



EXPANSIONS OF C-SETS AND D-SETS
HAVING JORDAN AUTOMORPHISM GROUPS AND
SOME RELATED QUESTIONS

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In the Name of Allâh, the Most Gracious, the Most Merciful.

All the praises and thanks be to Allâh, the Lord of the 'Alamîn (mankind, jinn and all that exists).

Quran 1:1



C E R T I F I C A T E

It is certified that the work contained in the thesis entitled '**Expansions of C -sets and D -sets having Jordan automorphism groups and some related questions**', by **Shabeena Ahmed**, a student in the Department of Mathematics, Indian Institute of Technology, Guwahati submitted for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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Keywords. *Semilinear order, C-relation, B-relation, D-relation, Jordan group, homogeneous structure, oligomorphic group, cycle type, uncountable cofinality.*

This thesis is about the interplay between permutation groups and some relational structures. Special emphasis is on a class of permutation groups called Jordan groups in which, in some non-degenerate way, the pointwise stabiliser of a subset is transitive on the complement. The relations considered are semilinear orders, C -relations, B -relations and D -relations. These relations are precisely the ones that appear in the study of Jordan groups. We try to answer the following natural question: Suppose we start with a relational structure whose automorphism group is a Jordan group. Can we expand this structure by imposing extra relations on it so that it still admits a Jordan automorphism group?

First, a C -set Ω and its automorphism group is studied. It is known from the literature that the automorphism group of a C -set is a Jordan group. Two types of automorphisms of (Ω, C) are defined. Roughly, a chain automorphism induces an order-preserving permutation on some chain of the underlying semilinear order Λ fixing setwise that particular chain. And a branch automorphism fixes a node of Λ inducing a permutation on the branches at that fixed node. A relation which restricts the class of branch automorphisms of (Ω, C) is called a branch relation. And a relation which restricts the class of chain automorphisms of (Ω, C) is called a chain relation. It is shown that any element of $\text{Aut}(\Omega, C)$ can be expressed as a product of these two types of automorphisms. A few other relations namely, a binary (2-place) relation \leq and quaternary (4-place) relations V , L and R are also



imposed on the C -set. The group $\text{Aut}(\Omega, C, \leq)$ is shown to be 2-homogeneous which is not 2-transitive and $\text{Aut}(\Omega, C, \leq, V)$ is 2-homogeneous but not relatively 3-transitive. Again, $\text{Aut}(\Omega, C, L)$ is 2-transitive but $\text{Aut}(\Omega, C, L, R)$ is not even 2-homogeneous. Further, it is shown that each of these automorphism groups are Jordan groups. For example, the Jordan sets of $\text{Aut}(\Omega, C, \leq)$ are of the forms $\Sigma_{\alpha, P} = \{\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mid q \in P\}$, where α is an element of Λ and P is either a singleton subset or an open interval of \mathbb{Q} . The point stabiliser of each of these groups is also determined. A minimal Jordan group G_0 is determined such that the imposition of any extra relation on the underlying structure no longer admits a Jordan automorphism group. It is shown that a branch relation which contains the class of translation branch automorphisms in its automorphism group can be imposed on the C -set and will still admit a Jordan group. And any chain relation imposed on the C -set will still admit a Jordan group.

The structures of (Ω, C) with the extra relations are then studied with reference to homogeneous structures and oligomorphic groups. It is shown that all the structures under consideration are homogeneous and their automorphism groups are oligomorphic. The realisable cycle types of non-identity elements of the automorphism groups are also determined.

Finally, a D -set Ψ and its automorphism group is studied. A few other relations namely, a ternary (3-place) relation K and 5-place relations T and R are imposed on Ψ . It is found that the automorphism groups of each of these structures are Jordan groups. For any element x_0 in Ψ , the relation K induces a linear order on $(\Psi \setminus \{x_0\}, C)$, the relation T induces a 4-place relation L and the 5-place relation R induces the corresponding 4-place relation R as defined in the C -set Ω . The last result obtained is on cofinality. Suppose we have an automorphism group A . If there exists a countable chain $H_0 < H_1 < H_2 < \dots$ of proper subgroups with union A then we say that A has countable cofinality. Otherwise we say that A has uncountable cofinality. It is shown that the group $\text{Aut}(\Psi, D)$ has uncountable cofinality.

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Chapter 1

Introduction

This thesis is about the interplay between permutation groups and some relational structures. The central idea is that certain classes of permutation groups can also be viewed as groups of automorphisms of relational structures of specific kinds. We shall be concentrating on a very special class of permutation groups called Jordan groups which can be defined as those groups in which, in some non-degenerate way, the pointwise stabiliser of a subset is transitive on the complement. There is the beautiful theory of relational structures (as developed by Adeleke & Neumann) which are defined to be structures that have some natural binary, ternary or quaternary relations defined on them, either as sets of nodes of the underlying tree or as sets of the ends (or the points at infinity) of the tree.

The relational structures that we consider are a semilinear order, a C -relation, a B -relation and a D -relation (Section 2.3.2). Briefly, a semilinearly ordered set or simply a semilinear order is a ‘connected’ partially ordered set in which for every point, the set of all points below it is linearly ordered; above it there may nevertheless be ‘branching’ or ‘ramification’. A linearly ordered subset of a partially ordered set is known as a chain; the maximal chains in a semilinearly ordered set form a rich and interesting structure in their own right. The behaviour of the maximal chains is described by a ternary relation which we call a C -relation. A B -relation is another ternary relation which captures the notion of a point

lying between two others in a tree-like structure. And finally a D -relation is a quaternary relation defined on the ends or the points at infinity of a set with a B -relation.

What the four concepts have in common is that these are precisely the structures which appear in a study of Jordan groups which we shall describe in detail later on. We explore this idea in a few directions and try to add to the body of the knowledge in this area by asking the simple and natural question: Suppose we start with a relational structure whose automorphism group is a Jordan group. Can we expand this set by imposing extra relations on it that preserve the Jordan property of the automorphism group? If yes, can we specify the kinds of relations that we can impose? If we start with a set equipped with a C -relation then the problem can be stated as follows (refer [BMMN][Problem 8]).

Problem 1.1 *Describe all ways in which a set with a C -relation can be expanded by imposing extra relations while preserving the Jordan property of the automorphism group.*

In trying to answer this question we impose extra relations on a set with a C -relation and study their automorphism groups. We do a parallel study on a set with a D -relation. We also study some other properties of the automorphism groups of these structures.

This chapter is divided into two sections. The first contains a brief idea of the motivation for the thesis. The second gives an outline of the chapters. The material presented in Chapter 2 is elementary and can be found in any standard book of the subject. The axioms of the relations have been taken from [AN2]. The references for the material in Chapter 3 are [AN2] and [BMMN]. The main reference for the work presented in Chapter 4 is [M2]. Other references are [A], [C1] and [M1]. For Chapter 5 references are [DM], [C2], [MP] and [SP]. And for Chapter 6 the references are [M2], [C1], [MN] and [DHM].

1.1 Motivation for the thesis

In the 1870's, Jordan developed the early theory of permutation groups now known as Jordan groups. The key result in the theory is Jordan's Theorem.

Theorem 1.1 [Jordan] *A finite primitive group having a proper Jordan set is 2-transitive. •*

In the finite case, all classes of primitive Jordan groups have been determined but the picture in the infinite case is quite different. Jordan's Theorem fails in the infinite case. The simplest counter-example is the group A of order-preserving permutations of \mathbb{Q} in which any open interval is a Jordan set. This group is 2-homogeneous but not 2-transitive. There are various symmetric groups, for example, and affine and projective groups can be defined over infinite fields with finite or infinite dimension. These examples retain much of the structure of their finite counterparts, but infinite multiply transitive groups can also exhibit behaviour that is just not possible for a finite group. New methods including ideas from model theory have been employed recently in an attempt to understand these infinite groups. There is an excellent survey on Jordan groups by Macpherson [M1].

There are many naturally occurring examples of Jordan sets. The ones that are of greatest interest to us are those that occur as automorphism groups of certain relational systems. The example A considered earlier can be varied in several ways. First, we can generalise from $\text{Aut}(\mathbb{Q}, \leq)$ to $\text{Aut}(\Omega, \leq)$, where (Ω, \leq) is an arbitrary relatively 2-transitive (refer Definition 2.8) dense linearly ordered set. The non-empty convex open subsets will always be primitive Jordan sets. Next, we can consider the cyclic order on \mathbb{Q} (a ternary relation - think of the rational points on the unit circle) and we can generalise to arbitrary 2-transitive dense cyclic order relations. Then there are the corresponding linear betweenness or cyclic separation relations: their automorphism groups are the groups of order-preserving or order-reversing transformations with respect to the original linear or



cyclic order respectively. In all these cases every primitive Jordan set is open and convex and conversely, every non-empty convex open set is a primitive Jordan set. Semilinear orders, C -relations, B -relations and D -relations are generalisations of these relations.

Some interesting relations that can be imposed on a set with a C -relation such that the automorphism group still has primitive Jordan sets have been discussed by Cameron [C1]. For example, suppose (P, \leq) is a countable 1- and 2-homogeneous semilinear order and suppose its nodes are coloured densely red or green. Pick a dense set M of maximal chains of (P, \leq) and interpret a C -relation C in the usual way on M . Also define a (binary) adjacency relation on M , putting two members of M adjacent if, regarding them as chains, the common upper bound of their intersection is a red point of P . Then the automorphism group of the C -relation preserving the adjacency relation is a primitive Jordan group. We started our work using these ideas and have expanded the list of relations that can be imposed while preserving the Jordan property of the automorphism group.

1.2 Outline of the chapters

The next chapter of this thesis contains definitions and concepts that are required for our work. This includes basic ideas from group theory and axiomatic definitions of the relations that are considered in the thesis. Two classification theorems (Theorem 2.1 and Theorem 2.2) that gives the connection between permutation groups and relational structures are stated. Chapters 3-6 constitute the main body of our work followed by the conclusion in Chapter 7.

In Chapter 3 we construct the semilinearly ordered set Λ and define the set of maximal chains Ω and impose a C -relation. The set Ω can be expressed as

$$\{n_1q_1 \dots n_kq_k \mid n_i \in \mathbb{Q}, 1 \leq i \leq k, n_j < n_{j+1}, q_j \in \mathbb{Q} \setminus \{0\}, 1 \leq j \leq k-1, q_k \in \mathbb{Q}\}.$$

The C -relation defined on Ω is the usual C -relation which is defined as

$$C(x; y, z) :\leftrightarrow (y = z \neq x) \vee (x \cap y < y \cap z),$$

where $x \cap y$ denotes the intersecting node of x, y with $x \neq y$ which is defined to be the greatest node of Λ such that the node is an element of both the chains x and y . The C -set Ω that we construct exhibit a high degree of symmetry with maximal chains isomorphic to \mathbb{Q} and branching number \aleph_0 at each node of the underlying semilinearly ordered set Λ .

We define two kinds of automorphisms of (Ω, C) . Roughly, a chain automorphism is an automorphism that induces an order-preserving permutation on some chain of the underlying semilinear order fixing setwise that particular chain. And a branch automorphism is an automorphism that fixes a node of the semilinear order inducing a permutation on the branches of the fixed node. (For precise definitions of chain automorphisms and branch automorphisms refer Definitions 3.2 - 3.3). We show that any element of $\text{Aut}(\Omega, C)$ is expressible as a composition of chain automorphisms and branch automorphisms (Theorem 3.2).

In Chapter 4 we impose extra relations on the C -set Ω that we construct in Chapter 3. The first relation that we impose is a linear order (Section 4.1). Suppose x, y are two elements of Ω with $\alpha = x \cap y$ and suppose x, y lie on the q -th, p -th branches at α respectively. We define the linear order \leq on Ω as

$$x \leq y :\leftrightarrow q \leq p,$$

where the second \leq denotes the linear order on \mathbb{Q} . The chain automorphisms of $\text{Aut}(\Omega, C, \leq)$ are the chain automorphisms of $\text{Aut}(\Omega, C)$ whereas the branch automorphisms are those branch automorphisms of $\text{Aut}(\Omega, C)$ that induce order-preserving permutations on branches at a node of Λ . Thus the imposition of the relation \leq on (Ω, C) restricts the branch automorphisms of $\text{Aut}(\Omega, C)$. We call such a relation a branch relation. We show that $\text{Aut}(\Omega, C, \leq)$ is a 2-homogeneous group (Theorem 4.3) which is not 2-transitive. It is a Jordan group (Theorem 4.8) with sets of the form $\Sigma_{\alpha, P} = \{\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mid q \in P\}$ as Jordan sets, where $\alpha \in \Lambda$ and P is either a singleton subset of \mathbb{Q} or an open interval (a, b) of \mathbb{Q} . For

any element x in Ω we also determine the point stabiliser of x in $\text{Aut}(\Omega, C, \leq)$ (Theorem 4.6).

With the imposition of the linear order \leq on Ω the set of branch automorphisms of the automorphism group becomes smaller. This means that we have some control over the branch automorphisms of (Ω, C) . We impose another relation V on (Ω, C, \leq) to get a more smaller set of branch automorphisms in $\text{Aut}(\Omega, C, \leq, V)$ (Section 4.2). Let x, y be two distinct elements of Ω and suppose α denote their intersection $x \cap y$. Let x, y lie on the q -th, p -th branches at α respectively. We define the distance between x and y denoted by d as $d(x, y) := q - p$. The quaternary relation V on Ω is defined as

$$V(x, y; z, w) :\leftrightarrow d(x, y) = d(z, w).$$

The branch automorphisms of $\text{Aut}(\Omega, C, \leq, V)$ induce order-preserving permutations of \mathbb{Q} of the form $q \mapsto aq + b, a \in \mathbb{Q}^+, b \in \mathbb{Q}$. We show that $\text{Aut}(\Omega, C, \leq, V)$ is a 2-homogeneous group (Theorem 4.10) which is not relatively 3-transitive. The automorphism group $\text{Aut}(\Omega, C, \leq, V)$ is a Jordan group (Theorem 4.12) with Jordan sets of the form $\Sigma_{\alpha, q} = \{\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k\}$, where $\alpha \in \Lambda$ and $q \in \mathbb{Q}$.

Next we define a quaternary relation L on Ω (Section 4.3). For any element $\alpha \in \Lambda$ with expression $n_1 q_1 n_2 q_2 \dots n_k$ we define the level of α , denoted by $L(\alpha)$ to be n_k . Given two elements $\alpha, \beta \in \Lambda$ we say that $\alpha L \beta :\leftrightarrow L(\alpha) = L(\beta)$. Using this binary relation L on Λ we define the quaternary relation L on Ω as

$$L(x, y; z, w) :\leftrightarrow L(x \cap y) = L(z \cap w).$$

The branch automorphisms of (Ω, C, L) are same as the branch automorphisms of (Ω, C) . On the other hand the chain automorphisms of (Ω, C, L) are those chain automorphisms of $\text{Aut}(\Omega, C)$ that induce order-preserving permutations on just the \mathbb{Q}_1 -branch of Ω . Here again, we show that $\text{Aut}(\Omega, C, L)$ is a 2-transitive group (Theorem 4.14) which is also a Jordan group (Theorem 4.16) with Jordan sets of the form $\Sigma_{\alpha, P} := \{\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mid q \in P\}$, where $\alpha \in \Lambda$ and P

is any subset of \mathbb{Q} . We also determine the point stabiliser of any element x in $\text{Aut}(\Omega, C, L)$ (Theorem 4.15).

The imposition of the relation L on (Ω, C) reduces the class of chain automorphisms of $\text{Aut}(\Omega, C)$. We call such a relation a chain relation. To obtain a smaller class of chain automorphisms we impose another quaternary relation R (Section 4.4). Consider dyadic and ternary fractions of \mathbb{Q} of the forms $\frac{m}{2^n}$ and $\frac{m}{3^n}$, $m \in \mathbb{Z}, n \in \mathbb{N}, n > 1$ respectively. We specify that integers are assumed to be neither dyadic nor ternary. We say that two rational numbers are of same type if both are dyadic or both ternary or both are non-dyadic, non-ternary. The quaternary relation R is defined on (Ω, C) as

$$R(x, y; z, w) :\leftrightarrow L(x \cap y) \text{ and } L(z \cap w) \text{ are of same type.}$$

Suppose $L(x, y; z, w)$ holds for elements x, y, z, w in Ω . Then $L(x \cap y) = L(z \cap w)$, which implies trivially that $L(x \cap y)$ and $L(z \cap w)$ are of same type which in turn implies that $R(x, y; z, w)$. But nodes of same type need not necessarily be in the same level. We have shown that chain automorphisms of $\text{Aut}(\Omega, C, L, R)$ can induce neither magnifications (Lemma 4.19) nor non-integer translations (Lemma 4.20) on the \mathbb{Q}_1 -branch. The group $\text{Aut}(\Omega, C, L, R)$ is a Jordan group (Theorem 4.19) which is not 2-homogeneous (Theorem 4.18).

Next, we combine all the relations \leq, V, L, R defined so far on the C -set Ω (Section 4.5). The chain automorphisms and the branch automorphisms of $\text{Aut}(\Omega, C, \leq, V, L, R)$ are the chain automorphisms of $\text{Aut}(\Omega, C, L, R)$ and the branch automorphisms of $\text{Aut}(\Omega, C, \leq, V)$ respectively. The automorphism group $\text{Aut}(\Omega, C, \leq, V, L, R)$ is also a Jordan group (Theorem 4.21). We show that G_0 , the group of translation branch automorphisms is a minimal Jordan group of $\text{Aut}(\Omega, C, \leq, V, L, R)$ (Theorem 4.22). Finally we show that imposition of a branch relation that contains the class of translation branch automorphisms in its automorphism group admits a Jordan automorphism group (Lemma 4.27). And imposition of any chain relation on (Ω, C) admits a Jordan automorphism group (Lemma 4.28).

In Chapter 5 we study the structures of (Ω, C) with the extra relations with reference to homogeneous structures (as in model theory, refer Definition 5.1) and oligomorphic groups (refer Definition 5.2). We also look at the cycle types of the automorphism groups. We show that the structure $(\Omega, C, \leq, V, L, R)$ is homogeneous (Theorem 5.4). We use the one-point extension property to show that a structure is homogeneous (refer [DM][Theorem 9.5A]). A structure is homogeneous if there is only one orbit on each isomorphism type on sets of size n of Ω and a homogeneous structure is oligomorphic if the number of isomorphism types on sets of size n in Ω is finite for every $n \in \mathbb{N}$. We show that all the structures under discussion are homogeneous. Hence to show that the automorphism groups of each of these structures is oligomorphic we show that the number of isomorphism types on sets of size n is finite, for every $n \in \mathbb{N}$. We determine the number of isomorphism types of sets of size n in (Ω, C) such that pairs of elements from the set determine m nodes, $1 \leq m \leq n-1$. We show that the number of these isomorphism types is finite which means that the group $\text{Aut}(\Omega, C)$ is oligomorphic (Theorem 5.5). We also show that each of the groups $\text{Aut}(\Omega, C, \leq)$, $\text{Aut}(\Omega, C, \leq, V)$, $\text{Aut}(\Omega, C, L)$ and $\text{Aut}(\Omega, C, L, R)$ are oligomorphic (Theorems 5.6 - 5.9). Further we show that the group $\text{Aut}(\Omega, C, \leq, V, L, R)$ is oligomorphic (Theorem 5.10).

In the last section of Chapter 5 we give the realisable cycle types of the automorphism groups. We show that the cycle types realised by non-identity automorphisms of the group $\text{Aut}(\Omega, C, \leq)$ are of the forms $\aleph_0^{\aleph_0} 1^k$, $0 \leq k \leq \aleph_0$ (Theorem 5.12). The cycle types realised by non-identity automorphisms of the groups $\text{Aut}(\Omega, C, \leq, V)$, $\text{Aut}(\Omega, C, \leq, V, L)$ and $\text{Aut}(\Omega, C, \leq, V, L, R)$ are found to be same (Theorems 5.13 - 5.15). And the cycle type realised by non-identity elements of G_0 is of the form $\aleph_0^{\aleph_0} 1^{\aleph_0}$ (Theorem 5.16).

In Chapter 6 we construct a B -set Γ and subsequently define the D -set Ψ which is a suitably defined subset of the set of directions of the B -set Γ . We define a ternary relation K (Section 6.6), a 5-place relation T (Section 6.7) and an-

other 5-place relation R on Ψ (Section 6.8). We find that $\text{Aut}(\Psi, D, K)$ is a 3-homogeneous group (Theorem 6.3) which is a Jordan group (Theorem 6.5). The group $\text{Aut}(\Psi, D, T)$ is a 3-transitive group (Theorem 6.6) which is also a Jordan group (Theorem 6.7). And finally $\text{Aut}(\Psi, D, T, R)$ is also a Jordan group (Section 6.8.3) but not 3-homogeneous (Lemma 6.23). For any point $x_0 \in \Psi$, the relation K induces a linear order on $(\Psi \setminus \{x_0\}, C)$ (Lemma 6.11), the relation T induces a 4-place relation L (Lemma 6.21) and the 5-place relation R induces the corresponding 4-place relation R (Lemma 6.24) as defined in the C -set Ω . We show that $\text{Aut}(\Psi, D, K, T, R)$ is a Jordan group (Theorem 6.8).

The last major result we present in Chapter 6 is on cofinality. Suppose we have an automorphism group A . If there exists a countable chain $H_0 < H_1 < H_2 < \dots$ of proper subgroups with union A then we say that A has countable cofinality. Otherwise we say that A has uncountable cofinality. We show that the group $\text{Aut}(\Psi, D)$ has uncountable cofinality (Theorem 6.9).

Chapter 7 constitutes the conclusion and briefly sums up our findings. It also contains a section that gives some interesting open problems related to our work.





Chapter 2

Preliminaries

This chapter is devoted to the concepts that are required for our work. It includes axiomatic descriptions of the relations that are considered in this thesis.

2.1 Notation

We shall normally use the following convention: capital Greek letters $\Omega, \Lambda, \Psi, \Gamma$ will denote sets (and their subsets) which carry some structure on them. Groups will act on these sets. Small Greek letters $\alpha, \beta, \gamma, \delta$ will denote elements of these sets. Capital italics G, H will denote groups; capital italics B, C, D, K, S will be used for relations. Conventional relational symbols such as $\leq, <$ will always stand for non-strict and strict order relations (partial or linear as the context requires), respectively. We shall use \parallel to denote incomparability of two elements of a partially ordered set. Punctuation will be used as a mnemonic to indicate symmetry properties of n -place relations: for example if we write $C(x; y, z)$ then it will indicate that the 3-place relation C is symmetric with respect to the second and third variables. In formulae we shall use the usual logical connectives $\rightarrow, \leftrightarrow, \vee, \wedge$ and \neg to mean 'if ... then ...', 'if and only if', 'or', 'and' and 'not' respectively. Symbols such as $\Rightarrow, \Leftrightarrow$ will be used only in explanations. And we shall use $:=$ and $:\leftrightarrow$ to indicate definitions.

2.2 Permutation groups

A permutation on a set Ω is a bijective map from Ω to Ω . If g is a permutation on Ω then we write ω^g for the image of ω under g , for $\omega \in \Omega$. The set of all permutations on a set Ω is a group under the operation of composition of mappings. We call this group the symmetric group on Ω and denote it by $\text{Sym}(\Omega)$.

Definition 2.1 A permutation group on a set Ω is a subgroup of $\text{Sym}(\Omega)$.

Definition 2.2 Let G be a permutation group on a set Ω . The group G is said to be k -transitive if for any two k -tuples $(\alpha_1, \alpha_2, \dots, \alpha_k)$ and $(\beta_1, \beta_2, \dots, \beta_k)$ of distinct elements of Ω there exists g in G such that $\alpha_i^g = \beta_i$ for $1 \leq i \leq k$. The group G is said to be k -homogeneous if for any two subsets Γ_1, Γ_2 of Ω of size k there exists an element g in G such that $\Gamma_1^g = \Gamma_2$.

For $k = 1$, we say the group is *transitive* and *homogeneous* respectively. The group G is said to be *highly transitive* (*highly homogeneous*) if it is k -transitive (k -homogeneous) for all $k \in \mathbb{N}$.

Example 2.1 The symmetric group S_n is n -transitive, since it contains all the permutations on a set of n elements.

Example 2.2 The alternating group A_n is however only $(n - 2)$ -transitive.

Example 2.3 The group $\text{Aut}(\mathbb{Q}, \leq)$ of order preserving automorphisms of the rationals is highly homogeneous.

Definition 2.3 For $\alpha \in \Omega$ we define the *stabiliser* of α in G to be the set

$$G_\alpha := \{g \in G \mid \alpha^g = \alpha\}.$$

It is easy to see that G_α is a subgroup of G . The stabiliser G_α is also called a point stabiliser. Similarly a k -point stabiliser $G_{\alpha_1, \alpha_2, \dots, \alpha_k}$ can be defined which fixes k distinct points $\alpha_1, \alpha_2, \dots, \alpha_k$ in Ω .

2.3 Relational structures

Definition 2.4 A transitive permutation group G is *primitive* if there are no non-trivial and proper G -invariant equivalence relations.

A G -invariant equivalence relation is said to be *non-trivial* if there is a class with more than one element and it is said to be *proper* if there is more than one class. A k -transitive permutation group ($k > 1$) is *k-primitive* if any $(k - 1)$ -point stabiliser is primitive on the complement of the $k - 1$ points on Ω .

Definition 2.5 Let G be a group acting on a set Ω . We say that G acts *faithfully* on Ω if the set $\{g \in G \mid \omega^g = \omega \text{ for all } \omega \in \Omega\}$ consists of only the identity element in G .

Let G and H be permutation groups acting faithfully on sets Δ and Γ respectively. We define $K := G^\Gamma = \{f \mid f : \Gamma \rightarrow G\}$ and we let H act on K by specifying that $f^h(\alpha) = f(\alpha h^{-1})$, for all $\alpha \in \Gamma$ and $h \in H$.

Definition 2.6 The *wreath product* W of G by H is defined to be the semidirect product of K by H , that is,

$$W := G \text{ Wr}_\Gamma H := \{(f, h) \mid f \in K, h \in H\}$$

with multiplication defined by $(f_1, h_1)(f_2, h_2) = (f_1 f_2^{h_1^{-1}}, h_1 h_2)$. It acts naturally on $\Delta \times \Gamma$ with the action defined as $(\delta, \gamma)(f, h) = (\delta f(\gamma), \gamma h)$. The group H is called the *top group*, G the *bottom group* and K the *base group* of the wreath product W . The factors ($\cong G$) of K are called the *co-ordinate subgroups* (indexed by Γ) of the base group K . The wreath product construction on permutation groups is associative.

2.3 Relational structures

For each integer $n \geq 1$, an n -place relation on a set Ω is a subset $\Lambda \subseteq \Omega^n$ where $\Omega^n = \Omega \times \Omega \times \dots \times \Omega$ (n times). Strictly speaking, the relations just defined are

finitary relations. It is also possible to define infinitary relations in an analogous way but relations in our study will always be finitary relations. We use the terms binary, ternary and quaternary to refer to the cases 2-place, 3-place and 4-place respectively. A *relational structure* (or simply a structure) is a set Ω together with a family of relations on Ω .

Definition 2.7 Let R be a set of relations on a non-empty set Ω . Now $\text{Sym}(\Omega)$ acts (componentwise) on Ω^k for each k , so we can consider the set G of all permutations on Ω which maps a related k -tuple to another related k -tuple. It is easily seen that G is a subgroup of $\text{Sym}(\Omega)$; G is called the group of *R -preserving permutations* on Ω , or the *automorphism group of the relational structure* (Ω, R) , and is denoted by $\text{Aut}(\Omega, R)$.

Definition 2.8 Let (Ω, R) be a relational structure and G be its automorphism group. The group G is defined to be *relatively k -transitive* if for any two k -tuples $(\alpha_1, \alpha_2, \dots, \alpha_k)$ and $(\beta_1, \beta_2, \dots, \beta_k)$ there exists $g \in \text{Aut}(\Omega, R)$ such that $\alpha_i^g = \beta_i$, $1 \leq i \leq k$. And the group G is said to be *relatively k -homogeneous* if for any two subsets Γ_1, Γ_2 of size k there exists $g \in G$ such that $\Gamma_1^g = \Gamma_2$.

2.3.1 Linear relational structures

In this section we give the definitions of linear orderings, linear betweenness relations, cyclic orderings and cyclic separation relations.

Definition 2.9 A structure (Λ, \leq) is a *partially ordered set* if the following conditions are satisfied:

$$(a1) \quad \alpha \leq \beta \wedge \beta \leq \alpha \leftrightarrow \alpha = \beta;$$

$$(a2) \quad \alpha \leq \beta \wedge \beta \leq \gamma \rightarrow \alpha \leq \gamma.$$

It is said to be a *linearly ordered set* if in addition

$$(a3) \quad \alpha \leq \beta \vee \beta \leq \alpha.$$

A linearly ordered set is also called a *chain* or referred to simply as a *linear order*. We say that (Λ, \leq) is *dense* and *without end-points* if

2.3 Relational structures

$$(a4) \quad \alpha < \beta \rightarrow (\exists \gamma)(\alpha < \gamma < \beta);$$

$$(a5) \quad (\forall \alpha)(\exists \beta)(\exists \gamma)(\beta < \alpha < \gamma).$$

Definition 2.10 A *linear betweenness relation* ρ is a ternary relation which is axiomatically defined by the following:

$$(b1) \quad \rho(\alpha; \beta, \gamma) \rightarrow \rho(\alpha; \gamma, \beta);$$

$$(b2) \quad \rho(\alpha; \beta, \gamma) \wedge \rho(\beta; \alpha, \gamma) \leftrightarrow \alpha = \beta;$$

$$(b3) \quad \rho(\alpha; \beta, \gamma) \rightarrow \rho(\alpha; \beta, \delta) \vee \rho(\alpha; \gamma, \delta);$$

$$(b4) \quad \rho(\alpha; \beta, \gamma) \vee \rho(\beta; \gamma, \alpha) \vee \rho(\gamma; \alpha, \beta).$$

Given a linearly ordered set (Λ, \leq) the derived linear betweenness relation ρ is defined by

$$(b^*) \quad \rho(\alpha; \beta, \gamma) :\leftrightarrow (\beta \leq \alpha \leq \gamma) \vee (\gamma \leq \alpha \leq \beta).$$

Definition 2.11 A *cyclic or circular order relation* is described by a ternary relation K satisfying the following conditions:

$$(K1) \quad K(\alpha, \beta, \gamma) \rightarrow K(\beta, \gamma, \alpha);$$

$$(K2) \quad K(\alpha, \beta, \gamma) \wedge K(\beta, \alpha, \gamma) \leftrightarrow \alpha = \beta \vee \beta = \gamma \vee \gamma = \alpha;$$

$$(K3) \quad K(\alpha, \beta, \gamma) \rightarrow K(\alpha, \beta, \delta) \vee K(\delta, \beta, \gamma);$$

$$(K4) \quad K(\alpha, \beta, \gamma) \vee K(\beta, \alpha, \gamma).$$

A circular order K may be derived from a binary relation \leq by the rule

$$(K^*) \quad K(\alpha, \beta, \gamma) :\leftrightarrow \alpha \leq \beta \leq \gamma \vee \beta \leq \gamma \leq \alpha \vee \gamma \leq \alpha \leq \beta.$$

Definition 2.12 A *separation relation* is an abstraction of the geometrical idea of two pairs of points on a circle separating each other. It is a four place relation S satisfying the following conditions:

$$(S1) \quad S(\alpha, \beta; \gamma, \delta) \rightarrow S(\beta, \alpha; \gamma, \delta) \wedge S(\gamma, \delta; \alpha, \beta);$$

$$(S2) \quad S(\alpha, \beta; \gamma, \delta) \wedge S(\alpha, \gamma; \beta, \delta) \leftrightarrow \beta = \gamma \vee \alpha = \delta;$$

$$(S3) \quad S(\alpha, \beta; \gamma, \delta) \rightarrow S(\alpha, \beta; \gamma, \epsilon) \vee S(\alpha, \beta; \delta, \epsilon);$$

$$(S4) \quad S(\alpha, \beta; \gamma, \delta) \vee S(\alpha, \gamma; \delta, \beta) \vee S(\alpha, \delta; \beta, \gamma).$$

From a circular order K a separation relation S can be derived by the rule

$$(S^*) \quad S(\alpha, \beta; \gamma, \delta) :\leftrightarrow (K(\alpha, \beta, \gamma) \wedge K(\alpha, \delta, \beta)) \vee (K(\alpha, \gamma, \beta) \wedge K(\alpha, \beta, \delta)).$$

2.3.2 Generalisations of linear relational structures

A semilinear order, betweenness relation, C -relation and a D -relation are generalisations of the four relations discussed in Section 2.3.1. We give the definitions of these generalised relations here. For a detailed description of the relations refer Adeleke & Neumann [AN2]. The pictorial representation of the relations are given in Fig. 2.1.

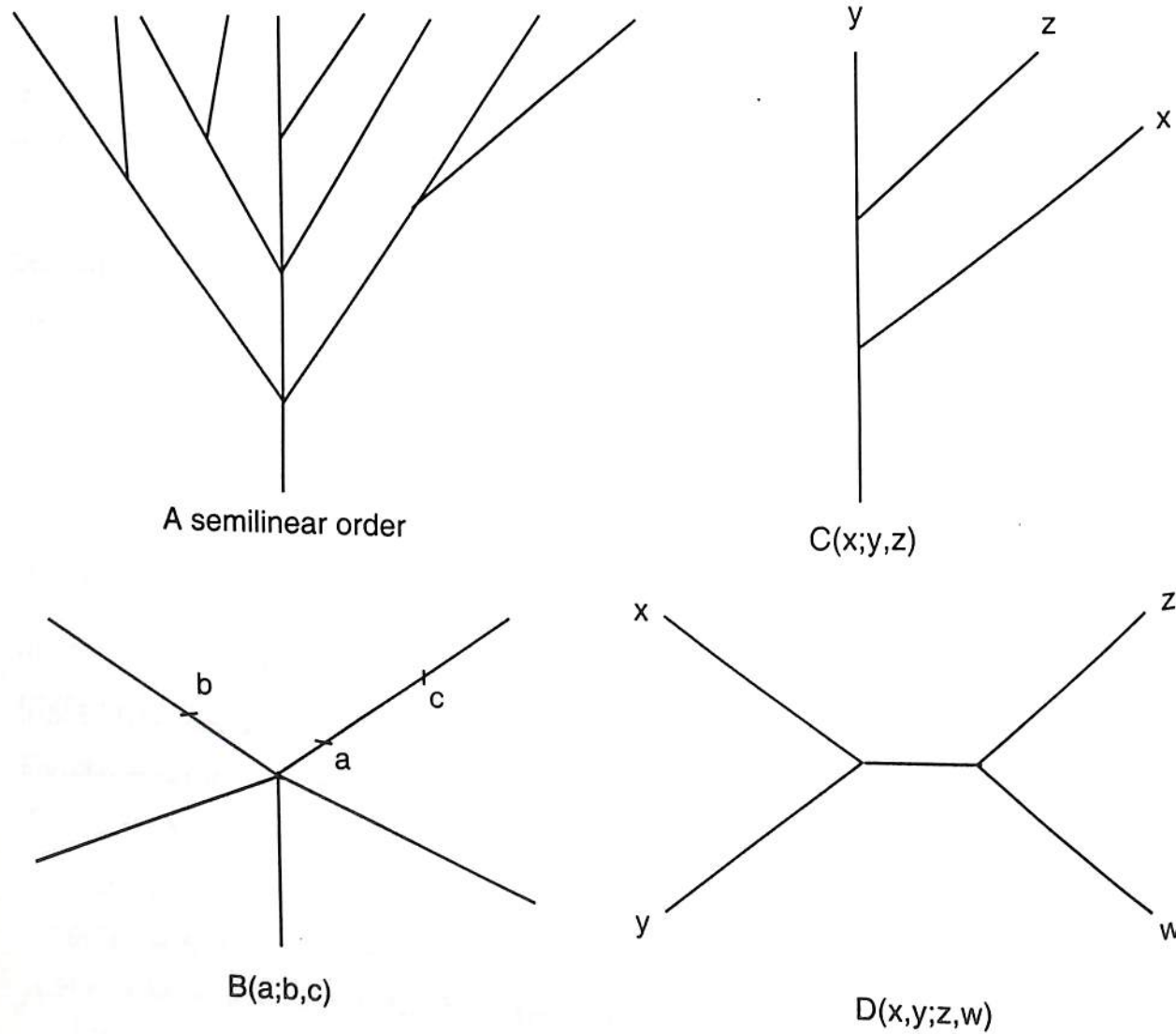


Figure 2.1:

2.3 Relational structures

Definition 2.13 A partially ordered set (Λ, \leq) is called a *semilinearly ordered set* (also referred to as a *semilinear order*) if the following conditions hold:

- (P1) $\beta \leq \alpha \wedge \gamma \leq \alpha \rightarrow \beta \leq \gamma \vee \gamma \leq \beta$;
- (P2) $(\exists \gamma)(\gamma \leq \alpha \wedge \gamma \leq \beta)$;
- (P3) (Λ, \leq) is not a total order.

The semilinear order is said to be *unbounded* or *without end-points* if

(P4) $(\forall \alpha)(\exists \beta)(\exists \gamma)(\beta < \alpha < \gamma)$;

and (Λ, \leq) is said to be *dense* if

(P5) $\alpha < \beta \rightarrow (\exists \gamma)(\alpha < \gamma < \beta)$ holds.

Definition 2.14 A ternary relation on a set Ω is called a C -relation if the following conditions hold:

- (C1) $C(x; y, z) \rightarrow C(x; z, y)$;
- (C2) $C(x; y, z) \rightarrow \neg C(y; x, z)$;
- (C3) $C(x; y, z) \rightarrow C(x; w, z) \vee C(w; y, z)$;
- (C4) $x \neq y \rightarrow C(x; y, y)$.

We then call Ω a C -set. In addition, if

- (C5) $(\forall y, z)(\exists x)C(x; y, z)$;
- (C6) $x \neq y \rightarrow (\exists z)(y \neq z \wedge C(x; y, z))$;

hold then the C -relation is said to be *proper*. The set Ω is said to be *dense* if

(C7) $C(x; y, z) \rightarrow (\exists w)(C(w; y, z) \wedge C(x; y, w))$.

Definition 2.15 A ternary relation on a set Γ is called a B -relation if the following conditions hold:

- (B1) $B(\alpha; \beta, \gamma) \rightarrow B(\alpha; \gamma, \beta)$;
- (B2) $B(\alpha; \beta, \gamma) \wedge B(\beta; \alpha, \gamma) \leftrightarrow \alpha = \beta$;
- (B3) $B(\alpha; \beta, \gamma) \rightarrow B(\alpha; \beta, \delta) \vee B(\alpha; \gamma, \delta)$.

