On the Spectra and the Laplacian Spectra of Graphs

by

Sasmita Barik

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI - 781039
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Sasmita Barik
Roll No. - 02612303

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI - 781039
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Certificate

It is certified that the work contained in this thesis entitled **"On the Spectra and the Laplacian Spectra of Graphs"** by **Sasmita Barik,** a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

November, 2006

Dr. Sukanta Pati
Assistant Professor
Department of Mathematics
Indian Institute of Technology Guwahati

To my parents

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Sasmita Barik
Department of Mathematics
Indian Institute of Technology Guwahati

Abstract

Throughout all graphs are assumed to be simple. Let $A(G)$ and $L(G)$ denote the adjacency and the Laplacian matrix corresponding to a graph G, respectively. The second smallest eigenvalue of $L(G)$ is called the *algebraic connectivity* of G and is denoted by $a(G)$. A corresponding eigenvector is called a *Fiedler vector* of G.

The study of *spectral integral variations in graphs* has been a subject of interest in the past few years (see Fan [21, 22, 23], Kirkland [46] and So [65]). We say that the spectral integral variation occurs in one place by adding an edge $e \notin G$ if (i) $L(G)$ and $L(G+e)$ have exactly $n-1$ eigenvalues in common and (ii) if λ is the other eigenvalue of $L(G)$, then $\lambda+2$ is the other eigenvalue of $L(G+e)$. We supply a characterization of connected graphs in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the algebraic connectivity. The results proved for that purpose suggest methods to construct graphs for which such variation occurs and also to construct graphs for which such variation never occurs. An argument showing that such a variation can never occur for the Laplacian spectral radius is supplied. Graphs with integer Laplacian spectrum has been a subject of study for many researchers, see, for example, Grone and Merris [34] and Grone, Merris and Sunder [33]. Probably the most common example is the star on $n \geq 3$ vertices with Laplacian eigenvalues 0, 1 (multiplicity $n-2$), and n. The characterization of trees with 1 as a Laplacian eigenvalue is rather difficult. It is known that the star is the only tree with algebraic connectivity 1. We take up the problem of characterizing trees that have 1 as the third smallest Laplacian eigenvalue. Furthermore, we characterize trees in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the third smallest Laplacian eigenvalue.

Graph operations and graph products play a crucial role in graph theory. They give rise to many important classes of graphs. (See [17, 25, 43, 51, 57] to know more.) We define a new operation by attaching copies of one graph to some vertices of another and investigate the Laplacian spectrum of the resulting graphs. Let F and H_v be two connected graphs, v be a specified vertex of H_v and $u_1, \ldots, u_k \in F$. Then the graph $G[F, u_1, \ldots, u_k, H_v]$ obtained by attaching a copy of H_v to each of the vertices u_i, $i = 1, \ldots, k$ (identify u_i with the vertex v of the ith copy) is called a *graph with k pockets*. This being a very general operation it is not possible to obtain the Laplacian spectrum of $G[F, u_1, \ldots, u_k, H_v]$ from the Laplacian spectra of F and H_v. Thus a natural question is 'how far the Laplacian spectrum of G can be described by using the Laplacian spectra of F and H_v.' We prove some results towards this and show that the complete Laplacian spectrum of G can also be described in some particular cases. As

applications one can generalize some well-known results of Neumann [59] and Guo [36].

Out of all graph products the cartesian product, categorical product, strong product and lexicographic product are probably the four most commonly used graph products. We do a study of some of the spectral properties corresponding to these four graph products, including some interesting results on the algebraic connectivity.

Let F be a connected graph on vertices $u_1, \ldots, u_n$ and H be any graph. Then the corona $F \circ H$ is defined as the graph obtained by taking n copies of H and for each i inserting edges between the ith vertex of F and each vertex of the ith copy of H (see Frucht and Harary [29]). Notice that the corona is a special case of $G[F, u_1, \ldots, u_n, H_v]$, where H_v is the graph obtained by taking a new vertex v and adding edges between v and each vertex of H. We do a study of both the ordinary spectrum (eigenvalues of the adjacency matrix) and the Laplacian spectrum of coronas. For a connected graph F and any r-regular graph H we provide complete information about the ordinary spectrum of $F \circ H$ using the ordinary spectra of F and H. Complete information about the Laplacian spectrum of $F \circ H$ is also provided even when H is not regular. As an application we show how to construct infinitely many pairs of nonisomorphic graphs with the same spectrum and the same Laplacian spectrum. We prove some structural results on the Fiedler vectors of the coronas and give an application.

