A few analytical solutions for predicting one-dimensional steady infiltration in heterogeneous soils

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DOCTOR OF PHILOSOPHY

by

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CERTIFICATE

This is to certify that the thesis entitled "**A few analytical solutions for predicting one-dimensional steady infiltration in heterogeneous soils**" submitted by Mr. Jagadish Talukdar (156104024) , to the Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy in Civil Engineering is a record of bonafide research work carried out by him under my supervision and guidance. It is also certified that the boundary value problems that have been solved in this report is based on a few mathematical results developed by me (Barua 2021; https://zenodo.org/record/4717255#.YMrnUUzhWM8). Mr. Talukdar has worked on these problems for about five years and the thesis is, in my opinion, worthy of consideration for the degree of Doctor of Philosophy in accordance with the rules and regulations of this Institute.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for award of any degree or diploma.

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STATEMENT

I do hereby declare that the matter embodied in the thesis is a result of research work carried out by me in the Department of Civil Engineering, Indian Institute of Technology Guwahati, Guwahati, Assam, India.

In keeping with the general practice and reporting scientific observations, due acknowledgements have been made wherever the work described is based on the findings of other investigators.

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LIST OF NOTATIONS USED IN CHAPTER 3 AND APPENDIX A

The following notations are used in Chapter 3 and Appendix A.

A	angle made by the infiltration column with the horizontal (degree)
$C_{p(q)}^{(j)}$	coefficient attached to the p^{th} term [Eq. (3.30)] of the q-domain solution to the problem obtained using the j^{th} approach ($p = 1, 2, 3, ...;$ $q = 1, 2, 3, ...;$ $j = 1, 2$)
D	specific capacity of soil (L^{-1})
$h(x)$	spatial distribution of suction head in $[0, L]$ (L)
h_L	suction head at the boundary $x = L$ (L)
$h_{(i)}^{(j)}(x)$	i-domain solution to the problem obtained using the j^{th} approach (L) ($i = 1, 2, 3, ...;$ $j = 1, 2$)
$K_s(x)$	spatial distribution of saturated hydraulic conductivity of soil in $[0, L]$ (LT^{-1})
$K[h(x)]$	hydraulic conductivity (Gardner) of soil as a function of suction head h (LT^{-1})
L	length of the infiltration column (L)
$L_1, L_2, ..., L_N$	N distinct points in $[0, L]$ (located at $x = L_1,$ $L_2, ..., L_N$) being used for generating the analytical solution utilizing either of the approaches
$h_c(x)$	Spatial variation of air entry value in Gardner's conductivity function for $[0, L]$(L)
$n(x)$	Spatial variation of pore size in Gardner's conductivity function for [0,L] (dimensionless)
q	infiltration flux at $x = 0$ (LT^{-1})
q_x	infiltration flux at a distance x from the start of an infiltration column (LT^{-1})
$S(x)$	spatial distribution of the sink term (i.e., the root-water extraction term) in $[0, L]$ (LL^{-1}T^{-1})
S_e	effective saturation of soil (dimensionless)
S_M	volume of water extracted by roots per unit cross-sectional area of a soil column per unit time in length L of the column (LT^{-1})
$S_D^{(j)}$	set of points used for generating a solution in $[0, L]$ using the j^{th} approach ($j = 1, 2$)
$S_{D(i)}^{(j)}$	set of points used for generating a solution in the i^{th} sub-domain of $[0, L]$ using the j^{th} approach ($i = 1, 2, 3, ...;$ $j = 1, 2$)
t	time variable (T)
x	space variable (L)
θ	volumetric moisture content of soil (dimensionless)
θ_r	residual moisture content of soil (dimensionless)

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θ_s	saturation moisture content of soil (dimensionless)
φ	total hydraulic head (L)
$\vec{\nabla}$	nabla operator

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LIST OF NOTATIONS USED IN CHAPTER 4 AND APPENDIX B

The following notations are used in Chapter 4 and Appendix B.

A	angle made by the infiltration column with the horizontal (degree)
$D_{p(q)}^{(j)}$	coefficient attached to the p^{th} term [Eq. (4.12)] of the q-domain solution to the problem obtained using the j^{th} approach ($p = 1, 2, 3, ...;$ $q = 1, 2, 3, ...;$ $j = 1, 2$)
$h(x)$	spatial distribution of suction head in $[0, L]$ (L)
h_L	suction head at the boundary $x = L$ (L)
$h_{(i)}^{(j)}(x)$	i-domain solution to the problem obtained using the j^{th} approach (L) ($i = 1, 2, 3, ...;$ $j = 1, 2$)
$K_s(x)$	spatial distribution of saturated hydraulic conductivity of soil in $[0, L]$ (LT^{-1})
$K_V[h(x)]$	hydraulic conductivity (van Genuchten) of soil as a function of suction head h (LT^{-1})
L	length of the infiltration column (L)
$L_1, L_2, ..., L_N$	N distinct points in $[0, L]$ (located at $x = L_1,\ L_2, ..., L_N$) being used for generating the analytical solution utilizing either of the approaches
$m(x)$	spatial distribution of Van Genuchten's m-parameter in [0,L] (dimensionless)
$n(x)$	spatial distribution of Van Genuchten's n-parameter in $[0, L]$ (dimensionless)
q	infiltration flux at $x = 0$ (LT^{-1})
$S(x)$	spatial distribution of the sink term (i.e., the root-water extraction term) in $[0, L]$ ($LL^{-1}T^{-1}$)
S_e	effective saturation of soil (dimensionless)
S_M	volume of water extracted by roots per unit cross-sectional area of a soil column per unit time in length L of the column (LT^{-1})
$S_D^{(j)}$	set of points used for generating a solution in $[0, L]$ using the j^{th} approach ($j = 1, 2$)
$S_{D(i)}^{(j)}$	set of points used for generating a solution in the i^{th} sub-domain of $[0, L]$ using the j^{th} approach ($i = 1, 2, 3, ...;$ $j = 1, 2$)
t	time variable (T)
x	space variable (L)
$\alpha(x)$	spatial distribution of Van Genuchten's α parameter in [0,L] [L^{-1}]
θ	volumetric moisture content of soil (dimensionless)
θ_r	residual moisture content of soil (dimensionless)
θ_s	saturation moisture content of soil (dimensionless)

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φ	total hydraulic head (L)
$\vec{\nabla}$	nabla operator

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ABSTRACT

Infiltration is an important component of the hydrologic cycle and the study of dynamics of water movement in the vadose zone has important applications in many areas of science and engineering including agriculture and ecosystem management. A correct description of infiltration in the vadose zone is also important in understanding fate and transport of agricultural pesticides and other contaminants through it. Because of the importance of this key hydrological process, it has been a subject of intense academic research in the field of hydro-science for several decades now. Infiltration through an unsaturated soil is mostly modeled using the Richards' equation, an equation that has been obtained by applying the law of conservation of mass and the generalized Darcy's law. It is actually a standard two phase air-water flow equation in a porous medium where the air phase is assumed as infinitely mobile. Water flow in the unsaturated zone of a soil may also be influenced by the root-water extractions of plants and the soil water dynamics associated with a soil-plant-water system may be substantially impacted by this parameter alone. Thus, it is essential that this important parameter is being aptly included while modeling flow through such a zone.

Infiltration modeling using the Richards' equation is generally done by making use of two widely used conductivity functions, namely the Gardner's conductivity model and the Mualem-Van Genuchten's conductivity model. Both the Gardner- and Mualem-van Genuchten-based infiltration equations, however, are highly nonlinear in nature for which analytical solutions are currently available for relatively simple infiltration settings only in homogeneous and in layered soils. Because of the extreme nonlinearities of these equations, they are mostly solved by numerical means; however, as pointed out by several researchers in the past, numerical solutions of these equations may have their own problems as well, the most notable being is the fact that this way of solving these equations may not always lead to converged solutions of these equations for all possible flow situations in soils. Analytical models of the Richards'-based infiltration equations, on the other hand, mostly do not have convergence issues as they solve these equations in an exact way. However, as mentioned before, both the Gardner- and Mualem-van Genuchten-based infiltration equations are currently analytically tractable for relatively infiltration situations only in homogeneous soils and not when the parameters of these equations are spatially changing in an infiltrating space. This is true both when a root-water extraction term

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(i.e., a sink term) is present in a flow space and when it is absent. Actually there is currently no analytical solution to either the Gardner-based or the Mualem-van Genuchten-based equation even for a homogeneous soil for all possible variation of parameters of these equations. Soils in nature are seldom homogeneous. Thus, there is a need to solve both the Gardner- and the Mualem-van Genuchten-based infiltration equations by treating the parameters of these equations as space variables. Further, as the presence of a root-water extraction term in a field soil is also mostly a norm rather than an exception, the sink term in these infiltration equations should also be properly included in the mathematical treatment of these equations. In view of the same, efforts have been made in this study to work out suitable analytical solutions to both the Gardner- and the Mualem-van Genuchten-based equations for a heterogeneous soil by treating the conductivity related parameters as well the sink term of these equations as spatial variables. In the development of these solutions, it is being assumed for simplicity that the flow is steady and one-dimensional. All these developed solutions are infinite series solutions the accuracy of which, like any other series solution of a differential equation, depends on the number of terms that are being considered in their development. Further, they are all new since, as mentioned before, there is currently no analytical solution to either the Gardner-based or the Mualem-van Genuchten-based infiltration equation for a heterogeneous soil with our without a sink term on an infiltrating space. The validity of the developed solutions is checked by comparing with the analytical works of others for a few simplified infiltration situations; also, a few numerical checks and experimental comparisons on them have also been carried out. These solutions can be used to predict infiltration behavior through any arbitrarily inclined heterogeneous soil space with or without a sink term. This is an important observation as with the existing solutions of either the Gardner-based or the Mualem-van Genuchten-based infiltration equation, infiltrations studies in heterogeneous soils with our without the sink term is not possible.

The study shows that infiltration on a heterogeneous Gardner or van-Genuchten soil is a highly complex process involving many variables and the spatial variations of these variables in such a soil may greatly influence the infiltration mechanics associated with it; this is true both when a root-water function is present in an infiltrating space and when it is absent. It has also come out of the study that infiltration hydraulics related to a heterogeneous Gardner or van-Genuchten soil is mostly due to the combined effect of all the players of the system and is not due to one or two infiltration variables of the system alone. As field soils are inherently heterogeneous and further

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as there is currently no analytical solution to either the Gardner or van-Genuchten-based infiltration equation for a heterogeneous soil with or without the sink term, it is hoped that the proposed solutions will be worthwhile additions to the collection of analytical solutions on the subject.

Keywords: Analytical solution; Gardner's conductivity function; van Genuchten's conductivity function; Root-water extraction function; Soil heterogeneity.

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CHAPTER 1

INTRODUCTION

1.1 General

Infiltration is an important hydrological process in nature and its accurate estimation is vital for proper soil and water management of a watershed (Philip 1957a, Serrano 2004; Leij et al. 2006; Assouline 2013; Herrada et al. 2014; Zhang et al. 2016, 2018, 2021; Broadbridge et al. 2017; Farthing and Ogden 2017; Gundalia 2018; Mahapatra et al. 2020; Singh 2018; Qing et al. 2020). It determines the fraction of rainfall that would result as surface runoff, has a crucial role in the water budget equation and is one of the main variables involved in transport processes in soils (Mishra et al. 2003; Singh 2010; Assouline 2013; Singh 2018). An accurate estimation of infiltrating is essential for predicting movement of chemicals and pollutants in soils (Prasad and Romkens 1982; Zadeh et al. 2007; Kirkham 2014; Gundalia 2018). To acknowledge the interaction between surface water and saturated groundwater, understanding water flow in the vadose zone is critical (Liang and Huwang 2003; Cho et al. 2010; Sulis et al. 2010; Ogden et al. 2015; Qing et al. 2020). Infiltration allows the soil to store and provide water for uptake by plants; it is also a source of water for soil organisms (Basha 1999; Varallyay 2008; Guswa 2010; Haghnazari et al. 2015). Infiltration also plays a crucial role in the carbon and nitrogen cycles of a soil-crop system (Golmohammadi et al. 2016; Liang et al. 2016, 2018). As infiltration is a major component of the hydrologic cycle, a thorough knowledge of the hydraulics associated with this widely prevailing natural process is expected to lead to a better estimate of runoff, soil moisture patterns and groundwater recharge associated with a watershed (Serrano 1998; Venkata et al. 2008; Padretti et al. 2012; Baiamonte 2020). Also, for better environmental management of a landscape, the necessity to do a quantitative analysis of infiltration has also got more augmented in recent times (Yeh et al. 1985a; Huang and Wu 2012; Wu et al. 2015). Phillip (1969) and Brutsaert (2005) defined infiltration as the process due to which surface water enters the soil and moves through it due to the influence of gravity. Further, apart from the gravity force, the movement of water in the vadose zone is also greatly influenced by capillary forces resulting due to passage of water through small-sized pores inside a soil. Further, infiltration is also influenced by the orientation of a soil profile on which it is taking place (Phillip 1969; Smith 1983; Oldenberg and Pruess 1993; Nicholl et al. 1994; Chan et al. 2004; Golding et al. 2013; Hayek 2016a).

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As advection, dispersion and adsorption in the vadose zone are also closely related to infiltration, the study of this important process is also expected to throw light on how the pollutants traverse in the unsaturated zone of a soil profile (van Genuchten and Jury 1987; Basha 1994; Hoven et al. 2003; Cavalcante and Zornberg 2017).

The study of water flow in unsaturated soils is not new and dates back to the early 19th century (Hillel 1998). Flow in the unsaturated zone is mostly modeled by making use of the celebrated Richards' equation (Parlange et al. 1999; Zhan and Ng 2004; DiCarlo 2005; Zlotnik et al. 2007; Crevoiser et al. 2009; Weill et al. 2009; Broadbridge et al. 2017; Tzavaras et al. 2017; Tavangarrad et al. 2018 – to cite a few). It is an equation based on continuum theory. It is a highly nonlinear equation involving complex relationships between the infiltration variables of interest and the associated hydraulic parameters (Serrano 1998; Kuraz et al. 2010; Hayek 2016a, Broadbridge et al. 2017). It can actually be considered as a simplification of the standard two-phase flow equation involving a gas phase and a water phase where the pore air pressure in a soil space is essentially treated to be at a constant atmospheric pressure owing to the excessive mobility of the air phase in comparison to the water phase (Nayagum et al. 2004; Wang 2011; Farthing and Ogden 2017). Flow in the vadose zone is also greatly influenced by evaporation and root-water extraction affects of plants and the entire groundwater dynamics associated with a soil-plant system may be substantially impacted by these parameters (Prasad 1988; Basha 1994; Ojha and Rai 1996; Mathur and Rao 1999; Wu et al. 1999; Li et al. 2001; Vrught et al. 2001; Zuo et al. 2004; Yadav and Mathur 2008; Ojha et al. 2009; Couvreur et al. 2012; Boughami et al. 2018). Thus, care need be exercised to see that these variables are suitably considered while analyzing subsurface flow behavior in a soil-plant system. From a hydrological point of view, spatial distribution of root-water uptakes may influence water flow to atmosphere as well as on to the deep groundwater zone of a watershed (Canadell et al. 1996). Further, interaction between roots and the surrounding soil-water system of a watershed may substantially attenuate the contaminant related affects on the subsurface water of the watershed (Clausnitzer and Hopmans 1994; Clothier and Green 1994). Thus, root-water extraction is an important variable in the hydraulics of water flow in variably saturated soils. This variable is generally modeled by considering it as a sink term – a term which can vary both in space and time – in the Richards' (1931) equation (Whistler et al. 1968; Molz and Remson 1970; Hoogland et al. 1981; Clausnitzer and Hopmans 1994; Mathur and Rao 1999; Feddes et al. 2001; Zuo et al. 2004; Yuan and Lu 2005; Ojha et al. 2009; Zeng and Decker 2009; Kuhlmann et al. 2012; Broadbridge et al. 2017).

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As the Richards' equation is highly nonlinear with its parameters mostly dependent on moisture content of a soil, an analytical solution of it is not easily tractable and for only relatively simple hydrologic situations only, analytical solution of it are currently available (Zhang 2015, 2021). Because of its highly nonlinear nature, generally numerical methods are being employed to solve this equation (Haverkamp et al. 1977; van Genuchten 1982; Milly 1985; Allen and Murphy 1986; Nielsen et al. 1986; Celia et al. 1987, 1990; Huwang et al. 1994; Rathfelder and Abriola 1994; Pan et al. 1996; Tocci et al. 1997; Bergamaschi and Putti 1999; Woodward and Dawson, 2000; Kavetski et al. 2001, 2002; Farthing et al. 2003; Bause and Knaber 2004; Miller et al. 2006; Simunek et al. 2008a, Simunek et al. 2008b; D'Haese et al. 2007; Li et al. 2007; Chen and Ren 2008; Casulli and Zanulli 2010; Kuraz et al. 2010; Kuznetsovetal et al. 2012; Juncu et al. 2012; Lai and Ogden 2015, Zhang et al. 2015, 2016, 2018, 2021; Vocciante et al. 2016; Zha et al. 2017, 2019; Berardi et al. 2018; Chavez-Negrete et al. 2018; Suk and Park 2019- to cite a few).

Based on the type of variable used, Richards' equation is generally expressed in three forms: the moisture content (i.e., θ-based) form, the pressure head form (i.e., h-based) and a combination of both moisture content and pressure head form (Hills et al. 1989; Brunone et al. 2003; Zha et al. 2013; Zhang et al. 2016, 2021; Suk and Park 2019). For simulating infiltration in uniform dry soils, the θ-based Richards' infiltration equation may be employed as this form of the Richards' equation gives good mass conservation results for such situations (Hills et al. 1989; Kirkland et al. 1992; Forsyth et al. 1995; Brunone et al. 2003; Zha et al. 2013; Zhang et al. 2021). Unfortunately, this form of the Richards' infiltration equation is not suitable for simulating water in non-uniform soils and/or in soils with mixed saturated and unsaturated regions (Yeh et al. 2015; Suk and Park 2019; Zhang 2021). For simulating flow in saturated and unsaturated multi-layered soils, the h-based Richards' infiltration equation is mostly adopted (Romano et al. 1998; Celia et al. 1990; Simunek et al. 2006; Suk and Park 2019; Zhang 2021). However, numerical schemes for the h-based Richards' infiltration equation may suffer from large mass balance and convergence problems (van Genuchten 1982; Milly 1984; Allen and Murphy 1986; Celia 1987; Hao et al. 2005; Lai and Ogden 2015; Zhang et al. 2018, 2021; Suk and Park 2019; Vrugt and Gao 2021). To capitalize on the advantages of both θ- and h-based forms of the Richards' equation, several numerical techniques based on a mixed form of the Richards' equation have also been developed for simulating moisture movement in variably saturated soils (Kirkland et al. 1992; Wu and Forsyth 2001; Zhang 2016, 2018, 2021). However, in many simulations

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considering the mixed form of the Richards' equation, the non-smooth transition between the primary variables may lead to unrealistic modeling results (Krabbenhøft 2007; Zha et al. 2017).

For tacking the nonlinearities of the Richards' equation, several transformation techniques have been used in the past (Ross 1990; Chen and Dai 2016; Williams et al. 2000). Of these, the Kirchhoff transformation probably is one of the most widely used transformation techniques for linearizing the Richards' equation (Ross and Bristow 1990; Bakker and Nieber 2004; Ji et al. 2008; Zhang et al. 2015; 2016; 2018; 2021; Suk and Park 2019). However, such a transformation may cause the transformed variable to become discontinuous at the interfaces between different soil layers which may in turn lead to numerical difficulties (Zha et al. 2013, 2019). Overall, due to the high nonlinearity of the Richards' equation, it is one of the most difficult equations to be solved accurately and reliably in the field of hydroscience (Farthing and Ogden 2015; Zha et al. 2019; Zhang et al. 2021). As mentioned before, several attempts have been made to solve this equation mathematically – both by analytical and numerical means – but an all encompassing solution of this equation valid concurrently for both saturated and unsaturated heterogeneous soils appears to be currently lagging. Numerical models have the advantage of handling complex differential equations with ease in both regular and irregular flow domains and also have the desirable quality of tackling difficult initial and boundary conditions (Brufau et al. 2002; Craig and Read 2010, Lewis 2013; Wang and Shi 2013; Ludvigsson et al. 2018). Being approximate methods, their convergence and stability, however, need to be thoroughly checked before they are being finally applied to analyze real field situations. Further, as their applications often involve processing of a vast data set, they may, at times, be quite tedious to use. Also, many a times, the data required for their use may be prohibitively expensive and/or difficult to acquire.

1.2 Motivations of the Study

In spite of many advantages of the numerical methods in solving the Richards' infiltration equation, an accurate and stable numerical solution of this equation for all possible moisture content distributions in a heterogeneous soil, as mentioned before, currently appears to be missing. Analytical models have some advantages over numerical models as they are mostly stable and do not have convergence issues. They are also more flexible than numerical models and most often they help in providing a better physical insight of a studied system as compared to numerical and experimental studies (Haitjema 2006; Sarva et al. 2010; Lewis 2013; Hayek 2016a). In addition, because of the inherent accuracy of analytical solutions, they are also being frequently employed for verifying complex numerical codes related to the

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study of various physical processes in a porous formation (Ross and Palange 1994; Haitjema 2006; Sarva et al. 2010; Huwang and Wu 2012; Hayek 2015, 2016a). Further, an analytical solution may also be used to estimate hydraulic parameters associated with a water flow equation if the solution can be suitably inverted analytically (Hayek 2016a). Also, with the advent of many new and powerful analytical tools (Heaselet and Alksne 1961; Clarkson and Kruskal 1989; Liao 1992, Adomian 1994; He 1999; Cantwel 2002) complex differential equations are also now been increasingly solved by analytical means. Hybrid models – combining the powers of both analytical and numerical models – are also now becoming quite popular in solving intricate problems involving transport processes in soils and other natural systems (McDermott et al. 2009; Craig and Read 2010; Morel-Seytoux 2015; Meunier et al. 2017; Gasparin et al. 2018, De Barros et al. 2019).

Because of many advantages of analytical models, several efforts have already been made to obtain analytical solutions of the Richards' equation for different flow situations in homogeneous and in layered soils (Gardner 1958; Phillip 1969, 1972, 1974, 1991; Parlange 1971b, 1972a; Raats 1972; Warrick 1974, 1975, 1988, 2003; Parlange et al. 1980, 1982, 1984, 1985, 1992, 1997, 1998, 1999; Warrick et al. 1985, 1990, 1991; Broadbridge et al. 1988, 2009, 2017; Broadbridge and White 1988; Sander et al. 1988; White and Broadbridge 1988; Zimmerman and Bodvarsson 1989; Warrick and Yeh 1990; Barry and Sander 1991; Srivastava and Yeh 1991; Mualem et al. 1993; Salvucci 1993; Smith et al. 1993; Shan and Stephens 1995; Basha 1994, 1999, 2000a 2000b, 2011; Serrano 1998, 2004; Warrick and Knight 2002, 2003, 2004; Zhu and Mohanty 2002; Assouline and Mualem 2003; Yuan and Lu 2005; Tracy 2006; Kuhlman and Warrick 2008; Triadis and Broadbridge 2010; Sadeghi et al. 2012; Hayek 2014, 2015, 2016a; Chen and Dai 2017; Baiamonte 2020; Moret-Fernández et al. 2020; Jaiswal et al. 2021; Leij et al. 2021; Vrugt and Gao 2021; Zhu and Xiao 2022 – to name a few). Most of these solutions, however, are based on some sort of linearization of the governing equation and by assuming the soil within each layer of a layered soil as homogeneous. However, as the Richards' infiltration equation is highly nonlinear, there is right-now no analytical solution of it for a heterogeneous soil where the properties of the soil are continuously changing in space (Crevoisier et al. 2009; Lai and Ogden 2015; Ogden et al. 2015; Zhang et al. 2015, Hayek 2016a; Farthing and Ogden 2017; Zeng et al. 2018; Suk and Park 2019).

Infiltration modeling using Richards' equation is generally done by making use of conductivity functions as proposed by Gardner (1958) or the conductivity function as proposed by van Genuchten (Mualem 1976; van Genuchten 1980). In this context, it to be

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noted that Gardner (1958) proposed mainly two conductivity functions relating the unsaturated hydraulic conductivity of a soil with its matric potential, the one in which conductivity has an exponential relation with suction head (named here as GEX) and the other – the more general one (named here as GHC) – in which conductivity is related to suction head as $K_s/\left[1+(h/h_c)^n\right]$, where h is the suction head, K_s is the saturated hydraulic conductivity, h_c is the air entry value of the suction head and n is a constant. The use of Gardner's (1958) exponential conductivity function (GEX) greatly helps in making the Richards' equation amenable for analytical treatment and many have obtained analytical solutions in the past utilizing the same for different flow situations in soil (Gardner 1958; Philip 1972; Raats 1972; Warrick 1974; Philip and Forrester 1975; Merrill et al. 1978; Philip 1989a, 1989b, 1998; Knight et al. 1989; Srivastava and Yeh 1991; Basha 1994, 2000, 2011; Shan and Stephens 1995; Tracy 1995; Warrick and Knight 2002, 2003, 2004; Yuan and Lu 2005; Barontini et al. 2007; Kuhlman and Warrick 2008; Warrick et al. 2008; Huang and Wu 2012; De Luca and Cepeda 2016; Wu et al. 2016; Broadbridge et al. 2017 – to name a few). However, because of the complexities of Gardner's general conductivity (GHC) model (1958), not many analytical solutions are currently available involving this model even for a homogeneous soil.

By making use of his GHC model, Gardner (1958) studied evaporation from a shallow water table for $n=1,\ 1.5,\ 2,\ 3$ and 4 of his model and Warrick (1988) provided an exact solution for this problem in the form of an incomplete Beta function and a hypergeometric series for all real $n>1$. Salvucci (1993) worked out an approximate closed-form solution for the steady one-dimensional GHC-based Richards' equation for predicting moisture movement through a homogeneous vertical soil column. This approximate solution appears to work quite well for coarse-textured soils but for fine-textured soils, however, it has been found to deviate from numerical results. Also, this solution is not for situations where a root-water extraction function exists in an infiltrating space. Basha (1999) formulated an approximate perturbation solution to the steady one-dimensional GHC-based infiltration equation for a homogeneous soil capable of handling any arbitrary root-water extraction function in an infiltration column. He also provided an exact solution to the problem for situations where n is an integer and where the root-water extraction term (i.e., the sink term) is not there. Zhu and Mohanty (2002) also provided an analytical solution to the problem considered by Basha (1999) but their solution is for zero-sink situations only and not when a sink term in the form of a root-

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water uptake function is present in an infiltrating space. However, the big advantage of this solution over that of Basha's (1999) solution is that it is exact for both integral and non-integral values of *n*. Further, Zhu and Mohanty (2002) also provided a solution to the problem based on Brooks and Corey's (1964) conductivity function as well.

Among the various models used for working out the unsaturated hydraulic conductivity of a soil column at its different levels of saturation, the Mualem-van Genuchten (MVG) conductivity model (Mualem 1976; van Genuchten 1980) has currently been found to be the most widely used model for such a purpose. Thus, this model can be considered as a benchmark model for simulating flow and transport in unsaturated soils (Ippisch et al. 2006; Schaap and van Genuchten 2006; Vereecken et al. 2010; Dettmann et al. 2014; Farthing and Ogden 2017; Shein et al. 2018; Latorre and Moret-Fernández 2019; Suk and Park 2019; De Melo et al. 2021 – to cite a few). To the best of our knowledge, there is currently no analytical solution to the Mualem-van Genuchten-based Richards' infiltration equation with a sink term even for a homogeneous soil. Without the sink term, however, Rockhold et al. (1997) provided an analytical solution of it for a layered vertical soil column. Actually, Rockhold et al.'s (1997) solution is not obtained by solving the MVG-based Richards' equation directly; instead, they obtained their solution by first approximating the logarithmic form of the MVG conductivity function as a series of finite piecewise-linear segments in a flow space and then integrating the governing differential equation in each of these segments. Perhaps, because of this, Hayek (2016a) decided not to consider this as an analytical solution of this equation in his infiltration study of 2016. Also, because of the inherent nature of Rockhold et al.'s (1997) solution, it is valid only for situations where the hydraulic properties within each layer of a layered soil are all homogeneous and not otherwise.

From the above reviews of various analytical solutions related to infiltration, it is thus clear that there is currently no analytical solution to either the GHC-based or the MVG-based infiltration equation for a heterogeneous soil. In fact, even for a homogeneous soil, there is currently no analytical solution to either the GHC-based or the MVG-based infiltration equation for all possible variations of parameters of these equations. Further, even for homogeneous soils, the existing analytical solutions for the GHC-based and the MVG-based infiltration equations are for infiltration through a vertical soil column only and not for infiltration through any arbitrarily inclined soil profile. It needs no mentioning that field soils are seldom homogeneous (Nielsen et al. 1973; Warrick and Nielsen 1980; Peck 1983; Mualem 1984; Greminger et al. 1985; Sharma et al. 1987; Loague and Gander 1990; Yeh and Harvey 1990; Logsdon and Jaynes 1996; Wildenschild and Jensen 1999; Elcateb et al. 2003;

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Williams and Houseman 2014; Saschidharan et al. 2019; Soraganvi et al. 2020). Thus, the importance of including soil heterogeneities while modeling the Richards' infiltration equation with a suitable conductivity function probably cannot be overemphasized (Yeh et al. 1985a,1985b, 1985c; Yeh 1989; Mantoglou and Gelhar 1987a, 1987b, 1987c; Zhu and Mohanty 2003, 2004; Neuweiler and Eichel 2006; Li et al. 2016; Ngo-Cong et al. 2020; Zhang et al. 2021). In view of the same and remembering the importance of working out, if possible, analytical solutions of hydro-geo problems, an effort is being made in this study to develop a few analytical models related to one-dimensional water movement in a heterogeneous unsaturated soil column. All these solutions are being developed by making use of a few mathematical results (related to solution of ordinary differential equations) as proposed by Barua (2021).

1.3 Objectives

(i) To develop an analytical solution to the one-dimensional steady-state Richards' equation for an arbitrarily inclined heterogeneous soil column with a sink term (i.e., with an arbitrary root-water extraction term) by utilizing Gardner's (1958) general hydraulic conductivity (GHC) function.

(ii) To develop an analytical solution for the same infiltration settings as mentioned in (i) but now by introducing Mualem-van Ganuchten's (1980) hydraulic conductivity (MVG) function in the Richards' equation.

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CHAPTER 2

LITERATURE REVIEW

An attempt will now be made to provide a brief review about the various mathematical works carried out in the past in the field of infiltration research. As our work in this report is related to analytical modeling of the Richards' infiltration equation, our reviews here will be confined to past analytical works on infiltration only. However, in this context it needs mentioning that, as may be observed, information on a few recent numerical modeling works on infiltration have already been given in Section 1.1 (i.e., in the 'Introduction' section) of the previous chapter.

2.1 Literature Review Related to Analytical Modeling of Infiltration

Using physical principles, Green and Ampt (1911) developed an infiltration model for predicting infiltration capacity rate of a soil. This model is still widely used for many infiltration studies. By using conservation principles, Richards (1931) developed an equation which can be used to predict moisture movement in variably saturated soils. This equation, because of its wide versatility, can be considered as the founding equation for studying vadose zone hydrology (Singh 2018). Kostiakov (1932) provided an empirical equation which can be utilized to estimate the infiltration capacity rate of a soil. Horton (1933, 1939) put forward a theory of infiltration based on hydrologic systems concept and tested the same on experimental plots in 1941 (Horton 1941). Philip (1957a, 1957b, 1957c, 1957d, 1958) was one of the pioneer's of infiltration studies and presented many solutions of the Richards' equation for a homogeneous soil under various unsaturated soil settings for both steady as well as transient flow situations. Gardner (1958) introduced a transformation which made it possible for him to obtain exact solutions to a few steady-state and approximate solutions to a few transient state infiltration problems. Brutsaert (1968) provided an analytical solution for vertical infiltration into a very dry soil by considering simple algebraic functions for the soil hydraulic properties. Childs and Bybordi (1969) made use of the Green and Ampt's (1911) infiltration law to study water movement in layered soils. Philip (1969) introduced a concept called "time of gravity" that defines the time when gravitational flow dominates over that of capillary flow in an infiltrating flow field.

Raat's (1970) made studies on infiltration resulting from an array of line sources in a homogeneous soil. Parlange (1971a, 1971b, 1971c, 1972a, 1972b, 1972c, 1972d, 1972e, 1973), in his series of work on infiltration, derived analytical solutions for steady infiltration

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in homogeneous soils under different unsaturated flow situations. Parlange (1971b) introduced the double integration technique to obtain solution to the Richards' equation and solved several infiltration problems with different boundary conditions using this technique (Parlange 1972a-1972d). Philip (1972, 1974) worked out analytical solutions for predicting steady infiltration resulting from surface, buried point and line sources in heterogeneous soils by assuming hydraulic conductivity to vary exponentially with both moisture content and depth. These solutions further assume the source and sink terms to be absent in the studied flow domains. By assuming hydraulic conductivity to vary exponentially with pressure head, Raats (1972) obtained analytical expressions for predicting steady infiltration from a single and arbitrary distribution of sources in a homogeneous soil of infinite extent. Talsma and Parlange (1972) showed that only two variables, namely sorptivity and saturated hydraulic conductivity of a soil, are good enough for applying Philip's (1957a) vertical infiltration equation for most practical situations. Braester (1973) provided approximate analytical solutions to the one-dimensional infiltration equation for situations when the soil column is semi-infinite and when a water table lies at a finite depth from the surface of the soil. In both of these solutions, diffusivity is assumed as constant, hydraulic conductivity to vary exponentially with suction head and the soil medium as homogeneous. Mein and Larson (1973) proposed a model for computing infiltration in a homogeneous soil under steady rainfall situations. Philip and Knight (1974), by utilizing the concept of flux-concentration relation (Philip 1973; Knight and Philip 1973), developed quasi-analytical solutions to a series of unsaturated flow problems in homogeneous soils. Warrick (1974) worked out steady-state analytical solutions for the one-dimensional infiltration equation for any arbitrary plant-water extraction function in a homogenous soil both for situations when the soil is of an infinite depth and when a water table is present at a finite depth from surface of the soil. In this analysis, Gardner's (1958) exponential hydraulic conductivity model is being used; also a constant flux boundary is assumed at the top of the soil. Further, in the analytical development, the sink term is assumed as an explicit function of depth only.

Warrick (1975) provided analytical solutions to the one-dimensional linearized moisture flow equation for different surface boundary conditions. In all these analytical developments, a homogeneous semi-infinite flow medium is assumed along with the assumption of a constant diffusivity coefficient and an exponentially varying hydraulic conductivity function with pressure head (Gardner 1958). Babu (1976a, 1976b, 1976c) used Boltzmann variables and perturbation techniques to study horizontal and vertical infiltration in a homogeneous soil by treating the boundaries as Dirichlet boundaries. This analysis is for situations where

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diffusivity is constant or where it varies exponentially with moisture content. Brutsaert (1976) made use of Parlange's (1971b) double integration technique to work out a general solution for horizontal infiltration and Smith and Parlange (1978) used this technique to develop a general hydrologic infiltration model. Lomen and Warrick (1976) worked out several solutions of the one-dimensional steady-state moisture flow equation for different soil-water extraction functions by making using of Gardner's (1958) exponential conductivity model. The advantage of this solution is that it considers the sink term as a function of both soil depth and matric potential of a soil. Lomen and Warrick (1978) presented analytical solutions for the one-dimensional infiltration equation for a homogeneous soil with point and line sources located at the surface or being buried in the soil by again resorting to Gardner's (1958) exponential model of hydraulic conductivity distribution. Batu (1978) gave three-dimensional analytical solutions for predicting steady infiltration from single and periodic strip sources in a homogenous soil by using Fourier analysis. In this analysis, Gardner's (1958) exponential hydraulic conductivity model is used for the analytical development. White et al. (1979), by making use of the flux-concentration relationship of Philip (1973), developed an analytical model for studying one-dimensional constant flux absorption of water in a horizontal homogeneous soil.

Parlange et al. (1980) obtained a solution to the one-dimensional diffusion equation by assuming a power law relation between diffusivity and moisture content. Warrick et al. (1980) derived a three-dimensional steady-state solution to simulate plant-water-uptake in a homogeneous soil by assuming unsaturated hydraulic conductivity to vary exponentially with pressure head (Gardner 1958). Clothier et al. (1981) worked out an analytical model for the constant diffusivity vertical infiltration equation by assuming a quadratic functional variation of hydraulic conductivity with moisture content and found their model predictions to match fairly well with experimental data. Perroux et al. (1981) extended the constant flux absorption solution of White et al. (1979) to constant-flux vertical infiltration by obtaining a solution to the relevant governing equation. Batu (1982) provided transient solutions for predicting two-dimensional infiltration and/or infiltration-evaporation from non-uniform and non-periodic sources in a homogeneous and isotropic soil by assuming diffusivity as constant and utilizing Gardner's (1958) exponential hydraulic conductivity function. Parlange et al. (1982) developed a quasi-exact implicit solution of the infiltration equation for describing cumulative vertical infiltration into a homogeneous soil with a uniform initial moisture content distribution. Prasad and Romkens (1982) suggested an approximate series solution to the one-dimensional Richards' equation for a homogeneous soil under changing boundary

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conditions. In the derivation of this solution, diffusivity and hydraulic conductivity are assumed to vary with moisture content in specific ways. Rogers et al.'s (1983) studied two-phase oil and water infiltration under a non-changing flux boundary. Boulier et al. (1984) provided a quasi-analytical solution of the infiltration equation for a constant flux condition at the surface of a soil. This solution can consider both uniform as well as nonuniform initial soil moisture profiles of a soil column for fluxes either smaller or greater than the saturated hydraulic conductivity of the soil. Parlange et al. (1984) worked out a perturbation solution of the nonlinear diffusion equation for a homogeneous soil where water content is expressed as a function of distance to the wetting front. Phillip (1984a, 1984b) made steady infiltration studies from cylindrical and spherical cavities in a homogeneous soil by making use of Gardner's (1958) exponential model. In these analyses, the moisture potential is assumed fixed at the surface of the cavities.

Parlange et al. (1985) extended Parlange et al.'s (1982) implicit infiltration solution to include different ponded conditions at the surface of a soil. This solution assumes a uniform initial water content in a soil; however, it can accommodate any well defined time varying ponding distribution at the top of a soil. Warrick et al. (1985), by utilizing a mathematical procedure as presented by Philip (1969), provided a general solution to the one-dimensional infiltration equation for a homogenous soil. This is a versatile solution as it has the capability of including different hydraulic conductivity functions in its fold including Brooks and Corey's (1964) and van Genuchten's (1980) hydraulic functions. Broadbridge and White (1988), by assuming diffusivity to vary in a particular way with the moisture content, proposed analytical models for describing infiltration in uniform soils. These models are capable of including soil properties extending from weakly nonlinear to highly nonlinear in a homogeneous soil medium. Broadbridge et al. (1988), by making use of the same assumptions as made by Broadbridge and White (1988), carried out extensive analytical studies on the nonlinear infiltration equation for a finite-sized soil underlain by an impervious barrier. Warrick (1988) provided analytical solutions for studying steady-state evaporation from a shallow water table in a homogeneous soil by making use of Brooks and Corey's (1964) conductivity function. White and Broadbridge (1988) tested the applications of Broadbridge and White's (1988) models for a wide range of soils in laboratory and field studies. Sander et al. (1988) showed that by a simple change in the space and time variables, Rogers et al.'s (1983) solution for two-phase oil and water infiltration under a non-changing flux boundary, can also be transformed to predict constant flux infiltration in a soil column provided the conductivity field is described in a certain way with moisture content. Philip

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(1989a, 1989b, 1998) studied seepage exclusion problems in cavities of different sizes and shapes in homogeneous soils by utilizing Gardner's (1958) exponential conductivity model. Using the same conductivity function, Knight et al. (1989) introduced a general theory of water inclusion for circular cylindrical cavities in a homogeneous soil. Zimmerman and Bodvarsson (1989) derived an approximate solution for infiltration in a horizontal variably saturated homogeneous soil column of semi-infinite domain by utilizing Brooks and Corey's (1964) diffusivity and conductivity relations.

Broadbridge and Rogers (1990), by assuming conductivity and diffusivity to vary in specific ways (Broadbridge and White 1988; White and Broadbridge 1988) with moisture content of a homogeneous soil, obtained analytical solutions of the one-dimensional nonlinear convective diffusion equation for vertical drainage and redistribution in the soil. Haverkamp et al. (1990) extended Parlange et al's (1982) solution of the infiltration equation for ponded conditions. This implicit infiltration equation can account for the possibility of an infinite diffusivity near saturation. Meyer and Warrick (1990) developed an analytical expression for soil water diffusivity by utilizing experimentally observed horizontal infiltration data. Singh and Yu (1990), using systems approach, developed a general model for infiltration and showed that infiltration models of Green and Ampt (1911), Kostiakov (1932), Horton (1939), Philip (1957a), Holtan (1961) and Overton (1964) are all special cases of this model. Warrick and Yeh (1990) developed an algorithm for studying steady one-dimensional vertical infiltration in a layered soil. This model can account for both upward and downward flow and can handle prescribed head or flux boundary conditions. Warrick et al. (1990) solved the one-dimensional Richards' equation for constant flux surface input incorporating drainage from partially and deeply wetted soil columns. Barry and Sander (1991) utilized the Bācklund transformation to work out an analytical solution for predicting one-dimensional infiltration in a homogenous soil for any arbitrary temporal flux distribution at the surface of the soil. Philip (1991) developed an analytical solution for a problem of infiltration into concave and convex topographies in a homogeneous isotropic soil with uniform initial moisture content. By making use of integral transformation methods and by assuming hydraulic conductivity as an exponential function of matric potential and a linear function of moisture content, Protopapas and Bras (1991) provided an analytical solution for the multidimensional transient infiltration problem for a heterogeneous soil. By making use of integral and Laplace transformations, Srivastava and Yeh (1991) developed analytical solutions to predict one-dimensional transient infiltration in single and layered soils underlain by a water table. These solutions employ Gardner's (1958) exponential conductivity model

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and an exponential relation of moisture content with suction head. Warrick et al. (1991) obtained a solution to the one-dimensional infiltration equation for situations where the surface flux is not a constant but varies in a piecewise continuous way with time. Parlange et al. (1992) extended the Heaselet and Alksne technique to solve the nonlinear diffusion equation for arbitrary diffusivity variations in a homogeneous soil. Warrick and Broadbridge (1992) explored the relationship between sorptivity and macroscopic capillary length (Philip 1985) of a homogeneous soil. Salvucci (1993) made use of Gardner's (1958) general hydraulic conductivity function to derive an approximate solution to the steady-state one-dimensional vertical infiltration equation for a homogeneous soil. Smith et al. (1993) presented a physically based rainfall infiltration model for complex storms in a homogeneous soil. This model extends Parlange et al.'s (1982) three-parameter infiltration model to soils with very high initial moisture content. Warrick and Hussen (1993), using scaling techniques and by utilizing Brooks and Corey's (1964) hydraulic conductivity function, developed one-dimensional infiltration models for a few flow situations in an unsaturated soil. Using Gardner's (1958) exponential conductivity model and Green's function method, Basha (1994) formulated integral solutions to multidimensional steady-state infiltration problems in a homogeneous soil that can handle flow situations related to surface/subsurface irrigation, root-water uptake and evaporation. Edwards and Broadbridge (1994) applied the Lie group symmetry analysis to obtain exact solutions to a class of transient nonlinear diffusion equations for both two- and three-dimensional flow situations. Haverkamp et al.'s (1994) provided quasi-exact infiltration equation for modeling one-dimensional infiltration in homogeneous soils. Ross and Parlange (1994) obtained transient solutions for one-dimensional infiltration and drainage situations by assuming power law relations of diffusivity and hydraulic conductivity with moisture content for homogeneous soils.

Barry et al. (1995) provided an explicit form of Parlanage et al's (1985) ponded infiltration solution of the Richards' equation for a homogeneous soil. Shan and Stephens (1995) provided analytical solutions for infiltration in single and two-layered soils using Gardner's (1958) exponential conductivity model. Assuming variations of moisture content with pressure head and relative hydraulic conductivity with pressure head in certain ways, Tracy (1995) obtained analytical solutions for a few multidimensional infiltration problems in an unsaturated soil. By introducing a transformed time variable, Salvucci (1996) obtained a series solution to the one-dimensional water transport equation in a homogeneous and unbounded soil with uniform initial moisture content. Parlange et al. (1997) worked out a general approximate solution to the one-dimensional Richards' equation by including gravity

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effects in Parlange et al.'s (1992) infiltration solution. Rockhold et al. (1997) developed a steady-state one-dimensional solution for predicting vertical infiltration in a variably saturated layered soil of finite thickness. The solution has the advantage of handling any arbitrary hydraulic properties of a variably saturated soil column. Witelski (1997) made use of the perturbation method to study horizontal and vertical infiltration in a layered soil underlain by an impervious boundary. In this analysis, hydraulic conductivity and diffusivity are assumed to vary in certain ways (Bear 1972; Witelski 1995) with moisture content of a soil. Liu et al. (1998) showed that the time compression approximation can be used to predict cumulative infiltration in linear soils in a very accurate way. Parlange et al. (1998) developed an approximate solution to the one-dimensional nonlinear diffusion equation for a homogeneous soil. This solution is applicable for arbitrary soil properties and boundary conditions. Serrano (1998) proposed an analytical decomposition method for solving the nonlinear infiltration equation with constant boundary and continuous wetting conditions in a homogeneous soil of infinite extent. Using the Green's function method, Basha (1999a) obtained a general analytical solution for studying unsteady multidimensional moisture movement in a semi-infinite homogeneous soil with any arbitrary initial boundary condition. Basha's (1999a) multidimensional infiltration solution is based on the assumption of an exponential dependence of hydraulic conductivity (Gardner 1958) and moisture content on matric potential of a soil. The advantage of Basha's (1999a) infiltration solution is that it has the ability to include any arbitrary root-water uptake function in its fold. Utilizing the perturbation method, Basha (1999b) provided a general mathematical infrastructure for working out approximate solutions to different steady-state infiltration problems in a homogeneous soil with any arbitrary root-water extraction function. The solution procedure utilizes Gardner's (1958) general conductivity model and yields exact results for all integral values of one of the parameters of this model when the root-water extraction function is taken as zero in the governing equation. Parlange et al. (1999) provided an approximate series solution to the classical one-dimensional Richards' equation for a homogeneous soil and claimed that this solution can be used to predict time to ponding in an accurate way.

Basha (2000a) made use of the Green function method to study multidimensional steady infiltration with arbitrary boundary conditions in a homogeneous soil underlain by a water table. In this analysis also, Gardner's (1958) exponential conductivity model is being used. Basha (2000b) again made use of Green's function to study multidimensional non-steady infiltration in a variably saturated homogeneous soil underlain by a shallow water table. Parlange et al. (2000) extended Liu et al.'s (1998) analysis of time compression

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approximation on cumulative infiltration from a linear soil to a more realistic soil using power law diffusivity variation. Chen et al. (2001a, 2001b) used integral transform for obtaining solution to a linearized form of the Richards' equation for both one and multidimensional moisture transport in a variably saturated soil with arbitrary time dependent surface fluxes before ponding. By using a series of transformations and Green's function, Basha (2002) provided a general solution to the one-dimensional Burgers' equation for a homogeneous soil of finite and semi-infinite depths for any arbitrary initial moisture distribution in the soil. Basha's (2002) solution can account for any time dependent flux boundary or a constant water content boundary at the surface of a semi-infinite soil; for a finite depth soil, this solution can account for a time dependent flux boundary or a constant water content boundary at the surface and a constant water content boundary at a finite depth from the surface of the soil. Parlange et al. (2002) utilized the Lambert W-function to convert the implicit forms of Green and Ampt (1911) and Parlange et al. (1982) infiltration formulas into their respective explicit forms. Warrick and Knight (2002) applied the analytical element method to study two-dimensional moisture movement through circular inclusions in a homogeneous soil. They used Gardner's (1958) exponential conductivity model in their analytical development. Zhu and Mohanty (2002) provided analytical solutions to the one-dimensional steady-state Richards' equation for describing vertical infiltration in an unsaturated homogeneous soil by independently considering Gardner's (1958) (rational power) and Brooks and Corey's (1964) conductivity models. Chen et al. (2003) extended this work to accommodate situations involving a variety of time dependent surface flux conditions after ponding. In all these solutions, hydraulic conductivity and moisture content are assumed to vary exponentially with pressure head. Using Gardner's (1958) exponential conductivity model, Warrick and Knight (2003) extended Raat's (1970) infiltration solution for flow from an array of line sources in a homogeneous soil to that of a layered one. Braddock and Parlange (2003), by utilizing a modified Gardner's (1958) exponential conductivity function, extended Warrick's (1974) one-dimensional steady-state infiltration solution for unsaturated soil situations to saturated situations as well. Warrick and Knight (2004) utilized Gardner's (1958) exponential conductivity model to develop an analytical solution for describing three-dimensional unsaturated movement of water through spherical inclusions in a homogeneous soil.

Chen and Tan (2005) studied two-dimensional moisture movement in a homogeneous soil underlain by a water table under ponding irrigation. Serrano (2004) provided simple models for simulating water movement in a homogeneous soil by utilizing approximate solutions of

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pressure-head version of the Richards' equation. Witelski (2005) studied one-dimensional motion of wetting fronts for different vertical infiltration problems in a homogeneous soil of semi-infinite extent by appropriately modeling the infiltration equation for the studied situations. In this analysis, situations with constant diffusivity and linear variation of conductivity with moisture content as well as diffusivity and conductivity variation as per Brooks and Corey's model (1964), are being considered for the analytical treatments. Yuan and Lu (2005) made use of the Kirchoff's transformation to linearize the one-dimensional Richards' equation for a homogeneous unsaturated soil and obtained analytical expressions for different infiltrating situations with arbitrary root-water uptake functions. In all their analysis, Gardner's (1958) exponential hydraulic conductivity model is utilized for mathematical convenience. Assuming relative hydraulic conductivity to vary exponentially with pressure head and linearly with moisture content, Tracy (2006) obtained analytical solutions of the multidimensional Richards' equation for a few steady and transient unsaturated flow situations in a homogeneous soil. Barontini et al. (2007) worked out an analytical solution to the one-dimensional Richards' equation for predicting vertical infiltration in a homogeneous soil by assuming saturated hydraulic conductivity of the soil to decrease in an exponential way with depth. In the derivation of this solution, Gardner's (1958) exponential conductivity model is used. Considering soil diffusivity as constant and assuming a linear variation of hydraulic conductivity with moisture content, Menziani et al. (2007) provided analytical solutions to the linearized Richards' equation for a variety of initial and boundary conditions. Mollerup and Hansen (2007) extended the power series solution of Philip's (1958) one-dimensional ponded flow infiltration problem from a constant ponded head to that of a falling ponded head (with evaporation) at the surface of a soil. Adopting a similar procedure as being followed by Mollerup and Hansen (2007), Mollerup (2007) also provided a power series solution to the more general infiltration problem of a variable ponded head at the surface of a homogeneous soil. Zlotnik et al. (2007) combined the mathematical approaches of Philip (1969) and Ross and Parlange (1994) to obtain an exact solution to the one-dimensional Richards' equation for a homogeneous soil of semi-infinite extension. This solution can accommodate any well defined initial condition in an infiltrating space.

Kuhlman and Warrick (2008) developed a two-dimensional infinite series solution of the steady-state Richards' equation for an elliptic-cylinder cavity for a homogeneous soil using Gardner's (1958) exponential conductivity function. Warrick et al. (2008) examined steady-state lateral flows through variably saturated inclined soils underlain by an impervious

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barrier. Broadbridge et al. (2009), by adopting a suitable linearizing procedure, developed an analytical solution to an integrable version of the one-dimensional time-dependent Richards' infiltration equation for a homogeneous soil. The solution is obtained by assuming the boundaries as constant water content boundaries. Singh (2010) proposed an entropy theory for modeling infiltration in unsaturated soils and derived the infiltration equations of Green and Ampt (1911), Kostiakov (1932), Horton (1939), Philip (1957a), Holtan (1961) and Overton (1964) using this theory. Asgari et al. (2011) made use of Brooks and Corey's (1964) diffusivity and conductivity models to obtain a new general analytical solution to the one-dimensional nonlinear Richards' equation for a homogeneous unsaturated soil of semi-infinite extent. Triadis and Broadbridge (2010) developed a series solution to the nonlinear one-dimensional Richards' infiltration equation for a homogeneous soil for constant moisture content boundary conditions. The main advantage of this solution is its ability to work well both in soils with initially low or high moisture content. Utilizing rational forms of hydraulic conductivity and moisture retention functions, Basha (2011) provided approximate solutions based on traveling and kinematic wave solutions of the infiltration equation, for predicting one-dimensional infiltration in a semi-infinite unsaturated homogeneous soil under different boundary conditions. Ghotbi et al. (2011) obtained an approximate analytical solution to the one-dimensional Richards' equation for a homogeneous soil by using the homotopy analysis method. In their analysis, Brooks and Corey's (1964) models of conductivity and diffusivity variations with moisture content are being used.