We then call Γ a B -set. If additionally (Γ, B) satisfies



$$(B4) \quad \neg B(\alpha; \beta, \gamma) \rightarrow (\exists \delta \neq \alpha)(B(\delta; \alpha, \beta) \wedge B(\delta; \alpha, \gamma))$$

then we shall call B a *betweenness relation*. If

$$(B5) \quad (\forall \alpha, \beta)(\exists \gamma \neq \alpha)B(\alpha; \beta, \gamma)$$

holds then we say that Γ is *unending in all directions*; and if

$$(B6) \quad \alpha \neq \beta \rightarrow (\exists \gamma \neq \alpha, \beta)B(\gamma; \alpha, \beta)$$

then we say that Γ is *dense*.

Definition 2.16 A quaternary relation on a set Ψ is a *D-relation* if the following conditions are satisfied:

$$(D1) \quad D(x, y; z, w) \rightarrow D(y, x; z, w) \wedge D(z, w; x, y);$$

$$(D2) \quad D(x, y; z, w) \rightarrow \neg D(x, z; y, w);$$

$$(D3) \quad D(x, y; z, w) \rightarrow D(u, y; z, w) \vee D(x, y; z, u);$$

$$(D4) \quad (x \neq z \wedge y \neq z) \rightarrow D(x, y; z, z).$$

We then call Ψ a *D-set*. We say that Ψ is a proper *D-set* if in addition,

$$(D5) \quad (x, y, z \text{ distinct}) \rightarrow (\exists w)(z \neq w \wedge D(x, y; z, w))$$

holds and we say that (Ψ, D) is *dense* if

$$(D6) \quad D(x, y; z, w) \rightarrow (\exists u)(D(u, y; z, w) \wedge D(x, u; z, w) \wedge D(x, y; u, w) \wedge D(x, y; z, u)).$$

2.4 Jordan groups

Let G be a transitive permutation group. A non-empty, non-singleton subset Γ of Ω is a *Jordan set* if for every α, β in Γ , there is a permutation in G mapping α to β and fixing every point of $\Omega \setminus \Gamma$. A Jordan set Γ of Ω is *proper* if $\Omega \setminus \Gamma$ is infinite, or if finite then G is not $(|\Omega \setminus \Gamma| + 1)$ -transitive. Thus a Jordan set is proper if its existence cannot be trivially derived from multiple transitivity of G . A Jordan set Γ is a *primitive Jordan set* if the action of the subgroup of G fixing every point of $\Omega \setminus \Gamma$ is primitive on Γ . Similarly n -transitive and n -primitive Jordan sets are defined. Suppose Γ is a Jordan set and $H \leq G_{(\Omega \setminus \Gamma)}$ is a subgroup of G

that fixes every point of $\Omega \setminus \Gamma$ and is transitive on Γ then H is called a *Jordan group associated* with the Jordan set Γ .

Definition 2.17 A transitive permutation group G on a set Ω is a *Jordan group* if Ω contains proper Jordan sets.

For example, given a set Ω , any subset of Ω of size greater than one is a Jordan set in $\text{Sym}(\Omega)$. The group of order-preserving permutations on (\mathbb{Q}, \leq) forms a Jordan group with non-empty open intervals as Jordan sets. The group induced on an open interval in (\mathbb{Q}, \leq) is highly homogeneous and hence an open interval in \mathbb{Q} is a primitive Jordan set of the primitive group $\text{Aut}(\mathbb{Q}, \leq)$. In a 2-homogeneous semilinear order, every 'cone' is a primitive Jordan set. The following theorem of Adeleke & Neumann [AN1][Theorem 3] classifies all primitive Jordan groups with primitive Jordan sets; this is the theorem which connects Jordan groups and the relations under our consideration.

Theorem 2.1 [Adeleke & Neumann] Suppose that G is a primitive permutation group that has primitive proper Jordan sets. If G is not highly transitive then there is a G -invariant relation R on Ω which is one of the following:

- (i) a linear order;
- (ii) a linear betweenness relation;
- (iii) a cyclic order;
- (iv) a cyclic separation relation;
- (v) a semilinear order;
- (vi) a general betweenness relation;
- (vii) a C -relation;
- (viii) a D -relation. •

The assumption that G has proper primitive Jordan sets imposes very strong conditions on the group. Adeleke & Macpherson [AM] have classified those primitive Jordan groups which do not have primitive Jordan sets and have thereby completed the classification of all primitive Jordan groups.

Theorem 2.2 [Adeleke & Macpherson] Suppose that G is an infinite primitive Jordan group which is not highly transitive. Then, either G preserves one of the relations (i) - (viii) of Theorem 2.1 or one of the following holds:

- (ix) G is a group of automorphisms of a Steiner k -system for some $k \geq 2$;
- (x) none of (i) - (ix) holds, and G preserves a limit of a sequence of Steiner systems, general betweenness relations or D -relations. •

Chapter 3

Construction of a C -set; its automorphism group

In this chapter we construct a C -set and study its automorphism group. We first construct the underlying semilinearly ordered set such that the maximal chains are isomorphic to \mathbb{Q} and the branches at each node of the semilinearly ordered set is indexed by \mathbb{Q} . Some constructions have been briefly mentioned in the literature for example [BMMN][Section 12.3], [A][Section 9], [AN2][Section 5] and [M1][Section 6].

If we choose our semilinearly ordered set such that the maximal chains are isomorphic to \mathbb{Q} and the branches at each node is indexed by \mathbb{Z} then the automorphism group of the maximal chains, with the C -relation and some of the other relations discussed in the following chapter, is not 2-homogeneous but is still a Jordan group. If we index the branches at a node by a finite set then we can neither expect 2-homogeneity nor the Jordan property of the automorphism group. Nor can we index the branches by a closed subset of \mathbb{Q} of the form $[a, b]$, $a, b \in \mathbb{Q}$ and expect the desired properties. However if we choose the semilinearly ordered set with maximal chains isomorphic to \mathbb{Z} and branches indexed by \mathbb{Q} then the automorphism group of the maximal chains is 2-homogeneous and is a Jordan group.



3.1 Construction

We begin with a vertical copy of the rational line \mathbb{Q} endowed with the usual linear order. The elements are arranged in an ascending order from bottom to top with $-\infty$ in the downward direction. We call this copy of \mathbb{Q} the \mathbb{Q}_1 -branch. In the first step we attach copies of the set $\{n \in \mathbb{Q} \mid n_1 < n\}$ indexed by \mathbb{Q} to each point n_1 of \mathbb{Q}_1 and call these the \mathbb{Q}_2 -branches at n_1 . The \mathbb{Q}_2 -branches are indexed such that the branches to the left of the \mathbb{Q}_1 -branch are negative and those to the right are positive. The indices of the branches become smaller from right to left with the \mathbb{Q}_2 -branch indexed by 0 being identified with the segment $\{n \in \mathbb{Q} \mid n_1 < n\}$ of the \mathbb{Q}_1 -branch. Any \mathbb{Q}_2 -branch at a point n_1 will carry the ordering of \mathbb{Q} and the elements of this branch are incomparable to the elements lying on any other \mathbb{Q}_2 -branch attached at the same point. Next we attach copies of the set $\{n \in \mathbb{Q} \mid n_2 < n\}$ at each point n_2 of the \mathbb{Q}_2 -branches and call these the \mathbb{Q}_3 -branches. The process is continued countably infinite number of times.

The structure that we obtain is clearly a connected rooted tree with $-\infty$ of the \mathbb{Q}_1 -branch as the root. There is a unique path from the root to any element which gives an expression for the element. Let α be an element such that the path of α from the root branches at n_1 on the \mathbb{Q}_1 -branch along the q_1 -th branch, then branches again at n_2 on the \mathbb{Q}_2 -branch along the q_2 -th branch and so on and then branches at n_{k-1} on the \mathbb{Q}_{k-1} -branch along the q_{k-1} -th branch and finally terminates at the point n_k on the \mathbb{Q}_k -branch. We call Λ , the set consisting of elements which can be expressed as a finite sequence of the form

$$n_1q_1n_2q_2 \dots n_k, n_i \in \mathbb{Q}, 1 \leq i \leq k, 1 \leq j \leq k-1, n_j < n_{j+1}, q_j \in \mathbb{Q} \setminus \{0\}.$$

For two elements α, β of Λ with expressions $n_1q_1n_2q_2 \dots n_k$ and $n'_1q'_1n'_2q'_2 \dots n'_k$ we define a binary relation \leq on Λ by saying $\alpha \leq \beta$ if and only if

- (i) $k \leq k'$,
- (ii) $n_i = n'_i, q_i = q'_i$ for $1 \leq i \leq k-1$ and
- (iii) $n_k \leq n'_k$.

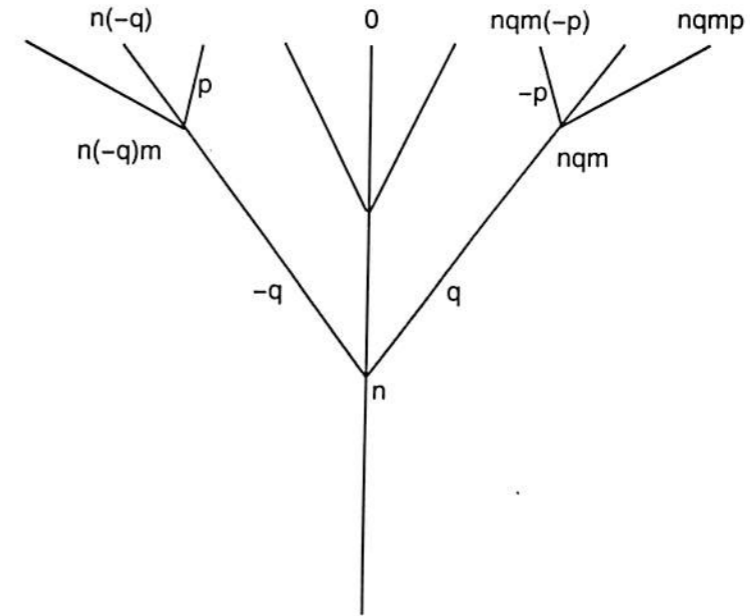


Figure 3.1:

This ordering is just a version of the familiar *lexicographic* order. Routine checking shows that this is a semilinear order on Λ (Definition 2.13). For any element $\alpha = n_1q_1n_2q_2 \dots n_k \in \Lambda$ and $q \in \mathbb{Q} \setminus \{0\}$, let αq denote the union of the set of all elements which are less than or equal to α and the set of all elements greater than α that lies on the q -th branch at α . The elements of the set αq are linearly ordered and so it is a chain of Λ . Moreover αq is not contained in any linearly ordered set and hence it is a maximal chain. Let 0 denote the maximal chain \mathbb{Q}_1 and Ω denote the set of maximal chains of Λ expressible as

$$\{0\} \cup \{\alpha q \mid \alpha \in \Lambda, q \in \mathbb{Q} \setminus \{0\}\}. \tag{3.1}$$

Some elements of Ω are given in Fig. 3.1. We can also express the \mathbb{Q}_1 -branch denoted by 0 as $n0$ for any $n \in \mathbb{Q}$ so that Ω can be expressed as

$$\{n_1q_1 \dots n_kq_k \mid n_i \in \mathbb{Q}, 1 \leq i \leq k, n_j < n_{j+1}, q_j \in \mathbb{Q} \setminus \{0\}, 1 \leq j \leq k-1, q_k \in \mathbb{Q}\}. \tag{3.2}$$

Any element of Ω has a unique expression in Eq. 3.1 but in Eq. 3.2 the uniqueness



of expressions is lost. For example, in Eq. 3.2 both the expressions $n_1 0$ and $n_2 0$ denote the 0 element for $n_1 \neq n_2$. Also $n_1 q_1$, for $n_1 \in \mathbb{Q}$, $q_1 \in \mathbb{Q} \setminus \{0\}$ can be expressed as $n_1 q_1 n_2 0$ for $n_1 < n_2 \in \mathbb{Q}$.

We call elements of Λ nodes and elements of Ω chains. For $x = n_1 q_1 n_2 q_2 \dots n_k q_k$ we call n_1, n_2, \dots, n_k the odd co-ordinates and q_1, q_2, \dots, q_k the even co-ordinates of x respectively. We say that the length of x is $2k$ and refer to $1, 2, \dots, k$ as the indices of the co-ordinates. Note that elements of Ω have finite length which is necessary to restrict the cardinality of Ω to \aleph_0 . A typical chain $n_1 q_1 n_2 q_2 \dots n_k q_k$ has \aleph_0 choices for each n_i and for each q_i for $1 \leq i \leq k$. This means that Ω is a countable product of countable sets and therefore Ω has countable cardinality.

Definition 3.1 For two distinct elements x, y of Ω we define the intersecting node denoted by $x \cap y$ to be the greatest node of Λ which is an element of both the chains x and y .

Note that the greatest node that is an element of both x and y exists because we assume Λ to be of positive type (refer Section 3.2). This means that the branches on which x and y lie at $x \cap y$ are distinct. Similarly we can define the intersecting node of two distinct elements of Λ .

A ternary relation C can be defined on Ω that satisfies the axioms of a C -relation (Definition 2.14) as follows. Given three elements $x, y, z \in \Omega$, we say

$$C(x; y, z) :\leftrightarrow (y = z \neq x) \vee (x \cap y < y \cap z). \tag{3.3}$$

We can also define the C -relation on Ω in terms of the co-ordinates of elements of Ω as follows. Let x, y, z be three elements of Ω . If $y = z \neq x$ then clearly $C(x; y, z)$ holds. Suppose x, y, z are all distinct with expressions as $n_1 q_1 n_2 q_2 \dots n_k q_k$, $y = m_1 p_1 m_2 p_2 \dots m_j p_j$ and $z = l_1 r_1 l_2 r_2 \dots l_i r_i$ respectively. Let d be the first index at which at least two of x, y, z differ. Suppose that d is odd then $C(x; y, z)$ holds if any of the following hold:

3.2 The C -set Ω

$$n_d < m_d = l_d \tag{Fig. 3.2(a)},$$

$$m_d = l_d < n_d, p_d = r_d \tag{Fig. 3.2(b)},$$

$$n_d < m_d, n_d < l_d, m_d \neq l_d \tag{Fig. 3.2(c)}.$$

Suppose that d is even then $C(x; y, z)$ holds if

$$p_d = r_d \tag{Fig. 3.2(d)}.$$

Note that if d is even and $q_d \neq p_d \neq r_d$ then the relation C does not hold for any

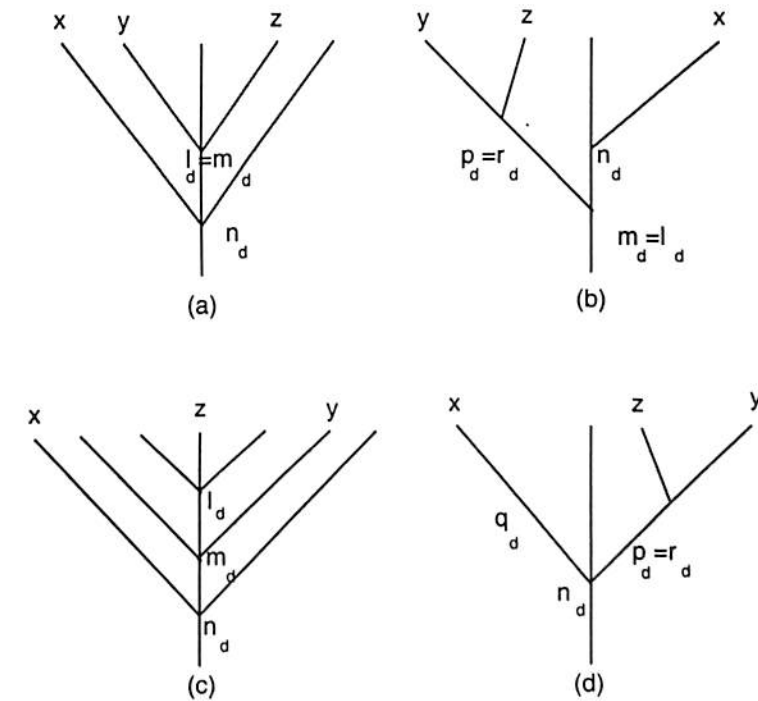


Figure 3.2:

orientation of x, y, z which means $\neg C(x; y, z)$ and $\neg C(y; z, x)$ and $\neg C(z; x, y)$. We denote this as $N(x, y, z)$.

3.2 The C -set Ω

The theory of semilinearly ordered sets and C -relations which are inter-twined is discussed in detail by Adeleke & Neumann [AN2][Part II and Part III]. We

give here a brief sketch of the material that is needed for our work. Note that semilinear orders in [AN2] are upper semilinear orders and those considered here are lower semilinear orders.

A subset Σ of Λ is called a *lower section* if it is bounded above (that is, there exists $\alpha \in \Lambda$ such that $\sigma \leq \alpha$ for all $\sigma \in \Sigma$) and is a filter (that is,

$$(\forall \sigma \in \Sigma)(\forall \omega \in \Lambda)(\omega \leq \sigma \rightarrow \omega \in \Sigma)).$$

Clearly a lower section is always a chain. If Σ is a lower section we define

$$\Lambda_{>}(\Sigma) := \{\omega \mid \sigma < \omega \text{ for all } \sigma \in \Sigma\},$$

and define a relation R on $\Lambda_{>}(\Sigma)$ by specifying that $(\omega_1, \omega_2) \in R$ if there exists $\omega \in \Lambda_{>}(\Sigma)$ with $\omega \leq \omega_1$ and $\omega \leq \omega_2$. The relation R is easily checked to be an equivalence relation. The equivalence classes are called the cones above Σ . The *ramification order* $\text{ramord}(\Sigma)$ of a lower section Σ is defined to be the number of cones above Σ . The ramification order is also called the *branching number*. If $\text{ramord}(\Sigma)$ is greater than one then Σ is called a *ramification point* or simply a *branch point*. Obviously if $\text{Aut}(\Lambda, \leq)$ (refer Section 3.3) is relatively 2-transitive then $\text{Aut}(\Lambda, \leq)$ is transitive on the ramification points and so the number of cones below a ramification point is independent of which point we choose. This cardinal number is known as the *ramification order* of (Λ, \leq) . The ramification order of the semilinearly ordered set Λ that we have constructed is \aleph_0 and each node of Λ is a branch point. Every branch point of our semilinear order is of *positive type* by which we mean that every branch point as a lower section of Λ has a greatest element.

3.3 Automorphism group of (Ω, C)

The C -set Ω that we have constructed exhibit a high degree of symmetry with maximal chains isomorphic to \mathbb{Q} and branching number \aleph_0 . The set Ω is dense as well. In this section we aim at understanding the structure of the automorphism

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group of (Ω, C) as a permutation group on Ω and give a list of its Jordan sets. Most of the material in this section is taken from Adeleke & Neumann [AN2].

Let G denote the automorphism group $\text{Aut}(\Omega, C)$ (which consists of the subgroup of $\text{Sym}(\Omega)$ that maps C -related sets onto C -related sets) and let H denote the automorphism group $\text{Aut}(\Lambda, \leq)$. We show that the groups H and G are isomorphic.

Theorem 3.1 *The group H is isomorphic to G .*

Proof. We first show that H is a subgroup of G . Given any automorphism h of (Λ, \leq) we can construct an automorphism of (Ω, C) as follows. Elements of Ω are of the form $\alpha q, \alpha \in \Lambda, q \in \mathbb{Q}$. The element $h \in H$ maps the element αq to the element $\alpha^h p$ of Ω where α^h is the image of the node α under the automorphism h and the nodes of Λ on the q -th branch at α are mapped to the nodes on the p -th branch at α^h . This is true because h preserves the semilinear ordering on Λ . For simplicity we denote the p -th branch at the node α^h of the above discussion as the q^h -th branch. We define $(\alpha q)^h = \alpha^h q^h$, where we make no distinction between the element of H and the induced mapping of G .

We now show that the induced mapping on Ω is one-one. Let $\alpha q, \beta p$ be two elements of Ω , $\alpha, \beta \in \Lambda$, $q, p \in \mathbb{Q}$, such that $\alpha q \neq \beta p$. We show that $\alpha^h q^h \neq \beta^h p^h$. Since αq and βp are distinct so either $(\alpha = \beta, q \neq p)$ or $(\alpha \neq \beta)$. Suppose $\alpha = \beta, q \neq p$ then since $h \in H$ so $\alpha^h = \beta^h$ and $q^h \neq p^h$. For a contradiction, suppose if possible $q^h = p^h$ then any element on the q^h -th branch at α^h is comparable to any other element on the p^h -th branch. Now h and hence h^{-1} are elements of $\text{Aut}(\Lambda, \leq)$ and elements on the q -th and the p -th branches at α are incomparable, which cannot happen. Hence $q^h \neq p^h$ so that $\alpha^h q^h \neq \beta^h p^h$. Similarly if $\alpha \neq \beta$ then again $\alpha^h \neq \beta^h$ and so $\alpha^h q^h \neq \beta^h p^h$. Thus h is one-one.

We now show that h on Ω is onto. Let αq be any element of Ω . Now h is onto so that there exists $\beta \in \Lambda$ and $p \in \mathbb{Q}$ such that $\beta^h = \alpha$ and all the nodes on the p -th branch at β are mapped to the nodes on the q -th branch at α , for h has to preserve the semilinear ordering on the p -th branch at β so that $\beta^h p^h = \alpha q$.

Hence h as a mapping of Ω is onto. This means that the induced mapping h is a permutation on Ω .

We show that h preserves the C -relation on Ω . Suppose α is any element of Λ then h maps elements in distinct branches at α to elements in distinct branches at α^h . This is because h preserves the semilinear order on Λ . Thus if $x = \alpha q, y = \alpha p, p \neq q \in \mathbb{Q}$, then $x^h = \alpha^h q^h, y^h = \alpha^h p^h$ and $q^h \neq p^h$ so that $x^h \cap y^h = \alpha^h = (x \cap y)^h$. Now let $C(x; y, z)$. This implies either $y = z \neq x$ or $x \cap y < y \cap z$. If $y = z \neq x$ then the images of the chains y and z are equal which is different from the image of the chain x so that $y^h = z^h \neq x^h$. If $x \cap y < y \cap z$ then $(x \cap y)^h < (y \cap z)^h \Rightarrow x^h \cap y^h < y^h \cap z^h \Rightarrow C(x^h; y^h, z^h)$ so that the induced h preserves C on Ω .

Conversely, given any automorphism g of (Ω, C) we can construct an automorphism h of (Λ, \leq) in the following way. Any element α of Λ can be written in the form $x \cap y$ where $x = \alpha q, y = \alpha p, q \neq p$. We define $\alpha^h := x^g \cap y^g$. We can then show that h is an automorphism of (Λ, \leq) which means that G is a subgroup of H . Therefore H is isomorphic to G . •

The importance of Theorem 3.1 is that to define an automorphism of (Ω, C) it is sufficient to define the images of the nodes on the chains in the set Ω and conversely. We can thus define an automorphism of (Ω, C) in at most a countable number of steps as follows. First we define the images of the nodes on the \mathbb{Q}_1 -branch (the 0-chain) such that the set of nodes on the chain is fixed setwise and the linear order of \mathbb{Q} is preserved. Next we define the images of the enumeration of the \mathbb{Q}_2 -branches adjoined at nodes of the \mathbb{Q}_1 -branch. After that we define the images of the nodes of the \mathbb{Q}_2 -branches such that each \mathbb{Q}_2 -branch is fixed setwise and the linear order of the chains is preserved. Next we define the images of the enumeration of the \mathbb{Q}_3 -branches adjoined at nodes of the \mathbb{Q}_2 -branches. We continue this process for at most a countable number of times. The process gives an automorphism of (Ω, C) . Therefore to define an automorphism of (Ω, C) we essentially need to define two kinds of automorphisms.

3.3 Automorphism group of (Ω, C)

Definition 3.2 A *chain automorphism* is an automorphism that induces an order-preserving permutation on some \mathbb{Q}_k -branch, fixes setwise that particular \mathbb{Q}_k -branch, fixes nodes on other \mathbb{Q}_k -branches and nodes on all \mathbb{Q}_i -branches for $i < k$. A chain automorphism preserves the length of the elements of Ω and induces the identity permutation on the branches at nodes of Λ .

For example, suppose α is the node on the \mathbb{Q}_{k-1} -branch on which the \mathbb{Q}_k -branch under consideration is attached, n is the last co-ordinate in the expression of α and let the \mathbb{Q}_k -branch lie on the q -th branch at α . Then the chain automorphism when regarded as an automorphism of (Λ, \leq) acts non-trivially on the set

$$\{\alpha q n_1 q_1 n_2 q_2 \dots n_k | n < n_1 < n_2 < \dots < n_k \in \mathbb{Q}, q_1, q_2, \dots, q_{k-1} \in \mathbb{Q}\}$$

and acts trivially on the remaining nodes of Λ .

Definition 3.3 A *branch automorphism* is an automorphism that fixes a node, induces a permutation on the branches at the fixed node, fixes all nodes smaller or incomparable to the fixed node and preserves the level (refer Section 4.3) of the nodes.

For example, suppose α is a node of Λ . A branch automorphism g that induces a permutation on the branches at the node α , acts non-trivially on the set

$$\{\alpha q n_1 q_1 n_2 q_2 \dots n_k | n < n_1 < n_2 < \dots < n_k \in \mathbb{Q}, q, q_1, q_2, \dots, q_{k-1} \in \mathbb{Q}\},$$

where n is the last co-ordinate in the expression of α and acts trivially on the remaining nodes of Λ .

3.3.1 Composition of automorphisms in $\text{Aut}(\Omega, C)$

We show that any element of $\text{Aut}(\Omega, C)$ can be expressed as a composition of chain automorphisms and branch automorphisms. Consider the set $\mathbb{Q} \times \mathbb{Q} = \{(n, q) | n \in \mathbb{Q}, q \in \mathbb{Q}\}$. Let $P(\mathbb{Q}_0 \times \mathbb{Q}_0)$ be the subset of the power set of $\mathbb{Q} \times \mathbb{Q}$ consisting of those subsets A which can be expressed as follows. Suppose A is an

element of $P(\mathbb{Q}_0 \times \mathbb{Q}_0)$ then each element $n \in \mathbb{Q}$ occurs as the first co-ordinate in one and only one element of A and the second co-ordinate of elements of A is zero for all but finitely many elements. In other words, the subsets A of $P(\mathbb{Q} \times \mathbb{Q})$ are expressible as

$$\{(n, q) \mid \{n \mid (n, q) \in A\} = \mathbb{Q} \text{ and } q = 0 \text{ for all but finitely many elements } \}.$$

Consider the following mapping:

$$\begin{aligned} \phi : \Omega &\rightarrow P(\mathbb{Q}_0 \times \mathbb{Q}_0) \\ n_1 q_1 n_2 q_2 \dots n_k q_k &\mapsto \{(n, q) \mid q = q_i \text{ for } n = n_i, 1 \leq i \leq k, q = 0 \text{ otherwise } \}. \end{aligned}$$

It can be easily checked that ϕ is a bijective mapping. Sometimes for convenience we will express x of Ω in the form $\phi(x)$. Expressing an element of Ω in the form $\phi(x)$ we get a list of the nodes of Λ which lie on the element x of Ω .

Let $\lambda \in \Lambda$ and $P(\lambda)$ denote the set of elements of Ω passing through λ which is nothing but the set of cones at λ . Let g be an automorphism of (Ω, C) . For elements λ, δ of Λ let $P(\lambda)^g, P(\delta)^g$ denote the images of $P(\lambda), P(\delta)$ respectively. It is trivial to check that if $\lambda \leq \delta$ with respect to the semilinear order in Λ then $P(\delta) \subseteq P(\lambda)$, that is, $P(\delta)^g \subseteq P(\lambda)^g$. And if $\lambda \parallel \delta$ then $P(\lambda) \cap P(\delta) = \emptyset$, that is, $P(\lambda)^g \cap P(\delta)^g = \emptyset$.

Theorem 3.2 Any element of $\text{Aut}(\Omega, C)$ can be expressed as a composition of chain automorphisms and branch automorphisms.

Proof. Let g be an element of $\text{Aut}(\Omega, C)$. Consider the \mathbb{Q}_1 -branch which is the 0 element of Ω . Let 0^g be its image under g . For each $n_1 \in \mathbb{Q}_1$ consider the set of cones $\{P(n_1) \mid n_1 \in \mathbb{Q}_1\}$ and let $\{P(n_1)^g \mid n_1 \in \mathbb{Q}_1\}$ be its image set. Now $P(n_1)^g$ is of the form $P(n'_1), n'_1 \in \mathbb{Q}_1^g$. Let us denote n'_1 by n_1^g so that $P(n_1)^g$ is denoted as $P(n_1^g)$. Suppose $n_1 = l_1 \in 0$ and the image $l_1^g = m_1 p_1 m_2 p_2 \dots m_j$, $j \geq 1$. Let a_1 be the permutation on \mathbb{Q}_1 defined as

$$a_1 : \mathbb{Q}_1 \rightarrow \mathbb{Q}_1$$

$$n_1 \mapsto \text{the last co-ordinate in the expression of } n_1^g.$$

For example, if $n_1 = l_1$ as above then $l_1^{a_1} = m_j$. The permutation a_1 preserves the linear order \leq on \mathbb{Q}_1 , because a_1 depends on g and the automorphism g of (Ω, C) preserves the set-inclusion of cones, by which we mean that if $\lambda \leq \delta$ then $P(\delta^g) \subseteq P(\lambda^g)$. The order-preserving permutation a_1 of \mathbb{Q}_1 can be extended to an automorphism g_1 of (Ω, C) as follows:

$$\begin{aligned} g_1 : \quad \Omega &\rightarrow \Omega \\ \{(n, q)\} &\mapsto \{(n^{a_1}, q)\}. \end{aligned}$$

Here we are expressing elements of Ω as the corresponding element in the set $P(\mathbb{Q}_0 \times \mathbb{Q}_0)$. The automorphism g_1 is a chain automorphism that induces an order-preserving permutation on the \mathbb{Q}_1 -branch.

Now we look at the set Ω^{g_1} which is nothing but Ω . We refer the set Ω as Ω^{g_1} simply because now we are looking at the image of the set Ω under the automorphism g_1 . For each node $n_1^{a_1} \in \mathbb{Q}_1$ consider the set

$$B(n_1^{a_1}) := \{n_1^{a_1} q \mid q \in \mathbb{Q}_1\} \subseteq \Omega^g,$$

which consists of an element from each cone in the set $P(n_1^{a_1})$. The automorphism g maps a cone in $P(n_1)$ to a cone in $P(n_1^g)$. Looking at these images we get a permutation on \mathbb{Q} associated with the node $n_1^{a_1} \in \mathbb{Q}$ defined as

$$\begin{aligned} b_{1, n_1^{a_1}} : \mathbb{Q} &\rightarrow \mathbb{Q} \\ q &\mapsto q' \end{aligned}$$

where q' occurs in $n_1^g q' n_{11} q_{11} n_{12} q_{12} \dots n_{1k} q_{1k}$, the expression of $(n_1 q)^g$. Note that for each $n_1^{a_1} \in \mathbb{Q}$ we get a permutation $b_{1, n_1^{a_1}}$. For simplicity we denote the permutation $b_{1, n_1^{a_1}}$ as simply b_1 . For $l_1 r_1 \in B(l_1)$ let the image under g be $m_1 p_1 m_2 p_2 \dots m_j p_j m_{j+1} p_{j+1} \dots m_{j+j'} p_{j+j'}$, $j + j' \geq 1$ which is equal to $l_1^g p_j m_{j+1} p_{j+1} \dots m_{j+j'} p_{j+j'}$, since we have already said that $l_1^g = m_1 p_1 m_2 p_2 \dots m_j$. We then have $r_1^{b_{1, n_1^{a_1}}} = r_1^{b_{1, m_j}} = p_j$. Combining all these permutations b_1 we define an automorphism h_1 on Ω^{g_1} as

$$\begin{aligned} h_1 : \quad \Omega^{g_1} &\rightarrow \Omega^{g_1} \\ \{(n, q)\} &\mapsto \{(n, q^{b_1})\}, \end{aligned}$$

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where b_n is equal to $b_{1,n}$ if $(n, q)^{g^{-1}}$ is a node on the \mathbb{Q}_1 -branch, otherwise b_n is the identity permutation. The automorphism $g_1 h_1$ maps the 0 element of Ω to the element 0^g . For each node $n_1 \in \mathbb{Q}_1$ we get a branch automorphism at the node n_1 when we restrict h_1 to the set $B(n_1)$.

Now we look at $\Omega^{g_1 h_1}$. For each node $n_1 \in 0$ consider an element of $B(n_1)$ say $n_1 q_1$. The image of $n_1 q_1$ under $g_1 h_1$ is $(n_1 q_1)^{g_1 h_1}$. Consider the segment of $(n_1 q_1)^{g_1 h_1}$ consisting of all points greater than n_1^g which we denote as the interval $(n_1^g, (n_1 q_1)^{g_1 h_1})$. We repeat the same steps in this segment as we have done for the 0-chain. Consider the set of cones $\{P(n_1 q_1 n_2) \mid n_1 q_1 n_2 \in (n_1, n_1 q_1 n_2)\}$. The image of the cones under the automorphism $g_1 h_1$ can be expressed as $\{P(n_1 q_1 n_2)^{g_1 h_1}\}$. Clearly $P((n_1 q_1 n_2)^{g_1 h_1}) \subset P(n_1^g)$ for each node $(n_1 q_1 n_2)^{g_1 h_1}$ in $(n_1^g, (n_1 q_1)^{g_1 h_1})$. Looking at the images of $P(n_1 q_1 n_2)^g$ we get the image of $(n_1 q_1 n_2)^{g_1 h_1}$ which corresponds to the node $n_1 q_1 n_2$. Let $a_{2, (n_1 q_1)^{g_1 h_1}}$ be the order-preserving permutation on $(n_1^g, (n_1 q_1)^{g_1 h_1})$ defined as

$$a_{2, (n_1 q_1)^{g_1 h_1}} : (n_1^g, (n_1 q_1)^{g_1 h_1}) \rightarrow (n_1^g, (n_1 q_1)^{g_1 h_1}) \\ (n_1 q_1 n_2)^{g_1 h_1} \mapsto (n_1 q_1)^{g_1 h_1} n_2^{a_2},$$

where $n_2^{a_2}$ is the last co-ordinate of $(n_1 q_1 n_2)^g$ with $(n_1 q_1)^{g_1 h_1} = (n_1 q_1)^g$. For $n_1 \in \mathbb{Q}_1$ and $q \in \mathbb{Q}$ we get a map $a_{2, (n_1 q_1)^{g_1 h_1}}$. We denote this map simply as a_2 , where a_2 is defined on some branch of $\Omega^{g_1 h_1}$ which is a \mathbb{Q}_2 -branch. Suppose $(l_1 r_1)^g = m_1 p_1 m_2 p_2 \dots m_j p_j m_{j+1} p_{j+1} \dots m_{j+j'} p_{j+j'}$, $j + j' \geq 1$ as before and $(l_1 r_1 l_2)^g = m_1 p_1 m_2 p_2 \dots m_j p_j m_{j+1} p_{j+1} \dots m_{j+j'} p_{j+j'} m_{j+j'+1} p_{j+j'+1} \dots m_{j+j'+j''}$, $j + j' + j'' \geq 2$. Then $((l_1 r_1 l_2)^{g_1 h_1})^{a_2} = (l_1 r_1)^{g_1 h_1} m_{j+j'+j''}$, $(l_1 r_1)^{g_1 h_1} = (l_1 r_1)^g$. We define an automorphism g_2 on $\Omega^{g_1 h_1}$ combining all the permutations of the form a_2 to get an automorphism g_2 on $\Omega^{g_1 h_1}$ as

$$g_2 : \Omega^{g_1 h_1} \rightarrow \Omega^{g_1 h_1} \\ \{(n, q)\} \mapsto \{(n^{a_n}, q)\},$$

where a_n is the identity mapping if $n^{h_1^{-1} g_1^{-1}}$ is an element of the \mathbb{Q}_1 -branch and otherwise a_n is the permutation $a_{2, nq}$ on the branch nq of $\Omega^{g_1 h_1}$.

We now look at $\Omega^{g_1 h_1 g_2}$. Consider the set $B(n_1 q_1 n_2)$ which has its image as

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$B((n_1 q_1 n_2)^{g_1 h_1 g_2})$ under the composition $g_1 h_1 g_2$. Elements of $B((n_1 q_1 n_2)^{g_1 h_1 g_2})$ are expressible as $\{(n_1 q_1 n_2)^{g_1 h_1 g_2} q \mid q \in \mathbb{Q}\}$. Looking at the images of $B(n_1 q_1 n_2)$ under the automorphism g we get a permutation on \mathbb{Q} associated with the node $(n_1 q_1 n_2)^{g_1 h_1 g_2}$ defined as

$$b_{2, (n_1 q_1 n_2)^{g_1 h_1 g_2}} : \mathbb{Q} \rightarrow \mathbb{Q} \\ q \mapsto q'',$$

where q'' is the co-ordinate succeeding $n_2^{a_2}$ in the expression of $(n_1 q_1 n_2 q)^g$ with $n_2^{a_2}$ as defined earlier. For simplicity we denote $b_{2, (n_1 q_1 n_2)^{g_1 h_1 g_2}}$ as b_2 , remembering that b_2 is defined on a node of the image of a \mathbb{Q}_2 -branch under $g_1 h_1 g_2$. Combining all the permutations of the form b_2 we define an automorphism h_2 on $\Omega^{g_1 h_1 g_2}$ as

$$h_2 : \Omega^{g_1 h_1 g_2} \rightarrow \Omega^{g_1 h_1 g_2} \\ \{(n, q)\} \mapsto \{(n, q^{b_n})\},$$

where b_n is equal to $b_{2, n}$ if $(n, q)^{g_2^{-1} h_1^{-1} g_1^{-1}}$ is a node on the \mathbb{Q}_2 -branch, otherwise b_n is the identity permutation. The composition $g_1 h_1 g_2 h_2$ maps each \mathbb{Q}_2 -branch onto its image under the automorphism g .

