STRONG LINEARIZATIONS OF POLYNOMIAL AND RATIONAL MATRICES AND RECOVERY OF SPECTRAL DATA

Ph.D. Thesis

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March 2019

STRONG LINEARIZATIONS OF POLYNOMIAL AND RATIONAL MATRICES AND RECOVERY OF SPECTRAL DATA

A Thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

by

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to the

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March 2019

DECLARATION

I do hereby declare that the work contained in this thesis entitled **"Strong Linearizations of Polynomial and Rational Matrices and Recovery of Spectral Data"** has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under the guidance of **Prof. Rafikul Alam** for the award of the degree of Doctor of Philosophy and that this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is certified that the work contained in this thesis entitled **“Strong Linearizations of Polynomial and Rational Matrices and Recovery of Spectral Data”** by **Ranjan Kumar Das,** a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

March 2019

Dr. Rafikul Alam
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Dedicated To

My Family

ACKNOWLEDGMENT

It is with immense gratitude that I acknowledge the individuals who in one way or another contributed and extended their unfailing support, encouragement, and guidance in the preparation and completion of this study.

First and foremost, I would like to express my deepest and most sincere gratitude to my thesis supervisor Prof. Rafikul Alam for providing me the opportunity to work with him and for making this thesis possible. His expertise and clarity of thought in conducting research as well as in its presentations have strongly motivated me. I am grateful to him for his invaluable guidance, constant encouragement and utmost care throughout this thesis work. I am deeply indebted to him for listening to my countless queries and providing insightful comments which helped to make my PhD experience productive and stimulating.

I acknowledge my gratitude to my doctoral committee members Prof. Rajen Kumar Sinha, Prof. Shreemayee Bora, and Dr. Anjan Kumar Chakrabarty, for their valuable comments and suggestions during this thesis work and for their personal concern towards my progress in research. I am thankful to Prof. Sukanta Pati, Prof. Bhaba Kumar Sarma, Prof. Swaroop Nandan Bora, Prof. Anupam Saikia and Dr. Jitendriya Swain, for their help and support during the tenure of the research. I acknowledge the help and cooperation of all the faculty members of the Department of Mathematics, IIT Guwahati.

I would like to thank Indian Institute of Technology Guwahati for the facilities provided to me during my research work and the Ministry of Human and Resource Development, Govt. of India, for offering me financial assistance for pursing PhD at IIT Guwahati. I thank all the staff members of Department of Mathematics, IIT Guwahati, for their assistance in administrative and technical matters.

I am extremely grateful to Prof. Parasar Mohanty, IIT Kanpur, for his encouragement and guidance during and after my M.Sc. days. I am thankful to him for providing me a project in my M.Sc. and supervising it. I am extremely thankful to Dr. Namita Behera for sharing ideas and motivating me all the time. I thank Dr. Arnab Ghosh, Dr. Punit Sharma and Biswajit for many fruitful discussions. I have been inspired by some excellent teachers during my school and college days. Special mention goes to Basudev Swain, Rama Chandra Mahapatra, Bhargav Chandra Naik, Narendra Satpathy, Asid Mishra, Jibadhan Naik and Shantanu Pradhan.

I am extremely thankful to Ramesh for sharing a great friendship. We together decided to pursue career in research, faced a lot of ups and downs, and succeeded at last. My special acknowledgment goes to Maneesh and Ashish for sharing a great friendship and making my time at IIT Guwahati enjoyable and memorable. I would like to give special thanks to Debasish for his company and for helping me with his expert command on LaTeX and MATLAB. My time was made enjoyable by many friends. Tuba, Debu, Mukan, Soumya, Nabin, Jogen, Deba, Nasim, Sheela, deserve a special mention. Some special words of gratitude go to my friends and labmates: Naba da, Jayanta da, Swarup da, Ankur, Bala, Barik, Ananda, Shiba, Pratap, Madhu, Tanu, Debu, Tamal, Devanand, Sanjoy, Jibrail, Nandita, Bandita, Gautam, Mohit, Trushna, Swarup and Saroj.

It is not possible to express in words my gratitude towards my family, my parents Dinabandhu Das and Mahima Das, my brother Bipin and sister-in-law Dipi, my sister Mami and brother-in-law Bijaya, for their unconditional love, care and belief on me. Without their support, I could have never completed this thesis. I extend my gratitude specially to Baby, Ani, Ashok and Gudu for their love and care. It is always joyful to talk with the two little hearts of my family Khushi and Siddi. Their meaningless talks and nonlinear behaviors amused me a lot.

Above all, I thank God for everything!

March 2019

With Regards,

Ranjan Kumar Das

ABSTRACT

Linearization is a classical technique widely used to deal with matrix polynomial. The main purpose of the thesis is to construct and analyze strong linearizations of polynomial and rational matrices. The first part of the thesis is devoted to construction of strong linearizations of matrix polynomials including structure-preserving strong linearizations and the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the linearizations. The second part of the thesis is devoted to construction of strong linearizations of rational matrices including structure-preserving strong linearizations and the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the linearizations.

Fiedler pencils (FPs), generalized Fiedler pencils (GFPs), Fiedler pencils with repetition (FPRs) and generalized Fiedler pencils with repetition (GFPRs) are important family of strong linearizations of matrix polynomials which have been studied extensively over the years. It is well known that the family of GFPRs of matrix polynomials subsumes the family of FPRs and is an important source of strong linearizations, especially structure-preserving strong linearizations of structured matrix polynomials. We propose a unified framework for analysis and construction of a family of Fiedler-like pencils, which we refer to as extended GFPRs (EGFPRs), that subsumes all the known classes of Fiedler-like pencils such as FPs, GFPs, FPRs and GFPRs of matrix polynomials. We show that the unified framework allows us to construct structure-preserving strong linearizations with additional properties such as banded pencils with low bandwidth and preservation of sign characteristic in the case of Hermitian matrix polynomials. Moreover, we describe the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the EGFPRs and show that the recovery is operation-free. In particular, we describe the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the FPRs and GFPRs which has been an open problem.

Rational matrices arise in many applications. Linearization of rational matrices has been introduced recently for solving rational eigenproblems. FPs, GFPs and FPRs for rational matrices have been constructed which are shown to be linearization of rational matrices. We introduce a strong linearization (referred to as Rosenbrock strong linearization) of rational matrices and show that structural indices of finite as well as infinite poles and zeros of rational matrices can be recovered from those of the strong linearizations. We show that FPs, GFPs and FPRs of rational matrices are in fact Rosenbrock strong linearizations. Also, we introduce a family of Fiedler-like pencils, which we refer to as generalized Fiedler pencils with repetition (GFPRs), for rational matrices and show that the GFPRs are strong linearizations. We show that the family of GFPRs is an important source of structure-preserving strong linearization of rational matrices. In fact, we utilize GFPRs to construct structure-preserving strong linearization of structured (symmetric, skew-symmetric, Hamiltonian, skew-Hamiltonian, Hermitian, para-Hermitian, etc.) rational matrices. We show that the Hermitian GFPRs preserve the Cauchy-Maslov index of Hermitian rational matrices. We describe the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the GFPRs and show that the recovery is operation-free. We also introduce affine spaces of strong linearization of rational matrices and describe the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the strong linearizations.

Contents

List of Symbols

$\mathbb{N}$	the set of the natural numbers
$\mathbb{Z}$	the set of the integers
$\mathbb{R}$	the field of the real numbers
$\mathbb{C}$	the field of the complex numbers
$\mathbb{C}^{m\times n}$	the space of complex matrices of size $m \times n$
$\mathbb{C}[\lambda]^{m\times n}$	the vector space of $m \times n$ matrix polynomials
$\mathbb{C}(\lambda)$	the field of rational functions
$\mathbb{C}(\lambda)^{m\times n}$	the vector space of $m \times n$ rational matrix
I_n	the identity matrix of size n
e_k	k-th column of the identity matrix I_n
A^T	the transpose of real A
A^*	complex conjugate transpose of A
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	the block diagonal matrix $diag(A, B)$
$A^{\mathcal{B}}$	block transpose of a matrix A
M_i^P	Fiedler matrix associated with $P(\lambda)$
$M_i(X)$	elementary matrix with matrix assignment X
$\mathbb{M}_i^{\mathcal{S}}$	Fiedler matrix associated with $\mathcal{S}(\lambda)$
$\mathbb{M}_i(X)$	elementary matrix with matrix assignment X

$\mathcal{N}_r(G)$	right nullspace of singular rational matrix $G(\lambda)$
$\mathcal{N}_l(G)$	left nullspace of singular $G(\lambda)$
FPs	Fiedler pencils
GFPs	generalized Fiedler pencils
PGFs	proper generalized Fiedler pencils
FPRs	Fiedler pencils with repetition
GFPRs	generalized Fiedler pencils with repetition
EGFPRs	extended generalized Fiedler pencils with repetition
SIP	Successor Infix Property
$csf(\beta)$	column standard form of an index tuple β
$rsf(\beta)$	row standard form of an index tuple β
$c_s(\alpha)$	consecutions of an index tuple α at an index s
$i_s(\alpha)$	inversions of an index tuple α at an index s
$c(\sigma)$	total number of consecutions of a permutation σ
$i(\sigma)$	total number of inversions of a permutation σ

1

Introduction

Nonlinear eigenvalue problems in general and polynomial and rational eigenvalue problems in particular arise in many applications such as in acoustic emissions of high speed trains, calculations of quantum dots, free vibration of plates with elastically attached masses, vibrations of fluid-solid structures, and in control theory, see [9, 45, 63, 49, 55, 53, 62, 40, 59, 56] and the references therein. For example, the 129×129 polynomial eigenvalue problem (PEP)

$$P(\lambda)u := (\lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0)u = 0$$

arises from a finite element solution of the equation for the modes of a planar waveguide using piecewise linear basis functions ϕ_i, $i = 0 : 128$. The coefficient matrices are defined by

$$A_1 = \frac{\delta^2}{4}\operatorname{diag}(-1, 0, 0, ..., 0, 0, 1), \;\; A_3 = \operatorname{diag}(1, 0, 0, ..., 0, 0, 1),$$

$$A_0(i,j) = \frac{\delta^4}{16}(\phi_i, \phi_j), \;\; A_2(i,j) = (\phi_i', \phi_j') - (q\phi_i, \phi_j), \;\; A_4(i,j) = (\phi_i, \phi_j),$$

where ϕ_i' is the derivative of ϕ_i; the parameter δ describes the difference in refractive index between the cover and the substrate of the waveguide; q is a function used in the derivation of the variational formulation and is constant in each layer; see [9] and the references therein. On the other hand, the rational eigenvalue problem (REP)

$$G(\lambda)x := -Kx + \lambda Fx + \sum_{j=1}^{k} \frac{\lambda}{\beta_j - \lambda} C_j x = 0,$$

arises in the study of mechanical vibrations of fluid-solid structures [49, 62, 63], where K and F are positive definite matrices and the matrices $C_j, j = 1 : k$, are Hermitian and have low ranks.

Linearization is a classical technique widely used for solving polynomial eigenproblems in which a matrix polynomial is transformed to a matrix pencil of larger size. Let $P(\lambda) := \sum_{i=0}^{m} \lambda^i A_i$ be an $n \times n$ matrix polynomial of degree m. Then an $mn \times mn$ matrix pencil $L(\lambda) := \mathcal{X} + \lambda\mathcal{Y}$ is said to be a linearization [33, 46] of $P(\lambda)$ if there exist $mn \times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ (i.e., $\det(U(\lambda))$ and $\det(V(\lambda))$ are nonzero constants independent of λ) such that

$$U(\lambda)L(\lambda)V(\lambda) = \mathrm{diag}(I_{(m-1)n},\ P(\lambda)) \text{ for } \lambda \in \mathbb{C}.$$

Additionally, if $\mathrm{rev}\, L(\lambda) := \mathcal{Y} + \lambda\mathcal{X}$ is a linearization of $\mathrm{rev}\, P(\lambda)$ then $L(\lambda)$ is said to be a *strong linearization* [46] of $P(\lambda)$, where $\mathrm{rev}\, P(\lambda) := \lambda^m P(1/\lambda)$. Construction of strong linearizations of matrix polynomials is an active area of research and several important families of strong linearizations of matrix polynomials such as Fiedler pencils (FPs), generalized Fiedler pencils (GFPs), Fiedler pencils with repetition (FPRs), generalized Fiedler pencils with repetition (GFPRs), vector space linearizations, block Kronecker pencils and block minimal basis pencils have been introduced and analyzed in the last one and half decades, see [6, 13, 17, 14, 15, 20, 19, 26, 27, 61, 46, 45, 25, 30, 42, 28, 51] and the references therein. By contrast, linearization of rational matrices is a very recent concept. With a view to providing a direct method for solving rational eigenproblems, a new concept of linearization of rational matrices has been introduced in [2]; see also [5]. Fiedler-like pencils (FPs, GFPs, FPRs) for rational matrices have been constructed in [2, 3, 4, 8] and are shown to be linearizations of rational matrices.

The main purpose of the thesis is to construct and analyze strong linearizations of polynomial and rational matrices. The thesis consists of two parts. The first part is devoted to construction of strong linearizations of matrix polynomials including structure-preserving strong linearizations and the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the linearizations. The second part of the thesis is devoted to construction of strong linearizations of rational matrices including structure-preserving strong linearizations and the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the linearizations.

The family of GFPRs of matrix polynomials subsumes the family of FPRs and is a very important source of strong linearizations, especially structure-preserving strong linearizations of matrix polynomials, and have been studied extensively in recent years [13, 14, 16, 17, 19, 20, 27, 61]. However, barring a small subset of FPRs known as type-1 FPRs [14], the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of FPRs and GFPRs is still an open problem. One of the main objectives of the thesis is to fill this gap. We describe the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the GFPRs and show that the recovery is operation-free, that is, eigenvectors and minimal bases can be recovered without performing any arithmetic operations and the minimal indices can be recovered by applying a uniform shift. The main building blocks of GFPRs are very special GFPs and the restrictions that come with the special GFPs limit the full potential of GFPRs. With a view to realizing full potentials of Fiedler-like pencils, we introduce a new family of extended GFPRs (EGFPRs) of matrix polynomials whose building blocks are the family of GFPs instead of very special GFPs and whose construction is operation-free as in the case of GFPRs. We show that the EGFPRs are strong linearizations of matrix polynomials and subsume FPs, GFPs, FPRs and GFPRs. Moreover, the EGFPRs substantially expand the arena in which to look for strong linearizations, especially structure-preserving strong linearizations, of matrix polynomials with additional features such as low bandwidth and preservation of sign characteristic in the case of Hermitian matrix polynomials. Most importantly, we describe the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the EGFPRs and show that the recovery is operation-free as in the case of GFPRs.

The concept of *Rosenbrock linearization* of rational matrices has been introduced in [2] by considering minimal realizations of rational matrices which generalizes linearization of matrix polynomials to the case of rational matrices. FPs, GFPs and FPRs of rational matrices have been constructed in [2, 4, 8] and are shown to be Rosenbrock linearization of rational matrices. The recovery of eigenvectors of rational matrices from those of the FPs, GFPs and type-1 FPRs has been described in [3, 4, 8]. However, the recovery of minimal bases and minimal indices of rational matrices from those of the FPs, GFPs and FPRs (including recovery of eigenvectors from FPRs) is still an open problem. Further, construction of structure-preserving Rosenbrock linearization of structured rational matrices is still an open problem. Furthermore, Rosenbrock linearization does not enable recovery of pole-zero structural indices at infinity of rational matrices. Our main objective in the second part of the thesis is to address all these problems. Firstly, we introduce *Rosenbrock strong linearization* of rational matrices and describe recovery of pole-zero structural indices at infinity of rational matrices from those of the Rosenbrock strong linearizations. Secondly, we introduce affine spaces of potential linearizations of rational matrices, which are analogues of vector space linearizations of matrix polynomials, and show that almost all pencils in the affine spaces are Rosenbrock strong linearizations. We also describe the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the Rosenbrock strong linearizations belonging to the affine spaces. Thirdly, we show that FPs, GFPs, and FPRs of rational matrices are indeed Rosenbrock strong linearizations. Further, we describe the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the FPs, GFPs and FPRs and show that the recovery is operation-free. Finally, we introduce the family of GFPRs of rational matrices and show that GFPRs are Rosenbrock strong linearizations. We describe the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the GFPRs and show that the recovery is operation-free. Most importantly, we show that the family of GFPRs of rational matrices is a rich source of structure-preserving Rosenbrock strong linearizations and construct structure-preserving linearizations for various structured rational matrices such as symmetric, skew-symmetric, Hermitian, skew-Hermitian, Hamiltonian, skew-Hamiltonian, para-Hermitian, and para-skew-Hermitian rational matrices. In particular, we show that the Hermitian GFPRs preserve the Cauchy-Maslov index of Hermitian rational matrices.

The thesis is organized as follows. The rest of this chapter is devoted to basic results on matrix polynomials, rational matrices, and LTI state-space systems. Chapter 2 describes the recovery of eigenvectors, minimal bases and minimal indices of a matrix polynomial from those of its GFPRs. Chapter 3 introduces a new family of linearizations, which we refer to as EGFPRs of matrix polynomials, and describes the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomial from those of its EGFPRs. Chapter 4 is devoted to constructing structure (symmetric, Hermitian, palindromic, anti-palindromic) preserving linearizations for corresponding structured matrix polynomials by utilizing the EGFPRs of matrix polynomials. Chapter 5 is devoted to Rosenbrock strong linearizations of rational matrices and construction of two affine spaces of linearizations for a rational matrix. It also describes the recovery of eigenvectors, minimal bases and minimal indices of a rational matrix from those of the Rosenbrock strong linearizations. Chapter 6 analyzes the recovery of minimal bases and minimal indices of a rational matrix from those of its FPs, GFPs and FPRs that has been studied in the literature. Chapter 7 is devoted to constructing structure (symmetric, skew-symmetric, Hermitian, skew-Hermitian, Hamiltonian, skew-Hamiltonian, para-Hermitian, para-skew-Hermitian) preserving Rosenbrock strong linearizations for corresponding structured rational matrices.

1.1 Preliminaries

Throughout this thesis, we use $\mathbb{C}^n$ and $\mathbb{C}^{m\times n}$ to denote the vector space of n-tuples $[x_1, \ldots, x_n]^T$, $x_i \in \mathbb{C}$, and the vector space of $m \times n$ matrices with entries from $\mathbb{C}$, respectively. Let $A \in \mathbb{C}^{m\times n}$. We denote the transpose (resp., conjugate transpose) of A by A^T (resp., A^*). We define the right and left null spaces of A by

$$\mathcal{N}_r(A) := \{x \in \mathbb{C}^n : Ax = 0\} \text{ and } \mathcal{N}_l(A) := \{x \in \mathbb{C}^m : x^T A = 0\},$$

respectively. The Kronecker product of matrices will be used frequently in the thesis.

Definition 1.1.1. *[54] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$. Then the Kronecker product (tensor product) of A and B is defined by*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}$$

where a_{ij} are the entries of A.

We denote the j-th column of the $n \times n$ identity matrix I_n by e_j. The Kronecker product satisfies the following.

- Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{r\times s}, C \in \mathbb{C}^{n\times p}$ and $D \in \mathbb{C}^{s\times t}$. Then

$$(A \otimes B)(C \otimes D) = AC \otimes BD \in \mathbb{C}^{mr\times pt}.$$

- For all A and B, $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)^* = A^* \otimes B^*$.
- If A and B are nonsingular then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Remark 1.1.2. *Let $\mathcal{X} \in \mathbb{C}^{mn\times mn}$ be an $m \times m$ block matrix with $n \times n$ blocks. Note that for $j = 1, \ldots, m$, $(e_j^T \otimes I_n)\mathcal{X}$ and $\mathcal{X}(e_j \otimes I_n)$, respectively, are the j-th block row and j-th block column of $\mathcal{X}$.*

Definition 1.1.3. *For $k, \ell \in \mathbb{Z}$, we use the following notation*

$$k : \ell := \begin{cases} k, k+1, \ldots, \ell & \text{if } k \leq \ell, \\ \emptyset & \text{if } k > \ell. \end{cases}$$

When $k \leq \ell$, $(k : \ell)$ is called a string of integers from k to ℓ. Further, we define $(\infty : \ell) := \emptyset$ for any integer $\ell \in \mathbb{Z}$.

1.2 Polynomial and rational matrices

We denote by $\mathbb{C}[\lambda]$ the polynomial ring over the complex field $\mathbb{C}$ and $\mathbb{C}[\lambda]^{m\times n}$ the vector space of $m \times n$ matrix polynomials over $\mathbb{C}$. Further, we denote by $\mathbb{C}(\lambda)$ the field of rational functions of the form $p(\lambda)/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are scalar polynomials in $\mathbb{C}[\lambda]$. We denote by $\mathbb{C}(\lambda)^n$ the vector space of column n-tuples over the field $\mathbb{C}(\lambda)$ and $\mathbb{C}(\lambda)^{m\times n}$ the vector space of m-by-n matrices over the field $\mathbb{C}(\lambda)$, that is, elements of $\mathbb{C}(\lambda)^{m\times n}$ are $m \times n$ rational matrices.

Since matrix polynomials are rational matrices, by default, many concepts which are defined in this section for rational matrices hold for matrix polynomials as well.

Let $G(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$. Then the rank of $G(\lambda)$ over the field $\mathbb{C}(\lambda)$ is called the *normal rank* of $G(\lambda)$ and is denoted by $\mathrm{nrank}(G)$. Equivalently, $\mathrm{nrank}(G) = \max_{\lambda} \mathrm{rank}(G(\lambda))$, where the maximum is taken over all λ which are not poles of $G(\lambda)$. If $\mathrm{nrank}(G) = n = m$ then $G(\lambda)$ is said to be ***regular***, otherwise $G(\lambda)$ is said to be ***singular.***

Definition 1.2.1 (Eigenvalue, [2, 40, 52])**.** *Let* $G(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$*. A complex number* $\lambda_0 \in \mathbb{C}$ *is said to be an* eigenvalue *(or,* eigenspectrum*) of* $G(\lambda)$ *if* $\mathrm{rank}(G(\lambda_0)) < \mathrm{nrank}(G)$*. We denote the set of eigenvalues of* $G(\lambda)$ *by* $\mathrm{eig}(G)$*.*

If $G(\lambda)$ is regular and $\mu \in \mathbb{C}$ is an eigenvalue of $G(\lambda)$, then the right null space $\mathcal{N}_r(G(\mu))$ and the left null space $\mathcal{N}_l(G(\mu))$ of $G(\mu)$ are given by

$$\begin{aligned}\mathcal{N}_r(G(\mu)) &:= \{x \in \mathbb{C}^n : G(\mu)x = 0\} \subset \mathbb{C}^n,\\ \mathcal{N}_l(G(\mu)) &:= \{y \in \mathbb{C}^m : y^T G(\mu) = 0\} \subset \mathbb{C}^m.\end{aligned}$$

If $G(\lambda)$ is singular, then the right null space $\mathcal{N}_r(G)$ and the left null space $\mathcal{N}_l(G)$ of $G(\lambda)$ are given by

$$\begin{aligned}\mathcal{N}_r(G) &:= \{x(\lambda) \in \mathbb{C}(\lambda)^n : G(\lambda)x(\lambda) = 0\} \subset \mathbb{C}(\lambda)^n,\\ \mathcal{N}_l(G) &:= \{y(\lambda) \in \mathbb{C}(\lambda)^m : y^T(\lambda)G(\lambda) = 0\} \subset \mathbb{C}(\lambda)^m.\end{aligned}$$

For $x(\lambda) := (x_1(\lambda), \ldots, x_k(\lambda))^T \in \mathbb{C}[\lambda]^k$, the *degree* of $x(\lambda)$, denoted by $\deg(x)$, is the greatest degree of its components $x_i(\lambda)$, i.e., $\deg(x) = \max_{1\leq i\leq k} \deg(x_i)$.

Definition 1.2.2. *[31, 40] Let* $G(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ *be singular. Let* $\mathcal{B} := \big(x_1(\lambda), \ldots, x_p(\lambda)\big)$ *be a polynomial basis of* $\mathcal{N}_r(G)$ *ordered so that* $\deg(x_1) \leq \cdots \leq \deg(x_p)$*, where* $x_i(\lambda)$*,* $i = 1 : p$*, are vector polynomials in* $\mathbb{C}[\lambda]^n$*. Then*

$$\mathrm{Ord}(\mathcal{B}) := \deg(x_1) + \cdots + \deg(x_p)$$

is called the order *of the basis* $\mathcal{B}$*.*

Definition 1.2.3. *[31, 40] Let* $G(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ *be singular and* $\mathcal{B} := \big(x_1(\lambda), \ldots, x_p(\lambda)\big)$ *be a polynomial basis of* $\mathcal{N}_r(G)$ *ordered so that* $\deg(x_1) \leq \cdots \leq \deg(x_p)$*. Then* $\mathcal{B}$ *is said to be a* minimal polynomial basis *of* $\mathcal{N}_r(G)$ *(or a* **right minimal basis** *of* $G(\lambda)$*) if* $\mathcal{E}$ *is any polynomial basis of* $\mathcal{N}_r(G)$ *then* $\mathrm{Ord}(\mathcal{E}) \geq \mathrm{Ord}(\mathcal{B})$*. If* $\mathcal{B}$ *is a minimal basis of* $\mathcal{N}_r(G)$*, then* $\deg(x_1) \leq \cdots \leq \deg(x_p)$ *are called the* **right minimal indices** *of* $G(\lambda)$*. A* **left minimal basis** *and the* **left minimal indices** *of* $G(\lambda)$ *are defined similarly.*

We say that a $k \times p$ matrix polynomial $Z(\lambda)$ is a minimal basis if the columns of $Z(\lambda)$ form a minimal basis of the subspace of $\mathbb{C}(\lambda)^k$ spanned (over the field $\mathbb{C}(\lambda)$) by the columns of $Z(\lambda)$.

A matrix polynomial $U(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ is said to be ***unimodular*** if $\det U(\lambda)$ (where det denotes the determinant) is a nonzero constant independent of λ. Next, we describe the Smith-McMillan form of a rational matrix $G(\lambda)$, which will be used to define structural indices associated with $G(\lambda)$.

Theorem 1.2.4 (Smith-McMillan form, [2, 8, 40, 52])**.** *Let* $G(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ *with normal rank* r*. Then there exist unimodular matrix polynomials* $U(\lambda)$ *and* $V(\lambda)$ *of sizes* $m \times m$ *and* $n \times n$*, respectively, such that* $G(\lambda) = U(\lambda)\,\mathbf{SM}(G(\lambda))\,V(\lambda)$*, where*

$$\mathbf{SM}(G(\lambda)) = \mathrm{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \frac{\phi_2(\lambda)}{\psi_2(\lambda)}, \ldots, \frac{\phi_r(\lambda)}{\psi_r(\lambda)}, 0, \ldots, 0\right) \tag{1.1}$$

is called the Smith-McMillan *form of* $G(\lambda)$*. The scalar polynomials* $\phi_i(\lambda)$ *and* $\psi_i(\lambda)$ *are monic (have the highest degree coefficient equal to 1), are pairwise coprime (do not have common divisors) for* $i = 1, 2, \ldots, r$ *and satisfy the divisibility properties:* $\phi_i(\lambda)/\phi_{i+1}(\lambda)$ *and* $\psi_{i+1}(\lambda)/\psi_i(\lambda)$*,* $i = 1, 2, \ldots, r-1$*. The polynomials* $\phi_1(\lambda), \ldots, \phi_r(\lambda)$ *and* $\psi_1(\lambda), \ldots, \psi_r(\lambda)$ *are called* invariant zero polynomials *and* invariant pole polynomials *of* $G(\lambda)$*, respectively. The* zero polynomial $\phi_G(\lambda)$ *and the* pole polynomial $\psi_G(\lambda)$ *of* $G(\lambda)$ *are defined by*

$$\phi_G(\lambda) := \phi_1(\lambda)\,\phi_2(\lambda)\cdots\phi_r(\lambda) \text{ and } \psi_G(\lambda) := \psi_1(\lambda)\,\psi_2(\lambda)\cdots\psi_r(\lambda).$$

If $G(\lambda)$ is a matrix polynomial then in (1.1) $\psi_i(\lambda) = 1$ for all $i = 1 : r$, and in that case (1.1) is called the *Smith* form of $G(\lambda)$.

Definition 1.2.5 (Zeros and poles, [2, 40, 52, 59])**.** *Let* $G(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$*. Let* $\phi_G(\lambda)$ *and* $\psi_G(\lambda)$ *be the zero and the pole polynomials of* $G(\lambda)$*, respectively. A complex number* λ *is said to be a* zero *of* $G(\lambda)$ *if* $\phi_G(\lambda) = 0$*. A complex number* λ *is said to be a* pole *of* $G(\lambda)$ *if* $\psi_G(\lambda) = 0$*. We denote the set of zeros and poles of* $G(\lambda)$ *by* $Sp(G)$ *and* $Poles(G)$*, respectively.*

Definition 1.2.6 (Eigenpole, [2, 8])**.** *A complex number* λ_0 *is said to be an* eigenpole *of* $G(\lambda)$ *if* λ_0 *is a pole of* $G(\lambda)$ *and there exists* $v(\lambda) \in \mathbb{C}[\lambda]^n$ *with* $v(\lambda_0) \neq 0$ *such that* $\lim\limits_{\lambda \to \lambda_0} G(\lambda)v(\lambda) = 0$. *We denote the set of eigenpoles of* $G(\lambda)$ *by* $\mathrm{eip}(G)$.

Remark 1.2.7. *For a rational matrix* $G(\lambda)$*, it is possible that* $Sp(G) \cap poles(G) \neq \emptyset$*. In fact, we have* $\mathrm{eig}(G) \subset Sp(G)$, $\mathrm{eip}(G) \subset Sp(G)$, $\mathrm{eig}(G) \cap \mathrm{eip}(G) = \emptyset$ *and* $Sp(G) = \mathrm{eig}(G) \cup \mathrm{eip}(G)$. *See [2, 8] for further details..*

We end this section by defining the structural indices associated with poles and zeros of a rational matrix.

Definition 1.2.8 (Partial multiplicities of zeros, [2, 8])**.** *Let* $\lambda_0 \in Sp(G)$*. Then* $\phi_G(\lambda_0) = 0$ *and* $\phi_i(\lambda) = (\lambda - \lambda_0)^{\gamma_i} d_i(\lambda)$ *with* $d_i(\lambda_0) \neq 0$ *and* $\gamma_i \geq 0$ *for* $i = 1 : r$*. The index tuple* $Ind_s(\lambda_0, G) := (\gamma_1, \ldots, \gamma_r)$ *is called the* multiplicity index *of* G *at* λ_0 *and satisfies the condition* $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_r$*. The nonzero components in* $Ind_s(\lambda_0, G)$ *are called the* partial multiplicities *of* λ_0 *as a zero of* $G(\lambda)$*. The factors* $(\lambda - \lambda_0)^{\gamma_i}$ *with* $\gamma_i \neq 0$ *are called the* elementary divisors *of* $G(\lambda)$ *at* λ_0*. The* algebraic multiplicity *of* λ_0 *is defined by* $m_s(\lambda_0) := \gamma_1 + \gamma_2 + \cdots + \gamma_r =$ *multiplicity of* λ_0 *as a root of* $\phi_G(\lambda)$*. If* $m_s(\lambda_0) = 1$ *then* λ_0 *is called a* simple eigenvalue *of* $G(\lambda)$.

Definition 1.2.9 (Partial multiplicities of poles, [2, 8])**.** *Let* $\lambda_0 \in Poles(G)$*. Then* $\psi_G(\lambda_0) = 0 \Rightarrow \psi_i(\lambda) = (\lambda - \lambda_0)^{\alpha_i} q_i(\lambda)$ *with* $q_i(\lambda_0) \neq 0$ *and* $\alpha_i \geq 0$ *for* $i = 1 : r$*. The index tuple* $Ind_p(\lambda_0, G) := (\alpha_r, \alpha_{r-1} \ldots, \alpha_1)$ *is called the* multiplicity index *of* G *at* λ_0 *and satisfies the condition* $\alpha_r < \alpha_{r-1} < \cdots < \alpha_1$*. The nonzero components in* $Ind_p(\lambda_0, G)$ *are called the* partial multiplicities *of* λ_0 *as a pole of* $G(\lambda)$*. The factors* $(\lambda - \lambda_0)^{\alpha_i}$ *with* $\alpha_i \neq 0$ *are called* elementary divisors *of* $G(\lambda)$ *at the pole* λ_0*. The* algebraic multiplicity *of* λ_0 *is defined by* $m_p(\lambda_0) := \alpha_1 + \alpha_2 + \cdots + \alpha_r =$ *multiplicity of* λ_0 *as a root of* $\psi_G(\lambda)$*. If* $m_p(\lambda_0) = 1$ *then* λ_0 *is called a* simple pole *of* $G(\lambda)$.

1.2.1 System matrix associated with a rational matrix

A rational matrix $G(\lambda)$ is said to be *proper* if $G(\lambda) \to D$ as $\lambda \to \infty$, where D is a matrix. On the other hand, $G(\lambda)$ is said to be *nonproper* if $\|G(\lambda)\| \to \infty$ as $\lambda \to \infty$. Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$. Then $G(\lambda)$ can be written uniquely as $G(\lambda) = P(\lambda) + Q(\lambda)$, where $P(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ and $Q(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$ is a *strictly proper* rational matrix, i.e., $Q(\lambda) \to 0$ as $\lambda \to \infty$. Further $Q(\lambda)$ can be written as $Q(\lambda) = C(\lambda E - A)^{-1}B$, where $\lambda E - A$ is an $r \times r$ matrix pencil with E being nonsingular, $C \in \mathbb{C}^{n\times r}$ and $B \in \mathbb{C}^{r\times n}$. Thus $G(\lambda)$ can be written as

$$G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B, \tag{1.2}$$

and is called a ***realization*** of $G(\lambda)$. A realization of $G(\lambda)$ is said to be ***minimal realization*** if the size of the pencil $\lambda E - A$ is the smallest among all the realizations of $G(\lambda)$, see [40]. The matrix polynomial $\mathcal{S}(\lambda)$ given by

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right] \tag{1.3}$$

is called the ***system matrix*** (or the ***Rosenbrock system matrix***) of $G(\lambda)$ associated with the realization (1.2) of $G(\lambda)$. The system matrix $\mathcal{S}(\lambda)$ is said to be *irreducible* if the realization (1.2) is minimal. Note that the system matrix $\mathcal{S}(\lambda)$ is irreducible if and only if $\operatorname{rank}\left(\begin{bmatrix} B & A - \lambda E \end{bmatrix}\right) = r = \operatorname{rank}\left(\begin{bmatrix} C^T & (A - \lambda E)^T \end{bmatrix}^T\right)$, see [2, 40, 52, 24].

Remark 1.2.10. *Unless stated otherwise throughout the thesis we consider only minimal realization (1.2) of the rational matrix $G(\lambda)$. Hence the system matrix $\mathcal{S}(\lambda)$ as given (1.3) is always irreducible.*

The following result establish the relation between the structural indices of $G(\lambda)$ and its associated system matrix $\mathcal{S}(\lambda)$.

Theorem 1.2.11. *[2, 40, 52] Let $G(\lambda)$ and $\mathcal{S}(\lambda)$ be as given in (1.2) and (1.3), respectively. Then the zeros of $G(\lambda)$ are the same as the eigenvalues of $\mathcal{S}(\lambda)$ and the poles of $G(\lambda)$ are the same as the eigenvalues of $\lambda E - A$.*

1.2.2 Linearizations of matrix polynomials

For a matrix polynomial $P(\lambda) := \sum_{i=0}^{k} \lambda^i A_i \in \mathbb{C}[\lambda]^{m \times n}$, the reversal of $P(\lambda)$ is defined by [46]

$$\mathit{rev}P(\lambda) := \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^i A_{k-i} \text{ for } \lambda \in \mathbb{C}. \tag{1.4}$$

Definition 1.2.12 (Eigenvalue at ∞, [46])**.** *Let $P(\lambda)$ be a regular matrix polynomial of degree $k \geq 1$. Then ∞ is said to be an eigenvalue of $P(\lambda)$ if 0 is an eigenvalue of $\mathit{rev}P(\lambda)$.*

Note that for a regular $P(\lambda)$, λ_0 is an eigenvalue of $P(\lambda)$ iff $1/\lambda_0$ is an eigenvalue of $\mathit{rev}P(\lambda)$ with 0 and ∞ are considered as reciprocals, see [46]. If $P(\lambda)$ is regular, then the polynomial eigenvalue problem (PEP) is to find eigenvalues $\mu \in \mathbb{C}$ and nonzero eigenvectors $x \in \mathbb{C}^n$, $y \in \mathbb{C}^m$ such that $P(\mu)x = 0$ and $y^T P(\mu) = 0$. If $P(\lambda)$ is singular, then the PEP is to find the minimal bases and minimal indices of $P(\lambda)$.

Linearization is a classical technique widely used for solving polynomial eigenproblems in which a matrix polynomial $P(\lambda)$ is transformed to a matrix pencil $L(\lambda) = \lambda X + Y$ of larger size with the same eigenvalues, and solve the problem for the pencil $L(\lambda)$, see [33, 46]. Formally, linearization is defined as follows.

Definition 1.2.13 (Linearization, [33, 46])**.** *Let* $P(\lambda)$ *be an* $n \times n$ *matrix polynomial of degree* m*. A pencil* $L(\lambda) = \lambda X + Y$ *with* $X, Y \in \mathbb{C}^{mn \times mn}$ *is called a linearization of* $P(\lambda)$ *if there exist* $mn \times mn$ *unimodular matrix polynomials* $U(\lambda)$ *and* $V(\lambda)$ *such that*

$$U(\lambda)L(\lambda)V(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & P(\lambda) \end{array}\right]$$

for all $\lambda \in \mathbb{C}$*. Additionally, if* $\mathrm{rev}L(\lambda) := \lambda Y + X$ *is a linearization of* $\mathrm{rev}P(\lambda)$ *then* $L(\lambda)$ *is said to be a* strong linearization *of* $P(\lambda)$*.*

It is clear from the above definition that $\mathrm{rank}(P(\lambda)) < \mathrm{nrank}(P) \Leftrightarrow \mathrm{rank}(L(\lambda)) < \mathrm{nrank}(L)$. This implies that $\mathrm{eig}(L) = \mathrm{eig}(P)$.

For a given matrix polynomial, there exist infinitely many linearizations. Fiedler pencils (FPs), generalized Fiedler pencils (GFPs), Fiedler pencils with repetition (FPRs), and generalized Fiedler pencils with repetition (GFPRs) are well known classes of strong linearizations of matrix polynomials which have been studied extensively in recent years, see [6, 17, 15, 13, 16, 19, 61] and the references therein. Next, we define FPs, GFPs, FPRs and GFPRs of a matrix polynomial. Henceforth, we assume that $P(\lambda)$ is an $n \times n$ matrix polynomial of degree m and is given by

$$P(\lambda) := \sum\nolimits_{i=0}^{m} \lambda^i A_i. \tag{1.5}$$

The Fiedler matrices M_i^P, $i = \pm 0, \pm 1, \ldots, \pm m$, associated with $P(\lambda)$ are defined as follows [6, 15, 61]:

$$M_0^P := \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix}, \quad M_{-m}^P := \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix},$$

$$M_i^P := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ and } M_{-i}^P := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}$$

for $i = 1, \ldots, m-1$, and $M_{-0}^P := (M_0^P)^{-1}$ and $M_m^P := (M_{-m}^P)^{-1}$. The Fiedler matrices $M_i^P, i = \pm 0, \pm 1, \ldots, \pm m$, satisfy the following properties, see [6, 15, 61].

- M_i^P is invertible for $i = \pm 1, \ldots, \pm (m-1)$.

- $M_{\pm m}^P$ (resp., $M_{\pm 0}^P$) is invertible if and only if A_m (resp., A_0) is invertible.

- $M_i^P M_j^P = M_j^P M_i^P$ for $||i| - |j|| > 1$.

Definition 1.2.14. *[4] Let N be a finite set. A bijection $\omega : N \to N$ is called a permutation of N. τ is said to be a sub-permutation of N if τ is a permutation of a subset of N. We denote the empty permutation by $\emptyset$. Two sub-permutations σ and τ of N are said to be disjoint if $\sigma \cap \tau = \emptyset$, where the intersection is defined to be the intersection of the underlying subsets of the sub-permutations. If $\omega := (\sigma, \tau)$ is a permutation of N then the sub-permutations σ and τ are said to be a partition of ω.*

For a sub-permutation $\alpha := (i_1, i_2, \ldots, i_p)$ of $\{\pm 0, \pm 1, \ldots, \pm m\}$, we define

$$M_\alpha^P := M_{i_1}^P M_{i_2}^P \cdots M_{i_p}^P \text{ if } \alpha \neq \emptyset, \text{ and } M_\alpha^P := I_{mn} \text{ if } \alpha = \emptyset.$$

A Fiedler pencil of $P(\lambda)$ is defined as follows.

Definition 1.2.15 (Fiedler pencils (FPs), [26])**.** *Let $M_0^P, \ldots, M_{m-1}^P$ and M_{-m}^P be the Fiedler matrices associated with $P(\lambda)$. Then, for given any permutation σ of $\{0 : m-1\}$, the $mn \times mn$ matrix pencil*

$$L_\sigma(\lambda) := \lambda M_{-m}^P - M_\sigma^P$$

is called the Fiedler pencil *(FP) of $P(\lambda)$ associated with σ.*

A Generalized Fiedler pencil of $P(\lambda)$ is defined as follows.

Definition 1.2.16 (Generalized Fiedler pencils (GFPs), [15])**.** *Let $\omega := (\omega_0, \omega_1)$ be a permutation of $\{0 : m\}$. Then the $mn \times mn$ matrix pencil*

$$T_\omega(\lambda) := \lambda M_{-\omega_1}^P - M_{\omega_0}^P$$

is said to be a generalized Fiedler pencil (GFP) of $P(\lambda)$ associated with ω. In particular, if $0 \in \omega_0$ and $m \in \omega_1$, then $T_\omega(\lambda)$ is said to be a proper generalized Fiedler (PGF) pencil of $P(\lambda)$. Otherwise, $T_\omega(\lambda)$ is said to be a non-proper generalized Fiedler (NPGF) pencil of $P(\lambda)$.

Sometime we write GFP as GF pencil. Note that FPs of $P(\lambda)$ are subclasses of the class of GFPs of $P(\lambda)$. It has been shown that FPs and GFPs of $P(\lambda)$ are strong linearizations of $P(\lambda)$ [26, 15].

1.2.3 Recovery of eigenvectors and minimal bases from FPs and GFPs

Recovery of eigenvectors, minimal bases and minimal indices of $P(\lambda)$ from those of the FPs and GFPs are well known [26, 15]. We present these results for ready reference.

Definition 1.2.17 (Index tuple, [4])**.** *A tuple* $\sigma := (j_1, j_2, \ldots, j_p) \in \mathbb{Z}^p$ *is said to be an index tuple containing indices from* $\mathbb{Z}$. *We define* $-\sigma := (-j_1, -j_2, \ldots, -j_p)$, $rev(\sigma) := (j_p, j_{p-1}, \ldots, j_2, j_1)$ *and* $\sigma + q := (j_1 + q, j_2 + q, , \ldots, j_p + q)$ *for* $q \in \mathbb{Z}$. *For any index tuples* $\alpha_1 := (i_1, \ldots, i_s)$ *and* $\alpha_2 := (j_1, \ldots, j_t)$, *we define* $\alpha_1 \cup \alpha_2 := (\alpha_1, \alpha_2) = (i_1, \ldots, i_s, j_1, \ldots, j_t)$.

Definition 1.2.18. *[17] (a) (Subtuple) Let* α *be an index tuple. An index tuple* β *is said to be a subtuple of* α *if* $\beta = \alpha$ *or if* β *can be obtained from* α *by deleting some indices in* α.

(b) Let α *be an index tuple and* β *be a subtuple of* α. *Then* β *is said to be the subtuple of* α *with indices* $\{i_1, \ldots, i_k\} \subset \alpha$, *if* β *is obtained from* α *by deleting all indices of* α *except* $i_1, \ldots, i_k$.

Example 1.2.19. *Consider an index tuple* α *given by* $\alpha = (1, 2, 0, 3, 0, 2)$. *Then* $(2, 3, 2)$ *and* $(0, 3, 0)$ *are subtuples of* α *but* $(2, 2, 3)$ *and* $(0, 0, 3)$ *are not subtuples of* α. *Further,* $(2, 0, 0, 2)$ *is the subtuple of* α *with indices* $\{0, 2\}$. ■

We now define ***consecutions*** and ***inversions*** of an index tuple which will play a crucial role in the recovery of eigenvectors and minimal bases and will be used extensively in the thesis.

Definition 1.2.20 (Consecutions and inversions)**.** *Let* α *be an index tuple containing indices from* $\{0 : k\}$ *(resp.,* $\{-k : -1\}$*) for some non-negative integer* k. *Suppose that* $t \in \alpha$. *Then we say that* α *has* p *consecutions at* t *if* $(t, t + 1, \ldots, t + p)$ *is a subtuple of* α *but* $(t, t + 1, \ldots, t + p, t + p + 1)$ *is not a subtuple of* α. *We denote the number of consecutions of* α *at* t *by* $c_t(\alpha)$. *If* $t \notin \alpha$ *then we define* $c_t(\alpha) := -1$. *We say that* α *has* q *inversions at* t *if* $(t + q, t + q - 1, \ldots, t)$ *is a subtuple of* α *but* $(t + q + 1, t + q, \ldots, t)$ *is not a subtuple of* α. *We denote the number of inversions of* α *at* t *by* $i_t(\alpha)$. *If* $t \notin \alpha$ *then we define* $i_t(\alpha) := -1$.

Example 1.2.21. *Let* $\alpha := (1, 0, 2, 1, 3, 2, 4, 1, 3, 2, 1)$. *Then* $c_0(\alpha) = 3$ *as* $(0, 1, 2, 3)$ *is a subtuple of* α *but* $(0, 1, 2, 3, 4)$ *is not a subtuple of* α. *Similarly,* $c_3(\alpha) = 1$ *as* $(3, 4)$ *is a subtuple of* α *but* $(3, 4, 5)$ *is not a subtuple of* α. *Further,* $i_0(\alpha) = 1$ *as* $(1, 0)$ *is a subtuple of* α *but* $(2, 1, 0)$ *is not a subtuple of* α. *Similarly, observe that* $i_1(\alpha) = 3$ *and* $i_3(\alpha) = 1$. ■

Remark 1.2.22. *It follows from Definition 1.2.20 that* $c_t(\alpha) = i_t(rev(\alpha))$ *for any index tuple* α *and for any index* t.

Remark 1.2.23. *It is to be noted here that, for any index tuple* α *and* $t \in \alpha$, *we have* $c_t(\alpha) > 0$ *(resp.,* $i_t(\alpha) > 0$*) does not imply that* $i_t(\alpha) = 0$ *(resp.,* $c_t(\alpha) = 0$*). By* *contrast, if* α *is a permutation then* $c_t(\alpha) > 0$ *(resp.,* $i_t(\alpha) > 0$*) implies* $i_t(\alpha) = 0$ *(resp.,* $c_t(\alpha) = 0$*).*

Definition 1.2.24. *Let* σ *be a permutation of* $\{0 : k\}$ *for some integer* $k \geq 1$*. Then the total number consecutions* $c(\sigma)$ *and inversions* $i(\sigma)$ *of* σ *are given by*

$$\begin{aligned} c(\sigma) &:= \textit{cardinality of the set } \{j \in \sigma : c_j(\sigma) \geq 1\}, \\ i(\sigma) &:= \textit{cardinality of the set } \{j \in \sigma : i_j(\sigma) \geq 1\}. \end{aligned}$$

Remark 1.2.25. *Note that, for a permutation* σ *of* $\{0 : k\}$*,* $k \geq 1$*, we have* $c(\sigma)+i(\sigma) = k$*. For example, let* $\sigma := (0, 1, 4, 3, 2, 5)$ *be a permutation of* $\{0 : 5\}$*. Then* $c(\sigma)+i(\sigma) = 5$ *since* $c(\sigma) = 3$ *and* $i(\sigma) = 2$*.*

The next result is a restatement of [15, Theorems 3.2, 4.1 and 4.2] which describes the recovery of eigenvectors, minimal bases and minimal indices of $P(\lambda)$ from those of the GF pencils. We present these results for ready reference.

Theorem 1.2.26. *[15] Let* $T(\lambda) := \lambda M^P_\tau - M^P_\sigma$ *be a GF pencil of* $P(\lambda)$ *associated with a permutation* (σ, ω) *of* $\{0 : m\}$*, where* $\tau := -\omega$*. Define*

$$F^{\text{GF}}(P) := \begin{cases} e^T_{m-c_0(\sigma)} \otimes I_n & \textit{if } 0 \in \sigma \textit{ and } c_0(\sigma) < m \\ A_m^{-1}(e_1^T \otimes I_n) & \textit{if } 0 \in \sigma \textit{ and } c_0(\sigma) = m \\ e^T_{m-s} \otimes I_n & \textit{if } \begin{cases} 0 \in \omega,\ i_0(\omega)+1 \in \sigma \textit{ and} \\ s := i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 < m \end{cases} \\ A_m^{-1}(e_1^T \otimes I_n) & \textit{if } \begin{cases} 0 \in \omega,\ i_0(\omega)+1 \in \sigma \textit{ and} \\ i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 = m \end{cases} \\ e^T_{m-i_0(\omega)} \otimes I_n & \textit{if } 0 \in \omega,\ i_0(\omega) < m \textit{ and } i_0(\omega)+1 \notin \sigma \\ A_0^{-1}(e_m^T \otimes I_n) & \textit{if } 0 \in \omega \textit{ and } i_0(\omega) = m \end{cases}$$

and

$$K^{\text{GF}}(P) := \begin{cases} e^T_{m-i_0(\sigma)} \otimes I_n & \textit{if } 0 \in \sigma \textit{ and } i_0(\sigma) < m \\ A_m^{-T}(e_1^T \otimes I_n) & \textit{if } 0 \in \sigma \textit{ and } i_0(\sigma) = m \\ e^T_{m-s} \otimes I_n & \textit{if } \begin{cases} 0 \in \omega,\ c_0(\omega)+1 \in \sigma \textit{ and} \\ s := c_0(\omega) + i_{c_0(\omega)+1}(\sigma) + 1 < m \end{cases} \\ A_m^{-T}(e_1^T \otimes I_n) & \textit{if } \begin{cases} 0 \in \omega,\ c_0(\omega)+1 \in \sigma \textit{ and} \\ c_0(\omega) + i_{c_0(\omega)+1}(\sigma) + 1 = m \end{cases} \\ e^T_{m-c_0(\omega)} \otimes I_n & \textit{if } 0 \in \omega,\ c_0(\omega) < m \textit{ and } c_0(\omega)+1 \notin \sigma \\ A_0^{-T}(e_m^T \otimes I_n) & \textit{if } 0 \in \omega \textit{ and } c_0(\omega) = m. \end{cases}$$

Define the maps $F^{\text{GF}}(P) : \mathcal{N}_r(T) \to \mathcal{N}_r(P),\ x \mapsto F^{\text{GF}}(P)x$, *and* $K^{\text{GF}}(P) : \mathcal{N}_l(T) \to \mathcal{N}_l(P),\ y \mapsto K^{\text{GF}}(P)y$. *Then we have the following.*

(I) (Regular $P(\lambda)$). *Suppose that $P(\lambda)$ is regular and $\mu \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$. If $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_p)$ are bases of $\mathcal{N}_r(T(\mu))$ and $\mathcal{N}_l(T(\mu))$, respectively, then $(F^{\text{GF}}(P)x_1, \ldots, F^{\text{GF}}(P)x_p)$ and $(K^{\text{GF}}(P)y_1, \ldots, K^{\text{GF}}(P)y_p)$ are bases of $\mathcal{N}_r(P(\mu))$ and $\mathcal{N}_l(P(\mu))$, respectively.*

(II) (Singular $P(\lambda)$). *Suppose that $P(\lambda)$ is singular. (Then note that $0 \in \sigma$ and $-m \in \tau$). Let τ be given by $\tau := (\tau_l, -m, \tau_r)$. Set $\alpha := (-rev(\tau_l), \sigma, -rev(\tau_r))$. Let $c(\alpha)$ and $i(\alpha)$, respectively, be the total number of consecutions and inversions of the permutation α of $\{0 : m-1\}$. (Note that in this case $F^{\text{GF}}(P) = e^T_{m-c_0(\sigma)} \otimes I_n$ and $K^{\text{GF}}(P) = e^T_{m-i_0(\sigma)} \otimes I_n$).*

Right minimal bases. *If $(x_1(\lambda), \ldots, x_p(\lambda))$ is a right minimal basis of $T(\lambda)$ then $(F^{\text{GF}}(P)x_1(\lambda), \ldots, F^{\text{GF}}(P)x_p(\lambda))$ is a right minimal basis of $P(\lambda)$. Further, if $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $T(\lambda)$ then $\varepsilon_1 - i(\alpha) \leq \cdots \leq \varepsilon_p - i(\alpha)$ are the right minimal indices of $P(\lambda)$.*

Left minimal bases. *If $(y_1(\lambda), \ldots, y_p(\lambda))$ is a left minimal basis of $T(\lambda)$ then $(K^{\text{GF}}(P)y_1(\lambda), \ldots, K^{\text{GF}}(P)y_p(\lambda))$ is a left minimal basis of $P(\lambda)$. Further, if $\xi_1 \leq \cdots \leq \xi_p$ are the left minimal indices of $T(\lambda)$ then $\xi_1 - c(\alpha) \leq \cdots \leq \xi_p - c(\alpha)$ are the left minimal indices of $P(\lambda)$.*

Remark 1.2.27. *In Theorem 1.2.26, if $T(\lambda)$ is a PGF pencil (i.e., $0 \in \sigma$ and $-m \in \tau$), then we denote $F^{\text{PGF}}(P) := F^{\text{GF}}(P)$ and $K^{\text{PGF}}(P) := K^{\text{GF}}(P)$, which are given by $F^{\text{PGF}}(P) = e^T_{m-c_0(\sigma)} \otimes I_n$ and $K^{\text{PGF}}(P) = e^T_{m-i_0(\sigma)} \otimes I_n$.*

The following result is a restatement of [15, Theorem 3.4] which describes the recovery of eigenvectors of $P(\lambda)$ corresponding to an eigenvalue at ∞ from those of the GF pencils of $P(\lambda)$.

Theorem 1.2.28. *(Theorem 3.4, [15]) Let $P(\lambda)$ be a regular matrix polynomial. Let $T(\lambda) := \lambda M^P_{-\tau} - M^P_\sigma$ be a GF pencil of $P(\lambda)$ associated with a permutation (σ, τ) of $\{0 : m\}$, where $m \in \tau$. Suppose that ∞ is an eigenvalue of $P(\lambda)$.*

Right eigenvectors at ∞. *Let $\big(x_1, \ldots, x_k\big)$ be a basis of the right eigenspace of $T(\lambda)$ at ∞. Then we have the following.*

(a) If $c_{-m}(\tau) < m$ then $\big((e^T_{c_{-m}(\tau)+1} \otimes I_n)x_1, \ldots, (e^T_{c_{-m}(\tau)+1} \otimes I_n)x_k\big)$ is a basis of the right eigenspaces of $P(\lambda)$ at ∞.

(b) If $c_{-m}(\tau) = m$ then $\big(A_0^{-1}(e^T_m \otimes I_n)x_1, \ldots, A_0^{-1}(e^T_m \otimes I_n)x_k\big)$ is a basis of the right eigenspaces of $P(\lambda)$ at ∞.

Left eigenvectors at ∞. *Let $(y_1, \ldots, y_k)$ be a basis of the left eigenspace of $T(\lambda)$ at ∞. Then we have the following.*

(c) If $i_{-m}(\tau) < m$ then $((e^T_{i_{-m}(\tau)+1} \otimes I_n)y_1, \ldots, (e^T_{i_{-m}(\tau)+1} \otimes I_n)y_k)$ is a basis of the left eigenspaces of $P(\lambda)$ at ∞.

(d) If $i_{-m}(\tau) = m$ then $(A_0^{-T}(e^T_m \otimes I_n)y_1, \ldots, A_0^{-T}(e^T_m \otimes I_n)y_k)$ is a basis of the left eigenspace of $P(\lambda)$ at ∞.

1.2.4 SIP, CSF, RSF, Operation free products

Next, we define Successor Infix Property (SIP), row standard form (rsf) and column standard form (csf) of an index tuple which will be used extensively in the thesis.

Definition 1.2.29 (Successor Infix Property, [61])**.** *Let $\alpha := (i_1, i_2, \ldots, i_t)$ be an index tuple containing indices from $\{0, 1, \ldots, h\}$ for some non-negative integer h. Then α is said to satisfy the Successor Infix Property (**SIP**) if for every pair of indices $i_a, i_b \in \alpha$ with $1 \leq a < b \leq t$ satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ such that $a < c < b$. Let β be an index tuple containing indices from $\{-h, -h+1, \ldots, -1\}$. Then β is said to satisfy the SIP if $\beta + h$ satisfies the SIP.*

Definition 1.2.30. *[61, 14] Let $\tau := (j_1, j_2, \ldots, j_q)$ be an index tuple containing indices from $\mathbb{Z}$ and $\sigma := (i_1, i_2, \ldots, i_t)$ be an index tuple containing indices from $\{0, 1, \ldots, h\}$ for some non-negative integer h. Then:*

(a) j_p is said to be a simple index of τ if $j_p \neq j_k$ for $k = 1 : q$ and $k \neq p$. We say that τ is simple if each index j_p is a simple index for $p = 1 : q$.

*(b) σ is said to be in **column standard form** if*

$$\sigma = (a_s : b_s, a_{s-1} : b_{s-1}, \ldots, a_2 : b_2, a_1 : b_1),$$

with $0 \leq b_1 < \cdots < b_{s-1} < b_s \leq h$ and $0 \leq a_j \leq b_j$, for all $j = 1, \ldots, s$. We denote the column standard form of σ by **csf**(σ)*. Let β be an index tuple containing indices from $\{-h, -h+1, \ldots, -1\}$. Then β is said to be in column standard form if $\beta + h$ is in column standard form.*

*(c) σ is said to be in **row standard form** if*

$$\sigma = (rev(a_1 : b_1), rev(a_2 : b_2), \ldots, rev(a_{s-1} : b_{s-1}), rev(a_s : b_s)),$$

with $0 \leq b_1 < \cdots < b_{s-1} < b_s \leq h$ and $0 \leq a_j \leq b_j$, for all $j = 1, \ldots, s$. We denote the row standard form of σ by **rsf**(σ)*. Let β be an index tuple containing indices* *from* $\{-h, -h+1, \ldots, -1\}$*. Then* β *is said to be in row standard form if* $\beta + h$ *is in row standard form.*

Definition 1.2.31. *[14, 20] Let* α *be a permutation of* $\{0, 1, \ldots, k\}$ *for* $k \geq 0$ *with csf(*α*) being the column standard form of* α*. Then an index* $s \in \{0, 1, \ldots, k-1\}$ *is said to be a right index of type-1 relative to* α *if there is a string* $(s : t)$ *in the csf(*α*) such that* $s < t$*.*

Definition 1.2.32 (Associated simple tuple, [20])**.** *Let* k *be a non-negative integer and* α *be a permutation of* $\{0 : k\}$*. Suppose that csf(*α*)* $= (\boldsymbol{b}_d, \boldsymbol{b}_{d-1}, \ldots, \boldsymbol{b}_1)$*, where* $\boldsymbol{b}_i = (a_{i-1} + 1 : a_i)$ *for* $i = 2 : d$ *and* $\boldsymbol{b}_1 = (0 : a_1)$*. If* s *is a right index of type-1 relative to* α *then the simple tuple associated with* (α, s) *is denoted by* $z_r(\alpha, s)$ *and is given by*

- $z_r(\alpha, s) := (\boldsymbol{b}_d, \boldsymbol{b}_{d-1}, \ldots, \boldsymbol{b}_{h+1}, \widetilde{\boldsymbol{b}}_h, \widetilde{\boldsymbol{b}}_{h-1}, \boldsymbol{b}_{h-2}, \ldots, \boldsymbol{b}_1)$ *if* $0 \neq s = a_{h-1} + 1$*, where*
$$\widetilde{\boldsymbol{b}}_h = (a_{h-1} + 2 : a_h) \text{ and } \widetilde{\boldsymbol{b}}_{h-1} = (a_{h-2} + 1 : a_{h-1} + 1).$$
- $z_r(\alpha, s) := (\boldsymbol{b}_d, \boldsymbol{b}_{d-1}, \ldots, \boldsymbol{b}_2, \widetilde{\boldsymbol{b}}_1, \widetilde{\boldsymbol{b}}_0)$ *if* $s = 0$*, where* $\widetilde{\boldsymbol{b}}_1 = (1 : a_1)$ *and* $\widetilde{\boldsymbol{b}}_0 = (0)$*.*

Definition 1.2.33 (Right and left index tuple of type-1, [20])**.** *Let* α *be a permutation of* $\{0 : k\}$*,* $k \geq 0$*. Let* β_1 *and* β_2 *be tuples with indices from* $\{0 : k-1\}$*.*

(a) We say that $\beta_2 := (s_1, \ldots, s_r)$ *is a right index tuple of type-1 relative to* α *if, for* $i = 1 : r$*,* s_i *is a right index of type-1 relative to* $z_r(\alpha, (s_1, \ldots, s_{i-1}))$*, where* $z_r(\alpha, (s_1, \ldots, s_{i-1})) := z_r(z_r(\alpha, (s_1, \ldots, s_{i-2})), s_{i-1})$ *for* $i > 2$*.*

(b) We say that β_1 *is a left index tuple of type-1 relative to* α *if rev(*β_1*) is a right index tuple of type-1 relative to rev(*α*). Define* $z_l(\beta_1, \alpha) := rev(z_r(rev(\alpha), rev(\beta_1)))$*.*

For an index tuple α containing indices either from $\{0 : m-1\}$ or $\{-m : -1\}$, M^P_α is said to be operation-free if each block entry of M^P_α is either $0, \pm I_n$ or $\pm A_j$ for $j = 0 : m$. In such a case, M^P_α can be computed directly from $P(\lambda)$ without performing any arithmetic operations.

Lemma 1.2.34. *[61, 14] Let* $M^P_{\pm i}, i = 0 : m$*, be the Fiedler matrices associated with* $P(\lambda)$*. Let* α *be an index tuple containing indices either from* $\{0 : m-1\}$ *or* $\{-m : -1\}$*. Then* M^P_α *is operation-free* $\Leftrightarrow$ α *satisfies the SIP* $\Leftrightarrow M^P_\alpha = M^P_{\mathrm{csf}(\alpha)} \Leftrightarrow M^P_\alpha = M^P_{\mathrm{rsf}(\alpha)}$*.*

2

Recovery of Eigenvectors and Minimal Bases of Matrix Polynomials from GFPRs

The main objective of this chapter is to derive recovery rules of eigenvectors and minimal bases of a matrix polynomial $P(\lambda)$ from those of the generalized Fiedler pencils with repetition (GFPRs) of $P(\lambda)$. We show that the eigenvectors and minimal bases of $P(\lambda)$ can be recovered from those of the GFPRs of $P(\lambda)$ without performing any arithmetic operations. Also, we describe the recovery of eigenvectors of $P(\lambda)$ corresponding to the eigenvalue at ∞ from those of the GFPRs. When $P(\lambda)$ is symmetric we describe a simplified rule for the recovery of eigenvectors and minimal bases of $P(\lambda)$ from those of the symmetric GFPRs of $P(\lambda)$. Further, when $P(\lambda)$ is skew-symmetric (resp., T-even, T-odd, T-palindromic), we describe alternative recovery rules for eigenvectors and minimal bases of $P(\lambda)$ from those of the skew-symmetric (resp., T-even, T-odd, T-palindromic) FPRs of $P(\lambda)$. These structured FPRs are known to be type-1 and hence the eigenvector recovery rules are well known [14]. However, our eigenvector recovery rules for these structured type-1 FPRs are simpler and different from the existing recovery rules and can be easily read off from the index tuples defining the FPRs. We mention that the recovery rule for type-1 FPRs given in [14] may not work for type-1 GFPRs, see Example 2.1.19. Finally, we present algorithms for constructing FPRs and GFPRs of $P(\lambda)$ (without going through the explicit multiplication of the elementary and Fiedler matrices).

2.1 Elementary matrices and GFPRs

For rest of this chapter, we consider $P(\lambda)$ as an $n \times n$ matrix polynomial having degree m and is given by $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$. We now define the Fiedler and elementary matrices associated with $P(\lambda)$ which will be used throughout the thesis. For an arbitrary matrix $X \in \mathbb{C}^{n\times n}$, we consider the following elementary matrices [17]:

$$M_0(X) := \begin{bmatrix} I_{(m-1)n} & \\ & X \end{bmatrix}, \quad M_{-m}(X) := \begin{bmatrix} X & \\ & I_{(m-1)n} \end{bmatrix},$$

$$M_i(X) := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & X & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ and } M_{-i}(X) := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & X & \\ & & & I_{(i-1)n} \end{bmatrix}$$

for $i = 1 : m-1$. Note that, for $i = 1 : m-1$, $M_i(X)$ and $M_{-i}(X)$ are invertible for any X. Moreover, $(M_i(X))^{-1} = M_{-i}(-X)$. On the other hand, the matrices $M_0(X)$ and $M_{-m}(X)$ are invertible if and only if X is invertible. We define $M_{-0}(X) := (M_0(X))^{-1}$ and $M_m(X) := (M_{-m}(X))^{-1}$. Note that $M_i(X)M_j(Y) = M_j(Y)M_i(X)$ holds for any matrices $X, Y \in \mathbb{C}^{n\times n}$ if $||i| - |j|| > 1$.

For $i \in \{-m : m-1\}$, we define [17]

$$M_i^P := \begin{cases} M_i(-A_i) & \text{if } i \geq 0, \\ M_i(A_{-i}) & \text{if } i < 0. \end{cases}$$

Recall from Section 1.2.2 that M_i^P, $i \in \{-m : m-1\}$, are the Fielder matrices associated with $P(\lambda)$ and are given by [17]:

$$M_0^P = \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix}, \quad M_i^P = \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ for } i = 1 : m-1,$$

$$M_{-m}^P = \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix}, \text{ and } M_{-i}^P = \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & A_i & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ for } i = 1 : m-1.$$

Note that M_0^P (resp., M_{-m}^P) is invertible if and only if A_0 (resp., A_m) is invertible. We define $M_{-0}^P := (M_0^P)^{-1}$ and $M_m^P := (M_{-m}^P)^{-1}$.

We now introduce some notation that will be used throughout the thesis and will be used to describe the recovery of eigenvectors and minimal bases of $P(\lambda)$ from those of the GFPRs of $P(\lambda)$.

Definition 2.1.1 (Matrix assignments, [17])**.** *Let* $\boldsymbol{t} = (t_1, t_2, \ldots, t_r)$ *be a nonempty index tuple containing indices from* $\{\pm 0, \pm 1, \ldots, \pm m\}$ *and* $X := (X_1, X_2, \ldots, X_r)$ *be a tuple of* $n \times n$ *matrices. We define*

$$M_{\boldsymbol{t}}(X) := M_{t_1}(X_1) M_{t_2}(X_2) \cdots M_{t_r}(X_r)$$

and say that X *is a matrix assignment for* $\boldsymbol{t}$*. Further, we say that the matrix* X_j *is assigned to the position* j *in* $\boldsymbol{t}$*. The matrix assignment* X *for* $\boldsymbol{t}$ *is said to be nonsingular if the matrices assigned by* X *to the positions in* $\boldsymbol{t}$ *occupied by the* ± 0 *and* $\pm m$ *indices are nonsingular.*

A matrix assignment X is said to be a symmetric (resp., Hermitian) matrix assignment if all the matrices in X are symmetric (resp., Hermitian).

If $\mathbf{t}$ is empty, by default we define $M_{\mathbf{t}}(X) := I_{mn}$. Further, if $X^1, \ldots, X^s$ are matrix assignments for the index tuples $\mathbf{t}_1, \ldots, \mathbf{t}_s$, respectively, then we define [17]

$$M_{(\mathbf{t}_1, \ldots, \mathbf{t}_s)}(X^1, \ldots, X^s) := M_{\mathbf{t}_1}(X^1) \cdots M_{\mathbf{t}_s}(X^s).$$

We define $M_{\mathbf{t}}^P := M_{t_1}^P M_{t_2}^P \cdots M_{t_r}^P$ if $\mathbf{t} := (t_1, t_2, \ldots, t_r)$ is a nonempty index tuple, and $M_{\mathbf{t}}^P := I_{mn}$ if $\mathbf{t}$ is an empty tuple. If $\mathbf{t}_1, \ldots, \mathbf{t}_s$ are index tuples then $M_{\mathbf{t}_1, \ldots, \mathbf{t}_s}^P = M_{\mathbf{t}_1}^P \cdots M_{\mathbf{t}_s}^P$, see [17] for further details.

Definition 2.1.2. *[17] Let* $\boldsymbol{t} = (t_1, t_2, \ldots, t_r)$ *be an index tuple containing indices from* $\{\pm 0, \pm 1, \ldots, \pm m\}$*. Then* $X := (X_1, X_2, \ldots, X_r)$ *is said to be the trivial matrix assignment associated with* $P(\lambda)$ *if* $M_{t_j}(X_j) = M_{t_j}^P$ *for* $j = 1 : r$*.*

Remark 2.1.3. *Note that if* X *is the trivial matrix assignment for* $\boldsymbol{t}$ *associated with* $P(\lambda)$*, then* $M_{\boldsymbol{t}}(X) = M_{\boldsymbol{t}}^P$*. For an index tuple* $\boldsymbol{t}$*,* $M_{\boldsymbol{t}}(X)$ *is invertible if* X *is an invertible matrix assignment.*

Definition 2.1.4. *[17] (a) We say that two nonnegative indices* i *and* j *in an index tuple commute if* $|i - j| \neq 1$*.*

(b) Let $\boldsymbol{t}$ *and* $\widehat{\boldsymbol{t}}$ *be index tuples of nonnegative integers. We say that* $\widehat{\boldsymbol{t}}$ *is obtained from* $\boldsymbol{t}$ *by a transposition if* $\widehat{\boldsymbol{t}}$ *is obtained from* $\boldsymbol{t}$ *by interchanging two distinct commuting indices in adjacent positions. If* i *and* $i+1$ *are the positions of the interchanged indices in* $\boldsymbol{t}$*, then we call the transposition that produces* $\widehat{\boldsymbol{t}}$ *from* $\boldsymbol{t}$ *to be the permutation of* $\{1 : |\boldsymbol{t}|\}$

obtained by interchanging i and $i+1$ in the identity permutation $(1 : |\boldsymbol{t}|)$, where $|\boldsymbol{t}|$ denotes the number of indices in $\boldsymbol{t}$.

(c) Given two index tuples $\boldsymbol{t}$ and $\widehat{\boldsymbol{t}}$ of nonnegative integers, we say that $\boldsymbol{t}$ is equivalent to $\widehat{\boldsymbol{t}}$ if $\boldsymbol{t} = \widehat{\boldsymbol{t}}$ or if $\widehat{\boldsymbol{t}}$ can be obtained from $\boldsymbol{t}$ by a sequence of transpositions. If $\boldsymbol{t}$ and $\widehat{\boldsymbol{t}}$ are index tuples of negative integers and k is the minimum index among the indices in $\boldsymbol{t}$ and $\widehat{\boldsymbol{t}}$, then we say that $\boldsymbol{t}$ is equivalent to $\widehat{\boldsymbol{t}}$ if $-k+\boldsymbol{t}$ is equivalent to $-k+\widehat{\boldsymbol{t}}$. If $\boldsymbol{t}$ and $\widehat{\boldsymbol{t}}$ are equivalent index tuples, we write $\boldsymbol{t} \sim \widehat{\boldsymbol{t}}$.

(d) Let $\boldsymbol{t}$ and $\widehat{\boldsymbol{t}}$ be two equivalent index tuples of nonnegative (resp., negative) integers. Suppose that $\widehat{\boldsymbol{t}}$ (resp., $-k+\widehat{\boldsymbol{t}}$, where k is the minimum index in $\boldsymbol{t}$) is obtained from $\boldsymbol{t}$ (resp., from $-k+\boldsymbol{t}$) by an ordered sequence of transpositions $\sigma_1, \ldots, \sigma_s$. Then we say that the composition $\sigma = \sigma_1 \circ \cdots \circ \sigma_s$ is the allowed permutation that transforms $\boldsymbol{t}$ to $\widehat{\boldsymbol{t}}$.

Let $\mathbf{t} := (t_1, t_2, \ldots, t_r)$ be an index tuple and $X := (X_1, X_2, \ldots, X_r)$ be a matrix assignment for $\mathbf{t}$. Then for any permutation α of $\{1 : r\}$, we define $\alpha(X) := (X_{\alpha(1)}, X_{\alpha(2)}, \ldots, X_{\alpha(r)})$. Let α and β be the allowed permutations that transfer $\mathbf{t}$ to $rsf(\mathbf{t})$ and $csf(\mathbf{t})$, respectively, where $rsf(\mathbf{t})$ is the row standard form of $\mathbf{t}$ and $csf(\mathbf{t})$ is the column standard form of $\mathbf{t}$. Then we denote $\alpha(X)$ by $rsf(X)$ and $\beta(X)$ by $csf(X)$. Similarly, we denote by $rev(X)$ the matrix assignment for $\mathbf{t}$ obtained from X by reversing the order of the matrices, i.e., $rev(X) := (X_r, X_{r-1}, \ldots, X_2, X_1)$.

Example 2.1.5. *Let $\boldsymbol{t} = (2,1,3,2,0,1)$ and $X = (X_1, X_2, \ldots, X_6)$ be a matrix assignment for $\boldsymbol{t}$. Then $rsf(\boldsymbol{t}) = (2,1,0,3,2,1)$. Let α be the allowed permutation that transfer $\boldsymbol{t}$ to $rsf(\boldsymbol{t})$. Then $rsf(X) = \alpha(X) = (X_{\alpha(1)}, X_{\alpha(2)}, \ldots, X_{\alpha(6)}) = (X_1, X_2, X_5, X_3, X_4, X_6)$.*

The following result will be frequently used in the thesis.

Lemma 2.1.6 ([17], Lemma 4.4)**.** *Let $\boldsymbol{t}$ be an index tuple containing indices from either $\{0 : m-1\}$ or $\{-m : -1\}$ and let X be a matrix assignment for $\boldsymbol{t}$. If $\boldsymbol{t} \sim \widehat{\boldsymbol{t}}$ (i.e., $\boldsymbol{t}$ is equivalent to $\widehat{\boldsymbol{t}}$) and α is the allowed permutation that transforms $\boldsymbol{t}$ into $\widehat{\boldsymbol{t}}$ then $M_{\boldsymbol{t}}(X) = M_{\widehat{\boldsymbol{t}}}(\alpha(X))$.*

For an index tuple α containing indices either from $\{0 : m-1\}$ or $\{-m : -1\}$, $M_\alpha(\mathcal{X})$ is said to be operation-free if each block entry of $M_\alpha(\mathcal{X})$ is either 0, $\pm I_n$, or any one of the matrices in the matrix assignment $\mathcal{X}$. In such a case, $M_\alpha(\mathcal{X})$ can be computed directly from $P(\lambda)$ and $\mathcal{X}$ without performing any arithmetic operations. The following result characterizes operation free products.

Lemma 2.1.7. *[61, 14, 17] Let α be an index tuple containing indices from $\{0 : m-1\}$ (resp., $\{-m : -1\}$). Then $M_\alpha(\mathcal{X})$ is operation-free $\Leftrightarrow$ α satisfies the SIP $\Leftrightarrow$ $M_\alpha(\mathcal{X}) = M_{\mathrm{csf}(\alpha)}(csf(\mathcal{X}))$ $\Leftrightarrow$ $M_\alpha(\mathcal{X}) = M_{\mathrm{rsf}(\alpha)}(rsf(\mathcal{X}))$.*

Remark 2.1.8. *Let α be an index tuple containing indices from $\{0 : m-1\}$ (resp., $\{-m : -1\}$) such that α satisfies the SIP. Then the positions of the block entries of $M_\alpha(\mathcal{X})$ do not depend upon the particular matrix assignment $\mathcal{X}$, that is, the positions of the block entries of $M_\alpha(\mathcal{X})$ depend only on α, see [17].*

The Fiedler pencils with repetition (FPRs) and generalized Fiedler pencils with repetition (GFPRs) of $P(\lambda)$ are defined as follows.

Definition 2.1.9 (FPRs and GFPRs, [61, 14, 17])**.** *Let $0 \leq h \leq m-1$ and let σ and τ be permutations of $\{0 : h\}$ and $\{-m : -h-1\}$, respectively. Let σ_j and τ_j, $j = 1, 2$, be index tuples containing indices from $\{0 : h-1\}$ and $\{-m : -h-2\}$, respectively, such that $(\sigma_1, \sigma, \sigma_2)$ and (τ_1, τ, τ_2) satisfy the SIP. Let X_1, X_2, Y_1 and Y_2 be nonsingular matrix assignments for $\sigma_1, \sigma_2, \tau_1$ and τ_2, respectively. Then the pencil*

$$L(\lambda) = M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) \tag{2.1}$$

is said to be a GFPR of $P(\lambda)$. In particular, if X_j and Y_j, for $j = 1, 2$, are the trivial matrix assignments then $L(\lambda) = M^P_{\tau_1}M^P_{\sigma_1}(\lambda M^P_\tau - M^P_\sigma)M^P_{\sigma_2}M^P_{\tau_2}$ is said to be an FPR of $P(\lambda)$.

Caution: In [61, 14, 17], the FPRs and GFPRs are defined by excluding the condition that X_1, X_2, Y_1 and Y_2 are nonsingular matrix assignments. Note that $L(\lambda)$ is not a linearization of $P(\lambda)$ when any of the matrix assignments X_1, X_2, Y_1 and Y_2 is singular. Hence for simplicity we always consider that X_1, X_2, Y_1 and Y_2 are nonsingular matrix assignments.

2.1.1 Recovery of eigenvectors and minimal bases

This section is devoted to describe the recovery of eigenvectors and minimal bases of a matrix polynomial from those of its FPRs and GFPRs. The following remark will be frequently used in the rest of this chapter.

Remark 2.1.10. *Let σ_ℓ and τ_ℓ, for $\ell = 1, 2$, be as in Definition 2.1.9. For $i \in \sigma_1 \cup \sigma_2$ and $j \in \tau_1 \cup \tau_2$, we have $||i| - |j|| > 2$. Hence σ_k and τ_k, $k = 1, 2$, commute with each other, i.e., $M_{\sigma_p}(X)M_{\tau_q}(Y) = M_{\tau_q}(Y)M_{\sigma_p}(X)$ for $p, q \in \{1, 2\}$, where X and Y are any arbitrary matrix assignments.*

The next result will play a crucial role in the recovery of eigenvectors and minimal bases of $P(\lambda)$ from those of the GFPRs of $P(\lambda)$. Recall from Definition 1.2.20 that $c_s(\alpha)$ (resp., $i_t(\alpha)$) denotes the number of consecutions (resp., inversions) of α at s (resp., t).

Lemma 2.1.11. *Let* $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M_{\tau}^{P} - M_{\sigma}^{P})M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ *be a GFPR of* $P(\lambda)$ *as given in (2.1). Then we have* $(e_{m-c_0(\sigma)}^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e_{m-c_0(\sigma,\sigma_2)}^T \otimes I_n$ *and* $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)\ (e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$.

Proof. Let Z be an $n \times n$ arbitrary matrix. Note that for $j = 0, 1, \ldots, m-2$, we have

$$(e_{m-j}^T \otimes I_n)M_{j+1}(Z) = (e_{m-j}^T \otimes I_n)\begin{bmatrix} I_{(m-j-2)n} & & & \\ & Z & I_n & \\ & I_n & 0 & \\ & & & I_{jn} \end{bmatrix} = e_{m-(j+1)}^T \otimes I_n,$$

and for $j = 0, 1, \ldots, m-1$ and $i \notin \{j, j+1\}$, we have

$$(e_{m-j}^T \otimes I_n)M_i(Z) = (e_{m-j}^T \otimes I_n)\begin{bmatrix} I_{(m-i-1)n} & & & \\ & Z & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} = e_{m-j}^T \otimes I_n.$$

This shows that

$$(e_{m-j}^T \otimes I_n)M_i(Z) = \begin{cases} e_{m-(j+1)}^T \otimes I_n & \text{for } i = j+1 \text{ and } j = 0 : m-2, \\ e_{m-j}^T \otimes I_n & \text{for } i \notin \{j, j+1\} \text{ and } j = 0 : m-1. \end{cases} \tag{2.2}$$

Similarly, for $j = 0, 1, \ldots, m-2$, we have

$$M_{j+1}(Z)(e_{m-j} \otimes I_n) = \begin{bmatrix} I_{(m-j-2)n} & & & \\ & Z & I_n & \\ & I_n & 0 & \\ & & & I_{jn} \end{bmatrix}(e_{m-j} \otimes I_n) = e_{m-(j+1)} \otimes I_n,$$

and for $j = 0, 1, \ldots, m-1$ and $i \notin \{j, j+1\}$, we have

$$M_i(Z)(e_{m-j} \otimes I_n) = \begin{bmatrix} I_{(m-i-1)n} & & & \\ & Z & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}(e_{m-j} \otimes I_n) = e_{m-j} \otimes I_n.$$

This shows that

$$M_i(Z)(e_{m-j} \otimes I_n) = \begin{cases} e_{m-(j+1)} \otimes I_n & \text{for } i = j+1 \text{ and } j = 0 : m-2, \\ e_{m-j} \otimes I_n & \text{for } i \notin \{j, j+1\} \text{ and } j = 0 : m-1. \end{cases} \tag{2.3}$$

Now we evaluate $(e^T_{m-c_0(\sigma)} \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ and $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n)$. Since σ is a permutation of $\{0 : h\}$, we have $c_0(\sigma) \leq h$ and $i_0(\sigma) \leq h$. Further, since τ_1 and τ_2 are index tuples containing indices from $\{-m : -(h+2)\}$, we have $M_{\tau_1}(Y_1) = \mathrm{diag}(*, I_{(h+1)n})$ and $M_{\tau_2}(Y_2) = \mathrm{diag}(*, I_{(h+1)n})$. Thus $(e^T_{m-c_0(\sigma)} \otimes I_n)M_{\tau_2}(Y_2) = e^T_{m-c_0(\sigma)} \otimes I_n$ and $M_{\tau_1}(Y_1)(e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma)} \otimes I_n$. Consequently, $(e^T_{m-c_0(\sigma)} \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = (e^T_{m-c_0(\sigma)} \otimes I_n)M_{\sigma_2}(X_2)$ and $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n) = M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n)$ since by Remark 2.1.10 we have $M_{\tau_k}(Y_k)M_{\sigma_k}(X_k) = M_{\sigma_k}(X_k)M_{\tau_k}(Y_k)$ for $k = 1, 2$. It remains to show that $(e^T_{m-c_0(\sigma)} \otimes I_n)M_{\sigma_2}(X_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$ and $M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$. Now there are two cases.

Case-I: Suppose that $h \geq 1$. First, we show that $(e^T_{m-c_0(\sigma)} \otimes I_n)M_{\sigma_2}(X_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$. Since σ has $c_0(\sigma)$ consecutions at 0, we have $\sigma \sim (\sigma^L, 0, 1, \ldots, c_0(\sigma))$ for some subpermutation σ^L of σ. Note that

$$(\sigma_1, \sigma, \sigma_2) \sim \big(\sigma_1, \sigma^L, 0, 1, \ldots, c_0(\sigma), \sigma_2\big) \text{ satisfies the SIP.} \tag{2.4}$$

Suppose that $c_0(\sigma) + 1 \notin \sigma_2$. Then, since $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP, it is clear from (2.4) that $c_0(\sigma) \notin \sigma_2$. Now as $c_0(\sigma), c_0(\sigma) + 1 \notin \sigma_2$, by (2.2), we have $(e^T_{m-c_0(\sigma)} \otimes I_n)M_{\sigma_2}(X_2) = e^T_{m-c_0(\sigma)} \otimes I_n$. Since $c_0(\sigma) + 1 \notin \sigma_2$, it follows from (2.4) that $c_0(\sigma, \sigma_2) = c_0(\sigma)$ which gives the desired result.

Next, suppose that $c_0(\sigma) + 1 \in \sigma_2$. Suppose that the $\mathrm{csf}(\sigma_2)$ is given by $\mathrm{csf}(\sigma_2) = \big((\ell_{h-1} : h-1), (\ell_{h-2} : h-2), \ldots, (\ell_1 : 1), (\ell_0 : 0)\big)$. Since $\sigma_2 \sim csf(\sigma_2)$, without loss of generality, we assume that $\sigma_2 = csf(\sigma_2)$. Let t be the largest integer such that $c_0(\sigma) + 1 \in (\ell_t : t)$. Then $\sigma_2 = \big((\ell_{h-1} : h-1), (\ell_{h-2} : h-2), \ldots, (\ell_t : t), \ldots, (\ell_1 : 1), (\ell_0 : 0)\big) =: (\alpha, (\ell_t : t), \beta)$ and $c_0(\sigma) + 1 \notin \alpha$. Since

$$(\sigma_1, \sigma, \sigma_2) \sim \big(\sigma_1, \sigma^L, 0, 1, \ldots, c_0(\sigma), \alpha, (\ell_t : t), \beta\big) \text{ satisfies the SIP} \tag{2.5}$$

and $c_0(\sigma) + 1 \notin \alpha$, we must have $\ell_t = c_0(\sigma) + 1$. Indeed, if $\ell_t < c_0(\sigma) + 1$ then $c_0(\sigma) \in (\ell_t, t)$. Since $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP, it is clear from (2.5) that $c_0(\sigma) + 1 \in \alpha$ which contradicts that $c_0(\sigma) + 1 \notin \alpha$. Further, since $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP and $c_0(\sigma) + 1 \notin \alpha$, it is clear from (2.5) that $c_0(\sigma) \notin \alpha$. Thus $\sigma_2 = \big(\alpha, (c_0(\sigma) + 1 : t), \beta\big)$, where $c_0(\sigma), c_0(\sigma) + 1 \notin \alpha$. We denote by $(*)$ any arbitrary matrix assignment. Then

$$\begin{aligned}
&(e^T_{m-c_0(\sigma)} \otimes I_n)\, M_\alpha(*)\, M_{(c_0(\sigma)+1:t)}(*)\, M_\beta(*)\\
&= (e^T_{m-c_0(\sigma)} \otimes I_n)\, M_{(c_0(\sigma)+1:t)}(*)\, M_\beta(*) \text{ by (2.2) as } c_0(\sigma), c_0(\sigma)+1 \notin \alpha\\
&= (e^T_{m-c_0(\sigma)} \otimes I_n) M_{c_0(\sigma)+1}(*) M_{c_0(\sigma)+2}(*) \cdots M_t(*)\, M_\beta(*)\\
&= (e^T_{m-t} \otimes I_n)\, M_\beta(*) \text{ by applying (2.2) repeatedly}\\
&= e^T_{m-t} \otimes I_n \text{ by (2.2) as } t, t+1 \notin \beta.
\end{aligned}$$

Thus $(e^T_{m-c_0(\sigma)} \otimes I_n)M_{\sigma_2}(X_2) = e^T_{m-t} \otimes I_n$ as $M_{\sigma_2}(X_2) = M_{(\alpha,(c_0(\sigma)+1:t),\beta)}(X_2)$. Now since $(\sigma, \sigma_2) \sim \big(\sigma^L, 0, 1, \ldots, c_0(\sigma), \alpha, (c_0(\sigma)+1 : t), \beta\big)$ and $c_0(\sigma)+1 \notin \alpha$, it follows that $(0, 1, \ldots, t)$ is a subtuple of (σ, σ_2) and $(0, 1, \ldots, t+1)$ is not a subtuple of (σ, σ_2). Hence $c_0(\sigma, \sigma_2) = t$ which gives the desired result.

Next, we show that $M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$. Since σ has $i_0(\sigma)$ inversions at 0, we have $\sigma \sim (i_0(\sigma), \ldots, 1, 0, \sigma^R)$ for some sub-permutation σ^R of σ. Note that

$$(\sigma_1, \sigma, \sigma_2) \sim \big(\sigma_1, i_0(\sigma), \ldots, 1, 0, \sigma^R, \sigma_2\big) \text{ satisfies the SIP.} \tag{2.6}$$

Suppose that $i_0(\sigma)+1 \notin \sigma_1$. Then, since $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP, it is clear from (2.6) that $i_0(\sigma) \notin \sigma_1$. Now as $i_0(\sigma), i_0(\sigma)+1 \notin \sigma_1$, by (2.3), we have $M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma)} \otimes I_n$. Since $i_0(\sigma)+1 \notin \sigma_1$, it follows from (2.6) that $i_0(\sigma_1, \sigma) = i_0(\sigma)$ which gives the desired result.

Next, suppose that $i_0(\sigma)+1 \in \sigma_1$. Suppose that the rsf$(\sigma_1)$ is given by rsf$(\sigma_1) = \big(rev(r_0 : 0), rev(r_1 : 1), \ldots, rev(r_{h-2} : h-2), rev(r_{h-1} : h-1)\big)$. Since $\sigma_1 \sim rsf(\sigma_1)$, without loss of generality, we assume that $\sigma_1 = rsf(\sigma_1)$. Let k be the largest integer such that $i_0(\sigma)+1 \in rev(r_k : k)$. Then $\sigma_1 = \big(rev(r_0 : 0), rev(r_1 : 1), \ldots, rev(r_k : k), \ldots, rev(r_{h-2} : h-2), rev(r_{h-1} : h-1)\big) =: (\gamma, rev(r_k : k), \delta)$ and $i_0(\sigma)+1 \notin \delta$. Since

$$(\sigma_1, \sigma, \sigma_2) \sim \big(\gamma, rev(r_k : k), \delta, i_0(\sigma), \ldots, 1, 0, \sigma^R, \sigma_2\big) \text{ satisfies the SIP} \tag{2.7}$$

and $i_0(\sigma)+1 \notin \delta$, we must have $r_k = i_0(\sigma)+1$. Indeed, if $r_k < i_0(\sigma)+1$ then $i_0(\sigma) \in rev(r_k, k)$. Since $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP, it is clear from (2.7) that $i_0(\sigma)+1 \in \delta$ which contradicts that $i_0(\sigma)+1 \notin \delta$. Further, since $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP and $i_0(\sigma)+1 \notin \delta$, it is clear from (2.7) that $i_0(\sigma) \notin \delta$. Thus $\sigma_1 = \big(\gamma, rev(i_0(\sigma)+1 : k), \delta\big)$, where $i_0(\sigma), i_0(\sigma)+1 \notin \delta$. Now we have

$$\begin{aligned}
&M_\gamma(*)\, M_{rev(i_0(\sigma)+1:k)}(*)\, M_\delta(*)\, (e_{m-i_0(\sigma)} \otimes I_n) \\
&= M_\gamma(*)\, M_{rev(i_0(\sigma)+1:k)}(*)\, (e_{m-i_0(\sigma)} \otimes I_n) \text{ by (2.3) as } i_0(\sigma), i_0(\sigma)+1 \notin \delta \\
&= M_\gamma(*)\, M_k(*)\, M_{k-1}(*) \cdots M_{i_0(\sigma)+1}(*)\, (e_{m-i_0(\sigma)} \otimes I_n) \\
&= M_\gamma(*)\, (e_{m-k} \otimes I_n) \text{ by applying (2.3) repeatedly} \\
&= e_{m-k} \otimes I_n \text{ by (2.3) as } k, k+1 \notin \gamma.
\end{aligned}$$

Thus $M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n) = e_{m-k} \otimes I_n$ as $M_{\sigma_1}(X_1) = M_{(\gamma,rev(i_0(\sigma)+1:k),\delta)}(X_1)$. Now since $(\sigma_1, \sigma) \sim \big(\gamma, k, k-1, \ldots, i_0(\sigma)+1, \delta, i_0(\sigma), \ldots, 1, 0, \sigma^R\big)$ and $i_0(\sigma)+1 \notin \delta$, it follows that $(k, \ldots, 1, 0)$ is a subtuple of (σ_1, σ) and $(k+1, k, \ldots, 1, 0)$ is not a subtuple of (σ_1, σ). Hence $i_0(\sigma_1, \sigma) = k$ which gives the desired result.

Case-II: Suppose that $h = 0$. Then $\sigma = (0)$ and $c_0(\sigma) = 0 = i_0(\sigma)$. Hence there are no choices for σ_1 and σ_2, i.e., $\sigma_1 = \emptyset$ and $\sigma_2 = \emptyset$. So $M_{\sigma_1}(X_1) = I_{mn} = M_{\sigma_2}(X_2)$. Hence $(e_m^T \otimes I_n)M_{\sigma_2}(X_2) = e_m^T \otimes I_n$ and $M_{\sigma_1}(X_1)(e_m \otimes I_n) = e_m \otimes I_n$. This completes the proof. $\square$

We now describe automatic recovery of minimal bases of $P(\lambda)$ from those of the GFPRs of $P(\lambda)$. Two $k \times k$ matrix pencils $T_1(\lambda)$ and $T_2(\lambda)$ are said to be equivalent if $T_1(\lambda) = \mathcal{A}T_2(\lambda)\mathcal{B}$ for some nonsingular matrices $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2.1.12. *Let* $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M_\tau^P - M_\sigma^P)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ *be a GFPR of* $P(\lambda)$. *Let* τ *be given by* $\tau := (\tau_l, -m, \tau_r)$. *Set* $\alpha := (-rev(\tau_l), \sigma, -rev(\tau_r))$. *Let* $c_L := c(\alpha)$ *and* $i_L := i(\alpha)$ *be the total number of consecutions and inversions of the permutation* α *of* $\{0, 1, \ldots, m-1\}$, *respectively. Suppose that* $P(\lambda)$ *is singular. Then we have the following.*

(a) **Right minimal bases.** *If* $x \in \mathcal{N}_r(L)$ *then* $(e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)x \in \mathcal{N}_r(P)$. *Moreover,* $F^{\text{GFPR}}(P) : \mathcal{N}_r(L) \to \mathcal{N}_r(P)$, $x \longmapsto (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)x$, *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_r(L)$ *to a minimal basis of* $\mathcal{N}_r(P)$. *Thus, if* $(u_1(\lambda), \ldots, u_p(\lambda))$ *is a right minimal basis of* $L(\lambda)$ *then* $\big((e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u_p(\lambda)\big)$ *is a right minimal basis of* $P(\lambda)$.

(b) If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $L(\lambda)$ *then* $\varepsilon_1 - i_L \leq \cdots \leq \varepsilon_p - i_L$ *are the right minimal indices of* $P(\lambda)$.

(c) **Left minimal bases.** *If* $y \in \mathcal{N}_l(L)$ *then* $(e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)y \in \mathcal{N}_l(P)$. *Moreover,* $K^{\text{GFPR}}(P) : \mathcal{N}_l(L) \to \mathcal{N}_l(P)$, $y \longmapsto (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)y$, *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_l(L)$ *to a minimal basis of* $\mathcal{N}_l(P)$. *Thus, if* $(v_1(\lambda), \ldots, v_p(\lambda))$ *is a left minimal basis of* $L(\lambda)$ *then* $\big((e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)v_1(\lambda), \ldots, (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)v_p(\lambda)\big)$ *is a left minimal basis of* $P(\lambda)$.

(d) If $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $L(\lambda)$ *then* $\eta_1 - c_L \leq \cdots \leq \eta_p - c_L$ *are the left minimal indices of* $P(\lambda)$.

Proof. We have $L(\lambda) = M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)T_\omega(\lambda)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$, where $T_\omega(\lambda) := \lambda M_\tau^P - M_\sigma^P$ is a GF pencil of $P(\lambda)$ associated with the permutation $\omega := (\sigma, -\tau)$ of $\{0, 1, \ldots, m\}$. Since $M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ is nonsingular, it is easily seen that the map $M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) : \mathcal{N}_r(L) \to \mathcal{N}_r(T_\omega)$, $x(\lambda) \mapsto \big(M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)\big)x(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(L)$ to a minimal basis of $\mathcal{N}_r(T_\omega)$. On the other hand, by Theorem 1.2.26, $F^{\text{GF}}(P) : \mathcal{N}_r(T_\omega) \to \mathcal{N}_r(P)$, $x(\lambda) \mapsto (e^T_{m-c_0(\sigma)} \otimes I_n)x(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(T_\omega)$ to a minimal basis of $\mathcal{N}_r(P)$. Hence $F^{\text{GF}}(P)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) : \mathcal{N}_r(L) \to \mathcal{N}_r(P)$ is an isomorphism and maps a minimal basis of $\mathcal{N}_r(L)$ to a minimal basis of $\mathcal{N}_r(P)$. Now, by Lemma 2.1.11, we have $F^{\text{GF}}(P)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) =$

$$(e^T_{m-c_0(\sigma)} \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n = F^{\text{GFPR}}(P)$$

which proves (a).

Now, let $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ be the right minimal indices of $L(\lambda)$. Since the GF pencil $T_\omega(\lambda)$ is strictly equivalent to $L(\lambda)$, $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are also the right minimal indices of $T_\omega(\lambda)$. Hence by Theorem 1.2.26, $\varepsilon_1 - i_L \leq \cdots \leq \varepsilon_p - i_L$ are the right minimal indices of $P(\lambda)$. This proves (b).

For the recovery of left minimal bases, observe that $\big(M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)\big)^T : \mathcal{N}_l(L) \to \mathcal{N}_l(T_\omega)$, $y(\lambda) \mapsto \big(M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)\big)^T y(\lambda)$ is an isomorphism and maps a minimal basis of $\mathcal{N}_l(L)$ to a minimal basis of $\mathcal{N}_l(T_\omega)$. Again by Theorem 1.2.26, $K^{\text{GF}}(P) : \mathcal{N}_l(T_\omega) \to \mathcal{N}_l(P)$, $y(\lambda) \mapsto (e^T_{m-i_0(\sigma)} \otimes I_n)y(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_l(T_\omega)$ to a minimal basis of $\mathcal{N}_l(P)$. Hence $K^{\text{GF}}(P)(M_{\tau_1}(Y_1)M_{\sigma_1}(X_1))^T : \mathcal{N}_l(L) \to \mathcal{N}_l(P)$ is an isomorphism and maps a minimal basis of $\mathcal{N}_l(L)$ to a minimal basis of $\mathcal{N}_l(P)$. Now, by Lemma 2.1.11, we have $K^{\text{GF}}(P)\big(M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)\big)^T =$

$$\big(M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n)\big)^T = e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n = K^{\text{GFPR}}(P)$$

which proves (c).

Finally, let $\eta_1 \leq \cdots \leq \eta_p$ be the left minimal indices of $L(\lambda)$. Since the GF pencil $T_\omega(\lambda)$ is strictly equivalent to $L(\lambda)$, $\eta_1 \leq \cdots \leq \eta_p$ are also the left minimal indices of $T_\omega(\lambda)$. Hence by Theorem 1.2.26, $\eta_1 - c_L \leq \cdots \leq \eta_p - c_L$ are the left minimal indices of $P(\lambda)$. This completes the proof. $\square$

A verbatim proof of Theorem 2.1.12 yields the following result.

Corollary 2.1.13. *Suppose that $P(\lambda)$ is regular and $\mu \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$. Let $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ be a GFPR of $P(\lambda)$. Then we have the following.*

(a) **Right eigenvectors.** *If $u \in \mathcal{N}_r(L(\mu))$ then $(e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u \in \mathcal{N}_r(P(\mu))$. In fact, the mapping $u \mapsto (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u$ is a linear isomorphism from $\mathcal{N}_r(L(\mu))$ to $\mathcal{N}_r(P(\mu))$.*

(b) **Left eigenvectors.** *If $v \in \mathcal{N}_l(L(\mu))$ then $(e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)v \in \mathcal{N}_l(P(\mu))$. In fact, the mapping $v \mapsto (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)v$ is a linear isomorphism from $\mathcal{N}_l(L(\mu))$ to $\mathcal{N}_l(P(\mu))$.*

Remark 2.1.14. *Notice that the proofs of Theorem 2.1.12 and Corollary 2.1.13 do not use the fact that the index tuple (τ_1, τ, τ_2) satisfies the SIP and hence these results remain valid for GFPR-like pencils of the form*

$$L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$$

in which only the index tuple $(\sigma_1, \sigma, \sigma_2)$ *satisfies the SIP but not the tuple* (τ_1, τ, τ_2)*. This shows the extent to which the index tuples* $(\sigma_1, \sigma, \sigma_2)$ *and* (τ_1, τ, τ_2) *influence operation-free recovery of eigenvectors and minimal bases of* $P(\lambda)$ *from those of* $L(\lambda)$*.*

We now illustrate eigenvector recovery rule for $P(\lambda)$ by considering an example.

Example 2.1.15. *Let* $P(\lambda) := \sum_{i=0}^{4} \lambda^i A_i$ *be regular. Consider* $\sigma = (1, 2, 3, 0)$*,* $\sigma_2 = (2, 1)$*,* $\tau = (-4)$ *and* $\sigma_1 = \tau_1 = \tau_2 = \emptyset$*. Let* (X, Y) *be a matrix assignment for* σ_2*. Then the GFPR* $L(\lambda) := \big(\lambda M^P_{-4} - M^P_{(1,2,3,0)}\big) M_{(2,1)}(X, Y)$ *of* $P(\lambda)$ *is given by*

$$L(\lambda) = \begin{bmatrix} \lambda A_4 + A_3 & -X & -Y & -I_n \\ A_2 & \lambda X - I_n & \lambda Y & \lambda I_n \\ A_1 & \lambda I_n & A_0 & 0 \\ -I_n & 0 & \lambda I_n & 0 \end{bmatrix}.$$

Let $u \in \mathcal{N}_r(L(\lambda))$ *and* $v \in \mathcal{N}_l(L(\lambda))$*. Define* $u_i := (e_i^T \otimes I_n)u$ *and* $v_i := (e_i^T \otimes I_n)v$ *for* $i = 1 : 4$*. We have* $c_0(\sigma, \sigma_2) = 1$ *and* $i_0(\sigma_1, \sigma) = 1$*. Hence by Corollary 2.1.13,* $(e_{4-1}^T \otimes I_n)u = u_3 \in \mathcal{N}_r(P(\lambda))$ *and* $(e_{4-1}^T \otimes I_n)v = v_3 \in \mathcal{N}_l(P(\lambda))$*. To verify the recovery rule, consider* $L(\lambda)u = 0$*. This gives*

$$(\lambda A_4 + A_3)u_1 - Xu_2 - Yu_3 - u_4 = 0 \tag{2.8}$$

$$A_2u_1 + (\lambda X - I_n)u_2 + \lambda Y u_3 + \lambda u_4 = 0 \tag{2.9}$$

$$A_1u_1 + \lambda u_2 + A_0u_3 = 0 \tag{2.10}$$

$$-u_1 + \lambda u_3 = 0 \tag{2.11}$$

From (2.11) we have $u_1 = \lambda u_3$*. Adding* λ *times (2.8) with (2.9) we have* $u_2 = (\lambda^2 A_4 + \lambda A_3 + A_2)u_1 = (\lambda^3 A_4 + \lambda^2 A_3 + \lambda A_2)u_3$*. Substituting the values of* u_1 *and* u_2 *in (2.10) we have* $(\lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0)u_3 = 0 \Rightarrow P(\lambda)u_3 = 0$*, i.e.,* $u_3 \in \mathcal{N}_r(P(\lambda))$*. Note that* $u_3 \neq 0$*. Indeed, if* $u_3 = 0$ *then it follows that* $u = 0$*. Similarly, we can verify that* $v_3^T P(\lambda) = 0$ *and* $v_3 \neq 0$*.*

We have the following recovery rule for FPRs of $P(\lambda)$ which follows from Theorem 2.1.12 and Corollary 2.1.13 by considering X_j and Y_j, $j = 1, 2$, as the trivial matrix assignments.

Corollary 2.1.16. *Let* $L(\lambda) = M^P_{\tau_1} M^P_{\sigma_1} (\lambda M^P_{\tau} - M^P_{\sigma}) M^P_{\sigma_2} M^P_{\tau_2}$ *be an FPR of* $P(\lambda)$*. Then we have the following.*

(I) (Regular $P(\lambda)$**).** *Suppose that* $P(\lambda)$ *is regular and* $\mu \in \mathbb{C}$ *is an eigenvalue of* $P(\lambda)$*. If* $(x_1, \ldots, x_p)$ *and* $(y_1, \ldots, y_p)$*, respectively, are bases of right and left eigenspaces* *of* $L(\mu)$, *then* $((e^T_{m-c_0(\sigma,\sigma_2)}\otimes I_n)x_1,\ldots,(e^T_{m-c_0(\sigma,\sigma_2)}\otimes I_n)x_p)$ *and* $((e^T_{m-i_0(\sigma_1,\sigma)}\otimes I_n)y_1,\ldots,(e^T_{m-i_0(\sigma_1,\sigma)}\otimes I_n)y_p)$, *respectively, are bases of right and left eigenspaces of* $P(\mu)$.

(II) (Singular $P(\lambda)$). *Suppose that* $P(\lambda)$ *is singular. Let* τ *be given by* $\tau := (\tau_l, -m, \tau_r)$. *Set* $\alpha := (-rev(\tau_l), \sigma, -rev(\tau_r))$. *Let* $c_L := c(\alpha)$ *and* $i_L := i(\alpha)$ *be the total number of consecutions and inversions of the permutation* α *of* $\{0 : m-1\}$.

(a) **Right minimal bases.** *If* $(x_1(\lambda),\ldots,x_p(\lambda))$ *is a right minimal basis of* $L(\lambda)$ *then* $((e^T_{m-c_0(\sigma,\sigma_2)}\otimes I_n)x_1(\lambda),\ldots,(e^T_{m-c_0(\sigma,\sigma_2)}\otimes I_n)x_p(\lambda))$ *is a right minimal basis of* $P(\lambda)$. *Further, if* $\varepsilon_1 \le \cdots \le \varepsilon_p$ *are the right minimal indices of* $L(\lambda)$ *then* $\varepsilon_1 - i_L \le \cdots \le \varepsilon_p - i_L$ *are the right minimal indices of* $P(\lambda)$.

(b) **Left minimal bases.** *If* $(y_1(\lambda),\ldots,y_p(\lambda))$ *is a left minimal basis of* $L(\lambda)$ *then* $((e^T_{m-i_0(\sigma_1,\sigma)}\otimes I_n)y_1(\lambda),\ldots,(e^T_{m-i_0(\sigma_1,\sigma)}\otimes I_n)y_p(\lambda))$ *is a left minimal basis of* $P(\lambda)$. *Further, if* $\eta_1 \le \cdots \le \eta_p$ *are the left minimal indices of* $L(\lambda)$ *then* $\eta_1 - c_L \le \cdots \le \eta_p - c_L$ *are the left minimal indices of* $P(\lambda)$.

The following example illustrates eigenvector recovery rule for $P(\lambda)$ from those of the FPRs of $P(\lambda)$.

Example 2.1.17. *Let* $P(\lambda)$, σ, σ_j, τ *and* τ_j *be as given in Example 2.1.15. Then the FPR* $L(\lambda) := \big(\lambda M^P_{-4} - M^P_{(1,2,3,0)}\big) M^P_{(2,1)}$ *is given by*

$$L(\lambda) = \begin{bmatrix} \lambda A_4 + A_3 & A_2 & A_1 & -I_n \\ A_2 & -\lambda A_2 - I_n & -\lambda A_1 & \lambda I_n \\ A_1 & \lambda I_n & A_0 & 0 \\ -I_n & 0 & \lambda I_n & 0 \end{bmatrix}.$$

Let $u \in \mathcal{N}_r(L(\lambda))$ *and* $v \in \mathcal{N}_l(L(\lambda))$. *Define* $u_i := (e_i^T \otimes I_n)u$ *and* $v_i := (e_i^T \otimes I_n)v$ *for* $i = 1:4$. *We have* $c_0(\sigma,\sigma_2) = 1$ *and* $i_0(\sigma_1,\sigma) = 1$. *Hence by Corollary 2.1.16,* $(e^T_{4-1}\otimes I_n)u = u_3 \in \mathcal{N}_r(P(\lambda))$ *and* $(e^T_{4-1}\otimes I_n)v = v_3 \in \mathcal{N}_l(P(\lambda))$ *which can be easily verified.*

Recall from Definition 1.2.33 the left and right index tuples of type-1 and the notations $z_r(.,.)$ and $z_l(.,.)$.

A GFPR $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ of $P(\lambda)$ is said to be type-1 when the index tuples σ_2 and $rev(\sigma_1)$ are right index tuples of type-1 relative to σ and $rev(\sigma)$, respectively, and the tuples τ_2 and $rev(\tau_1)$ are right index tuples of type-1 relative to τ and $rev(\tau)$, respectively; see [14] for type-1 FPRs. This restriction on the index tuples effectively reduces the type-1 GFPRs to be a small subclass of the class of GFPRs. For example, if $P(\lambda)$ is singular and of degree $m = 4$ then for $\sigma := (0,1,2,3)$ and $\tau := (-4)$, there are GFPRs but no type-1 GFPRs of $P(\lambda)$.

For a comparison, we now state the recovery rules for eigenvectors and minimal bases derived in [14] for type-1 FPRs.