Assuming hydraulic conductivity and moisture content to vary exponentially with pressure head, Huwang and Wu (2012) made use of Fourier integral transformation to develop analytical expressions for predicting one-dimensional horizontal and vertical infiltration in saturated as well as unsaturated homogeneous soils. These solutions have the capability of handling both changing flux and pressure head boundaries. Sadeghi et al. (2012) proposed an exact infinite series solution for predicting steady evaporation in a homogeneous soil underlain by a shallow water table by utilizing Brooks and Corey's (1964) hydraulic conductivity function. Apart from evaporation predictions, this solution can also be used for inverse modeling to determine the parameters of Brooks and Corey's (1964) hydraulic model. Triadis and Broadbridge (2012) suggested that infiltration behavior in a homogeneous soil can be determined by the nature of relationship between soil moisture potential and hydraulic conductivity in the soil. Hayek (2014), by taking resort to a general similarity transformation and by assuming the relative permeability and capillary pressure as independent power functions of the water saturation, developed exact solutions for a few infiltration problems in

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a semi-infinite homogeneous soil. In these analytical developments, three relative permeability functions, namely linear, quadratic and cubic, are being used. Hayek (2015) worked out an analytical solution for studying steady one-dimensional infiltration through an unsaturated homogeneous soil column underlain by a water table at a finite depth from the surface of the soil. The solution assumes power law variations of hydraulic conductivity and diffusivity with effective liquid saturation and considers the top boundary as a constant flux boundary and the bottom boundary as a constant saturation boundary. Latoree et al. (2015) applied an inverse estimation procedure to determine soil sorptivity and saturated hydraulic conductivity of a soil from measured cumulative infiltration data utilizing Parlange et al's (1982) infiltration solution. In this analysis, the beta parameter of the Parlange et al's (1982) solution is taken as one. De Luca and Cepeda (2016), using Gardner's (1958) exponential conductivity model, developed an analytical procedure for solving the one-dimensional Richards' equation for a layered soil. Hayek (2016a), by utilizing Brooks and Corey's (1964) water retention model and Mualem and Dagan's (1978) hydraulic conductivity function, obtained a general analytical model for predicting one-dimensional transient vertical infiltration in a homogeneous soil of semi-infinite extent. Hayek (2016a)'s analytical solution can account for both fixed water content as well as constant flux boundaries at the surface of a soil.

Hayek (2016b) provided an exact solution to the one-dimensional transient Richards' equation for a semi-infinite homogeneous soil by using the travelling wave solution approach. This solution is based on Gardner's (1958) exponential conductivity function. Using Gardner's (1958) exponential conductivity function and applying Green's function, Wu et al. (2016) obtained analytical solutions of the one-dimensional coupled infiltration and deformation problems in unsaturated soils under changing rainfall flux conditions. Solutions were obtained for both before and after ponding scenarios at the surface of a soil. Broadbridge et al. (2017) presented multidimensional transient solutions of the Richards' equation for predicting water movement in homogeneous unsaturated soils with root-water extractions for a few infiltration situations by taking recourse to the Kirchoff's transformation method. These models are developed by assuming an exponential variation of hydraulic conductivity (Gardner 1958) with suction head. Cavalcante and Zornberg (2017) developed a series of transient analytical solutions to the one-dimensional Richards' equation for a few infiltration problems in a homogenous soil of finite as well as semi-infinite extents. These analytical treatments assume a logarithmic relation between suction head and moisture content and a linear relationship between hydraulic conductivity and suction head. Chen and

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Dai (2017), using Boltzmann transformation, derived an approximate series solution of the horizontal infiltration problem for a homogeneous soil. This solution can account for any arbitrary soil diffusivity variation with moisture content. Moret-Fernández and Latorre (2017) showed that the parameter beta of Parlange et al's (1982) infiltration equation is strongly related to the textural properties of a soil. Su et al. (2017), by working out a new method based on the principle of least square and variational principle, obtained an approximate solution to the one-dimensional Richards' equation for simulating vertical water movement in a homogeneous soil of infinite extent. In their analysis, Brooks and Corey's (1964) water retention and conductivity equations are used. Latoree et al. (2018) studied the influence of Parlange et al's (1982) beta parameter on sorptivity and saturated hydraulic conductivity of a homogeneous soil. Rahmati et al. (2019) introduced a three-term approximate expansion of Haverkamp et al.'s (1994) quasi-exact infiltration equation for modeling one-dimensional infiltration in homogeneous soils. This solution gives a good fit of Haverkamp et al.'s (1994) infiltration equation; further, this solution can also be used to obtain accurate estimates of sorptivity and saturated hydraulic conductivity of a soil from experimental results (cumulative infiltration data) on the soil.

Baiamonte (2020) provided an exact solution to the one-dimensional gravity driven infiltration equation for a homogeneous soil under constant rainfall intensity. This solution is obtained by assuming the matric potential gradient as negligible in comparison with the gravitational potential gradient for vertical flow in a homogeneous soil. Further, Brooks and Corey's (1964) hydraulic function is also used in this analytical development. Moret-Fernández et al. (2020) presented three- and four-term approximate expansions of Haverkamp et al's (1994) infiltration equation for estimating sorptivity and saturated hydraulic conductivity of a homogeneous soil from disc infiltrometer measurements. Jaiswal et al. (2021), by making use of the secant method, provided an exact and computationally efficient solution of the three-parameter Parlange et al.'s (1982) infiltration equation for a homogeneous soil and Vrugt and Gao (2021) presented appropriate theory and algorithms for forward and inverse modeling of this solution. Leij et al. (2021) applied Cole-Hopf and Laplace transformations to simulate transient one-dimensional infiltration in a semi-infinite homogeneous soil by using Brooks and Corey's (1964) water retention and hydraulic conductivity functions. The solution is obtained by assuming a uniform initial water content in an infiltrating column, a fixed water content at the start of the column and a zero-gradient at an infinite distance in the column. Zhu and Xiao (2022) proposed a variable separation

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method for easy application of Parlange's (1972a) one-dimensional infiltration solution of the Richards' equation.

2.2 Research Gaps

The research gaps which have led to the objectives of this research work have already been mentioned in detail in Section 1.2 ("Motivations of the Study") of the previous chapter, To avoid repetitions, we will be touching the justification part of our research work lightly here.

Infiltration is ubiquitous in a vegetated ecosystem and so is the presence of different types of root distributions in such a system. Also, heterogeneity of a field soil is mostly a norm rather than an exception (Mualem 1984; Yeh and Harvey 1990; Wildenschild and Jensen 1999; Williams and Houseman 2014; Saschidharan et al. 2019; Soraganvi et al. 2020). Thus, when developing a model (whether analytical or numerical) for infiltration studies, care has to be exercised to see that heterogeneity of the soil as well as the root-water extraction (i.e., the sink term) term are being properly accounted for in the development of the model.

From the reviews of various analytical solutions as mentioned in Section 2.1 above related to infiltration problems for different hydro-geo settings, it is clear that there is right now no analytical solution to either the general Gardner-based (GHC-based) or the Mualem-van Genuchten-based (MVG-based) infiltration equation for a heterogeneous soil. Actually, even for a homogeneous soil, an analytical solution to either the GHC-based or the MVG-based infiltration equation with a root-water extraction term for all possible ranges of parameters of the soil, appears to be currently lagging.

Gardner (1958) carried out analytical modeling studies on evaporation from a shallow water table using his GHC conductivity function for values of n equal to 1, 2, 3 and 4 in a homogeneous soil. Warrick (1988) put forward an exact solution of the Gardner's GHC-based infiltration equation for all n (n need not be an integer) greater than one. Salvucci's (1993) developed an approximate solution for the GHC-based infiltration equation for a homogeneous soil. This solution is valid for zero-sink situations only; thus, it cannot be used when a root-water function exists in an infiltrating space. Basha's (1999) also developed an approximate analytical solution to this infiltration equation for a homogeneous soil. He also developed an exact analytical solution of the GHC-based infiltration equation for the situation where n is an integer and where the root-extraction term does not exist in a flow space. Zhu and Mohanty (2002) also provided an analytical solution of the GHC-based infiltration equation for a homogeneous soil. This solution also cannot accommodate a sink term in an

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infiltrating space. However, the main advantage of this solution is that it is valid for both integral as well as non-integral values of n.

An analytical solution to the MVG-based infiltration equation currently does not exist for a heterogeneous soil, with or without a sink term in an infiltrating space. Actually, as stated before, because of the extreme nonlinearity of this equation, a strictly analytical solution of this equation is currently not there even for a homogeneous soil both when a sink term is present in a flow space and when it is absent. Rockhold et al. (1997) attempted to provide an analytical solution to the MVG-infiltration equation for a layered soil by neglecting the sink term. However, this solution cannot be considered strictly as an analytical solution of this equation as it fails to include the Mualem-van Genuchten (1980) hydraulic conductivity model directly inside it (Hayek 2016a). In this connection, we would like to reiterate that Rockhold et al. (1997) obtained a solution to the MVG-based infiltration equation for a layered soil not by directly considering the MVG conductivity model; instead, they first approximated the logarithmic form of this conductivity function as a series of linear splines in an infiltrating space and then integrating the governing equation independently in each of the splines by utilizing the relevant conductivity variations within the soil splines.

All the existing analytical solutions for the GHC-based and the MVG-based infiltration equations as mentioned above are for infiltration through a vertical soil column only and not for infiltration through an arbitrarily inclined column. There thus lies a necessity to obtain suitable analytical models to both the GHC-based and the MVG-based infiltration equations for an arbitrarily inclined soil column as the existing solutions of these equations, even for homogeneous soils, are not valid for all possible soil parameter variations when a root-extraction term is present in an infiltrating space. Further, while obtaining analytical solutions of these equations, there is a need to also include soil heterogeneity and root-water extraction in the analytical developments as soils in nature are seldom homogeneous and the presence of a root-extraction term in a vegetated system is a certainty. This study attempts to address these needs by making use of a few mathematical results as proposed by Barua (2021).

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CHAPTER 3

ANALYTICAL MODELLING OF 1-D INFILTRATION WITH ARBITRARY ROOT-WATER UPTAKE USING GARDNER'S MODEL

In this chapter, a steady-state analytical solution is developed for simulating one-dimensional infiltration in a heterogeneous soil formation by utilizing the general form of Gardner's (1958) hydraulic conductivity model. The highly nonlinear Richards' equation with a spatially varying sink term is solved for a sloping aquifer for a variety of infiltrating situations by employing a few mathematical procedures as proposed by Barua (2021). Two approaches are presented – one in which the governing equation along with its first two derivatives are being used and the other, in which the governing equation alone is being used – to obtain a solution to the problem. The developed solution assumes a user specified flux (Neumann boundary) at the beginning of an infiltrating column and a specified suction head (Dirichlet boundary) at the end of it. The proposed solution is verified about its correctness by comparing it with the analytical and experimental works of others for a few simple infiltration situations. Further, a few numerical checks on it are also carried out utilizing the CHEMFLO-2000 (Nofziger and Wu 2000) numerical codes. The developed analytical solution to the Gardner-based infiltration equation (Zhu and Mohanty 2002) is an inclusive one as its use is not confined to homogeneous soils alone but to heterogeneous soils as well. Further, the solution is applicable both when a root-water term (i.e., a sink term) is present in an infiltrating space and when it is absent. Thus, it is hoped that the new and versatile infiltration model proposed here would lead to a better understanding of infiltration dynamics in an unsaturated soil as compared to relatively simple available models on the subject.

3.1 Mathematical Formulation and Solution

The governing equation for unsaturated flow in a heterogeneous soil column can be expressed as (Richards' equation; Molz and Remson 1970; Clausnitzer and Hopmans 1994)

$$\frac{\partial \theta}{\partial t} = \vec{\nabla} \bullet \left[K(h) \vec{\nabla} \varphi \right] - S \tag{3.1}$$

where $h(x)$ is the suction head (capillary tension), $K(h)$ is the unsaturated hydraulic conductivity [dependent on suction head $h(x)$], θ is the volumetric moisture content, φ is the total energy head, S is the root-water uptake (sink) term and $\vec{\nabla}$ is the nabla operator. This equation is based on the law of conservation of mass and the generalized Darcy's law. It is actually a standard two phase air-water flow equation in a porous medium where the air

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phase is assumed infinitely mobile (Nayagum et al. 2004; Wang et al. 2011; Tavangarrad et al. 2018). Eq. (3.1) can also be written as

$$\frac{\partial\theta}{\partial h}\frac{\partial h}{\partial t} = D\frac{\partial h}{\partial t} = \vec{\nabla}\bullet\left[K(h)\vec{\nabla}\varphi\right] - S \tag{3.2}$$

where $D = \frac{\partial\theta}{\partial h}$ is the specific capacity. For steady state $\frac{\partial h}{\partial t}$, naturally, would be zero; Eq. (3.2) would then be

$$\vec{\nabla}\bullet\left[K(h)\vec{\nabla}\varphi\right] - S = 0 \tag{3.3}$$

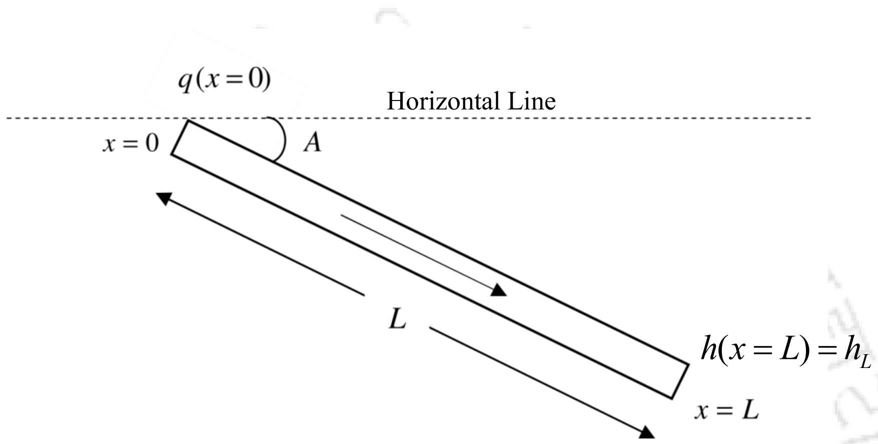

Fig. 3.1. Geometry of a one-dimensional infiltration model in an inclined soil column.

For one-dimensional infiltration in an inclined column of length L as shown in Fig. 3.1, Eq. (3.3) reduces to

$$\frac{d}{dx}\{K(h)\left[h'(x) - \sin A\right]\} - S = 0 \tag{3.4}$$

where A, as can be seen in Fig. 3.1, is the angle made by the soil column with the horizontal axis and $h(x) = \phi(x) + x\sin(A)$. We now propose to solve Eq. (3.4) by enforcing a Neumann boundary at $x = 0$ and a Dirichlet boundary at $x = L$ as under

$$h(x=L) = h_L \tag{3.5}$$

$$K(h)\left[h'(x) - \sin A\right]_{x=0} = -q \tag{3.6}$$

For a heterogeneous soil, Gardner's (1958) general hydraulic conductivity function, $K[h(x)]$, can be expressed as

$$K[h(x)] = \frac{K_s(x)}{1 + \left[\frac{h(x)}{h_c(x)}\right]^{n(x)}} \tag{3.7}$$

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where K_s is the saturated hydraulic conductivity of the soil, h is the suction head, h_c is the air entry value of the suction head and n is a constant.

Integrating Eq. (3.4) and plugging Eqs. (3.6) and (3.7) into it, we get

$$\left\{\frac{K_s(x)}{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}}\right\}\left[h'(x)-\sin A\right]+q=\int_0^x S(x)dx \tag{3.8}$$

Eq. (3.8) may also be expressed in a functional form; representing such a function as $\psi(x)$, we get

$$\psi(x)=K_s(x)\left[h'(x)-\sin A\right]+\left[q-\int_0^x S(x)dx\right]\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}=0 \tag{3.9}$$

With the $\psi(x)$ function thus defined, $h(x)$ – the solution of the differential equation for a given set of boundary and soil parameter conditions $\left[K_s(x),\ n(x),\ h_c(x)\right]$ – may next be obtained in a polynomial form to any desired degree of accuracy by utilizing the $\psi(x)$ function alone or in combination with its higher derivatives [i.e., by using $\psi(x)$ alone or $\psi(x)$ with $\psi'(x)$ or $\psi(x)$ with $\psi'(x)$ and $\psi''(x)$ and so on; see Barua (2021)] in certain ways. For completeness, however, we are presenting here two approaches for generating our solutions, the one where $\psi(x)$ with $\psi'(x)$ and $\psi''(x)$ functions are being used (as mentioned before, we can go beyond $\psi''(x)$ as well) and the other, where only $\psi(x)$ is being used. We are naming here the first methodology as the first approach and the second methodology as the second approach. We, however, once again emphasize that both the approaches can be used to obtain solution to a problem to any desired degree of accuracy (Barua 2021). Since in the first of our approaches, $\psi(x)$ along with first and second of its derivatives are needed, we proceed by first evaluating these derivatives; the first derivative of Eq. (3.9) gives

$$\psi'(x)=K_s'(x)\left[h'(x)-\sin A\right]+K_s(x)h''(x)+\left\{\frac{d}{dx}\left[q-\int_0^x S(x)dx\right]\right\}\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}$$

$$+\left[q-\int_0^x S(x)dx\right]\left\{\left\{\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}\left\{n'(x)\left\{\log_e\left[\frac{h(x)}{h_c(x)}\right]\right\}\right.\right.$$

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$$+n(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\Bigg]\Bigg\}\Bigg\}=0 \tag{3.10}$$

and from the second derivate, we get

$$\psi''(x)=K_s''(x)[h'(x)-\sin A]+2\ K_s''(x)h''(x)+K_s(x)h'''(x)$$

$$+\left\{\frac{d^2}{dx^2}\left[q-\int_0^x S(x)dx\right]\right\}\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}$$

$$+2\left\{\frac{d}{dx}\left[q-\int_0^x S(x)dx\right]\right\}\left\{\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}\left\{n'(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+n(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right\}$$

$$+\left[q-\int_0^x S(x)dx\right]\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\left\{\left\{n'(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+n(x)\left[\frac{h'(x)}{h(x)}--\frac{h_c'(x)}{h_c(x)}\right]\right\}^2\right.$$

$$+\left\{n''(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+2n'(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right.$$

$$+n(x)\left\{\frac{h''(x)h(x)-[h'(x)]^2}{[h(x)]^2}-\frac{h_c''(x)h_c(x)-[h_c'(x)]^2}{[h_c(x)]^2}\right\}\Bigg\}\Bigg\}\Bigg\}=0 \tag{3.11}$$

It should be noted that for an unsaturated infiltration situation, the ratio $\frac{h(x)}{h_c(x)}$ is always positive and hence the log terms in Eqs. (3.10) and (3.11) will survive all the time for all possible infiltration situations.

As the Neumann condition has already been incorporated into the governing equation [Eq. (3.9)], the solution of the problem requires generating a polynomial $h(x)$ including the Dirichlet condition only. Thus, the solution polynomial can initially be defined in a way such that the boundary at $x=L$ gets automatically satisfied; thus, it can be of the form

$$h(x)=C_1\left(\frac{L-x}{L}\right)+h_L\left(\frac{x}{L}\right)+C_2x(x-L)+\ldots,\ldots,\ldots f(x), \tag{3.12}$$

where $f(x)$ would be function of the type $C_{3N}\times\{(\text{function of } x^{3N} \text{ degree})\}$ or of the type $C_N\times\{(\text{function of } x^N \text{ degree})\}$ depending on whether the first or second of our approaches is being followed to work out our solution [where C_i s $(i=1 \text{ to } N)$ are all constants]. It should be noted that for $h(x)$ to satisfy the boundary condition at $x=L,$ $f(x)$ has to be chosen in a way such that it becomes zero at $x=L.$ Of course, the $h(x)$ polynomial can be

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expressed in a different way than that shown in Eq. (3.12); however, whatever way it is been expressed, care need to be exercised to see that the boundary condition at $x = L$ is being respected by it. In the analysis that we are carrying out here, we are, however, expressing $h(x)$ as the way as is been shown in Eq. (3.12). Following Barua (2021), we now proceed with the analysis of our flow situation by forcing $\psi(x)$, $\psi'(x)$ and $\psi''(x)$ (as per our first approach) zero or by forcing $\psi(x)$ alone (as per our second approach) zero at any arbitrarily chosen set of uniformly-spaced N points in [0, L]; this will give us $3N$ (if we follow our first approach) or N (if we follow our second approach) equations which can then be solved to get the constants of our desired polynomial. However, it is not necessary that all the N points need to be taken in one go; both the approaches for our flow situation can be initiated with a two-coefficient based polynomial of the nature

$$h(x)=C_1\left(\frac{L-x}{L}\right)+h_L\left(\frac{x}{L}\right)+C_2\left[x(x-L)\right] \tag{3.13}$$

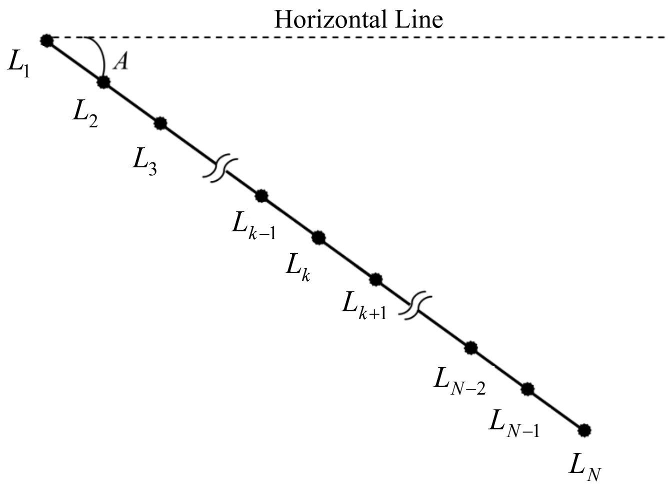

Fig. 3.2. Consideration of N uniformly-spaced points in $[0, L]$ for computation.

where C_1 and C_2 are the unknown coefficients. Using this polynomial, $\psi(x)$ can next be forced to zero at any two points in the flow domain, say at $x = L_1 = 0$ and L_2 (Fig. 3.2); this will give us the relations

$$\psi(L_1 = 0, C_1, C_2) = K_s(0)\left[C_1\left(-\frac{1}{L}\right)+\frac{h_L}{L}+C_2(-L)-\sin A\right] + q\left\{1+\left[\frac{C_1}{h_c(0)}\right]^{n(0)}\right\}=0 \tag{3.14}$$

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$$\psi(L_2, C_1, C_2) = K_s(L_2)\left[C_1\left(-\frac{1}{L}\right) + \frac{h_L}{L} + C_2(2L_2 - L) - \sin A\right]$$
$$+\left[q - \int_0^{L_2} S(x)dx\right]\left\{1 + \left[\frac{C_1\left(\frac{L-L_2}{L}\right) + h_L\left(\frac{L_2}{L}\right) + C_2 L_2(L_2 - L)}{h_c(L_2)}\right]^{n(L_2)}\right\} = 0 \qquad (3.15)$$

From Eqs. (3.14) and (3.15) above, as can be seen, the constants C_1 and C_2 of Eq. (3.13) can be calculated using the Newton-Raphson or some other method (say by Method of Iteration; Scarborough 1966). Thus, if $C_1^{(0)}$ and $C_2^{(0)}$ are being assumed as initial approximations of C_1 and C_2, following the Newton-Raphson procedure, the equations for generating the correction terms can then be written as

$$\psi(0, C_1^0 + \Delta C_1, C_2^0 + \Delta C_2) = \psi(0, C_1^0, C_2^0) + \Delta C_1\left[\frac{\partial\psi(0)}{\partial C_1}\right]_{C_1^0,\, C_2^0} + \Delta C_2\left[\frac{\partial\psi(0)}{\partial C_2}\right]_{C_1^0,\, C_2^0}$$

and

$$\psi(L_2, C_1^0 + \Delta C_1, C_2^0 + \Delta C_2) = \psi(L_2, C_1^0, C_2^0) + \Delta C_1\left[\frac{\partial\psi(L_2)}{\partial C_1}\right]_{C_1^0,\, C_2^0} + \Delta C_2\left[\frac{\partial\psi(L_2)}{\partial C_2}\right]_{C_1^0,\, C_2^0} \qquad (3.17)$$

where ΔC_1 and ΔC_2 are the corrections that can be determined using the relations

$$\Delta C_1 = \frac{\begin{vmatrix} -\psi(0, C_1^0, C_2^0) & \left[\frac{\partial\psi(0)}{\partial C_2}\right]_{C_1^0,\, C_2^0} \\ -\psi(L_1, C_1^0, C_2^0) & \left[\frac{\partial\psi(L_2)}{\partial C_2}\right]_{C_1^0,\, C_2^0} \end{vmatrix}}{\begin{vmatrix} \left[\frac{\partial\psi(0)}{\partial C_1}\right]_{C_1^0,\, C_2^0} & \left[\frac{\partial\psi(0)}{\partial C_2}\right]_{C_1^0,\, C_2^0} \\ \left[\frac{\partial\psi(L_2)}{\partial C_1}\right]_{C_1^0,\, C_2^0} & \left[\frac{\partial\psi(L_2)}{\partial C_2}\right]_{C_1^0,\, C_2^0} \end{vmatrix}} \qquad (3.18)$$

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$$\Delta C_2 = \frac{\begin{vmatrix} \left[\dfrac{\partial \psi(0)}{\partial C_1}\right]_{C_1^0,\, C_2^0} & -\psi(0, C_1^0, C_2^0) \\ \left[\dfrac{\partial \psi(L_2)}{\partial C_1}\right]_{C_1^0,\, C_2^0} & -\psi(L_2, C_1^0, C_2^0) \end{vmatrix}}{\begin{vmatrix} \left[\dfrac{\partial \psi(0)}{\partial C_1}\right]_{C_1^0,\, C_2^0} & \left[\dfrac{\partial \psi(0)}{\partial C_2}\right]_{C_1^0,\, C_2^0} \\ \left[\dfrac{\partial \psi(L_2)}{\partial C_1}\right]_{C_1^0,\, C_2^0} & \left[\dfrac{\partial \psi(L_2)}{\partial C_2}\right]_{C_1^0,\, C_2^0} \end{vmatrix}} \tag{3.19}$$

The ΔC_1 and ΔC_2 thus obtained can then be added to the initial $C_i^{(0)}$ s $(i = 1,2)$ to get the new values of them. Taking these new values of C_1 and C_2 as the starting values, the process is then repeated to get the new values of ΔC_1 and ΔC_2 which are then again added to the starting values of C_1 and C_2. This process can be continued till there is no further change in the C_1 and C_2 values. With the constants C_1 and C_2 thus determined, we can then next move to add the third term of $h(x)$; it will of the form

$$h(x) = (C_1 + \Delta C_1)\left(\frac{L - x}{L}\right) + h_L\left(\frac{x}{L}\right) + (C_2 + \Delta C_2)[x(x - L)] + C_3\left[x^2(x - L)\right] \tag{3.20}$$

In Eq. (3.20), as can be seen, C_3 is the constant attached to the newly added third term $\left[x^2(x - L)\right]$. Now, to get C_3 and further correction terms of C_1 and C_2, respectively, $\psi(x)$ can next be forced zero at L_1, L_2 and L_3; this will give us the relations

$$\psi\left[0, C_1 + \Delta C_1, C_2 + \Delta C_2, C_3\right] = K_s(0)\left[(C_1 + \Delta C_1)\left(-\frac{1}{L}\right) + \frac{h_L}{L} + (C_2 + \Delta C_2)(-L) - \sin A\right]$$
$$+ q\left\{1 + \left[\frac{C_1 + \Delta C_1}{h_c(0)}\right]^{n(0)}\right\} = 0 \tag{3.21}$$

$$\psi\left[L_2, C_1 + \Delta C_1, C_2 + \Delta C_2, C_3\right] = K_s(L_2)\left\{(C_1 + \Delta C_1)\left(-\frac{1}{L}\right) + \frac{h_L}{L} + (C_2 + \Delta C_2)(2L_2 - L)\right.$$
$$\left. + C_3\left[2L_2(L_2 - L) + L_2^{\,2}\right] - \sin A\right\}$$

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$$+\left[q-\int_0^{L_2} S(x)dx\right]\left\{1+\left[\frac{(C_1+\Delta C_1)\left(\frac{L-L_2}{L}\right)+h_L\left(\frac{L_2}{L}\right)}{h_c(L_2)}\right.\right.$$

$$\left.\left.+\frac{(C_2+\Delta C_2)L_2(L_2-L)+C_3L_2^{\,2}(L_2-L)}{h_c(L_2)}\right]^{n(L_2)}\right\}=0 \tag{3.22}$$

and

$$\psi\left[L_3,C_1+\Delta C_1,C_2+\Delta C_2,C_3\right]=K_s(L_3)\left\{(C_1+\Delta C_1)\left(-\frac{1}{L}\right)+\frac{h_L}{L}+(C_2+\Delta C_2)(2L_3-L)\right.$$

$$\left.+C_3\left[2L_3(L_3-L)+L_3^{\,2}\right]-\sin A\right\}$$

$$+\left[q-\int_0^{L_3} S(x)dx\right]\left\{1+\left[\frac{(C_1+\Delta C_1)\left(\frac{L-L_3}{L}\right)+h_L\left(\frac{L_3}{L}\right)}{h_c(L_3)}\right.\right.$$

$$\left.\left.+\frac{(C_2+\Delta C_2)L_3(L_3-L)+C_3L_3^{\,2}(L_3-L)}{h_c(L_3)}\right]^{n(L_3)}\right\}=0 \tag{3.23}$$

where C_1 and C_2 are the values obtained in the first step and C_3 is the value needed to be determined. If C_1 and C_2 as obtained in the first step are taken as initial values of these constants and ΔC_1 and ΔC_2 are their correction terms in this step and if C_3^0 is the initial guess value of the third coefficient with ΔC_3 as its correction term, then the three Newton-Raphson equations for this step can be represented as

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$$\psi\left[0,(C_1+\Delta C_1),(C_2+\Delta C_2),(C_3^0+\Delta C_3)\right] = \psi(0,C_1,C_2,C_3^0) + \Delta C_1\left[\frac{\partial\psi(0)}{\partial C_1}\right]_{C_1,C_2,C_3^0} + \Delta C_2\left[\frac{\partial\psi(0)}{\partial C_2}\right]_{C_1,C_2,C_3^0} + \Delta C_3\left[\frac{\partial\psi(0)}{\partial C_3}\right]_{C_1,C_2,C_3^0} = 0 \quad (3.24)$$

$$\psi\left[L_2,(C_1+\Delta C_1),(C_2+\Delta C_2),(C_3^0+\Delta C_3)\right] = \psi(L_2,C_1,C_2,C_3^0) + \Delta C_1\left[\frac{\partial\psi(L_2)}{\partial C_1}\right]_{C_1,C_2,C_3^0} + \Delta C_2\left[\frac{\partial\psi(L_2)}{\partial C_2}\right]_{C_1,C_2,C_3^0} + \Delta C_3\left[\frac{\partial\psi(L_2)}{\partial C_3}\right]_{C_1,C_2,C_3^0} = 0 \quad (3.25)$$

$$\psi\left[L_3,(C_1+\Delta C_1),(C_2+\Delta C_2),(C_3^0+\Delta C_3)\right] = \psi(L_3,C_1,C_2,C_3^0) + \Delta C_1\left[\frac{\partial\psi(L_3)}{\partial C_1}\right]_{C_1,C_2,C_3^0} + \Delta C_2\left[\frac{\partial\psi(L_3)}{\partial C_2}\right]_{C_1,C_2,C_3^0} + \Delta C_3\left[\frac{\partial\psi(L_3)}{\partial C_3}\right]_{C_1,C_2,C_3^0} = 0 \quad (3.26)$$

From the above three equations, the correction terms $\Delta C_1, \Delta C_2$ and ΔC_3 can then be evaluated as

$$\Delta C_1 = \frac{\begin{vmatrix} -\psi(0,C_1,C_2,C_3^0) & \left[\frac{\partial\psi(0)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(0)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ -\psi(L_2,C_1,C_2,C_3^0) & \left[\frac{\partial\psi(L_2)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(L_2)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ -\psi(L_3,C_1,C_2,C_3^0) & \left[\frac{\partial\psi(L_3)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(L_3)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \end{vmatrix}}{\begin{vmatrix} \left[\frac{\partial\psi(0)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(0)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(0)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ \left[\frac{\partial\psi(L_2)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(L_2)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(L_2)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ \left[\frac{\partial\psi(L_3)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(L_3)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\frac{\partial\psi(L_3)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \end{vmatrix}} \quad (3.27)$$

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$$\Delta C_2 = \frac{\begin{vmatrix} \left[\dfrac{\partial \psi(0)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & -\psi(0,C_1,C_2,C_3^0) & \left[\dfrac{\partial \psi(0)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ \left[\dfrac{\partial \psi(L_2)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & -\psi(L_2,C_1,C_2,C_3^0) & \left[\dfrac{\partial \psi(L_2)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ \left[\dfrac{\partial \psi(L_3)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & -\psi(L_3,C_1,C_2,C_3^0) & \left[\dfrac{\partial \psi(L_3)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \end{vmatrix}}{\begin{vmatrix} \left[\dfrac{\partial \psi(0)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(0)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(0)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ \left[\dfrac{\partial \psi(L_2)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_2)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_2)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ \left[\dfrac{\partial \psi(L_3)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_3)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_3)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \end{vmatrix}} \tag{3.28}$$

$$\Delta C_3 = \frac{\begin{vmatrix} \left[\dfrac{\partial \psi(0)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(0)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & -\psi(0,C_1,C_2,C_3^0) \\ \left[\dfrac{\partial \psi(L_2)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_2)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & -\psi(L_2,C_1,C_2,C_3^0) \\ \left[\dfrac{\partial \psi(L_3)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_3)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & -\psi(L_3,C_1,C_2,C_3^0) \end{vmatrix}}{\begin{vmatrix} \left[\dfrac{\partial \psi(0)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(0)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(0)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ \left[\dfrac{\partial \psi(L_2)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_2)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_2)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \\ \left[\dfrac{\partial \psi(L_3)}{\partial C_1}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_3)}{\partial C_2}\right]_{C_1,C_2,C_3^0} & \left[\dfrac{\partial \psi(L_3)}{\partial C_3}\right]_{C_1,C_2,C_3^0} \end{vmatrix}} \tag{3.29}$$

In the same way, $h(x)$ can be extended to a four degree polynomial by equating $\psi(x)$ to zero simultaneously at four distinct points L_1, L_2, L_3 and L_4 in $[0,L]$ and then solving the ensuing equations obtained out of it. In fact, the same procedure can be carryout to extend $h(x)$ to a polynomial of any desired degree. It is worth mentioning at this stage that, with our type of representation, $h(x)$ for a polynomial of twenty degree would of the form

$$\begin{aligned} h(x) = {} & C_1\left(\frac{L-x}{L}\right) + h_L\left(\frac{x}{L}\right) + C_2\left[x(x-L)\right] + C_3\left[x^2(x-L)\right] + C_4\left[x^2(x-L)^2\right] + C_5\left[x^3(x-L)^2\right] \\ & + C_6\left[x^3(x-L)^3\right] + C_7\left[x^4(x-L)^3\right] + C_8\left[x^4(x-L)^4\right] + C_9\left[x^5(x-L)^4\right] + C_{10}\left[x^5(x-L)^5\right] \\ & + C_{11}\left[x^6(x-L)^5\right] + C_{12}\left[x^6(x-L)^6\right] + C_{13}\left[x^7(x-L)^6\right] + C_{14}\left[x^7(x-L)^7\right] + C_{15}\left[x^8(x-L)^7\right] \end{aligned}$$

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$$+C_{16}\left[x^{8}(x-L)^{8}\right]+C_{17}\left[x^{9}(x-L)^{8}\right]+C_{18}\left[x^{9}(x-L)^{9}\right]+C_{19}\left[x^{10}(x-L)^{9}\right]$$

$$+C_{20}\left[x^{10}(x-L)^{10}\right] \tag{3.30}$$

Also, whatever has been just said also holds for the $\psi'(x)$ and $\psi''(x)$ expressions as well. That is, just like the way $\psi(x)$ has been used to extend the $h(x)$ polynomial, $\psi'(x)$ and $\psi''(x)$ functions can also be utilized in a similar way to increase the terms of this polynomial as well.

The mathematical procedure as presented here can accommodate any continuous distribution of root-water uptake functions; however, in our examples here only three root-water distribution functions are studied. They are uniform distribution and linearly increasing and decreasing distributions of $S(x)$ along the length of a soil column. It should be noted that it is not necessary for $S(x)$ to exist for the entire length of a soil column. For a soil column with a uniform root-water uptake functions (Fig. 3.3), $S(x)$ for the entire length L can be represented as

$$S(x)=\frac{S_M}{L} \tag{3.31}$$

where S_M is the total volume of water being extracted by roots per unit cross-sectional area of the soil column per unit time in length L of the soil column.

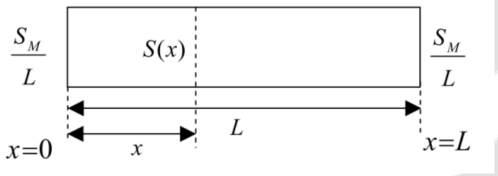

Fig. 3.3. Geometry of a soil column with a uniform root-water uptake function along the length of the soil.

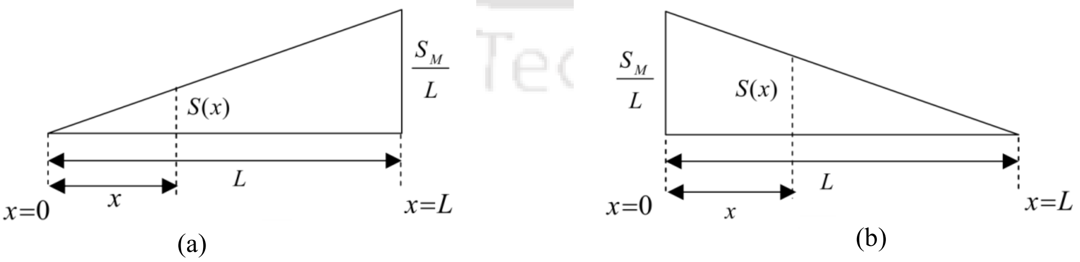

Fig. 3.4. Geometry of a soil column with (a) a linearly increasing root-water uptake function and (b) a linearly decreasing root-water uptake function.

For a linearly increasing root-water distribution as shown in Fig. 3.4(a), $S(x)$ naturally, will be of the form

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$$S(x) = \frac{S_M}{L^2} x \tag{3.32}$$

Also, for a linearly decreasing function root-water distribution as shown in Fig.3.(b), $S(x)$ can be expressed as

$$S(x) = \frac{S_M}{L} - \frac{S_M}{L^2} x \tag{3.33}$$

It should be noted that for these two cases, the total volume of water being extracted by the roots per unit cross section area over the whole domain L is $\frac{S_M}{2}$.

The root-water integral $\int_0^x S(x)\,dx$ of Eq. (3.9) then works out for the uniform, linearly increasing and linearly decreasing situations as $\left(\frac{S_M}{L}\right)x$, $\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)$, $\left(\frac{S_M}{L}\right)x - \frac{S_M}{L^2}\left(\frac{x^2}{2}\right)$, respectively. Plugging these $S(x)$ integrals in the $\psi(x)$, $\psi'(x)$ and $\psi''(x)$ expressions of Eqs. (3.9), (3.10) and (3.11), we find these functions for uniform $S(x)$ distribution as

$$\psi(x) = K_s(x)[h'(x) - \sin A] + \left[q - \frac{S_M}{L}x\right]\left\{1 + \left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\} = 0 \tag{3.34}$$

$$\begin{aligned}\psi'(x) = K_s'(x)\,[h'(x) - \sin A] + K_s(x)h''(x) + \left\{\frac{d}{dx}\left[q - \frac{S_M}{L}x\right]\right\}\left\{1 + \left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\} \\ + \left[q - \frac{S_M}{L}x\right]\left\{\left\{\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}\left\{n'(x)\left\{\log_e\left[\frac{h(x)}{h_c(x)}\right]\right\}\right.\right. \\ \left.\left. + n(x)\left[\frac{h'(x)}{h(x)} - \frac{h_c'(x)}{h_c(x)}\right]\right\}\right\} = 0\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\psi''(x) = K_s''(x)[h'(x) - \sin A] + 2\,K_s''(x)h''(x) + K_s(x)h'''(x) \\ + \left\{\frac{d^2}{dx^2}\left[q - \frac{S_M}{L}x\right]\right\}\left\{1 + \left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}\end{aligned}$$

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$$+2\left\{\frac{d}{dx}\left[q-\frac{S_M}{L}x\right]\right\}\left\{\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}\left\{n'(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+n(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right\}$$

$$+\left[q-\frac{S_M}{L}x\right]\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\left\{\left\{n'(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+n(x)\left[\frac{h'(x)}{h(x)}--\frac{h_c'(x)}{h_c(x)}\right]\right\}^2\right.$$

$$+\left\{n''(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+2n'(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right.$$

$$\left.\left.\left.+n(x)\left\{\frac{h''(x)h(x)-[h'(x)]^2}{[h(x)]^2}-\frac{h_c''(x)h_c(x)-[h_c'(x)]^2}{[h_c(x)]^2}\right\}\right\}\right\}=0 \qquad (3.36)$$

For linearly increasing $S(x)$ distribution, these functions work out as

$$\psi(x)=K_s(x)[h'(x)-\sin A]+\left[q-\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}=0 \qquad (3.37)$$

$$\psi'(x)=K_s'(x)\,[h'(x)-\sin A]+K_s(x)h''(x)+\left\{\frac{d}{dx}\left[q-\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\right\}\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}$$

$$+\left[q-\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\left\{\left\{\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}\left\{n'(x)\left\{\log_e\left[\frac{h(x)}{h_c(x)}\right]\right\}\right.\right.$$

$$\left.\left.+n(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right\}\right\}=0 \qquad (3.38)$$

and

$$\psi''(x)=K_s''(x)[h'(x)-\sin A]+2\,K_s''(x)h''(x)+K_s(x)h'''(x)$$

$$+\left\{\frac{d^2}{dx^2}\left[q-\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\right\}\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}$$

$$+2\left\{\frac{d}{dx}\left[q-\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\right\}\left\{\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}\left\{n'(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+n(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right\}$$

$$+\left[q-\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\left\{\left\{n'(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+n(x)\left[\frac{h'(x)}{h(x)}--\frac{h_c'(x)}{h_c(x)}\right]\right\}^2\right.$$

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$$+\left\{n''(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+2n'(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right.$$

$$\left.\left.\left.+n(x)\left\{\frac{h''(x)h(x)-[h'(x)]^2}{[h(x)]^2}-\frac{h_c''(x)h_c(x)-[h_c'(x)]^2}{[h_c(x)]^2}\right\}\right\}\right\}\right\}=0 \qquad (3.39)$$

Further, for linearly decreasing $S(x)$ distribution, we find these functions as

$$\psi(x)=K_s(x)[h'(x)-\sin A]+\left[q-\frac{S_M}{L}x+\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}=0 \qquad (3.40)$$

$$\psi'(x)=K_s'(x)\,[h'(x)-\sin A]+K_s(x)h''(x)+\left\{\frac{d}{dx}\left[q-\frac{S_M}{L}x+\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\right\}\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}$$

$$+\left[q-\frac{S_M}{L}x+\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\left\{\left\{\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}\left\{n'(x)\left\{\log_e\left[\frac{h(x)}{h_c(x)}\right]\right\}\right.\right.$$

$$\left.\left.+n(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right\}\right\}=0 \qquad (3.41)$$

and

$$\psi''(x)=K_s''(x)[h'(x)-\sin A]+2\,K_s''(x)h''(x)+K_s(x)h'''(x)$$

$$+\left\{\frac{d^2}{dx^2}\left[q-\frac{S_M}{L}x+\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\right\}\left\{1+\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}$$

$$+2\left\{\frac{d}{dx}\left[q-\frac{S_M}{L}x+\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\right\}\left\{\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\right\}$$

$$\times\left\{n'(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+n(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right\}$$

$$+\left[q-\frac{S_M}{L}x+\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right]\left[\frac{h(x)}{h_c(x)}\right]^{n(x)}\left\{\left\{n'(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+n(x)\left[\frac{h'(x)}{h(x)}--\frac{h_c'(x)}{h_c(x)}\right]\right\}^2\right.$$

$$+\left\{n''(x)\log_e\left[\frac{h(x)}{h_c(x)}\right]+2n'(x)\left[\frac{h'(x)}{h(x)}-\frac{h_c'(x)}{h_c(x)}\right]\right.$$

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$$+n(x)\left\{\frac{h''(x)h(x)-[h'(x)]^2}{[h(x)]^2}-\frac{h_c''(x)h_c(x)-[h_c'(x)]^2}{[h_c(x)]^2}\right\}\Bigg\}\Bigg\}=0 \tag{3.42}$$

As shown in Barua (2021), the solution polynomial for a differential equation can be obtained for the entire space in an extended way. However, the infiltration problem that we are considering here can also be solved by dividing a flow domain into smaller divisions and then solving the problem independently in each of these domains. This is possible since the nature of our infiltration problem considered here allows us to determine the infiltration flux (Neumann boundary) at any cross section of a flow domain. For many infiltration problems, this way of solving may greatly reduce the computational effort required in obtaining their solutions as compared to the calculation efforts required in obtaining their solutions considering the full space. This procedure will now be explained in detail considering a soil space with two divisions, both when $S(x)$ is zero and when it is a known function of space. The solution is first attempted in division I of the flow space which, as can be seen in Figs. 3.5(i) and 3.5(ii), starts from $x=l_1$ and ends in $x=L,$ all these distances being measured with respect to the origin $O.$ Thus, the flow space for division I is of $L-l_1$ length and X_1 – the space variable in domain I – is zero at $x=l_1,$ where $X_1=x-l_1.$ Further, if $S(x)=0,$ the infiltration flux at $X_1=0$ will then be the same q as at $x=0.$ Thus, we can see that the Neumann boundary condition for the whole space is also applicable for the soil division I as well. Also, it should be noted that the Dirichlet condition at $x=L$ also holds for X_1 at $x=L$ for division I of the flow domain. Thus, the differential equation as well as its boundary conditions can all be known for division I and hence the problem can be solved in this domain. Once the solution for this domain is known, as can be seen in Fig. 3.5(i), the pressure head at $X_1=0$ will then become the Dirichlet condition for the domain II at $x=l_1;$ hence, the problem in domain II can also be next solved as both the Neumann as well as Dirichlet boundaries for this flow space have also become known now. If a root-water uptake function exists [i.e., if $S(x)\neq 0$], then the infiltration flux available at a distance x from the origin can be evaluated by integrating the same within the domain of interest; thus, if $S(x)$ is uniform, we have q at x (named here as q_x) as

$$q_x=q-\int_0^x S(x)\,dx=q-\left(\frac{S_M}{L}\right)x \tag{3.43}$$

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In the same way, q_x for linearly increasing and linearly decreasing $S(x)$ situations can be worked out; they can be represented as

$$q_x = q - \int_0^x S(x)\,dx = q - \frac{S_M}{L^2}\left(\frac{x^2}{2}\right) \tag{3.44}$$

and

$$q_x = q - \int_0^x S(x)\,dx = q - \left[\left(\frac{S_M}{L}\right)x - \frac{S_M}{L^2}\left(\frac{x^2}{2}\right)\right] \tag{3.45}$$

Thus, if $S(x) \neq 0$, the infiltration flux available at the start of domain I (i.e., at $x = l_1$) will not be the q available at $x = 0$ but will be a quantity depending on the nature of variation of $S(x)$ in the flow space L. As can be seen from the equations above, the infiltration flux available at the start of domain I is $q - \left(\frac{S_M}{L}\right)l_1$, if $S(x)$ is uniform, $q - \frac{S_M}{L^2}\left(\frac{l_1^2}{2}\right)$, if $S(x)$ is linearly increasing and $q - \left[\left(\frac{S_M}{L}\right)l_1 - \frac{S_M}{L^2}\left(\frac{l_1^2}{2}\right)\right]$, if $S(x)$ is linearly decreasing. Further, the root-water uptake integral in the domains would be $\left(\frac{S_M}{L}\right)X_1$ for $0 \leq X_1 \leq L - l_1$ and $\left(\frac{S_M}{L}\right)x$ for $0 \leq x \leq l_1$ if the root-water uptake function is uniform; $\frac{S_M}{L^2}\left(l_1 X_1 + \frac{X_1^2}{2}\right)$ for $0 \leq X_1 \leq L - l_1$ and $\frac{S_M}{L^2}\left(\frac{x^2}{2}\right)$ for $0 \leq x \leq l_1$ if the function is linearly increasing and $\frac{S_M}{L^2}(L - l_1)X_1 - \left(\frac{S_M}{L^2}\right)\left(\frac{X_1^2}{2}\right)$ for $0 \leq X_1 \leq L - l_1$ and $\frac{S_M}{L}x - \frac{S_M}{L^2}\left(\frac{x^2}{2}\right)$ for $0 \leq x \leq l_1$ if the function is linearly decreasing.

For clarity, all these situations are also shown separately in Figs. 3.6, 3.7 and 3.8, respectively.

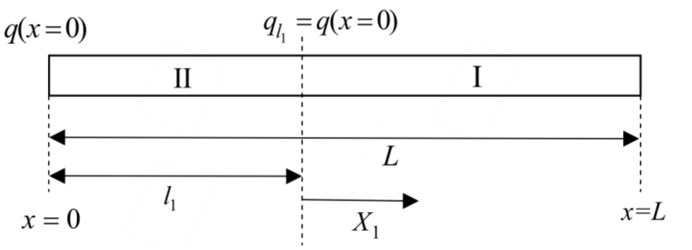

Fig. 3.5(i). Infiltration flux for domains I and II when $S(x) = 0$

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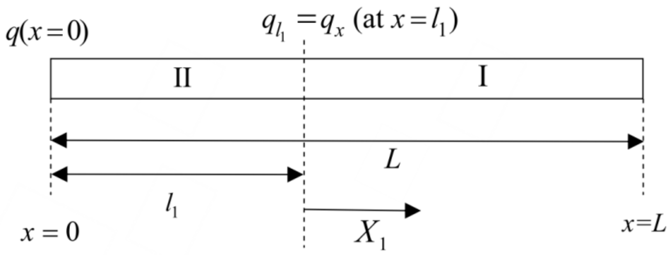

Fig. 3.5(ii). Infiltration flux for domains I and II when $S(x)$ varies along the length of a soil column.

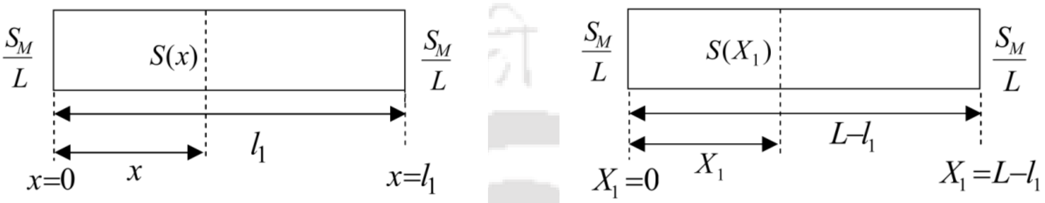

Fig. 3.6. Root-water uptake distributions in domains I and II when $S(x)$ varies uniformly along the length of a soil column.

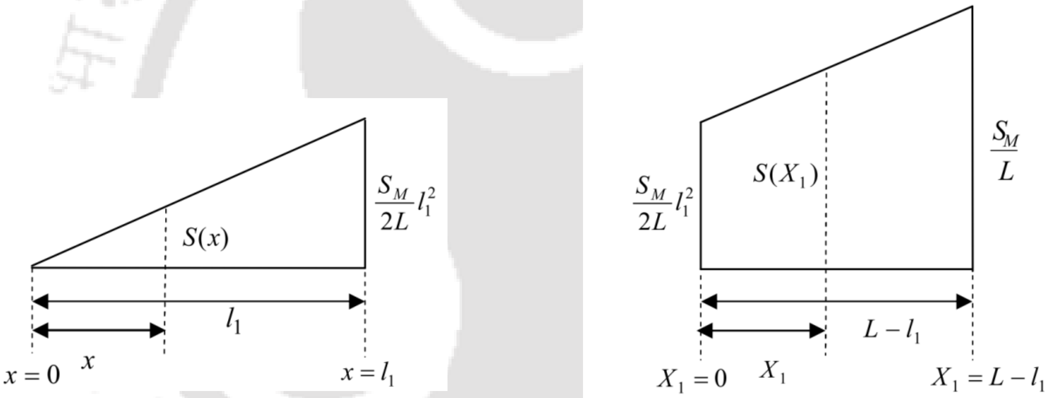

Fig. 3.7. Root-water uptake distributions in domains I and II when $S(x)$ varies in a linearly increasing manner along the length of a soil column.