We repeat this process of extracting chain automorphisms and branch automorphisms of g alternatively so that with suitable definitions of $g_3, h_3, g_4, h_4, \dots$ the composition $g_1 h_1 g_2 h_2 g_3 h_3$ gives the image of the \mathbb{Q}_3 -branches of Ω under g , the composition $g_1 h_1 g_2 h_2 g_3 h_3 g_4 h_4$ gives the image of the \mathbb{Q}_4 -branches of Ω under g . Carrying on in this manner we can show that any automorphism of (Ω, C) can be expressed as a composition of chain and branch automorphisms. •

3.3.2 Some permutations on the rational numbers

Theorem 3.2 gives an expression of elements of $\text{Aut}(\Omega, C)$ in terms of chain automorphisms and branch automorphisms of (Ω, C) . In other words, the knowledge of the types of chain automorphisms and branch automorphisms of (Ω, C) gives a picture of the elements of $\text{Aut}(\Omega, C)$. The elements of Ω when regarded as maximal chains of the underlying semilinear order are isomorphic to \mathbb{Q} and the branches at a node of the semilinear order are indexed by \mathbb{Q} . Therefore permu-

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tations of \mathbb{Q} denoted by $\text{Sym}(\mathbb{Q})$ and the subgroup of $\text{Sym}(\mathbb{Q})$ that preserve the linear order on \mathbb{Q} denoted by $\text{Aut}(\mathbb{Q}, \leq)$ play a dominant role in our study of automorphisms of $\text{Aut}(\Omega, C)$.

We give a list of some elements of $\text{Sym}(\mathbb{Q})$, the group of permutations on \mathbb{Q} .

(i) *Translation* mapping defined as $q \mapsto q + q_0, q_0 \in \mathbb{Q}$.

(ii) *Magnification* mapping defined as $q \mapsto qq_0, q_0 \in \mathbb{Q} \setminus \{0\}$. For $q_0 = -1$ we call the magnification a *reflection* and for $q_0 \in \mathbb{Q}^+$ (the positive rational numbers excluding 0) we call it a *positive magnification*.

(iii) Permutation on any finite subset of \mathbb{Q} and fixing the rest.

(iv) A *k-homogeneous automorphism* defined as follows ([BMMN][p. 26]).

Let $A = \{a_1, a_2, \dots, a_k\}, a_1 < a_2 < \dots < a_k$ and $B = \{b_1, b_2, \dots, b_k\}, b_1 < b_2 < \dots < b_k$ be two subsets of \mathbb{Q} with same size k . Define intervals $A_i := (a_i, a_{i+1})$ and $B_i := (b_i, b_{i+1})$ for each $i = 1, 2, \dots, k-1$, and set $A_0 := (-\infty, a_1), A_k := (a_k, \infty), B_0 := (-\infty, b_1), B_k := (b_k, \infty)$. Define $f_i : A_i \mapsto B_i$, for $i = 0, 1, 2, \dots, k$ as

$$f_0 : a \mapsto b_1 + (a - a_1),$$

$$f_i : a \mapsto b_i + \frac{(a - a_i)(b_{i+1} - b_i)}{(a_{i+1} - a_i)}, \quad 1 \leq i \leq k-1,$$

$$f_k : a \mapsto b_k + (a - a_k).$$

We define a mapping g on \mathbb{Q} as $a_i^g = b_i, i = 1, 2, \dots, k, g|_{A_i} = f_i, i = 0, 1, \dots, k$. The mapping g is called a *k-homogeneous automorphism* on \mathbb{Q} which maps the k -set A onto the k -set B .

Note that a *k-homogeneous automorphism* is a composition of translations and positive magnifications and both translations and positive magnifications preserve the linear order \leq on \mathbb{Q} . Therefore a *k-homogeneous automorphism* is an element of $\text{Aut}(\mathbb{Q}, \leq)$.

3.3.3 Properties of the automorphism group of (Ω, C)

We first look at the type of branch automorphisms and chain automorphisms that can be defined on (Ω, C) . For our reference in future we give names of the automorphisms that we define.

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Branch automorphisms of $\text{Aut}(\Omega, C)$

Branch automorphisms of $\text{Aut}(\Omega, C)$ can induce any permutation (of the rational numbers discussed in Section 3.3.2) on the branches at nodes of Λ .

Definition 3.4 Let α be a node of $\Lambda, q_0 \in \mathbb{Q}$ and let g be defined on Ω as

$$\omega \mapsto \omega, \text{ for all } \omega \text{ not passing through } \alpha,$$

$$\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mapsto \alpha(q + q_0) n_1 q_1 n_2 q_2 \dots n_k q_k.$$

Then g is a branch automorphism that induces a permutation on the branches at the node α . We call g a *translation branch automorphism*.

Definition 3.5 For $\alpha \in \Lambda$ and $q_0 \in \mathbb{Q} \setminus \{0\}$ let g be defined on Ω as

$$\omega \mapsto \omega, \text{ for all } \omega \text{ not passing through } \alpha,$$

$$\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mapsto \alpha(q q_0) n_1 q_1 n_2 q_2 \dots n_k q_k.$$

Then g is a branch automorphism that induces a permutation on the branches at the node α . We call g a *magnification branch automorphism*.

Definition 3.6 Let α be a node of Λ, P be a finite subset of \mathbb{Q} and a be a permutation on the set P . Let g be defined on Ω as

$$\omega \mapsto \omega, \text{ for } \omega \text{ not passing through } \alpha,$$

$$\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mapsto \alpha q^a n_1 q_1 n_2 q_2 \dots n_k q_k, \text{ where } q^a = q, \text{ if } q \notin P.$$

The automorphism g is a branch automorphism that induces permutation on the branches at the node α . We call g a *finite set permutation branch automorphism*.

Definition 3.7 Let α be a node of Λ and for $k \in \mathbb{N}$, let h_k be a *k-homogeneous* mapping of (\mathbb{Q}, \leq) . Let g be defined on Ω as

$$\omega \mapsto \omega, \text{ for all } \omega \text{ not passing through } \alpha,$$

$$\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mapsto \alpha q^{h_k} n_1 q_1 n_2 q_2 \dots n_k q_k.$$

The automorphism g is a branch automorphism that induces permutation on the branches at the node α . We call g a *k-homogeneous branch automorphism*.

Chain automorphisms of $\text{Aut}(\Omega, C)$

We now look at the chain automorphisms of (Ω, C) . Automorphisms of (Ω, C) must preserve the underlying semilinear order on Λ and so chain automorphisms must preserve the linear order of the chains of Λ . Therefore there cannot exist any finite set permutation chain mappings. Translation automorphisms and positive magnification chain automorphisms can occur in only the \mathbb{Q}_1 -branch. This is because chain automorphisms on a \mathbb{Q}_k -branch, $k > 1$ has at least one fixed point, namely the node on which the \mathbb{Q}_k -branch is attached.

Definition 3.8 For $n_0 \in \mathbb{Q}$ let g be defined on Ω as

$$n_1 q_1 n_2 q_2 \dots n_k q_k \mapsto (n_1 + n_0) q_1 (n_2 + n_0) q_2 \dots (n_k + n_0) q_k.$$

Then g is a chain automorphism that induces a translation of the nodes on the \mathbb{Q}_1 -branch. We call g a *translation chain automorphism*.

Definition 3.9 Let g_k be a k -homogeneous map on (\mathbb{Q}, \leq) . We define g on Ω as

$$n_1 q_1 n_2 q_2 \dots n_j q_j \mapsto n_1^{g_k} q_1 n_2^{g_k} q_2 \dots n_j^{g_k} q_j.$$

Then g is a chain automorphism that induces g_k on the \mathbb{Q}_1 -branch. For $i > 1$ let \mathbb{Q}_i be a branch at α for some node $\alpha \in \Lambda$ and let \mathbb{Q}_i be on the q -th branch at α . Let h_k be a k -homogeneous map on the nodes of the \mathbb{Q}_i -branch. We define h on Ω as to fix any x that does not pass through α , any x that passes through α but lie on a branch different from the q -th branch at α and on the remaining elements of Ω the automorphism h is defined as

$$\alpha q n_1 q_1 n_2 q_2 \dots n_j q_j \mapsto \alpha q n_1^{h_k} q_1 n_2^{h_k} q_2 \dots n_j^{h_k} q_j.$$

Then h is a chain automorphism that induces h_k on the \mathbb{Q}_i -branch. The automorphisms g and h are called *k -homogeneous chain automorphisms*.

Here the branch automorphisms and chain automorphisms are defined on Ω . However these same mappings can also be defined on Λ . We shall refer to chain automorphisms and branch automorphisms of Ω or Λ wherever suitable.

 3.3 Automorphism group of (Ω, C)

The theory of C -relations given in [AN2][Section14] is that in which the automorphism group is 2-transitive. We show that the C -set Ω that we have constructed admits a 2-transitive automorphism group. Before that we show that $\text{Aut}(\Lambda, \leq)$ is transitive.

Lemma 3.1 *The group $H = \text{Aut}(\Lambda, \leq)$ is transitive.*

Proof. Let α, β be distinct elements of Λ . We need to find h in H such that $\alpha^g = \beta$. Now the lengths of α and β are either equal or unequal.

First suppose that α and β are of equal length. To start with let the lengths of α and β be 1 so that $\alpha = n_1, \beta = n_2, n_1 \neq n_2$. The translation chain automorphism on the \mathbb{Q}_1 -branch translating nodes by the quantity $n_2 - n_1$ is the required automorphism. Next, suppose α, β are of equal length $2t - 1, t > 1$ so that expressions of α, β are of the forms $n_1 q_1 n_2 q_2 \dots n_{t-1} q_{t-1} n_t, m_1 p_1 m_2 p_2 \dots m_{t-1} p_{t-1} m_t$ respectively. We first define a t -homogeneous chain automorphism h that induces a permutation of the nodes on the \mathbb{Q}_1 -branch such that the nodes n_i are mapped onto the nodes $m_i, 1 \leq i \leq t$. And then define branch automorphisms f_1, f_2, \dots, f_{t-1} that induces 2-homogeneous mappings of the branches on each of the nodes $m_1, m_1 p_1 m_2, \dots, m_1 p_1 m_2 p_2 \dots m_{t-1}$ such that the q_j -th branch is mapped to the p_j -th branch, $1 \leq j \leq t - 1$ respectively. Then the composition $h f_1 f_2 \dots f_{t-1}$ (refer Section 3.3.1) maps α to β .

Next, suppose the lengths of α and β are unequal. Let $\alpha = n_1 q_1 n_2 q_2 \dots n_k$ and $\beta = m_1 p_1 m_2 p_2 \dots m_j$ be of lengths $2k - 1$ and $2j - 1$ respectively. Suppose $k < j$ in which case we express α as an element of length $2j - 1$ as

$$\alpha = n_1 q_1 n_2 q_2 \dots n_{k-1} q_{k-1} l_1 0 l_2 0 \dots l_{j-k} 0 n_k, n_{k-1} < l_1 < l_2 < \dots < l_{j-k} < n_k.$$

(If $j < k$ then we express β as an element of length $2k - 1$.) We now define a chain automorphism g that induces a j -homogeneous mapping of the \mathbb{Q}_1 -branch such that the nodes $n_1, n_2, \dots, n_{k-1}, l_1, l_2, \dots, l_{j-1}, n_k$ are mapped onto the nodes m_1, m_2, \dots, m_j . Then define branch automorphisms $f_i, 1 \leq i \leq k - 1$ that induces 2-homogeneous mappings of the branches at the nodes $m_1, m_1 p_1 m_2, \dots,$

$m_1p_1m_2p_2 \dots m_{k-1}$ such that the q_i -th branch is mapped onto the p_i -th branch respectively. We also define branch automorphisms f_l , $k \leq l \leq j-1$ that induces 2-homogeneous mappings of the branches at the nodes $m_1p_1m_2p_2 \dots m_{k-1}p_{k-1}m_k$, $m_1p_1 \dots m_{k-1}p_{k-1}m_kp_km_{k+1}$, \dots , $m_1p_1 \dots m_{k-1}p_{k-1}m_kp_km_{k+1}p_{k+1} \dots m_{j-1}$ that maps the 0-th branch to the p_l -th branch respectively. Then the composition $gf_1f_2 \dots f_{j-1}$ (refer Section 3.3.1) is the required automorphism that maps α to β . •

In Lemma 3.1 we have used a method to equalise the lengths of the expressions of two elements of Ω . In the following lemma we give another method to equalise the lengths of the expressions of elements of Ω . Equalising the lengths of the expressions of elements of Ω forces the uniqueness of the expressions of elements of Ω to be lost. Same procedures can be applied to equalise lengths of any number of elements.

Lemma 3.2 *Lengths of the expressions of any two elements of Ω can be equalised.*

Proof. Let x and y be any two elements of Ω with expressions $n_1q_1n_2q_2 \dots n_kq_k$ and $m_1p_1m_2p_2 \dots m_jp_j$ which are of lengths $2k$ and $2j$ respectively, $k \neq j$. We equalise the lengths of the expressions of x and y as follows. Let n_k 's and m_j 's be expressed in ascending order as l_1, l_2, \dots, l_i . We then express x and y as $l_1q'_1l_2q'_2 \dots l_iq'_i$ and $l_1p'_1l_2p'_2 \dots l_ip'_i$, $i \leq k+j$, respectively where $q'_{i_0} = q_{i_0}$, $1 \leq i_0 \leq i$ if l_{i_0} occurs in the expression of x and $q'_{i_0} = 0$ otherwise and $p'_{i_0} = p_{i_0}$, $1 \leq i_0 \leq i$ if l_{i_0} occurs in the expression of y and $p'_{i_0} = 0$ otherwise. •

Lemma 3.3 *The group $G = \text{Aut}(\Omega, C)$ is transitive on Ω .*

Proof. Let $x = n_1q_1n_2q_2 \dots n_kq_k$ and $y = m_1p_1m_2p_2 \dots m_jp_j$ be two elements of Ω . We need to find $g \in G$ such that $x^g = y$. We first express x and y with equalised lengths $2i$ (refer Lemma 3.2) as $l_1q'_1l_2q'_2 \dots l_iq'_i$ and $l_1p'_1l_2p'_2 \dots l_ip'_i$ respectively. We then define translation branch automorphisms f_t , $1 \leq t \leq i$ that induces translations of the branches at the nodes $l_1, l_1p'_1l_2, \dots, l_1p'_1l_2p'_2 \dots l_i$

respectively such that the q'_t -th branch is mapped to the p'_t -th branch. The composition of the branch automorphisms $f_1f_2 \dots f_i$ maps x to y . •

The branch automorphisms defined in Lemma 3.3 are not the only automorphisms that map α to β . Branch automorphisms that induce magnifications or permutations or k -homogeneous mappings will do the job as well.

Theorem 3.3 *The group $G = \text{Aut}(\Omega, C)$ is 2-transitive on Ω .*

Proof. Let (x, y) and (z, w) , $x \neq y$ and $z \neq w$ be two distinct pairs of elements of Ω such that $x \cap y = \alpha$ and $z \cap w = \beta$. We need to find $f \in G$ such that $x^f = z$ and $y^f = w$. Using Lemma 3.1 and Theorem 3.1 there exists an element g_1 in G that maps α to β . Let x^{g_1}, y^{g_1} be the images of x, y respectively under this automorphism. We need to define an automorphism that maps x^{g_1}, y^{g_1} to z, w respectively. Let the expressions of x^{g_1}, y^{g_1}, z, w be $\beta qn_1q_1n_2q_2 \dots n_kq_k$, $\beta pm_1p_1m_2p_2 \dots m_jp_j$, $\beta rl_1r_1l_2r_2 \dots l_ir_i$, and $\beta ts_1t_1s_2t_2 \dots s_vt_v$ respectively, where $q \neq p, r \neq t$. If either $x^{g_1} = z$ or $y^{g_1} = w$, say $x^{g_1} = z$ then using transitivity of G and the fact that the q -th branch (which is equal to the r -th branch) at β can be fixed when mapping the p -th branch to the t -th branch, the result follows. (Here of course translation branch automorphisms will not work.) Suppose $x^{g_1} \neq z$ and $y^{g_1} \neq w$. We first equalise the lengths of x^{g_1}, y^{g_1}, z, w with β occurring as the first node in each of the expressions. Then define 2-homogeneous branch automorphisms at each node occurring in the equalised expressions of x^{g_1}, y^{g_1}, z, w starting with β such that at each node the branches of x^{g_1}, y^{g_1} are mapped to the branches of z, w respectively. The composition of g_1 and these branch automorphisms will give an automorphism that maps x, y to z, w respectively. •

Note that in defining the branch automorphisms in Theorem 3.3 we can also define finite set permutation branch automorphisms and still get our result. This means that more than one element of $\text{Aut}(\Omega, C)$ will map an ordered tuple of Ω to another ordered tuple and hence the automorphism group is not sharply 2-transitive. We study a little more of the automorphism group $\text{Aut}(\Omega, C)$ and

proceed to give a lemma taken from Adeleke & Neumann [AN2][Lemma 14.2] which will be used very frequently.

The notion of a C -set describes the behaviour of maximal chains in a semilinearly ordered set. Let us fix an element x of Ω and define three binary relations T_x , R_x and S_x on $\Omega \setminus \{x\}$ by specifying that

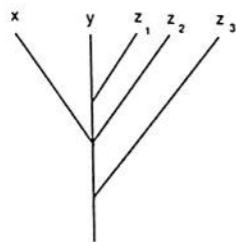


Figure 3.3:

$$\begin{aligned} yT_xz & :\leftrightarrow \neg C(y; z, x), \\ yR_xz & :\leftrightarrow \neg C(y; z, x) \wedge \neg C(z; y, x), \\ yS_xz & :\leftrightarrow C(x; y, z). \end{aligned}$$

In Fig. 3.3 we have yT_xz_i , $1 \leq i \leq 3$, yR_xz_j , $1 \leq j \leq 2$ and yS_xz_1 . A set which is an S_x -class for some x will be referred to generically as an S -class in Ω . The S -classes are like cones above branch points in semilinearly ordered sets. The number of S -classes at each node of x is \aleph_0 in our set Ω which is the branching number of Ω . Let Γ_x denote the set of all equivalence classes determined by R_x on $\Omega \setminus \{x\}$ and define a binary relation \leq_x on Γ_x by the rule $R_x(y) \leq_x R_x(z) :\leftrightarrow yT_xz$. Then T_x is a linear ordering of Γ_x , which has no maximum or minimum element. Moreover if (Ω, C) is dense then (Γ_x, T_x) is dense as a linearly ordered set [AN2][Lemma 10.2].

Lemma 3.4 [Adeleke & Neumann] *Let Σ be an S -class in the C -set (Ω, C) . Every automorphism of Σ extends to an automorphism of Ω that fixes $\Omega \setminus \Sigma$ pointwise. •*

Suppose we have a branch automorphism g that induces a permutation on the branches at a node α . Then we can suitably define an S -class Σ on the node

3.3 Automorphism group of (Ω, C)

α such that g fixes $\Omega \setminus \Sigma$ pointwise. Similarly given any chain automorphism on a \mathbb{Q}_k -branch we can suitably define an S -class on a node on the \mathbb{Q}_{k-1} -branch containing the specified \mathbb{Q}_k -branch.

Lemma 3.5 *The group $H = \text{Aut}(\Lambda, \leq)$ is relatively 2-transitive on Λ .*

Proof. Let (α, β) and (γ, δ) be two pairs of distinct elements of Λ such that $\alpha < \beta$ and $\gamma < \delta$. We need to find $h \in H$ such that $(\alpha, \beta)^h = (\gamma, \delta)$. By Lemma 3.1 there exists $h_1 \in H$ that maps α to γ . Next on the S -class at the node α^{h_1} we can define a composition of branch automorphisms and chain automorphisms such that β^{h_1} is mapped to δ . These maps that we need to construct are similar to those that we constructed to show transitivity of $\text{Aut}(\Lambda, \leq)$.

If on the other hand α, β and γ, δ are such that $\alpha \parallel \beta$ and $\gamma \parallel \delta$ we first need to define a map g_1 that maps $\alpha \cap \beta$ to $\gamma \cap \delta$, which is possible by Lemma 3.1. Then we define a branch automorphism g_2 on the node $\gamma \cap \delta$ that maps the branches of $\alpha^{g_1}, \beta^{g_1}$ to γ, δ respectively. Then we define a composition of branch automorphisms and chain automorphisms say g_3 on the S -class at the node $\gamma \cap \delta$ containing the nodes $\alpha^{g_1g_2}$ and γ such that $\alpha^{g_1g_2}$ is mapped to γ . Similarly we define a composition of branch automorphisms and chain automorphisms say g_4 on the S -class at the node $\gamma \cap \delta$ containing the nodes $\beta^{g_1g_2}$ and δ such that $\beta^{g_1g_2}$ is mapped to δ . Then the composition $g_1g_2g_3g_4$ maps (α, β) to (γ, δ) . •

We show that $\text{Aut}(\Omega, C)$ is relatively 3-transitive.

Theorem 3.4 *The group $G = \text{Aut}(\Omega, C)$ is relatively 3-transitive on Ω .*

Proof. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two 3-tuples such that $C(x_1; y_1, z_1)$ and $C(x_2; y_2, z_2)$ holds. We need to show that there exists an automorphism in G that maps (x_1, y_1, z_1) to (x_2, y_2, z_2) . Let $\alpha = x_1 \cap y_1$, $\beta = y_1 \cap z_1$, $\gamma = x_2 \cap y_2$ and $\delta = y_2 \cap z_2$. Because $C(x_1; y_1, z_1)$ and $C(x_2; y_2, z_2)$ so we must have $\alpha < \beta$ and $\gamma < \delta$. By relative 2-transitivity of H there exists $h_1 \in H$ and hence in G such that $(\alpha, \beta)^{h_1} = (\gamma, \delta)$. This means that $\gamma = x_1^{h_1} \cap y_1^{h_1}$, $\delta = y_1^{h_1} \cap z_1^{h_1}$. By transitivity of

G there exists $g_1 \in G$ such that $x_1^{h_1 g_1} = x_2$. Again using 2-transitivity on the S_{z_2} -class containing y_2 we can define an automorphism, say g_2 that maps $y^{h_1 g_1}, z^{h_1 g_1}$ to y_2, z_2 respectively. Note that the S_{z_2} -class containing y_2 does not contain x_2 and hence g_2 can be extended to an automorphism g_3 on (Ω, C) such that g_3 induces g_2 on the S_{z_2} -class containing y_2 and fixes the complement pointwise. Then $h_1 g_1 g_3$ is the required mapping that maps (x_1, y_1, z_1) to (x_2, y_2, z_2) . •

Lemma 3.6 *The group G is not 3-transitive.*

Proof. This is obvious because C is a ternary relation and so a C -related 3-tuple cannot be mapped to another 3-tuple which is not C -related. •

3.3.4 Jordan sets of $\text{Aut}(\Omega, C)$

We now look at the Jordan sets of $\text{Aut}(\Omega, C)$. Adeleke & Neumann have shown that in $\text{Aut}(\Omega, C)$ proper Jordan sets have to be any of the three types Σ [AN2][Theorem 14.9] described below. We use the notion of an S -class as defined in the previous Section.

Type I Σ is an S -class (Fig. 3.4(a)) or

Type II Σ is a union of two or more S -classes associated with some node of Λ (Fig. 3.4(b)) or

Type III there is an element $x \in \Omega$ and an open convex segment Δ of Γ_x such that $\Sigma = \{\omega \in \Omega \mid R_x(\omega) \in \Delta\}$, (refer Section 3.2) (Fig. 3.4(c)).

There is another type of Jordan set in $\text{Aut}(\Omega, C)$ which is mentioned as a remark by Macpherson [M1][p. 86] that can be described as

Type IV for any node α in Λ , Σ is the set of all chains not passing through α (Fig. 3.4(d)).

In Adeleke & Neumann's Theorem there is another type mentioned which reduces to a single S -class in our set Ω .

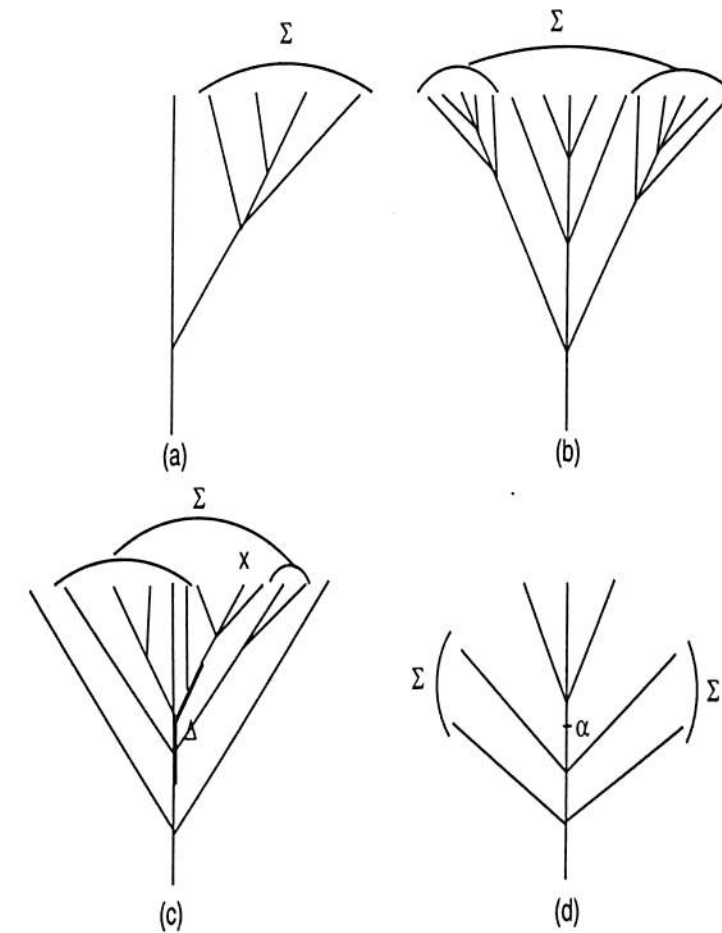


Figure 3.4:

Most of the theory of C -relations stated so far has been developed by Adeleke & Neumann. In the next chapter we impose extra relations on the C -set Ω and investigate whether the Jordan property is preserved in the automorphism groups with the extra relations.

Chapter 4

Expansions of C -sets

In this chapter we define some extra relations on the C -set constructed in the previous chapter. We study the automorphism groups of (Ω, C) with these extra relations and show that each of these automorphism groups contain Jordan sets. We also construct a minimal Jordan group.

4.1 A linear order on (Ω, C)

The first relation that we impose on (Ω, C) is a linear order. The idea of the linear order is the same as that defined by Adeleke [A][Section 9]. In defining the

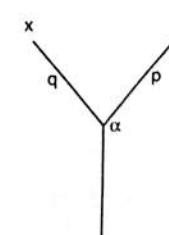


Figure 4.1:

linear order the underlying linear order of \mathbb{Q} will play a dominant role. Let x, y be two elements of Ω with $\alpha = x \cap y$ and let x, y lie on the q -th, p -th branches at

α respectively. We define the linear order \leq on Ω by

$$x \leq y : \leftrightarrow q \leq p,$$

where the first \leq denotes the linear ordering on Ω and the second \leq denotes the linear ordering on \mathbb{Q} (Fig. 4.1). As usual we use $<$ to denote strict inequality.

The linear order naturally induces a linear betweenness relation on Ω . Given three elements α, β, γ of \mathbb{Q} , the relation ρ on \mathbb{Q} (refer [BMMN][p. 107]) is defined as follows:

$$\rho(\alpha; \beta, \gamma) : \leftrightarrow (\beta \leq \alpha \leq \gamma) \vee (\gamma \leq \alpha \leq \beta),$$

where \leq is the linear order on \mathbb{Q} . The linear betweenness relation ρ on Ω is defined similarly. For any three elements x, y, z of Ω we define ρ as follows:

$$\rho(x; y, z) : \leftrightarrow (y \leq x \leq z) \vee (z \leq x \leq y).$$

It is easy to check that the linear order \leq on (Ω, C) is *dense* and unbounded. We have already seen that the set Ω is countable and because by Cantor's Theorem there is just one countable dense linear order without end-points, we can say that the set (Ω, C, \leq) is order-isomorphic to (\mathbb{Q}, \leq) .

4.1.1 Automorphism group of (Ω, C, \leq)

Here we look at the automorphism group of (Ω, C, \leq) which we denote as G_{\leq} for simplicity. We show that $\text{Aut}(\Omega, C, \leq)$ is a Jordan group. First we look at a result of Adeleke [A][p. 191] which we refer to as Adeleke's result. Let H_{\leq} be the subgroup of $\text{Aut}(\Lambda, \leq)$ which consists of elements that induce order-preserving permutations on the branches at each node. Then Adeleke's result is the following.

Adeleke's Result. *The group H_{\leq} is a subgroup of G_{\leq} and G_{\leq} is an order 2-transitive Jordan group which is not order 3-transitive.*
By order k -transitivity he means relative k -transitivity (refer Definition 2.8) with

4.1 A linear order on (Ω, C)

respect to the relations \leq and C . The difference between the C -sets of Adeleke and the one constructed here is that in Adeleke's C -set the maximal chains are isomorphic to \mathbb{Z} while in our C -set the maximal chains are isomorphic to \mathbb{Q} . Therefore Adeleke's result which is true in his example is only partially correct in our construction. In our setting we call the version of Adeleke's result as *generalised Adeleke's result*. Also we prove something more than what is in Adeleke's result: we prove that H_{\leq} is isomorphic to G_{\leq} .

Generalised Adeleke's result. *The group H_{\leq} is isomorphic to G_{\leq} which is not 2-transitive but relatively 3-homogeneous Jordan group.*

We prove that G_{\leq} is a Jordan group in Theorem 4.8. In this section we prove the remaining parts of generalised Adeleke's result.

Theorem 4.1 *The group H_{\leq} is isomorphic to G_{\leq} .*

Proof. We have already shown H to be isomorphic to G (Theorem 3.1). Let x, y be two elements of Ω such that $x < y$. We first show that every element of H_{\leq} preserves the linear order of x and y . Let h be an element of H_{\leq} which means that h is an element of H . This in turn means that there is an induced automorphism in (Ω, C) . Let us use the same notation h for the induced automorphism in (Ω, C) . Let $\alpha = x \cap y$ so that $x = \alpha q, y = \alpha p, q \neq p \in \mathbb{Q}$. Since $x < y$ we must have $q < p$. Let the images of x and y be $\alpha^h q^h$ and $\alpha^h p^h$ respectively. Because $q < p$ at the node α and h preserves the linear order amongst the branches at nodes of Λ so we must have $q^h < p^h$ which means $x^h < y^h$ as required.

Conversely, we show that G_{\leq} is a subgroup of H_{\leq} . Let g be an element of G_{\leq} so that g induces an automorphism of H . We need to show that $g \in H_{\leq}$. Let q, p be two distinct branches at any node α of Λ such that $q < p$. Then $x = \alpha q$ and $y = \alpha p$ are elements of Ω such that $x < y$. Since $g \in G_{\leq}$ so $x^g < y^g$. Now $x^g = \alpha^g q^g$ and $y^g = \alpha^g p^g$ so that $q^g < p^g$ and hence g preserves the ordering of the branches at any node of Λ . Therefore g is an element of H_{\leq} . •

The following lemmas give the types of branch automorphisms and chain auto-

morphisms of $\text{Aut}(\Omega, C, \leq)$.

Lemma 4.1 *The branch automorphisms of $\text{Aut}(\Omega, C, \leq)$ are such that the induced permutation on the branches at nodes preserve the linear ordering.* •

Lemma 4.2 *The chain automorphisms of $\text{Aut}(\Omega, C, \leq)$ can induce any order-preserving permutation on the nodes of \mathbb{Q}_k -branches, $k \in \mathbb{N}$.* •

Let G_ρ denote the subgroup of G that preserves the linear betweenness relation ρ on Ω . Let also H_ρ denote the subgroup of H that induces permutation on the branches at nodes that preserve the linear betweenness relation. The analogous version of Theorem 4.1 is the following.

Theorem 4.2 *The group H_ρ is isomorphic to G_ρ .* •

Any element of G which preserves the linear order on Ω also preserves the linear betweenness relation but the converse is not true. Consider for example the automorphism $g_0 \in G$ defined as

$$g_0 : n_1q_1n_2q_2 \dots n_kq_k \mapsto n_1(-q_1)n_2(-q_2) \dots n_k(-q_k).$$

Clearly g_0 is an element of G_ρ which does not belong to G_\leq . Hence G_\leq is a proper subgroup of G_ρ . The order of g_0 in G is 2 and the group G_ρ is generated by G_\leq and g_0 .

Theorem 4.3 *The group G_\leq is 2-homogeneous which is neither 2-transitive nor 3-homogeneous but relatively 3-transitive.*

Proof. The automorphism group G_\leq cannot be 2-transitive because the relation \leq is a binary relation on Ω . Neither is it 3-homogeneous because a 3-set which is C -related in some order cannot be mapped to another 3-set which is not C -related in any order. To show that G_\leq is 2-homogeneous we argue as in Theorem 3.3 with branch automorphisms that induce 2-homogeneous mappings of branches at nodes of Λ . To show relative 3-transitivity of G_\leq we again use similar arguments as in Theorem 3.4 with the branch automorphisms preserving the linear order of the branches at nodes of Λ . •

4.1 A linear order on (Ω, C)

Corollary 4.1 *The group G_\leq is primitive.* •

Theorem 4.4 *The group G_ρ is 2-transitive.*

Proof. Let $(x, y), (z, w)$ be any two ordered pairs of elements of Ω . To show 2-transitivity of G_ρ we need to find g in G_ρ such that $x^g = z$ and $y^g = w$. If $x < y$ and $z < w$ then there exists $g \in G_\leq$ and hence in G_ρ such that $x^g = z$ and $y^g = w$. If $x < y$ and $w < z$ then $y^{g_0} < x^{g_0}$ where g_0 is as earlier. By 2-homogeneity of G_\leq (Theorem 4.3) there exists h in G_\leq such that $(x^{g_0})^h = z$ and $(y^{g_0})^h = w$. Then g_0h maps x, y to z, w respectively. •

4.1.2 Point stabiliser of $\text{Aut}(\Omega, C, \leq)$

Let x be any element of Ω and G_x denote the point stabiliser of G , namely the subgroup of G that fixes x . Let α be any node on the chain x and let x be on the q_0 -th branch at α so that the expression of x is of the form $\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k$. Let n be the last co-ordinate in the expression of α . For $q_0 \neq p_0 \in \mathbb{Q}$, we define

$$\Sigma(\alpha, p_0) := \{\alpha p_0 m_1 p_1 m_2 p_2 \dots m_j p_j \mid n < m_1 < m_2 < \dots < m_j, p_1, p_2, \dots, p_j \in \mathbb{Q}\}.$$

Clearly $\Sigma(\alpha, p_0)$ is an S_x -class (refer Section 3.3.3) at the node α . Our C -set is relatively 3-transitive and so the S -classes are relatively 3-transitive (refer [AN2][Lemma 14.4]).

Lemma 4.3 *The group $\text{Aut}(\Sigma(\alpha, p_0), C)$ is isomorphic to $\text{Aut}(\Omega, C)$.* •

The following theorem [AN2][Theorem 14.6] gives the structure of a point stabiliser in $\text{Aut}(\Omega, C)$.

Theorem 4.5 [Adeleke & Neumann] *For $x \in \Omega$ let G_x denote the point stabiliser of x in G . Then $G_x \cong (G \text{ Wr } \text{Sym}(\mathbb{Q} \setminus \{q_0\})) \text{ Wr } \text{Aut}(\mathbb{Q}, \leq)$.* •

We now look at the point stabiliser of x in G_\leq . Let $G_{(\leq)x}$ denote the point stabiliser of x in G_\leq . Let $\Omega_{<}(x)$ and $\Omega_{>}(x)$ denote the sets $\{y \in \Omega \mid y < x\}$ and

$\{y \in \Omega \mid x < y\}$ respectively and for any $q \in \mathbb{Q}$, let $\text{Aut}(\mathbb{Q}_{<}(q), \leq)$, $\text{Aut}(\mathbb{Q}_{>}(q), \leq)$ denote the order-preserving permutations on the intervals $(-\infty, q)$, (q, ∞) respectively. Clearly $G_{(\leq)_x}$ fixes the subsets $\Omega_{<}(x)$ and $\Omega_{>}(x)$ setwise. Let α be any node of Λ on the chain x and let x be on the q_0 -th branch at α so that x is of the form $\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k$. Let $\Sigma_{>}(\alpha), \Sigma_{<}(\alpha)$ be the subsets of Ω defined as

$$\{\alpha q_1 m_1 p_1 m_2 p_2 \dots m_j p_j \mid q_1 > q_0\}, \{\alpha q_2 l_1 r_1 l_2 r_2 \dots l_i r_i \mid q_2 < q_0\}$$

respectively (Fig. 4.2) and for $q \in \mathbb{Q}$, let $\Sigma(\alpha, q)$ denote $\{\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k\}$. Then clearly $\Sigma_{>}(\alpha) = \bigcup_{q_1 > q_0} \Sigma(\alpha, q_1)$ and $\Sigma_{<}(\alpha) = \bigcup_{q_2 < q_0} \Sigma(\alpha, q_2)$. We have already seen in Lemma 4.3 that $\text{Aut}(\Sigma(\alpha, q), C)$ is isomorphic to $\text{Aut}(\Omega, C)$ for any $\alpha \in \Lambda, q \in \mathbb{Q}$. Hence when the linear order is imposed on $\Sigma(\alpha, q)$ and Ω , we have an isomorphism between $\text{Aut}(\Sigma(\alpha, q), C, \leq)$ and $\text{Aut}(\Omega, C, \leq)$. Now we consider $\Sigma_{>}(\alpha)$ and its automorphism group.

