive constant independent of ε) is chosen to be sufficiently large. Then, clearly

$$\phi^{\pm} \in \mathcal{C}^0(\overline{G}), \quad \phi^{\pm}(x,t) \leq 0, \; (x,t) \in \Gamma.$$

Now, from (6.10), (6.15) and employing the bounds on v and w^1, we have for $t > 0$

$$[\phi^{\pm}](\xi,t) \; = \; \pm[w^2](\xi,t) \pm [v](\xi,t) \; = \; 0,$$

and

$$\begin{aligned} \left[\frac{\partial \phi^{\pm}}{\partial x}\right](\xi,t) &= \pm\left[\frac{\partial w^2}{\partial x}\right](\xi,t) \pm \left[\frac{\partial v}{\partial x}\right](\xi,t) \\ &\quad - \left[\frac{\alpha}{\varepsilon}\Big(\pm v(\xi^+,t) - \mathfrak{M}\exp(2t)\Big) + \frac{1}{\sqrt{\varepsilon}}\Big(\pm v(\xi^-,t) - \mathfrak{M}\exp(2t)\Big)\right] \\ &= \pm\frac{\partial w^1}{\partial x}(\xi,t) + \left[\frac{\alpha}{\varepsilon}\Big(\mathfrak{M}\exp(2t) \mp v(\xi^+,t)\Big) + \frac{1}{\sqrt{\varepsilon}}\Big(\mathfrak{M}\exp(2t) \mp v(\xi^-,t)\Big)\right] \\ &\geq 0. \end{aligned}$$

Further, using the bounds on the derivatives of v, we obtain for $(x,t) \in G^-$,

$$\begin{aligned} L_{1,\varepsilon}\phi^{\pm}(x,t) &= \left[-\Big(\mathfrak{M}\exp(2t) \pm v(x,t)\Big) \pm \left(f - 2\sqrt{\varepsilon}\frac{\partial v}{\partial x}\right) - \mathfrak{M}b\exp(2t)\right] \\ &\quad \times \exp\big(-(\xi - x)/\sqrt{\varepsilon} \big) \\ &\leq 0, \end{aligned}$$

and for $(x,t) \in G^+$,

$$\begin{aligned} L_{2,\varepsilon}\phi^{\pm}(x,t) &= \frac{\alpha}{\varepsilon}\Big[(a-\alpha)\Big(\pm v(x,t) - \mathfrak{M}\exp(2t)\Big) \pm \varepsilon\Big(2\frac{\partial v}{\partial x} + \frac{f}{\alpha}\Big) - \mathfrak{M}\frac{\varepsilon b}{\alpha}\exp(2t)\Big] \\ &\quad \times \exp\big(-(x-\xi)\alpha/\varepsilon\big) \\ &\leq 0. \end{aligned}$$

Then, it follows from the maximum principle (Lemma 6.2.1) that $\big|\phi^{\pm}(x,t)\big| \leq 0,\ (x,t) \in \overline{G}$, which leads to the desired bound on w^2. The bounds on the derivatives of w^2 follow from the arguments given in [55] and hence, this completes the proof. ■

Remark 6.2.5. *The bounds obtained in Theorem 6.2.4 for the layer component w (defined in (6.8)) on G^+, are more sharper than that obtained by Brayanov [5]. This in fact preserve the same asymptotic behavior as that of the layer component of the exact solution to the stationary one-dimensional singularly perturbed convection diffusion problem, studied in [[54], Chapter 8].*

6.3 Numerical Approximation

This section provides a suitable mesh for the discretization of the domain to obtain an ε-uniformly convergent difference scheme and also explicitly describes the difference scheme used to discretize the problem (6.1)-(6.3).

6.3.1 Discretization of the domain

Consider the domain $\overline{G} = \overline{\Omega}\times[0,T] = [0,1]\times[0,T]$ and let $N \geq 16$ be an even positive integer. Here, we will construct a rectangular mesh $\overline{G}_{\varepsilon}^{N,M} = \overline{\Omega}_x^{N,\varepsilon} \times \mathbb{S}_t^M$, which is a combination of the piecewise-uniform Shishkin mesh condensed near to the boundary and the interior layers for the spatial variable and a uniform mesh for the temporal variable. Firstly, to define the piecewise-uniform Shishkin mesh, the spatial domain $\overline{\Omega}$ is subdivided into five subintervals as

$$\overline{\Omega} = [0,\sigma_1] \cup [\sigma_1, \xi-\sigma_1] \cup [\xi-\sigma_1, \xi] \cup [\xi, \xi+\sigma_2] \cup [\xi+\sigma_2, 1]$$

for some σ_1, σ_2 that satisfy $0 < \sigma_1 \leq \xi/4$, $0 < \sigma_2 \leq (1-\xi)/2$. On the subintervals $[0,\sigma_1], [\xi-\sigma_1,\xi]$ a uniform mesh with $N/8$ mesh-intervals is placed and a uniform mesh with $N/4$ mesh-intervals is placed on the subintervals $[\sigma_1, \xi-\sigma_1], [\xi, \xi+\sigma_2], [\xi+\sigma_2, 1]$ such that

$$\Omega_x^{N,\varepsilon} = \{x_i : 1 \leq i \leq N/2-1\} \bigcup \{x_i : N/2+1 \leq i \leq N-1\}$$

denotes the set of interior points of the mesh. Clearly $x_{N/2} = \xi$ and $\overline{\Omega}_x^{N,\varepsilon} = \{x_i\}_0^N$. Note that this mesh is uniform when $\sigma_1 = \xi/4,\ \sigma_2 = (1-\xi)/2$. It is fitted to the problem (6.1)-(6.3) by choosing σ_1 and σ_2 to be the following functions of N and ε

$$\sigma_1 = \min\left\{\frac{\xi}{4}, \tau_1\sqrt{\varepsilon}\ln N\right\}, \quad \sigma_2 = \min\left\{\frac{1-\xi}{2}, \frac{2\tau_2\varepsilon}{\alpha}\ln N\right\},$$

where τ_1, τ_2 are constants will be chosen later on. In the analysis, we shall assume that $\sigma_1 = \tau_1\sqrt{\varepsilon}\ln N, \sigma_2 = 2\tau_2\varepsilon\ln N/\alpha$, otherwise N^{-1} is exponentially small relatively to ε, which is very unlikely in practice (and in this case the method can be analyzed in the classical way). On the time domain $[0,T]$, the equidistant meshes are introduced in the temporal variable such that

$$\mathbb{S}_t^M = \{t_n = n\,\Delta t,\ n = 0,\ldots,M,\ t_0 = 0, t_M = T,\ \Delta t = T/M\},$$

where M denotes the number of mesh-intervals in the t-direction. Let the step sizes in space be denoted by

$$h_i = x_i - x_{i-1},\ i = 1,\ldots,N, \qquad \widehat{h}_i = h_i + h_{i+1},\ i = 1,\ldots,N-1.$$

Further, denote the mesh width h_i in the spatial direction as follows:

$$h_i = \begin{cases} h_{(l)} = 8\sigma_1/N, & \text{for } i = 1,\ldots,N/8, 3N/8+1,\ldots,N/2, \\ H_{(l)} = 4(\xi - 2\sigma_1)/N, & \text{for } i = N/8+1,\ldots,3N/8, \\ h_{(r)} = 4\sigma_2/N, & \text{for } i = N/2+1,\ldots,3N/4, \\ H_{(r)} = 4(1-\xi-\sigma_2)/N, & \text{for } i = 3N/4+1,\ldots,N. \end{cases}$$

6.3.2 The backward-Euler hybrid finite difference scheme

On the domain G^-, we discretize the problem (6.1)-(6.3) in space by the classical central difference scheme and for the time derivative, we employ the backward-Euler method. Whereas we propose a hybrid scheme on the domain G^+ for the spatial discretization. The hybrid scheme consists of the midpoint upwind scheme in the outer region $[\xi+\sigma_2, 1]$ and the classical central difference scheme in the interior layer region $(\xi, \xi+\sigma_2)$. At the point of discontinuity, a second-order one-sided difference approximations are used to keep the continuity of the spatial derivative. Then, the proposed numerical scheme takes the following form:

$$\begin{cases} L_{1,cen}^{N,M} U_i^{n+1} = f_i^{n+1}, & \text{for } i = 1,\ldots,N/2-1, \\ L_{2,cen}^{N,M} U_i^{n+1} = f_i^{n+1}, & \text{for } i = N/2+1,\ldots,3N/4-1, \\ L_{2,mu}^{N,M} U_i^{n+1} = f_{i+1/2}^{n+1}, & \text{for } i = 3N/4,\ldots,N-1, \\ D_x^F U_i^{n+1} - D_x^B U_i^{n+1} = 0, & \text{for } i = N/2, \end{cases} \tag{6.16}$$

where

$$\begin{cases} L_{1,cen}^{N,M} U_i^{n+1} &= D_t^- U_i^{n+1} - \varepsilon \delta_x^2 U_i^{n+1} + b_i^{n+1} U_i^{n+1}, \\ L_{2,cen}^{N,M} U_i^{n+1} &= -\varepsilon \delta_x^2 U_i^{n+1} - a_i^{n+1} D_x^0 U_i^{n+1} + b_i^{n+1} U_i^{n+1}, \\ L_{2,mu}^{N,M} U_i^{n+1} &= -\varepsilon \delta_x^2 U_i^{n+1} - a_{i+1/2}^{n+1} D_x^+ U_i^{n+1} + b_{i+1/2}^{n+1} U_{i+1/2}^{n+1}, \end{cases} \tag{6.17}$$

and

$$D_x^F U_i^n = (-U_{i+2}^n + 4U_{i+1}^n - 3U_i^n)/2h_{(r)}, \quad D_x^B U_i^n = (U_{i-2}^n - 4U_{i-1}^n + 3U_i^n)/2h_{(l)}. \tag{6.18}$$

After rearranging the terms in (6.16), we obtain the following form of the difference scheme on the mesh $\overline{G}_\varepsilon^{N,M}$:

$$\begin{cases} U_i^0 = s_0(x_i), \quad \text{for } i = 0, \ldots, N, \\ \begin{cases} L_\varepsilon^{N,M} U_i^{n+1} = \widetilde{f}_i^{n+1}, \quad \text{for } i = 1, \ldots, N-1, \\ U_0^{n+1} = s_1(t_{n+1}), \quad U_N^{n+1} = s_2(t_{n+1}), \end{cases} \\ \text{for } n = 0, \ldots, M-1, \end{cases} \tag{6.19}$$

where the finite difference operator $L_\varepsilon^{N,M}$ is defined as

$$L_\varepsilon^{N,M} U_i^{n+1} = \begin{cases} \left[r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1}\right] + \left[p_i^- U_{i-1}^n + p_i^0 U_i^n + p_i^+ U_{i+1}^n\right], \\ \qquad \text{for } i = 1, \ldots, N/2-1, N/2+1, \ldots, N-1, \\ q_i^{-,2} U_{i-2}^{n+1} + q_i^{-,1} U_{i-1}^{n+1} + q_i^0 U_i^{n+1} + q_i^{+,1} U_{i+1}^{n+1} + q_i^{+,2} U_{i+2}^{n+1}, \\ \qquad \text{for } i = N/2, \end{cases} \tag{6.20}$$

and the right hand side vector $\widetilde{f}^{n+1}$ as

$$\widetilde{f}_i^{n+1} = \begin{cases} \left[m_i^- f_{i-1}^{n+1} + m_i^0 f_i^{n+1} + m_i^+ f_{i+1}^{n+1}\right], \quad \text{for } i = 1, \ldots, N/2-1, \\ \qquad\qquad N/2+1, \ldots, N-1, \\ 0, \quad \text{for } i = N/2. \end{cases} \tag{6.21}$$

Here, for $i = 1, \ldots, N/2-1$, the coefficients are given by

$$\begin{cases} r_i^- = -\dfrac{2\varepsilon}{\widehat{h}_i h_i}, \\ r_i^0 = \left(\dfrac{2\varepsilon}{h_i h_{i+1}} + b_i^{n+1} + \dfrac{1}{\Delta t}\right), \\ r_i^+ = -\dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}}, \\ p_i^- = 0, \quad p_i^0 = -\dfrac{1}{\Delta t}, \quad p_i^+ = 0, \\ m_i^- = 0, \quad m_i^0 = 1, \quad m_i^+ = 0, \end{cases} \tag{6.22}$$

and for $i = N/2+1, \ldots, 3N/4-1$,

$$\begin{cases} r_i^- = -\left(\dfrac{2\varepsilon}{\widehat{h}_i h_i} - \dfrac{a_i^{n+1}}{\widehat{h}_i}\right), \\ r_i^0 = \left(\dfrac{2\varepsilon}{h_i h_{i+1}} + b_i^{n+1}\right), \\ r_i^+ = -\left(\dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}} + \dfrac{a_i^{n+1}}{\widehat{h}_i}\right), \\ p_i^- = 0, \quad p_i^0 = 0, \quad p_i^+ = 0, \\ m_i^- = 0, \quad m_i^0 = 1, \quad m_i^+ = 0, \end{cases} \tag{6.23}$$

and for $i = 3N/4, \ldots, N-1$,

$$\begin{cases} r_i^- = -\dfrac{2\varepsilon}{\widehat{h}_i h_i}, \\ r_i^0 = \left(\dfrac{2\varepsilon}{h_i h_{i+1}} + \dfrac{a_{i+1/2}^{n+1}}{h_{i+1}} + \dfrac{b_{i+1/2}^{n+1}}{2}\right), \\ r_i^+ = \left(-\dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}} - \dfrac{a_{i+1/2}^{n+1}}{h_{i+1}} + \dfrac{b_{i+1/2}^{n+1}}{2}\right), \\ p_i^- = 0, \quad p_i^0 = 0, \quad p_i^+ = 0, \\ m_i^- = 0, \quad m_i^0 = \dfrac{1}{2}, \quad m_i^+ = \dfrac{1}{2}, \end{cases} \tag{6.24}$$

and lastly, for $i = N/2$,

$$q_i^{-,2} = -\frac{1}{2h_{(l)}}, \quad q_i^{-,1} = \frac{2}{h_{(l)}}, \quad q_i^0 = -\frac{3}{2}\left(\frac{1}{h_{(r)}} + \frac{1}{h_{(l)}}\right), \quad q_i^{+,1} = \frac{2}{h_{(r)}}, \quad q_i^{+,2} = -\frac{1}{2h_{(r)}}. \tag{6.25}$$

6.4 Error Analysis

This section studies the stability of the proposed numerical scheme (6.19) and finally, the ε-uniform error estimate is obatined.

We see that the finite difference operator $L_\varepsilon^{N,M}$ defined in (6.20) does not satisfy the discrete maximum principle and accordingly, the present form (6.19) can not help us to find out the uniform stability criteria for the proposed numerical scheme (6.16). Due to this reason we need to transform the equation (6.19) for $i = N/2$, i.e.,

$$q_{N/2}^{-,2} U_{N/2-2}^{n+1} + q_{N/2}^{-,1} U_{N/2-1}^{n+1} + q_{N/2}^0 U_{N/2}^{n+1} + q_{N/2}^{+,1} U_{N/2+1}^{n+1} + q_{N/2}^{+,2} U_{N/2+2}^{n+1} = 0, \tag{6.26}$$

so that the operator associated with the new system of equations will be able to reflect the stability property of the proposed scheme (6.16). Now, from (6.19) we have

$$\begin{cases} U_{N/2-2}^{n+1} = \dfrac{h_l^2}{\varepsilon}\left[-f_{N/2-1}^{n+1} + r_{N/2-1}^0 U_{N/2-1}^{n+1} + r_{N/2-1}^+ U_{N/2}^{n+1} - \dfrac{1}{\Delta t} U_{N/2-1}^n\right], \\ U_{N/2+2}^{n+1} = \dfrac{2h_r^2}{2\varepsilon + h_r a_{N/2+1}}\left[-f_{N/2+1}^{n+1} + r_{N/2+1}^0 U_{N/2+1}^{n+1} + r_{N/2+1}^- U_{N/2}^n\right]. \end{cases} \tag{6.27}$$

Therefore, after substituting $U_{N/2-2}^{n+1}$ and $U_{N/2+2}^{n+1}$ from (6.26) using (6.27), the difference scheme (6.19) gets change into the following form on the mesh $\overline{G}_\varepsilon^{N,M}$:

$$\begin{cases} U_i^0 = s_0(x_i), \quad \text{for } i = 0, \ldots, N, \\ \begin{cases} L_H^{N,M} U_i^{n+1} = \widetilde{f_{H,i}}^{n+1}, \quad \text{for } i = 1, \ldots, N-1, \\ U_0^{n+1} = s_1(t_{n+1}), \quad U_N^{n+1} = s_2(t_{n+1}), \end{cases} \\ \text{for } n = 0, \ldots, M-1. \end{cases} \tag{6.28}$$

Here, the finite difference operator $L_H^{N,M}$ and the right hand side term $\widetilde{f_{H,i}}^{n+1}$ are respectively defined as

$$\begin{cases} L_H^{N,M} U_i^{n+1} = \left[r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1}\right] + \left[p_i^- U_{i-1}^n + p_i^0 U_i^n + p_i^+ U_{i+1}^n\right], \quad \text{and} \\ \widetilde{f_{H,i}}^{n+1} = \left[m_i^- f_{i-1}^{n+1} + m_i^0 f_i^{n+1} + m_i^+ f_{i+1}^{n+1}\right], \end{cases} \tag{6.29}$$

where for $i = 1, \ldots, N/2-1, N/2+1, \ldots, N-1$, the coefficients $r_i^-, r_i^0, r_i^+; p_i^-, p_i^0, p_i^+; m_i^-, m_i^0, m_i^+$ are described in (6.22)-(6.24) and for $i = N/2$,

$$\begin{cases} r_i^- = -\dfrac{1}{2h_l}\left[4 - \dfrac{2\left(2\varepsilon + h_l^2 b_{i-1}^{n+1} + \frac{h_l^2}{\Delta t}\right)}{2\varepsilon}\right], \\ r_i^0 = \left[\dfrac{1}{2h_r}\left(3 - \dfrac{2\varepsilon - h_r a_{i+1}}{2\varepsilon + h_r a_{i+1}}\right) + \dfrac{1}{h_l}\right], \\ r_i^+ = -\dfrac{1}{2h_r}\left[4 - \dfrac{2(2\varepsilon + h_r^2 b_{i+1}^{n+1})}{2\varepsilon + h_r a_{i+1}^{n+1}}\right], \\ p_i^- = -\dfrac{h_l}{2\varepsilon\Delta t}, \quad p_i^0 = 0, \quad p_i^+ = 0, \\ m_i^- = \dfrac{h_l}{2\varepsilon}, \quad m_i^0 = 0, \quad m_i^+ = \dfrac{h_r}{\left(2\varepsilon + h_r a_{i+1}^{n+1}\right)}. \end{cases} \tag{6.30}$$

Let $G_\varepsilon^{N,M} = \overline{G}_\varepsilon^{N,M} \bigcap G$ and $\Gamma_\varepsilon^{N,M} = \overline{G}_\varepsilon^{N,M} \setminus G_\varepsilon^{N,M}$.

Lemma 6.4.1. *Assume that $N \geq N_0$, where*

$$\frac{N_0}{\ln N_0} \geq 4\tau_2 \frac{\alpha^*}{\alpha}, \tag{6.31}$$

$$\frac{\alpha N_0}{2} \geq \|b\|_\infty \quad \textit{and} \quad \left(\|b\|_\infty + \Delta t^{-1}\right) \leq \frac{2\Bbbk N_0^2}{\ln^2 N_0}, \tag{6.32}$$

where $\Bbbk = \left(1/8\tau_1\right)^2$. *Then, the operator* $L_H^{N,M}$ *defined by (6.29) satisfies a* ***discrete minimum principle****, i.e., if the mesh function* Z *satisfies* $Z \geq 0$ *on* $\Gamma_\varepsilon^{N,M}$*, then* $L_H^{N,M} Z \geq 0$ *in* $G_\varepsilon^{N,M}$ *implies that* $Z \geq 0$ *at each point of* $\overline{G}_\varepsilon^{N,M}$*.*

Proof. On $G_\varepsilon^{N,M}$, set

$$L_H^{N,M} Z_i^{n+1} = \left[A_{i,i-1} Z_{i-1}^{n+1} + A_{i,i} Z_i^{n+1} + A_{i,i+1} Z_{i+1}^{n+1}\right] - \left[B_{i,i-1} Z_{i-1}^{n} + B_{i,i} Z_i^{n} + B_{i,i+1} Z_{i+1}^{n}\right],$$

where the matrices $\mathbf{A} := (A_{i,j})$ and $\mathbf{B} := (B_{i,j})$ are respectively defined by

$$\begin{cases} A_{i,i-1} = r_i^-, \; A_{i,i} = r_i^0, \; A_{i,i+1} = r_i^+, \quad \text{and} \\ B_{i,i-1} = -p_i^-, \; B_{i,i} = -p_i^0, \; B_{i,i+1} = -p_i^+. \end{cases}$$

Under the assumptions (6.31) and (6.32), the matrix $\mathbf{A}$ is an M-matrix. Also, the matrix $\mathbf{B} \geq 0$. Therefore, the proof follows from Lemma 3.8 of the book of Roos *et al*. [77]. ■

An immediate consequence of the discrete minimum principle is the following stability results.

Lemma 6.4.2. *Let* Z *be any mesh function such that* $Z = 0$ *on* $\Gamma_\varepsilon^{N,M}$ *and let the assumptions (6.31) and (6.32) of Lemma 6.4.1 hold. Then,*

$$\left|Z_i^n\right| \leq \frac{(1+T)}{\mu} \left\|L_H^{N,M} Z\right\|_{\infty, \overline{G}_\varepsilon^{N,M}}, \quad \textit{for } 0 \leq i \leq N,$$

where $\mu = \min\left\{1, \alpha/(1-\xi)\right\}$.

Proof. Consider the following discrete functions

$$\psi_i^{\pm,n} = \begin{cases} \dfrac{(1+t_n)}{\mu} \left\|L_H^{N,M} Z\right\|_\infty \pm Z_i^n, & \text{for } 0 \leq i \leq N/2, \\ \dfrac{(1+t_n)(1-x_i)}{\mu(1-\xi)} \left\|L_H^{N,M} Z\right\|_\infty \pm Z_i^n, & \text{for } N/2 < i \leq N. \end{cases}$$

Then, clearly $\psi_0^{\pm,n+1} \geq 0$, $\psi_N^{\pm,n+1} = 0$, $\psi_i^{\pm,0} \geq 0$ and $L_H^{N,M} \psi_i^{\pm,n+1} \geq 0$, for $i = 1, \ldots, N/2-1, N/2+1, \ldots, N-1$. Further,

$$\begin{aligned} L_H^{N,M} \psi_{N/2}^{\pm,n+1} \;\; &\geq \;\; -\left(D_x^F - D_x^B\right) \psi_{N/2}^{\pm,n+1} + m_i^- \left[\left\|L_H^{N,M} Z\right\|_\infty \pm L_H^{N,M} Z_{N/2-1}^{n+1}\right] \\ &\quad + m_i^+ \left[\left\|L_H^{N,M} Z\right\|_\infty \pm L_H^{N,M} Z_{N/2+1}^{n+1}\right] \\ &\geq \;\; -\left(D_x^F - D_x^B\right) \psi_{N/2}^{\pm,n+1} \pm \left[m_i^- L_H^{N,M} Z_{N/2-1}^{n+1} + m_i^+ L_H^{N,M} Z_{N/2+1}^{n+1}\right] \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{(1+t_{n+1})}{\mu(1-\xi)}\left\|L_H^{N,M}Z\right\|_\infty \\
&\quad \pm\Big[-(D_x^F-D_x^B)Z_{N/2}^{n+1}+m_i^-L_H^{N,M}Z_{N/2-1}^{n+1}+m_i^+L_H^{N,M}Z_{N/2+1}^{n+1}\Big] \\
&\geq \left\|L_H^{N,M}Z\right\|_\infty \pm L_H^{N,M}Z_{N/2}^{n+1}\geq 0.
\end{aligned}
$$

Therefore, applying Lemma 6.4.1 we obtain the required stability bound. ■

Lemma 6.4.3. *Let U be the solution of (6.28) and the assumptions (6.31) and (6.32) of Lemma 6.4.1 hold. Then,*

$$\left\|U\right\|_{\infty,\overline{G}_\varepsilon^{N,M}} \leq \left\|U\right\|_{\infty,\Gamma_\varepsilon^{N,M}}+\frac{(1+T)}{\mu}\left\|\widetilde{f_H}\right\|_{\infty,\overline{G}_\varepsilon^{N,M}},$$

where $\mu=\min\left\{1,\alpha/(1-\xi)\right\}$.