It is well known that a graph G is bipartite if and only if the negative of each ordinary eigenvalue of G is also an ordinary eigenvalue of G. In contrast we pose a similar question of characterizing all graphs for which the reciprocal of each ordinary eigenvalue is also an ordinary eigenvalue. A graph G is said to have *property* (R) if $\frac{1}{\lambda}$ is an ordinary eigenvalue of G whenever λ is an ordinary eigenvalue of G. Further, if λ and $\frac{1}{\lambda}$ have the same multiplicity, for each ordinary eigenvalue λ then it is said to have the *property* (SR). We first supply a family of bipartite graphs with property (R). Note that in general a graph with property (R) need not be a corona of two graphs and need not even be bipartite. We characterize all trees with property (SR) and show that such a tree is the corona of some tree and an isolated vertex. We give a combinatorial description of the inverse of the adjacency matrix of a bipartite graph with a unique perfect matching which generalizes a known result of Buckley, Doty and Harary [14], Godsil [30] and Pavlikova [62]. We obtain a crucial result about the behaviour of the smallest (ordinary) eigenvalue of a nonsingular tree under a particular graph operation, defined by Xu in [69]. The results are then used to classify the trees with property (R). It turns out that it is the same class of trees which have property (SR). Further we also give some more equivalent characterizations of this class.

Contents

Chapter 1

Introduction

1.1 General introduction

Throughout all graphs are assumed to be simple. Let G be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. The *adjacency matrix* $A(G)$ of G, is defined as $A(G) = [a_{ij}]_n$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are adjacent in } G, \\ 0, & \text{otherwise.} \end{cases}$$

Let $D(G)$ be the diagonal matrix with the ith diagonal entry equal to the degree of the vertex i in G. We call $D(G)$ as the *degree matrix* of G. The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$. It is known that $L(G)$ is a positive semidefinite matrix. The smallest eigenvalue of $L(G)$ is 0 with a corresponding eigenvector $\mathbb{1}$, the vector of all ones. It is also known that the eigenvalue 0 of $L(G)$ is simple if and only if G is connected. As $L(G)$ is a real symmetric matrix, we see that there are $n-1$ linearly independent eigenvectors of $L(G)$ which are orthogonal to $\mathbb{1}$. We shall refer to these eigenvectors as the *nontrivial eigenvectors* of $L(G)$. As an immediate consequence we see that: *when G is connected, each nontrivial eigenvector of $L(G)$ has at least one positive entry and one negative entry.*

There is an extensive literature available on the adjacency and Laplacian matrices of graphs. Spectra of these matrices are used to characterize a graph or obtain information about the graph. Graph spectra finds its applications in other fields such as chemistry. We refer the reader to a classical book by Cvetković, Doob, and Sachs [17] and two survey articles by Merris [52] and Mohar [57], for more background on these two matrices.

The adjacency matrix and its eigenvalues have been much more investigated in the past than the Laplacian matrix. The first recognizable appearance of $L(G)$ occurs in the well-known Matrix-Tree-Theorem, by G. Kirchhoff (1847) which states that: *the determinant of any cofactor of the Laplacian matrix is equal to the number of spanning trees in a graph.* (See Merris [52] and Mohar [57] for some more references on this theorem.) This put the study of the Laplacian matrix as an interesting subject. Since then several authors from different disciplines have enriched the subject. The Laplacian spectrum of a graph, in particular, the second smallest eigenvalue of the Laplacian matrix has received much attention in last two decades. Denote the spectrum of $L(G)$ by $S(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G))$, where $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ are eigenvalues of $L(G)$ arranged in nondecreasing order. M. Fiedler [25] proved that $\lambda_2 > 0$ *if and only if the graph is connected.* Thus, we see that the graph structure is already reflected in the spectrum. This observation led Fiedler to define the *algebraic connectivity* of G by $a(G) = \lambda_2(G)$, viewing it as a quantitative measure of connectivity. Subsequent works establish a close connection between the graph structure and the eigenvectors (known as *Fiedler vectors*) corresponding to the algebraic connectivity; see for example [3, 25, 26, 27, 28, 33, 34, 35, 47, 48, 49, 51, 52, 53, 54, 57, 64]. Let Y be a Fiedler vector of G. By $Y(v)$, we denote the coordinate of the vector Y corresponding to the vertex v. A vertex u is called a *characteristic vertex* (with respect to Y) if $Y(u) = 0$ and there is a vertex w adjacent to u satisfying $Y(w) \neq 0$. An edge $\{u, v\}$ is called a *characteristic edge* (with respect to Y) if $Y(u)Y(v) < 0$. The *characteristic set* $C(G, Y)$ is the collection of all characteristic vertices and characteristic edges of G with respect to Y. Let S be a nonempty proper subset of $V(G)$. A *branch* at S is a connected component of $G \setminus S$. The principal submatrix of $L(G)$ corresponding to a branch B is denoted by $\hat{L}(B)$. A branch B at S is called a *Perron branch of* G if the smallest eigenvalue $\tau(\hat{L}(B))$ of $\hat{L}(B)$ is less than or equal to $a(G)$. Let T be a tree. It is known that $|C(T, Y)| = 1$, for any Fiedler vector Y of T. If for some Fiedler vector $C(T, Y) = \Big\{\{u, v\}\Big\}$, then the multiplicity of $a(T)$ is known to be 1. If $C(T, Y) = \{u\}$, then the multiplicity of $a(T)$ is known to be $k - 1$, where k is the number of Perron branches of T at u. In general when a characteristic set $C(G, Y)$ consists only of vertices and if k is the number of Perron branches of G at $C(G, Y)$, then the multiplicity of $a(G)$ is known to be at least $k - 1$. In Section 1.2, we cite some useful results describing the case of equality.