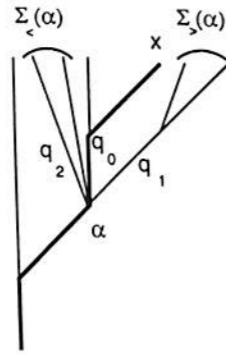


Figure 4.2:

Lemma 4.4 *The group $\text{Aut}(\Sigma_{>}(\alpha), C, \leq)$ is transitive but not primitive.*

Proof. Using branch automorphisms that preserve the linear order \leq on Ω and which fixes $\Sigma_{>}(\alpha)$ setwise we can show transitivity. For elements x, y with expressions $\alpha q' m_1 p_1 m_2 p_2 \dots m_j p_j$ and $y = \alpha q'' l_1 r_1 l_2 r_2 \dots l_i r_i$, $q' > q, q'' > q$ respectively, we define a binary relation \sim as $x \sim y \Leftrightarrow q' = q''$ which can be easily checked to be $\text{Aut}(\Sigma_{>}(\alpha), C, \leq)$ -invariant. Therefore $\text{Aut}(\Sigma_{>}(\alpha), C, \leq)$ is imprimitive and for each $q_0 > q$, $\Sigma(\alpha, q_0)$ is a block of imprimitivity. •

4.1 A linear order on (Ω, C)

Lemma 4.5 *For any $x \in \Omega$ and $\alpha \in \Lambda$ with q_0 the branch chosen by x at α and $\Sigma_{>}(\alpha), \Sigma_{<}(\alpha)$ having the usual notation*

$$\text{Aut}(\Sigma_{>}(\alpha), C, \leq) = \text{Aut}(\Sigma(\alpha, q), C, \leq) \text{Wr Aut}(\mathbb{Q}_{>}(q_0), \leq), \text{ for } q > q_0,$$

$$\text{Aut}(\Sigma_{<}(\alpha), C, \leq) = \text{Aut}(\Sigma(\alpha, q'), C, \leq) \text{Wr Aut}(\mathbb{Q}_{<}(q_0), \leq), \text{ for } q' < q_0.$$

Proof. To show $\text{Aut}(\Sigma_{>}(\alpha), C, \leq) \leq \text{Aut}(\Sigma(\alpha, q), C, \leq) \text{Wr Aut}(\mathbb{Q}_{>}(q_0), \leq)$ we use Theorem 8.5 of [BMMN]. Here $\text{Aut}(\Sigma_{>}(\alpha), C, \leq)$ is a transitive group of automorphisms on $\Sigma_{>}(\alpha)$ and \sim of Lemma 4.4 is a congruence on $\Sigma_{>}(\alpha)$ with $\Sigma(\alpha, q), q > q_0$ as the \sim -classes. The factor group on $\Sigma_{>}(\alpha)$ (see Note III of [BMMN][p. 72]) produced by the congruence \sim , denoted by $G_{\{\Sigma_{>}(\alpha)\}}/G_{(\Sigma_{>}(\alpha))}$ is isomorphic to $\text{Aut}(\Sigma_{>}(\alpha), C, \leq)$. The set $\Sigma_{>}(\alpha)/\sim$ of all \sim -classes is nothing but $\mathbb{Q}_{>}(q_0)$ and $\text{Aut}(\mathbb{Q}_{>}(q_0), \leq)$ is the group of permutations induced by G on $\mathbb{Q}_{>}(q_0)$. Thus we can identify $\Sigma_{>}(\alpha)$ with $\Sigma(\alpha, q) \times \mathbb{Q}_{>}(q_0)$ and so $\text{Aut}(\Sigma_{>}(\alpha), C, \leq) \leq \text{Aut}(\Sigma(\alpha, q), C, \leq) \text{Wr Aut}(\mathbb{Q}_{>}(q_0))$.

To show the converse part we note that $\text{Aut}(\Sigma_{>}(\alpha), C, \leq)$ acts on $\Sigma(\alpha, q)$ and $\text{Aut}(\mathbb{Q}_{>}(q_0), \leq)$ acts on $\mathbb{Q}_{>}(q_0)$. Now the mapping defined as

$$\Sigma(\alpha, q) \times \mathbb{Q}_{>}(q_0) \rightarrow \Sigma_{>}(\alpha)$$

$$(\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k, q_1) \mapsto \alpha q_1 n_1 q_1 n_2 q_2 \dots n_k q_k,$$

for any $q_1 > q_0$ identifies $\Sigma(\alpha, q) \times \mathbb{Q}_{>}(q_0)$ with $\Sigma_{>}(\alpha)$. Also $\Sigma(\alpha, q), q > q_0$ form the blocks of imprimitivity of $\text{Aut}(\Sigma_{>}(\alpha), C, \leq)$ and hence the group $\text{Aut}(\Sigma(\alpha, q), C, \leq) \text{Wr Aut}(\mathbb{Q}_{>}(q_0), \leq)$ acts on $\Sigma_{>}(\alpha)$. Moreover, $\text{Aut}(\Sigma(\alpha, q), C, \leq) \text{Wr Aut}(\mathbb{Q}_{>}(q_0), \leq)$ preserve C and \leq on $\Sigma_{>}(\alpha)$ and so $\text{Aut}(\Sigma(\alpha, q), C, \leq) \text{Wr Aut}(\mathbb{Q}_{>}(q_0), \leq) \leq \text{Aut}(\Sigma_{>}(\alpha), C, \leq)$. The last part follows similarly. •

We are now in a position to give the complete description of $G_{(\leq)_x}$.

Theorem 4.6 *For any element x in Ω and a node α on x ,*

$$G_{(\leq)_x} \cong (\text{Aut}(\Sigma_{<}(\alpha), C, \leq) \times \text{Aut}(\Sigma_{>}(\alpha), C, \leq)) \text{Wr Aut}(\mathbb{Q}, \leq).$$

Proof. Let $\Sigma(\alpha)$ denote $\Sigma_{>}(\alpha) \cup \Sigma_{<}(\alpha)$. The point stabiliser $G_{(\leq)_x}$ has 3 blocks on Ω namely $\{x\}$, $\Omega_{<}(x)$ and $\Omega_{>}(x)$. On fixing x , the group $G_{(\leq)_x}$ fixes setwise the set of all nodes of Λ that lie on the chain x . Thus if β is any node of Λ that lies on x and if α is mapped to β then $\Sigma(\alpha)$ is mapped to $\Sigma(\beta)$ such that $\Sigma_{>}(\alpha)$ and $\Sigma_{<}(\alpha)$ are mapped onto $\Sigma_{>}(\beta)$ and $\Sigma_{<}(\beta)$ respectively. Thus using Lemma 4.5 we have the result. •

Any element of the point stabiliser $G_{(\rho)_x}$ will either preserve or reverse the linear ordering of the branches at each node of Λ . Thus if β is any node of Λ that lies on the chain x then for each $g \in G_{\leq}(x)$ that maps α to β with $\Sigma_{>}(\alpha)$ and $\Sigma_{<}(\alpha)$ being mapped to $\Sigma_{>}(\beta)$ and $\Sigma_{<}(\beta)$ respectively, there exists $g' \in G_{(\rho)_x}$ such that $\Sigma_{>}(\alpha)$ and $\Sigma_{<}(\alpha)$ are mapped to $\Sigma_{<}(\beta)$ and $\Sigma_{>}(\beta)$ respectively. That is, g' interchanges $\Sigma_{>}(\alpha^g)$ and $\Sigma_{<}(\alpha^g)$ for each $\alpha \in \Lambda$ that lies on the chain x . Thus we have the following theorem.

Theorem 4.7 For $x \in \Omega$ and a node α on x we have $G_{(\rho)_x} \cong G_{(\leq)_x} \rtimes C_2$, where \rtimes stands for semidirect product, C_2 is the cyclic group generated by g_0 and g_0 is the element of G as defined earlier. •

4.1.3 Jordan sets of $\text{Aut}(\Omega, C, \leq)$

We first show that in the automorphism group of a linearly ordered set, any Jordan set (provided it exists) is convex.

Lemma 4.6 Any Jordan set of the automorphism group of a linearly ordered set is convex.

Proof. Let Ω be a linearly ordered set with a linear order \leq and let Σ be a Jordan set of $\text{Aut}(\Omega, \leq)$. We need to show that Σ is convex. Suppose on the contrary, $\alpha < \gamma < \beta$ and $\alpha, \beta \in \Sigma$, $\gamma \notin \Sigma$. Then there exists $g \in \text{Aut}(\Omega, \leq)$ such that $\alpha^g = \beta$ and g fixes γ . Then $\alpha < \gamma < \beta \Rightarrow \alpha^g < \gamma^g < \beta^g$, which means $\beta < \gamma$, a contradiction. Hence γ should belong to Σ so that Σ is convex. •

4.1 A linear order on (Ω, C)

Jordan sets of types III and IV from the list in Section 3.3.4 are non-convex. Sets of type I are convex and not all sets of type II are convex. Let α be any element of Λ and (a, b) be an open interval in \mathbb{Q} . Then the convex Jordan sets of type II can be written in the form

$$\Sigma_{\alpha, (a, b)} = \{\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mid q \in (a, b)\}.$$

We show that sets of the type $\Sigma_{\alpha, (a, b)}$ are Jordan sets in G_{\leq} . For any $\alpha \in \Lambda$ and an open convex interval (a, b) in \mathbb{Q} , we define a subgroup of G_{\leq} denoted by $J_{\alpha, (a, b)}$ consisting of branch automorphisms as follows. Let g be any element of $J_{\alpha, (a, b)}$. Then g induces k -homogeneous permutations on the branches at the node α which also fixes setwise the branches indexed by the interval (a, b) and fixes pointwise the branches indexed by $\mathbb{Q} \setminus (a, b)$. On any other node on $\Sigma_{\alpha, (a, b)}$ which is greater than α let g induce either a translation or a k -homogeneous automorphism. On the remaining nodes of Λ let g induce the identity permutation on the branches. To prove that G_{\leq} is a Jordan group we prove that $J_{\alpha, (a, b)}$ is the associated Jordan group of the set $\Sigma_{\alpha, (a, b)}$ (refer Section 2.4).

Theorem 4.8 The group G_{\leq} is a Jordan group.

Proof. Let us consider $\Sigma_{\alpha, (a, b)}$ and the subgroup $J_{\alpha, (a, b)}$ of G_{\leq} as defined in the preceding paragraph. Let $x = \alpha c m_1 p_1 m_2 p_2 \dots m_j p_j$ and $y = \alpha d l_1 r_1 l_2 r_2 \dots l_i r_i$ be two distinct elements of $\Sigma_{\alpha, (a, b)}$ such that $c, d \in (a, b)$. We first equalise the lengths of x and y as $\alpha c l_1 q'_1 l_2 q'_2 \dots l_k q'_k$ and $\alpha d l_1 p'_1 l_2 p'_2 \dots l_k p'_k$, with usual notation of the symbols (refer Lemma 3.2). We now consider the element g in G_{\leq} defined as follows. On the node α let g induce a 2-homogeneous branch automorphism on $((a, b), \leq)$ that maps the c -th branch to the d -th branch. On the nodes $\alpha d l_1, \alpha d l_1 p'_1 l_2, \dots, \alpha d l_1 p'_1 l_2 p'_2 \dots l_k$, let g induce translation mappings that maps the q'_t -th branch to the p'_t -th branch, $1 \leq t \leq k$ respectively. On the remaining nodes let g induce the identity permutation on the branches. Then clearly g is an element of $J_{\alpha, (a, b)}$ that maps x to y and fixes $\Omega \setminus \Sigma_{\alpha, (a, b)}$ pointwise. Therefore $J_{\alpha, (a, b)}$ is the associated Jordan group of $\Sigma_{\alpha, (a, b)}$ which proves that G_{\leq} is a Jordan group. •

From Theorem 4.8 it follows that to prove G_{\leq} is a Jordan group we require only branch automorphisms. Non-identity chain mappings are required to show 2-transitivity of Jordan sets of type I . Jordan sets of type I are primitive Jordan sets. When $|(a, b)| > 1$, each $q \in (a, b)$ gives a block of imprimitivity of $\Sigma_{\alpha, (a, b)}$. By the same reasoning of Lemma 4.6 we can show that there cannot exist any proper non-convex Jordan set for the automorphism group of a set with a linear betweenness relation. The Jordan sets of G_{ρ} are same as the Jordan sets of G_{\leq} .

4.2 A quaternary branch relation V on Ω

Any branch automorphism of $\text{Aut}(\Omega, C, \leq)$ induces order-preserving permutations of the branches at nodes of Λ . Therefore the imposition of the linear order \leq on Ω gave us some control over the types of branch automorphisms of (Ω, C) .

Definition 4.1 A relation is called a *branch relation* if the automorphism group with the new relation restricts the branch automorphisms of $\text{Aut}(\Omega, C)$.

The linear order defined in Section 4.1 is a branch relation. In this section we impose another branch relation which is a 4-place relation denoted by V on (Ω, C, \leq) to get a more smaller set of branch automorphisms. Let x, y be two distinct elements of Ω with $\alpha = x \cap y$. Let x, y lie on the q -th, p -th branches at α respectively. We define the distance between x and y denoted by d as $d(x, y) := q - p$. The quaternary relation V on Ω is defined as follows. For $x, y, z, w, x \neq y, z \neq w$ in Ω we define

$$V(x, y; z, w) :\leftrightarrow d(x, y) = d(z, w).$$

Note that the following axioms hold for the relation V :

- (i) $V(x, y; z, w) \rightarrow V(y, x; w, z) \wedge V(z, w; x, y)$,
- (ii) $V(x, y; z, w) \rightarrow \neg V(y, x; z, w)$,
- (iii) $V(x, y; z, w) \wedge V(z, w; u, v) \rightarrow V(x, y; u, v)$.

4.2 A quaternary branch relation V on Ω

4.2.1 Automorphism group of (Ω, C, \leq, V)

Let $G_{\leq, V}$ denote the group $\text{Aut}(\Omega, C, \leq, V)$. Clearly $G_{\leq, V}$ is a subgroup of G_{\leq} . The relation V gives a further restriction on the branch automorphisms of $\text{Aut}(\Omega, C)$. Suppose $H_{\leq, d}$ denotes the subgroup of H_{\leq} that consists of automorphisms that preserves the distance between pairs of elements of Ω . Then we have the following.

Theorem 4.9 *The group $H_{\leq, d}$ is isomorphic to $G_{\leq, V}$.*

Proof. We have shown H_{\leq} to be isomorphic to G_{\leq} (refer Theorem 4.1). Let $x, y, z, w, x \neq y, z \neq w$ be four chains such that $V(x, y; z, w)$. We first show that every element $h \in H_{\leq, d}$ preserves the relation V on the images of x, y, z, w . Let $\alpha = x \cap y, \beta = z \cap w$ and let x, y lie on the q -th, p -th branches at α respectively and z, w lie on the r -th, s -th branches at β respectively. Since $V(x, y; z, w)$ holds we must have $d(q, p) = d(r, s)$. Let the images of x, y be $\alpha^h q^h, \alpha^h p^h$ respectively and that of z, w be $\beta^h r^h, \beta^h s^h$ respectively. Since h preserves the distance between pairs of elements of Ω we must have $d(q^h, p^h) = d(r^h, s^h)$ and hence $V(x^h, y^h; z^h, w^h)$ which means $H_{\leq, d}$ is a subgroup of $G_{\leq, V}$. Conversely we can show that $G_{\leq, V}$ is a subgroup of $H_{\leq, d}$ which means that $H_{\leq, d}$ is isomorphic to $G_{\leq, V}$. •

Lemma 4.7 *If a branch automorphism in $\text{Aut}(\Omega, C, \leq, V)$ fixes the distance of pairs of elements of Ω with same intersecting node then the automorphism fixes the distance of all pairs of elements of Ω .*

Proof. Let g be an automorphism of $\text{Aut}(\Omega, C, \leq, V)$ that fixes the distance of pairs of elements of Ω with α as the intersecting node. Suppose β is another node of Λ and $\beta p, \beta q$ are chains, $p \neq q$. If possible let g map $\beta q, \beta p$ to $(\beta q)^g, (\beta p)^g$ such that $d(\beta q, \beta p) \neq d((\beta q)^g, (\beta p)^g)$. Now $\alpha q, \alpha p$ are such that $d(\alpha p, \alpha q) = d(\beta p, \beta q) = p - q$. By assumption g fixes the distance of branches at the node α and so we have $d((\alpha p)^g, (\alpha q)^g) = d(\alpha p, \alpha q) = d(\beta p, \beta q) \neq d((\beta q)^g, (\beta p)^g)$, which

implies that g does not preserve V on Ω . Therefore g must fix the distance of branches at the node β . And this must be true of all nodes of Λ . •

It is easy to check the following lemmas.

Lemma 4.8 Translation branch automorphisms and magnification branch automorphisms will preserve the relation V on (Ω, C) . •

Lemma 4.9 The chain automorphisms of $\text{Aut}(\Omega, C, \leq, V)$ are same as the chain automorphisms of $\text{Aut}(\Omega, C, \leq)$ (refer Lemma 4.2). •

Theorem 4.10 The group $G_{\leq, V}$ is 2-homogeneous on Ω .

Proof. Let $(x, y), (z, w)$ be two pairs of chains with $x < y, z < w$. We need to show that there exists $g \in G_{\leq, V}$ that maps (x, y) to (z, w) . To map (x, y) to (z, w) we first need to map $x \cap y$ to $z \cap w$ and then the corresponding branches of the images of (x, y) to the corresponding branches of (z, w) on the node $z \cap w$ and on the relevant nodes greater than $z \cap w$. For this we just require branch automorphisms that induce permutations on the branches at nodes that are of the form $q \mapsto aq + b$, where $a \in \mathbb{Q}^+, b \in \mathbb{Q}$ and \mathbb{Q}^+ denotes the positive rational numbers including 0. Clearly these automorphisms are elements of $G_{\leq, V}$. Therefore $G_{\leq, V}$ is 2-homogeneous. •

4.2.2 Point stabiliser of $\text{Aut}(\Omega, C, \leq, V)$

Let x be any element of Ω and $G_{(\leq, V)_x}$ denote the point stabiliser of $G_{\leq, V}$ in Ω . For any α in Λ and $q \in \mathbb{Q}$, let $\text{Aut}(\Sigma(\alpha, q), C, \leq, V)$ denote the automorphism group on $(\Sigma(\alpha, q), C)$ that preserves the relations \leq and V on branches at nodes on the set $\Sigma(\alpha, q)$. And let $M(\mathbb{Q})_{q_0}$ denote the automorphism group on \mathbb{Q} which consists of permutations of the form $q \mapsto aq + b$ that fixes the point $q_0, a \in \mathbb{Q}^+, b \in \mathbb{Q}$ and \mathbb{Q}^+ denotes the positive rational numbers including 0. When the chain x is fixed then for each node α on x , the branch on which x lie at the node α is fixed. The structure of a point stabiliser in $G_{\leq, V}$ is given by the following theorem which can be proved using similar arguments as in Theorem 4.6.

4.2 A quaternary branch relation V on Ω

Theorem 4.11 For any element x in Ω and a node α on x ,

$$G_{(\leq, V)_x} \cong (\text{Aut}(\Sigma(\alpha, q), C, V) \times M(\mathbb{Q})_{q_0}) \text{ Wr Aut}(\mathbb{Q}, \leq),$$

where q_0 is the branch chosen by x at α . •

4.2.3 Jordan sets of $\text{Aut}(\Omega, C, \leq, V)$

Jordan sets of $G_{\leq, V}$ (if they exist) have to be either one or both of the types of Jordan sets of G_{\leq} . We show that sets of type I are Jordan sets of $G_{\leq, V}$. For $q_0 \in \mathbb{Q}$ Jordan sets of type I are of the form $\Sigma_{\alpha, q_0} = \{\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k\}$. For any $\alpha \in \Lambda$ and $q_0 \in \mathbb{Q}$, we consider the group J_{α, q_0} that consists of branch automorphisms inducing translations on the branches at the node α and on all nodes greater than α that are on the set Σ_{α, q_0} . Clearly J_{α, q_0} is a subgroup of $G_{\leq, V}$. We prove that $G_{\leq, V}$ is a Jordan group with J_{α, q_0} the associated Jordan group of the Jordan set Σ_{α, q_0} .

Theorem 4.12 The group $G_{\leq, V}$ is a Jordan group.

Proof. Let us consider Σ_{α, q_0} and the subgroup J_{α, q_0} of $G_{\leq, V}$ as defined in the preceding paragraph. Let $x = \alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k$ and $y = \alpha q_0 m_1 p_1 m_2 p_2 \dots m_j p_j$ be two distinct elements of Σ_{α, q_0} . We first equalise the lengths of the expressions of x and y as $\alpha q_0 l_1 q'_1 l_2 q'_2 \dots l_i q'_i$ and $\alpha q_0 l_1 p'_1 l_2 p'_2 \dots l_i p'_i$, respectively with usual notation of the symbols (refer Lemma 3.2). We now consider the element g in $G_{\leq, V}$ that induces translations on the branches at the nodes $\alpha q_0 l_1, \alpha q_0 l_1 p'_1 l_2, \dots, \alpha q_0 l_1 p'_1 l_2 p'_2 \dots l_i$ mapping q'_t to $p'_t, 1 \leq t \leq i$, respectively. Then clearly g is an element of J_{α, q_0} that maps x to y and fixes $\Omega \setminus \Sigma_{\alpha, q_0}$ pointwise which proves that J_{α, q_0} is the associated Jordan group of Σ_{α, q_0} . •

We recall that $J_{\alpha, (a, b)}$ is the associated Jordan group of Jordan sets for G_{\leq} (refer Section 4.1.3). Using Lemma 4.7 we see that elements of J_{α, q_0} does not preserve V in (Ω, C, \leq) . Hence there cannot exist Jordan sets of type II in $G_{\leq, V}$.

4.3 A quaternary chain relation L on Ω

In this section we define another quaternary relation on Ω . Let x be an element of Ω and α be a node on x . The chain x is isomorphic to \mathbb{Q} with the underlying linear order \leq on \mathbb{Q} and so there is exactly one point on \mathbb{Q} to which the point α corresponds. Let the expression for α be $n_1q_1n_2q_2 \dots n_k$. Then the point of \mathbb{Q} to which α corresponds is n_k . We define the *level* of α to be n_k and denote it as $L(\alpha)$. Any maximal chain of Λ is isomorphic to \mathbb{Q} and so for any $q \in \mathbb{Q}$, there is exactly one point on any chain at that level. We state this fact in the following lemma.

Lemma 4.10 *Given any $q \in \mathbb{Q}$ and an element $x \in \Omega$ there is exactly one point on the chain x at level q . •*

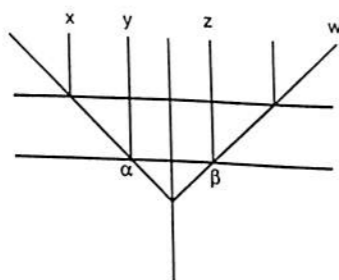


Figure 4.3:

Given two elements $\alpha = n_1q_1n_2q_2 \dots n_k$ and $\beta = m_1p_1m_2p_2 \dots m_j$ of Λ the level of the two elements are equal if $n_k = m_j$. We define a binary relation L on Λ as

$$\alpha L \beta \iff L(\alpha) = L(\beta).$$

The binary relation L on Λ is an equivalence relation. Each $q \in \mathbb{Q}$ determines an equivalence class, namely $\{\alpha \in \Lambda \mid L(\alpha) = q\}$. Therefore the number of equivalence classes of Λ determined by the relation L is \aleph_0 . Level on elements of Λ induces a *level* on pairs of elements of Ω defined as follows. Let $\{x, y\}$ be a

4.3 A quaternary chain relation L on Ω

2-set of Ω such that $x \neq y$. Then we define

$$L\{x, y\} = L(x \cap y) = L(\alpha), \text{ where } \alpha = x \cap y.$$

The binary relation L on Λ can then be considered to be a binary relation on pairs of distinct elements of Ω . Without introducing another notation, we call this relation on pairs of distinct elements of Ω as L . Let $\{x, y\}$ and $\{z, w\}$ be two pairs of distinct elements of Ω with $\alpha = x \cap y$ and $\beta = z \cap w$. The relation L when considered as a relation on Ω is defined as

$$L(x, y; z, w) \iff L(x \cap y) = L(z \cap w)$$

(Fig. 4.3). Let $\Omega_0^{\{2\}} = \{\{x, y\} \mid x, y \in \Omega, x \neq y\}$. Then L is an equivalence relation on pairs of distinct elements of Ω . For any $q \in \mathbb{Q}$, the set of pairs of distinct elements of Ω which meet at a node on level q forms an equivalence class.

4.3.1 Automorphism group of (Ω, C, L)

As usual let H_L denote the subgroup of H that preserves the equivalence relation “has the same level” on Λ and G_L denote the subgroup of G that preserves the equivalence relation on pairs of elements of Ω . We show that H_L is isomorphic to G_L .

Theorem 4.13 *The group H_L is isomorphic to G_L .*

Proof. We have already shown H to be isomorphic to G (refer Theorem 3.1). Let g be an element of H_L and $x, y, z, w \in \Omega$ such that $x \neq y, z \neq w$ and $L(x, y; z, w)$. Then $\alpha L \beta$, where $\alpha = x \cap y, \beta = z \cap w$. Since g preserves L on elements of Λ so $\alpha^g L \beta^g$ so that $L(x^g, y^g; z^g, w^g)$, where $\alpha^g = x^g \cap y^g$ and $\beta^g = z^g \cap w^g$. Thus the induced automorphism $g \in G$ preserves L on Ω and so H_L is a subgroup of G_L . Conversely we can show that G_L is a subgroup of H_L . •

Lemma 4.11 *The branch automorphisms of $\text{Aut}(\Omega, C, L)$ can induce any permutation on the branches at nodes of Λ . •*

Chain automorphisms of $\text{Aut}(\Omega, C, L)$ need to preserve the level of the nodes of Λ . Let g be any chain automorphism of $\text{Aut}(\Omega, C, L)$. If α is a node at level q and g maps α to β which is at level p , $p \neq q$, then g maps all nodes at level q to nodes at level p . Therefore the images of the level of nodes on the \mathbb{Q}_1 -branch completely determines the chain automorphism g . Hence we have the following.

Lemma 4.12 *The chain automorphisms of $\text{Aut}(\Omega, C, L)$ are those that induce order-preserving permutations on just the \mathbb{Q}_1 -branch. •*

Lemma 4.13 *The group G_L is transitive on Ω .*

Proof. Let x, y be distinct elements in Ω . We need to find $g \in G$ such that $x^g = y$. We first equalise the lengths of x and y so that the equalised length of x and y is $2i$ with expressions of x and y as $l_1q'_1l_2q'_2 \dots l_iq'_i$ and $l_1p'_1l_2p'_2 \dots l_ip'_i$ respectively, with notation as in Lemma 3.2. We define branch automorphisms g_1, g_2, \dots, g_i that induces permutations on the nodes $l_1, l_1p'_1l_2, \dots, l_1p'_1l_2p'_2 \dots l_i$ that maps the q'_t -th branch onto the p'_t -th branch, $1 \leq t \leq i$, respectively. The composition $g := g_1g_2 \dots g_i$ of branch automorphisms (refer Section 3.3.1) is the required automorphism that maps x to y . Note that each branch automorphism fixes the level of nodes and so g fixes the level of nodes and hence is an element of G_L . •

Theorem 4.14 *The group G_L is 2-transitive on Ω .*

Proof. Let (x, y) and (z, w) be two pairs of distinct elements of Ω . We need to find an element in G_L that maps (x, y) to (z, w) . Let $x \cap y = \alpha$ and $z \cap w = \beta$. By transitivity of H_L there exists an element g_1 in H_L and hence in G_L that maps α to β . Let g_2 be a branch automorphism that induces a permutation on the branches at the node β (which is equal to α^{g_1}) that maps the branches of x^{g_1}, y^{g_1} to the branches of z, w respectively. Then $x^{g_1g_2}, z$ are on the same branch at β which is also true of $y^{g_1g_2}$ and w . We equalise the lengths of $x^{g_1g_2}$ and z and then define branch automorphisms that induces permutations of the branches on the

nodes of the expressions of $x^{g_1g_2}$ and z such that the branches of $x^{g_1g_2}$ are mapped to the branches of z at each of these nodes. Each of these branch automorphisms will therefore fix the branch of $y^{g_1g_2}$ at β and hence fix the element $y^{g_1g_2}$. The composition of these branch automorphisms say, g_3 will map $x^{g_1g_2}$ to z and will fix $y^{g_1g_2}$. Similarly we equalise the lengths of the expressions of $y^{g_1g_2}$ and w and then define a composition map of branch automorphisms, say g_4 that maps $y^{g_1g_2}$ to w and fix $x^{g_1g_2}$. The composition map $g_1g_2g_3g_4$ will map x, y to z, w respectively. •

Corollary 4.2 *The group G_L is primitive in its action on Ω . •*

The group G_L can be shown to be relatively 3-transitive. This is true because chains are dense in Λ .

4.3.2 Point stabiliser of $\text{Aut}(\Omega, C, L)$

In this section we look at the point stabiliser of G_L . For any $x \in \Omega$ let $G_{(L)x}$ denote the point stabiliser of x in Ω . Let α be any node and $\Sigma(\alpha, q)$ denote the set of all elements of Ω on the q -th branch at α and $\Sigma(\alpha)$ denote the set of all chains that pass through the node α which can also be expressed as the union of the sets $\Sigma(\alpha, q)$, $q \in \mathbb{Q}$. It is seen that $\text{Aut}(\Sigma(\alpha, q), C)$ is isomorphic to $\text{Aut}(\Omega, C)$ (Lemma 4.3), for any $\alpha \in \Lambda, q \in \mathbb{Q}$. Therefore when the relation L is imposed on $\Sigma(\alpha, q)$ and Ω , it is trivial to check that $\text{Aut}(\Sigma(\alpha, q), C, L)$ is isomorphic to $\text{Aut}(\Omega, C, L)$. We give a result on the automorphism group of $\Sigma(\alpha)$.

Lemma 4.14 *The group $\text{Aut}(\Sigma(\alpha), C, L)$ is transitive but not primitive on $\Sigma(\alpha)$.*

Proof. Transitivity can be easily shown by using branch automorphisms as in Lemma 4.13. The binary relation \sim defined in Lemma 4.4 gives the blocks of imprimitivity of G_L . •

Let N denote the collection of branch automorphisms of (Ω, C) . We denote the collection of branch automorphisms restricted to the sets $\Sigma(\alpha), \Sigma(\alpha, q)$ as $N(\Sigma(\alpha)), N(\Sigma(\alpha, q))$ respectively. Note that N fixes the level of nodes of Λ and hence are elements of G_L .

Lemma 4.15 For any $\alpha \in \Lambda$ and $q \in \mathbb{Q}$, $N(\Sigma(\alpha)) = N(\Sigma(\alpha, q)) \text{ Wr Sym}(\mathbb{Q})$.

Proof. We just need to verify that $\Sigma(\alpha, q)$ are the blocks of imprimitivity of $N(\Sigma(\alpha))$. •

Theorem 4.15 For any $x \in \Omega$ with the node α on x and x on the q_0 -th branch at α , $G_{(L)_x} = (N(\Sigma(\alpha, q)) \text{ Wr Sym}(\mathbb{Q} \setminus \{q_0\})) \rtimes \text{Aut}(\mathbb{Q}, \leq)$.

Proof. We first show that any element of $G_{(L)_x}$ is also an element of $(N(\Sigma(\alpha, q)) \text{ Wr Sym}(\mathbb{Q} \setminus \{q_0\})) \rtimes \text{Aut}(\mathbb{Q}, \leq)$. Suppose g is an element of G_L that fixes x . This means the automorphism g will fix setwise the set of all nodes on the chain x . If β is a node on x and α is mapped to β then $\bigcup_{q \in \mathbb{Q} \setminus \{q_0\}} \Sigma(\alpha, q)$ will be mapped to $\bigcup_{p \in \mathbb{Q} \setminus \{p_0\}} \Sigma(\beta, p)$, where x lies on the p_0 -th branch at β . Moreover g preserves L on Ω and so chain automorphisms induce non-identity order-preserving permutations on just the \mathbb{Q}_1 -branch. This means that g is an element of $(N(\Sigma(\alpha, q)) \text{ Wr Sym}(\mathbb{Q} \setminus \{q_0\})) \rtimes \text{Aut}(\mathbb{Q}, \leq)$.

Conversely let g be an element of $(N(\Sigma(\alpha, q)) \text{ Wr Sym}(\mathbb{Q} \setminus \{q_0\})) \rtimes \text{Aut}(\mathbb{Q}, \leq)$. Then at each node α of the chain x the branch on which x lie will be fixed and hence g will fix x . Also elements of $N(\Sigma(\alpha, q))$ fixes the level of nodes of Λ and so g must preserve the relation L . Therefore $g \in G_{(L)_x}$. •

4.3.3 Jordan sets of $\text{Aut}(\Omega, C, L)$

Let α be a node and P be any subset of \mathbb{Q} . Consider

$$\Sigma_{\alpha, P} := \{\alpha q n_1 q_1 n_2 q_2 \dots n_k q_k \mid q \in P\},$$

which is a set of type *I* if $|P| = 1$ and a set of type *II* if $|P| > 1$ (refer Section 3.3.4). Also consider a set of type *IV* which can be expressed as

$$\Sigma_{\alpha} := \{x \in \Omega \mid \alpha \notin x\}.$$

We define a subgroup $J_{\alpha, P}$ to consist of branch automorphisms which can be described as follows. Let g be an element of $J_{\alpha, P}$ then g induces a permutation on

4.3 A quaternary chain relation L on Ω

the branches on the node α that fixes P setwise and $\mathbb{Q} \setminus P$ pointwise. On nodes greater than α which is on $\Sigma_{\alpha, P}$ the automorphism g induces any permutation on the branches. On the remaining nodes, g induces the identity permutation on the branches. We next define another subgroup J_{α} of $\text{Aut}(\Omega, C)$ which consists of chain automorphisms and branch automorphisms defined as follows. Let g be a chain automorphism of J_{α} . Then g induces a k -homogeneous order-preserving permutation on the \mathbb{Q}_1 -branch that fixes $(-\infty, L(\alpha))$ setwise and $[L(\alpha), \infty)$ pointwise. Let h be a branch automorphism of J_{α} . On the node α and on nodes greater than α let h induce the identity permutation on the branches. On any node β smaller than α let h induce permutations of \mathbb{Q} that fixes the q_0 -th branch, where q_0 is the branch of α at β . And on nodes incomparable to α let h induce any permutation on the branches. It is then clear that $J_{\alpha, P}$ and J_{α} are subgroups of G_L . To prove that G_L is a Jordan group we prove that $\Sigma_{\alpha, P}$ and Σ_{α} are Jordan sets with associated Jordan groups as $J_{\alpha, P}$ and J_{α} respectively.

Lemma 4.16 Sets of type *I* and *II* are Jordan sets of G_L .

Proof. Let us consider $\Sigma_{\alpha, P}$ and the subgroup $J_{\alpha, P}$ of G_L as defined in the preceding paragraph. Let $x = \alpha c m_1 p_1 m_2 p_2 \dots m_j p_j$ and $y = \alpha d l_1 r_1 l_2 r_2 \dots l_i r_i$ be two distinct elements of $\Sigma_{\alpha, P}$ such that $c, d \in P$. We first equalise the lengths of x and y as $\alpha c n_1 p'_1 n_2 p'_2 \dots n_k p'_k$ and $\alpha d n_1 r'_1 n_2 r'_2 \dots n_k r'_k$, with usual notation of the symbols (refer Lemma 3.2). We now consider a branch automorphism g that induces a permutation on the branches at the node α , maps the c -th branch to the d -th branch and fixes branches indexed by elements in the set $\mathbb{Q} \setminus P$. We also define branch automorphisms g_t , $1 \leq t \leq k$ that induces permutations on the branches at the nodes $\alpha d n_1, \alpha d n_1 r'_1 n_2, \dots, \alpha d n_1 r'_1 n_2 r'_2 \dots n_k$ that maps the p'_t -th branch to the r'_t -th branch respectively. Then clearly $g g_1 g_2 \dots g_k$ is an element of $J_{\alpha, P}$ that maps x to y and fixes $\Omega \setminus \Sigma_{\alpha, P}$ pointwise. Therefore $\Sigma_{\alpha, P}$ is a Jordan set and $J_{\alpha, P}$ is the associated Jordan group. •

Lemma 4.17 Sets of type *IV* are Jordan sets of G_L .



Proof. Let us consider Σ_α and the subgroup J_α of G_L as defined in the paragraph preceding Lemma 4.16. Let $x = \beta c m_1 p_1 m_2 p_2 \dots m_j p_j$ and $y = \gamma d l_1 r_1 l_2 r_2 \dots l_i r_i$ be two distinct elements of Σ_α such that $L(\beta) < L(\alpha)$ and $L(\gamma) < L(\alpha)$. We first define a chain automorphism g that induces an order-preserving permutation on the \mathbb{Q}_1 -branch that maps $L(\beta)$ to $L(\gamma)$ and fixes $(L(\alpha), \infty)$ pointwise. Next, we equalise the lengths of x^g and y . Then as in Lemma 4.16 we define branch automorphisms that induces permutations of the branches at nodes in the expressions of x^h and y with equalised lengths mapping the branches of x^h to the branches of y respectively. The composition of g and these branch automorphisms is the required automorphism that maps x to y . Clearly this automorphism is an element of J_α which fixes $\Omega \setminus \Sigma_\alpha$ pointwise. Therefore Σ_α is a Jordan set and J_α the associated Jordan group. •

The results of the two lemmas above prove the following.

Theorem 4.16 *The group G_L is a Jordan group.* •

Note that the subset P which occurs in the expression of a Jordan set of type II may be an open interval of \mathbb{Q} , that is, of the form (a, b) , $a \neq b$. Therefore Jordan sets of G_\leq as defined in Section 4.1.3 are Jordan sets of G_L as well.

4.4 A quaternary relation R on Ω

The chain automorphisms of elements of $\text{Aut}(\Omega, C)$ induce order-preserving permutations on any chain. With the imposition of the relation L on Ω the set of chain automorphisms became smaller.

Definition 4.2 A relation is called a *chain relation* if the automorphism group with the new relation restricts the chain automorphisms of (Ω, C) .

The relation L defined in Section 4.3 is a chain relation. In this section we define another chain relation that further reduces the class of chain automorphisms. Consider the rational line \mathbb{Q} . When we talk about \mathbb{Q} as the rational line we have

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in mind the underlying linear order on \mathbb{Q} . Consider the *dyadic fractions* of \mathbb{Q} of the form $\frac{m}{2^n}$, $m \in \mathbb{Z}, n \in \mathbb{N}, n > 1$. Here integers are not dyadic fractions. It can be checked that between any two rational numbers there is a dyadic fraction as well as a non-dyadic fraction. This property of \mathbb{Q} is called *dense codense*. That is, dyadic fractions are dense codense in the rational line. Bhattacharjee et al. [BMMN][p. 81] has talked about the dense codense property of \mathbb{Q} with two colours, which essentially uses dense codense property of subsets of rational numbers. Also consider the *ternary fractions* of the form $\frac{m}{3^n}$, $m \in \mathbb{Z}, n \in \mathbb{N}, n > 1$. Ternary fractions can also be checked to be dense codense in \mathbb{Q} . We say that two rational numbers are of *same type* if both the rational numbers are dyadic or both are ternary or both are non-dyadic, non-ternary.

Let x, y, z, w be four elements of Ω , $x \neq y, z \neq w$. We define a quaternary relation R on Ω as follows:

$$R(x, y; z, w) :\leftrightarrow L\{x, y\} \text{ and } L\{z, w\} \text{ are of same type .}$$

Suppose $L(x, y; z, w)$ holds for elements x, y, z, w in Ω . Then it means that $L\{x, y\} = L\{z, w\}$, which implies trivially that $L\{x, y\}$ and $L\{z, w\}$ are of same type which in turn implies that $R(x, y; z, w)$ holds. But nodes of same type need not necessarily be at the same level.