Proof. Consider the following discrete functions

$$\varphi_i^{\pm,n}=\left\|U\right\|_{\infty,\Gamma_\varepsilon^{N,M}}+\begin{cases}\dfrac{(1+t_n)}{\mu}\left\|\widetilde{f_H}\right\|_\infty \pm U_i^n, & \text{for } 0\leq i\leq N/2,\\ \dfrac{(1+t_n)(1-x_i)}{\mu(1-\xi)}\left\|\widetilde{f_H}\right\|_\infty \pm U_i^n, & \text{for } N/2<i\leq N.\end{cases}$$

Then, clearly $\varphi_0^{\pm,n+1}\geq 0,\ \varphi_N^{\pm,n+1}\geq 0,\ \varphi_i^{\pm,0}\geq 0$ and $L_H^{N,M}\varphi_i^{\pm,n+1}\geq 0,\quad$ for $i=1,\ldots,N/2-1,N/2+1,\ldots,N-1$. Further,

$$L_H^{N,M}\varphi_{N/2}^{\pm,n+1}\geq -(D_x^F-D_x^B)\varphi_{N/2}^{\pm,n+1}\geq \frac{(1+t_{n+1})}{\mu(1-\xi)}\left\|\widetilde{f_H}\right\|_\infty\geq 0,$$

and henceforth, the required bound follows from Lemma 6.4.1. ■

To estimate the nodal error $\left|U_i^{n+1}-u(x_i,t_{n+1})\right|$ separately outside and inside the layers, the discrete solution U is decomposed by applying the following technique. First, define the mesh functions V_L and V_R, which approximate v respectively to the left and to the right of the point of discontinuity $x=\xi$. Then, the mesh function W^1 is constructed to approximate w^1 and also, the mesh functions W_L^2 and W_R^2 are constructed to approximate w^2 respectively to the left and to the right of the point of discontinuity $x=\xi$. Thus, $W_L=W^1+W_L^2$ and $W_R=W_R^2$ approximate w on either side of $x=\xi$ so that the amplitude of the jump $W_R(\xi)-W_L(\xi)$ is determined by the size of the jump $\left|[v](\xi)\right|$.

Here, the mesh functions V_L and V_R are defined to be the solutions of the following discrete problems

$$\begin{cases} L_H^{N,M}V_{L,i}^{n+1}=\widetilde{f}_{H,i}^{\,n+1}, & \text{for } i=1,\ldots,N/2-1,\\ V_{L,0}^{n+1}=v(0,t_{n+1}),\quad V_{L,N/2}^{n+1}=v(\xi^-,t_{n+1}), & n\geq 0,\\ V_{L,i}^0=v(x_i,0),\quad i\leq N/2, \end{cases}$$

and

$$\begin{cases} L_H^{N,M} V_{R,i}^{n+1} = \widetilde{f_{H,i}}^{n+1}, \quad \text{for } i = N/2+1, \ldots, N-1, \\ V_{R,N/2}^{n+1} = v(\xi^+, t_{n+1}), \quad V_{R,N}^{n+1} = v(1, t_{n+1}), \quad n \geq 0, \\ V_{R,i}^{0} = v(x_i, 0), \quad i \geq N/2. \end{cases}$$

Also, define the mesh function W^1 to be the solution of the following discrete problem

$$\begin{cases} L_H^{N,M} W_i^{1,n+1} = 0, \quad \text{for } i = 1, \ldots, N/2-1, \\ W_0^{1,n+1} = w^1(0, t_{n+1}), \quad W_{N/2}^{1,n+1} = w^1(\xi, t_{n+1}), \quad n \geq 0, \\ W_i^{1,0} = w^1(x_i, 0), \quad i \leq N/2, \end{cases}$$

and the mesh functions W_L^2 and W_R^2 are defined to be the solutions of the following system of finite difference equations

$$\begin{cases} L_H^{N,M} W_{L,i}^{2,n+1} = 0, \quad \text{for } i = 1, \ldots, N/2-1, \\ L_H^{N,M} W_{R,i}^{2,n+1} = 0, \quad \text{for } i = N/2+1, \ldots, N-1, \\ W_{L,0}^{2,n+1} = 0, \quad W_{R,N}^{2,n+1} = 0, \\ W_{L,i}^{2,0} = 0, \quad i \leq N/2, \quad W_{R,i}^{2,0} = 0, \quad i \geq N/2, \\ W_{R,N/2}^{2,n+1} + V_{R,N/2}^{2,n+1} = W_{L,N/2}^{2,n+1} + V_{L,N/2}^{2,n+1}, \\ D_x^F W_{R,N/2}^{2,n+1} + D_x^F V_{R,N/2}^{2,n+1} = D_x^B W_{L,N/2}^{2,n+1} + D_x^B V_{L,N/2}^{2,n+1} + D_x^B W_{N/2}^{1,n+1}, \quad n \geq 0. \end{cases}$$

So, U is defined to be satisfied by the following decomposition

$$U_i^{n+1} = \begin{cases} V_{L,i}^{n+1} + W_{L,i}^{n+1}, & \text{for } i = 1, \ldots, N/2-1, \\ V_{L,i}^{n+1} + W_{L,i}^{n+1} = V_{R,i}^{n+1} + W_{R,i}^{n+1}, & \text{for } i = N/2, \\ V_{R,i}^{n+1} + W_{R,i}^{n+1}, & \text{for } i = N/2+1, \ldots, N-1. \end{cases}$$

First, the errors are analyzed separately for the smooth components V_L and V_R of the solution U in the region $G^- \cup G^+$.

Lemma 6.4.4. *Assume that $\varepsilon \leq N^{-1}$. Then under the assumptions (6.31) and (6.32) of Lemma 6.4.1, the errors associated to the smooth components satisfy the following estimates*

$$\begin{cases} \left| V_{L,i}^{n+1} - v(x_i, t_{n+1}) \right| \leq C\Big(N^{-2} + \Delta t\Big), & \textit{for } 1 \leq i \leq N/2-1, \\ \left| V_{R,i}^{n+1} - v(x_i, t_{n+1}) \right| \leq CN^{-2}, & \textit{for } N/2+1 \leq i \leq N-1. \end{cases}$$

Proof. The truncation error corresponding to the smooth component on the left side of the discontinuity satisfies the following estimate

$$\left|L_H^{N,M}\big(V_{L,i}^{n+1}-v(x_i,t_{n+1})\big)\right| \le \begin{cases} \left[\dfrac{\varepsilon}{3}\big(h_i+h_{i+1}\big)\left\|\dfrac{\partial^3 v}{\partial x^3}\right\|_\infty + \dfrac{\Delta t}{2}\left\|\dfrac{\partial^2 v}{\partial t^2}\right\|_\infty\right], \\ \qquad\qquad\qquad\qquad \text{for } i = N/8, 3N/8, \\ \left[\dfrac{\varepsilon}{12}h_i^2\left\|\dfrac{\partial^4 v}{\partial x^4}\right\|_\infty + \dfrac{\Delta t}{2}\left\|\dfrac{\partial^2 v}{\partial t^2}\right\|_\infty\right], \\ \qquad\qquad\qquad\qquad \text{for } i \neq N/8, 3N/8. \end{cases}$$

Then, using $h_i \le 4N^{-1}$ and the bounds on the derivatives of v given in Theorem 6.2.4, we have

$$\left|L_H^{N,M}\big(V_{L,i}^{n+1}-v(x_i,t_{n+1})\big)\right| \le \begin{cases} C\Big(\varepsilon N^{-1}+\Delta t\Big), & \text{for } i = N/8, 3N/8, \\ C\Big(N^{-2}+\Delta t\Big), & \text{for } i \neq N/8, 3N/8. \end{cases}$$

Define the following barrier function

$$\Psi_{L,i}^n = C\Big[2\xi\sigma_1 N^{-2}\rho(x_i) + t_n\big(N^{-2}+\Delta t\big)\Big], \quad \text{for } 0 \le i \le N/2,$$

where

$$\rho(x_i) = \begin{cases} \dfrac{x_i}{\sigma_1}, & \text{for } 0 \le i \le N/8, \\ 1, & \text{for } N/8 \le i \le 3N/8, \\ \dfrac{\xi - x_i}{\sigma_1}, & \text{for } 3N/8 \le i \le N/2. \end{cases}$$

Then, we have

$$\begin{aligned} L_H^{N,M}\Psi_{L,i}^{n+1} &\ge \begin{cases} C\Big(\varepsilon N^{-1}+N^{-2}+\Delta t\Big), & \text{for } i = N/8, 3N/8, \\ C\Big(N^{-2}+\Delta t\Big), & \text{for } i \neq N/8, 3N/8. \end{cases} \\ &\ge \left|L_H^{N,M}\big(V_{L,i}^{n+1}-v(x_i,t_{n+1})\big)\right|. \end{aligned}$$

Thus, applying the discrete minimum principle to $\Psi_{L,i}^n \pm \big(V_{L,i}^n - v(x_i,t_n)\big)$ over $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^-}$, we have

$$\left|V_{L,i}^{n+1} - v(x_i,t_{n+1})\right| \le \Psi_{L,i}^{n+1} \le C\Big(N^{-2}+\Delta t\Big), \quad \text{for } 1 \le i \le N/2-1.$$

On the other hand, introduce a new barrier function

$$\Psi_{R,i} = C\Big(N^{-2}+\Delta t\Big)(1-x_i), \quad \text{for } N/2 \le i \le N.$$

Now, the truncation error corresponding to the smooth component on the right hand side of the discontinuity satisfies the following estimate

$$\left|L_H^{N,M}\big(V_{R,i}^{n+1} - v(x_i,t_{n+1})\big)\right|$$

$$\le \begin{cases} C\Big[h_i^2\Big(\varepsilon\Big\|\dfrac{\partial^4 v}{\partial x^4}\Big\|_\infty + \Big\|\dfrac{\partial^3 v}{\partial x^3}\Big\|_\infty\Big)\Big], & \text{for } N/2+1 \le i \le 3N/4-1, \\ C\Big[\big(\varepsilon + h_{i+1}\big)\big(h_i + h_{i+1}\big)\Big\|\dfrac{\partial^3 v}{\partial x^3}\Big\|_\infty + h_{i+1}^2\Big(\Big\|\dfrac{\partial^2 v}{\partial x^2}\Big\|_\infty + \Big\|\dfrac{\partial v}{\partial x}\Big\|_\infty\Big)\Big], & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 3N/4 \le i \le N-1. \end{cases}$$

Then, using the assumption $\varepsilon \le N^{-1}$ and arguing similarly as $(V_L - v)$, it is easy to show that

$$\left|L_H^{N,M}\big(V_{R,i}^{n+1} - v(x_i,t_{n+1})\big)\right| \le C\Big(N^{-2} + \Delta t\Big) \le L_H^{N,M}\Psi_{R,i}, \quad \text{for } N/2+1 \le i \le N-1.$$

Therefore, we apply the discrete minimum principle to $\Psi_{R,i} \pm \big(V_{R,i}^n - v(x_i,t_n)\big)$ over $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^+}$, which leads to the following error bound

$$\left|V_{R,i}^{n+1} - v(x_i,t_{n+1})\right| \le \Psi_{R,i} \le C\Big(N^{-2} + \Delta t\Big), \quad \text{for } N/2+1 \le i \le N-1.$$

Hence the proof. ■

Lemma 6.4.5. *For $i = N/2, \ldots, N$, define the mesh function*

$$Q_i = \prod_{j=1}^{N-i}\Big(1 + \frac{\alpha\hbar_j}{2\varepsilon}\Big),$$

(with the usual convention that if $i = N$, then $Q_N = 1$) where

$$\hbar_j = \begin{cases} H_{(r)}, & \text{for } 1 \le j \le N/4, \\ h_{(r)}, & \text{for } N/4+1 \le j \le N/2. \end{cases}$$

Then for some constant C, we have

$$L_H^{N,M} Q_i \ge \begin{cases} \dfrac{C}{2\varepsilon + \alpha h_{(r)}} Q_i, & \text{for } N/2+1 \le i \le 3N/4-1, \\ \dfrac{C}{2\varepsilon + \alpha\hbar_{N-i}} Q_i, & \text{for } 3N/4 \le i \le N-1. \end{cases}$$

Proof. Set $\gamma = \alpha/2$. Since $(Q_{i+1} - Q_i) = -\dfrac{\gamma\hbar_{N-i}}{\varepsilon} Q_{i+1}$ and $a_i^{n+1} > \alpha$, it follows that

$$\begin{aligned} L_H^{N,M} Q_i &= \frac{\gamma}{h_{(r)}}(Q_{i+1} - Q_i) + a_i^{n+1}\Big[\frac{\gamma}{\varepsilon} + \frac{\gamma^2 h_{(r)}}{2\varepsilon^2}\Big] Q_{i+1} + b_i^{n+1} Q_i \\ &\ge \frac{\gamma}{\varepsilon} Q_{i+1}(a_i^{n+1} - \gamma) + a_i^{n+1}\frac{\gamma^2 h_{(r)}}{2\varepsilon^2} Q_{i+1} \ge \frac{C}{\varepsilon + \gamma h_{(r)}} Q_i, \quad \text{for } N/2+1 \le i \le 3N/4-1, \end{aligned}$$

and

$$L_H^{N,M} Q_i = \frac{2\gamma}{(h_i + h_{i+1})}(Q_{i+1} - Q_i) + a^{n+1}_{i+1/2}\frac{\gamma}{\varepsilon}Q_{i+1} + b^{n+1}_{i+1/2}Q_{i+1/2}$$

$$\geq \frac{\gamma}{\varepsilon}Q_{i+1}\left[a^{n+1}_{i+1/2} - \frac{2\gamma\hbar_{N-i}}{(h_i + h_{i+1})}\right] \geq \frac{C}{\varepsilon + \gamma\hbar_{N-i}}Q_i, \quad \text{for } 3N/4 \leq i \leq N-1. \qquad \blacksquare$$

Lemma 6.4.6. *Let $\tau_2 \geq 2$, then there exists a constant C such that*

$$\prod_{j=N-i+1}^{N/2}\left(1 + \frac{\alpha\hbar_j}{2\varepsilon}\right)^{-1} \leq CN^{-4(2i/N-1)}, \quad \textit{for } N/2+1 \leq i \leq 3N/4. \tag{6.33}$$

Proof. Here, the arguments as in the proof of [[88], Lemma 3.1] are used. For $N/2+1 \leq i \leq 3N/4$,

$$\begin{aligned}
\prod_{j=N-i+1}^{N/2}\left(1 + \frac{\alpha\hbar_j}{2\varepsilon}\right)^{-1} &= \prod_{j=N-i+1}^{N/2}\left(1 + \frac{\alpha h_{(r)}}{2\varepsilon}\right)^{-1} \leq \exp\big(-\alpha(x_i - \xi)/(2\varepsilon + \alpha h_{(r)})\big) \\
&= N^{-2\tau_2(2i/N-1)/(1+4\tau_2 N^{-1}\ln N)} \\
&= N^{-2\tau_2(2i/N-1)} N^{[8\tau_2^2(2i/N-1)(N^{-1}\ln N)/(1+4\tau_2 N^{-1}\ln N)]} \\
&\leq CN^{-2\tau_2(2i/N-1)},
\end{aligned}$$

because the sequence $N^{[8\tau_2^2(2i/N-1)(N^{-1}\ln N)/(1+4\tau_2 N^{-1}\ln N)]}$ is bounded. Henceforth, the result (6.33) follows from the assumption $\tau_2 \geq 2$. $\blacksquare$

Lemma 6.4.7. *The following inequality hold true:*

$$\exp\big(-(x_i - \xi)\alpha/2\varepsilon\big) \leq \prod_{j=N-i+1}^{N/2}\left(1 + \frac{\alpha\hbar_j}{2\varepsilon}\right)^{-1}, \quad \textit{for } N/2+1 \leq i \leq N-1. \tag{6.34}$$

Proof. Set $\gamma = \alpha/2$. Then inequality (6.34) follows from Lemma 4.4.8. $\blacksquare$

Now, we shall proceed to analyse the errors for the layer components W_L and W_R of the solution U in the region $\big((0, \xi - \sigma_1] \cup [\xi + \sigma_2, 1)\big) \times (0, T]$.

Lemma 6.4.8. *Let $\tau_1, \tau_2 \geq 2$. Then under the assumptions (6.31) and (6.32) of Lemma 6.4.1, the errors associated to the layer components satisfy the following estimates*

$$\left|W^{n+1}_{L,i} - w(x_i, t_{n+1})\right| \leq \begin{cases} C\Big(N^{-2}\ln^2 N + \Delta t\Big), & \textit{for } 1 \leq i \leq N/8 - 1, \\ C\Big(N^{-2} + \Delta t\Big), & \textit{for } N/8 \leq i \leq 3N/8, \end{cases} \tag{6.35}$$

and

$$\left|W^{n+1}_{R,i} - w(x_i, t_{n+1})\right| \leq CN^{-2}, \quad \textit{for } 3N/4 \leq i \leq N-1. \tag{6.36}$$

Proof. For $1 \le i \le N-1$, the error can be written as

$$\big(W_L - w\big) = \big(W^1 - w^1\big) + \big(W_L^2 - w^2\big).$$

First, consider the error $\big(W^1 - w^1\big)$ and it is straightforwad that

$$L_H^{N,M}(W^1 - w^1) = -\varepsilon\bigg(\frac{\partial^2}{\partial x^2} - \delta_x^2\bigg)w^1 + \bigg(\frac{\partial}{\partial t} - D_t^-\bigg)w^1.$$

Now, using the bounds of the derivatives of w^1 from Theorem 6.2.4, we have

$$\bigg|\varepsilon\bigg(\frac{\partial^2}{\partial x^2} - \delta_x^2\bigg)w^1(x_i,t_{n+1})\bigg| \le \begin{cases} \dfrac{\varepsilon}{12}h_i^2\bigg\|\dfrac{\partial^4 w^1}{\partial x^4}\bigg\|_\infty, & \text{for } 1 \le i \le N/8-1,\\ 2\varepsilon \max\limits_{x\in[x_{i-1},x_{i+1}]}\bigg|\dfrac{\partial^2 w^1}{\partial x^2}(x,t_{n+1})\bigg|, & \text{for } N/8 \le i \le N/2-1,\end{cases}$$

$$\le \begin{cases} C\dfrac{h_l^2}{\varepsilon}, & \text{for } 1 \le i \le N/8-1,\\ C\exp\big(-x_{i-1}/\sqrt{\varepsilon}\big), & \text{for } N/8 \le i \le N/2-1.\end{cases}$$

Thus, if $\tau_1 \ge 2$,

$$\bigg|L_H^{N,M}(W_i^{1,n+1} - w^1(x_i,t_{n+1}))\bigg| \le \begin{cases} C\Big(N^{-2}\ln^2 N + \Delta t\Big), & \text{for } 1 \le i \le N/8-1,\\ C\Big(N^{-2} + \Delta t\Big), & \text{for } N/8 \le i \le N/2-1.\end{cases}$$

and then applying Lemma 6.4.2 to the mesh function $\big(W^1 - w^1\big)$ over $\overline{G}_\varepsilon^{N,M}\bigcap\overline{G^-}$ implies that

$$\bigg|W_i^{1,n+1} - w^1(x_i,t_{n+1})\bigg| \le \begin{cases} C\Big(N^{-2}\ln^2 N + \Delta t\Big), & \text{for } 1 \le i \le N/8-1,\\ C\Big(N^{-2} + \Delta t\Big), & \text{for } N/8 \le i \le N/2-1.\end{cases} \tag{6.37}$$

Next, consider the error $\big(W_L^2 - w^2\big)$ and an analogous argument for w^2 leads to following estimates

$$\bigg|\varepsilon\bigg(\frac{\partial^2}{\partial x^2} - \delta_x^2\bigg)w^2(x_i,t_{n+1})\bigg| \le \begin{cases} C\exp\big(-(\xi - x_{i+1})/\sqrt{\varepsilon}\big), & \text{for } 1 \le i \le 3N/8,\\ C\dfrac{h_l^2}{\varepsilon}, & \text{for } 3N/8+1 \le i \le N/2-1,\end{cases}$$

and hence

$$\bigg|L_H^{N,M}(W_{L,i}^{2,n+1} - w^2(x_i,t_{n+1}))\bigg| \le \begin{cases} C\Big(N^{-2} + \Delta t\Big), & \text{for } 1 \le i \le 3N/8,\\ C\Big(N^{-2}\ln^2 N + \Delta t\Big), & \text{for } 3N/8+1 \le i \le N/2-1.\end{cases}$$

For $n \geq 0$, we define

$$\chi(x_i, t_n) = \begin{cases} W_{L,i}^{2,n} - w^2(x_i, t_n), & \text{for } i \neq N/2, \\ 0, & \text{for } i = N/2. \end{cases}$$

Now, applying Lemma 6.4.2 to the mesh function χ over $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^-}$, we obtain

$$\left| W_{L,i}^{2,n+1} - w^2(x_i, t_{n+1}) \right| \leq \begin{cases} C\Big(N^{-2} + \Delta t\Big), & \text{for } 1 \leq i \leq 3N/8, \\ C\Big(N^{-2} \ln^2 N + \Delta t\Big), & \text{for } 3N/8 + 1 \leq i \leq N/2 - 1. \end{cases} \tag{6.38}$$

Finally, the result (6.35) follows easily from (6.37) and (6.38). On the other hand, by definition W_R satisfies the following homogeneous difference equation

$$\begin{cases} L_H^{N,M} W_{R,i}^{n+1} = 0, \quad \text{for } N/2 + 1 \leq i \leq N - 1, \\ W_{R,0}^{n+1} = 0, \quad W_{R,i}^0 = 0, \quad i \geq N/2. \end{cases} \tag{6.39}$$

Using Lemma 6.4.3 and Theorem 6.2.4, we can easily obtain

$$|W_{R,N/2}^{n+1}| \leq C. \tag{6.40}$$

Therefore, it follows from (6.39), (6.40) and Lemma 6.4.5 that

$$\Phi_{R,i} = C\left[\prod_{j=1}^{N/2} \left(1 + \frac{\alpha \hbar_j}{2\varepsilon}\right)^{-1}\right] Q_i, \quad \text{for } N/2 \leq i \leq N,$$

is a barrier function for $\{W_{R,i}^n\}$. Thus, by the discrete minimum principle over $\overline{G}_\varepsilon^{N,M} \bigcap \overline{G^+}$, we have

$$|W_{R,i}^{n+1}| \leq C \prod_{j=N-i+1}^{N/2} \left(1 + \frac{\alpha \hbar_j}{2\varepsilon}\right)^{-1}, \quad \text{for } N/2 + 1 \leq i \leq N - 1. \tag{6.41}$$

Now, for $3N/4 \leq i \leq N - 1$, inequality (6.41) and Lemma 6.4.6 imply that

$$|W_{R,i}^{n+1}| \leq C \prod_{j=N/4+1}^{N/2} \left(1 + \frac{\alpha \hbar_j}{2\varepsilon}\right)^{-1} \leq C N^{-2}. \tag{6.42}$$

Next, from Theorem 6.2.4 and Lemmas 6.4.6 and 6.4.7 we obtain

$$|w^2(x_i, t_{n+1})| \leq C N^{-2}, \quad \text{for } 3N/4 \leq i \leq N - 1. \tag{6.43}$$

Therefore, invoke the triangle inequality to the error $\big(W_R - w\big)$ and use (6.42) and (6.43) to obtain the result (6.36). Hereby the proof is complete. ■

6.4.1 The main convergence result

Theorem 6.4.9. *Assume that $N \geq N_0$ satisfies the conditions given in (6.31), (6.32) and $\varepsilon \leq N^{-1}$. Then, if $\tau_1, \tau_2 \geq 2$, the respective solutions u and U of (6.1)-(6.3) and (6.28) satisfy the following error bounds at time level t_n:*

$$\Big|U_i^n - u(x_i,t_n)\Big| \leq \begin{cases} C\Big[N^{-2}\ln^2 N + \Delta t\Big], & \text{for } 1 \leq i \leq N/8-1, \\ C\Big[N^{-2} + \Delta t\Big], & \text{for } N/8 \leq i \leq 3N/8, \\ C\Big[N^{-2}\ln^3 N + N^{-1}\ln^2 N\Delta t + \Delta t\Big], & \text{for } 3N/8 < i < 3N/4, \\ CN^{-2}, & \text{for } 3N/4 \leq i \leq N-1. \end{cases} \tag{6.44}$$

Proof. The proof is splitted up into two different cases depending on the location of mesh point $x_i \in \overline{\Omega}_x^{N,\varepsilon}$.