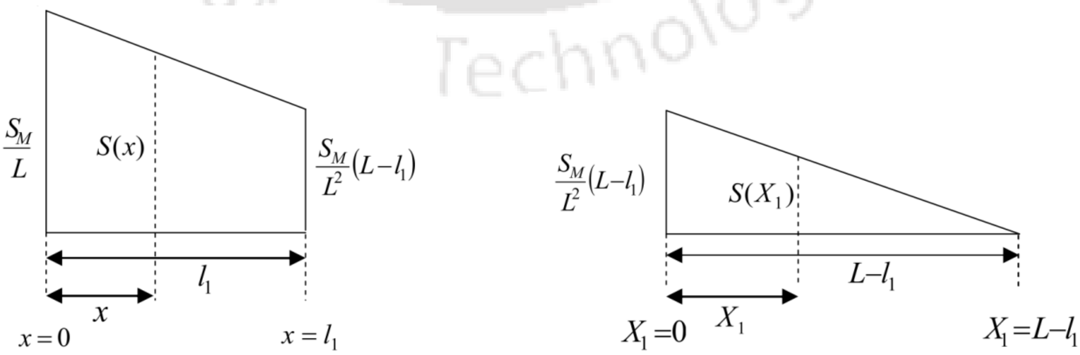

Fig. 3.8. Root-water uptake distributions in domains I and II when $S(x)$ varies in a linearly decreasing manner along the length of a soil column.

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The splitting procedure as explained above for a two-division soil column can also be very well extended in a similar way to any number of divisions that we may wish to split a soil column. This is because, for the type of infiltration problems that we are considering here, the infiltrating flux at the start of a soil subdivision and the root-water uptake distribution function in it, can always be determined irrespective of how many subdivisions a soil column is being split. This is an important observation, since as mentioned before, solving a boundary value problem by either of our approaches (Barua 2021) by splitting a flow domain may require much less computational effort than that required for solving the same by considering the full space. We would also like to state that the solutions of all the infiltration problems that have been solved here using our first approach are all in the form a fifteen degree polynomial of the type as shown in Eq. (3.30) up to its C_{15}^{th} term. Also, the solutions of all the infiltration problems that have been solved here using our second approach are all twenty degree polynomials of the type as shown in Eq. (3.30) up to its last term (i.e., up to the C_{20}^{th} term). The former solutions for all the cases have been obtained by forcing $\psi(x)$, $\psi'(x)$ and $\psi''(x)$ to be zero at five equally-spaced points of a flow domain covering its entire length (this will split the domain into four equal parts) and the latter solutions for all the cases have been obtained by forcing $\psi(x)$ alone to be zero at twenty equally-spaced points spanning the entire length of a domain (this will split the domain into nineteen equal parts). It is, however, to be noted that even though we are considering here the forced points to be equally-spaced in $[0, L]$ in both of our approaches in obtaining solution to a problem, it is not necessary that the forced points in $[0, L]$ be equally-spaced to apply our mathematical procedures. Solution of a problem by both the approaches can also be obtained by considering unequally-spaced forced points in a flow space as well. Further, as the accuracy of a solution by both the approaches depends on how closely the forced points are being considered in $[0, L]$, a solution of any desired accuracy pertaining to a problem can be obtained using either of the approaches by sufficiently reducing the largest gap in between any two adjacent forced points in $[0, L]$ (Barua 2021).

3.1.1 Methodological frame work and step by step procedure

For ease of obtaining solution to a Gardner-based infiltration problem using our proposed approaches, mathematical procedures associated with them will now also be presented in a step by step way.

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Suppose a Gardner-based infiltration problem is given in a heterogeneous soil of domain L with the boundary conditions q and h_L and the spatially varying parameters $K_S(x)$, $h_c(x)$, $n(x)$ and $S(x)$. Suppose also that a solution of this problem exists in the domain $[0, L]$. We will now list the various steps which may be followed in either of our approaches to obtain a series solution of this Gardner-based infiltration problem.

First Approach

Step 1:

Assume a $3N-$degree $(N = 2, 3, 4, ...;$ as mentioned before, all the examples that are solved here using this approach have been done by taking N as 5) solution polynomial for the problem as that shown in Eq. (3.30). Of course, the assumed polynomial may be of a different type and degree than that shown by Eq. (3.30) [Eq. (3.30) is for illustration only] but whatever way we may choose this polynomial, care has to be taken to see that it satisfies the boundary condition at $x = L$. It may be observed that Eq. (3.30) is satisfying the boundary condition at $x = L$ by its very definition.

Step 2:

This polynomial has $3N$ unknown constants, namely C_1, C_2, ..., C_{3N}; so we need to generate $3N$ equations to solve them. Choose N distinct points in $[0, L]$ – say L_1, L_2, ..., L_N – spreading across the entire length of the domain L (as already stated, in all the examples solved here using this approach, N is taken as 5 and L_i $(i = 1 \text{ to } 5)$ are taken as equally spaced in $[0, L]$). Equate $\psi(x)$ of Eq. (3.9) to zero at these points. This will give us N equations.

Step 3:

Equate $\psi'(x)$ of Eq. (3.10) to zero at these points; this will also give us N more equations.

Step 4:

Finally, also equate $\psi''(x)$ of Eq. (3.11) to zero at these points; this will give us a further N equations.

Step 5:

Solve these $3N$ equations by the Newton-Raphson or by some other method (Scarborough 1966) to get the constants C_1, C_2, ..., C_{3N}. Once we have these constants, we will then be having all the constants of our assumed solution polynomial and the problem will then thus

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stand solved. A general flow-chart for use of our first approach is as given in Fig. 3.9(i) below.

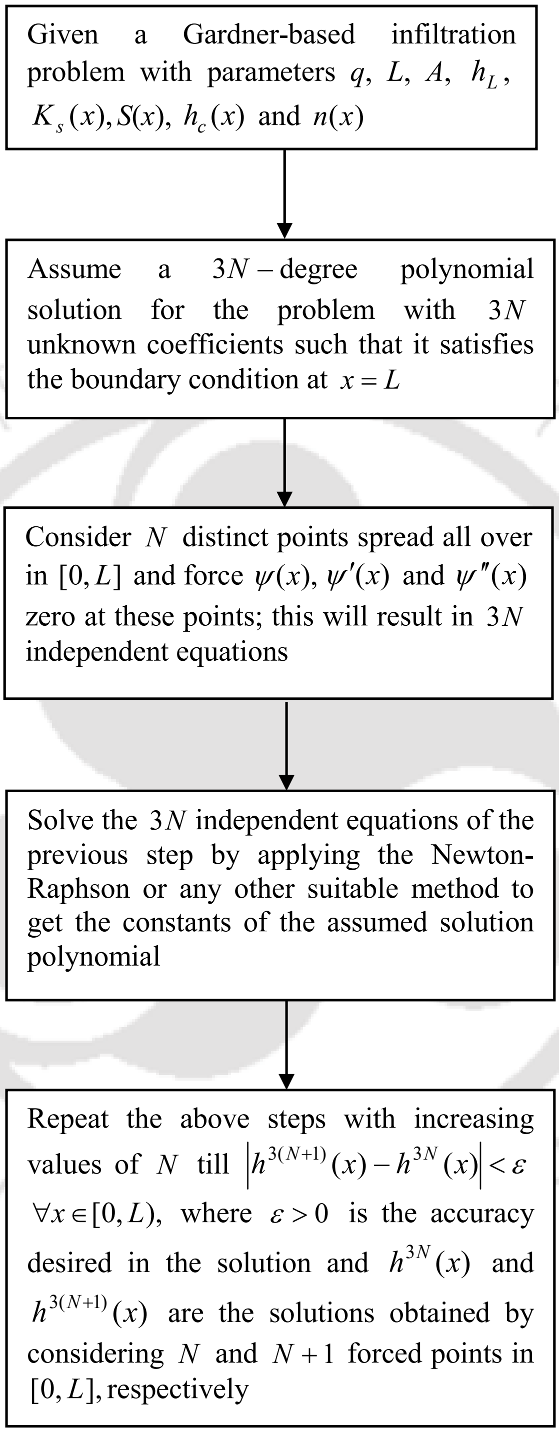


Fig. 3.9(i). Flow-chart for solving a Gardner-based infiltration problem by utilizing the first approach.

<u>Second Approach</u>

Step 1:

Assume a $N-$degree ($N = 2, 3, 4, \ldots$; as mentioned before, all the examples that are solved here using this approach are being done by taking N as 20) solution polynomial for the problem like that shown in Eq. (3.30). Of course, the assumed polynomial here may also be of a different type and degree than that shown by Eq. (3.30) and here also the chosen polynomial should be such that it satisfies the boundary condition at $x = L$.

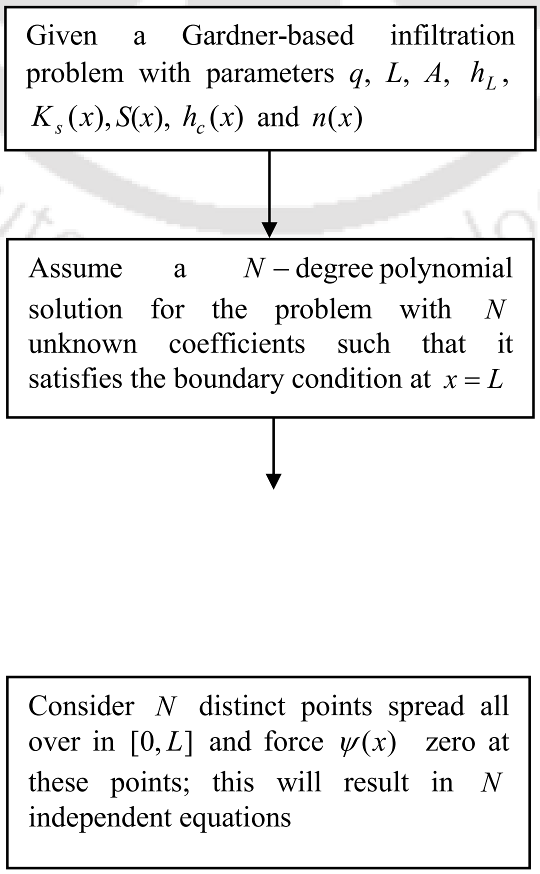
Step 2:

The assumed polynomial has now N unknown constants, namely C_1, C_2, …, C_N; so we need to generate N equations to solve them. Choose N distinct points in $[0, L]$ – say L_1, L_2, …, L_N – spreading across the entire length of the flow domain L (as already stated, in all the examples solved here using this approach, N is taken as 20 and L_is ($i = 1$ to 20) are taken as equally spaced in $[0, L]$). Equate $\psi(x)$ of Eq. (3.9) to zero at these points. This will give us N equations. We thus get the needed N equations from the $\psi(x)$ function itself.

Step 3:

Solve these N equations by the Newton-Raphson or by some other method (say, by the Method of Iteration; Scarborough 1966) to get the constant(s) C_1, C_2, …, C_N. Once we have these constants, our problem will then stand solved by this approach as well.

A general flow-chart explaining the application of our second approach is as given in Fig. 3.9(ii) below.



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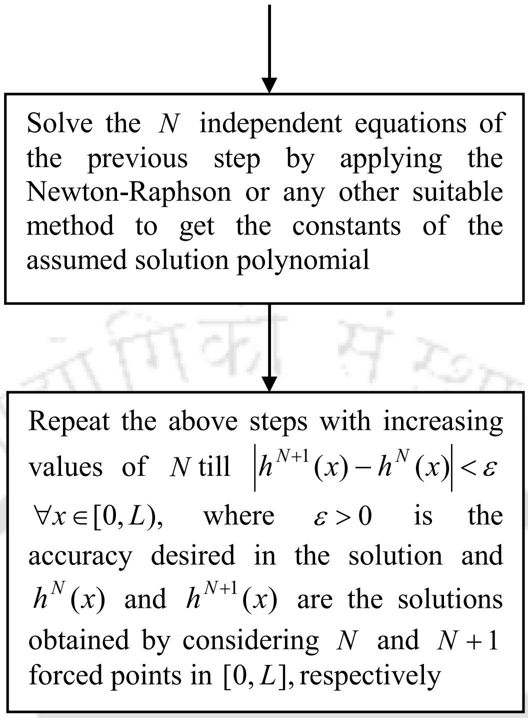

Fig. 3.9(ii). Flow-chart for solving a Gardner-based infiltration problem by utilizing the second approach.

To achieve the same accuracy in a solution, the first approach requires relatively fewer forced points in $[0,L]$ than that by the second approach but this approach is mathematically more demanding than that of the second approach. This is because its use involves differentiation of the governing equation twice [i.e., apart from $\psi(x)$, we also need $\psi'(x)$ and $\psi''(x)$] whereas in the second approach, the governing equation alone [i.e., Eq. (3.9)] can be used to get a solution of it. Further, as mentioned before, the accuracy of both of these approaches depends on how closely the forced points are being taken in an infiltrating space. Thus, a solution polynomial of any desired degree pertaining to an infiltration problem can be found by sufficiently reducing the largest gap in between any two adjacent forced points in $[0,L]$. Also, if we attempt to solve the Gardner-based infiltration problem here by splitting a domain, then also the steps as mentioned above for both the approaches will still be applicable for obtaining a solution of it. This is because, as mentioned in the previous section, the nature of our problem always allows us to estimate the boundary conditions (i.e., both the Neumann as well as the Dirichlet conditions) associated with any sub-domain of (i.e., for any sub-domain of $[0,L]$) of the problem.

3.2 Verification of Proposed Solutions

To check the veracity of our series solutions obtained from both the approaches, we will now make a few comparisons of our predictions with the experimental and analytical works of others for a few unsaturated flow situations. We first make a comparison with Moore's (1939) experimental evaporation data [see also Gardner and Fireman (1958)] from a water table located at about 105 cm from the surface of the soil. The other flow parameters of this flow situation are as shown in Figs. 3.10(i) and 3.10(ii). It should be noted that we have used four sub-domains in our first approach (i.e., where $\psi(x)$, $\psi'(x)$ and $\psi''(x)$ functions are being used for solution) and two sub-domains in our second approach (i.e., where only $\psi(x)$ function is being used for solution) for obtaining solutions for this flow situation. Further, if we name the set of 'developing points' for first of our approaches in the i^{th} sub-domain as $S_{D(i)}^{(1)}$ (the points at which $\psi(x)$ and its first and second derivatives are forced zero) and the set of developing points for second of our approaches in the i^{th} sub-domain as $S_{D(i)}^{(2)}$, then the solutions obtained from both the approaches for this flow situation can be expressed as shown in Appendix A. In this appendix, as may be observed, the sub-domain numbers are being represented with a subscript and the solution approach followed (i.e., whether we have employed first or second of our approaches) in solving a problem by a superscript. We will be following these notations in our solutions throughout this report. As may be observed, we have used a fifteen degree polynomial in the first approach and a twenty degree polynomial in the second approach to arrive at these solutions. These are illustrations only; in general a solution to the problem can be attempted through any combination of $\psi(x)$ and its higher derivatives or as shown, by using $\psi(x)$ alone. However, for the type of problems that is being considered here, in both the approaches, the 'developing points' should be spread in the entire flow space in such a way that the gap in between them should tend to zero. This is because for the nature of the differential equation that is being considered here, the accuracy of its solution obtained using the mathematical procedures as just mentioned (Barua 2021) depends on how closely the 'developing points' are being distributed in a flow space.

Figures 3.10(i) and 3.10(ii) and Table 3.1 show the comparison scenarios of predictions as obtained from our solutions with those obtained from Moore's (1939) experimental results. As may be observed, for the studied flow situation, the predictions as obtained from both of our solutions are in close agreement with Moore's experimental observations thereby showing that our solutions have been correctly developed.

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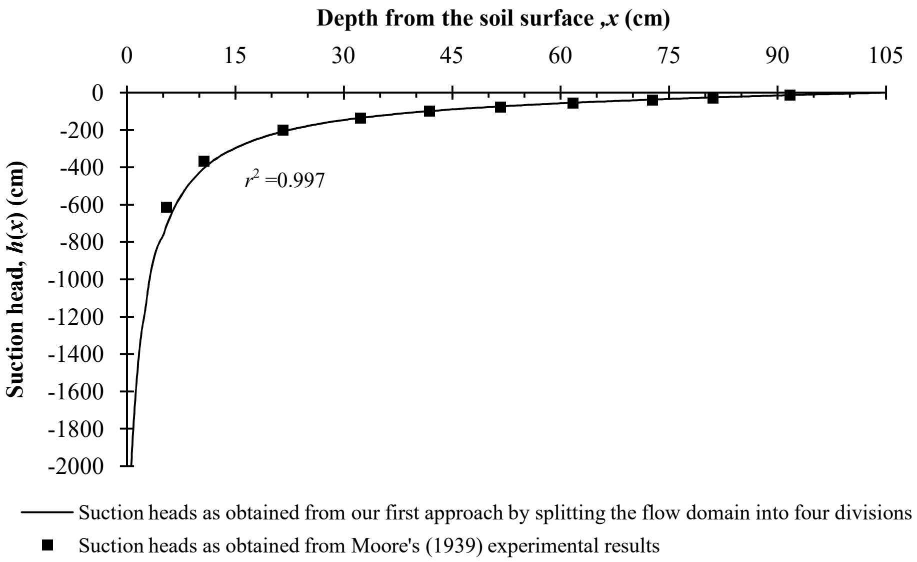

Fig. 3.10(i). Comparison of suction heads as obtained from $h_{(1)}^{(1)}(x)$, $h_{(2)}^{(1)}(X_1)$, $h_{(3)}^{(1)}(X_2)$ and $h_{(4)}^{(1)}(X_3)$ with the corresponding values as obtained from Moore's (1939) experimental data when the parameters of the flow situation are taken as $A = 90^0$, $S(x) = 0$ and $K(h) = \dfrac{a}{b + [h(x)]^n}$, a = 400 cm^3/day, $b = 400$ cm^2, $L = 105$ cm, $h_L = 0$ cm, $q = -0.08$ cm/day and $n = 2$ and the $S_{D(i)}^{(1)}$ set used for each i^{th} sub-domain is $\{0, \frac{L_i}{19} j$, where $j = 1,2,3,4\}$ where L_i is the length of each sub-domain.

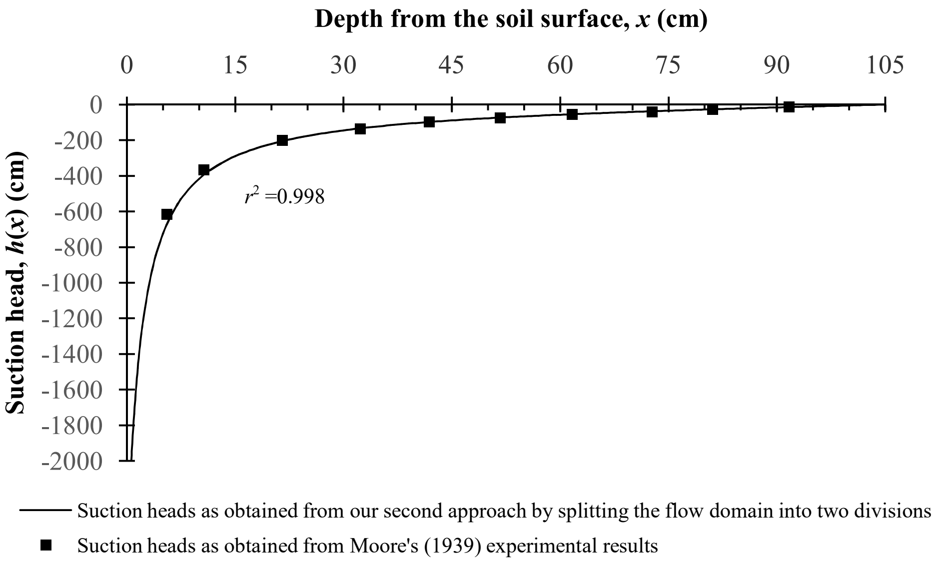

Fig. 3.10(ii). Comparison of suction heads as obtained from $h_{(1)}^{(2)}(x)$ and $h_{(2)}^{(2)}(X_1)$ with the corresponding values as obtained from Moore's (1939) experimental data when the parameters of the flow situation are taken as $A = 90^0$, $S(x) = 0$ and $K(h) = \dfrac{a}{b + [h(x)]^n}$, a = 400 cm^3/day, $b = 400$ cm^2, $L = 105$ cm, $h_L = 0$ cm, $q = -0.08$ cm/day and $n = 2$ and the $S_{D(i)}^{(2)}$ set used for each i^{th} sub-domain is $\{0, \frac{L_i}{19} j$, where $j = 1,2,3,4\}$ where L_i is the length of each sub-domain.

Table 3.1. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Moore's (1939) experimental data when the parameters of the flow situation are taken as shown in Figs. 3.10(i) and 3.10(ii)

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Depths as measured from the surface of the soil (cm)	Suction heads as obtained from Moore's (1939) experimental data (cm)	Suction heads as obtained from $h_{(1)}^{(1)}(x)$, $h_{(2)}^{(1)}(X_1)$, $h_{(3)}^{(1)}(X_2)$ and $h_{(4)}^{(1)}(X_3)$ (cm)	Suction heads as obtained from $h_{(1)}^{(2)}(x)$ and $h_{(2)}^{(2)}(X_1)$ (cm)
05.494	-614.515	-659.664	-675.501
10.619	-364.885	-390.070	-395.536
21.540	-200.993	-203.387	-205.005
32.274	-137.512	-133.781	-133.891
41.830	-98.369	-98.086	-98.206
51.666	-76.864	-73.247	-73.256
61.636	-55.351	-54.396	-54.407
72.676	-41.502	-37.795	-37.796
81.053	-26.697	-26.984	-26.986
91.695	-12.870	-14.556	-14.556

To have a further check on our solutions, we next compare them with Basha's (1999b) analytical works on the problem for a homogeneous soil. Figs. 3.11(i), 3.11(ii) and Tables 3.2a-3.2d show a few comparison results of our solutions with that of Basha's (1999b) exact solution. As can be seen, all these flow scenarios are related to vertical infiltration only and that all are also without a root-extraction term [i.e., with $S(x) = 0$] As may be observed, for all these comparison situations, our predictions are in very close agreement with those obtained from Basha's (1999b) exact solution thereby showing once again that our solutions have been correctly developed. It is worth noting here that, even for a homogeneous soil without the sink term, Basha's (1999b) exact solution is for integral values of *n* only and not otherwise. The solutions developed here by both the approaches, however, are valid for both integral and non-integral values of *n* both when a sink term is present in a flow space and when it is absent. Thus, even for a homogeneous soil without the extraction term, our solution to the problem obtained by either of the approaches has an advantage over that of Basha's (1999b) exact solution. To avoid overcrowding in the text, the details of these solutions, however, have not been included here.

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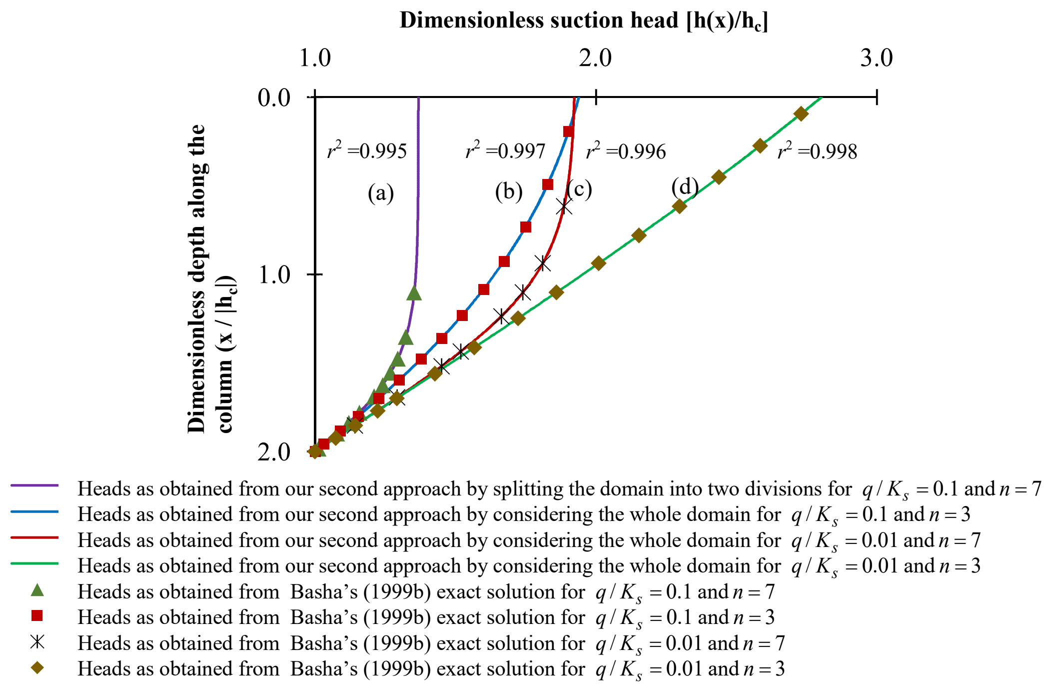

Fig. 3.11(i). Comparison of dimensionless suction heads as obtained from our first approach with the corresponding values as obtained from Basha's (1999b) exact solution when the parameters of the flow situations are taken as as $A = 90^0$, $S(x) = 0$, $L = 100$ cm, $h_L = -50$ cm, $h_c = -50$ cm and (a) $q/K_s = 0.1$ and $n = 7$ (by splitting the domain into two divisions, namely 0-50 cm and 50-100 cm), (b) $q/K_s = 0.1$ and $n = 3$, (c) $q/K_s = 0.01$ and $n = 7$ and (d) $q/K_s = 0.01$, $n = 3$ and the $S^{(2)}_{D(i)}$ set used for each i^{th} sub-domain is $\{0, \frac{L_i}{19} j$, where $j = 1,2,3,4\}$ where L_i is the length of each sub-domain.

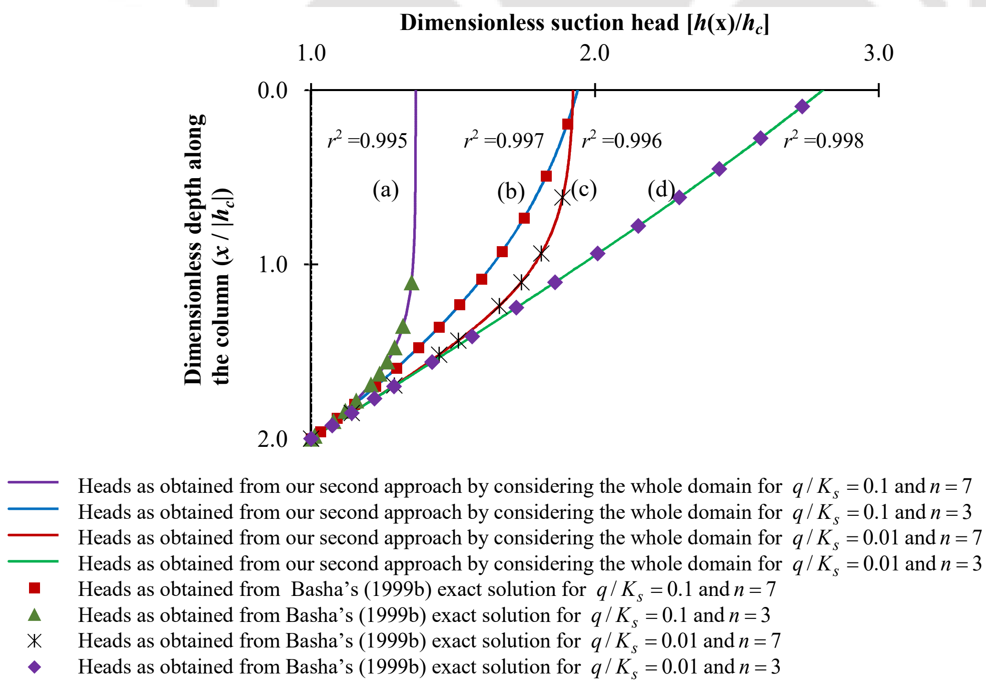

Fig. 3.11(ii). Comparison of dimensionless suction heads as obtained from our second approach with the corresponding values as obtained from Basha's (1999b) exact solution when the parameters of the flow situations are taken as $A = 90^0$, $S(x) = 0$, $L = 100$ cm, $h_L = -50$ cm, $h_c = -50$ cm and (a) $q/K_s = 0.1$ and $n = 7$, (b) $q/K_s = 0.1$ and $n = 3$, (c) $q/K_s = 0.01$ and $n = 7$ and (d) $q/K_s = 0.01$ and $n = 3$ and $S^{(2)}_D$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1,2 \ldots 19\}$.

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Table 3.2a. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Basha's (1999b) exact solution when the parameters of the flow situation are taken as shown in Figs. 3.11(i)(a) and 3.11(ii)(b)

Dimensionless depths along the column	Dimensionless suction heads as obtained from Basha's (1999b) exact solution	Dimensionless suction heads as obtained from our first approach	Dimensionless suction heads as obtained from our second approach
1.102	1.354	1.354	1.354
1.353	1.324	1.324	1.324
1.476	1.295	1.295	1.295
1.556	1.269	1.269	1.269
1.626	1.241	1.241	1.241
1.690	1.211	1.211	1.210
1.781	1.159	1.159	1.158
1.840	1.121	1.121	1.120
1.898	1.079	1.079	1.078
1.984	1.013	1.013	1.013
2.000	1.000	1.000	1.000

Table 3.2b. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Basha's (1999b) exact solution when the parameters of the flow situation are taken as shown in Figs. 3.11(i)(b) and 3.11(ii)(b)

Dimensionless depths along the column	Dimensionless suction heads as obtained from Basha's (1999b) exact solution	Dimensionless suction heads as obtained from our first approach	Dimensionless suction heads as obtained from our second approach
0.194	1.903	1.902	1.903
0.493	1.828	1.828	1.828
0.733	1.750	1.750	1.750
0.927	1.673	1.673	1.673
1.085	1.600	1.600	1.600
1.232	1.524	1.524	1.524
1.361	1.451	1.451	1.451
1.478	1.378	1.379	1.378
1.595	1.301	1.301	1.301
1.701	1.227	1.228	1.227
1.801	1.154	1.155	1.154
1.883	1.092	1.092	1.092
1.959	1.033	1.033	1.033
2.000	1.000	1.000	1.000

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Table 3.2c. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Basha's (1999b) exact solution when the parameters of the flow situation are taken as shown in Figs. 3.11(i)(c) and 3.11(ii)(c)

Dimensionless depths along the column	Dimensionless suction heads as obtained from Basha's (1999b) exact solution	Dimensionless suction heads as obtained from our first approach	Dimensionless suction heads as obtained from our second approach
0.616	1.886	1.885	1.886
0.938	1.811	1.809	1.811
1.103	1.741	1.740	1.741
1.238	1.663	1.665	1.663
1.437	1.519	1.521	1.519
1.519	1.451	1.453	1.451
1.695	1.293	1.296	1.293
1.853	1.143	1.143	1.143
2.000	1.000	1.000	1.000

Table 3.2d. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Basha's (1999b) exact solution when the parameters of the flow situation are taken as shown in Figs. 3.11(i)(d) and 3.11(ii)(d)

Dimensionless depths along the column	Dimensionless suction heads as obtained from Basha's (1999b) exact solution	Dimensionless suction heads as obtained from our first approach	Dimensionless suction heads as obtained from our second approach
0.094	2.730	2.730	2.730
0.276	2.584	2.584	2.584
0.452	2.438	2.438	2.438
0.616	2.297	2.297	2.297
0.780	2.153	2.153	2.153
0.938	2.010	2.010	2.010
1.103	1.860	1.860	1.860
1.249	1.723	1.723	1.723
1.413	1.568	1.567	1.568
1.560	1.427	1.427	1.427
1.701	1.292	1.291	1.292
1.771	1.223	1.223	1.223
1.853	1.143	1.143	1.143
1.924	1.075	1.075	1.075
2.000	1.000	1.000	1.000

Figs. 3.12(i), 3.12(ii) and Table 3.3 further show the comparison results of our solutions for a homogeneous soil infiltration scenario with those obtained from Basha's (1999b) approximate and numerical solutions to the problem. As can be seen, this infiltration situation is also related to vertical infiltration only but, unlike the flow situations of Figs. 3.11(i) and 3.11(ii), we now also have a sink term in the considered flow space. From these figures and table it can be seen that our predictions for these situations are having a close match with those obtained from Basha's (1999b) numerical work (Runga-Kutta fourth-order

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method). Further, they are also matching in an approximate way with the predictions obtained from his approximate solution. It is worth noting here that Basha's (1999b) perturbation solution for the infiltration problem for a homogeneous soil with a sink term with Gardner's (1958) general hydraulic conductivity model, is of an approximate nature (even though it is applicable for both integral and non-integral values of n); thus, the approximate matching of our predictions with this model's outputs is understandable. But the very close matching of our predictions with those obtained from Basha's (1999b) numerical solution clearly again demonstrates that our models have been rightly developed.

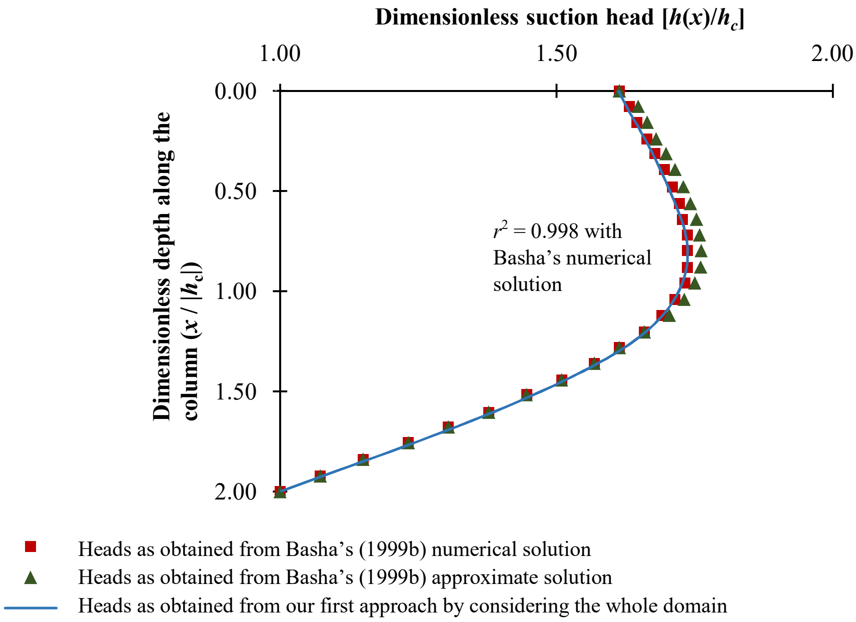

Fig. 3.12(i). Comparison of dimensionless suction heads as obtained from our first approach with the corresponding values as obtained from Basha's (1999b) approximate and numerical solutions when the parameters of the flow situations are taken as $A = 90^0$, $\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}, $L = 100$ cm, h_L = -50 cm, q / K_s = 0.1, $h_c = -50$ cm, $n = 5$ and $S_D^{(1)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{4} j$, where $j = 1, 2, 3, 4\}$.

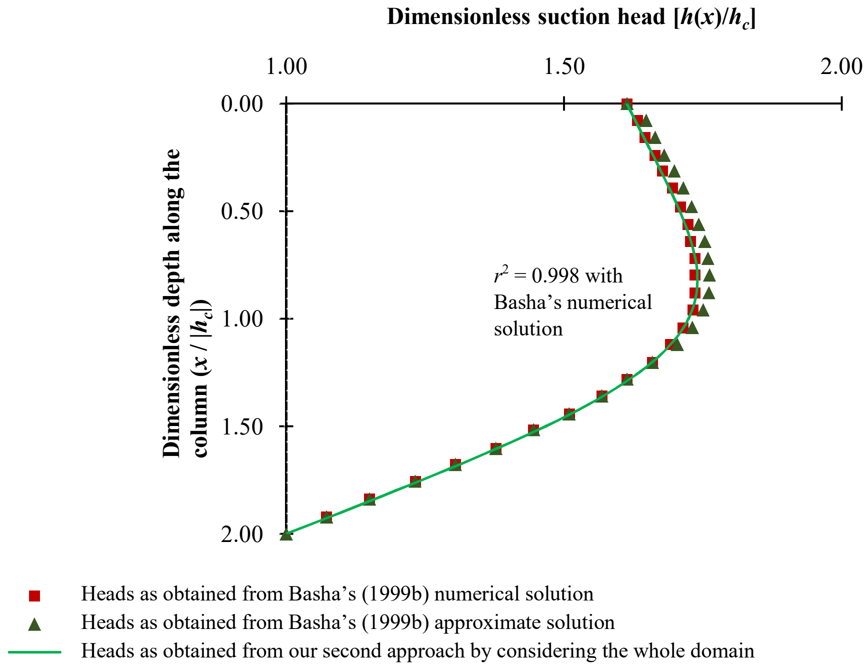

Fig. 3.12(ii). Comparison of dimensionless suction heads as obtained from our second approach with the corresponding values as obtained from Basha's (1999b) approximate and numerical solutions when the parameters of the flow situation are taken as $A = 90^0$, $\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}, $L = 100$ cm, h_L = -50 cm, q / K_s = 0.1, $h_c = -50$ cm, $n = 5$ and $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2 \ldots 19\}$.

Table 3.3. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Basha's (1999b) solution when the parameters of the flow situation are taken as shown in Figs. 3.12(i) and 3.12(ii)

Dimensionless depths along the column	Dimensionless suction heads as obtained from Basha's (1999b) approximate solution	Dimensionless suction heads as obtained from Basha's (1999b) numerical solution	Dimensionless suction heads as obtained from our first approach	Dimensionless suction heads as obtained from our second approach
0.000	1.614	1.614	1.612	1.613
0.078	1.648	1.632	1.627	1.628
0.157	1.664	1.645	1.643	1.643
0.240	1.680	1.664	1.660	1.660
0.313	1.698	1.677	1.674	1.674
0.392	1.715	1.695	1.688	1.689
0.479	1.730	1.709	1.702	1.705
0.562	1.742	1.723	1.715	1.718
0.641	1.753	1.727	1.726	1.729
0.719	1.759	1.736	1.734	1.736
0.797	1.762	1.736	1.737	1.740
0.880	1.761	1.736	1.735	1.738
0.959	1.750	1.732	1.728	1.731
1.041	1.731	1.714	1.713	1.715
1.120	1.704	1.691	1.692	1.692
1.203	1.659	1.659	1.661	1.659

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1.281	1.614	1.614	1.623	1.619
1.359	1.568	1.568	1.576	1.571
1.442	1.509	1.509	1.517	1.512
1.516	1.445	1.445	1.459	1.454
1.604	1.377	1.377	1.385	1.379
1.677	1.305	1.305	1.319	1.313
1.756	1.232	1.232	1.244	1.240
1.839	1.150	1.150	1.162	1.160
1.922	1.073	1.073	1.078	1.078
2.000	1.000	1.000	1.000	1.000

As a further check of our proposed solutions, comparisons of them with Zhu and Mohanty's (2002) solutions are next carried out for a few vertical infiltration scenarios in a homogeneous soil; Figs. 3.13(i), 3.13(ii) and Tables 3.4a, 3.4b and 3.4c show these comparison results. As can be seen from these figures and tables, in all these tested situations, the predictions from our solutions are in good agreement with those obtained from Zhu and Mohanty's (2002) analytical solutions thereby providing us with an additional proof about the veracity of our solutions.

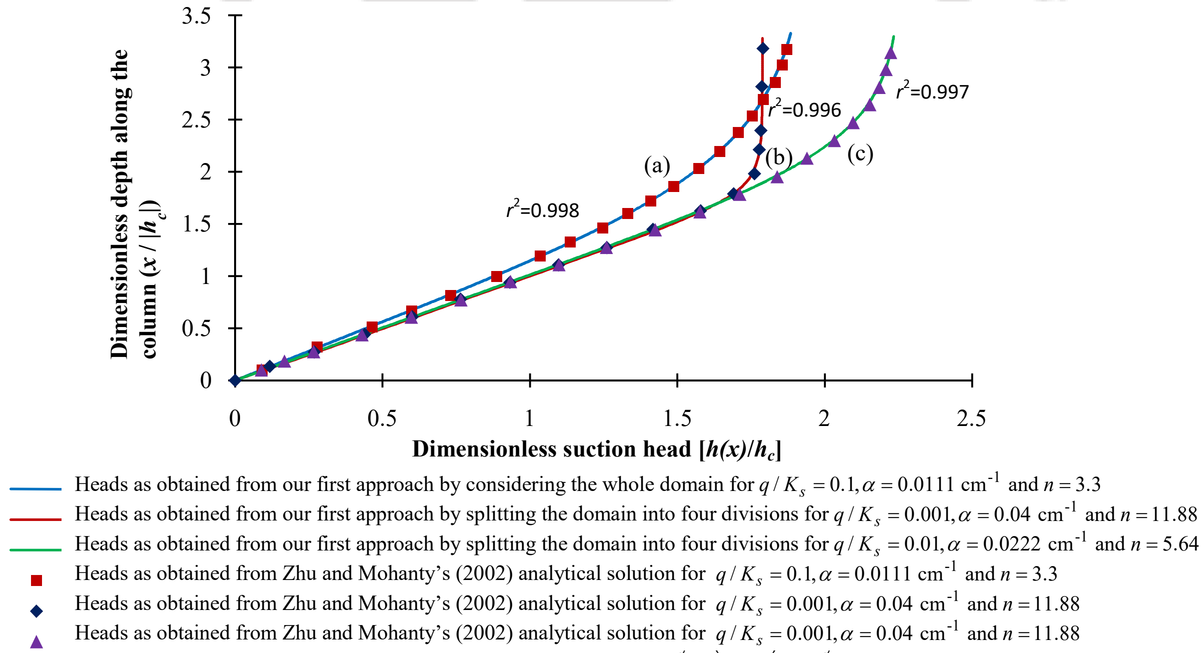

Fig. 3.13(i). Comparison of dimensionless suction heads as obtained from our first approach with the corresponding values as obtained from Zhu and Mohanty's (2002) analytical solution when the parameters of the flow situations are taken as $S(x) = 0$, $h_L = 0$ cm, $h_c = -1/\alpha$ and (a) $L = 300$ cm, $q/K_s = 0.1$, $\alpha = 0.0111\,\text{cm}^{-1}$ and $n = 3.3$, (b) $L = 82$ cm, $q/K_s = 0.001$, $\alpha = 0.04$ cm^{-1} and $n = 11.88$ (by splitting the domain into four divisions, namely, 0-10 cm, 10-20 cm, 20-41 cm and 41-82 cm) and (c) $L = 148.5$ cm, $q/K_s = 0.01$, $\alpha = 0.0222\,\text{cm}^{-1}$ and $n = 5.64$ (by splitting the domain into two divisions, namely 0-100 cm and 100-148.5 cm) and the $S_{D(i)}^{(2)}$ set used for each i^{th} sub-domain is $\{0, \frac{L_i}{19}j$, where $j = 1,2,3,4\}$ where L_i is the length of each sub-domain.

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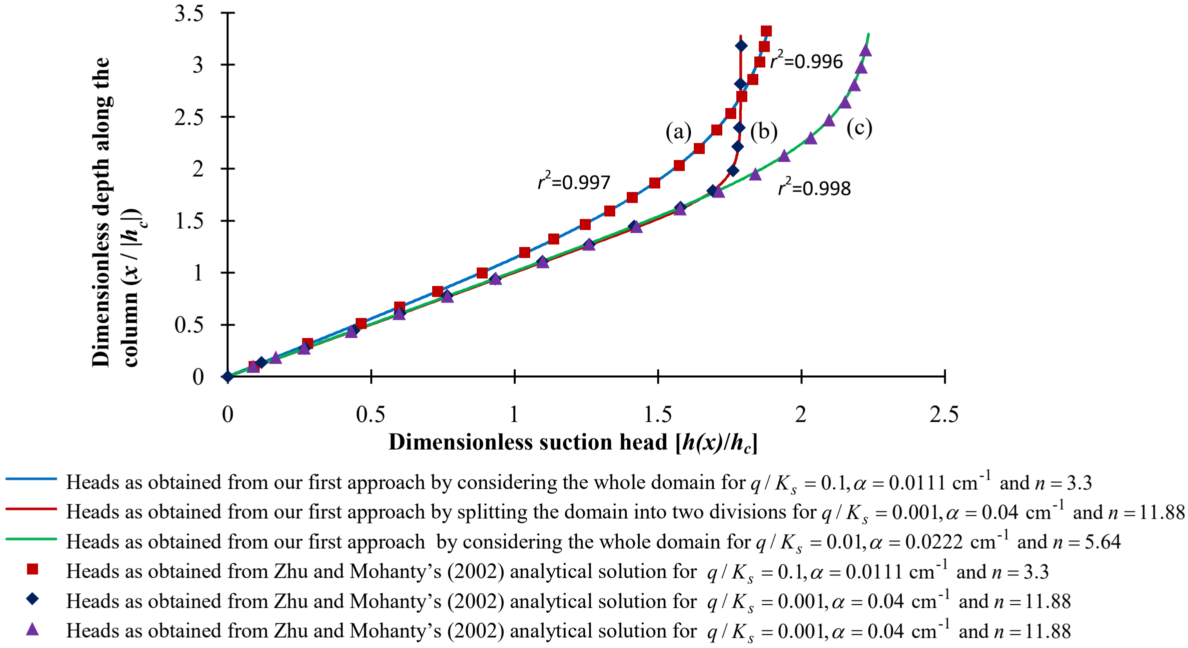

Fig. 3.13(ii). Comparison of dimensionless suction heads as obtained from our second approach with the corresponding values as obtained from Zhu and Mohanty's (2002) analytical solution when the parameters of the flow situations are taken as taken as $S(x)=0$, $h_L=0$ cm, $h_c=-1/\alpha$ and (a) L = 300 cm $q/K_s=0.1$, $\alpha=0.0111\,\text{cm}^{-1}$ and $n=3.3$, (b) L = 82 cm, $q/K_s=0.001$, $\alpha=0.04\,\text{cm}^{-1}$ and $n=11.88$ (by splitting the domain into two divisions, namely $0-41$ cm and $41-82$ cm) and (c) $L=148.5$ cm, $q/K_s=0.01$, $\alpha=0.0222\,\text{cm}^{-1}$ and $n=5.64$ and $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j$, where $j=1,2\ldots19\}$.

Table 3.4a. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Zhu and Mohanty's (2002) solution when the parameters of the flow situation are taken as shown in Figs. 3.13(i)(a) and 3.13(ii)(a)

Dimensionless depths along the column	Dimensionless suction heads as obtained from Zhu and Mohanty's (2002) solution	Dimensionless suction heads as obtained from our first approach	Dimensionless suction heads as obtained from our second approach
0.100	0.091	0.091	0.090
0.318	0.278	0.278	0.286
0.513	0.465	0.465	0.461
0.669	0.598	0.598	0.599
0.817	0.731	0.731	0.728
0.997	0.887	0.887	0.881
1.192	1.035	1.035	1.040
1.324	1.136	1.136	1.141
1.465	1.246	1.246	1.242
1.598	1.332	1.332	1.331
1.722	1.410	1.410	1.408
1.863	1.488	1.488	1.487
2.034	1.573	1.573	1.571
2.197	1.644	1.644	1.640
2.377	1.706	1.706	1.704
2.533	1.753	1.753	1.749
2.697	1.792	1.792	1.789

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2.861	1.831	1.831	1.821
3.025	1.855	1.855	1.847
3.172	1.870	1.870	1.866

Table 3.4b. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Zhu and Mohanty's (2002) solution when the parameters of the flow situation are taken as shown in Figs. 3.13(i)(b) and 3.13(ii)(b)

Dimensionless depths along the column	Dimensionless suction heads as obtained from Zhu and Mohanty's (2002) solution	Dimensionless suction heads as obtained from our first approach	Dimensionless suction heads as obtained from our second approach
0.000	0.000	0.000	0.000
0.135	0.118	0.136	0.134
0.273	0.269	0.278	0.273
0.441	0.440	0.445	0.440
0.610	0.600	0.614	0.609
0.778	0.765	0.778	0.777
0.939	0.932	0.940	0.938
1.107	1.097	1.106	1.105
1.272	1.260	1.268	1.269
1.446	1.417	1.435	1.436
1.627	1.579	1.586	1.590
1.787	1.691	1.665	1.690
1.981	1.761	1.739	1.755
2.213	1.778	1.773	1.781
2.395	1.783	1.784	1.786
2.817	1.787	1.788	1.788
3.182	1.791	1.788	1.788

Table 3.4c. Comparison of dimensionless suction heads as obtained from both of our approaches with the corresponding values as obtained from Zhu and Mohanty's (2002) solution when the parameters of the flow situation are taken as shown in Figs. 3.13(i)(c) and 3.13(ii)(c)

Dimensionless depths along the column	Dimensionless suction heads as obtained from Zhu and Mohanty's (2002) solution	Dimensionless suction heads as obtained from our first approach	Dimensionless suction heads as obtained from our second approach
0.101	0.089	0.100	0.100
0.185	0.167	0.183	0.183
0.275	0.266	0.272	0.272
0.436	0.430	0.432	0.432
0.607	0.598	0.601	0.601
0.773	0.766	0.765	0.765
0.946	0.934	0.936	0.936
1.108	1.099	1.094	1.094
1.275	1.260	1.255	1.255
1.444	1.425	1.414	1.414
1.615	1.577	1.567	1.567
1.783	1.712	1.706	1.705
1.952	1.839	1.830	1.830
2.129	1.940	1.942	1.942
2.298	2.033	2.028	2.028

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2.472 2.644 2.808 2.980 3.144	2.096 2.153 2.185 2.208 2.224	2.096 2.146 2.181 2.206 2.223	2.096 2.147 2.181 2.206 2.223

We also next make a comparison of the suction heads as obtained from our solutions with the corresponding values as obtained from the CHEMFLO-2000 (Nofziger and Wu 2000) numerical codes for a few infiltration situations; Figs. 3.14(i), 3.14(ii) and Table 3.5 show the comparison results for the tested situations. As can be seen, for these tested scenarios also our suction head predictions can match with those obtained from the CHEMFLO-2000 (Nofziger and Wu 2000) codes in a very accurate way thereby highlighting once again the validity of our proposed solutions.

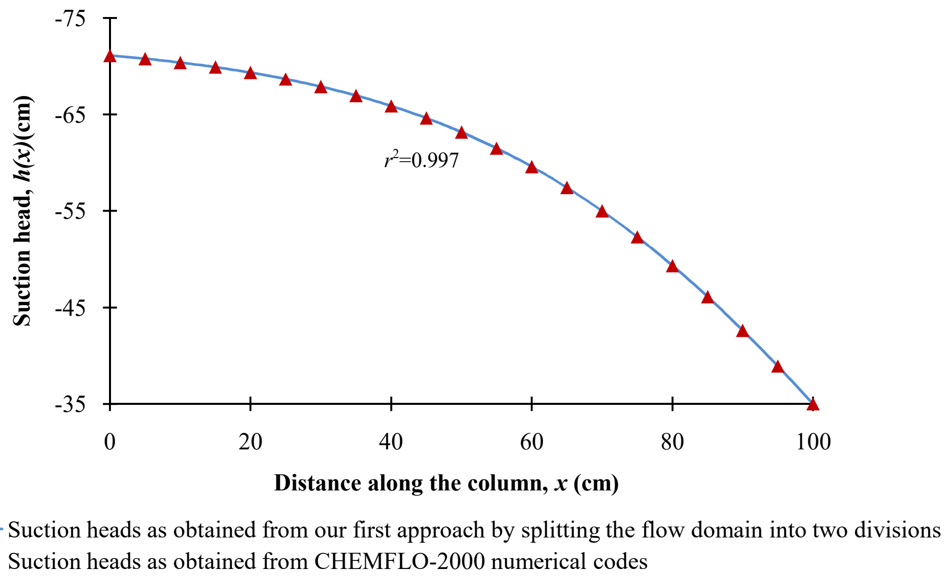

Fig. 3.14(i). Comparison of suction heads as obtained from our first approach (by splitting the domain into two divisions, namely, 0-50 cm and 50 to 100 cm) with the corresponding values as obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as $A=90^0$, $S(x)=0$, $K_S(x)=1$ cm/hr, $L=100$ cm, $h_L=-35$ cm, $q=0.1$ cm/hr, $h_C=-35$ cm, $n=3$ and $S_D^{(1)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{4}j$, where $j=1,2,3,4\}$.