Theorem 2.1.18 ([14], Theorem 3.6)**.** *Let* $L(\lambda) := M^P_{\tau_1} M^P_{\sigma_1}\big(\lambda M^P_{\tau} - M^P_{\sigma}\big) M^P_{\sigma_2} M^P_{\tau_2}$ *be a type-1 FPR of* $P(\lambda)$*.*

(a) Suppose that $P(\lambda)$ *is singular. Then we have the following.*

Right minimal basis: *If* $(x_1(\lambda), \ldots, x_k(\lambda))$ *is a right minimal basis* $L(\lambda)$ *then* $\big((e^T_{m-\alpha} \otimes I_n)x_1(\lambda), \ldots, (e^T_{m-\alpha} \otimes I_n)x_k(\lambda)\big)$ *is a right minimal basis of* $P(\lambda)$*, where* $\alpha := c_0(z_r(\sigma, \sigma_2))$*, where* $z_r(\sigma, \sigma_2)$ *is the simple tuple associated with* (σ, σ_2)*.*

Left minimal basis: *If* $(y_1(\lambda), \ldots, y_k(\lambda))$ *is a left minimal basis* $L(\lambda)$ *then* $\big((e^T_{m-\beta} \otimes I_n)y_1(\lambda), \ldots, (e^T_{m-\beta} \otimes I_n)y_k(\lambda)\big)$ *is a left minimal basis of* $P(\lambda)$*, where* $\beta := c_0(z_r(rev(\sigma), rev(\sigma_1)))$*.*

(b) Suppose that $P(\lambda)$ *is regular and* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $P(\lambda)$*.*

Right eigenvectors: *The map* $\mathcal{N}_r(L(\lambda)) \longrightarrow \mathcal{N}_r(P(\lambda)), v \longmapsto (e^T_{m-\alpha} \otimes I_n)v$*, is a linear isomorphism, where* $\alpha := c_0(z_r(\sigma, \sigma_2))$*.*

Left eigenvectors: *The map* $\mathcal{N}_l(L(\lambda)) \longrightarrow \mathcal{N}_l(P(\lambda)), u \longmapsto (e^T_{m-\beta} \otimes I_n)u$*, is a linear isomorphism, where* $\beta := c_0(z_r(rev(\sigma), rev(\sigma_1)))$*.*

A natural question that arises is this: *Can Theorem 2.1.18 be used to recover eigenvectors and minimal bases of* $P(\lambda)$ *from those of the type-1 GFPRs of* $P(\lambda)$*?* It turns out that Theorem 2.1.18 is not valid for GFPRs, that is, it cannot guarantee recovery of eigenvectors and minimal bases of $P(\lambda)$ from those of the type-1 GFPRs. We illustrate this by an example.

Example 2.1.19. *Let* $P(\lambda) := \sum_{i=0}^{4} \lambda^i A_i$ *be regular and* $A_0 = 0$*. Consider* $\sigma = (2, 3, 0, 1)$*,* $\sigma_2 = (2, 0, 1), \tau = (-4)$ *and* $\sigma_1 = \tau_1 = \tau_2 = \emptyset$*. Let* (X, Y, Z) *be a nonsingular matrix assignment for* σ_2*, i.e.,* Y *is nonsingular. Then the GFPR* $L(\lambda) := \big(\lambda M^P_{-4} - M^P_{(2,3,0,1)}\big) M_{(2,0,1)}(X, Y, Z)$ *of* $P(\lambda)$ *is given by*

$$L(\lambda) = \begin{bmatrix} \lambda A_4 + A_3 & -X & -Z & -I_n \\ A_2 & \lambda X + A_1 & \lambda Z - Y & \lambda I_n \\ -I_n & \lambda I_n & 0 & 0 \\ 0 & 0 & \lambda Y & 0 \end{bmatrix}.$$

Note that $L(\lambda)$ *is a type-1 GFPR. Let* $u \in \mathcal{N}_r(L(\lambda))$*. Define* $u_i := (e^T_i \otimes I_n)u$ *for* $i = 1:4$*. Note that* σ_2 *is a right index tuple of type-1 relative to* σ *and* $c_0(z_r(\sigma, \sigma_2)) = 1$*. Thus by Theorem 2.1.18,* $(e^T_{4-1} \otimes I_n)u = u_3 \in \mathcal{N}_r(P(\lambda))$*. It is easy to see that* $L(\lambda)u = 0$ *implies* $u_3 = 0$ *since* Y *is a nonsingular matrix. Thus* u_3 *cannot be an eigenvector of* $P(\lambda)$*. This shows that Theorem 2.1.18 is not applicable to type-1 GFPRs. Next, observe* *that* $c_0(\sigma,\sigma_2)=2$*. Thus, by Corollary 2.1.13,* $(e_{4-2}^T\otimes I_n)u=u_2\in\mathcal{N}_r(P(\lambda))$*. It is easily verified that* $u_2\neq 0$ *and* $P(\lambda)u_2=0$ *showing that* u_2 *is indeed an eigenvector of* $P(\lambda)$*.* $\blacksquare$

We now present an example to illustrate a case when our recovery rule coincides with that in Theorem 2.1.18.

Example 2.1.20. *Let* $P(\lambda):=\sum_{i=0}^{7}\lambda^i A_i$*. Consider* $\sigma=(4:6,2:3,0:1),\sigma_1=(2,1,3),\sigma_2=(2,4,5,3,4),\tau=(-7)$ *and* $\tau_1=\tau_2=\emptyset$*. Clearly, rev*(σ_1) *and* σ_2 *are right index tuples of type-1 relative to rev*(σ) *and* σ*, respectively. Consider the FPR* $L(\lambda):=M^P_{(2,1,3)}\big(\lambda M^P_{-7}-M^P_{(4:6,2:3,0:1)}\big)M^P_{(4,5,2,3,4)}$*. Let* $u\in\mathcal{N}_r(L(\lambda))$ *and* $v\in\mathcal{N}_l(L(\lambda))$*. Define* $u_i:=(e_i^T\otimes I_n)u$ *and* $v_i:=(e_i^T\otimes I_n)v$ *for* $i=1:7$*.*

We have $c_0(\sigma,\sigma_2)=4$ *and* $i_0(\sigma_1,\sigma)=2$*. Hence, by Corollary 2.1.13,* $(e_{7-4}^T\otimes I_n)u=u_3\in\mathcal{N}_r(P(\lambda))$ *and* $(e_{7-2}^T\otimes I_n)v=v_5\in\mathcal{N}_l(P(\lambda))$*.*

On the other hand, σ_2 *is a right index tuple of type-1 relative to* σ *and the simple tuple associated with* (σ,σ_2) *is given by* $z_r(\sigma,\sigma_2)=(6,5,0:4)$*. Thus* $c_0(z_r(\sigma,\sigma_2))=4$*. Hence by Theorem 2.1.18,* $(e_{7-4}^T\otimes I_n)u=u_3\in\mathcal{N}_r(P(\lambda))$*. Similarly, rev*$(\sigma_1)$ *is a right index tuple of type-1 relative to rev*(σ) *and the simple tuple associated with* $(rev(\sigma),rev(\sigma_1))$ *is* $z_r(rev(\sigma),rev(\sigma_1))=(6,5,4,3,0:2)$*. Thus* $c_0(z_r(rev(\sigma),rev(\sigma_1)))=2$*. Hence by Theorem 2.1.18,* $(e_{7-2}^T\otimes I_n)v=v_5\in\mathcal{N}_l(P(\lambda))$*.* $\blacksquare$

We remark that our recovery rules for right eigenvectors and right minimal bases of $P(\lambda)$ from those of GFPRs (and hence of FPRs) can be read off from $c_0(\sigma,\sigma_2)$ whereas the rules for left eigenvectors and left minimal bases can be read off from $i_0(\sigma_1,\sigma)$. Most importantly, $c_0(\sigma,\sigma_2)$ and $i_0(\sigma_1,\sigma)$ can be easily read off by looking at the index tuples σ,σ_2 and σ_1. By contrast, the recovery formulae for right (resp., left) eigenvectors and right (resp., left) minimal bases from those of the type-1 FPRs in Theorem 2.1.18 require the number of consecutions of the simple index tuple $z_r(\sigma,\sigma_2)$ (resp., $z_r(rev(\sigma),rev(\sigma_1))$) at 0. The simple index tuples $z_r(\sigma,\sigma_2)$ and $z_r(rev(\sigma),rev(\sigma_1))$ are defined recursively and hence cannot be read off readily from the index tuples σ,σ_2 and σ_1. Therefore, even though for certain type-1 FPRs our recovery rules coincide with those in Theorem 2.1.18, our recovery rules are automatic and can be easily read off from the index tuples σ,σ_1 and σ_2.

2.1.2 Eigenvalue at infinity and recovery of eigenvectors

We now describe the recovery of eigenvectors of $P(\lambda)$ corresponding to an eigenvalue at ∞ from those of the GFPRs of $P(\lambda)$. For this purpose, we need the following result.

Lemma 2.1.21. *Let α be an index tuple containing indices from $\{-m : -1\}$ such that α satisfies the SIP. Then we have the following.*

(a) If $-s \in \alpha$ and $c_{-s}(\alpha) = p$, then $\alpha \sim \big(\alpha^L, -s, -(s-1), \ldots, -(s-p), \alpha^R\big)$ for some index tuples α^L and α^R such that $-s \notin \alpha^L$ and $-(s-p), -(s-p-1) \notin \alpha^R$.

(b) if $-t \in \alpha$ and $i_{-t}(\alpha) = q$, then $\alpha \sim \big(\alpha^L, -(t-q), \ldots, -(t-1), -t, \alpha^R\big)$ for some index tuples α^L and α^R such that $-t \notin \alpha^R$ and $-(t-q), -(t-q-1) \notin \alpha^L$.

Proof. Since α satisfies the SIP, we can write α in the column standard form. Let the column standard form of α be given by

$$csf(\alpha) = \big((-a_1 : -1), \ldots, (-a_{k-1} : -(k-1)), (-a_k : -k), \ldots, (-a_m : -m)\big),$$

where $-k$ is the largest integer such that $-s \in (-a_k : -k)$. Then $(-a_k : -k) = (-a_k, -a_k+1, \ldots, -s-1, -s, \ldots, -k)$. Now, set

$$\alpha^L := \big((-a_1 : -1), \ldots, (-a_{k-1} : -(k-1)), -a_k, -a_k+1, \ldots, -s-1\big)$$

$$\text{and } \alpha^R := \big((-a_{k+1} : -(k+1)), \ldots, (-a_m : -m)\big).$$

Then $csf(\alpha) = \big(\alpha^L, -s, -(s-1), \ldots, -k, \alpha^R\big)$, where $-s \notin \alpha^L$ and $-k, -(k-1) \notin \alpha^R$. Thus $(-s, -(s-1), \ldots, -k)$ is a subtuple of α and $(-s, -(s-1), \ldots, -k, -(k-1))$ is not a subtuple of α. So $s - k = c_{-s}(\alpha)$, i.e., $s - k = p$. Thus $k = s - p$ and $-(s-p), -(s-p-1) \notin \alpha^R$. This proves (a) as $\alpha \sim csf(\alpha)$.

(b) Let the row standard form of α be given by

$$rsf(\alpha) = \big(rev(-b_m : -m), \ldots, rev(-b_k : -k), rev(-b_{k-1} : -(k-1)), \ldots, rev(-b_1 : -1)\big),$$

where $-k$ is the largest integer such that $-t \in rev(-b_k : -k)$. Then $rev(-b_k : -k) = (-k, \ldots, -t, -t-1, \ldots, -b_k+1, -b_k)$. Now, set

$$\alpha^L := \big(rev(-b_m : -m), \ldots, rev(-b_{k+1} : -(k+1))\big)$$

$$\text{and } \alpha^R := \big(-t-1, \ldots, -b_k+1, -b_k, rev(-b_{k-1} : -(k-1)), \ldots, rev(-b_1 : -1)\big).$$

Then $rsf(\alpha) = \big(\alpha^L, -k, \ldots, -(t-1), -t, \alpha^R\big)$, where $-t \notin \alpha^R$ and $-k, -(k-1) \notin \alpha^L$. Thus $(-k, \ldots, -(t-1), -t)$ is a subtuple of α and $(-(k-1), -k, \ldots, -(t-1), -t)$ is not a subtuple of α. So $t - k = i_{-t}(\alpha)$, i.e., $t - k = q$. Thus $k = t - q$ and $-(t-q), -(t-q-1) \notin \alpha^L$. This proves (b) as $\alpha \sim rsf(\alpha)$. ☐

We need the following result to derive the recovery of eigenvectors of $P(\lambda)$ corresponding to an eigenvalue at ∞ from those of the GFPRs of $P(\lambda)$.

Lemma 2.1.22. *Let* $L(\lambda) := M_{\tau_1}(Y_1)\, M_{\sigma_1}(X_1)\, (\lambda M^P_\tau - M^P_\sigma)\, M_{\sigma_2}(X_2)\, M_{\tau_2}(Y_2)$ *be a GFPR of* $P(\lambda)$*. Then we have the following.*

(a) $(e^T_{c_{-m}(\tau)+1} \otimes I_n) M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) = e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n.$

(b) $M_{\tau_1}(Y_1) M_{\sigma_1}(X_1)\, (e_{i_{-m}(\tau)+1} \otimes I_n) = e_{i_{-m}(\tau_1,\tau)+1} \otimes I_n.$

Proof. Let Z be an $n \times n$ arbitrary matrix. For $j = 1 : m-1$, we have

$$(e^T_{m-j} \otimes I_n) M_{-j}(Z) = (e^T_{m-j} \otimes I_n) \begin{bmatrix} I_{(m-j-1)n} & & & \\ & 0 & I_n & \\ & I_n & Z & \\ & & & I_{(j-1)n} \end{bmatrix} = e^T_{m-(j-1)} \otimes I_n$$

and

$$M_{-j}(Z)(e_{m-j} \otimes I_n) = \begin{bmatrix} I_{(m-j-1)n} & & & \\ & 0 & I_n & \\ & I_n & Z & \\ & & & I_{(j-1)n} \end{bmatrix} (e_{m-j} \otimes I_n) = e_{m-(j-1)} \otimes I_n.$$

This shows that

$$(e^T_{m-j} \otimes I_n) M_{-k}(Z) = \begin{cases} e^T_{m-(j-1)} \otimes I_n & \text{for } k = j \text{ and } j = 1 : m-1, \\ e^T_{m-j} \otimes I_n & \text{for } k \notin \{j, j+1\},\ j = 0 : m-1, \end{cases} \tag{2.12}$$

and

$$M_{-k}(Z)(e_{m-j} \otimes I_n) = \begin{cases} e_{m-(j-1)} \otimes I_n & \text{for } k = j \text{ and } j = 1 : m-1, \\ e_{m-j} \otimes I_n & \text{for } k \notin \{j, j+1\},\ j = 0 : m-1. \end{cases} \tag{2.13}$$

(a) Now we show that

$$(e^T_{c_{-m}(\tau)+1} \otimes I_n) M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) = e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n. \tag{2.14}$$

Since τ is a permutation of $\{-m : -(h+1)\}$, we have $c_{-m}(\tau) \leq m-h-1$ and $c_{-m}(\tau)+1 \leq m-h$. Further, since σ_2 is an index tuple containing indices from $\{0 : h-1\}$, we have $M_{\sigma_2}(X_2) = \mathrm{diag}(I_{(m-h)n}, \ast)$. Consequently, we have $(e^T_{c_{-m}(\tau)+1} \otimes I_n)\, M_{\sigma_2}(X_2) = e^T_{c_{-m}(\tau)+1} \otimes I_n$. Hence to prove (2.14) we only need to show that $(e^T_{c_{-m}(\tau)+1} \otimes I_n) M_{\tau_2}(Y_2) =$

$e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n$. Observe that $e^T_{c_{-m}(\tau)+1} \otimes I_n = e^T_{m-(m-c_{-m}(\tau)-1)} \otimes I_n$. Set $s := c_{-m}(\tau)$. We show that

$$(e^T_{m-(m-s-1)} \otimes I_n) M_{\tau_2}(Y_2) = e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n \tag{2.15}$$

which gives (2.14). Since τ has s consecutions at $-m$, we have $\tau \sim \big(\tau^L, -m, -(m-1), \ldots, -(m-s)\big)$. Note that

$$(\tau, \tau_2) \sim \big(\tau^L, -m, -(m-1), \ldots, -(m-s), \tau_2\big) \text{ satisfies the SIP.} \tag{2.16}$$

Case-I: Suppose that $-(m-s-1) \notin \tau_2$. Then, since (τ, τ_2) satisfies the SIP, it is clear from (2.16) that $-(m-s) \notin \tau_2$. Now, as $-(m-s), -(m-s-1) \notin \tau_2$, by (2.12), we have $(e^T_{m-(m-s-1)} \otimes I_n)\, M_{\tau_2}(Y_2) = e^T_{m-(m-s-1)} \otimes I_n = e^T_{s+1} \otimes I_n$. Since $-(m-s-1) \notin \tau_2$, it is clear from (2.16) that $c_{-m}(\tau, \tau_2) = s$. This proves (2.15).

Case-II: Suppose that $-(m-s-1) \in \tau_2$. Set $p := c_{-(m-s-1)}(\tau_2)$, i.e., τ_2 has p consecutions at $-(m-s-1)$. Then by Lemma 2.1.21,

$$\tau_2 \sim \big(\tau_2^L, -(m-s-1), -(m-s-2), \ldots, -(m-s-p-1), \tau_2^R\big),$$

where $-(m-s-1) \notin \tau_2^L$ and $-(m-s-p-1), -(m-s-p-2) \notin \tau_2^R$. By setting $t := m-s-p-1$, we have $-t, -(t-1) \notin \tau_2^R$. Note that

$$(\tau, \tau_2) \sim \big(\tau^L, -m : -(m-s), \tau_2^L, -(m-s-1) : -t, \tau_2^R\big) \text{ satisfies the SIP,} \tag{2.17}$$

where $-(m-s-1) \notin \tau_2^L$ and $-t, -(t-1) \notin \tau_2^R$. As $-(m-s-1) \notin \tau_2^L$, we have $-(m-s) \notin \tau_2^L$ since otherwise (τ, τ_2) would not satisfy the SIP which is clear from (2.17). We denote by $(*)$ any arbitrary matrix assignment. Then

$$\begin{aligned}
&(e^T_{m-(m-s-1)} \otimes I_n)\, M_{\tau_2^L}(*)\, M_{(-(m-s-1):-t)}(*)\, M_{\tau_2^R}(*) = \\
&\qquad (e^T_{m-(m-s-1)} \otimes I_n) M_{(-(m-s-1):-t)}(*)\, M_{\tau_2^R}(*) \left\{ \begin{array}{l} \text{by (2.12) since} \\ -(m-s), -(m-s-1) \notin \tau_2^L \end{array} \right. \\
&= \quad (e^T_{m-(t-1)} \otimes I_n)\, M_{\tau_2^R}(*) \text{ by applying (2.12) repeatedly} \\
&= \quad e^T_{m-(t-1)} \otimes I_n \text{ by (2.12) as } -t, -(t-1) \notin \tau_2^R.
\end{aligned}$$

Hence $(e^T_{m-(m-s-1)} \otimes I_n) M_{\tau_2}(Y_2) = e^T_{m-(t-1)} \otimes I_n$ as $\tau_2 \sim \big(\tau_2^L, -(m-s-1) : -t, \tau_2^R\big)$. It is clear from (2.17) that $c_{-m}(\tau, \tau_2) = m - t$, that is, $m - (t-1) = c_{-m}(\tau, \tau_2) + 1$. This proves (2.15) and hence (2.14) holds. This completes the proof of (a).

(b) Next, we show that

$$M_{\tau_1,\sigma_1}(Y_1, X_1)\, (e_{i_{-m}(\tau)+1} \otimes I_n) = e_{i_{-m}(\tau_1,\tau)+1} \otimes I_n. \tag{2.18}$$

Since τ is a permutation of $\{-m : -(h+1)\}$, we have $i_{-m}(\tau) \leq m-h-1$ and $i_{-m}(\tau)+1 \leq m-h$. Further, since σ_1 is an index tuple containing indices from $\{0 : h-1\}$, we have $M_{\sigma_1}(X_1) = \operatorname{diag}(I_{(m-h)n}, *)$. Consequently, we have $M_{\sigma_1}(X_1)\,(e_{i_{-m}(\tau)+1} \otimes I_n) = e_{i_{-m}(\tau)+1} \otimes I_n$. Thus, to prove (2.18) we only need to show that $M_{\tau_1}(Y_1)\,(e_{i_{-m}(\tau)+1} \otimes I_n) = e_{i_{-m}(\tau_1,\tau)+1} \otimes I_n$.

Observe that $e_{i_{-m}(\tau)+1} \otimes I_n = e_{m-(m-i_{-m}(\tau)-1)} \otimes I_n$. Set $s := i_{-m}(\tau)$. Next, we show that

$$M_{\tau_1}(Y_1)\,(e_{m-(m-s-1)} \otimes I_n) = e_{i_{-m}(\tau_1,\tau)+1} \otimes I_n \tag{2.19}$$

which gives (2.18). Since τ has s inversions at $-m$, we have $\tau \sim \big(-(m-s), \ldots, -(m-1), -m, \tau^R\big)$. Note that

$$(\tau_1, \tau) \sim \big(\tau_1, -(m-s), \ldots, -(m-1), -m, \tau^R\big) \text{ satisfies the SIP.} \tag{2.20}$$

Case-I: Suppose that $-(m-s-1) \notin \tau_1$. Then, since (τ_1, τ) satisfies the SIP, it is clear from (2.20) that $-(m-s) \notin \tau_1$. Now, as $-(m-s), -(m-s-1) \notin \tau_1$, by (2.13), we have $M_{\tau_1}(Y_1)\,(e_{m-(m-s-1)} \otimes I_n) = e_{m-(m-s-1)} \otimes I_n = e_{s+1} \otimes I_n$. Since $-(m-s-1) \notin \tau_1$, it is clear from (2.20) that $i_{-m}(\tau_1, \tau) = s$. This proves (2.19).

Case-II: Suppose that $-(m-s-1) \in \tau_1$. Set $p := i_{-(m-s-1)}(\tau_1)$, i.e., τ_1 has p inversions at $-(m-s-1)$. Then by Lemma 2.1.21,

$$\tau_1 \sim \big(\tau_1^L, -(m-s-p-1), \ldots, -(m-s-2), -(m-s-1), \tau_1^R\big),$$

where $-(m-s-1) \notin \tau_1^R$ and $-(m-s-p-1), -(m-s-p-2) \notin \tau_1^L$. Setting $t := m-s-p-1$, we have $-t, -(t-1) \notin \tau_1^L$. Note that

$$(\tau_1, \tau) \sim \big(\tau_1^L, -t, \ldots, -(m-s-1), \tau_1^R, -(m-s), \ldots, -m, \tau^R\big) \tag{2.21}$$

satisfies the SIP, where $-(m-s-1) \notin \tau_1^R$ and $-t, -(t-1) \notin \tau_1^L$. As $-(m-s-1) \notin \tau_1^R$, we have $-(m-s) \notin \tau_1^R$ since otherwise (τ_1, τ) would not satisfy the SIP which is clear from (2.21). Now we have

$$\begin{aligned}
& M_{\tau_1^L}(*)\, M_{(-t,\ldots,-(m-s-2),-(m-s-1))}(*)\, M_{\tau_1^R}(*)\, (e_{m-(m-s-1)} \otimes I_n) \\
= \; & M_{\tau_1^L}(*)\, M_{(-t,\ldots,-(m-s-2),-(m-s-1))}(*)\, (e_{m-(m-s-1)} \otimes I_n) \left\{ \begin{array}{l} \text{by (2.13) as } -(m-s), \\ -(m-s-1) \notin \tau_1^R \end{array} \right. \\
= \; & M_{\tau_1^L}(*)\, (e_{m-(t-1)} \otimes I_n) \text{ by applying (2.13) repeatedly} \\
= \; & e_{m-(t-1)} \otimes I_n \text{ by (2.13) as } -t, -(t-1) \notin \tau_1^L.
\end{aligned}$$

Hence $M_{\tau_1}(Y_1)\,(e_{m-(m-s-1)} \otimes I_n) = e_{m-(t-1)} \otimes I_n$ as $\tau_1 \sim \big(\tau_1^L, -t, \ldots, -(m-s-1), \tau_1^R\big)$. It is clear from (2.21) that $i_{-m}(\tau_1, \tau) = m-t$, that is, $m-(t-1) = i_{-m}(\tau_1, \tau)+1$. This proves (2.19) and hence (2.18) holds. This completes the proof of (b). $\square$

We are now ready to describe the recovery of eigenvectors of $P(\lambda)$ corresponding to an eigenvalue at ∞ from those of the GFPRs of $P(\lambda)$.

Theorem 2.1.23 (Eigenvector recovery at ∞). *Let $P(\lambda)$ be regular and let $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ be a GFPR of $P(\lambda)$. Suppose that ∞ is an eigenvalue of $P(\lambda)$. Then we have the following.*

Right eigenvectors. *If $\big(x_1, \ldots, x_k\big)$ is a basis of the right eigenspace of $L(\lambda)$ at ∞, then $\Big((e^T_{c-m(\tau,\tau_2)+1} \otimes I_n)x_1, \ldots, (e^T_{c-m(\tau,\tau_2)+1} \otimes I_n)x_k\Big)$ is a basis of the right eigenspace of $P(\lambda)$ at ∞.*

Left eigenvectors. *If $\big(y_1, \ldots, y_k\big)$ is a basis of the left eigenspace of $L(\lambda)$ at ∞, then $\Big((e^T_{i-m(\tau_1,\tau)+1} \otimes I_n)y_1, \ldots, (e^T_{i-m(\tau_1,\tau)+1} \otimes I_n)y_k\Big)$ is a basis of the left eigenspace of $P(\lambda)$ at ∞.*

Proof. We have $L(\lambda) = \lambda L_1 - L_0$, where $L_1 := M_{(\tau_1,\sigma_1)}(Y_1, X_1)M^P_\tau M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ and $L_0 := M_{(\tau_1,\sigma_1)}(Y_1, X_1)M^P_\sigma M_{(\sigma_2,\tau_2)}(X_2, Y_2)$. Note that ∞ is an eigenvalue of $L(\lambda) \iff 0$ is an eigenvalue of $rev(L(\lambda)) \iff 0$ is an eigenvalue of L_1. Since $M_{(\tau_1,\sigma_1)}(Y_1, X_1)$ is invertible, we have $\mathcal{N}_r(L_1) = \mathcal{N}_r\Big(M^P_\tau M_{(\sigma_2,\tau_2)}(X_2, Y_2)\Big)$. Further, since $M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ is nonsingular, it is easily seen that the map $\mathcal{N}_r\Big(M^P_\tau M_{(\sigma_2,\tau_2)}(X_2, Y_2)\Big) \longrightarrow \mathcal{N}_r(M^P_\tau),\ z \mapsto \Big(M_{(\sigma_2,\tau_2)}(X_2, Y_2)\Big)z$ is an isomorphism. Define $T(\lambda) := \lambda M^P_\tau - M^P_\sigma$. Then $\mathcal{N}_r(M^P_\tau) = \mathcal{N}_r\Big(rev(T(0))\Big)$. Now, by Theorem 1.2.28, the map

$$\mathcal{N}_r\Big(rev(T(0))\Big) \longrightarrow \mathcal{N}_r\Big(rev(P(0))\Big),\ u \mapsto (e^T_{c-m(\tau)+1} \otimes I_n)u$$

is an isomorphism. Hence the map

$$\mathcal{N}_r\Big(rev(L(0))\Big) \longrightarrow \mathcal{N}_r\Big(rev(P(0))\Big),\ x \mapsto \Big((e^T_{c-m(\tau)+1} \otimes I_n)M_{(\sigma_2,\tau_2)}(X_2, Y_2)\Big)x \quad (2.22)$$

is an isomorphism. Now, by Lemma 2.1.22, we have

$$(e^T_{c-m(\tau)+1} \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e^T_{c-m(\tau,\tau_2)+1} \otimes I_n.$$

Hence the desired result for the right eigenspace of $P(\lambda)$ at ∞ follows from (2.22).

Next, we prove the result for left eigenspace of $P(\lambda)$ at ∞. Since $M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ is invertible, we have $\mathcal{N}_l\Big(rev(L(0))\Big) = \mathcal{N}_l(L_1) = \mathcal{N}_l\Big(M_{(\tau_1,\sigma_1)}(Y_1, X_1)M^P_\tau\Big)$. Further, since $M_{(\tau_1,\sigma_1)}(Y_1, X_1)$ is nonsingular, the map $\mathcal{N}_l\Big(M_{(\tau_1,\sigma_1)}(Y_1, X_1)M^P_\tau\Big) \longrightarrow \mathcal{N}_l(M^P_\tau),\ z \mapsto \Big(M_{(\tau_1,\sigma_1)}(Y_1, X_1)\Big)^T z$ is an isomorphism. Recall that $T(\lambda) := \lambda M^P_\tau - M^P_\sigma$ and hence $\mathcal{N}_l(M^P_\tau) = \mathcal{N}_l\Big(rev(T(0))\Big)$. Now, by Theorem 1.2.28, the map

$$\mathcal{N}_l\Big(rev(T(0))\Big) \longrightarrow \mathcal{N}_l\Big(rev(P(0))\Big),\ v \mapsto (e^T_{i-m(\tau)+1} \otimes I_n)v$$

is an isomorphism. Hence the map

$$\mathcal{N}_l\big(rev(L(0))\big) \longrightarrow \mathcal{N}_l\big(rev(P(0))\big),\ y \mapsto \big((e^T_{i-m(\tau)+1} \otimes I_n)(M_{(\tau_1,\sigma_1)}(Y_1, X_1))^T\big)y \tag{2.23}$$

is an isomorphism. Now, by Lemma 2.1.22, we have

$$(e^T_{i-m(\tau)+1}\otimes I_n)\big(M_{(\tau_1,\sigma_1)}(Y_1,X_1)\big)^T = \big(M_{(\tau_1,\sigma_1)}(Y_1,X_1)\,(e_{i-m(\tau)+1}\otimes I_n)\big)^T = e^T_{i-m(\tau_1,\tau)+1}\otimes I_n.$$

Hence the result for the left eigenspace of $P(\lambda)$ at ∞ follows from (2.23). ☐

Remark 2.1.24. *Notice that the proof of Lemma 2.1.22 and Theorem 2.1.23 does not use the fact that the index tuple* $(\sigma_1, \sigma, \sigma_2)$ *satisfies the SIP and hence the results in Theorem 2.1.23 remain valid for GFPR-like pencils of the form*

$$L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$$

in which only the index tuple (τ_1, τ, τ_2) *satisfies the SIP but not the tuple* $(\sigma_1, \sigma, \sigma_2)$*. This shows that the SIP of* (τ_1, τ, τ_2) *is enough for operation-free recovery of eigenvectors of* $P(\lambda)$ *corresponding to the eigenvalue* ∞ *from those of* $L(\lambda)$*. Compare this with Remark 2.1.14.*

We illustrate the recovery of eigenvectors of $P(\lambda)$ corresponding to an eigenvalue at ∞ from those of the GFPRs of $P(\lambda)$ by considering an example.

Example 2.1.25. *Let* $P(\lambda) := \sum_{i=0}^{5} \lambda^i A_i$*. Suppose that* $P(\lambda)$ *is regular and* ∞ *is an eigenvalue of* $P(\lambda)$*. Let* $\sigma := (0,1)$, $\sigma_1 := \emptyset, \sigma_2 = (0), \tau := (-4,-5,-3,-2)$, $\tau_2 = (-4,-3)$ *and* $\tau_1 = \emptyset$*. Let* X *and* (Y,Z) *be any nonsingular matrix assignments for* σ_2 *and* τ_2*, respectively. Then the GFPR* $L(\lambda) = \big(\lambda M^P_{(-4,-5,-3,-2)} - M^P_{(0,1)}\big)M_0(X)M_{(-4,-3)}(Y,Z) =: \lambda L_1 - L_0$ *of* $P(\lambda)$ *is given by*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & A_5 & A_4 & 0 \\ I_n & 0 & Y & A_3 & 0 \\ 0 & I_n & Z & A_2 & 0 \\ 0 & 0 & 0 & 0 & X \end{bmatrix} - \begin{bmatrix} 0 & 0 & I_n & 0 & 0 \\ I_n & 0 & Y & 0 & 0 \\ 0 & I_n & Z & 0 & 0 \\ 0 & 0 & 0 & -A_1 & X \\ 0 & 0 & 0 & -A_0 & 0 \end{bmatrix}.$$

Let x *and* y*, respectively, be right and left eigenvectors of* $L(\lambda)$ *corresponding to the eigenvalue* ∞*. Define* $x_i := (e_i^T \otimes I_n)x$ *and* $y_i := (e_i^T \otimes I_n)y$, $i = 1:5$*. We have* $c_{-5}(\tau, \tau_2) = c_{-5}(-4,-5,-3,-2,-4,-3) = 2$*. Hence by Theorem 2.1.23,* $(e^T_{c-m(\tau,\tau_2)+1} \otimes I_n)x = (e_3^T \otimes I_n)x = x_3$ *is a right eigenvector of* $P(\lambda)$ *corresponding to the eigenvalue* ∞*.*

Similarly, $i_{-5}(\tau_1, \tau) = i_{-5}(-4, -5, -3, -2) = 1$. *Hence by Theorem 2.1.23,* $(e^T_{i_{-m}(\tau_1,\tau)+1} \otimes I_n)y = (e_2^T \otimes I_n)y = y_2$ *is a left eigenvector of* $P(\lambda)$ *corresponding to the eigenvalue* ∞.

To verify the recovery rule, consider $L_1 x = 0$. *This gives* $x_4 = 0 = x_5$ *and* $A_5 x_3 = 0$. *Further, if* $x_3 = 0$ *then* $x_1 = 0 = x_2$. *Thus* $x_3 = 0 \Rightarrow x = 0$. *Hence* $x_3 \neq 0$ *and is a right eigenvector of* $P(\lambda)$ *corresponding to the eigenvalue* ∞.

Similarly, $y^T L_1 = 0$ *implies that* $y_i^T = 0$ *for* $i = 3, 4, 5$, *and* $y_2^T A_5 = 0$. *Further, if* $y_2^T = 0$ *then* $y_1^T = 0$. *Thus* $y_2^T \Rightarrow y = 0$. *Hence* $y_2^T \neq 0$ *and is a left eigenvector of* $P(\lambda)$ *corresponding to the eigenvalue* ∞. ■

2.2 Recovery of eigenvectors and minimal bases from structured linearizations

By using GFPRs (specially, FPRs) of $P(\lambda)$, structure (symmetric, skew-symmetric, even, odd, palindromic, etc.) preserving linearizations have been constructed for structured $P(\lambda)$, see [13, 17, 19, 20]. This section is devoted to describing the recovery of eigenvectors and minimal bases of a structured $P(\lambda)$ from those of the structure-preserving linearizations.

First, we prove a result on minimal indices which will be useful in recovering the minimal indices of a structured $P(\lambda)$ from those of the structure-preserving linearizations of $P(\lambda)$. Note that if $P(\lambda)$ is singular then the left (resp., right) minimal indices of $P(\lambda)$ and $XP(\lambda)Y$ are the same for any nonsingular matrices X and Y.

Lemma 2.2.1. *Let* $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ *be a GFPR of* $P(\lambda)$. *Suppose that the left minimal indices of* $P(\lambda)$ *(resp.,* $L(\lambda)$*) are the same as the right minimal indices of* $P(\lambda)$ *(resp.,* $L(\lambda)$*). If* $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ *are the minimal (left and right) indices of* $L(\lambda)$ *then* $\varepsilon_1 - (m-1)/2 \leq \cdots \leq \varepsilon_k - (m-1)/2$ *are the minimal (left and right) indices of* $P(\lambda)$.

Proof. Let $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ be the minimal (left and right) indices of $L(\lambda)$. Since $T(\lambda) := \lambda M^P_\tau - M^P_\sigma$ is strictly equivalent to $L(\lambda)$, $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ are the minimal (left and right) indices of $T(\lambda)$. Note that $T(\lambda)$ is a PGF pencil of $P(\lambda)$. Let τ be given by $\tau = (\tau_\ell, -m, \tau_r)$. Define $\alpha := (-rev(\tau_\ell), \sigma, -rev(\tau_r))$. Then α is a permutation of $\{0, 1, \ldots, m-1\}$. Let c_L and i_L, respectively, be the total number of consecutions and inversions of α. Then by Theorem 1.2.26, $\varepsilon_1 - i_L \leq \cdots \leq \varepsilon_k - i_L$ and $\varepsilon_1 - c_L \leq \cdots \leq \varepsilon_k - c_L$, respectively, are the right and left minimal indices of $P(\lambda)$. Since the left and right minimal indices of $P(\lambda)$ are the same, we must have $i_L = c_L$. But $i_L + c_L = m - 1$. Consequently, we have $i_L = (m-1)/2 = c_L$ which yields the desired result. □

2.2.1 Symmetric linearizations

A matrix polynomial $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ is said to be symmetric if $P(\lambda)^T = P(\lambda)$ for $\lambda \in \mathbb{C}$, i.e., if $A_i^T = A_i$, $i = 0, 1, \ldots, m$. Since symmetric GFPRs are special cases of GFPRs of $P(\lambda)$, we show that simplified recovery rules hold for symmetric GFPRs which can be easily read off from the index tuples defining the symmetric GFPRs.

Let $\mathbf{t}$ be an index tuple. Recall that, an index tuple $\widehat{\mathbf{t}}$ is said to be a subtuple of $\mathbf{t}$ if $\widehat{\mathbf{t}} = \mathbf{t}$ or if $\widehat{\mathbf{t}}$ can be obtained from $\mathbf{t}$ by deleting some indices in $\mathbf{t}$. Also, recall that $\widehat{\mathbf{t}}$ is the subtuple of $\mathbf{t}$ with indices $\{t_1, \ldots, t_r\}$ if $\widehat{\mathbf{t}}$ is obtained from $\mathbf{t}$ by deleting all indices except $t_1, \ldots, t_r$.

Definition 2.2.2. *[17] (a) (Admissible tuple). Let $h \geq 0$ be an integer. We say that $\boldsymbol{w}$ is an admissible tuple of $\{0 : h\}$ if $\boldsymbol{w}$ is a permutation of $\{0 : h\}$ and*

$$csf(\boldsymbol{w}) = \big(h-1 : h, h-3 : h-2, \ldots, p+1 : p+2, 0 : p\big) \tag{2.24}$$

for some $0 \leq p \leq h$. We call p as the index of $\boldsymbol{w}$ and denote it by $Ind(\boldsymbol{w})$.
(b) (Symmetric complement). Let $h \geq 0$ be an integer and let $\boldsymbol{w}$ be an admissible tuple of $\{0 : h\}$ with index p. Then the symmetric complement of $\boldsymbol{w}$, denoted by $c_{\boldsymbol{w}}$, is defined by

$$\boldsymbol{c}_w := \begin{cases} \big(h-1, h-3, \ldots, p+3, p+1, (0:p)_{rev_c}\big) & \text{if } p \geq 1, \\ (h-1, h-3, \ldots, 1) & \text{if } p = 0 \text{ and } h > 0, \\ \emptyset & \text{if } h = 0, \end{cases}$$

where $(0 : p)_{rev_c} := (0 : p-1, 0 : p-2, \ldots, 0 : 1, 0)$.

For simplicity, we always consider an admissible tuple of the form (2.24). Clearly, for an integer $h \geq 0$, there exist a unique admissible tuple of $\{0 : h\}$ with index 0 or 1 [17].

Definition 2.2.3 (Simple admissible tuple)**.** *An admissible tuple $\boldsymbol{w}$ of $\{0 : h\}$, $h \geq 0$, is said to be the simple admissible tuple if $Ind(\boldsymbol{w}) = 0$ or $Ind(\boldsymbol{w}) = 1$.*

Note that for the simple admissible tuple $\mathbf{w}$ of $\{0 : h\}$, we have $Ind(\mathbf{w}) = 0$ (resp., $Ind(\mathbf{w}) = 1$) if h is even (resp., odd). We mention that in [17] the simple admissible tuple is referred as "the admissible tuple".

Definition 2.2.4. *[17] Given $h \geq 0$, we say that an index tuple $\boldsymbol{t}$ is in canonical form for h if $\boldsymbol{t}$ is of the form*

$$\Big(a_1 : h-2, a_2 : h-4, \ldots, a_{\lfloor \frac{h}{2} \rfloor} : h - 2\lfloor \tfrac{h}{2} \rfloor\Big)$$

with $a_i \geq 0$, $i = 1 : \lfloor \frac{h}{2} \rfloor$, where $\lfloor \cdot \rfloor$ stands for the greatest integer function.

Note that, if $h = 0, 1$, then an index tuple in canonical form for h is necessarily empty.

The following result characterizes all block symmetric GFPRs of $P(\lambda)$.

Theorem 2.2.5 ([17], Theorem 6.11)**.** *Let $0 \leq h < m$. Let $\boldsymbol{w}_h$ and $\boldsymbol{w}_{m-h-1}$ be the simple admissible tuples associated with h and $m-h-1$, respectively. Let $\boldsymbol{t}_{w_h}$ and $m + \boldsymbol{t}_{v_h}$ be index tuples in canonical form for h and $m-h-1$, respectively. Let $\mathcal{X}$ and $\mathcal{Y}$ be nonsingular matrix assignments for $\boldsymbol{t}_{w_h}$ and $\boldsymbol{t}_{v_h}$, respectively. Then*

$$L(\lambda) := M_{(\boldsymbol{t}_{w_h},\boldsymbol{t}_{v_h})}(\mathcal{X},\mathcal{Y})(\lambda M^P_{\boldsymbol{v}_h} - M^P_{\boldsymbol{w}_h}) M^P_{(\boldsymbol{c}_{w_h},\boldsymbol{c}_{v_h})} M_{(rev(\boldsymbol{t}_{w_h}),rev(\boldsymbol{t}_{v_h}))}(rev(\mathcal{X}),rev(\mathcal{Y})), \tag{2.25}$$

is a block symmetric GFPR of $P(\lambda)$, where $\boldsymbol{v}_h = -m + \boldsymbol{w}_{m-h-1}$, $\boldsymbol{c}_{w_h}$ be the symmetric complement of $\boldsymbol{w}_h$, $m + \boldsymbol{c}_{v_h}$ is the symmetric complement of $\boldsymbol{w}_{m-h-1}$. Moreover, any block symmetric GFPR of $P(\lambda)$ is of the form (2.25).

If the matrices in the matrix assignments $\mathcal{X}$ and $\mathcal{Y}$ in Theorem 2.2.5 are all symmetric then $L(\lambda)$ is symmetric for a symmetric $P(\lambda)$ [17]. It is well known that $L(\lambda)$ given in (2.25) is not a linearization of $P(\lambda)$ when $P(\lambda)$ is singular having even degree [13, 17].

Since $P(\lambda)$ is symmetric, the left and right minimal bases of $P(\lambda)$ are the same. Hence the left and right minimal indices of $P(\lambda)$ are also the same. We therefore refer to a left or a right minimal basis of $P(\lambda)$ as a minimal basis of $P(\lambda)$. The next result describes the recovery of minimal bases and minimal indices of $P(\lambda)$ from those of the symmetric GFPRs of $P(\lambda)$.

Theorem 2.2.6. *Let $P(\lambda)$ be symmetric and singular and let $L(\lambda)$ be as in Theorem 2.2.5. Then we have the following.*

(a) If $(u_1(\lambda), \ldots, u_k(\lambda))$ is a minimal basis of $L(\lambda)$ then $((e^T_{m-\alpha} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-\alpha} \otimes I_n)u_k(\lambda))$ is a minimal basis of $P(\lambda)$, where $\alpha = 0$ if $h = 0$, and $\alpha = 2 + c_2(rev(\boldsymbol{t}_{w_h}))$ if $h > 0$.

(b) If $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ are the minimal indices of $L(\lambda)$ then $\varepsilon_1 - (m-1)/2 \leq \cdots \leq \varepsilon_k - (m-1)/2$ are the minimal indices of $P(\lambda)$.

Proof. Suppose that $h = 0$. Then $\mathbf{w}_h = (0)$ and $\mathbf{c}_{w_h} = \emptyset = \mathbf{t}_{w_h}$. Thus $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$ $= 0$. Next, suppose that $h > 0$. Since $P(\lambda)$ is singular, A_m and A_0 are singular. Thus, we must have $0 \notin \mathbf{c}_{w_h} \cup \mathbf{t}_{w_h}$ and $-m \notin \mathbf{c}_{v_h} \cup \mathbf{t}_{v_h}$ since otherwise $L(\lambda)$ would not be a linearization of $P(\lambda)$. Note that if h is odd then $0 \in \mathbf{c}_{w_h}$. Hence h must be even. Since $\mathbf{w}_h$ is the simple admissible tuple of $\{0 : h\}$, we have $\mathbf{w}_h = (h-1 : h, h-3 : h-2, \ldots, 1 : 2, 0)$. This implies that $\mathbf{c}_{w_h} = (h-1, h-3, \ldots, 1)$. Hence $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) = 2 + c_2(rev(\mathbf{t}_{w_h}))$ (recall from Definition 1.2.20 that if $2 \notin rev(\mathbf{t}_{w_h})$ then $c_2(rev(\mathbf{t}_{w_h})) = -1$). Hence the proof of (a) follows from Theorem 2.1.12 with $\sigma = \mathbf{w}_h$, $\sigma_1 = \mathbf{t}_{w_h}$, $\sigma_2 = (\mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$, $\tau = \mathbf{v}_h$, $\tau_1 = \mathbf{t}_{v_h}$ and $\tau_2 = (\mathbf{c}_{v_h}, rev(\mathbf{t}_{v_h}))$.

Finally, (b) follows from Lemma 2.2.1 as $P(\lambda)$ and $L(\lambda)$ are both symmetric. $\square$

The next result describes the recovery of eigenvectors of a symmetric $P(\lambda)$ from those of the symmetric GFPRs of $P(\lambda)$.

Theorem 2.2.7. *Let $P(\lambda)$ be symmetric and regular. Let $L(\lambda)$ be as in Theorem 2.2.5. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$. Let ℓ be the index of $\boldsymbol{w}_h$ (note that $\ell = 0$ if h is even, and $\ell = 1$ if h is odd). If $(u_1, \ldots, u_k)$ is a basis of $\mathcal{N}_r(L(\lambda))$ then $((e^T_{m-\ell^*} \otimes I_n)u_1, \ldots, (e^T_{m-\ell^*} \otimes I_n)u_k)$ is a basis of $\mathcal{N}_r(P(\lambda))$ (also a basis of $\mathcal{N}_l(P(\lambda))$), where $\ell^* = \ell$ if $h = \ell$, and $\ell^* = \ell + 2 + c_{\ell+2}(rev(\boldsymbol{t}_{w_h}))$ if $h > \ell$.*

Proof. We have $\mathbf{w}_h = (h-1 : h, h-3 : h-2, \ldots, \ell+1 : \ell+2, 0 : \ell)$, where $\ell = 0$ if h is even, and $\ell = 1$ if h is odd. Suppose that $\ell = h$. Then $\mathbf{w}_h = (0 : \ell)$ and hence $c_0(\mathbf{w}_h) = \ell$. Note that $\mathbf{c}_{w_h}$ and $\mathbf{t}_{w_h}$ are index tuples with indices from $\{0 : h-1\}$ and $\{0 : h-2\}$, respectively. Thus $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) = \ell$. Next, suppose that $h > \ell$. Then $\mathbf{w}_h = (h-1 : h, h-3 : h-2, \ldots, \ell+1 : \ell+2, 0 : \ell)$ and $\mathbf{c}_{w_h} = (h-1, h-3, \ldots, \ell+1, (0 : \ell)_{rev_c})$. Thus $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) = \ell + 2 + c_{\ell+2}(rev(\mathbf{t}_{w_h}))$ (note that if $\ell + 2 \notin rev(\mathbf{t}_{w_h})$ then $c_{\ell+2}(rev(\mathbf{t}_{w_h})) = -1$). Hence the desired result follows from Corollary 2.1.13 with $\sigma = \mathbf{w}_h$, $\sigma_1 = \mathbf{t}_{w_h}$, $\sigma_2 = (\mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$, $\tau = \mathbf{v}_h$, $\tau_1 = \mathbf{t}_{v_h}$ and $\tau_2 = (\mathbf{c}_{v_h}, rev(\mathbf{t}_{v_h}))$. $\square$

The following example describes the recovery of eigenvectors from a symmetric GFPR.

Example 2.2.8. *Let $P(\lambda) := \sum_{i=0}^{5} \lambda^i A_i$ be symmetric. Suppose that $P(\lambda)$ is regular and μ is an eigenvalue of $P(\lambda)$. Consider $\boldsymbol{w}_h = (3:4, 1:2, 0)$, $\boldsymbol{t}_{w_h} = (1:2)$, $\boldsymbol{v}_h = (-5)$ and $\boldsymbol{t}_{v_h} = \emptyset$. Let (X, Y) be any arbitrary matrix assignment for $\boldsymbol{t}_{w_h}$. Then $L(\lambda) := M_{(1:2)}(X, Y)\big(\lambda M^P_{-5} - M^P_{(3:4,1:2,0)}\big) M^P_{(3,1)} M_{(2,1)}(Y, X) =$*

$$\begin{bmatrix} \lambda A_5 + A_4 & A_3 & -Y & -X & -I_n \\ A_3 & -\lambda A_3 + A_2 & \lambda Y + A_1 & \lambda X - I_n & \lambda I_n \\ -Y & \lambda Y + A_1 & -\lambda A_1 + A_0 & \lambda I_n & 0 \\ -X & \lambda X - I_n & \lambda I_n & 0 & 0 \\ -I_n & \lambda I_n & 0 & 0 & 0 \end{bmatrix}$$

is a block symmetric GFPR of $P(\lambda)$. Hence if X and Y are symmetric matrices then $L(\lambda)$ is symmetric whenever $P(\lambda)$ is symmetric.

Let $x \in \mathcal{N}_r(L(\mu))$. Define $x_i := (e^T_i \otimes I_n)x$ for $i = 1:5$. Since $h = 4$ is even, we have $\ell = 0$ in Theorem 2.2.7. Now $c_2(rev(\boldsymbol{t}_{w_h})) = c_2(2,1) = 0$ and $2 + c_2(rev(\boldsymbol{t}_{w_h})) = 2$. Hence by Theorem 2.2.7, we have $(e^T_{5-2} \otimes I_n)x = x_3 \in \mathcal{N}_r(P(\mu))$ which can be easily verified.

2.2.2 Skew-symmetric linearizations

A matrix polynomial $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ is said to be skew-symmetric if $P(\lambda)^T = -P(\lambda)$ for $\lambda \in \mathbb{C}$, i.e., if $A_i^T = -A_i$, $i = 0, 1, \ldots, m$. This section is devoted to describing the recovery of eigenvectors and minimal bases of a skew-symmetric $P(\lambda)$ from those of the skew-symmetric linearizations of $P(\lambda)$ that has been constructed in [20].

Definition 2.2.9. *[20] A matrix $S \in \mathbb{C}^{mn}$ is said to be a quasi-identity matrix if $S = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_m I_n$, where $\epsilon_i \in \{1, -1\}$ for $i = 1 : m$.*

Recall the left and right index tuples of type-1 from Definition 1.2.33. The following theorem gives skew-symmetric linearizations of $P(\lambda)$.

Theorem 2.2.10 ([20], Theorem 3.15)**.** *Let $P(\lambda)$ be skew-symmetric and let $0 \leq h \leq m-1$. Let $\boldsymbol{w}$ and $\widetilde{\boldsymbol{w}}$ be any admissible tuples of $\{0 : h\}$ and $\{0 : m-h-1\}$, respectively, and $\boldsymbol{c}_w$ and $\boldsymbol{c}_{\widetilde{w}}$ be the symmetric complement of $\boldsymbol{w}$ and $\widetilde{\boldsymbol{w}}$, respectively. Let $\boldsymbol{t}_w$ with indices from $\{0 : h-1\}$ and $\boldsymbol{t}_{\widetilde{w}}$ with indices from $\{0 : m-h-2\}$ be right index tuples of type-1 relative to $rev(\boldsymbol{w})$ and $rev(\widetilde{\boldsymbol{w}})$, respectively. Define $\boldsymbol{v} := -m + \widetilde{\boldsymbol{w}}$, $\boldsymbol{c}_v := -m + \boldsymbol{c}_{\widetilde{w}}$ and $\boldsymbol{t}_v := -m + \boldsymbol{t}_{\widetilde{w}}$. Let $L(\lambda) := M^P_{rev(\boldsymbol{t}_v)} M^P_{rev(\boldsymbol{t}_w)} (\lambda M^P_{\boldsymbol{v}} - M^P_{\boldsymbol{w}}) M^P_{\boldsymbol{c}_w} M^P_{\boldsymbol{t}_w} M^P_{\boldsymbol{c}_v} M^P_{\boldsymbol{t}_v}$. Then, up to multiplication by -1, there exists a unique quasi-identity matrix S such that $SL(\lambda)$ is skew-symmetric.*

It is well known that $L(\lambda)$ given in Theorem 2.2.10 is not a linearization of $P(\lambda)$ when $P(\lambda)$ is singular having even degree [20].

For a skew-symmetric $P(\lambda)$, we have $u(\lambda) \in \mathcal{N}_r(P) \Leftrightarrow u(\lambda) \in \mathcal{N}_l(P)$. Hence $\mathcal{N}_r(P(\lambda))$ and $\mathcal{N}_l(P(\lambda))$ are the same for an eigenvalue λ of a regular skew-symmetric $P(\lambda)$.

Theorem 2.2.11. *Let $P(\lambda)$ be skew-symmetric and regular. Let $L(\lambda)$ and S be as in Theorem 2.2.10. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$. Let ℓ be the index of $\boldsymbol{w}$. If $(u_1, \ldots, u_k)$ is a basis of $\mathcal{N}_r(SL(\lambda))$ then $((e^T_{m-\ell^*} \otimes I_n)u_1, \ldots, (e^T_{m-\ell^*} \otimes I_n)u_k)$ is a basis of $\mathcal{N}_r(P(\lambda))$ (also a basis of $\mathcal{N}_l(P(\lambda))$), where $\ell^* = \ell$ if $h = \ell$, and $\ell^* = \ell + 2 + c_{\ell+2}(\boldsymbol{t}_w)$ if $h > \ell$.*

Proof. The proof is similar to that of Theorem 2.2.7. □

Note that the left and right minimal bases of a skew-symmetric $P(\lambda)$ are the same. Hence the left and right minimal indices of $P(\lambda)$ are also the same. Now, we describe the recovery of minimal bases of $P(\lambda)$ from those of the skew-symmetric linearizations of $P(\lambda)$.

Theorem 2.2.12. *Let $P(\lambda)$ be skew-symmetric and singular. Let $L(\lambda)$ and S be as in Theorem 2.2.10. Then we have the following.*

(a) If $(u_1(\lambda), \ldots, u_k(\lambda))$ *is a minimal basis of* $SL(\lambda)$ *then* $((e^T_{m-\alpha} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-\alpha} \otimes I_n)u_k(\lambda))$ *is a minimal basis of* $P(\lambda)$*, where* $\alpha = 0$ *if* $h = 0$*, and* $\alpha = 2 + c_2(\boldsymbol{t}_w)$ *if* $h > 0$.

(b) If $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ *are the minimal indices of* $SL(\lambda)$ *then* $\varepsilon_1 - (m-1)/2 \leq \cdots \leq \varepsilon_p - (m-1)/2$ *are the minimal indices of* $P(\lambda)$.

Proof. The proof of (a) is similar to that of part (a) of Theorem 2.2.6.

Since S is nonsingular, the left (resp., right) minimal indices of $SL(\lambda)$ and $L(\lambda)$ are the same. Hence (b) follows from Lemma 2.2.1. □

2.2.3 Even and odd linearizations

A matrix polynomial $P(\lambda)$ is said to be T-even if $P(-\lambda) = P(\lambda)^T$ for $\lambda \in \mathbb{C}$. On the other hand, $P(\lambda)$ is said to be T-odd if $P(-\lambda) = -P(\lambda)^T$ for $\lambda \in \mathbb{C}$. We say $P(\lambda)$ is T-alternating if it is either T-even or T-odd. This section is devoted to describing the recovery of eigenvectors, minimal bases and minimal indices of T-alternating $P(\lambda)$ from those of the T-alternating linearizations of $P(\lambda)$ that has been constructed in [20].

The following result gives T-alternating linearizations of T-alternating $P(\lambda)$.

Theorem 2.2.13 ([20], Theorem 4.15)**.** *Let* $0 \leq h \leq m-1$ *be even. Let* $\boldsymbol{w}$ *and* $\widetilde{\boldsymbol{w}}$ *be any admissible tuples of* $\{0 : h\}$ *and* $\{0 : m-h-1\}$*, respectively, and* $\boldsymbol{c}_w$ *and* $\boldsymbol{c}_{\widetilde{w}}$ *be the symmetric complement of* $\boldsymbol{w}$ *and* $\widetilde{\boldsymbol{w}}$*, respectively. Define* $\boldsymbol{v} := -m + \widetilde{\boldsymbol{w}}$ *and* $\boldsymbol{c}_v := -m + \boldsymbol{c}_{\widetilde{w}}$*. Let* $L(\lambda) := \big(\lambda M^P_{\boldsymbol{v}} - M^P_{\boldsymbol{w}}\big) M^P_{\boldsymbol{c}_w} M^P_{\boldsymbol{c}_v}$*. Then, up to multiplication by* -1*, there exists a unique quasi-identity matrix* S *such that* $SL(\lambda)$ *is* T*-even (resp.,* T*- odd) when* $P(\lambda)$ *is* T*-even (resp.,* T*-odd).*

It is well known that $L(\lambda)$ given in Theorem 2.2.13 is not a linearization of $P(\lambda)$ when $P(\lambda)$ is singular having even degree [20].

For a T-alternating $P(\lambda)$, we have $u(\lambda) \in \mathcal{N}_r(P) \Leftrightarrow u(-\lambda) \in \mathcal{N}_l(P)$. Hence for an eigenvalue λ of a regular $P(\lambda)$, $\mathcal{B} := (u_1, \ldots, u_k)$ is a basis of $\mathcal{N}_r(P(\lambda))$ if and only if $\mathcal{B}$ is a basis of $\mathcal{N}_l(P(\lambda))$.

Theorem 2.2.14. *Let* $P(\lambda)$ *be regular and* T*-alternating and let* S *and* $L(\lambda)$ *be as in Theorem 2.2.13. Suppose that* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $P(\lambda)$*. Let* ℓ *be the index of* $\boldsymbol{w}$*. If* $(u_1, \ldots, u_k)$ *is a basis of* $\mathcal{N}_r(SL(\lambda))$ *then* $((e^T_{m-\ell^*} \otimes I_n)u_1, \ldots, (e^T_{m-\ell^*} \otimes I_n)u_k)$ *is a basis of* $\mathcal{N}_r(P(\lambda))$*, where* $\ell^* = \ell$ *if* $\ell = h$*, and* $\ell^* = \ell + 1$ *if* $\ell < h$.

Proof. A verbatim proof of Theorem 2.2.7 with $\mathbf{t}_{w_h} = \emptyset$ gives the desired result. □

Since $P(\lambda)$ is T-alternating, $(x_1(\lambda), \ldots, x_p(\lambda))$ is a right minimal basis of $P(\lambda)$ if and only if $(x_1(-\lambda), \ldots, x_p(-\lambda))$ is a left minimal basis of $P(\lambda)$. So the left and right minimal indices of $P(\lambda)$ are the same.

Theorem 2.2.15. *Let $P(\lambda)$ be T-alternating and singular and let S and $L(\lambda)$ be as in Theorem 2.2.13. Then we have the following.*

(a) Let $(u_1(\lambda), \ldots, u_k(\lambda))$ be a right minimal basis of $SL(\lambda)$. Then $((e_{m-\alpha}^T \otimes I_n)u_1(\lambda), \ldots, (e_{m-\alpha}^T \otimes I_n)u_k(\lambda))$ is a right minimal basis of $P(\lambda)$, where $\alpha = 0$ if $h = 0$, and $\alpha = 1$ if $h > 0$.

(b) If $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ are the minimal indices of $SL(\lambda)$ then $\varepsilon_1 - (m-1)/2 \leq \cdots \leq \varepsilon_k - (m-1)/2$ are the minimal indices of $P(\lambda)$.

Proof. A verbatim proof of Theorem 2.2.6 with $\mathbf{t}_{w_h} = \emptyset$ gives the desired results. $\square$

2.2.4 Palindromic linearizations

A matrix polynomial $P(\lambda)$ is said to be T-palindromic if $rev(P(\lambda)) = P(\lambda)^T$ for $\lambda \in \mathbb{C}$. This section is devoted to describing the recovery of eigenvectors, minimal bases and minimal indices of $P(\lambda)$ from the T-palindromic linearizations of $P(\lambda)$ that has been constructed in [19] by using the FPRs. For rest of this section, we define

$$R := \begin{bmatrix} 0 & & I_n \\ & \iddots & \\ I_n & & 0 \end{bmatrix} \in \mathbb{C}^{mn \times mn}.$$

Recall from Definition 1.2.33 the left and right index tuples of type-1 and the notations $z_r(.,.)$ and $z_l(.,.)$. The following result gives T-palindromic linearizations of $P(\lambda)$.

Theorem 2.2.16 ([19], Theorem 3.3)**.** *Let $P(\lambda)$ be T-palindromic of odd degree $m \geq 3$ and $h = (m-1)/2$. Let $\boldsymbol{q}$ be a permutation of $\{0, 1, \ldots, h\}$. Assume that $\boldsymbol{r}_q$ and $\boldsymbol{l}_q$, respectively, are right and left index tuples of type-1 relative to $\boldsymbol{q}$. Define*

$$L(\lambda) := M^P_{-m+rev(\boldsymbol{r}_q)} M^P_{\boldsymbol{l}_q} \big(\lambda M^P_{-m+rev(\boldsymbol{q})} - M^P_{\boldsymbol{q}}\big) M^P_{\boldsymbol{r}_q} M^P_{-m+rev(\boldsymbol{l}_q)}.$$

Then $SRL(\lambda)$ is T-palindromic, where $S = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_m I_n$ is a quasi-identity matrix with $\epsilon_i = \pm 1$ and

$$\epsilon_i = -1 \quad \text{if and only if} \quad \begin{cases} z_l(\boldsymbol{l}_q, \boldsymbol{q}) \text{ has an inversion at } i-1, \text{ or} \\ z_r(\boldsymbol{q}, \boldsymbol{r}_q) \text{ has a consecution at } m-i. \end{cases} \tag{2.26}$$

Since $P(\lambda)$ is T-palindromic, if $(x_1(\lambda), \ldots, x_k(\lambda))$ is a right (resp., left) minimal basis of $P(\lambda)$ then $(rev(x_1(\lambda)), \ldots, rev(x_k(\lambda)))$ is a left (resp., right) minimal basis of $P(\lambda)$. Hence the left and right minimal indices of $P(\lambda)$ are the same. It also follows that $\mathcal{N}_r(P(\lambda)) = \mathcal{N}_l(P(\lambda))$ when $P(\lambda)$ is regular and λ is an eigenvalue of $P(\lambda)$.

Theorem 2.2.17. *Let $P(\lambda)$ be T-palindromic of odd degree $m \geq 3$ and let S, R and $L(\lambda)$ be as in Theorem 2.2.16.*

(a) Suppose that $P(\lambda)$ is regular and $\lambda \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$. If $(u_1, \ldots, u_k)$ is a basis of $\mathcal{N}_r(SRL(\lambda))$ then $((e^T_{m-c_0(\boldsymbol{q},\boldsymbol{r}_q)} \otimes I_n)u_1, \ldots, (e^T_{m-c_0(\boldsymbol{q},\boldsymbol{r}_q)} \otimes I_n)u_k)$ is a basis of $\mathcal{N}_r(P(\lambda))$.

(b) Suppose that $P(\lambda)$ is singular. Let $(u_1(\lambda), \ldots, u_k(\lambda))$ be a right minimal basis of $SRL(\lambda)$. Then $((e^T_{m-c_0(\boldsymbol{q},\boldsymbol{r}_q)} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-c_0(\boldsymbol{q},\boldsymbol{r}_q)} \otimes I_n)u_k(\lambda))$ is a right minimal basis of $P(\lambda)$.

(c) If $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ are the minimal indices of $SRL(\lambda)$ then $\varepsilon_1 - (m-1)/2 \leq \cdots \leq \varepsilon_k - (m-1)/2$ are the minimal indices of $P(\lambda)$.

Proof. The proofs of (a) and (b), respectively, follow from Theorem 2.1.12 and Corollary 2.1.13 with $\sigma = \mathbf{q}$, $\sigma_1 = \mathbf{l}_q$, $\sigma_2 = \mathbf{r}_q$, $\tau = -m + rev(\mathbf{q})$, $\tau_1 = -m + rev(\mathbf{r}_q)$ and $\tau_2 = -m + rev(\mathbf{l}_q)$ and X_j and Y_j, $j = 1, 2$, are the trivial matrix assignments. Since S and R are nonsingular, the left (resp., right) minimal indices of $SRL(\lambda)$ and $L(\lambda)$ are the same. Hence (c) follows from Lemma 2.2.1. $\square$

Remark 2.2.18. *We mention that when $P(\lambda)$ is T-anti-palindromic, that is, $P(\lambda)^T = -rev(P(\lambda))$, the recovery formulae for eigenvectors, minimal bases and minimal indices of $P(\lambda)$ from those of the T-anti-palindromic linearizations of $P(\lambda)$ are the same as in the case of T-palindromic $P(\lambda)$. Indeed, let $P(\lambda)$ be T-anti-palindromic. It is well known that $P(\lambda)$ is T-anti-palindromic if and only if $P(-\lambda)$ is T-palindromic [27]. Suppose that $P(\lambda)$ is of odd degree $m \geq 3$. Define $Q(\lambda) := P(-\lambda)$. Consider the FPR $L(\lambda) := M^Q_{-m+rev(\boldsymbol{r}_q)} M^Q_{\boldsymbol{l}_q}(\lambda M^Q_{-m+rev(\boldsymbol{q})} - M^Q_{\boldsymbol{q}}) M^Q_{\boldsymbol{r}_q} M^Q_{-m+rev(\boldsymbol{l}_q)}$ of $Q(\lambda)$ as in Theorem 2.2.16, where M^Q_j's are the Fiedler matrices associated with $Q(\lambda)$. Then $SRL(-\lambda)$ is a T-anti-palindromic strong linearization of $P(\lambda)$, where S is a quasi-identity matrix as in (2.26).*

Since $rev(P(\lambda)) = -P(\lambda)^T$, we have $\mathcal{N}_r(revP) = \mathcal{N}_r(-P^T) = \mathcal{N}_l(P)$ and $\mathcal{N}_l(revP) = \mathcal{N}_l(-P^T) = \mathcal{N}_r(P)$. Thus, if $(x_1(\lambda), \ldots, x_k(\lambda))$ is a right (resp., left) minimal basis of $P(\lambda)$ then $(rev(x_1(\lambda)), \ldots, rev(x_k(\lambda)))$ is a left (resp., right) minimal basis of $P(\lambda)$. Also, the left and right minimal indices of $P(\lambda)$ are the same. Hence the recovery formulae for eigenvectors, minimal bases and minimal indices of a T-anti-palindromic $P(\lambda)$ from those of the T-anti-palindromic linearizations obtained from FPRs are the same as those of T-palindromic $P(\lambda)$ and T-palindromic linearizations obtained from FPRs.

2.3 Block structure of eigenvectors of GFPRs

Suppose that $P(\lambda)$ is regular. Let $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ be a GFPR of $P(\lambda)$. Let $\mathbf{v} \in \mathcal{N}_r(L(\lambda))$ be an eigenvector and v be the eigenvector of $P(\lambda)$ recovered from $\mathbf{v}$. Then the quotient $q(\mathbf{v}) := \|\mathbf{v}\|/\|v\|$ may be useful for comparing backward errors of $\mathbf{v}$ with that of v, when λ is simple, and the condition number of λ as an eigenvalue of $L(\lambda)$ with the condition number of λ as an eigenvalue of $P(\lambda)$; see [14]. Let $\mathbf{v} := [v_1, \ldots, v_m]^B$ be a partition of $\mathbf{v}$, where $v_j \in \mathbb{C}^n$ for $j = 1 : m$ and $\mathbf{x}^B$ denotes block transpose of $\mathbf{x}$. By Corollary 2.1.13, we have $v_j = v$ for $j = m - c_0(\sigma, \sigma_2)$. However, for obtaining lower and upper bounds of the quotient $q(\mathbf{v})$, it may be helpful to know the block structure of $\mathbf{v}$, that is, to know the explicit relationships between v and the blocks v_j for $j = 1 : m$.

The block structure of eigenvectors of Fiedler pencils has been determined in [26] and the block structure of eigenvectors of GFPs and type-1 FPRs has been determined in [14]. The block structure of eigenvectors has also been determined for pencils belonging to four special classes of block symmetric extended block Kronecker pencils of $P(\lambda)$ in [21]. Also, it has been shown in [21] that a block symmetric GFPR is block permutationally congruent to a pencil belonging to one of the four special classes of block symmetric extended block Kronecker pencils of $P(\lambda)$. However, the problem is that it is not known how to determine the block permutation whose existence is proved in [21]. Nevertheless, this result shows that block structures of eigenvectors of block symmetric GFPRs are amenable to determination modulo some permutations. However, for a GFPR $L(\lambda)$, the blocks v_j and v are unlikely to share a nice relation for $j = 1 : m$. This is because of the fact the pencil $\lambda M^P_\tau - M^P_\sigma$ in $L(\lambda)$ is a GF pencil and hence block structure of its eigenvectors is known. Now, if $\mathbf{u}$ is a right eigenvector of $\lambda M^P_\tau - M^P_\sigma$ then $(M_{\sigma_2}(X_2)M_{\tau_2}(Y_2))^{-1}\mathbf{u}$ is a right eigenvector of $L(\lambda)$. However, computation of $(M_{\sigma_2}(X_2)M_{\tau_2}(Y_2))^{-1}$ will not be easy and even when the inverse is available, the inverse is unlikely to produce a meaningful relation between blocks when multiplied to $\mathbf{u}$. Thus the blocks of $\mathbf{v} := (M_{\sigma_2}(X_2)M_{\tau_2}(Y_2))^{-1}\mathbf{u}$ are unlikely to share meaningful relations which can be harnessed to deduce bounds on the quotient $q(\mathbf{v})$. Assuming that the GFPR $L(\lambda)$ is also an extended block Kronecker pencil does not improve the situation either. We illustrate this by an example.

Example 2.3.1. *Let* $P(\lambda) := \sum_{i=0}^{4} \lambda^i A_i$. *Let* (X, Y, Z, W) *be a nonsingular matrix assignment for* $(1, 2, 1, 0)$. *Then* $L(\lambda) := \big(\lambda M^P_{-4} - M^P_{(1,2,3,0)}\big)M_{(1,2,1,0)}(X, Y, Z, W)$ *is a*

GFPR of $P(\lambda)$ *which is explicitly given by*

$$L(\lambda) = \left[\begin{array}{cc|cc} \lambda A_4 + A_3 & -Y & -Z & -W \\ A_2 & \lambda Y - X & \lambda Z - I_n & \lambda W \\ A_1 & A_0 + \lambda X & \lambda I_n & 0 \\ \hline -I_n & \lambda I_n & 0 & 0 \end{array}\right] =: \left[\begin{array}{c|c} M(\lambda) & K_2(\lambda) \\ \hline K_1(\lambda) & 0 \end{array}\right].$$

Note that $L(\lambda)$ *is an extended block-Kronecker pencil [18] since* $\begin{bmatrix} Z & W \\ I_n & 0 \end{bmatrix}$ *is nonsingular (as* W *is nonsingular) and* $K_2(\lambda) = \begin{bmatrix} -I_n & 0 \\ \lambda I_n & -I_n \\ 0 & \lambda I_n \end{bmatrix} \begin{bmatrix} Z & W \\ I_n & 0 \end{bmatrix}$. *It is easy to see that* $L(\lambda) = T(\lambda)U$, *where* $T(\lambda)$ *and* U *are given by*

$$T(\lambda) := \left[\begin{array}{cc|cc} \lambda A_4 + A_3 & 0 & -I_n & 0 \\ A_2 & 0 & \lambda I_n & -I_n \\ A_1 & A_0 & 0 & \lambda I_n \\ \hline -I_n & \lambda I_n & 0 & 0 \end{array}\right] \text{ and } U := \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & Y & Z & W \\ 0 & X & I_n & 0 \end{bmatrix}.$$

Let Π *be the block-permutation matrix given by* $\Pi_{i,j} = I_n$ *if and only if* $(i,j) \in \{(1,1), (2,3), (3,4), (4,2)\}$, *where* $\Pi_{i,j}$ *denotes the* (i,j)*-th block entry of* Π. *Then* $T(\lambda) = F_\sigma(\lambda)\Pi$, *where* $F_\sigma(\lambda) := \lambda M^P_{-4} - M^P_{(1,2,3,0)}$ *is the Fiedler pencil of* $P(\lambda)$ *associated with the permutation* $\sigma := (1,2,3,0)$ *of* $\{0:3\}$. *Now by [26, Theorem 5.7],* $v(\lambda) \in \mathcal{N}_r(P)$ *if and only if* $\mathbf{u}(\lambda) := \begin{bmatrix} \lambda v(\lambda) \\ \lambda P_1 v(\lambda) \\ \lambda P_2 v(\lambda) \\ v(\lambda) \end{bmatrix} \in \mathcal{N}_r(F_\sigma)$, *where* P_j *denotes the* j*-th Horner shift [26] of* $P(\lambda)$. *Hence* $v(\lambda) \in \mathcal{N}_r(P)$ *if and only if* $\Pi^{-1}\mathbf{u}(\lambda) \in \mathcal{N}_r(T)$. *Since* $L(\lambda) = T(\lambda)U$, *it follows that* $v(\lambda) \in \mathcal{N}_r(P)$ *if and only if*

$$\mathbf{v}(\lambda) := U^{-1}\Pi^{-1}\mathbf{u}(\lambda) = \begin{bmatrix} \lambda v(\lambda) \\ v(\lambda) \\ (\lambda P_2 - X)v(\lambda) \\ W^{-1}\big(\lambda P_1 - \lambda Z P_2 - Y + ZX\big)v(\lambda) \end{bmatrix} \in \mathcal{N}_r(L).$$

Notice that the block structure of $\mathbf{v}(\lambda)$ *is far from being helpful. Next, note that* $L(\lambda) = F_\sigma(\lambda)S$, *where* $S := M_{(1,2,1,0)}(X,Y,Z,W) = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & Y & Z & W \\ 0 & X & I_n & 0 \\ 0 & I_n & 0 & 0 \end{bmatrix}$. *In the present case, it is easy to compute* S^{-1}. *This shows that* $v(\lambda) \in \mathcal{N}_r(P)$ *if and only if* $\mathbf{u}(\lambda) \in \mathcal{N}_r(F_\sigma)$ *which in turn shows that* $v(\lambda) \in \mathcal{N}_r(P)$ *if and only if* $S^{-1}\mathbf{u}(\lambda) \in \mathcal{N}_r(L)$. *A little calculation shows that*

$$S^{-1}\mathbf{u}(\lambda) = \begin{bmatrix} \lambda v(\lambda) \\ v(\lambda) \\ (\lambda P_2 - X)v(\lambda) \\ W^{-1}\big(\lambda P_1 - \lambda Z P_2 - Y + ZX\big)v(\lambda) \end{bmatrix} = \mathbf{v}(\lambda) \in \mathcal{N}_r(L). \blacksquare$$

As the above example illustrates, the determination of block structure of eigenvectors of GFPRs is a difficult problem. Even the block structure of eigenvectors of type-1 FPRs obtained in [14] do not hold for type-1 GFPRs. The determination of block structure of eigenvectors of GFPRs under appropriate assumptions and the selection of optimal (in the sense of conditioning and backward errors) GFPRs are interesting open problems.

2.4 Algorithms for FPRs and GFPRs

A GFPR can be constructed explicitly without multiplying the Fiedler and elementary matrices associated with $P(\lambda)$, that is, one can easily construct a GFPR only by looking at the index tuples and the matrix assignments those are used to define the GFPR. In this section, we present algorithms which construct FPRs and GFPRs. For this purpose, we define some technical terms which will be useful to describe the algorithms.

Definition 2.4.1. *Let* α *be an index tuple containing indices from* $\{0 : d\}$, *for some integer* $d \geq 0$, *such that* α *satisfies the SIP. Suppose that* $t \in \alpha$. *Let the* $rsf(\alpha)$ *be given by* $rsf(\alpha) = \big(rev(a_0 : 0), rev(a_1 : 1), \ldots, rev(a_k : k), rev(a_{k+1} : k+1) \ldots, rev(a_d : d)\big)$. *Then we say that* α *has* p (≥ 0) *reverse consecutions at* t *if* $t \in rev(a_k : k)$ *and* $a_k = t - p$ *for smallest the integer* k. *We denote the number of reverse consecution of* α *at* t *by* $\mathfrak{rc}_t(\alpha)$. *Further, we define* $\alpha^R(t) := \big(rev(a_{k+1} : k+1), \ldots, rev(a_d : d)\big)$, *where* $t \in rev(a_k : k)$ *for the smallest integer* k.

Example 2.4.2. *Let* $\alpha = (3,1,0,2,4,1,3)$. *Then rsf*$(\alpha) = (1,0,3,2,1,4,3) = (rev(0 : 1), rev(1 : 3), rev(3 : 4)) =: \big(rev(a_0 : 0), rev(a_1 : 1), rev(a_2 : 2), rev(a_3 : 3), rev(a_4 : 4)\big)$, *where* $a_1 = 0, a_3 = 1, a_4 = 3$ *and* $a_0 = \infty = a_2$. *Then we have* $\mathfrak{rc}_2(\alpha) = 1$ *since* $2 \in rev(a_3 : 3)$ *and* $a_3 = 2 - 1$. *Further, we have* $\alpha^R(2) = rev(a_4 : 4) = rev(3 : 4)$. *Similarly, we have* $\mathfrak{rc}_1(\alpha) = 1$ *and* $\alpha^R(1) = (rev(1 : 3), rev(3 : 4))$.

The reverse consecutions for index tuple containing negative indices is defined as follows.

Definition 2.4.3. *Let* β *be an index tuple containing indices from* $\{-d : -1\}$, *for some integer* $d \geq 1$, *such that* β *satisfies the SIP. Suppose that* $-t \in \beta$. *Let the* $rsf(\beta)$ *be given by*

$$rsf(\beta) = \Big(rev(-a_d : -d), \ldots, rev(-a_{k+1} : -(k+1)), rev(-a_k : -k), \ldots, rev(-a_1 : -1)\Big).$$

Then we say that β *has* q (≥ 0) *reverse consecutions at* $-t$ *if* $-t \in rev(-a_k : -k)$ *and* $-a_k = -t - q$ *for the largest integer* k. *We denote the number of reverse consecution of* β *at* $-t$ *by* $\mathfrak{rc}_{-t}(\beta)$. *Further, we define* $\beta^R(-t) := (rev(-a_{k-1} : -(k-1)), \ldots, rev(-a_1 : -1))$, *where* $-t \in rev(-a_k : -k)$ *for the largest integer* k.

Definition 2.4.4. *Let* α *be an index tuple containing indices from* $\{-m : m-1\}$. *Suppose that* $s \in \alpha$. *Then the position of the first occurrence of* s *in* α *is denoted by* $p_s(\alpha)$.

For example, let $\alpha = (0, 1, 2, 3, 2, -1, -5, 4, 3, -1)$ be an index tuple and $X = (X_1, X_2, \ldots, X_{10})$ be an arbitrary matrix assignment for α. Then $p_2(\alpha) = 3$ and $p_{-1}(\alpha) = 6$ as the first occurrence of 2 and -1 appear in $3rd$ and $6th$ positions in α, respectively.

The following Algorithm-1 and Algorithm-2 construct an FPR of $P(\lambda)$, and Algorithm-3 and Algorithm-4 construct a GFPR of $P(\lambda)$. For proof see Appendix A.

Algorithm 1: $\lambda L_1 - L_0 := M^P_{\tau_1} M^P_{\sigma_1} (\lambda M^P_\tau - M^P_\sigma) M^P_{\sigma_2} M^P_{\tau_2}$ be an FPR of $P(\lambda)$

```
Input    : $\sigma, \tau, \sigma_j$ and $\tau_j$, $j = 1, 2$.
Output   : $L_0$
Notation: $\alpha := rsf(\sigma_1, \sigma, \sigma_2)$ and $\beta := rsf(\tau_1, \tau_2)$.
          $L_0^\ell := \ell^{th}$ block row of $L_0 = (e_\ell^T \otimes I_n) L_0$
for $i = 0 : h$ do
    if the subtuple of $\alpha$ with indices $\{i, i+1\}$ starts with $i+1$ then
        $L_0^{m-i} = e^T_{m-(i+1+c_{i+1}(\alpha))} \otimes I_n$
    else
        if $0 \neq i - \mathfrak{rc}_i(\alpha) =: p$, $\big(set\ \alpha^R := \alpha^R(i)\big)$ then
            $L_0^{m-i} = (e^T_{m-(p+c_p(\alpha^R))} \otimes I_n) - \sum_{j=p}^{i} e^T_{m-(j+1+c_{j+1}(\alpha^R))} \otimes A_j$
        else
            $L_0^{m-i} = -\sum_{j=0}^{i} e^T_{m-(j+1+c_{j+1}(\alpha^R))} \otimes A_j$
for $i = h+1 : m-1$ do
    if $-i, -(i+1) \notin \beta$ or the subtuple of $\beta$ with indices $\{-i, -(i+1)\}$ starts with $-i$ then
        $L_0^{m-i} = e^T_{m-(i-1-c_{-i}(\beta))} \otimes I_n$
    else
        if $-m \neq -(i + \mathfrak{rc}_{-(i+1)}(\beta) + 1) =: -q$, $\big(set\ \beta^R := \beta^R(-(i+1))\big)$ then
            $L_0^{m-i} = (e^T_{m-(q-1-c_{-q}(\beta^R))} \otimes I_n) + \sum_{j=i}^{q-1} \big(e^T_{m-(j-1-c_{-j}(\beta^R))} \otimes A_{j+1}\big)$
        else
            $L_0^{m-i} = \sum_{j=i}^{m-1} (e^T_{m-(j-1-c_{-j}(\beta^R))} \otimes A_{j+1})$
return $L_0$
```

```
Algorithm 2: $\lambda L_1 - L_0 := M^P_{\tau_1} M^P_{\sigma_1}(\lambda M^P_\tau - M^P_\sigma) M^P_{\sigma_2} M^P_{\tau_2}$ be an FPR of $P(\lambda)$
  Input    : $\sigma, \tau, \sigma_j$ and $\tau_j$ $j = 1, 2$.
  Output   : $L_1$
  Notation: $\alpha := rsf(\sigma_1, \sigma_2)$ and $\beta := rsf(\tau_1, \tau, \tau_2)$.
            $L_1^\ell := \ell^{th}$ block row of $L_1 = (e_\ell^T \otimes I_n) L_1$
  for $i = 0 : h - 1$ do
      if $i, i+1 \notin \alpha$ or the subtuple of $\alpha$ with indices $\{i, i+1\}$ starts with $i+1$ then
          $L_1^{m-i} = e^T_{m-(i+1+c_{i+1}(\alpha))} \otimes I_n$
      else
          if $0 \neq i - \mathfrak{rc}_i(\alpha) =: p$, $\big($set $\alpha^R := \alpha^R(i)\big)$ then
              $L_1^{m-i} = (e^T_{m-(p+c_p(\alpha^R))} \otimes I_n) - \sum_{j=p}^{i} (e^T_{m-(j+1+c_{j+1}(\alpha^R))} \otimes A_j)$
          else
              $L_1^{m-i} = -\sum_{j=0}^{i} (e^T_{m-(j+1+c_{j+1}(\alpha^R))} \otimes A_j)$
  for $i = h : m - 1$ do
      if the subtuple of $\beta$ with indices $\{-i, -(i+1)\}$ starts with $-i$ then
          $L_1^{m-i} = e^T_{m-(i-1-c_{-i}(\beta))} \otimes I_n$
      else
          if $-m \neq -(i + \mathfrak{rc}_{-(i+1)}(\beta) + 1) =: -q$, $\big($set $\beta^R := \beta^R(-(i+1))\big)$ then
              $L_1^{m-i} = (e^T_{m-(q-1-c_{-q}(\beta^R))} \otimes I_n) + \sum_{j=i}^{k-1} (e^T_{m-(j-1-c_{-j}(\beta^R))} \otimes A_{j+1})$
          else
              $L_1^{m-i} = \sum_{j=i}^{m-1} (e^T_{m-(j-1-c_{-j}(\beta^R))} \otimes A_{j+1})$
  return $L_1$
```

Algorithm 3: $\lambda L_1 - L_0 := M_{(\tau_1,\sigma_1)}(Y_1, X_1)(\lambda M^P_\tau - M^P_\sigma)M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ be a GFPR of $P(\lambda)$

```
Input    : $\sigma, \tau, \sigma_j$ and $\tau_j$, $j = 1, 2$. Matrix assignments $X_j$ and $Y_j$, $j = 1, 2$.
Output   : $L_0$
Notation: $\alpha := rsf(\sigma_1, \sigma, \sigma_2)$, $Z := rsf(X_1, P, X_2)$, where $P$ is the trivial matrix
          assignment for $\sigma$, $\beta := rsf(\tau_1, \tau_2)$ and $W := rsf(Y_1, Y_2)$.
          $L_0^\ell := \ell^{th}$ block row of $L_0 = (e_\ell^T \otimes I_n)L_0$.
for $i = 0 : h$ do
    if the subtuple of $\alpha$ with indices $\{i, i+1\}$ starts with $i+1$ then
        $L_0^{m-i} = e^T_{m-(i+c_{i+1}(\alpha)+1)} \otimes I_n$
    else
        if $0 \neq i - \mathfrak{rc}_i(\alpha) =: k$, $\big(set\ \alpha^R := \alpha^R(i)\big)$ then
            $L_0^{m-i} = e^T_{m-(k+c_k(\alpha^R))} \otimes I_n + \sum_{j=k}^{i} e^T_{m-(j+1+c_{j+1}(\alpha^R))} \otimes Z_{p_i(\alpha)+i-j}$
        else
            $L_0^{m-i} = \sum_{j=0}^{i} e^T_{m-(j+1+c_{j+1}(\alpha^R))} \otimes Z_{p_i(\alpha)+i-j}$
for $i = h+1 : m-1$ do
    if $-i, -(i+1) \notin \beta$ or the subtuple of $\beta$ with indices $\{-i, -(i+1)\}$ starts with
    $-i$ then
        $L_0^{m-i} = e^T_{m-(i-c_{-i}(\beta)-1)} \otimes I_n$
    else
        if $-m \neq -(i + \mathfrak{rc}_{-(i+1)}(\beta) + 1) =: -k$, $\big(set\ \beta^R := \beta^R(-(i+1))\big)$ then
            $L_0^{m-i} = e^T_{m-(k-c_{-k}(\beta^R)-1)} \otimes I_n + \sum_{j=i}^{k-1} e^T_{m-(j-c_{-j}(\beta^R)-1)} \otimes W_{p_{-(i+1)}(\beta)+i-j}$
        else
            $L_0^{m-i} = \sum_{j=i}^{m-1} e^T_{m-(j-c_{-j}(\beta^R)-1)} \otimes W_{p_{-(i+1)}(\beta)+i-j}$
return $L_0$
```

Algorithm 4: $\lambda L_1 - L_0 := M_{(\tau_1,\sigma_1)}(Y_1, X_1)(\lambda M^P_\tau - M^P_\sigma)M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ be a GFPR of $P(\lambda)$

```
Input    : $\sigma, \tau, \sigma_j$ and $\tau_j$, $j = 1, 2$. Matrix assignments $X_j$ and $Y_j$, $j = 1, 2$.
Output   : $L_1$
Notation: $\alpha := rsf(\sigma_1, \sigma_2)$, $Z := rsf(X_1, X_2)$, $\beta := rsf(\tau_1, \tau, \tau_2)$ and
          $W := rsf(Y_1, P, Y_2)$, where $P$ is the trivial matrix assignment for $\tau$.
          $L_1^\ell := \ell^{th}$ block row of $L_1 = (e_\ell^T \otimes I_n)L_1$.
for $i = 0 : h-1$ do
    if $i, i+1 \notin \alpha$ or the subtuple of $\alpha$ with indices $\{i, i+1\}$ starts with $i+1$ then
        $L_1^{m-i} := e^T_{m-(i+c_{i+1}(\alpha)+1)} \otimes I_n$
    else
        if $0 \neq i - \mathfrak{ic}_i(\alpha) =: k$, $\big(\text{set } \alpha^R := \alpha^R(i)\big)$ then
            $L_1^{m-i} := (e^T_{m-(k+c_k(\alpha^R))} \otimes I_n) + \sum_{j=k}^{i} (e^T_{m-(j+1+c_{j+1}(\alpha^R))} \otimes Z_{p_i(\alpha)+i-j})$
        else
            $L_1^{m-i} := \sum_{j=0}^{i} (e^T_{m-(j+1+c_{j+1}(\alpha^R))} \otimes Z_{p_i(\alpha)+i-j})$
for $i = h : m-1$ do
    if the subtuple of $\beta$ with indices $\{-i, -(i+1)\}$ starts with $-i$ then
        $L_1^{m-i} := e^T_{m-(i-c_{-i}(\beta)-1)} \otimes I_n$
    else
        if $-m \neq -(i + \mathfrak{ic}_{-(i+1)}(\beta) + 1) =: -k$, $\big(\text{set } \beta^R := \beta^R(-(i+1))\big)$ then
            $L_1^{m-i} := (e^T_{m-(k-c_{-k}(\beta^R)-1)} \otimes I_n) + \sum_{j=i}^{k-1} (e^T_{m-(j-c_{-j}(\beta^R)-1)} \otimes W_{p_{-(i+1)}(\beta)+i-j})$
        else
            $L_1^{m-i} := \sum_{j=i}^{m-1} (e^T_{m-(j-c_{-j}(\beta^R)-1)} \otimes W_{p_{-(i+1)}(\beta)+i-j})$
return $L_1$
```

3

EGFPRs of Matrix Polynomials

The basic building blocks of GFPRs of $P(\lambda)$ are special GFPs of $P(\lambda)$ of the form $\lambda M_\tau^P - M_\sigma^P$ as given in Definition 2.1.9 which in turn are determined by a sub-permutation $\{0 : h\}$ of $\{0 : m-1\}$. In this chapter, we show that by replacing $\lambda M_\tau^P - M_\sigma^P$ by any GFPs of $P(\lambda)$ results in a large family of pencils which we refer to as extended GFPRs (EGFPRs) of $P(\lambda)$ and show that the EGFPRs are strong linearizations of $P(\lambda)$. The EGFPRs have the same distinctive features as Fiedler pencils, namely, the construction of EGFPRs is operation-free and the recovery of eigenvectors, minimal bases and minimal indices of $P(\lambda)$ from those of the EGFPRs is also operation-free. Also, most of the structure-preserving strong linearizations that have been constructed in literature are obtained from GFPs and GFPRs. We show that the family of EGFPRs subsumes both GFPs and GFPRs and expands the arena in which to look for structure-preserving strong linearizations of $P(\lambda)$.

3.1 EGFPRs of matrix polynomials

Recall that $P(\lambda) := \sum_{j=0}^{m} \lambda^j A_j \in \mathbb{C}[\lambda]^{n\times n}$. We consider the elementary matrices $M_{\pm i}(X)$ and Fiedler matrices $M_{\pm i}^P$, for $i = 0 : m$, associated with $P(\lambda)$ as given in Chapter 2.

It is clear from the definitions of GFPs, FPRs and GFPRs of $P(\lambda)$ that not all the GFPs are FPRs, and vice versa. For example, consider the GFP $T(\lambda) = \lambda M^P_{(-4,-3,-1)} - M^P_{(0,2)}$ and the FPR $L(\lambda) = (\lambda M^P_{-4,-3} - M^P_{(0,1,2)})M_1^P$ of $P(\lambda) := \sum_{i=0}^{4} \lambda^i A_i$. Then $T(\lambda)$ is not an FPR and $L(\lambda)$ is not a GFP. Further, consider the pencil $F(\lambda) := (\lambda M^P_{(-4,-1)} - M^P_{(2,3,0)})M_2^P$. Then $F(\lambda)$ is neither a GFP nor an FPR of $P(\lambda)$. This motivates us to present a unified framework for construction of Fiedler-like pencils of $P(\lambda)$ which subsumes FPs, GFPs, FPRs, and GFPRs of $P(\lambda)$.

Definition 3.1.1 (EGFPR and EFPR of $P(\lambda)$)**.** *Let* (σ, ω) *be a permutation of* $\{0 : m\}$*. Set* $\tau := -\omega$*. Let* σ_1 *and* σ_2 *be index tuples containing indices from* $\sigma \setminus \{m-1, m\}$ *such that* $(\sigma_1, \sigma, \sigma_2)$ *satisfies the SIP. Similarly, let* τ_1 *and* τ_2 *be index tuples containing indices from* $\tau \setminus \{-1, -0\}$ *such that* (τ_1, τ, τ_2) *satisfies the SIP. Let* X_1*,* X_2*,* Y_1 *and* Y_2 *be any arbitrary matrix assignments for* σ_1*,* σ_2*,* τ_1 *and* τ_2*, respectively. Then the pencil*

$$L(\lambda) := M_{\tau_1}(Y_1)\, M_{\sigma_1}(X_1)\, (\lambda M^P_\tau - M^P_\sigma)\, M_{\sigma_2}(X_2)\, M_{\tau_2}(Y_2) \tag{3.1}$$

is said to be an extended generalized Fiedler pencil with repetition (EGFPR) of $P(\lambda)$*. In particular, if all the matrix assignments* X_j *and* Y_j*,* $j = 1, 2$*, are the trivial matrix assignments then the resulting pencil* $L(\lambda) := M^P_{\tau_1}\, M^P_{\sigma_1}\, (\lambda M^P_\tau - M^P_\sigma)\, M^P_{\sigma_2}\, M^P_{\tau_2}$ *is said to be an extended Fiedler pencil with repetition (EFPR) of* $P(\lambda)$*.*

It follows from Definition 3.1.1 that FPs, GFPs, FPRs and GFPRs of $P(\lambda)$ are subclasses of EGFPRs.

Example 3.1.2. *Let* $P(\lambda) := \sum_{i=0}^{6} \lambda^i A_i$ *and let* $\sigma := (0, 1, 4, 5)$*,* $\sigma_1 := \emptyset, \sigma_2 := (0, 4), \tau := (-2, -3, -6), \tau_1 := (-3)$ *and* $\tau_2 := \emptyset$*. Let* X *and* (Y, Z) *be any arbitrary matrix assignments for* (-3) *and* $(0, 4)$*, respectively. Then the pencil* $L(\lambda) := M_{-3}(X)\big(\lambda M^P_{(-2,-3,-6)} - M^P_{(0,1,4,5)}\big) M_{(0,4)}(Y, Z) =$

$$\begin{bmatrix} \lambda A_6 + A_5 & -Z & -I_n & 0 & 0 & 0 \\ A_4 & \lambda Z - I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n & \lambda I_n & 0 \\ -I_n & 0 & 0 & \lambda I_n - X & \lambda X & 0 \\ 0 & \lambda I_n & 0 & \lambda A_3 & \lambda A_2 + A_1 & -Y \\ 0 & 0 & 0 & 0 & A_0 & \lambda Y \end{bmatrix}$$

is an EGFPR of $P(\lambda)$*.* ■

Remark 3.1.3. *It follows from Definition 3.1.1 that* $(\tau_1, \sigma_1) \not\sim (\sigma_1, \tau_1)$ *and* $(\sigma_2, \tau_2) \not\sim (\tau_2, \sigma_2)$ *in general. So* $M_{\tau_1}(Y_1) M_{\sigma_1}(X_1) \neq M_{\sigma_1}(X_1) M_{\tau_1}(Y_1)$ *and* $M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) \neq M_{\tau_2}(Y_2) M_{\sigma_2}(X_2)$*. Thus, for* $L(\lambda) := M_{\tau_1}(Y_1) M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma) M_{\sigma_2}(X_2) M_{\tau_2}(Y_2)$ *and* $T(\lambda) := M_{\sigma_1}(X_1)\, M_{\tau_1}(Y_1)(\lambda M^P_\tau - M^P_\sigma) M_{\tau_2}(Y_2) M_{\sigma_2}(X_2)$*, we have* $L(\lambda) \neq T(\lambda)$*. By interchanging the positions of* $M_{\sigma_j}(X_j)$ *with* $M_{\tau_j}(Y_j)$ *in* $L(\lambda)$ *we obtain a new family of pencils for* $P(\lambda)$*.*

Note that for an index tuple α and a matrix assignment X of α, $M_\alpha(X)$ is operation free if and only if M^P_α is operation free, see Lemma 2.1.7 and Remark 2.1.8. Also note that M_α^P is not operation free if $m-1$ and m simultaneously belong to α, or if -1 and -0 simultaneously belong to α. The following result shows that a subclass of EGFPRs of $P(\lambda)$ is always operation free.

Proposition 3.1.4. *Let* $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M_\tau^P - M_\sigma^P)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) =: \lambda L_1 - L_0$ *be an EGFPR of* $P(\lambda)$ *such that* $m-1$ *and* m *simultaneously do not belong to* σ*, and* -1 *and* -0 *simultaneously do not belong to* τ*. Then* $L(\lambda)$ *is operation free.*

Proof. See Appendix B. □

The following result shows that, with some generic nonsingularity conditions, the EGFPRs are strong linearization of $P(\lambda)$.

Theorem 3.1.5. *If all the matrix assignments* X_j *and* Y_j*,* $j = 1, 2$*, are nonsingular, then the EGFPR* $L(\lambda)$ *as given in Definition 3.1.1 is a strong linearization of* $P(\lambda)$*.*

Proof. We have $L(\lambda) = M_{(\tau_1,\sigma_1)}(Y_1, X_1)\, T(\lambda)\, M_{(\sigma_2,\tau_2)}(X_2, Y_2)$, where $T(\lambda) := \lambda M_\tau^P - M_\sigma^P$ is a GF pencil of $P(\lambda)$. As X_j and Y_j, $j = 1, 2$, are nonsingular matrix assignments, $M_{(\tau_1,\sigma_1)}(Y_1, X_1)$ and $M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ are nonsingular. Since $T(\lambda)$ is a strong linearization of $P(\lambda)$, it follows that $L(\lambda)$ is a strong linearization of $P(\lambda)$. □

Assumption: Henceforth, for simplicity we assume that the matrix assignments are always nonsingular.

3.2 Recovery of minimal bases and minimal indices

The following lemma will play a crucial role in the proof of main result of this section.

Lemma 3.2.1. *Let* $L(\lambda) := M_{\tau_1}(Y_1)\, M_{\sigma_1}(X_1)\, (\lambda M_\tau^P - M_\sigma^P)\, M_{\sigma_2}(X_2)\, M_{\tau_2}(Y_2)$ *be an EGFPR of* $P(\lambda)$ *as given in (3.1). Suppose that* $0 \in \sigma$ *and* $-m \in \tau$*. Then*

$$(e^T_{m-c_0(\sigma)} \otimes I_n)\, M_{\sigma_2}(X_2)\, M_{\tau_2}(Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$$

and

$$M_{\tau_1}(Y_1)\, M_{\sigma_1}(X_1)\, (e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n.$$

Proof. For an arbitrary matrix $Z \in \mathbb{C}^{n\times n}$, from (2.2) and (2.3) we have

$$(e^T_{m-i} \otimes I_n)\, M_j(Z) = \begin{cases} e^T_{m-(i+1)} \otimes I_n & \text{for } j = i+1,\ i = 0 : m-2, \\ e^T_{m-i} \otimes I_n & \text{for } j \notin \{i, i+1\},\ i = 0 : m-1, \end{cases} \tag{3.2}$$

and

$$M_j(Z)(e_{m-i} \otimes I_n) = \begin{cases} e_{m-(i+1)} \otimes I_n & \text{for } j = i+1,\ i = 0 : m-2, \\ e_{m-i} \otimes I_n & \text{for } j \notin \{i, i+1\},\ i = 0 : m-1. \end{cases} \tag{3.3}$$

Given that $0 \in \sigma$ and $-m \in \tau$ (i.e., $m \notin \sigma$). Let h be the integer such that $0, 1, \ldots, h \in \sigma$ and $h+1 \notin \sigma$. Then $c_0(\sigma) \leq h$ and $i_0(\sigma) \leq h$. Further, we have $h, h+1 \notin \sigma_1 \cup \sigma_2$ as $h+1 \notin \sigma$ and $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP.

Let $\widehat{\sigma}$ and $\widehat{\sigma}_j$, $j = 1, 2$, respectively, be the subtuples of σ and σ_j with indices $\{0 : h\}$. Similarly, let $\widehat{\widehat{\sigma}}$ and $\widehat{\widehat{\sigma}}_j$, $j = 1, 2$, respectively, be the subtuples of σ and σ_j with indices $\{h+2 : m\}$. Then $\widehat{\sigma}$ and $\widehat{\widehat{\sigma}}$ commutes since for any indices $k \in \widehat{\sigma}$ and $\ell \in \widehat{\widehat{\sigma}}$ we have $|k - \ell| > 1$. Thus $\sigma \sim (\widehat{\sigma}, \widehat{\widehat{\sigma}}) \sim (\widehat{\widehat{\sigma}}, \widehat{\sigma})$. Similarly, $\sigma_j \sim (\widehat{\sigma}_j, \widehat{\widehat{\sigma}}_j) \sim (\widehat{\widehat{\sigma}}_j, \widehat{\sigma}_j)$ for $j = 1, 2$. Further, $(\widehat{\sigma}, \widehat{\widehat{\sigma}}_j) \sim (\widehat{\widehat{\sigma}}_j, \widehat{\sigma})$ for $j = 1, 2$. Since $h+1 \notin \sigma$, we have $i_0(\sigma) = i_0(\widehat{\sigma})$ and $c_0(\sigma) = c_0(\widehat{\sigma})$. Further, $c_0(\sigma, \sigma_2) = c_0(\widehat{\widehat{\sigma}}, \widehat{\sigma}, \widehat{\widehat{\sigma}}_2, \widehat{\sigma}_2) = c_0(\widehat{\widehat{\sigma}}, \widehat{\widehat{\sigma}}_2, \widehat{\sigma}, \widehat{\sigma}_2) = c_0(\widehat{\sigma}, \widehat{\sigma}_2)$, where the last equality holds as $0 \notin \widehat{\widehat{\sigma}} \cup \widehat{\widehat{\sigma}}_2$. Similarly, $i_0(\sigma_1, \sigma) = i_0(\widehat{\sigma}_1, \widehat{\sigma})$. Note that X_2 and Y_2 are arbitrary matrix assignments. We by denote $(*)$ any arbitrary matrix assignment. Then we have

$$\begin{aligned} & (e^T_{m-c_0(\widehat{\sigma})} \otimes I_n)\, M_{\widehat{\widehat{\sigma}}_2}(*) M_{\widehat{\sigma}_2}(*) M_{\tau_2}(*) \\ &= (e^T_{m-c_0(\widehat{\sigma})} \otimes I_n)\, M_{\widehat{\sigma}_2}(*) M_{\tau_2}(*) \text{ by (3.2) since } \begin{cases} c_0(\widehat{\sigma}) = c_0(\sigma) \leq h \text{ and } \widehat{\widehat{\sigma}}_2 \\ \text{contains indices from } \{h+2 : m\} \end{cases} \\ &= e^T_{m-c_0(\widehat{\sigma}, \widehat{\sigma}_2)} \otimes I_n = e^T_{m-c_0(\sigma, \sigma_2)} \otimes I_n \text{ by Lemma 2.1.11.} \end{aligned}$$

Thus $(e^T_{m-c_0(\sigma)} \otimes I_n)\, M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$.

Similarly, we have

$$\begin{aligned} & M_{\tau_1}(*) M_{\widehat{\sigma}_1}(*) M_{\widehat{\widehat{\sigma}}_1}(*)\, (e_{m-i_0(\widehat{\sigma})} \otimes I_n) \\ &= M_{\tau_1}(*) M_{\widehat{\sigma}_1}(*)\, (e_{m-i_0(\widehat{\sigma})} \otimes I_n) \text{ by (3.3) since } \begin{cases} i_0(\widehat{\sigma}) = i_0(\sigma) \leq h \text{ and } \widehat{\widehat{\sigma}}_1 \\ \text{contains indices from } \{h+2 : m\} \end{cases} \\ &= e_{m-i_0(\widehat{\sigma}_1, \widehat{\sigma})} \otimes I_n = e_{m-i_0(\sigma_1, \sigma)} \otimes I_n \text{ by Lemma 2.1.11.} \end{aligned}$$

Thus $M_{\tau_1}(Y_1) M_{\sigma_1}(X_1)\, (e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$. This completes the proof. ☐

We now describe the recovery of minimal bases and minimal indices of a singular $P(\lambda)$ from those of the EGFPRs of $P(\lambda)$.

Theorem 3.2.2. *Let* $L(\lambda) := M_{\tau_1}(Y_1)\, M_{\sigma_1}(X_1)\, (\lambda M^P_\tau - M^P_\sigma)\, M_{\sigma_2}(X_2)\, M_{\tau_2}(Y_2)$ *be an EGFPR of* $P(\lambda)$*. Suppose that* $P(\lambda)$ *is singular. Let* τ *be given by* $\tau := (\tau_l, -m, \tau_r)$.

Let $c_L := c(-rev(\tau_l), \sigma, -rev(\tau_r))$ *and* $i_L := i(-rev(\tau_l), \sigma, -rev(\tau_r))$, *respectively, be the total number of consecutions and inversions of the permutation* $(-rev(\tau_l), \sigma, -rev(\tau_r))$ *of* $\{0 : m-1\}$. *Then we have the following.*

(a) **Right minimal bases.** *If* $\big(u_1(\lambda), \ldots, u_p(\lambda)\big)$ *is a right minimal basis of* $L(\lambda)$ *then* $\big((e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u_p(\lambda)\big)$ *is a right minimal basis of* $P(\lambda)$.

(b) If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $L(\lambda)$ *then* $\varepsilon_1 - i_L \leq \cdots \leq \varepsilon_p - i_L$ *are the right minimal indices of* $P(\lambda)$.

(c) **Left minimal bases.** *If* $\big(v_1(\lambda), \ldots, v_p(\lambda)\big)$ *is a left minimal basis of* $L(\lambda)$ *then* $\big((e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)v_1(\lambda), \ldots, (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)v_p(\lambda)\big)$ *is a left minimal basis of* $P(\lambda)$.

(d) If $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $L(\lambda)$ *then* $\eta_1 - c_L \leq \cdots \leq \eta_p - c_L$ *are the left minimal indices of* $P(\lambda)$.

Proof. We have $L(\lambda) = \mathcal{A}\,T(\lambda)\mathcal{B}$, where $\mathcal{A} := M_{\tau_1}(Y_1)\,M_{\sigma_1}(X_1)$, $\mathcal{B} := M_{\sigma_2}(X_2)\,M_{\tau_2}(Y_2)$, and $T(\lambda) := \lambda M^P_\tau - M^P_\sigma$ is a GF pencil of $P(\lambda)$ associated with the permutation $\omega := (\sigma, -\tau)$ of $\{0 : m\}$ such that $0 \in \sigma$ and $-m \in \tau$. Since X_j and Y_j, $j = 1, 2$, are nonsingular matrix assignments, we have $\mathcal{A}$ and $\mathcal{B}$ are nonsingular constant matrices. Thus, the map $\mathcal{B} : \mathcal{N}_r(L) \to \mathcal{N}_r(T)$, $x(\lambda) \mapsto \mathcal{B}x(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(L)$ to a minimal basis of $\mathcal{N}_r(T)$. On the other hand, by Theorem 1.2.26, $F^{\text{GF}}(P) : \mathcal{N}_r(T) \to \mathcal{N}_r(P)$, $y(\lambda) \mapsto (e^T_{m-c_0(\sigma)} \otimes I_n)y(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(T)$ to a minimal basis of $\mathcal{N}_r(P)$. Consequently, we have $F^{\text{GF}}(P)\mathcal{B} : \mathcal{N}_r(L) \to \mathcal{N}_r(P)$, $z(\lambda) \mapsto (e^T_{m-c_0(\sigma)} \otimes I_n)\mathcal{B}z(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(L)$ to a minimal basis of $\mathcal{N}_r(P)$. By Lemma 3.2.1, we have

$$(e^T_{m-c_0(\sigma)} \otimes I_n)\mathcal{B} = (e^T_{m-c_0(\sigma)} \otimes I_n)\,M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$$

which proves (a).

Next, we prove the recovery of left minimal bases. Note that $\mathcal{A}^T : \mathcal{N}_l(L) \to \mathcal{N}_l(T)$, $x(\lambda) \mapsto \mathcal{A}^T x(\lambda)$, is a linear isomorphism and maps a minimal basis of $\mathcal{N}_l(L)$ to a minimal basis of $\mathcal{N}_l(T)$. By Theorem 1.2.26, $K^{\text{GF}}(P) : \mathcal{N}_l(T) \to \mathcal{N}_l(P)$, $y(\lambda) \mapsto (e^T_{m-i_0(\sigma)} \otimes I_n)y(\lambda)$, is a linear isomorphism and maps a minimal basis of $\mathcal{N}_l(T)$ to a minimal basis of $\mathcal{N}_l(P)$. Consequently, we have $K^{\text{GF}}(P)\,\mathcal{A}^T : \mathcal{N}_l(L) \to \mathcal{N}_l(P)$, $z(\lambda) \mapsto (e^T_{m-i_0(\sigma)} \otimes I_n)\mathcal{A}^T z(\lambda)$, is an isomorphism and maps a minimal basis $\mathcal{N}_l(L)$ to a minimal basis $\mathcal{N}_l(P)$. By Lemma 3.2.1, we have $(e^T_{m-i_0(\sigma)} \otimes I_n)\mathcal{A}^T = (e^T_{m-i_0(\sigma)} \otimes I_n)\big(M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)\big)^T = \big(M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{m-i_0(\sigma)} \otimes I_n)\big)^T = e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n$ which proves (c).

As $L(\lambda)$ is strictly equivalent to $T(\lambda)$, the left (resp., right) minimal indices of $L(\lambda)$ and $T(\lambda)$ are the same. Hence the desired results for minimal indices follow from Theorem 1.2.26. $\square$

3.3 Recovery of eigenvectors

We now describe the recovery of eigenvectors of $P(\lambda)$ corresponding to an eigenvalue $\mu \in \mathbb{C}$ from those of the EGFPRs of $P(\lambda)$.

Let $L(\lambda) = M_{(\tau_1,\sigma_1)}(Y_1, X_1)(\lambda M^P_\tau - M^P_\sigma)M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ be an EGFPR of $P(\lambda)$. First, we calculate $F^{\text{GF}}(P)\, M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ and $M_{(\tau_1,\sigma_1)}(Y_1, X_1)\, (K^{\text{GF}})^T$ for various cases of $F^{\text{GF}}(P)$ and $K^{\text{GF}}(P)$ as given in Theorem 1.2.26. This will enable us to recover the eigenvectors of $P(\lambda)$ from those of the EGFPRs of $P(\lambda)$. We proceed as follows.