4.4.1 Automorphism group of (Ω, C, L, R)

We now consider the automorphism group of (Ω, C) preserving L and R . As usual let $G_{L,R}$ denote the subgroup of G_L that also preserve the relation R and $H_{L,R}$ denote the subgroup of H_L that preserve the type of the level of nodes. Clearly $G_{L,R}$ is a subgroup of G_L and the following is trivial.

Theorem 4.17 *The group $H_{L,R}$ is isomorphic to $G_{L,R}$.* •

Lemma 4.18 *The branch automorphisms of $\text{Aut}(\Omega, C, L, R)$ are same as the branch automorphisms of $\text{Aut}(\Omega, C, L)$ (refer Lemma 4.11).* •

We show that chain automorphisms of $\text{Aut}(\Omega, C, L, R)$ can induce neither magnification nor non-integer translation on the \mathbb{Q}_1 -branch.

Lemma 4.19 *Chain automorphisms of $\text{Aut}(\Omega, C, L, R)$ cannot induce magnification on the \mathbb{Q}_1 -branch.*

Proof. Magnifications that preserve the linear order on \mathbb{Q} are only positive magnifications. Hence it is enough for us to show that chain automorphisms cannot induce positive magnifications. Let q_0 be any non-zero positive element of \mathbb{Q} . Let g be a chain automorphism that induces the magnification $q \mapsto q_0q$, on the nodes of the \mathbb{Q}_1 -branch. Suppose q_0 is dyadic and is of the form $\frac{m_1}{2^{n_1}}$, $m_1 \neq 2^m$ for any $m \in \mathbb{N}$. Consider the non-dyadic element $\frac{1}{m_1}$, then g maps the node $\frac{1}{m_1}$ to a node at level $\frac{1}{2^{n_1}}$, so that a non-dyadic fraction is mapped to a dyadic fraction. Hence if q_0 is dyadic then g does not preserve the relation R on (Ω, C, L) . The same result can be shown when q_0 is ternary or non-dyadic, non-ternary. Therefore g cannot induce a magnification on the \mathbb{Q}_1 -branch. •

Lemma 4.20 *Chain automorphisms of $\text{Aut}(\Omega, C, L, R)$ cannot induce non-integer translation on the \mathbb{Q}_1 -branch.*

Proof. Let q_0 be a non-integer element of \mathbb{Q} . Let g be a chain automorphism of $\text{Aut}(\Omega, C, L, R)$ that induces the non-integer translation $q \mapsto q + q_0$, on the nodes of the \mathbb{Q}_1 -branch. Suppose q_0 is dyadic and is of the form $\frac{m_1}{2^{n_1}}$. Consider the element $\frac{1}{3^{n_1}}$, which is ternary. Then g maps the node $\frac{1}{3^{n_1}}$ to a node at level $\frac{3^{n_1}m_1 + 2^{n_1}}{6^{n_1}}$, so that a ternary fraction is mapped to a non-ternary fraction. Similar result holds for the remaining types of q_0 . Therefore g cannot induce a non-integer translation on the \mathbb{Q}_1 -branch. •

However chain automorphisms of $\text{Aut}(\Omega, C, L, R)$ can induce integer translations on the \mathbb{Q}_1 -branch. In the structure (Ω, C, L, R) if we allow the set of dyadic, ternary and non-dyadic non-ternary nodes to be interchangeable then magnification chain automorphisms and non-integer translation chain automorphisms exist in $\text{Aut}(\Omega, C, L, R)$. However we are looking at the strongest possible structure

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on (Ω, C) such that the automorphism group is also a Jordan group. Hence we impose the condition that the elements of the automorphism group of (Ω, C, L, R) necessarily has to map a node to another node of the same type.

Lemma 4.21 *Chain automorphisms of $\text{Aut}(\Omega, C, L, R)$ can induce integer translations on the \mathbb{Q}_1 -branch. •*

We now look at some properties of $G_{L,R}$. Transitivity of $G_{L,R}$ can be shown by using the branch automorphisms of G_L as in Lemma 4.13.

Lemma 4.22 *The group $G_{L,R}$ is transitive. •*

Theorem 4.18 *The group $G_{L,R}$ is not 2-homogeneous on Ω .*

Proof. Let $\{x, y\}$ and $\{z, w\}$ be two pairs of elements of Ω such that $L\{z, w\} = L\{x, y\} + q$, where $q \in \mathbb{Q} \setminus \mathbb{N}$. Then because chain automorphisms of $G_{L,R}$ cannot be non-integer translations (Lemma 4.20), so $\{x, y\}$ cannot be mapped to $\{z, w\}$ in $G_{L,R}$. Hence $G_{L,R}$ is not 2-homogeneous. •

Lemma 4.23 *The group $G_{L,R}$ is imprimitive.*

Proof. For $q \in [0, 1), q \in \mathbb{Q}$, consider $A_q := \{\{x, y\} \mid L\{x, y\} = q\}$. Because chain automorphisms of $G_{L,R}$ can be integer translations so A_q is a block of imprimitivity of $G_{L,R}$ for each $q \in [0, 1)$. •

4.4.2 Jordan sets of $\text{Aut}(\Omega, C, L, R)$

In proving that $\Sigma_{\alpha, P} = \{\alpha q_1 n_1 q_2 n_2 q_3 \dots n_k q_k, q \in P\}$, (where α is any node of Λ and $P \subset \mathbb{Q}$) is a Jordan set of G_L (refer Section 4.3.3) we have defined the associated Jordan group to consist of branch automorphisms. Using the same arguments we can show that sets of the form $\Sigma_{\alpha, P}$, as defined above are Jordan sets of $G_{L,R}$ as well.

Theorem 4.19 *The group $G_{L,R}$ is a Jordan group. •*

4.5 Combining the relations \leq, V, L, R on (Ω, C)

In this section we combine the relations discussed so far, namely \leq, V, L and R on the C -set Ω and show that there exist Jordan sets in the automorphism group of $(\Omega, C, \leq, V, L, R)$. Let $G_{\leq, V, L, R}$ denote this automorphism group. We show that $G_{\leq, V, L, R}$ contains a minimal Jordan subgroup.

4.5.1 Properties of the group $\text{Aut}(\Omega, C, \leq, V, L, R)$

When we combine the relations \leq, V, L, R on (Ω, C) we have a very strong structure on Ω . Elements of $G_{\leq, V, L, R}$ will have to preserve all the four relations \leq, V, L, R which means that the automorphism group $G_{\leq, V, L, R}$ becomes much smaller. The group $G_{\leq, V, L, R}$ is contained in each of $G_{\leq, V}$ and $G_{L, R}$. The chain automorphisms of $G_{\leq, V, L, R}$ are nothing but the chain automorphisms of $G_{L, R}$ and the branch automorphisms of $G_{\leq, V, L, R}$ are the branch automorphisms of $G_{\leq, V}$. This is because L, R are chain relations and \leq, V are branch relations.

Lemma 4.24 *The branch automorphisms of $\text{Aut}(\Omega, C, \leq, V, L, R)$ are same as the branch automorphisms of $\text{Aut}(\Omega, C, \leq, V)$ (Lemma 4.8). •*

Lemma 4.25 *The chain automorphisms of $\text{Aut}(\Omega, C, \leq, V, L, R)$ are same as the chain automorphisms of $\text{Aut}(\Omega, C, L, R)$ (Lemma 4.21). •*

The group $G_{\leq, V, L, R}$ cannot be 2-homogeneous because the group $G_{L, R}$ is not 2-homogeneous (Theorem 4.18). In showing transitivity of $\text{Aut}(\Omega, C)$ (Lemma 3.3) we used branch automorphisms that induced translations on branches at nodes of Λ . These branch automorphisms are clearly elements of $G_{\leq, V, L, R}$ and hence transitivity holds in the automorphism group $G_{\leq, V, L, R}$.

Theorem 4.20 *The group $G_{\leq, V, L, R}$ is a transitive group. •*

4.6 A minimal Jordan group of $G_{\leq, V, L, R}$

4.5.2 Jordan sets of $\text{Aut}(\Omega, C, \leq, V, L, R)$

Jordan sets of $G_{\leq, V}$ are only of type I (refer Section 4.2.3) hence Jordan sets of $G_{\leq, V, L, R}$ (provided they exist) has to be only of type I . We show that there exist Jordan sets of type I in $G_{\leq, V, L, R}$. For α in Λ and q_0 in \mathbb{Q} let $\Sigma_{\alpha, q_0} = \{\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid L(\alpha) < n_1 < n_2 < \dots n_k \in \mathbb{Q}\}$ be a set of type I . Consider the subgroup J of $G_{\leq, V, L, R}$ which consists of branch automorphisms that induce translations on the branches at nodes greater than α and identity permutation on branches at the remaining nodes of Λ including α . We show that J is the associated Jordan group of Σ_{α, q_0} .

Theorem 4.21 *The group $G_{\leq, V, L, R}$ is a Jordan group.*

Proof. Let us consider Σ_{α, q_0} and the subgroup J of $G_{\leq, V, L, R}$ as defined in the preceding paragraph. Let $x = \alpha q_0 m_1 p_1 m_2 p_2 \dots m_j p_j$ and $y = \alpha q_0 l_1 r_1 l_2 r_2 \dots l_i r_i$ be two distinct elements of Σ_{α, q_0} . We first equalise the lengths of the expressions of x and y as $\alpha q_0 n_1 p'_1 n_2 p'_2 \dots n_k p'_k$ and $\alpha q_0 n_1 r'_1 n_2 r'_2 \dots n_k r'_k$ respectively with usual notation of the symbols (refer Lemma 3.2). We then consider the branch automorphisms $g_t, 1 \leq t \leq k$, that induce translations on the branches at the nodes $\alpha q_0 n_1, \alpha q_0 n_1 r'_1 n_2, \dots, \alpha q_0 n_1 r'_1 n_2 r'_2 \dots n_k$ which maps the p'_t -th branch to the r'_t -th branch respectively. Then clearly $g_1 g_2 \dots g_k$ is an element of J that maps x to y and fixes $\Omega \setminus \Sigma_{\alpha, q_0}$ pointwise. This means that J is the associated Jordan group of Σ_{α, q_0} which proves that $G_{\leq, V, L, R}$ is a Jordan group. •

4.6 A minimal Jordan group of $G_{\leq, V, L, R}$

Let G_0 denote the group that consists of branch automorphisms that induce translations on the branches at nodes of Λ . We can show (with similar arguments as in Section 4.5) that G_0 is a transitive group on Ω which is not 2-homogeneous. Also G_0 can be shown to be a Jordan group with Jordan sets of type I . We call a Jordan group *minimal* if there exists no Jordan subgroup of the group. We

show that G_0 is a minimal Jordan group of $G_{\leq, V, L, R}$. We first give a result on the automorphisms of (\mathbb{Q}, \leq) . Consider the subgroup T of $\text{Aut}(\mathbb{Q}, \leq)$ which consists of just the translations and the subgroup M of $\text{Aut}(\mathbb{Q}, \leq)$ which consists of just the positive magnifications. We show that the group generated by M and any non-trivial translation contains the group T .

Lemma 4.26 *All magnifications and a single translation of the group $\text{Aut}(\mathbb{Q}, \leq)$ generate all translations of $\text{Aut}(\mathbb{Q}, \leq)$.*

Proof. Let A be a subgroup of $\text{Aut}(\mathbb{Q}, \leq)$ that contains M , the group of all magnifications and the translation $q \mapsto q + k, k \in \mathbb{Q}, k \neq 0$. We show that A contains T the group of all translations. For any $j \in \mathbb{Q}, j \neq 0, j \neq k$ we get the translation $q \mapsto q + j$ by first magnifying q by $\frac{k}{j}$, then translating by k and then magnifying by $\frac{j}{k}$. Therefore A contains the translation $q \mapsto q + j$. Now j is chosen arbitrarily and hence A contains T . •

Theorem 4.22 *The group G_0 is a minimal Jordan group of $G_{\leq, V, L, R}$.*

Proof. The group G_0 consists of branch automorphisms that induce translations on the branches at nodes of Λ . Suppose there exists a subgroup G^* of G_0 which is a Jordan group. Consider a node α and chains $\alpha q, \alpha p, q \neq p$. Now G^* is a Jordan group therefore G^* is transitive on Ω and so there must exist some element in G^* that maps αq to αp . Now to map αq to αp we need a branch automorphism that induces a permutation on the branches at the node α which maps q to p and because G^* is a subgroup of G_{\leq} the induced permutation must preserve the linear order \leq on \mathbb{Q} . Now any magnification fixes the point 0 and so if one of p, q is 0 then we need the translation of the branches at the node α that maps q to p . By Lemma 4.26 it means that G^* induces all translations of the branches on the nodes of Λ . Varying $\alpha \in \Lambda$ we get branch automorphisms that induce translations on the branches at all nodes. This means that G^* cannot be a proper subgroup of G_0 . In other words, G_0 is a minimal Jordan group of $G_{\leq, V, L, R}$. •

4.7 Expansion of a C -set from another angle

So far we have imposed relations on (Ω, C) and investigated whether the automorphism groups of these expanded structures are Jordan groups. Now we attempt to address the question in the reverse direction. Suppose we have a C -set Ω and some relations, say R_1, R_2, \dots, R_n and suppose we know that $\text{Aut}(\Omega, C, R_1, R_2, \dots, R_n)$ is a Jordan group. Then can we say something about the relations R_1, R_2, \dots, R_n ? In this section we address this question.

Let G denote the group $\text{Aut}(\Omega, C, R_1, R_2, \dots, R_n)$ which is a Jordan group. Consider a node α and the chains $\alpha q, \alpha p, p \neq q$. Now G is a Jordan group and hence transitive on Ω so there must exist some element in G that maps αq to αp . Now to map αq to αp we need a branch automorphism that induces a permutation on the branches of the node α that maps the q -th branch to the p -th branch. We do not have any other restriction whatsoever on the automorphism. This task can be achieved by any permutation on \mathbb{Q} , the simplest being one that interchanges the q -th and the p -th branch at α and fixes all other branches. We therefore have no control on the branch automorphisms of G .

Now suppose we impose the restriction that one of the relations, say R_1 is \leq which means that G is a subgroup of G_{\leq} . Then we have some control on the branch automorphisms of G because the branch automorphisms have to preserve the linear order on the induced permutation on the branches at a node. We have already seen in Theorem 4.22 that G_0 is a minimal Jordan group of the automorphism group $\text{Aut}(\Omega, C, \leq, V, L, R)$, where \leq is one of the relations. Hence if we know that G contains the class of translation branch automorphisms then G contains G_0 as a subgroup, that is G contains a Jordan subgroup. This means that if we impose a branch relation on (Ω, C) and the branch relation is such that the automorphism group of the new structure contains the class of translation branch automorphisms then the automorphism group is a Jordan group.

Next, suppose R_2 is a chain relation. We have already seen that even if the automorphism group fixes the level of nodes of the underlying semilinear order

then it is still a Jordan group (refer Section 4.3.3). This means that imposition of chain relations makes no difference on the existence of Jordan sets. We state these results in the following lemmas.

Lemma 4.27 *Imposition of a branch relation on (Ω, C) that contains the class of translation branch automorphisms in its automorphism group admits a Jordan automorphism group. •*

Lemma 4.28 *Imposition of any chain relation on (Ω, C) admits a Jordan automorphism group. •*

Chapter 5

Some related questions on a C -set

In Chapter 4 we imposed extra relations on (Ω, C) and have shown that Jordan sets existed in the automorphism groups of these structures. In this chapter we look at the questions of homogeneous structures and oligomorphic groups of the structures considered in Chapter 4. We also look at the cycle types of the automorphism groups of these structures.

5.1 Homogeneous structures

The term “homogeneous” comes from model theory in logic. It has no relation to its use in permutation groups where we refer to the terms “ k -homogeneous” and “highly-homogeneous”.

Definition 5.1 A relational structure S is called *homogeneous* if for each embedding ϕ of a finite substructure U of S into S there exists $\psi \in \text{Aut}(S)$ such that ϕ equals the restriction ψ_U of ψ to U .

In other words, a structure is homogeneous if any embedding of a finite structure of S into S can be extended to an automorphism of S . The relational structure (\mathbb{Q}, \leq) is homogeneous. This can be shown as follows. Any finite substructure is an ordered set of rational numbers $a_1 < a_2 < \dots < a_k$ for some finite k . So an



embedding of this finite substructure is a mapping $\phi : a_i \mapsto b_i$, where $b_i \in \mathbb{Q}$ such that $b_1 < b_2 < \dots < b_k$. We can extend ϕ to an order-preserving permutation on \mathbb{Q} by using a k -homogeneous map as defined in Section 3.3.2 (iv). Since every embedding of a finite substructure can be extended to an automorphism, the relational structure is homogeneous. Note that as a permutation group, the group $\text{Aut}(\mathbb{Q}, \leq)$ is k -homogeneous for every $k \in \mathbb{N}$.

We state a simple criterion for recognising when a relational structure is homogeneous. Let Λ be a (possibly infinite) set and associate to each $\lambda \in \Lambda$ a non-negative integer n_λ . Then a relational structure of type $\langle n_\lambda \rangle_{\lambda \in \Lambda}$ is a set Ω together with an indexed family $R = \langle \rho_\lambda \rangle_{\lambda \in \Lambda}$ where ρ_λ is an n_λ -ary relation on Ω . We denote the relational structure by (Ω, R) or $(\Omega, \langle \rho_\lambda \rangle_{\lambda \in \Lambda})$. We call this structure *finite* or *countable* when Ω is, respectively, finite or countably infinite.

Let S and T be two relational structures of same type. We shall say that the *one-point extension property* (IPX) holds for S into T if when

(IPX) $U \subset V$ are finite substructures of S where V contains one more point than U then each embedding of U into T can be extended to an embedding of V into T .

Lemma 5.1 [Dixon & Mortimer] *If S is a countable relational structure and (IPX) holds for S into itself, then S is a homogeneous relational structure.* •

The proof follows by a recursive argument using a back-and-forth argument (refer [DM][Theorem 9.5A]). Elements of the C -set Ω are finite sequences of the form $n_1q_1n_2q_2 \dots n_kq_k$. So for each k , we have a countable choice for $n_1, n_2, \dots, n_k, q_1, q_2, \dots, q_k$, which is again countable. Therefore it is enough for us to check whether (IPX) holds in each of the relational structures. Before we proceed further we take a look at Λ , the underlying semilinearly ordered set of the C -set.

5.1.1 Homogeneity of (Λ, \leq)

Consider the set Λ with the semilinear order. We show that (Λ, \leq) is not homogeneous but can be made homogeneous by imposing an extra relation.

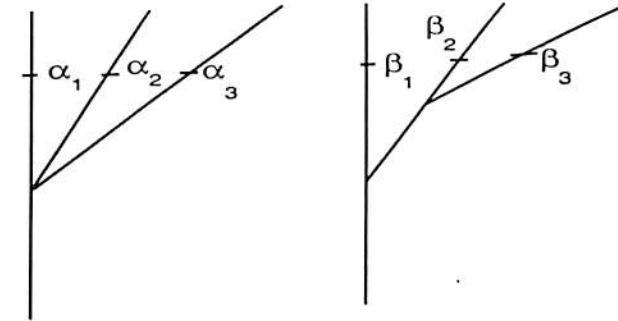


Figure 5.1:

Consider the sets $A = \{\alpha_1, \alpha_2, \alpha_3\}$ and $B = \{\beta_1, \beta_2, \beta_3\}$ of Λ as in Fig. 5.1. No pairs of elements of either A or B are comparable and so we can set up an isomorphism $\phi : A \rightarrow B$ defined by $\alpha_i \mapsto \beta_i, i = 1, 2, 3$. However, the isomorphism ϕ between A and B cannot be extended to an automorphism of (Λ, \leq) . This is because if it were possible then we should be able to find an automorphism g of (Λ, \leq) that maps α_i to $\beta_i, i = 1, 2, 3$. This would mean that g would map $\alpha_1 \cap \alpha_2$ to $\beta_1 \cap \beta_2$ and $\alpha_2 \cap \alpha_3$ to $\beta_2 \cap \beta_3$, where $\alpha_1 \cap \alpha_2$ denotes the intersecting node (refer Definition 3.1) of α_1 and α_2 . But $\alpha_1 \cap \alpha_2 = \alpha_2 \cap \alpha_3$ and $\beta_1 \cap \beta_2 \neq \beta_2 \cap \beta_3$ which contradicts the fact that g is an automorphism on Λ . Hence ϕ cannot be extended to an automorphism of (Λ, \leq) . Therefore (Λ, \leq) is not homogeneous.

We note that the extension of an isomorphism on finite sets to an automorphism of Λ becomes impossible only when the set consists of all incomparable elements such that the intersecting nodes of pairs of elements from the sets are not equal. For example if $\beta_1, \beta_2, \beta_3$ have the same configuration as $\alpha_1, \alpha_2, \alpha_3$ then the isomorphism ϕ can be easily extended to an automorphism of (Λ, \leq) . By *configuration* of $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ we mean the relation of the nodes

5.1 Homogeneous structures

$\alpha_1 \cap \alpha_2$, $\alpha_2 \cap \alpha_3$ and $\beta_1 \cap \beta_2$, $\beta_2 \cap \beta_3$ in Λ . Hence if we say that the configuration is same then we mean that the relation between $\alpha_1 \cap \alpha_2$ and $\alpha_2 \cap \alpha_3$ is same as the relation between $\beta_1 \cap \beta_2$ and $\beta_2 \cap \beta_3$ with respect to the semilinear order \leq of Λ . We therefore need to define the new relation in such a way that the behaviour of intersecting nodes of elements is captured by the relation. We define a ternary relation denoted by χ on incomparable elements α, β, γ of Λ as follows:

$$\chi(\alpha; \beta, \gamma) : \leftrightarrow \alpha \cap \beta < \beta \cap \gamma.$$

In other words, the relation χ uses the idea of the relation C defined on Ω , the set of maximal chains of Λ . Given three incomparable elements α, β, γ either any of $\chi(\alpha; \beta, \gamma)$, $\chi(\beta; \gamma, \alpha)$, $\chi(\gamma; \alpha, \beta)$, holds or none of these holds. We denote the latter case by $N(\alpha, \beta, \gamma)$. (Note that if $\chi(\alpha; \beta, \gamma)$ holds then $\chi(\alpha; \gamma, \beta)$ holds.) We verify that (IPX) holds in (Λ, \leq, χ) which will imply that (Λ, \leq, χ) is homogeneous.

Lemma 5.2 *The property (IPX) holds in (Λ, \leq, χ) .*

Proof. Let U, U' be finite isomorphic substructures of (Λ, \leq, χ) and let ϕ be an isomorphism of U into U' . Let V be another finite substructure of Ω such that V contains one more element than U does with $V = U \cup \{a\}$. We need to extend ϕ to V . For any $\alpha \in U$, we denote the isomorphic copy of α in U' as α' .

Suppose U contains just one element say α and let $V = \{\alpha, a\}$. If α, a are comparable with respect to the semilinear order \leq we extend ϕ to V such that α' and $\phi(a)$ are also comparable in the same way. If α, a are incomparable we extend ϕ to V such that α' and $\phi(a)$ are also incomparable.

Suppose U contains two elements then these might be either comparable or incomparable with respect to the semilinear order. Suppose both are comparable (which will give two structures depending on which is smaller). The third element of V which is not in U might be comparable to both or comparable to one and incomparable to the other. We define $\phi(\alpha)$ such that the elements of U' and $\phi(\alpha)$ are related in the same way as V . We define the image of α in the same way for the case when the two elements of U are incomparable.

5.1 Homogeneous structures

In general, suppose U contains n elements, say $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and V contains one more point say α_{n+1} . Then to extend ϕ to V we need to define $\phi(\alpha_{n+1})$ in such a way that $\phi(\alpha_{n+1})$ and the n elements of U' are related in the same way as the elements of V with respect to the relations \leq and χ . •

Theorem 5.1 *The structure (Λ, \leq, χ) is homogeneous.* •

The proof of Theorem 5.1 follows by using Lemma 5.1 and Lemma 5.2.

5.1.2 Homogeneity of (Ω, C) with the extra relations

We show that (IPX) holds in (Ω, C) which will imply that the structure (Ω, C) is homogeneous. We have talked about the intersecting node of two elements of Ω (Definition 3.1). Using the same idea we define the *intersecting node* of three or more elements of Ω to be the greatest node in Λ (with respect to the semilinear order) that is on each of the elements.

Lemma 5.3 *The property (IPX) holds in (Ω, C) .*

Proof. Let U, U' be finite isomorphic substructures of Ω and let ϕ be an isomorphism of U into U' . Let V be another finite substructure of Ω containing one more element than U . Let $V = U \cup \{x\}$ and let α, α' be the intersecting nodes of the elements in the sets U and U' respectively. Then the following cases arise.

Case (i) The intersecting node of x and some element in U is greater than α (Fig. 5.2(a)).

Case (ii) The intersecting node of x and u is equal to α , for all $u \in U$ (Fig. 5.2(b)).

Case (iii) The intersecting node of x and u is smaller than α , for all $u \in U$ (Fig. 5.2(c)).

We need to extend ϕ to V . Suppose x occurs as in Case (i). Let u_0 be an element in U such that for all $u \in U$, $x \cap u \leq x \cap u_0$. Such an element exists because U is finite and the set $\{x \cap u \mid u \in U\}$ is linearly ordered. By density of

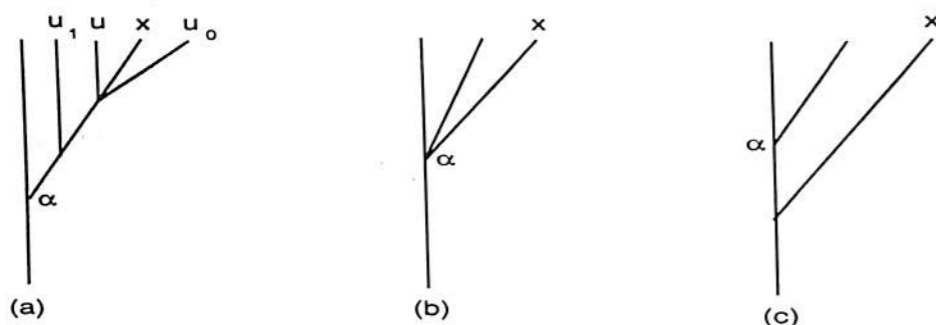


Figure 5.2:

Ω we can choose an element, say x' passing through the node α' such that every 3-element set in $U' \cup \{x'\}$ is C -related in the same way as the corresponding 3-element set in V and $x' \cap \phi(u) \leq x' \cap \phi(u_0)$, for all $u \in U$. We then set $x' = \phi(x)$ to get the required extension of ϕ to V .

Next, suppose x occurs as in Case (ii). We define $\phi(x)$ such that the intersecting node of $U' \cup \{\phi(x)\}$ is equal to α' .

Finally, suppose x occurs as in Case (iii). In this case we define $\phi(x)$ such that the intersecting node of $U' \cup \{\phi(x)\}$ is smaller than α' . Hence in any case (IPX) holds in the structure (Ω, C) . •

Theorem 5.2 *The structure (Ω, C) is homogeneous.* •

As in Lemma 5.3 we can show that (IPX) holds in each of the structures (Ω, C, \leq) , (Ω, C, \leq, V) , (Ω, C, L) and (Ω, C, L, R) so that the following is true.

Theorem 5.3 *The structures (Ω, C, \leq) , (Ω, C, \leq, V) , (Ω, C, L) and (Ω, C, L, R) are homogeneous.* •

Finally we can show that the structure $(\Omega, C, \leq, V, L, R)$ is homogeneous. To show this we need to extend the isomorphism between sets of size n to sets of size $(n+1)$ in such a way that the relations \leq and V are preserved amongst the branches at nodes and the relation L and type of the intersecting nodes of pairs of elements is preserved.

5.2 Oligomorphic groups

Theorem 5.4 *The structure $(\Omega, C, \leq, V, L, R)$ is homogeneous.* •

5.2 Oligomorphic groups

The term “oligomorphic” is intended to suggest “few shapes” of n -element subsets, referring to a structure on Ω admitting G as a group of automorphisms.

Definition 5.2 A permutation group G on an infinite set Ω is said to be *oligomorphic* if G has only finitely many orbits in its induced action on Ω^n , the set of n -tuples of elements of Ω , for every $n \in \mathbb{N}$.

The orbits of G on Ω^n refer to the orbits arising from the obvious induced action of G , defined by $(\alpha_1, \alpha_2, \dots, \alpha_n)^g = (\alpha_1^g, \alpha_2^g, \dots, \alpha_n^g)$, $\alpha_1, \alpha_2, \dots, \alpha_n \in \Omega$, $g \in G$. The distinct structures of Ω^n for $n \in \mathbb{N}$ up to isomorphism are called the *isomorphism types* on sets of size n in Ω . The number of isomorphism types on sets of size n in Ω is the number of distinct structures on Ω^n . The definition of oligomorphic groups can be stated with the help of isomorphism types of Ω^n as follows. Let G be a permutation group on an infinite set Ω and let $I(n)$ be the number of isomorphism types on sets of size n in Ω , $n \in \mathbb{N}$. Then G is said to be oligomorphic if $I(n)$ is finite and there are only finite number of orbits of G on each isomorphism type of Ω^n . Note that a structure is homogeneous if there is only one orbit on each isomorphism type on sets of size n in Ω . Therefore a homogeneous structure is oligomorphic if $I(n)$ is finite for every $n \in \mathbb{N}$. In Section 5.1 we have seen that the structure (Ω, C) and each of the structures with the imposed relations on (Ω, C) are homogeneous. Therefore to check whether $\text{Aut}(\Omega, C)$ is oligomorphic it is enough for us to check whether the isomorphism types of the structure (Ω, C) is finite. Similarly for the automorphism groups of (Ω, C) with the extra relations we need to check that the isomorphism types of the structures of (Ω, C) with the extra relations is finite.

Cameron [C1] has constructed some sequences which enumerates a class of trees sometimes with extra structures by the points at infinity. In Section 5.3 of

his paper, Cameron considered the structure with the set of leaves (vertices with valency 1) of a finite tree. By valency he means branching number at a node of the tree. He denotes this class of structures as T . As a first variation, instead of considering all finite trees he considers only those in which the valencies of the non-leaves do not exceed k , where k is a fixed integer which is at least 3. He uses the notation T_k to denote this class of structures. There is a unique boron tree (series-reduced tree with maximum valency 3) having 5 leaves and the sequence enumerating boron trees has been shown to be the sequence $M0327$ of [SP]. As a second variation he adds a binary relation symbol to the language, interpreted as a linear order and shows that the sequence enumerating LT (the structure T with the linear order) is $M3613$ and that which enumerates LT_3 is $M3002$ of [SP]. He has explored many other sequences related to such trees.

5.2.1 The group $\text{Aut}(\Omega, C)$

In this section we give the sequences enumerating the isomorphism types on sets of size n in $\text{Aut}(\Omega, C)$. The group $\text{Aut}(\Omega, C)$ is 2-transitive and so there is just one isomorphism type on sets of size 1 and sets of size 2 in Ω respectively. Given any set of size 3 in Ω the number of intersecting nodes determined by pairs of elements of the set can be one or two. In each case there can be just one isomorphism type. Hence the number of isomorphism types of sets of size 3 is two.

Given any set of size 4 in Ω the number of intersecting nodes determined by pairs of elements of the set is one, two or three. The number of isomorphism types in the first case is one and for the remaining two cases the number is two (Fig. 5.3) so that there are five isomorphism types of sets of size 4 in Ω . Again, given any set of size 5 the number of intersecting nodes of pairs of elements of the set is one, two, three or four. Here again, the number of isomorphism types in each of the cases is one, three, five and three respectively (Fig. 5.4) so that there are twelve isomorphism types of sets of size 5 in Ω . It turns out that given any n -set in Ω the number of intersecting nodes determined by pairs of elements of

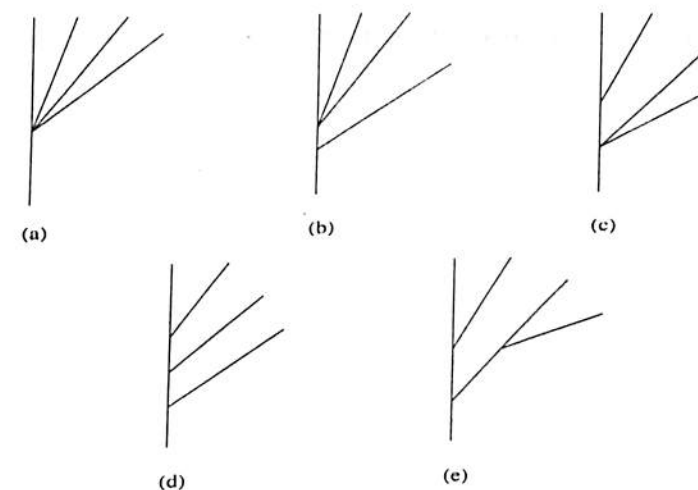


Figure 5.3:

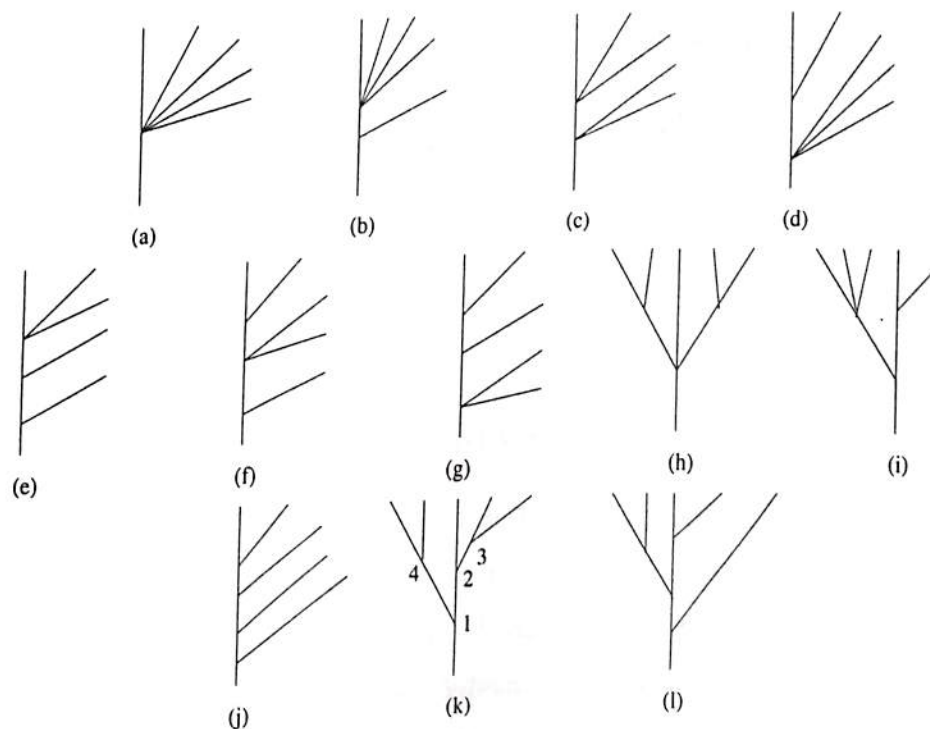


Figure 5.4:

the n -set can be any number between 1 and $n - 1$ including both these bounds. Therefore to prove that $\text{Aut}(\Omega, C)$ is oligomorphic we prove the following.

Lemma 5.4 *The number of isomorphism types of sets of size n in (Ω, C) , $n \in \mathbb{N}$ with m intersecting nodes, $1 \leq m \leq n - 1$, determined by pairs of elements of the n set is finite.*

Proof. Let $I(n, m)$ denote the number of isomorphism types of sets of size n with m intersecting nodes determined by pairs of elements of the n -set, $1 \leq m \leq n - 1$. Clearly for all $n \in \mathbb{N}$ we have $I(n, 1) = 1$. If there are two intersecting nodes determined by pairs of elements then nodes have to be linearly ordered. That is, determining $I(n, 2)$ means that we have to partition $n - 1$ into two numbers such that each is at least one. This can be done in $n - 2$ ways by partitioning n as $1, n - 2; 2, n - 3; \dots; n - 2, 1$. Therefore we have $I(n, 2) = n - 2$.

Now consider $I(n, 3)$. The three nodes can be either linearly ordered or can have two incomparable nodes as in Fig. 5.5. Let $I_1(n, 3)$ and $I_2(n, 3)$ denote

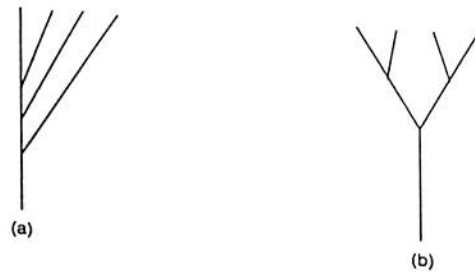


Figure 5.5:

the number of isomorphism types of sets of size n with three nodes as in Fig. 5.5(a) and Fig. 5.5(b) respectively. Clearly we have $I_1(4, 3) = I_2(4, 3) = 1$. We determine $I_1(n, 3)$ for $n > 4$. To determine $I_1(5, 3)$ we need to attach one new branch to any of the existing nodes in Fig. 5.5(a). Because the nodes are asymmetrical we can do it in three ways. To determine $I_1(6, 3)$ we need to attach two branches to any of the nodes. We can either add both the branches in a single node or attach the two branches in two different nodes. Hence there are $3 + 3 = 6$ ways of doing this. To determine $I_1(7, 3)$ we need to attach three branches to any of the nodes in Fig. 5.5(a). We can add all the three branches in one node or add two branches in one node and the remaining branch

in another node or add the three branches in three different nodes. This can be done in $3 + 6 + 1 = 10$ ways. Continuing in this way we get the sequence as $1, 3, 6, 10, 15, 21, 28, 36, \dots$, where the first term gives $I_1(4, 3)$ and the n -th term gives $I_1(n + 3, 3)$. This sequence occurs as M2535 of [SP]. Here $I_1(4, 3) = 1$ and for $n > 4$, $I_1(n, 3) = I_1(n - 1, 3) + (n - 3)$. We now determine $I_2(n, 3), n > 4$. Two nodes in Fig. 5.5(b) are symmetric and so we must have $I_2(n, 3) < I_1(n, 3), n > 4$. Actual calculation gives the sequence $I_2(n, 3)$ as $1, 2, 4, 6, 9, 12, 16, 20, \dots$, where the first term gives $I_2(4, 3)$ and the n -term gives $I_2(n + 3, 3)$. This sequence occurs as M0998 of [SP]. Here again, $I_2(4, 3) = 1$ and for $n > 4$,

$$I_2(n, 3) = \begin{cases} I_2(n - 1, 3) + \frac{n-3}{2}, & \text{if } n \text{ is odd,} \\ I_2(n - 1, 3) + \frac{n-2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

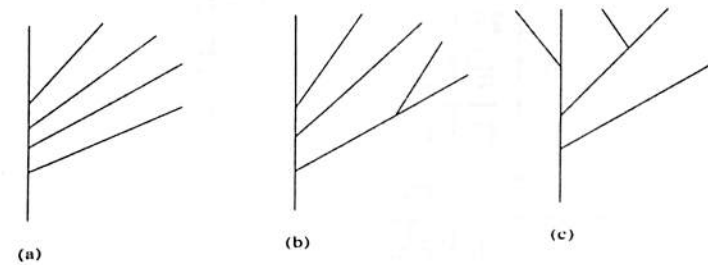


Figure 5.6:

Therefore $I(n, 3) < 2 \times I_1(n, 3)$, which is finite. We now compute $I(n, 4)$. There can be three structures of the four nodes as in Fig. 5.6. Let $I_1(n, 4), I_2(n, 4)$ and $I_3(n, 4)$ denote the number of isomorphism types of sets of size n with four nodes as in Fig. 5.6(a), Fig. 5.6(b) and Fig. 5.6(c) respectively. Clearly we have $I_1(5, 4) = I_2(5, 4) = I_3(5, 4) = 1$. The nodes in Fig. 5.6(a) and Fig. 5.6(b) are all asymmetric and two nodes in Fig. 5.6(c) are symmetric and so we must have $I_1(n, 4) = I_2(n, 4)$ and $I_3(n, 4) < I_1(n, 4), n > 5$. To determine $I_1(6, 4)$ we need to attach one new branch to any of the existing nodes in Fig. 5.6(a) which can be done in four ways. Similarly to determine $I_1(7, 4)$ we need to attach two new branches which can be done in ten ways. Continuing in this way we get the

Table 5.1: Isomorphism types of sets of size n of (Ω, C) with m nodes which are all comparable, $2 \leq n \leq 14$, $1 \leq m \leq 11$.