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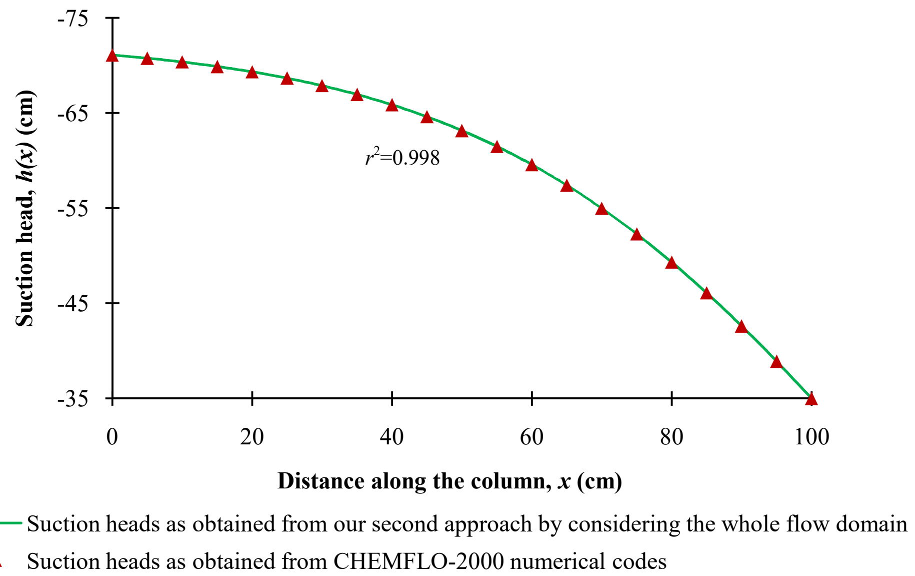

Fig. 3.14(ii). Comparison of suction heads as obtained from our second approach with the corresponding values as obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as $A=90^0$, $S(x)=0$, $K_S(x)=1$ cm/hr, $L=100$ cm, $h_L=-35$ cm, $q=0.1$ cm/hr, $h_C=-35$ cm, $n=3$ and $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j,$ where $j=1,2,\ldots,19\}$.

Table 3.5. Comparison of suction heads as obtained from both of our solution approaches with the corresponding values obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as shown in Figs. 3.14(i) and 3.14(ii)

Depths as measured from the surface of the soil (cm)	Suction heads as obtained from CHEMFLO-2000 numerical codes (cm)	Suction heads as obtained from our first approach (cm)	Suction heads as obtained from our second approach (cm)
0	-71.110	-71.108	-71.109
5	-70.774	-70.773	-70.774
10	-70.374	-70.373	-70.374
15	-69.899	-69.897	-69.899
20	-69.335	-69.334	-69.335
25	-68.668	-68.667	-68.668
30	-67.881	-67.881	-67.881
35	-66.957	-66.957	-66.957
40	-65.876	-65.876	-65.876
45	-64.618	-64.617	-64.618
50	-63.162	-63.161	-63.162
55	-61.489	-61.487	-61.488
60	-59.579	-59.577	-59.578
65	-57.416	-57.415	-57.416
70	-54.990	-54.989	-54.990
75	-52.293	-52.293	-52.293
80	-49.326	-49.327	-49.326
85	-46.096	-46.097	-46.096
90	-42.617	-42.618	-42.617
95	-38.909	-38.910	-38.909
100	-35.000	-35.000	-35.000

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As a final comparison of our mathematical procedure, we now also make a comparison of our solution for an infiltration problem with that obtained using the Picard's (Scarborough 1966) iterative method. For this, to have an easy evaluation of the Picard's integrals, we consider the governing equation (3.9) for an isotropic soil only with $S = 0$ and $n = 2$. For these situations, Eq. (3.9) can be expressed as

$$\frac{dh}{dx} = C_0\left[1 + (\alpha h)^2\right] \tag{3.46}$$

where $C_0 = -(q/K_s)$ and $\alpha = 1/h_c$. We now solve Eq. (3.46) in a space L by considering (again for easy evaluations of the Picards' integrals) $h = 0$ at $x = L$. With these assumptions in place, Picard's first integral in the solution of Eq. (3.46) can then be written as $\int_{h_1(x)}^{0} dh = C_0\int_{x}^{L}\left[1 + \alpha^2(h_0 = 0)^2\right]dx = C_0\int_{x}^{L}(1)dx,$ where $h_1(x)$ is the first approximation of $h(x)$ and $h_0 = 0$ is the initial guess value of the solution. Solution of this integral gives $h_1(x)$ as $h_1(x) = C_0(x - L).$ Now, using this first approximation, Picard's second integral can next be expressed; it works out as $\int_{h_2(x)}^{0} dh = C_0\int_{x}^{L}\left[1 + \alpha^2(h_1)^2\right]dx,$ where $h_2(x)$ is the second Picard's approximation of $h(x)$. From the solution of this equation, we find $h_2(x)$ as $h_2(x) = C_0(x - L) + \frac{1}{3}C_0^3\alpha^2(x - L)^3$. Continuing in the same manner, we find the first four terms of the solution polynomial as obtained from the Picard's method as

$$h(x) = C_0(x - L) + \left(\frac{1}{3}\right)C_0^3\alpha^2(x - L)^3 + \left(\frac{2}{15}\right)C_0^5\alpha^4(x - L)^5 + \left(\frac{17}{315}\right)C_0^7\alpha^6(x - L)^7 + \ldots \tag{3.47}$$

We will now make an attempt to solve Eq. (3.46) by making a direct application of Theorem-1A of Barua (2021). For that, we first assume the solution polynomial as

$$h(x) = C_1(x - L) + C_2(x - L)^2 + C_3(x - L)^3 + \ldots + C_7(x - L)^7 \tag{3.48}$$

where C_i $(i = 1 \text{ to } 7)$ are the unknown constants which need to be determined. Of course, we can choose the assumed polynomial in a different way as well but, as can be observed, this way choosing the assumed the polynomial has the advantage that it is automatically satisfying the stated boundary condition at $x = L$ [i.e., $h(x)$ is automatically becoming zero at $x = L$]. Further, as our intent here is to check our solution with Picard's seven degree polynomial solution [i.e., with the solution as given by Eq. (3.46)] to the problem, we have considered our assumed polynomial to be of that degree only; however, it needs to be noted

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that the mathematical procedure as explained here is a general one and can very well be used to generate a polynomial solution of any desired degree of the problem.

The $\psi(x)$ function for this differential equation can be written as

$$\psi(x) = \frac{dh}{dx} - C_0\left[1 + (\alpha h)^2\right] \tag{3.49}$$

Now, following Barua (2021), $h(x)$ of Eq. (3.48) can next be inputted in Eq. (3.49) and $\psi(x),\ \psi'(x), \psi''(x) \ldots \psi^{vi}(x)$ be forced zero at $x = L$ (or at any other point in [0,L]). This will generate seven equations which can then be solved to determine the constants of Eq. (3.48).

For example, $\psi(x=L) = \left\{\left(\frac{dh}{dx}\right) - C_0\left[1+(\alpha h)^2\right]\right\}_{x=L}$ gives $C_1 = C_0$; $\psi'(x=L) = \left[\left(\frac{d^2h}{dx^2}\right) - 2C_0 \alpha^2 h\left(\frac{dh}{dx}\right)\right]_{x=L} = 0$ gives $2C_2 - 0 = 0$, hence, $C_2 = 0$; $\psi''(x=L) = \left\{\left(\frac{d^3h}{dx^3}\right) - 2C_0\alpha^2 \left[h\left(\frac{d^2h}{dx^2}\right) + \left(\frac{dh}{dx}\right)^2\right]\right\}_{x=L} = 0$ gives $6C_3 - 2C_0C_1^2\alpha^2 = 0$, hence, $C_3 = \left(\frac{1}{3}\right)C_0^3\alpha^2$; $\psi'''(x=L) = \left\{\left(\frac{d^4h}{dx^4}\right) - 2C_0\alpha^2\left[h\left(\frac{d^3h}{dx^3}\right) + 3\left(\frac{dh}{dx}\right)\left(\frac{d^2h}{dx^2}\right)\right]\right\}_{x=L} = 0$ gives $24C_4 - 0 = 0$, hence $C_4 = 0$; $\psi^{iv}(x=L) = \left\{\left(\frac{d^5h}{dx^5}\right) - 2C_0\alpha^2\left[h\left(\frac{d^4h}{dx^4}\right) + 4\left(\frac{dh}{dx}\right)\left(\frac{d^3h}{dx^3}\right) + 3\left(\frac{d^2h}{dx^2}\right)^2\right]\right\}_{x=L} = 0$ gives $120C_5 - 48C_0C_1C_3\alpha^2 = 0$, hence $C_5 = \left(\frac{2}{15}\right)C_0^5\alpha^4$; $\psi^{v}(x=L) = \left\{\left(\frac{d^6h}{dx^6}\right) - 2C_0\alpha^2\left[h\left(\frac{d^5h}{dx^5}\right) + 5\left(\frac{dh}{dx}\right)\left(\frac{d^4h}{dx^4}\right) + 10\left(\frac{d^2h}{dx^2}\right)\left(\frac{d^3h}{dx^3}\right)\right]\right\}_{x=L} = 0$ gives $720C_6 - 0 = 0$, hence $C_6 = 0$; $\psi^{vi}(x=L) = \left\{\left(\frac{d^7h}{dx^7}\right) - 2C_0\alpha^2\left[h\left(\frac{d^6h}{dx^6}\right) + 6\left(\frac{dh}{dx}\right)\left(\frac{d^5h}{dx^5}\right) + 15\left(\frac{d^2h}{dx^2}\right)\left(\frac{d^4h}{dx^4}\right) + 10\left(\frac{d^3h}{dx^3}\right)^2\right]\right\}_{x=L} = 0$ gives $5040C_7 - 2C_0\alpha^2\left(720C_1C_5 + 360C_3^2\right) = 0$, hence $C_7 = \left(\frac{17}{315}\right)C_0^7\alpha^6$. If we substitute these C_i s ($i = 1$ to 7) in Eq. (3.48), we see that our solution polynomial is matching term-by-term with the one obtained using the Picard's method [i.e., with the polynomial of Eq. (3.47)]. This provides us with an additional verification of the mathematical procedure related to our second approach.

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3.3 Discussions

We will now analyze a few infiltration situations using the mathematical procedures (Barua 2021) as already mentioned. But before that, it is also worth mentioning here that the series solution proposed here for the one-dimensional infiltration problem with Gardner's (1958) general conductivity function can easily be extended to include Brooks and Corey's (1964) conductivity function in it as well as the general structure of both these conductivity functions are quite similar. This will become more obvious if we write down the Brooks and Corey's (1964) conductivity function; in our notations, this model for a heterogeneous soil can be expressed as

$$K(h) = \frac{K_s(x)}{\left[\left(\frac{h(x)}{h_c(x)}\right)\right]^{n(x)}} \tag{3.50}$$

As may be observed from the above equation, the only mathematical difference that is there in between Gardner's (1958) and Brooks and Corey's (1964) models is that, whereas there are two terms (1 and the fractional term) in the denominator of the former, the latter has only one term in it (i.e., only the fractional term). Thus, Brooks and Corey's (1964) conductivity model can be mathematically considered as a special case of Gardner's (1958) general hydraulic function and as such our Gardner-based series solution can be easily extended to incorporate this conductivity function as well. This can be done by simply neglecting one of the denominator terms in the Gardner's (1958) conductivity model. As the mathematical procedure for doing so is very similar to the one already mentioned in the development of the Gardner-based solution above, to avoid repetition, we are not presenting the same here. However, to highlight the fact that Barua's (2021) mathematical results can also be easily applied to solve the Brooks and Corey-based infiltration equation as well, we are also presenting here two examples with this conductivity function; Figs. 3.15(i) and 3.15(ii) show the details about the hydraulics of these infiltration situations. From these figures, we see that suction curves as obtained from our Brooks and Corey-based solutions for the studied infiltration situations are matching very closely with the corresponding ones obtained using the CHEMFLO-2000 (Nofziger and Wu 2000) numerical codes. This shows that our derived model for the Gardner-based infiltration equation can be easily adjusted to solve the Brooks and Corey-based infiltration equation too. This thus can be considered as an additional advantage of the developed solution as with Brooks and Corey's (1964) conductivity model

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also, there is currently no analytical solution of the Richards' infiltration equation for a heterogeneous soil.

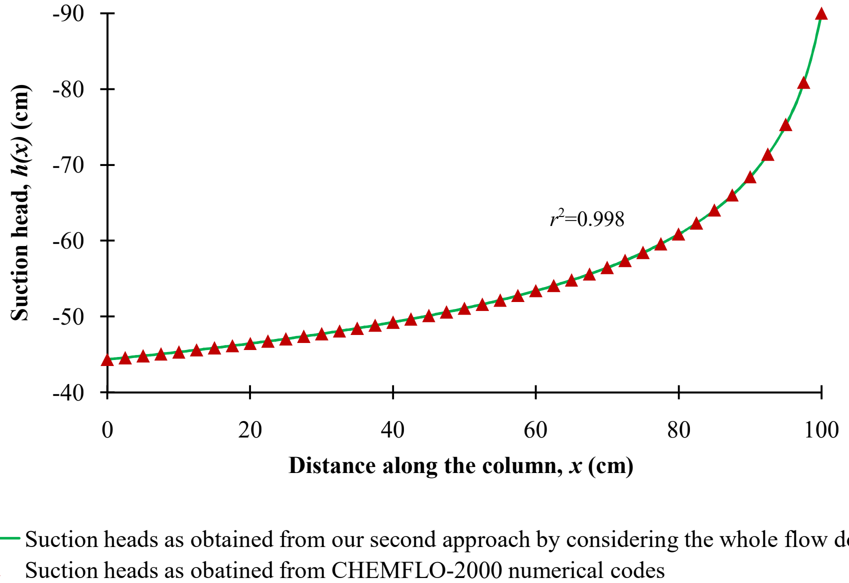

Fig. 3.15(i). Comparison of suction heads as obtained from our second approach with the corresponding values as obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as $A = 0^0$, $S(x) = 0$, $L = 100$ cm, $h_L = -45$ cm, $q/K_S = 0.1$, $h_c = -45$ cm, $n = 5.64$ and $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j,$ where $j = 1, 2, \ldots, 19\}$.

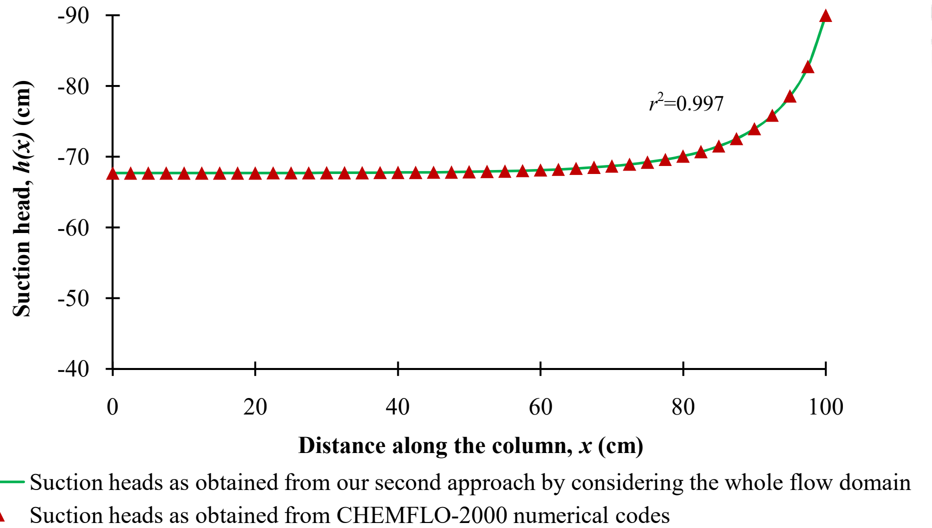

Fig. 3.15(ii). Comparison of suction heads as obtained from our second approach with the corresponding values as obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as $A = 90^0$, $S(x) = 0$, $L = 100$ cm, $h_L = -45$ cm, $q/K_S = 0.1$, $h_c = -45$ cm, $n = 5.64$ and $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j,$ where $j = 1, 2, \ldots, 19\}$.

We will now discuss the suction head profiles of a few infiltration situations that we have obtained using our second approach. The second approach [where only $\psi(x)$ function is being used to arrive at a solution] is being used to generate these profiles as, for these

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infiltration situations, it has been found to be computationally less demanding than the use of our first approach (where $\psi(x)$ as well its first and second derivatives are being used) in studying them.

3.3.1 Suction head profiles for different orientations of an infiltration column

Figs. 3.16 and 3.17 show the variation of suction head with the inclination of a soil profile when the flow parameters of Fig. 3.1 are as shown in these figures. As can be seen, for the studied infiltration situations, the magnitude of the suction heads in a flow space is increasing with the increase in inclination of a soil column, both when a sink term is present in the column and also when it is absent. These increases in suction heads, as can be seen from these figures, are more significant in flow locations lying close to the beginning of the infiltration columns. This is understandable since for an inclined soil, the total head pushing water through a soil column will also have a varying gravitational head along the length of the soil column (from 0 to $-L$ for a vertical soil column and from 0 to $-L/\sqrt{2}$ for a 45^0 inclined soil column) and the suction heads in an inclined infiltrating space are thus also getting influenced by this varying gravitational field associated with them apart from the other infiltration parameters of the system. It should be noted that gravity is not contributing to water movement in infiltration through a horizontal unsaturated soil column and hence the suction heads are not getting influenced by gravity for horizontal infiltration situations; this is true whether a sink is there or not in a horizontal infiltrating space.

The suction head profiles for the same orientations, as may be observed, are also dependent on the nature of distribution of the root-water extraction function in an infiltrating space. Further, with the increase in strength of the sink term, the tenacity with which water particles are being held in the soil pores of an infiltrating space would also increase. From these flow situations, it is clear that – considering all other parameters as non-changing – water is held more closely by soils in a vertically inclined infiltration column than that in a horizontal column. Thus, under otherwise similar flow situations, more work is needed by plant roots to extract water from a vertically infiltrating space than that from a horizontal one. This shows that the ease of water availability to plants may also be significantly dependent on the direction of movement of infiltrating water in a cropped field.

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Length of the column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L=50$ cm	$\frac{S(x)}{K_s}=0$	At $x=0$ $q/K_s=0.1$; at $x=L$ $h_L=-90$ cm	$h_c=-90$ cm, $n=3.3$
(b) $L=50$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L}\right)=\frac{0.1}{L}$ cm^{-1}		
(c) $L=100$ cm	$\frac{S(x)}{K_s}=0$		
(d) $L=100$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L}\right)=\frac{0.1}{L}$ cm^{-1}		

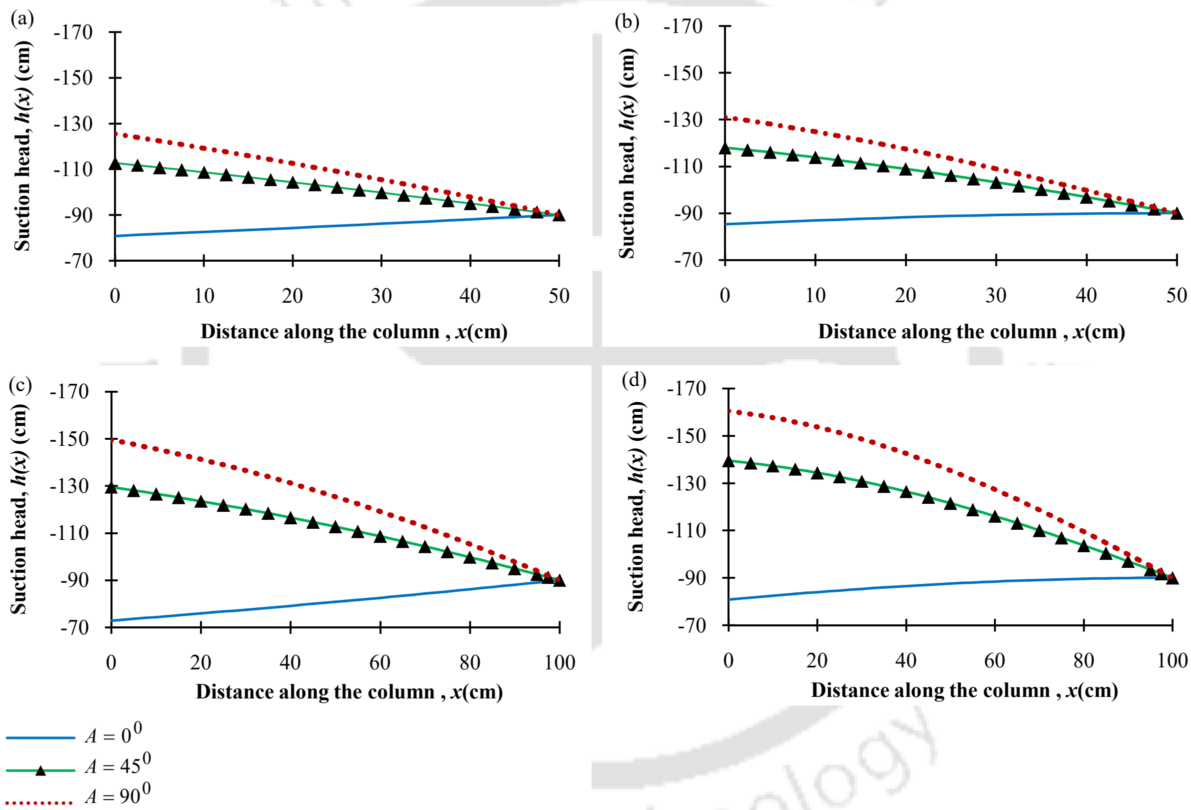

Fig. 3.16. Variation of suction head along the length of a soil profile for three values of A (namely $A=0^0, 45^0$ and 90^0) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j$, where $j=1,2,\ldots,19\}$.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{(S_M / K_s)}{L} = \frac{0.1}{L}$ cm^{-1}	At $x = 0$ $q / K_s = 0.1$; at $x = L$ $h_L = -90$ cm	$h_c = -90$ cm, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		

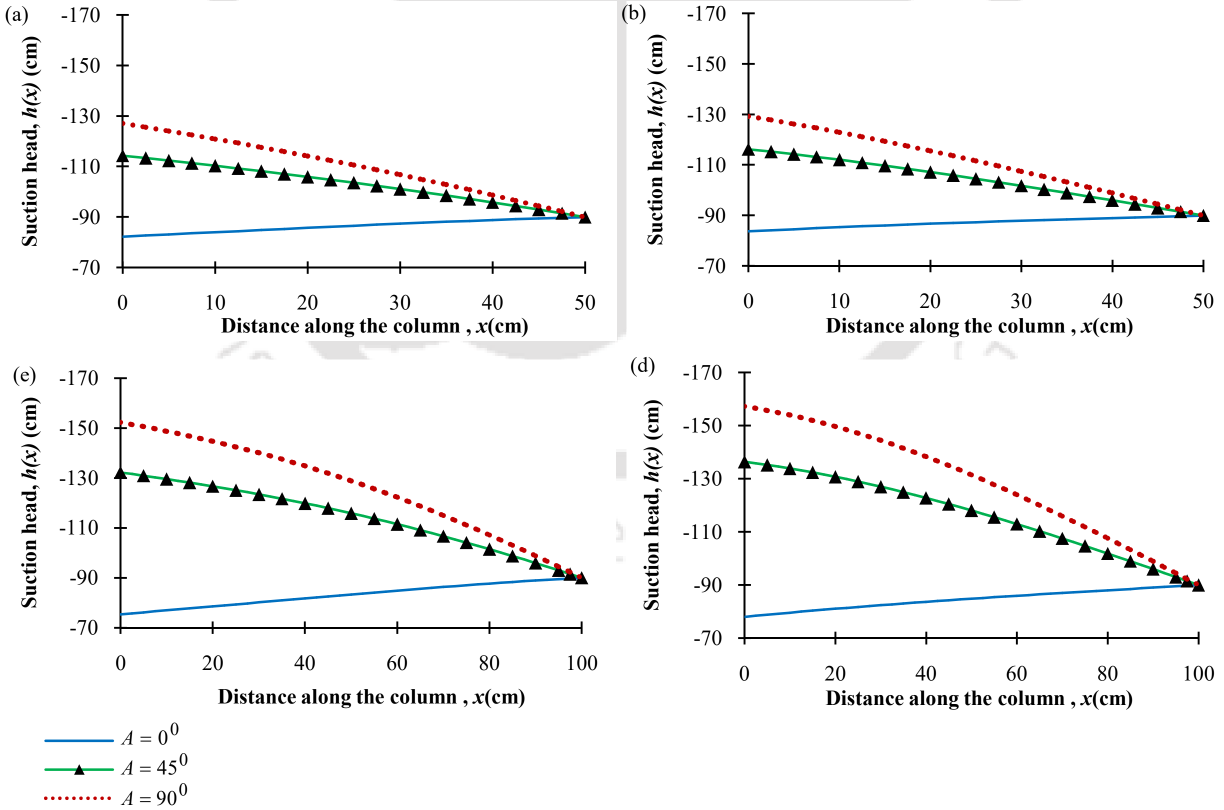

Fig. 3.17. Variation of suction head along the length of a soil profile for three values of A (namely $A = 0^0$, 45^0 and 90^0) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

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3.3.2 Suction head profiles for different q/K_S ratios in an infiltration column

From Figs. 3.18 to 3.21, we see that the suction head distribution in a one-dimensional unsaturated infiltrating soil column is highly influenced by the q/K_S ratio and the magnitude and distribution of the sink term in the column. As can be seen, for the studied horizontal flow situations here (Figs. 3.18 and 3.19), assuming other factors to remain the same, a low q/K_S ratio requires relatively less head gradient to drive water in an unsaturated horizontal soil column than for situations where this ratio is high; this is particularly true when the length of a soil column is relatively long. Also from zero-sink horizontal flow situations of Figs. 3.18(a) and (c), we see that the head variations are almost linear for these situations meaning that the hydraulic gradient specific to each of these situations at any cross-section in the flow domain is almost constant. However, when a sink term is being introduced in a horizontal infiltrating space – whether constant or varying – the suction head profile on it then has shown a propensity to bend upward (Figs. 3.18 and 3.19), particularly for low q/K_S ratio scenarios. A steeper suction head profile for $q/K_S = 0.1$ in comparison to $q/K_S = 0.01, 0.001$ profiles is understandable in the horizontal flow situations of Figs. 3.18 and 3.19 as a high q/K_S ratio here also means a high infiltration flux q (since K_s is kept constant) in an infiltration column and hence the need of a relatively high hydraulic gradient to drive this high q through the column.

It may also be observed in Figs. 3.18 to 3.20 that the introduction of a sink term alone may result in noticeable changes in the suction head profiles for both horizontal and vertical infiltration through an infiltration column particularly if the column is relatively long. For the vertical infiltration scenarios of Figs. 3.20 and 3.21, it can be seen that with the increase in q/K_S ratio, the role of gravity in driving water through an infiltration column is also increasing. This is happening because as K_s is kept constant for these situations, an increase of this ratio also means an increase in mass flow rate through an infiltration column and hence a greater role of gravity in pushing water through it.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary condition	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = L$ $h_L = -90$ cm	$A = 0^0$, $K_s = 1$ cmhr^1, $h_c = -90$ cm, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		

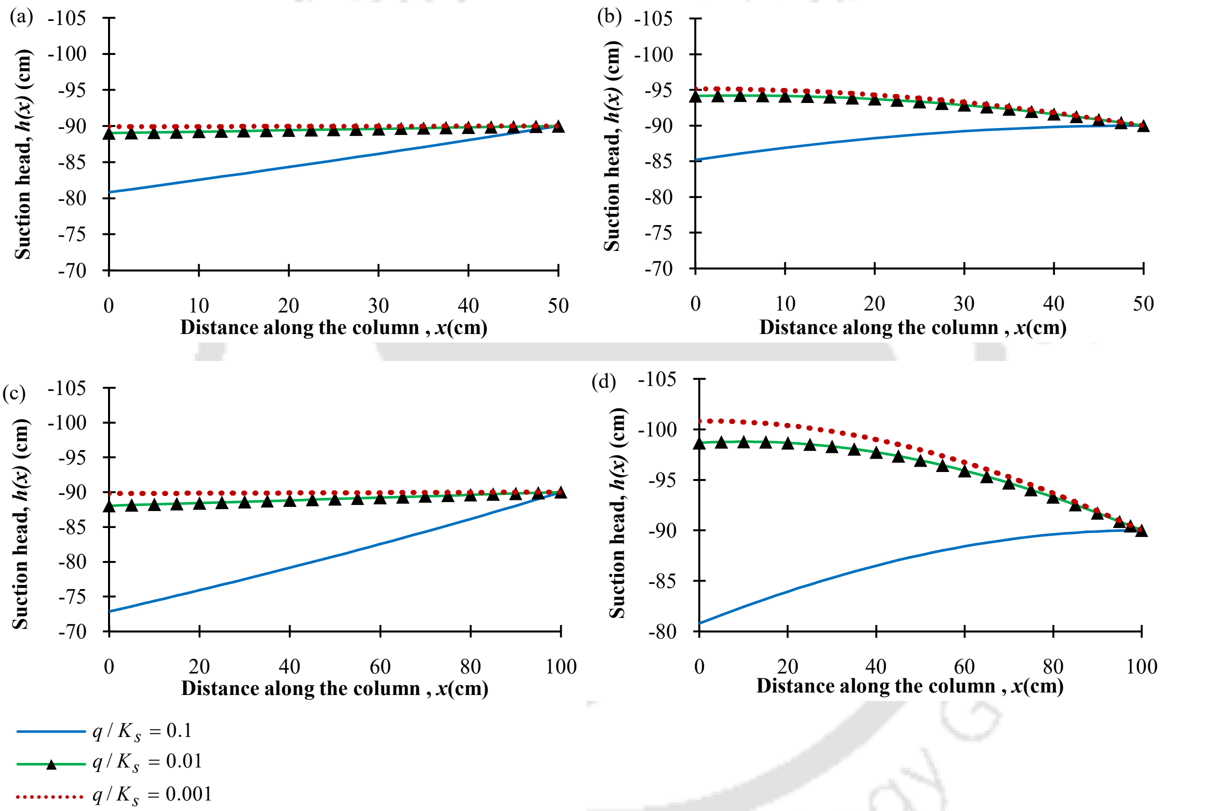

Fig. 3.18. Variation of suction head along the length of a soil profile for three values of q / K_s (namely $q / K_s = 0.1$, 0.01 and 0.001, where q is the flux at $x = 0$ and K_s is the saturated hydraulic conductivity) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

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Length of the column	Nature of root-water extraction function in [0, L]	Boundary condition	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}	At $x = L$, $h_L = -90$ cm	$A = 0^0$, $K_s = 1$ cmhr^{-1}, $h_c = -90$ cm, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		

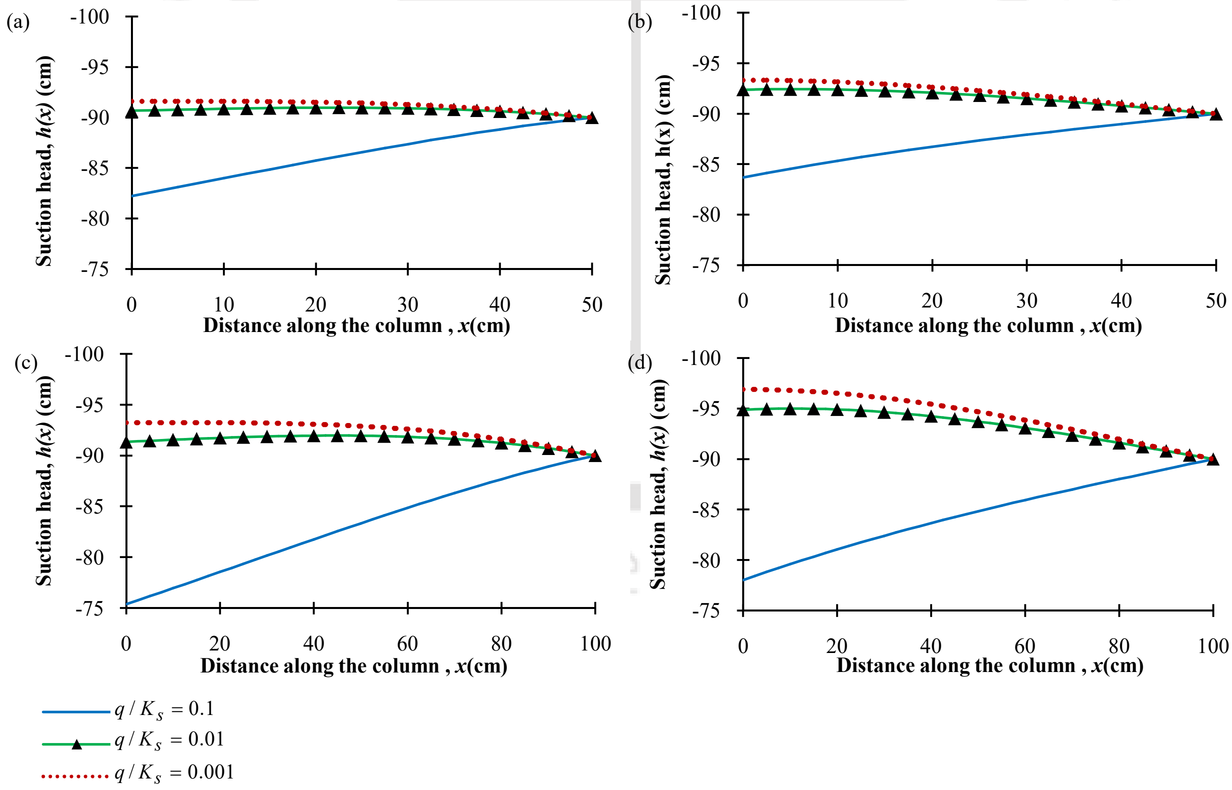

Fig. 3.19. Variation of suction head along the length of a soil profile for three values of q / K_s (namely $q / K_s = 0.1$, 0.01 and 0.001, where q is the flux at $x = 0$ and K_s is the saturated hydraulic conductivity) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary condition	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = L$, $h_L = -90$ cm	$A = 90^0$, $K_s = 1$ cmhr^{-1}, $h_c = -90$ cm, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		

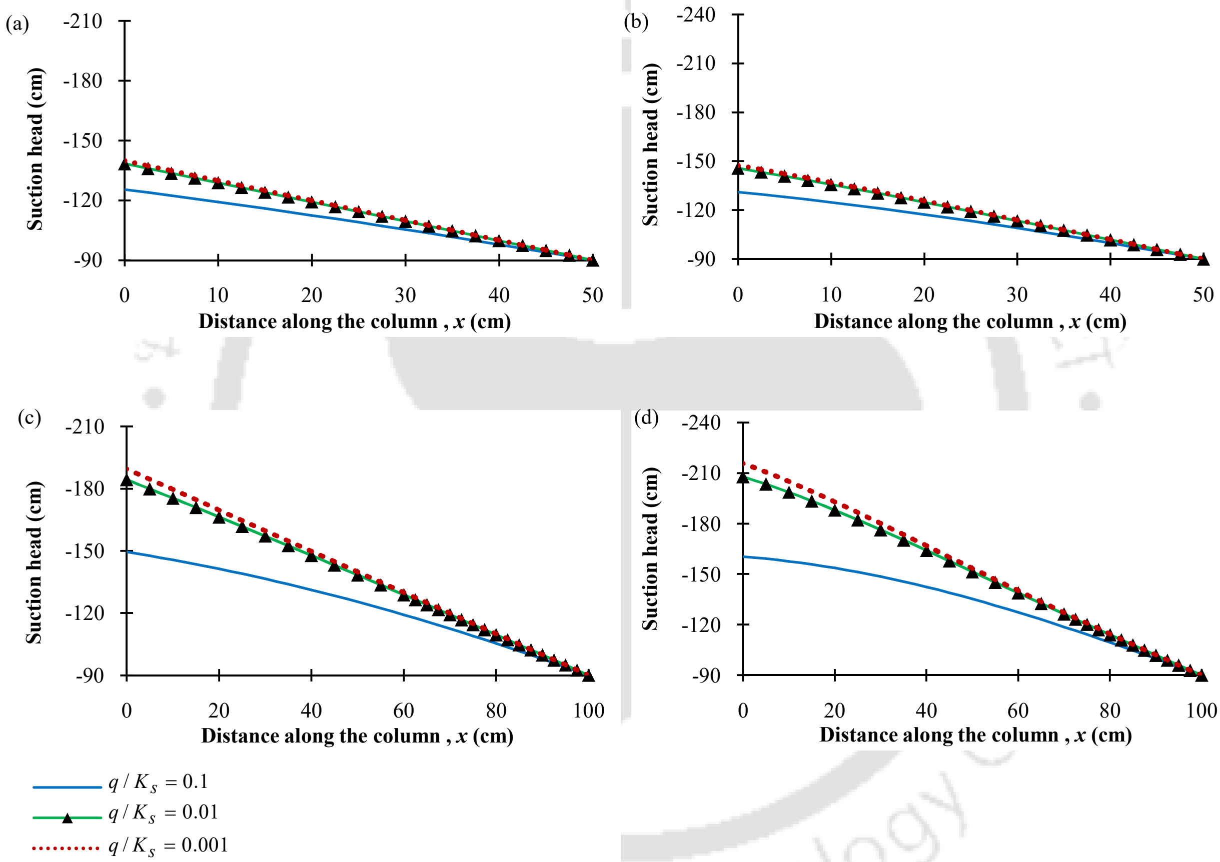

Fig. 3.20. Variation of suction head along the length of a soil profile for three values of q / K_s (namely $q / K_s = 0.1$, 0.01 and 0.001, where q is the flux at $x = 0$ and K_s is the saturated hydraulic conductivity) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j,$ where $j = 1, 2, \ldots, 19\}$.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary condition	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}	At $x = L$, $h_L = -90$ cm	$A = 90^0$, $K_s = 1$ cmhr^{-1}, $h_c = -90$ cm, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		

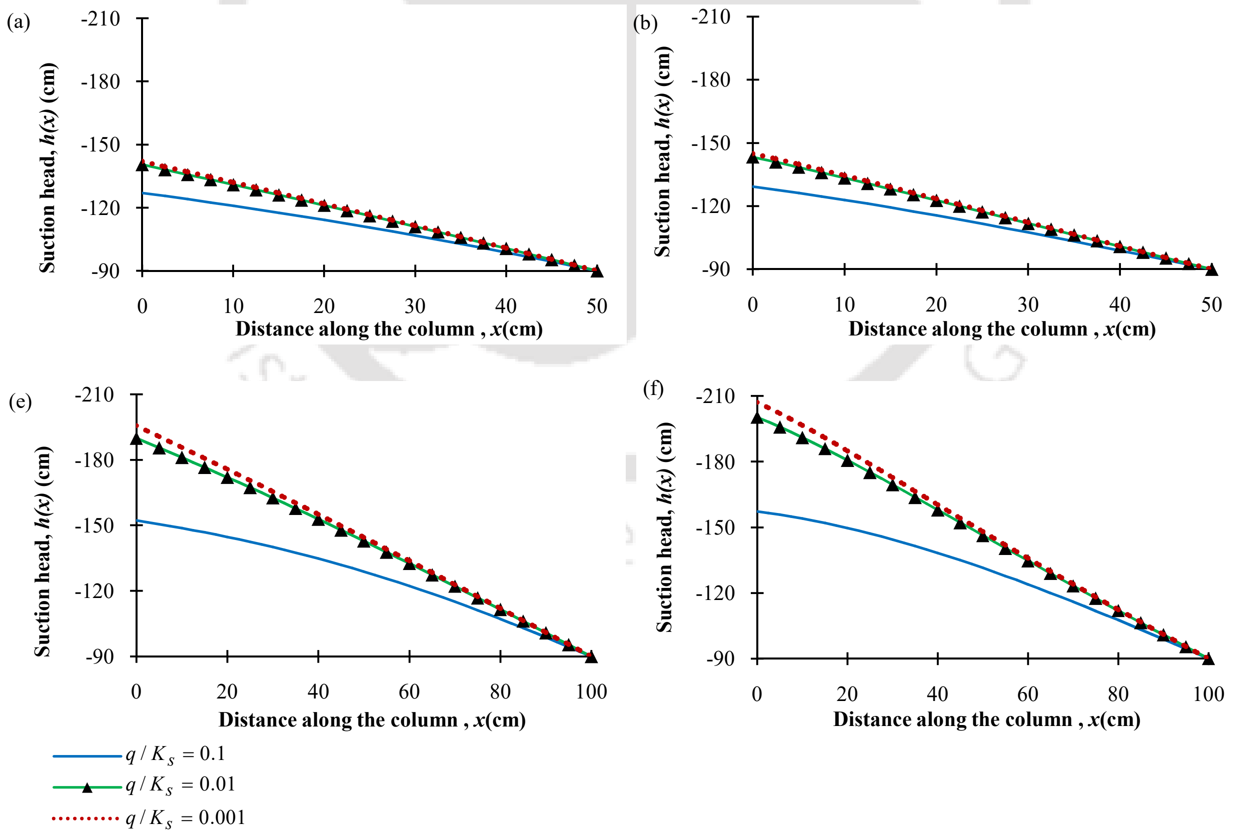

Fig. 3.21. Variation of suction head along the length of a soil profile for three values of q / K_s (namely $q / K_s = 0.1$, 0.01 and 0.001, where q is the flux at $x = 0$ and K_s is the saturated hydraulic conductivity) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

3.3.3 Suction head profiles for different values of n in an infiltration column

In Figs. 3.22 to 3.25, variations of suction head with distance have been studied for three different values of the grain size parameter n when the other parameters of the infiltration situations are as shown in these figures. As may be observed, among other factors remaining the same, these variations are quite dependent on the inclination of a soil column and for infiltration through a vertical soil, the suction head distribution on a flow space may vary appreciably with change in n values, particularly in areas close to the start of an infiltrating space. For horizontal infiltration scenarios that have been studied here (Figs. 3.22 and 3.23), suction heads are increasing in magnitude almost linearly with distance when the sink term is absent; however, with the sink terms, the suction head profiles are showing a tendency to change in an arched way with distance for all the three values of n (1.5, 3.3 and 5) considered in these examples.

For $n = 1.5$, as may be observed in Figs. 3.24 and 3.25, the suction head profiles are decreasing almost linearly with distance for all the vertical infiltration situations that have been considered here both in presence and absence of the sink term; however, for higher values of n, they can be seen to be arching upward mainly in locations close to the start of the soil column. A higher n value signifies a coarser soil and concurrently a higher conductivity but since the q/K_s ratio at $x = 0$ is kept the same in all these examples (i.e., $q/K_s = 0.1$), the discharge value at $x = 0$ is also relatively high when n is high. Thus, as may be observed in Figs. 3.24 and 3.25, with the increase in n, gravity's role on water movement in the infiltration columns of these flow situations is also increasing. It needs to be mentioned here that the head needed to push water in an infiltration column is dependent not only on the parameter n and the q/K_s ratio but on the other parameters of an infiltrating situation as well.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = 0$, $q/K_s = 0.1$; at $x = L$, $h_L = -90$ cm,	$A = 0^0$, $h_c = -90$ cm
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M/K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M/K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		

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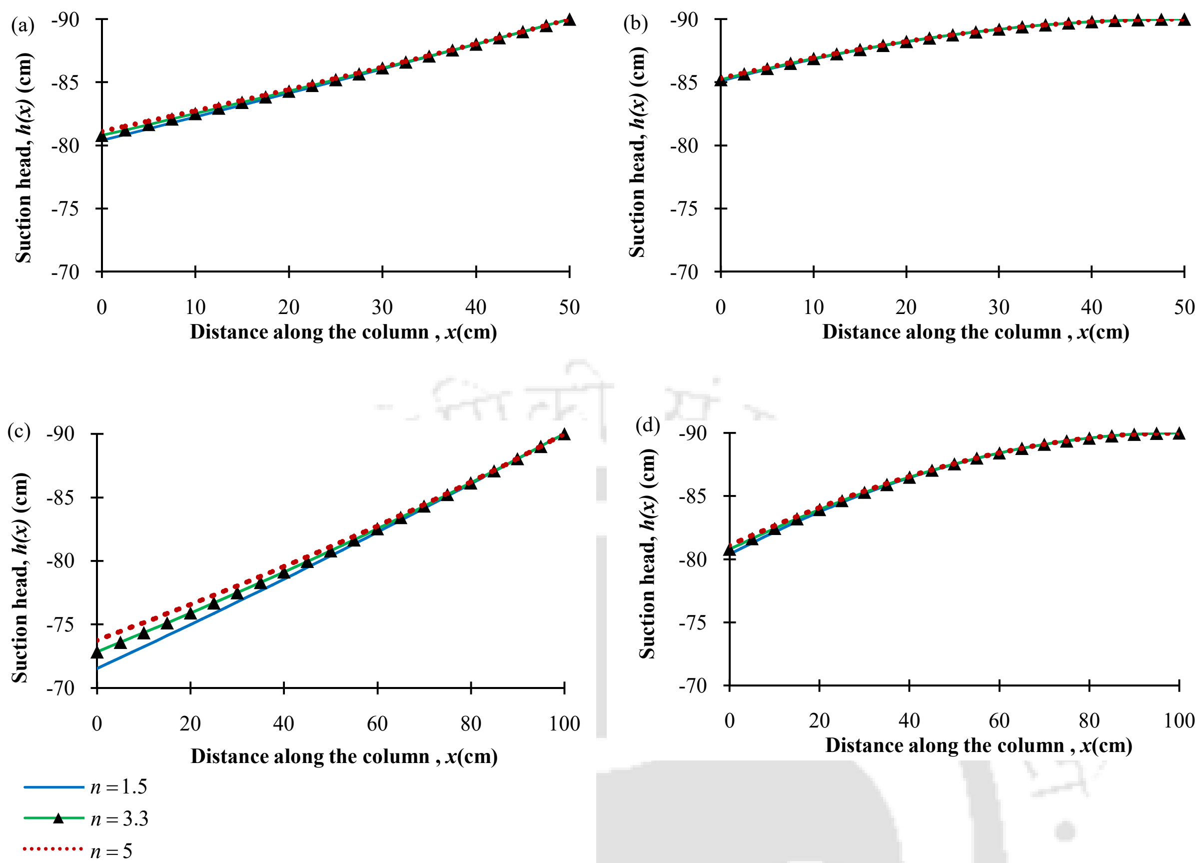

Fig. 3.22. Variation of suction head along the length of a soil profile for three values of n (namely $n = 1.5$, 3.3 and 5, where n is the grain size parameter) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

Length of the soil column	Nature of root-water extraction function in [0, *L*]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_S}{L^2} x \right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}	At $x = 0$, $q / K_s = 0.1$; at $x = L$, $h_L = -90$ cm	$A = 0^0$, $h_c = -90$ cm
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x \right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_S}{L^2} x \right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		

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(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		

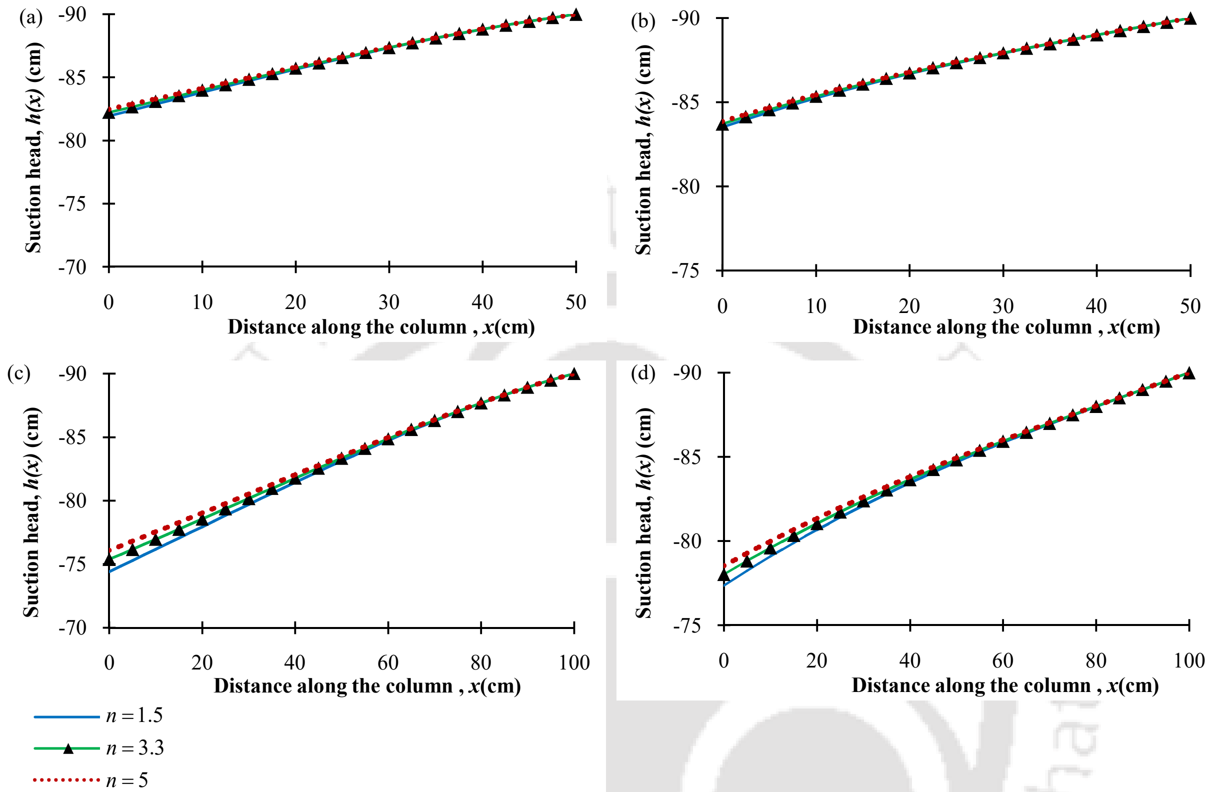

Fig. 3.23. Variation of suction head along the length of a soil profile for three values of n (namely $n = 1.5$, 3.3 and 5, where n is the grain size parameter) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = 0$, $q / K_s = 0.1$; at $x = L$, $h_L = -90$ cm	$A = 90^0$, $h_c = -90$ cm
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		

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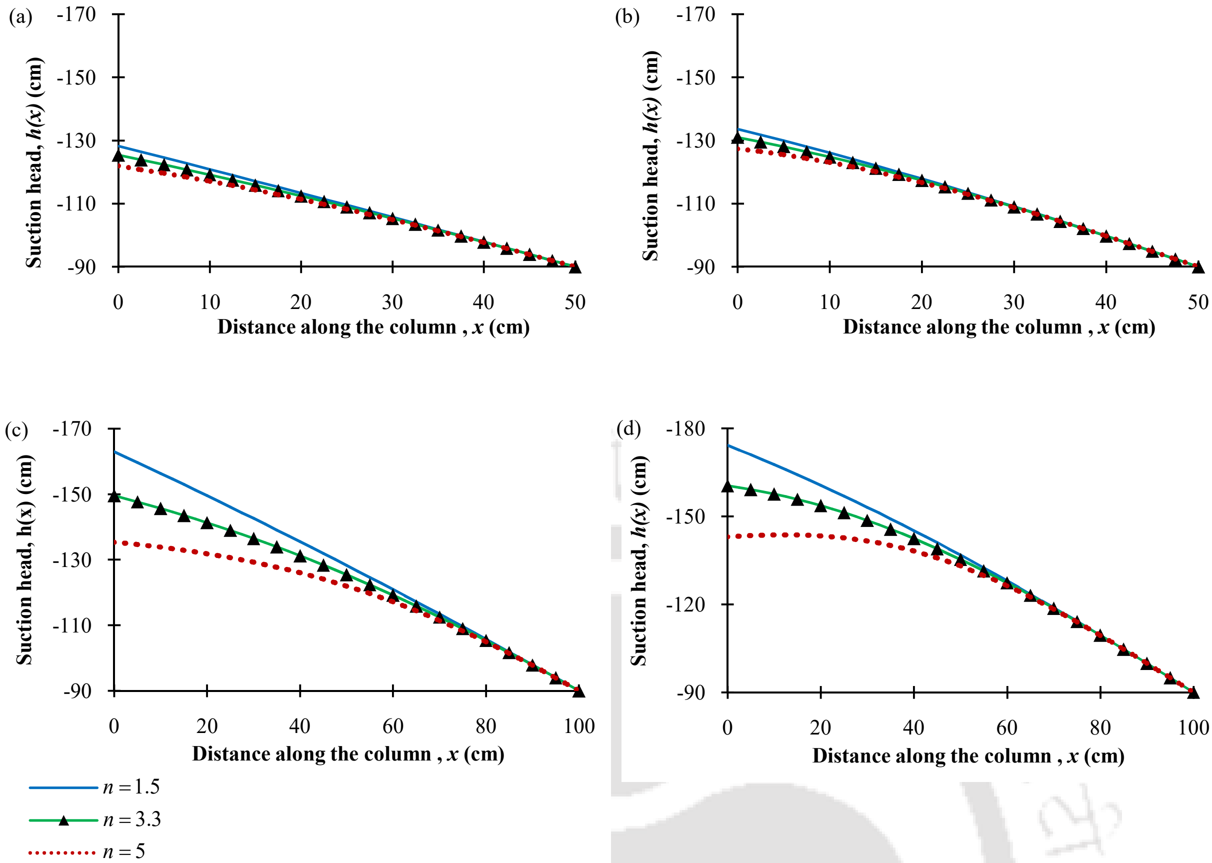

Fig. 3.24. Variation of suction head along the length of a soil profile for three values of n (namely $n=1.5$, 3.3 and 5, where n is the grain size parameter) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j$, where $j=1,2,\ldots,19\}$.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L=50$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_S}{L^2}x\right)$, where $\frac{S_M/K_s}{L}=\frac{0.1}{L}$ cm^{-1}	At $x=0$, $q/K_s=0.1$; at $x=L$, $h_L=-90$ cm	$A=90^0$, $h_c=-90$ cm
(b) $L=50$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L}-\frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L}=\frac{0.1}{L}$ cm^{-1}		
(c) $L=100$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_S}{L^2}x\right)$, where $\frac{S_M/K_s}{L}=\frac{0.1}{L}$ cm^{-1}		

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(d) $L=100$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L}-\frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L}=\frac{0.1}{L}$ cm^{-1}		

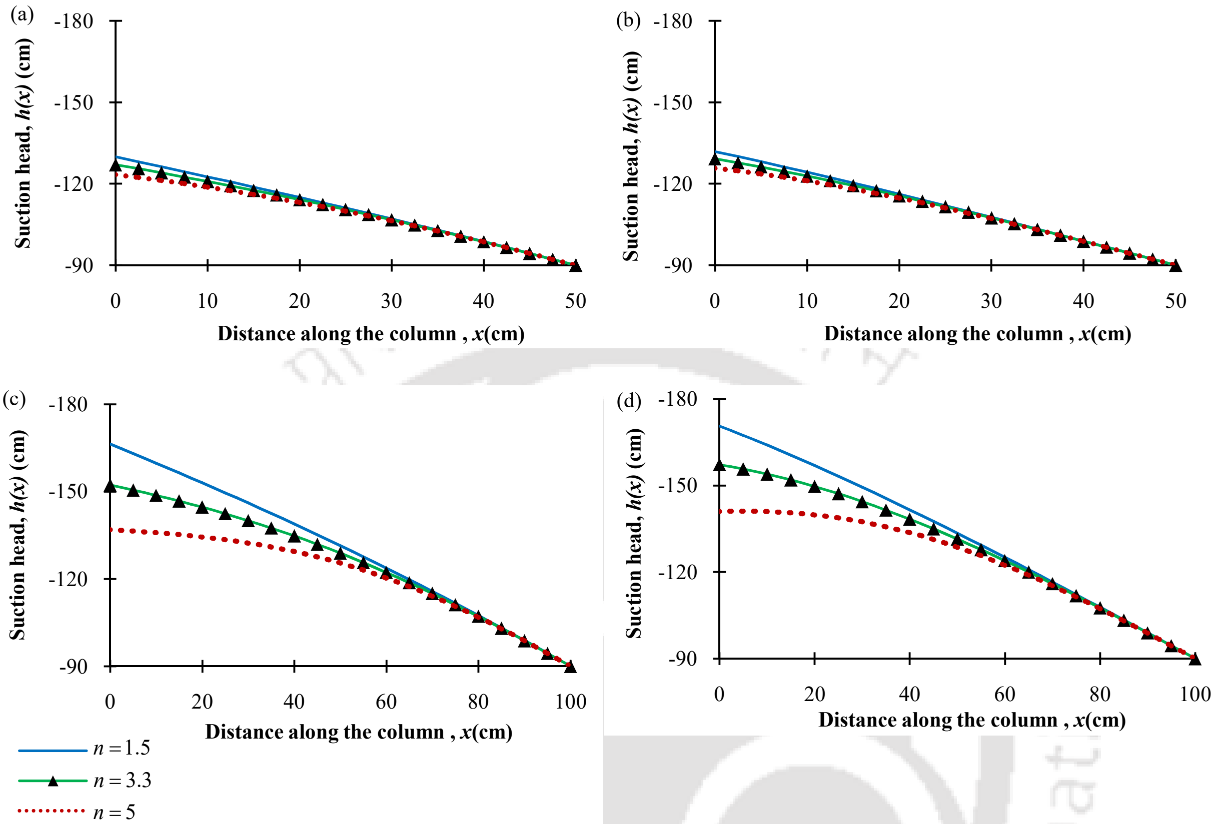

Fig. 3.25. Variation of suction head along the length of a soil profile for three values of n (namely $n=1.5$, 3.3 and 5, where n is the grain size parameter) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j$, where $j=1,2,\ldots,19\}$.