Lemma 3.3.1. *Let α be an index tuple containing indices from $\{0 : m\}$ such that α satisfies the SIP. Let $s \in \alpha$ and $c_s(\alpha)$ be the number of consecutions of α at s. Then*

$$\alpha \sim \big(\alpha^L, s, s+1, \ldots, s + c_s(\alpha), \alpha^R\big) \tag{3.4}$$

for some tuples α^L and α^R such that $s \notin \alpha^L$ and $s+c_s(\alpha), s+c_s(\alpha)+1 \notin \alpha^R$. Similarly, let $t \in \alpha$ and $i_t(\alpha)$ be the number of inversions of α at t. Then

$$\alpha \sim \big(\alpha^L, t + i_t(\alpha), \ldots, t + 1, t, \alpha^R\big) \tag{3.5}$$

for some tuples α^L and α^R such that $t \notin \alpha^R$ and $t + i_t(\alpha), t + i_t(\alpha) + 1 \notin \alpha^L$.

Proof. Since α satisfies the SIP, we can write α in the column standard form. Let the column standard form of α be given by

$$csf(\alpha) = \Big((a_m : m), \ldots, (a_{k+1} : k+1), (a_k : k), (a_{k-1} : k-1), \ldots, (a_1 : 1), (a_0 : 0)\Big),$$

where k is the largest integer such that $s \in (a_k : k) = (a_k, a_k + 1, \ldots, s-1, s, \ldots, k)$. Now, set

$$\alpha^L := \Big((a_m : m), \ldots, (a_{k+1} : k+1), a_k, a_k + 1, \ldots, s-1\Big) \text{ and}$$

$$\alpha^R := \Big((a_{k-1} : k-1), \ldots, (a_1 : 1), (a_0 : 0)\Big).$$

Then $csf(\alpha) = \big(\alpha^L, s, s+1, \ldots, k, \alpha^R\big)$, where $s \notin \alpha^L$ and for any index $j \in \alpha^R$, we have $j \leq k-1$. Hence $(s, s+1, \ldots, k)$ is a subtuple of α and $(s, s+1, \ldots, k, k+1)$ is not a subtuple of α. This implies that $c_s(\alpha) = k - s$, i.e., $k = s + c_s(\alpha)$ which gives (3.4).

Next, let the row standard form of α be given by $rsf(\alpha) =$

$$\Big(rev(b_0 : 0), \ldots, rev(b_{p-1} : p-1), rev(b_p : p), rev(b_{p+1} : p+1), \ldots, rev(b_m : m)\Big),$$

where p is the largest integer such that $t \in rev(b_p : p) = (p, p-1, \ldots, t, t-1, \ldots, b_p)$. Now, set

$$\alpha^L := \Big(rev(b_0 : 0), \ldots, rev(b_{p-1} : p-1)\Big) \text{ and}$$

$$\alpha^R := \Big(t-1, \ldots, b_p + 1, b_p, rev(b_{p+1} : p+1), \ldots, rev(b_m : m)\Big).$$

Then $rsf(\alpha) = (\alpha^L, p, \ldots, t+1, t, \alpha^R)$, where $t \notin \alpha^R$ and for any index $j \in \alpha^L$, we have $j \leq p-1$. Hence $(p, \ldots, t+1, t)$ is a subtuple of α and $(p+1, p, \ldots, t+1, t)$ is not a subtuple of α. This implies that $i_t(\alpha) = p - t$, i.e., $p = t + i_t(\alpha)$ which gives (3.5). □

Lemma 3.3.2. *Let α be an index tuple containing indices from $\{0 : m-1\}$ such that α satisfies the SIP. Let Z be an arbitrary matrix assignment for α. Let $0 \leq s \leq m-1$. Suppose that $s+1 \in \alpha$. Then we have the following.*

(a) If the subtuple of α with indices $\{s, s+1\}$ starts with $s+1$, then $(e_{m-s}^T \otimes I_n) M_\alpha(Z) = e_{m-(s+1+c_{s+1}(\alpha))}^T \otimes I_n$.

(b) If the subtuple of α with indices $\{s, s+1\}$ ends with $s+1$, then $M_\alpha(Z)(e_{m-s} \otimes I_n) = e_{m-(s+1+i_{s+1}(\alpha))} \otimes I_n$.

Proof. (a) Set $p := c_{s+1}(\alpha)$, i.e., α has p consecutions at $s+1$. Since α satisfies the SIP, by Lemma 3.3.1,

$$\alpha \sim (\alpha^L, s+1, s+2, \ldots, s+p+1, \alpha^R),$$

where $s+1 \notin \alpha^L$ and $s+p+1, s+p+2 \notin \alpha^R$. Further, since the subtuple of α with indices $\{s, s+1\}$ starts with $s+1$, we have $s \notin \alpha^L$. We denote by $(*)$ any arbitrary matrix assignment. Then we have

$$\begin{aligned}
&(e_{m-s}^T \otimes I_n)\, M_{\alpha^L}(*)\, M_{(s+1,s+2,\ldots,s+p+1)}(*)\, M_{\alpha^R}(*) \\
&= (e_{m-s}^T \otimes I_n)\, M_{(s+1,s+2,\ldots,s+p+1)}(*)\, M_{\alpha^R}(*) \text{ by (3.2) since } s, s+1 \notin \alpha^L \\
&= (e_{m-(s+p+1)}^T \otimes I_n)\, M_{\alpha^R}(*) \text{ by repeatedly applying (3.2)} \\
&= e_{m-(s+p+1)}^T \otimes I_n \text{ by (3.2) since } s+p+1, s+p+2 \notin \alpha^R.
\end{aligned}$$

Thus, in particular, we have $(e_{m-s}^T \otimes I_n) M_\alpha(Z) = e_{m-(s+p+1)}^T \otimes I_n$. Since $p = c_{s+1}(\alpha)$, the desired result follows.

(b) Set $q := i_{s+1}(\alpha)$, i.e., α has q inversions at $s+1$. Since α satisfies the SIP, by Lemma 3.3.1,

$$\alpha \sim \big(\alpha^L, s+q+1, \dots, s+2, s+1, \alpha^R\big),$$

where $s+1 \notin \alpha^R$ and $s+q+1, s+q+2 \notin \alpha^L$. Further, since the subtuple of α with indices $\{s, s+1\}$ ends with $s+1$, we have $s \notin \alpha^R$. Now we have

$$\begin{aligned}
& M_{\alpha^L}(*)\, M_{(s+1+q,\dots,s+2,s+1)}(*)\, M_{\alpha^R}(*)\, (e_{m-s} \otimes I_n) \\
& = M_{\alpha^L}(*)\, M_{(s+1+q,\dots,s+2,s+1)}(*)\, (e_{m-s} \otimes I_n) \text{ by (3.3) since } s, s+1 \notin \alpha^R \\
& = M_{\alpha^L}(*)\, (e_{m-(s+q+1)} \otimes I_n) \text{ by repeatedly applying (3.3)} \\
& = e_{m-(s+q+1)} \otimes I_n \text{ by (3.3) as } s+q+1, s+q+2 \notin \alpha^L.
\end{aligned}$$

Thus, in particular, we have $M_\alpha(Z)\,(e_{m-s} \otimes I_n) = e_{m-(s+q+1)} \otimes I_n$. Since $q = i_{s+1}(\alpha)$, the desired result follows. ☐

Lemma 3.3.3. *Let α be an index tuple containing indices from $\{-m : -1\}$ such that α satisfies the SIP. Let Z be an arbitrary matrix assignment for α. Let $1 \le s \le m-1$. Suppose that $-s \in \alpha$. If the subtuple of α with indices $\{-s, -(s+1)\}$ starts with $-s$, then $(e^T_{m-s} \otimes I_n) M_\alpha(Z) = e^T_{m-(s-c_{-s}(\alpha)-1)} \otimes I_n$.*

Similarly, if the subtuple of α with indices $\{-s, -(s+1)\}$ ends with $-s$, then $M_\alpha(Z)\,(e_{m-s} \otimes I_n) = e_{m-(s-i_{-s}(\alpha)-1)} \otimes I_n$.

Proof. For an arbitrary matrix $X \in \mathbb{C}^{n\times n}$, recall from (2.12) and (2.13) that

$$(e^T_{m-i} \otimes I_n)\; M_{-j}(X) = \begin{cases} e^T_{m-(i-1)} \otimes I_n & \text{if } j = i,\ i = 1 : m-1, \\ e^T_{m-i} \otimes I_n & \text{if } j \notin \{i, i+1\},\ i = 0 : m-1, \end{cases} \tag{3.6}$$

$$M_{-j}(X)\; (e_{m-i} \otimes I_n) = \begin{cases} e_{m-(i-1)} \otimes I_n & \text{if } j = i,\ i = 1 : m-1, \\ e_{m-i} \otimes I_n & \text{if } j \notin \{i, i+1\},\ i = 0 : m-1. \end{cases} \tag{3.7}$$

Set $p := c_{-s}(\alpha)$, i.e., α has p consecutions at $-s$. Since α satisfies the SIP, by Lemma 2.1.21,

$$\alpha \sim \big(\alpha^L, -s, -(s-1), \dots, -(s-p), \alpha^R\big),$$

where $-s \notin \alpha^L$ and $-(s-p), -(s-p-1) \notin \alpha^R$. Further, since the subtuple of α with indices $\{-s, -(s+1)\}$ starts with $-s$, we have $-(s+1) \notin \alpha^L$. We denote by $(*)$ any arbitrary matrix assignment. Then we have

$$\begin{aligned}
& (e^T_{m-s} \otimes I_n) M_{\alpha^L}(*)\, M_{(-s,-(s-1),\dots,-(s-p))}(*)\, M_{\alpha^R}(*) \\
& = (e^T_{m-s} \otimes I_n)\, M_{(-s,-(s-1),\dots,-(s-p))}(*)\, M_{\alpha^R}(*) \text{ by (3.6) since } -s, -(s+1) \notin \alpha^L
\end{aligned}$$

$$= (e^T_{m-(s-p-1)} \otimes I_n)\, M_{\alpha^R}(*) \text{ by repeatedly applying (3.6)}$$
$$= e^T_{m-(s-p-1)} \otimes I_n \text{ by (3.6) since } -(s-p), -(s-p-1) \notin \alpha^R.$$

Thus, in particular, we have $(e^T_{m-s} \otimes I_n) M_\alpha(Z) = e^T_{m-(s-p-1)} \otimes I_n$. Since $p := c_{-s}(\alpha)$, the desired result follows.

Next, set $q := i_{-s}(\alpha)$, i.e., α has q inversions at $-s$. Since α satisfies the SIP, by Lemma 2.1.21,

$$\alpha \sim \big(\alpha^L, -(s-q), \ldots, -(s-1), -s, \alpha^R\big),$$

where $-s \notin \alpha^R$ and $-(s-q), -(s-q-1) \notin \alpha^L$. Further, since the subtuple of α with indices $\{-s, -(s+1)\}$ ends with $-s$, we have $-(s+1) \notin \alpha^R$. Now

$$\begin{aligned}
& M_{\alpha^L}(*)\, M_{(-(s-q),\ldots,-(s-1),-s,)}(*)\, M_{\alpha^R}(*)\, (e_{m-s} \otimes I_n) \\
= \;& M_{\alpha^L}(*)\, M_{(-(s-q),\ldots,-(s-1),-s,)}(*)\, (e_{m-s} \otimes I_n) \text{ by (3.7) since } -s, -(s+1) \notin \alpha^R \\
= \;& M_{\alpha^L}(*)\, (e_{m-(s-q-1)} \otimes I_n) \text{ by repeatedly applying (3.7)} \\
= \;& e_{m-(s-q-1)} \otimes I_n \text{ by (3.7) since } -(s-q), -(s-q-1) \notin \alpha^L.
\end{aligned}$$

Thus, in particular, we have $M_\alpha(Z)\,(e_{m-s} \otimes I_n) = e_{m-(s-q-1)} \otimes I_n$. Since $q := i_{-s}(\alpha)$, we have the desired result. □

Lemma 3.3.4. *Let* $L(\lambda) := M_{\tau_1}(Y_1) M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma) M_{\sigma_2}(X_2) M_{\tau_2}(Y_2)$ *be an EGFPR of* $P(\lambda)$*. Suppose that* $0 \in \sigma$*. Then we have the following.*

(a) If $c_0(\sigma) < m$ *then* $(e^T_{m-c_0(\sigma)} \otimes I_n)\, M_{\sigma_2}(X_2)\, M_{\tau_2}(Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$*. Similarly, if* $i_0(\sigma) < m$ *then* $M_{\tau_1}(Y_1)\, M_{\sigma_1}(X_1)\, (e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$*.*

(b) If $c_0(\sigma) = m$ *then* $(e^T_1 \otimes I_n) M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) = e^T_1 \otimes I_n$*, and if* $i_0(\sigma) = m$ *then* $M_{\tau_1}(Y_1) M_{\sigma_1}(X_1)(e_1 \otimes I_n) = e_1 \otimes I_n$*.*

Proof. (a) First we prove that $(e^T_{m-c_0(\sigma)} \otimes I_n)\, M_{\sigma_2}(X_2)\, M_{\tau_2}(Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$. Since σ has $c_0(\sigma)$ $(\leq m-1)$ consecutions at 0, we have $\sigma \sim (\sigma^L, 0, 1, \ldots, c_0(\sigma))$, where either $m \in \sigma^L$ or $m \in -\tau$. If $-m \in \tau$ then the desired result follows from Lemma 3.2.1. But there is nothing special about $m \in \sigma$. If $m \in \sigma$ then by defining $\widetilde{\sigma} := \sigma \setminus \{m\}$ and $\widetilde{\tau} := (\tau, -m)$, we have $c_0(\sigma) = c_0(\widetilde{\sigma})$ and $c_0(\sigma, \sigma_2) = c_0(\widetilde{\sigma}, \sigma_2)$. Now by Lemma 2.1.11, $(e^T_{m-c_0(\widetilde{\sigma})} \otimes I_n)\, M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) = e^T_{m-c_0(\widetilde{\sigma},\sigma_2)} \otimes I_n$. Consequently, we have $(e^T_{m-c_0(\sigma)} \otimes I_n)\, M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) = (e^T_{m-c_0(\widetilde{\sigma})} \otimes I_n)\, M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) = e^T_{m-c_0(\widetilde{\sigma},\sigma_2)} \otimes I_n = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$.

Further, by similar arguments as above, we have $M_{\tau_1}(Y_1)\, M_{\sigma_1}(X_1)\, (e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$. This completes the proof of (a).

(b) Note that if $k \notin \{\pm(m-1), \pm m\}$, then $(e_1^T \otimes I_n)M_k(Z) = e_1^T \otimes I_n$ for any matrix $Z \in \mathbb{C}^{n\times n}$. Hence $(e_1^T \otimes I_n)M_{\sigma_2}(X_2) = e_1^T \otimes I_n$ and $M_{\sigma_1}(X_1)(e_1 \otimes I_n) = e_1 \otimes I_n$ as σ_j, $j = 1, 2$, contains indices from $\{0 : m-2\}$. Further, since $\tau_1 = \emptyset = \tau_2$, we have $M_{\tau_1}(Y_1) = I_{mn} = M_{\tau_2}(Y_2)$. Consequently, we have $(e_1^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e_1^T \otimes I_n$ and $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_1 \otimes I_n) = e_1 \otimes I_n$. This completes the proof of (b). □

Lemma 3.3.5. *Let* $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M_\tau^P - M_\sigma^P)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ *be an EGFPR of* $P(\lambda)$. *Suppose that* $0 \in \omega$ *(recall that* $\tau = -\omega$*). Then we have the following.*

(a) If $i_0(\omega) + 1 \in \sigma$ *and* $s := i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 < m$ *then*

$$(e_{m-s}^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e_{m-p}^T \otimes I_n,$$

where $p := i_0(\omega) + c_{i_0(\omega)+1}(\sigma, \sigma_2) + 1$.

(b) If $s = m$ *in part (a), then* $(e_1^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e_p^T \otimes I_n$, *where* $p := c_{-(m-1)}(\tau_2) + 2$.

(c) If $i_0(\omega) < m$ *and* $i_0(\omega) + 1 \notin \sigma$ *then*

$$(e_{m-i_0(\omega)}^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e_{m-p}^T \otimes I_n, \tag{3.8}$$

where $p := i_0(\omega) - c_{-i_0(\omega)}(\tau_2) - 1$.

(d) If $i_0(\omega) = m$ *in part (c), then* $(e_m^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = e_m^T \otimes I_n$.

Proof. (a) Given that $i_0(\omega) + 1 \in \sigma$ and $s := i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 < m$. Set $q := c_{i_0(\omega)+1}(\sigma)$. Then $s = i_0(\omega) + 1 + q$. Note that $0, 1, \ldots, i_0(\omega) \notin \sigma$. Since σ is a sub-permutation and has q consecutions at $i_0(\omega) + 1$, we have $\sigma \sim \big(\widehat{\sigma}, i_0(\omega) + 1, i_0(\omega) + 2, \ldots, i_0(\omega) + 1 + q\big)$, that is,

$$\sigma \sim \big(\widehat{\sigma}, i_0(\omega) + 1, i_0(\omega) + 2, \ldots, s\big). \tag{3.9}$$

Suppose that $s+1 \notin \sigma_2$. Then $s \notin \sigma_2$, since otherwise (σ, σ_2) would not satisfy the SIP which would contradict the condition that $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP. Thus $s, s+1 \notin \sigma_2$. Hence by (3.2), we have $(e_{m-s}^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = (e_{m-s}^T \otimes I_n)M_{\tau_2}(Y_2) = e_{m-s}^T \otimes I_n$, where the last equality follows from (3.6) since $s \in \sigma$ implies that $-s, -(s+1) \notin \tau_2$. Now, since $s+1 \notin \sigma_2$, it is clear from (3.9) that $c_{i_0(\omega)+1}(\sigma, \sigma_2) = c_{i_0(\omega)+1}(\sigma)$. This shows that $p = s$ which yields the desired result.

Next, suppose that $s+1 \in \sigma_2$. Set $r := c_{s+1}(\sigma_2)$, i.e., σ_2 has r consecutions at $s+1$. Then by Lemma 3.3.1, we have

$$\sigma_2 \sim \big(\sigma_2^L, s+1, s+2, \ldots, s+1+r, \sigma_2^R\big) \tag{3.10}$$

for some index tuples σ_2^L and σ_2^R, where $s+1 \notin \sigma_2^L$ and $s+1+r, s+2+r \notin \sigma_2^R$. Since (σ, σ_2) satisfies the SIP and $s+1 \notin \sigma_2^L$, it is clear from (3.9) and (3.10) that $s \notin \sigma_2^L$. So the subtuple of σ_2 with indices $\{s, s+1\}$ starts with $s+1$. Hence by Lemma 3.3.2, $(e_{m-s}^T \otimes I_n)M_{\sigma_2}(X_2) = e_{m-(s+1+r)}^T \otimes I_n$. Now since $s+1+r \in \sigma_2$, we have $s+1+r, s+2+r \in \sigma$. This implies $s+1+r, s+2+r \notin \tau_2$. Thus by (3.6), $(e_{m-(s+1+r)}^T \otimes I_n)M_{\tau_2}(Y_2) = e_{m-(s+1+r)}^T \otimes I_n$. Now, by (3.9) and (3.10), $c_{i_0(\omega)+1}(\sigma, \sigma_2) = s+r-i_0(\omega)$. Consequently, $s+r+1 = p$ which proves (a).

(b) Since σ_2 contains indices from $\{0 : m-2\}$, by (3.2), we have $(e_1^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = (e_1^T \otimes I_n)M_{\tau_2}(Y_2)$. As $i_0(\omega)+1 \in \sigma$ and $s = m$, we have $m \in \sigma$. Thus $-m \notin \tau$ and hence $-m \notin \tau_2$.

Suppose that $-(m-1) \notin \tau_2$. As $-m, -(m-1) \notin \tau_2$, by (3.6), we have $(e_1^T \otimes I_n)M_{\tau_2}(Y_2) = e_1^T \otimes I_n$. Hence the desired result follows from the fact that $-(m-1) \notin \tau_2$ implies $c_{-(m-1)}(\tau_2) = -1$. Next, suppose that $-(m-1) \in \tau_2$. Set $q := c_{-(m-1)}(\tau_2)$. As $-m \notin \tau_2$, the subtuple of τ_2 with indices $\{-m, -(m-1)\}$ starts with $-(m-1)$. Hence by Lemma 3.3.3, $(e_1^T \otimes I_n)M_{\tau_2}(Y_2) = (e_{m-(m-1)}^T \otimes I_n)M_{\tau_2}(Y_2) = e_{m-(m-q-2)}^T \otimes I_n = e_{q+2}^T \otimes I_n$ which proves (b).

(c) Let $i_0(\omega) < m$ and $i_0(\omega)+1 \notin \sigma$, i.e., $-i_0(\omega), -(i_0(\omega)+1) \in \tau$. This implies that $i_0(\omega), i_0(\omega)+1 \notin \sigma_2$. Thus by (3.2), we have

$$(e_{m-i_0(\omega)}^T \otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2) = (e_{m-i_0(\omega)}^T \otimes I_n)M_{\tau_2}(Y_2).$$

Since ω is a sub-permutation and has $i_0(\omega)$ inversions at 0, we have $\omega \sim (i_0(\omega), \ldots, 1, 0, \widehat{\omega})$, where $i_0(\omega)+1 \in \widehat{\omega}$. Consequently, we have (recall that $\tau = -\omega$)

$$\tau \sim \Big(-i_0(\omega), \ldots, -1, -0, \widehat{\tau}\Big), \text{ where } -(i_0(\omega)+1) \in \widehat{\tau}. \tag{3.11}$$

Suppose that $-i_0(\omega) \notin \tau_2$. Then $-(i_0(\omega)+1) \notin \tau_2$, since otherwise (τ, τ_2) would not satisfy the SIP which would contradict the condition that (τ_1, τ, τ_2) satisfies the SIP. As $-i_0(\omega), -(i_0(\omega)+1) \notin \tau_2$, by (3.6), we have $(e_{m-i_0(\omega)}^T \otimes I_n)M_{\tau_2}(Y_2) = e_{m-i_0(\omega)}^T \otimes I_n$. Again, since $-i_0(\omega) \notin \tau_2$, we have $c_{-i_0(\omega)}(\tau_2) = -1$ which gives (3.8).

Next, suppose that $-i_0(\omega) \in \tau_2$. Set $q := c_{-i_0(\omega)}(\tau_2)$. Then by Lemma 2.1.21,

$$\tau_2 \sim \Big(\tau_2^L, -i_0(\omega), -(i_0(\omega)-1), \ldots, -(i_0(\omega)-q), \tau_2^R\Big) \tag{3.12}$$

for some index tuples τ_2^L and τ_2^R such that $-i_0(\omega) \notin \tau_2^L$ and $-(i_0(\omega)-q), -(i_0(\omega)-q-1) \notin \tau_2^R$. As (τ, τ_2) satisfies the SIP and $-i_0(\omega) \notin \tau_2^L$, it is clear from (3.11) and (3.12) that $-(i_0(\omega)+1) \notin \tau_2^L$. Thus the subtuple of τ_2 with indices from $\{-i_0(\omega), -(i_0(\omega)+1)\}$ starts with $-i_0(\omega)$. Hence by Lemma 3.3.3, we have

$$(e_{m-i_0(\omega)}^T \otimes I_n)M_{\tau_2}(Y_2) = e_{m-(i_0(\omega)-q-1)}^T \otimes I_n,$$

which gives (3.8) as $q = c_{-i_0(\omega)}(\tau_2)$. This completes the proof of (c).

(d) Let $0 \in \omega$ and $i_0(\omega) = m$. Then $\sigma = \emptyset = \sigma_2$. Hence $M_{\sigma_2}(X_2) = I_{mn}$. Further, since τ_2 contains indices from $\{-m : -2\}$, we have $-0, -1 \notin \tau_2$. Thus $(e_m^T \otimes I_n)M_{\tau_2}(Y_2) = e_m^T \otimes I_n$ which proves (d). □

The following result is analogs to Lemma 3.3.5.

Lemma 3.3.6. *Let* $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M_\tau^P - M_\sigma^P)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ *be an EGFPR of* $P(\lambda)$ *as given in (3.1). Suppose that* $0 \in \omega$ *(recall that* $\tau = -\omega$*). Then we have the following.*

(a) If $c_0(\omega) + 1 \in \sigma$ *and* $s := c_0(\omega) + i_{c_0(\omega)+1}(\sigma) + 1 < m$ *then*

$$M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{m-s} \otimes I_n) = e_{m-p} \otimes I_n,$$

where $p := c_0(\omega) + i_{c_0(\omega)+1}(\sigma_1, \sigma) + 1$.

(b) If $s = m$ *in part (a), then* $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_1 \otimes I_n) = e_p \otimes I_n$*, where* $p := i_{-(m-1)}(\tau_1) + 2$.

(c) If $c_0(\omega) < m$ *and* $c_0(\omega) + 1 \notin \sigma$ *then*

$$M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{m-c_0(\omega)} \otimes I_n) = e_{m-p} \otimes I_n, \tag{3.13}$$

where $p := c_0(\omega) - i_{-c_0(\omega)}(\tau_1) - 1$.

(d) If $c_0(\omega) = m$ *in part (c), then* $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_m \otimes I_n) = e_m \otimes I_n$.

Proof. (a) Let $c_0(\omega) + 1 \in \sigma$ and $s := c_0(\omega) + i_{c_0(\omega)+1}(\sigma) + 1 < m$. Set $q := i_{c_0(\omega)+1}(\sigma)$, then $s = c_0(\omega) + 1 + q$. Note that $0, 1, \ldots, c_0(\omega) \notin \sigma$. Since σ is a sub-permutation and has q inversions at $c_0(\omega) + 1$, we have $\sigma \sim \big(c_0(\omega) + 1 + q, \ldots, c_0(\omega) + 2, c_0(\omega) + 1, \widehat{\sigma}\big)$, that is,

$$\sigma \sim \big(s, \ldots, c_0(\omega) + 2, c_0(\omega) + 1, \widehat{\sigma}\big). \tag{3.14}$$

Suppose that $s + 1 \notin \sigma_1$. Then $s \notin \sigma_1$, since otherwise (σ_1, σ) would not satisfy the SIP. As $s, s+1 \notin \sigma_1$ and $s \in \sigma$ implies that $-s, -(s+1) \notin \tau_1$, by (3.3) and (3.7), we have $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{m-s} \otimes I_n) = M_{\tau_1}(Y_1)(e_{m-s} \otimes I_n) = e_{m-s} \otimes I_n$. Now, since $s + 1 \notin \sigma_1$, it is clear from (3.14) that $i_{c_0(\omega)+1}(\sigma_1, \sigma) = i_{c_0(\omega)+1}(\sigma)$. This shows that $p = s$ which yields the desired result.

Next, suppose that $s + 1 \in \sigma_1$. Set $r := i_{s+1}(\sigma_1)$, i.e., σ_1 has r inversions at $s + 1$. Then by Lemma 3.3.1, we have

$$\sigma_1 \sim (\sigma_1^L, s + 1 + r, \ldots, s + 2, s + 1, \sigma_1^R) \tag{3.15}$$

for some tuples σ_1^L and σ_1^R, where $s+1 \notin \sigma_1^R$ and $s+1+r, s+2+r \notin \sigma_1^L$. Since (σ_1, σ) satisfies the SIP and $s+1 \notin \sigma_1^R$, it is clear from (3.14) and (3.15) that $s \notin \sigma_1^R$. So the subtuple of σ_1 with indices $\{s, s+1\}$ ends with $s+1$. Thus by Lemma 3.3.2, $M_{\sigma_1}(X_1)(e_{m-s} \otimes I_n) = e_{m-(s+1+r)} \otimes I_n$. Now since $s+1+r \in \sigma_1$, we have $s+1+r, s+2+r \in \sigma$. This implies that $s+1+r, s+2+r \notin \tau_1$. Hence by (3.7), $M_{\tau_1}(Y_1)(e_{m-(s+1+r)} \otimes I_n) = e_{m-(s+1+r)} \otimes I_n$. Now, by (3.14) and (3.15), we have $i_{c_0(\omega)+1}(\sigma_1, \sigma) = s + r - c_0(\omega)$. This shows that $s+r+1 = p$ which proves (a).

(b) Since σ_1 contains indices from $\{0 : m-2\}$, by (3.3), we have $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_1 \otimes I_n) = M_{\tau_1}(Y_1)(e_1 \otimes I_n)$. As $c_0(\omega)+1 \in \sigma$ and $s = m$, we have $m \in \sigma$. Thus $-m \notin \tau$ and $-m \notin \tau_1$.

Suppose that $-(m-1) \notin \tau_1$. As $-m, -(m-1) \notin \tau_1$, by (3.7), $M_{\tau_1}(Y_1)(e_1 \otimes I_n) = e_1 \otimes I_n$. Hence the desired result follows from the fact that $-(m-1) \notin \tau_1$ implies $i_{-(m-1)}(\tau_1) = -1$. Next, suppose that $-(m-1) \in \tau_1$. Set $q := i_{-(m-1)}(\tau_1)$. As $-m \notin \tau_1$, the subtuple of τ_1 with indices $\{-m, -(m-1)\}$ ends with $-(m-1)$. Hence by Lemma 3.3.3, we have $M_{\tau_1}(Y_1)(e_{m-(m-1)} \otimes I_n) = e_{m-(m-q-2)} \otimes I_n = e_{q+2} \otimes I_n$ which proves (b).

(c) Let $c_0(\omega) < m$ and $c_0(\omega)+1 \notin \sigma$, i.e., $-c_0(\omega), -(c_0(\omega)+1) \in \tau$. This implies that $c_0(\omega), c_0(\omega)+1 \notin \sigma_1$. Thus by (3.3), we have

$$M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{m-c_0(\omega)} \otimes I_n) = M_{\tau_1}(Y_1)(e_{m-c_0(\omega)} \otimes I_n).$$

Since ω is a sub-permutation and has $c_0(\omega)$ consecutions at 0, we have $\omega \sim (\widehat{\omega}, 0, 1, \ldots, c_0(\omega))$, where $c_0(\omega)+1 \in \widehat{\omega}$. Consequently, we have (recall that $\tau = -\omega$)

$$\tau \sim \Big(\widehat{\tau}, -0, -1, \ldots, -c_0(\omega)\Big), \text{ where } -(c_0(\omega)+1) \in \widehat{\tau}. \tag{3.16}$$

Suppose that $-c_0(\omega) \notin \tau_1$. Then $-(c_0(\omega)+1) \notin \tau_1$, since otherwise (τ_1, τ) would not satisfy the SIP. As $-c_0(\omega), -(c_0(\omega)+1) \notin \tau_1$, by (3.7), $M_{\tau_1}(Y_1)(e_{m-c_0(\omega)} \otimes I_n) = e_{m-c_0(\omega)} \otimes I_n$. Now, since $-c_0(\omega) \notin \tau_1$, we have $i_{-c_0(\omega)}(\tau_1) = -1$ which gives (3.13).

Next, suppose that $-c_0(\omega) \in \tau_1$. Set $q := i_{-c_0(\omega)}(\tau_1)$. Then by Lemma 2.1.21,

$$\tau_1 \sim \Big(\tau_1^L, -(c_0(\omega)-q), \ldots, -(c_0(\omega)-1), -c_0(\omega), \tau_1^R\Big)$$

for some tuples τ_1^L and τ_1^R such that $-c_0(\omega) \notin \tau_1^R$ and $-(c_0(\omega)-q), -(c_0(\omega)-q-1) \notin \tau_1^L$. As (τ_1, τ) satisfies the SIP and $-c_0(\omega) \notin \tau_1^R$, it is clear from (3.16) that $-(c_0(\omega)+1) \notin \tau_1^R$. Thus the subtuple of τ_1 with indices $\{-c_0(\omega), -(c_0(\omega)+1)\}$ ends with $-c_0(\omega)$. Hence by Lemma 3.3.3, $M_{\tau_1}(Y_1)(e_{m-c_0(\omega)} \otimes I_n) = e_{m-c_0(\omega)+q+1} \otimes I_n$ which gives (3.13). This completes the proof of (c).

(d) Then $\sigma = \emptyset = \sigma_1$ Hence $M_{\sigma_2}(X_2) = I_{mn}$. Further, since τ_1 contains indices from $\{-m : -2\}$, we have $-0, -1 \notin \tau_1$. Thus $M_{\tau_1}(Y_1)(e_m \otimes I_n) = e_m \otimes I_n$ which proves (d). $\square$

Now we are ready to describe the recovery of eigenvectors of a regular $P(\lambda)$ from those of the EGFPRs of $P(\lambda)$.

Theorem 3.3.7. *Suppose that $P(\lambda)$ is regular and $\mu \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$. Let $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ be an EGFPR of $P(\lambda)$. (Recall that $\omega = -\tau$). Define*

$$F^{\text{EGFPR}}(P) := \begin{cases} e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n & \textit{if } \; 0 \in \sigma \textit{ and } c_0(\sigma) < m \\ A_m^{-1}(e_1^T \otimes I_n) & \textit{if } \; 0 \in \sigma \textit{ and } c_0(\sigma) = m \\ e^T_{m-p} \otimes I_n & \textit{if } \begin{cases} 0 \in \omega,\ i_0(\omega)+1 \in \sigma \textit{ and} \\ s := i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 < m, \\ \textit{where } p := i_0(\omega) + c_{i_0(\omega)+1}(\sigma,\sigma_2) + 1 \end{cases} \\ A_m^{-1}(e_p^T \otimes I_n) & \textit{if } \begin{cases} 0 \in \omega,\ i_0(\omega)+1 \in \sigma \textit{ and} \\ i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 = m, \\ \textit{where } p := c_{-(m-1)}(\tau_2) + 2 \end{cases} \\ e^T_{m-p} \otimes I_n & \textit{if } \begin{cases} 0 \in \omega,\ i_0(\omega) < m \textit{ and } i_0(\omega)+1 \notin \sigma, \\ \textit{where } p := i_0(\omega) - c_{-i_0(\omega)}(\tau_2) - 1 \end{cases} \\ A_0^{-1}(e_m^T \otimes I_n) & \textit{if } \; 0 \in \omega \textit{ and } i_0(\omega) = m \end{cases}$$

and

$$K^{\text{EGFPR}}(P) := \begin{cases} e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n & \textit{if } \; 0 \in \sigma \textit{ and } i_0(\sigma) < m \\ A_m^{-T}(e_1^T \otimes I_n) & \textit{if } \; 0 \in \sigma \textit{ and } i_0(\sigma) = m \\ e^T_{m-p} \otimes I_n & \textit{if } \begin{cases} 0 \in \omega,\ \ c_0(\omega)+1 \in \sigma \textit{ and} \\ s := c_0(\omega) + i_{c_0(\omega)+1}(\sigma) + 1 < m, \\ \textit{where } p := c_0(\omega) + i_{c_0(\omega)+1}(\sigma_1,\sigma) + 1 \end{cases} \\ A_m^{-T}(e_p^T \otimes I_n) & \textit{if } \begin{cases} 0 \in \omega,\ \ c_0(\omega)+1 \in \sigma \textit{ and} \\ c_0(\omega) + i_{c_0(\omega)+1}(\sigma) + 1 = m, \\ \textit{where } p := i_{-(m-1)}(\tau_1) + 2 \end{cases} \\ e^T_{m-p} \otimes I_n & \textit{if } \begin{cases} 0 \in \omega,\ \ c_0(\omega) < m \textit{ and } c_0(\omega)+1 \notin \sigma, \\ \textit{where } p := c_0(\omega) - i_{-c_0(\omega)}(\tau_1) - 1 \end{cases} \\ A_0^{-T}(e_m^T \otimes I_n) & \textit{if } \; 0 \in \omega \textit{ and } c_0(\omega) = m. \end{cases}$$

Then the maps $F^{\text{EGFPR}}(P) : \mathcal{N}_r(L(\mu)) \to \mathcal{N}_r(P(\mu))$, $x \mapsto F^{\text{EGFPR}}(P)x$, *and* $K^{\text{EGFPR}}(P) : \mathcal{N}_l(L(\mu)) \to \mathcal{N}_l(P(\mu))$, $y \mapsto K^{\text{EGFPR}}(P)y$, *are linear isomorphisms. Thus, if* $(x_1, \ldots, x_p)$ *and* $(y_1, \ldots, y_p)$ *are bases of* $\mathcal{N}_r(L(\mu))$ *and* $\mathcal{N}_l(L(\mu))$, *respectively, then*

$$\Big(F^{\text{EGFPR}}(P)x_1, \ldots, F^{\text{EGFPR}}(P)x_p\Big) \text{ and } \Big(K^{\text{EGFPR}}(P)y_1, \ldots, K^{\text{EGFPR}}(P)y_p\Big)$$

are bases of $\mathcal{N}_r(P(\mu))$ *and* $\mathcal{N}_l(P(\mu))$, *respectively.*

Proof. We have $L(\lambda) = M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)T(\lambda)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$, where $T(\lambda) := \lambda M^P_\tau - M^P_\sigma$ is a GF pencil of $P(\lambda)$ associated with the permutation $(\sigma, -\tau)$ of $\{0 : m\}$. Since X_j and Y_j, $j = 1, 2$, are nonsingular matrix assignments, $M_{(\tau_1,\sigma_1)}(Y_1, X_1)$ and $M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ are nonsingular. Hence the map $M_{(\sigma_2,\tau_2)}(X_2, Y_2) : \mathcal{N}_r(L(\mu)) \to \mathcal{N}_r(T(\mu))$, $x \mapsto \big(M_{(\sigma_2,\tau_2)}(X_2, Y_2)\big)x$, is a linear isomorphism. On the other hand, by Theorem 1.2.26, $F^{\text{GF}}(P) : \mathcal{N}_r(T(\mu)) \to \mathcal{N}_r(P(\mu))$ is a linear isomorphism. Thus the map

$$\mathcal{N}_r(L(\mu)) \to \mathcal{N}_r(P(\mu)), \; x \mapsto \big(F^{\text{GF}}(P)M_{(\sigma_2,\tau_2)}(X_2, Y_2)\big)x,$$

is a linear isomorphism. Now the desired result for recovery of right eigenvectors follows from Theorem 1.2.26 and Lemmas 3.3.4 and 3.3.5.

Next we prove the recovery of left eigenvectors. As $(M_{(\sigma_1,\tau_1)}(Y_1, X_1))^T$ is nonsingular, $(M_{(\sigma_1,\tau_1)}(Y_1, X_1))^T : \mathcal{N}_l(L(\mu)) \to \mathcal{N}_l(T(\mu))$, $y \mapsto (M_{(\sigma_1,\tau_1)}(Y_1, X_1))^T y$, is a linear isomorphism. Also, by Theorem 1.2.26, $K^{\text{GF}}(P) : \mathcal{N}_l(T(\mu)) \longrightarrow \mathcal{N}_l(P(\mu))$ is a linear isomorphism. Thus the map

$$\mathcal{N}_l(L(\mu)) \to \mathcal{N}_l(P(\mu)), \; y \mapsto \Big(K^{\text{GF}}(P)(M_{(\sigma_1,\tau_1)}(Y_1, X_1))^T y\Big),$$

is a linear isomorphism. Now the desired result for recovery of left eigenvectors follows from Theorem 1.2.26 and Lemmas 3.3.4 and 3.3.6. □

We now illustrate eigenvector recovery rule for $P(\lambda)$ from those of the EGFPRs of $P(\lambda)$ by considering a few examples.

Example 3.3.8. *Let* $P(\lambda) := \sum_{i=0}^{6} \lambda^i A_i$. *Suppose that* $P(\lambda)$ *is regular and* $\mu \in \mathbb{C}$ *is an eigenvalue of* $P(\lambda)$. *Let* $\sigma := (1, 2, 5)$, $\sigma_1 := \emptyset, \sigma_2 := (1), \tau := (-6, -3, -4, -0), \tau_1 := (-4)$ *and* $\tau_2 := \emptyset$. *Let* X *and* Y *be any arbitrary matrix assignments for* τ_1 *and* σ_2, *respectively. Then the pencil* $L(\lambda) = M_{-4}(X)\big(\lambda M^P_{(-3,-4,-6,-0)} - M^P_{(1,2,5)}\big)M_1(Y) =$

$$\begin{bmatrix} \lambda A_6 + A_5 & -I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_n & \lambda I_n & 0 & 0 \\ -I_n & 0 & \lambda I_n - X & \lambda X & 0 & 0 \\ 0 & \lambda I_n & \lambda A_4 & \lambda A_3 + A_2 & -Y & -I_n \\ 0 & 0 & 0 & A_1 & \lambda Y - I_n & \lambda I_n \\ 0 & 0 & 0 & -I_n & -\lambda A_0^{-1} & 0 \end{bmatrix}$$

is an EGFPR of $P(\lambda)$.

Let $u \in \mathcal{N}_r(L)$ *and* $v \in \mathcal{N}_l(L)$. *Define* $u_i := (e_i^T \otimes I_n)u$ *and* $v_i := (e_i^T \otimes I_n)v$, $i = 1:6$. *Note that* $0 \in \omega$ *and* $i_0(\omega) = 0$. *Further,* $i_0(\omega) + 1 = 1 \in \sigma$ *and* $c_{i_0(\omega)+1}(\sigma) = 1$, *which implies that* $s := i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 = 2 < 6$. *As* $c_1(\sigma, \sigma_2) = 1$, *we have* $p := i_0(\omega) + c_{i_0(\omega)+1}(\sigma, \sigma_2) + 1 = 2$. *Thus by Theorem 3.3.7,* $(e_{6-2}^T \otimes I_n)u = u_4 \in \mathcal{N}_r(P(\mu))$. *To verify the recovery rule, consider* $L(\lambda)u = 0$. *This gives*

$$\begin{aligned} (\lambda A_6 + A_5)u_1 - u_2 &= 0 && (3.17) \\ -u_3 + \lambda u_4 &= 0 && (3.18) \\ -u_1 + (\lambda I_n - X)u_3 + \lambda X u_4 &= 0 && (3.19) \\ \lambda u_2 + \lambda A_4 u_3 + (\lambda A_3 + A_2)u_4 - Y u_5 - u_6 &= 0 && (3.20) \\ A_1 u_4 + (\lambda Y - I_n)u_5 + \lambda u_6 &= 0 && (3.21) \\ -u_4 - \lambda A_0^{-1} u_5 &= 0 && (3.22) \end{aligned}$$

From (3.18) we have $u_3 = \lambda u_4$. *Substituting* $u_3 = \lambda u_4$ *in (3.19) we have* $u_1 = \lambda^2 u_4$. *Adding* λ *times (3.20) with (3.21) and then substituting* $u_3 = \lambda u_4$ *we have* $\lambda^2 u_2 + (\lambda^3 A_4 + \lambda^2 A_3 + \lambda A_2 + A_1)u_4 - u_5 = 0$ *which together with (3.22) gives*

$$\lambda^3 u_2 + (\lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0)u_4 = 0. \qquad (3.23)$$

Now substituting the value of $\lambda^3 u_2$ *from (3.23) and* $u_1 = \lambda^2 u_4$ *in (3.17), we have* $P(\lambda)u_4 = 0$.

Next, consider $v \in \mathcal{N}_l(L)$. *Define* $v_i := (e_i^T \otimes I_n)v$, $i = 1:6$. *Note that* $0 \in \omega$ *and* $c_0(\omega) = 0$. *Further,* $c_0(\omega) + 1 = 1 \in \sigma$ *and* $i_1(\sigma) = 0$ *which implies that* $s := c_0(\omega) + i_{c_0(\omega)+1}(\sigma) + 1 = 1 < 6$. *As* $i_1(\sigma_1, \sigma) = 0$, *we have* $p := c_0(\omega) + i_{c_0(\omega)+1}(\sigma_1, \sigma) + 1 = 1$. *Hence by Theorem 3.3.7, we have* $(e_{6-1}^T \otimes I_n)v = v_5 \in \mathcal{N}_l(P(\mu))$ *which can be easily verified.* ■

Next, we consider an EGFPR which is not operation free (as -1 and -0 simultaneously belong to τ) but the recovery of eigenvector is operation free.

Example 3.3.9. *Let* $P(\lambda) := \sum_{i=0}^{5} \lambda^i A_i$. *Suppose that* $P(\lambda)$ *is regular and* $\mu \in \mathbb{C}$ *is an eigenvalue of* $P(\lambda)$. *Let* $\sigma := (4,2,3)$, $\sigma_1 := \emptyset, \sigma_2 := (2), \tau := (-5,-1,-0)$ *and* $\tau_1 = \emptyset = \tau_2$. *Let* X *be any arbitrary matrix assignment for* σ_2. *Then the EGFPR* $L(\lambda) = \big(\lambda M^P_{(-5,-1,-0)} - M^P_{(4,2,3)}\big) M_2(X)$ *of* $P(\lambda)$ *is given by*

$$L(\lambda) = \begin{bmatrix} \lambda A_5 + A_4 & A_3 & -X & -I_n & 0 \\ -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & A_2 & \lambda X - I_n & \lambda I_n & 0 \\ 0 & -I_n & 0 & 0 & -\lambda A_0^{-1} \\ 0 & 0 & \lambda I_n & 0 & -\lambda A_1 A_0^{-1} - I_n \end{bmatrix}.$$

Let $u \in \mathcal{N}_r(L)$ *and* $v \in \mathcal{N}_l(L)$. *Define* $u_i := (e_i^T \otimes I_n)u$ *and* $v_i := (e_i^T \otimes I_n)v$, $i = 1:5$. *Note that* $0 \in \omega$ *and* $i_0(\omega) = 1$. *Further,* $i_0(\omega) + 1 = 2 \in \sigma$ *and* $c_{i_0(\omega)+1}(\sigma) = 1$, *which imply that* $s := i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 = 3 < 5$. *As* $c_2(\sigma, \sigma_2) = 1$, *we have* $p := i_0(\omega) + c_{i_0(\omega)+1}(\sigma, \sigma_2) + 1 = 3$. *Thus by Theorem 3.3.7,* $(e_{5-3}^T \otimes I_n)u = u_2 \in \mathcal{N}_r(P(\mu))$ *which can be easily verified.*

Note that $0 \in \omega$ *and* $c_0(\omega) = 0 < 5$. *Further,* $c_0(\omega) + 1 = 1 \notin \sigma$. *As* $\tau_1 = \emptyset$, *we have* $i_{-c_0(\omega)}(\tau_1) = -1$. *So* $p := c_0(\omega) - i_{-c_0(\omega)}(\tau_1) - 1 = 0$. *Hence by Theorem 3.3.7,* $(e_{5-0}^T \otimes I_n)v = v_5 \in \mathcal{N}_l(P(\mu))$ *which can be easily verified.* ■

The EGFPR in the following example is not operation free (as -1 and -0 simultaneously belong to τ), but we can easily recover the eigenvectors of $P(\lambda)$ from those of the EGFPR.

Example 3.3.10. *Let* $P(\lambda) := \sum_{i=0}^{3} \lambda^i A_i$. *Suppose that* $P(\lambda)$ *is regular and* $\mu \in \mathbb{C}$ *is an eigenvalue of* $P(\lambda)$. *Let* $\sigma := (3)$, $\sigma_1 := \emptyset = \sigma_2, \tau := (-2,-1,-0), \tau_1 = \emptyset$ *and* $\tau_2 := (-2)$. *Then the EGFPR* $L(\lambda) = \big(\lambda M^P_{(-2,-1,-0)} - M^P_3\big) M^P_{-2}$ *is given by*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & -A_0^{-1} \\ 0 & I_n & -A_2 A_0^{-1} \\ I_n & A_2 & -A_1 A_0^{-1} \end{bmatrix} - \begin{bmatrix} 0 & A_3^{-1} & 0 \\ I_n & A_2 & 0 \\ 0 & 0 & I_n \end{bmatrix}.$$

Let $x \in \mathcal{N}_r(L(\mu))$ *and* $y \in \mathcal{N}_l(L(\mu))$. *Define* $x_i := (e_i^T \otimes I_n)x$ *and* $y_i := (e_i^T \otimes I_n)y$ *for* $i = 1:3$. *Note that* $0 \in \omega$ *and* $i_0(\omega) = 2$. *Further,* $i_0(\omega) + 1 = 3 \in \sigma$ *and* $c_{i_0(\omega)+1}(\sigma) = 0$, *which implies* $s := i_0(\omega) + c_{i_0(\omega)+1}(\sigma) + 1 = 3$, *i.e.,* $s = m$. *Now,* $c_{-(m-1)}(\tau_2) = c_{-2}(\tau_2) = 0$. *Thus by Theorem 3.3.7,* $A_m^{-1}(e_2^T \otimes I_n)x = A_3^{-1}x_2 \in \mathcal{N}_r(P(\mu))$ *which can be easily verified.*

For left eigenvector, we have $c_0(\omega)=0<m$, $c_0(\omega)+1=1\notin\sigma$ *and* $i_{-c_0(\omega)}(\tau_1)=-1$ *(as* $\tau_1=\emptyset$*). Thus* $p=c_0(\omega)-i_{-c_0(\omega)}(\tau_1)-1=0$*. Thus by Theorem 3.3.7,* $(e^T_{m-p}\otimes I_n)y=y_3\in\mathcal{N}_l(P(\mu))$ *which can be easily verified.* ■

3.4 Eigenvalue at infinity and recovery of eigenvectors

We now describe the recovery of eigenvectors of $P(\lambda)$ corresponding to an eigenvalue at ∞ from the eigenvectors of EGFPRs. Recall that ∞ is an eigenvalue of $P(\lambda)=\sum_{i=0}^m \lambda^i A_i$ if and only if 0 is an eigenvalue of $rev(P(\lambda))=\sum_{i=0}^m \lambda^i A_{m-i}$, i.e., 0 is an eigenvalue of A_m. Thus A_m is singular which implies that M^P_{-m} is singular, i.e., M^P_m does not exist as $M^P_m=(M^P_{-m})^{-1}$. So $-m$ always belongs to τ, that is, $m\notin\sigma$.

The following lemma will play a crucial role in the recovery of eigenvectors of $P(\lambda)$ corresponding to an eigenvalue ∞ from those of the EGFPRs.

Lemma 3.4.1. *Let* $L(\lambda):=M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ *be an EGFPR of* $P(\lambda)$ *such that* $-m\in\tau$.

(a) Suppose that $c_{-m}(\tau)\le m-1$ *and* $i_{-m}(\tau)\le m-1$*. Then*

$$(e^T_{c_{-m}(\tau)+1}\otimes I_n)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)=e^T_{c_{-m}(\tau,\tau_2)+1}\otimes I_n$$

and

$$M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(e_{i_{-m}(\tau)+1}\otimes I_n)=e_{i_{-m}(\tau_1,\tau)+1}\otimes I_n.$$

(b) If $c_{-m}(\tau)=m$ *then* $(e^T_m\otimes I_n)\,M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)=e^T_m\otimes I_n$*, and if* $i_{-m}(\tau)=m$ *then* $M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)\,(e_m\otimes I_n)=e_m\otimes I_n$.

Proof. (a) For an arbitrary $Z\in\mathbb{C}^{n\times n}$, from (2.12) and (2.13) we have

$$(e^T_j\otimes I_n)M_{-k}(Z)=e^T_j\otimes I_n \text{ for } \ k\notin\{m-j,m-j+1\},\ j=1:m, \tag{3.24}$$
$$M_{-k}(Z)(e_j\otimes I_n)=e_j\otimes I_n \text{ for } \ k\notin\{m-j,m-j+1\},\ j=1:m. \tag{3.25}$$

Case-I: Suppose that $\sigma\neq\emptyset$. Let $0\le h\le m-1$ be the integer such that $-m,-(m-1),\ldots,-(m-h)\in\tau$ and $-(m-h-1)\notin\tau$. This implies that $-(m-h),-(m-h-1)\notin\tau_1\cup\tau_2$ since $-(m-h-1)\notin\tau$ and (τ_1,τ,τ_2) satisfies the SIP. Further, we have $c_{-m}(\tau)\le h$ and $i_{-m}(\tau)\le h$.

Let $\widehat{\tau}$ and $\widehat{\tau}_j$, $j=1,2$, respectively, be the subtuples of τ and τ_j with indices $\{-m:-(m-h)\}$. Similarly, let $\widehat{\widehat{\tau}}$ and $\widehat{\widehat{\tau}}_j$, $j=1,2$, respectively, be the subtuples of τ and τ_j with indices $\{-(m-h-2):-0\}$, where $\{-a:-0\}:=\{-a,-(a-1),\ldots,-1,-0\}$ for any integer $a\ge 0$. Then $\widehat{\tau}$ and $\widehat{\widehat{\tau}}$ commutes since for any indices $k\in\widehat{\tau}$ and $\ell\in\widehat{\widehat{\tau}}$, we have $||k| - |\ell|| > 1$. Thus $\tau \sim (\widehat{\tau}, \widehat{\widehat{\tau}}) \sim (\widehat{\widehat{\tau}}, \widehat{\tau})$. Similarly, $\tau_j \sim (\widehat{\tau}_j, \widehat{\widehat{\tau}}_j) \sim (\widehat{\widehat{\tau}}_j, \widehat{\tau}_j)$ for $j = 1, 2$, as $\widehat{\tau}_j$ and $\widehat{\widehat{\tau}}_j$ commutes. Also we have $(\widehat{\tau}, \widehat{\widehat{\tau}}_j) \sim (\widehat{\widehat{\tau}}_j, \widehat{\tau})$ for $j = 1, 2$.

Since $-(m-h-1) \notin \tau$, we have $i_{-m}(\tau) = i_{-m}(\widehat{\tau})$ and $c_{-m}(\tau) = c_{-m}(\widehat{\tau})$. As $(\widehat{\tau}, \widehat{\tau}_2)$ contains indices from $\{-m : -(m-h)\}$, we have $c_{-m}(\widehat{\tau}, \widehat{\tau}_2) \leq h$. This shows that $c_{-m}(\widehat{\tau}, \widehat{\tau}_2, \widehat{\widehat{\tau}}_2) = c_{-m}(\widehat{\tau}, \widehat{\tau}_2)$ as $\widehat{\widehat{\tau}}_2$ contains indices from $\{-(m-h-2) : -0\}$. Consequently, we have $c_{-m}(\tau, \tau_2) = c_{-m}(\widehat{\widehat{\tau}}, \widehat{\tau}, \widehat{\tau}_2, \widehat{\widehat{\tau}}_2) = c_{-m}(\widehat{\tau}, \widehat{\tau}_2, \widehat{\widehat{\tau}}_2) = c_{-m}(\widehat{\tau}, \widehat{\tau}_2)$. Similarly, we have $i_{-m}(\tau_1, \tau) = i_{-m}(\widehat{\tau}_1, \widehat{\tau})$.

Note that X_2 and Y_2 are arbitrary matrix assignments. We denote by $(*)$ for any arbitrary matrix assignment. Now we have $(e^T_{c_{-m}(\tau)+1} \otimes I_n) M_{\sigma_2}(*) M_{\tau_2}(*) = (e^T_{c_{-m}(\widehat{\tau})+1} \otimes I_n)\, M_{\sigma_2}(*)\, M_{\widehat{\widehat{\tau}}_2}(*)\, M_{\widehat{\tau}_2}(*) =$

$$\begin{aligned}
&(e^T_{c_{-m}(\widehat{\tau})+1} \otimes I_n)\, M_{\widehat{\widehat{\tau}}_2}(*)\, M_{\widehat{\tau}_2}(*) \text{ by (3.2) since } \begin{cases} c_{-m}(\widehat{\tau}) \leq h \text{ and } \sigma_2 \text{ contains} \\ \text{indices from } \{0 : m-h-2\} \end{cases} \\
&= (e^T_{c_{-m}(\widehat{\tau})+1} \otimes I_n)\, M_{\widehat{\tau}_2}(*) \text{ by (3.24) since } \begin{cases} c_{-m}(\widehat{\tau}) \leq h \text{ and } \widehat{\widehat{\tau}}_2 \text{ contains} \\ \text{indices from } \{-(m-h-2) : -0\} \end{cases} \\
&= e^T_{c_{-m}(\widehat{\tau}, \widehat{\tau}_2)+1} \otimes I_n = e^T_{c_{-m}(\tau, \tau_2)+1} \otimes I_n \text{ by Lemma 2.1.22.}
\end{aligned}$$

Hence $(e^T_{c_{-m}(\tau)+1} \otimes I_n) M_{\sigma_2}(X_2) M_{\tau_2}(Y_2) = e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n$.

Similarly, we have $M_{\tau_1}(*)\, M_{\sigma_1}(*)\, (e_{i_{-m}(\tau)+1} \otimes I_n) =$

$$\begin{aligned}
&M_{\widehat{\tau}_1}(*)\, M_{\widehat{\widehat{\tau}}_1}(*)\, M_{\sigma_1}(*)\, (e_{i_{-m}(\widehat{\tau})+1} \otimes I_n) \\
&= M_{\widehat{\tau}_1}(*)\, M_{\widehat{\widehat{\tau}}_1}(*)\, (e_{i_{-m}(\widehat{\tau})+1} \otimes I_n) \text{ by (3.3) since } \begin{cases} i_{-m}(\widehat{\tau}) \leq h \text{ and } \sigma_1 \text{ contains} \\ \text{indices from } \{0 : m-h-2\} \end{cases} \\
&= M_{\widehat{\tau}_1}(*)\, (e_{i_{-m}(\widehat{\tau})+1} \otimes I_n) \text{ by (3.25) since } \begin{cases} i_{-m}(\widehat{\tau}) \leq h \text{ and } \widehat{\widehat{\tau}}_1 \text{ contains} \\ \text{indices from } \{-(m-h-2) : -0\} \end{cases} \\
&= e_{i_{-m}(\widehat{\tau}_1, \widehat{\tau})+1} \otimes I_n = e_{i_{-m}(\tau_1, \tau)+1} \otimes I_n \text{ by Lemma 2.1.22.}
\end{aligned}$$

Thus, we have $M_{\tau_1}(Y_1)\, M_{\sigma_1}(X_1)\, (e_{i_{-m}(\tau)+1} \otimes I_n) = e_{i_{-m}(\tau_1,\tau)+1} \otimes I_n$.

Case-II: Suppose that $\sigma = \emptyset$, i.e., τ is a permutation of $\{-m : -0\}$. This implies $\sigma_1 = \emptyset = \sigma_2$ and hence $M_{\sigma_1}(X_1) = I_{mn} = M_{\sigma_2}(X_2)$. Given that $c_{-m}(\tau) \leq m-1$ and $i_{-m}(\tau) \leq m-1$. By defining $\widetilde{\tau} := \tau \setminus \{-0\}$ and $\widetilde{\sigma} := (0)$, and by considering $h = m-1$, a verbatim proof of Case-I gives the desired result.

(b) If $c_{-m}(\tau) = m$ or $i_{-m}(\tau) = m$, then $\sigma = \emptyset$. So there are no choices for σ_1 and σ_2, i.e., $\sigma_1 = \emptyset = \sigma_2$. Hence $M_{\sigma_1}(X_1) = I_{mn} = M_{\sigma_2}(X_2)$. Further, since τ_j, $j = 1, 2$, contains indices from $\{-m : -2\}$, we have $(e^T_m \otimes I_n)\, M_{\tau_2}(Y_2) = e^T_m \otimes I_n$ and $M_{\tau_1}(Y_1)\, (e_m \otimes I_n) = e_m \otimes I_n$ which gives the desired result. $\square$

We now describe the recovery of eigenvectors of $P(\lambda)$ corresponding to the eigenvalue at ∞ from those of the EGFPRs of $P(\lambda)$.

Theorem 3.4.2. *Let* $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ *be an EGFPR of* $P(\lambda)$*. Suppose that* $P(\lambda)$ *is regular and* ∞ *is an eigenvalue of* $P(\lambda)$*.*

Right eigenvectors at ∞**.** *Let* $\big(x_1, \ldots, x_k\big)$ *be a basis of the right eigenspace of* $L(\lambda)$ *at* ∞*. Then we have the following.*

(a) If $c_{-m}(\tau) < m$*, then* $\big((e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n)x_1, \ldots, (e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n)x_k\big)$ *is a basis of the right eigenspace of* $P(\lambda)$ *at* ∞*.*

(b) If $c_{-m}(\tau) = m$*, then* $\big(A_0^{-1}(e^T_m \otimes I_n)x_1, \ldots, A_0^{-1}(e^T_m \otimes I_n)x_k\big)$ *is a basis of the right eigenspace of* $P(\lambda)$ *at* ∞*.*

Left eigenvectors at ∞**.** *Let* $\big(y_1, \ldots, y_k\big)$ *be a basis of the left eigenspace of* $L(\lambda)$ *at* ∞*. Then we have the following.*

(c) If $i_{-m}(\tau) < m$*, then* $\big((e^T_{i_{-m}(\tau_1,\tau)+1} \otimes I_n)y_1, \ldots, (e^T_{i_{-m}(\tau_1,\tau)+1} \otimes I_n)y_k\big)$ *is a basis of the left eigenspace of* $P(\lambda)$ *at* ∞*.*

(d) If $i_{-m}(\tau) = m$*, then* $\big(A_0^{-T}(e^T_m \otimes I_n)y_1, \ldots, A_0^{-T}(e^T_m \otimes I_n)y_k\big)$*is a basis of the left eigenspace of* $P(\lambda)$ *at* ∞*.*

Proof. Set $L_1 := M_{(\tau_1,\sigma_1)}(Y_1, X_1)\ M^P_\tau\ M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ and $L_0 := M_{(\tau_1,\sigma_1)}(Y_1, X_1)\ M^P_\sigma\ M_{(\sigma_2,\tau_2)}(X_2, Y_2)$. Then $L(\lambda) = \lambda L_1 - L_0$. Recall that ∞ is an eigenvalue of $L(\lambda) \Leftrightarrow 0$ is an eigenvalue of $rev(L(\lambda)) \Leftrightarrow 0$ is an eigenvalue of L_1. Since $M_{\tau_1,\sigma_1}(Y_1, X_1)$ is invertible, we have $\mathcal{N}_r(L_1) = \mathcal{N}_r(M^P_\tau M_{\sigma_2,\tau_2}(X_2, Y_2))$. Note that the map

$$\mathcal{N}_r(M^P_\tau M_{(\sigma_2,\tau_2)}(X_2, Y_2)) \to \mathcal{N}_r(M^P_\tau),\ z \mapsto (M_{(\sigma_2,\tau_2)}(X_2, Y_2))z,$$

is an isomorphism. Define $T(\lambda) := \lambda M^P_\tau - M^P_\sigma$. Then $\mathcal{N}_r(M^P_\tau) = \mathcal{N}_r\big(rev(T(0))\big)$.

(a) Now, by Theorem 1.2.28, the map

$$\mathcal{N}_r\big(rev(T(0))\big) \longrightarrow \mathcal{N}_r\big(rev(P(0))\big),\ u \mapsto (e^T_{c_{-m}(\tau)+1} \otimes I_n)u,$$

is an isomorphism. Hence the map

$$\mathcal{N}_r\big(rev(L(0))\big) \longrightarrow \mathcal{N}_r\big(rev(P(0))\big),\ x \mapsto \big((e^T_{c_{-m}(\tau)+1} \otimes I_n)M_{(\sigma_2,\tau_2)}(X_2, Y_2)\big)x,$$

is an isomorphism. Now by Lemma 3.4.1, we have $(e^T_{c_{-m}(\tau)+1} \otimes I_n)M_{(\sigma_2,\tau_2)}(X_2, Y_2) = e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n$. Hence the result for the right eigenspace of $P(\lambda)$ at ∞ follows.

The proof is similar for part (b).

(c) Next, we prove the results for left eigenspace of $P(\lambda)$ at ∞. Since $M_{(\sigma_2,\tau_2)}(X_2,Y_2)$ is invertible, we have $\mathcal{N}_l\big(rev(L(0))\big) = \mathcal{N}_l(L_1) = \mathcal{N}_l\big(M_{(\tau_1,\sigma_1)}(Y_1,X_1)M^P_\tau\big)$. Hence it follows that the map $\mathcal{N}_l\big(M_{(\tau_1,\sigma_1)}(Y_1,X_1)M^P_\tau\big) \longrightarrow \mathcal{N}_l(M^P_\tau)$, $z \mapsto (M_{\tau_1,\sigma_1}(Y_1,X_1))^T z$, is an isomorphism. We have $\mathcal{N}_l(M^P_\tau) = \mathcal{N}_l\big(rev(T(0))\big)$. Now, by Theorem 1.2.28, the map

$$\mathcal{N}_l\big(rev(T(0))\big) \longrightarrow \mathcal{N}_l\big(rev(P(0))\big),\ v \mapsto (e^T_{i_{-m}(\tau)+1} \otimes I_n)v,$$

is an isomorphism. Hence the map

$$\mathcal{N}_l\big(rev(L(0))\big) \longrightarrow \mathcal{N}_l\big(rev(P(0))\big),\ y \mapsto \big((e^T_{i_{-m}(\tau)+1} \otimes I_n)(M_{(\tau_1,\sigma_1)}(Y_1,X_1))^T\big)y,$$

is an isomorphism. Now by Lemma 3.4.1, we have $M_{(\tau_1,\sigma_1)}(Y_1,X_1)(e_{i_{-m}(\tau)+1} \otimes I_n) = e_{i_{-m}(\tau_1,\tau)+1} \otimes I_n$. Hence the result for the left eigenspaces of $P(\lambda)$ at ∞ follows.

The proof is similar for part (d). □

We now illustrate our recovery rule for eigenvectors of $P(\lambda)$ corresponding to an eigenvalue at ∞ from those of the EGFPRs of $P(\lambda)$ by considering an example.

Example 3.4.3. *Let* $P(\lambda) := \sum_{i=0}^{5} \lambda^i A_i$. *Suppose that* $P(\lambda)$ *is regular and* ∞ *is an eigenvalue of* $P(\lambda)$. *Let* $\sigma := (0,2)$, $\sigma_1 := \emptyset = \sigma_2, \tau := (-4,-5,-3,-1)$, $\tau_2 = (-4)$ *and* $\tau_1 = \emptyset$. *Let* X *be any arbitrary matrix assignment for* τ_2. *Then the EGFPR* $L(\lambda) = \big(\lambda M^P_{(-4,-5,-3,-1)} - M^P_{(0,2)}\big)M_{-4}(X) =: \lambda L_1 - L_0$ *of* $P(\lambda)$ *is given by*

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & I_n & 0 & 0 \\ 0 & A_5 & A_4 & 0 & 0 \\ I_n & X & A_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & I_n & A_1 \end{bmatrix} - \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ I_n & X & 0 & 0 & 0 \\ 0 & 0 & -A_2 & I_n & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_0 \end{bmatrix}.$$

Let x *and* y*, respectively, be right and left eigenvectors of* $L(\lambda)$ *corresponding to the eigenvalue* ∞. *Define* $x_i := (e^T_i \otimes I_n)x$ *and* $y_i := (e^T_i \otimes I_n)y$, $i = 1:5$. *We have* $c_{-m}(\tau) = c_{-5}(-4,-5,-3,-1) = 0 < 5$ *and* $c_{-5}(\tau,\tau_2) = c_{-5}(-4,-5,-3,-1,-4) = 1$. *Hence by Theorem 3.4.2,* $(e^T_{c_{-m}(\tau,\tau_2)+1} \otimes I_n)x = (e^T_2 \otimes I_n)x = x_2$ *is a right eigenvector of* $P(\lambda)$ *corresponding to the eigenvalue* ∞. *Similarly,* $i_{-5}(\tau_1,\tau) = i_{-5}(-4,-5,-3,-1) = 1$. *Hence by Theorem 3.4.2,* $(e^T_{i_{-m}(\tau_1,\tau)+1} \otimes I_n)y = (e^T_2 \otimes I_n)y = y_2$ *is a left eigenvector of* $P(\lambda)$ *corresponding to the eigenvalue* ∞.

To verify the recovery rule, consider $L_1 x = 0$. *This gives* $A_5 x_2 = 0$ *and* $x_i = 0$ *for* $i = 3,4,5$. *Now if* $x_2 = 0$ *then* $x_1 = 0$. *Thus* $x_2 = 0 \Rightarrow x = 0$. *Hence* $x_2 \neq 0$ *and is a right eigenvector of* $P(\lambda)$ *corresponding to the eigenvalue* ∞.

Similarly, $y^T L_1 = 0$ *implies* $y_2^T A_5 = 0$ *and* $y_i^T = 0$ *for* $i = 3, 4, 5$. *Now if* $y_2^T = 0$ *then* $y_1^T = 0$. *Thus* $y_2^T = 0 \Rightarrow y = 0$. *Hence* $y_2 \neq 0$ *and is a left eigenvector of* $P(\lambda)$ *corresponding to the eigenvalue* ∞. ■

3.5 Low bandwidth banded EGFPRs

Low bandwidth banded linearizations of $P(\lambda)$ are important from the point of numerical computations. As EGFPRs of $P(\lambda)$ substantially enlarge the arena in which to look for strong linearizations of $P(\lambda)$, it is natural to investigate the possibility to construct low bandwidth banded EGFPRs. In particular, we are interested to construct block tridiagonal and block penta-diagonal EGFPRs of $P(\lambda)$.

We make the conversion that $e_j = 0$ for $j \leq 0$ and $j > m$. The following result will be useful for constructing block penta-diagonal EGFPRs of $P(\lambda)$.

Proposition 3.5.1. *Let* α *be an index tuple containing indices from* $\{0 : m-1\}$ *such that* α *satisfies the SIP. Suppose that* $c_k(\alpha) \leq 1$ *and* $i_k(\alpha) \leq 1$ *for any index* $1 \leq k \leq m-1$. *Let* $\mathcal{X}$ *be a matrix assignment for* α. *Then* $M_\alpha(\mathcal{X})$ *is a block penta-diagonal matrix. Equivalently, we have*

$$(e_{m-k}^T \otimes I_n) M_\alpha(\mathcal{X}) = \sum_{j=-2}^{2} (e_{m-(k-j)}^T \otimes X_j) \text{ for } k = 0 : m-1, \tag{3.26}$$

where $X_j = 0$ *or* X_j *belongs to the matrix assignment* $\mathcal{X}$.

Proof. It is easily seen that the elementary matrices $M_j(W)$, $j = 0 : m-1$, satisfies $(e_{m-i}^T \otimes I_n)M_j(W) = (e_{m-i}^T \otimes W) + (e_{m-(i-1)}^T \otimes I_n)$ for $j = i$ and $i = 1 : m-1$, and $(e_m^T \otimes I_n)M_0(W) = e_m^T \otimes W$. Consequently, from (2.2) and (2.3) we have the following

$$(e_{m-i}^T \otimes I_n) M_j(W) = \begin{cases} e_{m-(i+1)}^T \otimes I_n & \text{for } \; j = i+1 \text{ and } i = 0 : m-2, \\ (e_{m-i}^T \otimes W) + (e_{m-(i-1)}^T \otimes I_n) & \text{for } j = i \text{ and } i = 1 : m-1, \\ e_m^T \otimes W & \text{for } j = i = 0, \\ e_{m-i}^T \otimes I_n & \begin{cases} \text{otherwise, i.e., when} \\ j \notin \{i, i+1\} \text{ for } i = 0 : m-1. \end{cases} \end{cases} \tag{3.27}$$

It is given that $c_k(\alpha) \leq 1$ and $i_k(\alpha) \leq 1$ for $1 \leq k \leq m-1$. This implies $c_0(\alpha) \leq 2$ and $i_0(\alpha) \leq 2$. Recall from the definition of consecutions and inversions of any index tuple β at any index t that if $t \notin \beta$ then $c_t(\beta) = -1 = i_t(\beta)$. Consequently, we have

$$\begin{cases} -1 \leq c_k(\alpha) \leq 1 \text{ and } -1 \leq i_k(\alpha) \leq 1 \text{ for } k = 1 : m-1, \\ -1 \leq c_0(\alpha) \leq 2 \text{ and } -1 \leq i_0(\alpha) \leq 2. \end{cases} \tag{3.28}$$

Consider $0 \leq k \leq m-1$. We now prove (3.26). There are two cases.

Case-I: Suppose that the subtuple of α with indices $\{k, k+1\}$ starts with $k+1$. Then by Lemma 2.1.21, we have $(e_{m-k}^T \otimes I_n)M_\alpha(\mathcal{X}) = e_{m-(k+1+c_{k+1}(\alpha))}^T \otimes I_n$. Since $-1 \leq c_{k+1}(\alpha) \leq 1$, we have (3.26).

Case-II: Suppose that the subtuple of α with indices $\{k, k+1\}$ starts with k. Let the row standard form of α be given by $rsf(\alpha) := (\beta, rev(a_k : k), \gamma) =$

$$\big(rev(a_0 : 0), rev(a_1 : 1), \ldots, rev(a_k : k), rev(a_{k+1} : k+1), \ldots, rev(a_{m-1} : m-1)\big). \tag{3.29}$$

Then we must have $rev(a_k : k) \neq \emptyset$, since otherwise it is clear from (3.29) that the subtuple of α with indices $\{k, k+1\}$ would start with $k+1$ which would contradict our assumption of Case-II. Since α is a tuple containing nonnegative indices, by using (3.28) it is clear from (3.29) that $a_j \geq j-1$ for all $j = \{0 : m-1\} \setminus \{2\}$ and $a_2 \geq 0$. Further, since $rev(a_k : k) \neq \emptyset$, we have $a_k \in \{k, k-1\}$ if $k \neq 2$, and $a_k \in \{k, k-1, k-2\}$ if $k = 2$.

Since $\alpha \sim rsf(\alpha)$, without loss of generality, we assume that $\alpha = rsf(\alpha)$. That is, $\alpha := (\beta, rev(a_k : k), \gamma)$. Let $\mathcal{Y}$ and $\mathcal{Z}$, respectively, be the corresponding matrix assignments for β and γ associated with $\mathcal{X}$. We denote by X^p, $p \in \{a_k : k\}$, the matrices associated with $\mathcal{X}$ and assigned to the index $p \in \{a_k : k\}$. That is, $\mathcal{X} = \big(\mathcal{Y}, rev(X^{a_k} \cdots X^k), \mathcal{Z}\big)$. Now we have

$$\begin{aligned}
(e_{m-k}^T \otimes I_n)M_\alpha(\mathcal{X}) &= (e_{m-k}^T \otimes I_n)M_\beta(\mathcal{Y})M_{rev(a_k:k)}(rev(X^{a_k} \cdots X^k))M_\gamma(\mathcal{Z}) \\
&= (e_{m-k}^T \otimes I_n)M_{rev(a_k:k)}(rev(X^{a_k} \cdots X^k))M_\gamma(\mathcal{Z}) \text{ by (3.27) as } k, k+1 \notin \beta \\
&= \begin{cases} \Big((e_{m-(a_k-1)}^T \otimes I_n) + \sum_{j=a_k}^{k}(e_{m-j}^T \otimes X^j)\Big)M_\gamma(\mathcal{Z}) & \text{if } a_k > 0, \\ \Big(\sum_{j=a_k}^{k}(e_{m-j}^T \otimes X^j)\Big)M_\gamma(\mathcal{Z}) & \text{if } a_k = 0 \end{cases} \\
&\qquad \text{by applying (3.27) repeatedly.}
\end{aligned} \tag{3.30}$$

Define $S := \{a_k - 1 : k\}$ if $a_k > 0$ and $S := \{a_k : k\}$ if $a_k = 0$. It is clear from (3.30) that for evaluating $(e_{m-k}^T \otimes I_n)M_\alpha(X)$ we need to calculate $(e_{m-\ell}^T \otimes I_n)M_\gamma(\mathcal{Z})$ for all $\ell \in S$. Since $\gamma := \big(rev(a_{k+1} : k+1), \ldots, rev(a_{m-1} : m-1)\big)$, it is clear that, for $\ell \in S$, we have either $\ell \notin \gamma$ or the subtuple of γ with indices $\{\ell, \ell+1\}$ starts with $\ell+1$. By Lemma 2.1.21, we have $(e_{m-\ell}^T \otimes I_n)M_\gamma(\mathcal{Z}) = e_{m-(\ell+1+c_{\ell+1}(\gamma))}^T \otimes I_n$. Hence from (3.30), we have $(e^T_{m-k}\otimes I_n)M_\alpha(\mathcal{X}) =$

$$\begin{cases} (e^T_{m-(a_k+c_{a_k}(\gamma))}\otimes I_n)+\sum_{j=a_k}^{k}(e^T_{m-(j+1+c_{j+1}(\gamma))}\otimes X^j) & \text{if } a_k>0,\\ \sum_{j=a_k}^{k}(e^T_{m-(j+1+c_{j+1}(\gamma))}\otimes X^j) & \text{if } a_k=0. \end{cases} \tag{3.31}$$

Since γ is a subtuple of α, we have $c_t(\gamma)\leq c_t(\alpha)$ for any index t. Hence $-1\leq c_{\ell+1}(\gamma)\leq 1$ for all $\ell\in S$. Further, since $a_k\in\{k,k-1\}$ if $k\neq 2$, and $a_k\in\{k,k-1,k-2\}$ if $k=2$, it is clear from (3.31) that (3.26) holds. Hence $M_\alpha(\mathcal{X})$ is a block penta-diagonal matrix. This completes the proof. $\square$

Analogous to Proposition 3.5.1 we have the following result for index tuple containing negative indices.

Proposition 3.5.2. *Let α be an index tuple containing indices from $\{-m:-1\}$ such that α satisfies the SIP. Suppose that $c_{-k}(\alpha)\leq 1$ and $i_{-k}(\alpha)\leq 1$ for any index $-(m-1)\leq -k\leq -1$ (this implies that $c_{-m}(\alpha)\leq 2$ and $i_{-m}(\alpha)\leq 2$). Let $\mathcal{X}$ be a matrix assignment for α. Then $M_\alpha(\mathcal{X})$ is a block penta-diagonal matrix. Equivalently, we have*

$$(e^T_{m-k}\otimes I_n)M_\alpha(\mathcal{X})=\sum_{j=-2}^{2}(e^T_{m-(k-j)}\otimes X_j) \text{ for } k=0:m-1, \tag{3.32}$$

where $X_j=0$ or X_j belongs to the matrix assignment $\mathcal{X}$.

Proof. It is easily seen that the elementary matrices $M_{-j}(W)$, $j=0:m$, satisfies

$$(e^T_{m-(m-1)}\otimes I_n)M_{-m}(W)=e^T_1\otimes W \text{ and}$$

$$(e^T_{m-i}\otimes I_n)M_{-j}(W)=(e^T_{m-i}\otimes W)+(e^T_{m-(i+1)}\otimes I_n) \text{ for } j=i+1, \text{ and } i=0:m-2.$$

Consequently, from (2.12) and (2.13) we have the following.

$$(e^T_{m-i}\otimes I_n)M_{-j}(W)=\begin{cases} (e^T_{m-i}\otimes W)+(e^T_{m-(i+1)}\otimes I_n) & \text{for } j=i+1, \text{ and } i=0:m-2\\ e^T_1\otimes W & \text{for } j=i+1,\ i=m-1\\ e^T_{m-(i-1)}\otimes I_n & \text{for } j=i, \text{ and } i=1:m-1\\ e^T_{m-i}\otimes I_n & \begin{cases}\text{otherwise, i.e., when}\\ j\notin\{i,i+1\} \text{ for } i=0:m-1.\end{cases} \end{cases} \tag{3.33}$$

The rest of the proof is similar to Proposition 3.5.1. $\square$

Proposition 3.5.3. *Let α be an index tuple containing indices from $\{0:m\}$ such that α satisfies the SIP. Suppose that $c_j(\alpha)\geq 2$ or $i_j(\alpha)\geq 2$ for some $1\leq j\leq m-1$. Let $\mathcal{X}$ be any arbitrary nonsingular matrix assignment for α. Then $M_\alpha(\mathcal{X})$ is not a block penta-diagonal matrix.*

Proof. Suppose that $c_j(\alpha) \geq 2$ for some $j \geq 1$. Let $1 \leq k \leq m-1$ be the smallest integer belongs to α such that $c_k(\alpha) \geq c_\ell(\alpha)$ for any index $\ell \in \alpha$ (i.e., $c_k(\alpha) \geq 2$). Then by Lemma 3.3.1, we have $\alpha \sim (\alpha^L, k, k+1, \ldots, k+c_k(\alpha), \alpha^R)$, where $k \notin \alpha^L$. Now we have the following cases.

(a) Assume that $k > 1$. Then $1 \notin \alpha^L$, since otherwise $k-1 \geq 1$ and $c_{k-1}(\alpha) > c_k(\alpha)$ which is a contradiction. This implies that the subtuple of α with indices $\{k-1, k\}$ starts with k. Hence by Lemma 3.3.2, we have $(e^T_{m-(k-1)} \otimes I_n)M_\alpha(\mathcal{X}) = e^T_{m-(k+c_k(\alpha))} \otimes I_n$. That is, $\big(m-(k+c_k(\alpha))\big)$-th block entry of $(m-(k-1))$-th block row is I_n. Since $c_k(\alpha) \geq 2$, we have $k + c_k(\alpha) > k+1 = (k-1)+2$. Hence $M_\alpha(\mathcal{X})$ is not a block penta-diagonal matrix.

(b) Assume that $k = 1$. Then $(\alpha^L, (1:1+c_1(\alpha)), \alpha^R)$, where $c_1(\alpha) \geq 2$. If $0 \notin \alpha^L$ then the subtuple of α with indices $\{0,1\}$ starts with 1. Hence by Lemma 3.3.2, we have $(e^T_m \otimes I_n)M_\alpha(\mathcal{X}) = e^T_{m-(1+c_1(\alpha))} \otimes I_n$. Since $c_1(\alpha) \geq 2$, we have $1 + c_1(\alpha) \geq 3$. Hence $M_\alpha(\mathcal{X})$ is not a block penta-diagonal matrix.

Let X^0 be the matrix assigned to the index $0 \in \alpha^L$ associated with $\mathcal{X}$, and let $\mathcal{X} = (\mathcal{X}^L, \mathcal{X}^M, \mathcal{X}^R)$, where $\mathcal{X}^L, \mathcal{X}^M$, and $\mathcal{X}^R$ are the matrix assignments associated with $\alpha^L, (1:1+c_1(\alpha))$ and α^R, respectively. Then $(e^T_m \otimes I_n)M_{\alpha^L}(\mathcal{X}^L) = e^T_m \otimes X^0$ by (3.27) as $0 \in \alpha^L$ and $1 \notin \alpha^L$. Now the subtuple of $\beta := (1:1+c_1(\alpha), \alpha^R)$ with indices $\{0,1\}$ starts with 1. Hence by Lemma 3.3.2, we have $(e^T_m \otimes)M_\beta(*) = e^T_{m-(1+c_1(\alpha))} \otimes I_n$, where $(*)$ denotes any arbitrary matrix assignment. Hence $(e^T_m \otimes I_n)M_\alpha(\mathcal{X}) = e^T_{m-(1+c_1(\alpha))} \otimes X^0$. Since $c_1(\alpha) \geq 2$, we have $1 + c_1(\alpha) \geq 3$. This shows that $M_\alpha(\mathcal{X})$ is not a block penta-diagonal matrix.

Similarly, if $i_j(\alpha) \geq 2$, for some $j \geq 1$, then it can be shown that $M_\alpha(\mathcal{X})$ is not a block penta-diagonal matrix. ☐

For index tuple containing negative indices we have the following result.

Proposition 3.5.4. *Let β be an index tuple containing indices from $\{-m : -0\}$ such that β satisfies the SIP. Suppose that $c_{-j}(\beta) \geq 2$ or $i_{-j}(\beta) \geq 2$ for some $1 \leq j \leq m-1$. Let $\mathcal{X}$ be a matrix assignment for β. Then $M_\beta(\mathcal{X})$ would never be a block penta-diagonal matrix.*

Proof. The proof is similar to Proposition 3.5.3. ☐

Definition 3.5.5. *Let α and β be sub-permutations of $\{0 : m-1\}$ and $\{-m : -1\}$, respectively. Then $k \in \alpha \setminus \{0\}$ is said to be an end index of α if $k-1 \notin \alpha$ or $k+1 \notin \alpha$. Similarly, $-t \in \beta \setminus \{-m\}$ is said to be an end index of β if $-(t-1) \notin \beta$ or $-(t+1) \notin \beta$.*

The following result characterizes all block penta-diagonal EGFPRs of $P(\lambda)$.

Theorem 3.5.6. *Let* $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_\tau - M^P_\sigma)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ *be an EGFPR of* $P(\lambda)$*. Suppose that* σ_j *(resp.,* τ_j*), for* $j = 1, 2$*, does not contain the end points of* σ *(resp.,* τ*). Then* $L(\lambda)$ *is block penta-diagonal if and only if* $c_t(\sigma_1, \sigma, \sigma_2) \leq 1$*,* $i_t(\sigma_1, \sigma, \sigma_2) \leq 1$*,* $c_{-t}(\tau_1, \tau, \tau_2) \leq 1$ *and* $i_{-t}(\tau_1, \tau, \tau_2) \leq 1$ *for any index* $1 \leq t \leq m - 1$*.*

Proof. ($\Longrightarrow$) The forward implication follows from Proposition 3.5.3 and Proposition 3.5.4.

($\Longleftarrow$) We have $L(\lambda) =: \lambda L_1 - L_0$, where $L_1 := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)M^P_\tau M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ and $L_0 := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)M^P_\sigma M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$. Note that $L(\lambda)$ is a block penta-diagonal pencil if and only if both L_0 and L_1 are block penta-diagonal matrices. We only prove that L_0 is a block penta-diagonal matrix. Similarly, L_1 is block penta-diagonal.

Since τ_1 and τ_2 do not contain the end points of τ, it follows that $||j| - |k|| \geq 2$ for $j \in (\tau_1, \tau_2)$ and $k \in (\sigma_1, \sigma, \sigma_2)$. Hence $\tau_j \sim (\sigma_1, \sigma, \sigma_2)$, $j = 1, 2$, and $L_0 = M_{\tau_1}(Y_1)M_{\tau_2}(Y_2)M_{\sigma_1}(X_1)M^P_\sigma\ M_{\sigma_2}(X_2) = M_{\sigma_1}(X_1)M^P_\sigma\ M_{\sigma_2}(X_2)M_{\tau_1}(Y_1)M_{\tau_2}(Y_2)$.

Suppose that $m \in \sigma$. Recall that, as σ_1 and σ_2 contain indices from $\{0 : m-2\}$, there are no repetition of $m - 1$ and m in the indices of $(\sigma_1, \sigma, \sigma_2)$. Hence $(\sigma_1, \sigma, \sigma_2) \sim (\beta, m)$ (resp., $(\sigma_1, \sigma, \sigma_2) \sim (m, \beta)$) if $m - 1 \in \sigma$ and σ has a consecution (resp., inversion) at $m - 1$, where $\beta := (\sigma_1, \sigma, \sigma_2) \setminus \{m\}$. Further, since $M_m(W) = \mathrm{diag}(W^{-1}, I_{(m-1)n})$ (for any nonsingular matrix $W \in \mathbb{C}^{n \times n}$) is a block diagonal matrix, without loss of generality we assume that $m \notin \sigma$, i.e., $m \notin (\sigma_1, \sigma, \sigma_2)$.

Now we prove that L_0 is a block penta-diagonal matrix. Note that to prove L_0 is a block penta-diagonal matrix it is enough to show that

$$(e^T_{m-k} \otimes I_n)L_0 = \sum_{j=-2}^{2} (e^T_{m-(k-j)} \otimes Z_j) \text{ for } k = 0 : m - 1, \tag{3.34}$$

where Z_j is any one of the matrices $0, \pm I_n, \pm A_0, \ldots, \pm A_m$ or the matrices in X_1, X_2, Y_1 and Y_2. Since $(\sigma, -\tau)$ is a permutation of $\{0 : m\}$, we have either $k \in \sigma$ or $k \in -\tau$.

Case-I: Suppose that $k \in \sigma$. Then $-k \notin \tau$. This implies that $-k, -(k + 1) \notin (\tau_1, \tau_2)$ since (τ_1, τ, τ_2) satisfies the SIP. Hence by (3.33), we have $(e^T_{m-k} \otimes I_n)M_{\tau_1}(Y_1)M_{\tau_2}(Y_2) = e^T_{m-k} \otimes I_n$. This implies that $(e^T_{m-k} \otimes I_n)L_0 = (e^T_{m-k} \otimes I_n)M_{\sigma_1}(X_1)M^P_\sigma M_{\sigma_2}(X_2)$. Now (3.34) follows from Proposition 3.5.1 by considering $\alpha := (\sigma_1, \sigma, \sigma_2)$ and $\mathcal{X} := (X_1, P, X_2)$, where P denotes the trivial matrix assignment for σ.

Case-II: Suppose that $k \in -\tau$. Then there are two cases.

(a) Suppose that $k + 1 \in \sigma$ (i.e., $-k$ is an end point of τ). Since τ_j, $j = 1, 2$, does not contain the end points of τ, we have $k, k + 1 \notin -(\tau_1, \tau_2)$. Now by following the similar arguments as given in Case-I we have (3.34).

(b) Suppose that $-(k + 1) \in \tau$. Then, since $(\sigma, -\tau)$ is a permutation of $\{0 : m\}$, we have $k, k + 1 \notin (\sigma_1, \sigma, \sigma_2)$. Hence by (3.27), we have $(e^T_{m-k} \otimes I_n)M_{\sigma_1}(X_1)M^P_\sigma M_{\sigma_2}(X_2) = e^T_{m-k} \otimes I_n$. Consequently, we have $(e^T_{m-k} \otimes I_n)L_0 = (e^T_{m-k} \otimes I_n)M_{\tau_1}(Y_1)M_{\tau_2}(Y_2)$. Now

(3.34) follows from Proposition 3.5.2 by considering $\beta := (\tau_1, \tau_2)$ and $\mathcal{X} := (Y_1, Y_2)$. Hence L_0 is a block penta-diagonal matrix. This completes the proof. $\square$

For constructing block tridiagonal EGFPRs of $P(\lambda)$ we need the following results.

Proposition 3.5.7. *Let α be an index tuple containing indices from $\{0 : m\}$ such that α satisfies the SIP. Suppose that $c_j(\alpha) \geq 1$ or $i_j(\alpha) \geq 1$ for some $1 \leq j \leq m-1$. Let $\mathcal{X}$ be any arbitrary nonsingular matrix assignment for α. Then $M_\alpha(\mathcal{X})$ is not a block-tridiagonal matrix.*

Similarly, let β be an index tuple containing indices from $\{-m : -0\}$ such that β satisfies the SIP. Suppose that $c_{-j}(\beta) \geq 1$ or $i_{-j}(\beta) \geq 1$ for some $1 \leq j \leq m-1$. Let $\mathcal{Y}$ be any arbitrary nonsingular matrix assignment for β. Then $M_\beta(\mathcal{Y})$ is not a block-tridiagonal matrix.

Proof. The proof is similar to those of Proposition 3.5.3 and Proposition 3.5.4. $\square$

The following theorem characterizes all block tridiagonal EGFPRs of $P(\lambda)$. Note that, for an index tuple α containing nonnegative indices and any index t, we have $c_t(\alpha) = 0$ and $i_t(\alpha) = 0$ imply that $t \in \alpha$ but $t-1, t+1 \notin \alpha$. Similar thing holds for index tuple containing negative indices.

Theorem 3.5.8. *Let $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M_\tau^P - M_\sigma^P)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ be an EGFPR of $P(\lambda)$. Then $L(\lambda)$ is block tridiagonal if and only if $c_t(\sigma_1, \sigma, \sigma_2) = 0$, $i_t(\sigma_1, \sigma, \sigma_2) = 0$, $c_{-t}(\tau_1, \tau, \tau_2) = 0$ and $i_{-t}(\tau_1, \tau, \tau_2) = 0$ for any index $1 \leq t \leq m-1$.*

Proof. ($\Longrightarrow$) The forward implication follows from Proposition 3.5.7.

($\Longleftarrow$) Define $\mathcal{A} := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)$ and $\mathcal{B} := M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$. Then we have $L(\lambda) =: \lambda L_1 - L_0$, where $L_1 := \mathcal{A}M_\tau^P\mathcal{B}$ and $L_0 := \mathcal{A}M_\sigma^P\mathcal{B}$. Note that $L(\lambda)$ is a block-tridiagonal pencil if and only if both L_0 and L_1 are block-tridiagonal matrices. We only proof that L_0 is a block tridiagonal matrix. Similarly, L_1 is block tridiagonal.

It is given that, for any index $1 \leq t \leq m-1$, if $t \in \sigma$ then we have $c_t(\sigma_1, \sigma, \sigma_2) = 0$ and $i_t(\sigma_1, \sigma, \sigma_2) = 0$. This implies that, for $1 \leq j \leq m-2$, if $j \in \sigma$ then $j+1 \notin \sigma$. Moreover, we have $\sigma_j = (0)$ or $\sigma_j = \emptyset$, $j = 1, 2$. Since $M_0(Z) = \operatorname{diag}(I_{(m-1)n}, Z)$, we have $M_{\sigma_j}(X_j) = \operatorname{diag}(I_{(m-1)n}, X_j)$ or $M_{\sigma_j}(X_j) = I_{mn}$ for $j = 1, 2$. Further, given that $c_{-p}(\tau_1, \tau, \tau_2) = 0$ and $i_{-p}(\tau_1, \tau, \tau_2) = 0$ for any index $-(m-1) \leq -p \leq -1$ and $-p \in \tau$. This implies that $\tau_j = (-m)$ or $\tau_j = \emptyset$, $j = 1, 2$. Since $M_{-m}(Z) = \operatorname{diag}(Z, I_{(m-1)n})$, we have $M_{\tau_j}(Y_j) = \operatorname{diag}(Y_j, I_{(m-1)n})$ or $M_{\tau_j}(Y_j) = I_{mn}$, $j = 1, 2$. Consequently, $\mathcal{A}$ and $\mathcal{B}$ are block diagonal matrices. Hence L_0 is block tridiagonal if and only if M_σ^P is block tridiagonal. Since, $|j-k| \geq 2$ for distinct $j, k \in \sigma$ with $1 \leq j, k \leq m-1$, it is clear from the Fiedler matrices M_ℓ^P, $\ell = 0 : m$, that M_σ^P is tridiagonal. This completes the proof. $\square$

4

Structured EGFPRs

In this chapter, we utilize EGFPRs to construct large families of structure-preserving strong linearizations of $P(\lambda)$ when $P(\lambda)$ is either Hermitian or palindromic. We show that EGFPRs allow construction of structured strong linearizations with additional properties such as low bandwidth (block tridiagonal and block penta-diagonal) pencils. We show that when $P(\lambda)$ is symmetric with degree greater than or equal to 4 (resp., 8), there does not exist a symmetric GFPR which is block tridiagonal (resp., block penta-diagonal). We show that the family of EGFPRs has no such shortcoming and enables us to construct symmetric block tridiagonal and symmetric block penta-diagonal EGFPRs when $P(\lambda)$ is symmetric. Moreover, for a Hermitian matrix polynomials $P(\lambda)$, EGFPRs provide us with a large class of Hermitian strong linearizations which also preserve the sign characteristic of $P(\lambda)$. We mention that the palindromic linearizations that we construct in this chapter contain infinite number of pencils in contrast to the palindromic linearizations that have been constructed by using GFPs in [27] and FPRs in [19]. We also mention that when $P(\lambda)$ is palindromic with degree greater than or equal to 5, there exist only three palindromic linearizations which is anti-block tridiagonal. On the other hand, the family of EGFPRs enables us to construct infinite number of anti-block tridiagonal palindromic linearizations of $P(\lambda)$ when $P(\lambda)$ is palindromic. Moreover, we characterize all anti-block tridiagonal palindromic linearizations that can be constructed by utilizing EGFPRs of $P(\lambda)$ when $P(\lambda)$ is palindromic.

4.1 Symmetric and Hermitian EGFPRs

A matrix polynomial $P(\lambda)$ is said to be symmetric (resp., Hermitian) if $P(\lambda)^T = P(\lambda)$ (resp., $P(\lambda)^* = P(\overline{\lambda})$, where $*$ denotes the conjugate transpose). In this section, we will construct symmetric EGFPRs for symmetric $P(\lambda)$. We mention that all the results of this section hold for Hermitian $P(\lambda)$ as well. Recall that $P(\lambda) := \sum_{j=0}^{m} \lambda^j A_j$. Consider the elementary matrices $M_{\pm i}(X)$ and the Fiedler matrices $M^P_{\pm i}$ associated with $P(\lambda)$ as given in Chapter 2.

We now extend the concept of admissible tuple, symmetric complement and index tuple in canonical form of $\{0 : h\}$, $h \geq 0$, to any $\{a : b\}$, where $0 \leq a \leq b$.

Definition 4.1.1 (Admissible tuple)**.** *Let $0 \leq a \leq b$ be integers. We say that $\boldsymbol{w}$ is an admissible tuple of $\{a : b\}$ if $\boldsymbol{w}$ is a permutation of $\{a : b\}$ and*

$$csf(\boldsymbol{w}) = \big(b-1 : b, b-3 : b-2, \ldots, a+p+1 : a+p+2, a : a+p\big) \tag{4.1}$$

for some $0 \leq p \leq b-a$. We call p the index of $\boldsymbol{w}$ and denote it by $Ind(\boldsymbol{w})$.

Example 4.1.2. *Let $a = 2$, $b = 6$, and $\boldsymbol{w}_1 = (5 : 6, 2 : 4)$, $\boldsymbol{w}_2 = (5 : 6, 3 : 4, 2)$. Then $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are admissible tuples of $\{2 : 6\}$ with $Ind(\boldsymbol{w}_1) = 2$ and $Ind(\boldsymbol{w}_2) = 0$.* ■

Note that $\mathbf{w}$ is an admissible tuple of $\{a : b\}$, $0 \leq a \leq b$, with index p if and only if $-a + \mathbf{w}$ is an admissible tuple of $\{0 : b-a\}$ with index p. For simplicity, we always consider an admissible tuple of the form (4.1).

Definition 4.1.3 (Simple admissible tuple)**.** *Let $\boldsymbol{w}$ be an admissible tuple of $\{a : b\}$, $0 \leq a \leq b$. Then $\boldsymbol{w}$ is said to be the simple admissible tuple of $\{a : b\}$ if $Ind(\boldsymbol{w}) = 0$ or $Ind(\boldsymbol{w}) = 1$. Equivalently, $\boldsymbol{w}$ is said to be the simple admissible tuple of $\{a : b\}$ if*

- $\boldsymbol{w} := \big(b-1 : b, \ldots, a+1 : a+2, a\big)$ *when $\{a : b\}$ contains odd number of elements,*
- $\boldsymbol{w} := \big(b-1 : b, \ldots, a+2 : a+3, a : a+1\big)$ *when $\{a : b\}$ contains even number of elements.*

Remark 4.1.4. *It clearly follows that, for $0 \leq a \leq b$, there exist a unique simple admissible tuple of $\{a : b\}$.*

Although we have defined any admissible tuple of $\{a : b\}$, $0 \leq a \leq b$, we will only use the simple admissible tuple in the rest of this chapter. The symmetric complement of any admissible tuple is defined as follows.

Definition 4.1.5. *Let $0 \leq a \leq b$ be integers. Then the reversal complement of $(a : b)$, denoted by $(a : b)_{rev_c}$, is defined as $(a : b)_{rev_c} := (a : b-1, a : b-2, \ldots, a : a+1, a)$. Let $\boldsymbol{w}$ be an admissible tuple of $\{a : b\}$ with index p. Then the symmetric complement of $\boldsymbol{w}$, denoted by $\mathbf{c}_{\boldsymbol{w}}$, is defined as follows:*

$$\boldsymbol{c}_w := \begin{cases} \big(b-1, b-3, \ldots, a+p+3, a+p+1, (a : a+p)_{rev_c}\big) & \text{if } p \geq 1, \\ \big(b-1, b-3, \ldots, a+1\big) & \text{if } p = 0 \text{ and } b > a, \\ \emptyset & \text{if } b = a. \end{cases}$$

Remark 4.1.6. *Let $\boldsymbol{w}$ be the simple admissible tuple of $\{a : b\}$, $0 \leq a \leq b$, and $\boldsymbol{c}_w$ be the symmetric complement of $\mathbf{w}$. Then by Definition 4.1.3 and Definition 4.1.5, we have*

$$\boldsymbol{c}_w = \begin{cases} \big(b-1, b-3, \ldots, a+1\big) & \text{if } \{a : b\} \text{ contains odd number of elements,} \\ \big(b-1, b-3, \ldots, a+2, a\big) & \text{if } \{a : b\} \text{ contains even number of elements.} \end{cases}$$

In Example 4.1.2, the symmetric complements of $\mathbf{w}_2$ is given by $(5, 3)$.

Remark 4.1.7. *Let $\boldsymbol{w}$ be the simple admissible tuple of $\{a : b\}$, $0 \leq a \leq b$, and $c_{\boldsymbol{w}}$ be the symmetric complement of $\mathbf{w}$. Then it follows from Remark 4.1.6 that $a \in \boldsymbol{c}_w$ if and only if $\{a : b\}$ contains even number of elements. Note that b never belongs to $\boldsymbol{c}_w$. Hence if $\{a : b\}$ contains odd number of elements then $a, b \notin \boldsymbol{c}_w$.*

Definition 4.1.8. *Let $0 \leq a \leq b$ be integers. An index tuple $\boldsymbol{t}$ is said to be in canonical form of $\{a : b\}$ if $\boldsymbol{t}$ is of the form*

$$\left(a_1 : b-2, a_2 : b-4, \ldots, a_{\lfloor \frac{b-a}{2} \rfloor} : b - 2\lfloor \tfrac{b-a}{2} \rfloor\right)$$

with $a \leq a_i$, $i = 1 : \lfloor \frac{b-a}{2} \rfloor$.

For example, let $a = 2$ and $b = 8$. Then $(2 : 6, 3 : 4, 2 : 2)$ with $a_1 = 2, a_2 = 3, a_3 = 2$, and $(3 : 6, 2 : 2)$ with $a_1 = 3, a_2 > 4, a_3 = 2$, are index tuples in canonical form of $\{2 : 8\}$.

Recall that an index tuple α is said to be symmetric index tuple if $\alpha \sim rev(\alpha)$. The following results will be useful for constructing symmetric EGFPRs.

Lemma 4.1.9 ([13], Lemma 3.11)**.** *Let $h \geq 0$. Let $\boldsymbol{w}$ be any admissible tuple of $\{0 : h\}$ and $\boldsymbol{c}_w$ be the symmetric complement of $\mathbf{w}$. Then $(\boldsymbol{w}, \boldsymbol{c}_w)$ and $\boldsymbol{c}_w$ are symmetric and satisfy the SIP.*

The following result generalizes Lemma 4.1.9.

Lemma 4.1.10. *Let $0 \leq a \leq b$ be integers. Let $\boldsymbol{w}$ be any admissible tuple of $\{a : b\}$ and $\boldsymbol{c}_w$ be the symmetric complement of $\mathbf{w}$. Then $(\boldsymbol{w}, \boldsymbol{c}_w)$ and $\boldsymbol{c}_w$ are symmetric and satisfy the SIP.*

Proof. An index tuple $\mathbf{t}$ containing indices from $\{a : b\}$ is symmetric (resp., satisfies the SIP) if and only if $-a + \mathbf{t}$ is symmetric (resp., satisfies the SIP). Let $\widetilde{\mathbf{w}} := -a + \mathbf{w}$ and $\widetilde{\mathbf{c}}_w := -a + \mathbf{c}_w$. Then $\widetilde{\mathbf{w}}$ is an admissible tuple of $\{0 : b - a\}$ and $\widetilde{\mathbf{c}}_w$ is the symmetric complement of $\widetilde{\mathbf{w}}$. Hence by Lemma 4.1.9, $(\widetilde{\mathbf{w}}, \widetilde{\mathbf{c}}_w)$ and $\widetilde{\mathbf{c}}_w$ are symmetric and satisfy the SIP. Consequently, $(\mathbf{w}, \mathbf{c}_w)$ and $\mathbf{c}_w$ are symmetric and satisfy the SIP. □

Recall that the basic building blocks of an EGFPR is a GF pencil and the repetition of indices in EGFPR depends on the indices defining the GF pencil. Next, we define a special kind of GF pencil and then by repeating indices we construct symmetric linearizations for a symmetric $P(\lambda)$.

Definition 4.1.11. *Let $k \geq 1$ be odd. Choose h_i, $i = 1 : k$, with $0 \leq h_1 < h_2 < h_3 < \cdots < h_{k-1} < h_k \leq m-1$ such that the string $(h_{j-1}+1 : h_j)$, $j = 2 : k$, contains odd number of elements. Set $h_{k+1} = m$. Define $H_1 := \{0 : h_1\}$ and $H_j := \{h_{j-1}+1 : h_j\}$ for $j = 2 : k+1$. Let $\boldsymbol{w}_j$ be the simple admissible tuple of H_j and $\boldsymbol{c}_{w_j}$ be the symmetric complement of $\boldsymbol{w}_j$ for $j = 1 : k+1$. Define $\boldsymbol{v}_j := -(h_j + h_{j-1} + 1) + \boldsymbol{w}_j$ and $\boldsymbol{c}_{v_j} := -(h_j + h_{j-1} + 1) + \boldsymbol{c}_{w_j}$ for $j = 2, 4, \ldots, k-1, k+1$. Let $\boldsymbol{t}_{w_j}$, $j = 1 : k+1$, be index tuple in canonical form of H_j and define $\boldsymbol{t}_{v_j} := -(h_j + h_{j-1} + 1) + \boldsymbol{t}_{w_j}$, $j = 2, 4, \ldots, k-1, k+1$. Define*

$$\boldsymbol{w} := (\boldsymbol{w}_1, \boldsymbol{w}_3, \ldots, \boldsymbol{w}_k), \ \ \boldsymbol{c}_w := (\boldsymbol{c}_{w_1}, \boldsymbol{c}_{w_3}, \ldots, \boldsymbol{c}_{w_k}), \ \ \boldsymbol{t}_w := (\boldsymbol{t}_{w_1}, \boldsymbol{t}_{w_3}, \ldots, \boldsymbol{t}_{w_k}) \text{ and}$$

$$\boldsymbol{v} := (\boldsymbol{v}_2, \boldsymbol{v}_4, \ldots, \boldsymbol{v}_{k+1}), \ \ \boldsymbol{c}_v := (\boldsymbol{c}_{v_2}, \boldsymbol{c}_{v_4}, \ldots, \boldsymbol{c}_{v_{k+1}}), \ \ \boldsymbol{t}_v := (\boldsymbol{t}_{v_2}, \boldsymbol{t}_{v_4}, \ldots, \boldsymbol{t}_{v_{k+1}}).$$

In Definition 4.1.11 it is clear that $(\mathbf{w}, -\mathbf{v})$ is a permutation of $\{0 : m\}$ with $0 \in \mathbf{w}$ and $-m \in \mathbf{v}$. Further, $\mathbf{w}$, $\mathbf{c}_w$, $\mathbf{v}$ and $\mathbf{c}_v$ are uniquely determined once h_j, $j = 1 : k$, are fixed.

Remark 4.1.12. *In Definition 4.1.11, we have $H_j := \{h_{j-1}+1 : h_j\}$, $j = 2 : k$, contains odd number of elements. Hence it follows from Remark 4.1.7 that $h_{j-1}+1, h_j \notin \boldsymbol{c}_{w_j}$ for $j = 2 : k$. This implies that $\boldsymbol{w}_i$ commutes with $\boldsymbol{c}_{w_j}$ for $i, j \in \{1 : k\}$ and $i \neq j$. Similarly, $\boldsymbol{v}_i$ commutes with $\boldsymbol{c}_{v_j}$ for $i \neq j$.*

Remark 4.1.13. *Let $\boldsymbol{h} := \{h_1, h_2, \ldots, h_k\}$, $\boldsymbol{w}$, $\boldsymbol{c}_w$, $\boldsymbol{t}_w$, $\boldsymbol{v}$, $\boldsymbol{c}_v$ and $\boldsymbol{t}_v$ be as in Definition 4.1.11. Then we have the following.*

(1) By Remark 4.1.7, it follows that $0 \in \boldsymbol{c}_w \Leftrightarrow \{0 : h_1\}$ contains even number of elements $\Leftrightarrow h_1$ is odd. In other words, $0 \notin \boldsymbol{c}_w \Leftrightarrow \{0 : h_1\}$ contains odd number of elements $\Leftrightarrow h_1$ is even.

(2) Similarly, by Remark 4.1.7, $-m \in \boldsymbol{c}_v \Leftrightarrow \{h_k + 1 : m\}$ contains even number of elements $\Leftrightarrow$ (a) $h_k + 1$ is odd and m is even, or (b) $h_k + 1$ is even and m is odd. Since k is odd and the strings $(h_{j-1}+1 : h_j)$, $j = 2 : k$, contain odd number of elements, it follows that $h_k + 1$ is odd (resp., even) when h_1 is even (resp., odd). Consequently, we have

- *$-m \in \boldsymbol{c}_v \Leftrightarrow$ (a) both h_1 and m are even, or (b) both h_1 and m are odd,*
- *$-m \notin \boldsymbol{c}_v \Leftrightarrow$ (a) h_1 is even and m is odd, or (b) h_1 is odd and m is even.*

The following result will be useful for constructing symmetric EGFPRs.