Case 1. For $1 \leq i \leq N/8-1$ $\big($**Boundary layer region**$\big)$ and for $N/8 \leq i \leq 3N/8 \bigcup 3N/4 \leq i \leq N-1$ $\big($**Outer region**$\big)$. Here, the estimate of $\big|U_i^n - u(x_i,t_n)\big|$ for $1 \leq i \leq 3N/8$ follows easily from Lemmas 6.4.4 and 6.4.8, by invoking the triangle inequality to the error

$$U - u = (V_L - v) + (W_L - w),$$

and the error estimate for $3N/4 \leq i \leq N-1$ follows similarly.

Case 2. $\big($**Interior layer region**$\big)$ Here, we need to find out the estimate of $\big|U_i^n - u(x_i,t_n)\big|$ for $3N/8 < i < 3N/4$. From **Case 1**, clearly we have

$$\Big|U_i^{n+1} - u(x_i,t_{n+1})\Big| \leq C\Big(N^{-2} + \Delta t\Big), \quad \text{for } i = 3N/8, 3N/4. \tag{6.45}$$

For $3N/8+1 \leq i \leq N/2-1$, using the bounds of the derivatives given in Theorem 6.2.4 we get

$$\Big|L_H^{N,M}\big(U_i^{n+1} - u(x_i,t_{n+1})\big)\Big| \leq \Big[\frac{\varepsilon}{12}h_i^2\Big\|\frac{\partial^4 u}{\partial x^4}\Big\|_\infty + \frac{\Delta t}{2}\Big\|\frac{\partial^2 u}{\partial t^2}\Big\|_\infty\Big] \leq C\Big(\frac{h_l^2}{\varepsilon} + \Delta t\Big). \tag{6.46}$$

Again, for $N/2+1 \leq i \leq 3N/4-1$, using Taylor's formula with the integral form of remainder as in [[41], Lemma 3.3] and the bounds of the derivatives given in Theorem 6.2.4 we deduce that

$$\begin{aligned} \Big|L_H^{N,M}\big(U_i^{n+1} - u(x_i,t_{n+1})\big)\Big| &\leq Ch_r\int_{x_{i-1}}^{x_{i+1}}\Big(\varepsilon\Big|\frac{\partial^4 u}{\partial s^4}(s,t_{n+1})\Big| + \Big|\frac{\partial^3 u}{\partial s^3}(s,t_{n+1})\Big|\Big)ds \\ &\leq \Big(Ch_r^2 + C\frac{h_r}{\varepsilon^2}\Big[\exp\big(-(\xi - x_{i+1})\alpha/\varepsilon\big) - \exp\big(-(\xi - x_{i-1})\alpha/\varepsilon\big)\Big]\Big) \\ &\leq C\Big[\Big(h_r^2 + \frac{h_r}{\varepsilon^2}\exp\big(-(\xi - x_i)\alpha/\varepsilon\big)\sinh(\alpha h/\varepsilon)\Big)\Big]. \end{aligned}$$

Clearly, the assumption (6.31) implies $\alpha h_r/\varepsilon \leq 2$ and since $\sinh\zeta \leq C\zeta$ for $0 \leq \zeta \leq 2$, so $\sinh(\alpha h_r/\varepsilon) \leq C\alpha h_r/\varepsilon$. Thus, for $N/2+1 \leq i \leq 3N/4-1$,

$$\left| L_H^{N,M}\left(U_i^{n+1} - u(x_i, t_{n+1})\right)\right| \leq C\frac{h_r^2}{\varepsilon^3}. \tag{6.47}$$

Now, at the mesh point $x_{N/2} = \xi$, using (6.46) and (6.47), we obtain the following estimate:

$$\begin{aligned}
&\left| m_{N/2}^- f_{N/2-1}^{n+1} + m_{N/2}^+ f_{N/2+1}^{n+1} - L_H^{N,M} u_{N/2}^{n+1}\right| \\
\leq\ & m_{N/2}^-\left| L_H^{N,M}\left(U_{N/2-1}^{n+1} - u(x_{N/2-1}, t_{n+1})\right)\right| + m_{N/2}^+\left| L_H^{N,M}\left(U_{N/2+1}^{n+1} - u(x_{N/2+1}, t_{n+1})\right)\right| \\
& + \left| \left(D_x^F - D_x^B\right) u_{N/2}^{n+1} - \left[\frac{\partial u}{\partial x}\right](x_{N/2}, t_{n+1})\right| \\
\leq\ & C\frac{h_l}{\varepsilon}\left(\frac{h_l^2}{\varepsilon} + \Delta t\right) + C\frac{h_r^2}{\varepsilon^3} + \left| D_x^F u_{N/2}^{n+1} - \frac{\partial u}{\partial x}(x_{N/2}, t_{n+1})\right| + \left| D_x^B u_{N/2}^{n+1} - \frac{\partial u}{\partial x}(x_{N/2}, t_{n+1})\right| \\
\leq\ & C\left[\frac{1}{\sqrt{\varepsilon}}\left(\frac{h_l^2}{\varepsilon} + \frac{h_l}{\sqrt{\varepsilon}}\Delta t\right) + \frac{h_r^2}{\varepsilon^3}\right].
\end{aligned} \tag{6.48}$$

Define the discrete function

$$\begin{aligned}
\Theta_i^n &= C\left(N^{-2}\ln^2 N + \Delta t\right)(1 + t_n) \\
&+ C\left(N^{-2}\ln^2 N + N^{-1}\ln N\Delta t\right)\begin{cases} \dfrac{1}{\tau_1\sqrt{\varepsilon}}\left[x_i - (\xi - \sigma_1)\right], & \text{for } 3N/8+1 \leq i \leq N/2-1, \\ \dfrac{\alpha}{2\tau_2\varepsilon}\left[(\xi + \sigma_2) - x_i\right], & \text{for } N/2+1 \leq i \leq 3N/4-1. \end{cases}
\end{aligned}$$

where C is chosen sufficiently large. Next, a direct calculation yields

$$L_H^{N,M}\Theta_i^{n+1} \geq \begin{cases} C\left(N^{-2}\ln^2 N + \Delta t\right), & \text{for } 3N/8+1 \leq i \leq N/2-1, \\ \dfrac{C}{\varepsilon}\left(N^{-2}\ln^2 N + N^{-1}\ln N\Delta t\right), & \text{for } N/2+1 \leq i \leq 3N/4-1. \end{cases} \tag{6.49}$$

Also, we deduce that

$$\begin{aligned}
L_H^{N,M}\Theta_{N/2}^{n+1} &\geq -\left(D_x^F - D_x^B\right)\Theta_{N/2}^{n+1} \\
&= C\left[\frac{1}{\tau_1\sqrt{\varepsilon}} + \frac{\alpha}{2\tau_2\varepsilon}\right]\left(N^{-2}\ln^2 N + N^{-1}\ln N\Delta t\right) \\
&\geq C\left[\frac{1}{\sqrt{\varepsilon}}\left(N^{-2}\ln^2 N + N^{-1}\ln N\Delta t\right) + \frac{1}{\varepsilon}\left(N^{-2}\ln^2 N\right)\right].
\end{aligned} \tag{6.50}$$

Therefore, it follows from (6.45)-(6.50) that

$$\begin{cases} L_H^{N,M}\Theta_i^{n+1} \ \geq \ \left|L_H^{N,M}\big(U_i^{n+1} - u(x_i, t_{n+1})\big)\right|, \quad \text{for } 3N/8 < i < 3N/4, \\ \Theta_i^{n+1} \ \geq \ \left|U_i^{n+1} - u(x_i, t_{n+1})\right|, \quad \text{for } i = 3N/8, 3N/4 \quad \text{and} \quad \Theta_i^0 \ \geq \ \left|U_i^0 - u(x_i, t_0)\right|. \end{cases}$$

Now, applying the discrete minimum principle to $\Theta_i^n \pm \big(U_i^n - u(x_i, t_n)\big)$ over the domain $\big(\overline{G}_\varepsilon^{N,M} \bigcap [\xi - \sigma_1, \xi + \sigma_2]\big) \times [0, T]$, we obtain that for $3N/8 < i < 3N/4$,

$$\left|U_i^n - u(x_i, t_n)\right| \leq C\bigg(N^{-2}\ln^3 N + N^{-1}\ln^2 N\Delta t + \Delta t\bigg).$$

Hence, this completes the proof. ■

6.5 Numerical Results

This section presents the numerical results obtained by the proposed scheme (6.16) to verify the theoretical results claimed in the previous section. To do this, the numerical experiments are profoundly carried out for the following test example on the piecewise-uniform rectangular mesh $\overline{G}_\varepsilon^{N,M}$. In all the cases, the numerical experiments are performed by choosing the constant $\tau_1 = \tau_2 = 2$, $\alpha = 0.9$ and $\Delta t = 0.8/N$, otherwise it is mentioned.

Example 6.5.1. *Consider the following parabolic IBVP:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon\dfrac{\partial^2 u}{\partial x^2} + x(1-x)u \ = \ 2(1+x^2)t^2, \quad (x,t) \in (0, 0.5) \times (0, 1], \\ -\varepsilon\dfrac{\partial^2 u}{\partial x^2} - \big(1 + x(1-x)\big)\dfrac{\partial u}{\partial x} + x(1-x)u \ = \ 3(1+x^2)t^2, \quad (x,t) \in (0.5, 1) \times (0, 1], \\ u(x, 0) = 0, \quad 0 \leq x \leq 1, \\ u(0, t) = t^2, \quad u(1, t) = 0, \quad 0 < t \leq 1. \end{cases} \tag{6.51}$$

As the exact solution of the IBVP (6.51) is not known, to demonstrate the numerical results obtained by the proposed scheme (6.16) for Example 6.5.1, we use the double mesh principle as described in Chapter 2.

In Table 6.1, we have presented the computed maximum point-wise errors $E_\varepsilon^{N,\Delta t}$ and the corresponding order of convergence $P_\varepsilon^{N,\Delta t}$ for Example 6.5.1. Clearly, from the results given in Table 6.1 we see that the computed ε-uniform errors $E^{N,\Delta t}$ decrease monotonically as N increases. This ensures that the proposed scheme (6.16) is ε-uniformly convergent. As a complement of this observation, we have plotted the maximum point-wise errors in Figure 6.3. In spite of this, in Figure 6.1 we have displayed the presence of interior layers near the point of discontinuity $x = 0.5$ and the boundary layer at $x = 0$ in the numerical solution of the IBVP (6.51), as the parameter ε decreases. Also to visualize the complete numerical solution of the IBVP (6.51) for $\varepsilon = 10^{-4}$, we have included surface plot for $N = 128$ in Figure 6.2.

Next, we see that the numerical results presented in Table 6.1 do not clearly reflect the actual theoretical order of convergence of the proposed scheme (6.16) for the spatial variable, as predicted by Theorem 6.4.9. Therefore, to justify the spatial order of convergence properly, the numerical experiments are carried out by taking $M = N^2$ and the maximum point-wise errors as well as the corresponding order of convergence are displayed in Table 6.2. Further, we have highlighted those errors in Figures 6.4. This in fact shows that the proposed scheme (6.16) is second-order spatial accurate outside the layer regions and almost second-order spatial accurate (reduced by the logarithmic factor) inside the layer regions as obtained in Theorem 6.4.9. Even this analysis also signifies the role of τ_1, τ_2 to fulfill our claim in the same Theorem.

6.6 Conclusion

In this chapter, a class of singularly perturbed mixed parabolic-elliptic problems is solved by using an efficient hybrid numerical scheme on a piecewise-uniform Shishkin mesh resolving both boundary and interior layers. For the parabolic part of the mixed type problem, the classical central difference scheme is used for the spatial discretization and the classical backward-Euler method is used for discretizing the time derivative. While for the elliptic part, the hybrid finite difference scheme (a proper combination of the midpoint upwind scheme in the outer region and the classical central difference scheme in the interior layer region) is proposed. It is proved that the newly proposed scheme is ε-uniform convergent with first-order accurate in time and almost second-order accurate in space.

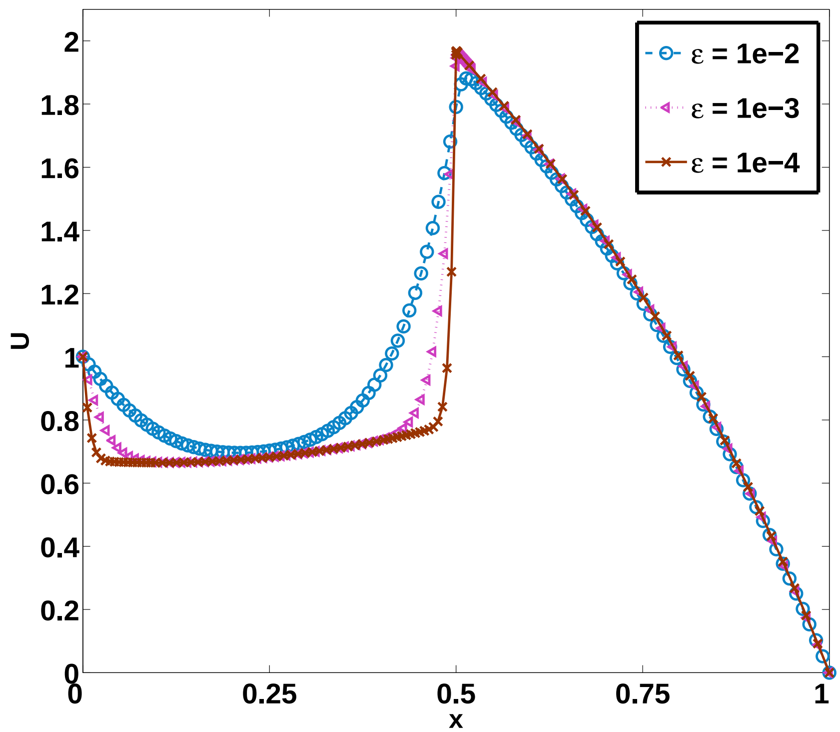


Figure 6.1: *Numerical solution at time* $t = 1$, $N = 128$ *for Example 6.5.1.*

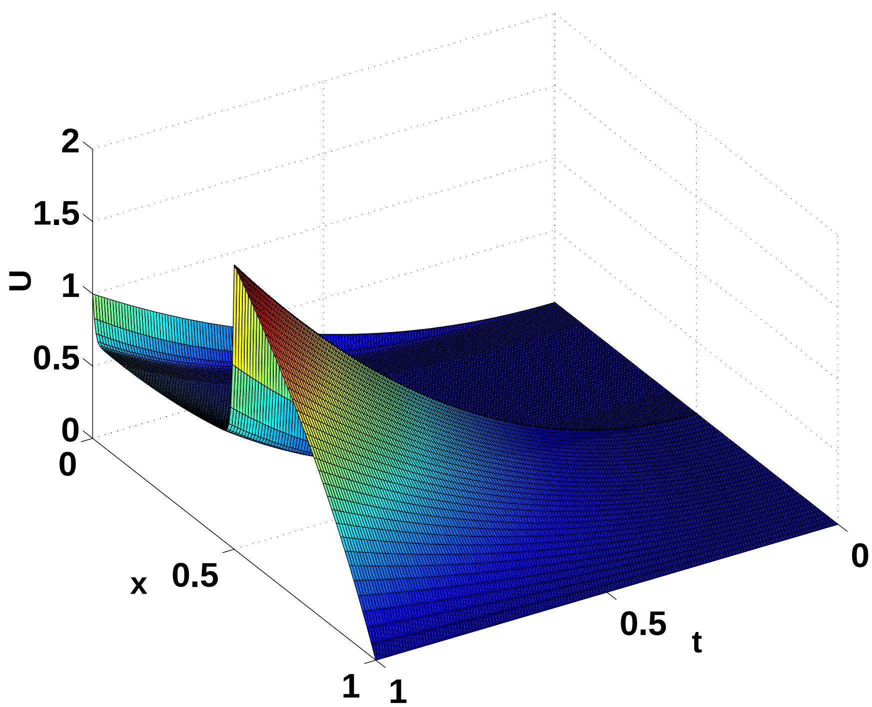


Figure 6.2: *Surface plot of the Numerical solution for* $\varepsilon = 1e-4$, $N = 128$ *for Example 6.5.1.*

Table 6.1: *Maximum point-wise errors and the corresponding order of convergence for Example 6.5.1.*

ε	Number of mesh intervals N					
	32	64	128	256	512	1024
$1e-1$	5.5328e-3	3.1077e-3	1.6788e-3	8.7806e-4	4.4993e-4	2.2787e-4
	0.8321	0.8883	0.9350	0.9646	0.9815	
$1e-2$	2.7425e-2	1.4890e-2	7.4364e-3	3.1797e-3	9.6870e-4	3.9862e-4
	0.8812	1.0016	1.2257	1.7148	1.2810	
$1e-3$	3.4708e-2	1.1989e-2	4.6311e-3	1.9700e-3	9.1034e-4	4.3948e-4
	1.5336	1.3723	1.2332	1.1137	1.0506	
$1e-4$	5.1792e-2	3.1488e-2	1.2468e-2	4.8015e-3	1.8360e-3	6.6975e-4
	0.7179	1.3366	1.3767	1.3869	1.4549	
$1e-5$	5.2002e-2	3.1389e-2	1.2425e-2	4.7725e-3	1.8297e-3	7.2699e-4
	0.7283	1.3370	1.3805	1.3831	1.3316	
$1e-6$	5.2065e-2	3.1353e-2	1.2410e-2	4.7626e-3	1.8277e-3	7.2671e-4
	0.7317	1.3370	1.3817	1.3817	1.3306	
$1e-7$	5.2085e-2	3.1341e-2	1.2406e-2	4.7596e-3	1.8274e-3	7.2598e-4
	0.7328	1.3371	1.3821	1.3811	1.3318	
$1e-8$	5.2091e-2	3.1337e-2	1.2405e-2	4.7591e-3	1.8188e-3	7.3962e-4
	0.7331	1.3370	1.3822	1.3877	1.2981	
$E^{N,\Delta t}$	**5.2091e-2**	**3.1488e-2**	**1.2468e-2**	**4.8015e-3**	**1.8360e-3**	**7.3962e-4**
$P^{N,\Delta t}$	**0.7262**	**1.3366**	**1.3767**	**1.3869**	**1.3117**	

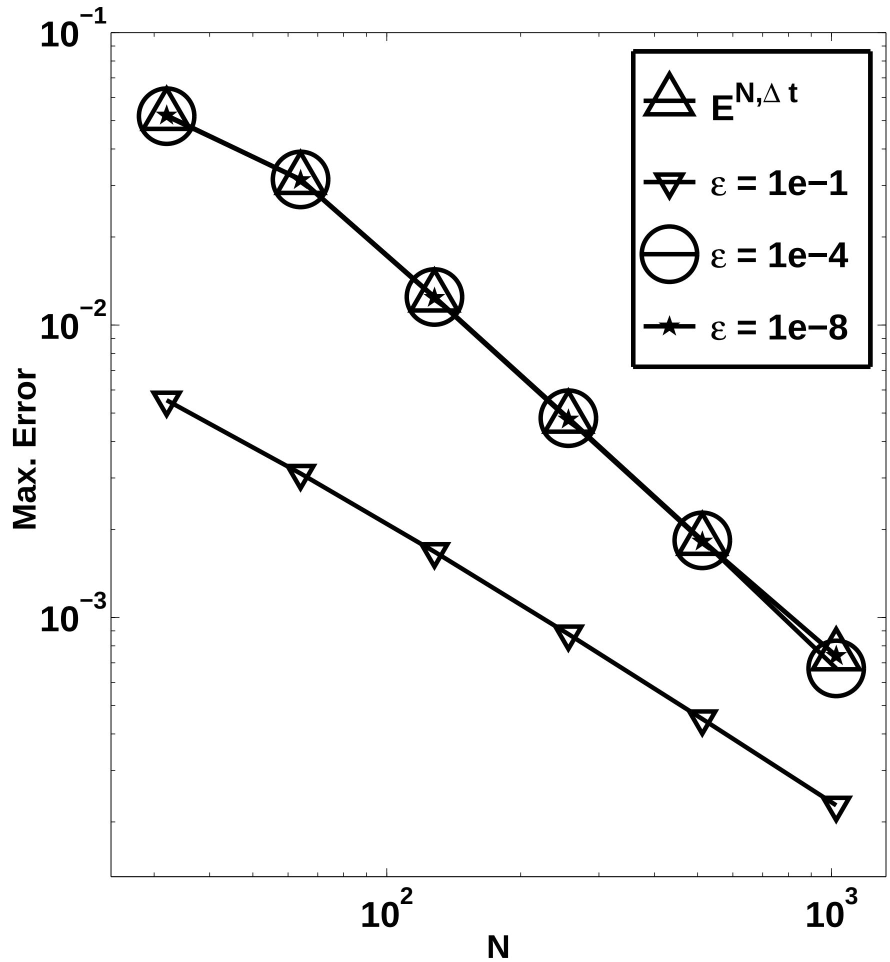


Figure 6.3: *Loglog plot of the maximum point-wise errors for Example 6.5.1.*

Table 6.2: *Maximum point-wise errors and the corresponding order of convergence calculated for Example 6.5.1 by taking* $M = N^2$.