No. of nodes (m) sets of size n	1	2	3	4	5	6	7	8	9	10	11
2	1										
3	1	1									
4	1	2	1								
5	1	3	3	1							
6	1	4	6	4	1						
7	1	5	10	10	5	1					
8	1	6	15	20	15	6	1				
9	1	7	21	35	35	21	7	1			
10	1	8	28	56	70	56	28	8	1		
11	1	9	36	84	126	126	84	36	9	1	
12	1	10	45	120	210	252	210	120	45	10	1
13	1	11	55	165	330	462	462	330	165	55	11
14	1	12	66	220	495	792	924	792	495	220	66

sequence as 1, 4, 10, 20, 35, 56, 84, ..., where the first term gives $I_1(n+4, 4)$. This sequence occurs as M3382 of [SP]. For $n \geq 6$, $I_1(n, 4) = I_1(n-1, 4) + I_1(n-1, 3)$, where $I_1(4, 4) = 0$. Now $I_2(n, 4) = I_1(n, 4)$ and $I_3(n, 3) < I_1(n, 4)$ therefore we must have $I(n, 4) < 3 \times I_1(n, 4)$, which is finite. Table 5.1 gives the number of isomorphism types of sets of size n in (Ω, C) with m intersecting nodes determined by pairs of elements of the set, $2 \leq n \leq 14$, $1 \leq m \leq 11$ such that the nodes are all linearly ordered. It is clear from the table that we can calculate $I_1(n, m)$, $n - m \geq 1$, from the relation

$$I_1(n, m) = \binom{n-2}{m-1}, n \geq 2, m \geq 1, \binom{0}{0} = 1, \text{ and}$$



5.2 Oligomorphic groups

$I_1(n, m) = I_1(n-1, m) + I_1(n-1, m-1)$, $I_1(m, m) = 0$, $I_1(m+1, m) = 1$, $I_1, \dots, -$, is the recurrence relation. Also we have

$$I(n, m) \leq I(m+1, m) \times I_1(n, m). \tag{5.1}$$

If we can show that $I(m+1, m)$ is finite then $I(n, m)$ will be finite for all $n, m \in \mathbb{N}$ and our lemma will be proved. We have calculated $I(2, 1)$, $I(3, 2)$, $I(4, 3)$ and $I(5, 4)$ as 1, 1, 2 and 3 respectively. Calculating some more terms we get the sequence as 1, 1, 2, 3, 6, 11, 22, 42, 84, 165, ..., where the first term gives $I(2, 1)$ and the n -th term gives $I(n+1, n)$. This sequence occurs as M0787 of [SP]. The n -th term is given by

$$I(n+1, n) = \begin{cases} I(n, n-1) + I(n-1, n-2) + \dots + I(\frac{n}{2} + 1, \frac{n}{2}), & \text{if } n \text{ is even,} \\ I(n, n-1) + I(n-1, n-2) + \dots + I(\frac{n+1}{2}, \frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

This means that $I(m+1, m)$ is finite and so $I(n, m)$ is finite. •

The following lemma gives an upper bound on the number of isomorphism types of sets of size n in Ω .

Lemma 5.5 Let $I(n)$ denote the number of isomorphism types of sets of size n in Ω then $I(n) \leq I_1(n, 1) \times I(2, 1) + I_1(n, 2) \times I(3, 2) + \dots + I_1(n, n-1) \times I(n, n-1)$.

Proof. Given any set of size n the number of intersecting nodes determined by pairs of elements of the set can be any of the numbers 1, 2, 3, ..., $n-1$. Therefore we have $I(n) = I(n, 1) + I(n, 2) + I(n, 3) + \dots + I(n, n-1)$. Using Eq. 5.1 we get the result. •

Using Theorem 5.2, Lemma 5.4 and Lemma 5.5 we have the following.

Theorem 5.5 The group $\text{Aut}(\Omega, C)$ is oligomorphic. •

5.2.2 Automorphism groups of (Ω, C) with the extra relations

Here we study whether the automorphism groups of (Ω, C) with the extra relations are oligomorphic.

Theorem 5.6 *The group $\text{Aut}(\Omega, C, \leq)$ is oligomorphic.*

Proof. Given any n -set in (Ω, C) there can be at most $n!$ number of ways in which the set can be ordered by the linear order \leq of Ω . The upper bound $n!$ is attained when there is just one intersecting node determined by pairs of elements of the n -set. Also $\text{Aut}(\Omega, C, \leq)$ is homogeneous (refer Theorem 5.3) and hence it follows that $\text{Aut}(\Omega, C, \leq)$ is oligomorphic. •

Theorem 5.7 *The group $\text{Aut}(\Omega, C, \leq, V)$ is oligomorphic.*

Proof. Given any n -set $\{x_1, x_2, \dots, x_n\}$ in (Ω, C, \leq) such that $x_1 < x_2 < \dots < x_n$, the number of structures of the n -set (depending on whether the pairs $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}$ are V -related) is finite. This means that the number of isomorphism types of sets of size n in (Ω, C, \leq, V) is finite. Therefore $\text{Aut}(\Omega, C, \leq, V)$ is oligomorphic. •

We now study whether the group $\text{Aut}(\Omega, C, L)$ is oligomorphic. The automorphism group $\text{Aut}(\Omega, C, L)$ is 2-transitive and so there is just one isomorphism type on sets of size 1 and sets of size 2 respectively in (Ω, C, L) . Consider the isomorphism types of sets of size 3 in (Ω, C) . Each of these isomorphism types in (Ω, C) will produce the same isomorphism type in (Ω, C, L) and so the isomorphism types of sets of size 3 in (Ω, C, L) are same as those of (Ω, C) .

Next, consider the isomorphism types of sets of size 4 in (Ω, C) as in Fig 5.3. The intersecting nodes determined by pairs of elements of the sets of size 4 as in Fig 5.3(a), Fig 5.3(b), Fig 5.3(c) and Fig 5.3(d) are all linearly ordered. Therefore the number of isomorphism types will remain the same in (Ω, C, L) . Consider the isomorphism type as in Fig 5.3(e). There are two incomparable nodes in this isomorphism type. The level of these two nodes can be either equal or unequal (any one could be smaller) and hence this isomorphism type produces three isomorphism types in (Ω, C, L) .

Next, consider the isomorphism types of sets of size 5 in (Ω, C) as in Fig 5.4. Here, except the isomorphism types in Fig 5.4(h), Fig 5.4(i), Fig 5.4(k) and Fig

5.4(l) all other isomorphism types will remain the same in (Ω, C, L) . Each of the isomorphism types in Fig 5.4(h), Fig 5.4(i) and Fig 5.4(l) have two incomparable nodes and hence will produce three isomorphism types in (Ω, C, L) , by a similar reasoning as for Fig 5.3(e). The structure in Fig 5.4(k) has four nodes with two pairs of incomparable nodes (2, 4) and (3, 4). Consider the isomorphism type of n -set, $n \geq 6$ with isomorphism type as in Fig 5.7. This isomorphism type also has four nodes with three mutually incomparable nodes. Let A, B denote the isomorphism types in Fig 5.4(k) and Fig 5.7 respectively. We calculate the number of isomorphism types produced by A and B in (Ω, C, L) . In the type B there is no restriction on the level of the nodes 2, 3, 4. But the same is not true of A . The nodes 2, 3 are linearly ordered and so once we determine 2, 3 we just have choices on the level of the node 4. Let $L(i)$ denote the level of the node i , $1 \leq i \leq 4$. Then $L(4)$ has to satisfy any one of the following relations,

$$L(1) < L(4) < L(2), L(4) = L(2), L(2) < L(4) < L(3), L(4) = L(3), L(3) < L(4).$$

Therefore A produces 5 isomorphism types in (Ω, C, L) . We now look at the type B . If the level of the nodes 2, 3, 4 are same then there is just one isomorphism type. If the nodes are in two different levels then there are $2^3 - 2 = 6$ isomorphism types. If the nodes are in 3 different levels then there are $3!$ isomorphism types. Hence the type B produces 13 isomorphism types in (Ω, C, L) which means that type B produces more isomorphism types than type A . This is true not only with four nodes but in general with any number of nodes greater than four. Now any finite set can determine at most a finite number of nodes and so to determine whether $\text{Aut}(\Omega, C, L)$ is oligomorphic we need to prove the following.

Lemma 5.6 *The number of isomorphism types produced by a finite set in (Ω, C, L) with $(m + 1)$ intersecting nodes determined by pairs of elements of the set such that m nodes are mutually incomparable, is finite.*

Proof. Consider the isomorphism type of any n -set, $n \geq 4$ with $(m + 1)$ intersecting nodes determined by pairs of elements of the n -set such that m nodes

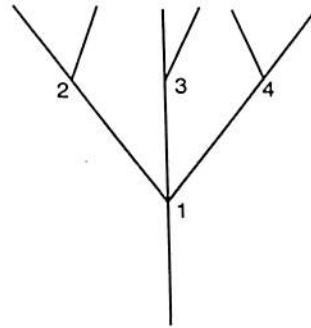


Figure 5.7:

are mutually incomparable. Let $J(n, m + 1)$ denote the sequence enumerating the number of these isomorphism types in (Ω, C, L) . We calculate the terms of the sequence $J(n, m + 1)$. Note that for $m = 0$ and $m = 1$, $J(n, m + 1)$ does not make sense because we are talking about incomparability of nodes and so we need at least 2 nodes. For $m = 2$ and $n \geq 4$, the two incomparable nodes can be on the same level or on two different levels with any one on the lower level. There is just one way of placing the nodes in one level and $2!$ ways of placing them in 2 different levels hence there can be three ways of ordering the levels of the incomparable nodes, so we have $J(n, 2 + 1) = 3$. For $m = 3$ and $n \geq 6$, the three incomparable nodes can be on the same level or on two different levels or on three different levels. There is just one way of placing the nodes in a single level, $2^3 - 2$ ways of placing them in two different levels and $3!$ ways of placing them in three different levels. Hence $J(n, 3 + 1) = 13$. Similarly for $m = 4$ and $n \geq 8$ we have $J(n, 4 + 1) = 75$. Calculating two more terms we get $J(n, 5 + 1) = 541$ and $J(n, 6 + 1) = 4683$. The sequence $J(n, m + 1)$ is $1, 3, 13, 75, 541, 4683, \dots$, where the first term enumerates $J(n, 1 + 1)$ and the r -th term gives $J(n, r + 1)$, $n \geq 2r$. This sequence is M2952 of [SP]. Hence the number of isomorphism types that we are looking for, is finite. •

Lemma 5.7 For any n -set with $m + 1$ intersecting nodes determined by pairs of elements of the n -set, the number of isomorphism types with m mutually incom-

5.3 Cycle types of the automorphism groups of (Ω, C)

parable nodes is greater than the number of isomorphism types with the (number of) mutually incomparable nodes less than m .

Proof. When the number of mutually incomparable nodes is more, we have more choices on the level of the nodes and so the lemma is immediate. •

Using Theorem 5.3, Lemma 5.6 and Lemma 5.7 we have the following.

Theorem 5.8 The group $\text{Aut}(\Omega, C, L)$ is oligomorphic. •

Theorem 5.9 The group $\text{Aut}(\Omega, C, L, R)$ is oligomorphic.

Proof. Given any n -set in (Ω, C, L) , each of the intersecting nodes determined by pairs of elements of the n -set can be any of the types dyadic, ternary and non-dyadic non-ternary. This means that the number of isomorphism types of an n -set in (Ω, C, L, R) is finite. Therefore $\text{Aut}(\Omega, C, L, R)$ is oligomorphic. •

Finally we look at the group $\text{Aut}(\Omega, C, \leq, V, L, R)$. The number of isomorphism types of sets of size n in each of the structures (Ω, C, \leq, V) and (Ω, C, L, R) are finite. Therefore when all the relations are combined the number of isomorphism types of sets of size n in $(\Omega, C, \leq, V, L, R)$ can be at most finite. We thus have the following.

Theorem 5.10 The group $\text{Aut}(\Omega, C, \leq, V, L, R)$ is oligomorphic. •

5.3 Cycle types of the automorphism groups of (Ω, C)

In this section we determine which cycle types are realised in the automorphism group $\text{Aut}(\Omega, C)$ and also in the automorphism groups of (Ω, C) with the extra relations under our consideration.

Definition 5.3 If g is a permutation on a set Σ then we say that g has cycle type $\aleph_0^{n_0} 1^{n_1} 2^{n_2} 3^{n_3} \dots$, $0 \leq n_i \leq |\Sigma|$, $i = 0, 1, 2, 3, \dots$ if its cycle decomposition

into disjoint cycles consists of n_0 cycles of length \aleph_0 and n_j cycles of length j , $j \in \mathbb{N}$. Cycles of g of length 1 are said to be *trivial* and all other cycles are said to be non-trivial.

Elements of the C -set that we have considered, when regarded as maximal chains of a semilinearly ordered set are order-isomorphic to \mathbb{Q} . Therefore results on cycle types of $\text{Aut}(\mathbb{Q}, \leq)$ will be helpful to our study.

Lemma 5.8 *If g is a non-identity automorphism of a dense linear order without end-points then*

- (i) g cannot contain any non-trivial finite cycle,
- (ii) g has infinitely many infinite cycles.

Proof. Without loss of generality we may assume the dense linear order without end-points to be (\mathbb{Q}, \leq) . Let g be a non-identity automorphism of (\mathbb{Q}, \leq) and suppose g contains a non-trivial finite cycle say $(\alpha_1 \alpha_2 \dots \alpha_n)$. Then $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha_1$, $\alpha_1 \neq \alpha_n$, which cannot happen. Therefore g cannot contain any non-trivial finite cycle. The proof of part (ii) follows directly from part (i) and [MP][Lemma 6.2]. •

Theorem 5.11 *The realisable cycle types of non-identity elements of the automorphism group of (\mathbb{Q}, \leq) are of the forms $\aleph_0^{\aleph_0} 1^n$, $0 \leq n \leq \aleph_0$.*

Proof. Let g be a non-identity automorphism of $\text{Aut}(\mathbb{Q}, \leq)$ with cycle type $\aleph_0^{a_0} 1^{a_1} 2^{a_2} 3^{a_3} \dots k^{a_k} \dots$ so that using Lemma 5.8 we have $a_0 = \aleph_0$ and $a_2 = a_3 = \dots = a_k = \dots = 0$. Suppose g fixes just one point b_0 . Then g fixes $(-\infty, b_0)$ and (b_0, ∞) setwise. Let b be any element in $(-\infty, b_0)$. Since g fixes no other point of \mathbb{Q} so b will lie on an infinite cycle on $(-\infty, b_0)$. And because \mathbb{Q} is dense so g will consist of infinite number of \aleph_0 cycles in each of $(-\infty, b_0)$ and (b_0, ∞) . This means that the cycle type of g is $\aleph_0^{\aleph_0} 1^1$. Suppose g fixes just two points, say $b_0, b_1, b_0 < b_1$. Then g fixes each of $(-\infty, b_0), (b_0, b_1), (b_1, \infty)$ setwise and in that case the cycle type of g is $\aleph_0^{\aleph_0} 1^2$. In general if g fixes n points, then the cycle

type of g is $\aleph_0^{\aleph_0} 1^n$, $n \in \mathbb{N}$. •

Cycle types of $\text{Aut}(\mathbb{Q}, \leq)$ are clearly realised in $\text{Sym}(\mathbb{Q})$ because $\text{Aut}(\mathbb{Q}, \leq)$ is a subgroup of $\text{Sym}(\mathbb{Q})$. Consider the cycle type $1^{a_1} 2^{a_2} \dots k^{a_k}$, where a_1, a_2, \dots, a_k are all finite. This cycle type obviously cannot be realised in $\text{Sym}(\mathbb{Q})$ because $a_1 + 2a_2 + \dots + ka_k$ is finite and the order of \mathbb{Q} is \aleph_0 . Any cycle type of the form $\aleph_0^{a_0} 1^{a_1} 2^{a_2} \dots$, provided $a_0 \aleph_0 + \sum_{i=1}^{\infty} ia_i = \aleph_0$ is realised in $\text{Sym}(\mathbb{Q})$.

5.3.1 Cycle types of the automorphism group $\text{Aut}(\Omega, C)$

We now look at the cycle types of the automorphism groups of the different structures on (Ω, C) . We start with (Ω, C) and first look at the cycle types of branch automorphisms and chain automorphisms, which have been defined in Section 3.3.

Lemma 5.9 *The realisable cycle types of non-identity chain automorphisms of (Ω, C) are $\aleph_0^{\aleph_0} 1^1$ and $\aleph_0^{\aleph_0} 1^{\aleph_0}$.*

Proof. Let g be a non-identity chain automorphism of (Ω, C) . Then g fixes some \mathbb{Q}_k -branch of Ω , where \mathbb{Q}_k is the branch on which g induces an order-preserving permutation and also g fixes all nodes on other \mathbb{Q}_k and \mathbb{Q}_i -branches for $i < k$. Now on the \mathbb{Q}_k -branch on which g acts non-trivially, the nodes might be all moved or some nodes might be fixed. If some nodes are fixed, say a number of nodes are fixed on the \mathbb{Q}_k -branch, $1 \leq a \leq \aleph_0$, then g will fix a countable number of elements because each fixed node will fix a countable number of branches arising from that point. If all nodes of the \mathbb{Q}_k -branch are moved then each branch at nodes of the \mathbb{Q}_k -branch will be on an infinite cycle so that there are \aleph_0 number of \aleph_0 cycles. If the \mathbb{Q}_k -branch is the \mathbb{Q}_1 -branch then g is a chain automorphism inducing a permutation on the \mathbb{Q}_1 branch which moves each node of the \mathbb{Q}_1 branch. In that case g will fix the \mathbb{Q}_1 branch. Each of the remaining elements of Ω will be on an infinite cycle. The automorphism g then realises the cycle type $1^1 \aleph_0^{\aleph_0}$. In all other cases, g realises the cycle type $1^{\aleph_0} \aleph_0^{\aleph_0}$. •



5.3 Cycle types of the automorphism groups of (Ω, C)

Lemma 5.9 gives an exhaustive list of the cycle types realised by chain automorphisms in (Ω, C) . But we cannot get an exhaustive list of realisable cycle types of branch automorphisms in (Ω, C) . We can only assert the following which easily follows from the definition of a branch automorphism.

Lemma 5.10 Suppose a branch automorphism of (Ω, C) has cycle type $\prod a_i^{n_i}$, $1 \leq a_i \leq \aleph_0$, $i \in \mathbb{N}$ then $n_i = \aleph_0$, for all $i \in \mathbb{N}$. •

For example let α be any node of Λ . Consider the branch automorphism g that induces a permutation on the branches at the fixed node α . Suppose the permutation has the cycle type $\aleph_0 1 a_1 a_2 \dots a_n$, $a_1, a_2, \dots, a_n \in \mathbb{N}$. Then the cycle type realised by g is $\aleph_0^{\aleph_0} 1^{\aleph_0} a_1^{\aleph_0} a_2^{\aleph_0} \dots a_n^{\aleph_0} 1^{\aleph_0}$, where the last \aleph_0 fixed points arise from the nodes smaller than and incomparable to α fixed by g . In other words, the cycle type realised by g is $\aleph_0^{\aleph_0} 1^{\aleph_0} a_1^{\aleph_0} a_2^{\aleph_0} \dots a_n^{\aleph_0}$, $a_n \in \mathbb{N}$.

5.3.2 Cycle types of the automorphism groups of (Ω, C) with the extra relations

Here we look at the cycle types realised by elements of the automorphism groups of (Ω, C) with the extra relations. We start with the linear order \leq on Ω . We could not give an exhaustive list of the cycle types of the branch automorphisms of $\text{Aut}(\Omega, C)$ but the moment we impose a linear order \leq on (Ω, C) we can give an exhaustive list of cycle types of the branch automorphisms. The cycle types of the chain automorphisms of $\text{Aut}(\Omega, C, \leq)$ will however be the same as those of $\text{Aut}(\Omega, C)$.

Lemma 5.11 The realisable cycle types of non-identity chain automorphisms of (Ω, C, \leq) are of the forms given in Lemma 5.9. •

Lemma 5.12 The cycle type realised by non-identity branch automorphisms of (Ω, C, \leq) is of the form $\aleph_0^{\aleph_0} 1^{\aleph_0}$.

5.3 Cycle types of the automorphism groups of (Ω, C)

Proof. Let g be a branch automorphism of (Ω, C, \leq) that induces a permutation on the branches at some node, say α of Λ . The permutation necessarily has to preserve the linear order on \mathbb{Q} . The cycle types realised by elements of $\text{Aut}(\mathbb{Q}, \leq)$ are of the forms $\aleph_0^{\aleph_0} 1^{a_1}$, $0 \leq a_1 \leq \aleph_0$ (refer Theorem 5.11). Each fixed branch at the node α (if there exists any) will give \aleph_0 fixed points in Ω . This means that the cycle type of g is $\aleph_0^{\aleph_0} 1^{\aleph_0} 1^{\aleph_0}$ where the last \aleph_0 fixed points arise from the nodes smaller than and incomparable to α fixed by g . Therefore the cycle type realised by g is $\aleph_0^{\aleph_0} 1^{\aleph_0}$. •

We now look at some other cycle types realised by elements of $\text{Aut}(\Omega, C, \leq)$

Lemma 5.13 The cycle types $\aleph_0^{\aleph_0} 1^k$, $1 < k < \aleph_0$ are realised in $\text{Aut}(\Omega, C, \leq)$.

Proof. For some $n \in \mathbb{Q}$, consider the node n in the \mathbb{Q}_1 -branch and consider a chain automorphism, say g_1 that induces an order-preserving permutation on the \mathbb{Q}_1 -branch that has no fixed points on the segment $(-\infty, n)$ and fixes all points on $[n, \infty)$. The cycle type of g_1 is then $\aleph_0^{\aleph_0} 1^{\aleph_0}$. Consider the branch automorphism, say g_2 that induces a permutation on the branches at the node n that fixes just the p -th and q -th branches, $p \neq q$ and has no other fixed branch. Then g_2 fixes np , nq . Any other element of Ω of the form nr , $r \neq q$, $r \neq p$ lie on an infinite cycle so that the cycle type of g_2 is $\aleph_0^{\aleph_0} 1^{\aleph_0}$. We now consider chain automorphisms g_3 and g_4 that induces order-preserving permutations on the p -th and q -th branches at the node n respectively so that all nodes on the respective branches are moved. Then g_3 has the cycle type $\aleph_0^{\aleph_0} 1^{\aleph_0}$, the fixed points being chains not passing through n and those chains on branches different from the p -th branch at n and most importantly, the chain np . Similarly the cycle type of g_4 is $\aleph_0^{\aleph_0} 1^{\aleph_0}$. Now the composition map $g_1 g_2 g_3 g_4$ has just the fixed points np , nq and any other chain lie on an infinite cycle. Therefore the cycle type of $g_1 g_2 g_3 g_4$ is $\aleph_0^{\aleph_0} 1^2$. Similarly we can produce an automorphism with cycle type $\aleph_0^{\aleph_0} 1^k$, $k > 2$. •

Lemma 5.14 The cycle type $\aleph_0^{\aleph_0}$ is realised in $\text{Aut}(\Omega, C, \leq)$.

5.3 Cycle types of the automorphism groups of (Ω, C)

Proof. Let g_1 be a chain automorphism as defined in Lemma 5.13, with notation as in the lemma. And let h_1 be a branch automorphism that induces a permutation on the branches at the node n that has no fixed points. Then h_1 has cycle type $\aleph_0^{\aleph_0} 1^{\aleph_0}$, the fixed points being chains not passing through n . Now the composition $g_1 h_1$ has no fixed point and each element of Ω lies in an infinite cycle under the action of $g_1 h_1$. Therefore the cycle type realised by the composition $g_1 h_1$ is $\aleph_0^{\aleph_0}$. •

Note that $\text{Aut}(\Omega, C, \leq)$ preserves a linear order and so finite cycles cannot be realised by elements of $\text{Aut}(\Omega, C, \leq)$. This means that elements of $\text{Aut}(\Omega, C, \leq)$ can only realise infinite cycles or else fix points of Ω . Also it is trivial to check that a finite number of infinite cycles cannot be realised by elements in $\text{Aut}(\Omega, C, \leq)$. Therefore the cycle types realised by elements in $\text{Aut}(\Omega, C, \leq)$ are just the types illustrated in Lemmas 5.11 - 5.14.

Theorem 5.12 *The cycle types realised by non-identity elements of $\text{Aut}(\Omega, C, \leq)$ are of the forms $\aleph_0^{\aleph_0} 1^k$, $0 \leq k \leq \aleph_0$. •*

We now look at the cycle types realised by non-identity elements of the group $\text{Aut}(\Omega, C, \leq, V)$. Since $\text{Aut}(\Omega, C, \leq, V)$ is a subgroup of $\text{Aut}(\Omega, C, \leq)$ therefore cycle types realised in $\text{Aut}(\Omega, C, \leq, V)$ have to be those from the list of $\text{Aut}(\Omega, C, \leq)$. It is easily verified that all the cycle types of $\text{Aut}(\Omega, C, \leq)$ are realised in $\text{Aut}(\Omega, C, \leq, V)$. Hence we have the following.

Theorem 5.13 *The cycle types realised by non-identity elements of the automorphism group $\text{Aut}(\Omega, C, \leq, V)$ are of the forms $\aleph_0^{\aleph_0} 1^k$, $0 \leq k \leq \aleph_0$. •*

We next look at the cycle types realised by elements in the automorphism group $\text{Aut}(\Omega, C, \leq, V, L)$. The cycle types realised by chain automorphisms and branch automorphisms of $\text{Aut}(\Omega, C, \leq, V, L)$ are same as those realised by $\text{Aut}(\Omega, C, \leq)$ and hence we have the following.

Theorem 5.14 *The cycle types realised by non-identity elements of the automorphism group of (Ω, C, \leq, V, L) are of the forms $\aleph_0^{\aleph_0} 1^k$, $0 \leq k \leq \aleph_0$. •*

5.3 Cycle types of the automorphism groups of (Ω, C)

The cycle types realised by branch automorphisms of $\text{Aut}(\Omega, C, \leq, V, L, R)$ are same as those realised by $\text{Aut}(\Omega, C, \leq)$. The cycle types realised by chain automorphisms of $\text{Aut}(\Omega, C, \leq, V, L, R)$ is only of the form $\aleph_0^{\aleph_0} 1^1$. This is because if the level of a node on the \mathbb{Q}_1 -branch is fixed by the chain automorphism then the level of all nodes has to be fixed since chain automorphisms of $\text{Aut}(\Omega, C, \leq, V, L, R)$ cannot induce magnifications and non-integer translations on the \mathbb{Q}_1 -branch (refer Lemma 4.19 and Lemma 4.20). However when the chain automorphisms and branch automorphisms are composed then it can be easily verified that all the cycle types of $\text{Aut}(\Omega, C, \leq)$ are realised by elements of $\text{Aut}(\Omega, C, \leq, V, L, R)$.

Theorem 5.15 *The cycle types realised by non-identity elements of the automorphism group $\text{Aut}(\Omega, C, \leq, V, L, R)$ are of the forms $\aleph_0^{\aleph_0} 1^k$, $0 \leq k \leq \aleph_0$. •*

Finally we look at the cycle type(s) realised by the non-identity elements of the minimal Jordan group G_0 as defined in Section 4.6. The chain automorphisms of G_0 induce identity order-preserving permutations on \mathbb{Q}_k -branches for each $k \in \mathbb{N}$ and hence there is no non-identity chain automorphism in the group G_0 . The cycle type(s) realised by elements of G_0 are therefore the cycle type(s) realised by branch automorphisms of G_0 . Now the cycle type realised by non-identity branch automorphisms of G_0 is $\aleph_0^{\aleph_0} 1^{\aleph_0}$. Therefore we have the following.

Theorem 5.16 *The cycle type realised by non-identity elements of the group G_0 is of the form $\aleph_0^{\aleph_0} 1^{\aleph_0}$. •*



Chapter 6

Expansions of D -sets

In Chapter 3 and Chapter 4 we constructed a C -set, imposed extra relations on it and investigated whether the Jordan property of the automorphism group was preserved. In this chapter we carry out a parallel study on a D -set. We first give a construction of a D -set and define extra relations on the D -set. We then study the existence of Jordan sets. We also show that the automorphism group of the D -set has uncountable cofinality.

6.1 Construction

The set on which the C -relation is imposed has \aleph_0 branches at each node of the tree, the branches of which are enumerated by \mathbb{Q} . Also the maximal chains of the C -set are isomorphic to \mathbb{Q} . To construct a D -set we glue suitably defined C -sets on an arbitrarily chosen point 0 as follows. Consider a set A , which consists of the rational numbers modulo 2. That is, A consists of the rational numbers in the interval $[0, 2)$. Let for any $a \in A$, Ω_a be defined as

$$\{0a n_1 q_1 \dots n_k q_k \mid 0 < n_i < n_{i+1} \in \mathbb{Q}, 1 \leq i \leq k-1, q_j \in A \setminus \{(a+1) \bmod 2\}, 1 \leq j \leq k\}.$$

The set Ω_a is a tree which has \aleph_0 -branches at each node such that the branches are indexed by rational numbers in the interval $\{[0, 2) \setminus \{(a+1) \bmod 2\}\}$. Some

branches of Ω_a for $a = 1$ is shown in Fig 6.1(a). We obtain our D -set by gluing the sets Ω_a , $a \in A$ on the chosen point 0 such that at the node 0 (and at all other nodes of the tree) the branches are enumerated by the rational numbers in the interval $[0, 2)$ which are arranged in a circular order (refer Section 6.6) around the point 0 (Fig 6.1(b)). Let Γ denote the set of all internal nodes of the

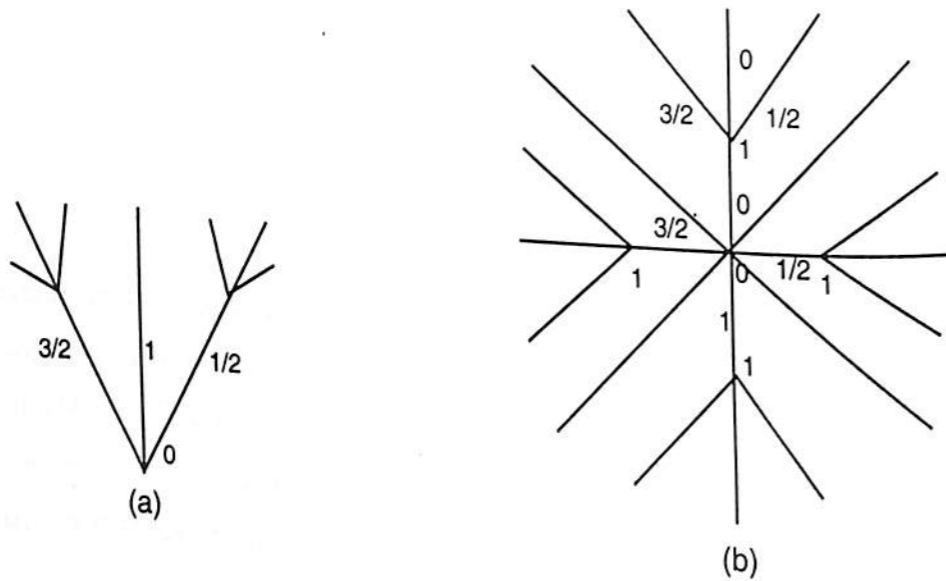


Figure 6.1:

structure that we have constructed and Ψ denote the set of all ends or directions of Γ . The set Γ is a connected tree with \aleph_0 branches at each node of Γ , branches indexed by elements of the set A such that the indices of the branches increases in a clockwise direction. The set Γ can be expressed as

$$\{0\} \cup \bigcup_{a \in A} \{0a n_1 q_1 \dots n_k | 0 < n_i < n_{i+1}, 1 \leq i \leq k-1, q_j \in A \setminus \{(a+1) \bmod 2\}, 1 \leq j \leq k\}$$

and the set Ψ can be expressed as

$$\bigcup_{a \in A} \{0a n_1 q_1 \dots n_k q_k | 0 < n_i < n_{i+1}, 1 \leq i \leq k-1, q_j \in A \setminus \{(a+1) \bmod 2\}, 1 \leq j \leq k\}$$

We call 0 the point of reference, $n_0, n_1, n_2, \dots, n_k$ the node co-ordinates and q_1, q_2, \dots, q_k the branch co-ordinates of the expressions of elements Γ and Ψ . For $a \in A$ the set Ω_a is countable and so the set Ψ is countable.

6.2 Some definitions and relations on Γ and Ψ

We call elements of Γ as *nodes* and elements of Ψ as *directions*. Let α be a node with expression $0q_0 n_1 q_1 n_2 q_2 \dots n_k$. We define the *segment* of α from 0 in Γ denoted by $[0, \alpha]$ to be the set of all nodes that lie on the path of α from 0 including both α and 0 (Fig 6.2(a)). The segment $[0, \alpha]$ can be expressed as

$$\{0\} \cup \{0q_0 n | 0 < n \leq n_1\} \cup \{0q_0 n_1 q_1 n | n_1 < n \leq n_2\} \cup \dots \cup \{0q_0 n_1 q_1 n_2 q_2 \dots n_{k-1} q_{k-1} n | n_{k-1} < n \leq n_k\}.$$

For any $\alpha \in \Gamma \setminus \{0\}$ the segment $[0, \alpha]$ carries a linear order. These linearly

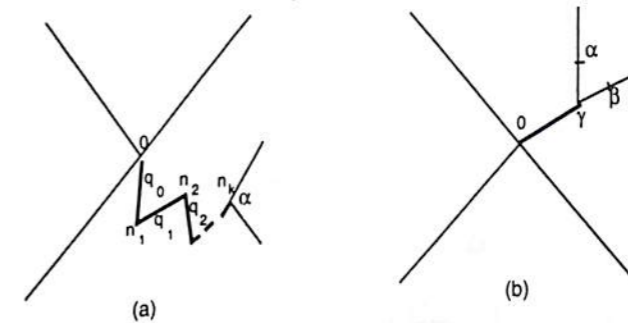


Figure 6.2:

ordered segments in Γ induce a linear betweenness relation on Γ which will be specified shortly. For $\alpha, \beta \in \Gamma \setminus \{0\}$ we define the *common segment* of α and β denoted by $\alpha \wedge \beta$ to be the longest segment $[0, \gamma]$ such that $[0, \gamma]$ occurs in both $[0, \alpha]$ and $[0, \beta]$ (Fig 6.2(b)). Note that $\alpha \wedge \beta$ can be just $\{0\}$. Next, we define the *segment* determined by α and β denoted by $[\alpha, \beta]$ to be the set of all nodes that lie on the path from α to β (or from β to α). Similarly for a direction $x = 0q_0 n_1 q_1 n_2 q_2 \dots n_k q_k$ we define the *half-line* $[0, x)$ to be the set of all nodes that lie on the path of x from 0. The half-line $[0, x)$ can be expressed as

$$[0, \alpha_0] \cup \{0q_0 n_1 q_1 n_2 q_2 \dots n_k q_k n | n_k < n\}$$

where $\alpha_0 = 0q_0 n_1 q_1 n_2 q_2 \dots n_k$. Likewise we can define a *half-line* $[\alpha, x)$ for any $\alpha \neq 0 \in \Gamma$. Figure 6.3 gives the picture of two half-lines. For a half-line $[\alpha, x)$, we



call α the node on the closed end of the half-line. Sometimes we do not include the point α in the half-line and call it an *open half-line*. We can also define the *common segment* of a node α and a direction x denoted by $\alpha \wedge x$ just as we defined the common segment of two nodes and likewise we can define the *common segment* of two directions x and y . A *line* which is comprised of two directions

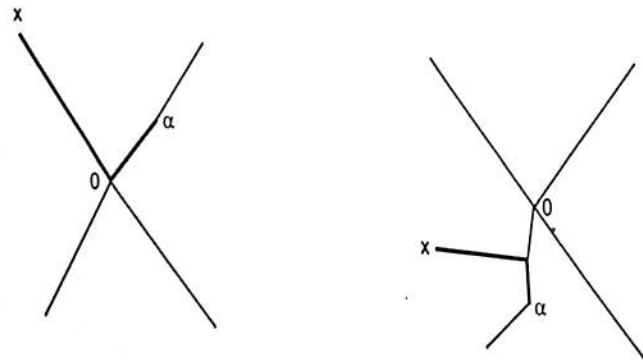


Figure 6.3:

x, y denoted by (x, y) (Fig. 6.4(a)) is defined as

$$\{[0, x] \cup [0, y] \setminus (x \wedge y)\} \cup \{\gamma\},$$

where γ is given by $x \wedge y = [0, \gamma]$. If $x = y$ then $(x, y) = \emptyset$, the empty set. Note that a line is a subset of Γ . We next introduce the term centroid with respect

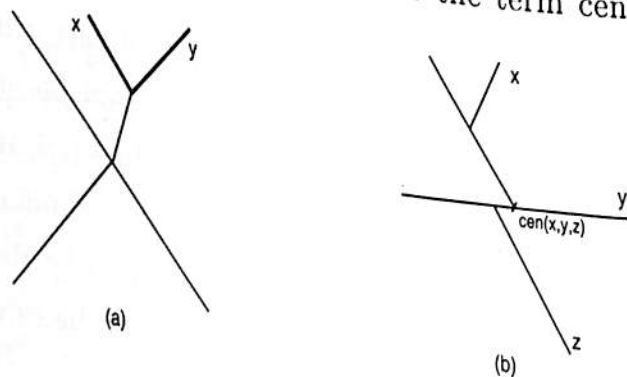


Figure 6.4:

to elements in Ψ . Adeleke and Neumann [AN2][Section 15] has used this term

with respect to elements in Γ and used the term structural partition for elements in Ψ . For three distinct elements x, y, z in Ψ , the centroid of x, y, z denoted by $cen(x, y, z)$ is the node in Γ that lies on each of the lines (x, y) , (y, z) and (z, x) . The pictorial representation of centroid is given in Fig. 6.4(b). We can also define the centroid of three distinct elements in Γ in a similar way. The following lemma gives a result of the centroid of elements in Ψ (refer [AN2][Theorem 24.2] for an idea of the proof).