Lemma 4.1.14. *Let* $\boldsymbol{w}$, $\boldsymbol{c}_w$, $\boldsymbol{t}_w$, $\boldsymbol{v}$, $\boldsymbol{c}_v$ *and* $\boldsymbol{t}_v$ *be as given in Definition 4.1.11. Then:*

(a) $(\boldsymbol{w}, \boldsymbol{c}_w)$ *and* $\boldsymbol{c}_w$ *are symmetric. Further,* $\boldsymbol{c}_w$ *commutes with* $\boldsymbol{v}$, $\boldsymbol{c}_v$ *and* $\boldsymbol{t}_v$*.*

(b) $(\boldsymbol{v}, \boldsymbol{c}_v)$ *and* $\boldsymbol{c}_v$ *are symmetric. Further,* $\boldsymbol{c}_v$ *commutes with* $\boldsymbol{w}$, $\boldsymbol{c}_w$ *and* $\boldsymbol{t}_w$*.*

Proof. (a) By Lemma 4.1.10, $(\mathbf{w}_j, \mathbf{c}_{w_j})$ and $\mathbf{c}_{w_j}$ are symmetric for all $j = 1 : k+1$. Thus $(\mathbf{w}, \mathbf{c}_w)$ is symmetric if $\mathbf{w}_i$ and $\mathbf{w}_j$ commute for $|i - j| > 1$ and $\mathbf{w}_i$ and $\mathbf{c}_{w_j}$ commute for $|i - j| > 1$. By Definition 4.1.11, it is easy to see that $\mathbf{w}_i$ and $\mathbf{w}_j$ commute, i.e., $(\mathbf{w}_i, \mathbf{w}_j) \sim (\mathbf{w}_j, \mathbf{w}_i)$ if $|i - j| > 1$. This implies that $(\mathbf{c}_{w_i}, \mathbf{c}_{w_j}) \sim (\mathbf{c}_{w_j}, \mathbf{c}_{w_i})$ if $|i - j| > 1$ since $\mathbf{c}_j \subset \mathbf{w}_j$, $j = 1 : k+1$. Moreover, $(\mathbf{w}_i, \mathbf{c}_{w_j}) \sim (\mathbf{c}_{w_j}, \mathbf{w}_i)$ if $|i - j| > 1$. Thus $(\mathbf{w}, \mathbf{c}_w)$ and $\mathbf{c}_w$ are symmetric.

Note that $(\mathbf{w}, -\mathbf{v})$ is a permutation of $\{0 : m\}$ which implies that $\mathbf{v}$ is a permutation of $-\{\{0 : m\} \setminus \mathbf{w}\}$. By Remark 4.1.12, it follows that if $i \in \mathbf{c}_w$ and $j \in \mathbf{v}$ then $||i| - |j|| > 1$. Hence $\mathbf{c}_w$ commutes with $\mathbf{v}$. Further, since $\mathbf{c}_v \subset \mathbf{v}$ and $\mathbf{t}_v \subset \mathbf{v}$, we have $\mathbf{c}_w$ commutes with both $\mathbf{c}_v$ and $\mathbf{t}_v$. This completes the proof of (a). The proof is similar for part (b). ☐

Definition 4.1.15. *Let* $\boldsymbol{h} := \{h_1, h_2, \ldots, h_k\}$, $\boldsymbol{w}$, $\boldsymbol{c}_w$, $\boldsymbol{t}_w$, $\boldsymbol{v}$, $\boldsymbol{c}_v$ *and* $\boldsymbol{t}_v$ *be as in Definition 4.1.11. Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be any nonsingular matrix assignments for* $\boldsymbol{t}_w$ *and* $\boldsymbol{t}_v$*, respectively. Define the pencil* $L(\lambda)$ *associated with* $\boldsymbol{w}$, $\boldsymbol{c}_w$, $\boldsymbol{t}_w$, $\boldsymbol{v}$, $\boldsymbol{c}_v$, $\boldsymbol{t}_v$, $\mathcal{X}$ *and* $\mathcal{Y}$ *by*

$$L(\lambda) := M_{\boldsymbol{t}_v}(\mathcal{Y}) M_{\boldsymbol{t}_w}(\mathcal{X}) \big(\lambda M^P_{\boldsymbol{v}} - M^P_{\boldsymbol{w}}\big) M^P_{\boldsymbol{c}_w} M_{rev(\boldsymbol{t}_w)}(rev(\mathcal{X})) M^P_{\boldsymbol{c}_v} M_{rev(\boldsymbol{t}_v)}(rev(\mathcal{Y})). \tag{4.2}$$

We denote $L(\lambda)$ *by* $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ *and by* $L_P(\boldsymbol{h}, \emptyset, \emptyset)$ *when* $\boldsymbol{t}_w = \emptyset = \boldsymbol{t}_v$*.*

Caution: As already mentioned in Definition 4.1.15 we always consider nonsingular matrix assignments $\mathcal{X}$ and $\mathcal{Y}$ in $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$.

Remark 4.1.16. *The pencil* $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ *in Definition 4.1.15 is uniquely determined by* $\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}$ *and* $\mathcal{Y}$*.*

It is clear from Definition 4.1.11 that $\big(\mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w)\big)$ and $\big(\mathbf{t}_v, \mathbf{v}, \mathbf{c}_v, rev(\mathbf{t}_v)\big)$ satisfy the SIP. Thus $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ is an EGFPR of $P(\lambda)$. Further, since $m \notin \mathbf{w}$ (resp, $-0 \notin \mathbf{v}$), by Proposition 3.1.4, we have $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ is an operation free EGFPR. Moreover, in the following result we will show that $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ is symmetric when $P(\lambda)$ is symmetric and all the matrices belonging to $\mathcal{X}$ and $\mathcal{Y}$ are symmetric.

Recall that a matrix assignment $\mathcal{X}$ is said to be a symmetric matrix assignment if all the matrices belonging to $\mathcal{X}$ are symmetric. Also, recall that an index tuple α is said to be symmetric if $\alpha \sim rev(\alpha)$.

Theorem 4.1.17. *Consider the EGFPR $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$, where $P(\lambda)$ is symmetric and $\mathcal{X}$ and $\mathcal{Y}$ are symmetric matrix assignments. Then $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is symmetric. Further, assume that $0 \notin \boldsymbol{c}_w$ when A_0 is singular and $-m \notin \boldsymbol{c}_v$ when A_m is singular. Then $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is a symmetric strong linearization of $P(\lambda)$.*

Proof. Since $P(\lambda)$ is symmetric and $\mathcal{X}$ and $\mathcal{Y}$ are symmetric matrix assignments, it is easy to see that $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ is symmetric if $\big(\mathbf{t}_v, \mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w), \mathbf{c}_v, rev(\mathbf{t}_v)\big)$ $=: \alpha$ and $\big(\mathbf{t}_v, \mathbf{t}_w, \mathbf{v}, \mathbf{c}_w, rev(\mathbf{t}_w), \mathbf{c}_v, rev(\mathbf{t}_v)\big) =: \beta$ are symmetric. By Lemma 4.1.14, $\mathbf{c}_v$ commutes with $\mathbf{t}_w$. Thus

$$\big(\mathbf{t}_v, \mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, rev(\mathbf{t}_w), \mathbf{c}_v, rev(\mathbf{t}_v)\big) \sim \big(\mathbf{t}_v, \mathbf{t}_w, \mathbf{w}, \mathbf{c}_w, \mathbf{c}_v, rev(\mathbf{t}_w), rev(\mathbf{t}_v)\big)$$

and

$$\big(\mathbf{t}_v, \mathbf{t}_w, \mathbf{v}, \mathbf{c}_w, rev(\mathbf{t}_w), \mathbf{c}_v, rev(\mathbf{t}_v)\big) \sim \big(\mathbf{t}_v, \mathbf{t}_w, \mathbf{v}, \mathbf{c}_w, \mathbf{c}_v, rev(\mathbf{t}_w), rev(\mathbf{t}_v)\big).$$

Hence α (resp., β) is symmetric if and only if $(\mathbf{w}, \mathbf{c}_w, \mathbf{c}_v)$ (resp., $(\mathbf{v}, \mathbf{c}_w, \mathbf{c}_v)$) is symmetric. By Lemma 4.1.14, $(\mathbf{w}, \mathbf{c}_w)$, $\mathbf{c}_w$, and $\mathbf{c}_v$ are symmetric. Further, $\mathbf{c}_v$ commutes with both $\mathbf{w}$ and $\mathbf{c}_w$. Hence $rev(\mathbf{w}, \mathbf{c}_w, \mathbf{c}_v) = (rev(\mathbf{c}_v), rev(\mathbf{w}, \mathbf{c}_w)) \sim (\mathbf{c}_v, \mathbf{w}, \mathbf{c}_w) \sim (\mathbf{w}, \mathbf{c}_w, \mathbf{c}_v)$, i.e., $(\mathbf{w}, \mathbf{c}_w, \mathbf{c}_v)$ is symmetric and so is α.

Similarly, by Lemma 4.1.14, $(\mathbf{v}, \mathbf{c}_v)$, $\mathbf{c}_v$, and $\mathbf{c}_w$ are symmetric. Further, $\mathbf{c}_w$ commutes with both $\mathbf{v}$ and $\mathbf{c}_v$. Hence $(\mathbf{v}, \mathbf{c}_w, \mathbf{c}_v) \sim rev(\mathbf{v}, \mathbf{c}_w, \mathbf{c}_v)$, i.e., $(\mathbf{v}, \mathbf{c}_w, \mathbf{c}_v)$ is symmetric and so is β. Hence $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ is symmetric.

Now, assume that $0 \notin \mathbf{c}_w$ when A_0 is singular and $-m \notin \mathbf{c}_v$ when A_m is singular. Then by taking $\sigma := \mathbf{w}$, $\tau := \mathbf{v}$, $\sigma_1 := \mathbf{t}_w$, $\sigma_2 := (\mathbf{c}_w, rev(\mathbf{t}_w))$, $\tau_1 := \mathbf{t}_v$ and $\tau_2 := (\mathbf{c}_v, rev(\mathbf{t}_v))$, it follows from Theorem 3.1.5 that $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ is a strong linearization of $P(\lambda)$. □

Remark 4.1.18. *Note that $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is a GFPR of $P(\lambda)$ when $\boldsymbol{h} = \{h_1\}$. Hence the class of symmetric GFPRs that has been constructed in [17] is a subclass of the class of symmetric EGFPRs given in Theorem 4.1.17.*

We now illustrate our construction of symmetric EGFPRs by considering an example.

Example 4.1.19. *Let $P(\lambda) := \sum_{i=0}^{8} \lambda^i A_i$. Consider $\boldsymbol{h} := \{h_1, h_2, h_3\}$, where $h_1 = 1$, $h_2 = 4$ and $h_3 = 7$. Let $\boldsymbol{t}_w = (5)$ and $\boldsymbol{t}_v = (-4)$, and let X and Y be any matrix assignments for $\boldsymbol{t}_w$ and $\boldsymbol{t}_v$, respectively. Then $L(\lambda) := L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y}) = M_{-4}(Y)M_5(X)\big(\lambda M^P_{(-8,-3:-2,-4)} - M^P_{(0:1,6:7,5)}\big)M^P_{(0,6)}M_5(X)M^P_{-3}M_{-4}(Y)$ is a block symmetric EGFPR of $P(\lambda)$ which is given by*

$$L(\lambda)=\begin{bmatrix} \lambda A_8+A_7 & A_6 & -X & 0 & -I_n & 0 & 0 & 0\\ A_6 & -\lambda A_6+A_5 & \lambda X-I_n & 0 & \lambda I_n & 0 & 0 & 0\\ -X & \lambda X-I_n & 0 & 0 & 0 & \lambda I_n & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -I_n & \lambda I_n & 0\\ -I_n & \lambda I_n & 0 & 0 & 0 & -Y & \lambda Y & 0\\ 0 & 0 & \lambda I_n & -I_n & -Y & \lambda A_4-A_3 & \lambda A_3 & 0\\ 0 & 0 & 0 & \lambda I_n & \lambda Y & \lambda A_3 & \lambda A_2+A_1 & A_0\\ 0 & 0 & 0 & 0 & 0 & 0 & A_0 & -\lambda A_0 \end{bmatrix}$$

Note that $L(\lambda)$ is not a GFPR. Moreover, if X and Y are symmetric matrices then $L(\lambda)$ is symmetric when $P(\lambda)$ is symmetric. ■

Remark 4.1.20. *Let $P(\lambda)$ and $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ be as given in Theorem 4.1.17. Since $\mathcal{X}$ and $\mathcal{Y}$ are nonsingular matrix assignments, it follows that $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is a symmetric strong linearization of $P(\lambda)$ if and only if $M^P_{\boldsymbol{c}_w}$ and $M^P_{\boldsymbol{c}_v}$ are nonsingular. Hence we have the following.*

- *If A_0 and A_m are nonsingular, then $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is a symmetric strong linearization of $P(\lambda)$ irrespective of m is even or odd.*
- *Assume that A_0 singular and A_m is nonsingular. It follows from Remark 4.1.13 that $0 \notin \boldsymbol{c}_w$ when h_1 is even. Consequently, $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is a symmetric strong linearization of $P(\lambda)$ when h_1 is even irrespective of m is even or odd.*
- *Assume that A_0 nonsingular and A_m is singular. Then $M^P_{\boldsymbol{c}_w}$ is always nonsingular. Further, $M^P_{\boldsymbol{c}_v}$ is nonsingular if and only if $-m \notin \boldsymbol{c}_v$. Consequently, it follows from Remark 4.1.13 that $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is a symmetric strong linearization of $P(\lambda)$ when either (a) h_1 is even and m is odd, or (b) h_1 is odd and m is even.*
- *Assume that A_0 and A_m are singular. Then $M^P_{\boldsymbol{c}_w}$ (resp., $M^P_{\boldsymbol{c}_v}$) is nonsingular if and only if $0 \notin \boldsymbol{c}_w$ (resp., $-m \notin \boldsymbol{c}_v$). Consequently, it follows from Remark 4.1.13 that $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is a symmetric strong linearization of $P(\lambda)$ when h_1 is even and m is odd. In other words, for a singular symmetric even degree $P(\lambda)$, Theorem 4.1.17 does not produce any symmetric linearization of $P(\lambda)$.*

Next, we present low bandwidth banded symmetric EGFPRs. We mention that there does not exist any block penta-diagonal symmetric GFPR for symmetric $P(\lambda)$ having degree $m \geq 8$. Indeed, any symmetric GFPR must be of the form $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ with $\mathbf{h} = \{h_1\}$ and symmetric matrix assignments $\mathcal{X}$ and $\mathcal{Y}$ [17]. Let $k \geq 4$ be an integer. Let $\mathbf{w}$ be the admissible tuple of $\{0:k\}$ and $\mathbf{r}_w$ be the symmetric complement of $\mathbf{w}$. Then $c_1(\mathbf{w},\mathbf{r}_w) \geq 2$ if k is even and $c_2(\mathbf{w},\mathbf{r}_w) \geq 2$ if k is odd. Now, suppose that $P(\lambda)$ has degree $m \geq 8$ and $0 \leq h_1 \leq m-1$. Then either $h_1 \geq 4$ or $m - h_1 - 1 \geq 4$. Hence it follows from Theorem 3.5.6 that the construction given in [17] does not produce any block penta-diagonal symmetric GFPR. Further, we mention that $(\lambda M^P_{(-5:-4,-7:-6)} - M^P_{(2:3,0:1)})M^P_{(2,0)}M^P_{(-5,-7)}$ is the only block penta-diagonal symmetric GFPR of symmetric $P(\lambda)$ having degree 7. On the other hand, the family of EGFPRs has no such shortcoming and enables us to construct symmetric block penta-diagonal EGFPRs for symmetric $P(\lambda)$.

Theorem 4.1.21. *Let h_i, $i = 1:k+1$, be as in Definition 4.1.11. Suppose that $h_1 \leq 3$, $m-4 \leq h_k$, and the string $(h_{j-1}+1 : h_j)$, $j = 2:k$, contains at most 3 elements. Let $\boldsymbol{t}_w = (0)$ or $\boldsymbol{t}_w = \emptyset$, and $\boldsymbol{t}_v = (-m)$ or $\boldsymbol{t}_v = \emptyset$. Then the symmetric EGFPR $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ in Theorem 4.1.17 is block penta-diagonal.*

Proof. Note that $M_{-m}(Z) = \operatorname{diag}(Z,\ I_{(m-1)n})$ and $M_0(Z) = \operatorname{diag}(I_{(m-1)n},\ Z)$, where $Z \in \mathbb{C}^{n\times n}$ is a nonsingular matrix. Since $\mathbf{t}_w = (0)$ or $\mathbf{t}_w = \emptyset$ and $\mathbf{t}_v = (-m)$ or $\mathbf{t}_v = \emptyset$, it follows that $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ is block penta-diagonal if and only if $L_P(\mathbf{h}, \emptyset, \emptyset) = \big(\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}}\big)M^P_{\mathbf{c}_w}M^P_{\mathbf{c}_v}$ is block penta-diagonal. Setting $\lambda T_1 - T_0 := \lambda M^P_{\mathbf{v}}M^P_{\mathbf{c}_w}M^P_{\mathbf{c}_v} - M^P_{\mathbf{w}}M^P_{\mathbf{c}_w}M^P_{\mathbf{c}_v}$, we have $L_P(\mathbf{h}, \emptyset, \emptyset)$ is penta-diagonal if both T_0 and T_1 are penta-diagonal. We show that T_0 is block penta-diagonal. The proof is similar for T_1.

By Definition 4.1.11, we have $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_3, \ldots, \mathbf{w}_k)$, where $\mathbf{w}_1$ is a permutation of $\{0:h_1\}$ and $\mathbf{w}_j$, $j = 1, 3, \ldots, k$, is a permutation of $\{h_{j-1}+1 : h_j\}$. That is, $\mathbf{w}$ is a permutation of $\{0:h_1\} \cup \{h_2+1 : h_3\} \cup \cdots \cup \{h_{k-1}+1 : h_k\}$. This implies that $|s-t| \geq 2$ for $s \in \mathbf{w}_i$ and $t \in \mathbf{w}_j$ and $i \neq j$. Note that $\mathbf{c}_w = (\mathbf{c}_{w_1}, \mathbf{c}_{w_3}, \ldots, \mathbf{c}_{w_k})$, where $\mathbf{c}_{w_j}$, $j = 1, 3, \ldots, k$, is a symmetric complement of $\mathbf{w}_j$. It is given in Definition 4.1.11 that $(h_{j-1}+1 : h_j) =: H_j$ contains odd number of elements. Further, since H_j contains at most 3 elements, we have $\mathbf{w}_j = (h_{j-1}+1)$ if H_j contains only one element and $\mathbf{w}_j = \big(h_{j-1}+2, h_j, h_{j-1}+1\big)$ if H_j contains 3 elements. This implies that $\mathbf{c}_{w_j} = \emptyset$ if H_j contains only one element and $\mathbf{c}_{w_j} = \big(h_{j-1}+2\big)$ if H_j contains 3 elements, where $h_{j-1}+1 < h_{j-1}+2 < h_j$. Hence $\mathbf{c}_w$ does not contain any end points of $\mathbf{w}$. Similarly, $\mathbf{c}_v$ does not contain any end points of $\mathbf{v}$. Moreover, it is easily seen that $c_s(\mathbf{w}, \mathbf{c}_w) \leq 1$, $i_s(\mathbf{w}, \mathbf{c}_w) \leq 1$, $c_s(\mathbf{c}_v) \leq 1$ and $i_s(\mathbf{c}_v) \leq 1$ for all $s \in \{1 : m-1\}$. Hence by Theorem 3.5.6, T_0 is block penta-diagonal. Consequently, $L_P(\mathbf{h}, \emptyset, \emptyset)$ is block penta-diagonal. $\square$

Example 4.1.22 (Block penta-diagonal)**.** *Let $P(\lambda) := \sum_{i=0}^{8} \lambda^i A_i$. Let $\boldsymbol{h} := \{h_1, h_2, h_3\}$, where $h_1 = 2$, $h_2 = 3$ and $h_3 = 4$. Consider $\boldsymbol{t}_w = (0)$, $\boldsymbol{t}_w = \emptyset$ and a nonsingular matrix X. Then the EGFPR $L(\lambda) := L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y}) = M_0(X)\big(\lambda M^P_{(-6:-5,-8:-7,-3)} -$*

$M^P_{(1:2,0,4)})M^P_1 M_0(X)M^P_{(-6,-8)}$

$$= \begin{bmatrix} -A_8 & 0 & \lambda A_8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_n & \lambda I_n & 0 & 0 & 0 & 0 \\ \lambda A_8 & -I_n & \lambda A_7 - A_6 & \lambda A_6 & 0 & 0 & 0 & 0 \\ 0 & \lambda I_n & \lambda A_6 & \lambda A_5 + A_4 & -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n & 0 & \lambda I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda I_n & \lambda A_3 + A_2 & A_1 & -X \\ 0 & 0 & 0 & 0 & 0 & A_1 & -\lambda A_1 + A_0 & \lambda X \\ 0 & 0 & 0 & 0 & 0 & -X & \lambda X & 0 \end{bmatrix}$$

is a block symmetric and block penta-diagonal pencil. Moreover, if X is symmetric then $L(\lambda)$ is symmetric when $P(\lambda)$ is symmetric. Note that $L(\lambda)$ is not a GFPR of $P(\lambda)$. ■

Next, we show that there does not exist any block tridiagonal symmetric GFPR for symmetric $P(\lambda)$ having degree $m \geq 4$. Let $k \geq 2$ be an integer. Let $\mathbf{w}$ be the admissible tuple of $\{0 : k\}$ and $\mathbf{r}_w$ be the symmetric complement of $\mathbf{w}$. Then $c_1(\mathbf{w}, \mathbf{r}_w) \geq 1$ or $i_1(\mathbf{w}, \mathbf{r}_w) \geq 1$. Now, suppose that $P(\lambda)$ has degree $m \geq 4$ and $0 \leq h_1 \leq m - 1$. Then either $h_1 \geq 2$ or $m - h_1 - 1 \geq 2$. Hence it follows from Theorem 3.5.8 that the construction given in [17] does not produce any block tridiagonal symmetric GFPR. On the other hand, the family of EGFPRs has no such shortcoming and enables us to construct symmetric block tridiagonal EGFPRs for symmetric $P(\lambda)$. Indeed, we have the following result.

Theorem 4.1.23. *Let h_i, $i = 1 : k$, be as in Definition 4.1.11. Suppose that $h_1 \leq 1$, $m - 2 \leq h_k$, and $h_j = h_{j-1} + 1$ for $j = 2 : k$. Then $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y}) = L_P(\boldsymbol{h}, \emptyset, \emptyset)$ is a block tridiagonal symmetric EGFPR when $P(\lambda)$ is symmetric.*

Proof. For the given choices of h_j, it follows from Definition 4.1.11 that $\mathbf{t}_w = \emptyset = \mathbf{t}_v$. Hence $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y}) = L_P(\mathbf{h}, \emptyset, \emptyset)$.

By using Theorem 3.5.8, the rest of the proof is similar to that of Theorem 4.1.21. □

Example 4.1.24. *For symmetric $P(\lambda)$ having degree 4, $(\lambda M^P_{(-4,-3,-1)} - M^P_{(0,2)})M^P_{-4}$ and $(\lambda M^P_{(-4,-2,)} - M^P_{(0,1,3)})M^P_0$ are block tridiagonal symmetric EGFPRs of $P(\lambda)$.* ■

We end this section with a remark that it is possible to construct a few more symmetric EGFPRs for symmetric $P(\lambda)$ by slightly modifying the indices given in Definition 4.1.11. It is clear from Definition 4.1.11 that $(\mathbf{w}, -\mathbf{v})$ is a permutation of $\{0 : m\}$ and $0 \in \mathbf{w}$ and $-m \in \mathbf{v}$. By considering $0 \in -\mathbf{v}$ or $m \in \mathbf{w}$ we have a few more symmetric EGFPRs. For this purpose, we consider the following index tuples.

Remark 4.1.25. *Recall that* $\boldsymbol{w}, \boldsymbol{c}_w, \boldsymbol{v}, \boldsymbol{c}_v, \boldsymbol{t}_w$ *and* $\boldsymbol{t}_v$ *in Definition 4.1.11 are uniquely determined once* h_j *and* H_j, $j = 1 : k+1$, *are fixed. Hence we only describe the changes in* h_j *and* H_j, $j = 1 : k+1$, *and the indices* $\boldsymbol{w}, \boldsymbol{c}_w, \boldsymbol{t}_w, \boldsymbol{v}, \boldsymbol{c}_v$ *and* $\boldsymbol{t}_v$ *are defined accordingly.*

Type-A: *Let* h_j *and* H_j *be as in Definition 4.1.11 such that* $h_1 \geq 1$ *is odd. Define* $H_1 = \{1 : h_1\}$. *Replace* $\boldsymbol{v}$ *by* $\boldsymbol{v} \cup \{-0\}$.

Type-B: *Let* h_j *and* H_j *be as in Definition 4.1.11 such that* $h_k \leq m-2$ *and the string* $(h_k+1 : m-1)$ *contains odd number of elements. Define* $H_{k+1} = \{h_k+1 : m-1\}$. *Replace* $\boldsymbol{w}$ *by* $\boldsymbol{w} \cup \{m\}$.

Type-C: *Let* h_j *and* H_j *be as in Definition 4.1.11 with* H_1 *and* H_{k+1} *as in Type-A and Type-B, respectively. Replace* $\boldsymbol{v}$ *by* $\boldsymbol{v} \cup \{-0\}$ *and* $\boldsymbol{w}$ *by* $\boldsymbol{w} \cup \{m\}$.

Theorem 4.1.26. *Let* $\boldsymbol{w}, \boldsymbol{c}_w, \boldsymbol{t}_w, \boldsymbol{v}, \boldsymbol{c}_v$ *and* $\boldsymbol{t}_v$ *be Type-A (resp., Type-B and Type-C) index tuples as given in Remark 4.1.25. Then the EGFPR* $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ *is symmetric if* $P(\lambda)$ *is symmetric and the matrix assignments* $\mathcal{X}$ *for* $\boldsymbol{t}_w$ *and* $\mathcal{Y}$ *for* $\boldsymbol{t}_v$ *are symmetric.*

Proof. A verbatim proof of Theorem 4.1.17 gives the desired result. □

The following example is a symmetric EGFPR where $\mathbf{w}$ and $\mathbf{v}$ are Type-A index tuples as in Remark 4.1.25.

Example 4.1.27 (Type-A)**.** *Let* $P(\lambda) := \sum_{i=0}^{8} \lambda^i A_i$, *where* A_0 *is nonsingular. Consider* $\boldsymbol{h} := \{h_1, h_2, h_3\}$, *where* $h_1 = 3$, $h_2 = 6$ *and* $h_3 = 7$. *Let* $\boldsymbol{t}_w = (1)$ *and* $\boldsymbol{t}_v = (-6)$, *and let* X *and* Y *be any matrix assignments for* $\boldsymbol{t}_w$ *and* $\boldsymbol{t}_v$, *respectively. Then* $L(\lambda) = M_{-6}(Y)M_1(X)\big(\lambda M^P_{(-8,-5:-4,-6,-0)} - M^P_{(2:3,1,7)}\big)M^P_2 M_1(X) M^P_{-5} M_{-6}(Y) =$

$$\begin{bmatrix}
\lambda A_8 + A_7 & 0 & -I_n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_n & \lambda I_n & 0 & 0 & 0 \\
-I_n & 0 & 0 & \lambda I_n - X & \lambda X & 0 & 0 & 0 \\
0 & -I_n & \lambda I_n - X & \lambda A_6 - A_5 & \lambda A_5 & 0 & 0 & 0 \\
0 & \lambda I_n & \lambda X & \lambda A_5 & \lambda A_4 + A_3 & A_2 & -Y & -I_n \\
0 & 0 & 0 & 0 & A_2 & -\lambda A_2 + A_1 & \lambda Y - I_n & \lambda I_n \\
0 & 0 & 0 & 0 & -Y & \lambda Y - I_n & -\lambda A_0^{-1} & 0 \\
0 & 0 & 0 & 0 & -I_n & \lambda I_n & 0 & 0
\end{bmatrix}$$

is a block symmetric EGFPR of $P(\lambda)$. *Moreover, if* X *and* Y *are symmetric matrices then* $L(\lambda)$ *is symmetric when* $P(\lambda)$ *is symmetric. Note that* $L(\lambda)$ *is not a GFPR of* $P(\lambda)$. ■

Remark 4.1.28. *Note that Types-A, B and C index tuples in Remark 4.1.25 are not defined when* M_0^P *and* M_{-m}^P *are singular. Hence for singular* $P(\lambda)$*, Types-A, B and C EGFPRs are not defined.*

4.2 Hermitian EGFPRs preserving the sign characteristic.

Hermitian matrix polynomials arise in many applications, and the sign characteristics of their real eigenvalues play an important role, see [1, 7, 11, 12, 16, 33, 34, 43, 44, 47, 48] and the references therein. For the rest of this section we assume that $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ is Hermitian with A_m being nonsingular. We characterize a subclass of Hermitian EGFPRs which preserve the sign characteristic of $P(\lambda)$.

When $P(\lambda)$ is Hermitian, the Jordan form J of $P(\lambda)$ is a direct sum of Jordan blocks associated with real eigenvalues and blocks of the type $\operatorname{diag}(J_r, \overline{J_r})$, where J_r is an $r \times r$ Jordan block associated with a nonreal eigenvalue λ and $\overline{J_r}$ is a Jordan block of the same size associated with $\overline{\lambda}$, see [1, 16, 33, 34] for details. For convenience we use the following notation. By $J_r(\lambda)$ we denote the Jordan block associated with the eigenvalue λ of size $r \times r$ if λ is real, and the direct sum of two Jordan blocks of size $r/2 \times r/2$, if λ is not real, in which case the first block corresponds to λ and the second to $\overline{\lambda}$ [34, 16]. To define the sign characteristic of $P(\lambda)$ we need the following notations.

Following [16] we define

$$C_P := \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & I_n \\ -A_m^{-1}A_0 & -A_m^{-1}A_1 & \cdots & \cdots & -A_m^{-1}A_{m-1} \end{bmatrix} \text{ and } B_P := \begin{bmatrix} A_1 & A_2 & \cdots & A_m \\ A_2 & \vdots & \iddots & \\ \vdots & A_m & & \\ A_m & & & 0 \end{bmatrix}.$$

Note that $\lambda I_{mn} - C_P$ is a strong linearization of $P(\lambda)$. For an integer $k \geq 1$, we define

$$R_k := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{C}^{k \times k}.$$

Theorem 4.2.1 ([16], Theorem 2.2)**.** *Let* $P(\lambda)$ *be a Hermitian matrix polynomial with nonsingular leading coefficient. Let* $\lambda_1, \ldots, \lambda_\alpha$ *be the real eigenvalues of* $P(\lambda)$ *and* $\lambda_{\alpha+1}, \ldots, \lambda_\beta$ *be the nonreal eigenvalues of* $P(\lambda)$ *from the upper half-plane. Then there* *exists a nonsingular matrix H such that*

$$J := H^{-1}C_P H = J_{\ell_1}(\lambda_1) \oplus \cdots \oplus J_{\ell_\alpha}(\lambda_\alpha) \oplus J_{\ell_{\alpha+1}}(\lambda_{\alpha+1}) \oplus \cdots \oplus J_{\ell_\beta}(\lambda_\beta)$$

and

$$P_{\epsilon,J} := H^* B_P H = \epsilon_1 R_{\ell_1} \oplus \cdots \oplus \epsilon_\alpha R_{\ell_\alpha} \oplus R_{\ell_{\alpha+1}} \oplus \cdots \oplus R_{\ell_\beta},$$

where $\epsilon = \{\epsilon_1, \ldots, \epsilon_\alpha\}$ is an ordered set of signs ± 1. The set ϵ is unique up to permutation of signs corresponding to Jordan blocks.

Definition 4.2.2. *[16] Let $P(\lambda)$ be as in Theorem 4.2.1. The set $\{\epsilon_1, \ldots, \epsilon_\alpha\}$ in Theorem 4.2.1 is called the sign characteristic of $P(\lambda)$.*

In the special case when λ_0 is a simple eigenvalue of $P(\lambda)$, the sign in the sign characteristic of $P(\lambda)$ corresponding to λ_0 is given by $sign(x^* P'(\lambda_0) x)$, where x is an eigenvector of $P(\lambda)$ associated with λ_0 and $P'(\lambda)$ is the first derivative of $P(\lambda)$, see [1, 16, 34].

Recall that a matrix assignment $\mathcal{X}$ is said to be a Hermitian matrix assignment if all the matrices belonging to $\mathcal{X}$ are Hermitian. Recall that, if $P(\lambda)$ is Hermitian and the matrix assignments $\mathcal{X}$ and $\mathcal{Y}$ are Hermitian, then the EGFPR

$$L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y}) =$$
$$M_{\mathbf{t}_v}(\mathcal{Y}) M_{\mathbf{t}_w}(\mathcal{X}) \big(\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}}\big) M^P_{\mathbf{c}_w} M_{rev(\mathbf{t}_w)}(rev(\mathcal{X})) M^P_{\mathbf{c}_v} M_{rev(\mathbf{t}_v)}(rev(\mathcal{Y})) \quad (4.3)$$

given in Theorem 4.1.17 is Hermitian. We refer the pencil (4.3) as a Hermitian EGFPR of $P(\lambda)$. In particular, when $\mathbf{t}_w = \emptyset = \mathbf{t}_v$, then the Hermitian EGFPR $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$ is denoted by $L_P(\mathbf{h}, \emptyset, \emptyset)$, that is, $L_P(\mathbf{h}, \emptyset, \emptyset) := \big(\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}}\big) M^P_{\mathbf{c}_w} M^P_{\mathbf{c}_v}$.

Caution: As already mentioned in Definition 4.1.15 we always consider nonsingular matrix assignments $\mathcal{X}$ and $\mathcal{Y}$ in the Hermitian EGFPR $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y})$.

Definition 4.2.3. *[44, 16] Two pencils $\lambda X + Y$ and $\lambda \widetilde{X} + \widetilde{Y}$ are said to be *congruent if there exists a nonsingular matrix Q such that $\lambda X + Y = Q(\lambda \widetilde{X} + \widetilde{Y})Q^*$.*

Lemma 4.2.4. *[44, 16] Let $L(\lambda) = \lambda L_1 - L_0$ and $\widehat{L} = \lambda \widehat{L}_1 - \widehat{L}_0$ be two complex Hermitian pencils such that L_1 and $\widehat{L}_1$ are nonsingular. Then $L(\lambda)$ and $\widehat{L}(\lambda)$ have the same elementary divisors and the same sign characteristic if and only if $L(\lambda)$ and $\widehat{L}(\lambda)$ are *congruent.*

The following result will be useful for characterizing a subclass of the Hermitian EGFPRs of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$.

Lemma 4.2.5. *Let $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ and $L_P(\boldsymbol{h}, \emptyset, \emptyset)$ be Hermitian EGFPRs of $P(\lambda)$ as in (4.3). Then $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ is *congruent to $L_P(\boldsymbol{h}, \emptyset, \emptyset)$. More precisely, $L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y}) = QL_P(\boldsymbol{h}, \emptyset, \emptyset)Q^*$, where $Q := M_{\mathbf{t}_v}(\mathcal{Y})M_{\mathbf{t}_w}(\mathcal{X})$ is nonsingular.*

Proof. Recall that $\mathbf{t}_w$ and $\mathbf{c}_v$ commute. Hence we have $L_P(\mathbf{h}, \mathbf{t}_w, \mathbf{t}_v, \mathcal{X}, \mathcal{Y}) =$

$$\begin{aligned}
&M_{\mathbf{t}_v}(\mathcal{Y})M_{\mathbf{t}_w}(\mathcal{X})\big(\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}}\big)M^P_{\mathbf{c}_w}M_{rev(\mathbf{t}_w)}(rev(\mathcal{X}))M^P_{\mathbf{c}_v}M_{rev(\mathbf{t}_v)}(rev(\mathcal{Y}))\\
&= M_{\mathbf{t}_v}(\mathcal{Y})M_{\mathbf{t}_w}(\mathcal{X})\big(\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}}\big)M^P_{\mathbf{c}_w}M^P_{\mathbf{c}_v}M_{rev(\mathbf{t}_w)}(rev(\mathcal{X}))M_{rev(\mathbf{t}_v)}(rev(\mathcal{Y}))\\
&= M_{\mathbf{t}_v}(\mathcal{Y})M_{\mathbf{t}_w}(\mathcal{X})L_P(\mathbf{h}, \emptyset, \emptyset)M_{rev(\mathbf{t}_w)}(rev(\mathcal{X}))M_{rev(\mathbf{t}_v)}(rev(\mathcal{Y})).
\end{aligned}$$

Let Z be an $n \times n$ arbitrary Hermitian matrix. Then $(M_i(Z))^* = M_i(Z^*) = M_i(Z)$ for $i = \pm 0, \pm 1, \ldots, \pm m$. As $\mathcal{X}$ and $\mathcal{Y}$ are Hermitian matrix assignments, we have

$$\big(M_{\mathbf{t}_v}(\mathcal{Y})M_{\mathbf{t}_w}(\mathcal{X})\big)^* = \big(M_{\mathbf{t}_w}(\mathcal{X})\big)^*\big(M_{\mathbf{t}_v}(\mathcal{Y})\big)^* = M_{rev(\mathbf{t}_w)}(rev(\mathcal{X}))\,M_{rev(\mathbf{t}_v)}(rev(\mathcal{Y})).$$

Hence the desired conclusion follows as $\mathcal{X}$ and $\mathcal{Y}$ are nonsingular matrix assignments. ☐

The following result characterizes a subclass of the Hermitian GFPRs which preserves the sign characteristic of $P(\lambda)$.

Theorem 4.2.6 ([16], Theorem 7.1)**.** *Let h be an even integer such that $0 \leq h \leq m-1$, and let $T(\lambda) = L_P(h, \boldsymbol{t}_w, \boldsymbol{t}_z, X, Y)$ be a Hermitian GFPR for $P(\lambda)$, where X and Y are nonsingular Hermitian matrix assignments for $\boldsymbol{t}_w$ and $\boldsymbol{t}_z$, respectively. Then $T(\lambda)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$. Thus, in particular, a Hermitian GFPR $L_P(h, \emptyset, \emptyset)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$.*

The following result will be useful for characterizing a subclass of the Hermitian EGFPRs of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$.

Proposition 4.2.7. *Let $0 \leq a < b < d \leq m-1$, where a is even, and $\{a+1 : b\}$ and $\{b+1 : d\}$ contain odd number of elements. Let $-(m+a+1)+\boldsymbol{x}$ be the simple admissible tuple of $\{a+1 : m\}$, $\boldsymbol{y}$ be the simple admissible tuple of $\{b+1 : d\}$, $-(a+b+1)+\boldsymbol{u}$ be the simple admissible tuple of $\{a+1 : b\}$ and $-(m+d+1)+\boldsymbol{s}$ be the simple admissible tuple of $\{d+1 : m\}$. Then we have*

$$M^P_{\boldsymbol{y}}M^P_{\boldsymbol{x}} = M^P_{\boldsymbol{u}}M^P_{\boldsymbol{s}} \text{ and } M^P_{\boldsymbol{c}_x}M^P_{rev(\boldsymbol{y})} = M^P_{\boldsymbol{c}_y}M^P_{\boldsymbol{c}_u}M^P_{\boldsymbol{c}_s}, \tag{4.4}$$

where $\boldsymbol{c}_x$, $\boldsymbol{c}_y$ $\boldsymbol{c}_u$ and $\boldsymbol{c}_s$ are the symmetric compliments of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{u}$ and $\boldsymbol{s}$, respectively.

Proof. Given that a is even. Since $\{a+1:b\}$ and $\{b+1:d\}$ contain odd number of elements, we have that b is odd and d is even. As $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{s}$ are the simple admissible tuples, we have

$$\begin{aligned}
\mathbf{y} &= \Big(d-1:d, d-3:d-2, \ldots, b+2:b+3, b+1\Big), \\
\mathbf{u} &= \Big(-(a+2):-(a+1), -(a+4):-(a+3), \ldots, -(b-1):-(b-2), -b\Big), \text{ and} \\
\mathbf{s} &= \begin{cases} \Big(-(d+2):-(d+1), \ldots, -(m-1):-(m-2), -m\Big) & \text{if } m \text{ is odd} \\ \Big(-(d+2):-(d+1), \ldots, -(m-2):-(m-3), -m:-(m-1)\Big) & \text{if } m \text{ is even.} \end{cases}
\end{aligned}$$

Further, we have $\mathbf{x} = (\alpha, \beta, \gamma)$, where

$$\begin{aligned}
\alpha &:= \Big(-(a+2):-(a+1), -(a+4):-(a+3), \ldots, -(b-1):-(b-2)\Big), \\
\beta &:= \Big(-(b+1):-b, -(b+3):-(b+2), \ldots, -(d-2):-(d-3), -d:-(d-1)\Big), \text{ and} \\
\gamma &:= \begin{cases} \Big(-(d+2):-(d+1), \ldots, -(m-1):-(m-2), -m\Big) & \text{if } m \text{ is odd} \\ \Big(-(d+2):-(d+1), \ldots, -(m-2):-(m-3), -m:-(m-1)\Big) & \text{if } m \text{ is even.} \end{cases}
\end{aligned}$$

Observe that $\mathbf{u} = (\alpha, -b)$ and $\mathbf{s} = \gamma$. Hence $\mathbf{x} = (\alpha, \beta, \mathbf{s})$. A straight forward calculation shows that $M^P_{\mathbf{y}} M^P_{\beta} = M^P_{-b}$. Thus we have

$$\begin{aligned}
M^P_{\mathbf{y}} M^P_{\mathbf{x}} &= M^P_{\mathbf{y}} M^P_{\alpha} M^P_{\beta} M^P_{\mathbf{s}} = M^P_{\alpha} M^P_{\mathbf{y}} M^P_{\beta} M^P_{\mathbf{s}} \text{ as } \alpha \text{ and } \mathbf{y} \text{ commute} \\
&= M^P_{\alpha} M^P_{-b} M^P_{\mathbf{s}} = M^P_{(\alpha,-b)} M^P_{\mathbf{s}} = M^P_{\mathbf{u}} M^P_{\mathbf{s}}
\end{aligned}$$

which yields the first part of (4.4).

Next, we show that $M^P_{\mathbf{c}_x} M^P_{rev(\mathbf{y})} = M^P_{\mathbf{c}_y} M^P_{\mathbf{c}_u} M^P_{\mathbf{c}_s}$. As $\mathbf{c}_x$ is the symmetric complement of $\mathbf{x}$, we have

$$\begin{aligned}
\mathbf{c}_x = \Big(&-(a+2), -(a+4), \ldots, -(b-1), -(b+1), -(b+3), \ldots, \\
&-(d-2), -d, -(d+2), -(d+4), \ldots, -(m^*-2), -m^*\Big),
\end{aligned}$$

where $m^* := m$ if m is even and $m^* := m-1$ if m is odd. Similarly, as $\mathbf{c}_y$, $\mathbf{c}_u$ and $\mathbf{c}_s$ are the symmetric complements of $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{s}$, respectively, we have

$$\mathbf{c}_y = \Big(d-1, d-3, \ldots, b+4, b+2\Big),$$

$$\begin{aligned}
\mathbf{c}_u &= \Big(-(a+2), -(a+4), \ldots, -(b-3), -(b-1)\Big), \text{ and} \\
\mathbf{c}_s &= \Big(-(d+2), -(d+4), \ldots, -(m^*-2), -m^*\Big),
\end{aligned}$$

where $m^* := m$ if m is even and $m^* := m-1$ if m is odd. Observe that $\mathbf{c}_x = \big(\mathbf{c}_u, \delta, \mathbf{c}_s\big)$, where $\delta := \big(-(b+1), -(b+3), \ldots, -(d-2), -d\big)$. A straight forward calculation shows that $M^P_\delta M^P_{rev(\mathbf{y})} = M^P_{\mathbf{c}_y}$. This implies that

$$\begin{aligned} M^P_{\mathbf{c}_x} M^P_{rev(\mathbf{y})} &= M^P_{\mathbf{c}_u} M^P_\delta M^P_{\mathbf{c}_s} M^P_{rev(\mathbf{y})} \\ &= M^P_{\mathbf{c}_u} M^P_\delta M^P_{rev(\mathbf{y})} M^P_{\mathbf{c}_s} \quad \text{as } \mathbf{y} \text{ and } \mathbf{c}_s \text{ commute} \\ &= M^P_{\mathbf{c}_u} M^P_{\mathbf{c}_y} M^P_{\mathbf{c}_s} = M^P_{\mathbf{c}_y} M^P_{\mathbf{c}_u} M^P_{\mathbf{c}_s} \quad \text{as } \mathbf{c}_y \text{ and } \mathbf{c}_u \text{ commute.} \end{aligned}$$

This completes the proof. ☐

The following result is a corollary to Theorem 4.1.17.

Corollary 4.2.8. *Let* $L(\lambda) := L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ *be a Hermitian EGFPR of* $P(\lambda)$ *such that* h_1 *is even, where* $\boldsymbol{h} = \{h_1, h_2, \ldots, h_k\}$*. Then* $L(\lambda)$ *is a Hermitian strong linearization of* $P(\lambda)$*.*

Proof. Since h_1 is even, it follows from Remark 4.1.13 that $0 \notin \mathbf{c}_w$. Given that $P(\lambda)$ is Hermitian with nonsingular A_m and the matrix assignments $\mathcal{X}$ and $\mathcal{Y}$ are nonsingular. Consequently, it follows by Theorem 4.1.17 that $L(\lambda)$ is a Hermitian strong linearization of $P(\lambda)$. ☐

The next result characterizes a subclass of the Hermitian EGFPRs which preserves the sign characteristic of $P(\lambda)$.

Theorem 4.2.9. *Let* $\boldsymbol{h} = \{h_1, h_2, \ldots, h_k\}$*, where* $k \geq 1$ *is odd,* $0 \leq h_1 < h_2 < \cdots < h_{k-1} < h_k \leq m-1$ *with even* h_1*. Let* $L(\lambda) := L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ *be a Hermitian EGFPR of* $P(\lambda)$*, where* $\mathcal{X}$ *and* $\mathcal{Y}$ *are nonsingular Hermitian matrix assignments for* $\boldsymbol{t}_w$ *and* $\boldsymbol{t}_v$*, respectively. Then* $L(\lambda)$ *is a Hermitian strong linearization of* $P(\lambda)$ *that preserves the sign characteristic of* $P(\lambda)$*.*

Proof. Let $T(\lambda) := (\lambda M^P_{\mathbf{z}} - M^P_{\mathbf{w}_1}) M^P_{\mathbf{c}_{w_1}} M^P_{\mathbf{c}_z}$, where $\mathbf{w}_1$ is the simple admissible tuple of $\{0 : h_1\}$, $-(m + h_1 + 1) + \mathbf{z}$ is the simple admissible tuple of $\{h_1 + 1 : m\}$, and $\mathbf{c}_{w_1}$ and $\mathbf{c}_z$ are the symmetric compliments of $\mathbf{w}_1$ and $\mathbf{z}$, respectively. Note that $T(\lambda)$ is a Hermitian GFPR of $P(\lambda)$. Hence by Theorem 4.2.6, $T(\lambda)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$. We will prove that $L(\lambda)$ is *congruent to $T(\lambda)$, which together with Corollary 4.2.8 and Lemma 4.2.4 will give the desired result.

By Lemma 4.2.5, $L(\lambda)$ is *congruent to $L_P(\mathbf{h}, \emptyset, \emptyset) = \big(\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}}\big) M^P_{\mathbf{c}_w} M^P_{\mathbf{c}_v}$. Hence we only need to show that $L_P(\mathbf{h}, \emptyset, \emptyset)$ is *congruent to $T(\lambda) = (\lambda M^P_{\mathbf{z}} - M^P_{\mathbf{w}_1}) M^P_{\mathbf{c}_{w_1}} M^P_{\mathbf{c}_z}$. Note that $L_P(\mathbf{h}, \emptyset, \emptyset) = \big(\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}}\big) M^P_{\mathbf{c}_w} M^P_{\mathbf{c}_v}$, where $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_3, \ldots, \mathbf{w}_k)$, $\mathbf{v} =$

$(\mathbf{v}_2, \mathbf{v}_4, \ldots, \mathbf{v}_{k+1})$, $\mathbf{c}_w = (\mathbf{c}_{w_1}, \mathbf{c}_{w_3}, \ldots, \mathbf{c}_{w_k})$ and $\mathbf{c}_v = (\mathbf{c}_{v_2}, \mathbf{c}_{v_4}, \ldots, \mathbf{c}_{v_{k+1}})$. We show that $QT(\lambda)Q^* = L_P(\mathbf{h}, \emptyset, \emptyset)$, i.e.,

$$Q(\lambda M^P_{\mathbf{z}} - M^P_{\mathbf{w}_1})M^P_{\mathbf{c}_{w_1}} M^P_{\mathbf{c}_z} Q^* = (\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}})M^P_{\mathbf{c}_w} M^P_{\mathbf{c}_v}, \tag{4.5}$$

where $Q := M^P_{\mathbf{w}_3,\mathbf{w}_5,\ldots,\mathbf{w}_k}$. It is enough to prove that $M^P_{\mathbf{c}_{w_1}} M^P_{\mathbf{c}_z} Q^* = M^P_{\mathbf{c}_w} M^P_{\mathbf{c}_v}$ and $QM^P_{\mathbf{z}} = M^P_{\mathbf{v}}$ which would imply (4.5). In other words, we show that

$$M^P_{\mathbf{w}_3,\mathbf{w}_5,\ldots,\mathbf{w}_k} M^P_{\mathbf{z}} = M^P_{\mathbf{v}_2,\mathbf{v}_4,\ldots,\mathbf{v}_{k+1}} \quad \text{and} \tag{4.6}$$

$$M^P_{\mathbf{c}_{w_1}} M^P_{\mathbf{c}_z} M^P_{rev(\mathbf{w}_3,\mathbf{w}_5,\ldots,\mathbf{w}_k)} = M^P_{(\mathbf{c}_{w_1},\mathbf{c}_{w_3},\ldots,\mathbf{c}_{w_k})} M^P_{(\mathbf{c}_{v_2},\mathbf{c}_{v_4},\ldots,\mathbf{c}_{v_{k+1}})}. \tag{4.7}$$

If $k = 1$ then $L(\lambda) = T(\lambda)$ and the desired result follows trivially. So we consider that $k > 1$. Note that k is odd. Since h_1 is even and by Definition 4.1.11, $\{h_{j-1}+1 : h_j\}$ contains odd number of elements for each $j = 2 : k$, we have $h_1, h_3, \ldots, h_k$ are even and $h_2, h_4, \ldots, h_{k-1}$ are odd. As $-(m + h_1 + 1) + \mathbf{z}$ is the simple admissible tuple of $\{h_1 + 1 : m\}$, we have $\mathbf{z} =$

$$\begin{cases} \left. \begin{array}{l} \Big(-(h_1+2) : -(h_1+1), -(h_1+4) : -(h_1+3), \ldots, \\ \hfill -(m-1) : -(m-2), -m \Big) \end{array} \right\} & \text{if } m \text{ is odd} \\ \left. \begin{array}{l} \Big(-(h_1+2) : -(h_1+1), -(h_1+4) : -(h_1+3), \ldots, \\ \hfill -(m-2) : (m-3), -m : -(m-1) \Big) \end{array} \right\} & \text{if } m \text{ is even} \end{cases}$$

We prove (4.6) and (4.7) by induction on number k of elements in the partition $\mathbf{h}$. If $k = 3$ (i.e., $\mathbf{h} = \{h_1, h_2, h_3\}$). Then $L_P(\mathbf{h}, \emptyset, \emptyset) = \big(\lambda M^P_{\mathbf{v}} - M^P_{\mathbf{w}}\big)M^P_{\mathbf{c}_w} M^P_{\mathbf{c}_v}$, where $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_3)$, $\mathbf{v} = (\mathbf{v}_2, \mathbf{v}_4)$, $\mathbf{c}_w = (\mathbf{c}_{w_1}, \mathbf{c}_{w_3})$ and $\mathbf{c}_v = (\mathbf{c}_{v_2}, \mathbf{c}_{v_4})$. The desired result follows from Proposition 4.2.7 by taking $\mathbf{x} = \mathbf{z}$, $\mathbf{y} = \mathbf{w}_3$, $\mathbf{u} = \mathbf{v}_2$, $\mathbf{s} = \mathbf{v}_4$, $a = h_1$, $b = h_2$ and $d = h_3$.

Assume that (4.6) and (4.7) hold for any partition $\mathbf{h}'$ of $\{0 : m\}$ that contains $k-2$ elements, i.e., $\mathbf{h}' = \{h'_1, h'_2, \ldots, h'_{k-2}\}$, where $0 \leq h'_1 < h'_2 < \cdots < h'_{k-2} \leq m-1$ with h'_1 even. Now we prove the result for $\mathbf{h} = \{h_1, h_2, \ldots, h_k\}$. Define $h'_j := h_j$ for $j = 1 : k-2$ and $\mathbf{h}' := \{h'_1, h'_2, \ldots, h'_{k-2}\}$. Then

$$L_P(\mathbf{h}', \emptyset, \emptyset) = (\lambda M^P_{\mathbf{v}'} - M^P_{\mathbf{w}'})M^P_{\mathbf{c}_{w'}} M^P_{\mathbf{c}_{v'}},$$

where $\mathbf{w}' = (\mathbf{w}'_1, \mathbf{w}'_3, \ldots, \mathbf{w}'_{k-2})$, $\mathbf{v}' = (\mathbf{v}'_2, \mathbf{v}'_4, \ldots, \mathbf{v}'_{k-1})$ and $\mathbf{c}_{w'}$ and $\mathbf{c}_{v'}$ are the symmetric complements of $\mathbf{w}'$ and $\mathbf{v}'$, respectively. Observe that $\mathbf{w}'_j = \mathbf{w}_j$ for $j = 1, 3, \ldots, k-2$, and $\mathbf{v}'_j = \mathbf{v}_j$ for $j = 2, 4, \ldots, k-3$. This imply that $\mathbf{c}_{w'_j} = \mathbf{c}_{w_j}$ for $j = 1, 3, \ldots, k-2$, and $\mathbf{c}_{v'_j} = \mathbf{c}_{v_j}$ for $j = 2, 4, \ldots, k-3$. Thus $L_P(\mathbf{h}', \emptyset, \emptyset) =$

$$\big(\lambda M^P_{(\mathbf{v}_2,\mathbf{v}_4,\ldots,\mathbf{v}_{k-3},\mathbf{v}'_{k-1})} - M^P_{(\mathbf{w}_1,\mathbf{w}_3,\ldots,\mathbf{w}_{k-2})}\big) M^P_{(\mathbf{c}_{w_1},\mathbf{c}_{w_3},\ldots,\mathbf{c}_{w_{k-2}})} M^P_{(\mathbf{c}_{v_2},\mathbf{c}_{v_4},\ldots,\mathbf{c}_{v_{k-3}},\mathbf{c}_{v'_{k-1}})},$$

where $-(m+h_{k-2}+1)+\mathbf{v}'_{k-1}$ is the simple admissible tuple of $\{h_{k-2}+1:m\}$. Now by induction hypothesis, we have

$$M^P_{\mathbf{w}_3,\mathbf{w}_5,\ldots,\mathbf{w}_{k-2}} M^P_{\mathbf{z}} = M^P_{\mathbf{v}'_2,\mathbf{v}'_4,\ldots,\mathbf{v}'_{k-3}} M^P_{\mathbf{v}'_{k-1}} = M^P_{\mathbf{v}_2,\mathbf{v}_4,\ldots,\mathbf{v}_{k-3}} M^P_{\mathbf{v}'_{k-1}} \quad \text{and}$$

$$\begin{aligned} M^P_{\mathbf{c}_{w_1}} M^P_{\mathbf{c}_z} M^P_{rev(\mathbf{w}_3,\mathbf{w}_5,\ldots,\mathbf{w}_{k-2})} &= M^P_{(\mathbf{c}_{w'_1},\mathbf{c}_{w'_3},\ldots,\mathbf{c}_{w'_{k-2}})} M^P_{(\mathbf{c}_{v'_2},\mathbf{c}_{v'_4},\ldots,\mathbf{c}_{v'_{k-3}},\mathbf{c}_{v'_{k-1}})} \\ &= M^P_{(\mathbf{c}_{w_1},\mathbf{c}_{w_3},\ldots,\mathbf{c}_{w_{k-2}})} M^P_{(\mathbf{c}_{v_2},\mathbf{c}_{v_4},\ldots,\mathbf{c}_{v_{k-3}},\mathbf{c}_{v'_{k-1}})}. \end{aligned}$$

This implies that

$$M^P_{\mathbf{w}_3,\mathbf{w}_5,\ldots,\mathbf{w}_{k-2},\mathbf{w}_k} M^P_{\mathbf{z}} = M^P_{\mathbf{v}_2,\mathbf{v}_4,\ldots,\mathbf{v}_{k-3}} M^P_{\mathbf{w}_k} M^P_{\mathbf{v}'_{k-1}} \quad \text{and}$$

$$M^P_{\mathbf{c}_{w_1}} M^P_{\mathbf{c}_z} M^P_{rev(\mathbf{w}_3,\mathbf{w}_5,\ldots,\mathbf{w}_{k-2},\mathbf{w}_k)} = M^P_{(\mathbf{c}_{w_1},\mathbf{c}_{w_3},\ldots,\mathbf{c}_{w_{k-2}})} M^P_{(\mathbf{c}_{v_2},\mathbf{c}_{v_4},\ldots,\mathbf{c}_{v_{k-3}})} M^P_{\mathbf{c}_{v'_{k-1}}} M^P_{rev(\mathbf{w}_k)}.$$

In view of (4.6) and (4.7), we only need to show that

$$M^P_{\mathbf{w}_k} M^P_{\mathbf{v}'_{k-1}} = M^P_{\mathbf{v}_{k-1}} M^P_{\mathbf{v}_{k+1}} \text{ and } M^P_{\mathbf{c}_{v'_{k-1}}} M^P_{rev(\mathbf{w}_k)} = M^P_{\mathbf{c}_{w_k}} M^P_{\mathbf{c}_{v_{k-1}}} M^P_{\mathbf{c}_{v_{k+1}}}. \tag{4.8}$$

Note that $\mathbf{w}_k$ is the simple admissible tuple of $\{h_{k-1}+1:h_k\}$, $-(h_{k-2}+h_{k-1}+1)+\mathbf{v}_{k-1}$ is the simple admissible tuple of $\{h_{k-2}+1:h_{k-1}\}$ and $-(m+h_k+1)+\mathbf{v}_{k+1}$ is the simple admissible tuple of $\{h_k+1:m\}$. Thus the desired result follows from Proposition 4.2.7 by taking $\mathbf{x}=\mathbf{v}'_{k-1}$, $\mathbf{y}=\mathbf{w}_k$, $\mathbf{u}=\mathbf{v}_{k-1}$, $\mathbf{s}=\mathbf{v}_{k+1}$, $a=h_{k-2}$, $b=h_{k-1}$ and $d=h_k$. ☐

Example 4.2.10. *Let* $P(\lambda) := \sum_{i=0}^{7} \lambda^i A_i$ *be Hermitian, where* A_7 *is nonsingular. Let* $\boldsymbol{h} := \{h_1,h_2,h_3\}$*, where* $h_1 = 0$*,* $h_2 = 3$ *and* $h_3 = 6$*. Let* $\boldsymbol{t}_w = (4)$*,* $\boldsymbol{t}_v = (-3)$*, and* X *and* Y *be any Hermitian matrices. Then by Theorem 4.2.9,* $L_P(\boldsymbol{h},\boldsymbol{t}_w,\boldsymbol{t}_v,X,Y) = M_{-3}(Y)M_4(X)\big(\lambda M^P_{(-7,-2,-1,-3)} - M^P_{(0,5,6,4)}\big)M^P_5 M_4(X) M^P_{-2} M_{-3}(Y)$

$$= \begin{bmatrix} \lambda A_7 + A_6 & A_5 & -X & 0 & -I_n & 0 & 0 \\ A_5 & -\lambda A_5 + A_4 & \lambda X - I_n & 0 & \lambda I_n & 0 & 0 \\ -X & \lambda X - I_n & 0 & 0 & 0 & \lambda I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_n & \lambda I_n \\ -I_n & \lambda I_n & 0 & 0 & 0 & -Y & \lambda Y \\ 0 & 0 & \lambda I_n & -I_n & -Y & \lambda A_3 - A_2 & \lambda A_2 \\ 0 & 0 & 0 & \lambda I_n & \lambda Y & \lambda A_2 & \lambda A_1 + A_0 \end{bmatrix}$$

is a Hermitian strong linearization of $P(\lambda)$ *and preserves the sign characteristic of* $P(\lambda)$*.* ■

We end this section with a discussion on the construction of low band-width banded Hermitian EGFPRs preserving the sign characteristic of $P(\lambda)$. Recall that there does not exist any block penta-diagonal Hermitian GFPR of $P(\lambda)$ when $P(\lambda)$ has degree $m \geq 8$. This implies that there does not exist any block penta-diagonal Hermitian GFPR which preserves the sign characteristic of $P(\lambda)$ when $m \geq 8$. Further, $T(\lambda) = (\lambda M^P_{(-5:-4,-7:-6)} - M^P_{(2:3,0:1)})M^P_{(2,0)}M^P_{(-5,-7)}$ is the only block penta-diagonal Hermitian GFPR of $P(\lambda)$ when $m = 7$. It is shown in [16, p. 266] that a Hermitian GFPR $L_P(h_1, \mathbf{t}_w, \mathbf{t}_z, \mathcal{X}, \mathcal{Y})$ with odd h_1 is a strong linearization of $P(\lambda)$ preserving the sign characteristic of $P(\lambda)$ if and only if the Hermitian GFPR $L_P(1, \emptyset, \emptyset)$ and the Hermitian GFPR $L_P(0, \emptyset, \emptyset)$ are *congruent, which is not true for all Hermitian $P(\lambda)$, see [16, Example 7.6]. Thus, in general, $T(\lambda)$ does not preserves the sign characteristic of $P(\lambda)$. Consequently, there does not exist any block penta-diagonal Hermitian GFPR which preserves the sign characteristic of $P(\lambda)$ when $m \geq 7$. On the other hand, the Hermitian EGFPR given in Example 4.2.10 is a strong linearization and preserves the sign characteristic of a Hermitian $P(\lambda)$. Moreover, we have the following result.

Corollary 4.2.11. *Let* $L(\lambda) := L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ *be the pencil given in Theorem 4.1.21 such that* h_1 *is even (i.e.,* $h_1 \in \{0, 2\}$*) and* $\mathcal{X}$ *and* $\mathcal{Y}$ *are nonsingular Hermitian matrix assignments for* $\boldsymbol{t}_w$ *and* $\boldsymbol{t}_v$*, respectively. Then* $L(\lambda)$ *is a Hermitian block penta-diagonal strong linearization of* $P(\lambda)$ *which preserves the sign characteristic of* $P(\lambda)$*.*

Proof. The desired result follows from Theorem 4.1.21 and Theorem 4.2.9. ☐

Recall that there does not exist any block tridiagonal Hermitian GFPR of $P(\lambda)$ when $P(\lambda)$ has degree $m \geq 4$. Further, $(\lambda M^P_{(-3,-2)} - M^P_{(0,1)})M^P_0 M^P_{-3}$ is the only block tridiagonal Hermitian GFPR of $P(\lambda)$ when $m = 3$. Now, by similar arguments as in the case of block penta-diagonal Hermitian GFPR it follows that there does not exist any block tridiagonal Hermitian GFPR which preserves the sign characteristic of $P(\lambda)$ when $m \geq 3$. For EGFPRs, we have the following result.

Corollary 4.2.12. *Consider the EGFPR* $L(\lambda) := L_P(\boldsymbol{h}, \boldsymbol{t}_w, \boldsymbol{t}_v, \mathcal{X}, \mathcal{Y})$ *as given in Theorem 4.1.23 such that* $h_1 = 0$*. Then* $L(\lambda)$ *is a Hermitian block tridiagonal strong linearization of* $P(\lambda)$ *which preserves the sign characteristic of* $P(\lambda)$*.*

Proof. The desired result follows from Theorem 4.1.23 and Theorem 4.2.9. ☐

For example, $(\lambda M^P_{(-4,-3,-1)} - M^P_{(0,2)})M^P_{-4}$ is a Hermitian EGFPR that preserves the sign characteristic $P(\lambda)$ when $P(\lambda)$ is Hermitian with degree 4.

4.3 Palindromic linearizations

This section is devoted to constructing T-palindromic (resp., T-anti-palindromic) strong linearizations of T-palindromic (resp., T-anti-palindromic) $P(\lambda)$ having odd degree $m \geq 3$ by using EGFPRs of $P(\lambda)$. Note that these linearizations contain infinite number of pencils in contrasts to the linearizations that have been constructed in [19, 27] by using the GFPs and FPRs of $P(\lambda)$. Moreover, for T-palindromic $P(\lambda)$, this construction enable us to generate T-palindromic linearizations which are low bandwidth banded pencils.

Assumption: For the rest of this section, we assume that $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ is T-palindromic with odd degree $m \geq 3$.

First, we define some special types of index tuples which we need for the construction of T-palindromic linearizations of $P(\lambda)$. Note that the column and row standard forms of a sub-permutation of $\{0 : m\}$ must be of the following form.

Let σ be a sub-permutation (also called, simple index tuple) of $\{0 : m\}$ such that a_0 and a_d, respectively, are the smallest and largest integer in σ. Then the column standard form $csf(\sigma)$ of σ is given by

$$csf(\sigma) = \big((a_{d-1}+k_{d-1} : a_d), (a_{d-2}+k_{d-2} : a_{d-1}), \ldots, (a_2+k_2 : a_3), (a_1+k_1 : a_2), (a_0 : a_1)\big),$$

where $a_0 \leq a_1$, $a_i < a_{i+1}$ and $1 \leq k_i$ for $i = 1 : d-1$. We call $(a_i+k_i : a_{i+1})$, $i = 0 : d-1$, as a string of $csf(\sigma)$, where $k_0 = 0$. Similarly, the row standard form $rsf(\sigma)$ of σ is given by

$$rsf(\sigma) = \big(rev(a_0 : a_1), rev(a_1+k_1 : a_2), \ldots, rev(a_{d-2}+k_{d-2} : a_{d-1}), rev(a_{d-1}+k_{d-1} : a_d)\big),$$

where $a_0 \leq a_1$, $a_i < a_{i+1}$ and $1 \leq k_i$ for $i = 1 : d-1$.

The following definitions generalize Definition 1.2.31, Definition 1.2.32 and Definition 1.2.33.

Definition 4.3.1 (Type-1 indices relative to a simple index tuple). *Let σ be a simple index tuple containing indices from $\{0 : m\}$. An index $s \in \sigma$ is said to be a right index of type-1 relative to σ if there is a string $(a_{h-1} + k_{h-1} : a_h)$ in the $csf(\sigma)$ such that $s = a_{h-1} + k_{h-1} < a_h$. Similarly s is said to be a left index of type-1 relative to σ if s is a right index of type-1 relative to $rev(\sigma)$.*

Definition 4.3.2 (Associated simple tuple). *Let σ be a simple index tuple containing indices from $\{0 : m\}$. Let $cfs(\sigma) = (\mathbf{b}_d, \mathbf{b}_{d-1}, \ldots, \mathbf{b}_1)$, where $\mathbf{b}_i = (a_{i-1} + k_{i-1} : a_i)$ for $i = 1 : d$ are the strings in $csf(\sigma)$. If $s \in \sigma$ is a right index of type-1 relative to σ, i.e., there exists a string $(a_{h-1}+k_{h-1} : a_h)$ in the $csf(\sigma)$ such that $s = a_{h-1}+k_{h-1} < a_h$, then the simple tuple associated with (σ, s), denoted by $z_r(\sigma, s)$, is the simple tuple given by*

- $z_r(\sigma, s) := (\boldsymbol{b}_d, \boldsymbol{b}_{d-1}, \ldots, \boldsymbol{b}_{h+1}, \widetilde{\boldsymbol{b}}_h, \widetilde{\boldsymbol{b}}_{h-1}, \boldsymbol{b}_{h-2}, \ldots, \boldsymbol{b}_1)$ *if* $s \neq 0$ *and* $k_{h-1} = 1$, *where*

$$\widetilde{\boldsymbol{b}}_h = (a_{h-1} + k_{h-1} + 1 : a_h), \text{ i.e., } \widetilde{\boldsymbol{b}}_h = (s + 1 : a_h) \text{ and}$$

$$\widetilde{\boldsymbol{b}}_{h-1} = (a_{h-2} + k_{h-2} : a_{h-1}, a_{h-1} + k_{h-1}), \text{ i.e., } \widetilde{\boldsymbol{b}}_{h-1} = (\boldsymbol{b}_{h-1}, s).$$

- $z_r(\sigma, s) := (\boldsymbol{b}_d, \boldsymbol{b}_{d-1}, \ldots, \boldsymbol{b}_{h+1}, \widetilde{\boldsymbol{b}}_h, s, \boldsymbol{b}_{h-1}, \ldots, \boldsymbol{b}_1)$ *if* $s \neq 0$ *and* $k_{h-1} > 1$, *where*

$$\widetilde{\boldsymbol{b}}_h = (a_{h-1} + k_{h-1} + 1 : a_h), \text{ i.e., } \widetilde{\boldsymbol{b}}_h = (s + 1 : a_h).$$

- $z_r(\sigma, s) := (\boldsymbol{b}_d, \boldsymbol{b}_{d-1}, \ldots, \boldsymbol{b}_2, \widetilde{\boldsymbol{b}}_1, \widetilde{\boldsymbol{b}}_0)$ *if* $s = 0$, *where* $\widetilde{\boldsymbol{b}}_1 = (1 : a_1)$ *and* $\widetilde{\boldsymbol{b}}_0 = (0)$.

An index tuple of type-1 is defined as follows.

Definition 4.3.3 (Right and left index tuple of type-1)**.** *Let* σ *be a simple index tuple containing indices from* $\{0 : m\}$. *Let* $\alpha := (s_1, \ldots, s_k)$ *and* β *be index tuples containing indices from* σ. *Then:*

- α *is said to be a right index tuple of type-1 relative to* σ *if, for* $i = 1 : k$, s_i *is a right index of type-1 relative to* $z_r(\sigma, (s_1, \ldots, s_{i-1}))$, *where* $z_r(\sigma, (s_1, \ldots, s_{i-1})) := z_r\big(z_r(\sigma, (s_1, \ldots, s_{i-2})), s_{i-1}\big)$ *for* $i > 2$.

- β *is said to be a left index tuple of type-1 relative to* σ, *if* $rev(\beta)$ *is a right index tuple of type-1 relative to* $rev(\sigma)$. *Moreover, if* β *is a left index tuple of type-1 relative to* σ, *we define* $z_l(\beta, \sigma) := rev\big(z_r(rev(\sigma), rev(\beta))\big)$.

Example 4.3.4. *Let* $\sigma = (5 : 7, 1 : 2, 0)$ *and* $\alpha = (5, 6, 5)$. *Then* σ *is a simple index tuple containing indices from* $\{0 : 7\}$. *The simple tuple associated with* $(\sigma, 5)$ *is given by* $z_r(\sigma, 5) = (6 : 7, 5, 1 : 2, 0)$. *Further,* $z_r(\sigma, (5, 6)) = (7, 5 : 6, 1 : 2, 0)$ *and* $z_r(\sigma, (5, 6, 5)) = (7, 6, 5, 1 : 2, 0)$. ■

We now define $\mathcal{P}$-admissible tuple which will play a central role in the construction of T-palindromic linearizations of $P(\lambda)$.

Definition 4.3.5 ($\mathcal{P}$-admissible tuple)**.** *Let* $P(\lambda)$ *be a matrix polynomial of odd degree* $m \geq 3$. *A sub-permutation* σ *of* $\{0 : m\}$ *is said to be a* $\mathcal{P}$*-admissible tuple relative to* $\{0 : m\}$ *if the following hold: (a)* $0 \in \sigma$ *(b) cardinality (i.e., the number of indices) of* σ *is* $(m+1)/2$, *and (c) if* $i \in \sigma$ *then* $m - i \notin \sigma$ *for* $i = 0 : m$.

Recall that for a tuple $X = (X_1, X_2, \ldots, X_r)$ of $n \times n$ arbitrary matrices, we have $-X = (-X_1, -X_2, \ldots, -X_r)$, $X^T = (X_1^T, X_2^T, \ldots, X_r^T)$ and $rev(X) = (X_r, \ldots, X_2, X_1)$.

Lemma 4.3.6. *Let σ be a $\mathcal{P}$-admissible tuple relative to $\{0 : m\}$. Let σ_r and σ_l, respectively, be a right and a left index tuple of type-1 relative to σ. Then $(\sigma_l, \sigma, \sigma_r)$ and $(-m + rev(\sigma_r), -m + rev(\sigma), -m + rev(\sigma_l))$ satisfy the SIP.*

Proof. As σ is a sub-permutation of $\{0 : m\}$, it is enough to show that (σ, σ_r) and (σ_l, σ) satisfy the SIP. By induction on the number k of indices of σ_r we prove that (σ, σ_r) satisfies the SIP. Let $k = 1$, i.e., $\sigma_r = (s_1)$. Since s_1 is a right index of type-1 relative to σ, there is a string $(s_1 : b)$ in $csf(\sigma)$ such that $s_1 < b$. Hence $s_1 + 1$ appears after s_1 in the indices of σ which implies that (σ, s_1) satisfies the SIP, i.e., (σ, σ_r) satisfies the SIP. Now suppose that $k > 0$ and $\sigma_r = (s_1, s_2 \ldots, s_{k-1}, s_k)$. Then by the definition of right index tuple of type-1 we have $\widetilde{\sigma}_r := (s_1 \ldots, s_{k-1})$ is a right index tuple of type-1 relative to σ and s_k is a right index of type-1 relative to $z_r(\sigma, (s_1, \ldots, s_{k-1}))$, i.e., $z_r(\sigma, \widetilde{\sigma}_r)$. By the induction hypothesis, $(\sigma, \widetilde{\sigma}_r)$ satisfies the SIP. Now since s_k is a right index of type-1 relative to $z_r(\sigma, \widetilde{\sigma}_r)$ there is a string $(s_k : b)$ in $z_r(\sigma, \widetilde{\sigma}_r)$ such that $s_k < b$. By Definition 4.3.2 and Definition 4.3.3 it follows that $s_k + 1$ appears to the right of the last index s_k in $(\sigma, \widetilde{\sigma}_r)$. This which implies that $(\sigma, \widetilde{\sigma}_r, s_k)$ satisfies the SIP, i.e., (σ, σ_r) satisfies the SIP.

Next we show that (σ_l, σ) satisfies the SIP. Note that an index tuple α satisfies the SIP if and only if $rev(\alpha)$ satisfies the SIP. Since σ_l is a left index tuple of type-1 relative to σ, $rev(\sigma_l)$ is a right index tuple of type-1 relative to $rev(\sigma)$. So by the above case, $(rev(\sigma), rev(\sigma_l))$ satisfies the SIP. Then $(\sigma_l, \sigma) = rev(rev(\sigma), rev(\sigma_l))$ satisfies the SIP.

Since $\beta := \big(-m + rev(\sigma_r), -m + rev(\sigma), -m + rev(\sigma_l)\big) = -m + rev\big(\sigma_l, \sigma, \sigma_r\big)$ and $\big(\sigma_l, \sigma, \sigma_r\big)$ satisfies the SIP, we have β satisfies the SIP. This completes the proof. ☐

Let σ be a $\mathcal{P}$-admissible tuple relative to $\{0 : m\}$. Let σ_r and σ_l, respectively, be right and left index tuples of type-1 relative to σ with the condition that if an index $j \in \sigma_r \cup \sigma_l$ then $j - 1 \in \sigma$ for $j = 1 : m$. Let X and Y be any arbitrary nonsingular matrix assignments for σ_r and σ_l, respectively. Define $L(\lambda) :=$

$$M_{-m+rev(\sigma_r)}(-rev(X^T))M_{\sigma_l}(Y)\big(\lambda M^P_{-m+rev(\sigma)} - M^P_\sigma\big)M_{\sigma_r}(X)M_{-m+rev(\sigma_l)}(-rev(Y^T)). \tag{4.9}$$

Then by Lemma 4.3.6, $L(\lambda)$ is an EGFPR of $P(\lambda)$. We use pencils $L(\lambda)$ of the form (4.9) to construct T-palindromic linearizations of $P(\lambda)$. Since arbitrary matrix assignments are used to define $L(\lambda)$, the family of T-palindromic linearizations of $P(\lambda)$ that we construct in this chapter contains infinite number of pencils.

Definition 4.3.7. *[19] A matrix $S \in \mathbb{C}^{mn}$ is said to be a quasi-identity matrix if $S = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_m I_n$, where $\epsilon_i \in \{1, -1\}$, $i = 1 : m$. We refer to $(\epsilon_1, \ldots, \epsilon_m)$ as the parameters of S. For $i = 1 : m$, we denote the i-th diagonal block of S by $S(i, i)$. Also, we denote the* *quasi-identity matrix whose only negative parameter is ϵ_i by S_i. By default we denote by S_0 and S_{m+1} the identity matrix I_{mn}, i.e., $S_0 := I_{mn} =: S_{m+1}$.*

For rest of this section, we define $R := \begin{bmatrix} & & I_n \\ & \iddots & \\ I_n & & \end{bmatrix} \in \mathbb{C}^{mn}$. Note that $RR = I_{mn}$, and $SS = I_{mn}$ for any quasi-identity matrix S.

The following result will play a crucial role in constructing T-palindromic linearizations of $P(\lambda)$. For simplicity of notation, for the rest of this section we set $M_i^T(X) := (M_i(X))^T$ for $i \in \{\pm 0, \pm 1, \ldots, \pm m\}$, where $X \in \mathbb{C}^{n\times n}$ is any arbitrary matrix.

Proposition 4.3.8. *Let $X \in \mathbb{C}^{n\times n}$ be an arbitrary matrix. Then the elementary matrices $M_{\pm i}(X)$, $i = 0 : m$, satisfy the following.*

$$RM_{-i}(X)R = S_{i+1}M^T_{m-i}(-X^T)S_i \text{ and}$$

$$RM_{m-i}(X)R = S_{m+1-i}M^T_{-i}(-X^T)S_{m-i} \text{ for } \ i = 1 : m.$$

If X is invertible then both equalities hold for $i = 0$.

Proof. Easy to prove. □

We need the following results for constructing T-palindromic linearizations of $P(\lambda)$.

Theorem 4.3.9 ([27], Theorem 4.8). *Let σ be a $\mathcal{P}$-admissible tuple relative to $\{0 : m\}$. Define $T(\lambda) := \lambda M^P_{-m+rev(\sigma)} - M^P_\sigma$. Then $SRT(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where S is the quasi-identity matrix given by*

$$S(i,i) := \begin{cases} -I & \text{if } \begin{cases} \sigma \text{ has an inversion at } i-1, \text{ or} \\ \sigma \text{ has a consecution at } m-i, \text{ or} \\ i \in \sigma \text{ and } i-1 \notin \sigma \end{cases} \\ I & \text{otherwise.} \end{cases}$$

Lemma 4.3.10 ([19], Lemma 3.1). *Let $P(\lambda)$ be a T-palindromic matrix polynomial. If $\mathbb{T}(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$ and $\mathbb{L}(\lambda) = Q_1\mathbb{T}(\lambda)Q_2$ for some constant nonsingular matrices Q_1 and Q_2, then $Q_2^TQ_1^{-1}\mathbb{L}(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$.*

By utilizing the EGFPRs, we now construct T-palindromic strong linearizations of $P(\lambda)$.

Lemma 4.3.11. *Let σ be a $\mathcal{P}$-admissible tuple relative to $\{0 : m\}$ and let σ_r be a right index tuple of type-1 relative to σ with the condition that if an index $j \in \sigma_r$ then*

$j-1 \in \sigma$ for $j = 1 : m$. Let X be a nonsingular matrix assignment for σ_r. Define $L(\lambda) := M_{-m+rev(\sigma_r)}(-rev(X^T))\big(\lambda M^P_{-m+rev(\sigma)} - M^P_{\sigma}\big)M_{\sigma_r}(X)$. Then $SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where S is the quasi-identity matrix given by

$$S(i,i) := \begin{cases} -I & \textit{if} \begin{cases} \sigma \textit{ has an inversion at } i-1, \textit{or} \\ z_r(\sigma,\sigma_r) \textit{ has a consecution at } m-i, \textit{or} \\ i \in \sigma \textit{ and } i-1 \notin \sigma \end{cases} \\ I & \textit{otherwise.} \end{cases} \tag{4.10}$$

Proof. We prove the result by induction on the number q of indices of σ_r. If $q = 0$, i.e., $\sigma_r = \emptyset$ then $z_r(\sigma,\sigma_r) = \sigma$, and hence the result follows from Theorem 4.3.9. Now suppose that $q > 0$. Assume that the result is true when σ_r contain $q-1$ indices. Suppose that $\sigma_r = (s_1,\ldots,s_{q-1},s_q)$, where s_i denotes the i-th index in σ_r and let $\widetilde{\sigma}_r = (s_1,\ldots,s_{q-1})$. Let $X = (X_1, X_2, \ldots, X_{q-1}, X_q)$. Set $\widetilde{X} = (X_1, X_2, \ldots, X_{q-1})$. Define $\widetilde{L}(\lambda) := M_{-m+rev(\widetilde{\sigma}_r)}(-rev(\widetilde{X}^T))\Big(\lambda M^P_{-m+rev(\sigma)} - M^P_{\sigma}\Big)M_{\widetilde{\sigma}_r}(\widetilde{X})$. By the induction hypothesis, $\widetilde{S}R\widetilde{L}(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where $\widetilde{S}$ is the quasi-identity matrix given by

$$\widetilde{S}(i,i) := \begin{cases} -I & \text{if} \begin{cases} \sigma \text{ has an inversion at } i-1, \text{or} \\ z_r(\sigma,\widetilde{\sigma}_r) \text{ has a consecution at } m-i, \text{or} \\ i \in \sigma \text{ and } i-1 \notin \sigma \end{cases} \\ I & \text{otherwise.} \end{cases} \tag{4.11}$$

Note that

$$L(\lambda) = M_{-m+s_q}(-X_q^T)\widetilde{L}(\lambda)M_{s_q}(X_q) = M_{-m+s_q}(-X_q^T)R\widetilde{S}\big(\widetilde{S}R\widetilde{L}(\lambda)\big)M_{s_q}(X_q)$$

since $RR = I$ and $\widetilde{S}\widetilde{S} = I$. Now, by Lemma 4.3.10,

$$M^T_{s_q}(X_q)\Big(M_{-m+s_q}(-X_q^T)R\widetilde{S}\Big)^{-1}L(\lambda) = M^T_{s_q}(X_q)\widetilde{S}R\Big(M_{-m+s_q}(-X_q^T)\Big)^{-1}L(\lambda)$$

is a T-palindromic strong linearization of $P(\lambda)$. Hence $SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where

$$S = M^T_{s_q}(X_q)\widetilde{S}R\Big(M_{-m+s_q}(-X_q^T)\Big)^{-1}R. \tag{4.12}$$

Thus we only need to prove that S is the quasi-identity matrix satisfying (4.10).

Case-I: Assume that $s_q > 0$. For $j = 1 : m-1$, we have $(M_j(Y))^{-1} = M_{-j}(-Y)$ for any matrix assignment Y. Thus by (4.12), $S = M^T_{s_q}(X_q)\widetilde{S}RM_{m-s_q}(X_q^T)R$. Now, by Proposition 4.3.8, we have

$$S = M^T_{s_q}(X_q)\widetilde{S}S_{m+1-s_q}M^T_{-s_q}(-X_q)S_{m-s_q}. \tag{4.13}$$

Now, since s_q is a right index of type-1 relative to $z_r(\sigma, \widetilde{\sigma}_r)$, there exist a string $(a_i + k_i : a_{i+1})$ in $z_r(\sigma, \widetilde{\sigma}_r)$ such that $s_q = a_i + k_i < a_{i+1}$. Next we show that $M_{s_q}^T(X_q)$ and $\widetilde{S}S_{m+1-s_q}$ commute. Since

$$M_{s_q}^T(X_q) = \begin{bmatrix} I_{(m-s_q-1)n} & & & \\ & X_q^T & I_n & \\ & I_n & 0 & \\ & & & I_{(s_q-1)n} \end{bmatrix},$$

it is enough to show that both $(m-s_q)$-th and $(m-s_q+1)$-th parameters of $\widetilde{S}S_{m-s_q+1}$ have the same sign. Recall from the definition of quasi-identity matrix that $(m-s_q+1)$-th parameter is the only negative parameter of S_{m-s_q+1}. Consequently, it is enough to show that both $(m-s_q)$-th and $(m-s_q+1)$-th parameters of $\widetilde{S}$ have the opposite sign. Since $s_q \in \sigma_r$, we have $s_q, s_q+1 \in \sigma$. Again, recall the assumption that if $j \in \sigma_r$ then $j-1 \in \sigma$ for $j = 1 : m-1$. Since $s_q \in \sigma_r$, by the assumption we have $s_q - 1 \in \sigma$. Thus $s_q-1, s_q, s_q+1 \in \sigma$. As σ is a $\mathcal{P}$-admissible tuple we have $m-s_q-1, m-s_q, m-s_q+1 \notin \sigma$. Hence by (4.11), the parameter of $\widetilde{S}$ at the position $(m-s_q)$-th (resp., $(m-s_q+1)$-th) is -1 if and only if $z_r(\sigma, \widetilde{\sigma}_r)$ has a consecution at s_q (resp., $s_q - 1$). Note that s_q is a right index of type-1 relative to $z_r(\sigma, \widetilde{\sigma}_r)$. Hence $z_r(\sigma, \widetilde{\sigma}_r)$ must be of the form

$$z_r(\sigma, \widetilde{\sigma}_r) = \Big((a_{d-1} + k_{d-1} : a_d), \ldots, (s_q : a_{i+1}), (a_{i-1} + k_{i-1} : a_i), \ldots, (a_0 : a_1)\Big)$$

with $s_q < a_{i+1}$. Thus $z_r(\sigma, \widetilde{\sigma}_r)$ has a consecution at s_q which implies that the parameters of $\widetilde{S}$ at the position $(m-s_q)$-th is -1. Since $s_q - 1 \in \sigma$ we have $a_i = s_q - 1$. Hence

$$z_r(\sigma, \widetilde{\sigma}_r) = \Big((a_{d-1} + k_{d-1} : a_d), \ldots, (s_q : a_{i+1}), (a_{i-1} + k_{i-1} : s_q - 1), \ldots, (a_0 : a_1)\Big).$$

Thus $z_r(\sigma, \widetilde{\sigma}_r)$ has an inversion at $s_q - 1$ which implies that $(m-s_q+1)$-th parameter of $\widetilde{S}$ is 1. Hence $(m-s_q)$-th and $(m-s_q+1)$-th parameters of $\widetilde{S}S_{m-s_q+1}$ are the same. Thus $M_{s_q}^T(X_q)$ and $\widetilde{S}S_{m+1-s_q}$ commute. Hence by (4.13), we have

$$S = \widetilde{S}S_{m+1-s_q}M_{s_q}^T(X_q)M_{-s_q}^T(-X_q)S_{m-s_q} = \widetilde{S}S_{m-s_q+1}S_{m-s_q}. \tag{4.14}$$

as $M_{s_q}^T(X_q)M_{-s_q}^T(-X_q) = I_{mn}$. Now we prove that S is the quasi-identity matrix satisfying (4.10). Observe that by (4.14), the quasi-identity matrices S and $\widetilde{S}$ have the same parameters except at the positions $(m-s_q)$ and $(m-s_q+1)$. So we only need to prove that S and $\widetilde{S}$ have opposite parameters at the positions $(m-s_q)$ and $(m-s_q+1)$. Now by Definition 4.3.2, we have

$$z_r(\sigma, \sigma_r) = \Big((a_{d-1} + k_{d-1} : a_d), \ldots, (s_q + 1 : a_{i+1}), (a_{i-1} + k_{i-1} : s_q), \ldots, (a_0 : a_1)\Big).$$

Thus the inversions and consecutions that occur in $z_r(\sigma, \widetilde{\sigma}_r)$ and $z_r(\sigma, \sigma_r)$ are the same except

- at s_q where $z_r(\sigma, \widetilde{\sigma}_r)$ has a consecution and $z_r(\sigma, \sigma_r)$ has an inversion, and
- at $s_q - 1$ where $z_r(\sigma, \widetilde{\sigma}_r)$ has an inversion and $z_r(\sigma, \sigma_r)$ has a consecution.

Hence we have

- $(m - s_q)$-th parameters of $\widetilde{S}$ and S are -1 and 1, respectively, and
- $(m - s_q + 1)$-th parameters of $\widetilde{S}$ and S are 1 and -1, respectively.

Thus the parameter of S and $\widetilde{S}$ at the positions $(m - s_q)$ and $(m - s_q + 1)$ have opposite sign.

Case-II: Assume that $s_q = 0$. Since X is a nonsingular matrix assignment for σ_r, X_q is a nonsingular matrix. This implies $M_0(X_q)$ is invertible. By (4.12), we have

$$S = M_0^T(X_q)\widetilde{S}R\Big(M_{-m}(-X_q^T)\Big)^{-1}R = M_0^T(X_q)\widetilde{S}RM_m(-X_q^T)R$$

as $M_m(Y) = (M_{-m}(Y))^{-1} = M_{-m}(Y^{-1})$ for any nonsingular $Y \in \mathbb{C}^{n\times n}$. Now, by Proposition 4.3.8, $S = M_0^T(X_q)\widetilde{S}S_{m+1}M_{-0}^T(X_q)S_m = M_0^T(X_q)\widetilde{S}M_{-0}^T(X_q)S_m$ as $S_{m+1} = I_{mn}$. Since $M_0^T(X_q) = \begin{bmatrix} I_{(m-1)n} & \\ & X_q^T \end{bmatrix}$, it trivially follows that $M_0^T(X_q)$ and $\widetilde{S}$ commute, which implies that $S = \widetilde{S}M_0^T(X_q)M_{-0}^T(X_q)S_m = \widetilde{S}S_m$ as $M_0^T(X_q)M_{-0}^T(X_q) = I_{mn}$. Since $s_q = 0$ is a right index of type-1 relative to $z_r(\sigma, \widetilde{\sigma}_r)$, we must have

$$z_r(\sigma, \widetilde{\sigma}_r) = \Big((a_{d-1} + k_{d-1} : a_d), \ldots, (a_1 + k_1 : a_2), (0 : a_1)\Big), \text{ where } a_1 > 0,$$

and

$$z_r(\sigma, \sigma_r) = \Big((a_{d-1} + k_{d-1} : a_d), \ldots, (a_1 + k_1 : a_2), (1 : a_1), 0\Big).$$

By similar arguments as in Case-I, it follows that S is the quasi-identity matrix satisfying (4.10). This completes the proof. □

Theorem 4.3.12. *Let σ be a $\mathcal{P}$-admissible tuple relative to $\{0 : m\}$. Let σ_r and σ_l, respectively, be a right and a left index tuple of type-1 relative to σ with the condition that if an index $j \in \sigma_r \cup \sigma_l$ then $j - 1 \in \sigma$ for $j = 1 : m$. Let X and Y be any arbitrary nonsingular matrix assignments for σ_r and σ_l, respectively. Define $L(\lambda) :=$*

$$M_{-m+rev(\sigma_r)}(-rev(X^T))M_{\sigma_l}(Y)\big(\lambda M^P_{-m+rev(\sigma)} - M^P_{\sigma}\big)M_{\sigma_r}(X)M_{-m+rev(\sigma_l)}(-rev(Y^T)).$$

Then $SRL(\lambda)$ *is a* T*-palindromic strong linearization of* $P(\lambda)$*, where* S *is the quasi-identity matrix given by*

$$S(i,i) := \begin{cases} -I & \textit{if} \begin{cases} z_l(\sigma_l,\sigma) \textit{ has an inversion at } i-1, \textit{or} \\ z_r(\sigma,\sigma_r) \textit{ has a consecution at } m-i, \textit{or} \\ i\in\sigma \textit{ and } i-1\notin\sigma \end{cases} \\ I & \textit{otherwise.} \end{cases} \tag{4.15}$$

Proof. We prove the result by induction on the number q of indices of σ_l. If $q=0$ then the result follows from Lemma 4.3.11. Now suppose that $q>0$. Assume that the result is true when σ_l contain $q-1$ indices. Suppose that $\sigma_l=(s_q,s_{q-1},\ldots,s_1)$, where s_i denotes an index in σ_l and let $\widetilde{\sigma}_l=(s_{q-1},\ldots,s_1)$. Let $Y=(Y_q,Y_{q-1},\ldots,Y_2,Y_1)$. Set $\widetilde{Y}=(Y_{q-1},\ldots,Y_2,Y_1)$. Define $\widetilde{L}(\lambda):=$

$$M_{-m+rev(\sigma_r)}(-rev(X^T))M_{\widetilde{\sigma}_l}(\widetilde{Y})\Big(\lambda M^P_{-m+rev(\sigma)}-M^P_{\sigma}\Big)M_{\sigma_r}(X)M_{-m+rev(\widetilde{\sigma}_l)}(-rev(\widetilde{Y}^T)).$$

By the induction hypothesis, $\widetilde{S}R\widetilde{L}(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where $\widetilde{S}$ is the quasi-identity matrix given by

$$\widetilde{S}(i,i) = \begin{cases} -I & \text{if} \begin{cases} z_l(\widetilde{\sigma}_l,\sigma) \text{ has an inversion at } i-1, \text{or} \\ z_r(\sigma,\sigma_r) \text{ has a consecution at } m-i, \text{or} \\ i\in\sigma \text{ and } i-1\notin\sigma \end{cases} \\ I & \text{otherwise.} \end{cases} \tag{4.16}$$

Recall the assumption that if an index $j\in\sigma_l\cup\sigma_r$ then $j-1\in\sigma$ for $j=1:m$. Let $i\in\sigma_l\cup\sigma_r$. Then $i-1,i,i+1\in\sigma$. As σ is a $\mathcal{P}$-admissible tuple we have $m-i-1,m-i,m-i+1\notin\sigma$. Hence $m-i-1,m-i,m-i+1\notin\sigma_l\cup\sigma_r$, i.e., $-i-1,-i,-i+1\notin -m+rev(\sigma_l)\cup -m+rev(\sigma_r)$. So σ_l and $-m+rev(\sigma_r)$ commute, and σ_r and $-m+rev(\sigma_l)$ commute. Thus $L(\lambda)=M_{s_q}(Y_q)\widetilde{L}(\lambda)M_{-m+s_q}(-Y_q^T)=M_{s_q}(Y_q)R\widetilde{S}\big(\widetilde{S}R\widetilde{L}(\lambda)\big)M_{-m+s_q}(-Y_q^T)$ as $RR=I$ and $\widetilde{S}\widetilde{S}=I$. Now, by Lemma 4.3.10,

$$M^T_{-m+s_q}(-Y_q^T)\Big(M_{s_q}(Y_q)R\widetilde{S}\Big)^{-1}L(\lambda)=M^T_{-m+s_q}(-Y_q^T)\widetilde{S}R\Big(M_{s_q}(Y_q)\Big)^{-1}L(\lambda)$$

is a T-palindromic strong linearization of $P(\lambda)$. Hence $SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where

$$S=M^T_{-m+s_q}(-Y_q^T)\widetilde{S}R\Big(M_{s_q}(Y_q)\Big)^{-1}R. \tag{4.17}$$

Thus we only need to prove that S is the quasi-identity matrix satisfying (4.15).

Case-I: Assume that $s_q > 0$. So $\Big(M_{s_q}(Y_q)\Big)^{-1} = M_{-s_q}(-Y_q)$ and hence by (4.17), $S = M^T_{-m+s_q}(-Y_q^T)\widetilde{S}RM_{-s_q}(-Y_q)R$. Now by Proposition 4.3.8, we have

$$S = M^T_{-m+s_q}(-Y_q^T)\widetilde{S}S_{s_q+1}M^T_{m-s_q}(Y_q^T)S_{s_q}. \tag{4.18}$$

Since s_q is a left index of type-1 relative to $z_l(\widetilde{\sigma}_l,\sigma)$, i.e., s_q is a right index of type-1 relative to $rev(z_l(\widetilde{\sigma}_l,\sigma))$, there exist a string $(a_i+k_i : a_{i+1})$ in $rev(z_l(\widetilde{\sigma}_l,\sigma))$ such that $s_q = a_i + k_i < a_{i+1}$. Next we show that $M^T_{m-s_q}(Y_q^T)$ and $\widetilde{S}S_{s_q+1}$ commute. Since

$$M^T_{m-s_q}(Y_q^T) = \begin{bmatrix} I_{(s_q-1)n} & & & \\ & Y_q & I_n & \\ & I_n & 0 & \\ & & & I_{(m-s_q-1)n} \end{bmatrix},$$

it is enough to show that both (s_q)-th and (s_q+1)-th parameters of $\widetilde{S}S_{s_q+1}$ have the same sign. Since (s_q+1)-th parameter is the only negative parameter of S_{s_q+1}, it is enough to show that (s_q)-th and (s_q+1)-th parameters of $\widetilde{S}$ have the opposite sign. Note that for any index $j \in \sigma_l$, we have $j, j+1 \in \sigma$. Thus, since $s_q \in \sigma_l$, we have $s_q, s_q+1 \in \sigma$. Again, recall the assumption that if $j \in \sigma_l$ then $j-1 \in \sigma$ for $j = 1 : m-1$. Since $s_q \in \sigma_l$, we have $s_q - 1 \in \sigma$. Thus $s_q-1, s_q, s_q+1 \in \sigma$. As σ is a $\mathcal{P}$-admissible tuple we have $m-(s_q-1), m-s_q, m-(s_q+1) \notin \sigma$. Hence by (4.16), the parameter of $\widetilde{S}$ at the position (s_q)-th (resp., (s_q+1)-th) is -1 if and only if $z_l(\widetilde{\sigma}_l,\sigma)$ has an inversion at s_q-1 (resp., s_q).

Note that s_q is a left index of type-1 relative to $z_l(\widetilde{\sigma}_l,\sigma)$, i.e., s_q is a right index of type-1 relative to $rev(z_l(\widetilde{\sigma}_l,\sigma))$. Hence $rev(z_l(\widetilde{\sigma}_l,\sigma))$ must be of the form

$$rev(z_l(\widetilde{\sigma}_l,\sigma)) = \Big((a_{d-1}+k_{d-1} : a_d), \ldots, (s_q : a_{i+1}), (a_{i-1}+k_{i-1} : a_i), \ldots, (a_0 : a_1)\Big),$$

with $s_q < a_{i+1}$. As $s_q - 1 \in \sigma$, we have $a_i = s_q - 1$. Hence $rev(z_l(\widetilde{\sigma}_l,\sigma))$ must be of the form

$$rev(z_l(\widetilde{\sigma}_l,\sigma)) = \Big((a_{d-1}+k_{d-1} : a_d), \ldots, (s_q : a_{i+1}), (a_{i-1}+k_{i-1} : s_q-1), \ldots, (a_0 : a_1)\Big). \tag{4.19}$$

Thus $rev(z_l(\widetilde{\sigma}_l,\sigma))$ has an inversion at s_q-1 and a consecution at s_q which imply that $z_l(\widetilde{\sigma}_l,\sigma)$ has a consecution at s_q-1 and an inversion at s_q. Hence the parameters of $\widetilde{S}$ at the position (s_q)-th is 1 and at the position (s_q+1)-th is -1, i.e., (s_q)-th and (s_q+1)-th parameters of $\widetilde{S}$ have the opposite sign. Thus $M^T_{m-s_q}(Y_q^T)$ and $\widetilde{S}S_{s_q+1}$ commute. By (4.18), we have

$$S = M^T_{-m+s_q}(-Y_q^T)M^T_{m-s_q}(Y_q^T)\widetilde{S}S_{s_q+1}S_{s_q} = \widetilde{S}S_{s_q+1}S_{s_q} \tag{4.20}$$

as $M^T_{-m+s_q}(-Y_q^T)M^T_{m-s_q}(Y_q^T) = I_{mn}$. Now we prove that S is the quasi-identity matrix satisfying (4.15). Note that by (4.20) the quasi-identity matrices S and $\widetilde{S}$ have the same parameters except at the positions s_q and (s_q+1). So we only need to prove that the parameters of S and $\widetilde{S}$ at the positions s_q and (s_q+1) have opposite sign.

By Definition 4.3.3, we have

$$\begin{aligned} rev(z_l(\sigma_l,\sigma)) &= z_r(rev(\sigma), rev(\sigma_l)) = z_r(z_r(rev(\sigma), rev(\widetilde{\sigma}_l)), s_q) \\ &= z_r(rev(z_l(\widetilde{\sigma}_l,\sigma)), s_q) \end{aligned} \tag{4.21}$$

Hence by (4.19) and (4.21), we have

$$rev(z_l(\sigma_l,\sigma)) = \Big((a_{d-1}+k_{d-1}:a_d),\ldots,(s_q+1:a_{i+1}),(a_{i-1}+k_{i-1}:s_q),\ldots,(a_0:a_1)\Big).$$

Note that the cosecutions and inversions that occur in $rev(z_l(\widetilde{\sigma}_l,\sigma))$ are the same as those that occur in $rev(z_l(\sigma_l,\sigma))$, except

- at s_q-1 where $rev(z_l(\widetilde{\sigma}_l,\sigma))$ has an inversion and $rev(z_l(\sigma_l,\sigma))$ has a consecution, i.e., $z_l(\widetilde{\sigma}_l,\sigma)$ has a consecution at s_q-1 and $z_l(\sigma_l,\sigma)$ has an inversion at s_q-1.
- at s_q where $rev(z_l(\widetilde{\sigma}_l,\sigma))$ has a consecution and $rev(z_l(\sigma_l,\sigma))$ has an inversion, i.e., $z_l(\widetilde{\sigma}_l,\sigma)$ has an inversion at s_q and $z_l(\sigma_l,\sigma)$ has a consecution at s_q.

Hence

- (s_q)-th parameters of $\widetilde{S}$ and S are 1 and -1, respectively, and
- (s_q+1)-th parameters of $\widetilde{S}$ and S are -1 and 1, respectively.

Hence the parameters of S and $\widetilde{S}$ at the positions s_q and s_q+1 have opposite sign.

Case II: Assume that $s_q=0$. Since Y is a nonsingular matrix assignment for σ_l, Y_q is a nonsingular matrix. So $M_0(Y_q)$ is invertible. By (4.17), $S = M^T_{-m}(-Y_q^T)\widetilde{S}R\Big(M_0(Y_q)\Big)^{-1}R$ $= M^T_{-m}(-Y_q^T)\widetilde{S}RM_{-0}(Y_q)R$. Now, by Proposition 4.3.8, $S = M^T_{-m}(-Y_q^T)\widetilde{S}S_1M^T_m(-Y_q^T)S_0$ $= M^T_{-m}(-Y_q^T)\widetilde{S}S_1M^T_m(-Y_q^T)$ as $S_0 = I_{mn}$. Since $M^T_{-m}(-Y_q^T) = \begin{bmatrix} -Y_q & \\ & I_{(m-1)n}\end{bmatrix}$, it is easy to see that $M^T_{-m}(-Y_q^T)$ and $\widetilde{S}S_1$ commute, which implies that

$$S = \widetilde{S}S_1M^T_{-m}(-Y_q^T)M^T_m(-Y_q^T) = \widetilde{S}S_1 \text{ as } M^T_{-m}(-Y_q^T)M^T_m(-Y_q^T) = I_{mn}.$$

Since $s_q=0$ is a left index of type-1 relative to $z_l(\widetilde{\sigma}_l,\sigma)$, i.e., 0 is a right index of type-1 relative to $rev(z_l(\widetilde{\sigma}_l,\sigma))$, $rev(z_l(\widetilde{\sigma}_l,\sigma))$ must be of the form

$$rev(z_l(\widetilde{\sigma}_l,\sigma)) = \Big((a_{d-1}+k_{d-1}:a_d),\ldots,(a_1+k_1:a_2),(0:a_1)\Big) \text{ with } a_1>0.$$

Then we have

$$rev(z_l(\sigma_l,\sigma)) = \Big((a_{d-1}+k_{d-1}:a_d),\ldots,(a_1+k_1:a_2),(1:a_1),0\Big).$$

By similar arguments as in Case-I, it follows that S is the quasi-identity matrix satisfying (4.15). This completes the proof. □

The following example construct a T-palindromic pencil of $P(\lambda)$.

Example 4.3.13. *Let* $P(\lambda) = \sum_{i=0}^{7} \lambda^i A_i$ *be* T*-palindromic. Consider* $\sigma = (4,0,1,2)$, $\sigma_r = (0,1,0)$ *and* $\sigma_2 = \emptyset$*. Then* σ *is a* $\mathcal{P}$*-admissible tuple relative to* $\{0:7\}$ *and* σ_r *is a right index tuple of type-1 relative to* σ*. Let* (X,Y,Z) *be a nonsingular matrix assignment for* σ_r*. Define*

$$L(\lambda) := M_{(-7,-6,-7)}(-Z^T,-Y^T,-X^T)\big(\lambda M^P_{(-5,-6,-7,-3)} - M^P_{(4,0,1,2)}\big)M_{(0,1,0)}(X,Y,Z).$$

Let S *be the quasi-identity matrix given by* $S = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_7 I_n$*, where* $\epsilon_i = -1$ *for* $i = 4$ *and* $\epsilon_i = 1$ *if* $i \neq 4$*, i.e., parameters of* S *are* $(1,1,1,-1,1,1,1)$*. Then by Lemma 4.3.11,* $SRL(\lambda)$ *is a* T*-palindromic strong linearization of* $P(\lambda)$ *and is given by*

$$\lambda \left[\begin{array}{ccc|cccc}
0 & 0 & 0 & 0 & 0 & X & 0 \\
0 & 0 & 0 & 0 & 0 & Y & Z \\
0 & 0 & 0 & I_n & A_3 & 0 & 0 \\
0 & 0 & 0 & 0 & -I_n & 0 & 0 \\
\hline
A_7 & A_6 & A_5 & 0 & 0 & 0 & 0 \\
0 & -X^T & -Y^T & 0 & 0 & 0 & 0 \\
0 & 0 & -Z^T & 0 & 0 & 0 & 0
\end{array}\right] + \left[\begin{array}{cccc|ccc}
0 & 0 & 0 & 0 & A_0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_1 & -X & 0 \\
0 & 0 & 0 & 0 & A_2 & -Y & -Z \\
\hline
0 & 0 & I_n & 0 & 0 & 0 & 0 \\
0 & 0 & A_4 & -I_n & 0 & 0 & 0 \\
X^T & Y^T & 0 & 0 & 0 & 0 & 0 \\
0 & Z^T & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Observe that $SRL(\lambda)$ *is an anti-block penta-diagonal pencil.* ■

Next, we consider an example where the matrix assignments X for σ_r and Y for σ_l in Theorem 4.3.12 are the trivial matrix assignment.

Example 4.3.14. *Let* $P(\lambda) = \sum_{i=0}^{7} \lambda^i A_i$ *be* T*-palindromic. Consider* $\sigma = (5:6,4,0)$ *and* $\sigma_r = 5$*. Let* $L(\lambda) := M^P_{-2}\big(\lambda M^P_{(-7,-3,-1,-2)} - M^P_{(5:6,4,0)}\big)M^P_5$*. Then by Theorem 4.3.12,*

$SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$ and is given by

$$\lambda\left[\begin{array}{ccc|cccc} 0 & 0 & 0 & 0 & I_n & A_2 & A_1 \\ 0 & 0 & 0 & I_n & 0 & A_3 & A_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_n \\ 0 & 0 & 0 & 0 & 0 & -I_n & 0 \\ \hline 0 & -I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_5 & I_n & 0 & 0 & 0 & 0 \\ A_7 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & A_0 \\ 0 & 0 & 0 & 0 & -I_n & -A_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ \hline 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ A_5 & A_4 & 0 & -I_n & 0 & 0 & 0 \\ A_6 & A_5 & -I_n & 0 & 0 & 0 & 0 \end{array}\right],$$

where $S = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_7 I_n$, with $\epsilon_i = 1$ for $i = 1, 2, 6, 7$, and $\epsilon_i = -1$ for $i = 3, 4, 5$. Observe that $SRL(\lambda)$ is an anti block penta-diagonal pencil. $\blacksquare$

The following result is a corollary to Theorem 4.3.12.

Corollary 4.3.15. *Let $P(\lambda)$ be a T-palindromic matrix polynomial of odd degree $m \geq 3$. Let σ be a $\mathcal{P}$-admissible tuple relative to $\{0 : m\}$. Let σ_l be a left index tuple of type-1 relative to σ with the condition that if an index $j \in \sigma_l$ then $j-1 \in \sigma$ for $j = 1 : m$. Let Y be any arbitrary nonsingular matrix assignment for σ_l. Define*

$$L(\lambda) := M_{\sigma_l}(Y)\big(\lambda M^P_{-m+rev(\sigma)} - M^P_\sigma\big) M_{-m+rev(\sigma_l)}(-rev(Y^T)).$$

Then $SRL(\lambda)$ is a T-palindromic strong linearization of $P(\lambda)$, where S is the quasi-identity matrix given by

$$S(i,i) := \begin{cases} -I & \text{if } \begin{cases} z_l(\sigma_l, \sigma) \text{ has an inversion at } i-1, \text{ or} \\ \sigma \text{ has a consecution at } m-i, \text{ or} \\ i \in \sigma \text{ and } i-1 \notin \sigma \end{cases} \\ I & \text{otherwise.} \end{cases}$$

More generally, we have the following result.

Corollary 4.3.16. *Let $P(\lambda)$ be a T-palindromic matrix polynomial of odd degree $m \geq 3$. Let σ be a $\mathcal{P}$-admissible tuple relative to $\{0 : m\}$. Let σ_r and σ_l, respectively, be a right and a left index tuple of type-1 relative to σ with the condition that if an index $j \in \sigma_r \cup \sigma_l$ then $j-1 \in \sigma$ for $j = 1 : m$. Let X and Y be any arbitrary matrix assignments for σ_r and σ_l, respectively. Define $L(\lambda) :=$*

$$M_{-m+rev(\sigma_r)}(-rev(X^T)) M_{\sigma_l}(Y)\big(\lambda M^P_{-m+rev(\sigma)} - M^P_\sigma\big) M_{\sigma_r}(X) M_{-m+rev(\sigma_l)}(-rev(Y^T)).$$

Then $L(\lambda)RS$ is a T-palindromic strong linearization of $P(\lambda)$ where S is the quasi-identity matrix given by

$$S(i,i) := \begin{cases} -I & \textit{if} \begin{cases} z_l(\sigma_l,\sigma) \textit{ has an inversion at } m-i, \textit{or} \\ z_r(\sigma,\sigma_r) \textit{ has a consecution at } i-1, \textit{or} \\ i \in \sigma \textit{ and } i-1 \notin \sigma \end{cases} \\ I & \textit{otherwise.} \end{cases} \tag{4.22}$$

Proof. Note that $P(\lambda)^T$ is a T-palindromic matrix polynomial. Also, σ_r (resp., σ_l) is a right (resp., left) index tuple of type-1 relative to σ if and only if $rev(\sigma_r)$ (resp., $rev(\sigma_l)$) is a left (resp., right) index tuple of type-1 relative to $rev(\sigma)$. This implies that $L(\lambda)^T$ is a strong linearization of $P(\lambda)^T$. Therefore, by Theorem 4.3.12, $SRL(\lambda)^T = (L(\lambda)RS)^T$ is a T-palindromic strong linearization of $P(\lambda)^T$, where S is the quasi-identity matrix given by

$$S(i,i) := \begin{cases} -I & \text{if} \begin{cases} z_l(rev(\sigma_r),rev(\sigma)) \text{ has an inversion at } i-1, \text{or} \\ z_r(rev(\sigma),rev(\sigma_l)) \text{ has a consecution at } m-i, \text{or} \\ i \in \sigma \text{ and } i-1 \notin \sigma \end{cases} \\ I & \text{otherwise.} \end{cases} \tag{4.23}$$

Since $(L(\lambda)RS)^T$ is a T-palindromic strong linearization of $P(\lambda)^T$, $L(\lambda)RS$ is a T-palindromic strong linearization of $P(\lambda)$. Hence the desired result follows as S given in (4.23) and (4.22) are the same. □

Next, we construct low bandwidth banded (anti-block tridiagonal and anti-block penta-diagonal) T-palindromic linearizations of $P(\lambda)$. The following result characterizes all anti-block penta-diagonal T-palindromic linearizations that can be obtained from the construction given in Theorem 4.3.12.