N	$\varepsilon = 10^{-4}$			
	boundary layer region $[0,\sigma_1)$	**left outer region** $[\sigma_1, \xi-\sigma_1]$	**interior layer region** $(\xi-\sigma_1, \xi+\sigma_2)$	**right outer region** $[\xi+\sigma_2, 1]$
32	1.2348e-2	5.1113e-4	3.9772e-2	1.5201e-3
	1.3103	2.0652	0.6283	1.9931
64	4.9790e-3	1.2214e-4	2.5729e-2	3.8186e-4
	1.3039	2.0128	1.4786	1.9809
128	2.0167e-3	3.0266e-5	9.2327e-3	9.6734e-5
	1.5376	2.0109	1.5208	1.9595
256	6.9466e-4	7.5095e-6	3.2176e-3	2.4872e-5
	1.6410	2.0101	1.6495	1.9198
512	2.2274e-4	1.8643e-6	1.0256e-3	6.5734e-6
	$\varepsilon = 10^{-7}$			
	boundary layer region $[0,\sigma_1)$	**left outer region** $[\sigma_1, \xi-\sigma_1]$	**interior layer region** $(\xi-\sigma_1, \xi+\sigma_2)$	**right outer region** $[\xi+\sigma_2, 1]$
32	1.2407e-2	5.1741e-4	3.9905e-2	1.5243e-3
	1.3097	2.0025	0.6438	2.0016
64	5.0051e-3	1.2913e-4	2.5540e-2	3.8063e-4
	1.3037	2.0006	1.4811	2.0004
128	2.0275e-3	3.2269e-5	9.1490e-3	9.5133e-5
	1.5376	2.0005	1.5174	2.0001
256	6.9840e-4	8.0643e-6	3.1960e-3	2.3782e-5
	1.6408	2.0005	1.5686	1.9999
512	2.2396e-4	2.0154e-6	1.0775e-3	5.9458e-6

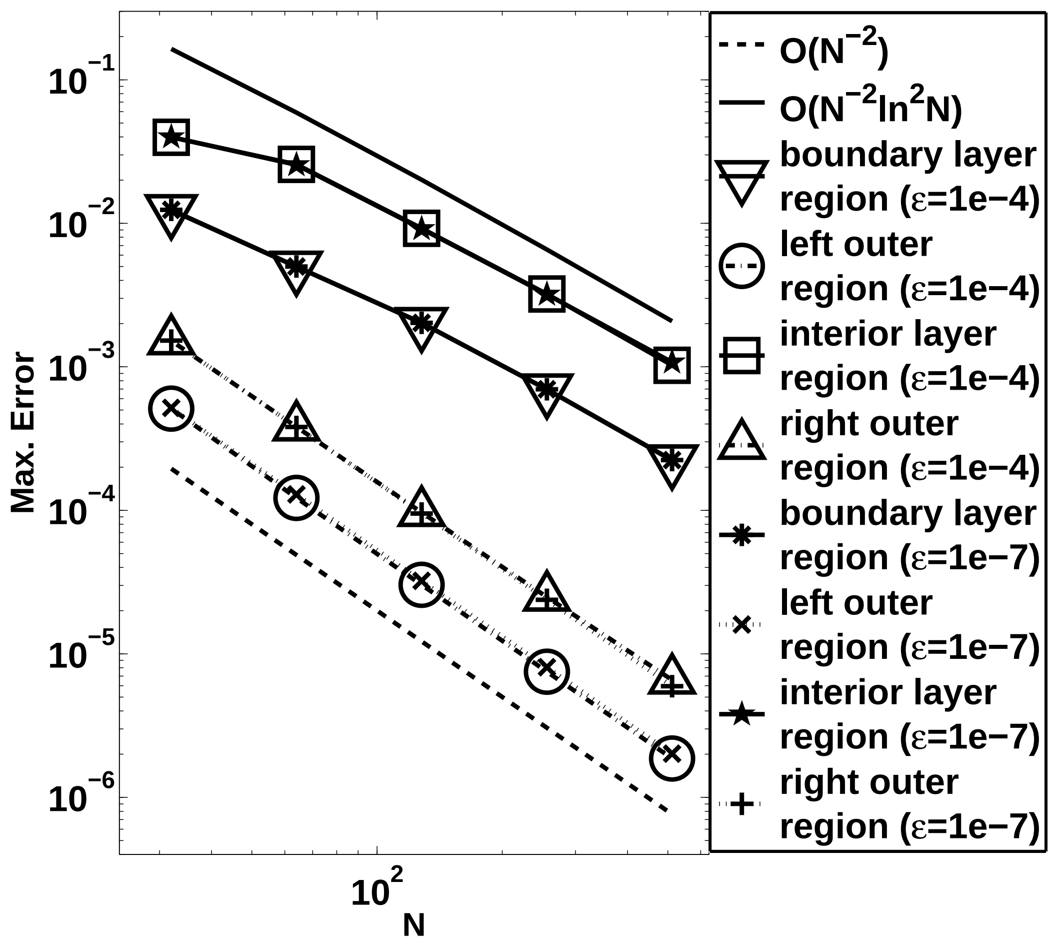


Figure 6.4: *Loglog plot of the spatial order of convergence for Example 6.5.1.*

Chapter 7

Parameter-Uniform Alternating Direction Hybrid Numerical Scheme for 2D Singularly Perturbed Parabolic Convection-Diffusion Problems

This chapter is devoted to develop and analyze an efficient numerical scheme for solving two-dimensional singularly perturbed parabolic convection-diffusion problems exhibiting a regular boundary layer. The numerical scheme comprises of the Peaceman and Rachford alternating direction method for the time discretization and a hybrid finite difference scheme (a proper combination of the midpoint upwind scheme in the outer region and the classical central difference scheme in the boundary layer region) for the spatial discretization. It is proved that the method converges uniformly with respect to the perturbation parameter ε, attaining almost second-order spatial accuracy in the discrete supremum norm, provided that the perturbation parameter ε satisfies $\varepsilon \leq N^{-1}$. This is accomplished by constructing a special rectangular mesh involving piecewise-uniform Shishkin meshes in the spatial directions.

7.1 Introduction

Consider the following 2D singularly perturbed parabolic convection-diffusion IBVP posed on the domain $\mathcal{G} = \mathcal{D} \times (0,T],\ \mathcal{D} = (0,1)^2,\ \mathbf{x} = (x,y) \in \mathbb{R}^2$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(\mathbf{x},t) + \mathcal{L}_\varepsilon u(\mathbf{x},t) \;=\; f(\mathbf{x},t), \quad (\mathbf{x},t) \in \mathcal{G}, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\mathcal{D}}, \\ u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\mathcal{D} \times (0,T], \end{cases} \tag{7.1}$$

where

$$\mathcal{L}_\varepsilon u \equiv -\varepsilon \triangle u + \mathbf{a}(\mathbf{x}) \cdot \nabla u + b(\mathbf{x})u,$$

$0 < \varepsilon \ll 1$ is a small parameter and the coefficients $\mathbf{a} = (a_1, a_2)$, b are sufficiently smooth functions such that

$$a_i(\mathbf{x}) \geq \alpha_i > 0,\ i = 1, 2, \quad b(\mathbf{x}) \geq 0, \quad \text{on } \overline{\mathcal{D}}. \tag{7.2}$$

Under sufficient smoothness and necessary compatibility conditions [[47], §4.5] imposed on the functions u_0 and f, the parabolic IBVP (7.1)-(7.2), admits a unique solution $u(\mathbf{x}, t)$ which exhibits a regular boundary layer of width $O(\varepsilon)$ at the sides $x = 1$ and $y = 1$ (see, e.g., [54]). One can see the asymptotic results about this kind of problem in [40].

The outline of this chapter is as follows: Section 7.7 recalls *a-priori* bounds on the derivatives of the analytical solution *via* decomposition. Section 7.3 studies the uniform convergence of the semidiscrete scheme based on the Peaceman and Ranchford method *via* an intermediate time semidiscretization step and state the asymptotic behavior of the solutions of the resulting semidiscrete problems and their spatial derivatives. Section 7.4 describes the piecewise-uniform Shishkin mesh and provides the detail construction of the newly proposed hybrid finite difference scheme for the spatial discretization of the semidiscrete problems resulting from the time discretization process. Here, the consistency, the stability and the uniform convergence of the proposed hybrid scheme are also studied. The fully discrete scheme is introduced in Section 7.5 and the main theoretical result, *i.e.*, the ε-uniform convergence of the fully discrete scheme, is proved. Finally in Section 7.6, extensive numerical experiments are conducted to verify the theoretical results and also to demonstrate the accuracy of this method.

7.2 Bounds on the Solution Decomposition

The stronger bounds on the derivatives of the analytical solution of the problem (7.1)-(7.2) are given in the following theorem. These bounds are obtained by decomposing the analytical solution into smooth and layer components.

Theorem 7.2.1. *The solution u of the problem (7.1)-(7.2) can be decomposed in the form*

$$u = \mathcal{U}_0 + \sum_{i=1}^{3} \mathcal{U}_i,$$

where $\mathcal{U}_0$ is the smooth component, $\mathcal{U}_1$ and $\mathcal{U}_2$ are the regular boundary layer functions near $x = 1$ and $y = 1$, respectively, and w is the corner layer function. Then, for all non-negative *integers* k_1, k_2, k_t, *such that* $k_s = k_1 + k_2$, $k_s + 2k_t \leq 6$, *and for* $(\mathbf{x}, t) \in \mathcal{G}$, *the components of* u *satisfy the following bounds:*

$$\left| \frac{\partial^{k_s+k_t}\mathcal{U}_0}{\partial x^{k_1}\partial y^{k_2}\partial t^{k_t}}(x,y,t) \right| \leq C,$$

$$\left| \frac{\partial^{k_s+k_t}\mathcal{U}_1}{\partial x^{k_1}\partial y^{k_2}\partial t^{k_t}}(x,y,t) \right| \leq C\varepsilon^{-k_1}\exp\big(-\alpha_1(1-x)/\varepsilon\big),$$

$$\left| \frac{\partial^{k_s+k_t}\mathcal{U}_2}{\partial x^{k_1}\partial y^{k_2}\partial t^{k_t}}(x,y,t) \right| \leq C\varepsilon^{-k_2}\exp\big(-\alpha_2(1-y)/\varepsilon\big),$$

$$\left| \frac{\partial^{k_s+k_t}\mathcal{U}_3}{\partial x^{k_1}\partial y^{k_2}\partial t^{k_t}}(x,y,t) \right| \leq C\varepsilon^{-k_s}\min\Big\{\exp\big(-\alpha_1(1-x)/\varepsilon\big),\ \exp\big(-\alpha_2(1-y)/\varepsilon\big)\Big\}.$$

Proof. The proof of these bounds are given by Clavero *et al.* in [[18], Appendix]. ■

7.3 The Time Semidiscretization

This section studies the time semidiscretization method which is essential for the convergence analysis of the fully discrete scheme, since it will contribute to the decomposition of the global error for the spatial and temporal variables. Also, the asymptotic behavior of the solutions of the semidiscrete problems is presented.

7.3.1 Discretization of the time domain

On the time domain $[0,T]$, the equidistant meshes are introduced with uniform time step Δt such that

$$\mathbb{S}_t^M = \{t_n = n\,\Delta t,\, n = 0,\dots,M,\ t_0 = 0, t_M = T,\ \Delta t = T/M\},$$

where M denotes the number of mesh-intervals in the t-direction.

7.3.2 The semidiscrete scheme

Let us split the convection-diffusion operator $\mathcal{L}_\varepsilon$ as follows: $\mathcal{L}_\varepsilon = \mathcal{L}_{1,\varepsilon} + \mathcal{L}_{2,\varepsilon}$, where

$$\begin{cases} \mathcal{L}_{1,\varepsilon} \equiv -\varepsilon\dfrac{\partial^2}{\partial x^2} + a_1(\mathbf{x})\dfrac{\partial}{\partial x} + b_1(\mathbf{x}), \\ \mathcal{L}_{2,\varepsilon} \equiv -\varepsilon\dfrac{\partial^2}{\partial y^2} + a_2(\mathbf{x})\dfrac{\partial}{\partial y} + b_2(\mathbf{x}), \end{cases}$$

with $b_i(\mathbf{x})$ such that $b(\mathbf{x}) = b_1(\mathbf{x}) + b_2(\mathbf{x})$ and $b_i(\mathbf{x}) \geq \beta_i \geq 0$, $i = 1,2$. Besides, we decompose the source term into two smooth terms as $f(\mathbf{x},t) = f_1(\mathbf{x},t) + f_2(\mathbf{x},t)$, such that

$$f_1(x,0,t) = f_1(x,1,t) = 0, \quad f_2(0,y,t) = f_2(1,y,t) = 0. \tag{7.3}$$

This property is required to study the asymptotic behavior of the exact solutions of the semidiscrete problems (7.8)-(7.9), as mentioned in [18].

Now, taking into account these decompositions, we discretize the problem (7.1)-(7.2) with respect to time by the classical Peaceman and Rachford alternating direction method. Then, we obtain the following semidiscrete scheme:

$$u^0(\mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\mathcal{D}},$$

$$\begin{cases} \left(I + \dfrac{\Delta t}{2}\,\mathcal{L}_{1,\varepsilon}\right) u^{n+1/2}(\mathbf{x}) = \widehat{f}_{1,u^n}(\mathbf{x}, t_n), \quad y \in (0,1), \\ u^{n+1/2}(0,y) = u^{n+1/2}(1,y) = 0, \end{cases} \tag{7.4}$$

$$\begin{cases} \left(I + \dfrac{\Delta t}{2}\,\mathcal{L}_{2,\varepsilon}\right) u^{n+1}(\mathbf{x}) = \widehat{f}_{2,u^{n+1/2}}(\mathbf{x}, t_{n+1/2}), \quad x \in (0,1), \\ u^{n+1}(x,0) = u^{n+1}(x,1) = 0, \end{cases} \tag{7.5}$$

where

$$\begin{cases} \widehat{f}_{1,u^n}(\mathbf{x}, t_n) = \left(I - \dfrac{\Delta t}{2}\,\mathcal{L}_{2,\varepsilon}\right) u^n(\mathbf{x}) + \dfrac{\Delta t}{2}(f_1(\mathbf{x}, t_{n+1/2}) + f_2(\mathbf{x}, t_n)), \\ \widehat{f}_{2,u^{n+1/2}}(\mathbf{x}, t_{n+1/2}) = \left(I - \dfrac{\Delta t}{2}\,\mathcal{L}_{1,\varepsilon}\right) u^{n+1/2}(\mathbf{x}) + \dfrac{\Delta t}{2}(f_1(\mathbf{x}, t_{n+1/2}) + f_2(\mathbf{x}, t_{n+1})), \end{cases} \tag{7.6}$$

and $u^n(\mathbf{x})$ is the semidiscrete approximation to the exact solution $u(\mathbf{x}, t)$ of the continuous problem (7.1)-(7.2) at time level $t_n = n\,\Delta t$.

7.3.3 Convergence analysis

Since the operators $\left(I + \dfrac{\Delta t}{2}\,\mathcal{L}_{i,\varepsilon}\right)$, $i = 1, 2$, satisfy a maximum principle, it is easy to show that

$$\left\| \left(I + \frac{\Delta t}{2}\,\mathcal{L}_{i,\varepsilon}\right)^{-1} \right\|_{\infty} \le \left(1 + \beta_i\,\frac{\Delta t}{2}\right)^{-1}. \tag{7.7}$$

This ensures that each step of the scheme (7.4)-(7.5) has a unique solution $u^{n+1}(\mathbf{x})$, which can be bounded independently of ε.

In order to analyze the consistency of the Peaceman and Rachford method, we define the local error e_{n+1} of the time semidiscretization scheme (7.4)-(7.5) by

$$e_{n+1} = u(\mathbf{x}, t_{n+1}) - \widehat{u}^{n+1}(\mathbf{x}),$$

where $\widehat{u}^{n+1}(\mathbf{x})$ is the solution obtained after one step of the semidiscrete scheme (7.4)-(7.5) by taking the exact value $u(\mathbf{x}, t_n)$, as the starting data $u^n(\mathbf{x})$, *i.e.*,

$$u^n(\mathbf{x}) = u(\mathbf{x}, t_n), \quad \mathbf{x} \in \overline{\mathcal{D}}.$$

Consequently, we have the following systems

$$\begin{cases} \left(I + \dfrac{\Delta t}{2}\,\mathcal{L}_{1,\varepsilon}\right)\widehat{u}^{n+1/2}(\mathbf{x}) = \widehat{f}_{1,u^n}(\mathbf{x}, t_n), \quad y \in (0,1), \\ \widehat{u}^{n+1/2}(0,y) = \widehat{u}^{n+1/2}(1,y) = 0, \end{cases} \tag{7.8}$$

$$\begin{cases} \left(I + \dfrac{\Delta t}{2}\,\mathcal{L}_{2,\varepsilon}\right)\widehat{u}^{n+1}(\mathbf{x}) = \widehat{f}_{2,\widehat{u}^{n+1/2}}(\mathbf{x}, t_{n+1/2}), \quad x \in (0,1), \\ \widehat{u}^{n+1}(x,0) = \widehat{u}^{n+1}(x,1) = 0, \end{cases} \tag{7.9}$$

where $f_1,\ f_2$ are defined by (7.6), respectively. Now, from [13] one can obtain the following consistency result.

Lemma 7.3.1. *Assume that*

$$\left|\frac{\partial^i}{\partial t^i} u(\mathbf{x}, t)\right| \leq C, \quad (\mathbf{x}, t) \in \overline{\mathcal{D}} \times [0, T], \quad 0 \leq i \leq 3.$$

Then, the local error corresponding to the scheme (7.4)-(7.5) satisfies

$$\left\|e_{n+1}\right\|_\infty \leq C(\Delta t)^3. \tag{7.10}$$

Next, we shall proceed to study the uniform convergence of this semidiscretization process. Let the global error E_n associated with the scheme (7.4)-(7.5) be defined by

$$E_n = u(\mathbf{x}, t_n) - u^n(\mathbf{x}).$$

Then, clearly we have $E_n = e_n + RE_{n-1}$, where

$$R \equiv \left(I + \frac{\Delta t}{2}\,\mathcal{L}_{2,\varepsilon}\right)^{-1}\left(I - \frac{\Delta t}{2}\,\mathcal{L}_{1,\varepsilon}\right)\left(I + \frac{\Delta t}{2}\,\mathcal{L}_{1,\varepsilon}\right)^{-1}\left(I - \frac{\Delta t}{2}\,\mathcal{L}_{2,\varepsilon}\right)$$

is the transition operator such that RE_{n-1} is the result obtained after one step of the scheme (7.4)-(7.5) taking $u^n = E_{n-1}$ as the starting data and zero values for the source terms f_1 and f_2. Using this recurrence, we easily get

$$E_n = \sum_{i=1}^{n} R^{n-i} e_i.$$

Thus, if the transition operator satisfies

$$\|R^i\|_\infty \leq C, \quad i = 1, \ldots, n, \tag{7.11}$$

it immediately follows that

$$\sup_{n \leq T/\Delta t} \|E_n\|_\infty \leq C(\Delta t)^2.$$

and hence, the semidiscrete scheme (7.4)-(7.5) will be uniformly convergent of second-order in time. But following the discussion rigorously made by Clevero *et al.* in [13], we obtain that

$$\|R^i\|_\infty \leq C/\Delta t, \quad i = 1, \dots, n, \tag{7.12}$$

assuming the commutativity of the operators $\mathcal{L}_{i,\varepsilon}$, $i = 1, 2$. Finally, combining the consistency result (7.10) with the stability result (7.12) of the scheme (7.4)-(7.5), we can conclude that the time semidiscretization process is uniformly convergent of only first-order in time.

Remark 7.3.2. *The utilization of (7.12) causes the theoretical order of convergence for the the Peaceman and Rachford method from two to one. However, this reduction is only theoretical, which does not appear in the numerical experiments performed in Section 7.6.*

7.3.4 Asymptotic behavior of the solutions of semidiscrete problems

In order to estimate the local truncation errors associated with the spatial discretizations of the resulting semidiscrete problems (7.8)-(7.9), in the next section, we need to know the asymptotic behavior of the exact solutions $\widehat{u}^{n+1/2}(\mathbf{x}), \widehat{u}^{n+1}(\mathbf{x})$ and their spatial derivatives, with respect to the perturbation parameter ε. This infact preserve the same behaviour as that of the exact solution u of the IBVP (7.1)-(7.2) and its spatial derivatives as shown in Theorem 7.2.1. This behavior is given by the following lemma.

Lemma 7.3.3. *The exact solutions of (7.8)-(7.9) can be decomposed as*

$$\begin{cases} \widehat{u}^{n+1/2}(\mathbf{x}) &= \eta_1 \widehat{w}^{n+1/2}(\mathbf{x}) + \widehat{z}^{n+1/2}(\mathbf{x}), \\ \widehat{u}^{n+1}(\mathbf{x}) &= \eta_2 \widehat{w}^{n+1}(\mathbf{x}) + \widehat{z}^{n+1}(\mathbf{x}), \end{cases} \tag{7.13}$$

where for $y \in (0,1)$, the components of $\widehat{u}^{n+1/2}(\mathbf{x})$ satisfy

$$\begin{cases} \widehat{w}^{n+1/2}(\mathbf{x}) &= \exp\big(-a_1(1,y)(1-x)/\varepsilon\big), \quad \eta_1 = \dfrac{\varepsilon}{a_1(1,y)} \dfrac{d\widehat{u}^{n+1/2}}{dx}(1,y), \\ \left|\dfrac{d^i \widehat{z}^{n+1/2}}{dx^i}\right| &\leq C\Big[1 + \varepsilon^{-i+1} \exp\big(-\alpha_1(1-x)/\varepsilon\big)\Big], \quad 0 \leq i \leq 4, \end{cases} \tag{7.14}$$

and for $x \in (0,1)$, the components of $\widehat{u}^{n+1}(\mathbf{x})$ satisfy

$$\begin{cases} \widehat{w}^{n+1}(\mathbf{x}) &= \exp\big(-a_2(x,1))(1-y)/\varepsilon\big), \quad \eta_2 = \dfrac{\varepsilon}{a_2(x,1)} \dfrac{d\widehat{u}^{n+1}}{dx}(x,1), \\ \left|\dfrac{d^i \widehat{z}^{n+1}}{dy^i}\right| &\leq C\Big[1 + \varepsilon^{-i+1} \exp\big(-\alpha_2(1-y)/\varepsilon\big)\Big], \quad 0 \leq i \leq 4. \end{cases} \tag{7.15}$$

Proof. The proof of this lemma is given in [18]. ■

7.4 The Spatial Discretization

This section deals with the spatial discretization of the 1D elliptic singularly perturbed problems (7.8)-(7.9). By constructing a suitable piecewise-uniform Shishkin mesh on the spatial domain, it is proved that the resulting numerical scheme is uniformly convergent .

7.4.1 The piecewise-uniform Shishkin mesh

Consider the spatial domain $\overline{\mathcal{D}} = [0,1]^2$ and let $N \geq 4$ be an even positive integer. Here, we consider the asymptotic behavior of the exact solutions of (7.8)-(7.9) obtained in Lemma 7.3.3, as the key for defining the appropriate Shishkin mesh on $\overline{\mathcal{D}}$. Let the rectangular mesh $\overline{\mathcal{D}}_{\varepsilon}^{N}$ be defined to be the tensor product of the 1D Shishkin meshes, *i.e.* $\overline{\mathcal{D}}_{\varepsilon}^{N} = \overline{I}_{1,\varepsilon}^{N} \times \overline{I}_{2,\varepsilon}^{N}$, which is constructed as follows.

First, define the transition parameters τ_ℓ, which are utilized to separate the coarse and fine portions of the meshes in both the spatial directions, by taking the following form

$$\tau_\ell = \min\left\{\frac{1}{2}, \tau_{\ell,0}\varepsilon \ln N\right\}, \quad \ell = 1,2, \tag{7.16}$$

where $\tau_{\ell,0}$, $\ell = 1,2$, are positive constants to be chosen later on. In the analysis, we shall assume that $\tau_\ell = \tau_{\ell,0}\varepsilon \ln N$. Note that if $\tau_\ell = 1/2$, then this mesh is uniform and N^{-1} is exponentially small relatively to ε, which is very unlikely in practice (and in this case the method can be analyzed in the classical way).