Harary and Schwenk [40] studied those graphs G such that $A(G)$ has integral spectrum. The analogous problem for $L(G)$ is also interesting, see, for example, Grone and Merris [34]. A graph G is said to be *Laplacian integral* if $S(G)$ consists entirely of integers. In [33] and [34], Grone, Merris and Sunder discussed some properties of trees with integer Laplacian eigenvalues and

Laplacian integral graphs, respectively. In [65], W. So considered the problem of preserving Laplacian integrality by adding an edge. He has given an equivalent condition for a graph with exactly one Laplacian eigenvalue moving up by an integer and others remaining invariant, when adding an edge. Fan [21] introduced the notion of *spectral integral variation* to study the general graphs (that is, the graphs where loops and parallel edges are allowed) with all changed Laplacian eigenvalues moving up by integers by adding an edge, and provided a method to construct a new Laplacian integral graph from a known one. He argued that if the spectral integral variation of a graph occurs by adding an edge, then it must occur either in one place or in two places. The problem for the former case was solved by So [65] for ordinary graphs and was solved by Fan [21] for general graphs. For the later case, Fan provided an equivalent condition in the case of a disconnected graph with two connected components, where the additional edge makes it connected. Recently, Kirkland [46] has characterized all graphs in which addition of a particular edge may result in increasing exactly two Laplacian eigenvalues by one each keeping the other Laplacian eigenvalues unchanged. Further, he has discussed the case where one of the changed eigenvalue is the algebraic connectivity. He has also characterized the graphs in which spectral integral variation occurs in two places, with the algebraic connectivity increasing from 1 to 2.

The *disjoint union* $F+H$ of two graphs F and H is the graph with the vertex set $V(F)\cup V(H)$ and the edge set $E(F)\cup E(H)$, where $V(F)$ and $V(H)$ are treated as disjoint sets. Their *join* is $F\vee H$ is the graph obtained from $F+H$ by inserting new edges from each vertex of F to every vertex of H. Fan [21] has shown that the spectral integral variation of $F+H$ occurs in one place by adding an edge $\{i,j\}$, $i\in F$, $j\in H$ if and only if both i and j are isolated vertices. It is obvious that in this case the changed eigenvalue of $F+H$ is the algebraic connectivity. Note that $F+H$ is a disconnected graph. We consider the problem of characterization of connected graphs in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the algebraic connectivity. The first part of the thesis is in this direction. Suppose that i and j are nonadjacent vertices of a connected graph G on n vertices. We show that spectral integral variation occurs in one place by adding an edge between i and j where the changed eigenvalue is $a(G)$ if and only if $G=G^*\vee(G_1+\{i\}+\{j\})$, where G^* is a graph of order k $(1\leq k\leq n-2)$, such that $a(G^*)$ is at least $2k-n$, and G_1 is any graph on $n-k-2$ vertices. We further explore the case when $a(G)$ is simple to see that such a variation occurs if and only if $G=K_n\setminus\{i,j\}$, where K_n is the complete graph of order n. The results suggest methods to construct graphs for which such variation occurs and also to construct graphs for which such variation never occurs. It has also been argued that it is not possible to get a graph for which such a variation occurs where the changed eigenvalue is the Laplacian spectral radius (the largest Laplacian eigenvalue).

As mentioned earlier graphs with integer Laplacian spectrum have been a subject of study for many researchers [20, 33, 34]. Probably the most common example is the star on $n \geq 2$ vertices with Laplacian eigenvalues 0, 1 (multiplicity $n-2$), and n. It is known that the star (on $n \geq 3$ vertices) is the only tree with algebraic connectivity 1. Interestingly it turns out that the star is the only tree in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the algebraic connectivity. Later it has been observed that the class of trees for which such variation occurs where the changed eigenvalue is the kth smallest Laplacian eigenvalue is a subclass of the class of trees with 1 as the kth smallest Laplacian eigenvalue. The characterization of trees with 1 as a Laplacian eigenvalue is rather difficult. We take up the problem of characterizing trees that have 1 as the third smallest Laplacian eigenvalue. This result is then used for a characterization of trees in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the third smallest Laplacian eigenvalue.