Theorem 4.3.17. *Let $P(\lambda), L(\lambda), S$ and R be as given in Theorem 4.3.12. Then $SRL(\lambda)$ is an anti-block penta-diagonal T-palindromic strong linearization of $P(\lambda)$ if and only if $c_t(\sigma_l,\sigma,\sigma_r) \leq 1$ and $i_t(\sigma_l,\sigma,\sigma_r) \leq 1$ for any index $1 \leq t \leq m-1$.*

Proof. Note that, for any index tuple α containing indices from $\{-m : m\}$, $M_\alpha(*)$ is a block penta-diagonal matrix if and only if $RM_\alpha(*)$ is an anti-block penta-diagonal matrix, where $(*)$ denotes any arbitrary matrix assignment for α. Since S is a quasi-identity matrix, it follows that $SRL(\lambda)$ is anti-block penta-diagonal if and only if $L(\lambda)$ is block penta-diagonal.

Recall the definition of an end index of a sub-permutation of $\{0 : m-1\}$ (resp., $\{-m : -1\}$) from Definition 3.5.5. It follows from the condition $j \in \sigma_r \cup \sigma_l \implies j-1 \in \sigma$, $j = 1 : m$, that σ_r and σ_l do not contain the end points of σ. This implies that $-m + rev(\sigma_r)$ and $-m + rev(\sigma_l)$ do not contain the end points of $-m + rev(\sigma)$. Hence the desired result follows from Theorem 3.5.6 by considering $\sigma_1 := \sigma_l$, $\sigma_2 := \sigma_r$, $\tau := -m + rev(\sigma)$, $\tau_1 := -m + rev(\sigma_r)$ and $\tau_2 := -m + rev(\sigma_l)$. $\square$

Note that, for any index tuple α containing indices from $\{-m : m\}$, $M_\alpha(*)$ is a block tridiagonal matrix if and only if $RM_\alpha(*)$ is an anti-block tridiagonal matrix, where $(*)$ denotes any arbitrary matrix assignment for α. The following result characterizes all anti-block tridiagonal T-palindromic linearizations that can be obtained from the construction given in Theorem 4.3.12.

Theorem 4.3.18. *Let $P(\lambda), L(\lambda), S$ and R be as given in Theorem 4.3.12. Then $SRL(\lambda)$ is an anti-block tridiagonal T-palindromic strong linearization of $P(\lambda)$ if and only if $c_t(\sigma_l, \sigma, \sigma_r) = 0$ and $i_t(\sigma_l, \sigma, \sigma_r) = 0$ for any index $1 \leq t \leq m-1$. Moreover, any anti-block tridiagonal $SRL(\lambda)$ in Theorem 4.3.12 must be one of the following pencils*

$$\begin{bmatrix} & & & & & & A_0 & \lambda X \\ & & & & & \lambda I_n & \lambda A_2 + A_1 & -X \\ & & & & I_n & 0 & -\lambda I_n & \\ & & & \lambda I_n & \lambda A_4 + A_3 & -I_n & & \\ & & I_n & 0 & -\lambda I_n & & & \\ & \iddots & \iddots & \iddots & & & & \\ \lambda A_m & \lambda A_{m-1} + A_{m-2} & -I_n & & & & & \\ X^T & -\lambda X^T & & & & & & \end{bmatrix} \tag{4.24}$$

for some nonsingular matrix X,

$$\begin{bmatrix} & & & & & & -Y & \lambda Y \\ & & & & & \lambda I_n & \lambda A_2 + A_1 & A_0 \\ & & & & I_n & 0 & -\lambda I_n & \\ & & & \lambda I_n & \lambda A_4 + A_3 & -I_n & & \\ & & I_n & 0 & -\lambda I_n & & & \\ & \iddots & \iddots & \iddots & & & & \\ -\lambda Y^T & \lambda A_{m-1} + A_{m-2} & -I_n & & & & & \\ Y^T & \lambda A_m & & & & & & \end{bmatrix} \tag{4.25}$$

for some nonsingular matrix Y *and*

$$\left[\begin{array}{cccccccccc}
 & & & & & & & & \lambda I_n & \lambda A_1 + A_0 \\
 & & & & & & & I_n & 0 & -\lambda I_n \\
 & & & & & & \lambda I_n & \lambda A_3 + A_2 & -I_n & \\
 & & & & & I_n & 0 & -\lambda I_n & & \\
 & & & \iddots & & \iddots & \iddots & & & \\
 & \lambda I_n & \lambda A_{m-2} + A_{m-3} & -I_n & & & & & & \\
I_n & 0 & -\lambda I_n & & & & & & & \\
\lambda A_m + A_{m-1} & -I_n & & & & & & & &
\end{array}\right]. \tag{4.26}$$

Proof. By Theorem 3.5.8 and similar arguments as in the proof of Theorem 4.3.17, it follows that $SRL(\lambda)$ is an anti-block tridiagonal T-palindromic strong linearization of $P(\lambda)$ if and only if $c_t(\sigma_l, \sigma, \sigma_r) = 0$ and $i_t(\sigma_l, \sigma, \sigma_r) = 0$ for any index $1 \leq t \leq m-1$.

We now show that any anti-block tridiagonal $SRL(\lambda)$ must be one of the pencils given in (4.24), (4.25) and (4.26). Since $c_t(\sigma_l, \sigma, \sigma_r) = 0$ and $i_t(\sigma_l, \sigma, \sigma_r) = 0$ for all $1 \leq t \leq m-1$, σ must be a permutation of either $\{0 :_2 m-1\}$ or $\{0, 1 :_2 m-2\}$, where $\{a :_2 b\} := \{a + 2j : j = 0 : \ell\}$ are all equispaced points in $[a, b]$ such that $a + 2\ell \leq b$. Then we have the following choices for σ_l and σ_r.

Case-I: Suppose that σ is a permutation of $\{0 :_2 m-1\}$. Then there are no choices for σ_l and σ_r (i.e., $\sigma_l = \emptyset = \sigma_r$) and $SRL(\lambda)$ is of the form (4.26), where the quasi identity matrix S is given by $S(i,i) = -1$ for $i \in \{2 :_2 m-1\}$ and $S(i,i) = 1$ for $i \notin \{2 :_2 m-1\}$. In this case $L(\lambda)$ is a GF pencil.

Case-II: Suppose that σ is a permutation of $\{0, 1 :_2 m-2\}$.

(a) Suppose that σ has a consecution at 0. Then $\sigma_l = \emptyset$, and $\sigma_r = (0)$ or $\emptyset$. If $\sigma_r = (0)$ then $SRL(\lambda)$ is of the form (4.24) for some nonsingular matrix assignment X of σ_r, where S is given by $S(i,i) = -1$ for $i \in \{3 :_2 m-2\}$ and $S(i,i) = 1$ for $i \notin \{3 :_2 m-2\}$. In this case $L(\lambda)$ is an EGFPR.

On the other hand, if $\sigma_r = \emptyset$, then $SRL(\lambda)$ is of the form (4.24) with $X = I_n$, where S is given by $S(i,i) = -1$ for $i \in \{3 :_2 m\}$ and $S(i,i) = 1$ for $i \notin \{3 :_2 m\}$. In this case $L(\lambda)$ is a GF pencil.

(b) Suppose that σ has an inversion at 0. Then $\sigma_r = \emptyset$, and $\sigma_l = (0)$ or $\emptyset$. If $\sigma_l = (0)$ then $SRL(\lambda)$ is of the form (4.25) for some nonsingular matrix assignment Y of σ_l, where S is given by $S(i,i) = -1$ for $i \in \{3 :_2 m-2\}$ and $S(i,i) = 1$ for $i \notin \{3 :_2 m-2\}$. In this case $L(\lambda)$ is an EGFPR.

On the other hand, if $\sigma_l = \emptyset$ then $SRL(\lambda)$ is of the form (4.25) with $Y = -I_n$, where S is given by $S(i,i) = -1$ for $i \in \{1 :_2 m-2\}$ and $S(i,i) = 1$ for $i \notin \{1 :_2 m-2\}$. In this case $L(\lambda)$ is a GF pencil. $\square$

Example 4.3.19. *Let* $P(\lambda) = \sum_{i=0}^{7} \lambda^i A_i$ *be* T*-palindromic. Consider the pencil* $L(\lambda) := M_{-7}(-X^T)\big(\lambda M^P_{(-6,-7,-4,-2)} - M^P_{(5,3,0:1)}\big)M_0(X)$. *Then by Lemma 4.3.11,* $SRL(\lambda)$ *is a* T*-palindromic strong linearization of* $P(\lambda)$ *and is given by*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & A_0 & \lambda X \\ 0 & 0 & 0 & 0 & \lambda I_n & \lambda A_2 + A_1 & -X \\ 0 & 0 & 0 & I_n & 0 & -\lambda I_n & 0 \\ 0 & 0 & \lambda I_n & \lambda A_4 + A_3 & -I_n & 0 & 0 \\ 0 & I_n & 0 & -\lambda I_n & 0 & 0 & 0 \\ \lambda A_7 & \lambda A_6 + A_5 & -I_n & 0 & 0 & 0 & 0 \\ X^T & -\lambda X^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $S = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_7 I_n$ *with* $\epsilon_i = 1$ *for* $i = 1,2,4,6,7,$ *and* $\epsilon_i = -1$ *for* $i = 3,5$. *Observe that* $SRL(\lambda)$ *is an anti-block tridiagonal pencil.* $\blacksquare$

Remark 4.3.20. *It follows from Theorem 4.3.18 that* $SRL(\lambda)$ *in Theorem 4.3.12 is not an anti-block tridiagonal pencil if* σ *is a permutation of* $\{0:2\}$. *Hence the construction given in [19] does not produce an anti-block tridiagonal* T*-palindromic linearization of a* T*-palindromic* $P(\lambda)$ *having odd degree* $m \geq 5$.

Remark 4.3.21 (T-anti-palindromic linearizations)**.** *Let* $P(\lambda)$ *be* T*-anti-palindromic. Recall from Chapter 2 that* $P(\lambda)$ *is* T*-anti-palindromic if and only if* $P(-\lambda)$ *is* T*-palindromic [27]. Suppose that* $P(\lambda)$ *is of odd degree* $m \geq 3$. *Define* $Q(\lambda) := P(-\lambda)$. *Consider the EGFPR* $L(\lambda) :=$

$$M_{-m+rev(\sigma_r)}(-rev(X^T))M_{\sigma_l}(Y)\big(\lambda M^Q_{-m+rev(\sigma)} - M^Q_\sigma\big)M_{\sigma_r}(X)M_{-m+rev(\sigma_l)}(-rev(Y^T))$$

of $Q(\lambda)$ *satisfying all conditions of Theorem 4.3.12. Then* $SRL(-\lambda)$ *is a* T*-anti-palindromic strong linearization of* $P(\lambda)$, *where* S *is the quasi-identity matrix as given in (4.15).*

We mention that the number of T-palindromic strong linearizations obtained from EGFPRs as compare to FPRs increases substantially when the degree of $P(\lambda)$ increases. We illustrate this by considering $m = 7$.

Example 4.3.22. *Let $P(\lambda)$ be T-palindromic with degree 7. Then Table 4.1 gives all the choices of σ, σ_l and σ_r which yield T-palindromic strong linearizations of $P(\lambda)$ excluding those that can be constructed from GFPs and FPRs. It follows from Theorem 4.3.17 and Theorem 4.3.18 that the last two choices of σ, σ_r and σ_l yield anti-block tridiagonal T-palindromic strong linearizations of $P(\lambda)$ and all other choices of σ, σ_r and σ_l yield anti-block penta-diagonal T-palindromic strong linearizations of $P(\lambda)$.* ■

σ_l	σ	σ_r	S
$\emptyset$	$(4,0:2)$	(0)	S_4S_6
$\emptyset$	$(4,0:2)$	$(0:1)$	S_4S_7
$\emptyset$	$(4,0:2)$	$(0:1,0)$	S_4
$\emptyset$	$(4,2,0:1)$	(0)	S_2S_4
(1)	$(4,2,0:1)$	$\emptyset$	$S_1S_4S_7$
$(0:1)$	$(4,2,0:1)$	$\emptyset$	S_4S_7
(1)	$(4,2,0:1)$	(0)	S_1S_4
$(0,1)$	$(4,2,0:1)$	(0)	S_4
$\emptyset$	$(4,1:2,0)$	(1)	$S_1S_4S_7$
$\emptyset$	$(4,1:2,0)$	$(1,0)$	S_1S_4
(0)	$(4,1:2,0)$	$\emptyset$	S_4S_6
(0)	$(4,1:2,0)$	(1)	S_4S_7
(0)	$(4,1:2,0)$	$(1,0)$	S_4
(0)	$(4,2,1,0)$	$\emptyset$	S_2S_4
$(1,0)$	$(4,2,1,0)$	$\emptyset$	S_1S_4
$(0:1,0)$	$(4,2,1,0)$	$\emptyset$	S_4
$\emptyset$	$(4:5,0:1)$	(0)	S_3S_4
$\emptyset$	$(5,4,0:1)$	(0)	S_4S_5
(0)	$(4:5,1,0)$	$\emptyset$	S_3S_4
(0)	$(5,4,1,0)$	$\emptyset$	S_4S_5
(5)	$(6,4,5,0)$	$\emptyset$	$S_3S_4S_5$
$\emptyset$	$(5:6,4,0)$	(5)	$S_3S_4S_5$
$\emptyset$	$(5,3,0:1)$	(0)	S_3S_5
(0)	$(5,3,1,0)$	$\emptyset$	S_3S_5

Table 4.1: Additional pencils when $m = 7$.

5

Affine Spaces of Strong Linearizations for Rational Matrices

The main aim of this chapter is to define Rosenbrock strong linearization of a rational matrix $G(\lambda)$. We describe the recovery of pole-zero structure of $G(\lambda)$ at infinity from the pole-zero structure at infinity of a Rosenbrock strong linearization of $G(\lambda)$. We construct affine spaces of potential linearizations of $G(\lambda)$ and show that almost all pencils in the affine spaces are Rosenbrock strong linearizations of $G(\lambda)$. These affine spaces of potential linearizations are constructed from the vector spaces of linearizations of matrix polynomials defined and analyzed by Mackey et al. in [SIAM J. Matrix Anal. Appl., 28(2006), pp.971-1004]. We show that eigenvectors, minimal bases and minimal indices of $G(\lambda)$ can be easily recovered from those of the Rosenbrock strong linearizations of $G(\lambda)$. In particular, we construct a symmetric Rosenbrock strong linearization of $G(\lambda)$ when $G(\lambda)$ is regular and symmetric.

5.1 Introduction

Recall that a rational matrix $G(\lambda)$ is said to be proper if $G(\lambda) \to D$ as $\lambda \to \infty$, where D is a matrix. On the other hand, $G(\lambda)$ is said to be nonproper if $\|G(\lambda)\| \to \infty$ as $\lambda \to \infty$. If $G(\lambda)$ is nonproper then it can be written as $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$, where $P(\lambda)$ is a matrix polynomial of degree $m \geq 1$ and $G_{sp}(\lambda)$ is *strictly proper*, that is, $G_{sp}(\lambda) \to 0$ as $\lambda \to \infty$, see [40, 52]. Rational matrices that arise in the context of rational eigenvalue problems are mostly nonproper, square and regular. On the other hand, rational matrices that arise as transfer functions of linear time-invariant (LTI) systems are mostly proper and may be rectangular. Since the primary aim of the thesis is to construct linearizations of rational matrices for solving rational eigenvalue problems, we consider only nonproper and square rational matrices.

Let $G(\lambda)$ be an $n \times n$ nonproper rational matrix. For constructing linearization of $G(\lambda)$, we follow the state-space model for rational matrices pioneered by Rosenbrock [52] and consider a realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ and the associated system matrix (also called Rosenbrock system matrix)

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right],$$

where $A - \lambda E$ is an $r \times r$ pencil with E being nonsingular, $P(\lambda)$ is an $n \times n$ matrix polynomial of degree m, $C \in \mathbb{C}^{n \times r}$ and $B \in \mathbb{C}^{r \times n}$. Note that $G(\lambda)$ is the transfer function and $\mathcal{S}(\lambda)$ is the system matrix of the LTI system in state-space form (SSF) given by [52]

$$\Sigma : \begin{array}{l} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + P(\frac{d}{dt})u(t), \end{array}$$

where $x(t)$ and $u(t)$ are state and control vectors, respectively. Recall that the realization of $G(\lambda)$ is said to be *minimal*, or equivalently, $\mathcal{S}(\lambda)$ is said to be *irreducible* if the LTI system Σ is controllable and observable. Equivalently, $\mathcal{S}(\lambda)$ is irreducible if and only if $\operatorname{rank}\left(\left[\begin{array}{cc} B & A - \lambda E \end{array}\right]\right) = r = \operatorname{rank}\left(\left[\begin{array}{cc} C^T & (A - \lambda E)^T \end{array}\right]^T\right)$ for all $\lambda \in \mathbb{C}$. Recall from Theorem 1.2.11 that the zeros of $G(\lambda)$ are the same as the eigenvalues of $\mathcal{S}(\lambda)$ when $\mathcal{S}(\lambda)$ is irreducible. See [2, 40, 52] for further details.

The concept of linearization of rational matrices introduced in [2] is based on the idea of linearization of the LTI system Σ. Indeed, by linearizing $P(\lambda)$, we wish to linearize the LTI system Σ by an LTI system Σ_L of the form

$$\Sigma_L : \begin{array}{l} K\dot{\mathbf{x}}(t) = H\mathbf{x}(t) + \mathcal{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{X}\mathbf{u}(t) - \mathcal{Y}\dot{\mathbf{u}}(t) \end{array}$$

such that the system Σ_L preserves the system characteristics (e.g., controllability and observability) of the system Σ and have the same poles and zeros (invariant zeros, transmission zeros, input/output decoupling zeros) as the system Σ, where H and K are $r \times r$ matrices with K being nonsingular, $\mathcal{X}$ and $\mathcal{Y}$ are $mn \times mn$ matrices and, $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are state and control vectors, respectively. Note that

$$\mathbb{L}(\lambda) := \left[\begin{array}{c|c} \mathcal{X} - \lambda\mathcal{Y} & \mathcal{C} \\ \hline \mathcal{B} & H - \lambda K \end{array}\right] \quad \text{and} \quad \mathbb{G}(\lambda) := \mathcal{X} - \lambda\mathcal{Y} + \mathcal{C}(\lambda K - H)^{-1}\mathcal{B}$$

are the system matrix and the transfer function of Σ_L, respectively. The Rosenbrock linearization of $\mathcal{S}(\lambda)$ introduced in [2] linearizes the LTI system Σ and is defined as follows.

Definition 5.1.1 (Rosenbrock linearization, [2, 4]). *Let $\mathcal{S}(\lambda)$ be the system matrix and $m := \deg(P) \geq 2$. Then an $(mn+r) \times (mn+r)$ system matrix $\mathbb{L}(\lambda)$ given by*

$$\mathbb{L}(\lambda) := \left[\begin{array}{c|c} \mathcal{X} - \lambda\mathcal{Y} & \mathcal{C} \\ \hline \mathcal{B} & H - \lambda K \end{array}\right],$$

where $H - \lambda K$ is an $r \times r$ pencil with K being nonsingular, is said to be a Rosenbrock linearization of $\mathcal{S}(\lambda)$ provided that there are $mn \times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$, and $r \times r$ nonsingular matrices U_0 and V_0 such that

$$\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & U_0 \end{array}\right] \mathbb{L}(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & V_0 \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right] \quad \textit{for all } \lambda \in \mathbb{C}. \tag{5.1}$$

If $\mathbb{L}(\lambda)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$ then it can be shown [4] that Σ_L is controllable (resp., observable) $\Longleftrightarrow$ Σ is controllable (resp., observable). Further, the LTI systems Σ_L and Σ have the same finite zeros (transmission zeros, invariant zeros and input/output decoupling zeros) and the same finite poles. Furthermore, for all $\lambda \in \mathbb{C}$, we have $U(\lambda)(\mathcal{X} - \lambda\mathcal{Y})V(\lambda) = \operatorname{diag}(I_{(m-1)n},\ P(\lambda))$ and

$$U(\lambda)\mathbb{G}(\lambda)V(\lambda) = \operatorname{diag}(I_{(m-1)n},\ G(\lambda)),$$

where $U(\lambda)$ and $V(\lambda)$ are unimodular matrix polynomials given in (5.1).

Thus, if $\mathbb{L}(\lambda)$ is a Rosenbrock linearization of $G(\lambda)$ then finite zero and finite pole structures of $\mathbb{L}(\lambda)$ are isomorphic to finite zero and finite pole structures of $G(\lambda)$. However, $\mathbb{L}(\lambda)$ and $G(\lambda)$ may not have the same pole and zero structures at infinity. We, therefore, define a ***Rosenbrock strong linearization*** of $G(\lambda)$ and show that the pole-zero structure of $G(\lambda)$ at infinity can be recovered from the pole-zero structure at infinity of a Rosenbrock strong linearization of $G(\lambda)$.

Linearization of matrix polynomials is a well developed area of research and there are rich sources of linearizations for matrix polynomials [33, 46, 45, 6, 61, 26, 25]. By contrast, linearization of rational matrices is a new emerging area of research. It is therefore desirable to extend linearizations of matrix polynomials to the case of rational matrices. However, whether any linearization of matrix polynomials can be extended to the case of rational matrices or not is still an open problem. We show that this is indeed possible for the vector spaces of linearizations of matrix polynomials defined and analyzed in [46].

We mention here that generalized state-space realization of rational matrices provides an alternative framework for constructing *strict linearizations* of rational matrices including rectangular rational matrices, see [60] and [40, Page 469]. Indeed, let

$G(\lambda) := P(\lambda) + G_{sp}(\lambda)$ be an $m \times n$ rational matrix, where $P(\lambda)$ is an $m \times n$ matrix polynomial and $G_{sp}(\lambda)$ is strictly proper. Let $G_{sp}(\lambda) = C_f(\lambda I_f - A_f)^{-1}B_f$ be a minimal realization of $G_{sp}(\lambda)$. Also, let $\lambda^{-1}P(\lambda^{-1}) = C_\infty(\lambda I_\infty - A_\infty)^{-1}B_\infty$ be a minimal realization of $\lambda^{-1}P(\lambda^{-1})$ so that $P(\lambda) = C_\infty(I_\infty - \lambda A_\infty)^{-1}B_\infty$. Here I_f and I_∞ are identity matrices. Then the pencil

$$\mathbb{S}(\lambda) := \left[\begin{array}{c|cc} 0 & C_\infty & C_f \\ \hline B_\infty & I_\infty - \lambda A_\infty & 0 \\ B_f & 0 & \lambda I_f - A_f \end{array}\right]$$

is the system matrix of $G(\lambda)$ and is called a strict linearization of $G(\lambda)$. The pencil $\mathbb{S}(\lambda)$ has the same zero and polar structures (finite, infinite, minimal indices) as $G(\lambda)$ [40]. However, generalized state-space realizations of rational matrices and hence strict linearizations of $G(\lambda)$ are potentially ill-posed problems which require special care for numerical computation [57].

We now present some basic results for ready reference. For a matrix polynomial $P(\lambda) := \sum_{j=0}^{m} \lambda^j A_j$, the pencils $C_1(\lambda) = \lambda X_1 + Y_1$ and $C_2(\lambda) = \lambda X_2 + Y_2$, respectively, are called the first and second companion pencils of $P(\lambda)$ [33], where $X_1 = X_2 = \mathrm{diag}(A_m,\ I_{(m-1)n})$,

$$Y_1 = \begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I_n & 0 \end{bmatrix}, \text{ and } Y_2 = \begin{bmatrix} A_{m-1} & -I_n & \cdots & 0 \\ A_{m-2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \cdots & 0 \end{bmatrix}.$$

Generalizing the companion pencils of $P(\lambda)$ two vector spaces $\mathcal{L}_1(P)$ and $\mathcal{L}_2(P)$ of $mn \times mn$ matrix pencils $L(\lambda) := \lambda X + Y$ are constructed in [46, 25]:

$$\begin{aligned} \mathcal{L}_1(P) &:= \{L(\lambda) = \lambda X + Y : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\, v \in \mathbb{C}^m\} \\ \mathcal{L}_2(P) &:= \left\{L(\lambda) = \lambda X + Y : (\Lambda^T \otimes I_n)L(\lambda) = w^T \otimes P(\lambda),\, w \in \mathbb{C}^m\right\}, \end{aligned} \tag{5.2}$$

where $\Lambda := [\lambda^{m-1}, \lambda^{m-2}, \ldots, \lambda, 1]^T \in \mathbb{C}^m$. The vectors v and w are called ***right ansatz*** and ***left ansatz*** vectors, respectively. Almost all pencils in $\mathcal{L}_1(P)$ and $\mathcal{L}_2(P)$ are linearizations of $P(\lambda)$ and eigenvectors, minimal bases and minimal indices of $P(\lambda)$ can be easily recovered from those of $L(\lambda)$, see [46, 25]. We utilize these ansatz spaces to construct Rosenbrock strong linearizations of $G(\lambda)$ and describe the recovery of eigenvectors, minimal bases and minimal indices of $G(\lambda)$ from those of the linearizations of $G(\lambda)$.

The following result, which is a restatement of [46, Corollary 3.7] and [25, Theorems 4.1 and 4.6], will play a crucial role in the subsequent development.

Theorem 5.1.2. *[46, 25] Let* $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ *with* $A_m \neq 0$ *be an* $n \times n$ *matrix polynomial (regular or singular) and* $L(\lambda) = \lambda X + Y$ *be an* $mn \times mn$ *pencil.*

(a) If $L(\lambda) \in \mathcal{L}_1(P)$ *and has right ansatz vector* e_1*, then*

$$X = \left[\begin{array}{c|c} A_m & X_{12} \\ \hline 0 & -Z \end{array}\right] \text{ and } Y = \left[\begin{array}{c|c} Y_{11} & A_0 \\ \hline Z & 0 \end{array}\right],$$

where $X_{12}, Y_{11} \in \mathbb{C}^{n \times (m-1)n}$ *are such that* $X_{12} + Y_{11} = [A_{m-1}\ A_{m-2}\ \cdots\ A_1]$ *and* $Z \in \mathbb{C}^{(m-1)n \times (m-1)n}$.

(b) If $L(\lambda) \in \mathcal{L}_1(P)$ *and has right ansatz vector* $v \neq 0$*, then there exists a nonsingular matrix* $M \in \mathbb{C}^{m \times m}$ *such that* $Mv = e_1$ *and the pencil* $(M \otimes I_n)L(\lambda)$ *is given by*

$$(M \otimes I_n)L(\lambda) = \lambda \left[\begin{array}{c|c} A_m & X_{12} \\ \hline 0 & -Z \end{array}\right] + \left[\begin{array}{c|c} Y_{11} & A_0 \\ \hline Z & 0 \end{array}\right], \tag{5.3}$$

for some $Z \in \mathbb{C}^{(m-1)n \times (m-1)n}$*, where* X_{12} *and* Y_{11} *are as in (a).*

(c) If $L(\lambda) \in \mathcal{L}_2(P)$ *and has left ansatz vector* $w \neq 0$*, then there exists a nonsingular matrix* $K \in \mathbb{C}^{m \times m}$ *such that* $K^T w = e_1$ *and the pencil* $L(\lambda)(K \otimes I_n)$ *is given by*

$$L(\lambda)(K \otimes I_n) = \lambda \left[\begin{array}{c|c} A_m & 0 \\ \hline X_{21} & -Z \end{array}\right] + \left[\begin{array}{c|c} Y_{11} & Z \\ \hline A_0 & 0 \end{array}\right], \tag{5.4}$$

where $X_{21}, Y_{11} \in \mathbb{C}^{(m-1)n \times n}$ *are such that* $X_{21} + Y_{11} = [A_{m-1}^T\ A_{m-2}^T\ \cdots\ A_1^T]^T$ *and* $Z \in \mathbb{C}^{(m-1)n \times (m-1)n}$.

All the matrices that can appear in the block labelled Z in (5.3) have the same rank, see [46, 25]. Ditto for the block labelled Z in (5.4). Thus, for $L(\lambda) \in \mathcal{L}_1(P)$ or $L(\lambda) \in \mathcal{L}_2(P)$, it makes sense to talk about the Z-rank of $L(\lambda)$.

Definition 5.1.3 (Z-rank, [25])**.** *The* Z*-rank of* $L(\lambda) \in \mathcal{L}_1(P)$ *is the rank of any matrix appearing in the block labelled* Z *in (5.3) under any reduction of* $L(\lambda)$ *of the form (5.3). If* Z *in (5.3) is nonsingular, then we say that* $L(\lambda) \in \mathcal{L}_1(P)$ *has full* Z*-rank. Similarly, the* Z*-rank of* $L(\lambda) \in \mathcal{L}_2(P)$ *is the rank of any matrix appearing in block* Z *in (5.4) under any reduction of* $L(\lambda)$ *of the form (5.4).*

5.2 Rosenbrock strong linearization

Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$ be nonproper and be given by $G(\lambda) := P(\lambda) + G_{sp}(\lambda)$, where $P(\lambda)$ is a matrix polynomial of degree m and $G_{sp}(\lambda)$ is a strictly proper rational matrix. Let $G_{sp}(\lambda) = C(\lambda E - A)^{-1}B$ be a minimal realization of $G_{sp}(\lambda)$, where $\lambda E - A$ is an $r \times r$ pencil with E being nonsingular, $C \in \mathbb{C}^{n\times r}$ and $B \in \mathbb{C}^{r\times n}$. Then

$$G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B \tag{5.5}$$

is a minimal realization of $G(\lambda)$. Consequently, the $(n+r)\times(n+r)$ Rosenbrock system matrix

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right] \tag{5.6}$$

associated with $G(\lambda)$ is irreducible. Recall that $\mathcal{S}(\lambda)$ is irreducible if and only if $\operatorname{rank}\left(\begin{bmatrix} B & A - \lambda E \end{bmatrix}\right) = r = \operatorname{rank}\left(\begin{bmatrix} C \\ (A - \lambda E) \end{bmatrix}\right)$ for all $\lambda \in \mathbb{C}$.

Assumption: *For the rest of this chapter, we assume that* $P(\lambda) := \sum_{i=0}^{m} \lambda^i A_i$ *with* $A_m \neq 0$ *and the realization* $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ *of* $G(\lambda)$ *given by (5.5) is minimal. The system matrix* $\mathcal{S}(\lambda)$ *associated with* $G(\lambda)$ *is given by (5.6).*

We now define *Rosenbrock strong linearization* of the $G(\lambda)$.

Definition 5.2.1 (Biproper rational matrix, [59]). *An* $n \times n$ *proper rational matrix* $F(\lambda)$ *is said to be* biproper *if* $F(\infty)$ *is nonsingular.*

Definition 5.2.2 (Rosenbrock strong linearization). *Let* $G(\lambda)$ *and* $\mathcal{S}(\lambda)$ *be as in (5.5) and (5.6), respectively. Also let* $\mathbb{L}(\lambda)$ *be an* $(mn+r)\times(mn+r)$ *irreducible system matrix associated with the transfer function* $\mathbb{G}(\lambda)$ *given by*

$$\mathbb{L}(\lambda) := \left[\begin{array}{c|c} \mathcal{X} - \lambda\mathcal{Y} & \mathcal{C} \\ \hline \mathcal{B} & H - \lambda K \end{array}\right] \quad \text{and } \mathbb{G}(\lambda) := \mathcal{X} - \lambda\mathcal{Y} + \mathcal{C}(\lambda K - H)^{-1}\mathcal{B},$$

where $H - \lambda K$ *is an* $r \times r$ *pencil with* K *being nonsingular,* $\mathcal{C} \in \mathbb{C}^{mn\times r}$, $\mathcal{B} \in \mathbb{C}^{r\times mn}$ *and* $\mathcal{X} - \lambda\mathcal{Y}$ *is an* $mn \times mn$ *pencil. Then* $\mathbb{L}(\lambda)$ *is said to be a Rosenbrock strong linearization of* $G(\lambda)$ *provided that the following conditions hold.*

(a) There exist $mn \times mn$ *unimodular matrix polynomials* $U(\lambda)$ *and* $V(\lambda)$, *and* $r \times r$ *nonsingular matrices* U_0 *and* V_0 *such that*

$$\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & U_0 \end{array}\right] \mathbb{L}(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & V_0 \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right] \quad \text{for all } \lambda \in \mathbb{C}.$$

(b) There exist $mn \times mn$ biproper rational matrices $\mathcal{O}_\ell(\lambda)$ and $\mathcal{O}_r(\lambda)$ such that

$$\mathcal{O}_\ell(\lambda)\lambda^{-1}\mathbb{G}(\lambda)\mathcal{O}_r(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}G(\lambda) \end{array}\right] \text{ for all } \lambda \in \mathbb{C}.$$

We also refer to $\mathbb{L}(\lambda)$ as a Rosenbrock strong linearization of $\mathcal{S}(\lambda)$.

We mention that Rosenbrock strong linearization can be defined for rectangular rational matrices by appropriately modifying Definition 5.2.2. We also mention that by [4, Theorem 3.5], the Rosenbrock strong linearization is equivalent to the strong linearization of rational matrices introduced in [5].

The condition (a) in Definition 5.2.2 ensures that the finite eigenstructure of $\mathbb{L}(\lambda)$ (resp., $H - \lambda K$) is the same as the finite zero (resp., pole) structure of $G(\lambda)$, see [2, 4]. On the other hand, the condition (b) ensures that the pole-zero structure of $G(\lambda)$ at infinity can be easily recovered from the pole-zero structure of $\mathbb{L}(\lambda)$ at infinity. The finite and infinite pole-zero structure of $\mathbb{L}(\lambda)$ can be computed from the Kronecker canonical form of $\mathbb{L}(\lambda)$, see [58, 57].

Two $n \times n$ rational matrices $G_1(\lambda)$ and $G_2(\lambda)$ are said to be *equivalent at infinity* if there exist biproper rational matrices $F_L(\lambda)$ and $F_R(\lambda)$ such that $F_L(\lambda)G_1(\lambda)F_R(\lambda) = G_2(\lambda)$, see [59]. We write $G_1(\lambda) \sim_{ei} G_2(\lambda)$ when $G_1(\lambda)$ and $G_2(\lambda)$ are equivalent at infinity. Note that if $G_1(\lambda) \sim_{ei} G_2(\lambda)$ then $G_1(\lambda)$ and $G_2(\lambda)$ have the same pole-zero structure at infinity. The pole-zero structure at infinity of a rational matrix is given by the Smith-McMillan form at infinity.

Theorem 5.2.3 ([59], Chapter 3)**.** *Let $H(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$ with normal rank ℓ. Then there exist biproper rational matrices $F_L(\lambda)$ and $F_R(\lambda)$ such that*

$$F_L(\lambda)H(\lambda)F_R(\lambda) = \mathrm{diag}(\lambda^{p_1}, \ldots, \lambda^{p_k}, \lambda^{-g_{k+1}}, \ldots, \lambda^{-g_\ell}, 0, \ldots, 0) =: SM_\infty(H(\lambda)),$$

where $p_1 \geq p_2 \geq \cdots \geq p_k \geq 0 > -g_{k+1} \geq \cdots \geq -g_\ell$ with p_i (resp., g_j) being nonnegative (resp., positive) integers for $i = 1 : k$ (resp., $j = k+1 : \ell$). Further, the pole-zero index at infinity given by $\mathrm{Ind}_\infty(H(\lambda)) := (p_1, \ldots, p_k, -g_{k+1}, \ldots, -g_\ell) \in \mathbb{Z}^\ell$ is a complete invariant under biproper equivalence. Furthermore, if $H(\lambda)$ is nonproper and m is the degree of the polynomial part of $H(\lambda)$ then $p_1 = m$.

The diagonal matrix $SM_\infty(H(\lambda))$ is the Smith-McMillan form of $H(\lambda)$ at infinity. If $p_i > 0$ then $H(\lambda)$ has a pole of order p_i at infinity and if $g_j > 0$ then $H(\lambda)$ has a zero of order g_j at infinity. If $H(\lambda)$ is a regular polynomial then $p_1 + \cdots + p_k \geq g_{k+1} + \cdots + g_\ell$, see [59, Corollary 3.87].

Remark 5.2.4. *Let* $G(\lambda)$, $\mathcal{S}(\lambda)$, $\mathbb{L}(\lambda)$ *and* $\mathbb{G}(\lambda)$ *be as in Definition 5.2.2. Then it is easily seen that* $\mathcal{S}(\lambda) \sim_{ei} \mathrm{diag}(\lambda I_r, G(\lambda))$ *and* $\mathbb{L}(\lambda) \sim_{ei} \mathrm{diag}(\lambda I_r, \mathbb{G}(\lambda))$ *showing that* $G(\lambda)$ *and* $\mathcal{S}(\lambda)$ *have the same zero structure at infinity and,* $\mathbb{G}(\lambda)$ *and* $\mathbb{L}(\lambda)$ *have the same zero structure at infinity. Indeed, we have*

$$\left[\begin{array}{c|c} I_n & C(\lambda E - A)^{-1} \\ \hline 0 & I_r \end{array}\right] \mathcal{S}(\lambda) \left[\begin{array}{c|c} I_n & 0 \\ \hline (\lambda E - A)^{-1}B & I_r \end{array}\right] = \left[\begin{array}{c|c} G(\lambda) & 0 \\ \hline 0 & A - \lambda E \end{array}\right]$$

which shows that $\mathcal{S}(\lambda) \sim_{ei} \mathrm{diag}(\lambda I_r, G(\lambda))$*. Similarly,* $\mathbb{L}(\lambda) \sim_{ei} \mathrm{diag}(\lambda I_r, \mathbb{G}(\lambda))$*. Now by Definition 5.2.2(b), we have* $\mathbb{G}(\lambda) \sim_{ei} \mathrm{diag}(\lambda I_{(m-1)n}, \lambda^{1-m} G(\lambda))$ *and hence* $\mathbb{L}(\lambda) \sim_{ei} \mathrm{diag}(\, \lambda I_{(m-1)n+r}, \lambda^{1-m} G(\lambda))$*. This shows that we can deduce pole-zero structure of* $\mathbb{L}(\lambda)$ *from those of* $G(\lambda)$ *and vice-versa.*

Theorem 5.2.5. *Let* $G(\lambda), \mathbb{G}(\lambda)$ *and* $\mathbb{L}(\lambda)$ *be as in Definition 5.2.2. Suppose that* q *is the normal rank of* $G(\lambda)$ *and* s *is the rank of* $\mathcal{Y}$*. Then we have*

$$SM_\infty(\mathbb{L}(\lambda)) = \lambda I_{r+s} \oplus \mathrm{diag}(\lambda^{-g_1}, \ldots, \lambda^{-g_\ell}) \oplus 0_{n-q}$$

where $0 \leq g_1 \leq \cdots \leq g_\ell$ *are integers and* $s + \ell = (m-1)n + q$ *with* $\ell < q$*. Furthermore,*

$$\begin{aligned} SM_\infty(\mathbb{G}(\lambda)) &= \lambda I_s \oplus \mathrm{diag}(\lambda^{-g_1}, \ldots, \lambda^{-g_\ell}) \oplus 0_{n-q}, \\ SM_\infty(G(\lambda)) &= \lambda^m I_{q-\ell} \oplus \mathrm{diag}\big(\lambda^{m-(g_1+1)}, \ldots, \lambda^{m-(g_\ell+1)}\big) \oplus 0_{n-q}. \end{aligned}$$

Equivalently, we have $\mathrm{Ind}_\infty(\mathbb{L}(\lambda)) = (1, \ldots, 1, -g_1, \ldots, -g_\ell)$*, where* 1 *appears* $r+s$ *times and* $0 \leq g_1 \leq \cdots \leq g_\ell$ *are integers, and*

$$\mathrm{Ind}_\infty(G(\lambda)) = \big(m, \ldots, m,\ m - (g_1+1), \ldots, m - (g_\ell + 1)\big),$$

where m *appears* $q - \ell$ *times.*

Proof. By [58, Theorem 2], $\mathbb{L}(\lambda)$ has exactly $r + s$ poles at infinity of order 1 and by [60, Corollary 1], $\mathbb{L}(\lambda)$ has no zero at infinity when $r + s = \mathrm{nrank}(\mathbb{L})$. By Remark 5.2.4, we have $\mathbb{L}(\lambda) \sim_{ei} \mathrm{diag}(\, \lambda I_{(m-1)n+r}, \lambda^{1-m} G(\lambda))$ which shows that $\mathbb{L}(\lambda)$ has at least $(m-1)n + r$ poles at infinity. Consequently, we have $s \geq (m-1)n$. Since $G(\lambda)$ has at least one pole at infinity and $\mathrm{nrank}(\mathbb{L}) = (m-1)n+r+q$, there exist a nonnegative integer $\ell < q$ such that $s+\ell = (m-1)n+q$ and $SM_\infty(\mathbb{L}(\lambda)) = \lambda I_{r+s} \oplus \mathrm{diag}(\lambda^{-g_1}, \ldots, \lambda^{-g_\ell}) \oplus 0_{n-q}$.

Next, $SM_\infty(\mathbb{G}(\lambda))$ follows from the fact that $\mathbb{L}(\lambda) \sim_{ei} \mathrm{diag}(\lambda I_r, \mathbb{G}(\lambda))$.

Finally, by Theorem 5.2.3, the highest order of a pole of $G(\lambda)$ at infinity is m. Since $SM_\infty(\lambda^{1-m} G(\lambda)) = \lambda^{1-m} SM_\infty(G(\lambda))$, by Remark 5.2.4, we have

$$\begin{aligned} \mathbb{L}(\lambda) &\sim_{ei} \mathrm{diag}\big(\lambda I_{(m-1)n+r},\ \lambda^{1-m} G(\lambda)\big) \sim_{ei} \mathrm{diag}\big(\lambda I_{(m-1)n+r},\ \lambda^{1-m} SM_\infty(G(\lambda))\big) \\ &= \lambda I_{r+s} \oplus \mathrm{diag}(\lambda^{-g_1}, \ldots, \lambda^{-g_\ell}) \oplus 0_{n-q} = SM_\infty(\mathbb{L}(\lambda)). \end{aligned}$$

Now equating the last n diagonal entries of $SM_\infty(\mathbb{L}(\lambda))$ to $\lambda^{1-m}SM_\infty(G(\lambda)$, we obtain the desired $SM_\infty(G(\lambda))$. The pole-zero indices $\mathrm{Ind}_\infty(\mathbb{L}(\lambda))$ and $\mathrm{Ind}_\infty(G(\lambda))$ follow immediately. $\square$

We mention that Theorem 5.2.5 also holds when $G(\lambda)$ is proper. In such a case, in Theorem 5.2.5, we have $\mathbb{L}(\lambda) = \mathcal{S}(\lambda)$, $s = 0$ and $\ell \leq q$.

The next result shows that the Smith-McMillan form of $G(\lambda)$ at infinity can be used to deduce Smith-McMillan forms of the system matrix as well as Rosenbrock strong linearizations of $G(\lambda)$.

Theorem 5.2.6. *Let $G(\lambda), \mathcal{S}(\lambda)$ and $\mathbb{L}(\lambda)$ be as in Definition 5.2.2. Suppose that $SM_\infty(G(\lambda)) = \mathrm{diag}(\lambda^{p_1}, \ldots, \lambda^{p_k}, I_t, \lambda^{-g_1}, \ldots, \lambda^{-g_\ell}) \oplus 0_{n-q}$, where q is the normal rank of $G(\lambda)$, $k + t + \ell = q$, $p_1 \geq \cdots \geq p_k > 0$ and $0 < g_1 \leq \cdots \leq g_\ell$ are natural numbers. Then we have the following*

$$\begin{aligned} SM_\infty(\mathcal{S}(\lambda)) &= \mathrm{diag}(\lambda^{p_1}, \ldots, \lambda^{p_k}, \lambda I_r, I_t, \lambda^{-g_1}, \ldots, \lambda^{-g_\ell}) \oplus 0_{n-q}, \\ SM_\infty(\mathbb{L}(\lambda)) &= \lambda I_u \oplus \mathrm{diag}(\lambda^{\hat{p}_2}, \ldots, \lambda^{\hat{p}_k},\ \lambda^{-(m-1)}I_t,\ \lambda^{-\hat{g}_1}, \ldots, \lambda^{-\hat{g}_\ell}) \oplus 0_{n-q}, \end{aligned}$$

where $u := (m-1)n + r + 1$, $\hat{p}_j := p_j + 1 - m \leq 1$ for $j = 2 : k$ and $\hat{g}_j := g_j + m - 1$ for $j = 1 : \ell$.

Proof. Note that by Theorem 5.2.3, we have $p_1 = m$. Now by Remark 5.2.4, we have $\mathcal{S}(\lambda) \sim_{ei} \mathrm{diag}(\lambda I_r,\ G(\lambda))$ and $\mathbb{L}(\lambda) \sim_{ei} \mathrm{diag}\big(\ \lambda I_{(m-1)n+r},\ \lambda^{1-m}G(\lambda)\big)$ which yield the desired results. $\square$

5.3 Affine spaces of strong linearizations for rational matrices

We now construct two affine spaces of pencils - a kind of ansatz spaces for $G(\lambda)$ - and show that almost all pencils in these affine spaces are Rosenbrock strong linearizations of $G(\lambda)$.

Consider the ansatz spaces $\mathcal{L}_1(P)$ and $\mathcal{L}_2(P)$ given in (5.2). For convenience, we write $(L(\lambda), v) \in \mathcal{L}_1(P)$ to mean that $L(\lambda) \in \mathcal{L}_1(P)$ with right ansatz vector v. Similarly, we write $(L(\lambda), w) \in \mathcal{L}_2(P)$ to mean that $L(\lambda) \in \mathcal{L}_2(P)$ with left ansatz vector w. We now define the affine spaces $\mathbb{L}_1(G)$ and $\mathbb{L}_2(G)$ as follows.

Definition 5.3.1. *Let* $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ *be as in (5.5). Define*

$$\mathbb{L}_1(G) := \left\{ \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right] : (L(\lambda), v) \in \mathcal{L}_1(P) \right\},$$

$$\mathbb{L}_2(G) := \left\{ \left[\begin{array}{c|c} L(\lambda) & e_m \otimes C \\ \hline w^T \otimes B & A - \lambda E \end{array}\right] : (L(\lambda), w) \in \mathcal{L}_2(P) \right\}.$$

We write $(\mathbb{T}(\lambda), v) \in \mathbb{L}_1(G)$ to mean that $\mathbb{T}(\lambda) \in \mathbb{L}_1(G)$ with right ansatz vector v when $\mathbb{T}(\lambda)$ is defined via $(L(\lambda), v) \in \mathcal{L}_1(P)$. Similarly, we write $(\mathbb{T}(\lambda), w) \in \mathbb{L}_2(G)$.

Now consider the vector spaces $\mathbb{V}_1(G)$ and $\mathbb{V}_2(G)$ given by

$$\mathbb{V}_1(G) := \left\{ \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline 0_{r\times mn} & 0_{r\times r} \end{array}\right] : (L(\lambda), v) \in \mathcal{L}_1(P) \right\},$$

$$\mathbb{V}_2(G) := \left\{ \left[\begin{array}{c|c} L(\lambda) & 0_{mn\times r} \\ \hline w^T \otimes B & 0_{r\times r} \end{array}\right] : (L(\lambda), w) \in \mathcal{L}_2(P) \right\}.$$

Some basic facts about the spaces $\mathbb{L}_i(G), i = 1, 2,$ are given in the following result.

Theorem 5.3.2. *(a) The set* $\mathbb{L}_i(G)$ *is an affine space with associated vector space* $\mathbb{V}_i(G)$ *and* $\dim \mathbb{L}_i(G) = \dim \mathcal{L}_i(P) = m(m-1)n^2 + m =: \ell$, *for* $i = 1, 2$.

(b) The maps

$$\mathbb{A}_1 : \mathcal{L}_1(P) \longrightarrow \mathbb{L}_1(G), \quad (L(\lambda), v) \longmapsto \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right],$$

$$\mathbb{A}_2 : \mathcal{L}_2(P) \longrightarrow \mathbb{L}_2(G), \quad (L(\lambda), w) \longmapsto \left[\begin{array}{c|c} L(\lambda) & e_m \otimes C \\ \hline w^T \otimes B & A - \lambda E \end{array}\right]$$

are affine bijections.

(c) Let $\{(X_1(\lambda), v_1), \ldots, (X_\ell(\lambda), v_\ell)\}$ *be a basis of* $\mathcal{L}_1(P)$. *Define*

$$\mathbb{X}_0(\lambda) := \mathbb{A}_1(0_{mn\times mn}, 0) = \left[\begin{array}{c|c} 0_{mn\times mn} & 0_{mn\times r} \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right],$$

$$\mathbb{X}_j(\lambda) := \mathbb{A}_1(X_j(\lambda), v_j) = \left[\begin{array}{c|c} X_j(\lambda) & v_j \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right], j = 1, 2, \ldots, \ell.$$

Then $\{\mathbb{X}_0(\lambda), \mathbb{X}_1(\lambda), \ldots, \mathbb{X}_\ell(\lambda)\}$ *is an affine basis (barycentric frame) of* $\mathbb{L}_1(G)$. *Similarly, let* $\{(Y_1(\lambda), w_1), \ldots, (Y_\ell(\lambda), w_\ell)\}$ *be a basis of* $\mathcal{L}_2(P)$. *Set* $\mathbb{Y}_0(\lambda) := \mathbb{A}_2(0_{mn\times mn}, 0)$ *and* $\mathbb{Y}_j(\lambda) := \mathbb{A}_2(Y_j(\lambda), w_j)$ *for* $j = 1, 2, \ldots, \ell$. *Then* $\{\mathbb{Y}_0(\lambda), \mathbb{Y}_1(\lambda), \ldots, \mathbb{Y}_\ell(\lambda)\}$ *is an affine basis of* $\mathbb{L}_2(G)$.

Proof. It follows that $\mathbb{L}_i(G)$ is an affine space with associated vector space $\mathbb{V}_i(G)$ for $i = 1, 2$. It is also easily seen that the maps

$$\mathbb{J}_1 : \mathcal{L}_1(P) \longrightarrow \mathbb{V}_1(G), \ \ (L(\lambda), v) \longmapsto \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline 0_{r\times mn} & 0_{r\times r} \end{array}\right],$$

$$\mathbb{J}_2 : \mathcal{L}_2(P) \longrightarrow \mathbb{V}_2(G), \ \ (L(\lambda), w) \longmapsto \left[\begin{array}{c|c} L(\lambda) & 0_{mn\times r} \\ \hline w^T \otimes B & 0_{r\times r} \end{array}\right]$$

are linear isomorphisms. Hence $\dim \mathbb{L}_i(G) = \dim \mathbb{V}_i(G) = \dim \mathcal{L}_i(P)$ for $i = 1, 2$. It is well known [46, Corollary 3.6] that $\dim \mathcal{L}_i(P) = \ell$, for $i = 1, 2$, which gives the desired result.

Since $\mathbb{A}_1(L(\lambda), v) = \mathbb{A}_1(0_{mn\times mn}, 0) + \mathbb{J}_1(L(\lambda), v)$ and $\mathbb{J}_1$ is a linear isomorphism, it follows that $\mathbb{A}_1$ is an affine bijection. The proof is similar for $\mathbb{A}_2$.

Finally, we have $\mathbb{X}_j(\lambda) - \mathbb{X}_0(\lambda) = \mathbb{J}_1(X_j(\lambda), v_j)$, for $j = 1, \ldots, \ell$, which shows that $\{\mathbb{X}_1(\lambda) - \mathbb{X}_0(\lambda), \ldots \mathbb{X}_\ell(\lambda) - \mathbb{X}_0(\lambda)\}$ is a basis of $\mathbb{V}_1(G)$. Hence $\{\mathbb{X}_0(\lambda), \mathbb{X}_1(\lambda), \ldots, \mathbb{X}_\ell(\lambda)\}$ is an affine basis, that is, a barycentric frame of $\mathbb{L}_1(G)$. The proof is similar for the affine basis of $\mathbb{L}_2(G)$. ☐

The affine spaces $\mathbb{L}_1(G)$ and $\mathbb{L}_2(G)$ can be thought of as affine extensions of the vector spaces $\mathcal{L}_1(P)$ and $\mathcal{L}_2(P)$, respectively. Thus, as in the case of the vector spaces $\mathcal{L}_1(P)$ and $\mathcal{L}_2(P)$, the affine spaces $\mathbb{L}_1(G)$ and $\mathbb{L}_2(G)$ can be characterized via column shifted sum $\boxplus\!\!\rightarrow$ and row shifted sum $\underset{\downarrow}{\boxplus}$ as described in [46]. Indeed, let $P(\lambda) := \sum_{j=0}^{m} A_j\lambda^j$. Then for all $v, w \in \mathbb{C}^m$, we have

$$\mathbb{L}_1(G) \ := \left\{ \left[\begin{array}{c|c} \lambda X + Y & v \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right] : X \boxplus\!\!\rightarrow Y = v \otimes \left[\begin{array}{cccc} A_m & A_{m-1} & \cdots & A_0 \end{array}\right] \right\},$$

$$\mathbb{L}_2(G) \ := \left\{ \left[\begin{array}{c|c} \lambda X + Y & e_m \otimes C \\ \hline w^T \otimes B & A - \lambda E \end{array}\right] : X \underset{\downarrow}{\boxplus} Y = w^T \otimes \left[\begin{array}{cccc} A_m^T & A_{m-1}^T & \cdots & A_0^T \end{array}\right]^T \right\}.$$

The pencils in $\mathcal{L}_1(P) \cup \mathcal{L}_2(P)$ are easily constructible from the data in $P(\lambda)$, see [46]. Consequently, given a realization of $G(\lambda)$, the pencils in $\mathbb{L}_1(G) \cup \mathbb{L}_2(G)$ are easily constructible from the data in $G(\lambda)$. Moreover, the affine maps $\mathbb{A}_1$ and $\mathbb{A}_2$ in Theorem 5.3.2 show that we can construct linearizations of $G(\lambda)$ directly from the linearizations of $P(\lambda)$.

We need the following result which is a restatement of [46, Theorem 4.1].

Theorem 5.3.3. *[46] Let $(L(\lambda), e_1) \in \mathcal{L}_1(P)$. If $L(\lambda)$ has full Z-rank then there exist $mn \times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that $U(\lambda)L(\lambda)V(\lambda) =$*

$\operatorname{diag}(I_{(m-1)n},\ P(\lambda))$ *for all* $\lambda \in \mathbb{C}$. *Further, we have* $U(\lambda)^{-1}(e_m \otimes I_n) = e_1 \otimes I_n$ *and* $(e_m^T \otimes I_n)V(\lambda)^{-1} = e_m^T \otimes I_n$.

We now show that almost all pencils in $\mathbb{L}_1(G)$ and $\mathbb{L}_2(G)$ are Rosenbrock strong linearizations of $\mathcal{S}(\lambda)$ and $G(\lambda)$.

Theorem 5.3.4. *Let* $\mathbb{T}(\lambda) := \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right] \in \mathbb{L}_1(G)$*, where* $(L(\lambda), v) \in \mathcal{L}_1(P)$ *and* $v \neq 0$. *If* $L(\lambda)$ *has full* Z*-rank then there exist* $mn \times mn$ *unimodular matrix polynomials* $U(\lambda)$ *and* $V(\lambda)$ *such that* $U(\lambda)^{-1}(e_m \otimes I_n) = v \otimes I_n$ *and* $(e_m^T \otimes I_n)V(\lambda)^{-1} = e_m^T \otimes I_n$. *Further, we have* $\operatorname{diag}(U(\lambda),\ I_r)\,\mathbb{T}(\lambda)\,\operatorname{diag}(V(\lambda),\ I_r) = \operatorname{diag}(I_{(m-1)n},\ \mathcal{S}(\lambda))$ *for all* $\lambda \in \mathbb{C}$. *Thus* $\mathbb{T}(\lambda)$ *is a Rosenbrock linearization of* $\mathcal{S}(\lambda)$.

Proof. Let $M \in \mathbb{C}^{m\times m}$ be a nonsingular matrix such that $Mv = e_1$. Then $(M \otimes I_n)L(\lambda) = e_1 \otimes P(\lambda)$ and hence by Theorem 5.1.2 the pencil $(M \otimes I_n)L(\lambda)$ can be written as

$$(M \otimes I_n)L(\lambda) = \lambda \left[\begin{array}{c|c} A_m & X_{12} \\ \hline 0 & -Z \end{array}\right] + \left[\begin{array}{c|c} Y_{11} & A_0 \\ \hline Z & 0 \end{array}\right],$$

with nonsingular $Z \in \mathbb{C}^{(m-1)n\times(m-1)n}$. Let $\widehat{L}(\lambda) := (M \otimes I_n)L(\lambda)$. Then by Theorem 5.3.3, there exist unimodular matrix polynomials $\widehat{U}(\lambda)$ and $\widehat{V}(\lambda)$ such that

$$\widehat{U}(\lambda)\widehat{L}(\lambda)\widehat{V}(\lambda) = \widehat{U}(\lambda)(M \otimes I_n)L(\lambda)\widehat{V}(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right],$$

$\widehat{U}(\lambda)^{-1}(e_m \otimes I_n) = e_1 \otimes I_n$ and $(e_m^T \otimes I_n)\widehat{V}(\lambda)^{-1} = e_m^T \otimes I_n$. Consequently, we have $\big(\widehat{U}(\lambda)(M \otimes I_n)\big)^{-1}(e_m \otimes I_n) = (M^{-1} \otimes I_n)\widehat{U}(\lambda)^{-1}(e_m \otimes I_n) = v \otimes I_n$. By setting $U(\lambda) := \widehat{U}(\lambda)(M \otimes I_n)$ and $V(\lambda) := \widehat{V}(\lambda)$ we have $\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \mathbb{T}(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right]$

$$= \left[\begin{array}{c|c} \widehat{U}(\lambda)(M \otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right] \left[\begin{array}{c|c} \widehat{V}(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} \widehat{U}(\lambda)(M \otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} L(\lambda) & \big(\widehat{U}(\lambda)(M \otimes I_n)\big)^{-1}(e_m \otimes C) \\ \hline (e_m^T \otimes B)\widehat{V}(\lambda)^{-1} & A - \lambda E \end{array}\right] \left[\begin{array}{c|c} \widehat{V}(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} \widehat{U}(\lambda)(M \otimes I_n)L(\lambda)\widehat{V}(\lambda) & e_m \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right]. \qquad \square$$

Let $L(\lambda) \in \mathcal{L}_1(P)$ with right ansatz vector v. Then $L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda)$ and $(\Lambda^T \otimes I_n)L(\lambda)^T = v^T \otimes P(\lambda)^T$. Hence $L(\lambda)^T \in \mathcal{L}_2(P^T)$ and has left ansatz vector v.

Similarly, for $L(\lambda) \in \mathcal{L}_2(P)$ with left ansatz vector v, we have $L(\lambda)^T \in \mathcal{L}_1(P^T)$ and has right ansatz vector v. Thus, by defining $\big(\mathbb{L}_1(G^T)\big)^T := \{\mathbb{T}(\lambda)^T : \mathbb{T}(\lambda) \in \mathbb{L}_1(G^T)\}$, we have $\mathbb{L}_2(G) = \big(\mathbb{L}_1(G^T)\big)^T$.

Theorem 5.3.5. *Let* $\mathbb{T}(\lambda) := \left[\begin{array}{c|c} L(\lambda) & e_m \otimes C \\ \hline w^T \otimes B & A - \lambda E \end{array}\right] \in \mathbb{L}_2(G)$, *where* $(L(\lambda), w) \in \mathcal{L}_2(P)$ *and* $w \neq 0$. *If* $L(\lambda)$ *has full* Z*-rank then there exist* $mn \times mn$ *unimodular matrix polynomials* $U(\lambda)$ *and* $V(\lambda)$ *such that* $U(\lambda)^{-1}(e_m \otimes I_n) = e_m \otimes I_n$ *and* $(e_m^T \otimes I_n)V(\lambda)^{-1} = w^T \otimes I_n$. *Further, we have* $\mathrm{diag}(U(\lambda),\ I_r)\,\mathbb{T}(\lambda)\,\mathrm{diag}(V(\lambda),\ I_r) = \mathrm{diag}(I_{(m-1)n},\ \mathcal{S}(\lambda))$ *for all* $\lambda \in \mathbb{C}$. *Thus* $\mathbb{T}(\lambda)$ *is a Rosenbrock linearization of* $\mathcal{S}(\lambda)$.

Proof. Let $K \in \mathbb{C}^{m \times m}$ be a nonsingular matrix such that $w^T K = e_1^T$. Then by Theorem 5.1.2, there is a nonsingular matrix $Z \in \mathbb{C}^{(m-1)n \times (m-1)n}$ such that $L(\lambda)(K \otimes I_n) = \lambda \left[\begin{array}{c|c} A_m & 0 \\ \hline X_{21} & -Z \end{array}\right] + \left[\begin{array}{c|c} Y_{11} & Z \\ \hline A_0 & 0 \end{array}\right]$. Hence we have

$$(K^T \otimes I_n)L^T(\lambda) = \lambda \left[\begin{array}{c|c} A_m^T & X_{21}^T \\ \hline 0 & -Z^T \end{array}\right] + \left[\begin{array}{c|c} Y_{11}^T & A_0^T \\ \hline Z^T & 0 \end{array}\right]. \tag{5.7}$$

Since $L(\lambda) \in \mathcal{L}_2(P)$, by (5.7), we have $L(\lambda)^T \in \mathcal{L}_1(P^T)$ having full Z-rank with w as right ansatz vector. By Theorem 5.3.4, there exist unimodular matrix polynomials $\widehat{U}(\lambda)$ and $\widehat{V}(\lambda)$ such that

$$\widehat{U}(\lambda)L(\lambda)^T\widehat{V}(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda)^T \end{array}\right] \Rightarrow \widehat{V}(\lambda)^T L(\lambda)\widehat{U}(\lambda)^T = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right]$$

and $\Big(\widehat{V}(\lambda)\Big)^{-T}(e_m \otimes I_n) = e_m \otimes I_n$ and $(e_m^T \otimes I_n)\Big(\widehat{U}(\lambda)\Big)^{-T} = w^T \otimes I_n$. By setting $U(\lambda) := \widehat{V}(\lambda)^T$ and $V(\lambda) := \widehat{U}(\lambda)^T$, the desired results follow. ☐

Observe that if $\mathbb{T}(\lambda) := \mathbb{A}_1(L(\lambda), v)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$ then $L(\lambda)$ is a linearization of $P(\lambda)$. On the other hand, if $L(\lambda) \in \mathcal{L}_1(P)$ is a linearization of $P(\lambda)$ then we cannot conclude that $\mathbb{T}(\lambda)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$ since $L(\lambda)$ may or may not have full Z-rank, see [25]. However, the reverse conclusion holds when $P(\lambda)$ is regular. The same conclusions hold when $\mathbb{T}(\lambda) := \mathbb{A}_2(L(\lambda), w)$.

Corollary 5.3.6. *Let* $(\mathbb{T}_i(\lambda), v_i) \in \mathbb{L}_i(G)$ *be given by* $\mathbb{T}_i(\lambda) := \mathbb{A}_i(L_i(\lambda), v_i)$ *with* $(L_i(\lambda), v_i) \in \mathcal{L}_i(P)$ *and* $v_i \neq 0, i = 1, 2$. *Suppose that* $P(\lambda)$ *is regular. Then* $\mathbb{T}_i(\lambda)$ *is a Rosenbrock linearization of* $\mathcal{S}(\lambda)$ $\Longleftrightarrow$ $L_i(\lambda)$ *is regular* $\Longleftrightarrow$ $L_i(\lambda)$ *is a linearization of* $P(\lambda), i = 1, 2$.

Proof. By [46, Theorem 4.3], $(L_i(\lambda), v_i) \in \mathcal{L}_i(P)$ has full Z-rank $\iff$ $L_i(\lambda)$ is regular $\iff$ $L_i(\lambda)$ is a linearization of $P(\lambda), i = 1, 2$. Hence if $L_1(\lambda)$ is regular then by Theorem 5.3.4, $\mathbb{T}_1(\lambda)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$. Conversely, if $\mathbb{T}_1(\lambda)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$ then by Theorem 5.3.4, we have $U(\lambda)L_1(\lambda)V(\lambda) = \operatorname{diag}(I_{(m-1)n}, P(\lambda))$, where $U(\lambda)$ and $V(\lambda)$ are unimodular matrix polynomials. This shows that $L_1(\lambda)$ is a linearization of $P(\lambda)$ and hence is regular. The proof is similar for $(\mathbb{T}_2(\lambda), v_2) \in \mathbb{L}_2(G)$. □

Next, we show that almost all pencils in $\mathbb{L}_1(G) \cup \mathbb{L}_2(G)$ are Rosenbrock strong linearizations of $G(\lambda)$. To that end, define

$$T(\lambda) := \begin{bmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^{m-1} \\ & 1 & \lambda & \ddots & \vdots \\ & & 1 & \ddots & \lambda^2 \\ & & & \ddots & \lambda \\ & & & & 1 \end{bmatrix} \otimes I_n \text{ and } R_m := \begin{bmatrix} & & I_n \\ & \iddots & \\ I_n & & \end{bmatrix} \in \mathbb{C}^{mn \times mn}.$$

Lemma 5.3.7. *Let* $\mathbb{T}(\lambda) := \left[\begin{array}{c|c} L(\lambda) & e_1 \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right] \in \mathbb{L}_1(G)$, *where* $(L(\lambda), e_1) \in \mathcal{L}_1(P)$ *with full Z-rank. Then there exist $mn \times mn$ biproper rational matrices $\mathcal{O}_\ell(\lambda)$ and $\mathcal{O}_r(\lambda)$ such that*

$$\mathcal{O}_\ell(\lambda)\lambda^{-1}\mathbb{G}(\lambda)\mathcal{O}_r(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}G(\lambda) \end{array}\right],$$

where $\mathbb{G}(\lambda) := L(\lambda) + (e_1 \otimes C)(\lambda E - A)^{-1}(e_m^T \otimes B)$ *is the transfer function of* $\mathbb{T}(\lambda)$.

Proof. For any pencil $L(\lambda) \in \mathcal{L}_1(P)$ with right ansatz vector e_1 and having full Z-rank, there exists a unimodular matrix polynomial $U(\lambda)$ such that

$$U(\lambda)L(\lambda)T(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & P(\lambda) \end{array}\right] \text{ with } U(\lambda) := \left[\begin{array}{c|c} 0 & Z^{-1} \\ \hline I_n & -W(\lambda)Z^{-1} \end{array}\right]$$

for some matrix polynomial $W(\lambda)$, see the proof of [46, Theorem 4.1].

Set $\widehat{L}(\lambda) := \operatorname{rev}L(\lambda)(R_m \otimes I_n)$. Then it is shown in the proof of [46, Theorem 4.1] that $\widehat{L}(\lambda)(\Lambda \otimes I_n) = e_1 \otimes \operatorname{rev}P(\lambda)$, that is, $\widehat{L}(\lambda) \in \mathcal{L}_1(\operatorname{rev}P)$ with right ansatz vector e_1 and having full $\widehat{Z}$-rank, where $\widehat{Z} := -Z(R_{m-1} \otimes I_n)$. Thus, there exists a unimodular matrix polynomials $\widehat{U}(\lambda)$ such that

$$\widehat{U}(\lambda)\widehat{L}(\lambda)T(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \operatorname{rev}P(\lambda) \end{array}\right] \text{ with } \widehat{U}(\lambda) := \left[\begin{array}{c|c} 0 & \widehat{Z}^{-1} \\ \hline I_n & -\widehat{W}(\lambda)\widehat{Z}^{-1} \end{array}\right]$$

for some matrix polynomial $\widehat{W}(\lambda)$. Hence $I_{(m-1)n} \oplus \operatorname{rev}P(\lambda) = \widehat{U}(\lambda)\,\operatorname{rev}L(\lambda)(R_m \otimes I_n)\,T(\lambda) = \widehat{U}(\lambda)\,\operatorname{rev}L(\lambda)\widehat{V}(\lambda)$, where $\widehat{V}(\lambda) := (R_m \otimes I_n)\,T(\lambda)$. Consequently,

$$\widehat{U}(1/\lambda)\,\lambda^{-1}L(\lambda)\,\widehat{V}(1/\lambda) = I_{(m-1)n} \oplus \lambda^{-m}P(\lambda). \tag{5.8}$$

Note that $\widehat{U}(1/\lambda)$ and $\widehat{V}(1/\lambda)$ are biproper rational matrices. Next, $\widehat{U}(1/\lambda)(e_1 \otimes I_n) = e_m \otimes I_n$ and $(e_m^T \otimes I_n)\widehat{V}(1/\lambda) = (e_m^T \otimes I_n)(R_m \otimes I_n)T(1/\lambda) = [1,\ \lambda^{-1}, \lambda^{-2}, \ldots, \lambda^{-(m-1)}] \otimes I_n$. Recall that $G_{sp}(\lambda) = C(\lambda E - A)^{-1}B$. Thus

$$\begin{aligned} &\lambda^{-1}\widehat{U}(1/\lambda)(e_1 \otimes C)\,(\lambda E - A)^{-1}\,(e_m^T \otimes B)\widehat{V}(1/\lambda) \\ &= \lambda^{-1}\widehat{U}(1/\lambda)(e_1 \otimes I_n)\,G_{sp}(\lambda)\,(e_m^T \otimes I_n)\widehat{V}(1/\lambda) \\ &= (e_m \otimes I_n)\,G_{sp}(\lambda)\,[\lambda^{-1},\ \lambda^{-2}, \ldots,\ \lambda^{-m}] \otimes I_n. \end{aligned} \tag{5.9}$$

Now by (5.8) and (5.9), we have $\widehat{U}(1/\lambda)\,\lambda^{-1}\mathbb{G}(\lambda)\,\widehat{V}(1/\lambda) =$

$$\begin{aligned} &(I_{(m-1)n} \oplus \lambda^{-m}P(\lambda)) + (e_m \otimes I_n)\,G_{sp}(\lambda)\,[\lambda^{-1},\ \lambda^{-2}, \ldots,\ \lambda^{-m}] \otimes I_n \\ = &\left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline [\lambda^{-1},\ \lambda^{-2}, \ldots,\ \lambda^{-(m-1)}] \otimes G_{sp}(\lambda) & \lambda^{-m}G(\lambda) \end{array}\right]. \end{aligned}$$

Hence by defining $\mathcal{O}_\ell(\lambda) := \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline [-\lambda^{-1},\ -\lambda^{-2}, \ldots, -\lambda^{-(m-1)}] \otimes G_{sp}(\lambda) & I_n \end{array}\right] \widehat{U}(1/\lambda)$ and $\mathcal{O}_r(\lambda) := \widehat{V}(1/\lambda)$, we have $\mathcal{O}_\ell(\lambda)\lambda^{-1}\mathbb{G}(\lambda)\mathcal{O}_r(\lambda) = I_{(m-1)n} \oplus \lambda^{-m}G(\lambda)$. Obviously, $\mathcal{O}_\ell(\lambda)$ and $\mathcal{O}_r(\lambda)$ are biproper rational matrices. This completes the proof. $\square$

The next result shows that pencils in $\mathbb{L}_1(G)$ having full Z-rank are Rosenbrock strong linearizations of $G(\lambda)$.

Theorem 5.3.8. *Let* $\mathbb{T}(\lambda) := \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right] \in \mathbb{L}_1(G)$ *with* $v \neq 0$, *where* $(L(\lambda), v) \in \mathcal{L}_1(P)$ *with full* Z*-rank. Then there exist biproper rational matrices* $\mathcal{O}_\ell(\lambda)$ *and* $\mathcal{O}_r(\lambda)$ *such that*

$$\mathcal{O}_\ell(\lambda)\lambda^{-1}\mathbb{G}(\lambda)\mathcal{O}_r(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}G(\lambda) \end{array}\right], \tag{5.10}$$

where $\mathbb{G}(\lambda) := L(\lambda) + (v \otimes C)(\lambda E - A)^{-1}(e_m^T \otimes B)$ *is the transfer function of* $\mathbb{T}(\lambda)$. *Thus* $\mathbb{T}(\lambda)$ *is a Rosenbrock strong linearization of* $G(\lambda)$.

Proof. Let $M \in \mathbb{C}^{m\times m}$ be a nonsingular matrix such that $Mv = e_1$. Then $\widehat{L}(\lambda) := (M\otimes I_n)L(\lambda) \in \mathcal{L}_1(P)$ with right ansatz vector e_1 and having full Z-rank. Thus by Lemma 5.3.7, there exist biproper rational matrices $\widehat{\mathcal{O}}_\ell(\lambda)$ and $\widehat{\mathcal{O}}_r(\lambda)$ such that

$$\widehat{\mathcal{O}}_\ell(\lambda)\lambda^{-1}\widehat{\mathbb{G}}(\lambda)\widehat{\mathcal{O}}_r(\lambda) = I_{(m-1)n} \oplus \lambda^{-m}G(\lambda), \tag{5.11}$$

where $\widehat{\mathbb{G}}(\lambda) = \widehat{L}(\lambda) + (e_1\otimes C)(\lambda E - A)^{-1}(e_m^T\otimes B) = \widehat{L}(\lambda) + (e_1\otimes I_n)G_{sp}(\lambda)(e_m^T\otimes I_n)$.

Now $\widehat{\mathcal{O}}_\ell(\lambda)\lambda^{-1}\widehat{L}(\lambda)\widehat{\mathcal{O}}_r(\lambda) = \widehat{\mathcal{O}}_\ell(\lambda)\lambda^{-1}(M\otimes I_n)L(\lambda)\widehat{\mathcal{O}}_r(\lambda) = \mathcal{O}_\ell(\lambda)\lambda^{-1}L(\lambda)\mathcal{O}_r(\lambda)$, where $\mathcal{O}_\ell(\lambda) := \widehat{\mathcal{O}}_\ell(\lambda)(M\otimes I_n)$ and $\mathcal{O}_r(\lambda) := \widehat{\mathcal{O}}_r(\lambda)$. Since $M\otimes I_n$ is nonsingular, $\mathcal{O}_\ell(\lambda)$ and $\mathcal{O}_r(\lambda)$ are biproper rational matrices. Similarly, we have

$$\begin{aligned}\widehat{\mathcal{O}}_\ell(\lambda)(e_1\otimes I_n)G_{sp}(\lambda)(e_m^T\otimes I_n)\widehat{\mathcal{O}}_r(\lambda) &= \widehat{\mathcal{O}}_\ell(\lambda)(Mv\otimes C)(\lambda E - A)^{-1}(e_m^T\otimes B)\widehat{\mathcal{O}}_r(\lambda)\\ &= \widehat{\mathcal{O}}_\ell(\lambda)(M\otimes I_n)(v\otimes C)(\lambda E - A)^{-1}(e_m^T\otimes B)\widehat{\mathcal{O}}_r(\lambda)\\ &= \mathcal{O}_\ell(\lambda)(v\otimes C)(\lambda E - A)^{-1}(e_m^T\otimes B)\mathcal{O}_r(\lambda).\end{aligned}$$

Hence by (5.11) we have $\mathcal{O}_\ell(\lambda)\lambda^{-1}\mathbb{G}(\lambda)\mathcal{O}_r(\lambda) = I_{(m-1)n}\oplus\lambda^{-m}G(\lambda)$.

By (5.10) and Theorem 5.3.4, $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. This completes the proof. $\square$

Now we show that pencils in $\mathbb{L}_2(G)$ having full Z-rank are Rosenbrock strong linearizations of $G(\lambda)$.

Theorem 5.3.9. *Let* $\mathbb{T}(\lambda) := \left[\begin{array}{c|c} L(\lambda) & e_m\otimes C \\ \hline w^T\otimes B & A-\lambda E\end{array}\right] \in \mathbb{L}_2(G)$ *with* $w \neq 0$, *where* $(L(\lambda), w) \in \mathcal{L}_2(P)$ *with full* Z*-rank. Then there exist biproper rational matrices* $\mathcal{O}_\ell(\lambda)$ *and* $\mathcal{O}_r(\lambda)$ *such that*

$$\mathcal{O}_\ell(\lambda)\lambda^{-1}\mathbb{G}(\lambda)\mathcal{O}_r(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}G(\lambda)\end{array}\right], \tag{5.12}$$

where $\mathbb{G}(\lambda) := L(\lambda) + (e_m\otimes C)(\lambda E - A)^{-1}(w^T\otimes B)$ *is the transfer function of* $\mathbb{T}(\lambda)$. *Thus* $\mathbb{T}(\lambda)$ *is a Rosenbrock strong linearization of* $G(\lambda)$.

Proof. We have $\mathbb{T}(\lambda)^T = \left[\begin{array}{c|c} L(\lambda)^T & w\otimes B^T \\ \hline e_m^T\otimes C^T & A^T - \lambda E^T\end{array}\right]$. Since $L(\lambda) \in \mathcal{L}_2(P)$, we have $L(\lambda)^T \in \mathcal{L}_1(P^T)$ having full Z-rank with w as the right ansatz vector [46]. Thus $\mathbb{T}(\lambda)^T \in \mathbb{L}_1(G^T)$. Hence by Theorem 5.3.8, there exist biproper rational matrices $\widehat{\mathcal{O}}_\ell(\lambda)$ and $\widehat{\mathcal{O}}_r(\lambda)$ such that $\widehat{\mathcal{O}}_\ell(\lambda)\lambda^{-1}\widehat{\mathbb{G}}(\lambda)\widehat{\mathcal{O}}_r(\lambda) = I_{(m-1)n}\oplus\lambda^{-m}G(\lambda)^T$, where $\widehat{\mathbb{G}}(\lambda) = L(\lambda)^T + (w\otimes B^T)(\lambda E^T - A^T)^{-1}(e_m^T\otimes C^T)$ is the transfer function of $\mathbb{T}(\lambda)^T$. Now by taking transpose, we have $\widehat{\mathcal{O}}_r(\lambda)^T \lambda^{-1}\widehat{\mathbb{G}}(\lambda)^T \widehat{\mathcal{O}}_\ell(\lambda)^T = I_{(m-1)n} \oplus \lambda^{-m}G(\lambda)$ and $\widehat{\mathbb{G}}(\lambda)^T = L(\lambda) + (e_m \otimes C)(\lambda E - A)^{-1}(w^T \otimes B) = \mathbb{G}(\lambda)$. Thus by defining $\mathcal{O}_\ell(\lambda) := \widehat{\mathcal{O}}_r(\lambda)^T$ and $\mathcal{O}_r(\lambda) := \widehat{\mathcal{O}}_\ell(\lambda)^T$ we obtain (5.12).

Finally, by (5.12) and Theorem 5.3.5, we conclude that $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. This completes the proof. □

It is shown in [46, 25] that almost all pencils in $\mathcal{L}_1(P)$ and $\mathcal{L}_2(P)$ have full Z-rank. Consequently, almost all pencils in $\mathbb{L}_1(G)$ and $\mathbb{L}_2(G)$ are Rosenbrock strong linearizations of $G(\lambda)$ and $\mathcal{S}(\lambda)$.

5.4 Symmetric linearization

We now describe a construction of a symmetric (resp., Hermitian) linearization of $G(\lambda)$ and $\mathcal{S}(\lambda)$ when $G(\lambda)$ and $\mathcal{S}(\lambda)$ are symmetric (resp., Hermitian). We define the adjoints of $\mathcal{S}(\lambda)$ by

$$\mathcal{S}^*(\lambda) := \left[\begin{array}{c|c} P^*(\lambda) & B^* \\ \hline C^* & A^* - \lambda E^* \end{array}\right] \text{ and } P^*(\lambda) := \sum\nolimits_{j=0}^{m} \lambda^j A_j^*$$

for all $\lambda \in \mathbb{C}$. Thus $\mathcal{S}(\lambda)$ is Hermitian if $\mathcal{S}^*(\lambda) = \mathcal{S}(\lambda)$ for $\lambda \in \mathbb{C}$. Similarly, $\mathcal{S}(\lambda)$ is symmetric if $\mathcal{S}(\lambda)^T = \mathcal{S}(\lambda)$ for $\lambda \in \mathbb{C}$. We say that the realization of $G(\lambda)$ in (5.5) is Hermitian (resp., symmetric) when $\mathcal{S}(\lambda)$ is Hermitian (resp., symmetric).

Let $\mathcal{H} := (\mathcal{H}_{ij})$ be a block $k \times \ell$ matrix, where each block $\mathcal{H}_{ij}$ is a $p \times q$ matrix. Recall that the *block transpose* of $\mathcal{H}$ is the block $\ell \times k$ matrix $\mathcal{H}^{\mathcal{B}}$ given by $(\mathcal{H}^{\mathcal{B}})_{ij} = \mathcal{H}_{ji}$. A block matrix $\mathcal{H}$ is said to be **block-symmetric** provided that $\mathcal{H}^{\mathcal{B}} = \mathcal{H}$, see [37].

The double ansatz space $\mathcal{DL}(P) := \mathcal{L}_1(P) \cap \mathcal{L}_2(P)$ consists of block-symmetric pencils. More precisely, if $L(\lambda) \in \mathcal{DL}(P)$ with right ansatz vector v and left ansatz vector w then $v = w$ and $L(\lambda)$ is block-symmetric. In particular, $\mathcal{DL}(P)$ contains symmetric (resp., Hermitian) linearizations of $P(\lambda)$ when $P(\lambda)$ is symmetric (resp., Hermitian) and regular, see [37, 46].

We define the block-transpose of a system matrix as follows.

Definition 5.4.1. *Let* $\mathcal{A} := \left[\begin{array}{c|c} A & u \otimes X \\ \hline v^T \otimes Y & Z \end{array}\right] \in \mathbb{C}^{(mn+r)\times(mn+r)}$, *where* $A = [A_{ij}]$ *is* $m \times m$ *block matrix with* $A_{ij} \in \mathbb{C}^{n\times n}, u, v \in \mathbb{C}^m, X \in \mathbb{C}^{n\times r}, Y \in \mathbb{C}^{r\times n}$ *and* $Z \in \mathbb{C}^{r\times r}$. *Define the block transpose of* $\mathcal{A}$ *by* $\mathcal{A}^{\mathcal{B}} := \left[\begin{array}{c|c} A^{\mathcal{B}} & v \otimes X \\ \hline u^T \otimes Y & Z \end{array}\right]$.

Observe that $\mathcal{A}$ is block-symmetric if and only if A is block-symmetric and $u = v$. Now consider the set of double ansatz pencils $\mathbb{DL}(G) := \mathbb{L}_1(G) \cap \mathbb{L}_2(G)$. We write $(\mathbb{T}_k(\lambda), v_k) \in \mathbb{L}_k(G)$ to mean that $\mathbb{T}_k(\lambda) := \mathbb{A}_k(L_k(\lambda), v_k)$ for $k = 1, 2$. Let $(\mathbb{T}(\lambda), v) \in \mathbb{L}_1(G)$ be given by

$$\mathbb{T}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right], \text{ where } (L(\lambda), v) \in \mathcal{L}_1(P).$$

Then it is easily seen that $(\mathbb{T}(\lambda), v) \in \mathbb{L}_1(G) \Leftrightarrow (\mathbb{T}(\lambda)^{\mathcal{B}}, v) \in \mathbb{L}_2(G)$ whenever $L(\lambda) \in \mathcal{DL}(P)$. Consequently, we have $(\mathbb{T}(\lambda), v) \in \mathbb{DL}(G) \Leftrightarrow (L(\lambda), v) \in \mathcal{DL}(P)$ and $v = e_m$. Since an ansatz vector v uniquely determines [46] a pencil in $\mathcal{DL}(P)$, we conclude that $\mathbb{DL}(G)$ consist of a single pencil $(\mathbb{T}(\lambda), e_m)$ given by

$$\mathbb{T}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & e_m \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right], \text{ where } (L(\lambda), e_m) \in \mathcal{DL}(P).$$

We refer to $\mathbb{T}(\lambda) \in \mathbb{DL}(G)$ as a double ansatz pencil of $G(\lambda)$ as well as of $\mathcal{S}(\lambda)$.

For a nonzero vector $v = [v_1, v_2, ..., v_m]^T \in \mathbb{C}^m$, the scalar polynomial

$$\mathbf{p}(x; v) = v_1 x^{m-1} + v_2 x^{m-2} + \cdots + v_{m-1} x + v_m$$

is referred to as the v-polynomial of the vector v. The convention is that $\mathbf{p}(x; v)$ has a root at ∞ whenever $v_1 = 0$, see [46].

Theorem 5.4.2 (Eigenvalue Exclusion Theorem, [46])**.** *Suppose that $P(\lambda)$ is a regular matrix polynomial and $L(\lambda) \in \mathcal{DL}(P)$ with ansatz vector v. Then $L(\lambda)$ is a linearization for $P(\lambda)$ if and only if no root of the v-polynomial $\mathbf{p}(x; v)$ is an eigenvalue of $P(\lambda)$.*

It is shown in [25, Theorem 6.1] that none of the pencils in $\mathcal{DL}(P)$ is a linearization of $P(\lambda)$ when $P(\lambda)$ is singular and $m \geq 2$. Therefore, for a double ansatz pencil in $\mathbb{DL}(G)$ to be a Rosenbrock strong linearization of $G(\lambda)$, the polynomial $P(\lambda)$ must necessarily be regular. In fact, we have the following result.

Theorem 5.4.3 (Structured linearization)**.** *Consider the system matrix $\mathcal{S}(\lambda)$ with $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ and $m > 1$. Then the double ansatz pencil*

$$\mathbb{T}(\lambda) := \lambda \left[\begin{array}{cccc|c} & & & A_m & \\ & & \iddots & A_{m-1} & \\ & & \iddots & \vdots & \\ & \iddots & \iddots & \vdots & \\ A_m & A_{m-1} & \cdots & A_1 & \\ \hline & & & & -E \end{array}\right] + \left[\begin{array}{cccc|c} & & & -A_m & \\ & & \iddots & -A_{m-1} & \\ & \iddots & \iddots & \vdots & \\ -A_m & -A_{m-1} & \cdots & -A_2 & \\ & & & A_0 & C \\ \hline & & & B & A \end{array}\right]$$

is block-symmetric with ansatz vector e_m. *Further,* $\mathbb{T}(\lambda)$ *is symmetric (resp., Hermitian) according as* $\mathcal{S}(\lambda)$ *is symmetric (resp., Hermitian).*

The pencil $\mathbb{T}(\lambda)$ *is a Rosenbrock strong linearization of* $G(\lambda)$ *if and only if* A_m *is nonsingular. Equivalently,* $\mathbb{T}(\lambda)$ *is a Rosenbrock strong linearization of* $G(\lambda)$ *if and only if* $\mathbb{T}(\lambda)$ *is regular.*

Proof. Note that $\mathbb{T}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & e_m \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right]$, where $L(\lambda) \in \mathcal{DL}(P)$ is the block-symmetric pencil with ansatz vector e_m, see [37]. Obviously $\mathbb{T}(\lambda)$ is block-symmetric and $\mathbb{T}(\lambda) \in \mathbb{DL}(G)$. Further, $\mathbb{T}(\lambda)$ is symmetric (resp., Hermitian) according as $\mathcal{S}(\lambda)$ is symmetric (resp., Hermitian).

Suppose that $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. Then there exist $mn \times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that

$$\mathrm{diag}(U(\lambda), I_r)\mathbb{T}(\lambda)\mathrm{diag}(V(\lambda), I_r) = \mathrm{diag}(I_{(m-1)n}, \mathcal{S}(\lambda)).$$

Hence it follows that $L(\lambda)$ is a linearization of $P(\lambda)$. Since $L(\lambda) \in \mathcal{DL}(P)$ and none of the pencils in $\mathcal{DL}(P)$ is a linearization when $P(\lambda)$ is singular [25, Theorem 6.1], the polynomial $P(\lambda)$ must be regular. Now, since $\mathbf{p}(x; e_m)$ has a root at ∞ and $L(\lambda) \in \mathcal{DL}(P)$ with ansatz vector e_m, by Theorem 5.4.2 we conclude that $P(\lambda)$ does not have an eigenvalue at ∞. This implies that A_m is nonsingular. Conversely, suppose that A_m is nonsingular. Then $P(\lambda)$ is regular and does not have an eigenvalue at ∞. Since $L(\lambda) \in \mathcal{DL}(P)$ with ansatz vector e_m, by Theorem 5.4.2, $L(\lambda)$ is a linearization of $P(\lambda)$. Consequently, by Corollary 5.3.6, $\mathbb{T}(\lambda)$ is a Rosenbrock linearization of $\mathcal{S}(\lambda)$. Hence by Theorem 5.3.8, $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$.

Next, observe that if A_m is singular then $\mathbb{T}(\lambda)$ is a singular pencil. Hence if $\mathbb{T}(\lambda)$ is regular then A_m is nonsingular. On the other hand, if A_m is nonsingular then $L(\lambda)$ is regular. Since $C(A - \lambda E)^{-1}B$ is strictly proper, it is easily seen that $\det(\mathbb{T}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. This shows that $\mathbb{T}(\lambda)$ is regular if and only if A_m is nonsingular. □

Notice that $\mathbb{DL}(G)$ does not contain a linearization of $G(\lambda)$ when $G(\lambda)$ is singular. By contrast, Fiedler and generalized Fiedler pencils of $G(\lambda)$ (regular or singular) constructed in [2, 4, 8] are shown to be Rosenbrock linearizations of $\mathcal{S}(\lambda)$. However, Fiedler and generalized Fiedler pencils cannot guarantee a structure-preserving linearization of $G(\lambda)$ when $G(\lambda)$ is structured. Indeed, unlike in the case of a symmetric matrix polynomial [6, 15], a symmetric $G(\lambda)$ does not have a symmetric generalized Fiedler pencil when $\deg(P)$ is even, see [4]. For example, the symmetric eigenproblem

$$G(\lambda)x := \Big(\lambda^2 A_2 + A_0 - \sum_{i=1}^{k} \frac{1}{1+\lambda b_i} L_i L_i^T\Big)x = 0, \tag{5.13}$$

which arises in the study of damped vibration of a structure [55], does not have a symmetric generalized Fiedler pencil [4], where L_i is an $n \times r_i$ full column rank matrix for $i = 1, \ldots, k$. Set $C := \begin{bmatrix} L_1 & L_2 & \cdots & L_k \end{bmatrix}$ and $E := \operatorname{diag}(-b_1 I_{r_1}, -b_2 I_{r_2}, \ldots, -b_k I_{r_k})$. Then we have a minimal symmetric realization of $G(\lambda)$ given by [55]

$$G(\lambda) = \lambda^2 A_2 + A_0 + C(\lambda E - I)^{-1} C^T.$$

The symmetric eigenproblem (5.13) can now be rewritten as a symmetric generalized eigenvalue problem [55]

$$\mathbb{H}(\lambda)u := \left(\lambda \left[\begin{array}{cc|c} & A_2 & \\ A_2 & & \\ \hline & & -E \end{array} \right] + \left[\begin{array}{cc|c} -A_2 & & \\ & A_0 & C \\ \hline & C^T & I \end{array} \right] \right) \begin{bmatrix} \lambda x \\ x \\ y \end{bmatrix} = 0,$$

where $y := (\lambda E - I)^{-1} C^T x$. Note that $\mathbb{H}(\lambda) \in \mathbb{DL}(G)$ with the ansatz vector e_2. Hence by Theorem 5.4.3, $\mathbb{H}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda) \iff A_2$ is nonsingular $\iff \mathbb{H}(\lambda)$ is regular. We mention that the symmetric pencil $\mathbb{H}(\lambda)$ is not a generalized Fiedler pencil of $G(\lambda)$.

5.5 Recovery of minimal bases and minimal indices

We now describe the recovery of minimal bases and minimal indices of $\mathcal{S}(\lambda)$ and $G(\lambda)$ from those of a Rosenbrock strong linearization $\mathbb{T}(\lambda)$ when $\mathbb{T}(\lambda) \in \mathbb{L}_i(G), i = 1, 2$.

The following result shows that the degree of a vector polynomial in the null space of a system matrix (in particular, $\mathcal{S}(\lambda)$) is determined by the first n components of the vector polynomial.

Theorem 5.5.1. *Let* $\mathcal{X}(\lambda) := \left[\begin{array}{c|c} X(\lambda) & X_{12} \\ \hline X_{21} & A - \lambda E \end{array} \right]$ *be an* $(n+r) \times (n+r)$ *system matrix, where* $E \in \mathbb{C}^{r \times r}$ *is a nonsingular matrix. Let* $u(\lambda) := \left[\begin{array}{c} u_1(\lambda) \\ \hline u_2(\lambda) \end{array} \right]$ *be a nonzero vector polynomial, where* $u_1(\lambda) \in \mathbb{C}[\lambda]^n$ *and* $u_2(\lambda) \in \mathbb{C}[\lambda]^r$*. Then we have the following.*

(a) If $u(\lambda) \in \mathcal{N}_r(\mathcal{X})$*, then* $\deg(u(\lambda)) = \deg(u_1(\lambda))$*. Moreover,* $\deg(u_2(\lambda)) < \deg(u_1(\lambda))$ *when* $\deg(u(\lambda)) \geq 1$*, and* $u_2(\lambda) \equiv 0$ *when* $\deg(u(\lambda)) = 0$*.*

(b) If $u(\lambda) \in \mathcal{N}_l(\mathcal{X})$*, then* $\deg(u(\lambda)) = \deg(u_1(\lambda))$*. Moreover,* $\deg(u_2(\lambda)) < \deg(u_1(\lambda))$ *when* $\deg(u(\lambda)) \geq 1$*, and* $u_2(\lambda) \equiv 0$ *when* $\deg(u(\lambda)) = 0$*.*

Proof. (a) We have $\mathcal{X}(\lambda)u(\lambda) = 0 \Rightarrow \left[\begin{array}{c|c} X(\lambda) & X_{12} \\ \hline X_{21} & A - \lambda E \end{array}\right] \left[\begin{array}{c} u_1(\lambda) \\ \hline u_2(\lambda) \end{array}\right] = 0$ which shows that

$$X(\lambda)u_1(\lambda) + X_{12}u_2(\lambda) = 0 \text{ and } X_{21}u_1(\lambda) + (A - \lambda E)u_2(\lambda) = 0. \tag{5.14}$$

Let $d := \deg(u(\lambda))$. Now suppose that $d \geq 1$ and $\deg(u_1(\lambda)) \leq \deg(u_2(\lambda))$. Then

$$u(\lambda) = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \lambda^d + \text{ lower order terms},$$

where $t_1 \in \mathbb{C}^n$ and $t_2 \in \mathbb{C}^r$ are constant vectors and $t_2 \neq 0$. Now by (5.14), we have

$$\left[\begin{array}{c} X_{21} \mid (A - \lambda E) \end{array}\right] u(\lambda) \equiv 0 \tag{5.15}$$

Extracting the coefficient of the highest degree (i.e., degree $d + 1$) term from the left hand side of (5.15), we see that $-Et_2 = 0$. Since E is nonsingular, we have $t_2 = 0$, which is a contradiction. Hence we must have $\deg(u_2(\lambda)) < \deg(u_1(\lambda))$.

Next, suppose that $d = 0$. Then $u(\lambda) = \left[\begin{array}{c} u_1 \\ \hline u_2 \end{array}\right] \in \mathcal{N}_r(\mathcal{X})$ is a constant vector, where $u_1 \in \mathbb{C}^n$ and $u_2 \in \mathbb{C}^r$. Then $\mathcal{X}(\lambda)u(\lambda) = 0 \Rightarrow \left[\begin{array}{c|c} X(\lambda) & X_{12} \\ \hline X_{21} & A - \lambda E \end{array}\right] \left[\begin{array}{c} u_1 \\ \hline u_2 \end{array}\right] = 0$ which gives $X(\lambda)u_1 + X_{12}u_2 = 0$ and $X_{21}u_1 + (A - \lambda E)u_2 = 0$ for all $\lambda \in \mathbb{C}$. This shows that $(X_{21}u_1 + Au_2) - \lambda E u_2 = 0$ for all $\lambda \in \mathbb{C}$. Consequently, we have $Eu_2 = 0 \Rightarrow u_2 = 0$ and that $\deg(u(\lambda)) = \deg(u_1(\lambda))$.

(b) Since $\mathcal{N}_l(\mathcal{X}) = \mathcal{N}_r(\mathcal{X}^T)$, the desired results follow from (a). $\square$

The next result analyzes isomorphisms between null spaces of $\mathcal{S}(\lambda)$ and its linearizations and the condition under which these isomorphisms map minimal bases to minimal bases.

Theorem 5.5.2. *Let $\mathbb{T}(\lambda)$ be an $(mn + r) \times (mn + r)$ matrix pencil such that*

$$\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{T}(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right], \tag{5.16}$$

where $U(\lambda)$ and $V(\lambda)$ are $mn \times mn$ unimodular matrix polynomials.

(a) The maps

$$\mathcal{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}) \longrightarrow \mathcal{N}_r(\mathbb{T}), \quad \begin{bmatrix} x(\lambda) \\ \hline y(\lambda) \end{bmatrix} \longmapsto \begin{bmatrix} V(\lambda)(e_m \otimes x(\lambda)) \\ \hline y(\lambda) \end{bmatrix},$$

$$\mathcal{F}(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}) \longrightarrow \mathcal{N}_r(\mathcal{S}), \quad \begin{bmatrix} u(\lambda) \\ \hline v(\lambda) \end{bmatrix} \longmapsto \begin{bmatrix} (e_m^T \otimes I_n)V^{-1}(\lambda)u(\lambda) \\ \hline v(\lambda) \end{bmatrix}$$

are linear isomorphisms and map vector polynomials to vector polynomials, where $x(\lambda) \in \mathbb{C}[\lambda]^n, u(\lambda) \in \mathbb{C}[\lambda]^{mn}$ *and* $y(\lambda), v(\lambda) \in \mathbb{C}[\lambda]^r$.

(b) Let k *a nonnegative integer. Then* $\deg\big(\mathcal{E}(\mathcal{S})x(\lambda)\big) = \deg(x(\lambda)) + k$ *for all nonzero* $x(\lambda) \in \mathcal{N}_r(\mathcal{S}) \cap \mathbb{C}[\lambda]^{n+r}$ *if and only if* $\deg\big(\mathcal{F}(\mathcal{S})y(\lambda)\big) = \deg(y(\lambda)) - k$ *for all nonzero* $y(\lambda) \in \mathcal{N}_r(\mathbb{T}) \cap \mathbb{C}[\lambda]^{mn+r}$. *In particular,* $\mathcal{E}(\mathcal{S})$ *is degree preserving (i.e.,* $k = 0$*) if and only if* $\mathcal{F}(\mathcal{S})$ *is degree preserving.*

(c) Suppose that $\deg\big(\mathcal{E}(\mathcal{S})x(\lambda)\big) = \deg(x(\lambda)) + k$ *for all nonzero* $x(\lambda) \in \mathcal{N}_r(\mathcal{S}) \cap \mathbb{C}[\lambda]^{n+r}$. *Then* $\mathcal{E}(\mathcal{S})$ *maps a minimal basis of* $\mathcal{N}_r(\mathcal{S})$ *to a minimal basis of* $\mathcal{N}_r(\mathbb{T})$ *and* $\mathcal{F}(\mathcal{S})$ *maps a minimal basis of* $\mathcal{N}_r(\mathbb{T})$ *to a minimal basis of* $\mathcal{N}_r(\mathcal{S})$. *Furthermore, if* $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $\mathbb{T}(\lambda)$ *then* $\varepsilon_1 - k \leq \cdots \leq \varepsilon_p - k$ *are the right minimal indices of* $\mathcal{S}(\lambda)$.

Proof. (a) Clearly we have $\mathcal{F}(\mathcal{S})\mathcal{E}(\mathcal{S}) = I_{n+r}$. Now, suppose that $\begin{bmatrix} x(\lambda) \\ \hline y(\lambda) \end{bmatrix} \in \mathcal{N}_r(\mathcal{S})$.

Then $\mathcal{S}(\lambda)\begin{bmatrix} x(\lambda) \\ \hline y(\lambda) \end{bmatrix} = 0 \Rightarrow \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right] \begin{bmatrix} e_m \otimes x(\lambda) \\ \hline y(\lambda) \end{bmatrix} = 0$. By (5.16) we have $\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{T}(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] \begin{bmatrix} e_m \otimes x(\lambda) \\ \hline y(\lambda) \end{bmatrix} = 0$ which gives

$$\mathbb{T}(\lambda)\begin{bmatrix} V(\lambda)(e_m \otimes x(\lambda)) \\ \hline y(\lambda) \end{bmatrix} = 0 \Rightarrow \mathbb{T}(\lambda)\mathcal{E}(\mathcal{S})\begin{bmatrix} x(\lambda) \\ \hline y(\lambda) \end{bmatrix} = 0.$$

This shows that $\mathcal{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}) \to \mathcal{N}_r(\mathbb{T})$ is well-defined. Let $u(\lambda) \in \mathcal{N}_r(\mathcal{S})$. Then $\mathcal{E}(\mathcal{S})u(\lambda) = 0 \Rightarrow \mathcal{F}(\mathcal{S})\mathcal{E}(S)u(\lambda) = 0 \Rightarrow u(\lambda) = 0$. Hence $\mathcal{E}(\mathcal{S}) : \mathcal{N}_r(S) \to \mathcal{N}_r(\mathbb{T})$ is one-one.

Next, suppose that $\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] \in \mathcal{N}_r(\mathbb{T})$. Then $\mathbb{T}(\lambda)\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0$ gives

$$\left[\begin{array}{c|c} U(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0$$

$$\Rightarrow \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0$$

$$\Rightarrow \mathcal{S}(\lambda)\left[\begin{array}{c} (e_m^T \otimes I_n)V(\lambda)^{-1}w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0 \Rightarrow \mathcal{S}(\lambda)\mathcal{F}(\mathcal{S})\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0.$$

This shows that $\mathcal{F}(S) : \mathcal{N}_r(\mathbb{T}) \to \mathcal{N}_r(\mathcal{S})$ is well-defined. Now, let $\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] \in \mathcal{N}_r(\mathbb{T})$ and $\mathcal{F}(\mathcal{S})\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0$. Then we have $\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = \left[\begin{array}{c} V(\lambda)^{-1}w(\lambda) \\ \hline z(\lambda) \end{array}\right] =$ $\left[\begin{array}{c} u(\lambda) \\ (e_m^T \otimes I_n)V(\lambda)^{-1}w(\lambda) \\ \hline z(\lambda) \end{array}\right] = \left[\begin{array}{c} u(\lambda) \\ \mathcal{F}(S)\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] \end{array}\right] = \left[\begin{array}{c} u(\lambda) \\ 0 \end{array}\right]$, for some $u(\lambda) \in \mathbb{C}[\lambda]^{(m-1)n}$.

On the other hand, $\mathbb{T}(\lambda)\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0$ gives

$$\left[\begin{array}{c|c} U(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0$$

$$\Rightarrow \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\left[\begin{array}{c} u(\lambda) \\ 0 \end{array}\right] = 0 \Rightarrow u(\lambda) = 0.$$

Hence $\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0 \Rightarrow \left[\begin{array}{c} w(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0$. This shows that $\mathcal{F}(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}) \to \mathcal{N}_r(\mathcal{S})$ is one-one.

Since $\mathcal{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}) \to \mathcal{N}_r(\mathbb{T})$ is one-one and $\mathcal{F}(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}) \to \mathcal{N}_r(\mathcal{S})$ is one-one, we have $\dim \mathcal{N}_r(\mathcal{S}) \leq \dim \mathcal{N}_r(\mathbb{T})$ and $\dim \mathcal{N}_r(\mathbb{T}) \leq \dim \mathcal{N}_r(\mathcal{S})$. Therefore $\dim \mathcal{N}_r(\mathbb{T}) = \dim \mathcal{N}_r(\mathcal{S})$. Hence $\mathcal{E}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}) \to \mathcal{N}_r(\mathbb{T})$ is an isomorphism and $\mathcal{F}(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}) \to \mathcal{N}_r(\mathcal{S})$ is an isomorphism. Since $V(\lambda)$ is unimodular, $\mathcal{E}(\mathcal{S})$ and $\mathcal{F}(\mathcal{S})$ map vector polynomials to vector polynomials.

(b) Suppose that $\deg\big(\mathcal{E}(\mathcal{S})x(\lambda)\big) = \deg(x(\lambda)) + k$ for all nonzero vector polynomial $x(\lambda) \in \mathcal{N}_r(\mathcal{S})$. Let $y(\lambda) \in \mathcal{N}_r(\mathbb{T})$ be a vector polynomial. Since $\mathcal{E}(\mathcal{S})$ is an isomorphism, we have $y(\lambda) = \mathcal{E}(\mathcal{S})z(\lambda)$ for some vector polynomial $z(\lambda) \in \mathcal{N}_r(\mathcal{S})$. So $\deg(y(\lambda)) = \deg\big(\mathcal{E}(\mathcal{S})z(\lambda)\big) = \deg(z(\lambda)) + k$. We have $\mathcal{F}(\mathcal{S})y(\lambda) = \mathcal{F}(\mathcal{S})\mathcal{E}(\mathcal{S})z(\lambda) = z(\lambda) \Rightarrow \deg\big(\mathcal{F}(\mathcal{S})y(\lambda)\big) = \deg(z(\lambda))$. Hence $\deg\big(\mathcal{F}(\mathcal{S})y(\lambda)\big) = \deg(y(\lambda)) - k$. On the other hand, suppose that $\deg\big(\mathcal{F}(\mathcal{S})y(\lambda)\big) = \deg(y(\lambda)) - k$ for all vector polynomial $y(\lambda) \in \mathcal{N}_r(\mathbb{T})$. Let $x(\lambda) \in \mathcal{N}_r(\mathcal{S})$ be a nonzero vector polynomial. Then $\mathcal{E}(\mathcal{S})x(\lambda) \in \mathcal{N}_r(\mathbb{T}) \Rightarrow \deg\big(\mathcal{F}(\mathcal{S})\mathcal{E}(\mathcal{S})x(\lambda)\big) = \deg\big(\mathcal{E}(\mathcal{S})x(\lambda)\big) - k$. But $\mathcal{F}(\mathcal{S})\mathcal{E}(\mathcal{S})x(\lambda) = x(\lambda)$. So $\deg\big(\mathcal{E}(\mathcal{S})x(\lambda)\big) = \deg(x(\lambda)) + k$. This proves (b).

(c) Let $\mathcal{B} := \big(z_1(\lambda), \ldots, z_p(\lambda)\big)$ be a right minimal basis of $\mathcal{S}(\lambda)$. Then by (a), $\mathcal{C} := \mathcal{E}(\mathcal{S})(\mathcal{B})$ is a polynomial basis of $\mathcal{N}_r(\mathbb{T})$. Since $\deg(\mathcal{E}(\mathcal{S})z_i(\lambda)) = \deg(z_i(\lambda)) + k$, for $i = 1, 2, \ldots, p$, we have $\mathrm{Ord}(\mathcal{C}) = \mathrm{Ord}(\mathcal{B}) + pk$. We claim that $\mathcal{C}$ is a minimal basis of $\mathbb{T}(\lambda)$. Indeed, let $\widetilde{\mathcal{C}}$ be any polynomial basis of $\mathcal{N}_r(\mathbb{T})$. Then by (a), $\widetilde{\mathcal{B}} := \mathcal{F}(\mathcal{S})\widetilde{\mathcal{C}}$ is a polynomial basis of $\mathcal{N}_r(\mathcal{S})$ and by (b), $\mathrm{Ord}(\widetilde{\mathcal{B}}) = \mathrm{Ord}(\widetilde{\mathcal{C}}) - pk \Rightarrow \mathrm{Ord}(\widetilde{\mathcal{C}}) = \mathrm{Ord}(\widetilde{\mathcal{B}}) + pk \geq \mathrm{Ord}(\mathcal{B}) + pk = \mathrm{Ord}(\mathcal{C})$. This shows that $\mathcal{C}$ is a minimal basis. Similarly, $\mathcal{F}(\mathcal{S})$ maps minimal bases of $\mathcal{N}_r(\mathbb{T})$ to minimal bases of $\mathcal{N}_r(\mathcal{S})$. The relationship between the right minimal indices of $\mathcal{S}(\lambda)$ and $\mathbb{T}(\lambda)$ is now immediate. This completes the proof. $\square$

The recovery of right minimal bases and right minimal indices of $\mathcal{S}(\lambda)$ from those of its linearizations in $\mathbb{L}_1(G)$ is given in the following result.