Now, we define the mesh $I_{1,\varepsilon}^{N}$ by dividing the interval $[0,1]$ into two subintervals $[0, 1-\tau_1]$ and $[1-\tau_1, 1]$ and on each subinterval a uniform mesh with $N/2$ mesh-intervals is placed such that $\overline{I}_{1,\varepsilon}^{N} = \{0 = x_0, x_1, \ldots, x_{N/2} = 1-\tau_1, \ldots, x_N = 1\}$. Analogously, we define $\overline{I}_{2,\varepsilon}^{N} = \{0 = y_0, y_1, \ldots, y_{N/2} = 1-\tau_2, \ldots, y_N = 1\}$. Here, the mesh widths in both the spatial directions will be denoted as

$$\begin{cases} h_{1,i} = x_i - x_{i-1},\ i = 1,\ldots,N, \quad \widehat{h}_{1,i} = h_{1,i} + h_{1,i+1},\ i = 1,\ldots,N-1, \\ h_{2,j} = y_j - y_{j-1},\ j = 1,\ldots,N, \quad \widehat{h}_{2,j} = h_{2,j} + h_{2,j+1},\ j = 1,\ldots,N-1. \end{cases}$$

Further, let $H_\ell = 2(1-\tau_\ell)/N$ and $h_\ell = 2\tau_\ell/N$, $\ell = 1,2$, be the mesh widths in $[0, 1-\tau_\ell]$ and $[1-\tau_\ell, 1]$ respectively. Then, it is easy to see that

$$N^{-1} \leq H_\ell \leq 2N^{-1}, \quad h_\ell = 2\tau_{\ell,0}\varepsilon N^{-1} \ln N, \quad \ell = 1,2.$$

7.4.2 The finite difference scheme

Let us denote $I_{\ell,\varepsilon}^{N} = \overline{I}_{\ell,\varepsilon}^{N} \cap (0,1)$, $\quad \ell = 1,2$. Before describing the scheme, for a given mesh function $v(x_i, y, t_n) = v_{x_i,y}^{n}$, $y \in I_{2,\varepsilon}^{N}$, define the forward, backward and central difference operators D_x^+, D_x^- and D_x^0 in space by

$$D_x^+ v_{x_i,y}^n = \frac{v_{x_{i+1},y}^n - v_{x_i,y}^n}{h_{1,i+1}}, \; D_x^- v_{x_i,y}^n = \frac{v_{x_i,y}^n - v_{x_{i-1},y}^n}{h_{1,i}}, \quad \text{and} \quad D_x^0 v_{x_i,y}^n = \frac{v_{x_{i+1},y}^n - v_{x_{i-1},y}^n}{\widehat{h}_{1,i}},$$

respectively and we define the second-order finite difference operator δ_x^2 in space by

$$\delta_x^2 v_{x_i,y}^n = \frac{2(D_x^+ v_{x_i,y}^n - D_x^- v_{x_i,y}^n)}{\widehat{h}_{1,i}}.$$

Similarly, for a given mesh function $v(x, y_j, t_n) = v_{x,y_j}^n$, $x \in I_{1,\varepsilon}^N$, we define the difference operators D_y^+, D_y^-, D_y^0 and δ_y^2. Also define

$$\begin{cases} v_{i-1/2}^n = \left(v_{x_{i-1},y}^n + v_{x_i,y}^n\right)/2, & y \in I_{2,\varepsilon}^N, \\ v_{j-1/2}^n = \left(v_{x,y_{j-1}}^n + v_{x,y_j}^n\right)/2, & x \in I_{1,\varepsilon}^N. \end{cases} \tag{7.17}$$

Let us denote $a_{1,i} = a_1(x_i, y)$, $a_{2,j} = a_2(x, y_j)$, $b_{1,i} = b_1(x_i, y), b_{2,j} = b_2(x, y_j)$ and we define $a_{1,i-1/2}$, $a_{2,j-1/2}$, $b_{1,i-1/2}$, $b_{2,j-1/2}$ similarly as in (7.17).

For spatial discretization of (7.8)-(7.9), we propose a hybrid scheme (see [59], for the details of the construction), which takes the following form: For $y \in I_{2,\varepsilon}^N$, $\widehat{U}_{0,y}^{n+1/2} = \widehat{U}_{1,y}^{n+1/2} = 0$,

$$\begin{cases} \widehat{U}_{i-1/2}^{n+1/2} + \dfrac{\Delta t}{2} \mathcal{L}_{1,mu}^N \widehat{U}_{x_i,y}^{n+1/2} = \dfrac{1}{2}\Big(\widehat{F}_{1,u^n}(x_i, y, t_n) + \widehat{F}_{1,u^n}(x_{i-1}, y, t_n)\Big), \\ \hfill \text{for } 1 \le i \le N/2, \\ \widehat{U}_{x_i,y}^{n+1/2} + \dfrac{\Delta t}{2} \mathcal{L}_{1,cen}^N \widehat{U}_{x_i,y}^{n+1/2} = \widehat{F}_{1,u^n}(x_i, y, t_n), \\ \hfill \text{for } N/2 < i \le N-1, \end{cases} \tag{7.18}$$

and for $x \in I_{1,\varepsilon}^N$, $\widehat{U}_{x,0}^{n+1} = \widehat{U}_{x,1}^{n+1} = 0$,

$$\begin{cases} \widehat{U}_{j-1/2}^{n+1} + \dfrac{\Delta t}{2} \mathcal{L}_{2,mu}^N \widehat{U}_{x,y_j}^{n+1} = \dfrac{1}{2}\Big(\widehat{F}_{2,\widehat{U}^{n+1/2}}(x, y_j, t_{n+1/2}) + \widehat{F}_{2,\widehat{U}^{n+1/2}}(x, y_{j-1}, t_{n+1/2})\Big), \\ \hfill \text{for } 1 \le j \le N/2, \\ \widehat{U}_{x,y_j}^{n+1} + \dfrac{\Delta t}{2} \mathcal{L}_{2,cen}^N \widehat{U}_{x,y_j}^{n+1} = \widehat{F}_{2,\widehat{U}^{n+1/2}}(x, y_j, t_{n+1/2}), \\ \hfill \text{for } N/2 < j \le N-1, \end{cases} \tag{7.19}$$

where

$$\begin{cases} \mathcal{L}_{1,mu}^N \widehat{U}_{x_i,y}^{n+1/2} \equiv -\varepsilon \delta_x^2 \widehat{U}_{x_i,y}^{n+1/2} + a_{1,i-1/2} D_x^- \widehat{U}_{x_i,y}^{n+1/2} + b_{1,i-1/2} \widehat{U}_{i-1/2}^{n+1/2}, \\ \mathcal{L}_{1,cen}^N \widehat{U}_{x_i,y}^{n+1/2} \equiv -\varepsilon \delta_x^2 \widehat{U}_{x_i,y}^{n+1/2} + a_{1,i} D_x^0 \widehat{U}_{x_i,y}^{n+1/2} + b_{1,i} \widehat{U}_{x_i,y}^{n+1/2}, \\ \mathcal{L}_{2,mu}^N \widehat{U}_{x,y_j}^{n+1} \equiv -\varepsilon \delta_y^2 \widehat{U}_{x,y_j}^{n+1} + a_{2,j-1/2} D_y^- \widehat{U}_{x,y_j}^{n+1} + b_{2,j-1/2} \widehat{U}_{j-1/2}^{n+1}, \\ \mathcal{L}_{2,cen}^N \widehat{U}_{x,y_j}^{n+1} \equiv -\varepsilon \delta_y^2 \widehat{U}_{x,y_j}^{n+1} + a_{2,j} D_y^0 \widehat{U}_{x,y_j}^{n+1} + b_{2,j} \widehat{U}_{x,y_j}^{n+1}, \end{cases} \tag{7.20}$$

and

$$\begin{cases} \widehat{F}_{1,u^n}(x_i,y,t_n) = \left(I - \dfrac{\Delta t}{2}\overline{\mathcal{L}}_{2,\varepsilon}^N\right)u(x_i,y,t_n) \\ \qquad\qquad + \dfrac{\Delta t}{2}\big(f_1(x_i,y,t_{n+1/2}) + f_2(x_i,y,t_n)\big), \\ \widehat{F}_{2,\widehat{U}^{n+1/2}}(x,y_j,t_{n+1/2}) = \left(I - \dfrac{\Delta t}{2}\overline{\mathcal{L}}_{1,\varepsilon}^N\right)\widehat{U}_{x,y_j}^{n+1/2} \\ \qquad\qquad + \dfrac{\Delta t}{2}\big(f_1(x,y_j,t_{n+1/2}) + f_2(x,y_j,t_{n+1})\big), \end{cases} \tag{7.21}$$

with the operators $\overline{\mathcal{L}}_{i,\varepsilon}^N,\ i=1,2$, which are defined by

$$\begin{cases} \overline{\mathcal{L}}_{2,\varepsilon}^N u(x,y,t_n) = \mathcal{L}_{2,\varepsilon}u(x,y,t_n), \\ \overline{\mathcal{L}}_{1,\varepsilon}^N \widehat{U}_{x,y}^{n+1/2} = -\overline{\mathcal{L}}_{2,\varepsilon}^N u(x,y,t_n) - \dfrac{2}{\Delta t}\big(\widehat{U}_{x,y}^{n+1/2} - u(x,y,t_n)\big) \\ \qquad\qquad + \big(f_1(x,y,t_{n+1/2}) + f_2(x,y,t_n)\big). \end{cases} \tag{7.22}$$

After rearranging the terms in (7.18)-(7.19), we obtain the following form of the difference schemes: For $y \in I_{2,\varepsilon}^N$, $\quad \widehat{U}_{0,y}^{n+1/2} = \widehat{U}_{1,y}^{n+1/2} = 0$,

$$\begin{cases} \mathcal{L}_{1,\varepsilon}^N \widehat{U}_{x_i,y}^{n+1/2} \equiv r_i^{1,-}\widehat{U}_{x_{i-1},y}^{n+1/2} + r_i^{1,0}\widehat{U}_{x_i,y}^{n+1/2} + r_i^{1,+}\widehat{U}_{x_{i+1},y}^{n+1/2} \\ \qquad = \dfrac{1}{2}\left(\widehat{F}_{1,u^n}(x_i,y,t_n) + \widehat{F}_{1,u^n}(x_{i-1},y,t_n)\right), \quad \text{for } 1 \le i \le N/2, \\ \mathcal{L}_{1,\varepsilon}^N \widehat{U}_{x_i,y}^{n+1/2} \equiv r_i^{1,-}\widehat{U}_{x_{i-1},y}^{n+1/2} + r_i^{1,0}\widehat{U}_{x_i,y}^{n+1/2} + r_i^{1,+}\widehat{U}_{x_{i+1},y}^{n+1/2} \\ \qquad = \widehat{F}_{1,u^n}(x_i,y,t_n), \quad \text{for } N/2 < i \le N-1, \end{cases} \tag{7.23}$$

and for $x \in I_{1,\varepsilon}^N$, $\quad \widehat{U}_{x,0}^{n+1} = \widehat{U}_{x,1}^{n+1} = 0$,

$$\begin{cases} \mathcal{L}_{2,\varepsilon}^N \widehat{U}_{x,y_j}^{n+1} \equiv r_j^{2,-}\widehat{U}_{x,y_{j-1}}^{n+1} + r_j^{2,0}\widehat{U}_{x,y_j}^{n+1} + r_j^{2,+}\widehat{U}_{x,y_{j+1}}^{n+1} \\ \qquad = \dfrac{1}{2}\left(\widehat{F}_{2,\widehat{U}^{n+1/2}}(x,y_j,t_{n+1/2}) + \widehat{F}_{2,\widehat{U}^{n+1/2}}(x,y_{j-1},t_{n+1/2})\right), \\ \qquad\qquad \text{for } 1 \le j \le N/2, \\ \mathcal{L}_{2,\varepsilon}^N \widehat{U}_{x,y_j}^{n+1} \equiv r_j^{2,-}\widehat{U}_{x,y_{j-1}}^{n+1} + r_j^{2,0}\widehat{U}_{x,y_j}^{n+1} + r_j^{2,+}\widehat{U}_{x,y_{j+1}}^{n+1} \\ \qquad = \widehat{F}_{2,\widehat{U}^{n+1/2}}(x,y_j,t_{n+1/2}), \quad \text{for } N/2 < j \le N-1, \end{cases} \tag{7.24}$$

where the coefficients $r_k^{\ell,\bullet},\ \ell=1,2,\ 1 \le k \le N-1$, are given by

$$r_k^{\ell,-} = \frac{\Delta t}{2}r_{mu,k}^{\ell,-} + \frac{1}{2}, \quad r_k^{\ell,0} = \frac{\Delta t}{2}r_{mu,k}^{\ell,0} + \frac{1}{2}, \quad r_k^{\ell,+} = \frac{\Delta t}{2}r_{mu,k}^{\ell,+}, \quad \text{for } 1 \le k \le N/2, \tag{7.25}$$

and

$$r_k^{\ell,-} = \frac{\Delta t}{2}r_{cen,k}^{\ell,-}, \quad r_k^{\ell,0} = \frac{\Delta t}{2}r_{cen,k}^{\ell,0} + 1, \quad r_k^{\ell,+} = \frac{\Delta t}{2}r_{cen,k}^{\ell,+}, \quad \text{for } N/2 < k \le N-1, \tag{7.26}$$

here

$$\begin{cases} r^{\ell,-}_{mu,k} &= -\dfrac{2\varepsilon}{\widehat{h}_{\ell,k}h_{\ell,k}} - \dfrac{a_{\ell,k-1/2}}{h_{\ell,k}} + \dfrac{b_{\ell,k-1/2}}{2}, \\ r^{\ell,0}_{mu,k} &= \dfrac{2\varepsilon}{h_{\ell,i}h_{\ell,k+1}} + \dfrac{a_{\ell,k-1/2}}{h_{\ell,k}} + \dfrac{b_{\ell,k-1/2}}{2}, \\ r^{\ell,+}_{mu,k} &= -\dfrac{2\varepsilon}{\widehat{h}_{\ell,k}h_{\ell,k+1}}, \end{cases} \quad \text{and} \quad \begin{cases} r^{\ell,-}_{cen,k} &= -\dfrac{2\varepsilon}{\widehat{h}_{\ell,k}h_{\ell,k}} - \dfrac{a_{\ell,k}}{\widehat{h}_{\ell,k}}, \\ r^{\ell,0}_{cen,k} &= \dfrac{2\varepsilon}{h_{\ell,k}h_{\ell,k+1}} + b_{\ell,k}, \\ r^{\ell,+}_{cen,k} &= -\dfrac{2\varepsilon}{\widehat{h}_{\ell,k}h_{\ell,k+1}} + \dfrac{a_{\ell,k}}{\widehat{h}_{\ell,k}}. \end{cases} \tag{7.27}$$

7.4.3 Error analysis

First, the stability of the difference schemes (7.23)-(7.24) is obtained in the following lemma.

Lemma 7.4.1. *Assume that $N \geq N_0$, where for $\ell = 1, 2$,*

$$\frac{N_0}{\ln N_0} \geq \tau_{\ell,0}\|a_\ell\|_\infty \quad \textit{and} \tag{7.28}$$

$$\left(\|b_\ell\|_\infty + 2\Delta t^{-1}\right) \leq \alpha_\ell N_0. \tag{7.29}$$

Then, we have

$$\begin{cases} r^{\ell,-}_k < 0, \ r^{\ell,+}_k < 0, & \text{for } 1 \leq k \leq N-1, \\ |r^{\ell,0}_1| - |r^{\ell,+}_1| \geq 0, \ |r^{\ell,0}_k| - |r^{\ell,-}_k| - |r^{\ell,+}_i| \geq 0, & \text{for } 1 < k \leq N/2, \\ |r^{\ell,0}_{N-1}| - |r^{\ell,-}_{N-1}| > 0, \ |r^{\ell,0}_k| - |r^{\ell,-}_k| - |r^{\ell,+}_k| > 0, & \text{for } N/2 < k < N-1. \end{cases}$$

Proof. The proof follows from Lemma 2.4.1. ■

Remark 7.4.2. *Lemma 7.4.1 shows that under the assumptions (7.28) and (7.29), the matrix associated with the difference operator $\mathcal{L}^N_{\ell,\varepsilon}$ is an M-matrix, for each ℓ and therefore, the operator $\mathcal{L}^N_{\ell,\varepsilon}$ satisfies a* ***discrete maximum principle****. Hence, the method is uniformly stable in the supremum norm.*

Before starting the error analysis, some technical lemmas which will be used to prove the uniform convergence of the proposed hybrid scheme (7.18)-(7.19) are stated below. Let us define the following two mesh functions $\phi_{\ell,k}(\gamma_\ell)$, $\ell = 1, 2$,

$$\begin{cases} \phi_{\ell,k}(\gamma_\ell) &= \displaystyle\prod_{J=k+1}^{N} \mu^{-1}_{\ell,J}(\gamma_\ell), \quad k = 0, \dots, N-1, \\ \mu_{\ell,k}(\gamma_\ell) &= 1 + \gamma_\ell h_{\ell,k}/\varepsilon, \ 1 \leq k \leq N, \ \text{and } \phi_{\ell,N}(\gamma_\ell) = 1, \end{cases}$$

where γ_ℓ is a positive constant such that $\gamma_\ell < \alpha_\ell/2$.

Lemma 7.4.3. *If $\gamma_\ell < \alpha_\ell/2,\ \ell = 1,2$, then for some constant C, we have*

$$\mathcal{L}^N_{\ell,\varepsilon}\phi_{\ell,k}(\gamma_\ell) \geq \begin{cases} \dfrac{C\Delta t}{\max\{\varepsilon, h_{\ell,k}\}}\phi_{\ell,k}(\gamma_\ell), & \textit{for } 1 \leq k \leq N/2, \\ \dfrac{C\Delta t}{\varepsilon}\phi_{\ell,k}(\gamma_\ell), & \textit{for } N/2 < k \leq N-1. \end{cases} \tag{7.30}$$

Proof. For the sake of simplicity, we shall use the notation $\phi_{\ell,k}$ instead of $\phi_{\ell,k}(\gamma_\ell)$. Firstly, apply the operator $\mathcal{L}^N_{\ell,\varepsilon}$ to the discrete function $\phi_{\ell,k}$ and invoke (7.25)-(7.26) with (7.20), to obtain

$$\begin{aligned}
\mathcal{L}^N_{\ell,\varepsilon}\phi_{\ell,k} &= r^{\ell,-}_k\phi_{\ell,k-1} + r^{\ell,0}_k\phi_{\ell,k} + r^{\ell,+}_k\phi_{\ell,k+1} \\
&= \begin{cases} \dfrac{\Delta t}{2}\Big(r^{\ell,-}_{mu,k}\phi_{\ell,k-1} + r^{\ell,0}_{mu,k}\phi_{\ell,k} + r^{\ell,+}_{mu,k}\phi_{\ell,k+1}\Big) + \Big(\phi_{\ell,k} + \phi_{\ell,k-1}\Big)/2, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \leq k \leq N/2, \\ \dfrac{\Delta t}{2}\Big(r^{\ell,-}_{cen,k}\phi_{\ell,k-1} + r^{\ell,0}_{cen,k}\phi_{\ell,k} + r^{\ell,+}_{cen,k}\phi_{\ell,k+1}\Big) + \phi_{\ell,k}, \quad \text{for } N/2 < k \leq N-1, \end{cases} \\
&= \begin{cases} \dfrac{\Delta t}{2}\mathcal{L}^N_{\ell,mu}\phi_{\ell,k} + \phi_{\ell,k}\big(1 + \mu^{-1}_{\ell,k}\big)/2, & \text{for } 1 \leq k \leq N/2, \\ \dfrac{\Delta t}{2}\mathcal{L}^N_{\ell,cen}\phi_{\ell,k} + \phi_{\ell,k}, & \text{for } N/2 < k \leq N-1. \end{cases}
\end{aligned}$$

Afterwards, the result (7.30) can be directly deduced by following the technique as in Lemma 2.4.6. Here, we used the following inequality

$$h_\ell/\varepsilon \leq 2/\|a_\ell\|_\infty, \tag{7.31}$$

which follows from the assumption (7.28) and hereby, this completes the proof. ■

Lemma 7.4.4. *The following inequalities hold true:*

(i) *If* $\gamma_\ell < \alpha_\ell/2,\ \ell = 1,2,$

$$\begin{cases} \exp\big(-\alpha_1(1-x_i)/\varepsilon\big) \leq \phi_{1,i}(\gamma_1), & i = 0,\ldots,N-1, \\ \exp\big(-\alpha_2(1-y_j)/\varepsilon\big) \leq \phi_{2,j}(\gamma_2), & j = 0,\ldots,N-1. \end{cases} \tag{7.32}$$

(ii) *There exists a constant C such that* $\phi_{\ell,N/2}(\gamma_\ell) \leq CN^{-\gamma_\ell\tau_{\ell,0}},\ \ell = 1,2.$ (7.33)

Proof. The proof of (i) follows from [[88], *Lemma 2.5*] and the proof of (ii) follows from [[89], *Lemma 3.1*]. ■

Now, we shall proceed to analyze the uniform convergence of the solutions of the 1D discrete problems (7.23)-(7.24), to the corresponding exact solutions of the semidiscrete problems (7.8)-(7.9). To do this, we shall start our analysis by estimating the local error obtained due to the discretization the problem (7.8) by the numerical scheme (7.18) in the x-direction. In the following, we concise the expressions by omitting the dependence on $y \in I_{2,\varepsilon}^N$.

Then, using (7.8), (7.21) and (7.25)-(7.26) for $\ell = 1$, we define the local truncation error for the scheme (7.23) as

$$\begin{aligned}
\tau^1_{\widehat{u}^{n+1/2}}(x_i) &= \mathcal{L}^N_{1,\varepsilon}[\widehat{u}^{n+1/2}_{x_i} - \widehat{U}^{n+1/2}_{x_i}] \\
&= \begin{cases} r_i^{1,-}\widehat{u}^{n+1/2}_{x_{i-1}} + r_i^{1,0}\widehat{u}^{n+1/2}_{x_i} + r_i^{1,+}\widehat{u}^{n+1/2}_{x_{i+1}} - \dfrac{1}{2}\Big(I + \dfrac{\Delta t}{2}\mathcal{L}_{1,\varepsilon}\Big)\widehat{u}^{n+1/2}(x_{i-1}) \\ -\dfrac{1}{2}\Big(I + \dfrac{\Delta t}{2}\mathcal{L}_{1,\varepsilon}\Big)\widehat{u}^{n+1/2}(x_i), \quad \text{for } 1 \le i \le N/2, \\ r_i^{1,-}\widehat{u}^{n+1/2}_{x_{i-1}} + r_i^{1,0}\widehat{u}^{n+1/2}_{x_i} + r_i^{1,+}\widehat{u}^{n+1/2}_{x_{i+1}} \\ -\Big(I + \dfrac{\Delta t}{2}\mathcal{L}_{1,\varepsilon}\Big)\widehat{u}^{n+1/2}(x_i), \quad \text{for } N/2 < i \le N-1, \end{cases} \\
&= \frac{\Delta t}{2}\tau^x_{\widehat{u}^{n+1/2}}(x_i), \qquad (7.34)
\end{aligned}$$

where

$$\tau^x_{\widehat{u}}(x_i) = \begin{cases} \mathcal{L}^N_{1,mu}\widehat{u}_{x_i} - (\mathcal{L}_{1,\varepsilon}\widehat{u})_{i-1/2}, & \text{for } 1 \le i \le N/2, \\ \mathcal{L}^N_{1,cen}\widehat{u}_{x_i} - (\mathcal{L}_{1,\varepsilon}\widehat{u})(x_i), & \text{for } N/2 < i \le N-1, \end{cases}$$

is the truncation error corresponding to the hybrid scheme used for the stationary 1D singularly perturbed convection-diffusion problem, whose coefficients are given by (7.27), for $\ell = 1$.