3.3.4 Suction head profiles for different values of h_c in an infiltration column

Figs. 3.26 to 3.29 show the variation of suction heads with distance for three different values of the parameter h_c when the other parameters of the flow situations are as shown in these figures. Here also, as may be observed from these profiles, a change in h_c alone may greatly affect the distribution of suction head profiles in an infiltrating soil column irrespective of whether a sink term is present in the flow space or not. It is also interesting to see from the flow situations of Figs. 3.26 and 3.27 that even though $h_c=-50$ cm signifies a relatively coarser soil than a soil with $h_c=-70$ cm or $h_c=-90$ cm, the energy gradient required in moving water in the horizontal infiltration scenarios for $h_c=-50$ cm are greater than that of the other two h_c situations. This is happening because, as the q/K_s ratio is also kept

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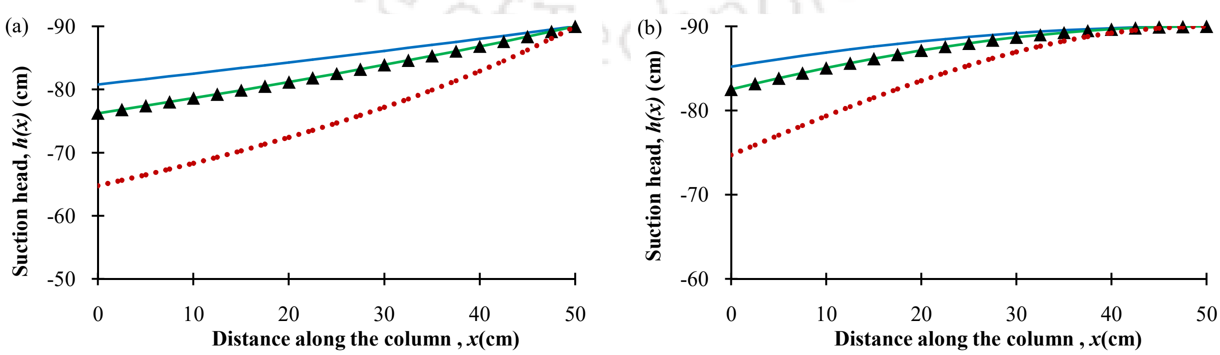
constant as 0.1 in these situations, an increase in K_s is also causing the infiltration flux q to proportionately increase in these cases. These increase in fluxes for the $h_c = -50$ cm situations in turn are requiring relatively higher energy gradients for them to be pushed through the infiltrating spaces as compared to the other two h_c cases considered in these examples.

It can also be observed from the vertical infiltration scenarios of Fis. 3.28 and 3.29 that gravity's role on water movement in these flow situations is progressively increasing with the decrease in magnitude of h_c, particularly in locations close to the start of the infiltration columns. This again is due to the fact a low (in magnitude) value of the soil parameter h_c signifies for these flow situations a relatively high mass flow rate and hence a greater influence of gravity in infiltration through such soils. It is also clear from the examples of Figs. 3.26 and 3.28 that the variation of suction head profiles for different h_cs in an infiltrating column with no sink may be profoundly different from those when a sink term is being introduced in the column; this again is true whether the column is a horizontal or an inclined one.

Length of the column	Nature of root-water extraction function in [0, *L*]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = 0$, $q / K_s = 0.1$; at $x = L$, $h_L = -90$ cm	$A = 0^0$, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		



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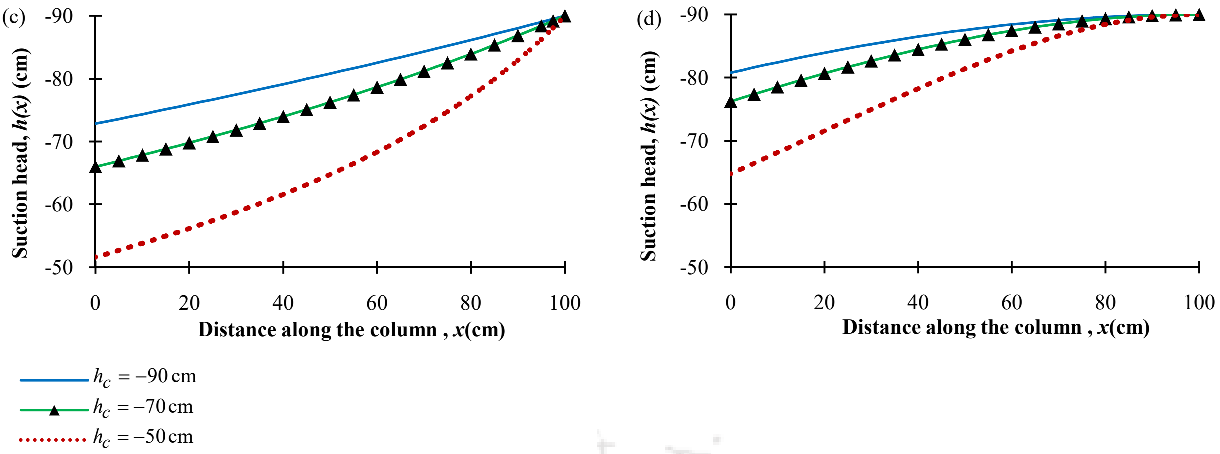
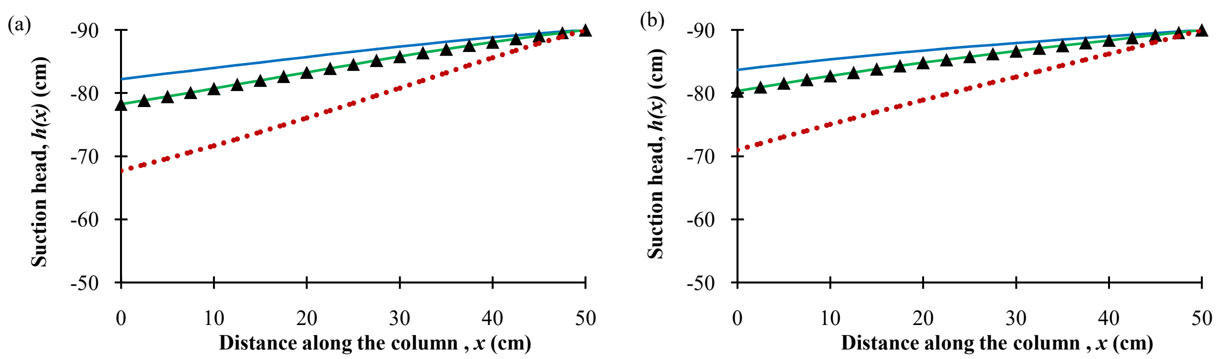


Fig. 3.26. Variation of suction head along the length of a soil profile for three values of h_c (namely $h_c = -90$ cm, -70 cm and -50 cm, where h_c is the air entry parameter) when the other parameters of the situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

Length of the soil column	Nature of root-water extraction function in [0, *L*]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}	At $x = 0$, $q / K_s = 0.1$; at $x = L$ $h_L = -90$ cm	$A = 0^0$, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		



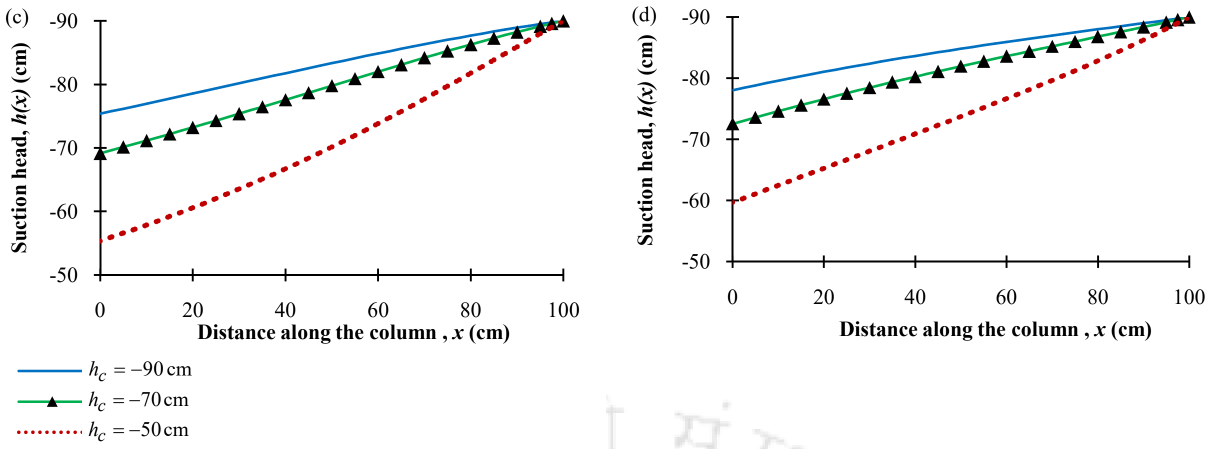
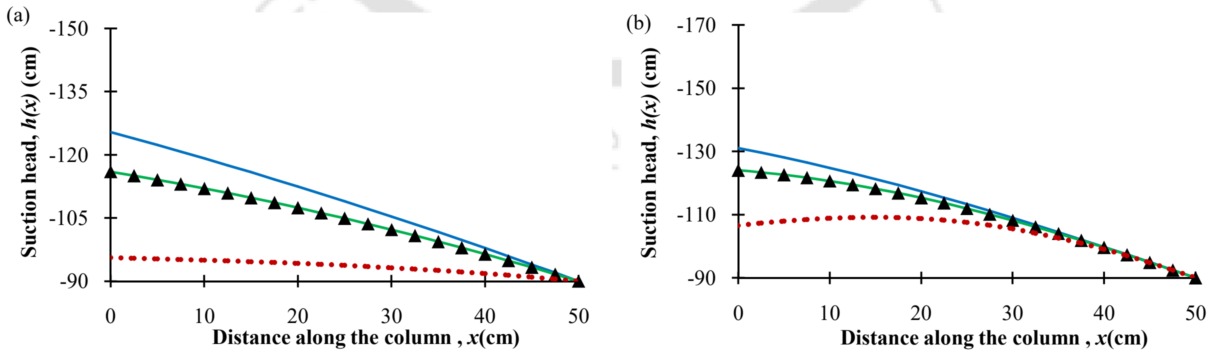


Fig. 3.27. Variation of suction head along the length of a soil profile for three values of h_c (namely $h_c = -90$ cm, -70 cm and -50 cm, where h_c is the air entry parameter) when the other parameters of the situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

Length of the column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = 0$, $q / K_s = 0.1$; at $x = L$ $h_L = -90$ cm	$A = 90^0$, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.1}{L}$ cm^{-1}		



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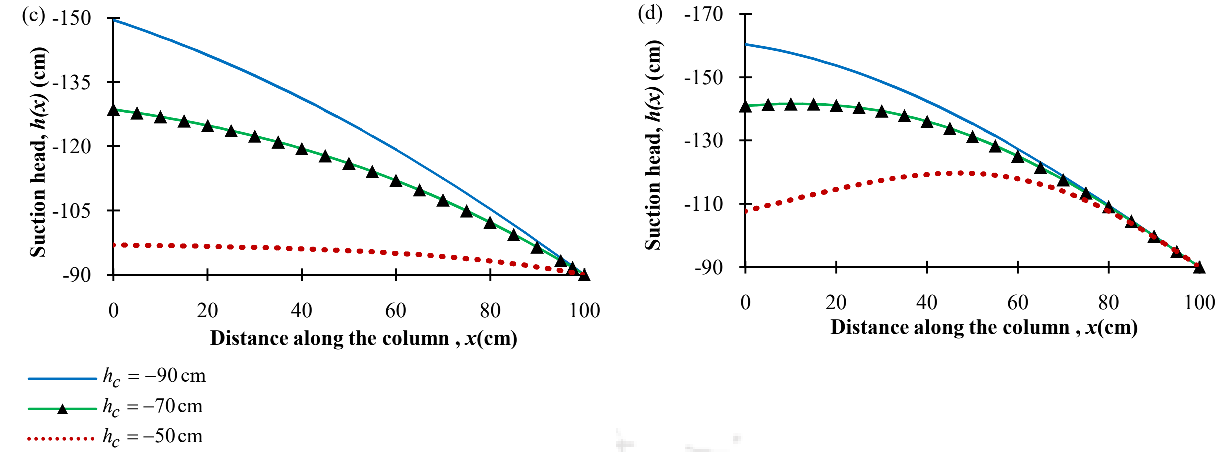
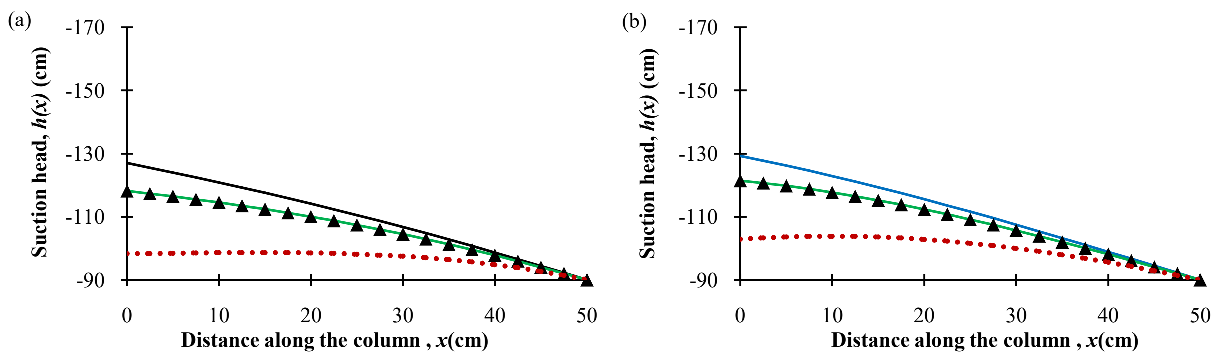


Fig. 3.28. Variation of suction head along the length of a soil profile for three values of h_c (namely $h_c = -90$ cm, -70 cm and -50 cm, where h_c is the air entry parameter) when the other parameters of the situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

Length of the soil column	Nature of root-water extraction function in [0, *L*]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}	At $x = 0$, $q / K_s = 0.1$; at $x = L$ $h_L = -90$ cm	$A = 90^0$, $n = 3.3$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.1}{L}$ cm^{-1}		



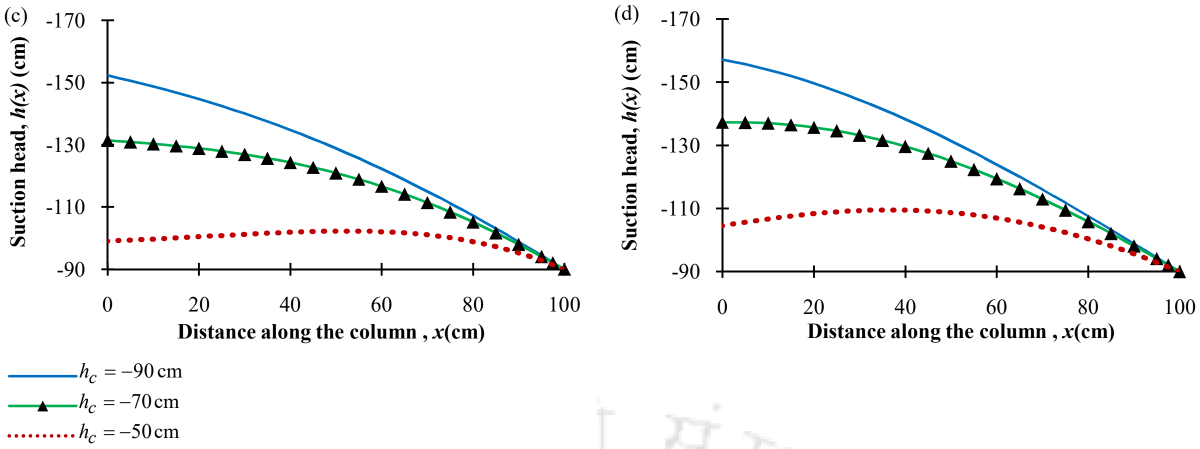


Fig. 3.29. Variation of suction head along the length of a soil profile for three values of h_c (namely $h_c = -90$ cm, -70 cm and -50 cm, where h_c is the air entry parameter) when the other parameters of the situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j,$ where $j = 1, 2, \ldots, 19\}$.

3.3.5 Suction head profiles for a few heterogeneous infiltration situations

As a final demonstration showing the general nature of our developed model, a few fully heterogeneous infiltration situations are also analysed (Fig. 3.30) by making use of this model. In all these situations, Gardner's (1958) hydraulic model is been used; however, as mentioned above, all these situations can be easily studied using Brooks and Corey's (1964) conductivity function as well by a slight adjustment of our solution. We are using the second of our approaches to obtain solutions to the infiltration problems considered in these examples; this is because we are finding this approach to be computationally less demanding for these situations as compared to our first approach. It needs to be mentioned here that the exponential conductivity variations that have been chosen in a few of these examples have also been considered by others (Philip 1972; Srivastava and Yeh 1991; Marshall et al. 2000; Barontini et al. 2007) in their infiltration studies in homogeneous and in layered soils. The constant and linear root-extraction functions considered in these examples are also on the basis of available literature (Feddes et al. 1988; Prasad 1988) on the subject. In this context, it should, however, be noted that these are illustrative examples only – the proposed model can accommodate any valid distribution of Gardner's parameters in an infiltrating space. Further, it can also accommodate any valid root-water extraction function in an infiltration column.

Figure 3.30 shows that the effect of the root-water extraction on infiltration hydraulics may be considerable. With the decrease in strength of the $S(x)$ term – among other factors

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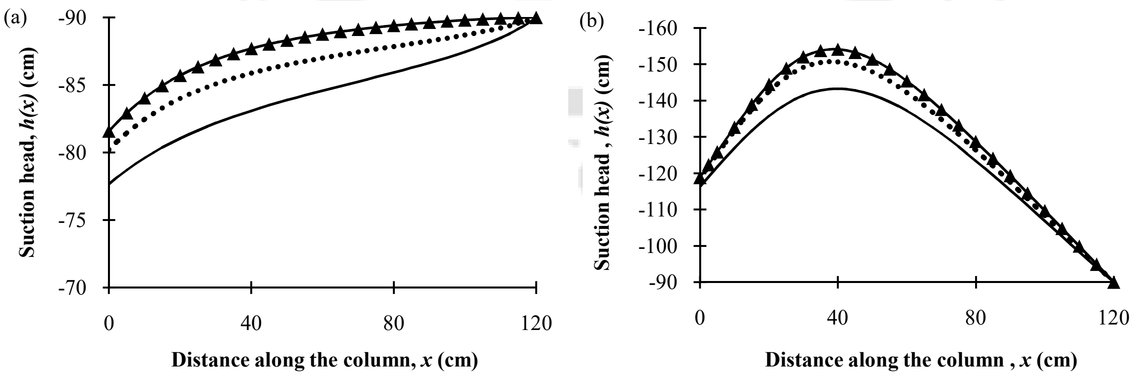
remaining the same – the suction head profiles have a tendency to shift downward. This means that, compared to a low-strength $S(x)$ (magnitude wise) infiltration situation, water particles in the voids of a soil will be held more tightly in a high-strength $S(x)$ situation, assuming of course that all other parameters of the situation are non-changing. Thus, the distribution of $S(x)$ alone in an infiltration space may impact the hydraulics associated with it in a substantial way. These flow situations also show that the nature of variation of saturated hydraulic conductivity of a soil in an infiltration column alone can have a profound impact on the infiltration dynamics in the column. From these examples, it has also become amply clear that gravity's influence on water movement in an unsaturated soil may be considerable and the hydraulics of flow in a horizontal infiltrating space may be considerably different from that of a vertical one. From Fig. 3.30, it is also clear that heterogeneity of a soil may greatly affect the infiltration hydraulics associated with it and since heterogeneity in a soil is mostly a norm than an exception, modeling infiltration in a soil with the homogeneity assumption may lead to serious error at times.

From the above, it is clear that the general Gardner-based infiltration model developed here depends on the various parameters of the model in a soil in a complex way and that the head distribution on an infiltrating space is mostly a result of intricate interplay of all these parameters of the system. Thus, the use of this model for infiltration studies requires that the parameters of the model are being first accurately determined for a studied soil as an incorrect estimation of even one of the parameters of the model may lead to significant error in the modeling results.



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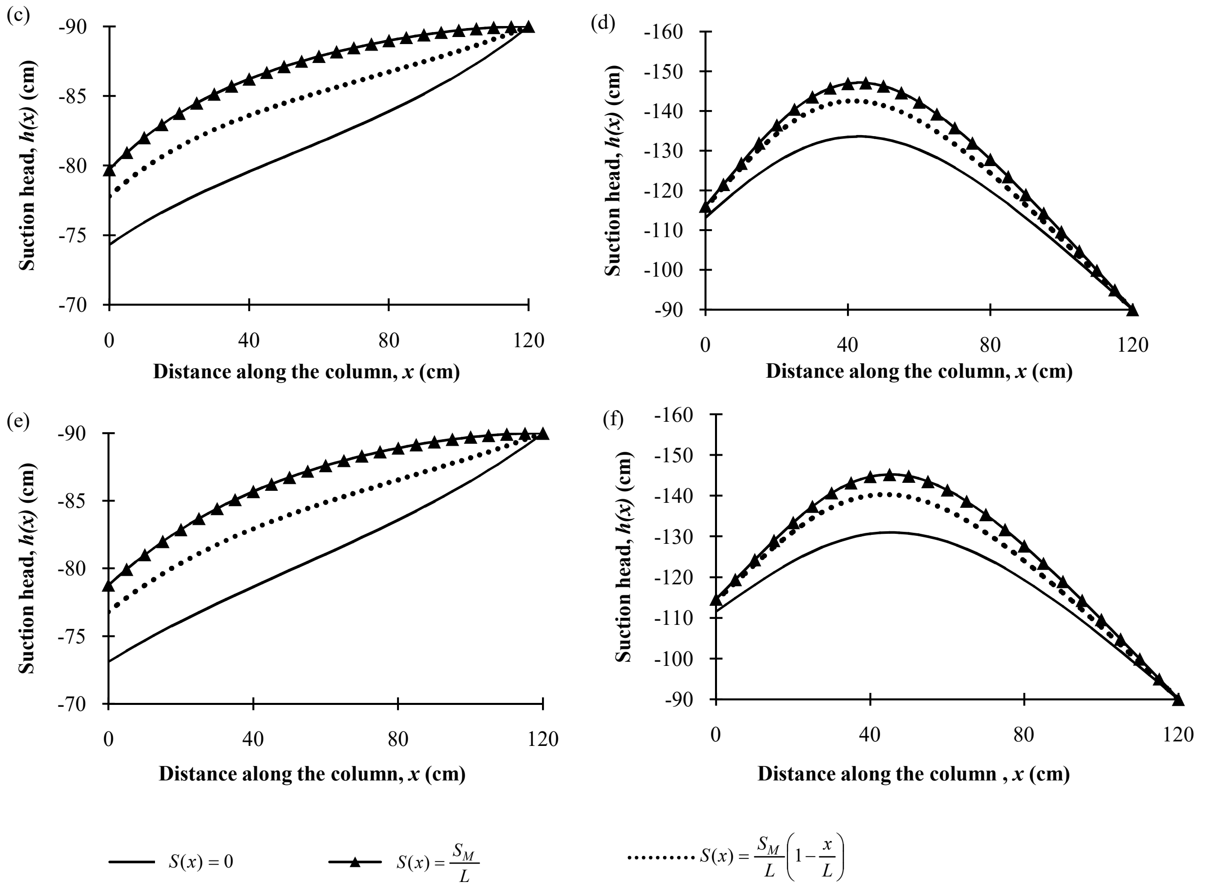

Fig. 3.30. Variation of suction head along the length of a soil profile when the parameters of the flow situations are taken as $A = 90^0$, $K_{s0} = 29.38$ cm/day, $K_{sL} = 2.938$ cm/day, $L = 100$ cm, $h_L = -90$ cm, $q = 0.2938$ cm/day, $S_M / L = 0.2938 / L$ cm/cm.day, $h_{c0} = -45$ cm, $h_{cL} = -90$ cm, $n_0 = 5.64$, $n_L = 3.3$ and (a) $K_s(x) = f_1(x)$, $h_c(x) = f_2(x)$, $n(x) = f_3(x)$ and $A = 0^0$, (b) $K_s(x) = f_1(x)$, $h_c(x) = f_2(x)$, $n(x) = f_3(x)$ and $A = 90^0$, (c) $K_s(x) = f_4(x)$, $h_c(x) = f_2(x)$, $n(x) = f_3(x)$ and $A = 0^0$, (d) $K_s(x) = f_4(x)$, $h_c(x) = f_2(x)$, $n(x) = f_3(x)$ and $A = 90^0$, (e) $K_s(x) = f_4(x)$, $h_c(x) = f_5(x)$, $n(x) = f_6(x)$ and $A = 0^0$, (f) $K_s(x) = f_4(x)$, $h_c(x) = f_5(x)$, $n(x) = f_6(x)$ and $A = 90^0$, where $f_1(x) = K_{s0} + \left(\frac{K_{sL} - K_{s0}}{L}\right)x$, $f_2(x) = h_{c0} + \left(\frac{h_{cL} - h_{c0}}{L}\right)x$, $f_3(x) = n_0 + \left(\frac{n_L - n_0}{L}\right)x$, $f_4(x) = K_{s0}\exp(\beta x)$, $f_5(x) = h_{c0}\exp(\gamma x)$, $f_6(x) = n_0\exp(\delta x)$, $\beta = \frac{1}{L}\log_e\left(\frac{K_{sL}}{K_{s0}}\right)$, $\gamma = \frac{1}{L}\log_e\left(\frac{h_{cL}}{h_{c0}}\right)$ and $\delta = \frac{1}{L}\log_e\left(\frac{n_L}{n_0}\right)$ and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j$, where $j = 1, 2 \ldots 19\}$.

3.4 Conclusions

1. An analytical solution has been obtained for the stead-state one-dimensional Gardner-based infiltration equation for studying water movement in a variably saturated heterogeneous soil by making use of a few mathematical results as proposed by Barua (2021). The infiltration problem that has been solved here is based on the general Gardner's

(1958) conductivity function but since Brooks and Corey's (1964) conductivity function is mathematically very similar to that of the Gardner's (1958) function and as such can be considered as a special case of it, the series solution proposed here can also be easily modified to accommodate Brooks and Corey's (1964) model as well. The proposed solution is new as there currently does not exist any analytical solution to the Gardner-based infiltration equation for a heterogeneous soil with our without a sink term. In fact, right-now there is no analytical solution of this equation even for a homogeneous soil for all possible variations of parameters of the problem.

2. Two approaches have been presented to solve the boundary value problem considered in the study – one in which the governing equation needs to be differentiated twice and the other where the governing equation can be directly used. The first approach requires going up to the second derivative of the governing equation (i.e., up to the $\psi''(x)$ function) to obtain a solution but in the second approach, the application of the governing equation alone is sufficient to get it. However, for achieving the same level of accuracy in a solution obtained through the first approach, the forced points needed in $[0, L]$ will be relatively less than that required using the second approach. Further, the accuracy of a solution obtained through either of the approaches depends on how many terms that are being considered in its development and a polynomial solution of any desired accuracy from either of the approaches can be obtained by sufficiently increasing the number of terms of the polynomial. The accuracy of the solutions obtained from both the approaches have been checked by comparing them with the solutions of others for a few relatively simple Gardner-based infiltration situations; very high r^2 values (mostly > 0.99) in the comparison examples show that the solutions have been correctly developed.

3. If a solution to the Gardner-based infiltration problem with a sink term exists for a set of parameters of the problem, then either of the approaches as mentioned above can be applied to work out a converging series solution to the problem. This is an important observation as numerical modeling of the Richards' equation – which generally are the de-facto models for modeling this equation – are not always assured of providing a converged solution to this equation for all possible soil properties and water content distributions in soils. It needs to be noted here that if a solution does not exist for a chosen set of parameters of the problem in an infiltrating space, then of course, solution to the problem, whether by analytical or numerical means, cannot be obtained in the considered space.

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4. The study shows that heterogeneity of a soil may greatly influence the infiltration hydraulics associated with such a system and that the application of an infiltration model being developed with the homogeneity assumption to study infiltration dynamics in a heterogeneous soil may, in many instances, lead to an erroneous reading of the studied system. The study also shows that infiltration through a heterogeneous soil depends on the spatial distributions of the soil hydraulic parameters and the sink term in a complex way and that the change of even one of these variables may alter the infiltration behavior through the soil in a significant way.

5. From the study it has also come out that, among other factors remaining the same, orientation of an infiltrating column may affect the infiltration dynamics through the column in a significant way and infiltration behavior through a horizontal column may be substantially different from that of infiltration through a vertical column. Also, gravity's influence on an infiltration column increases, as expected, with the increase in inclination of the column. This is particularly true for infiltration situations with relatively high infiltration rates in coarser soils. However, for low-rate infiltration flows in fine textured soils, matric potential distribution in an infiltration column may also play a significant role in infiltration mechanics through the column irrespective of any orientation of the column.

6. The tenacity with which water particles are being held in the soil voids of an infiltrating space may be significantly influenced by the presence and nature of distribution of the sink term in the flow space. Thus, the sink term must be properly accounted for while modeling infiltration in a cropped field where the presence of this term cannot be ruled out.

7. The proposed analytical solution obtained by either of the approaches of the Gardner-based infiltration equation is a much versatile solution of the equation as it can be used to study infiltration dynamics through any arbitrarily inclined heterogeneous soil column both when a sink term is present in the soil column and when it is absent. As the developed solution is analytical, it has also in it all the inherent advantages of analytical solutions. Further, as convergence and stability of the proposed analytical solution are always assured for all valid settings of the Gardner-based infiltration equation, it can also be used as a reference solution for verifying numerical schemes related to this equation. In addition, as soils in nature are mostly heterogeneous, the proposed infiltration model is also expected to provide a more realistic picture of water movement in the unsaturated zone of a field soil as compared to relatively simple available models on the subject.

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CHAPTER 4

ANALYTICAL MODELLING OF 1-D INFILTRATION WITH ARBITRARY ROOT-WATER UPTAKE USING VAN GENUCHTEN'S MODEL

This chapter is concerned with the development of an analytical solution to the one-dimensional steady-state infiltration equation being derived by making use of van Genuchten's (1980) conductivity function for a heterogeneous soil. The sink term of the equation is considered as any arbitrary root-water extracting function in an infiltrating space. Like in the solution of the infiltration equation with Gardner's (1958) conductivity model (Chapter 3), here also a few of Barua's (2021) mathematical results are being used to develop an analytical solution to the problem. An experimental and an analytical check on the developed model are also carried out; also, a few numerical checks on it are also performed utilizing the CHEMFLO-2000 (Nofziger and Wu 2000) numerical environment. All these checks show that the developed model has been correctly developed. As the developed model is of a very general nature, it can be easily reduced to tackle diverse infiltration scenarios by playing with the parameters of the problem as well as the sink term. Thus, it can be used for studying a wide range of infiltration situations in both homogeneous and heterogeneous soils. The salient details about the development of our solution will now be presented; further, discussions on a few applications of this solution will also be given.

4.1 Mathematical Formulation and Solution

The steady-state Richards' equation for one-dimensional heterogeneous soil has already been mentioned in Chapter 3 [Eq. (3.1)]; however, for ready reference, we are now again giving the same as under

$$\frac{\partial \theta}{\partial t} = \vec{\nabla} \bullet \left[K_V(h) \vec{\nabla} \varphi \right] - S \tag{4.1}$$

where $h(x)$ is the suction head, $K_V(h)$ is the unsaturated hydraulic conductivity [dependent on suction head $h(x)$], θ is the volumetric moisture content, φ is the total energy head, S is the root-water uptake (sink) term and $\vec{\nabla}$ is the nebula operator. For steady one-dimensional flow, Eq. (4.1) reduces to

$$\frac{d}{dx}\{K_V[h(x)][h'(x) - \sin A]\} - S(x) = 0 \tag{4.2}$$

where $K_V[h(x)]$ is the van Genuchten's (1980) soil hydraulic function for a heterogeneous soil.

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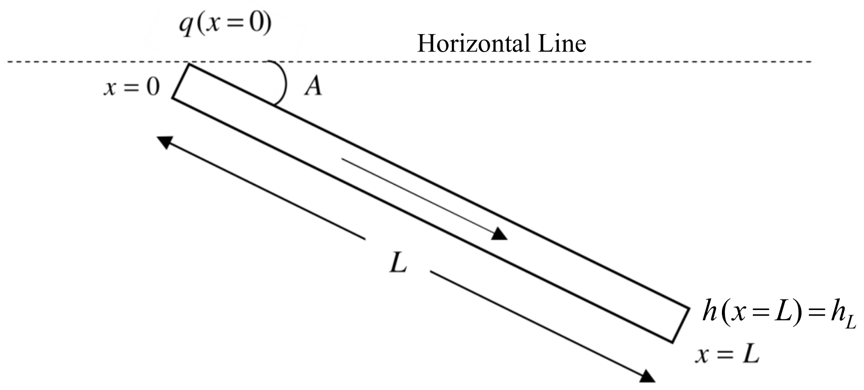

Fig. 4.1. Geometry of a one-dimensional infiltration model in an inclined soil column.

Van Genuchten provided an empirical expression for estimating the water retention function of a homogeneous soil. This function (named here as VG) for a homogeneous soil can be expressed as (van Genuchten 1980; Terleev et al. 2017)

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \left[1 + (-\alpha h)^n\right]^{-m},\ h < 0; \qquad = 1,\ h \geq 0 \tag{4.3}$$

where S_e (dimensionless) is the effective saturation, m and n are dimensionless parameters, α is a parameter having unit of L^{-1} and θ_s (dimensionless) and θ_r (dimensionless) are the saturated and residual volumetric water content of the soil, respectively. Also, as shown by Mualem (1976), suction head of a soil can be linked to its hydraulic conductivity as

$$\frac{K_V}{K_s} = (S_e)^{1/2}\left[\frac{\int_0^{S_e} \frac{dx}{h(x)}}{\int_0^1 \frac{dx}{h(x)}}\right]^2 \tag{4.4}$$

where K_s and the ratio K_V / K_s (dimensionless) are the saturated hydraulic conductivity and the relative hydraulic conductivity of the soil, respectively. From Eqs. (4.3) and (4.4), a relation for the K_V / K_s ratio can be obtained as (van Genuchten 1980; Terleev et al. 2017)

$$\frac{K_V}{K_s} = (S_e)^{1/2}\left[\frac{\int_0^{(S_e)^{1/m}} y^{m-1+1/n}(1-y)^{-1/n}\,dy}{\int_0^1 y^{m-1+1/n}(1-y)^{-1/n}\,dy}\right]^2 \tag{4.5}$$

where $y = x^{1/m} = (S_e)^{1/m} = [1 + (-\alpha h)^n]^{-1}$.

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The above integrals are not easily solvable for any arbitrary combination of the parameters m and n; however for $m = 1 - 1/n$ $(n > 1)$ as assumed by van Genucten (1980), they can be readily solved yielding an expression for K_V / K_s as (van Genuchten 1980; Terleev et al. 2017)

$$\frac{K_V}{K_s} = (S_e)^{1/2}\{1 - (-\alpha h)^{n-1}[1 + (-\alpha h)^n]^{-m}\}^2, \; h < 0;$$
$$= 1, \; h \geq 0 \tag{4.6}$$

From Eqs. (4.3) and (4.6), we see that for a heterogeneous soil, van Genuchten's (1980) conductivity function can be expressed for $h < 0$ as

$$K_V[h(x)] = K_S(x)\frac{\left\{1 - [-\alpha(x)h(x)]^{n(x)-1}\left\{1 + [-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\right\}^2}{\left\{1 + [-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{m(x)}{2}}} \tag{4.7}$$

where $K_s(x)$, $\alpha(x)$ and $n(x)$ are the spatial variations of these parameters in the soil and $m(x) = 1 - \frac{1}{n(x)}$. Substituting $K_V[h(x)]$ of Eq. (4.7) in Eq. (4.2) and then integrating the resultant expression, we get

$$\left\{K_S(x)\frac{\left\{1 - [-\alpha(x)h(x)]^{n(x)-1}\left\{1 + [-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\right\}^2}{\left\{1 + [-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{m(x)}{2}}}[h'(x) - \sin A]\right\} + q = \int_0^x S(x)\,dx \tag{4.8}$$

where $q = -K[h(x)][h'(x) - \sin A]_{x=0}$ is the infiltration flux at $x = 0$. We propose to obtain a solution of Eq. (4.8) using the boundary condition at $x = L$ as

$$h(x = L) = h_L \tag{4.9}$$

Eq. (4.9) can also be expressed as

$$K_s(x)\left\{1 - \left[-\alpha(x)h(x)\right]^{n(x)-1}\left\{1 + \left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}\right\}^2\left\{1 + \left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}}$$
$$\times\left[h'(x) - \sin A\right] + q = \int_0^x S(x)dx \tag{4.10}$$

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We will now make an attempt to solve Eq. (4.10) using the mathematical procedure as put forward by Barua (2021). Also, like in the solution of the Gardner-based (1958) based infiltration equation, we will be presenting here two approaches for the solution of Eq. (4.10) – one in which $\psi(x)$ along with its first and second derivatives will be used and the other where $\psi(x)$ alone will be used. Of course, as mentioned earlier, the solution to the problem can very well be attempted by considering still more higher derivatives of $\psi(x)$ than up to the second that has been considered in our first approach here (Barua 2021); however, as we can get a solution to the problem to any desired accuracy by using $\psi(x)$ alone or by using $\psi(x)$ and its first two derivatives only, we will be confining to these approaches only in this study. We name the $\psi(x)$ function and its derivates for this problem here with a subscript V to make them distinct from corresponding terms that are being associated with the solution of the Gardner-based infiltration equation. Now, the $\psi_V(x)$ function for this van Genuchten-based infiltration equation, in view of Eq. (4.10), can be expressed as

$$\psi_V(x) = K_s(x)\left\{1-[-\alpha(x)h(x)]^{n(x)-1}\left\{[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\right\}^2\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}$$

$$\times[h'(x)-\sin A]\} + q - \int_0^x S(x)dx = 0 \qquad (4.11)$$

As we also need the first and second derivatives of this equation for our first approach, we need to differentiate Eq. (4.11) two times. As these derivatives are somewhat long, for continuity in the text here, we are putting them in an Appendix Section (Appendix B).

Like in the previous chapter, here also we can start with the solution of the problem by assuming the suction head $h(x)$ as a polynomial of the form

$$h(x) = D_1\left(\frac{L-x}{L}\right) + h_L\left(\frac{x}{L}\right) + D_2 x(x-L) + \ldots + f(x) \qquad (4.12)$$

where D_i s are constants to be determined and $f(x) =$ $D_{3N}\{(\text{function of } x^{3N} \text{ degree})\}$ $(N \geq 1)$ or $f(x) =$ $D_N\{(\text{function of } x^N \text{ degree})\}$ $(N \geq 1)$ depending on whether $\psi(x)$ with first two of its derivatives (the first of our approaches) are being used or $\psi(x)$ alone is being used (the second our approaches) to arrive at a solution. It should be noted that for both the approaches, in order that $h(x)$ satisfies the boundary condition at $x = L,$ $f(x)$ has to be chosen such that it becomes zero at $x = L.$ As the solution approaches for the current problem are very similar to the ones that we have explained in the development of the

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solution of the Gardner-based Richards' equation, to avoid repetition, we are not giving the details of these approaches here again. Here, however, it is important to note that, instead of Gardner's (1958) conductivity function, we need to use van Genuchten's (1980) conductivity function to work out a solution of the governing equation (4.2). Thus, in our first approach here, we need to use $\psi_V(x)$, $\psi'_V(x)$ and $\psi''_V(x)$ as given by Eqs. (4.11), (B1) and (B2) to obtain a solution of our problem and in our second approach, we need to use only $\psi_V(x)$ of Eq. (4.11) to obtain a solution for the same. Further, the sink term of Eq. (4.1) can also be tackled in a similar way as has been done while obtaining solution of the Gardner-based infiltration equation. Here also, any continuous distribution of $S(x)$ on an infiltrating space is permissible; however, like in the previous case, here also only three $S(x)$ distributions, namely uniform and linearly increasing and linearly decreasing, are being considered in the analytical treatment of Eq. (4.2). As can be seen from our earlier derivations (Chapter 3), the $S(x)$ integral for these situations work out as $S(x)=\dfrac{S_M}{L}$, $S(x)=\dfrac{S_M}{L^2}x$ and $S(x)=\dfrac{S_M}{L}-\dfrac{S_M}{L^2}x$, respectively, where S_M is the volume of water being extracted by the roots in the length L of the soil column per unit area of the soil per unit time. It should be noted that the two cases of increasing and decreasing root-water distribution, total volume of water being extracted by the roots per unit cross section area over the whole domain L is $\dfrac{S_M}{2}$.

Like in the solution of the previous problem (Chapter 3), here also the constants of Eq. (4.12) specific to a problem can be worked out in steps using the $\psi_V(x)$ function by first considering only three terms of the equation and then forcing the function to be zero at any two development points – say at $x = L_1 = 0$ and L_2 – in the flow space [0, L]. It should be noted that the resultant equations obtained by equating $\psi_V(x)$ to zero at $x = L_1$ and L_2 can be solved by using a standard Newton-Raphson procedure or by some other method (Scarborough 1966). Once D_1 and D_2 are being obtained, a fourth term can next be added to the polynomial with the constants of the equation now treated as $D_1+\Delta D_1$, $D_2+\Delta D_2$ (where D_1 and D_2 are known from the previous step and ΔD_1 and ΔD_2 are the correction terms) and D_3. ΔD_1, ΔD_2 and D_3 can next be evaluated by forcing $\psi_V(x)$ zero at $x = L_1$, L_2 and at a third point L_3 in the in the flow space $[0, L]$. The procedure can be continued and the

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first N (i.e., D_1, $D_2,..., D_N$) constants of Eq. (4.12) can thus be determined using $\psi_V(x)$ at all the points of the developing set $S_D = \{L_1, L_2, L_3,..., L_N\}$. If we are using the first of our approaches, the same procedure can next be repeated with the $\psi'_V(x)$ and $\psi''_V(x)$ functions to work out the remaining constants of the x^{3N} polynomial using again the points of the developing set S_D. If, however, we are following the second of our approaches – where as mentioned before only $\psi_V(x)$ function is being used to evaluate the constants – then we can stop at the end of the first step itself. But this will generate a solution polynomial of x^N degree only. This should not be problem since a polynomial solution of any degree that we wish can still be obtained by simply increasing N to any desired value and concurrently increasing the points of the developing set S_D.

4.1.1 Methodological frame work and step by step procedure

For ease of application, mathematical procedures associated with our approaches will now also be presented in a step by step way. The steps to be followed in both the approaches are exactly similar to the ones already mentioned while obtaining solution to a Gardner-based infiltration problem. The only difference is that here we need to make use of the $\psi_V(x)$ function and its two derivatives in the first approach and the function $\psi_V(x)$ alone in the second approach, to get our solutions instead of their Gardner-based counterparts of the previous chapter. Still, for easy application of both of our approaches, we are now listing below the salient steps which may be followed in each of them for obtaining a series solution of Eq. (4.11).

Suppose a van Genuchten-based infiltration problem is given in a heterogeneous soil of domain L with the boundary conditions q and h_L and the spatially varying parameters $K_S(x)$, $\alpha(x)$, $n(x)$, $m(x)$ and $S(x)$. Suppose also that a solution of this problem exists in the domain $[0, L]$. The steps that may be followed in obtaining solution to the problem by either of our approaches can be listed as below.

<u>First Approach</u>

Step 1:

Assume a 15th degree solution polynomial of the problem [Eq. (4.12)] of the type as shown in Eq. (3.30) of the previous chapter with the last term of the polynomial as $D_{15}\left[x^8(x-L)^7\right]$. We would like to mention here that Eq. (4.12) is very similar to that of Eq.

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(3.30) except that the constants of this equation are now being represented by D_is to make them different from the C_is of the previous chapter. Of course, the assumed polynomial may be of a different type and degree than that shown by Eq. (4.12) [Eq. (4.12) is for illustration only] but whatever way we may choose this polynomial, care has to be seen that it satisfies the boundary condition at $x = L$. It may be observed that Eq. (4.12) is indeed satisfying the boundary condition at $x = L$.

Step 2:

This polynomial has fifteen unknown constants, namely D_1, D_2, …, D_{15}; so we need to generate fifteen equations to solve them. Choose five distinct points in $[0, L]$ – say L_1, L_2, … L_5 – and equate $\psi_V(x)$ of Eq. (4.11) to zero at these points. This will give us five equations.

Step 3:

Equate $\psi'_V(x)$ of Eq. (B1) to zero at these points; this will also give us five additional equations.

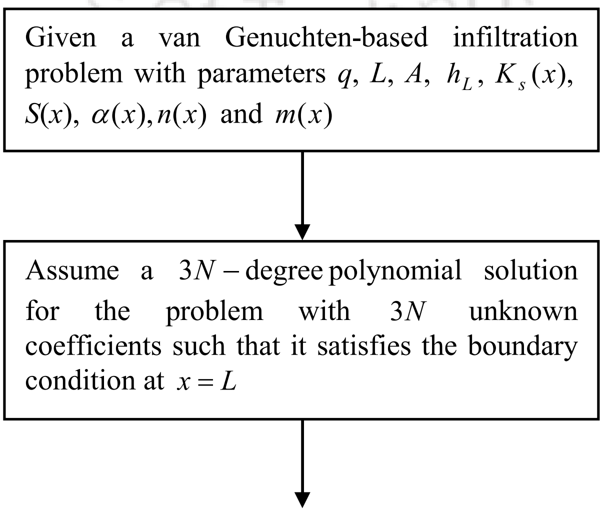
Step 4:

Finally, also equate $\psi''_V(x)$ of Eq. (B2) to zero at these points; this will give us five more equations.

Step 5:

Solve these fifteen equations by the Newton-Raphson or by some other method (Scarborough 1966) to get the constants D_1, D_2, …, D_{15}. Once we have these constants, our problem will then stand solved.

For ease of application, the various steps associated with this approach are also listed below in a flow-chart form.



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Consider N distinct points spread all over in $[0, L]$ and force $\psi(x)$, $\psi'(x)$ and $\psi''(x)$ zero at these points; this will result in $3N$ independent equations

Solve the $3N$ independent equations of the previous step by applying the Newton-Raphson or any other suitable method to get the constants of the assumed solution polynomial

Repeat the above steps with increasing values of N till $\left|h^{3(N+1)}(x) - h^{3N}(x)\right| < \varepsilon \quad \forall x \in [0, L)$, where $\varepsilon > 0$ is the accuracy desired in the solution and $h^{3N}(x)$ and $h^{3(N+1)}(x)$ are the solutions obtained by considering N and $N+1$ forced points in $[0, L]$, respectively

Fig. 4.2(i). Flow-chart for solving a van Genuchten-based infiltration problem by utilizing the first approach.

Second Approach

Step 1:

Assume a 20th degree solution polynomial of the problem of the type as shown in Eq. (4.12) [Eq. (3.30) of the previous chapter]. Of course, the assumed polynomial here may also be of a different type and degree than that shown by Eq. (4.12) and here also the chosen polynomial should be such that it satisfies the boundary condition at $x = L$.

Step 2:

The assumed polynomial has now twenty unknown constants, namely D_1, D_2, …, D_{20}; so we need to generate twenty equations to solve them. Choose twenty distinct points in $[0, L]$ – say L_1, L_2, … L_{20} – and equate $\psi_V(x)$ of Eq. (4.11) to zero at these points. This will give us twenty equations. We thus have the needed twenty equations from the $\psi(x)$ function itself and there is no need of any other function to generate any more equation.

Step 3:

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Solve these twenty equations by the Newton-Raphson or by some other method (Scarborough 1966) to get the constants D_1, D_2, ..., D_{20}. Once we have these constants, our problem will then stand solved by this approach as well.

For convenience of application, for this approach also we are presenting below the steps associated with this approach in a flow-chart form.

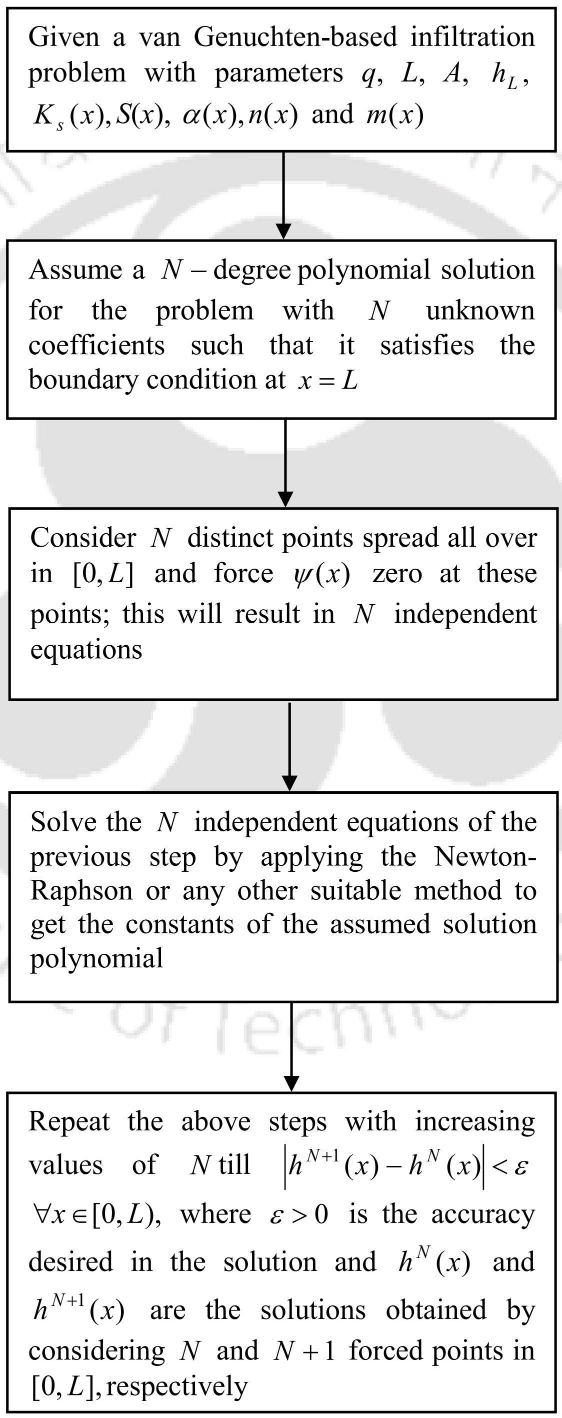

Fig.4.2(ii). Flow-chart for solving a van Genuchten-based infiltration problem by utilizing the second approach.