Theorem 5.5.3. *Let* $\mathbb{T}(\lambda) := \left[\begin{array}{c|c} L(\lambda) & v \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right] \in \mathbb{L}_1(G)$, *where* $(L(\lambda), v) \in \mathcal{L}_1(P)$ *with* $v \neq 0$ *and* $L(\lambda)$ *having full* Z*-rank. Then the following results hold:*

(a) Let $\left[\begin{array}{c} x \\ \hline y \end{array}\right] \in \mathcal{N}_r(\mathbb{T}) \cap \mathbb{C}[\lambda]^{mn+r}$ *with* $x \in \mathbb{C}[\lambda]^{mn}$ *and* $y \in \mathbb{C}[\lambda]^r$. *Then* $\left[\begin{array}{c} (e_m^T \otimes I_n)x \\ \hline y \end{array}\right] \in \mathcal{N}_r(\mathcal{S})$ *and* $\deg\left(\left[\begin{array}{c} (e_m^T \otimes I_n)x \\ \hline y \end{array}\right]\right) = \deg\left(\left[\begin{array}{c} x \\ \hline y \end{array}\right]\right) - (m-1)$. *Further, the mapping*

$$\mathbb{F}_1(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}) \longrightarrow \mathcal{N}_r(\mathcal{S}), \quad \left[\begin{array}{c} x \\ \hline y \end{array}\right] \longmapsto \left[\begin{array}{c} (e_m^T \otimes I_n)x \\ \hline y \end{array}\right]$$

is a linear isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{T})$ *to a minimal basis of* $\mathcal{N}_r(\mathcal{S})$, *where* $x \in \mathbb{C}[\lambda]^{mn}$ *and* $y \in \mathbb{C}[\lambda]^r$.

(b) If $\left[\begin{array}{c} u \\ \hline w \end{array}\right] \in \mathcal{N}_r(\mathcal{S})$ *then* $\left[\begin{array}{c} \Lambda \otimes u \\ \hline w \end{array}\right] \in \mathcal{N}_r(\mathbb{T})$, *where* $u \in \mathbb{C}[\lambda]^n$ *and* $w \in \mathbb{C}[\lambda]^r$.

Furthermore, the mapping $\mathbb{E}_1(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}) \longrightarrow \mathcal{N}_r(\mathbb{T})$, $\left[\begin{array}{c} u \\ \hline w \end{array}\right] \longmapsto \left[\begin{array}{c} \Lambda \otimes u \\ \hline w \end{array}\right]$ *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_r(\mathcal{S})$ *to a minimal basis of* $\mathcal{N}_r(\mathbb{T})$, *where* $u \in \mathbb{C}[\lambda]^n$ *and* $w \in \mathbb{C}[\lambda]^r$.

(c) If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $\mathbb{T}(\lambda)$ *then* $\varepsilon_1 - (m-1) \leq \cdots \leq \varepsilon_p - (m-1)$ *are the right minimal indices of* $\mathcal{S}(\lambda)$.

Proof. By Theorem 5.3.4 there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that $\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{T}(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]$, $V(\lambda)(e_m \otimes I_n) = \Lambda \otimes I_n$ and $(e_m^T \otimes I_n)V(\lambda)^{-1} = e_m^T \otimes I_n$. Now $\mathbb{F}_1(\mathcal{S}) \left[\begin{array}{c} x \\ \hline y \end{array}\right] = \left[\begin{array}{c} (e_m^T \otimes I_n)x \\ \hline y \end{array}\right] = \left[\begin{array}{c} (e_m^T \otimes I_n)V(\lambda)^{-1}x \\ \hline y \end{array}\right]$ $= \mathcal{F}(\mathcal{S}) \left[\begin{array}{c} x \\ \hline y \end{array}\right]$ and $\mathbb{E}_1(\mathcal{S}) \left[\begin{array}{c} u \\ \hline w \end{array}\right] = \left[\begin{array}{c} \Lambda \otimes u \\ \hline w \end{array}\right] = \left[\begin{array}{c} V(\lambda)(e_m \otimes u) \\ \hline w \end{array}\right] = \mathcal{E}(\mathcal{S}) \left[\begin{array}{c} u \\ \hline w \end{array}\right]$, where $\mathcal{E}(\mathcal{S})$ and $\mathcal{F}(\mathcal{S})$ are given in Theorem 5.5.2. Finally, let $z := \left[\begin{array}{c} u \\ \hline w \end{array}\right] \in \mathcal{N}_r(\mathcal{S})$, where $u \in \mathbb{C}[\lambda]^n$ and $w \in \mathbb{C}[\lambda]^r$. By Theorem 5.5.1, we have $\deg(z) = \deg(u)$ and $\deg\big(\mathbb{E}_1(\mathcal{S})z\big) = (m-1) + \deg(z)$. Hence the desired results follow from Theorem 5.5.2. ☐

The recovery of left minimal bases and left minimal indices of $\mathcal{S}(\lambda)$ from those of its linearizations in $\mathbb{L}_2(G)$ is given in the following result.

Theorem 5.5.4. *Let* $\mathbb{T}(\lambda) := \left[\begin{array}{c|c} L(\lambda) & e_m \otimes C \\ \hline w^T \otimes B & A - \lambda E \end{array}\right] \in \mathbb{L}_2(G)$, *where* $(L(\lambda), w) \in \mathcal{L}_2(P)$ *with* $w \neq 0$ *and* $L(\lambda)$ *having full* Z*-rank. Then the following results hold:*

(a) Let $\left[\begin{array}{c} x \\ \hline y \end{array}\right] \in \mathcal{N}_l(\mathbb{T}) \cap \mathbb{C}[\lambda]^{mn+r}$ *with* $x \in \mathbb{C}[\lambda]^{mn}$ *and* $y \in \mathbb{C}[\lambda]^r$. *Then* $\left[\begin{array}{c} (e_m^T \otimes I_n)x \\ \hline y \end{array}\right] \in \mathcal{N}_l(\mathcal{S})$ *and* $\deg\left(\left[\begin{array}{c} (e_m^T \otimes I_n)x \\ \hline y \end{array}\right]\right) = \deg\left(\left[\begin{array}{c} x \\ \hline y \end{array}\right]\right) - (m-1)$. *Further, the mapping* $\mathcal{N}_l(\mathbb{T}) \longrightarrow \mathcal{N}_l(\mathcal{S})$, $\left[\begin{array}{c} x \\ \hline y \end{array}\right] \longmapsto \left[\begin{array}{c} (e_m^T \otimes I_n)x \\ \hline y \end{array}\right]$ *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_l(\mathbb{T})$ *to a minimal basis of* $\mathcal{N}_l(\mathcal{S})$, *where* $x \in \mathbb{C}[\lambda]^{mn}$ *and* $y \in \mathbb{C}[\lambda]^r$.

(b) If $\left[\begin{array}{c} u \\ \hline v \end{array}\right] \in \mathcal{N}_l(\mathcal{S})$ *then* $\left[\begin{array}{c} \Lambda \otimes u \\ \hline v \end{array}\right] \in \mathcal{N}_l(\mathbb{T})$, *where* $u \in \mathbb{C}[\lambda]^n$ *and* $v \in \mathbb{C}[\lambda]^r$.

Furthermore, the mapping $\mathcal{N}_l(\mathcal{S}) \longrightarrow \mathcal{N}_l(\mathbb{T})$, $\left[\begin{array}{c} u \\ \hline v \end{array}\right] \longmapsto \left[\begin{array}{c} \Lambda \otimes u \\ \hline v \end{array}\right]$ *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_l(\mathcal{S})$ *to a minimal basis of* $\mathcal{N}_l(\mathbb{T})$, *where* $u \in \mathbb{C}[\lambda]^n$ *and* $w \in \mathbb{C}[\lambda]^r$.

(c) If $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $\mathbb{T}(\lambda)$ *then* $\eta_1 - (m-1) \leq \cdots \leq \eta_p - (m-1)$ *are the left minimal indices of* $\mathcal{S}(\lambda)$.

Proof. Note that $\mathbb{T}(\lambda)^T \in \mathbb{L}_1(G^T)$ with right ansatz vector w. Since $\mathcal{N}_l(\mathcal{S}) = \mathcal{N}_r(\mathcal{S}^T)$ and $\mathcal{N}_l(\mathbb{T}) = \mathcal{N}_r(\mathbb{T}^T)$, the desired results follow from Theorem 5.5.3. ☐

Next we describe the recovery of right minimal bases and right minimal indices of $\mathcal{S}(\lambda)$ from those of its linearizations in $\mathbb{L}_2(G)$.

Theorem 5.5.5. *Let* $\mathbb{T}(\lambda)$ *be as in Theorem 5.5.4 with left ansatz vector* w.

(a) Let $\left[\begin{array}{c} x \\ \hline y \end{array}\right] \in \mathcal{N}_r(\mathbb{T}) \cap \mathbb{C}[\lambda]^{mn+r}$ *with* $x \in \mathbb{C}[\lambda]^{mn}$ *and* $y \in \mathbb{C}[\lambda]^r$. *Then we have* $\left[\begin{array}{c} (w^T \otimes I_n)x \\ \hline y \end{array}\right] \in \mathcal{N}_r(\mathcal{S})$ *and* $\deg\left(\left[\begin{array}{c} (w^T \otimes I_n)x \\ \hline y \end{array}\right]\right) = \deg\left(\left[\begin{array}{c} x \\ \hline y \end{array}\right]\right)$. *Further, the mapping* $\mathbb{F}_2(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}) \longrightarrow \mathcal{N}_r(\mathcal{S})$, $\left[\begin{array}{c} x \\ \hline y \end{array}\right] \longmapsto \left[\begin{array}{c} (w^T \otimes I_n)x \\ \hline y \end{array}\right]$ *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_r(\mathbb{T})$ *to a minimal basis of* $\mathcal{N}_r(\mathcal{S})$, *where* $x \in \mathbb{C}[\lambda]^{mn}$ *and* $y \in \mathbb{C}[\lambda]^r$.

(b) The right minimal indices of $\mathbb{T}(\lambda)$ *and* $\mathcal{S}(\lambda)$ *are the same, i.e., if* $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $\mathbb{T}(\lambda)$ *then* $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are also the right minimal indices of* $\mathcal{S}(\lambda)$.

Proof. By Theorem 5.3.5 there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that $\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{T}(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]$ and $(e_m^T \otimes I_n)V(\lambda)^{-1} = w^T \otimes I_n$. Consequently, we have $\mathbb{F}_2(\mathcal{S}) = \mathcal{F}(\mathcal{S})$, where $\mathcal{F}(\mathcal{S})$ is defined in Theorem 5.5.2.

We now show that $\mathbb{F}_2(\mathcal{S})$ is degree preserving. Let $\left[\begin{array}{c} y(\lambda) \\ \hline z(\lambda) \end{array}\right] \in \mathcal{N}_r(\mathbb{T})$ be a nonzero vector polynomial. We show that $\deg\left(\left[\begin{array}{c} y(\lambda) \\ \hline z(\lambda) \end{array}\right]\right) = \deg\left(\left[\begin{array}{c} (w^T \otimes I_n)y(\lambda) \\ \hline z(\lambda) \end{array}\right]\right)$.

Since $\left[\begin{array}{c} y(\lambda) \\ \hline z(\lambda) \end{array}\right] \in \mathcal{N}_r(\mathbb{T})$ and $\mathbb{T}(\lambda)$ is a Rosenbrock system matrix, by Theorem 5.5.1, we have $\deg\left(\begin{bmatrix} y(\lambda) \\ \hline z(\lambda) \end{bmatrix}\right) = \deg(y(\lambda))$ and $\deg(z(\lambda)) < \deg(y(\lambda))$. Again, since $\begin{bmatrix} (w^T \otimes I_n)y(\lambda) \\ \hline z(\lambda) \end{bmatrix}$ $\in \mathcal{N}_r(\mathcal{S})$ and $\mathcal{S}(\lambda)$ is a system matrix, by Theorem 5.5.1,

$$\deg\left(\begin{bmatrix} (w^T \otimes I_n)y(\lambda) \\ \hline z(\lambda) \end{bmatrix}\right) = \deg((w^T \otimes I_n)y(\lambda))$$

and $\deg(z(\lambda)) < \deg((w^T \otimes I_n)y(\lambda))$. Hence the problem boils down to showing that $\deg((w^T \otimes I_n)y(\lambda)) = \deg(y(\lambda))$. Let $K \in \mathbb{C}^{m\times m}$ be a nonsingular matrix such that $w^T K = e_1^T$. Now observe that

$$\begin{bmatrix} (w^T \otimes I_n)y(\lambda) \\ \hline z(\lambda) \end{bmatrix} = \begin{bmatrix} (w^T \otimes I_n)(K \otimes I_n)(K^{-1} \otimes I_n)y(\lambda) \\ \hline z(\lambda) \end{bmatrix}$$
$$= \begin{bmatrix} (e_1^T \otimes I_n)(K^{-1} \otimes I_n)y(\lambda) \\ \hline z(\lambda) \end{bmatrix} = \begin{bmatrix} (e_1^T \otimes I_n)\widehat{y}(\lambda) \\ \hline z(\lambda) \end{bmatrix},$$

where $\widehat{y}(\lambda) := (K^{-1} \otimes I_n)y(\lambda)$. Let $\widehat{y}(\lambda) = \begin{bmatrix} \widehat{y}_1(\lambda) \\ \hline \vdots \\ \hline \widehat{y}_m(\lambda) \end{bmatrix}$, where $\widehat{y}_j(\lambda) \in \mathbb{C}[\lambda]^n$. Then $\deg((e_1^T \otimes I_n)\widehat{y}(\lambda)) = \deg(\widehat{y}_1(\lambda))$.

On the other hand, since $(K^{-1} \otimes I_n)$ is an invertible matrix, we have $\deg(y(\lambda)) = \deg((K^{-1} \otimes I_n)y(\lambda)) = \deg(\widehat{y}(\lambda))$. We claim that $\deg(\widehat{y}(\lambda)) = \deg(\widehat{y}_1(\lambda))$. Obviously $\deg(\widehat{y}_1(\lambda)) \leq \deg(\widehat{y}(\lambda))$. We now show that $\deg(\widehat{y}_1(\lambda)) < \deg(\widehat{y}(\lambda))$ is impossible. Set $d := \deg(\widehat{y}(\lambda))$ and suppose that $\deg(\widehat{y}_1(\lambda)) < \deg(\widehat{y}(\lambda))$. Then we can write

$$\widehat{y}(\lambda) = \begin{bmatrix} 0 \\ t \end{bmatrix} \lambda^d + \text{ lower order terms},$$

where $t \in \mathbb{C}^{(m-1)n}$ is a nonzero vector. Since $\deg\left(\begin{bmatrix} y(\lambda) \\ \hline z(\lambda) \end{bmatrix}\right) = \deg(y(\lambda))$ and $\deg(z(\lambda)) < \deg(y(\lambda))$, we have $\deg\left(\begin{bmatrix} \widehat{y}(\lambda) \\ \hline z(\lambda) \end{bmatrix}\right) = \deg(\widehat{y}(\lambda))$ and $\deg(z(\lambda)) < \deg(\widehat{y}(\lambda))$.

Thus $\left[\begin{array}{c} \widehat{y}(\lambda) \\ \hline z(\lambda) \end{array}\right] = \left[\begin{array}{c} 0 \\ t \\ \hline 0 \end{array}\right] \lambda^d +$ lower order terms. Since $\left[\begin{array}{c} y(\lambda) \\ \hline z(\lambda) \end{array}\right] \in \mathcal{N}_r(\mathbb{T})$, we have

$$\left[\begin{array}{c|c} L(\lambda) & e_m \otimes C \\ \hline w^T \otimes B & A - \lambda E \end{array}\right] \left[\begin{array}{c|c} K \otimes I_n & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c} \widehat{y}(\lambda) \\ \hline z(\lambda) \end{array}\right] = \mathbb{T}(\lambda) \left[\begin{array}{c} y(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0$$

$$\Rightarrow \left[\begin{array}{c|c} L(\lambda)(K \otimes I_n) & e_m \otimes C \\ \hline (w^T \otimes B)(K \otimes I_n) & A - \lambda E \end{array}\right] \left[\begin{array}{c} \widehat{y}(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0.$$

By Theorem 5.1.2, we have $\left[\begin{array}{c|c} \lambda \left[\begin{array}{c|c} A_m & 0 \\ \hline X_{21} & -Z \end{array}\right] + \left[\begin{array}{c|c} Y_{11} & Z \\ \hline A_0 & 0 \end{array}\right] & e_m \otimes C \\ \hline e_1^T \otimes B & A - \lambda E \end{array}\right] \left[\begin{array}{c} \widehat{y}(\lambda) \\ \hline z(\lambda) \end{array}\right] = 0,$

where $Z \in \mathbb{C}^{(m-1)n \times (m-1)n}$ is nonsingular. Hence for all $\lambda \in \mathbb{C}$, we have

$$\left[\begin{array}{c|c} \lambda \left[\begin{array}{c|c} A_m & 0 \\ \hline X_{21} & -Z \end{array}\right] + \left[\begin{array}{c|c} Y_{11} & Z \\ \hline A_0 & 0 \end{array}\right] & e_m \otimes C \\ \hline e_1^T \otimes B & A - \lambda E \end{array}\right] \left\{ \left[\begin{array}{c} 0 \\ t \\ \hline 0 \end{array}\right] \lambda^d + \text{lower order terms} \right\} = 0$$

$$\Rightarrow \left[\begin{array}{c|c} \lambda \left[\begin{array}{c|c} A_m & 0 \\ \hline X_{21} & -Z \end{array}\right] + \left[\begin{array}{c|c} Y_{11} & Z \\ \hline A_0 & 0 \end{array}\right] & e_m \otimes C \end{array}\right] \left\{ \left[\begin{array}{c} 0 \\ t \\ \hline 0 \end{array}\right] \lambda^d + \text{lower order terms} \right\} = 0.$$

Equating the coefficient of the highest degree (i.e. degree $d+1$) terms, we have

$$\left[\begin{array}{c|c} \left[\begin{array}{c|c} A_m & 0 \\ \hline X_{21} & -Z \end{array}\right] & 0 \end{array}\right] \left[\begin{array}{c} 0 \\ t \\ \hline 0 \end{array}\right] = 0 \Rightarrow \left[\begin{array}{c|c} A_m & 0 \\ \hline X_{21} & -Z \end{array}\right] \left[\begin{array}{c} 0 \\ t \end{array}\right] = 0 \Rightarrow \left[\begin{array}{c} 0 \\ \hline -Zt \end{array}\right] = 0.$$

Since Z is nonsingular, we have $t = 0$ which is a contradiction. Hence $\deg(\widehat{y}_1(\lambda)) = \deg(\widehat{y}(\lambda)) = \deg(y(\lambda))$ proving that $\mathbb{F}_2(\mathcal{S})$ is degree preserving. Now the desired results in (a) and (b) follow from Theorem 5.5.2. □

The recovery of left minimal bases and left minimal indices of $\mathcal{S}(\lambda)$ from those of its linearizations in $\mathbb{L}_1(G)$ is given in the following result.

Theorem 5.5.6. *Let $\mathbb{T}(\lambda)$ be as in Theorem 5.5.3 with right ansatz vector v.*

(a) Let $\left[\begin{array}{c} x \\ \hline y \end{array}\right] \in \mathcal{N}_l(\mathbb{T}) \cap \mathbb{C}[\lambda]^{mn+r}$ *with* $x \in \mathbb{C}[\lambda]^{mn}$ *and* $y \in \mathbb{C}[\lambda]^r$. *Then we*

have $\left[\begin{array}{c} (v^T\otimes I_n)x \\ \hline y \end{array}\right] \in \mathcal{N}_l(\mathcal{S})$ *and* $\deg\left(\left[\begin{array}{c} (v^T\otimes I_n)x \\ \hline y \end{array}\right]\right) = \deg\left(\left[\begin{array}{c} x \\ \hline y \end{array}\right]\right)$. *Further, the mapping* $\mathcal{N}_l(\mathbb{T}) \longrightarrow \mathcal{N}_l(\mathcal{S})$, $\left[\begin{array}{c} x \\ \hline y \end{array}\right] \longmapsto \left[\begin{array}{c} (v^T\otimes I_n)x \\ \hline y \end{array}\right]$ *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_l(\mathbb{T})$ *to a minimal basis of* $\mathcal{N}_l(\mathcal{S})$, *where* $x \in \mathbb{C}[\lambda]^{mn}$ *and* $y \in \mathbb{C}[\lambda]^r$.

(b) The left minimal indices of $\mathbb{T}(\lambda)$ *and* $\mathcal{S}(\lambda)$ *are the same, i.e., if* $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $\mathbb{T}(\lambda)$ *then* $\eta_1 \leq \cdots \leq \eta_p$ *are also the left minimal indices of* $\mathcal{S}(\lambda)$.

Proof. Note that $\mathbb{T}(\lambda)^T \in \mathbb{L}_2(G^T)$ with left ansatz vector v. Since $\mathcal{N}_l(\mathcal{S}) = \mathcal{N}_r(\mathcal{S}^T)$ and $\mathcal{N}_l(\mathbb{T}) = \mathcal{N}_r(\mathbb{T}^T)$, the desired results follow from Theorem 5.5.5. $\square$

The next result describes the recovery of left and right eigenvectors of a regular system matrix $\mathcal{S}(\lambda)$ from those of its linearizations.

Corollary 5.5.7. *Let* $X := \begin{bmatrix} X_{mn} \\ X_r \end{bmatrix}$ *be an* $(mn+r)\times p$ *matrix such that* $\operatorname{rank}(X) = p$, *where* X_{mn} *has* mn *rows and* X_r *has* r *rows. Let* $(\mathbb{T}(\lambda), v) \in \mathbb{L}_1(G)$ *and* $(\mathbb{H}(\lambda), w) \in \mathbb{L}_2(G)$ *be Rosenbrock strong linearization of* $G(\lambda)$. *Let* $\mu \in \mathbb{C}$ *be an eigenvalue of* $\mathcal{S}(\lambda)$.

(a) If X *is a basis of the right (resp., left) eigenspace of* $\mathbb{T}(\lambda)$ *corresponding to* μ *then* $\begin{bmatrix} (e_m^T\otimes I_n)X_{mn} \\ X_r \end{bmatrix}$ *(resp.,* $\begin{bmatrix} (v^T\otimes I_n)X_{mn} \\ X_r \end{bmatrix}$*) is a basis of the right (resp., left) eigenspace of* $\mathcal{S}(\lambda)$ *corresponding to* μ.

(b) If X *is a basis of the right (resp., left) eigenspace of* $\mathbb{H}(\lambda)$ *corresponding to* μ *then* $\begin{bmatrix} (w^T\otimes I_n)X_{mn} \\ X_r \end{bmatrix}$ *(resp.,* $\begin{bmatrix} (e_m^T\otimes I_n)X_{mn} \\ X_r \end{bmatrix}$*) is a basis of the right (resp., left) eigenspace of* $\mathcal{S}(\lambda)$ *corresponding to* μ.

We have described the recovery of eigenvectors, minimal bases and minimal indices of $\mathcal{S}(\lambda)$ from those of the Rosenbrock strong linearizations from $\mathbb{L}_1(G)\cup\mathbb{L}_2(G)$. We now describe the recovery of eigenvectors, minimal bases and minimal indices of $G(\lambda)$ from those of $\mathcal{S}(\lambda)$. This would complete the investigation on the recovery of eigenvectors, minimal bases and minimal indices of $G(\lambda)$ from those of the Rosenbrock strong linearizations from $\mathbb{L}_1(G)\cup\mathbb{L}_2(G)$.

Recall that an $n\times p$ matrix polynomial $X(\lambda)$ is said to be a minimal basis if the columns of $X(\lambda)$ form a minimal basis of the subspace of $\mathbb{C}(\lambda)^n$ spanned (over the field $\mathbb{C}(\lambda)$) by the columns of $X(\lambda)$.

The next result describes the recovery of minimal bases and minimal indices of $G(\lambda)$ from those of $\mathcal{S}(\lambda)$. We mention that (b) and (c) in Theorem 5.5.8 are proved in [60, Theorem 2] for a generalized state-space realization of $G(\lambda)$. For a state-space realization of $G(\lambda)$ as in (5.5), these results are stated to be true but proofs are not provided. We prove a result in Theorem 5.5.8(a), which is of independent interest, from which the results in (b) and (c) follow.

Theorem 5.5.8. *Let $G(\lambda)$ and $\mathcal{S}(\lambda)$ be as in (5.5) and (5.6), respectively. Then we have the following.*

(a) Let $x(\lambda) \in \mathcal{N}_r(G) \cap \mathbb{C}[\lambda]^n$ and $y(\lambda) \in \mathcal{N}_l(G) \cap \mathbb{C}[\lambda]^n$. Then

$$\begin{aligned}\mathbf{u}(\lambda) &:= \left[\begin{array}{c} x(\lambda) \\ \hline (\lambda E - A)^{-1}Bx(\lambda)\end{array}\right] \in \mathcal{N}_r(\mathcal{S}) \cap \mathbb{C}[\lambda]^{n+r} \text{ and } \deg(x(\lambda)) = \deg(\mathbf{u}(\lambda)),\\ \mathbf{v}(\lambda) &:= \left[\begin{array}{c} y(\lambda) \\ \hline (\lambda E - A)^{-T}C^T y(\lambda)\end{array}\right] \in \mathcal{N}_l(\mathcal{S}) \cap \mathbb{C}[\lambda]^{n+r} \text{ and } \deg(y(\lambda)) = \deg(\mathbf{v}(\lambda)).\end{aligned}$$

(b) Let $Z(\lambda) := \begin{bmatrix} Z_n(\lambda) \\ Z_r(\lambda)\end{bmatrix}$ be a matrix polynomial, where $Z_n(\lambda)$ has n rows and $Z_r(\lambda)$ has r rows. If $Z(\lambda)$ is a right (resp., left) minimal basis of $\mathcal{S}(\lambda)$ then $Z_n(\lambda)$ is a right (resp., left) minimal basis of $G(\lambda)$. Conversely, if $Z_n(\lambda)$ is a right (resp., left) minimal basis of $G(\lambda)$ then $Z(\lambda)$ with $Z_r(\lambda) := (\lambda E - A)^{-1}BZ_n(\lambda)$ (resp., $Z_r(\lambda) := (\lambda E - A)^{-T}C^T Z_n(\lambda)$) is a right (resp., left) minimal basis of $\mathcal{S}(\lambda)$.

(c) The minimal (left as well as right) indices of $G(\lambda)$ are the same as the minimal indices of $\mathcal{S}(\lambda)$, that is, $G(\lambda)$ and $\mathcal{S}(\lambda)$ have the same minimal indices.

Proof. (a) We prove the result for $\mathbf{u}(\lambda)$. The proof is similar for $\mathbf{v}(\lambda)$. It is easy to see that the linear map $\mathcal{N}_r(G) \longrightarrow \mathcal{N}_r(\mathcal{S}),\ x(\lambda) \longmapsto \left[\begin{array}{c} x(\lambda) \\ \hline (\lambda E - A)^{-1}Bx(\lambda)\end{array}\right]$ is an isomorphism. We now show that $\mathbf{u}(\lambda) \in \mathbb{C}[\lambda]^{n+r}$ and $\deg(x(\lambda)) = \deg(\mathbf{u}(\lambda))$.

Since $\mathcal{S}(\lambda)$ is irreducible, the pencil $\left[\begin{array}{c} C \\ \hline A - \lambda E\end{array}\right]$ has full column rank for all $\lambda \in \mathbb{C}$ and hence has a polynomial left inverse $\begin{bmatrix} X(\lambda) & Y(\lambda)\end{bmatrix}$, where $X(\lambda) \in \mathbb{C}[\lambda]^{r \times n}$ and $Y(\lambda) \in \mathbb{C}[\lambda]^{r \times r}$, see [40].

Thus,

$$\begin{aligned}\mathcal{S}(\lambda)\mathbf{u}(\lambda)=0 \;\;&\Rightarrow\;\; \begin{bmatrix} X(\lambda) & Y(\lambda)\end{bmatrix}\left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A-\lambda E\end{array}\right]\left[\begin{array}{c} x(\lambda) \\ \hline (\lambda E-A)^{-1}Bx(\lambda)\end{array}\right]=0\\ &\Rightarrow\;\; \left[\begin{array}{c|c} X(\lambda)P(\lambda)+Y(\lambda)B & I_r\end{array}\right]\left[\begin{array}{c} x(\lambda) \\ \hline (\lambda E-A)^{-1}Bx(\lambda)\end{array}\right]=0\\ &\Rightarrow\;\; \Big(X(\lambda)P(\lambda)+Y(\lambda)B\Big)x(\lambda)+(\lambda E-A)^{-1}Bx(\lambda)=0.\end{aligned}$$

Since $\Big(X(\lambda)P(\lambda)+Y(\lambda)B\Big)x(\lambda)\in\mathbb{C}[\lambda]^n$, we have $(\lambda E-A)^{-1}Bx(\lambda)\in\mathbb{C}[\lambda]^r$ which shows that $\mathbf{u}(\lambda)\in\mathcal{N}_r(\mathcal{S})\cap\mathbb{C}[\lambda]^{n+r}$. Hence by Theorem 5.5.1, $\deg(\mathbf{u}(\lambda))=\deg(x(\lambda))$. This proves (a). Now (b) and (c) follow from (a). $\square$

Now the recovery of eigenvectors, minimal bases and minimal indices of $G(\lambda)$ from those of the Rosenbrock strong linearizations from $\mathbb{L}_1(G)\cup\mathbb{L}_2(G)$ follow from Theorem 5.5.3, Theorem 5.5.4, Theorem 5.5.5, Theorem 5.5.6 and Theorem 5.5.8.

Corollary 5.5.9. *Let* $X(\lambda):=\begin{bmatrix} X_{mn}(\lambda)\\ X_r(\lambda)\end{bmatrix}$ *be an* $(mn+r)\times p$ *matrix polynomial, where* $X_{mn}(\lambda)$ *has* mn *rows and* $X_r(\lambda)$ *has* r *rows. Let* $(\mathbb{T}(\lambda),v)\in\mathbb{L}_1(G)$ *and* $(\mathbb{H}(\lambda),w)\in\mathbb{L}_2(G)$ *be Rosenbrock strong linearizations of* $G(\lambda)$.

(a) If $X(\lambda)$ *is a right (resp., left) minimal basis of* $\mathbb{T}(\lambda)$ *then* $(e_m^T\otimes I_n)X_{mn}(\lambda)$ *(resp.,* $(v^T\otimes I_n)X_{mn}(\lambda)$*) is a right (resp., left) minimal basis of* $G(\lambda)$.

(b) If $X(\lambda)$ *is a right (resp., left) minimal basis of* $\mathbb{H}(\lambda)$ *then* $(w^T\otimes I_n)X_{mn}(\lambda)$ *(resp.,* $(e_m^T\otimes I_n)X_{mn}(\lambda)$*) is a right (resp., left) minimal basis of* $G(\lambda)$.

(c) If $\varepsilon_1\leq\cdots\leq\varepsilon_p$ *are the right (resp., left) minimal indices of* $\mathbb{T}(\lambda)$ *(resp.,* $\mathbb{H}(\lambda)$*) then* $\varepsilon_1-(m-1)\leq\cdots\leq\varepsilon_p-(m-1)$ *are the right (resp., left) minimal indices of* $G(\lambda)$*. On the other hand, the left (resp., right) minimal indices of* $\mathbb{T}(\lambda)$ *(resp.,* $\mathbb{H}(\lambda)$*) and* $G(\lambda)$ *are the same.*

Corollary 5.5.10. *Let* $X:=\begin{bmatrix} X_{mn}\\ X_r\end{bmatrix}$ *be an* $(mn+r)\times p$ *matrix such that* $\operatorname{rank}(X)=p$, *where* X_{mn} *has* mn *rows and* X_r *has* r *rows. Let* $(\mathbb{T}(\lambda),v)\in\mathbb{L}_1(G)$ *and* $(\mathbb{H}(\lambda),w)\in\mathbb{L}_2(G)$ *be Rosenbrock strong linearizations of* $G(\lambda)$*. Let* $\mu\in\mathbb{C}$ *be an eigenvalue of* $G(\lambda)$.

(a) If X *is a basis of the right (resp., left) eigenspace of* $\mathbb{T}(\lambda)$ *corresponding to* μ *then* $(e_m^T\otimes I_n)X_{mn}$ *(resp.,* $(v^T\otimes I_n)X_{mn}$*) is a basis of the right (resp., left) eigenspace of* $G(\lambda)$ *corresponding to* μ.

(b) If X is a basis of the right (resp., left) eigenspace of $\mathbb{H}(\lambda)$ corresponding to μ then $(w^T \otimes I_n)X_{mn}$ (resp., $(e_m^T \otimes I_n)X_{mn}$) is a basis of the right (resp., left) eigenspace of $G(\lambda)$ corresponding to μ.

5.6 Linearizations via non-monomial polynomial bases

In this section we construct linearizations of $\mathcal{S}(\lambda)$ and $G(\lambda)$ corresponding to a nonmonomial polynomial basis and describe the recovery of minimal bases and minimal indices of $\mathcal{S}(\lambda)$ and $G(\lambda)$ from those of their linearizations. Although it is customary to consider polynomials in standard monomial basis $\{1, \lambda, \ldots, \lambda^j, \ldots\}$, there are applications in which it is useful to consider other polynomial bases. Let $\{\phi_0(\lambda), \phi_1(\lambda), \ldots, \phi_{m-1}(\lambda)\}$ be a basis for the space of scalar polynomials of degree at most $m-1$. Define $\Lambda := [\lambda^{m-1}, \lambda^{m-2}, \ldots, \lambda, 1]^T$ and $\Lambda_\phi := [\phi_0(\lambda), \phi_1(\lambda), \ldots, \phi_{m-1}(\lambda)]^T$. Let Φ be the unique nonsingular constant matrix such that $\Phi\Lambda = \Lambda_\phi$.

It is shown in [25] that the vector spaces $\mathcal{L}_1(P)$ and $\mathcal{L}_2(P)$ can be generalized by replacing the monomial basis Λ by the basis Λ_ϕ as follows. Define

$$\widetilde{\mathcal{L}}_1(P) = \left\{\widetilde{L}(\lambda) : \ \widetilde{L}(\lambda)(\Lambda_\phi \otimes I_n) = v \otimes P(\lambda), \ v \in \mathbb{C}^m\right\},$$
$$\widetilde{\mathcal{L}}_2(P) = \left\{\widetilde{L}(\lambda) : \ (\Lambda_\phi^T \otimes I_n)\widetilde{L}(\lambda) = w^T \otimes P(\lambda), \ w \in \mathbb{C}^m\right\},$$

where $\widetilde{L}(\lambda)$ is an $mn \times mn$ matrix pencil. Here v is the right ansatz vector for $\widetilde{L}(\lambda) \in \widetilde{\mathcal{L}}_1(P)$ and w is the left ansatz vector for $\widetilde{L}(\lambda) \in \widetilde{\mathcal{L}}_2(P)$. Note that if $\widetilde{L}(\lambda) \in \widetilde{\mathcal{L}}_1(P)$ then $L(\lambda) := \widetilde{L}(\lambda)(\Phi \otimes I_n) \in \mathcal{L}_1(P)$. In fact, it is easily seen that the map

$$\widetilde{\mathcal{L}}_1(P) \longrightarrow \mathcal{L}_1(P), \, \widetilde{L}(\lambda) \longmapsto \widetilde{L}(\lambda)(\Phi \otimes I_n)$$

is a linear isomorphism [25].

This shows that the pencils in $\mathcal{L}_1(P)$ and $\widetilde{\mathcal{L}}_1(P)$ are strictly equivalent. Hence a pencil $L(\lambda) \in \mathcal{L}_1(P)$ is a linearization of $P(\lambda)$ if and only if the corresponding pencil $\widetilde{L}(\lambda) \in \widetilde{\mathcal{L}}_1(P)$ is a linearization of $P(\lambda)$. We define $\widetilde{\mathbb{L}}_1(G)$ and $\widetilde{\mathbb{L}}_2(G)$ as follows:

$$\widetilde{\mathbb{L}}_1(G) \;:=\; \left\{ \left[\begin{array}{c|c} \widetilde{L}(\lambda) & v \otimes C \\ \hline e_m^T\Phi^{-1} \otimes B & A - \lambda E \end{array}\right] : (\widetilde{L}(\lambda), v) \in \widetilde{\mathcal{L}}_1(P)\right\},$$

$$\widetilde{\mathbb{L}}_2(G) \;:=\; \left\{ \left[\begin{array}{c|c} \widetilde{L}(\lambda) & \Phi^{-T}e_m \otimes C \\ \hline w^T \otimes B & A - \lambda E \end{array}\right] : (\widetilde{L}(\lambda), w) \in \widetilde{\mathcal{L}}_2(P)\right\}.$$

Theorem 5.6.1. *Let* $\widetilde{\mathbb{T}}(\lambda) := \left[\begin{array}{c|c} \widetilde{L}(\lambda) & v\otimes C \\ \hline e_m^T\Phi^{-1}\otimes B & A-\lambda E \end{array}\right] \in \widetilde{\mathbb{L}}_1(G)$*, where* $(\widetilde{L}(\lambda), v) \in \widetilde{\mathcal{L}}_1(P)$ *with* $v \neq 0$*. Then* $\mathbb{T}(\lambda) := \widetilde{\mathbb{T}}(\lambda)\left[\begin{array}{c|c} \Phi\otimes I_n & \\ \hline & I_r \end{array}\right] \in \mathbb{L}_1(G)$*. In fact, the map* $\mathbb{J} : \widetilde{\mathbb{L}}_1(G) \longrightarrow \mathbb{L}_1(G),\ \widetilde{\mathbb{T}}(\lambda) \longmapsto \widetilde{\mathbb{T}}(\lambda)\left[\begin{array}{c|c} \Phi\otimes I_n & \\ \hline & I_r \end{array}\right]$ *is a linear isomorphism.*

Proof. Note that $L(\lambda) := \widetilde{L}(\lambda)(\Phi\otimes I_n) \in \mathcal{L}_1(P)$ with right ansatz vector v. Hence $\widetilde{\mathbb{T}}(\lambda)\left[\begin{array}{c|c} \Phi\otimes I_n & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} \widetilde{L}(\lambda)(\Phi\otimes I_n) & v\otimes C \\ \hline e_m^T\otimes B & A-\lambda E \end{array}\right] \in \mathbb{L}_1(G)$. Obviously $\mathbb{J}$ is linear and injective. Since $\mathbb{T}(\lambda)\left[\begin{array}{c|c} \Phi^{-1}\otimes I_n & \\ \hline & I_r \end{array}\right] \in \widetilde{\mathbb{L}}_1(G)$ and $\mathbb{J}$ maps it to $\mathbb{T}(\lambda)$ for all $\mathbb{T}(\lambda) \in \mathbb{L}_1(G)$, we conclude that $\mathbb{J}$ is onto. This completes the proof. □

Analogously, the map $\mathbb{J} : \widetilde{\mathbb{L}}_2(G) \to \mathbb{L}_2(G),\ \widetilde{\mathbb{T}}(\lambda) \mapsto \left[\begin{array}{c|c} \Phi^T\otimes I_n & \\ \hline & I_r \end{array}\right]\widetilde{\mathbb{T}}(\lambda)$ is a linear isomorphism.

Observe that $\widetilde{\mathbb{T}}(\lambda) \in \widetilde{\mathbb{L}}_1(G)$ is a Rosenbrock strong linearization of $G(\lambda)$ if and only if $\mathbb{T}(\lambda) := \widetilde{\mathbb{T}}(\lambda)\left[\begin{array}{c|c} \Phi\otimes I_n & \\ \hline & I_r \end{array}\right] \in \mathbb{L}_1(G)$ is a Rosenbrock strong linearization of $G(\lambda)$. Similarly, $\widetilde{\mathbb{H}}(\lambda) \in \widetilde{\mathbb{L}}_2(G)$ is a Rosenbrock strong linearization of $G(\lambda)$ if and only if $\mathbb{H}(\lambda) := \left[\begin{array}{c|c} \Phi^T\otimes I_n & \\ \hline & I_r \end{array}\right]\widetilde{\mathbb{H}}(\lambda) \in \mathbb{L}_2(G)$ is a Rosenbrock strong linearization of $G(\lambda)$. Hence minimal bases and minimal indices of $\mathcal{S}(\lambda)$ and $G(\lambda)$ can be recovered from pencils in $\widetilde{\mathbb{L}}_1(G) \cup \widetilde{\mathbb{L}}_2(G)$ as follows.

Theorem 5.6.2. *Let* $\widetilde{\mathbb{T}}(\lambda) := \left[\begin{array}{c|c} \widetilde{T}(\lambda) & v\otimes C \\ \hline e_m^T\Phi^{-1}\otimes B & A-\lambda E \end{array}\right] \in \widetilde{\mathbb{L}}_1(G)$*, where* $(\widetilde{T}(\lambda), v) \in \widetilde{\mathcal{L}}_1(P)$ *with* $v \neq 0$ *and* $\widetilde{T}(\lambda)$ *having full* Z*-rank. Also let* $\widetilde{\mathbb{H}}(\lambda) := \left[\begin{array}{c|c} \widetilde{H}(\lambda) & \Phi^{-T}e_m\otimes C \\ \hline w^T\otimes B & A-\lambda E \end{array}\right] \in \widetilde{\mathbb{L}}_2(G)$*, where* $(\widetilde{H}(\lambda), w) \in \widetilde{\mathcal{L}}_2(P)$ *with* $w \neq 0$ *and* $\widetilde{H}(\lambda)$ *having full* Z*-rank.*

Let $X(\lambda) := \begin{bmatrix} X_{mn}(\lambda) \\ X_r(\lambda) \end{bmatrix}$ *be an* $(mn+r)\times p$ *matrix polynomial, where* $X_{mn}(\lambda)$ *has* mn *rows and* $X_r(\lambda)$ *has* r *rows.*

(a) If $X(\lambda)$ is a right (resp., left) minimal basis of $\widetilde{\mathbb{T}}(\lambda)$ then $\begin{bmatrix} (e_m^T\Phi^{-1}\otimes I_n)X_{mn}(\lambda) \\ X_r(\lambda) \end{bmatrix}$ *(resp.,* $\begin{bmatrix} (v^T\otimes I_n)X_{mn}(\lambda) \\ X_r(\lambda) \end{bmatrix}$*) is a right (resp., left) minimal basis of $\mathcal{S}(\lambda)$ and $(e_m^T\Phi^{-1}\otimes I_n)X_{mn}(\lambda)$ (resp., $(v^T\otimes I_n)X_{mn}(\lambda)$) is a right (resp., left) minimal basis of $G(\lambda)$.*

(b) If $X(\lambda)$ is a right (resp., left) minimal basis of $\widetilde{\mathbb{H}}(\lambda)$ then $\begin{bmatrix} (w^T\otimes I_n)X_{mn}(\lambda) \\ X_r(\lambda) \end{bmatrix}$ *(resp.,* $\begin{bmatrix} (e_m^T\Phi^{-1}\otimes I_n)X_{mn}(\lambda) \\ X_r(\lambda) \end{bmatrix}$*) is a right (resp., left) minimal basis of $\mathcal{S}(\lambda)$ and $(w^T\otimes I_n)X_{mn}(\lambda)$ (resp., $(e_m^T\Phi^{-1}\otimes I_n)X_{mn}(\lambda)$) is a right (resp., left) minimal basis of $G(\lambda)$.*

(c) If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are the right (resp., left) minimal indices of $\widetilde{\mathbb{T}}(\lambda)$ (resp., $\widetilde{\mathbb{H}}(\lambda)$) then $\varepsilon_1-(m-1) \leq \cdots \leq \varepsilon_p-(m-1)$ are the right (resp., left) minimal indices of $\mathcal{S}(\lambda)$ and $G(\lambda)$. On the other hand, the left (resp., right) minimal indices of $\widetilde{\mathbb{T}}(\lambda)$ (resp., $\widetilde{\mathbb{H}}(\lambda)$), $\mathcal{S}(\lambda)$ and $G(\lambda)$ are the same.

Proof. It is easy to see that $\left[\begin{array}{c|c} \Phi^{-1}\otimes I_n & \\ \hline & I_r \end{array}\right] : \mathcal{N}_r(\widetilde{\mathbb{T}}) \longrightarrow \mathcal{N}_r(\mathbb{T})$ and $\left[\begin{array}{c|c} \Phi^{-T}\otimes I_n & \\ \hline & I_r \end{array}\right] : \mathcal{N}_l(\widetilde{\mathbb{H}}) \longrightarrow \mathcal{N}_l(\mathbb{H})$ are degree preserving linear isomorphisms of vector polynomials and that $\widetilde{\mathbb{T}}(\lambda)$ and $\widetilde{\mathbb{H}}(\lambda)$ are Rosenbrock strong linearizations of $\mathcal{S}(\lambda)$ and $G(\lambda)$, where $\mathbb{T}(\lambda) := \widetilde{\mathbb{T}}(\lambda)\left[\begin{array}{c|c} \Phi\otimes I_n & \\ \hline & I_r \end{array}\right] \in \mathbb{L}_1(G)$ and $\mathbb{H}(\lambda) := \left[\begin{array}{c|c} \Phi^T\otimes I_n & \\ \hline & I_r \end{array}\right]\widetilde{\mathbb{H}}(\lambda) \in \mathbb{L}_2(G)$. Hence the desired results follow from Theorem 5.5.3, Theorem 5.5.4, Theorem 5.5.5, Theorem 5.5.6 and Theorem 5.5.8. □

6

Recovery of Minimal Bases and Minimal Indices of Rational Matrices from Fiedler-like Pencils

With a view to constructing linearizations of $G(\lambda)$, Fiedler-like pencils (such as Fiedler pencils, generalized Fiedler pencils and Fiedler pencils with repetition) of a rational matrix $G(\lambda)$ have been proposed recently which are shown to be linearizations of $G(\lambda)$ [2, 3, 4, 8], that is, Fiedler-like pencils satisfy condition (a) of Definition 5.2.2). We show that the Fiedler-like pencils allow operation-free recovery of eigenvectors and minimal bases of $G(\lambda)$, that is, eigenvectors and minimal bases of $G(\lambda)$ can be recovered from those of the Fiedler-like pencils of $G(\lambda)$ without performing any arithmetic operations. Further, we show that the minimal indices of $G(\lambda)$ can be easily recovered from those of the Fiedler-like pencils of $G(\lambda)$.

6.1 FPs of rational matrices

For the rest of this chapter, we assume that $P(\lambda) := \sum_{i=0}^{m} \lambda^i A_i$ with $A_m \neq 0$ and the realization $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1} B$ of $G(\lambda)$ given by (5.5) is minimal. The system matrix $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right]$ associated with $G(\lambda)$ is given by (5.6).

Recall from Section 2.1 that the Fiedler matrices M_i^P, $i \in \{\pm 0, \pm 1, \ldots, \pm m\}$, associated with $P(\lambda)$ are given by [6, 15, 61]:

$$M_0^P := \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix}, \quad M_i^P := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ for } i = 1 : m-1,$$

$$M^P_{-m} := \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix}, \; M^P_{-i} := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & A_i & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ for } i = 1 : m-1,$$

$M^P_{-0} := (M^P_0)^{-1}$ and $M^P_m := (M^P_{-m})^{-1}$.

The $(mn+r) \times (mn+r)$ Fiedler matrices $\mathbb{M}^{\mathcal{S}}_{\pm 0}, \ldots, \mathbb{M}^{\mathcal{S}}_{\pm m}$ associated with $\mathcal{S}(\lambda)$ are defined as [2, 4]:

$$\mathbb{M}^{\mathcal{S}}_0 := \left[\begin{array}{c|c} M^P_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right], \; \mathbb{M}^{\mathcal{S}}_{-m} := \left[\begin{array}{c|c} M^P_{-m} & 0 \\ \hline 0 & -E \end{array}\right], \; \mathbb{M}^{\mathcal{S}}_i := \left[\begin{array}{c|c} M^P_i & 0 \\ \hline 0 & I_r \end{array}\right], i = 1 : m-1,$$

$\mathbb{M}^{\mathcal{S}}_m := (\mathbb{M}^{\mathcal{S}}_{-m})^{-1}$ and $\mathbb{M}^{\mathcal{S}}_{-i} := (\mathbb{M}^{\mathcal{S}}_i)^{-1}$, $i = 0 : m-1$, where $M^P_{\pm 0}, \ldots, M^P_{\pm m}$ are the Fiedler matrices associated with $P(\lambda)$. The matrices $\mathbb{M}^{\mathcal{S}}_i$, $i \in \{\pm 0, \pm 1, \ldots, \pm m\}$, are also referred to as the Fiedler matrices of $G(\lambda)$.

Remark 6.1.1. *[4] We have* $\mathbb{M}^{\mathcal{S}}_i \mathbb{M}^{\mathcal{S}}_j = \mathbb{M}^{\mathcal{S}}_j \mathbb{M}^{\mathcal{S}}_i$ *for* $||i| - |j|| > 1$, *except for* $||i| - |j|| = m$.

Definition 6.1.2. *For any index tuple* $\alpha := (j_1, j_2, \ldots, j_k)$ *containing indices from* $\{\pm 0, \pm 1, \ldots, \pm m\}$, *we define*

$$\mathbb{M}^{\mathcal{S}}_\alpha := \mathbb{M}^{\mathcal{S}}_{j_1} \mathbb{M}^{\mathcal{S}}_{j_2} \cdots \mathbb{M}^{\mathcal{S}}_{j_k} \text{ if } \alpha \neq \emptyset, \text{ and } \mathbb{M}^{\mathcal{S}}_\alpha := I_{mn+r} \text{ if } \alpha = \emptyset.$$

A Fiedler pencil of a rational matrix is defined as follows.

Definition 6.1.3 (Fiedler pencil, [2])**.** *Let* $\mathbb{M}^{\mathcal{S}}_0, \mathbb{M}^{\mathcal{S}}_1, \ldots, \mathbb{M}^{\mathcal{S}}_{m-1}, \mathbb{M}^{\mathcal{S}}_{-m}$ *be the Fiedler matrices associated with* $G(\lambda)$. *Let* σ *be a permutation of* $\{0, 1, \ldots, m-1\}$. *Then* $\mathbb{L}_\sigma(\lambda) := \lambda \mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_\sigma$ *is called a Fiedler pencil (FP) of* $G(\lambda)$ *associated with* σ. *The pencil* $\mathbb{L}_\sigma(\lambda)$ *is also referred to as a Fiedler pencil of* $\mathcal{S}(\lambda)$.

We will prove in Chapter 7 that Fiedler pencils of $G(\lambda)$ are Rosenbrock strong linearizations of $G(\lambda)$. We mention that the operation free recovery rule of eigenvectors of $\mathcal{S}(\lambda)$ as well as $G(\lambda)$ from eigenvectors of the FPs of $G(\lambda)$ are derived in [3]. We now describe the recovery of minimal bases and minimal indices of $G(\lambda)$ from those of the Fiedler pencils of $G(\lambda)$.

Remark 6.1.4. *In view of Theorem 5.5.8, the investigation on the recovery of minimal bases and minimal indices of* $G(\lambda)$ *from those of the Fiedler pencils of* $G(\lambda)$ *will complete once we describe the recovery of minimal bases and minimal indices of* $\mathcal{S}(\lambda)$ *from those of the Fiedler pencils.*

The following definitions and results will be used in the subsequent development.

Definition 6.1.5 (Horner shift, [26])**.** *Let* $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ *be a matrix polynomial of degree* m*. For* $k = 0, 1, \ldots, m$*, the degree* k *Horner shift of* $P(\lambda)$ *is defined as the matrix polynomial given by* $P_k(\lambda) := A_{m-k} + \lambda A_{m-k+1} + \cdots + \lambda^k A_m$*.*

Note that the Horner shifts satisfy the following: $P_0(\lambda) = A_m$, $P_{k+1}(\lambda) = \lambda P_k(\lambda) + A_{m-k-1}$, for $0 \leq k \leq m-1$, and $P_m(\lambda) = P(\lambda)$.

Definition 6.1.6. *[26] Let* α *be a permutation of* $\{0, 1, \ldots, m-1\}$*. Then the consecution-inversion structure sequence of* α*, denoted by CISS(*α*), is the tuple* $(c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$*, where* σ *has* c_1 *consecutive consecutions at* $0, 1, \ldots, c_1 - 1$*;* i_1 *consecutive inversions at* $c_1, c_1 + 1, \ldots, c_1 + i_1 - 1$ *and so on, up to* i_ℓ *inversions at* $m - 1 - i_\ell, \ldots, m - 2$*.*

Recalling the cosecutions and inversions of an index tuple α at any index $t \in \alpha$, we have the following definition of CIP(α) when α is a permutation of $\{0 : m-1\}$.

Definition 6.1.7. *Let* α *be a permutation of* $\{0 : m-1\}$*. Suppose that* α *has* c_0 *consecutions at* 0 *and* i_0 *inversions at* 0*. Then we define CIP(*α*)* $:= (c_0, i_0)$*.*

Definition 6.1.8. *[26] Let* σ *be a permutation of* $\{0, 1, \ldots, m-1\}$ *having* $CISS(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$*.*

(a) Define $s_0 := 0$*,* $m_0 := 0$*,* $s_j := \sum_{p=1}^{j}(c_p + i_p)$*, and* $m_j := \sum_{p=1}^{j} i_p$ *for* $j = 1, 2, \ldots, \ell$*. Observe that,* $s_\ell = m - 1$ *and* $m_\ell = i(\sigma)$*.*

(b) Define $\Lambda_{\sigma,j}(P)$ *for* $j = 1, \ldots, \ell$*, and* $\widehat{\Lambda}_{\sigma,j}(P)$ *for* $j = 1, \ldots, \ell - 1$*, as follows:*

$$\Lambda_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ P_{m-s_{j-1}-c_j} \\ \vdots \\ P_{m-s_{j-1}-2} \\ P_{m-s_{j-1}-1} \end{bmatrix} \text{ and } \widehat{\Lambda}_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j-1} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ P_{m-s_{j-1}-c_j} \\ \vdots \\ P_{m-s_{j-1}-2} \\ P_{m-s_{j-1}-1} \end{bmatrix} \quad \text{if} \, c_1 \geq 1. \tag{6.1}$$

If $c_1 = 0$*, then define*

$$\Lambda_{\sigma,1}(P) := [\lambda^{i_1} I_n, \ldots, \lambda I_n, I_n]^{\mathcal{B}}, \ \widehat{\Lambda}_{\sigma,1}(P) := [\lambda^{i_1-1} I_n, \ldots, \lambda I_n, I_n]^{\mathcal{B}}$$

and $\Lambda_{\sigma,j}(P)$*,* $\widehat{\Lambda}_{\sigma,j}(P)$ *as in (6.1) for* $j > 1$*.*

We need the following result which will play a crucial role in the recovery of minimal bases and minimal indices of $G(\lambda)$ from those of its Fiedler pencils.

Theorem 6.1.9. *Let* $\mathbb{L}_\sigma(\lambda) := \lambda \mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_{\sigma}$ *be the Fiedler pencil of* $\mathcal{S}(\lambda)$ *associated with a permutation* σ *of* $\{0, 1, \ldots, m-1\}$. *Then there exist* $mn \times mn$ *unimodular matrix polynomials* $U_\sigma(\lambda)$ *and* $V_\sigma(\lambda)$ *such that for all* $\lambda \in \mathbb{C}$, *we have*

$$\left[\begin{array}{c|c} U_\sigma(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{L}_\sigma(\lambda) \left[\begin{array}{c|c} V_\sigma(\lambda) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]. \tag{6.2}$$

(a) Consider the map $E_\sigma(P) : \mathbb{C}(\lambda)^n \longrightarrow \mathbb{C}(\lambda)^{mn}, x(\lambda) \longmapsto V_\sigma(\lambda)(e_m \otimes I_n)x(\lambda)$. *Then* $E_{rev(\sigma)}(P^T)x(\lambda) = U_\sigma(\lambda)^T(e_m \otimes I_n)x(\lambda)$. *Suppose that* $CISS(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$. *Then we have*

$$E_\sigma(P) = \begin{bmatrix} \lambda^{m_{\ell-1}} \Lambda_{\sigma,\ell}(P) \\ \lambda^{m_{\ell-2}} \widehat{\Lambda}_{\sigma,\ell-1}(P) \\ \vdots \\ \lambda^{m_1} \widehat{\Lambda}_{\sigma,2}(P) \\ \widehat{\Lambda}_{\sigma,1}(P) \end{bmatrix} \quad \text{if } \ell > 1, \tag{6.3}$$

and $E_\sigma(P) = \Lambda_{\sigma,1}(P)$ *if* $\ell = 1$, *where* $\Lambda_{\sigma,j}(P)$ *and* $\widehat{\Lambda}_{\sigma,j}(P)$ *are as in Definition 6.1.8(b).*

(b) Define $\mathbb{E}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} E_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right]$ *and* $\mathbb{H}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} E_{rev(\sigma)}(P^T) & 0 \\ \hline 0 & I_r \end{array}\right]$. *Then the linear mappings* $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}) \longrightarrow \mathcal{N}_r(\mathbb{L}_\sigma)$, $\left[\begin{array}{c} x(\lambda) \\ \hline y(\lambda) \end{array}\right] \longmapsto \left[\begin{array}{c} E_\sigma(P)x(\lambda) \\ \hline y(\lambda) \end{array}\right]$, *and* $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}) \longrightarrow \mathcal{N}_l(\mathbb{L}_\sigma)$, $\left[\begin{array}{c} x(\lambda) \\ \hline y(\lambda) \end{array}\right] \longmapsto \left[\begin{array}{c} E_{rev(\sigma)}(P^T)x(\lambda) \\ \hline y(\lambda) \end{array}\right]$, *are isomorphisms, where* $x(\lambda) \in \mathbb{C}(\lambda)^{mn}$ *and* $y(\lambda) \in \mathbb{C}(\lambda)^r$. *Also* $\mathbb{H}_\sigma(\mathcal{S}) = \mathbb{E}_{rev(\sigma)}(\mathcal{S}^T)$.

(c) Suppose that CIP$(\sigma) = (c_0, i_0)$. *Define*

$$\mathbb{F}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} e^T_{m-c_0} \otimes I_n & 0 \\ \hline 0 & I_r \end{array}\right] \text{ and } \mathbb{K}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} e^T_{m-i_0} \otimes I_n & 0 \\ \hline 0 & I_r \end{array}\right].$$

Then the linear maps $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma) \to \mathcal{N}_r(\mathcal{S})$, $\left[\begin{array}{c} x(\lambda) \\ \hline y(\lambda) \end{array}\right] \longmapsto \left[\begin{array}{c} (e^T_{m-c_0} \otimes I_n)x(\lambda) \\ \hline y(\lambda) \end{array}\right]$, *and* $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma) \to \mathcal{N}_l(\mathcal{S})$, $\left[\begin{array}{c} x(\lambda) \\ \hline y(\lambda) \end{array}\right] \longmapsto \left[\begin{array}{c} (e^T_{m-i_0} \otimes I_n)x(\lambda) \\ \hline y(\lambda) \end{array}\right]$, *are isomorphisms,* *where* $x(\lambda) \in \mathbb{C}(\lambda)^{mn}$ *and* $y(\lambda) \in \mathbb{C}(\lambda)^{r}$. *Also* $\mathbb{F}_\sigma(\mathcal{S})\mathbb{E}_\sigma(\mathcal{S}) = I_{n+r} = \mathbb{K}_\sigma(\mathcal{S})\mathbb{H}_\sigma(\mathcal{S})$ *and* $\mathbb{K}_\sigma(\mathcal{S}) = \mathbb{F}_{rev(\sigma)}(\mathcal{S}^T)$.

Proof. The equality in (6.2) is proved in ([2], Theorem 4.13) and the results in (a) are proved in [26]. The results in (b) and (c) can be found in ([3], Theorem 3.3 and Theorem 3.15); see also ([8], Theorem 3.2.7 and Theorem 3.2.8). We mention that the recovery in part (c) uses $\mathrm{CIP}(\sigma) := (c_0, i_0)$ whereas Theorem 3.15 in [3] uses (c_1, i_1) from $\mathrm{CISS}(\sigma)$. However, the recovery rules remain the same. Note that c_1 in $\mathrm{CISS}(\sigma)$ denotes the total number of consecutive consecutions at 0 and i_1 denotes the total number consecutive inversions at c_1. Hence $c_1 = c_0$. ☐

The following lemma will be useful in the subsequent developments.

Lemma 6.1.10. *Let* $P_j(\lambda)$, *for* $j = 0, 1, \ldots, m-1$, *be the Horner shift of* $P(\lambda)$ *as in Definition 6.1.5. Let* $u(\lambda) := \begin{bmatrix} u_1(\lambda) \\ \hline u_2(\lambda) \end{bmatrix} \in \mathcal{N}_r(\mathcal{S})$ *be a nonzero vector polynomial, where* $u_1(\lambda) \in \mathbb{C}[\lambda]^n$ *and* $u_2(\lambda) \in \mathbb{C}[\lambda]^r$. *If* $P_j(\lambda)u_1(\lambda) \neq 0$ *then* $\deg\big(P_j(\lambda)u_1(\lambda)\big) \leq \deg\big(u(\lambda)\big) - 1$ *for* $j = 0, 1, \ldots, m-1$.

Proof. We have $\mathcal{S}(\lambda)u(\lambda) = 0$ which gives $P(\lambda)u_1(\lambda) + Cu_2(\lambda) = 0$ and $Bu_1(\lambda) + (A - \lambda E)u_2(\lambda) = 0$. Now for $0 \leq j \leq m-1$, we have

$$\begin{aligned}
\lambda^{m-j}P_j(\lambda)u_1(\lambda) &= \lambda^{m-j}P_j(\lambda)u_1(\lambda) - \Big(P(\lambda)u_1(\lambda) + Cu_2(\lambda)\Big) \\
&= \Big(\lambda^{m-j}P_j(\lambda) - P(\lambda)\Big)u_1(\lambda) - Cu_2(\lambda) \\
&= -\big(\lambda^{m-j-1}A_{m-j-1} + \cdots + \lambda A_1 + A_0\big)u_1(\lambda) - Cu_2(\lambda).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\deg\big(\lambda^{m-j}P_j(\lambda)u_1(\lambda)\big) &\leq \max\ \big\{(m-j-1) + \deg(u_1(\lambda)),\ \deg(Cu_2(\lambda))\big\} \\
&\leq \max\ \big\{(m-j-1) + \deg(u_1(\lambda)),\ \deg(u_2(\lambda))\big\} \\
&\leq \max\ \big\{(m-j-1) + \deg(u(\lambda)),\ \deg(u(\lambda))\big\} \\
&= (m-j-1) + \deg(u(\lambda)).
\end{aligned}$$

This shows that $(m-j) + \deg\big(P_j(\lambda)u_1(\lambda)\big) \leq\ (m-j-1) + \deg(u(\lambda))$ and hence $\deg\big(P_j(\lambda)u_1(\lambda)\big) \leq\ \deg\big(u(\lambda)\big) - 1$ for $j = 0, 1, \ldots, m-1$. ☐

The next result establishes degree-shifting property of the maps $\mathbb{E}_\sigma(\mathcal{S})$ and $\mathbb{H}_\sigma(\mathcal{S})$ given in Theorem 6.1.9 which will play an important role in the recovery of minimal indices of $\mathcal{S}(\lambda)$.

Theorem 6.1.11. *Let σ be a permutation of $\{0, 1, \ldots, m-1\}$. Let $c(\sigma)$ and $i(\sigma)$, respectively, be the total number of consecutions and inversions of σ. Let $\mathbb{L}_\sigma(\lambda), \mathbb{E}_\sigma(\mathcal{S})$ and $\mathbb{H}_\sigma(\mathcal{S})$ be as in Theorem 6.1.9. Then we have the following.*

(a) For every nonzero vector polynomial $u(\lambda) \in \mathcal{N}_r(\mathcal{S})$, we have

$$\deg\big(\mathbb{E}_\sigma(\mathcal{S})u(\lambda)\big) = \ i(\sigma) + \deg(u(\lambda)) = \deg\Big(\big[e_1^T \otimes I_n \mid \ 0\big]\mathbb{E}_\sigma(\mathcal{S})u(\lambda)\Big). \qquad (6.4)$$

(b) For every nonzero vector polynomial $v(\lambda) \in \mathcal{N}_l(\mathcal{S})$, we have

$$\deg\big(\mathbb{H}_\sigma(\mathcal{S})u(\lambda)\big) = \ c(\sigma) + \deg u(\lambda). \qquad (6.5)$$

Proof. (a) Let $u(\lambda) := \begin{bmatrix} u_1(\lambda) \\ \hline u_2(\lambda) \end{bmatrix} \in \mathcal{N}_r(\mathcal{S})$ be nonzero, where $u_1(\lambda) \in \mathbb{C}[\lambda]^n$ and $u_2(\lambda) \in \mathbb{C}[\lambda]^r$. Then by Theorem 5.5.1, $\deg(u(\lambda)) = \deg(u_1(\lambda))$. Now since $u(\lambda)$ is nonzero, $\mathbb{E}_\sigma(S)u(\lambda) \neq 0$ and by Theorem 6.1.9, $\mathbb{E}_\sigma(\mathcal{S})u(\lambda) \in \mathcal{N}_r(\mathbb{L}_\sigma)$. Since $\mathbb{L}_\sigma(\lambda)$ is itself a system matrix, by Theorem 5.5.1, we have $\deg(\mathbb{E}_\sigma(\mathcal{S})u(\lambda)) = \deg(\big(E_\sigma(P)u_1(\lambda)\big))$. Hence to determine the degree of $\mathbb{E}_\sigma(\mathcal{S})u(\lambda)$, we have to find the degree of $E_\sigma(P)u_1(\lambda)$.

In view of (6.3), there are only two different types of blocks in $E_\sigma(P)$.

Type-I: $\lambda^p I$ with $0 \leq p \leq i(\sigma)$, which results in $\lambda^p u_1(\lambda)$ blocks in $E_\sigma(P)u_1(\lambda)$. The maximum degree among all blocks of this type in $E_\sigma(P)u_1(\lambda)$ is $i(\sigma) + \deg(u_1(\lambda)) = \deg\Big((e_1^T \otimes I_n)E_\sigma(P)u_1(\lambda)\Big)$.

Type-II: $\lambda^q P_j(\lambda)$ *with* $0 \leq q \leq m_{\ell-1} \leq i(\sigma)$. Blocks of the form $\lambda^q P_j(\lambda)u_1(\lambda)$ are either 0 or by Lemma 6.1.10 we have

$$\deg(\lambda^q P_j(\lambda)u_1(\lambda)) \leq i(\sigma) + \deg(u_1(\lambda)) - 1 < i(\sigma) + \deg(u_1(\lambda)).$$

Thus $\deg\big(\mathbb{E}_\sigma(\mathcal{S})u(\lambda)\big) = i(\sigma) + \deg(u(\lambda)) = \deg\Big(\big[e_1^T \otimes I_n \mid \ 0\big]\mathbb{E}_\sigma(\mathcal{S})u(\lambda)\Big)$.

(b) For simplicity of notation, we write $\mathbb{L}_\sigma(\mathcal{S})$ and $\mathbb{L}_{rev(\sigma)}(\mathcal{S}^T)$ to denote the Fiedler pencil of $\mathcal{S}(\lambda)$ and $\mathcal{S}^T(\lambda)$ associated with the permutation σ and $rev(\sigma)$, respectively. Let $v(\lambda) \in \mathcal{N}_l(\mathcal{S})$. We have $\mathcal{N}_l(\mathcal{S}) = \mathcal{N}_r(\mathcal{S}^T) \xrightarrow{\mathbb{E}_{rev(\sigma)}(\mathcal{S}^T)} \mathcal{N}_r\big(\mathbb{L}_{rev(\sigma)}(\mathcal{S}^T)\big) = \mathcal{N}_l\Big(\big(\mathbb{L}_{rev(\sigma)}(\mathcal{S}^T)\big)^T\Big)$ $= \mathcal{N}_l\big(\mathbb{L}_\sigma(\mathcal{S})\big)$, where $\mathbb{E}_{rev(\sigma)}(\mathcal{S}^T)$ is as in Theorem 6.1.9. Since $\mathbb{H}_\sigma(\mathcal{S}) = \mathbb{E}_{rev(\sigma)}(\mathcal{S}^T)$, by part (a), we have $\deg\big(\mathbb{H}_\sigma(\mathcal{S})v(\lambda)\big) = \ i(rev(\sigma)) + \deg(v(\lambda)) = c(\sigma) + \deg(v(\lambda))$. ☐

Remark 6.1.12. *By Theorem 6.1.11, we have* $\deg\big(\mathbb{E}_\sigma(\mathcal{S})x(\lambda)\big) = \deg(x(\lambda)) + i(\sigma)$ *for all nonzero vector polynomial $x(\lambda) \in \mathcal{N}_r(\mathcal{S})$. Now, let $y(\lambda) \in \mathcal{N}_r(\mathbb{L}_\sigma)$ be a vector* *polynomial. Since* $\mathbb{E}_\sigma(\mathcal{S})$ *is an isomorphism, we have* $y(\lambda) = \mathbb{E}_\sigma(\mathcal{S})z(\lambda)$ *for some vector polynomial* $z(\lambda) \in \mathcal{N}_r(\mathcal{S})$*. So* $\deg(y(\lambda)) = \deg\big(\mathbb{E}_\sigma(\mathcal{S})z(\lambda)\big) = \deg(z(\lambda)) + i(\sigma)$*. Now* $\mathbb{F}_\sigma(\mathcal{S})y(\lambda) = \mathbb{F}_\sigma(\mathcal{S})\mathbb{E}_\sigma(\mathcal{S})z(\lambda) = z(\lambda) \Rightarrow \deg\big(\mathbb{F}_\sigma(\mathcal{S})y(\lambda)\big) = \deg(z(\lambda))$*. Thus* $\deg\big(\mathbb{F}_\sigma(\mathcal{S})y(\lambda)\big) = \deg(y(\lambda)) - i(\sigma)$ *for all nonzero vector polynomial* $y(\lambda) \in \mathcal{N}_r(\mathbb{L}_\sigma)$*.*

We are now ready to describe the recovery of minimal bases and minimal indices of $\mathcal{S}(\lambda)$ from those of the Fiedler pencils of $\mathcal{S}(\lambda)$.

Theorem 6.1.13. *Let* $\mathbb{L}_\sigma(\lambda)$ *be the Fiedler pencil of* $\mathcal{S}(\lambda)$ *associated with a permutation* σ *of* $\{0, 1, \ldots, m-1\}$*. Suppose that CIP*$(\sigma) = (c_0, i_0)$*. Let* $c(\sigma)$ *and* $i(\sigma)$*, respectively, be the total number of consecutions and inversions of* σ*. Let* $\mathbb{F}_\sigma(\mathcal{S})$ *and* $\mathbb{K}_\sigma(\mathcal{S})$ *be as in Theorem 6.1.9.*

(a) **Right minimal bases.** $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma) \to \mathcal{N}_r(\mathcal{S}),\ \begin{bmatrix} u(\lambda) \\ \hline v(\lambda) \end{bmatrix} \mapsto \begin{bmatrix} (e^T_{m-c_0} \otimes I_n)u(\lambda) \\ \hline v(\lambda) \end{bmatrix},$ *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_r(\mathbb{L}_\sigma)$ *to a minimal basis of* $\mathcal{N}_r(\mathcal{S})$*, where* $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ *and* $v(\lambda) \in \mathbb{C}(\lambda)^r$*. Thus, if* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a right minimal basis of* $\mathbb{L}_\sigma(\lambda)$ *then* $\big(\mathbb{F}_\sigma(\mathcal{S})w_1(\lambda), \ldots, \mathbb{F}_\sigma(\mathcal{S})w_p(\lambda)\big)$ *is a right minimal basis of* $\mathcal{S}(\lambda)$*.*
(b) If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $\mathbb{L}_\sigma(\lambda)$ *then* $\varepsilon_1 - i(\sigma) \leq \cdots \leq \varepsilon_p - i(\sigma)$ *are the right minimal indices of* $\mathcal{S}(\lambda)$*.*

(c) **Left minimal bases.** $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma) \to \mathcal{N}_l(\mathcal{S}),\ \begin{bmatrix} u(\lambda) \\ \hline v(\lambda) \end{bmatrix} \mapsto \begin{bmatrix} (e^T_{m-i_0} \otimes I_n)u(\lambda) \\ \hline v(\lambda) \end{bmatrix},$ *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_l(\mathbb{L}_\sigma)$ *to a minimal basis of* $\mathcal{N}_l(\mathcal{S})$*, where* $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ *and* $v(\lambda) \in \mathbb{C}(\lambda)^r$*. Thus, if* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a left minimal basis of* $\mathbb{L}_\sigma(\lambda)$ *then* $\big(\mathbb{K}_\sigma(\mathcal{S})w_1(\lambda), \ldots, \mathbb{K}_\sigma(\mathcal{S})w_p(\lambda)\big)$ *is a left minimal basis of* $\mathcal{S}(\lambda)$*.*
(d) If $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $\mathbb{L}_\sigma(\lambda)$ *then* $\eta_1 - c(\sigma) \leq \cdots \leq \eta_p - c(\sigma)$ *are the left minimal indices of* $\mathcal{S}(\lambda)$*.*

Proof. By Theorem 6.1.9, the map $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma) \longrightarrow \mathcal{N}_r(\mathcal{S})$ is an isomorphism. Now we show that $\mathbb{F}_\sigma(\mathcal{S})$ maps a right minimal basis of $\mathbb{L}_\sigma(\lambda)$ to a right minimal basis of $\mathcal{S}(\lambda)$. Let $\mathcal{B} := \big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ be a right minimal basis of $\mathbb{L}_\sigma(\lambda)$ and set $\mathcal{C} := \mathbb{F}_\sigma(\mathcal{S})(\mathcal{B})$. Let $\mathcal{C}$ be given by $\mathcal{C} = \big(x_1(\lambda), \ldots, x_p(\lambda)\big)$. Then $\mathcal{C}$ is a basis of $\mathcal{N}_r(\mathcal{S})$ and by Remark 6.1.12 we have $\deg\big(\mathbb{F}_\sigma(\mathcal{S})w_j(\lambda)\big) = \deg(w_j(\lambda)) - i(\sigma),\ j = 1 : p$. So $\mathrm{Ord}(\mathcal{C}) = \mathrm{Ord}(\mathcal{B}) - p\, i(\sigma)$. We claim that $\mathcal{C}$ is a minimal basis of $\mathcal{N}_r(\mathcal{S})$. Indeed, let $\widetilde{\mathcal{C}} := \big(y_1(\lambda), \ldots, y_p(\lambda)\big)$ be any polynomial basis of $\mathcal{N}_r(\mathcal{S})$. Then $\widetilde{\mathcal{B}} := \mathbb{E}_\sigma(\mathcal{S})(\widetilde{\mathcal{C}})$ is a polynomial basis of $\mathcal{N}_r(\mathbb{L}_\sigma)$. Since $\deg\big(\mathbb{E}_\sigma(\mathcal{S})y_j(\lambda)\big) = \deg(y_j(\lambda)) + i(\sigma),\ j = 1 : p$, we have $\mathrm{Ord}(\widetilde{\mathcal{B}}) = \mathrm{Ord}(\widetilde{\mathcal{C}}) + p\, i(\sigma)$

which implies $\mathrm{Ord}(\widetilde{\mathcal{C}}) = \mathrm{Ord}(\widetilde{\mathcal{B}}) - p\, i(\sigma) \geq \mathrm{Ord}(\mathcal{B}) - p\, i(\sigma) = \mathrm{Ord}(\mathcal{C})$. Hence $\mathcal{C}$ is a minimal basis of $\mathcal{N}_r(\mathcal{S})$. This proves (a).

By part (a), we have that $\mathcal{C}$ is a right minimal basis of $\mathcal{S}(\lambda)$ and $\deg(x_j(\lambda)) = \deg\big(\mathbb{F}_\sigma(\mathcal{S})w_j(\lambda)\big) = \deg(w_j(\lambda)) - i(\sigma)$ for $j = 1 : p$. This shows that if $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $\mathbb{L}_\sigma(\lambda)$ then $\varepsilon_1 - i(\sigma) \leq \cdots \leq \varepsilon_p - i(\sigma)$ are the right minimal indices of $\mathcal{S}(\lambda)$. This proves (b).

The results for left minimal bases and left minimal indices, respectively, follow from right minimal bases and right minimal indices in view of the following facts. First, $x(\lambda) \in \mathcal{N}_l(\mathbb{L}_\sigma) \iff x(\lambda) \in \mathcal{N}_r(\mathbb{L}_\sigma^T)$. Second, $(\mathbb{M}_j^{\mathcal{S}})^T$'s are Fiedler matrices of $\mathcal{S}(\lambda)^T$. Third, if $\mathbb{L}_\sigma(\lambda)$ is the Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a permutation σ then $\mathbb{L}_\sigma(\lambda)^T$ is the Fiedler pencil of $\mathcal{S}(\lambda)^T$ associated with the permutation $rev(\sigma)$. Fourth, $\mathrm{CIP}(\sigma) = (c_0, i_0) \iff \mathrm{CIP}(rev(\sigma)) = (i_0, c_0)$. Fifth, we have $i(\sigma) = c(rev(\sigma))$. This completes the proof. $\square$

As a consequence of Theorem 6.1.13 and Theorem 5.5.8 we have the following result.

Theorem 6.1.14. *Let $\mathbb{L}_\sigma(\lambda)$ be a Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a permutation σ of $\{0, 1, \ldots, m-1\}$. Let $c(\sigma)$ and $i(\sigma)$, respectively, be the total number of consecutions and inversions of σ. Suppose that $CIP(\sigma) = (c_0, i_0)$. Let $w_i(\lambda) := \left[\begin{array}{c} u_i(\lambda) \\ \hline v_i(\lambda) \end{array}\right] \in \mathbb{C}[\lambda]^{mn+r}$, where $u_i(\lambda) \in \mathbb{C}[\lambda]^{mn}$ and $v_i(\lambda) \in \mathbb{C}[\lambda]^r$ for $i = 1 : p$.*

(a) ***Right minimal bases.*** *If $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ is a right minimal basis of $\mathbb{L}_\sigma(\lambda)$ then $\big((e^T_{m-c_0} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-c_0} \otimes I_n)u_p(\lambda)\big)$ is a right minimal basis of $G(\lambda)$. Further, if $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $\mathbb{L}_\sigma(\lambda)$ then $\varepsilon_1 - i(\sigma) \leq \cdots \leq \varepsilon_p - i(\sigma)$ are the right minimal indices of $G(\lambda)$.*

(b) ***Left minimal bases.*** *If $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ is a left minimal basis of $\mathbb{L}_\sigma(\lambda)$ then $\big((e^T_{m-i_0} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-i_0} \otimes I_n)u_p(\lambda)\big)$ is a left minimal basis of $G(\lambda)$. Further, if $\eta_1 \leq \cdots \leq \eta_p$ are the left minimal indices of $\mathbb{L}_\sigma(\lambda)$ then $\eta_1 - c(\sigma) \leq \cdots \leq \eta_p - c(\sigma)$ are the left minimal indices of $G(\lambda)$.*

6.2 GFPs of rational matrices

Generalized Fiedler pencils (GFPs) of $G(\lambda)$ are defined and studied in [4, 8]. We mention that the operation free recovery rule of eigenvectors of $\mathcal{S}(\lambda)$ as well as $G(\lambda)$ from eigenvectors of the GFPs of $G(\lambda)$ are derived in [4]. In this section, we describe the recovery of minimal bases and minimal indices of $G(\lambda)$ from those of the GFPs of $G(\lambda)$. There are two types of GFPs, namely, proper generalized Fiedler (PGF) pencils and non-proper generalized Fiedler (NPGF) pencils [4]. Consider the Fiedler matrices $M^P_{\pm i}$

and $\mathbb{M}^{\mathcal{S}}_{\pm i}$ for $i = 0 : m$, associated with $P(\lambda)$ and $G(\lambda)$, respectively, as given in Section 6.1.

A permutation $w := (w_0, w_1)$ of $\{0, 1, \ldots, m\}$ is said to be *proper* if $0 \in w_0$ and $m \in w_1$, otherwise w is said to be *non-proper*.

Definition 6.2.1 (PGF pencil, [4]). *Let $\omega := (\omega_0, \omega_1)$ be a permutation of $\{0, 1, \ldots, m\}$. Define $\mathbb{T}_\omega(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-\omega_1} - \mathbb{M}^{\mathcal{S}}_{\omega_0}$. If ω is a proper permutation, then $\mathbb{T}_\omega(\lambda)$ is called a proper generalized Fiedler (PGF) pencil of $G(\lambda)$ (or PGF pencil of $\mathcal{S}(\lambda)$) associated with ω. Otherwise $\mathbb{T}_\omega(\lambda)$ is called a non-proper generalized Fiedler (NPGF) pencil of $G(\lambda)$ (or NPGF pencil of $\mathcal{S}(\lambda)$) associated with ω. Moreover, $\mathbb{T}_\omega(\lambda)$ is said to be a Type-I NPGF pencil of $G(\lambda)$ if $0 \in w_0$ and $m \in w_0$.*

Remark 6.2.2. *Note that $(\mathbb{M}^{\mathcal{S}}_{-m})^{-1}$ does not exist when $\mathcal{S}(\lambda)$ is singular and hence in such a case Type-I NPGF pencil is not defined. Also, since computation of $(\mathbb{M}^{\mathcal{S}}_{0})^{-1}$ would involve a substantial amount of work, for simplicity, we consider only PGF pencils of $G(\lambda)$.*

Let $\mathbb{T}_\omega(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-\omega_1} - \mathbb{M}^{\mathcal{S}}_{\omega_0}$ be the PGF pencil of $G(\lambda)$ associated with a permutation $\omega := (\omega_0, \omega_1)$ of $\{0 : m\}$. Let ω_1 be given by $\omega_1 := (\sigma_1, m, \sigma_2)$. Set $\sigma := (rev(\sigma_1), \omega_0, rev(\sigma_2))$. Then σ is a permutation of $\{0 : m-1\}$. Consider the Fiedler pencil $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_{\sigma}$ of $G(\lambda)$ associated with σ. Then $\mathbb{L}_\sigma(\lambda) = \mathbb{M}^{\mathcal{S}}_{rev(\sigma_1)}\mathbb{T}_\omega(\lambda)\mathbb{M}^{\mathcal{S}}_{rev(\sigma_2)}$, where $\mathbb{M}^{\mathcal{S}}_{rev(\sigma_1)}$ and $\mathbb{M}^{\mathcal{S}}_{rev(\sigma_2)}$ are nonsingular. This implies that any PGF pencil is strictly equivalent to a Fiedler pencil. Hence the left (resp., right) minimal indices of $\mathbb{T}_\omega(\lambda)$ and the left (resp., right) minimal indices of $\mathbb{L}_\sigma(\lambda)$ are the same. Moreover, any PGF pencil $\mathbb{T}_\omega(\lambda)$ is strictly equivalent to a Fiedler pencil that preserves the consecution of ω_0 at 0 (see Theorem 6.2.3) except for the following particular case:

$$\mathbb{T}_\delta(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-m}\mathbb{M}^{\mathcal{S}}_{-(m-1)}\cdots\mathbb{M}^{\mathcal{S}}_{-(c_0+1)} - \mathbb{M}^{\mathcal{S}}_{0}\mathbb{M}^{\mathcal{S}}_{1}\cdots\mathbb{M}^{\mathcal{S}}_{c_0} = \lambda\mathbb{M}^{\mathcal{S}}_{-\delta_1} - \mathbb{M}^{\mathcal{S}}_{\delta_0}, \tag{6.6}$$

where $c_0 \in \{0, 1, \ldots, m-2\}$. In (6.6) the permutation $\delta := (\delta_0, \delta_1)$ is defined by

$$\delta_0 := (0, 1, \ldots, c_0) \text{ and } \delta_1 := (m, m-1, \ldots, c_0+1). \tag{6.7}$$

Theorem 6.2.3. *[4] Let $\mathbb{T}_\omega(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-\omega_1} - \mathbb{M}^{\mathcal{S}}_{\omega_0}$ be the PGF pencil of $G(\lambda)$ associated with a permutation $\omega := (\omega_0, \omega_1)$ of $\{0, 1, \ldots, m\}$. Suppose that CIP$(\omega_0) = (c_0, i_0)$ and $\omega_1 \neq (m, m-1, \ldots, c_0+1)$. Let ω_1 be given by $\omega_1 := (\sigma_1, m, \sigma_2)$. If $c_0 + 1 \in \omega_0 \cup \sigma_1$ then define $\xi_1 := rev(\sigma_1)$ and $\xi_2 := rev(\sigma_2)$. On the other hand, if $c_0 + 1 \in \sigma_2$ and σ_2 has p consecutive inversions at c_0+1 then set $\tau_1 := (c_0+p+1, c_0+p, \ldots, c_0+1)$. Then $\mathbb{M}^{\mathcal{S}}_{\sigma_2} = \mathbb{M}^{\mathcal{S}}_{(\tau_1, \tau_2)}$ for some sub-permutation τ_2. Define $\xi_1 := rev(\sigma_1, \tau_1)$ and $\xi_2 := rev(\tau_2)$. Then ξ_1 and ξ_2 are sub-permutations of $\{1, 2, \ldots, m-1\}$ such that the following hold.*

(a) $c_0 \notin \xi_2$ *and* $c_0 + 1 \notin \xi_2$.

(b) $\sigma := (\xi_1, \omega_0, \xi_2)$ *is a permutation of* $\{0, 1, \ldots, m-1\}$ *and has* c_0 *consecutions at* 0*, that is,* σ *and* ω_0 *have the same number of consecutions at* 0.

(c) The Fiedler pencil $\mathbb{L}_\sigma(\lambda) := \lambda \mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_{\sigma}$ *of* $G(\lambda)$ *associated with* σ *is such that* $\mathbb{L}_\sigma(\lambda) = \mathbb{M}^{\mathcal{S}}_{\xi_1} \mathbb{T}_\omega(\lambda) \mathbb{M}^{\mathcal{S}}_{\xi_2}$ *and that* $\mathbb{M}^{\mathcal{S}}_{\xi_i} = \operatorname{diag}(M^P_{\xi_i}, I_r), i = 1, 2,$ *are nonsingular.*

Now we describe the recovery of minimal bases and minimal indices of $\mathcal{S}(\lambda)$ from those of the PGF pencils of $\mathcal{S}(\lambda)$.

Theorem 6.2.4. *Let* $\mathbb{T}_\omega(\lambda) := \lambda \mathbb{M}^{\mathcal{S}}_{-\omega_1} - \mathbb{M}^{\mathcal{S}}_{\omega_0}$ *be the PGF pencil of* $\mathcal{S}(\lambda)$ *associated with a proper permutation* $\omega := (\omega_0, \omega_1)$ *of* $\{0, 1, \ldots, m\}$*. Suppose that* $CIP(\omega_0) = (c_0, i_0)$*. Let* ω_1 *be given by* $\omega_1 := (\sigma_1, m, \sigma_2)$*. Let* $i_{\mathbb{T}} := i(rev(\sigma_1), \omega_0, rev(\sigma_2))$ *and* $c_{\mathbb{T}} := c(rev(\sigma_1), \omega_0, rev(\sigma_2))$*, respectively, be the total number of inversions and consecutions of the permutation* $(rev(\sigma_1), \omega_0, rev(\sigma_2))$ *of* $\{0 : m-1\}$.

(I) Right minimal bases and right minimal indices.

(a) $\mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}_\omega) \to \mathcal{N}_r(\mathcal{S}),\ \left[\begin{array}{c} u(\lambda) \\ \hline v(\lambda) \end{array}\right] \mapsto \left[\begin{array}{c} (e^T_{m-c_0} \otimes I_n) u(\lambda) \\ \hline v(\lambda) \end{array}\right]$*, is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_r(\mathbb{T}_\omega)$ *to a minimal basis of* $\mathcal{N}_r(\mathcal{S})$*, where* $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ *and* $v(\lambda) \in \mathbb{C}(\lambda)^r$*. Thus, if* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a right minimal basis of* $\mathbb{T}_\omega(\lambda)$ *then* $\big(\mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}) w_1(\lambda), \ldots, \mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}) w_p(\lambda)\big)$ *is a right minimal basis of* $\mathcal{S}(\lambda)$.

(b) If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $\mathbb{T}_\omega(\lambda)$ *then* $\varepsilon_1 - i_{\mathbb{T}} \leq \cdots \leq \varepsilon_p - i_{\mathbb{T}}$ *are the right minimal indices of* $\mathcal{S}(\lambda)$.

(II) Left minimal bases and left minimal indices.

(c) $\mathbb{K}^{\text{PGF}}_\omega(\mathcal{S}) : \mathcal{N}_l(\mathbb{T}_\omega) \to \mathcal{N}_l(\mathcal{S}),\ \left[\begin{array}{c} u(\lambda) \\ \hline v(\lambda) \end{array}\right] \mapsto \left[\begin{array}{c} (e^T_{m-i_0} \otimes I_n) u(\lambda) \\ \hline v(\lambda) \end{array}\right]$*, is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_l(\mathbb{T}_\omega)$ *to a minimal basis of* $\mathcal{N}_l(\mathcal{S})$*, where* $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ *and* $v(\lambda) \in \mathbb{C}(\lambda)^r$*. Thus, if* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a left minimal basis of* $\mathbb{T}_\omega(\lambda)$ *then* $\big(\mathbb{K}^{\text{PGF}}_\omega(\mathcal{S}) w_1(\lambda), \ldots, \mathbb{K}^{\text{PGF}}_\omega(\mathcal{S}) w_p(\lambda)\big)$ *is a left minimal basis of* $\mathcal{S}(\lambda)$.

(d) If $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $\mathbb{T}_\omega(\lambda)$ *then* $\eta_1 - c_{\mathbb{T}} \leq \cdots \leq \eta_p - c_{\mathbb{T}}$ *are the left minimal indices of* $\mathcal{S}(\lambda)$.

Proof. Any PGF pencil is strictly equivalent to a Fiedler pencil. Thus, there exist sub-permutations ξ_1 and ξ_2 of $\{1, \ldots, m-1\}$ such that $\mathbb{L}_\sigma(\lambda) := \mathbb{M}^{\mathcal{S}}_{\xi_1} \mathbb{T}_\omega(\lambda) \mathbb{M}^{\mathcal{S}}_{\xi_2} = \lambda \mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_{\sigma}$ is a Fiedler pencil of $\mathcal{S}(\lambda)$ associated with the permutation $\sigma := (\xi_1, \omega_0, \xi_2)$ of $\{0, 1, \ldots, m-1\}$. Since $\mathbb{M}^{\mathcal{S}}_{\xi_2}$ is nonsingular, it is easily seen that the map $(\mathbb{M}^{\mathcal{S}}_{\xi_2})^{-1} : \mathcal{N}_r(\mathbb{T}_\omega) \to \mathcal{N}_r(\mathbb{L}_\sigma), x(\lambda) \mapsto \big(\mathbb{M}^{\mathcal{S}}_{\xi_2}\big)^{-1} x(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{T}_\omega)$ to a minimal basis of $\mathcal{N}_r(\mathbb{L}_\sigma)$. On the other hand, by Theorem 6.1.13, $\mathbb{F}_\sigma(\mathcal{S})$:

$\mathcal{N}_r(\mathbb{L}_\sigma) \to \mathcal{N}_r(\mathcal{S})$ is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{L}_\sigma)$ to a minimal basis of $\mathcal{N}_r(\mathcal{S})$. Hence $\mathbb{F}_\sigma(\mathcal{S})(\mathbb{M}^{\mathcal{S}}_{\xi_2})^{-1} : \mathcal{N}_r(\mathbb{T}_\omega) \to \mathcal{N}_r(\mathcal{S})$ is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{T}_\omega)$ to a minimal basis of $\mathcal{N}_r(\mathcal{S})$. Note that $\mathbb{M}^{\mathcal{S}}_{\xi_2} = \mathrm{diag}(M_{\xi_2}, I_r)$.

Case-1: If $\omega_1 \neq (m, m-1, \ldots, c_0+1)$ then by Theorem 6.2.3, we can choose ξ_1 and ξ_2 such that σ preserves the consecution of ω_0 at 0, i.e., σ has c_0 consecutive consecutions at 0. Hence if $j \in \xi_2$ then $j \geq c_0 + 2$. Thus $M^P_{\xi_2} = \mathrm{diag}(*, I_{(c_0+1)n})$. Hence $M^{-1}_{\xi_2} = \mathrm{diag}(*, I_{(c_0+1)n})$, which shows that $(e^T_{m-c_0} \otimes I_n)(M^P_{\xi_2})^{-1} = e^T_{m-c_0} \otimes I_n$. Since σ has c_0 consecutive consecutions at 0, by Theorem 6.1.13, $\mathbb{F}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} e^T_{m-c_0} \otimes I_n & \\ \hline & I_r \end{array}\right]$. Consequently, we have

$$\mathbb{F}_\sigma(\mathcal{S})(\mathbb{M}^{\mathcal{S}}_{\xi_2})^{-1} = \left[\begin{array}{c|c} (e^T_{m-c_0} \otimes I_n)(M^P_{\xi_2})^{-1} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} e^T_{m-c_0} \otimes I_n & \\ \hline & I_r \end{array}\right] = \mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}).$$

Since $i_{\mathbb{T}} = i(\sigma)$, the total number of inversions of σ, the desired results for right minimal bases and right minimal indices follow from Theorem 6.1.13.

Case-2: Suppose that $\omega_1 = (m, m-1, \ldots, c_0+1)$. Then by choosing $\xi_1 = \emptyset$ and $\xi_2 = (c_0+1, \ldots, m-1)$, we have $\sigma := (\xi_1, \omega_0, \xi_2) = (0, 1, \ldots, m-1)$ and the Fiedler pencil $\mathbb{L}_\sigma(\lambda) = \mathbb{T}_\omega(\lambda)\mathbb{M}^{\mathcal{S}}_{\xi_2}$. Hence σ has $m-1$ consecutive consecutions at 0. Thus by Theorem 6.1.13, $\mathbb{F}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} e^T_{m-(m-1)} \otimes I_n & \\ \hline & I_r \end{array}\right]$. By (2.12), we have $(e^T_{m-j} \otimes I_n)(M^P_k)^{-1} = e^T_{m-(j-1)} \otimes I_n$ for $k = j$, $j = 1 : m-1$. Hence it is easy to see that $(e^T_{m-(m-1)} \otimes I_n)(M^P_{\xi_2})^{-1} = (e^T_{m-(m-1)} \otimes I_n)(M^P_{m-1})^{-1}(M^P_{m-2})^{-1} \cdots (M^P_{c_0+1})^{-1} = e^T_{m-c_0} \otimes I_n$. Consequently, we have

$$\mathbb{F}_\sigma(\mathcal{S})(\mathbb{M}^{\mathcal{S}}_{\xi_2})^{-1} = \left[\begin{array}{c|c} (e^T_{m-(m-1)} \otimes I_n)(M^P_{\xi_2})^{-1} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} e^T_{m-c_0} \otimes I_n & \\ \hline & I_r \end{array}\right] = \mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}).$$

Again, since $i_{\mathbb{T}} = m - 1 = i(\sigma)$, the total number of inversions of σ, the desired results for right minimal bases and right minimal indices follow from Theorem 6.1.13.

The proof is similar for left minimal bases and left minimal indices in view of the following facts. First, $x(\lambda) \in \mathcal{N}_l(\mathbb{T}_\omega) \iff x(\lambda) \in \mathcal{N}_r(\mathbb{T}^T_\omega)$. Second, $(\mathbb{M}^{\mathcal{S}}_j)^T$'s are Fiedler matrices of $\mathcal{S}(\lambda)^T$. Third, if $\mathbb{T}_\omega(\lambda)$ is the PGF pencil of $\mathcal{S}(\lambda)$ associated with a permutation $\omega := (\omega_0, \omega_1)$ then $\mathbb{T}_\omega(\lambda)^T$ is the PGF pencil of $\mathcal{S}(\lambda)^T$ associated with the permutation $\widehat{\omega} := (rev(\omega_0), rev(\omega_1))$. Fourth, $\mathrm{CIP}(\omega_0) = (c_0, i_0) \iff \mathrm{CIP}(rev(\omega_0)) = (i_0, c_0)$. Fifth, $i_{\mathbb{T}} = c_{\mathbb{T}^T}$, where $c_{\mathbb{T}^T} := c(\sigma_2, rev(\omega_0), \sigma_1)$. Indeed, we have $c(\sigma_2, rev(\omega_0), \sigma_1) = i\big(rev((\sigma_2, rev(\omega_0), \sigma_1))\big) = i(rev(\sigma_1), \omega_0, rev(\sigma_2)) = i_{\mathbb{T}}$. This completes the proof. $\square$

As a consequence of Theorem 6.2.4 and Theorem 5.5.8, we have the following result for minimal bases and minimal indices of $G(\lambda)$.

Theorem 6.2.5. *Consider the PGF pencil* $\mathbb{T}_\omega(\lambda) := \lambda \mathbb{M}^{\mathcal{S}}_{-\omega_1} - \mathbb{M}^{\mathcal{S}}_{\omega_0}$ *of* $G(\lambda)$ *associated with a proper permutation* $\omega := (\omega_0, \omega_1)$ *of* $\{0, 1, \ldots, m\}$. *Suppose that* $CIP(\omega_0) = (c_0, i_0)$. *Let* ω_1 *be given by* $\omega_1 := (\sigma_1, m, \sigma_2)$. *Let* $i_{\mathbb{T}} := i(rev(\sigma_1), \omega_0, rev(\sigma_2))$ *and* $c_{\mathbb{T}} := c(rev(\sigma_1), \omega_0, rev(\sigma_2))$ *be the total number of inversions and consecutions of* $(rev(\sigma_1), \omega_0, rev(\sigma_2))$, *respectively. Let* $w_i(\lambda) := \begin{bmatrix} u_i(\lambda) \\ \hline v_i(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{mn+r}$, *where* $u_i(\lambda) \in \mathbb{C}[\lambda]^{mn}$ *and* $v_i(\lambda) \in \mathbb{C}[\lambda]^r$ *for* $i = 1 : p$. *Then we have the following.*

(a) **Right minimal bases.** *If* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a right minimal basis of* $\mathbb{T}_\omega(\lambda)$ *then* $\big((e^T_{m-c_0} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-c_0} \otimes I_n)u_p(\lambda)\big)$ *is a right minimal basis of* $G(\lambda)$. *Further, if* $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $\mathbb{T}_\omega(\lambda)$ *then* $\varepsilon_1 - i_{\mathbb{T}} \leq \cdots \leq \varepsilon_p - i_{\mathbb{T}}$ *are the right minimal indices of* $G(\lambda)$.

(b) **Left minimal bases.** *If* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a left minimal basis of* $\mathbb{T}_\omega(\lambda)$ *then* $\big((e^T_{m-i_0} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-i_0} \otimes I_n)u_p(\lambda)\big)$ *is a left minimal basis of* $G(\lambda)$. *Further, if* $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $\mathbb{T}_\omega(\lambda)$ *then* $\eta_1 - c_{\mathbb{T}} \leq \cdots \leq \eta_p - c_{\mathbb{T}}$ *are the left minimal indices of* $G(\lambda)$.

6.3 FPRs of rational matrices

Fiedler pencils with repetition (FPRs) of $G(\lambda)$ are defined and studied in [8]. First, we show that an FPR of $G(\lambda)$ can be constructed directly from an FPR of $P(\lambda)$. Then we describe the recovery of eigenvectors, minimal bases and minimal indices of $G(\lambda)$ from those of the FPRs of $G(\lambda)$. We mention that the recovery of eigenvectors of $G(\lambda)$ from those of the FPRs of $G(\lambda)$ is known only for a small subclass of FPRs that corresponds to type-1 indices [8].

An FPR of $G(\lambda)$ is defined as follows.

Definition 6.3.1 (FPR, [8]). *Let* $0 \leq h \leq m-1$, *and let* σ *and* τ *be permutations of* $\{0, 1, \ldots, h\}$ *and* $\{-m, -m+1, \ldots, -h-1\}$, *respectively. Let* σ_1 *and* σ_2 *be index tuples with elements from* $\{1, 2, \ldots, h-1\}$ *such that* $(\sigma_1, \sigma, \sigma_2)$ *satisfies the SIP. Similarly, let* τ_1 *and* τ_2 *be index tuples with elements from* $\{-m+1, -m+2, \ldots, -h-2\}$ *such that* (τ_1, τ, τ_2) *satisfies the SIP. Then* $\mathbb{L}(\lambda) := \mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1}(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}$ *is called a Fiedler pencil with repetition (FPR) of* $G(\lambda)$ *(also referred to as FPR of* $\mathcal{S}(\lambda)$*).*

The following result shows that an FPR of $G(\lambda)$ can be constructed directly from an FPR of $P(\lambda)$, the polynomial part of $G(\lambda)$.

Theorem 6.3.2. *Let* $\mathbb{L}(\lambda) := \mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1}(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}$ *and* $L(\lambda) := M^{P}_{\tau_1}M^{P}_{\sigma_1}(\lambda M^{P}_{\tau} - M^{P}_{\sigma})M^{P}_{\sigma_2}M^{P}_{\tau_2}$ *be FPRs of* $G(\lambda)$ *and* $P(\lambda)$*, respectively. Then*

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & e_{m-i_0(\sigma_1,\sigma)} \otimes C \\ \hline e^T_{m-c_0(\sigma,\sigma_2)} \otimes B & A - \lambda E \end{array}\right].$$

Thus, the map $\mathrm{FPR}(P) \longrightarrow \mathrm{FPR}(G)$, $L(\lambda) \longmapsto \left[\begin{array}{c|c} L(\lambda) & e_{m-i_0(\sigma_1,\sigma)} \otimes C \\ \hline e^T_{m-c_0(\sigma,\sigma_2)} \otimes B & A - \lambda E \end{array}\right]$ *is a bijection, where FPR(P) and FPR(G) denote the set of FPRs of* $P(\lambda)$ *and* $G(\lambda)$*, respectively.*

Proof. We will prove a more general result in Theorem 7.2.4. ☐

We now describe the recovery of minimal bases and minimal indices of $\mathcal{S}(\lambda)$ from those of the FPRs of $\mathcal{S}(\lambda)$. Recall from Definition 1.2.20 that $c_t(\alpha)$ (resp., $i_t(\alpha)$) denotes the consecutions (resp., inversions) of an index tuple α at an index t.

Theorem 6.3.3. *Let* $\mathbb{L}(\lambda) := \mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1}(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}$ *be an FPR of* $\mathcal{S}(\lambda)$*. Let* τ *be given by* $\tau := (\tau_l, -m, \tau_r)$*. Set* $\alpha := \big(-rev(\tau_l), \sigma, -rev(\tau_r)\big)$*. Let* $c_{\mathbb{L}} := c(\alpha)$ *and* $i_{\mathbb{L}} := i(\alpha)$ *be the total number of consecutions and inversions of the permutation* α *of* $\{0, 1, \ldots, m-1\}$*. Then we have the following.*

(I) Right minimal bases and right minimal indices.

(a) $\mathbb{F}^{\scriptscriptstyle FPR}_{\mathbb{L}}(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}) \to \mathcal{N}_r(\mathcal{S})$, $\left[\begin{array}{c} u(\lambda) \\ \hline v(\lambda) \end{array}\right] \mapsto \left[\begin{array}{c} (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u(\lambda) \\ \hline v(\lambda) \end{array}\right]$, *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_r(\mathbb{L})$ *to a minimal basis of* $\mathcal{N}_r(\mathcal{S})$*, where* $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ *and* $v(\lambda) \in \mathbb{C}(\lambda)^r$*. Thus, if* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a right minimal basis of* $\mathbb{L}(\lambda)$ *then* $\big(\mathbb{F}^{\scriptscriptstyle FPR}_{\mathbb{L}}(\mathcal{S})w_1(\lambda), \ldots, \mathbb{F}^{\scriptscriptstyle FPR}_{\mathbb{L}}(\mathcal{S})w_p(\lambda)\big)$ *is a right minimal basis of* $\mathcal{S}(\lambda)$*.*

(b) If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right minimal indices of* $\mathbb{L}(\lambda)$ *then* $\varepsilon_1 - i_{\mathbb{L}} \leq \cdots \leq \varepsilon_p - i_{\mathbb{L}}$ *are the right minimal indices of* $\mathcal{S}(\lambda)$*.*

(II) Left minimal bases and left minimal indices.

(c) $\mathbb{K}^{\scriptscriptstyle FPR}_{\mathbb{L}}(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}) \to \mathcal{N}_l(\mathcal{S})$, $\left[\begin{array}{c} u(\lambda) \\ \hline v(\lambda) \end{array}\right] \mapsto \left[\begin{array}{c} (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)u(\lambda) \\ \hline v(\lambda) \end{array}\right]$, *is a linear isomorphism and maps a minimal basis of* $\mathcal{N}_l(\mathbb{L})$ *to a minimal basis of* $\mathcal{N}_l(\mathcal{S})$*, where* $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ *and* $v(\lambda) \in \mathbb{C}(\lambda)^r$*. Thus, if* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a left minimal basis of* $\mathbb{L}(\lambda)$ *then* $\big(\mathbb{K}^{\scriptscriptstyle FPR}_{\mathbb{L}}(\mathcal{S})w_1(\lambda), \ldots, \mathbb{K}^{\scriptscriptstyle FPR}_{\mathbb{L}}(\mathcal{S})w_p(\lambda)\big)$ *is a left minimal basis of* $\mathcal{S}(\lambda)$*.*

(d) If $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $\mathbb{L}(\lambda)$ *then* $\eta_1 - c_{\mathbb{L}} \leq \cdots \leq \eta_p - c_{\mathbb{L}}$ *are the left minimal indices of* $\mathcal{S}(\lambda)$*.*

Proof. We have $\mathbb{L}(\lambda) = \mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1}\mathbb{T}_{\omega}(\lambda)\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}$, where $\mathbb{T}_{\omega}(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma}$ is a PGF pencil of $G(\lambda)$ associated with the permutation $\omega := (\sigma, -\tau)$ of $\{0, 1, \ldots, m\}$. Since $\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}$ is nonsingular, it is easily seen that the map $\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2} : \mathcal{N}_r(\mathbb{L}) \to \mathcal{N}_r(\mathbb{T}_{\omega})$, $x(\lambda) \mapsto \big(\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}\big)x(\lambda)$ is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{L})$ to a minimal basis of $\mathcal{N}_r(\mathbb{T}_{\omega})$. On the other hand, by Theorem 6.2.4, $\mathbb{F}^{\text{PGF}}_{\omega}(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}_{\omega}) \to \mathcal{N}_r(\mathcal{S})$, $\left[\begin{array}{c} u(\lambda) \\ \hline v(\lambda) \end{array}\right] \mapsto \left[\begin{array}{c} (e^T_{m-c_0(\sigma)} \otimes I_n)u(\lambda) \\ \hline v(\lambda) \end{array}\right]$ is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{T}_{\omega})$ to a minimal basis of $\mathcal{N}_r(\mathcal{S})$, where $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ and $v(\lambda) \in \mathbb{C}(\lambda)^r$. Hence $\mathbb{F}^{\text{PGF}}_{\omega}(\mathcal{S})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2} : \mathcal{N}_r(\mathbb{L}) \to \mathcal{N}_r(\mathcal{S})$, $y(\lambda) \mapsto \big(\mathbb{F}^{\text{PGF}}_{\omega}(\mathcal{S})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}\big)y(\lambda)$ is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{L})$ to a minimal basis of $\mathcal{N}_r(\mathcal{S})$. Now, by Lemma 2.1.11, we have

$$\mathbb{F}^{\text{PGF}}_{\omega}(\mathcal{S})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2} = \left[\begin{array}{c|c} (e^T_{m-c_0(\sigma)} \otimes I_n)M^P_{\sigma_2}M^P_{\tau_2} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n & \\ \hline & I_r \end{array}\right] = \mathbb{F}^{\text{FPR}}_{\mathbb{L}}(\mathcal{S}),$$

which yields the desired result.

Now, let $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ be the right minimal indices of $\mathbb{L}(\lambda)$. Since the PGF pencil $\mathbb{T}_{\omega}(\lambda)$ is strictly equivalent to $\mathbb{L}(\lambda)$, $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are also the right minimal indices of $\mathbb{T}_{\omega}(\lambda)$. Hence by Theorem 6.2.4, $\varepsilon_1 - i_{\mathbb{L}} \leq \cdots \leq \varepsilon_p - i_{\mathbb{L}}$ are the right minimal indices of $\mathcal{S}(\lambda)$.

For the recovery of left minimal bases, observe that $(\mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1})^T : \mathcal{N}_l(\mathbb{L}) \to \mathcal{N}_l(\mathbb{T}_{\omega})$, $x(\lambda) \mapsto (\mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1})^T x(\lambda)$ is an isomorphism and maps a minimal basis of $\mathcal{N}_l(\mathbb{L})$ to a minimal basis of $\mathcal{N}_l(\mathbb{T}_{\omega})$. Again by Theorem 6.2.4, $\mathbb{K}^{\text{PGF}}_{\omega}(\mathcal{S}) : \mathcal{N}_l(\mathbb{T}_{\omega}) \to \mathcal{N}_l(\mathcal{S})$, $\left[\begin{array}{c} u(\lambda) \\ \hline v(\lambda) \end{array}\right] \longmapsto \left[\begin{array}{c} (e^T_{m-i_0(\sigma)} \otimes I_n)u(\lambda) \\ \hline v(\lambda) \end{array}\right]$ is an isomorphism and maps a minimal basis of $\mathcal{N}_l(\mathbb{T}_{\omega})$ to a minimal basis of $\mathcal{N}_l(\mathcal{S})$, where $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ and $v(\lambda) \in \mathbb{C}(\lambda)^r$. Hence $\mathbb{K}^{\text{PGF}}_{\omega}(\mathcal{S})(\mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1})^T : \mathcal{N}_l(\mathbb{L}) \to \mathcal{N}_l(\mathcal{S})$, $y(\lambda) \mapsto \big(\mathbb{K}^{\text{PGF}}_{\omega}(\mathcal{S})(\mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1})^T\big)y(\lambda)$ is an isomorphism and maps a minimal basis of $\mathcal{N}_l(\mathbb{L})$ to a minimal basis of $\mathcal{N}_l(\mathcal{S})$. Now,

$$\mathbb{K}^{\text{PGF}}_{\omega}(\mathcal{S})(\mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1})^T = \left[\begin{array}{c|c} (e^T_{m-i_0(\sigma)} \otimes I_n)(M^P_{\tau_1}M^P_{\sigma_1})^T & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} \big(M^P_{\tau_1}M^P_{\sigma_1}(e_{m-i_0(\sigma)} \otimes I_n)\big)^T & \\ \hline & I_r \end{array}\right]$$

and by Lemma 2.1.11, we have $M^P_{\tau_1}M^P_{\sigma_1}(e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$, which yields the desired result.