Lemma 7.4.5. *Let $\widehat{u}^{n+1/2}$ and $\{\widehat{U}^{n+1/2}\}$ be the exact and the discrete solutions of (7.8) and (7.23) respectively, and suppose that $N \ge N_0$ satisfies the conditions (7.28) and (7.29), for $\ell = 1$. Then, if $\gamma_1 < \alpha_1/2$ and $\varepsilon \le N^{-1}$, we have the following bounds*

$$\big|\widehat{u}^{n+1/2}_{x_i} - \widehat{U}^{n+1/2}_{x_i}\big| \le \begin{cases} C\Big(N^{-2} + N^{-\gamma_1\tau_{1,0}}\Big), & \textit{for } 1 \le i \le N/2, \\ C\Big(\tau^2_{1,0}N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}}\Big), & \textit{for } N/2 < i \le N-1. \end{cases} \qquad (7.35)$$

Proof. Firstly, using (7.14) and following the arguments given in Lemma 2.4.4, we can easily deduce from (7.34) that the local truncation error of (7.23) satisfies the estimates

$$\big|\tau^1_{\widehat{u}^{n+1/2}}(x_i)\big| \le \mathfrak{B}_{\varepsilon,\Delta t}\Big(x_i, h_{1,i}, \alpha_1, h_1\Big), \qquad (7.36)$$

where

$$\mathfrak{B}_{\varepsilon,\Delta t}\Big(x_i, h_{1,i}, \alpha_1, h_1\Big) = \begin{cases} C\Delta t\Big[(\varepsilon + h_{1,i})h_{1,i} + \dfrac{1}{\max\{\varepsilon, h_{1,i}\}}\exp\big(-\alpha_1(1-x_{i+1})/\varepsilon\big)\Big], \\ \hfill \text{for } 1 \le i < N/2, \\ C\Delta t\Big[(\varepsilon + h_{1,i})h_{1,i} + \varepsilon^{-1}\exp\big(-\alpha_1(1-x_{i+1})/\varepsilon\big)\Big], \\ \hfill \text{for } i = N/2, \\ C\Delta t\Big[h_1^2 + h_1^2\varepsilon^{-3}\exp\big(-\alpha_1(1-x_i)/\varepsilon\big)\Big], \\ \hfill \text{for } N/2 < i \le N-1. \end{cases}$$

Now, consider the following barrier function

$$\psi_{1,i}(\gamma_1) = C\Big[(\varepsilon + h_{1,i})h_{1,i}(1+x_i) + \mu_{1,i+1}(\gamma_1)\phi_{1,i}(\gamma_1)\Big], \quad \text{for } 0 \le i \le N.$$

Then, using the result (7.30) obtained in Lemma 7.4.3 for $l = 1$, inequality (7.32) and the esimate (7.36), we get

$$\mathcal{L}^N_{1,\varepsilon}\psi_{1,i}(\gamma_1) \ge \big|\tau^1_{\widehat{u}^{n+1/2}}(x_i)\big|, \quad \text{for } 1 \le i \le N-1.$$

Thus, by applying the discrete maximum principle for the operator $\mathcal{L}^N_{1,\varepsilon}$, we obtain that

$$\big|\widehat{u}^{n+1/2}_{x_i} - \widehat{U}^{n+1/2}_{x_i}\big| \le \psi_{1,i}(\gamma_1), \quad \text{for } 1 \le i \le N-1.$$

Now, using $H_1 \le 2N^{-1}$, $\varepsilon \le N^{-1}$ and invoking the inequalities (7.31), (7.33) for $l = 1$, finally we have

$$\big|\widehat{u}^{n+1/2}_{x_i} - \widehat{U}^{n+1/2}_{x_i}\big| \le C\Big(N^{-2} + N^{-\gamma_1\tau_{1,0}}\Big), \quad \text{for } 1 \le i \le N/2.$$

On the other hand, for $N/2 < i \le N-1$, consider the new barrier function

$$\varphi_{1,i}(\gamma_1) = C\Big[(N^{-2} + N^{-\gamma_1\tau_{1,0}})(1+x_i) + h_1^2\varepsilon^{-2}\phi_{1,i}(\gamma_1)\Big].$$

Then, it follows that

$$\begin{cases} \mathcal{L}^N_{1,\varepsilon}\varphi_{1,i}(\gamma_1) \ge \big|\tau^1_{\widehat{u}^{n+1/2}}(x_i)\big|, \quad \text{for } N/2 < i \le N-1, \\ \varphi_{1,N/2}(\gamma_1) \ge \big|\widehat{u}^{n+1/2}_{x_{N/2}} - \widehat{U}^{n+1/2}_{x_{N/2}}\big|, \quad \text{and} \quad \varphi_{1,N}(\gamma_1) \ge \big|\widehat{u}^{n+1/2}_{x_N} - \widehat{U}^{n+1/2}_{x_N}\big|. \end{cases}$$

Henceforth, using $h_1 = 2\tau_{1,0}\varepsilon N^{-1}\ln N$ and applying the discrete maximum principle for the operator $\mathcal{L}^N_{1,\varepsilon}$ on $[1-\tau, 1]$, we obtain

$$\big|\widehat{u}^{n+1/2}_{x_i} - \widehat{U}^{n+1/2}_{x_i}\big| \le \varphi_{1,i}(\gamma_1) \le C\Big(\tau^2_{1,0}N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}}\Big), \quad \text{for } N/2 < i \le N-1. \qquad \blacksquare$$

Next, we shall estimate the local error obtained due to the discretization of the problem (7.9) by the numerical scheme (7.19) in the y-direction. According to the requirement, we mention the dependence on $x \in I_{1,\varepsilon}^N$ in the following expressions, otherwise it is omitted. Now, for $x \in I_{1,\varepsilon}^N$, from (7.9) and (7.21) we get

$$\begin{aligned} & \widehat{F}_{2,\widehat{U}^{n+1/2}}(x, y_j, t_{n+1/2}) \\ = & \left(I - \frac{\Delta t}{2}\,\overline{\mathcal{L}}_{1,\varepsilon}^N\right)\widehat{U}_{y_j}^{n+1/2} - \left(I - \frac{\Delta t}{2}\,\mathcal{L}_{1,\varepsilon}\right)\widehat{u}^{n+1/2}(y_j) + \left(I + \frac{\Delta t}{2}\,\mathcal{L}_{2,\varepsilon}\right)\widehat{u}^{n+1}(y_j) \\ = & \mathcal{R}(y_j), \text{ say.} \end{aligned}$$

Then, with this and using (7.25)-(7.26) for $\ell = 2$, we define the local truncation error for the scheme (7.24) as

$$\begin{aligned} \tau_{\widehat{u}^{n+1}}^2(y_j) &= \mathcal{L}_{2,\varepsilon}^N[\widehat{u}_{y_j}^{n+1} - \widehat{U}_{y_j}^{n+1}] \\ &= \begin{cases} r_j^{2,-}\widehat{u}_{y_{j-1}}^{n+1} + r_j^{2,0}\widehat{u}_{y_j}^{n+1} + r_j^{2,+}\widehat{u}_{y_{j+1}}^{n+1} - \dfrac{1}{2}\big(\mathcal{R}(y_j) + \mathcal{R}(y_{j-1})\big), & \text{for } 1 \le j \le N/2, \\ r_j^{2,-}\widehat{u}_{y_{j-1}}^{n+1} + r_j^{2,0}\widehat{u}_{y_j}^{n+1} + r_j^{2,+}\widehat{u}_{y_{j+1}}^{n+1/2} - \mathcal{R}(y_j), & \text{for } N/2 < j \le N-1, \end{cases} \\ &= \frac{\Delta t}{2}\,\tau_{\widehat{u}^{n+1}}^y(y_j) + \begin{cases} \dfrac{1}{2}\displaystyle\sum_{k=1}^{2}\Big[\big(\widehat{u}_{y_{j-k+1}}^{n+1/2} - \widehat{U}_{y_{j-k+1}}^{n+1/2}\big) + \dfrac{\Delta t}{2}\big(\overline{\mathcal{L}}_{1,\varepsilon}^N\widehat{U}_{y_{j-k+1}}^{n+1/2} - \mathcal{L}_{1,\varepsilon}\widehat{u}_{y_{j-k+1}}^{n+1/2}\big)\Big], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \le j \le N/2, \\ \big(\widehat{u}_{y_j}^{n+1/2} - \widehat{U}_{y_j}^{n+1/2}\big) + \dfrac{\Delta t}{2}\big(\overline{\mathcal{L}}_{1,\varepsilon}^N\widehat{U}_{y_j}^{n+1/2} - \mathcal{L}_{1,\varepsilon}\widehat{u}_{y_j}^{n+1/2}\big), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } N/2 < j \le N-1, \end{cases} \end{aligned} \tag{7.37}$$

where

$$\tau_{\widehat{u}}^y(y_j) = \begin{cases} \mathcal{L}_{2,mu}^N\widehat{u}_{y_j} - (\mathcal{L}_{2,\varepsilon}\widehat{u})_{j-1/2}, & \text{for } 1 \le j \le N/2, \\ \mathcal{L}_{2,cen}^N\widehat{u}_{y_j} - (\mathcal{L}_{2,\varepsilon}\widehat{u})(y_j), & \text{for } N/2 < j \le N-1, \end{cases}$$

is the truncation error corresponding to the hybrid scheme used for the stationary 1D singularly perturbed convection-diffusion problem, whose coefficients are given by (7.27), for $\ell = 2$.

Theorem 7.4.6. *Let $\widehat{u}^{n+1}$ and $\{\widehat{U}^{n+1}\}$ be the exact and the discrete solutions of (7.9) and (7.24) respectively, and suppose that $N \ge N_0$ satisfies the conditions (7.28) and (7.29), for*

$\ell = 2$. Then, if $\gamma_2 < \alpha_2/2$ and $\varepsilon \le N^{-1}$, we have the following bounds

$$\left|\widehat{u}^{n+1}_{x_i,y_j} - \widehat{U}^{n+1}_{x_i,y_j}\right| \le \begin{cases} C\Big(N^{-2} + N^{-\gamma_1\tau_{1,0}} + N^{-\gamma_2\tau_{2,0}}\Big), \\ \qquad\qquad \textit{for } (x_i,y_j) \in (0,1-\tau_1] \times (0,1-\tau_2], \\ C\Big(\tau_{2,0}^2 N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}} + N^{-\gamma_2\tau_{2,0}}\Big), \\ \qquad\qquad \textit{for } (x_i,y_j) \in (0,1-\tau_1] \times (1-\tau_2,1), \\ C\Big(\tau_{1,0}^2 N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}} + N^{-\gamma_2\tau_{2,0}}\Big), \\ \qquad\qquad \textit{for } (x_i,y_j) \in (1-\tau_1,1) \times (0,1-\tau_2], \\ C\Big((\tau_{1,0}^2+\tau_{2,0}^2) N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}} + N^{-\gamma_2\tau_{2,0}}\Big), \\ \qquad\qquad \textit{for } (x_i,y_j) \in (1-\tau_1,1) \times (1-\tau_2,1). \end{cases} \tag{7.38}$$

Proof. Firstly, from (7.8) and (7.22), we have

$$\big(\overline{\mathcal{L}}^N_{1,\varepsilon}\widehat{U}^{n+1/2}_y - \mathcal{L}_{1,\varepsilon}\widehat{u}^{n+1/2}_y\big) = \frac{2}{\Delta t}\big(\widehat{u}^{n+1/2}_y - \widehat{U}^{n+1/2}_y\big). \tag{7.39}$$

Then, employing (7.37) in (7.39) yields

$$\begin{aligned} \tau^2_{\widehat{u}^{n+1}}(y_j) &= \frac{\Delta t}{2}\,\tau^y_{\widehat{u}^{n+1}}(y_j) + \begin{cases} \displaystyle\sum_{k=1}^{2}\big(\widehat{u}^{n+1/2}_{y_{j-k+1}} - \widehat{U}^{n+1/2}_{y_{j-k+1}}\big), & \text{for } 1 \le j \le N/2, \\ 2\big(\widehat{u}^{n+1/2}_{y_j} - \widehat{U}^{n+1/2}_{y_j}\big), & \text{for } N/2 < j \le N-1, \end{cases} \\ &= (I) + (II), \quad \text{say.} \end{aligned} \tag{7.40}$$

Now, (I) can be estimated in the similar way as in the previous lemma and also, we can apply the bounds (7.35) obtained in Lemma 7.4.5 to estimate (II). This leads the local truncation error of (7.24) to satisfy the following estimates for $1 \le j \le N-1$,

$$\left|\tau^2_{\widehat{u}^{n+1}}(x,y_j)\right| \le \begin{cases} \mathfrak{B}_{\varepsilon,\Delta t}\Big(y_j, h_{2,j}, \alpha_2, h_2\Big) + C\Big(N^{-2} + N^{-\gamma_1\tau_{1,0}}\Big), \\ \qquad\qquad \text{for } x \in I^N_{1,\varepsilon}\bigcap(0,1-\tau_1], \\ \mathfrak{B}_{\varepsilon,\Delta t}\Big(y_j, h_{2,j}, \alpha_2, h_2\Big) + C\Big(\tau_{1,0}^2 N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}}\Big), \\ \qquad\qquad \text{for } x \in I^N_{1,\varepsilon}\bigcap(1-\tau_1,1). \end{cases} \tag{7.41}$$

In rest of this proof we shall consider different cases depending on the location of mesh point $x \in I^N_{1,\varepsilon}$.

Case 1. For $x \in I^N_{1,\varepsilon}\bigcap(0,1-\tau_1]$. Consider the following barrier function

$$\psi_{2,j}(\gamma_2) = C\Big[\big(N^{-2} + N^{-\gamma_1\tau_{1,0}}\big)(1+y_j) + \mu_{2,j+1}(\gamma_2)\phi_{2,j}(\gamma_2)\Big], \quad \text{for } 0 \le j \le N.$$

Then, arguing in the same way as like Lemma 7.4.5 for $l = 2$, and the esimate (7.41), leads to

$$\left|\widehat{u}^{n+1}_{y_j} - \widehat{U}^{n+1}_{y_j}\right| \le \psi_{2,j}(\gamma_2), \quad \text{for } 1 \le j \le N-1.$$

Finally, invoking the inequalities (7.31) and (7.33) for $l = 2$, we have

$$\left|\widehat{u}^{n+1}_{y_j} - \widehat{U}^{n+1}_{y_j}\right| \le C\Big(N^{-2} + N^{-\gamma_1\tau_{1,0}} + N^{-\gamma_2\tau_{2,0}}\Big), \quad \text{for } 1 \le j \le N/2.$$

On the other hand, for $N/2 < j \le N-1$, consider the new barrier function

$$\varphi_{2,j}(\gamma_2) = C\Big[\big(N^{-2} + N^{-\gamma_1\tau_{1,0}} + N^{-\gamma_2\tau_{2,0}}\big)(1+y_j) + h_2^2\varepsilon^{-2}\phi_{2,j}(\gamma_2)\Big],$$

and follow the similar technique as in Lemma 7.4.5 on $[1-\tau_2, 1]$, to obtain the error bound

$$\left|\widehat{u}^{n+1}_{y_j} - \widehat{U}^{n+1}_{y_j}\right| \le \varphi_{2,j}(\gamma_2) \le C\Big(\tau^2_{2,0}N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}} + N^{-\gamma_2\tau_{2,0}}\Big), \quad \text{for } N/2 < j \le N-1,$$

where $h_2 = 2\tau_{2,0}\varepsilon N^{-1}\ln N$ is used.

Case 2. For $x \in I^N_{1,\varepsilon}\bigcap(1-\tau_1, 1)$. Considering the following barrier functions

$$\begin{cases} \psi_{2,j}(\gamma_2) = C\Big[\big(\tau^2_{1,0}N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}}\big)(1+y_j) + \mu_{2,j+1}(\gamma_2)\phi_{2,j}(\gamma_2)\Big], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \le j \le N. \\ \varphi_{2,j}(\gamma_2) = C\Big[\big(\tau^2_{1,0}N^{-2}\ln^2 N + N^{-\gamma_1\tau_{1,0}} + N^{-\gamma_2\tau_{2,0}}\big)(1+y_j) + h_2^2\varepsilon^{-2}\phi_{2,j}(\gamma_2)\Big], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } N/2 < j \le N-1. \end{cases}$$

and arguing similarly as in the previous case, the desired results follows. ■

Corollary 7.4.7. *Theorem 7.4.6 implies that for fixed $\tau_{\ell,0} \ge 2/\gamma_\ell$, $\ell = 1, 2$, there exists a constant C such that*

$$\left|\widehat{u}^{n+1}_{x_i,y_j} - \widehat{U}^{n+1}_{x_i,y_j}\right| \le \begin{cases} CN^{-2}, & \textit{for } (x_i, y_j) \in (0, 1-\tau_1] \times (0, 1-\tau_2], \\ CN^{-2}\ln^2 N, & \textit{otherwise}, \end{cases} \tag{7.42}$$

Therefore, the method (7.23)-(7.24) is in practice uniformly convergent of order almost two with respect to the spatial variables but the error constant C depends only on the value of $\tau_{\ell,0}$ (which is independent of ε).

Corollary 7.4.8. *If we take $N^{-q} \le C(\Delta t)^2$ with $0 < q < 1$, then from (7.42) we obtain*

$$\left|\widehat{u}^{n+1}_{x_i,y_j} - \widehat{U}^{n+1}_{x_i,y_j}\right| \le \begin{cases} C\Delta t\, N^{-2+q}, & \textit{for } (x_i, y_j) \in (0, 1-\tau_1] \times (0, 1-\tau_2], \\ C\Delta t\, N^{-2+q}\ln^2 N, & \textit{otherwise}. \end{cases} \tag{7.43}$$

This bound is required to prove the uniform convergence of the fully discrete scheme.

7.5 Uniform Convergence of the Fully Discrete Scheme

Combining the time semidiscretization schemes (7.8)-(7.9) and the spatial discretization technique given in Section 7.4, the following fully discrete scheme is deduced on the mesh $\overline{\mathcal{G}}_{\varepsilon}^{N,M} = \overline{\mathcal{D}}_{\varepsilon}^{N} \times \mathbb{S}_t^M$:

$$
\begin{cases}
U^0_{x_i,y_j} = u_0(x_i,y_j), \quad 0 \le i,j \le N,\\
\overline{\mathcal{L}}^N_{2,\varepsilon} U^0_{x_i,y_j} = \mathcal{L}_{2,\varepsilon} u_0(x_i,y_j), \quad 0 \le i,j \le N,\\
\begin{cases}
\mathcal{L}^N_{1,\varepsilon} U^{n+1/2}_{x_i,y} & \equiv r_i^{1,-} U^{n+1/2}_{x_{i-1},y} + r_i^{1,0} U^{n+1/2}_{x_i,y} + r_i^{1,+} U^{n+1/2}_{x_{i+1},y}\\
& = \dfrac{1}{2}\Big(\widehat{F}_{1,U^n}(x_i,y,t_n) + \widehat{F}_{1,U^n}(x_{i-1},y,t_n)\Big), \quad \text{for } 1 \le i \le N/2,\\
\mathcal{L}^N_{1,\varepsilon} U^{n+1/2}_{x_i,y} & \equiv r_i^{1,-} U^{n+1/2}_{x_{i-1},y} + r_i^{1,0} U^{n+1/2}_{x_i,y} + r_i^{1,+} U^{n+1/2}_{x_{i+1},y}\\
& = \widehat{F}_{1,U^n}(x_i,y,t_n), \quad \text{for } N/2 < i \le N-1,\\
U^{n+1/2}_{0,y} = U^{n+1/2}_{1,y} & = 0, \quad y \in I^N_{2,\varepsilon},
\end{cases}\\
\overline{\mathcal{L}}^N_{1,\varepsilon} U^{n+1/2}_{x,y} = -\overline{\mathcal{L}}^N_{2,\varepsilon} U^n_{x,y} - \dfrac{2}{\Delta t}\big(U^{n+1/2}_{x,y} - U^n_{x,y}\big) + \big(f_1(x,y,t_{n+1/2}) + f_2(x,y,t_n)\big),\\
\begin{cases}
\mathcal{L}^N_{2,\varepsilon} U^{n+1}_{x,y_j} & \equiv r_j^{2,-} U^{n+1}_{x,y_{j-1}} + r_j^{2,0} U^{n+1}_{x,y_j} + r_j^{2,+} U^{n+1}_{x,y_{j+1}}\\
& = \dfrac{1}{2}\Big(\widehat{F}_{2,U^{n+1/2}}(x,y_j,t_{n+1/2}) + \widehat{F}_{2,U^{n+1/2}}(x,y_{j-1},t_{n+1/2})\Big),\\
& \qquad \text{for } 1 \le j \le N/2,\\
\mathcal{L}^N_{2,\varepsilon} U^{n+1}_{x,y_j} & \equiv r_j^{2,-} U^{n+1}_{x,y_{j-1}} + r_j^{2,0} U^{n+1}_{x,y_j} + r_j^{2,+} U^{n+1}_{x,y_{j+1}}\\
& = \widehat{F}_{2,U^{n+1/2}}(x,y_j,t_{n+1/2}), \quad \text{for } N/2 < j \le N-1,\\
U^{n+1}_{x,0} = U^{n+1}_{x,1} & = 0, \quad x \in I^N_{1,\varepsilon},
\end{cases}\\
\overline{\mathcal{L}}^N_{2,\varepsilon} U^{n+1}_{x,y} = -\overline{\mathcal{L}}^N_{1,\varepsilon} U^{n+1/2}_{x,y} - \dfrac{2}{\Delta t}\big(U^{n+1}_{x,y} - U^{n+1/2}_{x,y}\big) + \big(f_1(x,y,t_{n+1/2}) + f_2(x,y,t_{n+1})\big),\\
\text{for } n = 0,\dots,M-1,
\end{cases}
\tag{7.44}
$$

where the coefficients $r_k^{\ell,\bullet}$, $\ell = 1,2$, $1 \le k \le N-1$ are described in (7.25)-(7.27) and $U^n_{x_i,y_j} = U(x_i,y_j,t_n)$ is the fully discrete approximation to the exact solution $u(\mathbf{x},t)$ of (7.1)-(7.2) at the mesh point $(x_i,y_j,t_n) \in \overline{\mathcal{G}}_{\varepsilon}^{N,M}$.