Lemma 6.1 [Adeleke & Neumann] *The centroid of any three distinct elements of Ψ is unique. Moreover each of the three elements lie on distinct branches of the centroid.* •

6.2.1 A ternary relation B on Γ

For three elements α, β, γ of Γ a ternary relation B is defined as

$$B(\alpha; \beta, \gamma) :\leftrightarrow \alpha \in [\beta, \gamma] \text{ (Fig 6.5(a)).}$$

If either $\alpha = \beta$ or $\alpha = \gamma$ then $B(\alpha; \beta, \gamma)$. It can be checked that the relation B defined here satisfies the axioms of a B -relation (Definition 2.15).

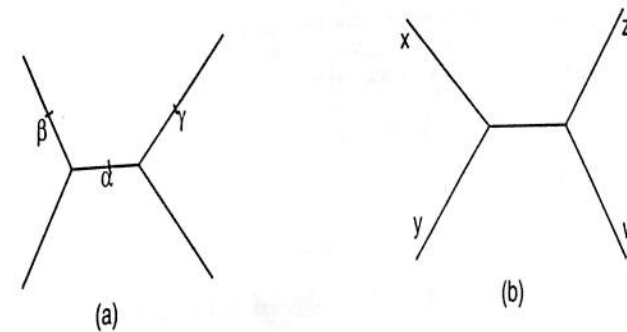


Figure 6.5:



6.3 The D -set Ψ

6.2.2 A quaternary relation D on Ψ

Let x, y, z, w be four elements of Ψ . We say that

$$D(x, y; z, w) :\leftrightarrow (x, y) \cap (z, w) = \emptyset, \text{ (Fig 6.5(b))}$$

where (x, y) denotes the line determined by x and y . If $x = y$ or $z = w$ then $D(x, y; z, w)$ holds. Here again, the relation D satisfies the axioms of a D -relation (Definition 2.16). In terms of centroids,

$$D(x, y; z, w) :\leftrightarrow (cen(x, y, z) = cen(x, y, w)) \neq (cen(x, z, w) = cen(y, z, w)).$$

6.3 The D -set Ψ

We take a quick look at the theory of B -relations and D -relations to make our presentation easier. A detailed theory is presented by Adeleke & Neumann [AN2][Part IV and Part V].

For any three elements α, β, γ of the B -set Γ we use the notation $L(\alpha, \beta, \gamma)$ to denote $(B(\alpha; \beta, \gamma) \vee B(\beta; \alpha, \gamma) \vee B(\gamma; \alpha, \beta))$; we then say that α, β, γ are *collinear*. A subset Σ of Γ is called a *linear set* if α, β, γ are collinear for all α, β, γ in Σ . If Σ is a linear subset of Γ then the restriction of B on Σ is a linear betweenness relation. A linear set Σ will be said to be *bounded* if there exist α, β such that $\Sigma \subseteq [\alpha, \beta]$. There are precisely two linear order relations \leq on Σ which are compatible with B in the sense that $(\beta \leq \alpha \leq \gamma) \rightarrow B(\alpha; \beta, \gamma)$; moreover each is the reverse of the other. If $\alpha \neq \beta$ then the two linear orderings of $[\alpha, \beta]$ can be distinguished by whether $\alpha < \beta$ or $\beta < \alpha$, and when convenient this may be used to distinguish the two linear orderings of a bounded linear set Σ . There is a close connection between B -sets and semilinear orderings. There are many ways to get a semilinear ordering from a B -set, but in the other direction there is a natural way to define a B -relation on a semilinearly ordered set which we define as $B(\alpha; \beta, \gamma)$ to hold precisely when

$$((\alpha \leq \beta) \wedge \neg(\alpha \leq \gamma)) \vee (\neg(\alpha \leq \beta) \wedge (\alpha \leq \gamma)) \vee (\alpha = glb\{\beta, \gamma\}),$$

6.3 The D -set Ψ

where $glb\{\beta, \gamma\}$ denote the greatest lower bound of β and γ with respect to semilinear order. The glb of two elements will exist if the semilinear order under consideration is of positive type (refer Section 3.2).

Let Σ be a subset of Γ . We will say that Σ is a *component* of Γ if the following two conditions are satisfied:

- (i) both Σ and $\Gamma \setminus \Sigma$ are non-empty and convex;
- (ii) if $\alpha \notin \Sigma, \beta_1, \beta_2 \in \Sigma$ then $||[\alpha, \beta_1] \cap [\alpha, \beta_2] \cap \Sigma| > 1$.

The components of B -sets play a similar role to cones in semilinear orderings. Each component is a semilinearly ordered set. We now pass on to the theory of D -relations. The following lemma (which is Theorem 22.1 of [AN2] with slight notational changes) shows how a C -set can be obtained from a D -set.

Lemma 6.2 [Adeleke & Neumann] Let (Ψ, D) be a D -set, x_0 be any element of Ψ and let $\Omega := \Psi \setminus \{x_0\}$. If C is the ternary relation on Ω such that $C(x; y, z)$ if and only if $D(x_0, x; y, z)$ holds in Ψ then (Ω, C) is a C -set. Moreover,

- (i) if Ψ is a proper D -set then Ω is a proper C -set.
- (ii) if Ψ is dense as D -set then Ω is dense as C -set. •

We now state how to create a D -set from a C -set [AN2][Theorem 23.4].

Lemma 6.3 [Adeleke & Neumann] Let (Ω, C) be a C -set. Define $\Psi := \Omega \cup \{\infty\}$, where $\infty \notin \Omega$, and define a quaternary relation D in Ψ by the rule $D(x, y; z, w)$ if trivially $(x = y, x \neq z, x \neq w)$ or $(z = w, x \neq z, y \neq z)$ or when x, y, z, w are distinct and at least one of the following holds:

- (i) $x = \infty \wedge C(y; z, w)$;
- (ii) $y = \infty \wedge C(x; z, w)$;
- (iii) $z = \infty \wedge C(w; x, y)$;
- (iv) $w = \infty \wedge C(z; x, y)$;
- (v) $C(x; z, w) \wedge C(y; z, w)$;
- (vi) $C(z; x, y) \wedge C(w; x, y)$.

Then (Ψ, D) is a D -set. If (Ω, C) is proper then (Ψ, D) is proper and if (Ω, C) is dense then (Ψ, D) is dense. •



6.4 Automorphism groups of (Γ, B) and (Ψ, D)

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A partition of a D -set Ψ as a disjoint union $\bigcup\{\Sigma \mid \Sigma \in S\}$ of non-empty subsets is associated with an equivalence relation λ . The partition, or the equivalence relation will be called a *structural partition* with sectors Σ if

- (i) $|S| \geq 3$
- (ii) $(\forall \Sigma \in S)(w_1, w_2 \in \Sigma \wedge w_3, w_4 \notin \Sigma \rightarrow D(w_1, w_2; w_3, w_4))$;
- (iii) w_1, w_2, w_3, w_4 distinct mod $\lambda \rightarrow N(w_1, w_2, w_3, w_4)$,

where $N(w_1, w_2, w_3, w_4)$ means w_1, w_2, w_3, w_4 are not D -related in any order.

6.4 Automorphism groups of (Γ, B) and (Ψ, D)

In this section we look at the group of automorphisms on Ψ that preserve the relation D . Let us set $G := \text{Aut}(\Psi, D)$ and let H denote the group $\text{Aut}(\Gamma, B)$. We show that the groups H and G are isomorphic.

Theorem 6.1 *The groups H and G are isomorphic.*

Proof. We first show that the group H is a subgroup of G . Let the action of H on Γ be $(\alpha, g) \mapsto \alpha^g$. Let x be an element of Ψ which has expression αq . We define a map from $\Psi \times H$ to Ψ as $(x, g) \mapsto x^g$, where $x^g = (\alpha q)^g = \alpha^g q^g$ with α^g as the image of α and the nodes on the q -th branch at α being mapped to the nodes on the q^g -th branch at α^g under the action of $g \in H$. This map is clearly induced by the action of H on Γ .

It is easy to check that the map defined satisfies the two conditions of an action, namely for the identity automorphism 1 of Γ and elements g, h of H , $x^1 = x$ and $(x^g)^h = x^{gh}$ holds. Therefore, H induces an action on Ψ .

Finally we show that g preserves the D -relation on Ψ . Before proving this we note that if $g \in H$ and α is any node of Γ then g maps elements on distinct branches at α to elements on distinct branches at α^g . Thus if $x = \alpha q, y = \alpha p, z = \alpha r, p \neq q \neq r \in A$, then $x^g = \alpha^g q^g, y^g = \alpha^g p^g, z^g = \alpha^g r^g, q^g \neq p^g \neq r^g$ so that $\text{cen}(x^g, y^g, z^g) = \alpha^g = (\text{cen}(x, y, z))^g$. Now let $D(x, y; z, w)$ so that $(x, y) \wedge (z, w) = \emptyset$. This implies for every $\alpha \in (x, y)$ we have $\alpha \notin (z, w)$. Because

6.4 Automorphism groups of (Γ, B) and (Ψ, D)

g preserves the B -relation so for all $\alpha^g \in (x^g, y^g)$ we have $\alpha^g \notin (z^g, w^g)$. Therefore the induced action of H on Ψ preserves the D -relation. Conversely we can show that G is a subgroup of H . •

We construct an automorphism of Ψ as follows. We choose a line say (x, y) and first define the images of the nodes on the line (x, y) such that the line is fixed setwise and the linear betweenness relation of the line is preserved. Next we choose a node say, α on the image of the line (x, y) and define the images of the enumeration of the branches at the node α . Each of the branches at α are segments of the form $[\alpha, \alpha q), q \in A$. In the next step we define the images of the nodes on the half lines $[\alpha, \alpha q), q \in A$ such that each half-line is fixed setwise and the linear betweenness relation is preserved with the node on the closed end of the half-line, namely α being fixed. In the successive step we define the images of the branches at nodes on the branches $[\alpha, \alpha q), q \in A$ such that the branch leading from the node (on which the images of the branches is defined) to the node α , namely the $(q + 1) \bmod z$ -th branch is fixed. We continue the constructions on segments of the form $[\alpha q \beta, \alpha q \beta p), [\alpha q \beta p \gamma, \alpha q \beta p \gamma r)$ and so on with usual meaning of the symbols $\beta, \gamma, \dots \in \Gamma, p, r, \dots \in A$. After a countable number of steps we obtain an automorphism of (Ψ, D) . Therefore to define an automorphism of Ψ we need three kinds of automorphisms specified as follows.

Definition 6.1 A *line automorphism* is an automorphism that induces a linear betweenness preserving permutation on some line (x, y) and fixes that line setwise.

Definition 6.2 A *half-line automorphism* is an automorphism that induces a linear order-preserving permutation on some half-line $[\alpha, \alpha q)$ (induced by the linear betweenness relation on Γ), fixes the node α , fixes nodes on all other half-lines of the form $[\alpha, \alpha p), p \neq q$.

Definition 6.3 A *branch automorphism* is an automorphism that fixes a node, induces a permutation on the branches at the fixed node and preserves the distance of the nodes from the fixed node.

Our interest is in $G = \text{Aut}(\Psi, D)$ as a permutation group on Ψ and the Jordan sets of G . We have seen in Lemma 6.2 that the point stabiliser G_{x_0} is a group of automorphisms of the C -set naturally induced on $\Psi \setminus \{x_0\}$ and so the knowledge of the automorphism group of C -sets can be used here. We look at some properties of $\text{Aut}(\Psi, D)$. Because the structure (Ψ, D) and (Γ, B) are inter-related we also look at related properties of $\text{Aut}(\Gamma, B)$. It is trivial to check that $\text{Aut}(\Gamma, B)$ is transitive and that $\text{Aut}(\Psi, D)$ is 3-transitive. We give brief sketches of the proofs to get an idea of the automorphisms of the respective groups.

Lemma 6.4 *The group $H = \text{Aut}(\Gamma, B)$ is transitive on Γ .*

Proof. Let α, β be two distinct elements of Γ . We need to show that there exists an element in H that maps α to β . Let $\gamma \in [\alpha, \beta]$, $\gamma \neq \alpha, \gamma \neq \beta$ and let α, β lie on the p -th, r -th branches at γ respectively. For $q_0 \in \mathbb{Q}$, $q_0 \neq p, q_0 \neq r$, the set $\Psi \setminus \{\gamma q_0\}$ forms a C -set with the C -relation defined as

$$C(x; y, z) :\leftrightarrow D(\gamma q_0, x; y, z).$$

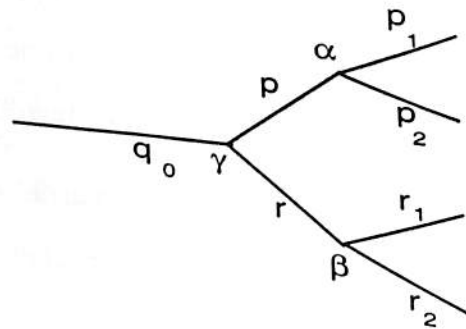


Figure 6.6:

Let p_1, p_2 be branches at α different from the branch leading α to γ and let r_1, r_2 be branches at β different from the branch leading β to γ (Fig. 6.6). Then $\alpha p_1 \cap \alpha p_2 = \alpha$ and $\beta r_1 \cap \beta r_2 = \beta$. By 2-transitivity of the C -set $\Psi \setminus \{\gamma q_0\}$ (refer Theorem 3.3) there exists $g \in \text{Aut}(\Psi \setminus \{\gamma q_0\}, C)$ that maps $(\alpha p_1, \alpha p_2)$ to $(\beta r_1, \beta r_2)$ which means g maps α to β (using Theorem 6.1). The automorphism

g can then be extended to Ψ and hence to Γ such that the node γ and all nodes on the q_0 -th branch at γ is fixed. •

Lemma 6.5 *The group $G = \text{Aut}(\Psi, D)$ is 3-transitive on Ψ .*

Proof. Let (x, y, z) and (u, v, w) be two 3-tuples of elements in Ψ . We need to show that there exists an element in G that maps (x, y, z) to (u, v, w) . Let $\text{cen}(x, y, z) = \alpha$ and $\text{cen}(u, v, w) = \beta$. By Lemma 6.4 there exists $g \in \text{Aut}(\Gamma, B)$ such that $\alpha^g = \beta$. Let x^g, y^g, z^g and u, v, w be on the q_1, q_2, q_3 and p_1, p_2, p_3 branches at β respectively. Let g_1 be a branch automorphism of (Ψ, D) that induces a permutation of the branches at the node β such that the branches q_1, q_2, q_3 are mapped onto the branches p_1, p_2, p_3 (Fig. 6.7). Now x^{gg_1} and u are

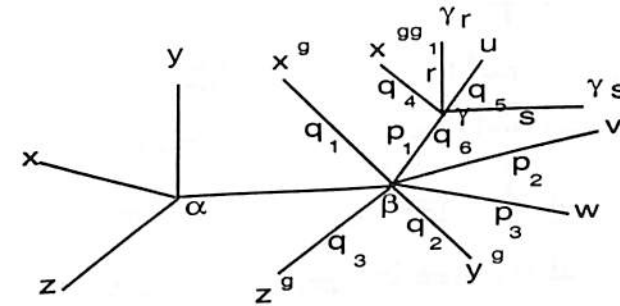


Figure 6.7:

on the p_1 -th branch at β . Let $\text{cen}(x^{gg_1}, u, v) = \gamma$ and let x^{gg_1}, u, v be on the q_4, q_5, q_6 branches at γ respectively. Now consider the set Ω_{p_1} (with notation as in Section 6.1) with the reference point as β instead of 0. Then Ω_{p_1} is clearly a C -set. Let $\gamma r, \gamma s$ be two elements of Ω_{p_1} such that each of r and s are different from both q_4 and $q_5 (= p_1)$ branches at γ , (since γr and γs are elements of Ω_{p_1} so both r and s are different from $q_6 (= (p_1 + 1)_{\text{mod} 2})$). By 2-transitivity of the automorphism group of Ω_{p_1} (Theorem 3.3) there exists an automorphism say g_2 that maps $(\gamma r, x^{gg_1})$ to $(\gamma s, u)$. This automorphism fixes γ and all nodes smaller than γ in the C -set. Moreover we specify that g_2 fixes all nodes incomparable to γ on the C -set. We then extend this automorphism to Ψ by defining g_2 to act

trivially on $\Psi \setminus \Omega_{p_1}$. This will ensure that y^{gg_1} and z^{gg_1} are fixed. Similarly we get automorphisms g_3, g_4 that maps y^{gg_1}, z^{gg_1} to v, w respectively. Then $gg_1g_2g_3g_4$ is the required automorphism in (Ψ, D) that maps (x, y, z) to (u, v, w) . •

The proof of Lemma 6.5 gives us a method of constructing automorphisms in (Ψ, D) that maps a 3-tuple to another 3-tuple. There are many automorphisms that do this. For example in constructing the branch automorphisms at the node β we just need to map the branches leading to x^g, y^g, z^g onto the branches leading to u, v, w respectively. We have lots of choices on the remaining branches on the domain set. And this is true of all the other automorphisms that we construct. When we impose some extra relation on (Ψ, D) the automorphism group of the new structure will be contained in the automorphism group of (Ψ, D) . Hence our knowledge of $\text{Aut}(\Psi, D)$ will be helpful to the study of the automorphism groups of the new structures. In the remaining part of this section we show that $\text{Aut}(\Gamma, B)$ is 2-transitive and that $\text{Aut}(\Psi, D)$ is relatively 4-transitive.

Lemma 6.6 *The group $H = \text{Aut}(\Gamma, B)$ is 2-transitive on Γ .*

Proof. Let (α, β) and (γ, δ) be two pairs of elements of Γ . We need to find an automorphism in (Γ, B) that maps α, β to γ, δ respectively. Let h be an automorphism in H that maps α to γ (refer Lemma 6.4). We then construct an automorphism that maps β^h to δ and fixes $\alpha^h = \gamma$. Now there are two cases. The first being that β^h and δ lie on the same branch at γ (Fig. 6.8(a)) and the second being that β^h and δ lie on different branches at γ (Fig. 6.8(b)). We treat the two cases separately. Suppose β^h and δ are on the same branch at γ , which is say, the q -th branch. Consider the set Σ of all directions that lie on the q -th branch at γ . The set Σ forms a C -set with the C -relation defined as $C(x; y, z) :\leftrightarrow D(\gamma p, x; y, z)$, where $p \neq q, x, y, z \in \Sigma$. This in turn means that the nodes on the set Σ (which excludes the node γ) carries a semilinear order. Therefore there exists an automorphism, say f on the set of nodes of Σ that preserves the semilinear order and which maps β^h to δ (by transitivity of the automorphism group on the semilinearly ordered set under our consideration,

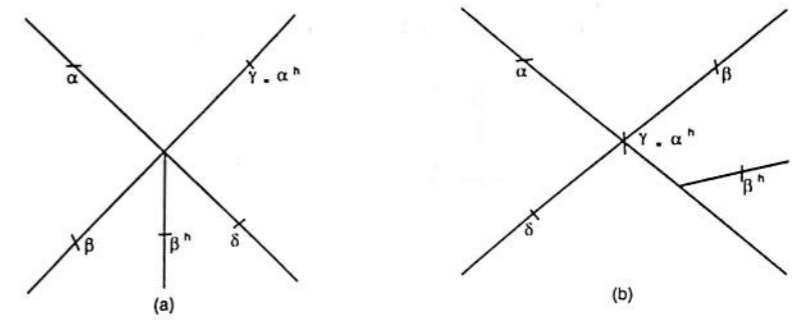


Figure 6.8:

refer Lemma 3.1). Let g be the extension of f on the nodes of Ψ that induces f on the nodes of Σ and which fixes the nodes on $\Psi \setminus \Sigma$ pointwise. Then the composition hg is the required automorphism that maps (α, β) to (γ, δ) .

Next, suppose the branches leading to β^h and δ from γ are different. We define a branch automorphism say g_1 on the node γ that maps the branch of β^h to the branch of δ so that β^{hg_1} and δ are as in the previous case. Let g_2 be a map similar to g as defined in the previous case. Then the composition hg_1g_2 is the required automorphism that maps α, β to γ, δ respectively. •

Theorem 6.2 *The group $G = \text{Aut}(\Psi, D)$ is relatively 4-transitive on Ψ .*

Proof. Let (x, y, z, w) and (u, v, a, b) be two sets of 4-tuples of elements in Ψ such that $D(x, y; z, w)$ and $D(u, v; a, b)$ holds. We need to show that there exists an automorphism in G that maps (x, y, z, w) to (u, v, a, b) . Let $\text{cen}(x, y, z) = \alpha, \text{cen}(x, z, w) = \beta, \text{cen}(u, v, a) = \gamma, \text{cen}(u, a, b) = \delta$. By 2-transitivity of $\text{Aut}(\Gamma, B)$ (Lemma 6.6) there exists an automorphism, say h that maps α, β to γ, δ respectively so that x, y, z, w are mapped to x^h, y^h, z^h, w^h respectively. We now have $\text{cen}(x^h, y^h, z^h) = \gamma$ and $\text{cen}(x^h, z^h, w^h) = \delta$. Let $\eta \in [\gamma, \delta], \eta \neq \gamma, \eta \neq \delta$ and let γ, δ lie on the q -th, p -th branches at η respectively (Fig. 6.9). Consider the subsets Σ_1 and Σ_2 of Ψ defined as follows. The set Σ_1 consists of all elements of Ψ that lie on the q -th branch at η and Σ_2 consists of all elements that lie on the p -th branch at η . Now on each of the sets Σ_1 and Σ_2 , we can define a C -relation

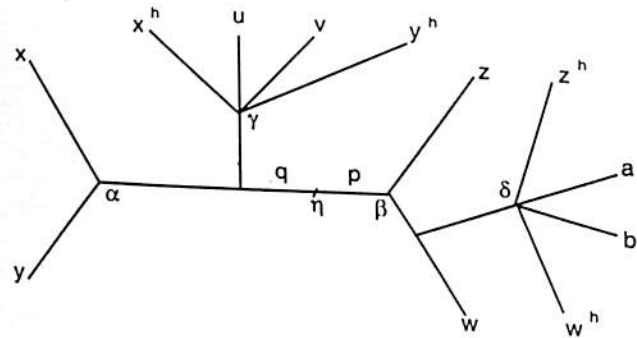


Figure 6.9:

induced by the D -relation on Ψ . By 2-transitivity of the automorphism group of the C -set Σ_1 (refer Theorem 3.3) there exists an element say g_1 on Σ_1 that maps x^h, y^h to u, v respectively which can be extended to an automorphism say f_1 on Ψ that induces g_1 on Σ_1 and fixes $\Psi \setminus \Sigma_1$ pointwise and hence fixes z^h, w^h . Similarly there exists an element say, f_2 on Ψ that maps z^h, w^h to a, b respectively and fixes $\Psi \setminus \Sigma_2$ pointwise. Then the composition hf_1f_2 is the required automorphism that maps (x, y, z, w) to (u, v, a, b) . •

6.5 Jordan sets of $\text{Aut}(\Psi, D)$

We now look at the Jordan sets of $\text{Aut}(\Psi, D)$. A complete list of proper Jordan sets for the automorphism group of a proper D -set has been given in [AN2][Section 28]. Jordan sets of $\text{Aut}(\Psi, D)$ has been shown to be of three types. In our terminology the proper Jordan sets of $\text{Aut}(\Psi, D)$ can be described as sets of the form Σ where Σ is any of the following three types.

Type I For any $\alpha \in \Gamma$ and $q_0 \in A$,
 $\Sigma = \{\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A \setminus \{(q_0 + 1) \bmod 2\}, 1 \leq i \leq k\}$.

Type II For any $\alpha \in \Gamma$ and A_0 a proper non-empty subset of A ,
 $\Sigma = \{\alpha p_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A, 1 \leq i \leq k, p_0 \in A_0\}$.

Type III For any line (x_0, y_0) in Γ and Δ be an open convex segment of (x_0, y_0)

in Γ , we have $\Sigma = \{z \in \Psi \mid \text{cen}(x_0, y_0, z) \in \Delta\}$.

Note that if A_0 as defined in type II is a singleton set then a set of type II is reduced to a set of type I.

Lemma 6.7 Any Jordan set of type I is 2-transitive.

Proof. For $\alpha \in \Gamma$ and $q_0 \in A$, let

$$\Sigma = \{\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A \setminus \{(q_0 + 1) \bmod 2\}, 1 \leq i \leq k\}$$

be a Jordan set of type I. Let H be the Jordan group associated with Σ . We show that H is 2-transitive on Σ . Let $(x, y), (z, w)$ be two pairs of elements of Σ . We define an element $g \in H$ that maps (x, y) to (z, w) . Let p_0 be an element of A such that $p_0 \neq q_0$. Let $\beta = \text{cen}(x, y, \alpha p_0)$, $\gamma = \text{cen}(z, w, \alpha p_0)$ (Fig. 6.10) and let g fix elements of the form $\alpha p m_1 p_1 m_2 p_2 \dots m_j p_j, p \neq q_0$. That is, g fixes elements which lie on branches at α different from the q_0 -th branch. Therefore g must also fix α . We now define the images of the elements which lie on the

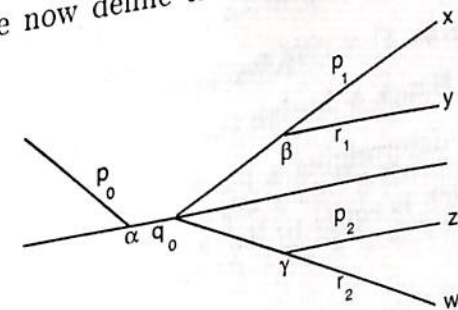


Figure 6.10:

q_0 -th branch at α . Let x, y be on the p_1 -th, r_1 -th branches at β and z, w be on the p_2 -th, r_2 -th branches at γ respectively. We define g on the q_0 -th branch at α in such a way that the segment $[\alpha, \beta]$ is mapped to the segment $[\alpha, \gamma]$ with the B -relation being preserved and the p_1 -th, r_1 -th branches at β being mapped to the p_2 -th, r_2 -th branches at γ respectively. Also the half-lines $[\beta, x), [\beta, y)$ are

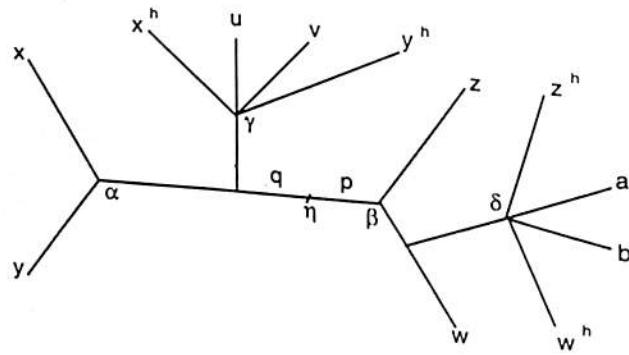


Figure 6.9:

induced by the D -relation on Ψ . By 2-transitivity of the automorphism group of the C -set Σ_1 (refer Theorem 3.3) there exists an element say g_1 on Σ_1 that maps x^h, y^h to u, v respectively which can be extended to an automorphism say f_1 on Ψ that induces g_1 on Σ_1 and fixes $\Psi \setminus \Sigma_1$ pointwise and hence fixes z^h, w^h . Similarly there exists an element say, f_2 on Ψ that maps z^h, w^h to a, b respectively and fixes $\Psi \setminus \Sigma_2$ pointwise. Then the composition hf_1f_2 is the required automorphism that maps (x, y, z, w) to (u, v, a, b) . •

6.5 Jordan sets of $\text{Aut}(\Psi, D)$

We now look at the Jordan sets of $\text{Aut}(\Psi, D)$. A complete list of proper Jordan sets for the automorphism group of a proper D -set has been given in [AN2][Section 28]. Jordan sets of $\text{Aut}(\Psi, D)$ has been shown to be of three types. In our terminology the proper Jordan sets of $\text{Aut}(\Psi, D)$ can be described as sets of the form Σ where Σ is any of the following three types.

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 $\Sigma = \{\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A \setminus \{(q_0 + 1) \bmod 2\}, 1 \leq i \leq k\}$.

Type II For any $\alpha \in \Gamma$ and A_0 a proper non-empty subset of A ,
 $\Sigma = \{\alpha p_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A, 1 \leq i \leq k, p_0 \in A_0\}$.

Type III For any line (x_0, y_0) in Γ and Δ be an open convex segment of (x_0, y_0)

in Γ , we have $\Sigma = \{z \in \Psi \mid cen(x_0, y_0, z) \in \Delta\}$.

Note that if A_0 as defined in type II is a singleton set then a set of type II is reduced to a set of type I.

Lemma 6.7 Any Jordan set of type I is 2-transitive.

Proof. For $\alpha \in \Gamma$ and $q_0 \in A$, let

$$\Sigma = \{\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A \setminus \{(q_0 + 1) \bmod 2\}, 1 \leq i \leq k\}$$

be a Jordan set of type I. Let H be the Jordan group associated with Σ . We show that H is 2-transitive on Σ . Let $(x, y), (z, w)$ be two pairs of elements of Σ . We define an element $g \in H$ that maps (x, y) to (z, w) . Let p_0 be an element of A such that $p_0 \neq q_0$. Let $\beta = cen(x, y, \alpha p_0)$, $\gamma = cen(z, w, \alpha p_0)$ (Fig. 6.10) and let g fix elements of the form $\alpha p m_1 p_1 m_2 p_2 \dots m_j p_j, p \neq q_0$. That is, g fixes elements which lie on branches at α different from the q_0 -th branch. Therefore g must also fix α . We now define the images of the elements which lie on the

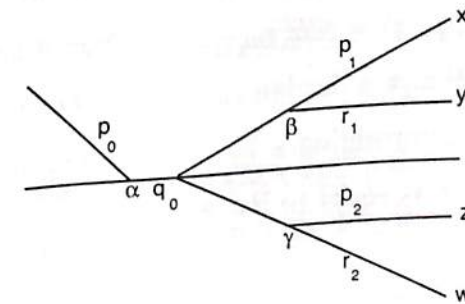


Figure 6.10:

q_0 -th branch at α . Let x, y be on the p_1 -th, r_1 -th branches at β and z, w be on the p_2 -th, r_2 -th branches at γ respectively. We define g on the q_0 -th branch at α in such a way that the segment $[\alpha, \beta]$ is mapped to the segment $[\alpha, \gamma]$ with the B -relation being preserved and the p_1 -th, r_1 -th branches at β being mapped to the p_2 -th, r_2 -th branches at γ respectively. Also the half-lines $[\beta, x), [\beta, y)$ are the p_1 -th, r_1 -th branches at β respectively.

6.6 A circular order on (Ψ, D)

circular order K and the separation relation S on A induces a circular order and a separation relation on Ψ respectively. We use the same notation used in A for the respective relations in Ψ which are defined as follows. For any three elements x, y, z of Ψ , let $cen(x, y, z)$ be the centroid of x, y, z and let x, y, z lie on the q -th, p -th and r -th branches at the node $cen(x, y, z)$ respectively. The circular order K on Ψ is defined as

$$K(x, y, z) : \leftrightarrow K(q, p, r) \text{ (refer Fig. 6.11),}$$

where $K(q, p, r)$ denotes the circular order on A . The circular order is said to

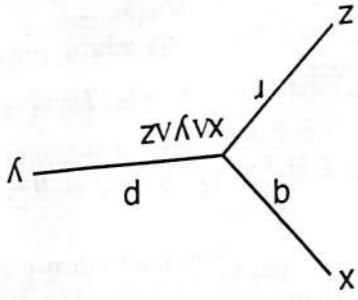


Figure 6.11:

be dense if, for any two distinct elements x and y there exists another element z with $K(x, y, z)$. The separation relation S on Ψ is defined as

$$S(x, y; z, w) : \leftrightarrow (K(x, y, z) \wedge K(x, z, y)) \vee (K(x, w, y) \wedge K(x, y, w)).$$

6.6.1 Automorphism group of (Ψ, D, K)

In this section we look at the automorphism group of (Ψ, D, K) and correspondingly the automorphism group of (Ψ, D, S) . Let G_K denote the subgroup of $G = \text{Aut}(\Psi, D)$ that preserve the circular order K on Ψ and H_K denote the subgroup of $H = \text{Aut}(\Gamma, B)$ that preserve the circular order of A on branches at nodes of Γ . Then as a consequence of Theorem 6.1 we have the following.

6.6 A circular order on (Ψ, D)

mapped to $[\gamma, z], [\gamma, w]$ respectively with the B -relation being preserved so that g maps the ordered pair (x, y) to the ordered pair (z, w) .

Lemma 6.8 Any Jordan set of either type II or type III is imprimitive.

Proof. For $\alpha \in \Gamma$ and A_0 a proper subset of A , let

$$\Sigma_1 = \{ \alpha p_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A, 1 \leq i \leq k, p_0 \in A_0 \}$$

be a Jordan set of type II with H_1 as the associated Jordan group. We define a binary relation \sim on Σ_1 as follows. For two elements $u = \alpha q_1 n_1 q_2 \dots n_k q_k$ and $v = \alpha p_1 n_1 m_2 p_2 \dots m_j p_j$ we say $u \sim v : \leftrightarrow q = p$. Then it can be easily checked that \sim is a H_1 -congruence on Ψ . Hence a Jordan set of type II is imprimitive with blocks consisting of elements in a single branch at α . The number of blocks is equal to the number of elements in A_0 .

Next, let (x_0, y_0) be a line in Γ and Δ be an open convex segment of (x_0, y_0) in Γ and let $\Sigma_2 = \{ z \in \Psi \mid cen(x_0, y_0, z) \in \Delta \}$ be a Jordan set of type III. Let H_2 be the associated Jordan group of Σ_2 . We define another binary relation ρ on Σ_2 as $zpw : \leftrightarrow cen(x_0, y_0, z) = cen(x_0, y_0, w)$. Then it can be checked that ρ is a H_2 -congruence on Ψ . Hence a Jordan set of type III is imprimitive with each node in the segment Δ determining a block. The number of blocks is equal to the cardinality of Δ which is equal to \aleph_0 .

6.6 A circular order on (Ψ, D)

The first relation that we impose on (Ψ, D) is a circular order. There is a natural ternary relation called a circular order K on $A = [0, 2)$ defined as

$$K(a, b, c) : \leftrightarrow (a \leq b \leq c) \vee (b \leq c \leq a) \vee (c \leq a \leq b),$$

where a, b, c are elements of A and \leq is the linear order of \mathbb{Q} restricted to A . The circular order K induces a quaternary relation called a separation relation S on A as $S(a, b, c, d) : \leftrightarrow (K(a, b, c) \wedge K(a, d, b)) \vee (K(a, c, b) \wedge K(a, b, d))$. The



6.6 A circular order on (Ψ, D)

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be a Jordan set of type II with H_1 as the associated Jordan group. We define a binary relation \sim on Σ_1 as follows. For two elements $u = \alpha q n_1 q_1 n_2 q_2 \dots n_k q_k$ and $v = \alpha p m_1 p_1 m_2 p_2 \dots m_j p_j$ we say $u \sim v : \Leftrightarrow q = p$. Then it can be easily checked that \sim is a H_1 -congruence on Ψ . Hence a Jordan set of type II is imprimitive with blocks consisting of elements in a single branch at α . The number of blocks is equal to the number of elements in A_0 .

Next, let (x_0, y_0) be a line in Γ and Δ be an open convex segment of (x_0, y_0) in Γ and let $\Sigma_2 = \{z \in \Psi \mid cen(x_0, y_0, z) \in \Delta\}$ be a Jordan set of type III. Let H_2 be the associated Jordan group of Σ_2 . We define another binary relation ρ on Σ_2 as $z \rho w : \Leftrightarrow cen(x_0, y_0, z) = cen(x_0, y_0, w)$. Then it can be checked that ρ is a H_2 -congruence on Ψ . Hence a Jordan set of type III is imprimitive with each node in the segment Δ determining a block. The number of blocks is equal to the cardinality of Δ which is equal to \aleph_0 . •

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6.6 A circular order on (Ψ, D)

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$$K(x, y, z) : \Leftrightarrow K(q, p, r) \text{ (refer Fig. 6.11),}$$

where $K(q, p, r)$ denotes the circular order on A . The circular order is said to

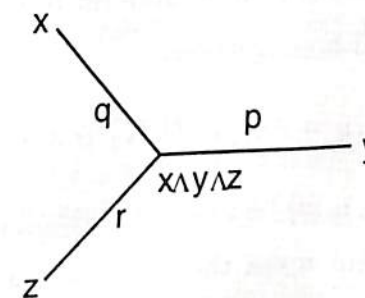


Figure 6.11:

be dense if, for any two distinct elements x and y there exists another element z with $K(x, y, z)$. The separation relation S on Ψ is defined as

$$S(x, y; z, w) : \Leftrightarrow (K(x, y, z) \wedge K(x, w, y)) \vee (K(x, z, y) \wedge K(x, y, w)).$$

6.6.1 Automorphism group of (Ψ, D, K)

In this section we look at the automorphism group of (Ψ, D, K) and correspondingly the automorphism group of (Ψ, D, S) . Let G_K denote the subgroup of $G = \text{Aut}(\Psi, D)$ that preserve the circular order K on Ψ and H_K denote the subgroup of $H = \text{Aut}(\Gamma, B)$ that preserve the circular order of A on branches at nodes of Γ . Then as a consequence of Theorem 6.1 we have the following.