Finally, let $\eta_1 \leq \cdots \leq \eta_p$ be the left minimal indices of $\mathbb{L}(\lambda)$. Since the PGF pencil $\mathbb{T}_{\omega}(\lambda)$ is strictly equivalent to $\mathbb{L}(\lambda)$, $\eta_1 \leq \cdots \leq \eta_p$ are also the left minimal indices of

$\mathbb{T}_\omega(\lambda)$. Hence by Theorem 6.2.4, $\eta_1 - c_{\mathbb{L}} \leq \cdots \leq \eta_p - c_{\mathbb{L}}$ are the left minimal indices of $\mathcal{S}(\lambda)$. This completes the proof. $\square$

We have the following recovery rules for eigenvectors of $\mathcal{S}(\lambda)$ when $\mathcal{S}(\lambda)$ is regular.

Theorem 6.3.4. *Suppose that $\mathcal{S}(\lambda)$ is regular and $\mu \in \mathbb{C}$ is an eigenvalue of $\mathcal{S}(\lambda)$. Let $\mathbb{L}(\lambda) := \mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1}(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}$ be an FPR of $\mathcal{S}(\lambda)$. Then:*

(a) **Right eigenvectors.** *If $\left[\begin{array}{c} u \\ \hline v \end{array}\right] \in \mathcal{N}_r(\mathbb{L}(\mu))$ then $\left[\begin{array}{c} (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u \\ \hline v \end{array}\right] \in \mathcal{N}_r(\mathcal{S}(\mu))$, where $u \in \mathbb{C}^{mn}$ and $v \in \mathbb{C}^r$. In fact, the mapping $\left[\begin{array}{c} u \\ \hline v \end{array}\right] \mapsto \left[\begin{array}{c} (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u \\ \hline v \end{array}\right]$ is a linear isomorphism between $\mathcal{N}_r(\mathbb{L}(\mu))$ and $\mathcal{N}_r(\mathcal{S}(\mu))$. Thus, if $(w_1, \ldots, w_p)$ is a basis of $\mathcal{N}_r(\mathbb{L}(\mu))$ then $\big(\mathbb{F}^{\mathrm{FPR}}_{\mathbb{L}}(\mathcal{S})w_1, \ldots, \mathbb{F}^{\mathrm{FPR}}_{\mathbb{L}}(\mathcal{S})w_p\big)$ is a basis of $\mathcal{N}_r(\mathcal{S}(\mu))$, where $\mathbb{F}^{\mathrm{FPR}}_{\mathbb{L}}$ is as given in Theorem 6.3.3.*

(b) **Left eigenvectors.** *If $\left[\begin{array}{c} u \\ \hline v \end{array}\right] \in \mathcal{N}_l(\mathbb{L}(\mu))$ then $\left[\begin{array}{c} (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)u \\ \hline v \end{array}\right] \in \mathcal{N}_l(\mathcal{S}(\mu))$, where $u \in \mathbb{C}^{mn}$ and $v \in \mathbb{C}^r$. In fact, the mapping $\left[\begin{array}{c} u \\ \hline v \end{array}\right] \mapsto \left[\begin{array}{c} (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)u \\ \hline v \end{array}\right]$ is a linear isomorphism between $\mathcal{N}_l(\mathbb{L}(\mu))$ and $\mathcal{N}_l(\mathcal{S}(\mu))$. Thus, if $(w_1, \ldots, w_p)$ is a basis of $\mathcal{N}_l(\mathbb{L}(\mu))$ then $\big(\mathbb{K}^{\mathrm{FPR}}_{\mathbb{L}}(\mathcal{S})w_1, \ldots, \mathbb{K}^{\mathrm{FPR}}_{\mathbb{L}}(\mathcal{S})w_p\big)$ is a basis of $\mathcal{N}_l(\mathcal{S}(\mu))$, where $\mathbb{K}^{\mathrm{FPR}}_{\mathbb{L}}$ is as given in Theorem 6.3.3.*

Proof. A verbatim proof of Theorem 6.3.3 yields the desired results. $\square$

As a consequence of Theorem 6.3.3 and Theorem 5.5.8, we have the following recovery rules for minimal bases and minimal indices of $G(\lambda)$ from those of the FPRs of $G(\lambda)$.

Theorem 6.3.5. *Let $\mathbb{L}(\lambda) := \mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1}(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}$ be an FPR of $G(\lambda)$. Let τ be given by $\tau := (\tau_l, -m, \tau_r)$. Set $\alpha := (-rev(\tau_l), \sigma, -rev(\tau_r))$. Let $c_{\mathbb{L}} := c(\alpha)$ and $i_{\mathbb{L}} := i(\alpha)$ be the total number of consecutions and inversions of the permutation α of $\{0, 1, \ldots, m-1\}$. Let $w_i(\lambda) := \left[\begin{array}{c} u_i(\lambda) \\ \hline v_i(\lambda) \end{array}\right] \in \mathbb{C}[\lambda]^{mn+r}$, where $u_i(\lambda) \in \mathbb{C}[\lambda]^{mn}$ and $v_i(\lambda) \in \mathbb{C}[\lambda]^r$ for $i = 1 : p$. Then we have the following.*

(a) **Right minimal bases.** *If $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ is a right minimal basis of $\mathbb{L}(\lambda)$ then $\big((e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u_p(\lambda)\big)$ is a right minimal basis of $G(\lambda)$. Further, if $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $\mathbb{L}(\lambda)$ then $\varepsilon_1 - i_{\mathbb{L}} \leq \cdots \leq \varepsilon_p - i_{\mathbb{L}}$ are the right minimal indices of $G(\lambda)$.*

(b) **Left minimal bases.** *If* $\big(w_1(\lambda), \ldots, w_p(\lambda)\big)$ *is a left minimal basis of* $\mathbb{L}(\lambda)$ *then* $\big((e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)u_1(\lambda), \ldots, (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)u_p(\lambda)\big)$ *is a left minimal basis of* $G(\lambda)$*. Further, if* $\eta_1 \leq \cdots \leq \eta_p$ *are the left minimal indices of* $\mathbb{L}(\lambda)$ *then* $\eta_1 - c_{\mathbb{L}} \leq \cdots \leq \eta_p - c_{\mathbb{L}}$ *are the left minimal indices of* $G(\lambda)$*.*

Similarly, we have the following recovery rules for eigenvectors of $G(\lambda)$ from those of the FPRs of $G(\lambda)$.

Theorem 6.3.6. *Suppose that* $G(\lambda)$ *is regular and* $\mu \in \mathbb{C}$ *is an eigenvalue of* $G(\lambda)$*. Let* $\mathbb{L}(\lambda) := \mathbb{M}^{\mathcal{S}}_{\tau_1}\mathbb{M}^{\mathcal{S}}_{\sigma_1}(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}^{\mathcal{S}}_{\sigma_2}\mathbb{M}^{\mathcal{S}}_{\tau_2}$ *be an FPR of* $G(\lambda)$*. Let* $w_i := \begin{bmatrix} u_i \\ \hline v_i \end{bmatrix} \in \mathbb{C}^{mn+r}$*, where* $u_i \in \mathbb{C}^{mn}$ *and* $v_i \in \mathbb{C}^r$ *for* $i = 1 : p$*. Then we have the following.*

(a) **Right eigenvectors.** *If* $(w_1, \ldots, w_p)$ *is a basis of* $\mathcal{N}_r(\mathbb{L}(\mu))$ *then* $\big((e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u_1, \ldots, (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)u_p\big)$ *is a basis of* $\mathcal{N}_r(G(\mu))$*.*

(b) **Left eigenvectors.** *If* $(w_1, \ldots, w_p)$ *is a basis of* $\mathcal{N}_l(\mathbb{L}(\mu))$ *then* $\big((e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)u_1, \ldots, (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n)u_p\big)$ *is a basis of* $\mathcal{N}_l(G(\mu))$*.*

Proof. Since $\mathbb{L}(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$, we have $\mathrm{eig}(\mathcal{S}) = \mathrm{eig}(\mathbb{L})$. Again since $\mathcal{S}(\lambda)$ is irreducible, we have $\mathrm{eig}(G) \subset \mathrm{eig}(\mathcal{S})$. Hence as a consequence of Theorem 6.3.4 and Theorem 5.5.8, the desired results follow. ☐

We now illustrate eigenvector recovery rule for $G(\lambda)$ by considering an example

Example 6.3.7. *Consider* $G(\lambda) := \sum_{i=0}^{4} \lambda^i A_i + C(\lambda E - A)^{-1}B$*. Now choose* $\sigma := (1, 2, 3, 0), \sigma_1 := \emptyset, \sigma_2 := (2, 1), \tau := (-4), \tau_1 = \emptyset$ *and* $\tau_2 := \emptyset$*. The FPR of* $G(\lambda)$ *associated with* $(\sigma_1, \sigma, \sigma_2)$ *and* (τ_1, τ, τ_2) *is given by* $\mathbb{L}(\lambda) := \big(\lambda\mathbb{M}^{\mathcal{S}}_{-4} - \mathbb{M}^{\mathcal{S}}_{(1,2,3,0)}\big)\mathbb{M}^{\mathcal{S}}_{(2,1)=}$

$$\left[\begin{array}{cccc|c} \lambda A_4 + A_3 & A_2 & A_1 & -I_n & 0 \\ A_2 & -\lambda A_2 - I_n & -\lambda A_1 & \lambda I_n & 0 \\ A_1 & \lambda I_n & A_0 & 0 & C \\ -I_n & 0 & \lambda I_n & 0 & 0 \\ \hline 0 & 0 & B & 0 & A - \lambda E \end{array}\right]$$

Let $\begin{bmatrix} u \\ - \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}(\lambda))$ *and* $\begin{bmatrix} x \\ - \\ y \end{bmatrix} \in \mathcal{N}_l(\mathbb{L}(\lambda))$ *be such that* $\{u, x\} \subset \mathbb{C}^{4n}$ *and* $\{v, y\} \subset \mathbb{C}^r$*.*

We have $c_0(\sigma, \sigma_2) = 1$ *and* $i_0(\sigma_1, \sigma) = 1$*. Hence by Theorem 6.3.4,* $\begin{bmatrix} (e^T_{4-1} \otimes I_n)u \\ \hline v \end{bmatrix} =$

$\left[\begin{array}{c} (e_3^T \otimes I_n)u \\ \hline v \end{array}\right] \in \mathcal{N}_r(\mathcal{S}(\lambda))$ *and* $\left[\begin{array}{c} (e_3^T \otimes I_n)x \\ \hline y \end{array}\right] \in \mathcal{N}_l(\mathcal{S}(\lambda))$. *To verify the recovery rule, define* $u_i := (e_i^T \otimes I_n)u$ *and* $x_i := (e_i^T \otimes I_n)x$, *for* $i = 1:4$, *and consider* $\mathbb{L}(\lambda) \left[\begin{array}{c} u \\ \hline v \end{array}\right] = 0$. *This gives*

$$(\lambda A_4 + A_3)u_1 + A_2u_2 + A_1u_3 - u_4 = 0 \tag{6.8}$$

$$A_2u_1 + (-\lambda A_2 - I_n)u_2 - \lambda A_1u_3 + \lambda u_4 = 0 \tag{6.9}$$

$$A_1u_1 + \lambda u_2 + A_0u_3 + Cv = 0 \tag{6.10}$$

$$-u_1 + \lambda u_3 = 0 \tag{6.11}$$

$$Bu_3 + (A - \lambda E)v = 0 \tag{6.12}$$

From (6.11) we have $u_1 = \lambda u_3$. *Now adding* λ *times (6.8) with (6.9) we have* $(\lambda^2 A_4 + \lambda A_3 + A_2)u_1 - u_2 = 0 \Rightarrow u_2 = (\lambda^2 A_4 + \lambda A_3 + A_2)u_1 = (\lambda^3 A_4 + \lambda^2 A_3 + \lambda A_2)u_3$. *Substituting the values of* u_1 *and* u_2 *in (6.10), we have* $(\lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0)u_3 + Cv = 0 \Rightarrow P(\lambda)u_3 + Cv = 0$. *Hence by (6.12), we have* $\mathcal{S}(\lambda) \left[\begin{array}{c} u_3 \\ \hline v \end{array}\right] = 0$, *i.e.,* $\left[\begin{array}{c} u_3 \\ \hline v \end{array}\right] \in \mathcal{N}_r(\mathcal{S}(\lambda))$. *Similarly, we can verify that* $\left[\begin{array}{c} x_3 \\ \hline y \end{array}\right] \in \mathcal{N}_l(\mathcal{S}(\lambda))$. *It is easily seen that* $G(\lambda)u_3 = 0$ *and* $x_3^T G(\lambda) = 0$. ■

7

Structured Strong Linearizations of Structured Rational Matrices

Structured rational matrices such as symmetric, skew-symmetric, Hamiltonian, skew-Hamiltonian, Hermitian, skew-Hermitian, para-Hermitian and para-skew-Hermitian rational matrices arise in many applications. For structured rational matrices, it is desirable to construct structure-preserving linearizations so as to preserve the symmetry in the eigenvalues and poles of the rational matrices. The primary aim of this chapter is to construct structure-preserving Rosenbrock strong linearizations of structured (symmetric, skew-symmetric, Hamiltonian, skew-Hamiltonian, Hermitian, skew-Hermitian, para-Hermitian and para-skew-Hermitian) rational matrices. For this purpose, we propose an infinite family of Fiedler-like pencils (which we refer to as generalized Fiedler pencils with repetition (GFPRs)) and show that the family of GFPRs is a rich source of structure-preserving strong linearizations of structured rational matrices. We construct symmetric, skew-symmetric, Hamiltonian, skew-Hamiltonian, Hermitian, skew-Hermitian, para-Hermitian and para-skew-Hermitian strong linearizations of a rational matrix $G(\lambda)$ when $G(\lambda)$ has the same structure. Further, when $G(\lambda)$ is real and symmetric, we show that the real symmetric linearizations of $G(\lambda)$ preserve the Cauchy-Maslov index of $G(\lambda)$. We describe the recovery of eigenvectors, minimal bases and minimal indices of $G(\lambda)$ from those of the linearizations of $G(\lambda)$ and show that the recovery is operation-free. We also show that FPs, GFPs and FPRs of $G(\lambda)$ constructed in Chapter 6 are Rosenbrock strong linearizations of $G(\lambda)$.

7.1 Introduction

Structured rational matrices such as symmetric, skew-symmetric, Hamiltonian, skew-Hamiltonian, Hermitian, skew-Hermitian, para-Hermitian and para-skew-Hermitian rational matrices arise in many applications, see [38, 41, 36, 32, 49, 55, 64, 35] and the references therein. For example, the Hermitian rational eigenvalue problem

$$G(\lambda)u := \Big(\lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1+\lambda b_i}\Delta K_i\Big)u = 0$$

arises in the study of damped vibration of a structure, where M and K are positive definite, b_i is a relaxation parameter and ΔK_i is an assemblage of element stiffness matrices [49, 55]. Also various structured rational matrices arise as transfer functions of linear time-invariant (LTI) systems, see [38, 41, 36, 32, 50, 64]. For structured rational matrices, it is desirable to construct structure-preserving linearizations so as to preserve the symmetry in the eigenvalues and poles of the rational matrices.

Our main aim in this chapter is to construct structure-preserving strong linearizations of structured rational matrices and to recover eigenvectors, minimal bases and minimal indices of rational matrices from those of the linearizations. We consider the following structures:

$$\begin{array}{ll|ll}
\text{symmetric}: & G(\lambda)^T = G(\lambda) & \text{Hermitian}: & G(\lambda)^* = G(\bar{\lambda}) \\
\text{skew-symmetric}: & G(\lambda)^T = -G(\lambda) & \text{skew-Hermitian}: & G(\lambda)^* = -G(\bar{\lambda}) \\
\text{Hamiltonian}: & G(\lambda)^T = G(-\lambda) & \text{para-Hermitian}: & G(\lambda)^* = G(-\bar{\lambda}) \\
\text{skew-Hamiltonian}: & G(\lambda)^T = -G(-\lambda) & \text{para-skew-Hermitian}: & G(\lambda)^* = -G(-\bar{\lambda}),
\end{array} \tag{7.1}$$

where X^T (resp., X^*) denotes the transpose (resp., conjugate transpose) of a matrix X and $\bar{\lambda}$ denotes the conjugate of λ. For more on these structured rational matrices, we refer to [38, 41, 36, 49, 50, 64, 32, 55] and the references therein.

The Cauchy-Maslov index [23] (also known as the matrix Cauchy index [10]) of a real symmetric rational matrix plays an important role in applications such as in networks of linear systems, see [22, 39] and the references therein. If $G(\lambda)$ is real symmetric then the Cauchy-Maslov index of $G(\lambda)$ is defined as $\mathbf{Ind}_{\text{CM}}(G) :=$ (# eigenvalues of $G(\lambda)$ which jump from $-\infty$ to $+\infty$) $-$ (# eigenvalues of $G(\lambda)$ which jump from $+\infty$ to $-\infty$) as the real parameter λ traverses from $-\infty$ to $+\infty$, see [10]. It is therefore desirable to construct real symmetric linearizations of $G(\lambda)$ that preserve the Cauchy-Maslov index of $G(\lambda)$.

We mention that there is a slight difference in the naming convention between some of the structured rational matrices and structured matrix polynomials. The Hamiltonian (resp., skew-Hamiltonian) structure for rational matrices is known as T-even (resp., T-odd) structure for matrix polynomials [45]. On the other hand, para-Hermitian (resp., para-skew-Hermitian) structure for rational matrices is known as $*$-even (rep., $*$-odd) structure for matrix polynomials [45]. We follow both the naming conventions in the rest of the chapter without any bias.

7.2 GFPRs of rational matrices

We now introduce a new family of Fiedler-like pencils for rational matrices which we refer to as generalized Fiedler pencils with repetition (GFPRs). We proceed as follows.

For the rest of this section, we assume that $P(\lambda) := \sum_{i=0}^{m} \lambda^i A_i$ with $A_m \neq 0$ and the realization $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ of $G(\lambda)$ given by (5.5) is minimal. The system matrix

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right] \tag{7.2}$$

given by (5.6). Recall from Section 1.2.2 that the Fiedler matrices $M^P_{\pm i}$, $i \in \{-m : m-1\}$, and the elementary matrices $M_{\pm i}(X)$, $i \in \{-m : m-1\}$, associated with $P(\lambda)$ are given by

$$M_0(X) := \begin{bmatrix} I_{(m-1)n} & \\ & X \end{bmatrix}, \; M_i(X) := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & X & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ for } i = 1 : m-1,$$

$$M_{-m}(X) := \begin{bmatrix} X & \\ & I_{(m-1)n} \end{bmatrix}, \; M_{-i}(X) := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & X & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ for } i = 1 : m-1,$$

$$\text{and } M_i^P := \begin{cases} M_i(-A_i) & \text{if } i \geq 0, \\ M_i(A_{-i}) & \text{if } i < 0. \end{cases}$$

Recall that $M_i(X)M_j(Y) = M_j(Y)M_i(X)$ holds for any matrices $X, Y \in \mathbb{C}^{n\times n}$ if $||i| - |j|| > 1$, see [17].

For an arbitrary matrix $X \in \mathbb{C}^{n\times n}$, we define $(mn+r)\times(mn+r)$ elementary matrices $\mathbb{M}_i(X)$ by

$$\mathbb{M}_i(X) := \left[\begin{array}{c|c} M_i(X) & \\ \hline & I_r \end{array}\right] \quad \text{for } \ i \in \{-m : m-1\}.$$

Note that $\mathbb{M}_i(X)$ and $\mathbb{M}_{-i}(X)$ are invertible and $(\mathbb{M}_i(X))^{-1} = \mathbb{M}_{-i}(-X)$ for $i = 1 : m-1$. On the other hand, the matrices $\mathbb{M}_0(X)$ and $\mathbb{M}_{-m}(X)$ are invertible if and only if X is invertible. For any arbitrary matrices $X, Y \in \mathbb{C}^{n\times n}$, we have $\mathbb{M}_i(X)\mathbb{M}_j(Y) = \mathbb{M}_j(Y)\mathbb{M}_i(X)$ if $||i| - |j|| > 1$.

Recall form Chapter 6.1 that the Fiedler matrices $\mathbb{M}^{\mathcal{S}}_{\pm 0}, \ldots, \mathbb{M}^{\mathcal{S}}_{\pm m}$ associated with $\mathcal{S}(\lambda)$ are given by:

$$\mathbb{M}^{\mathcal{S}}_0 := \left[\begin{array}{c|c} M^P_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right], \ \mathbb{M}^{\mathcal{S}}_{-m} := \left[\begin{array}{c|c} M^P_{-m} & 0 \\ \hline 0 & -E \end{array}\right], \ \mathbb{M}^{\mathcal{S}}_i := \left[\begin{array}{c|c} M^P_i & 0 \\ \hline 0 & I_r \end{array}\right], \ i = 1 : m-1,$$

and $\mathbb{M}^{\mathcal{S}}_{-i} := (\mathbb{M}^{\mathcal{S}}_i)^{-1}$ for $i = 1 : m-1$. The matrices $\mathbb{M}^{\mathcal{S}}_i$ are also referred to as Fiedler matrices of $G(\lambda)$. We have $\mathbb{M}^{\mathcal{S}}_i\mathbb{M}^{\mathcal{S}}_j = \mathbb{M}^{\mathcal{S}}_j\mathbb{M}^{\mathcal{S}}_i$ for $||i| - |j|| > 1$, except for $||i| - |j|| = m$. For convenience in defining Fiedler-like pencils, we define

$$\mathbb{M}^P_i := \left[\begin{array}{c|c} M^P_i & \\ \hline & I_r \end{array}\right] \quad \text{for } i \in \{-m : m-1\}. \tag{7.3}$$

Remark 7.2.1. *Note that* $\mathbb{M}^{\mathcal{S}}_i = \mathbb{M}^P_i$, *for* $i = \pm 1, \ldots, \pm(m-1)$, *and* $\mathbb{M}^{\mathcal{S}}_0 \neq \mathbb{M}^P_0$ *and* $\mathbb{M}^{\mathcal{S}}_{-m} \neq \mathbb{M}^P_{-m}$. *The utility of the notation* $\mathbb{M}^P_i$ *will be clear when we analyze Fiedler-like pencils.*

Let $\mathbf{t} := (t_1, t_2, \ldots, t_k)$ be an index tuple containing indices from $\{-m : m-1\}$ and $X := (X_1, X_2, \ldots, X_k)$ be a tuple of $n \times n$ matrices. Like before, we define $\mathbb{M}_{\mathbf{t}}(X) := \mathbb{M}_{t_1}(X_1)\mathbb{M}_{t_2}(X_2)\cdots\mathbb{M}_{t_k}(X_k)$, $\mathbb{M}^{\mathcal{S}}_{\mathbf{t}} := \mathbb{M}^{\mathcal{S}}_{t_1}\mathbb{M}^{\mathcal{S}}_{t_2}\cdots\mathbb{M}^{\mathcal{S}}_{t_k}$, and $\mathbb{M}^P_{\mathbf{t}} := \mathbb{M}^P_{t_1}\mathbb{M}^P_{t_2}\cdots\mathbb{M}^P_{t_k}$.

Definition 7.2.2 (GFPRs). *Let* $0 \leq h \leq m-1$, *and let* σ *and* τ *be permutations of* $\{0 : h\}$ *and* $\{-m : -h-1\}$, *respectively. Let* σ_1 *and* σ_2 *be index tuples containing indices from* $\{0 : h-1\}$ *such that* $(\sigma_1, \sigma, \sigma_2)$ *satisfies the SIP. Similarly, let* τ_1 *and* τ_2 *be index tuples containing indices from* $\{-m : -h-2\}$ *such that* (τ_1, τ, τ_2) *satisfies the SIP. Let* X_1, X_2, Y_1 *and* Y_2 *be any arbitrary matrix assignments for* $\sigma_1, \sigma_2, \tau_1$ *and* τ_2, *respectively. Then the pencil*

$$\mathbb{L}(\lambda) := \mathbb{M}_{\tau_1}(Y_1)\,\mathbb{M}_{\sigma_1}(X_1)\,(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\,\mathbb{M}_{\sigma_2}(X_2)\,\mathbb{M}_{\tau_2}(Y_2) \tag{7.4}$$

is said to be a generalized Fiedler pencil with repetition (GFPR) of $G(\lambda)$. *We also refer to* $\mathbb{L}(\lambda)$ *as a GFPR of* $\mathcal{S}(\lambda)$.

Note that if all the matrix assignments X_1, X_2, Y_1 and Y_2 in Definition 7.2.2 are the trivial matrix assignments then $\mathbb{L}(\lambda) = \mathbb{M}^P_{\tau_1}\mathbb{M}^P_{\sigma_1}(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}^P_{\sigma_2}\mathbb{M}^P_{\tau_2}$ is an FPR of $G(\lambda)$ as defined in Chapter 6, also see [8]. Hence the family of FPRs of $G(\lambda)$ is a subclass of the family of GFPRs of $G(\lambda)$.

Example 7.2.3. *Let* $G(\lambda) := \sum_{i=0}^{4} \lambda^i A_i + C(\lambda E - A)^{-1}B$. *Consider* $\sigma := (1, 2, 3, 0)$, $\tau := (-4)$, $\sigma_2 := (2, 1)$ *and* $\sigma_1 = \tau_1 = \tau_2 = \emptyset$. *Then*

$$\left(\lambda\mathbb{M}^{\mathcal{S}}_{-4} - \mathbb{M}^{\mathcal{S}}_{(1,2,3,0)}\right)\mathbb{M}_{(2,1)}(X, Y) = \left[\begin{array}{cccc|c} \lambda A_4 + A_3 & -X & -Y & -I_n & 0 \\ A_2 & \lambda X - I_n & \lambda Y & \lambda I_n & 0 \\ A_1 & \lambda I_n & A_0 & 0 & C \\ -I_n & 0 & \lambda I_n & 0 & 0 \\ \hline 0 & 0 & B & 0 & A - \lambda E \end{array}\right]$$

is a GFPR of $G(\lambda)$, *where* (X, Y) *is an arbitrary matrix assignment for* σ_2.

Recall from Chapter 2 that $L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P_{\tau} - M^P_{\sigma})M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ is called a GFPR of $P(\lambda)$, where σ, τ, σ_j and τ_j, $j = 1, 2$, are as given in Definition 7.2.2. In particular, if X_1, X_2, Y_1 and Y_2 are the trivial matrix assignments then $L(\lambda) := M^P_{\tau_1}M^P_{\sigma_1}(\lambda M^P_{\tau} - M^P_{\sigma})M^P_{\sigma_2}M^P_{\tau_2}$ is called an FPR of $P(\lambda)$.

We now show that a GFPR of $G(\lambda)$ can be constructed directly from a GFPR of $P(\lambda)$ without performing any arithmetic operations. Recall from Definition 1.2.20 that $c_s(\alpha)$ (resp., $i_t(\alpha)$) denotes the number of consecutions (resp., inversions) of α at s (resp., t).

Theorem 7.2.4. *Let* $\mathbb{L}(\lambda) := \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1)\,(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\,\mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2)$ *and* $L(\lambda) := M_{(\tau_1,\sigma_1)}(Y_1, X_1)\,(\lambda M^P_{\tau} - M^P_{\sigma})\,M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ *be GFPRs of* $G(\lambda)$ *and* $P(\lambda)$, *respectively. Then*

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & e_{m-i_0(\sigma_1,\sigma)} \otimes C \\ \hline e^T_{m-c_0(\sigma,\sigma_2)} \otimes B & A - \lambda E \end{array}\right].$$

Thus, the map $\mathrm{GFPR}(P) \to \mathrm{GFPR}(G)$, $L(\lambda) \mapsto \left[\begin{array}{c|c} L(\lambda) & e_{m-i_0(\sigma_1,\sigma)} \otimes C \\ \hline e^T_{m-c_0(\sigma,\sigma_2)} \otimes B & A - \lambda E \end{array}\right]$ *is a bijection, where GFPR(P) and GFPR(G) denote the set of GFPRs of* $P(\lambda)$ *and* $G(\lambda)$, *respectively.*

Proof. Let σ be given by $\sigma \sim (\delta_1, 0, \delta_2)$. Then we have

$$\begin{aligned} \mathbb{L}(\lambda) &= \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1)\,\left(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma}\right)\,\mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2) \\ &= \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1)\,\left(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\delta_1}\mathbb{M}^{\mathcal{S}}_{0}\mathbb{M}^{\mathcal{S}}_{\delta_2}\right)\,\mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2) \end{aligned}$$

$$= \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1,X_1)\left(\lambda \mathbb{M}^{\mathcal{S}}_{\tau} - \left[\begin{array}{c|c} M^P_{\delta_1} & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} M^P_0 & -e_m \otimes C \\ \hline -e^T_m \otimes B & -A \end{array}\right]\left[\begin{array}{c|c} M^P_{\delta_2} & 0 \\ \hline 0 & I_r \end{array}\right]\right)\mathbb{M}_{(\sigma_2,\tau_2)}(X_2,Y_2)$$

$$= \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1,X_1)\left(\lambda \left[\begin{array}{c|c} M^P_{\tau} & 0 \\ \hline 0 & -E \end{array}\right] - \left[\begin{array}{c|c} M^P_{\delta_1}M^P_0 M^P_{\delta_2} & M^P_{\delta_1}(-e_m \otimes C) \\ \hline (-e^T_m \otimes B)M^P_{\delta_2} & -A \end{array}\right]\right)\mathbb{M}_{(\sigma_2,\tau_2)}(X_2,Y_2)$$

$$= \left[\begin{array}{c|c} M_{(\tau_1,\sigma_1)}(Y_1,X_1) & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} \lambda M^P_{\tau} - M^P_{\sigma} & M^P_{\delta_1}(e_m \otimes C) \\ \hline (e^T_m \otimes B)M^P_{\delta_2} & A - \lambda E \end{array}\right]\left[\begin{array}{c|c} M_{(\sigma_2,\tau_2)}(X_2,Y_2) & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} L(\lambda) & M_{(\tau_1,\sigma_1)}(Y_1,X_1)M^P_{\delta_1}(e_m \otimes C) \\ \hline (e^T_m \otimes B)M^P_{\delta_2}M_{(\sigma_2,\tau_2)}(X_2,Y_2) & A - \lambda E \end{array}\right]. \tag{7.5}$$

Hence we only need to show that $(e^T_m \otimes I_n)M^P_{\delta_2}M_{(\sigma_2,\tau_2)}(X_2,Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$ and $M_{(\tau_1,\sigma_1)}(Y_1,X_1)M^P_{\delta_1}(e_m \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$. Let Z be an $n \times n$ arbitrary matrix. Recall that from (2.2) and (2.3), we have

$$M_i(Z)(e_{m-j} \otimes I_n) = \begin{cases} e_{m-(j+1)} \otimes I_n & \text{for } i = j+1 \text{ and } j = 0 : m-2, \\ e_{m-j} \otimes I_n & \text{for } i \notin \{j, j+1\} \text{ and } j = 0 : m-1, \end{cases} \tag{7.6}$$

and

$$(e^T_{m-j} \otimes I_n)M_i(Z) = \begin{cases} e^T_{m-(j+1)} \otimes I_n & \text{for } i = j+1 \text{ and } j = 0 : m-2, \\ e^T_{m-j} \otimes I_n & \text{for } i \notin \{j, j+1\} \text{ and } j = 0 : m-1. \end{cases} \tag{7.7}$$

Case-I: Suppose that $c_0(\sigma) > 0$. Then $i_0(\sigma) = 0$ since σ is a permutation. Since $\sigma \sim (\delta_1, 0, \delta_2)$, we have $\{1, 2, \ldots, c_0(\sigma)\} \subset \delta_2$ and if $c_0(\sigma)+1 \in \delta_2$ then δ_2 has an inversion at $c_0(\sigma)$. Thus by the commutative property of $\mathbb{M}^{\mathcal{S}}_j$ and by moving $\mathbb{M}^{\mathcal{S}}_j$ to $\mathbb{M}^{\mathcal{S}}_{\delta_1}$, if necessary, we can assume that $\delta_2 = (1, 2, \ldots, c_0(\sigma))$. In other words, $\sigma \sim (\delta_1, 0, 1, 2, \ldots, c_0(\sigma))$. Hence, if $j \in \delta_1$ then $j \geq c_0(\sigma) + 1$. Consequently, we have $M^P_{\delta_1} = \left[\begin{array}{c|c} * & \\ \hline & I_{c_0(\sigma)n} \end{array}\right]$. This shows that $M^P_{\delta_1}(e_m \otimes I_n) = e_m \otimes I_n$. Since $\delta_2 = (1, 2, \ldots, c_0(\sigma))$, by applying (7.7) repeatedly we have $(e^T_m \otimes I_n)M^P_{\delta_2} = e^T_{m-c_0(\sigma)} \otimes I_n$.

Case-II: Suppose that $i_0(\sigma) > 0$. Then $c_0(\sigma) = 0$ since σ is a permutation. Further, Since $\sigma \sim (\delta_1, 0, \delta_2)$, we have $\{i_0(\sigma), \ldots, 2, 1\} \subset \delta_1$ and if $i_0(\sigma) + 1 \in \delta_1$ then δ_1 has a consecution at $i_0(\sigma)$. By the commutative property of $\mathbb{M}^{\mathcal{S}}_j$ and by moving $\mathbb{M}^{\mathcal{S}}_j$ to $\mathbb{M}^{\mathcal{S}}_{\delta_2}$, if necessary, we can assume that $\delta_1 = (i_0(\sigma), \ldots, 2, 1)$. In other words, $\sigma \sim (i_0(\sigma), \ldots, 2, 1, 0, \delta_2)$. Thus, if $j \in \delta_2$ then $j \geq i_0(\sigma) + 1$. Consequently, we have $M^P_{\delta_2} = \left[\begin{array}{c|c} * & \\ \hline & I_{i_0(\sigma)n} \end{array}\right]$. This shows that $(e^T_m \otimes I_n)M^P_{\delta_2} = e^T_m \otimes I_n$. As $\delta_1 = (i_0(\sigma), \ldots, 2, 1)$, by applying (7.6) repeatedly we have $M^P_{\delta_1}(e_m \otimes I_n) = e_{m-i_0(\sigma)} \otimes I_n$.

Case-III: Suppose that $c_0(\sigma) = 0$ and $i_0(\sigma) = 0$. Then we must have $1 \notin \sigma$, i.e., $\sigma = (0)$. Hence $\delta_1 = \emptyset = \delta_2$ and $M^P_{\delta_1} = I_{mn} = M^P_{\delta_2}$. Hence $M^P_{\delta_1}(e_m \otimes I_n) = e_m \otimes I_n$ and $(e^T_m \otimes I_n)M^P_{\delta_2} = e^T_m \otimes I_n$.

Hence $M^P_{\delta_1}(e_m \otimes I_n) = e_{m-i_0(\sigma)} \otimes I_n$ and $(e^T_m \otimes I_n)M^P_{\delta_2} = e^T_{m-c_0(\sigma)} \otimes I_n$ in all the above cases. Now, by Lemma 2.1.11, we have $(e^T_m \otimes I_n)M^P_{\delta_2}M_{(\sigma_2,\tau_2)}(X_2, Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n$ and $M_{(\tau_1,\sigma_1)}(Y_1, X_1)M^P_{\delta_1}(e_m \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$. Hence the desired form of $\mathbb{L}(\lambda)$ follows from (7.5). ☐

7.2.1 Fiedler-like pencils are Rosenbrock strong linearizations

We now show that Fiedler pencils (FPs), generalized Fiedler pencils (GFPs) and GFPRs of $G(\lambda)$ are Rosenbrock strong linearizations of $G(\lambda)$. First, we show that the FPs of $G(\lambda)$ introduced in [2] (see Chapter 6) are Rosenbrock strong linearizations of $G(\lambda)$.

Recall that $\mathbb{L}_\sigma(\lambda) := \lambda \mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_{\sigma}$ is called a Fiedler pencil (FP) of $G(\lambda)$, where σ is a permutation of $\{0 : m-1\}$. We now define the reverse consecution-inversion structure sequence of a permutation which we need in order to prove that a Fielder pencil is a Rosenbrock strong linearization of $G(\lambda)$.

Definition 7.2.5. *Let α be a permutation of $\{0 : m-1\}$. Then the tuple RCISS$(\alpha) := (c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$ is called the reverse consecution-inversion structure sequence of α when α has consecutions at $m - c_1 - 1, m - c_1, \ldots, m-2$; inversions at $m - c_1 - i_1 - 1, m - c_1 - i_1, \ldots, m - c_1 - 2$ and so on, consecutions at $i_\ell, i_\ell + 1, \ldots, i_\ell + c_\ell - 1$; inversions at $0, 1, \ldots, i_\ell - 1$.*

Remark 7.2.6. *It is easy to see that RCISS$(\alpha) =$ rev$\big($CISS$($rev$(\alpha))\big)$, where CISS(α) is the consecution-inversion structure sequence of α defined in Definition 6.1.6 (also, see [26]).*

Example 7.2.7. *Let $m = 11$, and let α and β be permutations of $\{0 : 10\}$ given by $\alpha = (8 : 10, 7, 6, 5, 2 : 4, 1, 0)$ and $\beta = (10, 9, 5 : 8, 3 : 4, 2, 0 : 1)$. Then RCISS$(\alpha) = (2, 4, 2, 2)$ since α has consecutions at $8, 9$; inversions at $4, 5, 6, 7$; consecutions at $2, 3$; inversions at $0, 1$. Similarly, we have RCISS$(\beta) = (0, 2, 3, 1, 1, 2, 1, 0)$.*

Let α be a permutation of $\{0 : m-1\}$ with RCISS$(\alpha) = (c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$. We define

$$m_0 := 0, \; n_0 := 0, \text{ and } m_p := \sum\nolimits_{j=1}^{p} c_j \text{ and } n_p := \sum\nolimits_{j=1}^{p} i_j \text{ for } p = 1 : \ell. \tag{7.8}$$

Observe that $m_\ell = c(\sigma)$ and $n_\ell = i(\sigma)$, that is, m_ℓ is the total number of consecutions of α and n_ℓ is the total number of inversions of α. Thus $m_\ell + n_\ell = m - 1$ (follows from

Remark 1.2.25). Further, we define

$$s_0 := 0 \text{ and } s_p := \sum\nolimits_{j=1}^{p} (c_j + i_j) \text{ for } p = 1 : \ell. \tag{7.9}$$

Observe that $s_\ell = m_\ell + n_\ell = m - 1$.

For $i \geq 0$ and $j \geq 0$, we define $\widehat{\Lambda}_{i,j}(\lambda)$, $\Lambda_{i,j}(\lambda)$, $\widehat{\Omega}_{i,j}(\lambda)$, and $\Omega_{i,j}(\lambda)$ as follows:

$$\widehat{\Lambda}_{i,j}(\lambda) := \begin{bmatrix} I_n \\ \lambda I_n \\ \lambda^2 I_n \\ \vdots \\ \lambda^{i-1} I_n \\ 0_{jn\times n} \end{bmatrix} \in \mathbb{C}[\lambda]^{(i+j)n\times n}, \ \Lambda_{i,j}(\lambda) := \begin{bmatrix} I_n \\ \lambda I_n \\ \lambda^2 I_n \\ \vdots \\ \lambda^{i-1} I_n \\ 0_{jn\times n} \\ \lambda^i I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(i+1+j)n\times n}, \tag{7.10}$$

$$\widehat{\Omega}_{i,j}(\lambda) := \begin{bmatrix} 0_{in\times n} \\ I_n \\ \lambda I_n \\ \lambda^2 I_n \\ \vdots \\ \lambda^{j-1} I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(i+j)n\times n} \text{ and } \Omega_{i,j}(\lambda) := \begin{bmatrix} 0_{in\times n} \\ I_n \\ \lambda I_n \\ \lambda^2 I_n \\ \vdots \\ \lambda^{j-1} I_n \\ \lambda^j I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(i+j+1)n\times n}. \tag{7.11}$$

Note that $\Lambda_{i,j}(\lambda) = \begin{bmatrix} \widehat{\Lambda}_{i,j}(\lambda) \\ \lambda^i I_n \end{bmatrix}$ and $\Omega_{i,j}(\lambda) = \begin{bmatrix} \widehat{\Omega}_{i,j}(\lambda) \\ \lambda^j I_n \end{bmatrix}$. Further, $\Lambda_{0,j}(\lambda) = \begin{bmatrix} 0_{jn\times n} \\ I_n \end{bmatrix}$ and $\Omega_{i,0}(\lambda) = \begin{bmatrix} 0_{in\times n} \\ I_n \end{bmatrix}$. For simplicity, we write $\widehat{\Lambda}_{i,j}$, $\Lambda_{i,j}$, $\widehat{\Omega}_{i,j}$ and $\Omega_{i,j}$ for $\widehat{\Lambda}_{i,j}(\lambda)$, $\Lambda_{i,j}(\lambda)$, $\widehat{\Omega}_{i,j}(\lambda)$ and $\Omega_{i,j}(\lambda)$, respectively.

Remark 7.2.8. *It follows from (7.10) and (7.11) that* $(e_k^T \otimes I_n)\widehat{\Lambda}_{i,j}(\lambda) = 0 \iff \widehat{\Omega}_{i,j}(\lambda)^{\mathcal{B}}(e_k \otimes I_n) \neq 0$ *for any* $i \geq 0$, $j \geq 0$ *and* $1 \leq k \leq i + j$.

Let $\mathcal{H} := (\mathcal{H}_{ij})$ be a block $k \times \ell$ matrix, where each block $\mathcal{H}_{ij}$ is a $p \times q$ matrix. Recall that the block transpose of $\mathcal{H}$ is the block $\ell \times k$ matrix $\mathcal{H}^{\mathcal{B}}$ given by $(\mathcal{H}^{\mathcal{B}})_{ij} = \mathcal{H}_{ji}$, see [26].

Definition 7.2.9. *Let* α *be a permutation of* $\{0, 1, \ldots, m-1\}$ *with RCISS*$(\alpha) = (c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$*. We define* $\Lambda_\alpha(\lambda) \in \mathbb{C}[\lambda]^{mn\times n}$ *and* $\Omega_\alpha(\lambda) \in \mathbb{C}[\lambda]^{n\times mn}$ *as follows:*

$$\Lambda_\alpha(\lambda) := \begin{bmatrix} \widehat{\Lambda}_{c_1,i_1}(\lambda) \\ \lambda^{m_1}\widehat{\Lambda}_{c_2,i_2}(\lambda) \\ \vdots \\ \lambda^{m_{\ell-2}}\widehat{\Lambda}_{c_{\ell-1},i_{\ell-1}}(\lambda) \\ \lambda^{m_{\ell-1}}\Lambda_{c_\ell,i_\ell}(\lambda) \end{bmatrix} \text{ if } \ell > 1, \tag{7.12}$$

and $\Lambda_\alpha(\lambda) := \Lambda_{c_1,i_1}(\lambda)$ *if* $\ell = 1$,

$$\Omega_\alpha(\lambda) := \begin{bmatrix} \widehat{\Omega}_{c_1,i_1}(\lambda) \\ \lambda^{n_1}\widehat{\Omega}_{c_2,i_2}(\lambda) \\ \vdots \\ \lambda^{n_{\ell-2}}\widehat{\Omega}_{c_{\ell-1},i_{\ell-1}}(\lambda) \\ \lambda^{n_{\ell-1}}\Omega_{c_\ell,i_\ell}(\lambda) \end{bmatrix}^{\mathcal{B}} \text{ if } \ell > 1, \tag{7.13}$$

and $\Omega_\alpha(\lambda) := \left(\Omega_{c_1,i_1}(\lambda)\right)^{\mathcal{B}}$ *if* $\ell = 1$.

Remark 7.2.10. *Let* α *be a permutation of* $\{0 : m-1\}$ *with RCISS*$(\alpha) = (c_1, i_1, \ldots, c_\ell, i_\ell)$*. Since* $\widehat{\Lambda}_{c_j,i_j}(\lambda)$ *and* $\widehat{\Omega}_{c_j,i_j}(\lambda)$ *are the basic building blocks of* $\Lambda_\alpha(\lambda)$ *and* $\Omega_\alpha(\lambda)$*, respectively, it follows from Remark 7.2.8 that* $(e_k^T \otimes I_n)\Lambda_\alpha(\lambda) = 0 \iff \Omega_\alpha(\lambda)(e_k \otimes I_n) \neq 0$ *for any* $k \in \{1 : m-1\}$*. Further, note that* $(e_m^T \otimes I_n)\Lambda_\alpha(\lambda) = \lambda^{m_\ell} I_n$ *and* $\Omega_\alpha(\lambda)(e_m \otimes I_n) = \lambda^{n_\ell} I_n$*.*

Caution: The same notations $m_j, n_j, s_j, \Lambda_\alpha(\lambda)$ and $\widehat{\Lambda}_{i,j}(\lambda)$ are used in Chapter 6 for different purposes.

Recall from Chapter 6 that $P_k(\lambda) := A_{m-k} + \lambda A_{m-k+1} + \cdots + \lambda^k A_m$, $k = 0 : m$, is called the Horner shift of $P(\lambda)$ of degree k [26]. For $1 \leq i \leq m-1$, we consider the following $mn \times mn$ unimodular matrix polynomials [26]

$$Q_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & \lambda I_n & \\ & 0_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix}$$

and

$$R_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & 0_n & I_n & \\ & I_n & P_i(\lambda) & \\ & & & I_{(m-i-1)n} \end{bmatrix} = R_i^{\mathcal{B}}(\lambda).$$

Observe that $R_i(\lambda)$ depends on the Horner shifts of $P(\lambda)$ whereas $Q_i(\lambda)$ does not. For simplicity, we write Q_i and R_i for $Q_i(\lambda)$ and $R_i(\lambda)$, respectively.

The following results will be useful for proving that the FPs of $G(\lambda)$ are Rosenbrock strong linearizations of $G(\lambda)$.

Lemma 7.2.11. *Let $P(\lambda)$ be a matrix polynomial of degree m and α be a permutation of $\{0 : m-1\}$. Suppose that RCISS$(\alpha) = (c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$. For $j = 1 : \ell$, set*

$$U_{(c_j,i_j)} := R^{\mathcal{B}}_{s_{j-1}+c_j+i_j} \cdots R^{\mathcal{B}}_{s_{j-1}+c_j+1} Q^{\mathcal{B}}_{s_{j-1}+c_j} \cdots Q^{\mathcal{B}}_{s_{j-1}+1} \tag{7.14}$$

$$\text{and } V_{(c_j,i_j)} := R_{s_{j-1}+1} \cdots R_{s_{j-1}+c_j} Q_{s_{j-1}+c_j+1} \cdots Q_{s_{j-1}+c_j+i_j}. \tag{7.15}$$

Let $U(\lambda)$ and $V(\lambda)$ be given by $U(\lambda) := U_{(c_\ell,i_\ell)} U_{(c_{\ell-1},i_{\ell-1})} \cdots U_{(c_2,i_2)} U_{(c_1,i_1)}$ and $V(\lambda) := V_{(c_1,i_1)} V_{(c_2,i_2)} \cdots V_{(c_{\ell-1},i_{\ell-1})} V_{(c_\ell,i_\ell)}$. Then

$$U(\lambda)(e_1 \otimes I_n) = \Lambda_\alpha(\lambda) \text{ and } (e_1^T \otimes I_n)V(\lambda) = \Omega_\alpha(\lambda),$$

where $\Lambda_\alpha(\lambda)$ and $\Omega_\alpha(\lambda)$ are as given in Definition 7.2.9.

Proof. For $t \in \{1 : m-1\}$, we have the following:

$$Q_t^{\mathcal{B}}(e_i \otimes I_n) = \begin{cases} (e_i \otimes I_n) + (e_{i+1} \otimes \lambda I_n) & \text{if } t = i \text{ for } i = 1 : m-1, \\ e_i \otimes I_n & \text{if } t \neq i \text{ for } i = 1 : m, \end{cases} \tag{7.16}$$

$$R_t^{\mathcal{B}}(e_i \otimes I_n) = \begin{cases} e_{i+1} \otimes I_n & \text{if } t = i \text{ for } i = 1 : m-1, \\ e_i \otimes I_n & \text{if } t \notin \{i, i-1\} \text{ for } i = 1 : m. \end{cases} \tag{7.17}$$

Let $1 \leq k \leq m-1$ and $p \geq 0,\ q \geq 0$ be such that $k + p + q - 1 \leq m - 1$. Consider

$$Z(\lambda) := \underbrace{R^{\mathcal{B}}_{k+p+q-1} \cdots R^{\mathcal{B}}_{k+p+1} R^{\mathcal{B}}_{k+p}}_{Y(\lambda)} \underbrace{Q^{\mathcal{B}}_{k+p-1} \cdots Q^{\mathcal{B}}_{k+1} Q^{\mathcal{B}}_{k}}_{X(\lambda)}.$$

Then $X(\lambda)$ (resp., $Y(\lambda)$) is a product of p (resp., q) $Q^{\mathcal{B}}$'s (resp., $R^{\mathcal{B}}$'s). We show that

$$Z(\lambda)(e_k \otimes I_n) = \begin{bmatrix} 0_{(k-1)n \times n} \\ \widehat{\Lambda}_{p,q} \\ \lambda^p I_n \\ 0_{(m-k-p-q)n \times n} \end{bmatrix}. \tag{7.18}$$

We have

$$\begin{aligned}
X(\lambda)\,(e_k \otimes I_n) &= \left(Q^{\mathcal{B}}_{k+p-1}\cdots Q^{\mathcal{B}}_{k+1}\right)\left((e_{k+1}\otimes \lambda I_n) + (e_k \otimes I_n)\right) \text{ by (7.16)}\\
&= \left(Q^{\mathcal{B}}_{k+p-1}\cdots Q^{\mathcal{B}}_{k+2}\right)\left((e_{k+2}\otimes \lambda^2 I_n) + (e_{k+1}\otimes \lambda I_n) + (e_k \otimes I_n)\right) \text{ by (7.16)}\\
&= Q^{\mathcal{B}}_{k+p-1}\left((e_{k+p-1}\otimes \lambda^{p-1} I_n) + \sum\nolimits_{j=k}^{k+p-2}(e_j\otimes \lambda^{j-k}I_n)\right) \text{ by repeatedly applying (7.16)}\\
&= (e_{k+p}\otimes \lambda^p I_n) + (e_{k+p-1}\otimes \lambda^{p-1}I_n) + \sum\nolimits_{j=k}^{k+p-2}(e_j\otimes \lambda^{j-k}I_n) \text{ by (7.16)}\\
&= (e_{k+p}\otimes \lambda^p I_n) + \sum\nolimits_{j=k}^{k+p-1}(e_j\otimes \lambda^{j-k}I_n).
\end{aligned}$$

Then $Z(\lambda)(e_k \otimes I_n) =$

$$\begin{aligned}
&\left(R^{\mathcal{B}}_{k+p+q-1}\cdots R^{\mathcal{B}}_{k+p+1}R^{\mathcal{B}}_{k+p}\right)\left((e_{k+p}\otimes \lambda^p I_n) + \sum\nolimits_{j=k}^{k+p-1}(e_j\otimes \lambda^{j-k}I_n)\right)\\
&= (e_{k+p+q}\otimes \lambda^p I_n) + \sum\nolimits_{j=k}^{k+p-1}(e_j\otimes \lambda^{j-k}I_n) \text{ by applying (7.17) repeatedly}\\
&= \begin{bmatrix} 0_{(k-1)n\times n}\\ \widehat{\Lambda}_{p,q}\\ \lambda^p I_n\\ 0_{(m-k-p-q)n\times n}\end{bmatrix}, \text{ which proves (7.18).}
\end{aligned}$$

We now prove that $U(\lambda)(e_1\otimes I_n) = \Lambda_\alpha(\lambda)$. Recall the definitions of $\Lambda_\alpha(\lambda)$, m_j and s_j associated with $\mathrm{RCISS}(\alpha) = (c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$. If $\ell = 1$ then by (7.18) we have $U(\lambda)(e_1\otimes I_n) = U_{(c_1,i_1)}(e_1\otimes I_n) = \Lambda_\alpha(\lambda)$. Next, if $\ell > 1$ then by using (7.16), (7.17) and (7.18) repeatedly we have $U(\lambda)(e_1\otimes I_n) = \Lambda_\alpha(\lambda)$. Indeed, we have the following. Recall that $\widehat{\Lambda}_{c_j,i_j} \in \mathbb{C}[\lambda]^{(c_j+i_j)n\times n}$. We denote by $\mathbf{0}$ the zero matrix of an appropriate size. Then we have

$$\begin{aligned}
U(\lambda)(e_1\otimes I_n) &= U_{(c_\ell,i_\ell)}\cdots U_{(c_2,i_2)}U_{(c_1,i_1)}(e_1\otimes I_n)\\
&= U_{(c_\ell,i_\ell)}\cdots U_{(c_2,i_2)}\begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ \lambda^{c_1}I_n\\ \mathbf{0}\end{bmatrix} \text{ by (7.18) since } s_0 = 0\\
&= U_{(c_\ell,i_\ell)}\cdots U_{(c_2,i_2)}\left((e_{s_1+1}\otimes \lambda^{c_1}I_n) + \begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ 0\\ \mathbf{0}\end{bmatrix}\right) \text{ since } s_1 = c_1 + i_1
\end{aligned}$$

$$= U_{(c_\ell,i_\ell)}\cdots U_{(c_2,i_2)}(e_{s_1+1}\otimes \lambda^{c_1}I_n) + \begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ 0\\ 0\end{bmatrix} \quad \text{by } (7.14), (7.16) \text{ and } (7.17)$$

$$= U_{(c_\ell,i_\ell)}\cdots U_{(c_3,i_3)}\begin{bmatrix}0_{s_1n\times n}\\ \lambda^{c_1}\widehat{\Lambda}_{c_2,i_2}\\ \lambda^{c_1}\lambda^{c_2}I_n\\ 0\end{bmatrix} + \begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ 0\\ 0\end{bmatrix} \quad \text{by } (7.18)$$

$$= U_{(c_\ell,i_\ell)}\cdots U_{(c_3,i_3)}(e_{s_2+1}\otimes \lambda^{c_1+c_2}I_n) + \begin{bmatrix}0_{s_1n\times n}\\ \lambda^{c_1}\widehat{\Lambda}_{c_2,i_2}\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ 0\\ 0\end{bmatrix} \quad \begin{matrix}\text{by } (7.14), (7.16)\\ \text{and } (7.17)\end{matrix}$$

$$= U_{(c_\ell,i_\ell)}\cdots U_{(c_3,i_3)}(e_{s_2+1}\otimes \lambda^{m_2}I_n) + \begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ \lambda^{m_1}\widehat{\Lambda}_{c_2,i_2}\\ 0\\ 0\end{bmatrix} \quad \text{since } m_1 = c_1 \text{ and } m_2 = c_1 + c_2$$

$$= U_{(c_\ell,i_\ell)}(e_{s_{\ell-1}+1}\otimes \lambda^{m_{\ell-1}}I_n) + \begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ \lambda^{m_1}\widehat{\Lambda}_{c_2,i_2}\\ \vdots\\ \lambda^{m_{\ell-2}}\widehat{\Lambda}_{c_{\ell-1},i_{\ell-1}}\\ 0\end{bmatrix} \quad \begin{matrix}\text{by repeated application of}\\ (7.16), (7.17) \text{ and } (7.18)\end{matrix}$$

$$= \begin{bmatrix}0_{s_{\ell-1}n\times n}\\ \lambda^{m_{\ell-1}}\widehat{\Lambda}_{c_\ell,i_\ell}\\ \lambda^{m_{\ell-1}}\lambda^{c_\ell}I_n\end{bmatrix} + \begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ \lambda^{m_1}\widehat{\Lambda}_{c_2,i_2}\\ \vdots\\ \lambda^{m_{\ell-2}}\widehat{\Lambda}_{c_{\ell-1},i_{\ell-1}}\\ 0\end{bmatrix} = \begin{bmatrix}\widehat{\Lambda}_{c_1,i_1}\\ \lambda^{m_1}\widehat{\Lambda}_{c_2,i_2}\\ \vdots\\ \lambda^{m_{\ell-2}}\widehat{\Lambda}_{c_{\ell-1},i_{\ell-1}}\\ \lambda^{m_{\ell-1}}\Lambda_{c_\ell,i_\ell}\end{bmatrix} = \Lambda_\alpha(\lambda) \text{ by } (7.18).$$

This proves that $U(\lambda)(e_1\otimes I_n) = \Lambda_\alpha(\lambda)$.

Next we prove that $(e_1^T \otimes I_n)V(\lambda) = \Omega_\alpha(\lambda)$. For $t \in \{1 : m-1\}$, we have

$$(e_i^T \otimes I_n)Q_t = \begin{cases} (e_i^T \otimes I_n) + (e_{i+1}^T \otimes \lambda I_n) & \text{if } t = i \text{ for } i = 1 : m-1, \\ e_i^T \otimes I_n & \text{if } t \neq i \text{ for } i = 1 : m, \end{cases} \tag{7.19}$$

$$(e_i^T \otimes I_n)R_t = \begin{cases} e_{i+1}^T \otimes I_n & \text{if } t = i \text{ for } i = 1 : m-1, \\ e_i^T \otimes I_n & \text{if } t \notin \{i, i-1\} \text{ for } i = 1 : m. \end{cases} \tag{7.20}$$

Let $1 \leq k \leq m-1$. Let $p \geq 0$ and $q \geq 0$ be such that $k+p+q-1 \leq m-1$. Consider $W(\lambda) := R_k R_{k+1} \cdots R_{k+p-1} Q_{k+p} Q_{k+p+1} \cdots Q_{k+p+q-1}$. Then (7.19) and (7.20) and similar arguments as those in the proof of (7.18) give

$$(e_k^T \otimes I_n)W(\lambda) = \begin{bmatrix} 0_{(k-1)n \times n} \\ \widehat{\Omega}_{p,q} \\ \lambda^q I_n \\ 0_{(m-k-p-q)n \times n} \end{bmatrix}^{\mathcal{B}} \tag{7.21}$$

Hence by (7.21), (7.19), (7.20) and by similar arguments as those in the proof of $U(\lambda)(e_1 \otimes I_n) = \Lambda_\alpha(\lambda)$, we have $(e_1^T \otimes I_n)V(\lambda) = \Omega_\alpha(\lambda)$. □

Proposition 7.2.12. *Let* $X(\lambda) := [\mathbf{x}_1 \ \ \mathbf{x}_2 \ \ \cdots \ \ \mathbf{x}_m]^{\mathcal{B}}$ *and* $Y(\lambda) := [\mathbf{y}_1 \ \ \mathbf{y}_2 \ \ \cdots \ \ \mathbf{y}_m]$, *where* $\mathbf{x}_i = 0$ *or* $\mathbf{x}_i = \lambda^{p_i} I_n$, *and* $\mathbf{y}_i = 0$ *or* $\mathbf{y}_i = \lambda^{q_i} I_n$, *for some* $p_i \geq 0$ *and* $q_i \geq 0$, $i = 1 : m$. *Suppose that* $\mathbf{x}_i \mathbf{y}_i = 0$ *for* $i = 1 : m-1$. *Then there exist an* $m \times m$ *lower block-triangular matrix polynomial* $L(\lambda)$ *with diagonal blocks* I_n *and an* $m \times m$ *upper block-triangular matrix polynomial* $U(\lambda)$ *with diagonal blocks* I_n *such that*

$$L(\lambda)\left(\left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & 0 \end{array} \right] + X(\lambda)Y(\lambda) \right) U(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathbf{x}_m \mathbf{y}_m \end{array} \right]. \tag{7.22}$$

Proof. Define $Z := \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & 0 \end{array} \right] + X(\lambda)Y(\lambda)$. Let $Z_{i,j}$ be the (i,j)-th block entry of Z. Since $\mathbf{x}_i \mathbf{y}_i = 0$, we have $Z_{i,i} = I_n$ for $i = 1 : m-1$ and $Z_{m,m} = \mathbf{x}_m \mathbf{y}_m$. Further, note that we have either $Z_{i,j} = \lambda^{p_i+q_j} I_n$ or $Z_{i,j} = 0$ for all $i \neq j$. Hence we have $Z = \begin{bmatrix} I_n & Z_{1,2} & \cdots & Z_{1,m-1} & Z_{1,m} \\ Z_{2,1} & I_n & \cdots & Z_{2,m-1} & Z_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Z_{m-1,1} & Z_{m-1,2} & \cdots & I_n & Z_{m-1,m} \\ Z_{m,1} & Z_{m,2} & \cdots & Z_{m,m-1} & Z_{m,m} \end{bmatrix}$. Now define $L(\lambda)$ and $U(\lambda)$ by

$$L(\lambda) := \begin{bmatrix} I_n & & & \\ -Z_{2,1} & I_n & & \\ \vdots & \cdots & \ddots & \\ -Z_{m,1} & -Z_{m,2} & \cdots & I_n \end{bmatrix} \text{ and } U(\lambda) := \begin{bmatrix} I_n & -Z_{1,2} & \cdots & -Z_{1,m} \\ & I_n & & -Z_{2,m} \\ & & \ddots & \vdots \\ & & & I_n \end{bmatrix}.$$

Note that $\mathbf{x}_i\mathbf{y}_i = 0 \Rightarrow \mathbf{y}_i\mathbf{x}_i = 0$ for $i = 1 : m-1$. Hence it follows that $Z_{i,j}Z_{j,k} = \mathbf{x}_i\mathbf{y}_j\mathbf{x}_j\mathbf{y}_k = 0$ for $i, k \in \{1 : m\}$ and $j \in \{1 : m-1\}$. Consequently, by block Gaussian elimination, we have $L(\lambda)ZU(\lambda) = \operatorname{diag}(I_{(m-1)n},\ \mathbf{x}_m\mathbf{y}_m)$. ☐

As an immediate corollary we have the following result.

Corollary 7.2.13. *Let α be a permutation of $\{0, 1, \ldots, m-1\}$ with RCISS$(\alpha) = (c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$. Consider $\Lambda_\alpha(\lambda)$ and $\Omega_\alpha(\lambda)$ associated with RCISS(α) as given in Definition 7.2.9. Then there exist an $m \times m$ lower block-triangular matrix polynomial $T_1(\lambda)$ with diagonal blocks I_n and an $m \times m$ upper block-triangular matrix polynomial $T_2(\lambda)$ with diagonal blocks I_n such that*

$$T_1(\lambda)\left(\left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & 0 \end{array}\right] + \Lambda_\alpha(\lambda)\,\Omega_\alpha(\lambda)\right) T_2(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{m-1}I_n \end{array}\right].$$

Proof. Note that we have $(e_m^T \otimes I_n)\,\Lambda_\alpha(\lambda) = \lambda^{m_\ell}I_n$ and $\Omega_\alpha(\lambda)\,(e_m \otimes I_n) = \lambda^{n_\ell}I_n$ and that $m_\ell + n_\ell = m-1$. By Remark 7.2.10, it follows that $\Lambda_\alpha(\lambda)$ and $\Omega_\alpha(\lambda)$ satisfy the conditions of Proposition 7.2.12. Hence the result follows from Proposition 7.2.12. ☐

We now prove that Fiedler pencils are Rosenbrock strong linearizations of $G(\lambda)$. Recall that the reversal of a matrix polynomial $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ is defined by $\operatorname{rev} P(\lambda) := \sum_{j=0}^{m} \lambda^j A_{m-j}$.

Theorem 7.2.14. *Let $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_{\sigma}$ be the Fiedler pencil of $G(\lambda)$ associated with a permutation σ of $\{0 : m-1\}$. Then $\mathbb{L}_\sigma(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. More precisely, we have the following.*

(a) There exist $mn \times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$, and $r \times r$ nonsingular matrices U_0 and V_0 such that

$$\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & U_0 \end{array}\right] \mathbb{L}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & V_0 \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right] \quad \text{for all } \lambda \in \mathbb{C}.$$

(b) There exist biproper rational matrices $\mathcal{O}_\ell(\lambda)$ and $\mathcal{O}_r(\lambda)$ such that

$$\mathcal{O}_\ell(\lambda)\,\lambda^{-1}\mathbb{G}(\lambda)\,\mathcal{O}_r(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}G(\lambda) \end{array}\right], \tag{7.23}$$

where $\mathbb{G}(\lambda) = L_\sigma(\lambda) + (e_{m-i_0(\sigma)} \otimes C)(\lambda E - A)^{-1}(e^T_{m-c_0(\sigma)} \otimes B)$ *is the transfer function of* $\mathbb{L}_\sigma(\lambda)$ *and* $L_\sigma(\lambda) := \lambda M^P_{-m} - M^P_\sigma$ *is the Fiedler pencil of* $P(\lambda)$ *associated with* σ.

Proof. Part (a) is proved in [2, Theorem 4.13]. Hence we only prove (b).

By Theorem 7.2.4, we have $\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & e_{m-i_0(\sigma)} \otimes C \\ \hline e^T_{m-c_0(\sigma)} \otimes B & A - \lambda E \end{array}\right]$. Hence $\mathbb{G}(\lambda)$ is the transfer function of $\mathbb{L}_\sigma(\lambda)$. Let σ be given by $\sigma = (\delta_1, 0, \delta_2)$. Then we have $L_\sigma(\lambda) = \lambda M^P_{-m} - M^P_\sigma = \lambda M^P_{-m} - M^P_{\delta_1} M^P_0 M^P_{\delta_2}$. It is shown in the proof of [26, Theorem 4.6] that $-\text{rev}L_\sigma(\lambda)$ is strictly equivalent to $-\text{rev}P(\lambda)$. More precisely, $J(M^P_{\delta_1})^{-1}\big(-\text{rev}L_\sigma(\lambda)\big)(M^P_{\delta_2})^{-1}J =: L_\alpha(\lambda)$ is a Fiedler pencil of $-\text{rev}P(\lambda)$, where $\alpha = \big(m - rev(\delta_1), 0, m - rev(\delta_2)\big)$ and $J := \begin{bmatrix} & & I_n \\ & \iddots & \\ I_n & & \end{bmatrix} \in \mathbb{C}^{mn \times mn}$. Hence $L_\alpha(\lambda)$ is a linearization of $-\text{rev}P(\lambda)$. Thus there exist unimodular matrix polynomials $\widehat{U}(\lambda)$ and $\widehat{V}(\lambda)$ such that

$$\widehat{U}(\lambda)\, L_\alpha(\lambda)\, \widehat{V}(\lambda) = \left[\begin{array}{c|c} -I_{(m-1)n} & 0 \\ \hline 0 & -revP(\lambda) \end{array}\right],$$

where $\widehat{U}(\lambda)$ and $\widehat{V}(\lambda)$ are given by [26]

$$\widehat{U}(\lambda) := U_0 U_1 \cdots U_{m-3} U_{m-2}, \text{ with } U_j = \begin{cases} Q^{\mathcal{B}}_{m-1-j} & \text{if } \alpha \text{ has a consecution at } j, \\ R^{\mathcal{B}}_{m-1-j} & \text{if } \alpha \text{ has an inversion at } j, \end{cases} \tag{7.24}$$

$$\widehat{V}(\lambda) := V_{m-2} V_{m-3} \cdots V_1 V_0, \text{ with } V_j = \begin{cases} R_{m-1-j} & \text{if } \alpha \text{ has a consecution at } j, \\ Q_{m-1-j} & \text{if } \alpha \text{ has an inversion at } j. \end{cases} \tag{7.25}$$

Note that $R_i^{'}s$ in (7.24) and (7.25) are associated with the matrix polynomial $-\text{rev}P(\lambda)$. Thus we have

$$\left[\begin{array}{c|c} -I_{(m-1)n} & 0 \\ \hline 0 & -\text{rev}\, P(\lambda) \end{array}\right] = \widehat{U}(\lambda) L_\alpha(\lambda) \widehat{V}(\lambda) = \widehat{U}(\lambda) J (M^P_{\delta_1})^{-1}\big(-\text{rev}L_\sigma(\lambda)\big)(M^P_{\delta_2})^{-1} J \widehat{V}(\lambda)$$

$$\Rightarrow \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m} P(\lambda) \end{array}\right] = \widehat{U}(1/\lambda)\, J (M^P_{\delta_1})^{-1}\Big(\lambda^{-1} L_\sigma(\lambda)\Big)(M^P_{\delta_2})^{-1} J\, \widehat{V}(1/\lambda). \tag{7.26}$$

Next we evaluate

$$\widehat{U}(\lambda) J (M^P_{\delta_1})^{-1}(e_{m-i_0(\sigma)} \otimes I_n) \text{ and } (e^T_{m-c_0(\sigma)} \otimes I_n)(M^P_{\delta_2})^{-1} J \widehat{V}(\lambda). \tag{7.27}$$

Recall that $M^P_{-t} = \begin{bmatrix} I_{(m-t-1)n} & & & \\ & 0 & I_n & \\ & I_n & A_t & \\ & & & I_{(t-1)n} \end{bmatrix}$ for $t = 1 : m-1$. Hence we have

$$(e^T_{m-q} \otimes I_n) M^P_{-t} = \begin{cases} e^T_{m-(q-1)} \otimes I_n & \text{for } t = q \text{ and } q = 1 : m-1, \\ e^T_{m-q} \otimes I_n & \text{for } t \notin \{q, q+1\},\ q = 0 : m-1, \end{cases} \tag{7.28}$$

and

$$M^P_{-t}(e_{m-q} \otimes I_n) = \begin{cases} e_{m-(q-1)} \otimes I_n & \text{for } t = q \text{ and } q = 1 : m-1, \\ e_{m-q} \otimes I_n & \text{for } t \notin \{q, q+1\},\ q = 0 : m-1. \end{cases} \tag{7.29}$$

Case-I: Suppose that $c_0(\sigma) > 0$. Then $i_0(\sigma) = 0$. Since σ has $c_0(\sigma)$ consecutions at 0, we have $\sigma \sim (\sigma^L, 0, 1, 2, \ldots, c_0(\sigma))$. Without loss of generality, we assume that $\sigma = (\delta_1, 0, \delta_2) = (\delta_1, 0, 1, 2, \ldots, c_0(\sigma))$, that is, $\delta_2 = (1, 2, \ldots, c_0(\sigma))$. Then by repeated application of (7.28) we have

$$(e^T_{m-c_0(\sigma)} \otimes I_n)(M^P_{\delta_2})^{-1} = (e^T_{m-c_0(\sigma)} \otimes I_n) M^P_{-c_0(\sigma)} M^P_{-(c_0(\sigma)-1)} \cdots M^P_{-2} M^P_{-1} = e^T_m \otimes I_n.$$

Hence $(e^T_{m-c_0(\sigma)} \otimes I_n)(M^P_{\delta_2})^{-1} J = (e^T_m \otimes I_n) J = e^T_1 \otimes I_n$.

Further, since $i_0(\sigma) = 0$ and $0, 1 \notin \delta_1$, by (7.29) we have $(M^P_{\delta_1})^{-1}(e_{m-i_0(\sigma)} \otimes I_n) = (M^P_{\delta_1})^{-1}(e_m \otimes I_n) = e_m \otimes I_n$. Hence $J(M^P_{\delta_1})^{-1}(e_{m-i_0(\sigma)} \otimes I_n) = J(e_m \otimes I_n) = e_1 \otimes I_n$.

Case-II: Suppose that $i_0(\sigma) > 0$. Then $c_0(\sigma) = 0$. Since σ has $i_0(\sigma)$ inversions at 0, we have $\sigma \sim (i_0(\sigma), \ldots, 2, 1, 0, \sigma^R)$. Without loss of generality, we assume that $\sigma = (\delta_1, 0, \delta_2) = (i_0(\sigma), \ldots, 2, 1, 0, \delta_2)$, that is, $\delta_1 = (i_0(\sigma), \ldots, 2, 1)$. Then by repeated application of (7.29) we have

$$(M^P_{\delta_1})^{-1}(e_{m-i_0(\sigma)} \otimes I_n) = M^P_{-1} M^P_{-2} \cdots M^P_{-(i_0(\sigma)-1)} M^P_{-i_0(\sigma)} (e_{m-i_0(\sigma)} \otimes I_n) = e_m \otimes I_n.$$

Hence we have $J(M^P_{\delta_1})^{-1}(e_{m-i_0(\sigma)} \otimes I_n) = J(e_m \otimes I_n) = e_1 \otimes I_n$.

Further, since $c_0(\sigma) = 0$ and $0, 1 \notin \delta_2$, by (7.28) we have $(e^T_{m-c_0(\sigma)} \otimes I_n)(M^P_{\delta_2})^{-1} = (e^T_m \otimes I_n)(M^P_{\delta_2})^{-1} = e^T_m \otimes I_n$. Hence $(e^T_{m-c_0(\sigma)} \otimes I_n)(M^P_{\delta_2})^{-1} J = (e^T_m \otimes I_n) J = e^T_1 \otimes I_n$.

Thus in both the cases, we have

$$\begin{aligned} \widehat{U}(\lambda) J (M^P_{\delta_1})^{-1}(e_{m-i_0(\sigma)} \otimes I_n) &= \widehat{U}(\lambda)(e_1 \otimes I_n) \\ (e^T_{m-c_0(\sigma)} \otimes I_n)(M^P_{\delta_2})^{-1} J \widehat{V}(\lambda) &= (e^T_1 \otimes I_n) \widehat{V}(\lambda). \end{aligned} \tag{7.30}$$

Next, we calculate $\widehat{U}(\lambda)(e_1 \otimes I_n)$ and $(e^T_1 \otimes I_n)\widehat{V}(\lambda)$. Note that α is a permutation of $\{0 : m-1\}$. Let RCISS(α) be given by RCISS$(\alpha) = (c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$. Recall from

(7.8) and (7.9) the definitions of m_j, n_j and s_j, for $j = 0 : \ell$, associated with RCISS(α). By (7.24) and (7.25), we have

$$\widehat{U}(\lambda) = \widehat{U}_{(c_\ell,i_\ell)}\widehat{U}_{(c_{\ell-1},i_{\ell-1})}\cdots\widehat{U}_{(c_2,i_2)}\widehat{U}_{(c_1,i_1)} \text{ and}$$

$$\widehat{V}(\lambda) = \widehat{V}_{(c_1,i_1)}\widehat{V}_{(c_2,i_2)}\cdots\widehat{V}_{(c_{\ell-1},i_{\ell-1})}\widehat{V}_{(c_\ell,i_\ell)},$$

where $\widehat{U}_{(c_j,i_j)} = R^{\mathcal{B}}_{s_{j-1}+c_j+i_j}\cdots R^{\mathcal{B}}_{s_{j-1}+c_j+1}Q^{\mathcal{B}}_{s_{j-1}+c_j}\cdots Q^{\mathcal{B}}_{s_{j-1}+1}$ and

$$\widehat{V}_{(c_j,i_j)} = R_{s_{j-1}+1}\cdots R_{s_{j-1}+c_j}Q_{s_{j-1}+c_j+1}\cdots Q_{s_{j-1}+c_j+i_j}.$$

Hence by Lemma 7.2.11, we have $\widehat{U}(\lambda)(e_1\otimes I_n) = \Lambda_\alpha(\lambda)$ and $(e_1^T\otimes I_n)\widehat{V}(\lambda) = \Omega_\alpha(\lambda)$, where $\Lambda_\alpha(\lambda)$ and $\Omega_\alpha(\lambda)$ are as given in Definition 7.2.9. Now by (7.30) we have

$$\widehat{U}(\lambda)J(M^P_{\delta_1})^{-1}(e_{m-i_0(\sigma)}\otimes I_n) = \Lambda_\alpha(\lambda) \text{ and } (e^T_{m-c_0(\sigma)}\otimes I_n)(M^P_{\delta_2})^{-1}J\widehat{V}(\lambda) = \Omega_\alpha(\lambda). \tag{7.31}$$

Define $\widehat{\mathcal{O}}_\ell(\lambda) := \widehat{U}(1/\lambda)J(M^P_{\delta_1})^{-1}$ and $\widehat{\mathcal{O}}_r(\lambda) := (M^P_{\delta_2})^{-1}J\widehat{V}(1/\lambda)$. Then $\widehat{\mathcal{O}}_\ell(\lambda)$ and $\widehat{\mathcal{O}}_r(\lambda)$ are biproper rational matrices. Set $G_{sp}(\lambda) := C(\lambda E - A)^{-1}B$. Then

$$\begin{aligned}
&\widehat{\mathcal{O}}_\ell(\lambda)\lambda^{-1}\mathbb{G}(\lambda)\widehat{\mathcal{O}}_r(\lambda)\\
&= \widehat{\mathcal{O}}_\ell(\lambda)\lambda^{-1}L_\sigma(\lambda)\widehat{\mathcal{O}}_r(\lambda) + \widehat{\mathcal{O}}_\ell(\lambda)\Big((e_{m-i_0(\sigma)}\otimes I_n)\lambda^{-1}G_{sp}(\lambda)(e^T_{m-c_0(\sigma)}\otimes I_n)\Big)\widehat{\mathcal{O}}_r(\lambda)\\
&= \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}P(\lambda)\end{array}\right] + \Lambda_\alpha(1/\lambda)\ \lambda^{-1}G_{sp}(\lambda)\ \Omega_\alpha(1/\lambda) \text{ by (7.26) and (7.31)}\\
&= \left[\begin{array}{c|c} 0_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}P(\lambda)\end{array}\right] + \underbrace{\left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & 0\end{array}\right] + \Lambda_\alpha(1/\lambda)\ \lambda^{-1}G_{sp}(\lambda)\ \Omega_\alpha(1/\lambda)}_{W(\lambda)}.
\end{aligned} \tag{7.32}$$

Let $T_1(\lambda)$ and $T_2(\lambda)$ be the matrix polynomials given in Corollary 7.2.13. Then $T_1(1/\lambda)$ and $T_2(1/\lambda)$ are biproper rational matrices. Let $\widehat{T}_j(1/\lambda)$, $j = 1, 2$, denote the matrix obtained by multiplying each off diagonal block of $T_j(1/\lambda)$ by $-\lambda^{-1}G_{sp}(\lambda)$. Then by Corollary 7.2.13 we have

$$\widehat{T}_1(1/\lambda)\ W(\lambda)\ \widehat{T}_2(1/\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-(m-1)}\lambda^{-1}G_{sp}(\lambda)\end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}G_{sp}(\lambda)\end{array}\right]. \tag{7.33}$$

Obviously, $\widehat{T}_1(1/\lambda)$ and $\widehat{T}_2(1/\lambda)$ are biproper rational matrices. Hence by defining $\mathcal{O}_\ell(\lambda) := \widehat{T}_1(1/\lambda)\widehat{\mathcal{O}}_\ell(\lambda)$ and $\mathcal{O}_r(\lambda) := \widehat{\mathcal{O}}_r(\lambda)\widehat{T}_2(1/\lambda)$, the equality in (7.23) follows from (7.32) and (7.33). This completes the proof of (b). ☐

Next, we show that GFPRs of $G(\lambda)$ are Rosenbrock strong linearizations of $G(\lambda)$. We need the following result.

Proposition 7.2.15. *Let* $\mathbb{T}(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_{\alpha}$ *be the Fiedler pencil of* $G(\lambda)$ *associated with a permutation* α *of* $\{0 : m-1\}$*. Let* $\mathbb{L}(\lambda)$ *be a pencil given by* $\mathbb{L}(\lambda) := \operatorname{diag}(\mathcal{X}, X_0)\mathbb{T}(\lambda)\operatorname{diag}(\mathcal{Y}, Y_0)$*, where* $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{mn\times mn}$ *and* $X_0, Y_0 \in \mathbb{C}^{r\times r}$ *are nonsingular matrices. Then* $\mathbb{L}(\lambda)$ *is a Rosenbrock strong linearization of* $G(\lambda)$*.*

Proof. Since $\mathbb{T}(\lambda)$ is a Fiedler pencil of $G(\lambda)$, by Theorem 7.2.14, $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. Hence there exist $mn\times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$, and $r\times r$ nonsingular matrices U_0 and V_0 such that

$$\begin{aligned}\operatorname{diag}\big(I_{(m-1)n}, \mathcal{S}(\lambda)\big) &= \operatorname{diag}\big(U(\lambda), U_0\big)\,\mathbb{T}(\lambda)\operatorname{diag}\big(V(\lambda), V_0\big)\\ &= \operatorname{diag}\big(U(\lambda)\mathcal{X}^{-1}, U_0X_0^{-1}\big)\,\mathbb{L}(\lambda)\operatorname{diag}\big(\mathcal{Y}^{-1}V(\lambda), Y_0^{-1}V_0\big). \qquad (7.34)\end{aligned}$$

By Theorem 7.2.4, we have $\mathbb{T}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & e_{m-i_0(\alpha)}\otimes C \\ \hline e^T_{m-c_0(\alpha)}\otimes B & A-\lambda E\end{array}\right]$, where $L(\lambda) = \lambda M^P_{-m} - M^P_{\alpha}$ is the Fiedler pencil of $P(\lambda)$ associated with α. Then

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} \mathcal{X}L(\lambda)\mathcal{Y} & \mathcal{X}(e_{m-i_0(\alpha)}\otimes C)Y_0 \\ \hline X_0(e^T_{m-c_0(\alpha)}\otimes B)\mathcal{Y} & X_0(A-\lambda E)Y_0\end{array}\right]$$

and $\mathbb{G}_{\mathbb{L}}(\lambda) := \mathcal{X}L(\lambda)\mathcal{Y} + \mathcal{X}(e_{m-i_0(\alpha)}\otimes C)(\lambda E - A)^{-1}(e^T_{m-c_0(\alpha)}\otimes B)\mathcal{Y}$ is the transfer function of $\mathbb{L}(\lambda)$. Since $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$, there exist biproper rational matrices $\mathcal{O}_\ell(\lambda)$ and $\mathcal{O}_r(\lambda)$ such that

$$\mathcal{O}_\ell(\lambda)\,\lambda^{-1}\mathbb{G}_{\mathbb{T}}(\lambda)\,\mathcal{O}_r(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}G(\lambda)\end{array}\right], \qquad (7.35)$$

where $\mathbb{G}_{\mathbb{T}}(\lambda) = L(\lambda) + (e_{m-i_0(\alpha)}\otimes C)(\lambda E - A)^{-1}(e^T_{m-c_0(\alpha)}\otimes B)$ is the transfer function of $\mathbb{T}(\lambda)$. Since $\mathcal{X}^{-1}\mathbb{G}_{\mathbb{L}}(\lambda)\mathcal{Y}^{-1} = \mathbb{G}_{\mathbb{T}}(\lambda)$, it follows from (7.35) that

$$\mathcal{O}_\ell(\lambda)\mathcal{X}^{-1}\lambda^{-1}\mathbb{G}_{\mathbb{L}}(\lambda)\mathcal{Y}^{-1}\mathcal{O}_r(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \lambda^{-m}G(\lambda)\end{array}\right]. \qquad (7.36)$$

Note that $\mathcal{O}_\ell(\lambda)\mathcal{X}^{-1}$ and $\mathcal{Y}^{-1}\mathcal{O}_r(\lambda)$ are biproper rational matrices. Hence it follows from (7.34) and (7.36) that $\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. $\square$

Now we prove that the GFPRs of $G(\lambda)$ are Rosenbrock strong linearizations.

Theorem 7.2.16. *Let* $\mathbb{L}(\lambda) := \mathbb{M}_{\tau_1}(Y_1)\,\mathbb{M}_{\sigma_1}(X_1)(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}_{\sigma_2}(X_2)\,\mathbb{M}_{\tau_2}(Y_2)$ *be a GFPR of* $G(\lambda)$ *as given in Definition 7.2.2, where all the matrix assignments* X_j *and* Y_j*,* $j = 1, 2$*, are nonsingular. Then* $\mathbb{L}(\lambda)$ *is a Rosenbrock strong linearization of* $G(\lambda)$*.*

Proof. Let τ be given by $\tau = (\beta, -m, \gamma)$. Define $\alpha := (-\mathrm{rev}\,(\beta), \sigma, -\mathrm{rev}\,(\gamma))$ and $\mathbb{T}(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-m} - \mathbb{M}^{\mathcal{S}}_{\alpha}$. Then $\mathbb{T}(\lambda)$ is a Fiedler pencil of $G(\lambda)$ associated with the permutation α of $\{0 : m-1\}$. It is easily seen that $\mathbb{L}(\lambda) = \mathcal{A}\mathbb{T}(\lambda)\mathcal{B}$, where $\mathcal{A} = \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1)\mathbb{M}^{\mathcal{S}}_{\beta} = \left[\begin{array}{c|c} M_{(\tau_1,\sigma_1)}(Y_1, X_1)M^P_{\beta} & \\ \hline & I_r \end{array}\right]$ and $\mathcal{B} = \mathbb{M}^{\mathcal{S}}_{\gamma}\mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2) = \left[\begin{array}{c|c} M^P_{\gamma}M_{(\sigma_2,\tau_2)}(X_2, Y_2) & \\ \hline & I_r \end{array}\right]$. Since X_j and Y_j, $j = 1, 2$, are nonsingular matrix assignments, the matrices $M_{(\tau_1,\sigma_1)}(Y_1, X_1)$ and $M_{(\sigma_2,\tau_2)}(X_2, Y_2)$ are nonsingular. Hence by Proposition 7.2.15, $\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. ☐

Finally, we show that the GFPs of $G(\lambda)$ are also Rosenbrock strong linearizations.

Theorem 7.2.17. *Let* $\mathbb{T}_{\omega}(\lambda) := \lambda\mathbb{M}^{\mathcal{S}}_{-\omega_1} - \mathbb{M}^{\mathcal{S}}_{\omega_0}$ *be a GFP of* $G(\lambda)$*, where* $0 \in \omega_0$*. Then* $\mathbb{T}_{\omega}(\lambda)$ *is a Rosenbrock strong linearization of* $G(\lambda)$*.*

Proof. It is shown in [4, Theorem 2.13] that $\mathbb{T}_{\omega}(\lambda) = \mathrm{diag}(\mathcal{X},\ X_0)\ \mathbb{F}(\lambda)\ \mathrm{diag}(\mathcal{Y},\ Y_0)$ for some nonsingular matrices $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{mn\times mn}$ and $X_0, Y_0 \in \mathbb{C}^{r\times r}$, where $\mathbb{F}(\lambda)$ is a Fiedler pencil of $G(\lambda)$. Hence by Proposition 7.2.15, $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. ☐

7.3 Structure-preserving strong linearizations

This section is devoted to construction of structure-preserving strong linearizations of structured rational matrices. We consider only symmetric, skew-symmetric, Hamiltonian and skew-Hamiltonian rational matrices and construct their structure-preserving strong linearizations. The construction of structure-preserving strong linearizations of Hermitian, skew-Hermitian, para-Hermitian and para-skew-Hermitian rational matrices is similar. We show that the family of GFPRs of $G(\lambda)$ is a rich source of structure-preserving strong linearizations of $G(\lambda)$. For the rest of this section, we assume that $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$, where $P(\lambda) := \sum_{j=0}^{m} A_j\lambda^j$ with $A_m \neq 0$ and $G_{sp}(\lambda)$ is strictly proper, that is, $G_{sp}(\lambda) \to 0$ as $\lambda \to \infty$.

7.3.1 Symmetric GFPRs

Suppose that $G(\lambda)$ is symmetric, that is, $G(\lambda)^T = G(\lambda)$. Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$, it follows that both $P(\lambda)$ and $G_{sp}(\lambda)$ are symmetric. As $G_{sp}(\lambda)$ is strictly proper and symmetric, there exists a minimal symmetric realization of $G(\lambda)$ given by $G_{sp}(\lambda) = B^T(\lambda I_r - A)^{-1}B$, where A is a symmetric matrix [32, 36, 38]. Hence $G(\lambda) = P(\lambda) + B^T(\lambda I_r - A)^{-1}B$ is a minimal symmetric realization of $G(\lambda)$. The system matrix $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & B^T \\ \hline B & A - \lambda I_r \end{array}\right]$ is then symmetric and irreducible. Also, there exists a minimal symmetric realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + B^T(\lambda E - A)^{-1}B$, where A and E are symmetric matrices with E being nonsingular [29]. The system matrix

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & B^T \\ \hline B & A - \lambda E \end{array}\right] \tag{7.37}$$

is obviously symmetric and irreducible.

Recall that a block matrix $\mathcal{H}$ is said to be block-symmetric provided that $\mathcal{H}^{\mathcal{B}} = \mathcal{H}$, see [26]. Also, recall that the block transpose of a system matrix $\mathcal{A} := \left[\begin{array}{c|c} A & u \otimes X \\ \hline v^T \otimes Y & Z \end{array}\right]$ is defined by $\mathcal{A}^{\mathcal{B}} := \left[\begin{array}{c|c} A^{\mathcal{B}} & v \otimes X \\ \hline u^T \otimes Y & Z \end{array}\right]$, where $A = [A_{ij}]$ is an $m \times m$ block matrix with $A_{ij} \in \mathbb{C}^{n \times n}, u, v \in \mathbb{C}^m, X \in \mathbb{C}^{n \times r}, Y \in \mathbb{C}^{r \times n}$ and $Z \in \mathbb{C}^{r \times r}$. Observe that $\mathcal{A}$ is block-symmetric if and only if A is block-symmetric and $u = v$.

From Chapter 2, recall the following definitions of admissible tuple, simple admissible tuple, symmetric complement of an admissible tuple, and index tuple in canonical form.

Definition 7.3.1. *[17] (a) Let $h \geq 0$ be an integer. We say that $\boldsymbol{w}$ is an admissible tuple of $\{0 : h\}$ if $\boldsymbol{w}$ is a permutation of $\{0 : h\}$ and*

$$csf(\boldsymbol{w}) = \big(h-1 : h, h-3 : h-2, \ldots, p+1 : p+2, 0 : p\big) \tag{7.38}$$

for some $0 \leq p \leq h$. We call p the index of $\boldsymbol{w}$ and denote it by $Ind(\boldsymbol{w})$.

(b) Let $h \geq 0$ be an integer and let $\boldsymbol{w}$ be an admissible tuple of $\{0 : h\}$ with index p. Then the symmetric complement of $\boldsymbol{w}$, denoted by $c_{\boldsymbol{w}}$, is defined by

$$\boldsymbol{c}_w := \begin{cases} \big(h-1, h-3, \ldots, p+3, p+1, (0:p)_{rev_c}\big) & \text{if } p \geq 1, \\ (h-1, h-3, \ldots, 1) & \text{if } p = 0 \text{ and } h > 0, \\ \emptyset & \text{if } h = 0, \end{cases}$$

where $(0 : p)_{rev_c} := (0 : p-1, 0 : p-2, \ldots, 0 : 1, 0)$.

For simplicity, we always consider an admissible tuple of the form (7.38). Clearly, for an integer $h \geq 0$, there exists a unique admissible tuple of $\{0 : h\}$ with index 0 or 1 [17].

Definition 7.3.2. *An admissible tuple* $\boldsymbol{w}$ *of* $\{0:h\}$, $h \geq 0$, *is said to be the simple admissible tuple if* $Ind(\boldsymbol{w}) = 0$ *or* $Ind(\boldsymbol{w}) = 1$.

Note that for the simple admissible tuple $\mathbf{w}$ of $\{0:h\}$, we have $Ind(\mathbf{w}) = 0$ (resp., $Ind(\mathbf{w}) = 1$) if h is even (resp., odd).

Remark 7.3.3. *Let* $\boldsymbol{v}$ *be an admissible tuple of* $\{0:k\}$, $k \geq 0$, *and let* $\boldsymbol{c}_v$ *be the symmetric complement of* $\boldsymbol{v}$. *Then it follows from Definition 7.3.1 that* $0 \in \boldsymbol{c}_v$ *if and only if* $Ind(\boldsymbol{v}) \geq 1$. *In particular, for the simple admissible tuple* $\boldsymbol{w}$ *of* $\{0:h\}$, *we have* $0 \in \boldsymbol{c}_w$ *(resp.,* $0 \notin \boldsymbol{c}_w$*) if* h *is odd (resp., even), where* $\boldsymbol{c}_w$ *is the symmetric complement of* $\boldsymbol{w}$.

Definition 7.3.4. *[17] Given* $h \geq 0$, *we say that an index tuple* $\boldsymbol{t}$ *is in canonical form for* h *if* $\boldsymbol{t}$ *is of the form*

$$\Big(a_1 : h-2, a_2 : h-4, \ldots, a_{\lfloor \frac{h}{2} \rfloor} : h - 2\lfloor \tfrac{h}{2} \rfloor\Big)$$

with $a_i \geq 0$, $i = 1 : \lfloor \frac{h}{2} \rfloor$, *where* $\lfloor \cdot \rfloor$ *stands for the greatest integer function.*

Note that an index tuple in canonical form for h is necessarily empty for $h = 0, 1$.

The following result characterizes all symmetric GFPRs of a matrix polynomial.

Theorem 7.3.5 ([17], Theorem 6.11). *Let* $0 \leq h < m$. *Let* $\boldsymbol{w}_h$ *and* $\boldsymbol{v}_h + m$ *be the simple admissible tuples of* $\{0:h\}$ *and* $\{0:m-h-1\}$, *respectively. Let* $\boldsymbol{t}_{w_h}$ *and* $\boldsymbol{t}_{v_h} + m$ *be index tuples in canonical form for* h *and* $m-h-1$, *respectively. Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be nonsingular matrix assignments for* $\boldsymbol{t}_{w_h}$ *and* $\boldsymbol{t}_{v_h}$, *respectively. Then*

$$L(\lambda) := M_{(\boldsymbol{t}_{v_h}, \boldsymbol{t}_{w_h})}(\mathcal{Y}, \mathcal{X})(\lambda M^P_{\boldsymbol{v}_h} - M^P_{\boldsymbol{w}_h}) M^P_{(\boldsymbol{c}_{w_h}, \boldsymbol{c}_{v_h})} M_{(rev(\boldsymbol{t}_{w_h}), rev(\boldsymbol{t}_{v_h}))}(rev(\mathcal{X}), rev(\mathcal{Y})), \tag{7.39}$$

is a block symmetric GFPR of $P(\lambda)$, *where* $\boldsymbol{c}_{w_h}$ *and* $\boldsymbol{c}_{v_h} + m$ *are the symmetric complements of* $\boldsymbol{w}_h$ *and* $\boldsymbol{v}_h + m$, *respectively. Moreover, any block symmetric GFPR of* $P(\lambda)$ *is of the form (7.39). Furthermore, if all the matrices in the matrix assignments* $\mathcal{X}$ *and* $\mathcal{Y}$ *are symmetric, then* $L(\lambda)$ *is symmetric when* $P(\lambda)$ *is symmetric.*

The pencil in (7.39) is denoted by $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{X}, \mathcal{Y})$ and is uniquely determined by $h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}$, $\mathcal{X}$ and $\mathcal{Y}$, see [17].

Definition 7.3.6. *Let* $h, \boldsymbol{w}_h, \boldsymbol{c}_{w_h}, \boldsymbol{t}_{w_h}, \boldsymbol{v}_h, \boldsymbol{c}_{v_h}, \boldsymbol{t}_{v_h}$, $\mathcal{X}$ *and* $\mathcal{Y}$ *be as given in Theorem 7.3.5. Then we define*

$$\mathbb{L}(\lambda) := \mathbb{M}_{(\boldsymbol{t}_{v_h}, \boldsymbol{t}_{w_h})}(\mathcal{Y}, \mathcal{X})(\lambda \mathbb{M}^{\mathcal{S}}_{\boldsymbol{v}_h} - \mathbb{M}^{\mathcal{S}}_{\boldsymbol{w}_h}) \mathbb{M}^P_{(\boldsymbol{c}_{w_h}, \boldsymbol{c}_{v_h})} \mathbb{M}_{(rev(\boldsymbol{t}_{w_h}), rev(\boldsymbol{t}_{v_h}))}(rev(\mathcal{X}), rev(\mathcal{Y})). \tag{7.40}$$

The pencil $\mathbb{L}(\lambda)$ *in (7.40) is uniquely determined by* $h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}$, $\mathcal{X}$ *and* $\mathcal{Y}$. *We denote* $\mathbb{L}(\lambda)$ *by* $\mathbb{L}_{\mathcal{S}}(h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}, \mathcal{X}, \mathcal{Y})$.

The following result characterizes all block-symmetric GFPRs of $G(\lambda)$.

Theorem 7.3.7. *Let $\mathcal{S}(\lambda)$ be given in (7.2). Let $0 \leq h \leq m-1$ be even. Consider the GFPR $\mathbb{L}(\lambda) := \mathbb{L}_{\mathcal{S}}(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{X}, \mathcal{Y})$ associated with $\mathcal{S}(\lambda)$. Then $\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{X}, \mathcal{Y}) & e_{m-i_0(\mathbf{t}_{w_h}, \mathbf{w}_h)} \otimes C \\ \hline e^T_{m-c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))} \otimes B & A - \lambda E \end{array}\right]$. Further, we have the following:*

(a) $\mathbb{L}(\lambda)$ is a block symmetric GFPR of $\mathcal{S}(\lambda)$. Further, any block symmetric GFPR of $\mathcal{S}(\lambda)$ must be of the form $\mathbb{L}_{\mathcal{S}}(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{X}, \mathcal{Y})$ for some even $0 \leq h \leq m-1$.

(b) If m is odd then $\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $\mathcal{S}(\lambda)$. If m is even then $\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $\mathcal{S}(\lambda)$ when the leading coefficient of $P(\lambda)$ is nonsingular.

Proof. By substituting $\sigma = \mathbf{w}_h$, $\sigma_1 = \mathbf{t}_{w_h}$, $\sigma_2 = (\mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$, $\tau = \mathbf{v}_h, \tau_1 = \mathbf{t}_{v_h}$ and $\tau_2 = (\mathbf{c}_{v_h}, rev(\mathbf{t}_{v_h}))$ in Theorem 7.2.4, we have

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{X}, \mathcal{Y}) & e_{m-i_0(\mathbf{t}_{w_h}, \mathbf{w}_h)} \otimes C \\ \hline e^T_{m-c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))} \otimes B & A - \lambda E \end{array}\right]. \tag{7.41}$$

By Theorem 7.3.5, $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{X}, \mathcal{Y})$ is a block symmetric pencil. Hence it follows that $\mathbb{L}(\lambda)$ is block symmetric if and only if $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) = i_0(\mathbf{t}_{w_h}, \mathbf{w}_h)$. Next, we show that $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) = i_0(\mathbf{t}_{w_h}, \mathbf{w}_h)$.

Case-I: Suppose that $h = 0$. Then $\mathbf{w}_h = (0)$ and $\mathbf{c}_{w_h} = \emptyset = \mathbf{t}_{w_h}$. Hence $i_0(\mathbf{t}_{w_h}, \mathbf{w}_h) = 0 = c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h}))$.

Case-II: Suppose that $h > 0$. Since h is even and $\mathbf{w}_h$ is the simple admissible tuple of $\{0 : h\}$, we have $\mathbf{w}_h = (h-1 : h, h-3 : h-2, \ldots, 1 : 2, 0)$ and $\mathbf{c}_{w_h} = (h-1, h-3, \ldots, 3, 1)$. Thus $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) = 2 + c_2(rev(\mathbf{t}_{w_h}))$ and $i_0(\mathbf{t}_{w_h}, \mathbf{w}_h) = 2 + i_2(\mathbf{t}_{w_h})$. (Recall that for any index tuple β and for any index t, if $t \notin \beta$ then $c_t(\beta) = -1 = i_t(\beta)$). Hence $\mathbb{L}(\lambda)$ is block-symmetric since $i_t(\beta) = c_t(rev(\beta))$ for any index tuple β and any index t. This proves the first part of (a).

Next we prove that, if h is odd, then $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) \neq i_0(\mathbf{t}_{w_h}, \mathbf{w}_h)$. Then it follows from (7.41) that $\mathbb{L}(\lambda)$ is not a block symmetric GFPR of $\mathcal{S}(\lambda)$. This will prove the second part of (a).

Let $h \geq 0$ be odd. If $h = 1$ then $\mathbf{w}_h = (0, 1)$, $\mathbf{c}_{w_h} = (0)$ and $\mathbf{t}_{w_h} = \emptyset$. Thus $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) = 1$ and $i_0(\mathbf{t}_{w_h}, \mathbf{w}_h) = 0$. Hence $\mathbb{L}(\lambda)$ is not block symmetric.

Next, suppose that $h > 1$. Then $\mathbf{w}_h = (h-1 : h, h-3 : h-2, \ldots, 2 : 3, 0 : 1)$ and $\mathbf{c}_{w_h} = (h-1, h-3, \ldots, 2, 0)$. Thus $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) = 3 + c_3(rev(\mathbf{t}_{w_h})) = 3 + i_3(\mathbf{t}_{w_h})$ and $i_0(\mathbf{t}_{w_h}, \mathbf{w}_h) = 1 + i_1(\mathbf{t}_{w_h})$. We show that $3 + i_3(\mathbf{t}_{w_h}) \neq 1 + i_1(\mathbf{t}_{w_h})$. Let $i_1(\mathbf{t}_{w_h}) = p$. If $p = -1$ or $p = 0$ then $1 + i_1(\mathbf{t}_{w_h}) < 2 \leq 3 + i_3(\mathbf{t}_{w_h})$ and hence the desired result follows. Suppose that $p \geq 1$. Note that $\mathbf{t}_{w_h}$ is in canonical form for h ($h > 1$ is odd), i.e.,

$$\mathbf{t}_{w_h} = \big(a_1 : h-2, a_2 : h-4, \ldots, a_{\frac{h-1}{2}-1} : 3, a_{\frac{h-1}{2}} : 1\big). \tag{7.42}$$

We call $(a_j : h-2j)$, $j = 1, 2, \ldots, \frac{h-1}{2}$, as the strings of $\mathbf{t}_{w_h}$ and $h-2j$ as the right end point of the string $(a_j : h-2j)$. Since $i_1(\mathbf{t}_{w_h}) = p$, $(p+1, p, \ldots, 3, 2, 1)$ is a subtuple of $\mathbf{t}_{w_h}$ and $(p+2, p+1, p, \ldots, 2, 1)$ is not a subtuple of $\mathbf{t}_{w_h}$. It is clear from (7.42) that each index of the subtuple $(p+1, p, \ldots, 2, 1)$ of $\mathbf{t}_{w_h}$ belongs to distinct string of $\mathbf{t}_{w_h}$. By collecting all those strings we have a subtuple

$$\Big((p+1 : b_{p+1}), (p : b_p), \ldots, (3 : b_3), (2 : b_2), (1 : b_1)\Big)$$

of $\mathbf{t}_{w_h}$, where b_j's are the right end points of the collected strings. Hence $b_j \in \{1, 3, 5, \ldots, h-4, h-2\}$ for $j = 1 : p+1$ is such that $b_{p+1} > b_p > \cdots > b_3 > b_2 > b_1$. This implies that $b_2 \geq 3$ and hence $3 \in (2 : b_2)$, $b_3 \geq 5$ and hence $4 \in (3 : b_3)$, and so on $p+1 \in (p : b_p)$ and $p+2 \in (p+1 : b_{p+2})$. Consequently, $(p+2, p+1, \ldots, 4, 3)$ is a subtuple of $\mathbf{t}_{w_h}$ and $i_3(\mathbf{t}_{w_h}) \geq p-1$. So $3 + i_3(\mathbf{t}_{w_h}) \geq p+2 > p+1 = 1 + i_1(\mathbf{t}_{w_h})$. Hence $c_0(\mathbf{w}_h, \mathbf{c}_{w_h}, rev(\mathbf{t}_{w_h})) \neq i_0(\mathbf{t}_{w_h}, \mathbf{w}_h)$ and $\mathbb{L}(\lambda)$ is not a block symmetric GFPR of $\mathcal{S}(\lambda)$. This completes the proof of the second part of (a).

(b) Since h is even, by Remark 7.3.3 we have $0 \notin \mathbf{c}_{w_h}$. This implies that the matrix assignment for $\mathbf{c}_{w_h}$ is nonsingular. Further, it is given that $\mathcal{X}$ and $\mathcal{Y}$ are nonsingular matrix assignments for $\mathbf{t}_{w_h}$ and $\mathbf{t}_{v_h}$, respectively. Consequently, by taking $\sigma := \mathbf{w}_h, \tau := \mathbf{v}_h, \sigma_1 := \mathbf{t}_{w_h}, \sigma_2 := (\mathbf{c}_{w_h}, \mathrm{rev}\,(\mathbf{t}_{w_h}))$, $\tau_1 := \mathbf{t}_{v_h}$ and $\tau_2 := (\mathbf{c}_{v_h}, \mathrm{rev}\,(\mathbf{t}_{v_h}))$, it follows from Theorem 7.2.16 that $\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $\mathcal{S}(\lambda)$ if the matrix assignment for $\mathbf{c}_{v_h}$ is nonsingular. Suppose that m is odd. Then $m-h-1$ is even (since h is even) and by Remark 7.3.3, it follows that $0 \notin \mathbf{c}_{v_h} + m \implies -m \notin \mathbf{c}_{v_h}$. Hence the matrix assignment for $\mathbf{c}_{v_h}$ is nonsingular. On the other hand, if the leading coefficient of $P(\lambda)$ is nonsingular then the matrix assignment for $\mathbf{c}_{v_h}$ is nonsingular irrespective of m being even or odd. Hence $\mathbb{L}(\lambda)$ is a block symmetric Rosenbrock strong linearization of $\mathcal{S}(\lambda)$. $\square$

Corollary 7.3.8. *Let $G(\lambda)$ be symmetric and $\mathcal{S}(\lambda)$ be as given in (7.37). Let $0 \leq h \leq m-1$ be even. Consider the GFPR*

$$\mathbb{L}(\lambda) := \mathbb{L}_{\mathcal{S}}(h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}, \mathcal{X}, \mathcal{Y}) = \left[\begin{array}{c|c} L_P(h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}, \mathcal{X}, \mathcal{Y}) & e_{m-\alpha} \otimes B^T \\ \hline e_{m-\alpha}^T \otimes B & A - \lambda E \end{array}\right]$$

associated with $\mathcal{S}(\lambda)$, where $\alpha := i_0(\boldsymbol{t}_{w_h}, \boldsymbol{w}_h)$, $\mathcal{X}$ and $\mathcal{Y}$ are nonsingular matrix assignments and all the matrices in $\mathcal{X}$ and $\mathcal{Y}$ are symmetric. If m is odd then $\mathbb{L}(\lambda)$ is a symmetric Rosenbrock strong linearization of $G(\lambda)$. If m is even then $\mathbb{L}(\lambda)$ is a symmetric Rosenbrock strong linearization of $G(\lambda)$ when the leading coefficient of $P(\lambda)$ is nonsingular. Also the transfer function $\mathbb{G}(\lambda) := L(\lambda) + (e_{m-\alpha} \otimes B^T)(\lambda E - A)^{-1}(e_{m-\alpha}^T \otimes B)$ of $\mathbb{L}(\lambda)$ is symmetric, where $L(\lambda) := L_P(h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}, \mathcal{X}, \mathcal{Y})$.

Proof. By considering $C = B^T$ it follows from the proof of Theorem 7.3.7 that

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{X}, \mathcal{Y}) & e_{m-\alpha} \otimes B^T \\ \hline e_{m-\alpha}^T \otimes B & A - \lambda E \end{array}\right] \tag{7.43}$$

is a block symmetric Rosenbrock strong linearization of $\mathcal{S}(\lambda)$, where $\alpha := i_0(\mathbf{t}_{w_h}, \mathbf{w}_h)$. Since $P(\lambda)$ is symmetric and all the matrices in the matrix assignments $\mathcal{X}$ and $\mathcal{Y}$ are symmetric, by Theorem 7.3.5, we have $L_P(h, \mathbf{t}_{w_h}, \mathbf{t}_{v_h}, \mathcal{X}, \mathcal{Y})$ is symmetric. Further, since A and E are symmetric, it follows from (7.43) that $\mathbb{L}(\lambda)$ and $\mathbb{G}(\lambda)$ are symmetric. $\square$

Example 7.3.9. *Let $G(\lambda) = \sum_{i=0}^{5} \lambda^i A_i + B^T(\lambda E - A)^{-1}B$ be symmetric. Consider $h = 2$, $\boldsymbol{t}_{w_h} = (0)$ and $\boldsymbol{t}_{v_h} = (-5)$. Let X and Y be any arbitrary nonsingular symmetric matrices. Then the GFPR*

$$\mathbb{L}_{\mathcal{S}}(h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}, \mathcal{X}, \mathcal{Y}) = \left[\begin{array}{ccccc|c} 0 & -Y & \lambda Y & 0 & 0 & 0 \\ -Y & \lambda A_5 - A_4 & \lambda A_4 & 0 & 0 & 0 \\ \lambda Y & \lambda A_4 & \lambda A_3 + A_2 & A_1 & -X & 0 \\ 0 & 0 & A_1 & -\lambda A_1 + A_0 & \lambda X & B^T \\ 0 & 0 & -X & \lambda X & 0 & 0 \\ \hline 0 & 0 & 0 & B & 0 & A - \lambda E \end{array}\right]$$

is a symmetric Rosenbrock strong linearization of $\mathcal{S}(\lambda)$. Note that $\mathbb{L}_{\mathcal{S}}(h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}, \mathcal{X}, \mathcal{Y})$ is a block penta-diagonal pencil.

We now show that a real symmetric strong linearization preserves the Cauchy-Maslov index of a real symmetric rational matrix.

Definition 7.3.10. *[10] The Cauchy-Maslov index of a real symmetric rational matrix $G(\lambda)$ is defined by $\mathbf{Ind}_{\mathrm{CM}}(G) :=$ (# eigenvalues of $G(\lambda)$ which jump from $-\infty$ to $+\infty$) $-$ (# eigenvalues of $G(\lambda)$ which jump from $+\infty$ to $-\infty$) as the real parameter λ traverses from $-\infty$ to $+\infty$.*

The Cauchy-Maslov index of a real symmetric rational matrix plays an important role in many applications such as in networks of linear systems, see [10, 23, 22, 39] and the references therein. It is therefore desirable to construct real symmetric linearizations of $G(\lambda)$ that preserve the Cauchy-Maslov index of $G(\lambda)$.

Theorem 7.3.11. *Let $G(\lambda)$ be real symmetric and $\mathcal{S}(\lambda)$ be as given in (7.37). Let $\mathbb{L}(\lambda) := \mathbb{L}_{\mathcal{S}}(h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}, \mathcal{X}, \mathcal{Y})$ be a symmetric Rosenbrock strong linearization of $G(\lambda)$*

as given in Corollary 7.3.8. Let $\mathbb{G}(\lambda)$ *be the associated transfer function of* $\mathbb{L}(\lambda)$*. Then* $\mathbb{G}(\lambda)$ *is real symmetric and has the same Cauchy-Maslov index as* $G(\lambda)$*, that is,* $\mathbf{Ind}_{\mathrm{CM}}(G) = \mathbf{Ind}_{\mathrm{CM}}(\mathbb{G})$*.*

Proof. By Corollary 7.3.8 we have $\mathbb{G}(\lambda) = L(\lambda) + (e_{m-\alpha} \otimes B^T)(\lambda E - A)^{-1}(e_{m-\alpha}^T \otimes B)$ is symmetric, where $\alpha := i_0(\mathbf{t}_{w_h}, \mathbf{w}_h)$ and $L(\lambda)$ are as given in Corollary 7.3.8.