Theorem 7.5.1. ***(Global error)*** *Let $u(\mathbf{x},t_n)$ be the exact solution of (7.1)-(7.2) and $\{U^n\}$ be the discrete solution of the fully discrete scheme (7.44), at time level $t_n = n\Delta t$. Assume that $N \ge N_0$ satisfies the conditions (7.28), (7.29) and $\varepsilon \le N^{-1}$, $N^{-q} \le C(\Delta t)^2$ with $0 < q < 1$. Then, if $\gamma_\ell < \alpha_\ell/2$ and $\tau_{\ell,0} \ge 2/\gamma_\ell$, $\ell = 1,2$, the error associated with the total* *discretization of (7.1)-(7.2) by the fully discrete scheme (7.44) at time level t_n satisfies*

$$\left\|\left\{u(x_i,y_j,t_n)\right\}_{i,j}-\left\{U^n_{x_i,y_j}\right\}_{i,j}\right\|_\infty \le \begin{cases} C\Big(\Delta t+N^{-2+q}\Big), & \\ \qquad \textit{for } (x_i,y_j)\in(0,1-\tau_1]\times(0,1-\tau_2], & \\ C\Big(\Delta t+N^{-2+q}\ln^2 N\Big), & \textit{otherwise.}\end{cases} \tag{7.45}$$

Proof. Here, we shall present the proof for the outer region *i.e.*, for $(x_i,y_j)\in(0,1-\tau_1]\times(0,1-\tau_2]$ only. Let the vector of the global errors at time level t_n be denoted by $E_n^N=\left\{u(x_i,y_j,t_n)\right\}_{i,j}-\left\{U^n_{x_i,y_j}\right\}_{i,j}$, the vector of the local errors by $e_n^N=\left\{u(x_i,y_j,t_n)\right\}_{i,j}-\left\{\widehat{u}(x_i,y_j,t_n)\right\}_{i,j}$ and the residual by $d_n^N=\left\{\widehat{u}(x_i,y_j,t_n)\right\}_{i,j}-\left\{\widehat{U}(x_i,y_j,t_n)\right\}_{i,j}$. Then, splitting the global error E_n^N, we have

$$E_n^N=e_n^N+d_n^N+R_NE_{n-1}^N, \tag{7.46}$$

where R_N is the transition operator associated to the fully discrete scheme (7.44), which is defined in the following way: $R_NE_{n-1}^N$ is the result obtained after one step of the scheme (7.44) with $U^n=E_{n-1}^N$ and the source term f_1,f_2 equal to zero. From the recurrence (7.46), we deduce the following form

$$E_n^N=\sum_{i=1}^{n}R_N^{n-i}e_i^N+\sum_{i=1}^{n}R_N^{n-i}d_i^N. \tag{7.47}$$

Afterwards, taking into account that the same uniform bounds hold for R_N^j that we have obtained for R^j and also applying the estimates given by (7.10) and (7.43) in the equation (7.47), we obtain that

$$\begin{aligned}\|E_n^N\|_\infty &\le \sum_{i=1}^{n}\|R_N^{n-i}\|_\infty\big(\|e_i^N\|_\infty+\|d_i^N\|_\infty\big) \le \frac{C}{\Delta t}\sum_{i=1}^{n}\big((\Delta t)^3+(\Delta t)^2N^{-2+q}\big)\\ &\le C\big(\Delta t+N^{-2+q}\big).\end{aligned}$$

The proof for rest of the regions follows analogously. ■

7.6 Numerical Results

This section presents the numerical results obtained by the fully discrete scheme (7.44) to verify the theoretical results claimed in the previous section. To do this, the numerical experiments are carried out for the following test example on the piecewise-uniform mesh $\overline{\mathcal{G}}_\varepsilon^{N,M}$. In all the cases, the numerical experiments are performed by choosing the constant

$\tau_{1,0} = \tau_{2,0} = 4.2$, and $\Delta t = 1.6/N$, otherwise it is mentioned. Note that in the following example, we decompose the source term $f(\mathbf{x}, t)$ in the form

$$f_2(\mathbf{x}, t) = f(x, 0, t) + y\big(f(x, 1, t) - f(x, 0, t)\big), \quad f_1(\mathbf{x}, t) = f(\mathbf{x}, t) - f_2(\mathbf{x}, t),$$

so that the property (7.3) is satisfied under the following compatibility condition:

$$f(0, 0, t) = f(0, 1, t) = f(1, 0, t) = f(1, 1, t) = 0, \quad \text{for } t \in [0, T].$$

Example 7.6.1. *Consider the following parabolic IBVP:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \triangle u + (1 + x(1-x))\dfrac{\partial u}{\partial x} + (1 + y(1-y))\dfrac{\partial u}{\partial y} = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathcal{D} \times (0, 1], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\mathcal{D}}, \\ u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\mathcal{D} \times [0, 1], \end{cases} \tag{7.48}$$

where $\mathcal{D} = (0, 1)^2$, the source term $f(\mathbf{x}, t)$ and the initial data $u(\mathbf{x}, 0)$ are such that the exact solution of the above problem is

$$u(\mathbf{x}, t) = \big(1 - \exp(-t)\big)\Big(m_1 + m_2 x + \exp\big(-(1-x)/\varepsilon\big)\Big)\Big(m_1 + m_2 y + \exp\big(-(1-y)/\varepsilon\big)\Big),$$

with $m_1 = -\exp(-1/\varepsilon)$, $m_2 = -1 - m_1$. As the exact solution of the IBVP (7.48) is known, for each ε, the maximum point-wise error is calculated by

$$e_\varepsilon^{N,\Delta t} = \max_{(x_i, y_j, t_n) \in \overline{\mathcal{G}}_\varepsilon^{N,M}} \Big| u(x_i, y_j, t_n) - U^{N,\Delta t}(x_i, y_j, t_n) \Big|,$$

where $u(x_i, y_j, t_n)$ and $U^{N,\Delta t}(x_i, y_j, t_n)$ denote the exact and the numerical solution obtained on the mesh $\overline{\mathcal{G}}_\varepsilon^{N,M}$ with N mesh-intervals in the x- and y-directions, and M mesh-intervals in the t-direction such that $\Delta t = T/M$ is the uniform time step. In addition, the corresponding order of convergence is determined by

$$p_\varepsilon^{N,\Delta t} = \log_2\left(\frac{e_\varepsilon^{N,\Delta t}}{e_\varepsilon^{2N,\Delta t/2}}\right).$$

Here, for each N and Δt, the quantities $e^{N,\Delta t}$ and $p^{N,\Delta t}$ are defined analogously as defined in Chapter 2. The calculated maximum point-wise errors $e_\varepsilon^{N,\Delta t}$ and the corresponding order of convergence $p_\varepsilon^{N,\Delta t}$ for Example 7.6.1 are presented in Table 7.1 for various values of ε and N.

From the results given in Table 7.1 we see that the computed ε-uniform errors $E^{N,\Delta t}$ decrease monotonically as N increases. This ensures that the proposed scheme (7.44) is ε-uniformly convergent. As a complement of this observation, we have plotted the maximum point-wise errors in Figure 7.3. In order to have the complete understanding of the numerical solution of the IBVP (7.48), we have included two surface plots as well as the corresponding contour plots at the final time for $\varepsilon = 10^{-2}, 10^{-4}$ and $N = 128$ in Figure 7.1. Moreover, we have graphically shown the surface plots the errors corresponding to $\varepsilon = 10^{-2}, 10^{-4}$ in Figure 7.2. We observe that the errors are very high inside the layer regions (particularly in the corner region), where the gradient of the solutions becomes steep for the small values of ε as shown in Figure 7.1.

Next, we see that the numerical results presented in Table 7.1 do not clearly reflect the actual theoretical order of convergence of the proposed scheme (7.44) for the spatial variable, as predicted by Theorem 7.5.1. Therefore, to justify the spatial order of convergence properly, the numerical experiments are carried out by taking $M = N$ and the maximum point-wise errors as well as the corresponding order of convergence are displayed in Table 7.2. Further, we have highlighted those errors in Figures 7.4. This in fact shows that the proposed scheme (7.44) is second-order spatial accurate outside the layer regions. Whereas the almost second-order convergence in space (reduced by the logarithmic factor) is reflected inside the layer regions, as derived in Corollary 7.4.7. Even this analysis also signifies the role of $\tau_{\ell,0},\ \ell = 1, 2$ to fulfill our claim in the same corollary.

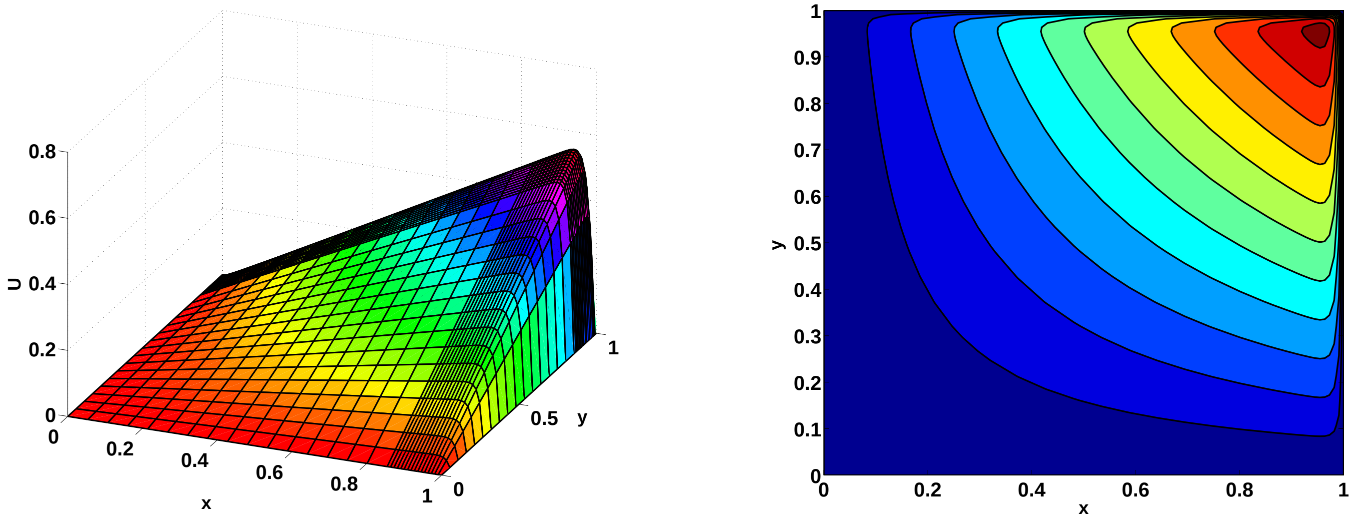
Remark 7.6.2. *From a practical point of view, we would like to mention that though we have used the weak stability property* $\|R_N^i\|_\infty \leq C/\Delta t$ *to obtain the estimate (7.45) in Theorem 7.5.1, the above numerical experiments correspond to the bound* $\|R_N^i\|_\infty \leq C$ *(a discrete version of (7.11)). However, if we suppose that this stronger stability property is satisfied, then the restriction* $N^{-q} \leq C(\Delta t)^2$ *can be weakened to* $N^{-q} \leq C(\Delta t)$ *and hereby, we can avoid the theoretical order reduction in time by one, for the proposed scheme (7.44).*

7.7 Conclusion

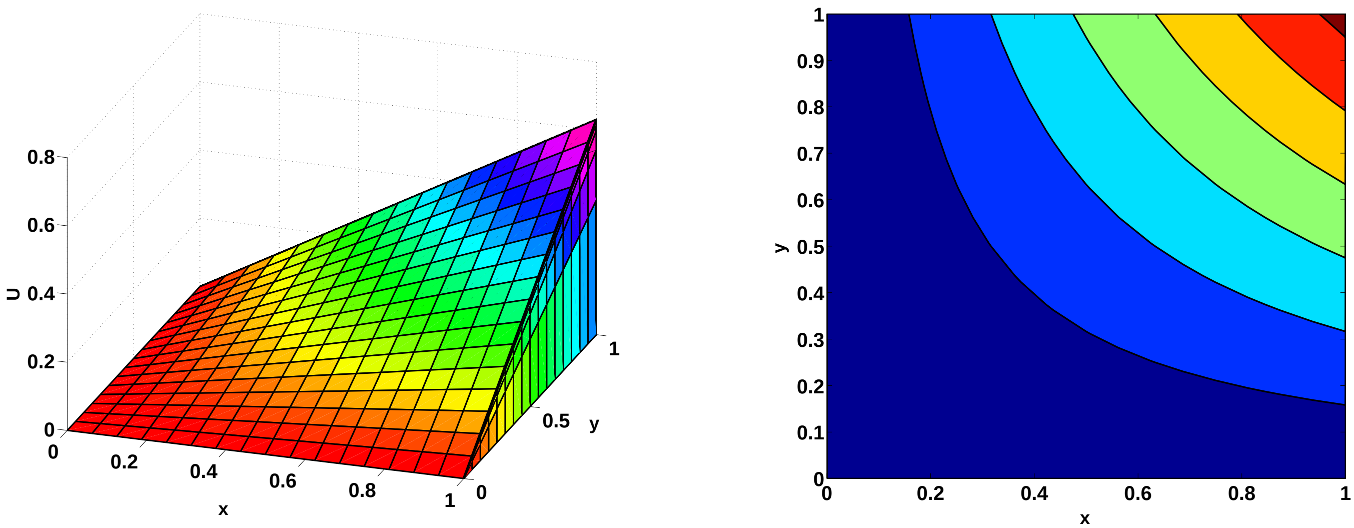
In this chapter, two-dimensional singularly perturbed parabolic convection-diffusion IBVP exhibiting a regular boundary layer is solved by using an efficient hybrid numerical scheme on a spacial rectangular mesh. For the spatial discretization, the hybrid scheme utilizes a proper combination of the midpoint upwind scheme and the classical central difference scheme and for discretizing the time derivative, the Peaceman and Rachford method scheme is used. It has been theoretically proved that the newly prosed scheme is ε-uniform convergent with almost second-order accurate in space. It is computationally observed that the scheme is also second-order accurate in time, though the theoretical order reduction in time is still there. This method has an advantage of solving only the tridiagonal linear systems instead of the block tridiagonal system, which therefore reduces the computational cost. In fact in this chapter, we have successfully extended the hybrid scheme proposed for one-dimensional case in Chapter 2.

Table 7.1: *Maximum point-wise errors and the corresponding order of convergence for Example 7.6.1.*

ε	Number of mesh intervals N				
	16	32	64	128	256
$1e-1$	7.3165e-3	1.6942e-3	3.9650e-4	9.4858e-5	2.8085e-5
	2.1105	2.0953	2.0635	1.7560	
$1e-2$	7.4781e-2	2.2646e-2	7.9849e-3	2.6983e-3	8.7976e-4
	1.7234	1.5039	1.5652	1.6169	
$1e-3$	7.6813e-2	2.3311e-2	8.2673e-3	2.7843e-3	9.0823e-4
	1.7203	1.4955	1.5701	1.6162	
$1e-4$	7.7092e-2	2.3381e-2	8.2957e-3	2.7929e-3	9.1109e-4
	1.7212	1.4949	1.5706	1.6161	
$1e-5$	7.7127e-2	2.3393e-2	8.2993e-3	2.7938e-3	9.1137e-4
	1.7211	1.4950	1.5708	1.6161	
$1e-6$	7.7131e-2	2.3395e-2	8.2998e-3	2.7939e-3	9.1141e-4
to $1e-8$	1.7211	1.4950	1.5708	1.6161	
$e^{N,\Delta t}$	**7.7131e-2**	**2.3395e-2**	**8.2998e-3**	**2.7939e-3**	**9.1141e-4**
$p^{N,\Delta t}$	**1.7211**	**1.4950**	**1.5708**	**1.6161**	

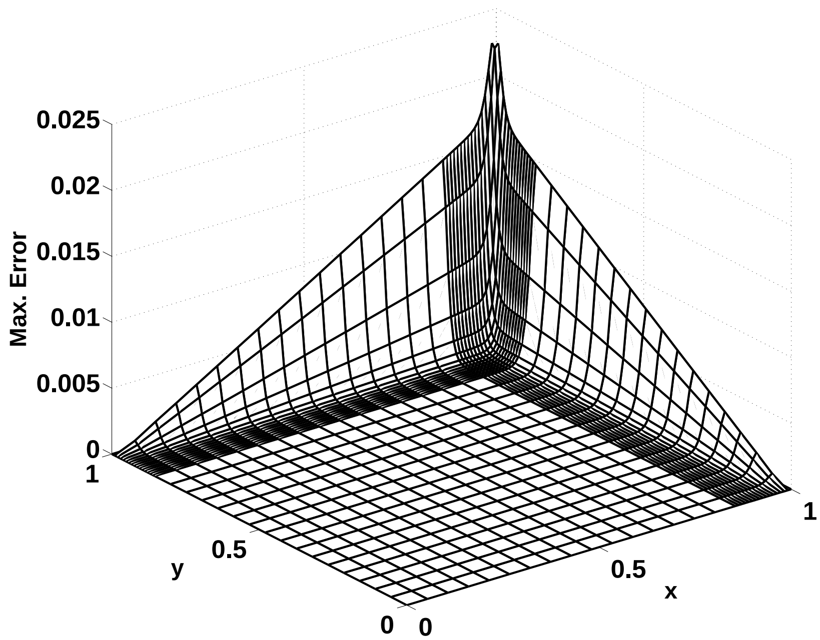


(a) $\varepsilon = 1e-2$.

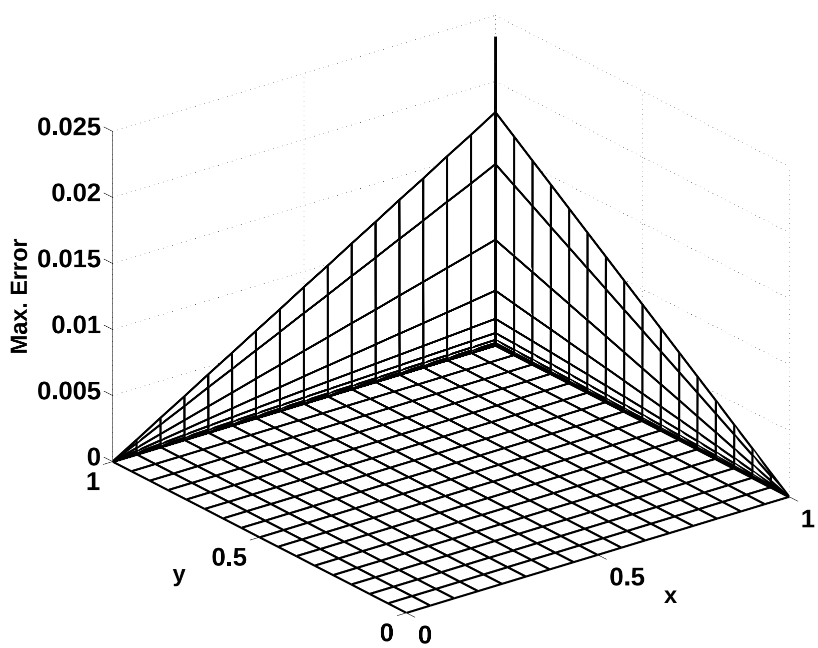


(b) $\varepsilon = 1e-4$.

Figure 7.1: *Surface and contour plots of the Numerical solution* U *at* $t = 1$ *and* $N = 32$ *for Example 7.6.1.*



(a) $\varepsilon = 1e - 2$.



(b) $\varepsilon = 1e - 4$.

Figure 7.2: *Surface plots of Error*$=|u - U|$ *at* $t = 1$ *and* $N = 32$ *for Example 7.6.1.*

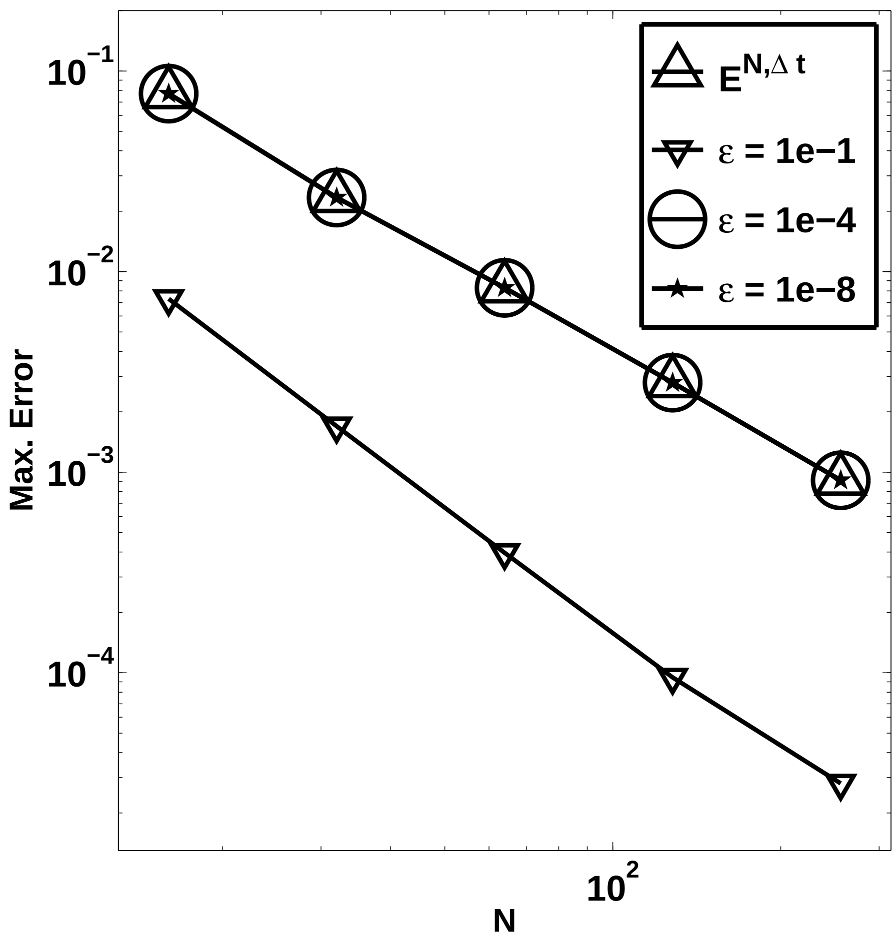


Figure 7.3: *Loglog plot of the maximum point-wise errors for Example 7.6.1.*

Table 7.2: *Maximum point-wise errors and the corresponding order of convergence calculated for Example 7.6.1 by taing $M = N$.*

N	$\varepsilon = 10^{-4}$			
	outer region $[0, 1-\tau_1]\times[0, 1-\tau_2]$	**right boundary layer region** $(1-\tau_1, 1]\times[0, 1-\tau_2]$	**top boundary layer region** $[0, 1-\tau_1]\times(1-\tau_2, 1]$	**corner layer region** $(1-\tau_1, 1]\times(1-\tau_2, 1]$
16	4.3288e-5 2.0207	4.7838e-2 1.4395	4.7827e-2 1.4392	7.7006e-2 1.7201
32	1.0668e-5 1.9973	1.7638e-2 1.5730	1.7637e-2 1.5730	2.3374e-2 1.4947
64	2.6720e-6 1.9989	5.9282e-3 1.5814	5.9282e-3 1.5814	8.2944e-3 1.5706
128	6.6852e-7 2.0016	1.9810e-3 1.6307	1.9810e-3 1.6307	2.7926e-3 1.6161
256	1.6694e-7	6.3972e-4	6.3973e-4	9.1100e-4
	$\varepsilon = 10^{-8}$			
	outer region	**right boundary layer region**	**top boundary layer region**	**corner layer region**
16	4.3584e-5 2.0191	4.7896e-2 1.4391	4.7882e-2 1.4387	7.7047e-2 1.7202
32	1.0753e-5 1.9948	1.7664e-2 1.5725	1.7663e-2 1.5724	2.3384e-2 1.4947
64	2.6978e-6 1.9969	5.9391e-3 1.5809	5.9391e-3 1.5809	8.2978e-3 1.5706
128	6.7591e-7 1.9993	1.9852e-3 1.6302	1.9852e-3 1.6302	2.7935e-3 1.6161
256	1.6906e-7	6.4130e-4	6.4130e-4	9.1132e-4

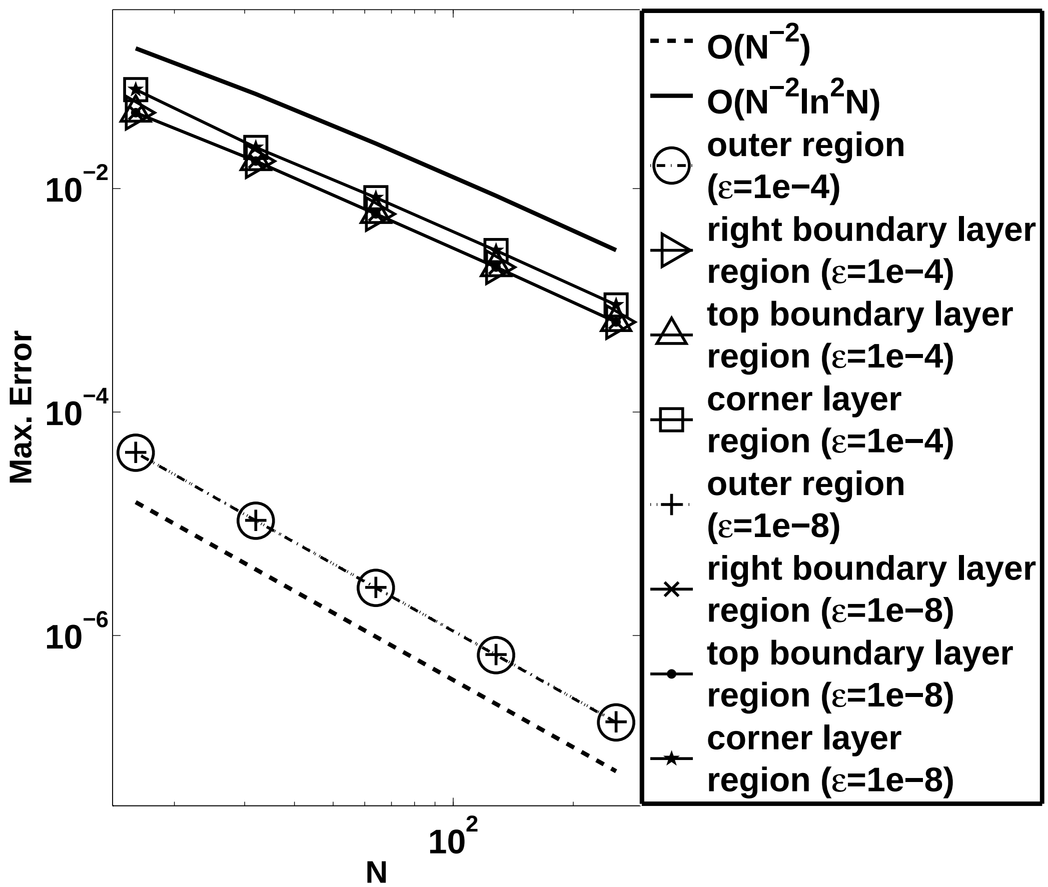


Figure 7.4: *Loglog plot of the spatial order of convergence for Example 7.6.1.*

Chapter 8

Conclusions

This chapter presents the summary of the contributed results made in this thesis, followed by the future scopes for possible extensions of the present works.