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Like the first approach of obtaining solution to a Gardner-based infiltration problem, to achieve the same accuracy in a solution, the first approach for obtaining solution to a van Genuchten-based problem also requires relatively fewer forced points in $[0, L]$ than that by the second approach. This method of working out a solution, however, is computationally more demanding as compared to the second method as it involves determining the first and the second derivatives of the highly nonlinear van Genuchten-based differential equation as given by Eq. (4.11) [i.e., of the $\psi_V(x)$ function]. The application of the second approach, however, involves only $\psi_V(x)$ and is thus computationally less demanding than that of the first approach in obtaining solution to the problem.

Further, as mentioned before, the accuracy of both of these approaches depends on how closely the forced points are being taken in an infiltrating space. Thus, a solution polynomial of any desired degree pertaining to a van Geunuchten infiltration problem can be found – just like that in case of a Gardner-based problem – by sufficiently reducing the largest gap in between any two adjacent forced points in $[0, L]$. Also, if we attempt to solve the van Genuchten-based infiltration problem by splitting a domain, then also the steps as mentioned above for both the approaches will still be applicable for obtaining a solution of it. This is because, as already mentioned in the previous chapter, for the type of problems that we are dealing with here, the boundary conditions (i.e., both the Neumann as well as the Dirichlet conditions) associated with any sub-domain of $[0, L]$ can always be determined.

We will now carry out a few tests to ascertain the correctness of our solution. We will first make an analytical comparison of our solution; next an experimental and a few numerical checks on it will also be performed.

4.2 Verification of Proposed Solution

By approximating the logarithmic of the hydraulic conductivity function (i.e., the $\log_e K_V[h(x)]$ function) as a series of piecewise-linear splines, Rockhold et al. (1997) provided an analytical solution to the one-dimensional steady vertical infiltration problem for a variably saturated layered soil by assuming the hydraulic properties within each layer as uniform. Thus, as mentioned before, Rockhold et al.'s (1997) did not obtain solution to the van Genuchten-based infiltration equation in a direct way but obtained their solution by first approximating the logarithmic of the van Genuchten conductivity function as a series of linear splines. Also, this solution is for a layered soil only when the soil properties within the layers are unvarying with space and not when they are changing with distance within the layers. Further, this solution also does not consider the root-water function and, as mentioned

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before, is applicable for a vertically inclined soil column only. This solution, however, as may be observed (Rockhold et al. 1997), can be treated as a special case of our solution and hence a comparison of it with our solution is still possible for relatively simplified infiltration situations. One such situation is being shown in Figs. 4.3(i) and 4.3(ii) and Table 4.2. As can be seen, this vertical infiltration scenario of about 585 cm is made up of 29 alternate layers of fine sand (Berino loamy fine sand) and clay loam soils (Gelndale silty clay loam). The details about the depth of the sub-layers of this soil column are as shown in Table 4.1. From Table 4.2, the points at which predictions from our analytical solutions are being compared with the corresponding predictions obtained from Wierenga et al.'s (1988) (as reported by Porro et al. 1993) experimental results and Rockhold et al.'s (1997) analytical solution can be seen. From Figs. 4.3(i), 4.3(ii) and Table 4.2 it is clear that suction head predictions as obtained from both of our approaches for the studied infiltration situation are matching very closely with the corresponding results as obtained from Rockhold et al.'s (1997) analytical solution thereby showing that our solutions have been correctly developed. With the experimental results also [Wierenga et al. 1988 (as reported by Porro et al. 1993)], the matching of our solution can be considered as quite satisfactory. In this context, it should be noted that the constitutive equation that is being used to convert the suction heads to moisture contents here (Mualem, 1976; Rockhold et al., 1976) is

$$\theta(h) = \theta_r + (\theta_s - \theta_r)\left[1 + (\alpha h)^n\right]^{-m} \tag{4.13}$$

where θ_r and θ_s are the residual and saturated water contents, respectively.

Table 4.1. Soil properties [Wierenga et al. 1988; (as reported by Porro et al. 1993); see also Hills et al. 1989] of the infiltration problem as considered in Figs. 4.3(i) and 4.3(ii) and the points taken in $S_D^{(1)}$ and $S_D^{(2)}$ for its solution

Soil type		Soil properties	
Berino loamy fine sand (sand)		$K_s = 6.26\times10^{-3}$ cm s^{-1}, $\alpha = 0.028$ cm^{-1}, $n = 2.239$, $\theta_r = 0.0286$ and $\theta_s = 0.3658$	
Glendale silty clay loam (loam)		$K_s = 1.52\times10^{-4}$ cm s^{-1}, $\alpha = 0.0104$ cm^{-1}, $n = 1.3954$, $\theta_r = 0.1060$ and $\theta_s = 0.4686$	
Depth of the soil layers as measured from the surface of the soil (cm)	Soil type	Thickness of soil layers (cm)	Developing points considered for computation
1-20	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4}i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19}j, \text{ where } j = 1,2,\ldots,19\right\}$
20-40	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4}i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19}j, \text{ where } j = 1,2,\ldots,19\right\}$

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40-60	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
60-80	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
80-100	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
100-120	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
120-140	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
140-160	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
160-180	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
180-200	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
200-219	sand	19	$S_D^{(1)} = \left\{0, \frac{19}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{19}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
219-240	loam	21	$S_D^{(1)} = \left\{0, \frac{21}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{21}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
240-260	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
260-280	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
280-300	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
300-319	loam	19	$S_D^{(1)} = \left\{0, \frac{19}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{19}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
319-340	sand	21	$S_D^{(1)} = \left\{0, \frac{21}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{21}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
340-360	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
360-380	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
380-400	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
400-420	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
420-440	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
440-460	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$
460-480	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1,2,\ldots,19\right\}$

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480-500	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1, 2, \ldots, 19\right\}$
500-520	loam	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1, 2, \ldots, 19\right\}$
520-540	sand	20	$S_D^{(1)} = \left\{0, \frac{20}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{20}{19} j, \text{ where } j = 1, 2, \ldots, 19\right\}$
540-558	loam	18	$S_D^{(1)} = \left\{0, \frac{18}{4} i, \text{ where } i = 1,2,3,4\right\}$; $S_D^{(2)} = \left\{0, \frac{18}{19} j, \text{ where } j = 1, 2, \ldots, 19\right\}$
558-585	sand	27	For 580-585 cm, $S_D^{(1)} = \left\{0, \frac{5}{4} i, \text{ where } i = 1,2,3,4\right\}$ and for 558-580 cm, $S_D^{(1)} = \left\{0, \frac{22}{4} i \text{ where } i = 1,2,3,4\right\}$; for 558 -585 cm, $S_D^{(2)} = \left\{0, \frac{27}{19} k \text{ where } k = 1,2,\ldots,19\right\}$

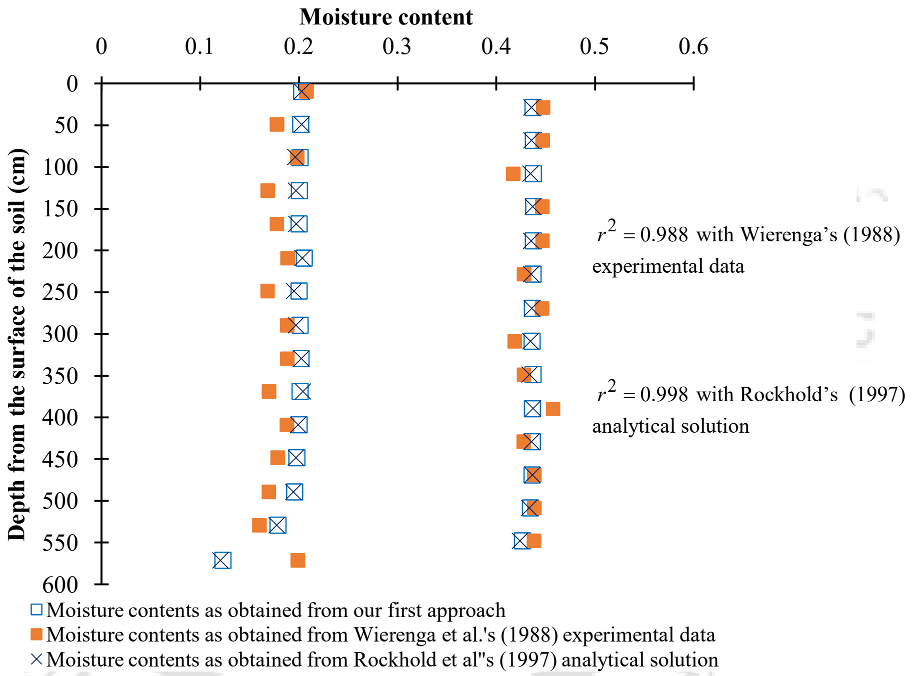

Fig. 4.3(i). Comparison of moisture contents as obtained from our first approach with the corresponding values as obtained from Wierenga et al.'s (1988) experimental data and with Rockhold et al.'s (1997) analytical solution when the soil parameters and the $S_D{}^{(1)}$ points for different layers are as shown in Table 4.1 and the other parameters of the flow situation are taken as $A = 90^0$, $S(x) = 0$ and $q = 2.38 \times 10^{-5}$ cm s^{-1}.

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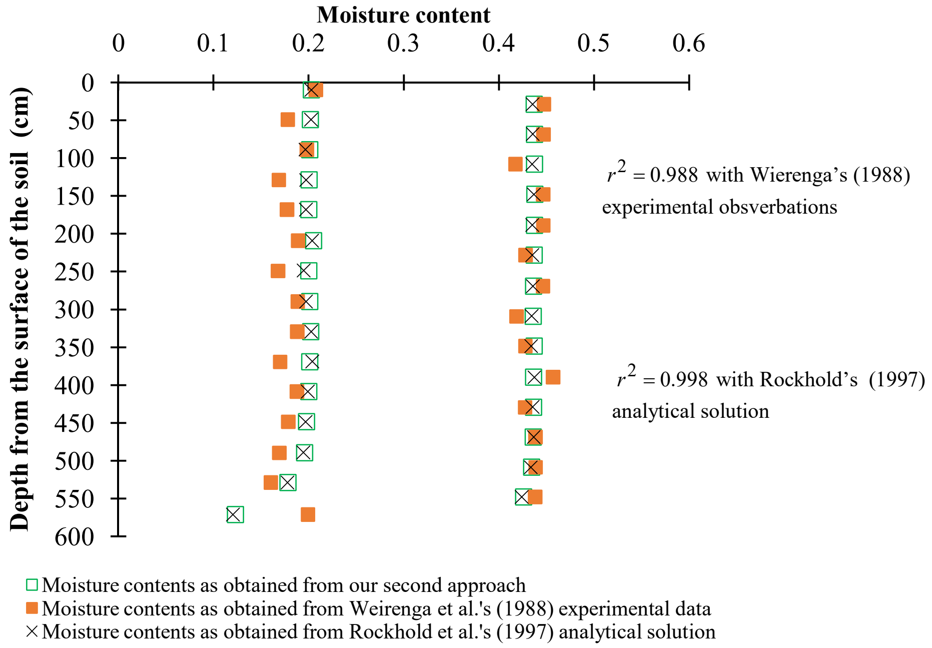

Fig. 4.3(ii). Comparison of moisture contents as obtained from our second approach with the corresponding values as obtained from Wierenga et al.'s (1988) experimental data and with Rockhold et al.'s (1997) analytical solution when the soil parameters and the $S_D^{(2)}$ points for different layers are as shown in Table 4.1 and the other parameters of the flow situation are taken as $A = 90^0$, $S(x) = 0$ and $q = 2.38\times10^{-5}$ cm s^{-1}.

Table 4.2. Comparison of moisture contents as obtained from both of our approaches with the corresponding values as obtained from Wierenga et al.'s (1988) (also see Rockhold et al. 1997) experimental data and Rockhold's (1997) analytical solution when the parameters of the flow situation are as mentioned in Figs. 4.3(i) and 4.3(ii)

Depths as measured from the surface of the soil (cm)	Moisture contents as obtained from Wierenga et al.'s (1988) experimental observation	Moisture contents as obtained from Rockhold et al.'s (1997) analytical solution	Moisture contents as obtained from our first approach	Moisture contents as obtained from our second approach
9.57	0.208	0.203	0.203	0.203
28.72	0.447	0.435	0.437	0.437
49.15	0.178	0.203	0.202	0.202
68.36	0.447	0.435	0.437	0.437
88.84	0.198	0.196	0.201	0.201
108.05	0.417	0.435	0.438	0.438
128.53	0.168	0.197	0.200	0.200
147.73	0.447	0.437	0.438	0.438
168.06	0.177	0.197	0.200	0.200
188.49	0.446	0.435	0.438	0.438
209.22	0.189	0.203	0.205	0.205
228.17	0.428	0.435	0.437	0.437
248.66	0.168	0.195	0.200	0.200
269.19	0.446	0.436	0.437	0.437
289.66	0.188	0.197	0.201	0.201
308.87	0.419	0.434	0.436	0.436
329.35	0.188	0.202	0.202	0.202

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348.56	0.428	0.434	0.437	0.437
369.02	0.170	0.204	0.201	0.201
389.57	0.457	0.437	0.437	0.437
408.72	0.188	0.199	0.200	0.200
429.26	0.427	0.436	0.437	0.437
448.15	0.179	0.197	0.198	0.198
468.68	0.439	0.436	0.436	0.436
489.17	0.169	0.194	0.196	0.196
508.37	0.438	0.433	0.434	0.434
528.91	0.160	0.177	0.179	0.178
548.08	0.438	0.424	0.426	0.426
571.30	0.198	0.122	0.123	0.122
585.00	0.068	0.068	0.068	0.068

As a further check on our solution, we next compare our results with those obtained numerically using the CHEMFLO-2000 (Nofziger and Wu 2000) numerical codes when the flow parameters of the infiltrating situations are taken as shown in Figs. 4.4(i), 4.4(ii), 4.5(i) and 4.5(ii), respectively. Here also, as can be seen in these figures and Tables 4.3 and 4.4, our predictions from both of our approaches could successfully reproduce the ones as obtained by numerical means thereby emphasizing once again the rightness of these solutions. To have a better idea about the nature of these solutions, their details are also given in an appendix section (Eqs. B1 and B2 of Appendix B).

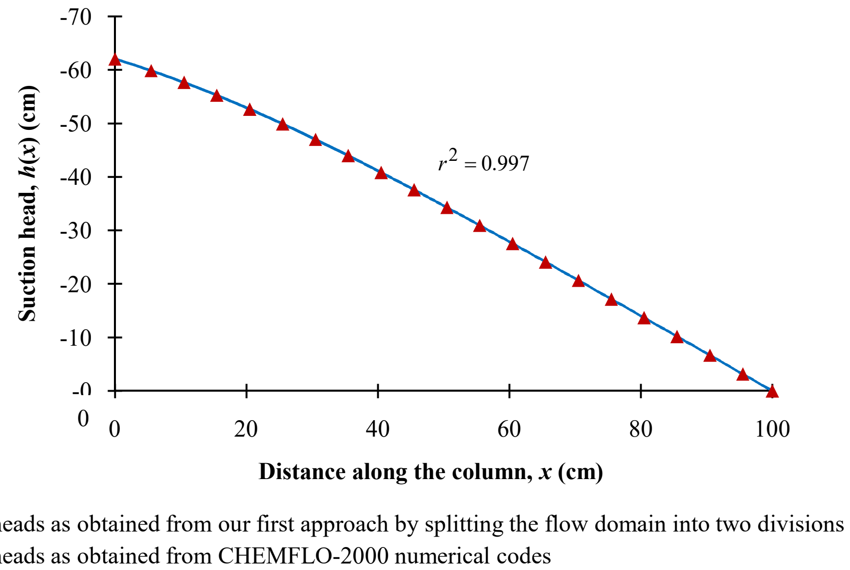

Fig. 4.4(i). Comparison of suction heads as obtained from our first approach [by splitting the flow domain into two divisions, namely 0-50 cm (with $S_{D(1)}^{(1)} = \{0, 12.5, 25, 37.5, 50\}$) and 50-100 cm (with $S_{D(2)}^{(1)} = \{0, 12.5, 25, 37.5, 49.5\}$] with the corresponding values as obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as $A = 45^0$, $S(x) = 0$, $K_s = 22.536$ cm hr^{-1}, $L = 100$ cm, $h_L = 0$ cm, $q = 0.08568$ cm hr^{-1}, $\alpha = 0.028$ cm^{-1} and $n = 2.239$.

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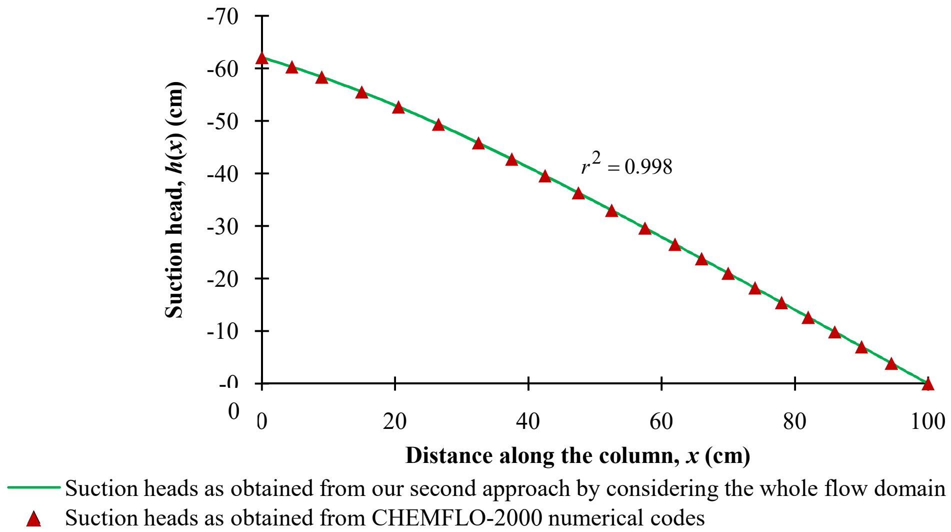

Fig. 4.4(ii). Comparison of suction heads as obtained from our second approach by considering the whole flow domain (with $S_D^{(2)}$ = {0, 5.26, 10.526, 15.789, 21.052, 26.315, 31.578, 36.842, 42.105, 47.368, 52.631, 57.894, 63.157, 68.421, 73.684, 78.947, 84.210, 89.473, 94.736, 100}) with the corresponding values as obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as $A = 45^0$, $S(x) = 0$, $K_s = 22.536$ cm hr^{-1}, $L = 100$ cm, $h_L = 0$ cm, $q = 0.08568$ cm hr^{-1}, $\alpha = 0.028$ cm^{-1} and $n = 2.239$.

Table 4.3. Comparison of suction heads as obtained from both of our approaches with the corresponding values as obtained from CHEMFLO-2000 (Nofziger and Wu 2000) numerical codes when the parameters of the flow situation are as mentioned in Figs. 4.4(i) and 4.4(ii)

Depths as measured from the surface of the soil (cm)	Suction heads as obtained from CHEMFLO-2000 numerical codes (cm)	Suction heads as obtained from $h_{(1)}^{(1)}(x)$ and $h_{(2)}^{(1)}(x)$ (cm)	Suction heads as obtained from $h^{(2)}(x)$ (cm)
0.00	-62.070	-62.073	-62.069
5.00	-60.092	-60.093	-60.091
10.00	-57.908	-57.909	-57.907
15.00	-55.524	-55.526	-55.524
20.00	-52.955	-52.953	-52.954
25.00	-50.214	-50.212	-50.214
30.00	-47.323	-47.316	-47.322
35.00	-44.301	-44.291	-44.300
40.00	-41.168	-41.158	-41.167
45.00	-37.944	-37.937	-37.944
50.00	-34.648	-34.644	-34.647
55.00	-31.293	-31.291	-31.293
60.00	-27.894	-27.892	-27.894
65.00	-24.462	-24.459	-24.461
70.00	-21.003	-21.002	-21.003
75.00	-17.526	-17.525	-17.526
80.00	-14.036	-14.036	-14.035
85.00	-10.535	-10.536	-10.535
90.00	-7.028	-7.029	-7.027
95.00	-3.515	-3.516	-3.515
100.00	0.000	0.000	0.000

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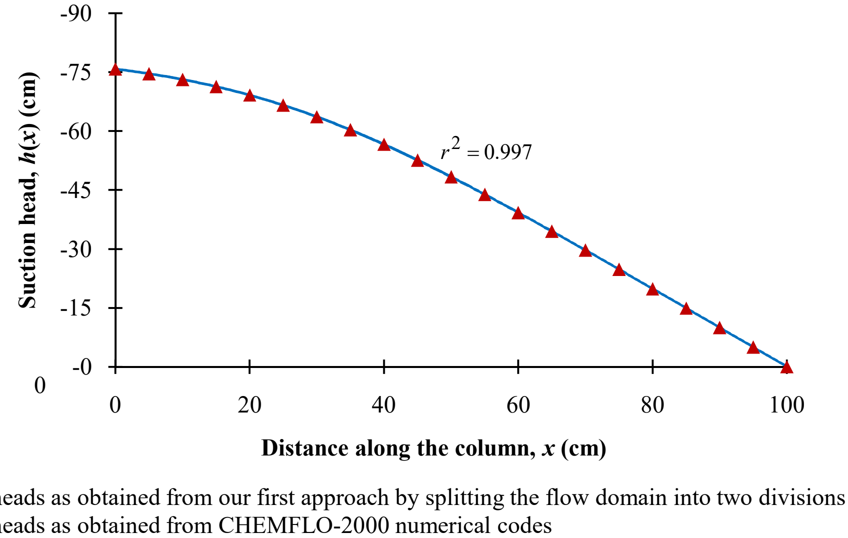

Fig. 4.5(i). Comparison of suction heads as obtained from our first approach [by splitting the flow domain into two divisions, namely 0-50 cm (with $S_{D(1)}^{(1)}$ = {0,12.5,25,37.5,50}) and 50-100 cm (with $S_{D(2)}^{(1)}$ = {0,12.5,25,37.5,49.5}] with the corresponding values as obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as $A = 90^0$, $S(x) = 0$, $K_s = 22.536$ cm hr^{-1}, $L = 100$ cm, $h_L = 0$ cm, $q = 0.08568$ cm hr^{-1}, $\alpha = 0.028$ cm^{-1} and $n = 2.239$.

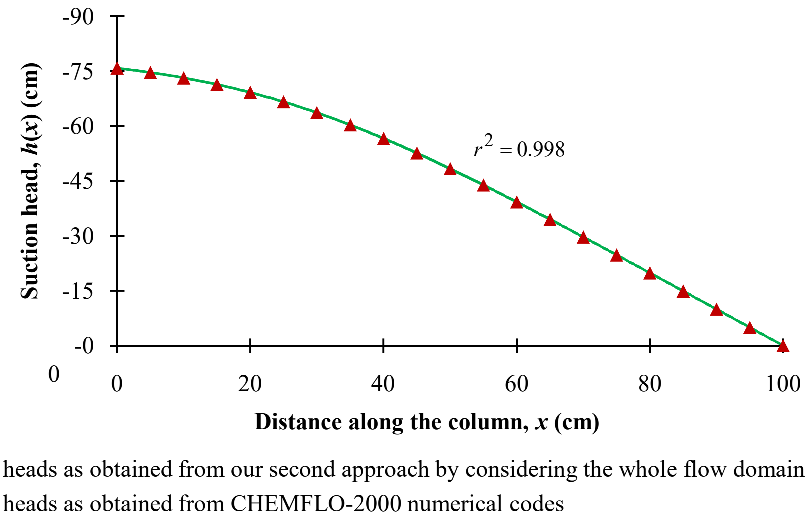

Fig. 4.5(ii). Comparison of suction heads as obtained from our second approach by considering the whole flow domain (with $S_D^{(2)}$ = {0, 5.26, 10.526, 15.789, 21.052, 26.315, 31.578, 36.842, 42.105, 47.368, 52.631, 57.894, 63.157, 68.421, 73.684, 78.947, 84.210, 89.473, 94.736, 100}) with the corresponding values as obtained from CHEMFLO-2000 numerical codes when the parameters of the flow situation are taken as $A = 90^0$, $S(x) = 0$, $K_s = 22.536$ cm hr^{-1}, $L = 100$ cm, $h_L = 0$ cm, $q = 0.08568$ cm hr^{-1}, $\alpha = 0.028$ cm^{-1} and $n = 2.239$.

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Table 4.4. Comparison of suction heads as obtained from both of our approaches with the corresponding values as obtained from CHEMFLO-2000 (Nofziger and Wu 2000) numerical codes when the parameters of the flow situation are as mentioned in Figs. 4.5(i) and 4.5(ii)

Depths as measured from the surface of the soil (cm)	Suction heads as obtained from CHEMFLO-2000 numerical codes (cm)	Suction heads as obtained from $h_{(1)}^{(1)}(x)$ and $h_{(2)}^{(1)}(x)$ (cm)	Suction heads as obtained from our second approach (cm) (cm)
0.00	-75.832	-75.829	-75.831
5.00	-74.643	-74.640	-74.643
10.00	-73.166	-73.162	-73.166
15.00	-71.359	-71.354	-71.358
20.00	-69.185	-69.180	-69.184
25.00	-66.623	-66.617	-66.622
30.00	-63.666	-63.661	-63.665
35.00	-60.326	-60.320	-60.325
40.00	-56.632	-56.626	-56.632
45.00	-52.628	-52.621	-52.627
50.00	-48.363	-48.356	-48.363
55.00	-43.890	-43.885	-43.890
60.00	-39.259	-39.254	-39.259
65.00	-34.510	-34.506	-34.510
70.00	-29.679	-29.678	-29.679
75.00	-24.792	-24.791	-24.792
80.00	-19.867	-19.870	-19.867
85.00	-14.918	-14.923	-14.918
90.00	-9.953	-9.959	-9.953
95.00	-4.979	-4.982	-4.979
100.00	0.000	0.000	0.000

We will now be studying a few infiltration scenarios using our second approach. We are using our second approach for analysing these flow situations as we are finding this approach computationally less demanding than making a mathematical study of these situations using our first approach.

4.3 Discussions

4.3.1 Suction head profiles for different orientations of an infiltration column

Here also, like in the analysis of the Gardner-based infiltration solution, we start our discussion by studying the variation of suction head profiles with inclination of an infiltration column for a few infiltration situations. Figures 4.6 and 4.7 show these infiltration scenarios. As may be observed, in case of infiltration in a van Genuchten soil [i.e., a soil with a $K(h)$ variation as given by van Genuchten's (1980) conductivity function], the suction heads are getting more negative with increase in distance when infiltration is taking place on a horizontal column. This shows that the hydraulic gradient at any location in a horizontal infiltration column with a van Genuchten soil is – as it should be – due to change in matric potential of the soil only along the length of the infiltration column. However, for infiltration

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through inclined columns, because of the presence of the gravitational head, the suction head curves can be seen to be sloping downward along the length of the soil columns. The same trends can be seen whether a sink term is there or not in an inclined infiltration column.

From Figs. 4.6 and 4.7, we see that suction head profiles in an infiltration column may be widely influenced by the gravitational head associated with it, mainly in locations close to the start of the column. An inclined infiltration column will have a varying gravitational field connected to it (from 0 to $-L$ for a vertical soil and from 0 to $-L/\sqrt{2}$ for a 45^0 inclined soil and so on) and the distribution of this field in the column may play an important part in the infiltration dynamics through it. Infiltration through a horizontal soil column, however, as already mentioned, is not impacted by gravity as the gravitational head attached to every point of the flow domain for such a situation would be the same. From the studied inclined flow situations of Figs. 4.5 and 4.6, it is also apparent that the hydraulic gradient needed to drive water in the infiltration columns of these situations is being provided mostly by gravity in locations adjacent to the start of the columns with the gravity effect, understandably, decreasing with the decrease in inclination of the soil columns. From these flow situations, it is also clear that suction head profiles are also sensitive to the nature of distribution of the sink term in an infiltrating space and with the increase in strength of the sink term, the water particles are also getting more tightly bound in the voids of soil of an infiltration column. Thus, like in infiltration through a Gardner soil (i.e., a soil respecting the general Gardner's (1958) conductivity function), considering other factors as same, more work is needed to be done by the roots of plants to extract water from a vertically infiltrated van Genuchten soil than that when infiltration is taking place through the soil on a horizontal plane. Thus, like in a Gardner's soil, the ease of water availability to plants from infiltrating water in a van Genuchten soil also may depend, to a large extent, on the direction infiltration through the soil.

Length of the soil column	Nature of root-water extraction function in [0, *L*]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = 0$ $q/K_s = 0.01$; at $x = L$ $h_L = -45$ cm	$\alpha = 0.02\text{cm}^{-1}$, $n = 1.41$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M/K_s}{L}\right) = \frac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M/K_s}{L}\right) = \frac{0.01}{L}$ cm^{-1}		

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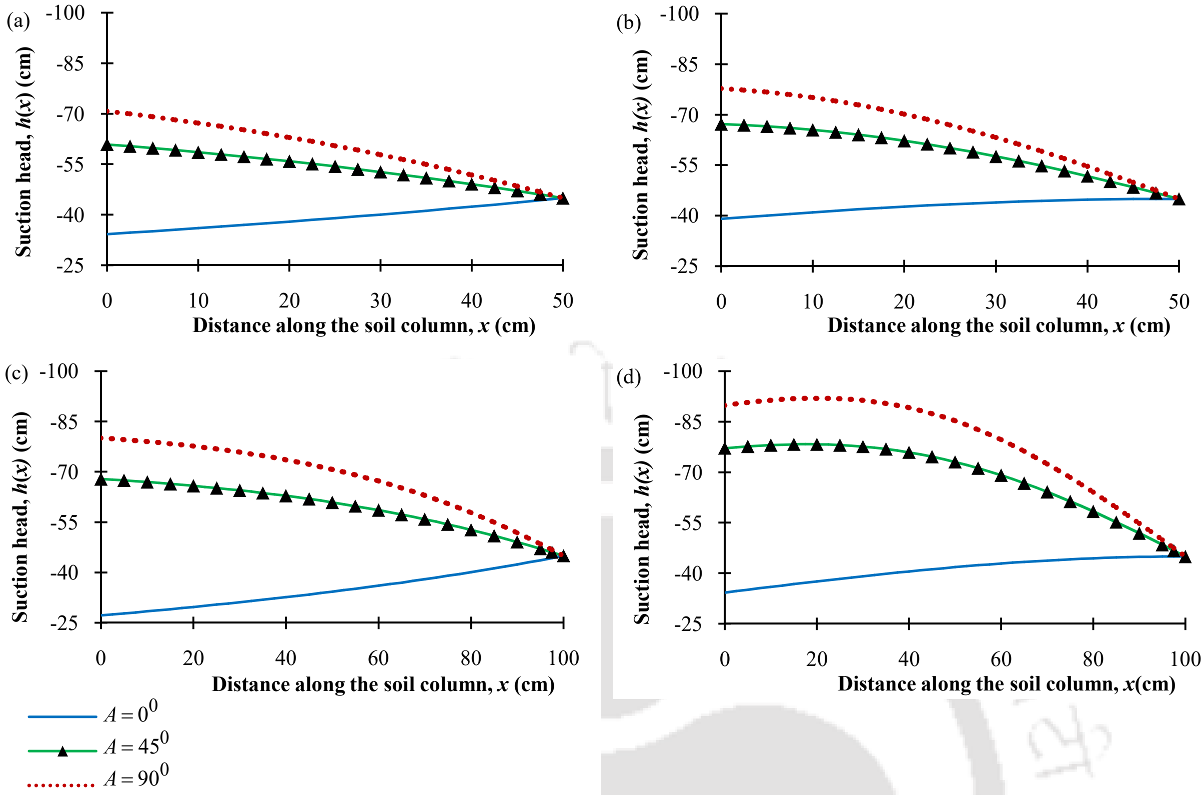

Fig. 4.6. Variation of suction head along the length of an infiltration column for three values of A (namely $A=0^0, 45^0$ and 90^0) when the other parameters of the flow situations are taken as above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j$, where $j=1,2,\ldots,19\}$.

Length of the soil column	Nature of root-water extraction function in [0, *L*]	Boundary conditions	Parameters
(a) $L=50$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L}=\frac{0.01}{L}$ cm^{-1}	At $x=0$ $q/K_s=0.01$; at $x=L$ $h_L=-45$ cm	$\alpha=0.02\text{cm}^{-1}$, $n=1.41$
(b) $L=50$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L}-\frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L}=\frac{0.01}{L}$ cm^{-1}		
(c) $L=100$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L}=\frac{0.01}{L}$ cm^{-1}		
(d) $L=100$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L}-\frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L}=\frac{0.01}{L}$ cm^{-1}		

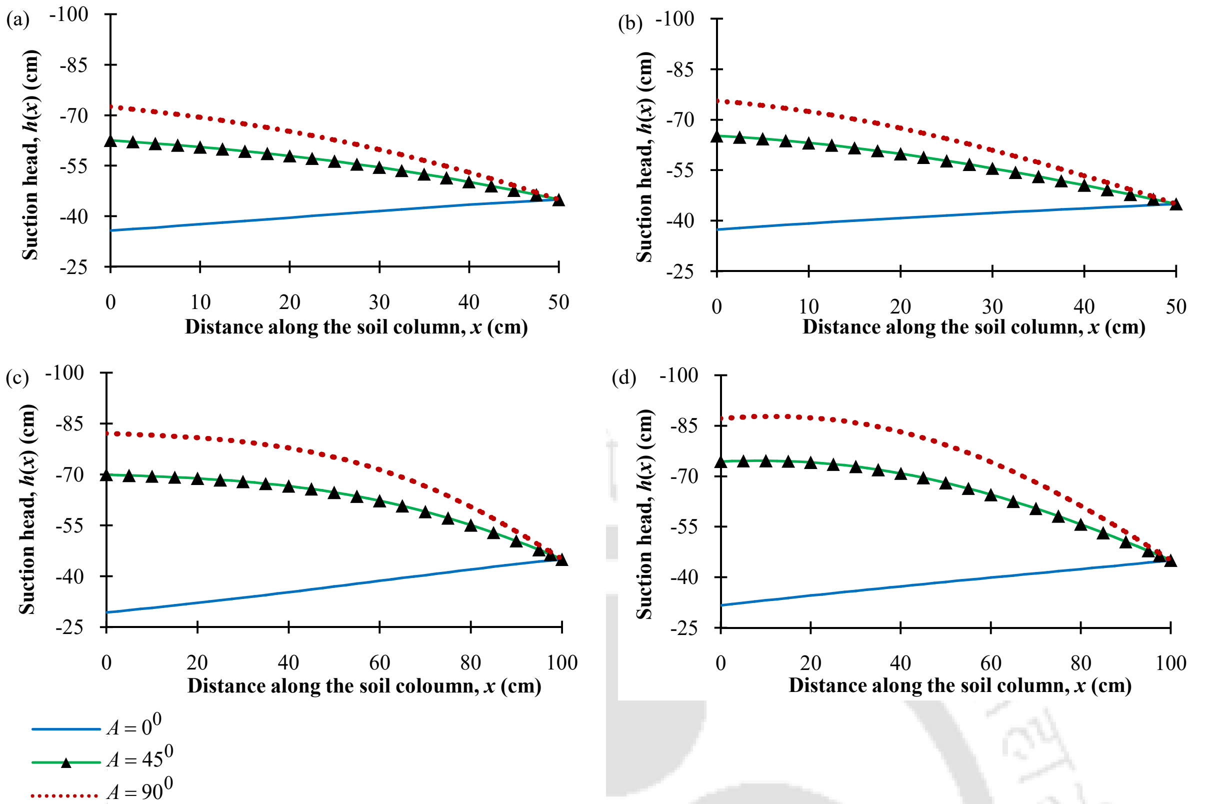

Fig. 4.7. Variation of suction head along the length of an infiltration column for three values of A (namely $A = 0^0, 45^0$ and 90^0) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j, \text{ where } j = 1, 2, \ldots, 19\}$.

4.3.2 Suction head profiles for different q/K_s ratios in an infiltration column

From Figs. 4.8 – 4.11, it is also clear that the suction head profiles along the length of an infiltration column are highly influenced by the q/K_s ratio and the angle of inclination of the column, both when a sink term is present in the column and when it is absent. From the horizontal infiltration scenarios of Figs. 4.8 and 4.9 it may be observed that, for large q/K_s ratios, the hydraulic gradient required to move water in an infiltration column with a van Genuchten soil may be quite high, particularly in locations close to the end of the column. This is understandable as a higher q/K_s ratio – assuming all other parameters as non-changing – means relatively a higher infiltration flux here (since K_S is kept fixed here) in an infiltrating space and hence the need of a higher energy gradient to push this flux through it. For infiltration in an inclined column, however, gravity will come into picture and the infiltration dynamics through the column (Figs. 4.10 and 4.11) can then get drastically changed from that of horizontal infiltration through the column. This is particularly noticeable when a root-water function is present in the column.

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It may also be observed from Figs. 4.10 and 4.11 that, as compared to the $q/K_S = 0.001$ situation, gravity's influence on water movement in the infiltration columns of the $q/K_S = 0.05$ and 0.01 situations is relatively quite high for virtually the entire length of flow of these situations. This again can be attributed to the fact that a higher q/K_S ratio means a relatively higher mass flow rate (i.e., infiltration flux) of water (since K_S is fixed here) through an infiltrating space and consequently a higher role of gravity on moisture movement through the flow space. Even for $q/K_S = 0.001$ situations for which the infiltration flux is relatively quite low, gravity's part on water movement through the infiltrating spaces of the concerned situations, as may be seen, can still be significant, particularly in locations close to the start of the flow spaces. The flow situations of Figs. 4.8 – 4.11 also show that the presence of a root-extraction function in a van Genuchten soil may alter the infiltration hydraulics in the soil in a marked way when compared to zero-sink infiltration situations in the soil; further, the nature of the root-water extraction function itself may have a profound affect on the infiltration mechanics in the soil. Thus, if modelling of infiltration is being attempted in an agricultural field, care should be taken to see that this variable is being suitably included in the model as a cropped field is most likely to be associated with a root-water uptake function of some nature.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\dfrac{S(x)}{K_s} = 0$	At $x = L$ $h_L = -45$ cm	$A = 0^0$, $K_s = 1\,\text{cm hr}^{-1}$, $\alpha = 0.02\,\text{cm}^{-1}$, $n = 1.41$
(b) $L = 50$ cm	$\dfrac{S(x)}{K_s} = \left(\dfrac{S_M / K_s}{L}\right) = \dfrac{0.01}{L}\ \text{cm}^{-1}$		
(c) $L = 100$ cm	$\dfrac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\dfrac{S(x)}{K_s} = \left(\dfrac{S_M / K_s}{L}\right) = \dfrac{0.01}{L}\ \text{cm}^{-1}$		

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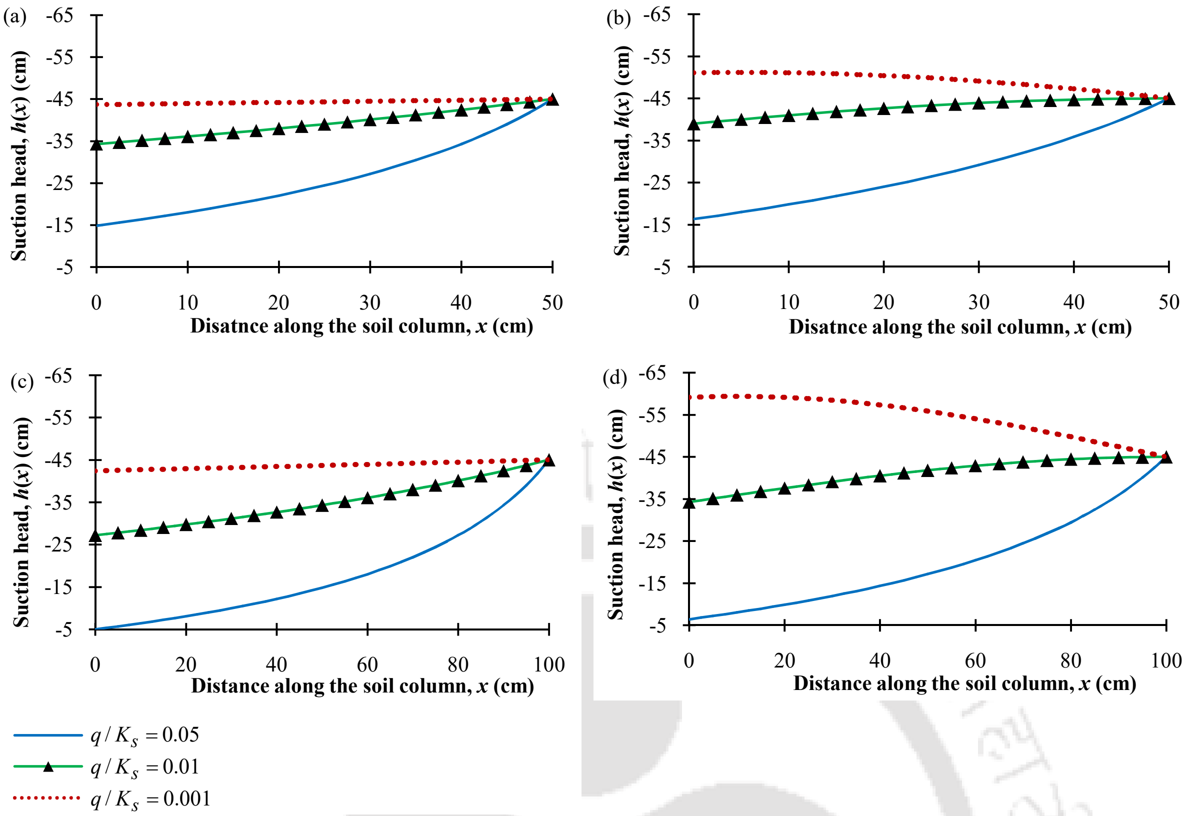

Fig. 4.8. Variation of suction head along the length of an infiltration column for three values of q/K_s (namely $q/K_s = 0.05$, 0.01 and 0.001, where q is the flux at $x = 0$ and K_s is the saturated hydraulic conductivity) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / L}{L} = 0.01$ cm^{-1}	At $x = L$ $h_L = -45$ cm	$A = 0^0$, $K_s = 1\,\text{cm hr}^{-1}$, $\alpha = 0.02\,\text{cm}^{-1}$, $n = 1.41$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / L}{L} = 0.01$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / L}{L} = 0.01$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / L}{L} = 0.01$ cm^{-1}		

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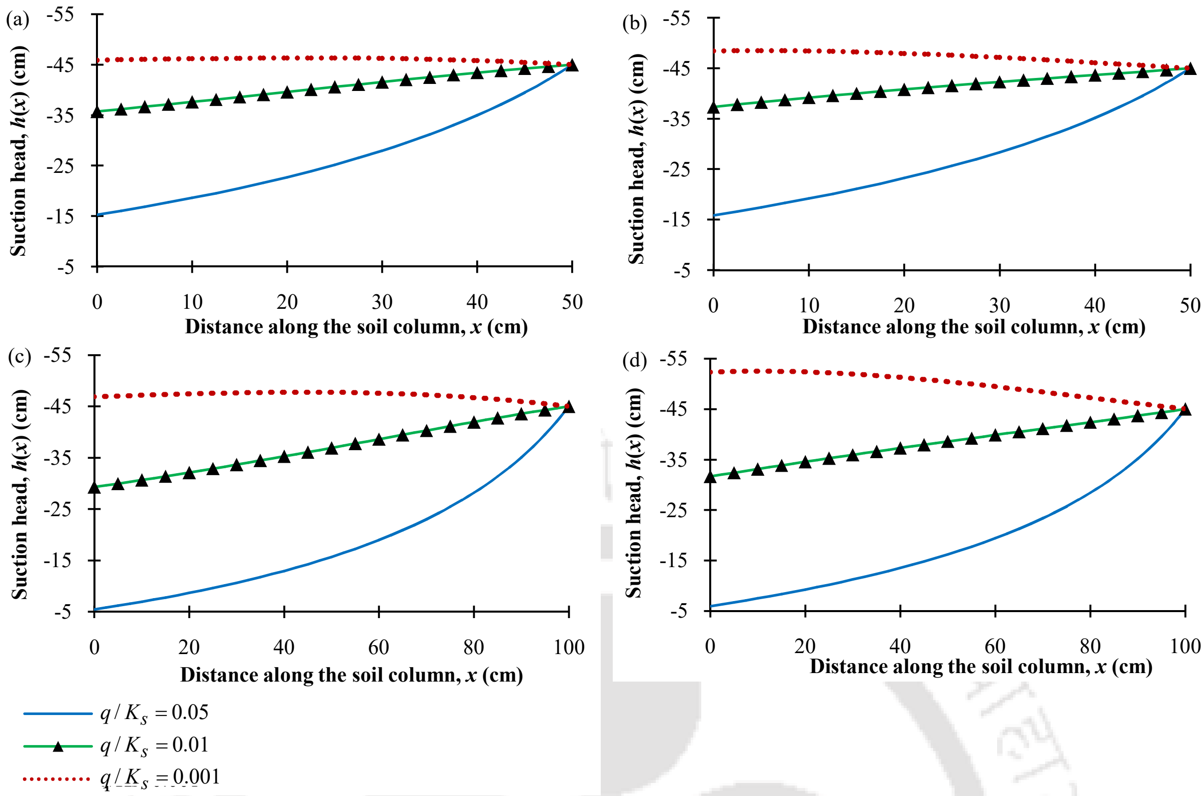
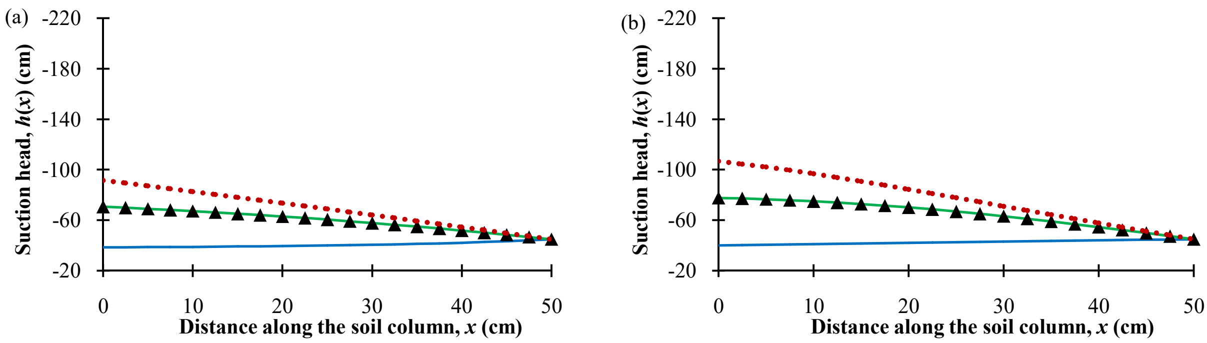


Fig. 4.9. Variation of suction head along the length of an infiltration column for three values of q/K_s (namely $q/K_s = 0.05$, 0.01 and 0.001, where q is the flux at $x = 0$ and K_s is the saturated hydraulic conductivity) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = L$ $h_L = -45$ cm	$A = 90^0$, $K_s = 1\,\text{cm hr}^{-1}$, $\alpha = 0.02\,\text{cm}^{-1}$, $n = 1.41$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.01}{L}$ cm^{-1}		



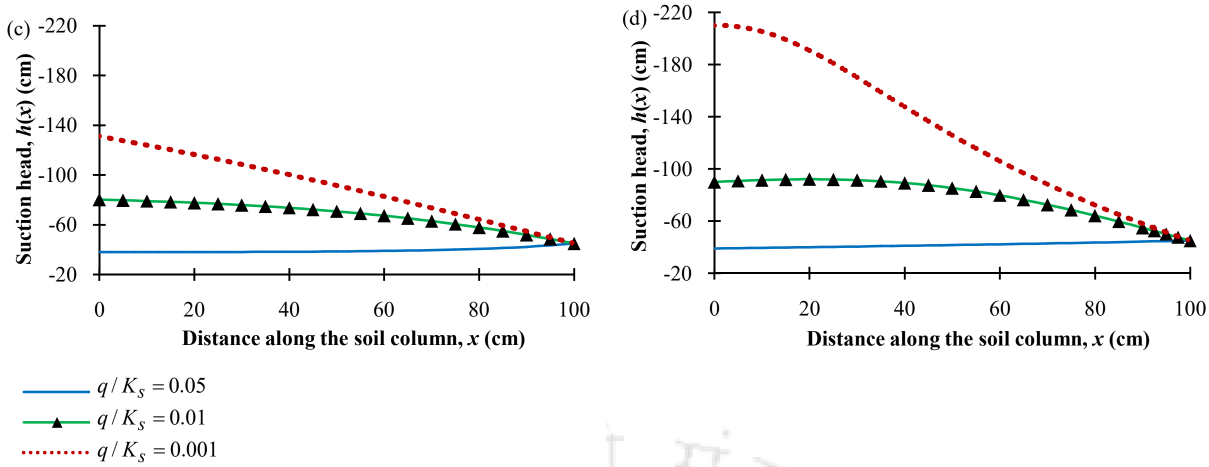
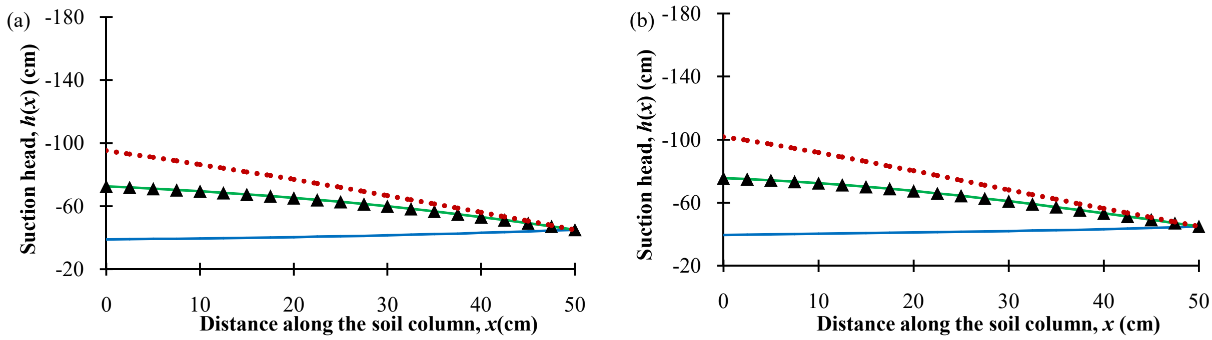


Fig. 4.10. Variation of suction head along the length of an infiltration column for three values of q/K_s (namely $q/K_s = 0.05$, 0.01 and 0.001, where q is the flux at $x = 0$ and K_s is the saturated hydraulic conductivity) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j$, where $j = 1, 2, \ldots, 19\}$.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/L}{L} = 0.1\text{ cm}^{-1}$	At $x = L$ $h_L = -45$ cm	$A = 90^0$, $K_s = 1\,\text{cm hr}^{-1}$, $\alpha = 0.02\,\text{cm}^{-1}$, $n = 1.41$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M/K_s}{L} - \frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L} = \frac{0.01}{L}\text{ cm}^{-1}$		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L} = \frac{0.01}{L}\text{ cm}^{-1}$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M/K_s}{L} - \frac{S_M/K_s}{L^2}x\right)$, where $\frac{S_M/K_s}{L} = \frac{0.01}{L}\text{ cm}^{-1}$		



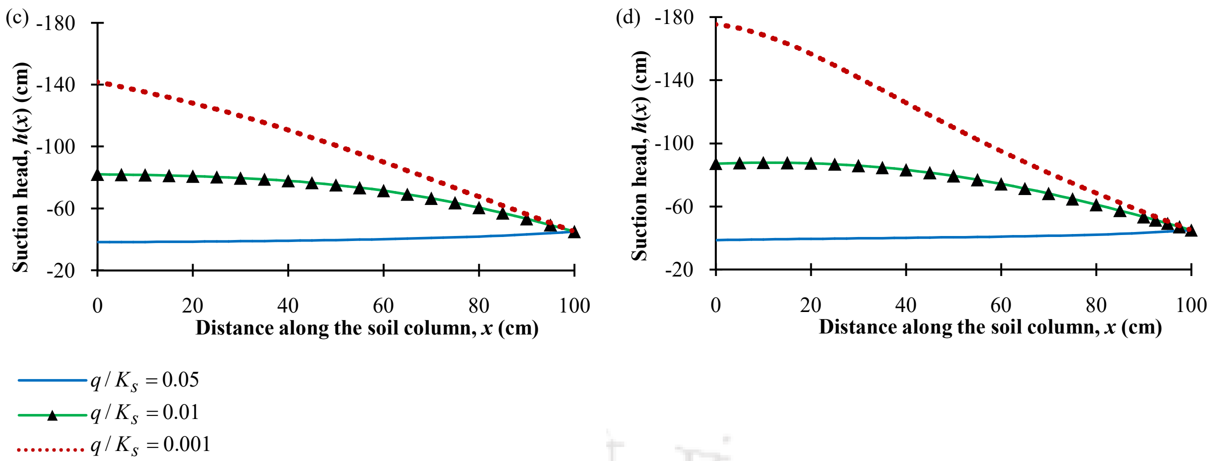


Fig. 4.11. Variation of suction head along the length of an infiltration column for three values of q/K_s (namely $q/K_s = 0.05$, 0.01 and 0.001, where q is the flux at $x = 0$ and K_s is the saturated hydraulic conductivity) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j,$ where $j = 1, 2, \ldots, 19\}$.