Next, we show that $\mathbf{Ind}_{\mathrm{CM}}(G) = \mathbf{Ind}_{\mathrm{CM}}(\mathbb{G})$. Set $G_{sp}(\lambda) := B^T(\lambda E - A)^{-1}B$. Then we have $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$ and

$$\begin{aligned} \mathbb{G}(\lambda) &= L(\lambda) + (e_{m-\alpha} \otimes B^T)(\lambda E - A)^{-1}(e_{m-\alpha}^T \otimes B) \\ &= L(\lambda) + (e_{m-\alpha} \otimes I_n)B^T(\lambda E - A)^{-1}B(e_{m-\alpha}^T \otimes I_n) \\ &= L(\lambda) + \mathrm{diag}(0, \ldots, 0, \underbrace{G_{sp}(\lambda)}_{(m-\alpha)\text{-th position}}, 0, \ldots, 0). \end{aligned} \tag{7.44}$$

Since $L(\lambda)$ is a matrix pencil, it follows from (7.44) that the contribution in $\mathbf{Ind}_{\mathrm{CM}}(\mathbb{G})$ comes only from $\mathrm{diag}(0, \ldots, 0, G_{sp}(\lambda), 0, \ldots, 0)$. Hence we have

$$\begin{aligned} \mathbf{Ind}_{\mathrm{CM}}(\mathbb{G}) &= \mathbf{Ind}_{\mathrm{CM}}\big(\mathrm{diag}(0, \ldots, 0, G_{sp}(\lambda), 0, \ldots, 0)\big) \\ &= \mathbf{Ind}_{\mathrm{CM}}\big(G_{sp}(\lambda)\big) = \mathbf{Ind}_{\mathrm{CM}}(G). \end{aligned}$$

This completes the proof. ☐

Remark 7.3.12. *Although the Cauchy-Maslov index is defined for real symmetric rational matrices, it can be extended to Hermitian rational matrices.*

Remark 7.3.13. *Recall that for a symmetric rational matrix* $G(\lambda) = P(\lambda) + B^T(\lambda I_r - A)^{-1}B$ *with* $P(\lambda) = \sum_{j=0}^{m} A_j\lambda^j$ *and* $m > 1$*, Theorem 5.4.3 construct only one symmetric linearization* $\mathbb{T}(\lambda)$ *which is given by*

$$\mathbb{T}(\lambda) := \lambda \left[\begin{array}{cccc|c} & & & A_m & \\ & & \iddots & A_{m-1} & \\ & & \iddots & \vdots & \\ & \iddots & \iddots & \vdots & \\ A_m & A_{m-1} & \cdots & A_1 & \\ \hline & & & & -E \end{array}\right] + \left[\begin{array}{cccc|c} & & & -A_m & \\ & & \iddots & -A_{m-1} & \\ & \iddots & \iddots & \vdots & \\ -A_m & -A_{m-1} & \cdots & -A_2 & \\ & & & A_0 & C \\ \hline & & & B & A \end{array}\right]$$

Further, $\mathbb{T}(\lambda)$ *is a Rosenbrock strong linearization of* $G(\lambda)$ *if and only if* A_m *is nonsingular. By contrast, the family of GFPRs enables us to construct an infinite number of symmetric strong linearizations of* $G(\lambda)$*. In fact, by considering* $h = 0$*,* $\boldsymbol{t}_{w_h} = \emptyset$ *and* $\boldsymbol{t}_{v_h} = -m + (0 : m-3, 0 : m-5, \ldots)$*, we have* $\mathbb{L}_{\mathcal{S}}(h, \boldsymbol{t}_{w_h}, \boldsymbol{t}_{v_h}, \mathcal{X}, \mathcal{Y}) = \mathbb{T}(\lambda)$*, where* $\mathcal{X}$ *and* $\mathcal{Y}$ *are the trivial matrix assignments for* $\boldsymbol{t}_{w_h}$ *and* $\boldsymbol{t}_{v_h}$*, respectively.*

7.3.2 Hamiltonian linearizations

Recall that a rational matrix $G(\lambda)$ is said to be Hamiltonian (i.e., T-even) if $G(-\lambda)^T = G(\lambda)$. Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$, it follows that if $G(\lambda)$ is T-even then both $P(\lambda)$ and $G_{sp}(\lambda)$ are T-even. We now construct T-even Rosenbrock strong linearizations of $G(\lambda)$. We proceed as follows.

For the rest of this chapter, we define $J := \begin{bmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{bmatrix}$ when $r = 2\ell$. Note that $J^T = J^{-1} = -J$. Further, we define $\mathbb{J}_{k,r} := \operatorname{diag}(I_k, J)$ for any integer $k \geq 1$ when $r = 2\ell$.

Definition 7.3.14. *[38] A matrix $X \in \mathbb{C}^r$ with $r := 2\ell$ is said to be Hamiltonian (resp., skew-Hamiltonian) if JX is symmetric (resp., JX is skew-symmetric), that is, $(JX)^T = JX$ (resp., $(JX)^T = -JX$).*

If X is Hamiltonian then $(JX)^T = JX \Rightarrow (XJ)^T = XJ$. Similarly, if X is skew-Hamiltonian then we have $(XJ)^T = -XJ$.

Definition 7.3.15. *Let $G(\lambda)$ be a Hamiltonian (i.e., T-even) rational matrix.*

(a) A realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ is said to be a Hamiltonian realization of $G(\lambda)$ if $P(\lambda)$ is T-even, A is Hamiltonian with $r = 2\ell$ and $JB = C^T$.

(b) A system matrix $\mathcal{S}(\lambda)$ of the form $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda I_r \end{array}\right]$ is said to be a Hamiltonian system matrix if $r = 2\ell$ and $\mathbb{J}_{n,r}\mathcal{S}(\lambda)$ is T-even, that is, if $\big(\mathbb{J}_{n,r}\,\mathcal{S}(-\lambda)\big)^T = \mathbb{J}_{n,r}\,\mathcal{S}(\lambda)$, where $\mathbb{J}_{n,r} := \operatorname{diag}(I_n, J)$.

(c) A realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ with E being nonsingular is said to be a T-even realization of $G(\lambda)$ if $C = B^T$ and both $P(\lambda)$ and $\lambda E - A$ are T-even.

Note that the system matrix $\mathcal{S}(\lambda)$ associated with a T-even realization of $G(\lambda)$ is T-even, that is, $\mathcal{S}(-\lambda)^T = \mathcal{S}(\lambda)$.

Remark 7.3.16. *Observe that $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ is a Hamiltonian realization of $G(\lambda)$ if and only if $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda I_r \end{array}\right]$ is a Hamiltonian system matrix of $G(\lambda)$. On the other hand, $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ is a T-even realization of $G(\lambda)$ if and only if $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right]$ is a T-even system matrix of $G(\lambda)$.*

For convenience, we often refer to $\mathcal{S}(\lambda)$ as a T-even (resp., Hamiltonian) realization of $G(\lambda)$ when $\mathcal{S}(\lambda)$ is T-even (resp., Hamiltonian).

Proposition 7.3.17. *Suppose that $G(\lambda)$ is Hamiltonian (i.e., T-even). Then we have the following.*

(a) There exists a minimal Hamiltonian realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ with $r = 2\ell$ and $JB = C^T$. Thus the associated system matrix $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & B^T J^T \\ \hline B & A - \lambda I_r \end{array}\right]$ is Hamiltonian.

(b) There exists a minimal T-even realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + B^T(\lambda E - A)^{-1}B$. Thus the system matrix $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & B^T \\ \hline B & A - \lambda E \end{array}\right]$ is T-even.

Proof. Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$ is T-even, we have $P(\lambda)$ and $G_{sp}(\lambda)$ are T-even. Also since $G_{sp}(\lambda)$ is strictly proper and T-even, there exists a minimal Hamiltonian realization of $G_{sp}(\lambda)$ of the form $G_{sp}(\lambda) = C(\lambda I_r - A)^{-1}B$ with $r = 2\ell$ and $JB = C^T$; see [38]. Hence $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ is a minimal Hamiltonian realization of $G(\lambda)$. Obviously the system matrix $\mathcal{S}(\lambda)$ is Hamiltonian, that is, $(\mathbb{J}_{n,r}\,\mathcal{S}(-\lambda))^T = \mathbb{J}_{n,r}\,\mathcal{S}(\lambda)$, where $\mathbb{J}_{n,r} := \mathrm{diag}(I_n, J)$. This proves (a).

The results in (b) follow from (a). Indeed, by part (a) we have $G(\lambda) = P(\lambda) + B^T J^T(\lambda I_r - A)^{-1}B = P(\lambda) + B^T(\lambda J - AJ)^{-1}B$. Since A is Hamiltonian, it follows that $\lambda J - AJ$ is T-even. Hence setting $E := J$ and redefining $A := AJ$, it follows that $G(\lambda) := P(\lambda) + B^T(\lambda E - A)^{-1}B$ is a minimal T-even realization of $G(\lambda)$. Evidently, the system matrix $\mathcal{S}(\lambda)$ is T-even, that is, $\mathcal{S}(-\lambda)^T = \mathcal{S}(\lambda)$. This proves (b). ☐

We construct T-even (resp., Hamiltonian) linearizations of $G(\lambda)$ corresponding to a T-even (resp., Hamiltonian) realization of $G(\lambda)$. We proceed as follows.

Definition 7.3.18. *[20] A matrix $Q \in \mathbb{C}^{mn\times mn}$ is said to be a quasi-identity matrix if $Q = \epsilon_1 I_n \oplus \cdots \oplus \epsilon_m I_n$, where $\epsilon_i \in \{\pm 1\}$ for $i = 1 : m$. We refer to ϵ_j, $j = 1 : m$, as the j-th parameter of Q.*

We need the following result which is a particular case of [20, Theorem 4.15].

Theorem 7.3.19 ([20], Theorem 4.15)**.** *Let $0 \leq h \leq m-1$ be even. Let $\boldsymbol{w}$ be the simple admissible tuple of $\{0 : h\}$ and $\boldsymbol{c}_w$ be the symmetric complement of $\boldsymbol{w}$. Let $\boldsymbol{z}+m$ be any admissible tuple of $\{0 : m-h-1\}$ and $\boldsymbol{c}_z + m$ be the symmetric complement of $\boldsymbol{z}+m$. Let $L(\lambda) := \big(\lambda M^P_{\boldsymbol{z}} - M^P_{\boldsymbol{w}}\big)M^P_{\boldsymbol{c}_w}M^P_{\boldsymbol{c}_z}$. Then, up to multiplication by -1, there exists a unique quasi-identity matrix Q such that $QL(\lambda)$ is T-even (resp., T- odd) when $P(\lambda)$ is T-even (resp., T-odd).*

We refer to [20, Algorithm 4.14] for more on the construction of the quasi-identity matrix Q. The next result provides T-even linearizations of $G(\lambda)$.

Theorem 7.3.20. *Let $G(\lambda)$ be T-even and $\mathcal{S}(\lambda)$ be a T-even realization of $G(\lambda)$ as given in Proposition 7.3.17(b). Let $h, \boldsymbol{w}, \boldsymbol{c}_w, \boldsymbol{z}$ and $\boldsymbol{c}_z$ be as in Theorem 7.3.19. Consider the GFPR $\mathbb{L}(\lambda) := \big(\lambda \mathbb{M}^{\mathcal{S}}_{\boldsymbol{z}} - \mathbb{M}^{\mathcal{S}}_{\boldsymbol{w}}\big)\mathbb{M}^{P}_{\boldsymbol{c}_w}\mathbb{M}^{P}_{\boldsymbol{c}_z}$ associated with $\mathcal{S}(\lambda)$. Then there exists a unique quasi-identity matrix $\mathbb{Q} := \operatorname{diag}(\mathbf{s}\, Q, I_r)$ such that*

$$\mathbb{Q}\mathbb{L}(\lambda) = \left[\begin{array}{c|c} \mathbf{s}\, QL(\lambda) & e_{m-i_0(\boldsymbol{w})} \otimes B^T \\ \hline e^T_{m-i_0(\boldsymbol{w})} \otimes B & A - \lambda E \end{array}\right]$$

is T-even, where Q and $L(\lambda)$ are as in Theorem 7.3.19 and $\mathbf{s}$ is the $(m - i_0(\boldsymbol{w}))$-th parameter of Q.

Assume that $Ind(\boldsymbol{z} + m) = 0$ when the leading coefficient of $P(\lambda)$ is singular. Then $\mathbb{Q}\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. The transfer function $\mathbb{G}(\lambda) := \mathbf{s}\, QL(\lambda) + (e_{m-i_0(\boldsymbol{w})} \otimes B^T)(\lambda E - A)^{-1}(e^T_{m-i_0(\boldsymbol{w})} \otimes B)$ of $\mathbb{Q}\mathbb{L}(\lambda)$ is T-even.

Proof. By Theorem 7.2.4, we have

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & e_{m-i_0(\mathbf{w})} \otimes B^T \\ \hline e^T_{m-c_0(\mathbf{w},\mathbf{c}_w)} \otimes B & A - \lambda E \end{array}\right],$$

where $L(\lambda)$ is as given in Theorem 7.3.19. Since h is even and $\mathbf{w}$ is the simple admissible tuple of $\{0 : h\}$, we have $\mathbf{w} = (h-1 : h, \ldots, 3 : 4, 1 : 2, 0)$ and $\mathbf{c}_w = (h-1, h-3, \ldots, 1)$. This implies that $i_0(\mathbf{w}) = c_0(\mathbf{w}, \mathbf{c}_w) = 0$ if $h = 0$, and $i_0(\mathbf{w}) = c_0(\mathbf{w}, \mathbf{c}_w) = 1$ if $h > 0$. By Theorem 7.3.19, $\mathbf{s}\, QL(\lambda)$ is T-even. Set $\alpha := i_0(\mathbf{w})$. Then $Q(e_{m-\alpha} \otimes I_n) = \mathbf{s}(e_{m-\alpha} \otimes I_n)$. Note that $\mathbf{s}\mathbf{s} = 1$. Consequently, we have

$$\mathbb{Q}\mathbb{L}(\lambda) = \left[\begin{array}{c|c} \mathbf{s}\, QL(\lambda) & \mathbf{s}\, Q(e_{m-\alpha} \otimes B^T) \\ \hline e^T_{m-\alpha} \otimes B & A - \lambda E \end{array}\right] = \left[\begin{array}{c|c} \mathbf{s}\, QL(\lambda) & e_{m-\alpha} \otimes B^T \\ \hline e^T_{m-\alpha} \otimes B & A - \lambda E \end{array}\right]. \tag{7.45}$$

Since $\mathbf{s}\, QL(\lambda)$ and $A - \lambda E$ are T-even, it follows from (7.45) that $\mathbb{Q}\mathbb{L}(\lambda)$ is T-even.

Since h is even, by Remark 7.3.3 we have $0 \notin \mathbf{c}_w$. This implies that the matrix assignment for $\mathbf{c}_w$ is nonsingular. Hence by taking $\sigma := \mathbf{w}, \tau := \mathbf{z}, \sigma_1 := \emptyset, \sigma_2 := \mathbf{c}_w$, $\tau_1 := \emptyset$ and $\tau_2 := \mathbf{c}_z$, it follows from Theorem 7.2.16 that $\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$ if the matrix assignment for $\mathbf{c}_z$ is nonsingular. If the leading coefficient of $P(\lambda)$ is nonsingular then the matrix assignment for $\mathbf{c}_z$ is nonsingular. On the other hand, if the leading coefficient of $P(\lambda)$ is singular and $Ind(\mathbf{z} + m) = 0$, then by Remark 7.3.3, we have $0 \notin \mathbf{c}_z + m \Rightarrow -m \notin \mathbf{c}_z$. Hence the matrix assignment for $\mathbf{c}_z$ is nonsingular. Thus, $\mathbb{Q}\mathbb{L}(\lambda)$ is a T-even Rosenbrock strong linearization of $G(\lambda)$. Obviously the transfer function $\mathbb{G}(\lambda)$ is T-even. □

Example 7.3.21. *Let* $G(\lambda) := \sum_{i=0}^{5} \lambda^i A_i + B^T(\lambda E - A)^{-1}B$ *be a* T*-even realization of* $G(\lambda)$ *and* $\mathcal{S}(\lambda)$ *be as in Proposition 7.3.17(b). Consider the GFPR* $\mathbb{L}(\lambda) = (\lambda \mathbb{M}^{\mathcal{S}}_{(-4:-3,-5)} - \mathbb{M}^{\mathcal{S}}_{(1:2,0)})\mathbb{M}^P_1 \mathbb{M}^P_{-4}$ *and* $\mathbb{Q} = diag(I_n, I_n, -I_n, I_n, -I_n, I_r)$*. Then*

$$\mathbb{QL}(\lambda) = \left[\begin{array}{ccccc|c} 0 & -I_n & \lambda I_n & 0 & 0 & 0 \\ -I_n & \lambda A_5 - A_4 & \lambda A_4 & 0 & 0 & 0 \\ -\lambda I_n & -\lambda A_4 & -\lambda A_3 - A_2 & -A_1 & I_n & 0 \\ 0 & 0 & A_1 & -\lambda A_1 + A_0 & \lambda I_n & B^T \\ 0 & 0 & I_n & -\lambda I_n & 0 & 0 \\ \hline 0 & 0 & 0 & B & 0 & A - \lambda E \end{array}\right]$$

is a T*-even Rosenbrock strong linearization of* $G(\lambda)$*. Observe that* $\mathbb{QL}(\lambda)$ *is a block penta-diagonal pencil.*

Next, we consider a Hamiltonian realization of $G(\lambda)$ and construct a Hamiltonian strong linearization of $G(\lambda)$.

Theorem 7.3.22. *Let* $G(\lambda)$ *be Hamiltonian and* $\mathcal{S}(\lambda)$ *be a Hamiltonian realization of* $G(\lambda)$ *as given in Proposition 7.3.17(a). Assume that* $Ind(\boldsymbol{z} + m) = 0$ *when the leading coefficient of* $P(\lambda)$ *is singular, where* $\boldsymbol{z}$ *is as given in Theorem 7.3.20. Then* $\mathbb{T}(\lambda) := \left[\begin{array}{c|c} \mathbf{s}\, QL(\lambda) & e_{m-i_0(\boldsymbol{w})} \otimes B^T J^T \\ \hline e^T_{m-i_0(\boldsymbol{w})} \otimes B & A - \lambda I_r \end{array}\right]$ *is Hamiltonian and is a Rosenbrock strong linearization of* $G(\lambda)$*, where* $\boldsymbol{w}$ *and* $\mathbf{s}\, QL(\lambda)$ *are as given in Theorem 7.3.20.*

The transfer function $\mathbb{G}(\lambda) := \mathbf{s}QL(\lambda) + (e_{m-i_0(\boldsymbol{w})} \otimes B^T J^T)(\lambda I_r - A)^{-1}(e^T_{m-i_0(\boldsymbol{w})} \otimes B)$ *of* $\mathbb{T}(\lambda)$ *is Hamiltonian.*

Proof. Define $\widehat{\mathcal{S}}(\lambda) := \mathbb{J}_{n,r}\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & B^T J^T \\ \hline JB & JA - \lambda J \end{array}\right]$. Since A is Hamiltonian, we have $JA - \lambda J$ is T-even. This shows that $\widehat{\mathcal{S}}(\lambda)$ is a T-even realization of $G(\lambda)$. Hence by Theorem 7.3.20,

$$\widehat{\mathbb{L}}(\lambda) := \left[\begin{array}{c|c} \mathbf{s}\, QL(\lambda) & e_{m-i_0(\mathbf{w})} \otimes B^T J^T \\ \hline e^T_{m-i_0(\mathbf{w})} \otimes JB & JA - \lambda J \end{array}\right] \tag{7.46}$$

is a T-even Rosenbrock strong linearizations of $\widehat{\mathcal{S}}(\lambda)$. Note that $\widehat{\mathbb{L}}(\lambda) = \mathbb{J}_{mn,r}\mathbb{T}(\lambda)$, where $\mathbb{J}_{mn,r} := \text{diag}(I_{mn}, J)$. Since $\widehat{\mathbb{L}}(\lambda)$ is T-even, it follows that $\mathbb{T}(\lambda)$ is Hamiltonian, that is, $(\mathbb{J}_{mn,r}\mathbb{T}(-\lambda))^T = \mathbb{J}_{mn,r}\mathbb{T}(\lambda)$. Further, since $\widehat{\mathbb{L}}(\lambda)$ is a Rosenbrock strong linearization of $\widehat{\mathcal{S}}(\lambda)$ and $\widehat{\mathcal{S}}(\lambda) = \mathbb{J}_{n,r}\mathcal{S}(\lambda)$, it follows that $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $\mathcal{S}(\lambda)$. Obviously the transfer function $\mathbb{G}(\lambda)$ is Hamiltonian. ☐

7.3.3 Skew-Hamiltonian linearizations

Recall that a rational matrix $G(\lambda)$ is said to be skew-Hamiltonian (i.e., T-odd) if $G(-\lambda)^T = -G(\lambda)$.

Proposition 7.3.23. *Let $G(\lambda)$ be T-odd. Then there exists a minimal T-odd realization of $G(\lambda)$ of the form $G(\lambda) := P(\lambda)+B^T(\lambda I_r - A)^{-1}B$, where $P(\lambda)$ and $\lambda I_r - A$ are T-odd. Thus the system matrix $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & -B^T \\ \hline B & \lambda I_r - A \end{array}\right]$ is T-odd.*

Proof. Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$ is T-odd, it follows that both $P(\lambda)$ and $G_{sp}(\lambda)$ are T-odd. Since $G_{sp}(\lambda)$ is T-odd and strictly proper, there exists a minimal T-odd realization of $G_{sp}(\lambda)$ of the form $G_{sp}(\lambda) = B^T(\lambda I_r - A)^{-1}B$, where A is skew-symmetric; see [38]. Since A is skew-symmetric, we have $\lambda I_r - A$ is T-odd. This shows that $G(\lambda) = P(\lambda) + B^T(\lambda I_r - A)^{-1}B$ is a minimal T-odd realization of $G(\lambda)$ and that the system matrix $\mathcal{S}(\lambda)$ is T-odd. □

The next result gives T-odd Rosenbrock strong linearizations of $G(\lambda)$.

Theorem 7.3.24. *Let $G(\lambda)$ be T-odd and $\mathcal{S}(\lambda)$ be as given in Proposition 7.3.23. Let $h, \boldsymbol{w}, \boldsymbol{c}_w, \boldsymbol{z}$ and $\boldsymbol{c}_z$ be as in Theorem 7.3.19. Consider the GFPR $\mathbb{L}(\lambda) := \big(\lambda\mathbb{M}^{\mathcal{S}}_{\boldsymbol{z}} - \mathbb{M}^{\mathcal{S}}_{\boldsymbol{w}}\big)\mathbb{M}^P_{\boldsymbol{c}_w}\mathbb{M}^P_{\boldsymbol{c}_z}$ associated with $\mathcal{S}(\lambda)$. Then there exists a unique quasi-identity matrix $\mathbb{Q} := \mathrm{diag}(\mathbf{s}\,Q, I_r)$ such that*

$$\mathbb{Q}\mathbb{L}(\lambda) = \left[\begin{array}{c|c} \mathbf{s}QL(\lambda) & -e_{m-i_0(\boldsymbol{w})} \otimes B^T \\ \hline e^T_{m-i_0(\boldsymbol{w})} \otimes B & \lambda I_r - A \end{array}\right]$$

is T-odd, where Q and $L(\lambda)$ are as in Theorem 7.3.19 and $\mathbf{s}$ is the $(m - i_0(\boldsymbol{w}))$-th parameter of Q.

Assume that $Ind(\boldsymbol{z}+m) = 0$ when leading coefficient of $P(\lambda)$ is singular. Then $\mathbb{Q}\mathbb{L}(\lambda)$ is a T-odd Rosenbrock strong linearization of $G(\lambda)$. The transfer function $\mathbb{G}(\lambda) := \mathbf{s}\,QL(\lambda) + (e_{m-i_0(\boldsymbol{w})} \otimes B^T)(\lambda I_r - A)^{-1}(e^T_{m-i_0(\boldsymbol{w})} \otimes B)$ of $\mathbb{Q}\mathbb{L}(\lambda)$ is T-odd.

Proof. By Theorem 7.2.4, we have

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & e_{m-i_0(\mathbf{w})} \otimes (-B^T) \\ \hline e^T_{m-c_0(\mathbf{w},\mathbf{c}_w)} \otimes B & \lambda I_r - A \end{array}\right],$$

where $L(\lambda)$ is as given in Theorem 7.3.19. It is shown in the proof of Theorem 7.3.20 that $i_0(\mathbf{w}) = c_0(\mathbf{w}, \mathbf{c}_w)$. Set $\alpha := i_0(\mathbf{w})$. Then $Q(e_{m-\alpha} \otimes I_n) = \mathbf{s}\,(e_{m-\alpha} \otimes I_n)$. Note that

$\mathbf{ss} = 1$. Consequently, we have

$$\mathbb{QL}(\lambda) = \left[\begin{array}{c|c} \mathbf{s}\,QL(\lambda) & \mathbf{s}\,Q(e_{m-\alpha}\otimes(-B^T)) \\ \hline e_{m-\alpha}^T\otimes B & \lambda I_r - A \end{array}\right] = \left[\begin{array}{c|c} \mathbf{s}\,QL(\lambda) & e_{m-\alpha}\otimes(-B^T) \\ \hline e_{m-\alpha}^T\otimes B & \lambda I_r - A \end{array}\right]. \tag{7.47}$$

By Theorem 7.3.19, $\mathbf{s}\,QL(\lambda)$ is T-odd. Since $\lambda I_r - A$ is T-odd, it follows from (7.47) that $\mathbb{QL}(\lambda)$ is T-odd.

By the same arguments as given in the proof of Theorem 7.3.20, it follows that $\mathbb{QL}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. Obviously, the transfer function $\mathbb{G}(\lambda)$ is T-odd. □

Example 7.3.25. *Let $G(\lambda) = \sum_{i=0}^{5}\lambda^i A_i + B^T(\lambda I_r - A)^{-1}B$ be T-odd realization of $G(\lambda)$ and $\mathcal{S}(\lambda)$ be as given in Proposition 7.3.23. Consider $\mathbb{Q} = \operatorname{diag}(I_n, -I_n, I_n, -I_n, -I_n, I_r)$ and the GFPR $\mathbb{L}(\lambda) := (\lambda\mathbb{M}^{\mathcal{S}}_{(-4:-3,-5)} - \mathbb{M}^{\mathcal{S}}_{(1:2,0)})\mathbb{M}^P_1\mathbb{M}^P_{-4}$. Then*

$$\mathbb{QL}(\lambda) = \left[\begin{array}{ccccc|c} 0 & -I_n & \lambda I_n & 0 & 0 & 0 \\ I_n & -\lambda A_5 + A_4 & -\lambda A_4 & 0 & 0 & 0 \\ \lambda I_n & \lambda A_4 & \lambda A_3 + A_2 & A_1 & -I_n & 0 \\ 0 & 0 & -A_1 & \lambda A_1 - A_0 & -\lambda I_n & -B^T \\ 0 & 0 & I_n & -\lambda I_n & 0 & 0 \\ \hline 0 & 0 & 0 & B & 0 & \lambda I_r - A \end{array}\right]$$

is a T-odd Rosenbrock strong linearization of $G(\lambda)$. Notice that $\mathbb{QL}(\lambda)$ is a block penta-diagonal pencil.

7.3.4 Skew-symmetric linearizations

Suppose that $G(\lambda)$ is skew-symmetric, that is, $G(\lambda)^T = -G(\lambda)$. Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$, it follows that $P(\lambda)$ and $G_{sp}(\lambda)$ are skew-symmetric.

Definition 7.3.26. *Suppose that $G(\lambda)$ is skew-symmetric.*

(a) A realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ is said to be a skew-Hamiltonian realization of $G(\lambda)$ if $P(\lambda)$ is skew-symmetric, A is skew-Hamiltonian with $r = 2\ell$ and $C^T = JB$.

(b) A system matrix $\mathcal{S}(\lambda)$ of the form $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & -C \\ \hline B & \lambda I_r - A \end{array}\right]$ is said to be a skew-Hamiltonian system matrix if $r = 2\ell$ and $\left(\mathbb{J}_{n,r}\,\mathcal{S}(\lambda)\right)^T = -\mathbb{J}_{n,r}\,\mathcal{S}(\lambda)$, where $\mathbb{J}_{n,r} := \operatorname{diag}(I_n, J)$.

(c) *A realization of* $G(\lambda)$ *of the form* $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ *with* E *being nonsingular is said to be a skew-symmetric realization of* $G(\lambda)$ *if* $C = B^T$ *and both* $P(\lambda)$ *and* $\lambda E - A$ *are skew-symmetric.*

Remark 7.3.27. *Observe that* $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ *is a skew-Hamiltonian realization of* $G(\lambda)$ *if and only if* $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & -C \\ \hline B & \lambda I_r - A \end{array}\right]$ *is a skew-Hamiltonian system matrix of* $G(\lambda)$*. On the other hand,* $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ *is a skew-symmetric realization of* $G(\lambda)$ *if and only if* $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & -C \\ \hline B & \lambda E - A \end{array}\right]$ *is a skew-symmetric system matrix of* $G(\lambda)$*.*

For convenience, we often refer to $\mathcal{S}(\lambda)$ as a skew-symmetric (resp., skew-Hamiltonian) realization of $G(\lambda)$ when $\mathcal{S}(\lambda)$ is skew-symmetric (resp., skew-Hamiltonian).

Proposition 7.3.28. *Suppose that* $G(\lambda)$ *is skew-symmetric. Then we have the following.*

(a) *There exists a minimal skew-Hamiltonian realization of* $G(\lambda)$ *of the form* $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ *with* $r = 2\ell$ *and* $JB = C^T$*. Thus the system matrix* $\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & -B^TJ^T \\ \hline B & \lambda I_r - A \end{array}\right]$ *associated with* $G(\lambda)$ *is skew-Hamiltonian.*

(b) *There exists a minimal skew-symmetric realization of* $G(\lambda)$ *of the form* $G(\lambda) = P(\lambda) + B^T(\lambda E - A)^{-1}B$*. Thus the system matrix* $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & -B^T \\ \hline B & \lambda E - A \end{array}\right]$ *associated with* $G(\lambda)$ *is skew-symmetric.*

Proof. Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$ is skew-symmetric, we have both $P(\lambda)$ and $G_{sp}(\lambda)$ are skew-symmetric. Also since $G_{sp}(\lambda)$ is strictly proper and skew-symmetric, there exists a minimal skew-Hamiltonian realization of $G_{sp}(\lambda)$ of the form $G_{sp}(\lambda) = C(\lambda I_r - A)^{-1}B$ with $r = 2\ell$ and $JB = C^T$; see [38]. Hence $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ is a minimal skew-Hamiltonian realization of $G(\lambda)$. Obviously the system matrix $\mathcal{S}(\lambda)$ is skew-Hamiltonian, that is, $(\mathbb{J}_{n,r}\,\mathcal{S}(\lambda))^T = -\mathbb{J}_{n,r}\,\mathcal{S}(\lambda)$, where $\mathbb{J}_{n,r} := \operatorname{diag}(I_n, J)$. This proves (a).

By part (a), $G(\lambda) = P(\lambda) + B^TJ^T(\lambda I_r - A)^{-1}B = P(\lambda) + B^T(\lambda J - AJ)^{-1}B$. Since A is skew-Hamiltonian, it follows that $\lambda J - AJ$ is skew-symmetric. Hence setting $E := J$ and redefining $A := AJ$, it follows that $G(\lambda) = P(\lambda) + B^T(\lambda E - A)^{-1}B$ is a minimal skew-symmetric realization of $G(\lambda)$. Evidently, the system matrix $\mathcal{S}(\lambda)$ is skew-symmetric, that is, $\mathcal{S}(\lambda)^T = -\mathcal{S}(\lambda)$. This proves (b). □

Recall from Definition 1.2.33 the type-1 left and right index tuples. We need the following result which is a particular case of [20, Theorem 3.15].

Theorem 7.3.29. *[20] Let $P(\lambda)$ be skew symmetric and let $0 \leq h \leq m-1$ be even. Let $\boldsymbol{w}$ be the simple admissible tuple of $\{0:h\}$ and $\boldsymbol{c}_w$ be the symmetric complement of $\boldsymbol{w}$. Let $\boldsymbol{z}+m$ be any admissible tuple of $\{0:m-h-1\}$. Let $\boldsymbol{c}_z+m$ be the symmetric complement of $\boldsymbol{z}+m$. Let $\boldsymbol{t}_w$ containing indices from $\{0:h-1\}$ and $\boldsymbol{t}_z+m$ containing indices from $\{0:m-h-2\}$ be right index tuples of type-1 relative to $rev(\boldsymbol{w})$ and $rev(\boldsymbol{z}+m)$, respectively. Consider*

$$L(\lambda) := M^P_{rev(\boldsymbol{t}_z)}M^P_{rev(\boldsymbol{t}_w)}\big(\lambda M^P_{\boldsymbol{z}} - M^P_{\boldsymbol{w}}\big)M^P_{\boldsymbol{c}_w}M^P_{\boldsymbol{t}_w}M^P_{\boldsymbol{c}_z}M^P_{\boldsymbol{t}_z}.$$

Then, up to multiplication by -1, there exists a unique quasi-identity matrix Q such that $QL(\lambda)$ is skew-symmetric.

We now construct skew-symmetric Rosenbrock strong linearizations of $G(\lambda)$.

Theorem 7.3.30. *Let $G(\lambda)$ be skew-symmetric and $\mathcal{S}(\lambda)$ be a skew-symmetric realization of $G(\lambda)$ as in Proposition 7.3.28(b). Let $h, \boldsymbol{w}, \boldsymbol{c}_w, \boldsymbol{t}_w, \boldsymbol{z}, \boldsymbol{c}_z$ and $\boldsymbol{t}_z$ be as in Theorem 7.3.29. Consider the GFPR $\mathbb{L}(\lambda) := \mathbb{M}^P_{rev(\boldsymbol{t}_z)}\mathbb{M}^P_{rev(\boldsymbol{t}_w)}\big(\lambda\mathbb{M}^{\mathcal{S}}_{\boldsymbol{z}} - \mathbb{M}^{\mathcal{S}}_{\boldsymbol{w}}\big)\mathbb{M}^P_{\boldsymbol{c}_w}\mathbb{M}^P_{\boldsymbol{t}_w}\mathbb{M}^P_{\boldsymbol{c}_z}\mathbb{M}^P_{\boldsymbol{t}_z}$ associated with $\mathcal{S}(\lambda)$. Then there exists a unique quasi-identity matrix $\mathbb{Q} := \mathrm{diag}(\mathbf{s}\,Q, I_r)$ such that*

$$\mathbb{Q}\mathbb{L}(\lambda) = \left[\begin{array}{c|c} \mathbf{s}\,QL(\lambda) & -e_{m-\alpha}\otimes B^T \\ \hline e^T_{m-\alpha}\otimes B & \lambda E - A \end{array}\right],$$

is skew-symmetric, where Q and $L(\lambda)$ are as in Theorem 7.3.29 and $\mathbf{s}$ is the $(m-\alpha)$-th parameter of Q with $\alpha := c_0(\boldsymbol{w}, \boldsymbol{c}_w, \boldsymbol{t}_w)$.

Assume that $Ind(\boldsymbol{z}+m) = 0$ when the leading coefficient A_m of $P(\lambda)$ is singular. Further, suppose that $0 \notin \boldsymbol{t}_w$ (resp., $-m \notin \boldsymbol{t}_z$) when A_0 (resp., A_m) is singular. Then $\mathbb{Q}\mathbb{L}(\lambda)$ is a skew-symmetric Rosenbrock strong linearization of $G(\lambda)$. The transfer function $\mathbb{G}(\lambda) := \mathbf{s}\,QL(\lambda) + (e_{m-\alpha}\otimes B^T)(\lambda E - A)^{-1}(e^T_{m-\alpha}\otimes B)$ of $\mathbb{Q}\mathbb{L}(\lambda)$ is skew-symmetric.

Proof. By Theorem 7.2.4, we have

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & e_{m-i_0(rev(\mathbf{t}_w),\mathbf{w})}\otimes(-B^T) \\ \hline e^T_{m-c_0(\mathbf{w},\mathbf{c}_w,\mathbf{t}_w)}\otimes B & \lambda E - A \end{array}\right],$$

where $L(\lambda)$ is as in Theorem 7.3.29. Next, we show that $i_0(rev(\mathbf{t}_w), \mathbf{w}) = c_0(\mathbf{w}, \mathbf{c}_w, \mathbf{t}_w)$. If $h = 0$ then $\mathbf{w} = (0)$ and $\mathbf{c}_w = \emptyset = \mathbf{t}_w$. Thus $i_0(rev(\mathbf{t}_w), \mathbf{w}) = 0 = c_0(\mathbf{w}, \mathbf{c}_w, \mathbf{t}_w)$.

Next, suppose that $h > 0$. Then we have $\mathbf{w} = (h-1 : h, h-3 : h-2, \ldots, 1 : 2, 0)$ and $\mathbf{c}_w = (h-1, h-3, \ldots, 3, 1)$. This implies that $c_0(\mathbf{w}, \mathbf{c}_w, \mathbf{t}_w) = 2 + c_2(\mathbf{t}_w)$ and $i_0(rev(\mathbf{t}_w), \mathbf{w}) = 2 + i_2(rev(\mathbf{t}_w)) = 2 + c_2(\mathbf{t}_w)$. Hence $i_0(rev(\mathbf{t}_w), \mathbf{w}) = c_0(\mathbf{w}, \mathbf{c}_w, \mathbf{t}_w)$.

By Theorem 7.3.29, we have $\mathbf{s}\, QL(\lambda)$ is skew-symmetric. Note that $Q(e_{m-\alpha} \otimes I_n) = \mathbf{s}\,(e_{m-\alpha} \otimes I_n)$ and $\mathbf{ss} = 1$. Consequently, we have

$$\mathbb{QL}(\lambda) = \left[\begin{array}{c|c} \mathbf{s}\, QL(\lambda) & \mathbf{s}\, Q(e_{m-\alpha} \otimes (-B^T)) \\ \hline e_{m-\alpha}^T \otimes B & \lambda E - A \end{array}\right] = \left[\begin{array}{c|c} \mathbf{s}\, QL(\lambda) & e_{m-\alpha} \otimes (-B^T) \\ \hline e_{m-\alpha}^T \otimes B & \lambda E - A \end{array}\right]. \tag{7.48}$$

Since $\mathbf{s}\, QL(\lambda)$ and $\lambda E - A$ are skew-symmetric, it follows from (7.48) that $\mathbb{QL}(\lambda)$ is skew-symmetric.

Since $0 \notin \mathbf{t}_w$ (resp., $-m \notin \mathbf{t}_z$) when A_0 (resp., A_m) is singular, the matrix assignments of $\mathbf{t}_w, \operatorname{rev}(\mathbf{t}_w), \mathbf{t}_z$ and $\operatorname{rev}(\mathbf{t}_z)$ are nonsingular. Hence by taking $\sigma := \mathbf{w}, \tau := \mathbf{z}, \sigma_1 := \operatorname{rev}(\mathbf{t}_w), \sigma_2 := (\mathbf{c}_w, \mathbf{t}_w)$, $\tau_1 := \operatorname{rev}(\mathbf{t}_z)$ and $\tau_2 := (\mathbf{c}_z, \mathbf{t}_z)$, it follows from Theorem 7.2.16 that $\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $\mathcal{S}(\lambda)$ if the matrix assignments for $\mathbf{c}_w$ and $\mathbf{c}_z$ are nonsingular. By the similar arguments as given in the proof Theorem 7.3.20, it follows that the matrix assignments for $\mathbf{c}_w$ and $\mathbf{c}_z$ are nonsingular. $\square$

Example 7.3.31. *Let* $G(\lambda) = \sum_{i=0}^{5} \lambda^i A_i + B^T(\lambda E - A)^{-1}B$ *be skew-symmetric and* $\mathcal{S}(\lambda)$ *be as in Proposition 7.3.28(b). Define* $\mathbb{L}(\lambda) := \left(\lambda \mathbb{M}^{\mathcal{S}}_{(-4:-3,-5)} - \mathbb{M}^{\mathcal{S}}_{(1:2,0)}\right)\mathbb{M}^P_1 \mathbb{M}^P_{-4}$ *and* $\mathbb{Q} := \operatorname{diag}(I_n, -I_n, -I_n, -I_n, I_n, I_r)$. *Then*

$$\mathbb{QL}(\lambda) = \left[\begin{array}{ccccc|c} 0 & -I_n & \lambda I_n & 0 & 0 & 0 \\ I_n & -\lambda A_5 + A_4 & -\lambda A_4 & 0 & 0 & 0 \\ -\lambda I_n & -\lambda A_4 & -\lambda A_3 - A_2 & -A_1 & I_n & 0 \\ 0 & 0 & -A_1 & \lambda A_1 - A_0 & -\lambda I_n & -B^T \\ 0 & 0 & -I_n & \lambda I_n & 0 & 0 \\ \hline 0 & 0 & 0 & B & 0 & \lambda E - A \end{array}\right]$$

is a skew-symmetric Rosenbrock strong linearization of $G(\lambda)$. *Observe that* $\mathbb{QL}(\lambda)$ *is a block penta-diagonal pencil.*

Next, we construct skew-Hamiltonian strong linearizations of $G(\lambda)$.

Theorem 7.3.32. *Let* $G(\lambda)$ *be skew-symmetric and* $\mathcal{S}(\lambda)$ *be a skew-Hamiltonian realization of* $G(\lambda)$ *as in Proposition 7.3.28(a). Let* $\boldsymbol{w}, \boldsymbol{c}_w, \boldsymbol{t}_w, \boldsymbol{z}, \boldsymbol{c}_z$ *and* $\boldsymbol{t}_z$ *be as in Theorem 7.3.30. Suppose that* $0 \notin \boldsymbol{t}_w$ *(resp.,* $-m \notin \boldsymbol{t}_z$*) when* A_0 *(resp.,* A_m*) is singular.*

Assume that $Ind(\boldsymbol{z}+m)=0$ *when* A_m *is singular. Then*

$$\mathbb{T}(\lambda) := \left[\begin{array}{c|c} \mathbf{s}\,QL(\lambda) & -e_{m-\alpha}\otimes B^TJ^T \\ \hline e^T_{m-\alpha}\otimes B & \lambda I_r - A \end{array}\right]$$

is a skew-Hamiltonian Rosenbrock strong linearization of $G(\lambda)$*, where* α *and* $\mathbf{s}\,QL(\lambda)$ *are as in Theorem 7.3.30. The transfer function* $\mathbb{G}(\lambda) := \mathbf{s}\,QL(\lambda)+(e_{m-\alpha}\otimes B^TJ^T)(\lambda I_r - A)^{-1}(e^T_{m-\alpha}\otimes B)$ *of* $\mathbb{T}(\lambda)$ *is skew-symmetric.*

Proof. Define $\widehat{\mathcal{S}}(\lambda) := \mathbb{J}_{n,r}\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & -B^TJ^T \\ \hline JB & \lambda J - JA \end{array}\right]$. Since A is skew-Hamiltonian, we have $\lambda J - JA$ is skew-symmetric. Hence $\widehat{\mathcal{S}}(\lambda)$ is skew-symmetric as $P(\lambda)$ and $\lambda J - JA$ are skew-symmetric. Now by Theorem 7.3.30,

$$\widehat{\mathbb{L}}(\lambda) := \left[\begin{array}{c|c} \mathbf{s}\,QL(\lambda) & -e_{m-\alpha}\otimes B^TJ^T \\ \hline e^T_{m-\alpha}\otimes JB & \lambda J - JA \end{array}\right] \tag{7.49}$$

is a skew-symmetric Rosenbrock strong linearizations of $\widehat{\mathcal{S}}(\lambda)$, where α and $\mathbf{s}\,QL(\lambda)$ are as in Theorem 7.3.30. Note that $\widehat{\mathbb{L}}(\lambda) = \mathbb{J}_{mn,r}\mathbb{T}(\lambda)$. Since $\widehat{\mathbb{L}}(\lambda)$ is skew-symmetric, it follows that $\mathbb{T}(\lambda)$ is skew-Hamiltonian, that is, $(\mathbb{J}_{mn,r}\mathbb{T}(\lambda))^T = -\mathbb{J}_{mn,r}\mathbb{T}(\lambda)$. Further, since $\widehat{\mathbb{L}}(\lambda)$ is a Rosenbrock strong linearization of $\widehat{\mathcal{S}}(\lambda)$ and $\widehat{\mathcal{S}}(\lambda) = \mathbb{J}_{n,r}\mathcal{S}(\lambda)$, it follows that $\mathbb{T}(\lambda)$ is a Rosenbrock strong linearization of $\mathcal{S}(\lambda)$. Obviously $\mathbb{G}(\lambda)$ is skew-symmetric and is the transfer function of $\mathbb{T}(\lambda)$. ☐

7.4 Recovery of eigenvectors and minimal bases

We now describe the recovery of eigenvectors, minimal bases and minimal indices of $G(\lambda)$ from those of the GFPRs of $G(\lambda)$. In view of Theorem 5.5.8 we only need to describe the recovery of eigenvectors, minimal bases and minimal indices of $\mathcal{S}(\lambda)$ from those of the GFPRs of $\mathcal{S}(\lambda)$.

For the rest of this chapter, we only consider GFPRs with nonsingular matrix assignments. Thus, if $\mathbb{L}(\lambda) := \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1,X_1)\ (\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}_{(\sigma_2,\tau_2)}(X_2,Y_2)$ is a GFPR of $\mathcal{S}(\lambda)$ then we assume that X_j and Y_j, $j=1,2$, are nonsingular matrix assignments.

Theorem 7.4.1. *Let* $\mathbb{L}(\lambda) := \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1,X_1)(\lambda\mathbb{M}^{\mathcal{S}}_{\tau} - \mathbb{M}^{\mathcal{S}}_{\sigma})\mathbb{M}_{(\sigma_2,\tau_2)}(X_2,Y_2)$ *be a GFPR of* $\mathcal{S}(\lambda)$*. Let* $Z(\lambda) := \left[\begin{array}{c} Z_{mn}(\lambda) \\ \hline Z_r(\lambda) \end{array}\right]$ *be an* $(mn+r)\times p$ *matrix polynomial, where* $Z_{mn}(\lambda)$ *has* mn *rows and* $Z_r(\lambda)$ *has* r *rows. Then we have the following.*

(a) If $Z(\lambda)$ *is a right (resp., left) minimal basis of* $\mathbb{L}(\lambda)$ *then* $\left[\begin{array}{c} (e^T_{m-c_0(\sigma,\sigma_2)}\otimes I_n)Z_{mn}(\lambda) \\ \hline Z_r(\lambda) \end{array}\right]$

(resp., $\begin{bmatrix} (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n) Z_{mn}(\lambda) \\ \hline Z_r(\lambda) \end{bmatrix}$*) is a right (resp., left) minimal basis of* $\mathcal{S}(\lambda)$*.*

(b) Let τ *be given by* $\tau := (\tau_l, -m, \tau_r)$*. Set* $\alpha := \big(-rev(\tau_l), \sigma, -rev(\tau_r)\big)$*. Let* $c(\alpha)$ *and* $i(\alpha)$ *be the total number of consecutions and inversions of the permutation* α*. If* $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ *are the right (resp., left) minimal indices of* $\mathbb{L}(\lambda)$ *then* $\varepsilon_1 - i(\alpha) \leq \cdots \leq \varepsilon_p - i(\alpha)$ *(resp.,* $\varepsilon_1 - c(\alpha) \leq \cdots \leq \varepsilon_p - c(\alpha)$*) are the right (resp., left) minimal indices of* $\mathcal{S}(\lambda)$*.*

Proof. We have $\mathbb{L}(\lambda) = U\, \mathbb{T}_\omega(\lambda)\, V$, where $\mathbb{T}_\omega(\lambda) := \lambda \mathbb{M}^{\mathcal{S}}_\tau - \mathbb{M}^{\mathcal{S}}_\sigma$ is a PGF pencil of $G(\lambda)$ associated with the permutation $\omega := (\sigma, -\tau)$ of $\{0 : m\}$, and $U := \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1)$ and $V := \mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2)$. Since V is a nonsingular matrix, it is easily seen that the map $V : \mathcal{N}_r(\mathbb{L}) \to \mathcal{N}_r(\mathbb{T}_\omega),\ z(\lambda) \mapsto V z(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{L})$ to a minimal basis of $\mathcal{N}_r(\mathbb{T}_\omega)$. On the other hand, by Theorem 6.2.4, $\mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}) : \mathcal{N}_r(\mathbb{T}_\omega) \to \mathcal{N}_r(\mathcal{S}),\ \begin{bmatrix} x(\lambda) \\ \hline y(\lambda) \end{bmatrix} \mapsto \begin{bmatrix} (e^T_{m-c_0(\sigma)} \otimes I_n) x(\lambda) \\ \hline y(\lambda) \end{bmatrix}$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{T}_\omega)$ to a minimal basis of $\mathcal{N}_r(\mathcal{S})$, where $x(\lambda) \in \mathbb{C}(\lambda)^{mn}$ and $y(\lambda) \in \mathbb{C}(\lambda)^r$. Consequently, $\mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}) V : \mathcal{N}_r(\mathbb{L}) \to \mathcal{N}_r(\mathcal{S}),\ z(\lambda) \mapsto \mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}) V z(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{L})$ to a minimal basis of $\mathcal{N}_r(\mathcal{S})$. Now, by Lemma 2.1.11, we have $\mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}) V = \mathbb{F}^{\text{PGF}}_\omega(\mathcal{S}) \mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2) =$

$$\left[\begin{array}{c|c} (e^T_{m-c_0(\sigma)} \otimes I_n)\, M_{(\sigma_2,\tau_2)}(X_2, Y_2) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n & \\ \hline & I_r \end{array}\right],$$

and hence the desired result for the recovery of right minimal bases follows.

Now we describe the recovery of left minimal bases. Since U is a nonsingular matrix, it is easily seen that the map $U^T : \mathcal{N}_l(\mathbb{L}) \to \mathcal{N}_l(\mathbb{T}_\omega),\ z(\lambda) \mapsto U^T z(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_l(\mathbb{L})$ to a minimal basis of $\mathcal{N}_l(\mathbb{T}_\omega)$. On the other hand, by Theorem 6.2.4, $\mathbb{K}^{\text{PGF}}_\omega(\mathcal{S}) : \mathcal{N}_l(\mathbb{T}_\omega) \to \mathcal{N}_l(\mathcal{S}),\ \begin{bmatrix} x(\lambda) \\ \hline y(\lambda) \end{bmatrix} \mapsto \begin{bmatrix} (e^T_{m-i_0(\sigma)} \otimes I_n) x(\lambda) \\ \hline y(\lambda) \end{bmatrix}$, is an isomorphism and maps a minimal basis of $\mathcal{N}_l(\mathbb{T}_\omega)$ to a minimal basis of $\mathcal{N}_l(\mathcal{S})$, where $x(\lambda) \in \mathbb{C}(\lambda)^{mn}$ and $y(\lambda) \in \mathbb{C}(\lambda)^r$. Consequently, $\mathbb{K}^{\text{PGF}}_\omega(\mathcal{S}) U^T : \mathcal{N}_l(\mathbb{L}) \to \mathcal{N}_l(\mathcal{S}),\ z(\lambda) \mapsto \mathbb{K}^{\text{PGF}}_\omega(\mathcal{S}) U^T z(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_l(\mathbb{L})$ to a minimal basis of $\mathcal{N}_l(\mathcal{S})$. Now $\mathbb{K}^{\text{PGF}}_\omega(\mathcal{S}) U^T = \mathbb{K}^{\text{PGF}}_\omega(\mathcal{S}) (\mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1))^T =$

$$\left[\begin{array}{c|c} (e^T_{m-i_0(\sigma)} \otimes I_n)\big(M_{(\tau_1,\sigma_1)}(Y_1, X_1)\big)^T & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} \Big(M_{(\tau_1,\sigma_1)}(Y_1, X_1)(e_{m-i_0(\sigma)} \otimes I_n)\Big)^T & \\ \hline & I_r \end{array}\right].$$

By Lemma 2.1.11, we have $M_{(\tau_1,\sigma_1)}(Y_1, X_1)\,(e_{m-i_0(\sigma)} \otimes I_n) = e_{m-i_0(\sigma_1,\sigma)} \otimes I_n$. Hence the desired result for recovery of left minimal bases follows.

Finally, let $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ be the right (resp., left) minimal indices of $\mathbb{L}(\lambda)$. Since the PGF pencil $\mathbb{T}_\omega(\lambda)$ is strictly equivalent to $\mathbb{L}(\lambda)$, $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are also the right (resp., left) minimal indices of $\mathbb{T}_\omega(\lambda)$. Hence by Theorem 6.2.4, $\varepsilon_1 - i(\alpha) \leq \cdots \leq \varepsilon_p - i(\alpha)$ (resp., $\varepsilon_1 - c(\alpha) \leq \cdots \leq \varepsilon_p - c(\alpha)$) are the right (resp., left) minimal indices of $\mathcal{S}(\lambda)$. ☐

Next we describe the recovery of eigenvectors of $\mathcal{S}(\lambda)$ from those of the GFPRs of $\mathcal{S}(\lambda)$ when $\mathcal{S}(\lambda)$ is regular. For this purpose, we need the following result.

Theorem 7.4.2. *[4] Let $\mathbb{T}_\omega(\lambda) := \lambda \mathbb{M}^{\mathcal{S}}_{-\omega_1} - \mathbb{M}^{\mathcal{S}}_{\omega_0}$ be the GF pencil of $\mathcal{S}(\lambda)$ associated with a permutation $\omega := (\omega_0, \omega_1)$ of $\{0 : m\}$, where $0 \in \omega_0$ and $m \in \omega_1$. Suppose that $\mathcal{S}(\lambda)$ is regular and $\mu \in \mathbb{C}$ is an eigenvalue of $\mathcal{S}(\lambda)$. Let $Z := \left[\begin{array}{c} Z_{mn} \\ \hline Z_r \end{array}\right]$ be an $(mn + r) \times p$ matrix such that $\operatorname{rank}(Z) = p$, where Z_{mn} has mn rows and Z_r has r rows. If Z is a basis of $\mathcal{N}_r(\mathbb{T}_\omega(\mu))$ (resp., $\mathcal{N}_l(\mathbb{T}_\omega(\mu))$) then $\left[\begin{array}{c} (e^T_{m-c_0(\omega_0)} \otimes I_n) Z_{mn} \\ \hline Z_r \end{array}\right]$ (resp., $\left[\begin{array}{c} (e^T_{m-i_0(\omega_0)} \otimes I_n) Z_{mn} \\ \hline Z_r \end{array}\right]$) is a basis of $\mathcal{N}_r(\mathcal{S}(\mu))$ (resp., $\mathcal{N}_l(\mathcal{S}(\mu))$).*

Theorem 7.4.3. *Let $\mathbb{L}(\lambda) := \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1)(\lambda \mathbb{M}^{\mathcal{S}}_\tau - \mathbb{M}^{\mathcal{S}}_\sigma)\mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2)$ be a GFPR of $\mathcal{S}(\lambda)$. Suppose that $\mathcal{S}(\lambda)$ is regular and $\mu \in \mathbb{C}$ is an eigenvalue of $\mathcal{S}(\lambda)$. Let $Z := \left[\begin{array}{c} Z_{mn} \\ \hline Z_r \end{array}\right]$ be an $(mn + r) \times p$ matrix such that $\operatorname{rank}(Z) = p$, where Z_{mn} has mn rows and Z_r has r rows. If Z is a basis of $\mathcal{N}_r(\mathbb{L}(\mu))$ (resp., $\mathcal{N}_l(\mathbb{L}(\mu))$) then $\left[\begin{array}{c} (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n) Z_{mn} \\ \hline Z_r \end{array}\right]$ (resp., $\left[\begin{array}{c} (e^T_{m-i_0(\sigma_1,\sigma)} \otimes I_n) Z_{mn} \\ \hline Z_r \end{array}\right]$) is a basis of $\mathcal{N}_r(\mathcal{S}(\mu))$ (resp., $\mathcal{N}_l(\mathcal{S}(\mu))$).*

Proof. A verbatim proof of Theorem 7.4.1 together with Theorem 7.4.2 yields the desired results. ☐

Next, we briefly describe the recovery of eigenvectors, minimal bases and minimal indices of a structured $G(\lambda)$ from those of the structured linearizations discussed in Section 7.3.

Note that if $G(\lambda)$ is singular then the left (resp., right) minimal indices of $G(\lambda)$ and $XG(\lambda)Y$ are the same for any nonsingular matrices X and Y. Hence it follows that if

$G(\lambda)$ is symmetric (resp., skew-symmetric, Hamiltonian, skew-Hamiltonian) then the left minimal indices of $G(\lambda)$ are the same as the right minimal indices of $G(\lambda)$. Consequently, if $\mathbb{L}(\lambda)$ is a structure-preserving linearization of $G(\lambda)$ considered in Section 7.3 then the left minimal indices of $\mathbb{L}(\lambda)$ are the same as the right minimal indices of $\mathbb{L}(\lambda)$. Since $\mathbb{L}(\lambda)$ is strictly equivalent to a GFPR $\mathbb{T}(\lambda) := \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1)(\lambda \mathbb{M}_\tau^{\mathcal{S}} - \mathbb{M}_\sigma^{\mathcal{S}})\mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2)$ of $G(\lambda)$, the left and right minimal indices of $\mathbb{T}(\lambda)$ are the same. Let τ be given by $\tau = (\tau_\ell, -m, \tau_r)$. Define $\alpha := (-rev(\tau_\ell), \sigma, -rev(\tau_r))$. Then α is a permutation of $\{0 : m-1\}$. Let $c(\alpha)$ and $i(\alpha)$, respectively, be the total number of consecutions and inversions of α. Let $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ be the minimal (left and right) indices of $\mathbb{T}(\lambda)$. Then by Theorem 7.4.1, $\varepsilon_1 - i(\alpha) \leq \cdots \leq \varepsilon_k - i(\alpha)$ and $\varepsilon_1 - c(\alpha) \leq \cdots \leq \varepsilon_k - c(\alpha)$, respectively, are the right and left minimal indices of $G(\lambda)$. Since the left and right minimal indices of $G(\lambda)$ are the same, we must have $i(\alpha) = c(\alpha)$. But $i(\alpha) + c(\alpha) = m - 1$. Consequently, we have $i(\alpha) = (m-1)/2 = c(\alpha)$ which shows that $\varepsilon_1 - (m-1)/2 \leq \cdots \leq \varepsilon_k - (m-1)/2$ are the minimal (left and right) indices of $G(\lambda)$. Recall that $\mathbb{L}(\lambda)$ is not a linearization of $G(\lambda)$ if m is even.

Thus, if $\mathbb{L}(\lambda)$ is a structure-preserving linearization of $G(\lambda)$ considered in Section 7.3 then the left minimal indices of $\mathbb{L}(\lambda)$ are the same as the right minimal indices of $\mathbb{L}(\lambda)$. Moreover, if $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ are the minimal (left and right) indices of $\mathbb{L}(\lambda)$ then $\varepsilon_1 - (m-1)/2 \leq \cdots \leq \varepsilon_k - (m-1)/2$ are the minimal (left and right) indices of $G(\lambda)$. Hence we only need to comment on the recovery of eigenvectors and minimal bases of $G(\lambda)$ from those of the $\mathbb{L}(\lambda)$.

Note that the left minimal bases of $G(\lambda)$ are the same as the right minimal bases of $G(\lambda)$ when $G(\lambda)$ is symmetric (resp., Hamiltonian, skew-Hamiltonian, skew-symmetric). Hence if $\mathbb{L}(\lambda)$ is a structure-preserving linearization of $G(\lambda)$ considered in Section 7.3 then the left minimal bases of $\mathbb{L}(\lambda)$ are the same as the right minimal bases of $\mathbb{L}(\lambda)$. Consequently, minimal bases and eigenvectors of $G(\lambda)$ can be recovered from those of $\mathbb{L}(\lambda)$ as special cases of Theorem 7.4.1 and Theorem 7.4.3. Indeed, for structure-preserving linearizations, we have $c_0(\sigma, \sigma_2) = 0$ when $h = 0$ and, $c_0(\sigma, \sigma_2)$ is given in Table 7.1 when $h > 0$.

Structure	symmetric	T-even/odd	skew-symmetric
$c_0(\sigma, \sigma_2)$	$2 + i_2(\mathbf{t}_{w_h})$	1	$2 + c_2(\mathbf{t}_w)$

Table 7.1: Value of $c_0(\sigma, \sigma_2)$ when $h > 0$.

8

Conclusions

The main purpose of this thesis was to construct and analyze strong linearizations of polynomial and rational matrices, and describe the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials and rational matrices from those of the linearizations.

The family of generalized Fiedler pencils with repetition (GFPRs) is an important source of strong linearizations of matrix polynomials, especially structure-preserving strong linearizations of structured matrix polynomials. Even though GFPRs have been studied extensively over the years, the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the GFPRs was an open problem. We have addressed this problem in Chapter-2 and derived recovery rules for eigenvectors, minimal basses and minimal indices of matrix polynomials from those of the GFPRs. We have shown that the recovery rules can be easily read off by looking at the index tuples defining the GFPRs. Further, we have derived simplified recovery rules for eigenvectors and minimal basses for structured (symmetric, skew-symmetric, even, odd and palindromic) matrix polynomials from those of the structure-preserving GFPRs.

Fiedler-like pencils (FPs, GFPs, FPRs, and GFPRs) of matrix polynomials are privileged among other linearizations since these pencils are easily constructible from the coefficients of matrix polynomials, enable us to construct structure-preserving linearizations for structured matrix polynomials, and allow easy recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the linearizations. With a view to presenting a unified framework and to constructing structure-preserving linearizations with a additional properties (banded pencils with low bandwidth), in Chapter-3 we have introduced a new family of extended GFPRs (EGFPRs) of matrix polynomials that subsumes FPs, GFPs, FPRs and GFPRs and have shown that the EGFPRs are strong linearizations of matrix polynomials. We have also described the recovery of eigenvectors, minimal bases and minimal indices of matrix polynomials from those of the EGFPRs and have shown that the recovery is operation-free. We have constructed structure-preserving (symmetric, Hermitian, palindromic) strong linearizations of structured matrix polynomials in Chapter-4 by utilizing the EGFPRs. The family of palindromic linearizations contains an infinite number of pencils in contrast to the finite family of palindromic linearizations that are available in the literature. Moreover, we have constructed a large family of Hermitian EGFPRs that preserves the sign characteristic of Hermitian matrix polynomials having nonsingular leading coefficients.

Linearization of rational matrices is an emerging area of research. With a view to computing (finite and infinite) pole and zero structures of rational matrices, we have introduced a strong linearization (referred to as Rosenbrock strong linearization) of rational matrices in Chapter-5 and have shown that the structural indices of finite as well as infinite poles and zeros of rational matrices can be recovered from those of the Rosenbrock strong linearizations. We have constructed two affine spaces of strong linearizations of rational matrices and described the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the strong linearizations. We have also constructed Rosenbrock strong linearizations of rational matrices corresponding to a non-monomial polynomial basis and have described the recovery of minimal bases and minimal indices of rational matrices from those of the linearizations.

Fiedler-like pencils (FPs, GFPs and FPRs) of rational matrices have been constructed recently for solving rational eigenvalue problems. We have shown in Chapter-6 that the FPs, GFPs and FPRs of rational matrices are Rosenbrock strong linearizations of rational matrices. Also, we have described the recovery of minimal bases and minimal indices of rational matrices from those of the FPs, GFPs and FPRs and have shown that the recovery is operation-free.

Structured (symmetric, skew-symmetric, Hermitian, skew-Hermitian, Hamiltonian, skew-Hamiltonian, para-Hermitian, and para-skew-Hermitian) rational matrices arise in many applications. Construction of structure-preserving linearizations for structured rational matrices has not been explored adequately in the literature. We have addressed this problem in Chapter-7. We have introduced a new family of pencils (GFPRs) of rational matrices and have shown that the GFPRs are Rosenbrock strong linearizations. We have constructed structure-preserving Rosenbrock strong linearizations of structured (symmetric, skew-symmetric, Hermitian, skew-Hermitian, Hamiltonian, skew-Hamiltonian, para-Hermitian, and para-skew-Hermitian) rational matrices by utilizing the GFPRs. We have shown that the Hermitian GFPRs preserve the Cauchy-Maslov index of Hermitian rational matrices. Also, we have described the recovery of eigenvectors, minimal bases and minimal indices of rational matrices from those of the GFPRs and have shown that the recovery is operation-free.

Publications from the Thesis

Published:

(1) R. K. Das and R. Alam, *Automatic recovery of eigenvectors and minimal bases of matrix polynomials from generalized Fiedler pencils with repetition*, Linear Algebra Appl., 569 (2019), pp. 78–112.

(2) R. K. Das and R. Alam, *Affine spaces of strong linearizations for rational matrices and the recovery of eigenvectors and minimal bases*, Linear Algebra Appl., 569 (2019), pp. 335–368.

(3) R. K. Das and R. Alam, *Recovery of minimal bases and minimal indices of rational matrices from Fiedler-like pencils*, Linear Algebra Appl., 566 (2019), pp. 34–60.

Submitted/Under preparation:

(1) R. K. Das and R. Alam, *Structured strong linearizations of structured rational matrices.* (Submitted).

(2) R. K. Das and R. Alam, *Extended generalized Fiedler pencils with repetition of matrix polynomials and recovery of eigenvectors, minimal bases and minimal indices.* (Under preparation).

(3) R. K. Das and R. Alam, *Palindromic linearizations of palindromic matrix polynomials.* (Under preparation).

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A

Appendix

A.1 Proof of Algorithms

For proving Algorithm-1 and Algorithm-2 we need the following results. Recall that the Fiedler matrices M_j^P, $j \in \{-m : m-1\}$, satisfies the following:

$$(e_{m-i}^T \otimes I_n) M_j^P = \begin{cases} e_{m-(i+1)}^T \otimes I_n & \text{if } j = i+1,\ i = 0 : m-2 \\ (e_{m-i}^T \otimes (-A_i)) + (e_{m-(i-1)}^T \otimes I_n) & \text{if } j = i,\ i = 1 : m-1 \\ e_m^T \otimes (-A_0) & \text{if } j = i = 0 \\ e_{m-i}^T \otimes I_n & \begin{array}{l} \text{otherwise, i.e., when} \\ j \notin \{i, i+1\},\ i = 0 : m-1, \end{array} \end{cases} \tag{A.1}$$

$$(e_{m-i}^T \otimes I_n) M_{-j}^P = \begin{cases} (e_{m-i}^T \otimes A_{i+1}) + (e_{m-(i+1)}^T \otimes I_n) & \text{if } j = i+1,\ i = 0 : m-2 \\ e_1^T \otimes A_m & \text{if } j = i+1,\ i = m-1 \\ e_{m-(i-1)}^T \otimes I_n & \text{if } j = i \text{ and } i = 1 : m-1 \\ e_{m-i}^T \otimes I_n & \begin{array}{l} \text{otherwise, i.e., when} \\ j \notin \{i, i+1\},\ i = 0 : m-1. \end{array} \end{cases} \tag{A.2}$$

We now present some technical results which will be used to prove Algorithms-1, 2, 3, 4. The following result follows from Lemma 3.3.2. We denote by $(*)$ any arbitrary matrix assignment.

Proposition A.1.1. *Let α be an index tuple containing indices from $\{0 : m-1\}$ such that α satisfies the SIP. Let $0 \leq s \leq m-1$. Suppose that $s, s+1 \notin \alpha$ or the subtuple of α with indices $\{s, s+1\}$ starts with $s+1$. Then it follows from Lemma 3.3.2 that $(e_{m-s}^T \otimes I_n) M_\alpha(*) = e_{m-k}^T \otimes I_n$, where $k = s + 1 + c_{s+1}(\alpha)$.*

Recall the notations $\mathfrak{rc}_t(\alpha)$ and $\alpha^R(t)$ from Definition 2.4.1.

Proposition A.1.2. *Let α be an index tuple containing indices from $\{0:m-1\}$ such that α satisfies the SIP. Let $j \in \{0:m-1\}$. If the subtuple of α with indices $\{j, j+1\}$ starts with j then $(e^T_{m-j}\otimes I_n)M^P_\alpha =$*

$$\begin{cases} (e^T_{m-(p+c_p(\beta))}\otimes I_n) - \sum\limits_{k=p}^{j}(e^T_{m-(k+1+c_{k+1}(\beta))}\otimes A_k) & \textit{if } p := j-\mathfrak{rc}_j(\alpha)\neq 0,\\ -\sum\limits_{k=0}^{j}(e^T_{m-(k+1+c_{k+1}(\beta))}\otimes A_k) & \textit{if } j-\mathfrak{rc}_j(\alpha)=0,\end{cases}$$

where $\beta := \alpha^R(j)$.

Proof. Suppose that the subtuple of α with indices $\{j, j+1\}$ starts with j. Let $rsf(\alpha) =$

$$\Big(rev(a_0:0),\ldots,rev(a_j:j),\ldots,rev(a_{m-1}:m-1)\Big) =: (\gamma, rev(a_j:j), \beta). \tag{A.3}$$

Then $rev(a_j:j)\neq\emptyset$, since otherwise the subtuple of α with indices $\{j, j+1\}$ would start with $j+1$ and contradict our assumption. Now we have

$$\begin{aligned}
&(e^T_{m-j}\otimes I_n)M^P_\alpha = (e^T_{m-j}\otimes I_n)\,M^P_\gamma M^P_{rev(a_j:j)}M^P_\beta \qquad\qquad (A.4)\\
&= (e^T_{m-j}\otimes I_n)\,M^P_{rev(a_j:j)}M^P_\beta \text{ by } (A.1) \text{ as } j, j+1\notin\gamma\\
&= (e^T_{m-j}\otimes I_n)\;M^P_j M^P_{j-1}\cdots M^P_{a_j}M^P_\beta\\
&= \big(-(e^T_{m-j}\otimes A_j) + (e^T_{m-(j-1)}\otimes I_n)\big)\;M^P_{j-1}\cdots M^P_{a_j}M^P_\beta \text{ by } (A.1)\\
&= \Big(-\sum_{k=a_j+1}^{j}(e^T_{m-k}\otimes A_k) + (e^T_{m-a_j}\otimes I_n)\Big)M^P_{a_j}M^P_\beta \text{ by applying } (A.1) \text{ repeatedly}\\
&= \left.\begin{cases} \Big((e^T_{m-(a_j-1)}\otimes I_n) - \sum\limits_{k=a_j}^{j}(e^T_{m-k}\otimes A_k)\Big)\;M^P_\beta & \text{if } a_j\neq 0\\ \Big(-\sum\limits_{k=0}^{j}(e^T_{m-k}\otimes A_k)\Big)\;M^P_\beta & \text{if } a_j = 0\end{cases}\right\}\text{ by } (A.1)
\end{aligned}$$

It is clear from (A.3) that $a_j = j - \mathfrak{rc}_j(\alpha) =: p$. Thus $(e^T_{m-j}\otimes I_n)M^P_\alpha =$

$$= \begin{cases} \Big((e^T_{m-(p-1)}\otimes I_n) - \sum_{k=p}^{j}(e^T_{m-k}\otimes A_k)\Big)M^P_\beta & \text{if } p\neq 0\\ \Big(-\sum_{k=0}^{j}(e^T_{m-k}\otimes A_k)\Big)M^P_\beta & \text{if } p = 0.\end{cases}$$

Define $\mathbb{S} := \{p-1, p, \ldots, j\}$ if $p\neq 0$ and $\mathbb{S} := \{0, 1, \ldots, j\}$ if $p = 0$. It is clear from (A.3) that $\beta = \alpha^R(j)$. It remains to show that $(e^T_{m-\ell}\otimes I_n)M^P_\beta = e^T_{m-(\ell+1+c_{\ell+1}(\beta))}\otimes I_n$ for all $\ell\in\mathbb{S}$.

We have $\beta = \big(rev(a_{j+1} : j+1), \ldots, rev(a_{m-1} : m-1)\big)$. Let $\ell \in \mathbb{S}$. Then $\ell \leq j$. Suppose that $\ell \in \beta$. Then $\ell \in rev(a_q : q) = (q, q-1, \ldots, a_q)$ for some $q \in \{j+1, j+2, \ldots, m-1\}$ which implies $\ell+1 \in \beta$ and the subtuple of β with indices $\{\ell, \ell+1\}$ starts with $\ell+1$. In other words, for $\ell \in \mathbb{S}$, if $\ell+1 \notin \beta$ then $\ell \notin \beta$. Hence, for $\ell \in \mathbb{S}$, we have either $\ell, \ell+1 \notin \beta$ or the subtuple of β with indices from $\{\ell, \ell+1\}$ starts with $\ell+1$. Hence by Proposition A.1.1, $(e^T_{m-\ell} \otimes I_n)M^P_\beta = e^T_{m-(\ell+1+c_{\ell+1}(\beta))} \otimes I_n$ which gives the desired result. ☐

The following result follows from Lemma 3.3.3.

Proposition A.1.3. *Let β be an index tuple containing indices from $\{-m : -1\}$ such that β satisfies the SIP. Let $1 \leq s \leq m-1$. Suppose that $-s, -(s+1) \notin \beta$ or the subtuple of β with indices $\{-s, -(s+1)\}$ starts with $-s$. Then it follows from Lemma 3.3.3 that $(e^T_{m-s} \otimes I_n)M_\alpha = e^T_{m-t} \otimes I_n$, where $t = s-1-c_{-s}(\alpha)$.*

Proposition A.1.4. *Let α be an index tuple containing indices from $\{-m : -1\}$ such that α satisfies the SIP. Let $j = 0 : m-1$. If the subtuple of α with indices $\{-j, -(j+1)\}$ starts with $-(j+1)$ then $(e^T_{m-j} \otimes I_n)M^P_\alpha =$*

$$\begin{cases} \left.\begin{array}{l} (e^T_{m-(p-1-c_{-p}(\beta))} \otimes I_n) \\ \quad + \displaystyle\sum_{k=j}^{p-1} e^T_{m-(k-1-c_{-k}(\beta))} \otimes A_{k+1} \end{array}\right\} & \text{if } -p := -(j+1+\mathfrak{rc}_{-(j+1)}(\alpha)) \neq -m, \\ \displaystyle\sum_{k=j}^{m-1} e^T_{m-(k-1-c_{-k}(\beta))} \otimes A_{k+1} & \text{if } -(j+1+\mathfrak{rc}_{-(j+1)}(\alpha)) = -m, \end{cases}$$

where $\beta := \alpha^R(-(j+1))$.

Proof. Suppose that the subtuple of α with indices $\{-j, -(j+1)\}$ starts with $-(j+1)$. Let $rsf(\alpha) =$

$$\Big(rev(-a_m : -m), \ldots, rev(-a_{j+1} : -(j+1)), rev(-a_j : -j), \ldots, rev(-a_1 : -1)\Big).$$

Then $rev(-a_{j+1} : -(j+1)) \neq \emptyset$, since otherwise the subtuple of α with indices $\{-j, -(j+1)\}$ would start with $-(j+1)$ and contradict our assumption. Note that $\alpha^R(-(j+1)) = \big(rev(-a_j : -j), rev(-a_{j-1} : -(j-1)), \ldots, rev(-a_1 : -1)\big)$ and $\gamma := \big(rev(-a_m : -m), \ldots, rev(-a_{j+2} : -(j+2))\big)$. Set $\beta := \alpha^R(-(j+1))$. Now we have

$$\begin{aligned} (e^T_{m-j} \otimes I_n)M^P_\alpha &= (e^T_{m-j} \otimes I_n)\, M^P_\gamma M^P_{rev(-a_{j+1}:-(j+1))} M^P_\beta \qquad \text{(A.5)} \\ &= (e^T_{m-j} \otimes I_n)\, M^P_{rev(-a_{j+1}:-(j+1))} M^P_\beta \text{ by (A.2) as } -j, -(j+1) \notin \gamma \\ &= (e^T_{m-j} \otimes I_n) M^P_{-(j+1)} M^P_{-(j+2)} \cdots M^P_{-a_{j+1}} M^P_\beta \\ &= \Big((e^T_{m-j} \otimes A_{j+1}) + (e^T_{m-(j+1)} \otimes I_n)\Big) M^P_{-(j+2)} \cdots M^P_{-a_{j+1}} M^P_\beta \text{ by (A.2)} \end{aligned}$$

$$= \left\{ \begin{array}{ll} \Big((e^T_{m-a_{j+1}} \otimes I_n) + \sum\limits_{k=j}^{a_{j+1}-1} (e^T_{m-k} \otimes A_{k+1}) \Big) M^P_\beta & \text{if } -a_{j+1} \neq -m \\ \Big(\sum\limits_{k=j}^{m-1} (e^T_{m-k} \otimes A_{k+1}) \Big) M^P_\beta & \text{if } -a_{j+1} = -m \end{array} \right\}$$

$$\text{by applying } (A.2) \text{ repeatedly.} \tag{A.6}$$

Note that $\mathfrak{rc}_{-(j+1)}(\alpha) = a_{j+1} - j - 1$. Set $p := a_{j+1}$. Then by (A.6), we have $(e^T_{m-j} \otimes I_n) M^P_\alpha =$

$$\begin{cases} \Big((e^T_{m-p} \otimes I_n) + \sum_{k=j}^{p-1} (e^T_{m-k} \otimes A_{k+1}) \Big) M^P_\beta & \text{if } -p \neq -m \\ \Big(\sum_{k=j}^{m-1} (e^T_{m-k} \otimes A_{k+1}) \Big) M^P_\beta & \text{if } -p = -m. \end{cases}$$

Define $\mathbb{S} := \{j, j+1, \ldots, p\}$ if $p \neq m$ and $\mathbb{S} := \{j, j+1, \ldots, m-1\}$ if $p = m$. It remains to show that $(e^T_{m-\ell} \otimes I_n) M^P_\beta = e^T_{m-(\ell-1-c_{-\ell}(\beta))} \otimes I_n$ for all $\ell \in \mathbb{S}$.

We have $\beta = (rev(-a_j : -j), \ldots, rev(-a_1 : -1))$. Let $\ell \in \mathbb{S}$. Suppose that $-(\ell+1) \in \beta$. Then $-(\ell+1) \in rev(-a_q : -q) = (-q, -(q+1), \ldots, -a_q)$ for some $q \in \{j, j-1, \ldots, 1\}$. Thus $-\ell \in \beta$ and the subtuple of β with indices $\{-(\ell+1), -\ell\}$ starts with $-\ell$. In other words, for $\ell \in \mathbb{S}$, if $-\ell \notin \beta$ then $-(\ell+1) \notin \beta$. Thus by Proposition A.1.3, $(e^T_{m-\ell} \otimes I_n) M^P_\beta = e^T_{m-(\ell-1-c_{-\ell}(\beta))} \otimes I_n$ which gives the desired result. □

A.1.1 Proof of Algorithm-1 and Algorithm-2

Proof. Let $\lambda L_1 - L_0 := M^P_{\tau_1} M^P_{\sigma_1} (\lambda M^P_\tau - M^P_\sigma) M^P_{\sigma_2} M^P_{\tau_2}$ be an FPR of $P(\lambda)$. Recall that σ, σ_1 and σ_2 (resp., σ_1 and σ_2) commute with τ_1 and τ_2 (resp., τ, τ_1 and τ_2). Also, recall that σ_j (resp., τ_j) contains indices from $\{0 : h-1\}$ (resp., $\{-m : -(h-2)\}$). Hence by (A.1) and (A.2), we have

$$(e^T_{m-j} \otimes I_n) L_0 = \begin{cases} (e^T_{m-j} \otimes I_n) M^P_{(\sigma_1, \sigma, \sigma_2)} & \text{if } j \leq h, \\ (e^T_{m-j} \otimes I_n) M^P_{(\tau_1, \tau_2)} & \text{if } j \geq h+1, \end{cases}$$

$$\text{and} \quad (e^T_{m-j} \otimes I_n) L_1 = \begin{cases} (e^T_{m-j} \otimes I_n) M^P_{(\sigma_1, \sigma_2)} & \text{if } j \leq h-1, \\ (e^T_{m-j} \otimes I_n) M^P_{(\tau_1, \tau, \tau_2)} & \text{if } j \geq h. \end{cases}$$

Now we have the following:

- Set $\alpha := (\sigma_1, \sigma, \sigma_2)$. Let $j \in \{0 : h\}$. If the subtuple of α with indices $\{j, j+1\}$ starts with $j+1$ then, by Proposition A.1.1, $(e^T_{m-j} \otimes I_n) L_0 = e^T_{m-(j+1+c_{j+1}(\alpha))} \otimes I_n$.

  Suppose that the subtuple of α with indices $\{j, j+1\}$ starts with j. Set $p := j - \mathfrak{rc}_j(\alpha)$ and $\alpha^R := \alpha^R(j)$. Then by Proposition A.1.2, we have $(e^T_{m-j} \otimes I_n) L_0 =$

$(e^T_{m-(p+c_p(\alpha^R))} \otimes I_n) - \sum_{k=p}^{j} e^T_{m-(k+1+c_{k+1}(\alpha^R))} \otimes A_k$ if $p \neq 0$, and $(e^T_{m-j} \otimes I_n)L_0 = -\sum\limits_{k=0}^{j} e^T_{m-(k+1+c_{k+1}(\alpha^R))} \otimes A_k$ if $p = 0$.

- Set $\beta := (\tau_1, \tau_2)$. Let $j \in \{h+1 : m-1\}$. If $-j, -(j+1) \notin \beta$ or the subtuple of β with indices $\{-j, -(j+1)\}$ starts with $-j$, then by PropositionA.1.3 we have $(e^T_{m-j} \otimes I_n)L_0 = e^T_{m-(j-1-c_{-j}(\beta))} \otimes I_n$.

  Suppose that the subtuple of β with indices $\{-j, -(j+1)\}$ starts with $-(j+1)$. Set $-q := -(j + \mathfrak{rc}_{-(j+1)}(\beta) + 1)$ and $\beta^R := \beta^R(-(j+1))$. Then by Proposition A.1.4 we have $(e^T_{m-j} \otimes I_n)L_0 = (e^T_{m-(q-1-c_{-q}(\beta^R))} \otimes I_n) + \sum_{k=j}^{q-1} \left(e^T_{m-(k-1-c_{-k}(\beta^R))} \otimes A_{k+1}\right)$ if $-q \neq -m$, and $(e^T_{m-j} \otimes I_n)L_0 = \sum_{k=j}^{m-1} (e^T_{m-(k-1-c_{-k}(\beta^R))} \otimes A_{k+1})$ if $-q = -m$. This completes the proof of Algorithm-1.

Similar proof for Algorithm-2. $\square$

Next we prove Algorithm-3 and Algorithm-4 For this purpose we need the following results. Recall that $M_j(X)$, $j \in \{-m : m-1\}$, satisfies the following:

$$(e^T_{m-i} \otimes I_n)M_j(X) = \begin{cases} e^T_{m-(i+1)} \otimes I_n & \text{if } j = i+1, \text{ for } i = 0 : m-2 \\ (e^T_{m-i} \otimes X) + (e^T_{m-(i-1)} \otimes I_n) & \text{if } j = i, \text{ for } i = 1 : m-1 \\ e^T_m \otimes X & \text{if } j = i = 0 \\ e^T_{m-i} \otimes I_n & \begin{cases} \text{otherwise, i.e., when} \\ j \notin \{i, i+1\}, \text{ for } i = 0 : m-1. \end{cases} \end{cases} \tag{A.7}$$

$$(e^T_{m-i} \otimes I_n)M_{-j}(X) = \begin{cases} (e^T_{m-i} \otimes X) + (e^T_{m-(i+1)} \otimes I_n) & \text{for } j = i+1, \text{ for } i = 0 : m-2 \\ e^T_1 \otimes X & \text{if } j = i+1, \ i = m-1 \\ e^T_{m-(i-1)} \otimes I_n & \text{if } j = i, \text{ for } i = 1 : m-1 \\ e^T_{m-i} \otimes I_n & \begin{cases} \text{otherwise, i.e., when} \\ j \notin \{i, i+1\}, \text{ for } i = 0 : m-1. \end{cases} \end{cases} \tag{A.8}$$

Let α be an index tuple containing indices from $\{0 : m-1\}$ (resp., $\{-m : -1\}$) such that α satisfies the SIP. Then the positions of the block entries of $M_\alpha(\mathcal{X})$ do not depend on the particular matrix assignment $\mathcal{X}$, that is, the positions of the block entries of $M_\alpha(\mathcal{X})$ only depend on α, see Remark 2.1.8. Hence the proof of the following result is analogs to Proposition A.1.2.

Proposition A.1.5. *Let* α *be an index tuple containing indices from* $\{0 : m-1\}$ *such that* α *satisfies the SIP. Suppose that* α *is in the column standard form. Let* $Z = (Z_1, Z_2, \ldots, Z_{|\alpha|})$ *be a matrix assignment for* α*, where* $|\alpha|$ *denotes the total number of indices in* α*. Let* $j \in \{0 : m-1\}$*. If the subtuple of* α *with indices* $\{j, j+1\}$ *starts with* j *then* $(e_{m-j}^T \otimes I_n)M_\alpha(Z) =$

$$\begin{cases} \left.\begin{array}{l} e^T_{m-(q+c_q(\beta))} \otimes I_n \\ \quad + \sum\limits_{k=q}^{j} (e^T_{m-(k+1+c_{k+1}(\beta))} \otimes Z_{p_j(\alpha)+j-k}) \end{array}\right\} & \text{if } q := j - \mathfrak{rc}_j(\alpha) \neq 0, \\ \sum\limits_{k=0}^{j} (e^T_{m-(k+1+c_{k+1}(\beta))} \otimes Z_{p_j(\alpha)+j-k} & \text{if } j - \mathfrak{rc}_j(\alpha) = 0, \end{cases} \tag{A.9}$$

where $\beta := \alpha^R(j)$ *and* $p_j(\alpha)$ *is the position of the first occurrence of* j *in* α*.*

Proof. It follows that the evaluation of $(e_{m-j}^T \otimes I_n)M_\alpha^P$ in Proposition A.1.2 depends only on $\alpha = (\gamma, rev(a_j : j), \beta)$ and the matrices assigned to $rev(a_j : j)$, i.e., the matrices assigned to γ and β have no role in the evaluation of $(e_{m-j}^T \otimes I_n)M_\alpha^P$. In other words, any matrices assigned to γ and β have no role in the evaluation of $(e_{m-j}^T \otimes I_n)M_\alpha^P$ in Proposition A.1.2. Further, (A.4) can be rewritten as

$$(e_{m-j}^T \otimes I_n)M_\alpha^P = (e_{m-j}^T \otimes I_n)\, M_\gamma^P\, M_{rev(a_j:j)}\big(-A_j, -A_{j-1}, \ldots, -A_{a_j}\big)\, M_\beta^P,$$

i.e., $\big(-A_j, -A_{j-1}, \ldots, -A_{a_j}\big)$ is assigned to $rev(a_j : j)$.

We now prove (A.9). Let $p_j(\alpha)$ be the position of the first occurrence of j in α. Let $\alpha = (\gamma, rev(a_j : j), \beta)$ be as in (A.3). Then $Z = \big(Z^\ell, Z^m, Z^r\big)$, where $Z^m := \big(Z_{p_j(\alpha)}, Z_{p_j(\alpha)+1}, \ldots, Z_{p_j(\alpha)+\mathfrak{rc}_j(\alpha)}\big)$ and Z^ℓ, Z^r are some matrices associated with Z. In other words, Z^ℓ, Z^m and Z^r are the matrix assignments for γ, $rev(a_j : j)$ and β, respectively. Now, by using (A.7) and (A.8), the prove of (A.9) is verbatim to that of Proposition A.1.2. $\square$

Similarly, the following result is analogs to Proposition A.1.4.

Proposition A.1.6. *Let* α *be an index tuple containing indices from* $\{-m : -1\}$ *such that* α *satisfies the SIP. Suppose that* α *is in the column standard form. Let* Z *be a matrix assignment for* α*. Let* $j = 0 : m-1$*. If the subtuple of* α *with indices* $\{-j, -(j+1)\}$ *starts with* $-(j+1)$ *then* $(e_{m-j}^T \otimes I_n)M_\alpha(Z) =$

$$\begin{cases} \left.\begin{array}{l} \big(e^T_{m-(q-1-c_{-q}(\beta))} \otimes I_n\big) \\ \quad + \sum\limits_{k=j}^{q-1} e^T_{m-(k-1-c_{-k}(\beta))} \otimes Z_{p_{-(j+1)}(\alpha)+k-j} \end{array}\right\} & \text{if } -q := -(j+1+\mathfrak{rc}_{-(j+1)}(\alpha)) \neq -m, \\ \sum\limits_{k=j}^{m-1} e^T_{m-(k-1-c_{-k}(\beta))} \otimes Z_{p_{-(j+1)}(\alpha)+k-j} & \text{if } -(j+1+\mathfrak{rc}_{-(j+1)}(\alpha)) = -m, \end{cases}$$

where $\beta := \alpha^R(-(j+1))$ *and* $p_{-(j+1)}(\alpha)$ *is the position of the first occurrence of* $-(j+1)$ *in* α.

A.1.2 Proof of Algorithm-3 and Algorithm-4

Proof. Let $\lambda L_1 - L_0 := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M_\tau^P - M_\sigma^P)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)$ be a GFPR of $P(\lambda)$. Recall that σ, σ_1 and σ_2 (resp., σ_1 and σ_2) commute with τ_1 and τ_2 (resp., τ, τ_1 and τ_2). Also, recall that σ_j (resp., τ_j) contains indices from $\{0 : h-1\}$ (resp., $\{-m : -(h-2)\}$). Hence by using (A.7) and (A.8) we have

$$(e_{m-i}^T \otimes I_n)L_0 = \begin{cases} (e_{m-i}^T \otimes I_n)M_{\sigma_1}(X_1)M_\sigma^P M_{\sigma_2}(X_2) & \text{if } i \leq h, \\ (e_{m-i}^T \otimes I_n)M_{\tau_1}(Y_1)M_{\tau_2}(Y_2) & \text{if } i \geq h+1, \end{cases}$$

$$\text{and} \quad (e_{m-i}^T \otimes I_n)L_1 = \begin{cases} (e_{m-i}^T \otimes I_n)M_{\sigma_1}(X_1)M_{\sigma_2}(X_2) & \text{if } i \leq h-1, \\ (e_{m-i}^T \otimes I_n)M_{\tau_1}(Y_1)M_\tau^P M_{\tau_2}(Y_2) & \text{if } i \geq h. \end{cases}$$

Now, the proof follows by similar arguments given in the proof of Algorithm-1 and Algorithm-2 and by using Propositions A.1.1, A.1.3, A.1.5, A.1.6. ☐

B

Appendix

B.1 Proof of Proposition 3.1.4

First, we present some technical results which will be used to prove that EGFPRs are operation free pencils. We denote by $(*)$ any arbitrary matrix assignment.

Lemma B.1.1. *Let* α *be an index tuple containing indices from* $\{0 : m-1\}$ *such that* α *satisfies the SIP. Let* $0 \leq s \leq m-1$*. Suppose that the subtuple of* α *with indices* $\{s+1, s+2\}$ *starts with* $s+2$*. Let* $\alpha = (\beta, s+2, \gamma)$*, where* $s+1, s+2 \notin \beta$*. Further, suppose that the subtuple of* γ *with indices* $\{s, s+1\}$ *starts with* $s+1$*. Then* $(e^T_{m-(s+1)} \otimes I_n)M_\alpha(*) \neq (e^T_{m-s} \otimes I_n)M_\gamma(*)$*. Moreover,* $(e^T_{m-(s+1)} \otimes I_n)M_\alpha(*) = e^T_{m-k} \otimes I_n$ *and* $(e^T_{m-s} \otimes I_n)M_\gamma(*) = e^T_{m-\ell} \otimes I_n$ *with* $k > \ell$*, where* $k = s+2+c_{s+2}(\alpha)$ *and* $\ell = s+1+c_{s+1}(\gamma)$*.*

Proof. Since the subtuple of α with indices $\{s+1, s+2\}$ starts with $s+2$, by Remark A.1.1, we have $(e^T_{m-(s+1)} \otimes I_n)M_\alpha(*) = e^T_{m-k} \otimes I_n$, where $k = s+2+c_{s+2}(\alpha)$. Similarly, since the subtuple of γ with indices $\{s, s+1\}$ starts with $s+1$, by Remark A.1.1, we have $(e^T_{m-s} \otimes I_n)M_\gamma(*) = e^T_{m-\ell} \otimes I_n$, where $\ell = s+1+c_{s+1}(\gamma)$. Now let $c_{s+1}(\gamma) = p$, i.e., $(s+1, s+2, \ldots, s+p+1)$ is a subtuple of γ. Since $\alpha = (\beta, s+2, \gamma)$ satisfies the SIP, it follows that $(s+2, \ldots, s+p+1, s+p+2)$ must be a subtuple of α. Hence $c_{s+2}(\alpha) \geq p$ and $k = s+2+c_{s+2}(\alpha) \geq s+p+2 > s+p+1 = \ell$ which gives the desired result. ☐

The following result is analogous to Lemma B.1.1.

Lemma B.1.2. *Let* α *be an index tuple containing indices from* $\{-m : -1\}$ *such that* α *satisfies the SIP. Suppose that the subtuple of* α *with indices* $\{-s, -(s+1)\}$ *starts with* $-s$*. Let* $\alpha = (\beta, -s, \gamma)$*, where* $-s, -(s+1) \notin \beta$*. Further, suppose that the subtuple of* γ *with indices* $\{-(s+1), -(s+2)\}$ *starts with* $-(s+1)$*. Then* $(e^T_{m-s} \otimes$

$I_n)M_\alpha(*) \neq (e^T_{m-(s+1)} \otimes I_n)M_\gamma(*)$. Moreover, $(e^T_{m-s} \otimes I_n)M_\alpha(*) = e^T_{m-k} \otimes I_n$ and $(e^T_{m-(s+1)} \otimes I_n)M_\gamma(*) = e^T_{m-\ell} \otimes I_n$ with $k < \ell$, where $k = s - 1 - c_{-s}(\alpha)$ and $\ell = s - c_{-(s+1)}(\gamma)$.

Recall that, for $a, b, q \in \mathbb{Z}$, we denote $\{a :_q b\} := \{a, a+q, a+2q, \ldots, b\}$. The following facts will be used to prove that EGFPRs are operation free pencils.

Remark B.1.3. *Let $1 \leq h_1 \leq h_2 < h_2 + 1 \leq h_3 \leq m - 1$. Let α be an index tuple containing indices from $\{-h_1 :_{-1} -h_2\} \cup \{h_2 + 1 : h_3\}$. Let β (resp., γ) be the subtuple of α with indices $\{-h_1 :_{-1} -h_2\}$ (resp., $\{h_2 + 1 : h_3\}$). Suppose that β and γ satisfy the SIP. Then from (A.1) and (A.2) we have the following.*

- *Let $h_1 - 1 \leq j \leq h_2 - 1$. If the subtuple of α with indices $\{-j, -(j+1)\}$ starts with $-j$ then the indices $h_2 + 1 : h_3$ in α are redundant for evaluating $(e^T_{m-j} \otimes I_n)M^P_\alpha$, i.e., $(e^T_{m-j} \otimes I_n)M^P_\alpha = (e^T_{m-j} \otimes I_n)M^P_\beta$.*
- *If the the subtuple of α with indices $\{-h_2, h_2 + 1\}$ starts with $-h_2$ then similarly as above the indices $h_2 + 1 : h_3$ in α are redundant for evaluating $(e^T_{m-h_2} \otimes I_n)M^P_\alpha$. On the other hand, if the subtuple of α with indices $\{-h_2, h_2+1\}$ starts with h_2+1 then the indices $-h_1 :_{-1} -h_2$ are redundant for evaluating $(e^T_{m-h_2} \otimes I_n)M^P_\alpha$, i.e., $(e^T_{m-h_2} \otimes I_n)M^P_\alpha = (e^T_{m-h_2} \otimes I_n)M^P_\gamma$.*
- *Let $h_2 + 1 \leq j \leq h_3$. If the subtuple of α with indices $\{j, j+1\}$ starts with $j + 1$ then the indices $-h_1 :_{-1} -h_2$ are redundant for evaluating $(e^T_{m-j} \otimes I_n)M^P_\alpha$, i.e, $(e^T_{m-j} \otimes I_n)M^P_\alpha = (e^T_{m-j} \otimes I_n)M^P_\gamma$.*

The following result will be useful for proving that EGFPRs are operation free.

Lemma B.1.4. *Let $1 \leq h_1 \leq h_2 < h_2 + 1 \leq h_3 \leq m - 1$. Let α be an index tuple containing indices from $\{-h_1 :_{-1} -h_2\} \cup \{h_2 + 1 : h_3\}$. Define $\beta :=$ the subtuple of α with indices $\{-h_1 :_{-1} -h_2\}$ and γ be the subtuple of α with indices $\{h_2 + 1 : h_3\}$. Suppose that β and γ satisfies the SIP. Then $M_\alpha(*)$ is operation free.*

Proof. We will prove that M^P_α is operation free. The proof depends only on the indices of α and does not depend on the matrix assignments for α. Hence $M_\alpha(*)$ is operation free.

For proving M^P_α is operation free it is equivalent to show that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free for all $j = 0 : m-1$. It follows from (A.1) and (A.2) that $(e^T_{m-j} \otimes I_n)M^P_\alpha = e^T_{m-j} \otimes I_n$ for all $j \in \{0 : h_1 - 2\} \cup \{h_3 + 1 : m - 1\}$. Hence it remains to show that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free for all $j = h_1 - 1 : h_3$. We proceed as follows.

(a) Let $h_1 - 1 \leq j \leq h_2 - 1$. We prove that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

Case-I: Suppose that $-j, -(j+1) \notin \alpha$ or the subtuple of α with indices $\{-j, -(j+1)\}$ starts with $-j$. Then by Remark A.1.3 and Remark B.1.3, we have $(e^T_{m-j} \otimes I_n)M^P_\alpha = e^T_{m-k} \otimes I_n$, where $k = j - 1 - c_{-j}(\alpha)$. Hence $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

Case-II: Suppose that the subtuple of α with indices $\{-j, -(j+1)\}$ starts with $-(j+1)$. The following steps show that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

Step-1: Let $\alpha = (\delta^j, -(j+1), \alpha^j)$, where $-j, -(j+1) \notin \delta^j$. Then we have

$$\begin{aligned}(e^T_{m-j} \otimes I_n)M^P_\alpha &= (e^T_{m-j} \otimes I_n)M^P_{(-(j+1),\alpha^j)} \text{ by } (A.1) \text{ and } (A.2) \text{ since } -j, -(j+1) \notin \delta^j \\ &= \Big((e^T_{m-j} \otimes A_{j+1}) + (e^T_{m-(j+1)} \otimes I_n)\Big)M^P_{\alpha^j} \text{ by } (A.2) \\ &= (e^T_{m-j} \otimes A_{j+1})M^P_{\alpha^j} + (e^T_{m-(j+1)} \otimes I_n)M^P_{\alpha^j}. \end{aligned} \tag{B.1}$$

Now, since $\alpha = (\delta^j, -(j+1), \alpha^j)$ satisfies the SIP, we have either $-j, -(j+1) \notin \alpha^j$ or the subtuple of α^j with indices $\{-j, -(j+1)\}$ starts with $-j$. Then by Remark A.1.3 and Remark B.1.3, we have $(e^T_{m-j} \otimes I_n)M^P_{\alpha^j} = e^T_{m-k_j} \otimes I_n$, where $k_j := j - 1 - c_{-j}(\alpha^j)$. Hence from (B.1) we have

$$(e^T_{m-j} \otimes I_n)M^P_\alpha = (e^T_{m-k_j} \otimes A_{j+1}) + (e^T_{m-(j+1)} \otimes I_n)M^P_{\alpha^j}. \tag{B.2}$$

The evaluation of $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is completed if any one of the following cases hold. Otherwise, we move to Step-2.

- Suppose that $-(j+1), -(j+2) \notin \alpha^j$ or the subtuple of α^j with indices $\{-(j+1), -(j+2)\}$ starts with $-(j+1)$. Then by Remark A.1.3 and Remark B.1.3 we have $(e^T_{m-(j+1)} \otimes I_n)M^P_{\alpha^j} = e^T_{m-k_{j+1}} \otimes I_n$, where $k_{j+1} := j - c_{-(j+1)}(\alpha^j)$. It follows by Lemma B.1.2 that $k_j < k_{j+1}$. Hence it follows from (B.2) that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

- Suppose that $j + 1 = h_2$. If the subtuple of α^j with indices $\{-h_2, h_2 + 1\}$ starts with $-h_2$ then by similar argument as above it follows that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free. On the other hand, if the subtuple of subtuple of α^j with indices $\{-h_2, h_2+1\}$ starts with h_2+1 then by Remark A.1.1 and Remark B.1.3, we have $(e^T_{m-h_2} \otimes I_n)M^P_{\alpha^j} = e^T_{m-\ell} \otimes I_n$, where $\ell := h_2 + 1 + c_{h_2+1}(\alpha^j)$. Hence it follows from (B.2) that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

Step-2: Suppose that the subtuple of α^j with indices $\{-(j+1), -(j+2)\}$ starts with $-(j+2)$. Then repeat Step-1 with α replaced by α^j and j replaced by $j+1$.

Since $\{h_1 - 1 : h_2 - 1\}$ contains $s := h_2 - h_1 + 1$ number of indices, we must stop before s numbers of steps. This completes the proof of $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

(b) Let $j = h_2$. If the subtuple of α with indices $\{-h_2, h_2 + 1\}$ starts with $h_2 + 1$ then by Remark A.1.1 and Remark B.1.3, we have $(e^T_{m-h_2} \otimes I_n)M^P_\alpha = e^T_{m-\ell} \otimes I_n$, where

$\ell := h_2 + 1 + c_{h_2+1}(\alpha)$. Hence $(e^T_{m-h_2} \otimes I_n)M^P_\alpha$ is operation free. On the other hand if the the subtuple of α with indices $\{-h_2, h_2+1\}$ starts with $-h_2$ then by Remark A.1.3 and Remark B.1.3, we have $(e^T_{m-h_2} \otimes I_n)M^P_\alpha = e^T_{m-\ell} \otimes I_n$, where $\ell := h_2 - 1 - c_{-h_2}(\alpha)$. Hence $(e^T_{m-h_2} \otimes I_n)M^P_\alpha$ is operation free.

(c) Let $h_2 + 1 \leq j \leq h_3$. We prove that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

Case-I: Suppose that $j, j+1 \notin \alpha$ or the subtuple of α with indices $\{j, j+1\}$ starts with $j+1$. Then by Remark A.1.1 and Remark B.1.3, we have2 $(e^T_{m-j} \otimes I_n)M_\alpha = (e^T_{m-j} \otimes I_n)M_\gamma = e^T_{m-k} \otimes I_n$ which is operation free, where $k = j + 1 + c_{j+1}(\gamma)$.

Case-II: Suppose that the subtuple of α with indices $\{j, j+1\}$ starts with j. Then the following steps show that $(e^T_{m-j} \otimes I_n)M_\alpha$ is operation free.

Step-1: Let $\alpha = (\xi^j, j, \alpha^j)$, where $j, j+1 \notin \xi^j$. Then we have

$$\begin{aligned}
(e^T_{m-j} \otimes I_n)M^P_\alpha &= (e^T_{m-j} \otimes I_n)M^P_{(j,\alpha^j)} \text{ by } (A.1) \text{ and } (A.2) \text{ since } j, j+1 \notin \xi^j \\
&= \Big((e^T_{m-j} \otimes (-A_j)) + (e^T_{m-(j-1)} \otimes I_n)\Big)M^P_{\alpha^j} \text{ by } (A.1) \\
&= (e^T_{m-j} \otimes (-A_j))M^P_{\alpha^j} + (e^T_{m-(j-1)} \otimes I_n)M^P_{\alpha^j} \qquad \text{(B.3)}
\end{aligned}$$

Since $\alpha = (\xi^j, j, \alpha^j)$ satisfies the SIP, we have either $j, j+1 \notin \alpha^j$ or the subtuple of α^j with indices $\{j, j+1\}$ starts with $j+1$. Hence by Remark A.1.1 and Remark B.1.3, we have $(e^T_{m-j} \otimes I_n)M^P_{\alpha^j} = e^T_{m-k_j} \otimes I_n$, where $k_j := j + 1 + c_{j+1}(\alpha^j)$. Hence from (B.3) we have

$$(e^T_{m-j} \otimes I_n)M^P_\alpha = (e^T_{m-k_j} \otimes (-A_j)) + (e^T_{m-(j-1)} \otimes I_n)M^P_{\alpha^j}. \qquad \text{(B.4)}$$

The evaluation of $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is completed if any one of the following cases hold. Otherwise, we move to Step-2.

- Suppose that $j, j-1 \notin \alpha^j$ or the subtuple of α^j with indices $\{j, j-1\}$ starts with j. Then by Remark A.1.1 and Remark B.1.3, we have $(e^T_{m-(j-1)} \otimes I_n)M^P_{\alpha^j} = e^T_{m-k_{j-1}} \otimes I_n$, where $k_{j-1} := j + c_j(\alpha^j)$. By Lemma B.1.1 we have $k_j > k_{j-1}$. Hence it follows from (B.4) that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

- Suppose that $j - 1 = h_2$. If the subtuple of α^j with indices $\{-h_2, h_2+1\}$ starts with $h_2 + 1$ then by similar arguments as above it follows that it follows that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free. On the other hand, if the subtuple of subtuple of α^j with indices $\{-h_2, h_2+1\}$ starts with $-h_2$ then by Remark A.1.3 and Remark B.1.3, we have $(e^T_{m-h_2} \otimes I_n)M^P_{\alpha^j} = e^T_{m-\ell} \otimes I_n$, where $\ell := h_2 - 1 + c_{-h_2}(\alpha^j)$. Hence it follows from (B.4) that $(e^T_{m-j} \otimes I_n)M^P_\alpha$ is operation free.

Step-2: Suppose that the subtuple of α^j with indices $\{j, j-1\}$ starts with $j-1$. Then repeat Step-1 with α replaced by α^j and j replaced by $j-1$.

Since $\{h_1 + 1 : h_3\}$ contains $s := h_3 - h_2$ number of indices, we must stop before s numbers of steps. This prove that $(e_{m-j}^T \otimes I_n)M_\alpha^P$ is operation free. $\square$

B.1.1 Proof of Proposition 3.1.4

Proof. Recall that (σ, ω) is a permutation of $\{0 : m\}$ and $\tau = -\omega$. Since $M_{-0}(Z)$ is a block diagonal matrix and $-0, -1 \notin \tau_1 \cup \tau_2$ (i.e., -0 and -1 do not repeat), without loss of generality we assume that $0 \in \sigma$. Similarly, since $M_m(Z)$ is a block diagonal matrix and $m-1, m \notin \sigma_j$, $j = 1, 2$, (i.e., $m-1$ and m do not repeat), without loss of generality we assume that $m \in -\tau$.

Now, since $(\sigma, -\tau)$ is a permutation of $\{0 : m\}$ with $0 \in \sigma$ and $m \in -\tau$, there exist $0 \leq h_1 < h_2 < \cdots < h_{k-1} < h_k \leq m - 1$ (with odd k) such that σ is a permutation of $\{0 : h_1\} \cup \{h_2 + 1 : h_3\} \cup \cdots \cup \{h_{k-1} + 1 : h_k\}$ and $-\tau$ is a permutation of $\{h_1 + 1 : h_2\} \cup \{h_3 + 1 : h_4\} \cup \cdots \cup \{h_k + 1 : m\}$. This implies that σ_1 and σ_2 contain indices from $\{0 : h_1 - 1\} \cup \{h_2 + 1 : h_3 - 1\} \cup \cdots \cup \{h_{k-1} + 1 : h_k - 1\}$ since $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP, and $-\tau_1$ and $-\tau_2$ contain indices from $\{h_1 + 2 : h_2\} \cup \{h_3 + 2 : h_4\} \cup \cdots \cup \{h_k + 2 : m\}$ since (τ_1, τ, τ_2) satisfies the SIP.

Set $\alpha := (\tau_1, \sigma_1, \sigma, \sigma_2, \tau_2)$. We now show that $M_\alpha(*)$ is operation free (i.e., L_0, given in Proposition 3.1.4, is operation free). From the above paragraph, it is clear that α contains indices from $\{0 : h_1\} \cup \{-(h_1 + 2) :_{-1} -h_2, h_2 + 1 : h_3\} \cup \{-(h_3 + 2) :_{-1} -h_4, h_4 + 1 : h_5\} \cup \cdots \cup \{-(h_{k-2} + 2) :_{-1} -h_{k-1}, h_{k-1} + 1 : h_k\} \cup \{h_k + 2 : m\} =: H_1 \cup H_3 \cup H_5 \cup \cdots \cup H_k \cup H_{k+1}$. Note that, for indices $i \in H_s$ and $j \in H_t$, we have $||i| - |j|| \geq 2$ if $s \neq t$. Hence $M_\alpha(*)$ is operation free if $M_{\alpha^{H_j}}(*)$ is operation free for all $j = 1, 3, \ldots, k + 1$, where α^{H_j} is the subtuple of α with indices from H_j. Since $\tau_1 \cup \tau_2$ (resp., $(\sigma_1, \sigma, \sigma_2)$) does not contains any index of H_1 (resp., H_{k+1}), we have $M_{\alpha^{H_1}}(*)$ (resp., $M_{\alpha^{H_{k+1}}}(*)$) is operation free. Further, by Lemma B.1.4, we have $M_{\alpha^{H_j}}(*)$ is operation free for all $j = 3, 5, \ldots, k$. Hence $M_\alpha(*)$ is operation free, i.e., L_0 is operation free.

Similar proof for $M_\beta(*)$ is operation free, where $\beta := (\tau_1, \sigma_1, \tau, \sigma_2, \tau_2)$, i.e., L_1, given in Proposition 3.1.4, is operation free. Hence the EGFPR $L(\lambda)$ is operation free.

$\square$