Lemma 6.9 *The group H_K is isomorphic to G_K . •*

Analogously, if G_S denotes the subgroup of G that preserves the separation relation of Ψ and H_S denotes the subgroup of H that preserves the separation relation amongst the branches at nodes of Γ then we have the following.

Lemma 6.10 *The group H_S is isomorphic to G_S and $H_K < H_S$. •*

The importance of the preceding two lemmas is that to construct elements of G_K and G_S it is enough to define elements of H that preserve the circular order and the separation relation of the branches respectively at nodes of Γ .

The group G_K cannot be 3-transitive because K is a proper ternary relation on Ψ . We show that G_K is 3-homogeneous.

Theorem 6.3 *The group $G_K = \text{Aut}(\Psi, D, K)$ is 3-homogeneous.*

Proof. Let $\{x, y, z\}$ and $\{u, v, w\}$ be two sets of size 3 in Ψ . We need to construct an automorphism in G_K that maps the 3-set $\{x, y, z\}$ onto the 3-set $\{u, v, w\}$. This can be easily constructed as in Lemma 6.5 with the following additional conditions on the choice of mappings. On the branches at a node the mappings (in addition to being bijective) have to preserve the circular order and if any mapping has a fixed branch then it has to preserve the linear order on the remaining branches induced by the circular order. •

Theorem 6.4 *The group $G_S = \text{Aut}(\Psi, D, S)$ is 3-transitive.*

Proof. Let (x, y, z) and (u, v, w) be two 3-tuples in Ψ . Suppose (x, y, z) and (u, v, w) are K -related in the same order (by which we mean that if $K(x, y, z)$ holds then $K(u, v, w)$ holds and if $K(y, x, z)$ holds then $K(v, u, w)$ holds). Then there exists $g \in G_K$ by Theorem 6.3 such that $(x, y, z)^g = (u, v, w)$. Now $G_K < G_S$ and so $g \in G_S$. On the other hand if (x, y, z) and (u, v, w) are K -related in two different orders (say, for example $K(x, y, z)$ and $K(v, u, w)$) then because G_S can reverse the circular order in Ψ there still exists $g \in G_S$ such that $(x, y, z)^g = (u, v, w)$. Therefore G_S is 3-transitive. •

6.6.2 Point stabiliser of $\text{Aut}(\Psi, D, K)$

We have already seen in Lemma 6.2 that the D -relation on Ψ induces a C -relation on $\Psi \setminus \{x_0\}$, for any element $x_0 \in \Psi$. Again it is well-known that a circular order on any set induces a linear order on the same set with one point removed from it. This means that the point stabiliser of G_K will preserve a linear order on $(\Psi \setminus \{x_0\}, C)$. Also we know that a separation relation on a set induces a linear betweenness relation on the set with one point removed. This means that the point stabiliser of G_S will preserve a linear betweenness relation on $(\Psi \setminus \{x_0\}, C)$. Hence the following are trivial.

Lemma 6.11 *For any $x_0 \in \Psi$, the point stabiliser of x_0 in G_K preserves a linear order on $(\Psi \setminus \{x_0\}, C)$.*

Proof. For $x, y \in \Psi \setminus \{x_0\}, x \neq y$ we define a relation \leq as $x \leq y \Leftrightarrow K(x_0, x, y)$. The relation \leq can be checked to be a linear order. •

Lemma 6.12 *For any $x_0 \in \Psi$, the point stabiliser of x_0 in G_S preserves a linear betweenness relation on $(\Psi \setminus \{x_0\}, C)$.*

Proof. For any $x, y, z \in \Psi \setminus \{x_0\}$ we define $\rho(x; y, z) \Leftrightarrow S(x_0, x; y, z)$. Here again it can be checked that ρ is a linear betweenness relation on $\Psi \setminus \{x_0\}$. •

6.6.3 Jordan sets of $\text{Aut}(\Psi, D, K)$

Jordan sets of $\text{Aut}(\Omega, C, \leq)$ are convex (refer Lemma 4.6) and hence Jordan sets of $\text{Aut}(\Psi, D, K)$ can be either of type I or type II from the list of Jordan sets of $\text{Aut}(\Psi, D)$ (refer Section 6.5) with the restriction that A_0 as defined in the Jordan set of type II is open and convex. We give a brief outline of the proofs.

Lemma 6.13 *A set of type I is a Jordan set for G_K .*

Proof. Let $\alpha \in \Gamma$ and q_0 be an element in A and

$$\Sigma = \{\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A \setminus \{(q_0 + 1) \bmod 2\}, 1 \leq i \leq k\}$$

6.7 A relation on Ψ induced by a metric on Γ

be a set of type I . We consider a subgroup J_{α, q_0} of G_K consisting of branch automorphisms defined as follows. Any element $g \in J_{\alpha, q_0}$ induces identity permutation on the branches at α and also at any node β which does not lie on elements of Σ . At any node γ which is on an element of Σ let g induce a circular order-preserving permutation on the branches that fixes the branch from γ to α . We need to show that J_{α, q_0} is the associated Jordan group of Σ . And this can be easily shown because Σ carries a C -relation and the automorphism group of a C -set is transitive. •

Lemma 6.14 A set of type II is a Jordan set for G_K .

Proof. Let $\alpha \in \Gamma$ and A_0 be a proper subset of A which is open and convex and

$$\Sigma = \{\alpha p_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_i \in \mathbb{Q}, q_i \in A, 1 \leq i \leq k, p_0 \in A_0\}$$

be a set of type II . We define a subgroup J_{α, A_0} of G_K consisting of branch automorphisms defined as follows. Any element $g \in J_{\alpha, A_0}$ induces identity permutation on the branches at any node β which does not lie on elements of Σ . At the node α elements of J_{α, A_0} induce circular order-preserving permutation on the branches that fixes elements in $A \setminus A_0$ pointwise. And finally at any node γ which is on an element of Σ let g induce a circular order-preserving permutation on the branches that fixes the branch from γ to α . Then it can be shown that J_{α, A_0} is the associated Jordan group of Σ . •

Note that the complement of a Jordan set of type I is a Jordan set of type II . Combining Lemma 6.13 and Lemma 6.14 we have the following.

Theorem 6.5 The group G_K is a Jordan group. •

6.7 A relation on Ψ induced by a metric on Γ

Let α be any element of $\Gamma \setminus \{0\}$ with expression $0q n_1 q_1 n_2 q_2 \dots n_k$. We define distance of α from 0 denoted by $d(0, \alpha)$ as

$$d(0, \alpha) := n_k.$$

6.7 A relation on Ψ induced by a metric on Γ

Here 0 is the point of reference of Γ . The element n_k is positive and so $d(0, \alpha)$ is always positive. We also define $d(0, 0) = 0$. Consider any other element $\beta = 0p m_1 p_1 m_2 p_2 \dots m_j$. We define distance of α and β denoted by $d(\alpha, \beta)$ as follows. Suppose $0 \in [\alpha, \beta]$ then $d(\alpha, \beta) := d(0, \alpha) + d(0, \beta)$. If $0 \notin [\alpha, \beta]$ and if $(\alpha \wedge \beta) = [0, \gamma]$ then $d(\alpha, \beta) := d(0, \alpha) + d(0, \beta) - 2 \times d(0, \gamma)$. We check that d is a metric on Γ .

(i) For any $\alpha \in \Gamma \setminus \{0\}$, we have $d(0, \alpha) \geq 0$. Consider another element $\beta \in \Gamma \setminus \{0\}$. If $0 \in [\alpha, \beta]$ then $d(\alpha, \beta) \geq 0$. If $0 \notin [\alpha, \beta]$ and $\gamma = (\alpha \wedge \beta)$ then $d(0, \alpha) \geq d(0, \gamma), d(0, \beta) \geq d(0, \gamma)$ so that $d(0, \alpha) + d(0, \beta) - 2 \times d(0, \gamma) \geq 0$. Hence in all cases $d(\alpha, \beta) \geq 0$.

(ii) For $\alpha \in \Gamma, d(0, \alpha) = 0 \Leftrightarrow \alpha = 0$. Consider $\alpha, \beta \in \Gamma$ such that $0 \in [\alpha, \beta]$. Then $d(\alpha, \beta) = 0 \Rightarrow d(0, \alpha) + d(0, \beta) = 0 \Rightarrow d(0, \alpha) = 0, d(0, \beta) = 0$, since $d(0, \alpha) \geq 0, d(0, \beta) \geq 0$. This means $\alpha = 0 = \beta \Rightarrow \alpha = \beta$. If $0 \notin [\alpha, \beta]$ then $d(\alpha, \beta) = 0 \Rightarrow d(0, \alpha) + d(0, \beta) - 2 \times d(0, \gamma) = 0 \Rightarrow d(0, \alpha) + d(0, \beta) = 2 \times d(0, \gamma) \Rightarrow d(0, \alpha) = d(0, \gamma), d(0, \beta) = d(0, \gamma)$, and so $\alpha = \gamma = \beta$. Hence in any case since $d(0, \alpha) \geq d(0, \gamma), d(0, \beta) \geq d(0, \gamma)$, and so $\alpha = \gamma = \beta$. Hence in any case $d(\alpha, \beta) = 0$ implies $\alpha = \beta$. Conversely, if $\alpha = \beta$ then $d(\alpha, \beta) = 0$.

(iii) For any $\alpha, \beta \in \Gamma$, we have $d(\alpha, \beta) = d(\beta, \alpha)$.

(iv) For $\alpha, \beta, \gamma \in \Gamma$ we have $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\gamma, \beta)$ with equality holding

when $\gamma \in [\alpha, \beta]$. Therefore d is a metric on Γ . We now look at a relation on Ψ induced by the

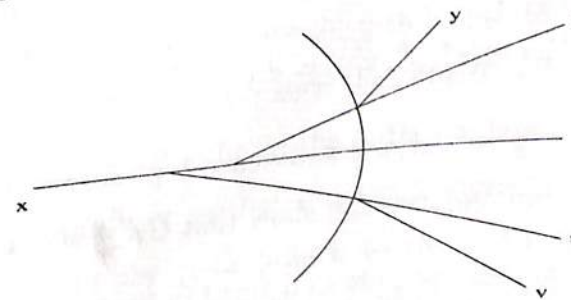


Figure 6.12:

metric d on Γ . Let x, y, z, u, v be elements of Ψ . We define a 5-place relation T on Ψ as

$$T(x; y, z; u, v) :\leftrightarrow d(\text{cen}(x, y, z), \text{cen}(x, y, u)) = d(\text{cen}(x, u, v), \text{cen}(x, y, u))$$

(Fig. 6.12). The relation T captures the notion of distance of two pairs of elements of Ψ from some arbitrarily chosen element of Ψ in the following sense. Suppose $T(x; y, z; u, v)$ holds then it means that the nodes at which the element x meets $y \wedge z$ and $u \wedge v$ is equidistant from the node $x \wedge y \wedge u$.

6.7.1 Automorphism group of (Ψ, D, T)

As usual let G_T denote the subgroup of $G = \text{Aut}(\Psi, D)$ that preserve the relation T . Let H_T denote the subgroup of H that preserve the distance of the nodes. We prove the following.

Lemma 6.15 *The group H_T is isomorphic to G_T .*

Proof. We have already seen that H is isomorphic to G (refer Theorem 6.1). Therefore we only need to show that elements in H_T preserves the relation T on Ψ and conversely elements in G_T preserve the distance of nodes of Γ . Let x, y, z, u, v be elements in Ψ such that $T(x; y, z; u, v)$ and let h be an element in H_T . Now $T(x; y, z; u, v) \Rightarrow d(\text{cen}(x, y, z), \text{cen}(x, y, u)) = d(\text{cen}(x, u, v), \text{cen}(x, y, u))$, and because h preserves the distance of nodes in Γ so we must have

$$d(\text{cen}(x^h, y^h, z^h), \text{cen}(x^h, y^h, u^h)) = d(\text{cen}(x^h, u^h, v^h), \text{cen}(x^h, y^h, u^h)).$$

This means that $T(x^h; y^h, z^h; u^h, v^h)$ and hence h preserves the relation T on Ψ . Therefore $H_T \leq G_T$. Similarly we can show that $G_T \leq H_T$. •

Consider the branch automorphisms as defined in Section 6.4. We show that G_T contains the class of branch automorphisms.

Lemma 6.16 *Branch automorphisms are elements of G_T .*

Proof. Let g be a branch automorphism of Ψ . We have defined a branch automorphism to fix a node and preserve the distance of the nodes from the fixed node. Therefore branch automorphisms preserve the distance of nodes and so using Lemma 6.15 branch automorphisms preserve the relation T on Ψ . •

We now look at transitivity properties of H_T and G_T .

Lemma 6.17 *The group H_T is transitive on Γ .*

Proof. Let α, β be two elements of Γ . We need to show that there exists $g \in H_T$ such that $\alpha^g = \beta$. Let γ be the point on the segment $[\alpha, \beta]$ such that $d(\alpha, \gamma) = d(\gamma, \beta)$. (Such an element in Γ exists because nodes are indexed by positive rational numbers.) Let α, β be on the q -th, p -th branches on the node γ (refer Fig. 6.13). Let g_1 be a branch automorphism that induces a permutation

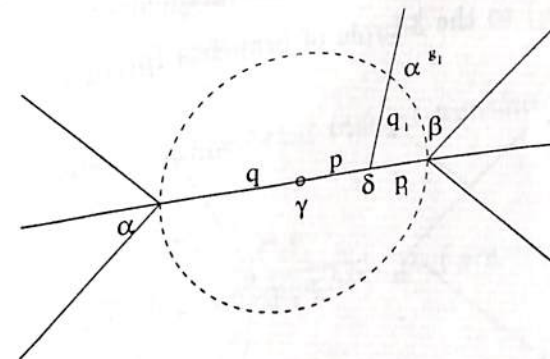


Figure 6.13:

on the branches at the node γ that maps the q -th branch to the p -th branch. Now α^{g_1} is a node on the p -th branch at γ such that $d(\gamma, \alpha^{g_1}) = d(\gamma, \beta)$. Let δ be the centroid of $\alpha^{g_1}, \beta, \gamma$ and let α^{g_1}, β be on the q_1 -th, p_1 -th branches at δ . We define another branch automorphism g_2 that induces a permutation on the branches at the node δ that maps the q_1 -th branch to the p_1 -th branch. We continue this process of constructing branch automorphisms so that ultimately the node α^{g_1} has the image β . Note that Ψ consists of finitely branching elements and Γ consists of nodes on the directions of Ψ so the process of constructing branch

automorphisms must terminate at a finite number of steps, say at the i -th step. The composition of the automorphisms $g_1 g_2 \dots g_i$ is the required automorphism that maps α to β . Therefore H_T is transitive on Γ . •

Theorem 6.6 *The group G_T is 3-transitive.*

Proof. Let (x, y, z) and (u, v, w) be two 3-tuples in Ψ . We need to show that there exists $g \in G_T$ such that $(x, y, z)^g = (u, v, w)$. Let $\alpha = cen(x, y, z)$ and $\beta = cen(u, v, w)$ be the centroids of x, y, z and u, v, w respectively. Let g be the map in H_T (which exists by Lemma 6.17) that maps α to β . Let x^g, y^g, z^g be on the q_1 -th, q_2 -th, q_3 -th branches and u, v, w be on the p_1 -th, p_2 -th, p_3 -th branches on β respectively (refer Fig. 6.14). Note that (q_1, q_2, q_3) and (p_1, p_2, p_3) are 3-tuples in A with no two elements in a tuple same. We define a branch automorphism say, h that induces a permutation on the branches at the node β that maps the 3-tuple of branches (q_1, q_2, q_3) to the 3-tuple of branches (p_1, p_2, p_3) . Now x^{gh} and u are

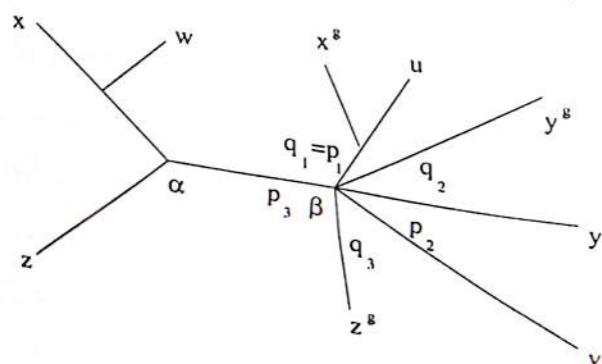


Figure 6.14:

elements on the p_1 -th branch at β . We define another branch automorphism that induces a permutation on the branches at the node $cen(x^{gh}, u, v)$ that maps the branch of x^{gh} to the branch of u . We continue this process so that ultimately the image of x^{gh} is u . We similarly construct branch automorphisms that maps y^g to v and z^g to w . The composition of all these automorphisms maps (x, y, z) to (u, v, w) . •

Corollary 6.1 *The group G_T is 2-primitive on Ψ . •*

So far we have just dealt with branch automorphisms of $G = \text{Aut}(\Psi, D)$, which we have shown to be elements of G_T . Let us now look at line automorphisms and half-line automorphisms of Ψ . Half-line automorphisms induces a permutation on some half-line, say $[\alpha, x)$ of Ψ and fixes all nodes on branches which are different from the branch on which the half-line $[\alpha, x)$ lie on the node α . This means that non-trivial half-line automorphisms does not preserve the distance of nodes of Γ . Passing on to line automorphisms, by similar observation we can say that only those line-automorphisms that induce distance preserving permutation on the nodes of some line preserve the relation T on Ψ . Therefore line automorphisms of G_T can be only those automorphisms that induce translations on lines.

Lemma 6.18 *A non-trivial half-line automorphism does not preserve the relation T on Ψ . •*

Lemma 6.19 *A line automorphism that induces translation on some line preserves T on Ψ . •*

Lemma 6.20 *A non-trivial line automorphism that preserves T on Ψ cannot fix more than two points on Ψ .*

Proof. Let g be a non-trivial line automorphism that induces a translation on some line. Then g cannot fix any node on the line. This means that all directions that branch out from the line will be moved. The only elements of Ψ that will remain fixed are the two directions of the line. •

6.7.2 Point stabiliser of $\text{Aut}(\Psi, D, T)$

Let $x_0 = 0q_0n_1q_1n_2q_2 \dots n_kq_k$ be an element of Ψ and let L_0 be the relation on $\Psi \setminus \{x_0\}$ induced by T which is defined as

$$L_0(y, z; u, v) : \leftrightarrow T(x_0; y, z; u, v).$$

We define the level of nodes of Γ denoted by L as

$$L(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0 \\ -d(0, \alpha), & \text{if } \alpha \in (0, x_0) \\ d(0, \alpha), & \text{if } \alpha = 0r_1l_1r_2l_2r_3l_3 \dots l_i, r \neq q_0 \\ -d(0, \beta) + d(\beta, \alpha), & \text{if } \alpha \notin (0, x_0) \text{ and } \exists \beta \in (0, x_0) \text{ with } \beta \in (0, \alpha), \end{cases}$$

where d is as defined in the beginning of this section. Using this level L of nodes we can define the relation L on Ψ as we defined in Section 4.3. It can be easily verified that $L_0(y, z; u, v)$ holds if and only if $L(y, z; u, v)$ holds in $\Psi \setminus \{x_0\}$. This means that the point stabiliser of x in G_T that preserves the relation L_0 will also preserve the relation L on $\Psi \setminus \{x\}$.

Lemma 6.21 For any $x \in \Psi$ the point stabiliser of x in G_T preserves the relation L (as defined in Section 4.3) on $(\Psi \setminus \{x\}, C)$. •

6.7.3 Jordan sets of $\text{Aut}(\Psi, D, T)$

We now look at the Jordan sets of $\text{Aut}(\Psi, D, T)$. We show that sets of type I and type II from the list in Section 6.5 are Jordan sets of the group G_T . Let $\alpha \in \Gamma$ and $q_0 \in A$ and

$$\Sigma = \{\alpha q_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_1, n_2, \dots, n_k \in \mathbb{Q}, q_1, q_2, \dots, q_k \in A \setminus \{(q_0 + 1) \bmod 2\}\}$$

be a set of type I. Consider the subgroup H of G_T which consists of the branch automorphisms that fixes the node α and induces permutation of the branches at the node α and which fixes $\Psi \setminus \Sigma$ pointwise. Then it can be shown that H is the associated Jordan group of Σ so that a set of type I is a Jordan set of G_T . Again, let $\alpha \in \Gamma$ and A_0 be a proper subset of A and

$$\Sigma = \{\alpha p_0 n_1 q_1 n_2 q_2 \dots n_k q_k \mid n_1, n_2, \dots, n_k \in \mathbb{Q}, q_1, q_2, \dots, q_k \in A, p_0 \in A_0\}$$

be a set of type II. Here again, consider the subgroup H of G_T which consists of the branch automorphisms that fixes the node α and induces permutation of the

6.8 A 5-place relation R on Ψ

branches at α and which fixes $\Psi \setminus \Sigma$ pointwise. Then it can be shown that H is the associated Jordan group of Σ . Therefore the following holds.

Theorem 6.7 The automorphism group G_T is a Jordan group. •

6.8 A 5-place relation R on Ψ

In this section we define a 5-place relation R on (Ψ, D) . For any five elements x, y, z, u, v in Ψ let R be defined as

$$R(x; y, z, u, v) :\leftrightarrow \text{cen}(x, y, z) \text{ and } \text{cen}(x, u, v) \text{ are of same type,}$$

where type of nodes is defined as in Section 4.4.

6.8.1 Automorphism group of (Ψ, D, T, R)

Consider the automorphism group on (Ψ, D, T) preserving R . Let $G_{T,R}$ denote the subgroup of G_T that also preserve the relation R . Let $H_{T,R}$ denote the subgroup of H_T that preserve the type of the nodes. Then the following is true.

Lemma 6.22 The group $H_{T,R}$ is isomorphic to $G_{T,R}$. •

Let $\{x, y, z\}$ and $\{u, v, w\}$ be two sets of size 3 in Ψ such that the nodes $\text{cen}(x, y, z)$ and $\text{cen}(u, v, w)$ are of different types. Then there cannot exist any element in $G_{T,R}$ that maps $\{x, y, z\}$ to $\{u, v, w\}$. Therefore we have the following.

Lemma 6.23 The group $G_{T,R}$ is not 3-homogeneous. •

6.8.2 Point stabiliser of $\text{Aut}(\Psi, D, T, R)$

For any $x_0 \in \Psi$, let R_0 be a relation on $\Psi \setminus \{x_0\}$ induced by the relation R on the structure (Ψ, D, T) defined as

$$R_0(x, y; z, w) :\leftrightarrow R(x_0; x, y, z, w).$$



The relation R that we defined on the C -set Ω (Section 4.4) was defined by taking into consideration the type and level of the nodes of the underlying semilinear order. The relation R on Ψ is also defined using the type of nodes of Γ . We have already shown that the relation T defined on Ψ induces the relation L on $\Psi \setminus \{x_0\}$ (Section 6.7.2). Hence it follows that the relation R on (Ψ, D, T) that induces the relation R_0 on $\Psi \setminus \{x_0\}$ is same as the relation R on (Ω, C, L) (Section 4.4).

Lemma 6.24 For $x \in \Psi$, the point stabiliser of x in $G_{T,R}$ preserves the relation R (as defined in Section 4.4) on $(\Psi \setminus \{x\}, C)$, in addition to the relation L . •

6.8.3 Jordan sets of $\text{Aut}(\Psi, D, T, R)$

Line automorphisms of (Ψ, D, T, R) can be only those automorphisms that induce integer translations of the nodes on a line (using Lemma 4.21). Therefore Jordan sets of type I and type II exist at the node 0. At any other node $\alpha \neq 0$, a Jordan set of type I is a cone on a branch different from the branch that leads α to 0. Similarly a Jordan set of type II on a node $\alpha \neq 0$ cannot contain the branch that leads α to 0.

6.9 Combining the relations K, T, R on (Ψ, D)

So far we have imposed extra relations on the relational structure (Ψ, D) one after another and verified whether the Jordan property of the automorphism group was preserved. In this section we combine all the relations considered so far and show that we still have Jordan sets in the automorphism group of (Ψ, D, K, T, R) . Let $G_{K,T,R}$ denote the automorphism group of (Ψ, D, K, T, R) . Elements of $G_{K,T,R}$ will have to preserve all the three relations K, T, R which means that the automorphism group $G_{K,T,R}$ is a subgroup of G_K (which we have considered in Section 6.6) and is also a subgroup of $G_{T,R}$ (which we have considered in Section 6.8). We have already seen in the respective sections that elements of G_K preserve the circular order of the branches on nodes, elements of G_T preserve the distance

6.10 Cofinality of $\text{Aut}(\Psi, D)$

of the nodes and elements of G_R preserve the type of the nodes. Consider the branch automorphisms of (Ψ, D) defined on the node 0 that preserve the circular order on Ψ . These automorphisms fix the distance of the nodes from the node 0. This means that trivially the distance of the nodes and the type of the nodes are preserved. Therefore these branch automorphisms are elements of $G_{K,T,R}$ which means that $G_{K,T,R}$ is a non-trivial group of automorphisms.

The group $G_{K,T,R}$ cannot be 3-homogeneous because the group $G_{T,R}$ is not 3-homogeneous. Using results in Section 6.6, Section 6.7 and Section 6.8 we can check the following.

Lemma 6.25 For any $x \in \Psi$ the point stabiliser of x in $G_{K,T,R}$ preserves a linear order (Section 4.1), a 4-place relation L (Section 4.3) and another 4-place relation R (Section 4.4) on $\Psi \setminus \{x\}$ induced by the circular order K and the 5-place relations T and R on Ψ respectively. •

For some $q_0 \in A$ and a convex open subset A_0 of A , consider sets of the forms $\{0q_0n_1q_1 \dots n_kq_k\}$ and $\{0p_0n_1q_1 \dots n_kq_k \mid p_0 \in A_0\}$. Also for any $0 \neq \alpha = 0qn_1q_1 \dots n_k \in \Gamma$ and $A'_0 \subset A \setminus \{(q+1) \bmod 2\}$ consider sets of the forms $\{\alpha p_0m_1p_1 \dots m_jp_j \mid p_0 \neq (q+1) \bmod 2\}$ and $\{\alpha p_0m_1p_1 \dots m_jp_j \mid p_0 \in A'_0\}$. It can shown that sets of these forms are Jordan sets of $G_{K,T,R}$ with associated Jordan groups comprising of branch automorphisms that induce permutations on the branches at nodes that preserve the circular order.

Theorem 6.8 The group $G_{K,T,R}$ is a Jordan group. •

6.10 Cofinality of $\text{Aut}(\Psi, D)$

We devote this section to a question on cofinality of $G = \text{Aut}(\Psi, D)$.

Definition 6.4 An automorphism group A is said to be of *countable cofinality* if there exists a countable chain $H_0 < H_1 < H_2 < \dots$ of proper subgroups of A with union A . Otherwise we say that A has *uncountable cofinality*.

Hodges et al. [HHLS] have given a construction of groups of uncountable cofinality. Macpherson & Neumann [MN] have shown that the group of all permutations of a countable set has uncountable cofinality. Gourion [G] has shown that the group of order-preserving permutations of \mathbb{Q} has uncountable cofinality. We prove that $\text{Aut}(\Psi, D)$ has uncountable cofinality [Theorem 6.9].

Theorem 6.9 *If $(H_i)_{i \in I}$ is a chain of proper subgroups of $G = \text{Aut}(\Psi, D)$ such that $\bigcup_{i \in I} H_i = G$ then $|I| > \aleph_0$.*

The proof is given at the end of this section. If $\Sigma \subseteq \Psi$ and $|\Sigma| = |\Psi \setminus \Sigma|$ then we call Σ a *moiety* of Ψ . If H is an automorphism group on Ψ (that is to say, a subgroup of $\text{Aut}(\Psi, D)$) and $\Delta \subseteq \Psi$ then $H_{\{\Delta\}}, H_{(\Delta)}$ denote respectively the setwise and pointwise stabilisers of Δ in H , and H^Δ denotes the group induced on Δ by $H_{\{\Delta\}}$. If $H^\Delta = \text{Aut}(\Delta, D_\Delta)$ where D_Δ is the relation induced by D on Δ , then we say that Δ is *full* for H . We state two lemmas which will be needed for the proof of Theorem 6.9. Analogous proofs of these two lemmas with $G = \text{Aut}(\Psi, D)$ replaced by $\text{Sym}(\Psi)$, where Ψ is an infinite set are given in [N][Note 3(iii) of §4] and [MN][Lemma 2.3] respectively. In our case $G = \text{Aut}(\Psi, D)$, where Ψ is countably infinite and so is a subgroup of $\text{Sym}(\Psi)$. Therefore these lemmas can be shown to hold here.

Lemma 6.26 *If all moieties of Ψ are full for a subgroup H of G then $H = G$. •*

Lemma 6.27 *Let H be a subgroup of G and let Σ_1, Σ_2 be subsets of Ψ that are full for H . If $|\Sigma_1 \cap \Sigma_2| = \aleph_0$ and $\Sigma_1 \cup \Sigma_2 = \Psi$ then $H = G$. •*

Let us call sets of type I from the list of Jordan sets of G (Section 6.5) as *cones*. And let us denote cones at any node α of Γ , $a \in A$ as $\Sigma(\alpha, a)$, so that cones at the node 0 are denoted by $\Sigma(0, a)$. For any $a \in A$, the cone $\Sigma(0, a)$ carries a C -relation induced by the D -relation on Ψ . By [DHM][Theorem 1.3] there exists a maximal normal subgroup of $\text{Aut}(\Sigma(0, a), C)$ which we call N and which can be expressed as

$$N := \{g \in \text{Aut}(\Sigma(0, a), C) \mid \exists \alpha \in \Sigma(0, a) \text{ and } \forall \beta < \alpha, \beta^g = \beta\}.$$

Consider the order-preserving permutation s on $[0, \infty)$ defined as

$$n \mapsto 2n, 0 < n.$$

We extend s to an automorphism h on Ψ defined as

$$\begin{aligned} 0a n_1 q_1 n_2 q_2 \dots n_k q_k &\mapsto 0a(2n_1)q_1(2n_2)q_2 \dots n_k q_k, \\ 0q n_1 q_1 n_2 q_2 \dots n_k q_k &\mapsto 0q n_1 q_1 n_2 q_2 \dots n_k q_k, \quad q \neq a. \end{aligned}$$

It is clear that $h \notin N$ because for any node α in the cone $\Sigma(0, a)$, h moves all points smaller than α with respect to the semilinear order induced on $\Sigma(0, a)$ by the B -relation on Γ . This means that h is an element of $\text{Aut}(\Sigma(0, a), C) \setminus N$ because N is a maximal normal subgroup of $\text{Aut}(\Sigma(0, a), C)$ it follows that the group $\langle N, h \rangle$ is the full group acting on $\Sigma(0, a)$ which fixes $\Psi \setminus \Sigma(0, a)$ pointwise.

Lemma 6.28 *If there exists a full moiety for H in Ψ then there exists $g \in G$ such that $G = \langle H, g \rangle$.*

Proof. Consider the cone $\Sigma(0, a)$ at the node $0 \in \Gamma$, for some $a \in A$. We have already seen in the discussion preceding this lemma that $\Sigma(0, a)$ is full for $\langle N, h \rangle = H$ in G . Consider a line automorphism $g_1 \in G$ that induces a permutation of the nodes on the line $(0(a+1) \bmod 2, 0a)$ that maps the node 0 to the node $0a1$ and a branch automorphism g_2 that induces a permutation on the branches at the node $0a1$ that interchanges the a -th and $(a+1) \bmod 2$ -th branches. Consider the subset B of Ψ defined as $B := \{z \in \Psi \mid \text{cen}(0a, 0(a+1) \bmod 2, z) = \alpha \text{ and } B(\alpha; 0, 0a1)\}$. It is clear that $\Sigma(0, a) \cap \Sigma(0, a)^{g_1 g_2} = B$ with $|B| = \aleph_0$ and $\Sigma(0, a) \cup \Sigma(0, a)^{g_1 g_2} = \Psi$. Now both $\Sigma(0, a)$ and $\Sigma(0, a)^{g_1 g_2}$ are full for $\langle H, g_1 g_2 \rangle$ because $\Sigma(0, a)$ is full for H and $g_1 g_2$ maps $\Sigma(0, a)$ to $\Sigma(0, a)^{g_1 g_2}$. And therefore by Lemma 6.27 we have $\langle H, g_1 g_2 \rangle = G$. That is, $G = \langle N, h, g_1 g_2 \rangle$. •

Proof of Theorem 6.9. Suppose that $(H_i)_{i \in I}$ is a chain of proper subgroups with union G such that $\bigcup H_i = G$. No cone can be a full moiety for H_i for any i , because if some cone say, $\Sigma(0, a)$ is a full moiety for H_i , for some i then there would exist some $j > i$ such that H_j would contain g and then $G = H_j$, using Lemma 6.28, which is a contradiction. Now suppose $|I| \leq \aleph_0$. Partition

Ψ as a disjoint union $\bigcup \Sigma(0, i)$ of moieties $\Sigma(0, i)$, $i \in I$ at the node 0. Let C_i be the induced C -relation on $\Sigma(0, i)$ induced by D on Ψ . For each $i \in I$ choose $f_i \in H_i \setminus H_i^{\Sigma(0, i)}$ and let f be the element of G which induces f_i on $\Sigma(0, i)$ for each $i \in I$. Then clearly $f \notin \bigcup H_i$ contrary to our assumption and this contradiction proves our theorem. •

Chapter 7

Conclusion

In this final chapter we give a brief summary of the work that has been presented in this thesis. We end by stating some open problems close to our area of work.

7.1 A brief summary of the work

We constructed a C -set Ω and studied the automorphism group of the C -set. We have shown that any element of the automorphism group of the C -set can be expressed as a product of chain automorphisms and branch automorphisms. We then imposed extra relations \leq, V, L and R on the C -set. We studied some properties of the automorphism groups of the C -set Ω with the extra relations. From literature the automorphism group of a C -set is known to be a Jordan group. We have shown that the automorphism groups of the structures with the extra relations imposed are also Jordan groups. We have determined a minimal Jordan group G_0 such that the imposition of extra relation on the underlying structure no longer admits a Jordan automorphism group. Further, we have shown that a branch relation that contains the class of translation branch automorphisms in its automorphism group can be imposed on the C -set and still admit a Jordan automorphism group. And any chain relation can be imposed on the C -set and still admit a Jordan automorphism group.

The next part of our work was on some related questions of the structures of the C -set with the extra relations. We looked into the questions whether the structures are homogeneous and whether their automorphism groups are oligomorphic. We have found positive answers to both these questions. We have also determined the realisable cycle types of the automorphism groups. We have shown that the cycle types realised by non-identity automorphisms of the groups $\text{Aut}(\Omega, C, \leq)$, $\text{Aut}(\Omega, C, \leq, V)$, $\text{Aut}(\Omega, C, \leq, V, L)$ and $\text{Aut}(\Omega, C, \leq, V, L, R)$ are of the forms $\aleph_0^{N_0} 1^k$, $0 \leq k \leq N_0$. And the cycle type realised by non-identity elements of the minimal Jordan group G_0 is of the form $\aleph_0^{N_0} 1^{N_0}$.

The last part of our work was on a D -set. We constructed a D -set Ψ and studied its automorphism group. We imposed extra relations K , T and R on the D -set and have found that the automorphism groups of these structures are Jordan groups. For any element x in Ψ we have found that the relation K induces a linear order on $(\Psi \setminus \{x_0\}, C)$, the relation T induces a 4-place relation L and the 5-place relation R induces the corresponding 4-place relation R as defined in the C -set Ω . An analogue for Theorem 3.2 should be true for D -sets but has not been worked out explicitly as the notation becomes very complicated. Another major result that we have got is the result on cofinality. We have found that the group $\text{Aut}(\Psi, D)$ has uncountable cofinality.

7.2 Some problems related to our work

Consider the group $G = \text{Aut}(\Omega, C)$ as considered in Chapter 3. We define subgroups H and K of G generated by chain automorphisms and branch automorphisms respectively. This means the subgroups H , K are expressible as

$$\langle h \mid h = h_1 h_2 \dots h_k, \text{ where each } h_k \text{ is a chain automorphism, } k \in \mathbb{N} \rangle,$$

$$\langle f \mid f = f_1 f_2 \dots f_j, \text{ where each } f_j \text{ is a branch automorphism, } j \in \mathbb{N} \rangle$$

respectively. Then one can ask whether the following is true.

Problem 7.1 The group $G = \text{Aut}(\Omega, C)$ is the direct product of H and K defined above. •

As said earlier we started our work with the problem of finding all possible expansions of a C -relation that preserve the Jordan property of the automorphism group. To have an idea of the types of relations that can be imposed on a C -relation we started with some actual relations. In the course of our study we realised that trying to find a list of the relations in the general sense of relations might not be plausible. So we defined two kinds of relations, namely chain relations and branch relations and answered the problem in terms of these kinds of relations. However, there could be other kinds of relations that can be imposed on a C -relation which preserves the Jordan property of the automorphism group. So the following problem remains open.

Problem 7.2 Does there exist any other kind of relations (other than chain relations and branch relations) that can be imposed on a C -relation while preserving the Jordan property of the automorphism group.

There could be another very important application in the field of character recognition. The problem of optical character recognition by reducing the data rate of the input image data can be investigated by formulating the problem in terms of C -sets with extra relations. The nodes of the underlying semilinear order can be considered as the input image data.

The class of oligomorphic permutation groups has many links with other topics, notably model theory. The main connection between oligomorphic permutation groups and model theory is given by the theorem of Ryll-Nardzewski which can be loosely stated as follows. For a more precise statement of the theorem refer [C2][Section 2.5].

Theorem 7.1 [Ryll-Nardzewski] A countable (first-order) structure is axiomatisable (that is, characterised, up to isomorphism, as a countable structure) if and only if its automorphism group is oligomorphic. •

In Chapter 5 we have shown that the automorphism groups of the structures of (Ω, C) with the extra relations are oligomorphic. Therefore by Ryll-Nardzewski's Theorem each of the structures should be axiomatisable.

Problem 7.3 *Axiomatise the structures described in Chapter 4.*

Questions on homogeneous structures and oligomorphic groups of the structures discussed in Chapter 6 can also be taken up for further work. Moreover the cycle types of these automorphism groups and cofinality of $\text{Aut}(\Psi, D)$ with the extra relations can also be studied. Our present work has shed light on some of the questions that can be asked about Jordan groups. We shall continue our work and hope that in time, we shall be able to find answers to more of these questions.

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