4.3.3 Suction head profiles for different values of α in an infiltration column

As can be seen in Figs. 4.12 – 4.15, the variation of α has a significant impact on the suction head distribution in an infiltration column both when a sink term is present and when it is absent in the column. Also, from the horizontal infiltration situations considered in these examples (Figs. 4.12 and 4.13), it is clear that for a relatively higher value of α – which signifies a relatively coarser soil – the energy gradient required at any point in an infiltrating domain may be relatively quite high when compared with a low α situation when the q/K_s ratio is kept constant in the flow space. This is because a higher α means a more conductive soil and since q/K_s is constant here, a higher q at $x = 0$. Thus, for these horizontal flow situations, a high K_s means a high steady infiltration flux at the start of an infiltration column and for $S(x) = 0$, the same flux would continue for the entire length of the soil. This, however, is not the case if $S(x) \neq 0$ as q at any distance x in the soil column is also dependent on the $S(x)$ distribution in the column. From these figures, it is also clear that the suction head profile in an infiltration column may be widely influenced by the presence and distribution of the $S(x)$ function in the column, and that for the same α, head profile for a $S(x) = 0$ infiltration situation in an infiltrating space may be widely different from that of a $S(x) \neq 0$ situation.

These flow situations also show that α's influence on vertical infiltration on a van Genuchten soil may be even more than that on horizontal infiltration through the soil. This is

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true both when a sink term is present in the soil and when it is absent. From these infiltration situations it is also clear that infiltration dynamics in a van Genuchten soil is highly sensitive to α as even a very small change of this parameter in the soil may transform the infiltration behaviour in it in a considerable way. Thus, while carrying out infiltration modelling in a van Genuchten soil, care must be exercised to see that this parameter is been correctly inputted into the model as a small error in this value may lead to substantial error in the modelling results.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = 0$ $q / K_s = 0.01$; at $x = L$ $h_L = -45$ cm	$A = 0^0$, $n = 1.41$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.01}{L}$ cm^{-1}		

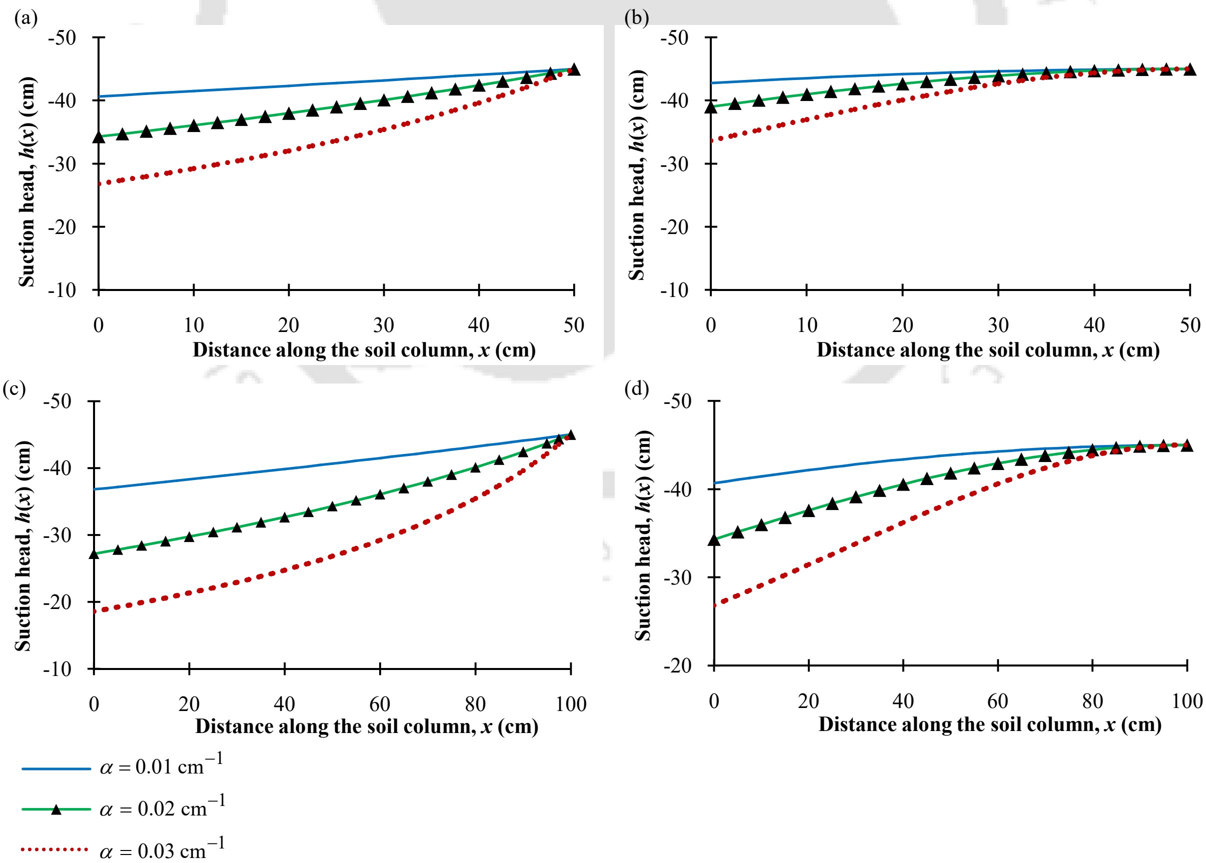

Fig. 4.12. Variation of suction head along the length of an infiltration column for three values of α (namely $\alpha = 0.01$ cm^{-1}, 0.02 cm^{-1}and 0.03 cm^{-1}, where α is a soil parameter) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}	At $x = 0$ $q / K_s = 0.01$; at $x = L$ $h_L = -45$ cm	$A = 0^0$, $n = 1.41$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		

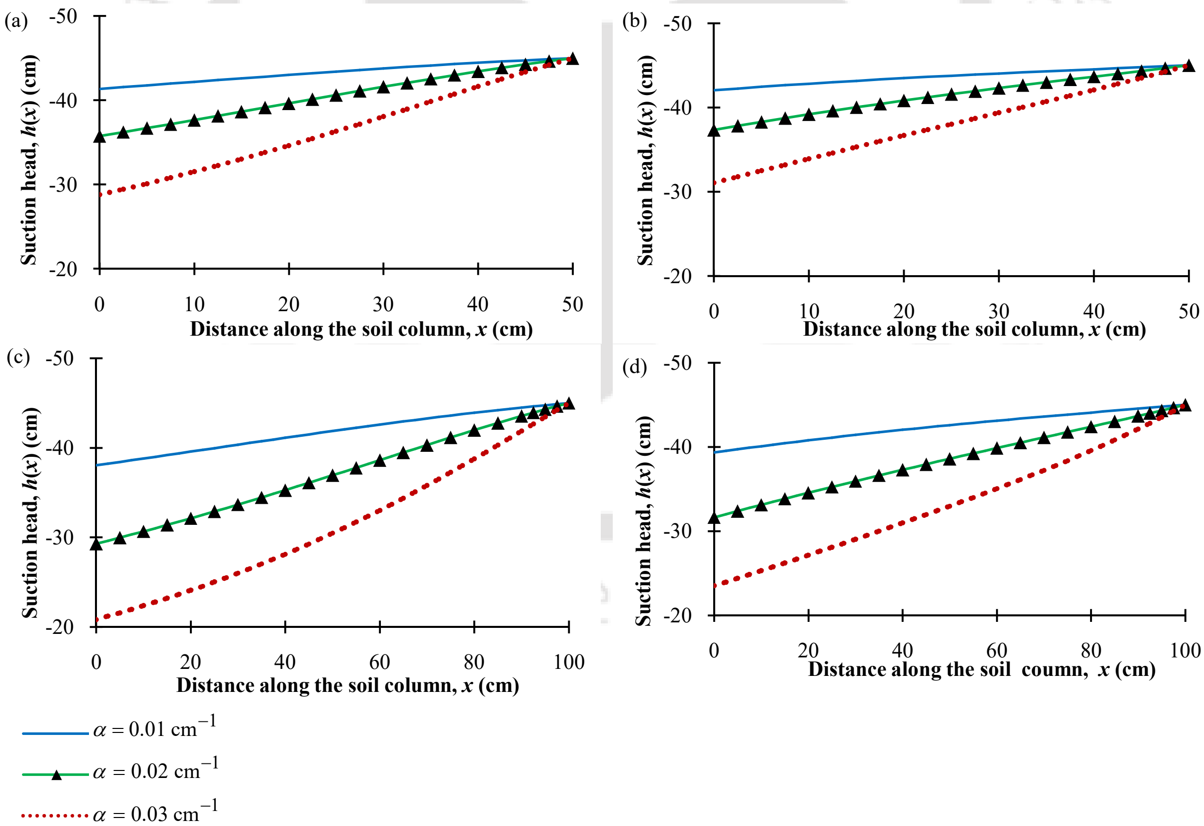

Fig. 4.13. Variation of suction head along the length of an infiltration column for three values of α (namely $\alpha = 0.01$ cm^{-1}, 0.02 cm^{-1}and 0.03 cm^{-1}, where α is a soil parameter) when the other parameters of the flow situations are taken shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j,$ where $j = 1, 2, \ldots, 19\}$.

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Length of the soil column	Nature of root-water extraction function in [0, *L*]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\dfrac{S(x)}{K_s} = 0$	At $x = 0$ $q / K_s = 0.01$; at $x = L$ $h_L = -45$ cm	$A = 90^0$, $n = 1.41$
(b) $L = 50$ cm	$\dfrac{S(x)}{K_s} = \left(\dfrac{S_M / K_s}{L}\right) = \dfrac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\dfrac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\dfrac{S(x)}{K_s} = \left(\dfrac{S_M / K_s}{L}\right) = \dfrac{0.01}{L}$ cm^{-1}		

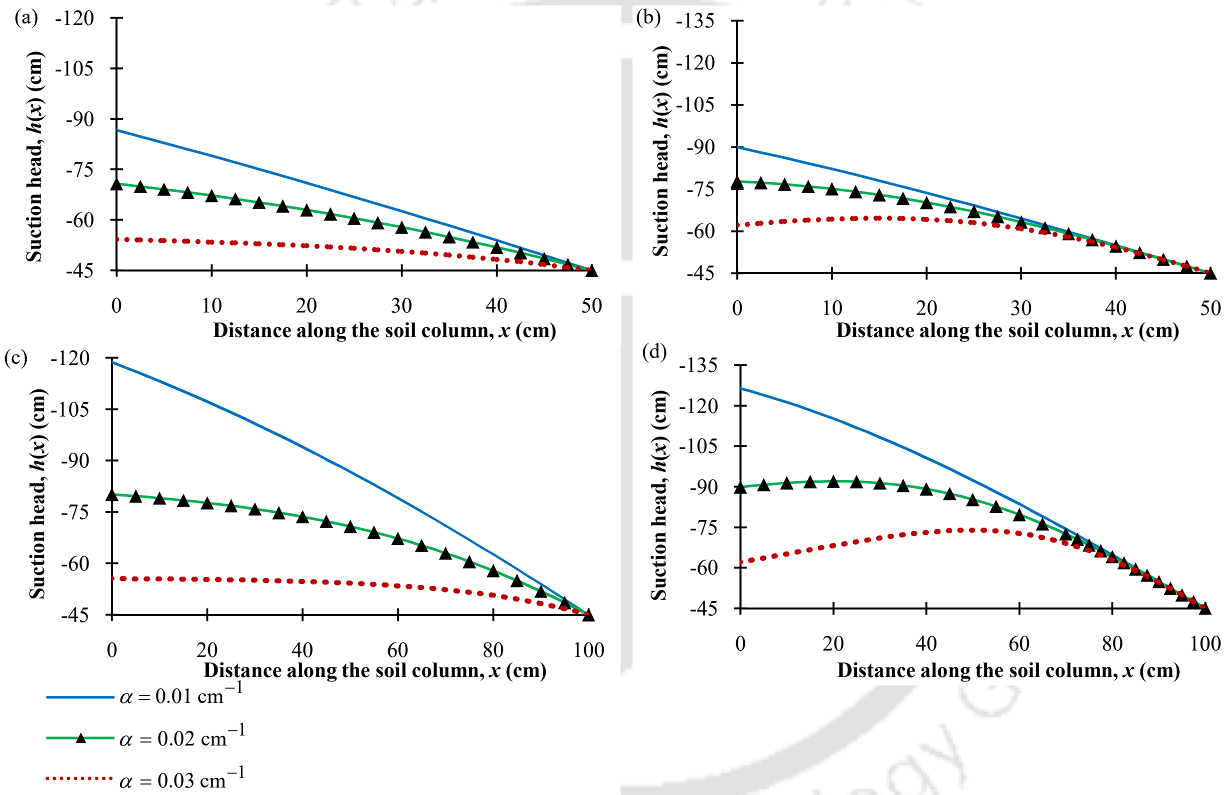

Fig. 4.14. Variation of suction head along the length of an infiltration column for three values of α (namely $\alpha = 0.01$ cm^{-1}, 0.02 cm^{-1}and 0.03 cm^{-1}, where α is a soil parameter) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \dfrac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}	At $x = 0$ $q / K_s = 0.01$; at $x = L$ $h_L = -45$ cm	$A = 90^0$, $n = 1.41$
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		

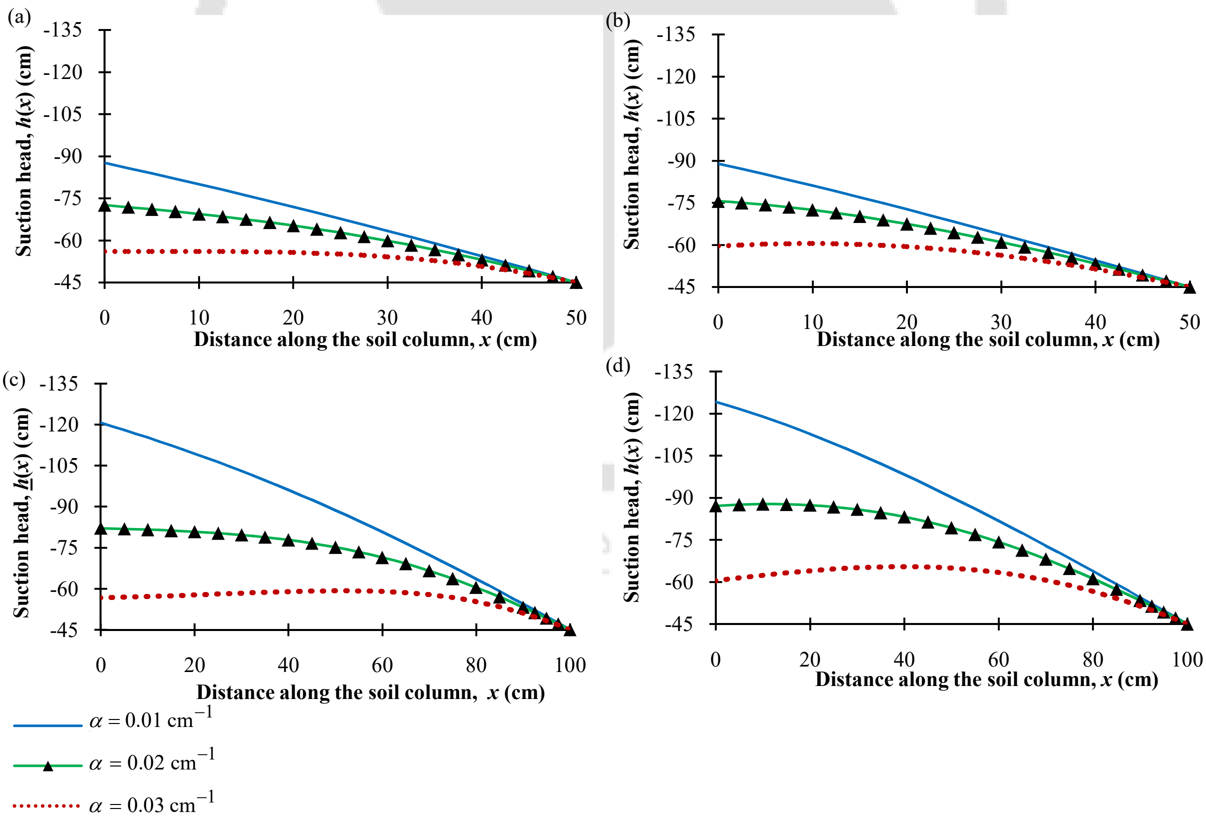

Fig. 4.15. Variation of suction head along the length of an infiltration column for three values of α (namely $\alpha = 0.01$ cm^{-1}, 0.02 cm^{-1}and 0.03 cm^{-1}, where α is a soil parameter) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

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4.3.4 Suction head profiles for different values of n in an infiltration column

Figs. 4.16 – 4.19 show the effect of the grain size parameter n on the studied infiltration situations. As can be observed, the hydraulic gradients required to move water in the infiltration columns for all the $n = 2$ horizontal flow situations considered here are relatively less than that of $n = 1.41$ and $n = 1.25$ situations. This is happening because in a fine grained soil with a low value of *n*, water in the voids of the soil are being more held tightly by the capillary forces than that in a coarse soil with a high value of *n*. Thus, to move water though the voids of a fine grained soil, relatively more energy is required than that required to move water through the voids of a coarse grained soil. As we are keeping the q / K_s ratio constant (equal to 0.01) in all these situations, the increase in *n*, however, is not only causing the hydraulic conductivity to increase but also the infiltration flux as well at $x = 0$. Thus, increase in n is bringing about two contrasting effects so far as energy requirements in moving water through these infiltration columns are concerned; one in which an increase in *n* is causing the energy requirement to decrease because of increase in K_S and the other in which an increase in n is causing the energy requirement to increase due to increase in q. It can also be observed in Figs. 4.18 and 4.19 that gravity's role on vertical infiltration through a van Genuchten soil may be quite significant mainly in locations close to the beginning of an infiltration column. This can be true for a van Genuchten soil with a high or low value of *n*.

The infiltration scenarios of Figs. 4.16 – 4.19 also show that the suction head profiles for different *n*'s in presence of a root-water distribution function may be significantly different from those without this function, particularly if an infiltrating space is relatively long. Thus, the effect on the infiltration dynamics due to change in *n* cannot be foretold in a concrete way as this effect is dependent on the choice of other parameters of the problem as well as has been clearly demonstrated in these figures. Hence, for the same n variation in a van Genuchten soil, the infiltration hydraulics associated with it may get noticeably changed if even a single other hydraulic parameter of the soil is been changed.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = 0$	At $x = 0$ $q/K_s = 0.01$; at $x = L$ $h_L = -45$ cm	$A = 0^0$, $\alpha = 0.02$ cm^{-1}
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = 0$		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L}\right) = \frac{0.01}{L}$ cm^{-1}		

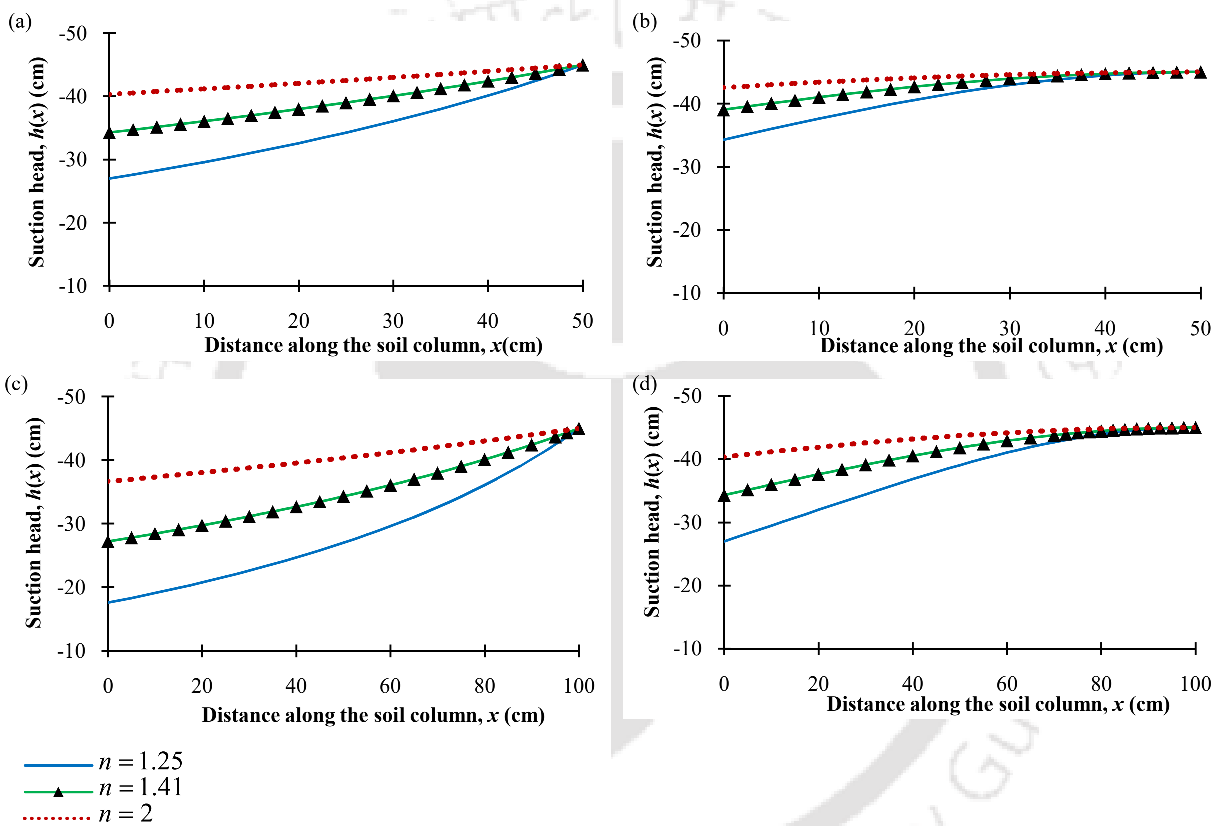

Fig. 4.16. Variation of suction head along the length of an infiltration column for three values of n (namely $n = 1.25$, 1.41 and 2, where n is the grain size parameter) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}	At $x = 0$ $q / K_s = 0.01$; at $x = L$ $h_L = -45$ cm	$A = 0^0$, $\alpha = 0.02$ cm^{-1}
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$ where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		

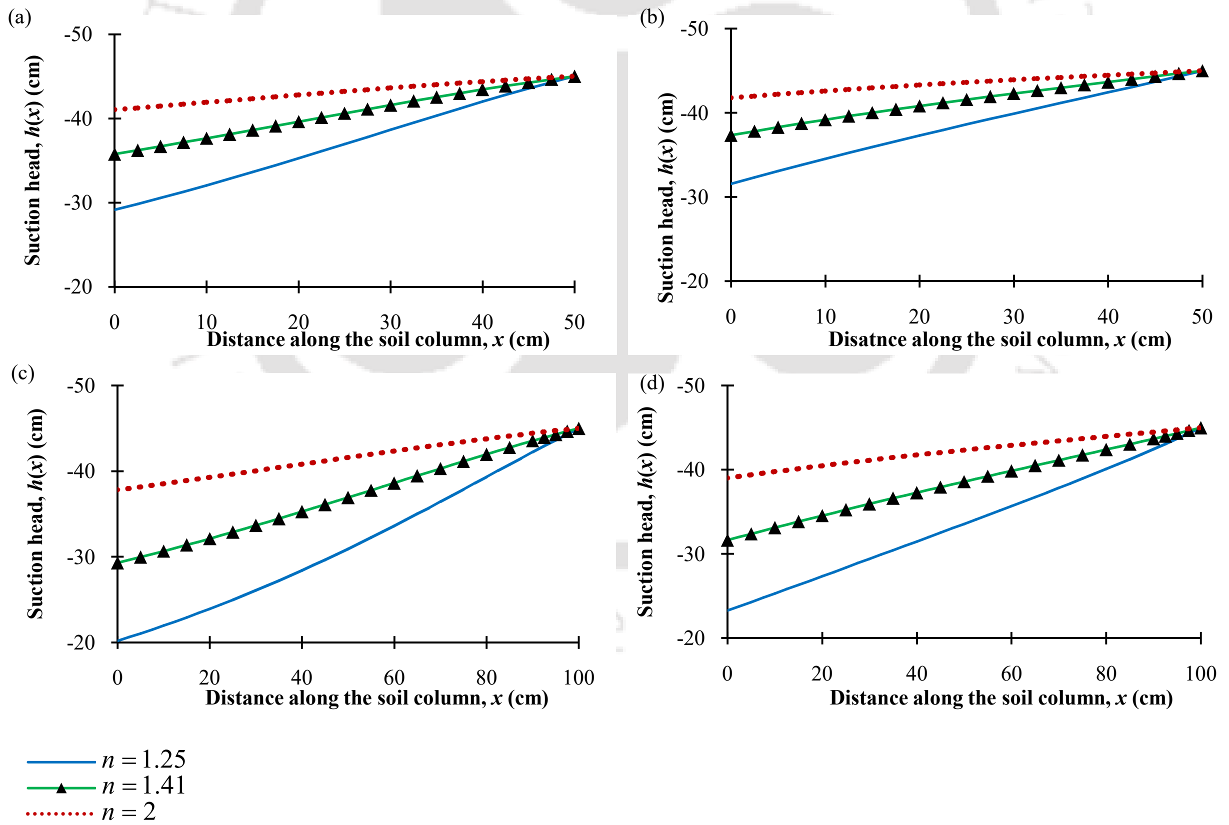

Fig. 4.17. Variation of suction head with length of a soil profile for three values of n (namely $n = 1.25$, 1.41 and 2, where n is the grain size parameter) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L=50$ cm	$\frac{S(x)}{K_s}=0$	At $x=0$ $q/K_s=0.01$; at $x=L$ $h_L=-45$ cm	$A=90^0$, $\alpha=0.02$ cm^{-1}
(b) $L=50$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L}\right)=\frac{0.01}{L}$ cm^{-1}		
(c) $L=100$ cm	$\frac{S(x)}{K_s}=0$		
(d) $L=100$ cm	$\frac{S(x)}{K_s}=\left(\frac{S_M/K_s}{L}\right)=\frac{0.01}{L}$ cm^{-1}		

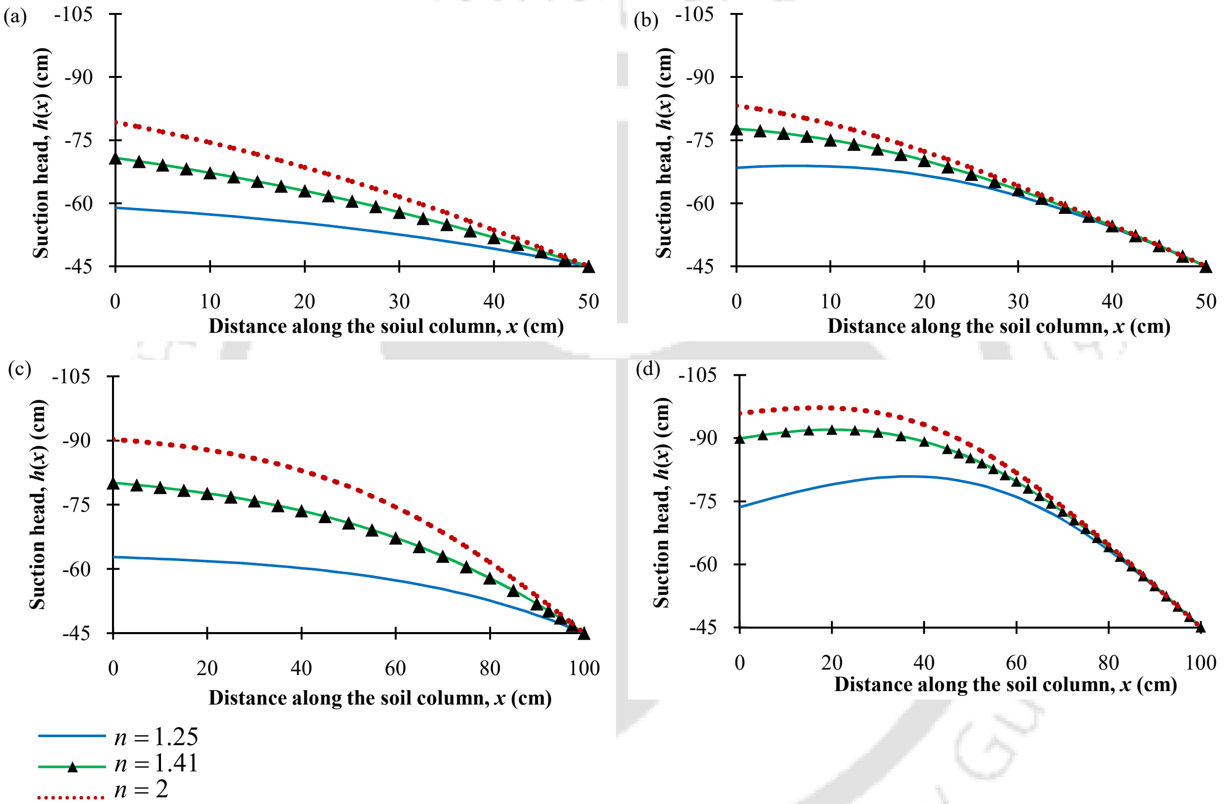

Fig. 4.18. Variation of suction head along the length of an infiltration column for three values of n (namely $n=1.25$, 1.41 and 2, where n is the grain size parameter) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19}j$, where $j=1,2,\ldots,19\}$.

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Length of the soil column	Nature of root-water extraction function in [0, L]	Boundary conditions	Parameters
(a) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}	At $x = 0$ $q / K_s = 0.01$; at $x = L$ $h_L = -45$ cm	$A = 90^0$, $\alpha = 0.02$ cm^{-1}
(b) $L = 50$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		
(c) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		
(d) $L = 100$ cm	$\frac{S(x)}{K_s} = \left(\frac{S_M / K_s}{L} - \frac{S_M / K_s}{L^2} x\right)$, where $\frac{S_M / K_s}{L} = \frac{0.01}{L}$ cm^{-1}		

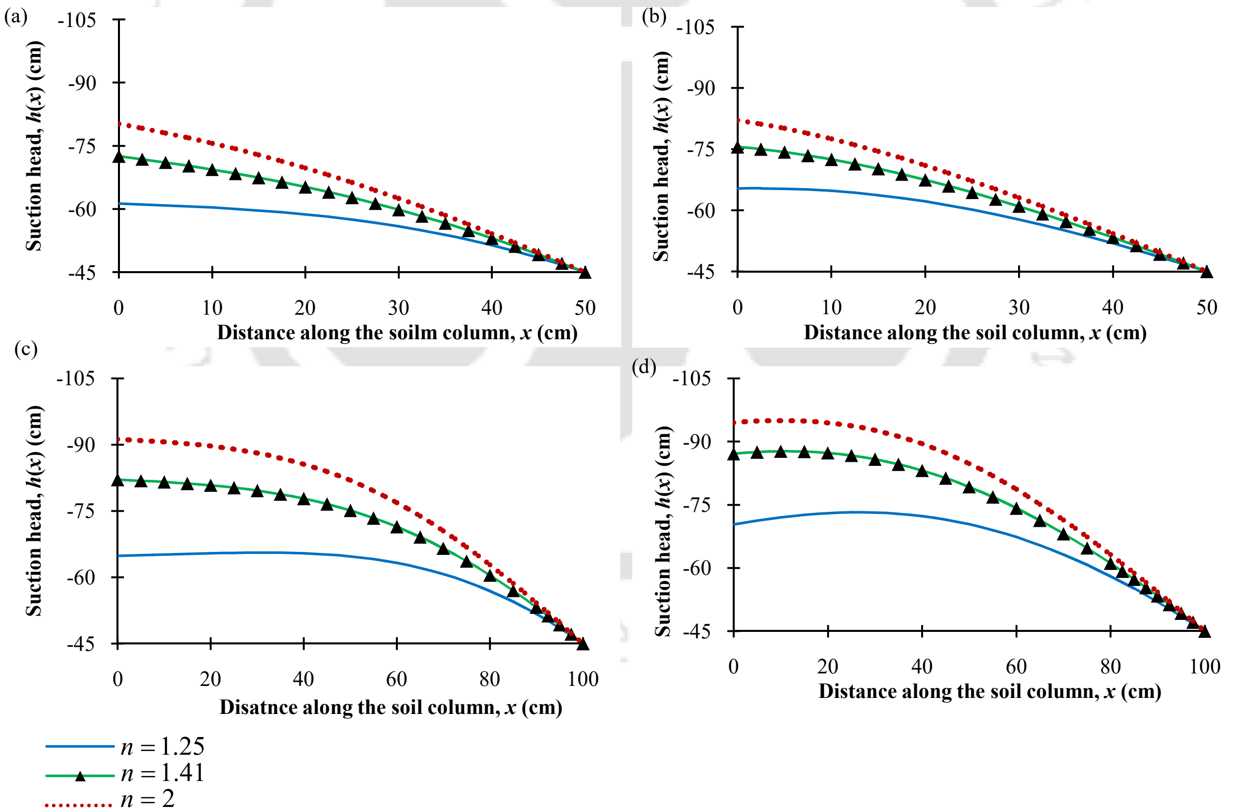

Fig. 4.19. Variation of suction head along the length of an infiltration column for three values of n (namely $n = 1.25$, 1.41 and 2, where n is the grain size parameter) when the other parameters of the flow situations are taken as shown above and the $S_D^{(2)}$ set used for obtaining solutions to these flow situations is $\{0, \frac{L}{19} j$, where $j = 1, 2, \ldots, 19\}$.

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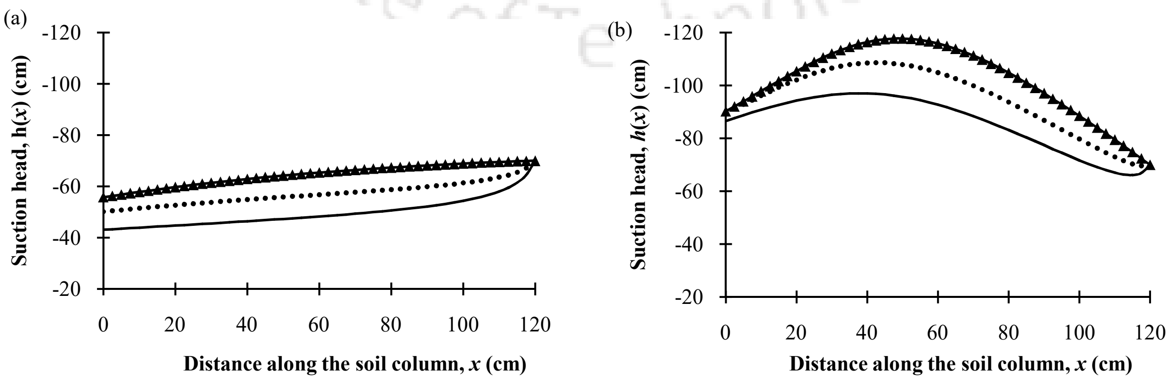
4.3.5 Suction head profiles for a few heterogeneous infiltration situations

In Fig. 4.20, suction head profiles corresponding to a few horizontal and vertical infiltration scenarios in heterogeneous soils are shown. As may be observed, in these situations van Genuchten's parameters are all changing in space in specific ways along with $S(x)$ distributions as shown. These are illustrative examples only; our solution obtained by either of the approaches, as already mentioned, is a general one capable of tackling any valid functional variations of the parameters (including that of the sink term) of the van Genuchten-based infiltration equation in an infiltrating space.

From these profiles, it is clear that variations of soil heterogeneities and the presence of a sink term in an infiltration column may alter the infiltration dynamics associated with it in a major way. Further, gravity's influence on infiltration mechanics may also be considerable judging from what can be seen from these flow situations. These profiles are also bringing forth some obvious facts: suction heads (in magnitude) in an infiltration column increase with the increase in strength of a sink term and decrease with the increase in water transmitting capacity of a soil – these observations naturally mainly hold when the other playing variables of the system are kept constant. It has also become evident from these flow situations that infiltration through a soil may be greatly impacted by a mere change in the saturated hydraulic conductivity of the soil, both in presence as well as absence of a sink term in the soil. Thus, saturated conductivity variation alone in an infiltrating space may affect the overall hydraulics associated with it in a significant way. Also, as already mentioned, infiltration dynamics in a van Genuchten soil may be greatly influenced by the nature and distribution of heterogeneity in the soil. Thus, a mathematical model developed for studying infiltration dynamics in a field situation should have the ability to tackle heterogeneities of a soil layer since field soils are mostly heterogeneous in nature.



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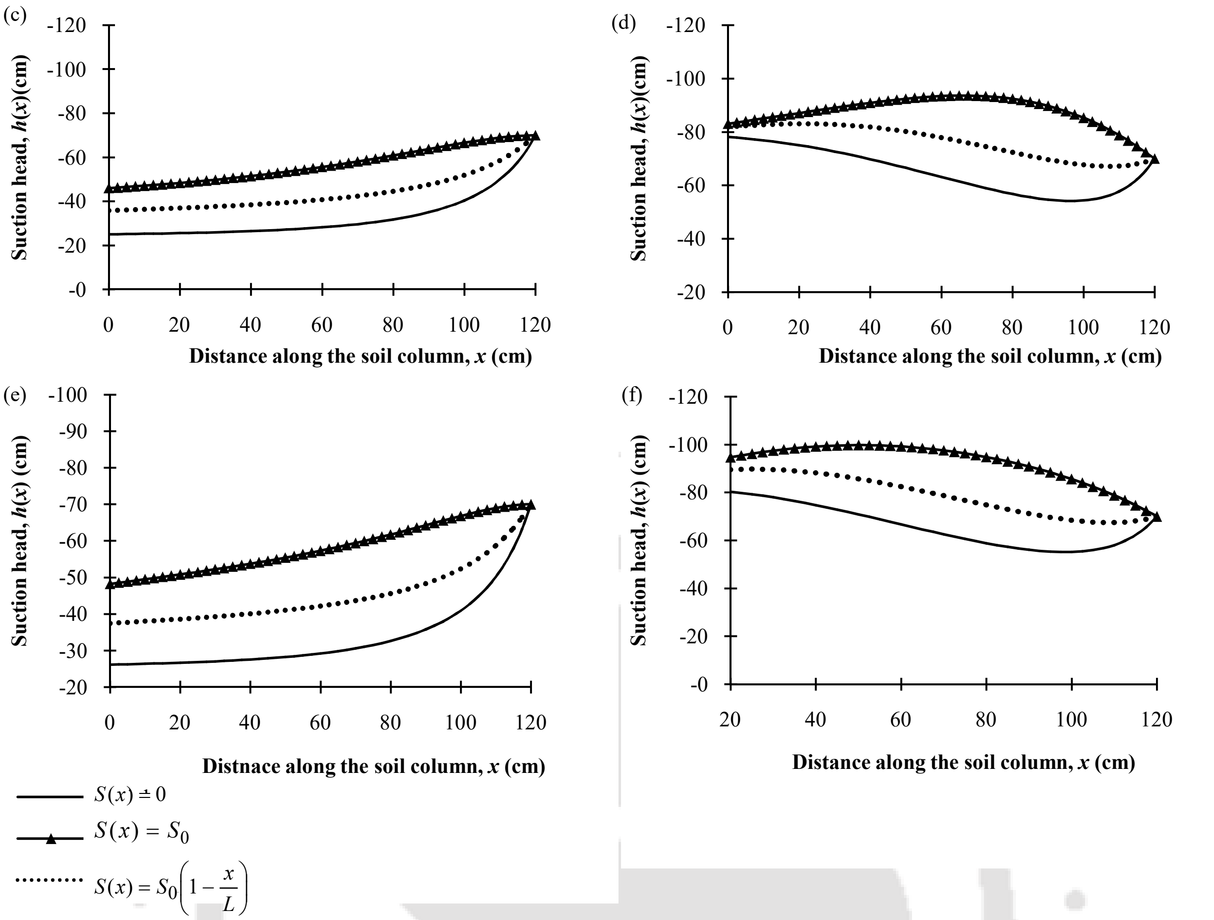

Fig. 4.20. Variation of suction head with length of a soil profile when the parameters of the flow situations are taken as $K(x=0) = K_{s0} = 6.26\times10^{-3}$ cm s^{-1}, $K(x=L) = K_{sL} = 1.52\times10^{-4}$ cm s^{-1}, $L = 120$ cm, $h_L =$ -70 cm, $q = 2.38\times10^{-5}$ cms^{-1} $\alpha(x=0) = \alpha_0 = 0.028$ cm^{-1}, $\alpha(x=L) = \alpha_L = 0.0104$ cm^{-1}, $n_0 = 2.239$, $n_L = 1.3954$, $S_0 = 2.38\times10^{-5}/L$ cm cm^{-1} s^{-1} and (a) $K_s(x) = f_1(x)$, $\alpha(x) = f_2(x)$, $n(x) = f_3(x)$ and $A = 0^0$, (b) $K_s(x) = f_1(x)$, $\alpha(x) = f_2(x)$, $n(x) = f_3(x)$ and $A = 90^0$, (c) $K_s(x) = f_4(x)$, $\alpha(x) = f_2(x)$, $n(x) = f_3(x)$ and $A = 0^0$, (d) $K_s(x) = f_4(x)$, $\alpha(x) = f_2(x)$, $n(x) = f_3(x)$ and $A = 90^0$, (e) $K_s(x) = f_4(x)$, $\alpha(x) = f_5(x)$, $n(x) = f_6(x)$ and $A = 0^0$, (f) $K_s(x) = f_4(x)$, $\alpha(x) = f_5(x)$, $n(x) = f_6(x)$ and $A = 45^0$, (h) $K_s(x) = f_4(x)$, $\alpha(x) = f_5(x)$, $n(x) = f_6(x)$ and $A = 90^0$, where $f_1(x) = K_{s0} + \left(\frac{K_{sL} - K_{s0}}{L}\right)x$, $f_2(x) = \alpha_0 + \left(\frac{\alpha_L - \alpha_0}{L}\right)x$, $f_3(x) = n_0 + \left(\frac{n_L - n_0}{L}\right)x$, $f_4(x) = K_{s0}\exp(\beta x)$, $f_5(x) = \alpha_0\exp(\gamma x)$, $f_6(x) = n_0\exp(\delta x)$, $\beta = \frac{1}{L}\log_e\left(\frac{K_{sL}}{K_{s0}}\right)$, $\gamma = \frac{1}{L}\log_e\left(\frac{\alpha_L}{\alpha_0}\right)$, $\delta = \frac{1}{L}\log_e\left(\frac{n_L}{n_0}\right)$ and the S_D set used for obtaining solutions to these flow situations is $\left\{0, \frac{L}{19}i, \text{ where } i = 1,2,\ldots,19\right\}$.

4.4 Conclusions

The major conclusions of the study carried out in this chapter can be summarized as below.

1. An analytical model has been developed for simulating steady one-dimensional infiltration in an inclined van Genuchten heterogeneous soil column by utilizing a mathematical

procedure as provided by Barua (2021). The developed model can account for not only any valid spatial distribution of the van Genuchten parameters in a flow space but can also accommodate any valid root-water uptake function along the length of an infiltrating column. The proposed solution of the van Genuchten-based Richards' infiltration equation is new as there is currently no analytical solution to this equation even for a homogeneous soil for all possible spatial variations of the parameters of the problem. As mentioned before, Rockhold et al.'s (1997) obtained an analytical solution to the van Genuchten-based infiltration equation for simulating infiltration in a layered vertical soil column under the assumption that the hydraulic properties within each layer of the soil are uniform. Thus, this solution is strictly not for a heterogeneous soil where the properties of the soil are continuously varying in space. Further, this solution also cannot accommodate the presence of a sink term in an infiltrating space. Hence, even for a layered soil, our solution of the van Genuchten-based infiltration equation with a sink term is of a much general nature than that of Rockhold et al.'s (1997) solution to the problem.

2. Like in the solution of the Gardner-based infiltration equation, here also two approaches have been presented to solve the boundary value problem considered in the study. In the first approach, the governing equation needs to be differentiated twice and in the second approach the governing equation alone can be used to obtain a solution of it. As the van Genuchten-based infiltration equation for a heterogeneous soil [i.e., Eq. (4.11)] involves many spatially varying terms, double differentiation of this equation may be somewhat tedious. Thus, the computational requirements needed in this approach to solve Eq. (4.11) for many infiltration settings may be a bit more than that by the use of the second approach. This is because, as mentioned before, in the second approach the governing equation alone can be utilized to obtain a solution of it. However, one major advantage of the first approach is that, for achieving the same level of accuracy in a solution, comparatively lesser number of forced points in $[0, L]$ are required in an infiltrating space as compared to the forced points required using the second method. The accuracy of a solution obtained through either of the approaches depends on the number of terms that are being considered in its development and a solution of any expected accuracy specific to a problem can be obtained from both the approaches by taking sufficient number of terms in the solution polynomial.

3. If a solution to the van Genuchten-based infiltration problem exists for a set of parameters of the problem, then either of the approaches guarantees a converging series solution to the problem. As mentioned before, the van Genuchten-based infiltration equation is a highly

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nonlinear differential equation for which currently no analytical solutions exists even for a homogeneous soil for all possible parameter variations of the problem. The h-based Richards' equation – the one that has been considered here – is a more versatile equation than its θ-based counterpart as, unlike the latter, it can be used to study variably saturated flow in heterogeneous soils (Brunone et al. 2003; Mao et al. 2011; Yeh et al. 2015; Zhang et al. 2015, 2018, 2021 – to cite a few) with relative ease. Numerical attempts to solve the h-based Richards' equation are many but as pointed out by several researchers in the past, numerical solutions of this equation may suffer from large mass balance errors and are thus not always assured of providing a converge solution to the problem for all possible soil properties and water content distributions in soils (Milly 1985; Allen and Murphy 1986; van Genuchten 1982; Celia 1987, 1990; Kirkland 1992; Hao 2005; Assouline 2013; Lai and Ogden 2015, Ogden et al. 2015; Zhang et al. 2015, 2018, 2021; Suk and Park 2019 – to mention a few). For a valid set of flow parameters, however, either of our approaches can be used to obtain a converging series solution to the problem. Further, any desired accuracy in the solution can be obtained by considering sufficient number of terms (depending on the extent of accuracy desired) in the solution.

4. The study shows that the spatial variations of the van Geunchen's (1980) parameters may affect the dynamics of infiltrations in an infiltration space in a major way – this is true both in presence as well absence of a sink term in the infiltrating space – and that the gravitational head associated with infiltration in an inclined soil column may greatly alter the infiltration hydraulics associated with such a system. Further, infiltration behaviour on a vertical soil column may be substantially different from that of a horizontal one where gravity is not taking any part in moving water though it. The study also shows that infiltration on a heterogeneous soil with a root-water extraction function may be substantially influenced by the nature of this function alone. Thus, due care should be taken to see that this function is being properly included in a model meant for carrying out infiltration study in a field situation like that of an agricultural field, where the presence of this function is a reality.

5. The study also shows that a change in the orientation of an infiltration column alone may alter the mechanics of flow in the column in a significant way. It is seen that, considering all other parameters of a van Genuchten infiltration problem as non-changing, water particles in the soil pores of a vertically inclined infiltration column will be held more tenaciously than that in a horizontal infiltration column. Thus, assuming all other factors as same, in the former case, more work is needed by plants to extract water from infiltrating water than that needed in the latter case.

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6. The proposed analytical solution of the van Genuchten-based infiltration equation is a much versatile solution as it can handle both soil heterogeneity and root-water extraction dynamics of an infiltrating space. As soils in nature are inherently heterogeneous (Yeh and Harvey 1990; Wildenschild and Hensen 1999; Sasidharan et al. 2019) and the presence of a root-extraction term for most field situations is also a reality, the proposed infiltration model is thus expected to provide a more realistic picture of infiltration in a field soil as compared to relatively simple available models on the subject.

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CHAPTER 5
SUMMARY AND CONCLUSIONS

5.1 Summary

A proper understanding of hydraulics of water movement in the unsaturated zone of a soil is essential for a better management of the environmental resources of a region (Melloul and Wollman 2003; Narula and Gosain 2013; Wang et al. 2014, 2016; Li et al. 2017; Zhang et al. 2016; 2018). Soil water dynamics in the unsaturated zone plays a key role in the hydrologic cycle of vegetated ecosystems and a know-how of mechanics of water movement in the vadose zone is essential for having a proper assessment about fate and transport of contaminants in soils (Daly and Porporato 2005; Broadbridge et al. 2017). Water flow in the unsaturated zone of a soil is generally been modeled using the Richards' equation – an equation which is an outcome of the law of conservation of mass and the Darcy's law. Mathematical modeling of this equation has a long past and has been the subject of intense research for several decades now (Celia et al. 1987; Li et al. 2016; Suk and Park 2019; Zhang et al. 2016, 2018, 2021). However, as this equation is extremely nonlinear, working out an accurate and reliable solution of this equation has always been a challenge (Farthing and Ogden 2017, Zhang et al. 2021). Two approaches – namely analytical and numerical – are generally followed to obtain solution to the Richards' equation. However, because of the high nonlinearity of this equation, analytical solutions of this equation have till now been possible for relatively simple flow situations only (Zhang et al. 2016, 2018). Several attempts were made by many in the past to solve the Richards' equation for different flow settings by utilizing numerical means (Celia et al. 1990; Kirkland et al. 1992; Kavetski et al. 2002; Miller et al. 2006; Wu 2010; Zhang et al. 2016; 2021; Ngo-Cong et al. 2020 – to cite a few). However, as several studies have pointed out (Milly 1985; Allen and Murphy 1986; Celia 1987, 1990; Kirkland 1992; Hao 2005; Assouline 2013; Lai and Odgen 2015, Ogden et al. 2015; Zhang et al. 2015, 2021; Suk and Park 2019; Vrugt and Gao 2021 – to mention a few), numerical modeling of the Richards'-based infiltration equation has its own drawbacks as well – namely this way of obtaining solution to the equation may suffer from large mass balance errors and is not always assured of convergence for all possible soil properties and water content distributions in soils.

There is currently no analytical solution to the Gardner or Mualem-van Genuchten-based steady-state one-dimensional infiltration equation for a heterogeneous soil with or without a sink term. a

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analytical solutions of the steady-state one-dimensional Richards' equation for a few relatively simple flow situations in homogeneous and in layered soils are available (Gardner 1958; Warrick 1974, 1988, 1991, 2003, 2005; Lomen and Warrick 1976; Warrick and Yeh 1990; Salvucci 1993; Basha 1994, 1999; Rockhold et al. 1997; Baker 2000; Gastȯ et al. 2002; Zhu and Mohanty 2002; Jury and Horton 2004; Lu et al. 2007; Mohanty and Zhu 2007; Szymkiewicz 2009; Huang and Wu 2012; Sadeghi et al. 2012 – to cite a few). These solutions, as mentioned before, are valid for homogeneous and/or layered soils only and not for a purely heterogeneous soils where properties of the soil are continually varying in space. In view of the same, an effort has been made in this study to obtain analytical solutions to both the Gardner- and Mualem-van Gunuchten-based steady-state one-dimensional infiltration equations for a heterogeneous soil where the sink term of these equations has also been taken as a spatial variable. All these solutions have been obtained based on a mathematical procedure as proposed by Barua (2021). The details of all these solutions and the hydraulic inferences obtained from them have already been comprehensibly described in the third and the fourth chapters of this report. However, for ready reference, the salient conclusions of the study are now again been presented below in a point-wise form.

5.2 Conclusions

The notable contributions of the study can be summarized as follows.