8.1 Summary of the Results

The work carried out in this thesis has achieved many interesting and good results which are concerned with solving singularly perturbed parabolic PDEs (with smooth and non-smooth data) as well as mixed parabolic-elliptic PDEs by efficient ε-uniformly convergent numerical methods. The results of this thesis with some important observations are precisely highlighted below:

- A uniformly convergent hybrid numerical scheme is proposed and analyzed for one-dimensional singularly perturbed parabolic convection-diffusion IBVPs with smooth and non-smooth data. It is proved that the method is almost second-order spatial accurate in the discrete supremum norm, provided the perturbation parameter ε satisfies $\varepsilon \leq N^{-1}$ (the most demandable case from the practical point of view).

- In case of the class of singularly perturbed parabolic IBVPs with discontinuous convection coefficients, an optimal order of convergence (up to a logarithmic factor) is obtained inside the layer regions for the newly proposed hybrid scheme. Besides, a clue has been provided to make the error estimate optimal corresponding to a hybrid scheme [10] for a class of singularly perturbed convection-diffusion BVPs possessing strong interiors.

- The idea behind the hybrid scheme is also successfully extended to the class of singularly perturbed mixed parabolic-elliptic type IBVPs exhibiting both boundary and interior layers. It is found that the method converges ε uniformly with an almost second-order spatial accuracy.

In the above results, it is observed that the use of the backward-Euler method for the time discretization does not reduce the spatial order of convergence of these new proposed hybrid schemes.

- Next, a second-order (both in space and time) uniformly convergent hybrid numerical scheme on a spacial rectangular mesh (tensor-product of 1D piecewise-uniform Shishkin meshs) is devised for two-dimensional singularly perturbed parabolic convection-diffusion IBVP, utilizing the Peaceman and Rachford method for the time discretization. This method has an advantage of solving only the tridiagonal linear systems instead of the block tridiagonal system and by this it reduces the cost of computation.

  Although, the theoretical order reduction in time is still there, it is computationally observed that the proposed method is second-order time accurate.

- Again, a unified theory for characterization of Shishkin-type meshes (including the piecewise-uniform Shishkin mesh and the Bakhalov-Shishkin mesh) resolving strong interior layers is provided for a class of singularly perturbed parabolic IBVPs having discontinuous convection coefficients and with this theoretical set up, it is shown that the optimal error bound obtained for the implicit upwind scheme on the Bakhalov-Shishkin mesh is in fact more accurate than that on the piecewise-uniform Shishkin mesh.

- Finally, through a post-processing technique (Richardson extrapolation) analyzed on a piecewise-uniform Shishkin mesh for singularly perturbed parabolic convection-diffusion IBVP with a regular boundary layer, the almost first-order convergence of the standard upwind scheme is converted into almost second-order convergence in the discrete supremum norm.

  Nevertheless, an easier approach has been provided to obtain the estimate corresponding to the smooth part of the error produced after extrapolation.

- Besides, an explicit proof of the ε-uniform error estimate of the classical implicit upwind scheme is also provided for singularly perturbed parabolic convection-diffusion IBVP exhibiting a layer on the right boundary. This in fact an optimal error estimate corresponding to the implicit upwind scheme in comparison with the proof given in [54] for the stationary convection-diffusion problem.

8.2 Future Scopes

This thesis has attempted to develop and analyze the robustness of the upwind based numerical methods through domain decomposition and post-processing techniques for different kinds of singularly perturbed time-dependent PDEs exhibiting boundary and interior layers. But the chances of examining the potential of these methods still remain for studying the numerical aspects of more complex PDEs. A brief outline, describing the possible extensions of the present works to be carried out in the future with suitable model problems, are presented below:

The hybrid numerical scheme, which is proposed and analyzed in **Chapter 2** for 1D singularly perturbed linear parabolic PDE with a regular boundary layer, can also be used to study the following 1D singularly perturbed parabolic Burgers' equation, posed on the domain $G = \Omega \times (0, T],\ \Omega = (0, 1)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \varepsilon\dfrac{\partial^2 u}{\partial x^2}, \quad (x,t) \in G, \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = u(1,t) = 0, \quad t \in (0,T], \end{cases} \tag{8.1}$$

The solution $u(x, t)$ of the parabolic IBVP (8.1), in general, possesses a regular boundary layer of width $O(\varepsilon)$ at $x = 1$.

Chapter 3 has provided the ε-uniform convergence analsis of Richardson extrapolation technique, which is applied to the standard upwind scheme for solving singularly perturbed parabolic PDE considered in **Chapter 2.** It will be much interesting to extend this analysis for the following singularly perturbed coupled system of parabolic IBVPs:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{L}_{x,\varepsilon}\mathbf{u} = \mathbf{f}, \quad (x,t) \in G, \\ \mathbf{u}(x,0) = \mathbf{0}, \quad x \in \overline{\Omega}, \\ \mathbf{u}(0,t) = \mathbf{u}(1,t) = \mathbf{0}, \quad t \in (0,T], \end{cases} \tag{8.2}$$

where

$$\mathbf{L}_{x,\varepsilon}\mathbf{u} \equiv -\mathrm{diag}(\varepsilon)\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}\mathbf{u},$$

with

$$\mathbf{A} = \begin{pmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix},$$

and the source term $\mathbf{f}$ is defined by $\mathbf{f} = (f_1(x,t), f_2(x,t))^{\mathrm{T}}$. Here, we assume that $\mathbf{B} = (b_{i,j})_{i,j=1}^2$ is an L_0-matrix (*i.e.*, whose diagonal entries are positive and off-diagonal entries are nonnegative) such that

$$\min_{x \in \overline{\Omega}} \left[b_{11}(x) + b_{12}(x), b_{21}(x) + b_{22}(x)\right] > \beta > 0.$$

and also assume that

$$a_1(x) > 2\alpha > 0, \quad a_2(x) > 2\alpha > 0, \quad x \in \overline{\Omega}.$$

Under these assumptions, the solution $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ of the system of IBVPs (8.2), in general, exhibits a boundary layer of width $O(\varepsilon \ln \varepsilon)$ at $x = 1$. It is to be noted that Deb and Natesan [20] recently studied Richardson extrapolation for singularly perturbed coupled system of convection-diffusion BVPs.

In **Chapter 4**, a uniformly convergent hybrid numerical scheme is proposed and analyzed for a class of singularly perturbed parabolic PDEs with discontinuous convection coefficients. One can further examine the efficiency of this scheme by applying it to the following class of singularly perturbed parabolic convection-diffusion IBVPs with discontinuous source term, posed on the domain $G^- \cup G^+$; $G^- = (0, \xi) \times (0, T]$, $G^+ = (\xi, 1) \times (0, T]$:

$$\begin{cases} L_\varepsilon u(x,t) \equiv \left(\varepsilon \dfrac{\partial^2 u}{\partial x^2} + a(x)\dfrac{\partial u}{\partial x} - b(x)u - \dfrac{\partial u}{\partial t}\right)(x,t) = f(x,t), \quad (x,t) \in G^- \cup G^+, \\ u(x,0) = s_0(x), \quad x \in \overline{\Omega}, \\ u(0,t) = s_1(t), \quad u(1,t) = s_2(t), \quad t \in (0,T], \end{cases} \tag{8.3}$$

with the following interface conditions

$$[u] = 0, \quad \left[\frac{\partial u}{\partial x}\right] = 0, \quad \text{at } x = \xi. \tag{8.4}$$

Here, we assume that the functions a, b, f satisfy the conditions

$$\begin{cases} a(x) > \alpha > 0, \quad b(x) \geq 0 \quad \text{on } \overline{\Omega}, \\ \big|[f]\big| \leq C, \quad \text{at } x = \xi. \end{cases}$$

Under these assumptions, the solution $u(x,t)$ of the parabolic IBVP (8.3)-(8.4), in general, possesses a boundary layer at $x = 0$ and a weak interior layer in the region to right side of the point $x = \xi$. Here, the width of both the layers is $O(\varepsilon)$.

Again, the theoretical framework given in **Chapter 5**, which unifies the ε-uniform convergence analysis of the implicit upwind scheme on Shishkin-type meshes, can also be carried out in future for the parabolic IBVP (8.3)-(8.4).

Next, consider the class of singularly perturbed mixed parabolic-elliptic IBVPs of the following form:

$$\begin{cases} L_{1,\varepsilon} u(x,t) \equiv \left(\dfrac{\partial u}{\partial t} - \varepsilon \dfrac{\partial^2 u}{\partial x^2} + b(x,t)u\right)(x,t) = f(x,t), \quad (x,t) \in G^-, \\ L_{2,\varepsilon} u(x,t) \equiv \left(-\varepsilon \dfrac{\partial^2 u}{\partial x^2} - a(x,t)\dfrac{\partial u}{\partial x} + b(x,t)u\right)(x,t) = f(x,t), \quad (x,t) \in G^+, \end{cases} \tag{8.5}$$

subject to the initial condition

$$u(x,0) = s_0(x), \quad x \in \overline{\Omega}, \tag{8.6}$$

and the following Robin type boundary conditions

$$\begin{cases} \lambda_1 u(0,t) + \varepsilon\gamma_1 \dfrac{\partial u}{\partial x}(0,t) = s_1(t), \\ \lambda_2 u(1,t) + \varepsilon\gamma_2 \dfrac{\partial u}{\partial x}(1,t) = s_2(t), \quad t \in (0,T], \end{cases} \tag{8.7}$$

where $\lambda_1, \lambda_2, \gamma_1, \gamma_2$ are all constants such that

$$\lambda_i, \gamma_i \geq 0, \quad \text{and} \quad \lambda_i + \gamma_i > 0, \quad i = 1, 2.$$

In addition, the solution $u(x,t)$ satisfies the following interface conditions

$$[u] = 0, \quad \left[\frac{\partial u}{\partial x}\right] = 0, \quad \text{at } x = \xi. \tag{8.8}$$

Here, we assume that the functions a, b, f satisfy the conditions

$$\begin{cases} a(x,t) > \alpha > 0, \ x > \xi, \quad b(x,t) \geq 0 \quad \text{on } \overline{G}, \\ \big|[f]\big| \leq C, \quad \text{at } x = \xi. \end{cases}$$

In general, under these assumptions, the solution $u(x,t)$ of the IBVP (8.5)-(8.8) possesses a boundary layer at $x = 0$ and strong interior layers of different widths in the neighborhood of the point $x = \xi$. The application of the hybrid scheme proposed in **Chapter 7** to the IBVP (8.5)-(8.8) can be chosen as another interesting future work.

Finally, in **Chapter 7**, an efficient uniformly convergent numerical scheme is developed for 2D singularly perturbed parabolic PDE with homogeneous Dirichlet boundary condition. One can possibly extend the convergence analysis of this newly developed scheme to the following 2D singularly perturbed parabolic convection-diffusion IBVP with nonhomogeneous Robin type boundary conditions, posed on the domain $\mathcal{G} = \mathcal{D} \times (0,T]$, $\mathcal{D} = (0,1)^2$, $\mathbf{x} = (x,y) \in \mathbb{R}^2$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(\mathbf{x},t) + \mathcal{L}_\varepsilon u(\mathbf{x},t) \ = \ f(\mathbf{x},t), \quad (\mathbf{x},t) \in \mathcal{G}, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\mathcal{D}}, \\ \lambda(\mathbf{x},t)u(\mathbf{x},t) + \varepsilon\gamma(\mathbf{x},t)\dfrac{\partial u}{\partial \mathrm{n}}(\mathbf{x},t) = s(\mathbf{x},t), \quad (\mathbf{x},t) \in \partial\mathcal{D} \times (0,T], \end{cases} \tag{8.9}$$

where

$$\mathcal{L}_\varepsilon u \ \equiv \ -\varepsilon \triangle u + \mathbf{a}(\mathbf{x}) \cdot \nabla u + b(\mathbf{x})u,$$

and $\dfrac{\partial u}{\partial \mathrm{n}}$ denotes the derivative of u in the outward normal direction to the boundary $\partial\mathcal{G} = \partial\mathcal{D} \times (0,T]$. Here, the coefficients $\mathbf{a} = (a_1, a_2)$, b, and the functions $\alpha(\mathbf{x},t), \beta(\mathbf{x},t)$ are assumed to satisfy the conditions

$$\begin{cases} a_i(\mathbf{x}) \geq \alpha_i > 0, \ i = 1, 2, \quad b(\mathbf{x}) \geq 0 \quad \text{on } \overline{\mathcal{D}}, \\ \lambda(\mathbf{x},t), \gamma(\mathbf{x},t) \geq 0, \quad \lambda(\mathbf{x},t) + \gamma(\mathbf{x},t) > 0 \quad \text{on } \partial\mathcal{D} \times (0,T]. \end{cases}$$

In general, under these assumptions, the solution $u(\mathbf{x}, t)$ of the parabolic IBVP (8.9) exhibits a regular boundary layer of width $O(\varepsilon)$ at the sides $x = 1$ and $y = 1$. In this regard, we want to mention that Hemkar *et al.* [35] utilized Defect correction technique to obtain high-order time-accurate numerical schemes for 1D singularly perturbed parabolic convection-diffusion IBVP with Robin boundary conditions.

Moreover, it is important to note that most of the convergence results obtained in this thesis are based on the layer resolving piecewise-uniform Shishkin meshes. It could be a more interesting and challenging work to establish those results using adaptive grids technique. This approach is based on the equidistribution principle and has the advantage that it can be applied to SPPs without prior information about the location and width of the boundary (or interior) layers.

Appendix A

The Proof of ε-Uniform Error Estimate of the Classical Upwind Scheme

In this Appendix, we shall comply the approach introduced in [27] with suitable modifications to estimate the ε-uniform error occurred, while discretizing the singularly perturbed parabolic convection-diffusion problem (3.1)-(3.2) exhibiting a layer on the right boundary, by the classical implicit upwind scheme on the domain $G = (0, 1) \times (0, T]$. In the following proof, we avoid the complexity in the symbols by omitting the superscripts of $U^{N,\Delta t}$.

First, we shall estimate the error associated to the smooth component V of the solution U using the following classical argument. Now, the truncation error satisfies the following estimate

$$\left| L_\varepsilon^{N,M}\big(V - v\big)(x_i, t_{n+1}) \right| \le \left[\frac{\varepsilon}{3}\big(h_i + h_{i+1}\big) \left\| \frac{\partial^3 v}{\partial x^3} \right\|_\infty + \frac{h_i}{2} a(x_i) \left\| \frac{\partial^2 v}{\partial x^2} \right\|_\infty + \frac{\Delta t}{2} \left\| \frac{\partial^2 v}{\partial t^2} \right\|_\infty \right].$$

Then, using $h_i \le 2N^{-1}$ and the bounds of the derivatives of v given in Theorem 3.2.1, we have

$$\left| L_\varepsilon^{N,M}\big(V - v\big)(x_i, t_{n+1}) \right| \le C\Big(N^{-1} + \Delta t\Big), \qquad \text{for } 1 \le i \le N - 1.$$

Thus, applying the discrete minimum principle (Lemma 3.3.1) for $L_\varepsilon^{N,M}$ to the discrete functions $C\big[x_i N^{-1} + t_n \Delta t\big] \pm \big(V - v\big)(x_i, t_n)$, we obtain

$$\left| \big(V - v\big)(x_i, t_{n+1}) \right| \le C\Big(N^{-1} + \Delta t\Big), \qquad \text{for } 1 \le i \le N - 1. \tag{A.1}$$

Next, we shall estimate the error associated to the layer component W by separately providing the proofs in two different spatial subregions depending on the location of mesh point $x_i \in \overline{\Omega}_x^{N,\tau}$. We now assume that $\tau = (2\varepsilon \ln N)/\alpha$ and set $\gamma = \alpha/2$.

Case 1. (**Outer region**) For $1 \le i \le N/2$. First, we shall obtain the bound for $|W(x_i, t_{n+1})|$. We define the discrete function Y_i to be the solution of the following discrete problem

$$\begin{cases} \Big(\varepsilon \delta_x^2 - \gamma D_x^-\Big) Y_i = 0, \quad 1 \le i \le N - 1, \\ Y_0 = 0, \quad Y_N = 1. \end{cases} \tag{A.2}$$

Then, we have

$$Y_i = \begin{cases} Y_{N/2} r_i, & 0 \le i \le N/2, \\ 1 + \big(Y_{N/2} - 1\big)\ell_i, & N/2 \le i \le N, \end{cases} \tag{A.3}$$

where

$$\begin{cases} r_i = \dfrac{\Lambda^i - 1}{\Lambda^{N/2} - 1}, & \Lambda = 1 + \dfrac{\gamma H}{\varepsilon}, \\ \ell_i = \dfrac{1 - \lambda^{i-N}}{1 - \lambda^{-N/2}}, & \lambda = 1 + \dfrac{\gamma h}{\varepsilon}, \end{cases}$$

and $Y_{N/2}$ satisfies

$$\Big(\varepsilon \delta_x^2 - \gamma D_x^-\Big) Y_{N/2} = 0. \tag{A.4}$$

From (A.3) and (A.4), we get

$$Y_{N/2} = \frac{D_x^+ \ell_{N/2}}{\big[D_x^+ \ell_{N/2} - (1/2)(\lambda + \Lambda) D_x^+ r_{N/2}\big]}. \tag{A.5}$$

Now, since for all $N \ge 1$,

$$\left(1 + \frac{2 \ln N}{N}\right) \le 2N^{-1},$$

which implies that $\lambda^{-N/2} \le 2N^{-1}$ and hence for all $N \ge 4$, it follows from (A.4) and (A.5) that

$$\begin{cases} 0 \le -\dfrac{\varepsilon}{\gamma} D_x^+ \ell_{N/2} = \dfrac{\lambda^{-N/2}}{1 - \lambda^{-N/2}} \le \dfrac{2N^{-1}}{1 - 2N^{-1}} \le 4N^{-1}, \\ \dfrac{\varepsilon}{\gamma} D_x^- r_{N/2} = \dfrac{\Lambda^{N/2-1}}{\Lambda^{N/2} - 1} \ge \dfrac{1}{\Lambda}, \\ 0 \le Y_{N/2} \le 8N^{-1}, \end{cases}$$

and

$$D_x^- Y_i \ge 0, \quad 1 \le i \le N.$$

Consider the discrete functions

$$\phi_i^{\pm,n} = \big|W(1, t_n)\big| Y_i \pm W(x_i, t_n).$$

Then, clearly $\phi_0^{\pm,n+1} = 0$, $\phi_N^{\pm,n+1} \ge 0$, $\phi_i^{\pm,0} = 0$ and

$$L_\varepsilon^{N,M} \phi_i^{\pm,n+1} = \big|W(1, t_n)\big| \Big[\big(a - \gamma\big) D_x^- Y_i + b Y_i\Big] \ge 0.$$

Therefore, applying the discrete minimum principle over $\overline{G}_\tau^{N,M}$ and using Theorem 3.2.1, we obtain for $1 \le i \le N/2$,

$$\big|W(x_i, t_{n+1})\big| \le \big|W(1, t_n)\big| Y_i \le \big|w(1, t_n)\big| Y_{N/2} \le C N^{-1},$$

Henceforth, invoking the triangle inequality to the error $\big(W_R - w\big)$ and using Theorem 3.2.1 lead to the estimate

$$\Big|\big(W-w\big)(x_i,t_{n+1})\Big| \le CN^{-1}, \quad \text{for } 1 \le i \le N/2. \tag{A.6}$$

Case 2. $\big($**Inner region**$\big)$ When $N/2 < i \le N-1$. From the truncation error and Theorem 3.2.1, we have

$$\Big|L_\varepsilon^{N,M}\big(W-w\big)(x_i,t_{n+1})\Big| \le C\Big[\tau\varepsilon^{-2}N^{-1}\exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)+\Delta t\Big]. \tag{A.7}$$

For $N/2 \le i \le N$, consider the discrete functions

$$\Phi_i^{\pm,n} = C\left[\frac{\exp\big(2\gamma h/\varepsilon\big)}{\gamma(\alpha-\gamma)}\tau\varepsilon^{-1}N^{-1}\mathbf{Y}_i + t_n\Delta t\right] + CN^{-1} \pm \big(W-w\big)(x_i,t_n),$$

where

$$\mathbf{Y}_i = \frac{\lambda^{i-N/2}-1}{\lambda^{N/2}-1}, \quad \lambda = 1+\frac{\gamma h}{\varepsilon}. \tag{A.8}$$

Now, it is easy to check that

$$D_x^-\mathbf{Y}_i \ge \frac{\gamma}{\varepsilon}\exp\big(-\gamma(1-x_{i-1})/\varepsilon\big) > 0. \tag{A.9}$$

Hence, $\mathbf{Y}_i$ increases monotonically with $0 \le \mathbf{Y}_i \le 1$. Then, clearly $\Phi_{N/2}^{\pm,n+1} \ge 0$, $\Phi_N^{\pm,n+1} > 0$, $\Phi_i^{\pm,0} \ge 0$ and employing (A.8), (A.9) we get

$$\begin{aligned}
L_\varepsilon^{N,M}\Phi_i^{\pm,n+1} &\ge C\left[\frac{\exp\big(2\gamma h/\varepsilon\big)}{\gamma(\alpha-\gamma)}\tau\varepsilon^{-1}N^{-1}\big(a(x_i)-\gamma\big)D_x^-\mathbf{Y}_i + \Delta t\right] \pm L_\varepsilon^{N,M}\big(W-w\big)(x_i,t_{n+1}) \\
&\ge C\tau\varepsilon^{-2}N^{-1}\left[\frac{\big(a(x_i)-\gamma\big)}{(\alpha-\gamma)}\exp\big(-\gamma(1-x_{i+1})/\varepsilon\big) - \exp\big(-\alpha(1-x_{i+1})/\varepsilon\big)\right] \\
&\ge 0.
\end{aligned}$$

Thus, applying the discrete minimum principle for $L_\varepsilon^{N,M}$ over the domain $\big(\overline{G}_\varepsilon^{N,M}\bigcap[1-\sigma,1]\big)\times[0,T]$ we have

$$\Big|\big(W-w\big)(x_i,t_{n+1})\Big| \le C\Big(N^{-1}\ln N + \Delta t\Big), \quad \text{for } N/2+1 \le i \le N-1. \tag{A.10}$$

Therefore, the result (3.11) stated in Theorem 3.3.2 follows from (A.1), (A.6) and (A.10).■

Remark A.0.1. *The estimate (3.11) can also be obtained by choosing* $\tau = (\varepsilon \ln N)/\alpha$, *provided the convection coefficient* $a(x)$ *satisfies* $a(x) > \alpha > 0$.

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List of Publications/Communicated

1. K. Mukherjee and S. Natesan. An efficient numerical scheme for singularly perturbed parabolic problems with interior layers. *Neural, Parallel, and Scientific Computations*, **16**:405–418, 2008.

2. K. Mukherjee and S. Natesan. Parameter-uniform hybrid numerical scheme for time-dependent convection-dominated initial-boundary-value problems. *Computing*, **84**(3-4):209–230, 2009.

3. K. Mukherjee and S. Natesan. ε-uniform error estimate of hybrid numerical scheme for singularly perturbed parabolic problems with interior layers (Communicated).

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