1. A series solution has been proposed to the general Gardner-based infiltration equation for predicting steady infiltration in an arbitrarily inclined unsaturated heterogeneous soil column with a sink term. The solution assumes a steady flux at the start of an infiltrating column (Neumann boundary) and a fixed piezometric head at the end of it (Dirichlet boundary). The proposed model is a general one capable of handling not only any valid spatial variation of the parameters of the Gardner-based infiltration equation on an infiltrating space but can also concurrently handle any valid spatial variation of a root-extraction term on the flow space. The mathematical procedure that has been used in obtaining solution to the Gardner-based infiltration equation, as shown in Chapter 3, can be easily adjusted to obtain solution to the Brooks and Corey's-based infiltration equation as well. This is because, as mentioned before (Chapter 3), Brooks and Corey's (1964) conductivity function is mathematically very similar to that of the general Gardner's (1958) conductivity function, differing from the latter by a constant term only in the denominator of the function. The proposed analytical solutions for both the Gardner- and

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Mualem-van Genuchten-based infiltration equations for a heterogeneous soil are all new as there is currently no analytical solution to either of these equations for a heterogeneous soil. In fact, as mentioned before, for a homogeneous soil also, an analytical solution to either of these infiltration equations is currently not available for all possible variations of parameters of these equations.

2. An analytical solution has also been worked out for the Mualem-van Genuchten-based infiltration equation for a heterogeneous solution. This solution can also account for any valid spatial variation of the parameters of the Mualem-van Genuchten-based infiltration equation in an infiltrating space including any valid spatial variation of the root-water extraction term on the flow space.

3. The mathematical procedure (Barua 2021) adopted in obtaining solutions to both the Gardner- and Mualem-van Genuchten-based infiltration equations ensures that for any valid set of parameters (i.e., for a set of parameters of either of these equations for which solution to an infiltration problem exists) of these equations in $[0, L]$, the proposed solutions for these equations are guaranteed to converge at all points in $[0, L]$. Further, the accuracy of both of these solutions [i.e., accuracy of the solutions of the Gardner and van Genuchten-based differential equations] pertaining to an infiltration problem can be increased to any desired level by sufficiently increasing the number of terms of these solutions.

4. Studies on both the Gardner- and Mualem-van Genuchten-based infiltration equations show that the hydraulics of flow associated with water movement in an unsaturated soil is highly impacted by the heterogeneities of a soil column, both in presence as well as absence of a sink term in an infiltrating space and that neglecting soil heterogeneity in modeling water flow in unsaturated soils may lead to serious modeling errors at times. The study also shows that spatial variation of saturated hydraulic conductivity alone in an infiltrating space may affect the dynamics of infiltration in it in a major way – this is true for infiltration in a Gardner as well as a van Genuchten soil.

5. It has also come out of the study that, among other factors remaining the same, water particles in the voids of soil of a vertical infiltration column may be held with a much greater force than that of a horizontal infiltration column. Thus, considering all other factors to remain as unchanging, more work is needed by the roots of plants to extract water from a vertically infiltrating space than that from a horizontally infiltrating one.

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6. It has also come out of the study that the hydraulics of flow associated with movement of water on the unsaturated zone of a heterogeneous soil may get substantially influenced by the presence of a sink term alone (i.e., a root-water extraction term) in the soil. Thus, the nature of a root-water extraction function alone in a vegetated soil may impact the infiltration dynamics in such a soil in a significant way.

7. As the proposed solutions can handle soil heterogeneity and root-water extraction with ease, they are expected to provide a more realistic description of infiltration in a field soil as compared to relatively simple available solutions to these problems. This is because, a field soil is seldom homogeneous and existence of a root-water extraction term in a field soil is more of a norm than an exception.

8. Another advantage of these solutions is that for them convergence is never an issue and for all valid infiltration settings, these solutions are guaranteed to give converged results at all points in an infiltrating space $[0, L]$. Further, the outputs obtained from both these solutions can also be considered as reliable as by sufficiently increasing the number of terms of these solutions, the accuracy of both of these solutions corresponding to an infiltration problem can be increased to any desired extent. Thus, because of their inherent stability and accuracy, they may also be employed for verifying complex numerical codes related to movement of water in variably saturated porous formations.

5.3 Highlights of Major Research Contributions

The major highlights of the study are as given below.

1. An analytical solution has been developed to the steady-state one-dimensional Richards' equation for an arbitrarily inclined heterogeneous soil column by making use of Gardner's (1958) general hydraulic conductivity function for such a soil. The solution can account for not only any valid spatial distributions of the parameters of the Gardner-based infiltration equation but also any valid root-water uptake distribution function in an infiltrating space. This solution is new as there is currently no analytical solution to the Gardner-based infiltration equation for a heterogeneous soil. In fact, for a homogeneous soil also, there is presently no analytical solution to this equation for all possible variations of the parameters of the equation in an infiltrating space, including variations of the sink term. The development of an analytical solution to the Gardner-based infiltration equation for a heterogeneous soil can thus be considered as one of the contributions of this study.

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2. An analytical solution has also been developed for the steady-state one-dimensional Richards' equation for a heterogeneous soil by utilizing Mualem-van Genuchten's hydraulic conductivity model for such a soil. This solution is also applicable for any orientation of an infiltrating space and it can also account for a spatially varying sink term in an infiltrating space. This solution is also new as there is currently no analytical solution to the Mualem-van Genuchten-based infiltration equation for a heterogeneous soil. Actually, for a homogeneous soil also, an analytical solution to the equation for all possible variations of the parameters of the equation is currently lacking. The development of an analytical solution to the Mualem-van Genuchten-based infiltration for a heterogeneous soil can thus also be considered as a worthwhile contribution of this study.

5.4 Limitations of the Study

In spite of the general nature of our proposed models, there are a few limitations of these models which we believe should be pointed out. They are listed as below.

1. In the derivations of the Gardner- and Mualem-van Genuchten-based infiltration equations, one of the assumptions, as mentioned before, is that the flow in an infiltration column is Darcian. In Darcian flow, the infiltration flux is assumed to bear a linear relationship with the hydraulic gradient. However, for a few porous and fractured media flow situations, the flow may be non-Darcian in nature both at high as well as low velocities (Basak 1976; Soni et al. 1978; Sen 1987, 1989, 1990; Bordier and Zimmer 2000; Wu 2001, 2002a, 2002b; Birpinar and Sen 2004; Moutsopoulus and Tsihromtzos 2005; Wen et al 2006, 2008, 2011, 2016; Shi et al. 2018 – to name a few). A flow is said to be non-Darcian if the specific flux does not exhibit a linear relationship with the hydraulic gradient. For low rate flows in clay and silt aquitards, non-Darcian flow may be due to electro-chemical effect among the surfaces of soil particles and pore fluids (Teh and Nie 2002; Wen et al. 2006, 2008; Shi et al. 2018). For high flow rates in coarse grained soils, non-Darcian flow may be attributed to inertial effects and possible onset of turbulent flow in these soils (Wen et al. 2006, 2008a, 2008b; Shi et al. 2018). As our work here are based on Darcian flow only the models thus developed for the Gardner- and Mualem-van Genuchten-based infiltration equations are for this type of flow only and not for situations where the linearity relation between specific discharge and hydraulic gradient does not hold. This thus can be considered as a limitation of our model since, as just mentioned, flow in a few porous flow situations may fail to be Darcian in nature.

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2. Mualem-van Genuchten (MVG) relative conductivity model (Mualem 1976; van Genuchten 1980) – the model that has been used in deriving our Mualem-van Genuchten-based infiltration equation [Eq. (4.6)] – is currently the de-facto model for studying flow and transport processes in unsaturated soils (Schaap et al. 2006; Ippisch et al. 2006; Vereecken et al. 2010; Dettmann et al. 2014; Farthing and Ogden 2017; Shein et al. 2018, Latorre et al. 2019; Suk and Park 2019; De Malo et al. 2021 – to name a few). But, as clearly pointed out by Terleev et al. (2017), the assumption of $m = 1 - 1/n\,(n > 1)$ made in the derivation of the MVG relative hydraulic conductivity model is inducing several limitations in it making the application of this model in a heavy soil of less textural homogeneity [like say that of the Beit Netofa Clay (BNC); please see Fig. 8 of Van Genuchten 1980; see also the figure given in Terleev et al. 2017)] quite questionable (Terleev et al. 2017, 2021). Also, the parameter α of the MVG model is often mentioned in the literature as being inversely linked with the air-entry pressure of a soil (van Genuchten 1980; Schaap and van Genuchten 2006; Vereecken et al. 2010; Aschonitis et al. 2015; Lai and Ogden 2015; Pan et al. 2019 – to cite a few). But as shown by Terleev et al. (2021), this linkage of α with the air-entry pressure of a soil cannot possibly be true since with this linkage, van Genuchten's (VG) water retention function [Eq. (4.3)] itself will not be valid at the capillary pressure corresponding to the air-entry pressure of a soil. This is because for such a scenario, the left- hand side of Eq. (4.3) would be equal to one but the right-hand side would be equal to $(2)^{-(1-1/n)}\ (n > 1)$ – as can be seen, a mismatch (Terleev et al. 2021). Also, as lucidly pointed out by Terleev et al. (2017), when $m = 1$, the parameters n and α of the VG model [(Eq. (4.3)] can then be treated as independent interpolating variables; however, when m is taken as $m = 1 - 1/n\ (n > 1)$ these parameters can then no longer be taken as free variables for interpolation with the relation between them becoming increasingly more prominent with the decrease in n values. Thus, the VG model may not work very well for a low n-value soil with the $m = 1 - 1/n\ (n > 1)$ assumption as the regressed parameters n and α may not be then fully independent of each other. Further, for such a situation, the interpolation equation of Eq. (4.3) (which can be obtained by minimizing the square of difference between the measured and estimated values of θ at different capillary pressure in a soil) probably would not be then always very accurate (Terleev 2017). In this context, it is worth noting that for a heavy soil of less textural homogeneity, n is relatively much lower as compared to a light soil of relatively

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good textural homogeneity. Thus, the MVG model – which, as mentioned before, is based on the VG model with the $m = 1 - 1/n$ $(n > 1)$ assumption – may not work very well in a heavy soil of heterogeneous texture as a low n for such a soil would mean a higher dependence of the parameter α on n. However, for light sandy and loamy soils of good textural homogeneity, the dependence of the parameters n on α is relatively less as compared to heavy soils of low textural homogeneity and hence the VG model [with the $m = 1 - 1/n$ $(n > 1)$ assumption] and consequently the MVG model, are all expected to work in an accurate way in these soils. The infiltration works of van Genuchten (1980) show that this is actually what is happening where, in spite of limitations of van Genuchten's (VG) model being brought on it due to the $m = 1 - 1/n$ $(n > 1)$ assumption, it has still been found to predict experimentally observed $\theta - h$ variations in light sandy and loamy soils (Van Genuchten 1980) in a precise way. Thus, the application of the MVG model – which, as already mentioned, is based on the VG model with the $m = 1 - 1/n$ $(n > 1)$ assumption – in estimating the $K - h$ variation in these soils can be considered as quite justifiable. However, while applying the MVG model in a soil, care has to be taken to see that it is being applied primarily in the $\theta - h$ zone of the soil for which the parameters of the model has been estimated through interpolation and not in a zone different from it. As the application of the MVG model in a heavy soil of less textural homogeneity is somewhat dubious and as our Mualem-van Genuchten-based infiltration equation is being built based on this model, the application of our Mualem-van Genuchten-based solution for infiltration studies in a heavy soil of dissimilar particle sizes probably cannot be always justified. This thus can be considered as a limitation of our Mualem-van Genuchten-based infiltration solution.

3. In the derivations of both of our Gardner- and Mualem-van Genuchten-based infiltration equations, we have, as mentioned before, apart from the Darcian assumption, also assumed the flow to be steady and one-dimensional in a heterogeneous soil space. These assumptions were also made by many in the past (Gardner 1958; Warrick 1974, 1988, 1991, 2003, 2005; Lomen and Warrick 1976; Warrick and Yeh 1990; Salvucci 1993; Basha 1994, 1999; Rockhold et al. 1997; Baker 2000; Gastȯ et al. 2002; Zhu and Mohanty 2002; Jury and Horton 2004; Lu et al. 2007; Mohanty and Zhu 2007; Szymkiewicz 2009; Huang and Wu 2012; Sadeghi et al. 2012 – to cite a few) while obtaining solutions to these infiltration equations for homogeneous and layered soils under different flow situations. As mentioned before, our solutions are of a general nature

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as they can handle any valid spatial variations of the parameters of these equations including the spatial distribution of a sink term in an infiltrating space. Even then, however, because of the assumptions inherent in their derivations, they cannot be applied to study transient and/or multi-dimensional infiltration situations. These thus can be considered as additional drawbacks of these equations. It needs, however, to be mentioned here that there are currently no analytical solutions to the Gardner- and Mualem-van Genuchten-based infiltration equations for a purely heterogeneous soil (as a matter of fact, even for a homogeneous soil, there are right now no analytical solutions of these equations for all possible variations of parameters of these equations) even for steady one-dimensional flow situations. We thus believe that even with the assumptions intrinsic in their derivations, our infiltration solutions are still expected to provide a better picture of infiltration mechanics in a field soil than that provided by relatively simple available solutions on the subject. This is because a field soil is seldom homogeneous in nature.

5.5 Scope for Further Study

1. The mathematical models that have been developed here are for Darcian flows only where the infiltration flux has been assumed to be related to the hydraulic gradient in a linear way. As mentioned before, there may be low and high flow rate situations in porous formations where the flow may fail to be Darcian. Naturally, for these situations our Darcian models would not be applicable. For flow in the non-Darcian domain, the relationship between hydraulic gradient and specific flux are generally described by two well known equations, namely Forchheimer and Izbash equations (Wen et al. 2006; Sidiropoulou et al. 2007; Moutsopoulos et al. 2009; Wen et al. 2008a, 2008b; Shi et al. 2018). In the former equation, the hydraulic gradient is expressed as a second degree polynomial function of the specific discharge and in the latter equation the hydraulic gradient is expressed as a power function of the specific discharge. Efforts may be made in future to solve both the boundary value problems considered in the present study for non-Darcian flows as well by making use of Forchheimer and/or Izbash equation(s).

2. As mentioned before, the MVG model – the model that has been used to derive the Mualem-van Genuchten-based infiltration equation in our study – works quite well for light sandy and loamy soils. However, as clearly pointed out by Terleev et al. (2017), the use of this conductivity model in a heavy soil of less textural homogeneity [like say that of the Beit Netofa Clay (BNC) soil; Fig. 8 of Van Genuchten 1980; also see Terleev et al. 2017] is somewhat questionable. With a view to improve some of the shortcomings of this model, Terleev et al. (2017, 2021) put

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forward an improved version of the MVG model capable of tackling both light as well as heavy-textured soils in a satisfactory way. The big advantage of this relative conductivity model lies in its inherent inclusivity as it has the capability to describe hydraulic conductivity variations in light as well as heavy soils of homogeneous or less homogeneous texture in an accurate way. This, however, is not always true of the MVG model which works mostly well, as mentioned before, for sandy and loamy soils of good textural homogeneity but not in a heavy-textured soil like that of the BNC soil. We believe, efforts should also be made in future to solve the Richards'-based infiltration equation by utilizing Terleev et al.'s (2917, 2021) conductivity model as well.

3. Efforts may also be made in future to relax the steady-state assumption made in the derivations of both the flow problems considered in the study and attempts may be made to solve these problems for transient situations as well. Further, solutions of these problems for two- and three-dimensional infiltration situations may also be attempted in future under both steady as well as transient conditions.

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Manuscript Published from the Thesis Work Till Now

The following manuscript has till now been published (currently on an on-line mode) from the work carried out in the thesis.

Talukdar, J. and Barua, G. (2022). An analytical solution of the one-dimensional steady-state Van Genuchten-based infiltration equation for a heterogeneous soil with a root-water extraction function. *Eurasian Soil Science* (Soil Physics Division), Pleiades Publishing, Springer, DOI:10.1134/S106422932206014X
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APPENDIX A

Coefficients of the Series Solutions as obtained from the Proposed Approaches for the Infiltration Problem as Shown in Fig. 3.10

The solution as obtained from of our first approach – where $\psi_V(x)$, $\psi'_V(x)$ and $\psi''_V(x)$ are being used to arrive at the solution – for the flow situation of Fig. 3.10 can be expressed as

$$h_{(1)}^{(1)}(x)=C_{1(1)}^{(1)}\left(\frac{L_{(1)}-x}{L_{(1)}}\right)+h_{L_{(1)}}\left(\frac{x}{L_{(1)}}\right)+C_{2(1)}^{(1)}\left[x(x-L_{(1)})\right]+C_{3(1)}^{(1)}\left[x^2(x-L_{(1)})\right]+C_{4(1)}^{(1)}\left[x^2(x-L_{(1)})^2\right]$$
$$+C_{5(1)}^{(1)}\left[x^3(x-L_{(1)})^2\right]+C_{6(1)}^{(1)}\left[x^3(x-L_{(1)})^3\right]+C_{7(1)}^{(1)}\left[x^4(x-L_{(1)})^3\right]+C_{8(1)}^{(1)}\left[x^4(x-L_{(1)})^4\right]$$
$$+C_{9(1)}^{(1)}\left[x^5(x-L_{(1)})^4\right]+C_{10(1)}^{(1)}\left[x^5(x-L_{(1)})^5\right]+C_{11(1)}^{(1)}\left[x^6(x-L_{(1)})^5\right]+C_{12(1)}^{(1)}\left[x^6(x-L_{(1)})^6\right]$$
$$+C_{13(1)}^{(1)}\left[x^7(x-L_{(1)})^6\right]+C_{14(1)}^{(1)}\left[x^7(x-L_{(1)})^7\right]+C_{15(1)}^{(2)}\left[x^8(x-L_{(1)})^7\right] \quad \text{(A1)}$$

for $0 \le x \le 5$ cm, where $L_{(1)} = 5$ cm, $h_{L_{(1)}}$ = -764.672 cm, $D_{(1)}^{(1)}$ = {0, 1.5, 2.5, 3.75, 5} and $C_{1(1)}^{(1)}$ = -2.611248e+03, $C_{2(1)}^{(1)}$ = -1.990977e+02, $C_{3(1)}^{(1)}$ = 2.976846e+01, $C_{4(1)}^{(1)}$ = -2.960910e+01, $C_{5(1)}^{(1)}$ = 5.005323e+00, $C_{6(1)}^{(1)}$ = 6.292101e-01, $C_{7(1)}^{(1)}$ = -6.896328e-02, $C_{8(1)}^{(1)}$ = 8.346156e+00, $C_{9(1)}^{(1)}$ = -4.571722e-01, $C_{10(1)}^{(1)}$ = 3.551316e+00, $C_{11(1)}^{(1)}$ = -9.512429e-02, $C_{12(1)}^{(1)}$ = 5.764092e-01, $C_{13(1)}^{(1)}$ = -5.755960e-03, $C_{14(1)}^{(1)}$ = 3.317729e-02, $C_{15(1)}^{(1)}$ = 9.888049e-06;

$h_{(2)}^{(1)}(X_1) = f(X_1)$ for 5 cm $\le x \le$ 15 cm, where $f(X_1)$ bears the same functional form as the right-hand-side of Eq. (A1) except that for this domain x and $L_{(1)}$ of Eq. (A1) are now X_1 $X_1 = x - 5$ cm and $L_{(2)}$, respectively and $L_{(2)} = 10$ cm, $h_{L_{(2)}}$ = -297.415 cm, $S_{D(2)}^{(1)}$ = {0, 2.5, 5, 7.5, 10} and $C_{1(2)}^{(1)}$ = -7.646720e+02, $C_{2(2)}^{(1)}$ = -7.129898e+00, $C_{3(2)}^{(1)}$ = 4.334450e-01, $C_{4(2)}^{(1)}$ = -8.272926e-02, $C_{5(2)}^{(1)}$ = 5.624140e-03, $C_{6(2)}^{(1)}$ = -1.185245e-04, $C_{7(2)}^{(1)}$ = 1.777501e-05, $C_{8(2)}^{(1)}$ = 1.226685e-03, $C_{9(2)}^{(1)}$ = -4.685022e-06, $C_{10(2)}^{(1)}$ = 1.535270e-04, $C_{11(2)}^{(1)}$ = -3.416105e-07, $C_{12(2)}^{(1)}$ = 6.804551e-06, $C_{13(2)}^{(1)}$ = -8.833827e-09, $C_{14(2)}^{(1)}$ = 1.027571e-07, $C_{15(2)}^{(1)}$ = -2.121659e-11;

$h_{(3)}^{(1)}(X_2) = f(X_2)$ for 15 cm $\le x \le$ 35 cm, where $f(X_2)$ has the same functional form as that of the right-hand-side of Eq. (A1) but x and $L_{(1)}$ of Eq. (A1) are now $X_2 = x - 15$ cm are $L_{(3)}$, respectively where $L_{(3)} = 20$ cm, $h_{L_{(3)}}$ = -123.654 cm, $S_{D(3)}^{(1)}$ = {0, 5, 10, 15, 20} and $C_{1(3)}^{(1)}$ = -2.974152e+02, $C_{2(3)}^{(1)}$ = -5.041545e-01, $C_{3(3)}^{(1)}$ = 1.383268e-02, $C_{4(3)}^{(1)}$ = -9.207213e-04, $C_{5(3)}^{(1)}$ = 2.710922e-05, $C_{6(3)}^{(1)}$ = -7.828771e-07, $C_{7(3)}^{(1)}$ = 2.873352e-08, $C_{8(3)}^{(1)}$ = 7.538289e-07,

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$C^{(1)}_{9(3)}$ = 2.286022e-09, $C^{(1)}_{10(3)}$ = 2.523246e-08, $C^{(1)}_{11(3)}$ = 2.022762e-11, $C^{(1)}_{12(3)}$ = 2.882684e-10, $C^{(1)}_{13(3)}$ = -1.508706e-14, $C^{(1)}_{14(3)}$ = 1.105069e-12, $C^{(1)}_{15(3)}$ = -1.111833e-16;

$h^{(1)}_{(4)}(X_3) = f(X_3)$ for $35\,\text{cm} \le x \le 105$ cm, where $f(X_3)$ has the same functional form as that of the right- hand-side of Eq. (A1) but x and $L_{(1)}$ of Eq. (2.46) are now $X_3 = x - 35$ cm and $L_{(4)}$, respectively where $L_{(4)} = 70$ cm, $h_{L_{(4)}} = 0\,\text{cm}$, $S^{(1)}_{D(4)} = \{0, 17.5, 35, 52.5, 70\}$ and $C^{(1)}_{1(4)}$ = -1.236538e+02, $C^{(1)}_{2(4)}$ = -3.387958e-02, $C^{(1)}_{3(4)}$ = 3.438954e-04, $C^{(1)}_{4(4)}$ = -9.333396e-06, $C^{(1)}_{5(4)}$ = 9.174577e-08, $C^{(1)}_{6(4)}$ = -2.124469e-09, $C^{(1)}_{7(4)}$ = 2.009488e-11, $C^{(1)}_{8(4)}$ = 5.765185e-11, $C^{(1)}_{9(4)}$ = 9.916770e-14, $C^{(1)}_{10(4)}$ = 1.659507e-13, $C^{(1)}_{11(4)}$ = 7.990829e-17, $C^{(1)}_{12(4)}$ = 1.577834e-16, $C^{(1)}_{13(4)}$ = 8.447671e-21, $C^{(1)}_{14(4)}$ = 4.979434e-20, $C^{(1)}_{15(4)}$ = -6.472014e-25.

For this problem, a solution can also be obtained using our second approach; calling this solution as $h^{(2)}(x)$, it can be written as

$$\begin{aligned} h^{(2)}_{(1)}(x) = {} & C^{(2)}_{1(1)}\left(\frac{L-x}{L}\right) + h_{L_{(1)}}\left(\frac{x}{L}\right) + C^{(2)}_{2(1)}\left[x(x-L)\right] + C^{(2)}_{3(1)}\left[x^2(x-L)\right] + C^{(2)}_{4(1)}\left[x^2(x-L)^2\right] \\ & + C^{(2)}_{5(1)}\left[x^3(x-L)^2\right] + C^{(2)}_{6(1)}\left[x^3(x-L)^3\right] + C^{(2)}_{7(1)}\left[x^4(x-L)^3\right] + C^{(2)}_{8(1)}\left[x^4(x-L)^4\right] \\ & + C^{(2)}_{9(1)}\left[x^5(x-L)^4\right] + C^{(2)}_{10(1)}\left[x^5(x-L)^5\right] + C^{(2)}_{11(1)}\left[x^6(x-L)^5\right] + C^{(2)}_{12(1)}\left[x^6(x-L)^6\right] \\ & + C^{(2)}_{13(1)}\left[x^7(x-L)^6\right] + C^{(2)}_{14(1)}\left[x^7(x-L)^7\right] + C^{(2)}_{15(1)}\left[x^8(x-L)^7\right] + C^{(2)}_{16(1)}\left[x^8(x-L)^8\right] \\ & + C^{(2)}_{17(1)}\left[x^9(x-L)^8\right] + C^{(2)}_{18(1)}\left[x^9(x-L)^9\right] + C^{(2)}_{19(1)}\left[x^{10}(x-L)^9\right] + C^{(2)}_{20(1)}\left[x^{10}(x-L)^{10}\right] \end{aligned} \quad \text{(A2)}$$

for $0 \le x \le 15\,\text{cm}$, where $L_{(1)} = 15\,\text{cm}$, $h_{L_{(1)}} = -290.129\,\text{cm}$, $S^{(2)}_{D(1)}$ = {0,0.798,1.578, 2.368, 3.157, 3.947,4.736, 5.526, 6.315, 7.105, 7.894, 8.684, 9.437, 10.263,11.052, 11.842, 12.631, 13.421, 14.210, 15} and $C^{(2)}_{1(1)}$ = -2.662015e+03, $C^{(2)}_{2(1)}$ = -8.401461e+01, $C^{(2)}_{3(1)}$ = 4.977266e+00, $C^{(2)}_{4(1)}$ = -2.629643e+00, $C^{(2)}_{5(1)}$ = 1.553906e-01, $C^{(2)}_{6(1)}$ = -7.877121e-02, $C^{(2)}_{7(1)}$ = 4.598427e-03, $C^{(2)}_{8(1)}$ = -2.104329e-03, $C^{(2)}_{9(1)}$ = 1.190164e-04, $C^{(2)}_{10(1)}$ = -4.617153e-05, $C^{(2)}_{11(1)}$ = 2.459693e-06, $C^{(2)}_{12(1)}$ = -7.735672e-07, $C^{(2)}_{13(1)}$ = 3.736272e-08, $C^{(2)}_{14(1)}$ = -9.272973e-09, $C^{(2)}_{15(1)}$ = 3.833036e-10, $C^{(2)}_{16(1)}$ = -7.376152e-11, $C^{(2)}_{17(1)}$ = 2.346703e-12, $C^{(2)}_{18(1)}$ = -3.445412e-13, $C^{(2)}_{19(1)}$ = 6.438923e-15, $C^{(2)}_{20(1)}$ = -7.064379e-16;

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$h_{(2)}^{(2)}(X_1) = f(X_1)$ for $15 \leq x \leq 105$ cm, where $f(X_1)$ has the same functional form as that of the right-hand-side of Eq. (A2) but x and $L_{(1)}$ of Eq. Eq. (A2) are now X_1 (where $X_1 = x - 15$ cm and $L_{(2)}$, respectively and $L_{(2)} = 90$ cm, $h_{L_{(2)}} = 0$ cm, $S_{D(2)}^{(2)} = \{0, 22.5, 45, 67.5, 90\}$ and $C_{1(2)}^{(2)} =$ -2.901288e+02, $C_{2(2)}^{(2)} =$ -1.632366e-01, $C_{3(2)}^{(2)} =$ 1.549088e-03, $C_{4(2)}^{(2)} =$ -9.086161e-05, $C_{5(2)}^{(2)} =$ 8.507784e-07, $C_{6(2)}^{(2)} =$ -4.979686e-08, $C_{7(2)}^{(2)} =$ 4.638761e-10, $C_{8(2)}^{(2)} =$ -2.611278e-11, $C_{9(2)}^{(2)} =$ 2.393819e-13, $C_{10(2)}^{(2)} =$ -1.229111e-14, $C_{11(2)}^{(2)} =$ 1.082116e-16, $C_{12(2)}^{(2)} =$ -4.778207e-18, $C_{13(2)}^{(2)} =$ 3.886729e-20, $C_{14(2)}^{(2)} =$ -1.405904e-21, $C_{15(2)}^{(2)} =$ 9.955315e-24, $C_{16(2)}^{(2)} =$ -2.845717e-25, $C_{17(2)}^{(2)} =$ 1.575884e-27, $C_{18(2)}^{(2)} =$ -3.455840e-29, $C_{19(2)}^{(2)} =$ 1.142947e-31, $C_{20(2)}^{(2)} =$ -1.862806e-33.

In all these equations above, it is to be noted that $h_{(i)}^{(j)}$ denotes the i-domain solution to the problem obtained using our j^{th} approach. Further, in these solutions, $C_{p(q)}^{(r)}$ denotes the coefficient attached to the p^{th} term [Eq. (3.30)] of the q-domain solution to the problem obtained using the r^{th} approach.

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APPENDIX B

B.1 First and Second Differentiations of the $\psi_V(x)$ Function

The first derivative of Eq. (4.10) works out as

$$\psi_V'(x) = K_s'(x)\left\{1-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\right\}^2$$

$$\times\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}[h'(x)-\sin A]+2K_s(x)\left\{1-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}\right.$$

$$\left.\times\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\right\}\left\{-(-1)[-\alpha(x)h(x)]^{-2}[-\alpha'(x)h(x)-\alpha(x)h'(x)]\right.$$

$$\times[-\alpha(x)h(x)]^{n(x)}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}$$

$$\times\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}$$

$$-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}$$

$$\times\left\{-m'(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}-\frac{m(x)[-\alpha(x)h(x)]^{n(x)}}{1+[-\alpha(x)h(x)]^{n(x)}}\right.$$

$$\left.\left.\times\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\right\}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}$$

$$\times[h'(x)-\sin A]$$

$$+K_s(x)\left\{1-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\right\}^2$$

$$\times\left\{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left\{-\frac{m'(x)}{2}\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}\right.\right.$$

$$-\frac{\frac{m(x)}{2}\left[-\alpha(x)h(x)\right]^{n(x)}}{1+\left[-\alpha(x)h(x)\right]^{n(x)}}\times\left\{n'(x)\log_e\left[-\alpha(x)h(x)\right]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\left[h'(x)-\sin A\right]$$

$$\left.+\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}}h''(x)\right\} \qquad \text{(B1)}$$

Also, differentiation of Eq. (B1) gives $\psi_V''(s)$ as

$$\psi_V''(x)=K_s''(x)\left\{1-\left[-\alpha(x)h(x)\right]^{-1}\left[-\alpha(x)h(x)\right]^{n(x)}\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}\right\}^2$$

$$\times\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left[h'(x)-\sin A\right]+4K_s'(x)$$

$$\times\left\{1-\left[-\alpha(x)h(x)\right]^{-1}\left[-\alpha(x)h(x)\right]^{n(x)}\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}\right\}$$

$$\times\left\{-(-1)\left[-\alpha(x)h(x)\right]^{-2}\left[-\alpha'(x)h(x)-\alpha(x)h'(x)\right]\left[-\alpha(x)h(x)\right]^{n(x)}\right.$$

$$\times\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}-\left[-\alpha(x)h(x)\right]^{-1}$$

$$\times\left\{\left[-\alpha(x)h(x)\right]^{n(x)}\left\{n'(x)\log_e\left[-\alpha(x)h(x)\right]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}$$

$$\times\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}-\left[-\alpha(x)h(x)\right]^{-1}\left[-\alpha(x)h(x)\right]^{n(x)}$$

$$\times\left\{\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}\left\{-m'(x)\log_e\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}-\frac{m(x)}{1+\left[-\alpha(x)h(x)\right]^{n(x)}}\right.\right.$$

$$\left.\left.\left.\times\left\{\left[-\alpha(x)h(x)\right]^{n(x)}\left\{n'(x)\log_e\left[-\alpha(x)h(x)\right]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\right\}\right\}\right\}$$

$$\times\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left[h'(x)-\sin A\right]+2K_s'(x)$$

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$$\times\left\{1-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\right\}^{2}$$

$$\times\left\{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left\{\frac{-m'(x)}{2}\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}\right.\right.$$

$$\left.\left.-\frac{0.5m(x)[-\alpha(x)h(x)]^{n(x)}}{1+[-\alpha(x)h(x)]^{n(x)}}\times\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}$$

$$\left.\times[h'(x)-\sin A]+\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}h''(x)\right\}$$

$$+2K_s(x)\times\left\{-(-1)[-\alpha(x)h(x)]^{-2}[-\alpha'(x)h(x)-\alpha(x)h'(x)][-\alpha(x)h(x)]^{n(x)}\right.$$

$$\times\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}-[-\alpha(x)h(x)]^{-1}$$

$$\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}$$

$$\times\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}$$

$$\times\left\{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\left\{-m'(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}-\frac{m(x)}{1+[-\alpha(x)h(x)]^{n(x)}}\right.\right.$$

$$\left.\left.\left.\left.\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\right\}\right\}\right\}^{2}$$

$$\times\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}[h'(x)-\sin A]$$

$$+2K_s(x)\left\{1-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\right\}$$

$$\times\left\{-\left\{(-1)(-2)[-\alpha(x)h(x)]^{-3}[-\alpha'(x)h(x)-\alpha(x)h'(x)]^{2}\right.\right.$$

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$$+(-1)[-\alpha(x)h(x)]^{-2}[-\alpha''(x)h(x)-2\alpha'(x)h'(x)-\alpha(x)h''(x)]\Big\}[-\alpha(x)h(x)]^{n(x)}$$

$$\times\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}-[-\alpha(x)h(x)]^{-1}$$

$$\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}^2\right.$$

$$+[-\alpha(x)h(x)]^{n(x)}\left\{n''(x)\log_e[-\alpha(x)h(x)]+2n'(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right.$$

$$\left.\left.+n(x)\left\{\frac{\alpha''(x)\alpha(x)-[\alpha'(x)]^2}{[\alpha(x)]^2}+\frac{h''(x)h(x)-[h'(x)]^2}{[h(x)]^2}\right\}\right\}\right\}\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}$$

$$-[-\alpha(x)h(x)]^{-1}[-\alpha(x)h(x)]^{n(x)}$$

$$\times\left\{\left\{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\left\{-m'(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}-\frac{m(x)}{1+[-\alpha(x)h(x)]^{n(x)}}\right.\right.\right.$$

$$\left.\left.\left.\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\right\}\right\}\right\}$$

$$\times\left\{-m'(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}-\frac{m(x)[-\alpha(x)h(x)]^{n(x)}}{1+[-\alpha(x)h(x)]^{n(x)}}\right.$$

$$\left.\times\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}$$

$$+\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\left\{-m''(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}\right.$$

$$-\frac{2m'(x)[-\alpha(x)h(x)]^{n(x)}}{1+[-\alpha(x)h(x)]^{n(x)}}\times\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}$$

$$+\frac{m(x)}{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^2}\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]\right.\right.$$

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$$+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\Bigg\}\Bigg\}^2 - \frac{m(x)}{1+[-\alpha(x)h(x)]^{n(x)}}$$

$$\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}^2\right.$$

$$+[-\alpha(x)h(x)]^{n(x)}\left\{n''(x)\log_e[-\alpha(x)h(x)]+2n'(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right.$$

$$+n(x)\left\{\frac{\alpha''(x)\alpha(x)-[\alpha'(x)]^2}{[\alpha(x)]^2}+\frac{h''(x)h(x)-[h'(x)]^2}{[h(x)]^2}\right\}\Bigg\}\Bigg\}\Bigg\}\Bigg\}\Bigg\}$$

$$-2(-1)[-\alpha(x)h(x)]^{-2}[-\alpha'(x)h(x)-\alpha(x)h'(x)]$$

$$\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}$$

$$\times\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)} -2(-1)[-\alpha(x)h(x)]^{-2}$$

$$\times[-\alpha'(x)h(x)-\alpha(x)h'(x)][-\alpha(x)h(x)]^{n(x)}$$

$$\times\left\{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\left\{-m'(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}-\frac{m(x)}{1+[-\alpha(x)h(x)]^{n(x)}}\right.\right.$$

$$\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\Bigg\}\Bigg\}$$

$$-[-\alpha(x)h(x)]^{-1}\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}$$

$$\times\left\{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{-m(x)}\left\{-m'(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}-\frac{m(x)}{1+[-\alpha(x)h(x)]^{n(x)}}\right.\right.$$

$$\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\Bigg\}\Bigg\}\Bigg\}$$

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$$\times\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left[h'(x)-\sin A\right]$$

$$+4K_s(x)\left\{1-\left[-\alpha(x)h(x)\right]^{-1}\left[-\alpha(x)h(x)\right]^{n(x)}\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}\right\}$$

$$\times\left\{-(-1)\left[-\alpha(x)h(x)\right]^{-2}\left[-\alpha'(x)h(x)-\alpha(x)h'(x)\right]\left[-\alpha(x)h(x)\right]^{n(x)}\right.$$

$$\times\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}-\left[-\alpha(x)h(x)\right]^{-1}$$

$$\times\left\{\left[-\alpha(x)h(x)\right]^{n(x)}\left\{n'(x)\log_e\left[-\alpha(x)h(x)\right]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}$$

$$\times\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}-\left[-\alpha(x)h(x)\right]^{-1}\left[-\alpha(x)h(x)\right]^{n(x)}$$

$$\times\left\{\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}\left\{-m'(x)\log_e\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}\right.\right.$$

$$\left.\left.\left.-\left\{\frac{m(x)\left[-\alpha(x)h(x)\right]^{n(x)}}{1+\left[-\alpha(x)h(x)\right]^{n(x)}}\left\{n'(x)\log_e\left[-\alpha(x)h(x)\right]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\right\}\right\}\right\}$$

$$\times\left\{\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left\{\frac{-m'(x)}{2}\log_e\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}\right.\right.$$

$$\left.\left.-\frac{\frac{m(x)}{2}\left[-\alpha(x)h(x)\right]^{n(x)}}{1+\left[-\alpha(x)h(x)\right]^{n(x)}}\times\left\{n'(x)\log_e\left[-\alpha(x)h(x)\right]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\right\}$$

$$\left.\times\left[h'(x)-\sin A\right]+\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}}h''(x)\right\}$$

$$+K_s(x)\left\{1-\left[-\alpha(x)h(x)\right]^{-1}\left[-\alpha(x)h(x)\right]^{n(x)}\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{-m(x)}\right\}^{2}$$

$$\times\left\{\left\{\left\{\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left\{-0.5m'(x)\log_e\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}\right.\right.\right.\right.$$

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$$-\left\{\frac{0.5m(x)[-\alpha(x)h(x)]^{n(x)}}{1+[-\alpha(x)h(x)]^{n(x)}}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\Bigg\}\Bigg\}$$

$$\times\left\{-0.5m'(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}-\frac{0.5m(x)[-\alpha(x)h(x)]^{n(x)}}{1+[-\alpha(x)h(x)]^{n(x)}}\right.$$

$$\left.\times\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}$$

$$+\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left\{-0.5m''(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}\right.$$

$$-\frac{m'(x)[-\alpha(x)h(x)]^{n(x)}}{1+[-\alpha(x)h(x)]^{n(x)}}\times\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}$$

$$+\frac{0.5m(x)}{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^2}\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]\right.\right.$$

$$\left.\left.+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}^2-\frac{0.5m(x)}{1+[-\alpha(x)h(x)]^{n(x)}}$$

$$\times\left\{[-\alpha(x)h(x)]^{n(x)}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}^2\right.$$

$$+[-\alpha(x)h(x)]^{n(x)}\left\{n''(x)\log_e[-\alpha(x)h(x)]+2n'(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right.$$

$$\left.\left.\left.+n(x)\left\{\frac{\alpha''(x)\alpha(x)-[\alpha'(x)]^2}{[\alpha(x)]^2}+\frac{h''(x)h(x)-[h'(x)]^2}{[h(x)]^2}\right\}\right\}\right\}\right\}[h'(x)-\sin A]$$

$$+2\left\{\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}^{\frac{-m(x)}{2}}\left\{-0.5m'(x)\log_e\left\{1+[-\alpha(x)h(x)]^{n(x)}\right\}\right.\right.$$

$$\left.\left.-\left\{\frac{0.5m(x)[-\alpha(x)h(x)]^{n(x)}}{1+[-\alpha(x)h(x)]^{n(x)}}\left\{n'(x)\log_e[-\alpha(x)h(x)]+n(x)\left[\frac{h'(x)}{h(x)}+\frac{\alpha'(x)}{\alpha(x)}\right]\right\}\right\}\right\}\right\}h''(x)$$

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$$+\left\{1+\left[-\alpha(x)h(x)\right]^{n(x)}\right\}^{\frac{-m(x)}{2}} h'''(x)\Bigg\} \tag{B2}$$

B.2 Single and Two-Spline Analytical Solutions for the Infiltration Situations as Shown in Figs. 4.4 and 4.5

The two-spline solution as obtained from of our first approach – where $\psi_V(x)$, $\psi_V'(x)$ and $\psi_V''(x)$ are being used to arrive at the solutions – for the flow situation of Fig. 4.4 can be expressed for $0 \le x \le 50\,\text{cm}$ as

$$\begin{aligned} h_{(1)}^{(1)}(x) = {} & D_{1(1)}^{(1)}\left(\frac{L_{(1)}-x}{L_{(1)}}\right) + h_{L_{(1)}}\left(\frac{x}{L_{(1)}}\right) + D_{2(1)}^{(1)}\left[x(x-L_{(1)})\right] + D_{3(1)}^{(1)}\left[x^2(x-L_{(1)})\right] \\ & + D_{4(1)}^{(1)}\left[x^2(x-L_{(1)})^2\right] + D_{5(1)}^{(1)}\left[x^3(x-L_{(1)})^2\right] + D_{6(1)}^{(1)}\left[x^3(x-L_{(1)})^3\right] \\ & + D_{7(1)}^{(1)}\left[x^4(x-L_{(1)})^3\right] + D_{8(1)}^{(1)}\left[x^4(x-L_{(1)})^4\right] + D_{9(1)}^{(1)}\left[x^5(x-L_{(1)})^4\right] \\ & + D_{10(1)}^{(1)}\left[x^5(x-L_{(1)})^5\right] + D_{11(1)}^{(1)}\left[x^6(x-L_{(1)})^5\right] + D_{12(1)}^{(1)}\left[x^6(x-L_{(1)})^6\right] \\ & + D_{13(1)}^{(1)}\left[x^7(x-L_{(1)})^6\right] + D_{14(1)}^{(1)}\left[x^7(x-L_{(1)})^7\right] + D_{15(1)}^{(2)}\left[x^8(x-L_{(1)})^7\right] \end{aligned} \tag{B3}$$

where $h_{(1)}^{(1)}$ is the suction head for the domain $0 \le x \le 50\,\text{cm}$, $L_{(1)} = 50\,\text{cm}$, $h_{L_{(1)}}$ =-34.644 cm, $S_{D(1)}^{(1)} = \{0, 12.5, 25, 37.5, 50\}$ and $D_{1(1)}^{(1)}$ =-6.207374e+01, $D_{2(1)}^{(1)}$ =3.478904e-03, $D_{3(1)}^{(1)}$ =-2.277651e-05, $D_{4(1)}^{(1)}$ =-1.870500e-07, $D_{5(1)}^{(1)}$ =3.329695e-09, $D_{6(1)}^{(1)}$ =2.500082e-11, $D_{7(1)}^{(1)}$ =-9.048954e-13, $D_{8(1)}^{(1)}$ =5.743037e-12, $D_{9(1)}^{(1)}$ =9.270131e-15, $D_{10(1)}^{(1)}$ =3.171875e-14, $D_{11(1)}^{(1)}$ = 1.651240e-17, $D_{12(1)}^{(1)}$ =5.882284e-17, $D_{13(1)}^{(1)}$ =5.798867e-21, $D_{14(1)}^{(1)}$ =3.634609e-20, $D_{15(1)}^{(1)}$ = 2.293135e-24.

The suction head distribution, $h_{(2)}^{(1)}(X_1)$, on the second division of the flow space (i.e., for $50 \le x \le 100$ cm) can be expressed as $h_{(2)}^{(1)}(X_1) = f(X_1)$, where $f(X_1)$ bears the same form as that of the right-hand-side of Eq. (B3) except that x and $L_{(1)}$ of this equation are now $X_1 = x - 50\,\text{cm}$ and $L_{(2)} = 50$ cm, respectively, and $h_{L_{(2)}} = 0$ cm, $S_{D(2)}^{(1)} = \{0, 12.5, 25, 37.5, 49.5\}$ and $D_{1(2)}^{(1)}$ =-3.464479e+01, $D_{2(1)}^{(1)}$ =5.462558e-04, $D_{3(1)}^{(1)}$ =-6.763595e-06, $D_{4(1)}^{(1)}$ =1.096377e-07, $D_{5(1)}^{(1)}$ =-1.158709e-09, $D_{6(1)}^{(1)}$ =-1.267879e-11, $D_{7(1)}^{(1)}$ =-5.695674e-12, $D_{8(1)}^{(1)}$ =2.136544e-

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12, $D^{(1)}_{9(1)}$ = 2.842264e-14, $D^{(1)}_{10(1)}$ = -1.157870e-14, $D^{(1)}_{11(1)}$ = -5.045524e-17, $D^{(1)}_{12(1)}$ = -2.128104e-17, $D^{(1)}_{13(1)}$ = -3.612504e-20, $D^{(1)}_{14(1)}$ = -1.306597e-20, $D^{(1)}_{15(1)}$ = -9.310269e-24.

We would like to mention once again here that the integer within the superscript in a constant signifies the approach that has been used to obtain it (i.e., it is 1 if the first approach is being used and 2 if the second approach is being adopted) and the integer within the subscript in a constant signifies the spline number to which it is being attached to.

For this problem, a solution can also be obtained using our second approach without the necessity of dividing the domain into smaller parts; calling this solution as $h^{(2)}(x)$, it can be written for the whole space $0 \le x \le 100\,\text{cm}$ as

$$\begin{aligned} h^{(2)}(x) = {} & D^{(2)}_1\left(\frac{L-x}{L}\right) + h_{L_{(1)}}\left(\frac{x}{L}\right) + D^{(2)}_2\left[x(x-L)\right] + D^{(2)}_3\left[x^2(x-L)\right] + D^{(2)}_4\left[x^2(x-L)^2\right] \\ & + D^{(2)}_5\left[x^3(x-L)^2\right] + D^{(2)}_6\left[x^3(x-L)^3\right] + D^{(2)}_7\left[x^4(x-L)^3\right] + D^{(2)}_8\left[x^4(x-L)^4\right] \\ & + D^{(2)}_9\left[x^5(x-L)^4\right] + D^{(2)}_{10}\left[x^5(x-L)^5\right] + D^{(2)}_{11}\left[x^6(x-L)^5\right] + D^{(2)}_{12}\left[x^6(x-L)^6\right] \\ & + D^{(2)}_{13}\left[x^7(x-L)^6\right] + D^{(2)}_{14}\left[x^7(x-L)^7\right] + D^{(2)}_{15}\left[x^8(x-L)^7\right] + D^{(2)}_{16}\left[x^8(x-L)^8\right] \\ & + D^{(2)}_{17}\left[x^9(x-L)^8\right] + D^{(2)}_{18}\left[x^9(x-L)^9\right] + D^{(2)}_{19}\left[x^{10}(x-L)^9\right] + D^{(2)}_{20}\left[x^{10}(x-L)^{10}\right] \end{aligned} \quad \text{(B4)}$$

where $L = 100\,\text{cm}$, $h_L = 0$ cm, $S^{(2)}_D$ = {0, 5.26, 10.526, 15.789, 21.052, 26.315, 31.578, 36.842, 42.105, 47.368, 52.631, 57.894, 63.157, 68.421, 73.684, 78.947, 84.210, 89.473, 94.736, 100} and $D^{(2)}_1$ = -6.206954e+01, $D^{(2)}_2$ = 2.459681e-03, $D^{(2)}_3$ = 1.633586e-05, $D^{(2)}_4$ = 8.181397e-09, $D^{(2)}_5$ = 7.524982e-10, $D^{(2)}_6$ = -1.157517e-11, $D^{(2)}_7$ = 1.446560e-16, $D^{(2)}_8$ = 1.200241e-15, $D^{(2)}_9$ = -2.399497e-17, $D^{(2)}_{10}$ = -2.724640e-19, $D^{(2)}_{11}$ = -8.182601e-21, $D^{(2)}_{12}$ = -1.182182e-22, $D^{(2)}_{13}$ = -3.870820e-24, $D^{(2)}_{14}$ = -4.947000e-26, $D^{(2)}_{15}$ = -1.000069e-27, $D^{(2)}_{16}$ = -1.384145e-29, $D^{(2)}_{17}$ = -1.498263e-31, $D^{(2)}_{18}$ = -2.116308e-33, $D^{(2)}_{19}$ = -1.008747e-35, $D^{(2)}_{20}$ = -1.425660e-37.

Similarly, for the flow situation as shown in Figs. 4.5(i) and 4.5(ii), the parameters of the solution as obtained from our first approach in the domain $0 \le x \le 50$ cm can be expressed as $L_{(1)} = 50\,\text{cm}$, $h_{L_{(1)}}$ = -48.35574cm, $G^{(1)}_{(1)}$ = {0, 12.5, 25, 37.5, 50} and $D^{(1)}_{1(1)}$ = -7.582865e+01, $D^{(1)}_{2(1)}$ = 6.753272e-03, $D^{(1)}_{3(1)}$ = -4.653297e-06, $D^{(1)}_{4(1)}$ = -8.337282e-07, $D^{(1)}_{5(1)}$ = -4.240856e-10, $D^{(1)}_{6(1)}$ = 1.530659e-10, $D^{(1)}_{7(1)}$ = 1.928271e-13, $D^{(1)}_{8(1)}$ = -7.702062e-13, $D^{(1)}_{9(1)}$ = -2.691805e-15,

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$D^{(1)}_{10(1)}$ = -4.066318e-15, $D^{(1)}_{11(1)}$ = -4.161608e-18, $D^{(1)}_{12(1)}$ = -7.547172e-18, $D^{(1)}_{13(1)}$ = -1.832567e-21, $D^{(1)}_{14(1)}$ = -4.663185e-21, $D^{(1)}_{15(1)}$ = -1.151302e-25 and in the domain $50 \le x \le 100$ cm, as $X_1 = x - 50$ cm and $L_{(2)} = 50$ cm, respectively, $h_{L_{(2)}} = 0$ cm, $G^{(1)}_{(2)} = \{0, 12.5, 25, 37.5, 49.5\}$ and $D^{(1)}_{1(2)}$ = -4.835574e+01, $D^{(1)}_{2(1)}$ = 1.832526e-03, $D^{(1)}_{3(1)}$ = -2.501710e-05, $D^{(1)}_{4(1)}$ = 4.366251e-07, $D^{(1)}_{5(1)}$ = -3.204337e-09, $D^{(1)}_{6(1)}$ = -6.296087e-11, $D^{(1)}_{7(1)}$ = -4.938138e-12, $D^{(1)}_{8(1)}$ = -5.580166e-12, $D^{(1)}_{9(1)}$ = -3.434261e-14, $D^{(1)}_{10(1)}$ = -3.044146e-14, $D^{(1)}_{11(1)}$ = -6.019594e-17, $D^{(1)}_{12(1)}$ = -5.609660e-17, $D^{(1)}_{13(1)}$ = -4.042027e-20, $D^{(1)}_{14(1)}$ = -3.446846e-20, $D^{(1)}_{15(1)}$ = -1.315682e-23.

Also, for this flow situation, the parameters of the full-domain solution (i.e., for $0 \le x \le 100$ cm) obtained using our second approach can be written as

$L = 100$ cm, $h_L = 0$ cm, $S^{(2)}_D = \{0,$ 5.26, 10.526, 15.789, 21.052, 26.315, 31.578, 36.842, 42.105, 47.368, 52.631, 57.894, 63.157, 68.421, 73.684, 78.947, 84.210, 89.473, 94.736, 100$\}$ and $D^{(2)}_1$ = -7.583090e+01, $D^{(2)}_2$ = 5.466185e-03, $D^{(2)}_3$ = -3.087294e-05, $D^{(2)}_4$ = -3.655463e-07, $D^{(2)}_5$ = 4.404691e-09, $D^{(2)}_6$ = 2.829426e-11, $D^{(2)}_7$ = -6.250584e-13, $D^{(2)}_8$ = -1.320437e-16, $D^{(2)}_9$ = 3.657149e-17, $D^{(2)}_{10}$ = -1.301595e-19, $D^{(2)}_{11}$ = -2.755609e-20, $D^{(2)}_{12}$ = 1.674464e-22, $D^{(2)}_{13}$ = -9.524598e-24, $D^{(2)}_{14}$ = -1.177062e-25, $D^{(2)}_{15}$ = -2.232260e-27, $D^{(2)}_{16}$ = -4.981993e-29, $D^{(2)}_{17}$ = -3.014206e-31, $D^{(2)}_{18}$ = -1.508803e-32, $D^{(2)}_{19}$ = 1.589086e-35, $D^{(2)}_{20}$ = -1.307654e-36.

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