Graph operations and graph products play a crucial role in graph theory. They give rise to many important classes of graphs. Many interesting graphs are obtained by combining pairs of graphs or operating on a single graph in some way. For example, the $m \times n$ *lattice graph* is just the cartesian product of the paths P_m and P_n. For certain families of graphs it is possible to identify a graph by looking at the spectrum. More generally, this is not possible. In some cases the spectrum / Laplacian spectrum of a relatively larger graph can be described in terms of the spectra / Laplacian spectra of some smaller (and simpler) graphs using some simple graph operations (like the disjoint union, the join of graphs, deletion / insertion of an edge, the complement etc) and graph products like cartesian product. To know more about this we refer the reader to the survey articles by Merris and Mohar [51, 57, 58]. The study of Laplacian spectrum under these operations and products is another matter of interest to us. The second part of the thesis is in this direction, where we define a new operation by attaching copies of one graph to some vertices of another and investigate the Laplacian spectrum of the resulting graphs. Let F and H_v be two connected graphs, v be a specified vertex of H_v and $u_1, \ldots, u_k \in F$. Then the graph $G[F, u_1, \ldots, u_k, H_v]$ obtained by attaching a copy of H_v to each of the vertices u_i, $i = 1, \ldots, k$ (identify u_i with the vertex v of the ith copy) is called a *graph with k pockets*. This being a very general operation it is not possible to obtain the Laplacian spectrum of

$G[F, u_1, \ldots, u_k, H_v]$ from the Laplacian spectra of F and H_v. Thus, a natural question is 'how far the Laplacian spectrum of G can be described by using the Laplacian spectra of F and H_v.' We prove some results towards this and show that the complete Laplacian spectrum of G can also be described in some particular cases. As an application we generalize some well-known results of Molitierno and Neumann [59] and Guo [36].

Given two graphs G and H one can define several graph products (see [38, 41, 43, 55, 63, 68]). Imrich and Klavžar [43] have introduced graph products and studied many graph invariants on the product graphs. Out of all graph products the cartesian product, categorical product (tensor product), strong product and lexicographic product are probably the four most commonly used graph products. We investigate the Laplacian spectra of the product graphs obtained by the above mentioned four graph products. Some interesting results on the algebraic connectivity are also supplied for some special classes of graphs. We prove some results that help us to construct new Laplacian integral graphs from the known ones using the graph products.

Let F be a connected graph on vertices $u_1, \ldots, u_n$ and H be any graph. Take a new vertex v and add edges between v and each vertex of H to obtain H_v. The graph $G = G[F, u_1, \ldots, u_n, H_v]$ is known as the *corona* (see Frucht and Harary [29]) of F and H and is denoted by $F \circ H$. In this specific setting we do a study of both the ordinary spectrum (eigenvalues of the adjacency matrix) and the Laplacian spectrum of coronas. It turns out that one can get complete information about the ordinary spectrum of the corona $F \circ H$ when F is connected and H is r-regular. One can also get complete information about the Laplacian spectrum of $F \circ H$ when F is connected and H is any graph (not necessarily r-regular). This information is then combined to construct infinitely many pairs of nonisomorphic graphs with the same ordinary spectrum and the same Laplacian spectrum. Results exhibiting the relationship between the graph structure and the Fiedler vectors are proved. Let G be any graph and Y a Fiedler vector. It is known that if $C(G, Y) = \{u\}$, then for any Fiedler vector Z the set $C(G, Z) = \{u\}$. A graph is said to be of *Type I* if the characteristic set consists of just one vertex. Constructing Type I trees with nonisomorphic Perron branches has been studied by Kirkland in [45]. As an application to our results we show how to construct graphs with nonisomorphic Perron branches.

It is well-known that a graph is bipartite if and only if the negative of each of its ordinary eigenvalue (eigenvalue of the adjacency matrix) is also an ordinary eigenvalue. In contrast, we pose an analogous question of characterizing all graphs for which the reciprocal of each ordinary eigenvalue is also an ordinary eigenvalue. A graph G is said to have *property* (R) if $\frac{1}{\lambda}$ is an ordinary eigenvalue of G whenever λ is an ordinary eigenvalue of G. Characterizing graphs with property (R) is a rather difficult task. Thus, one may ask for a characterization of all such trees. One could further add one more restriction to look for a smaller class in the following way: Characterize such trees T with property (R) so that each eigenvalue of T and its reciprocal have the same multiplicity (such trees are called trees with *property* (SR)). The answer to this question reveals a startling relationship with the graph operations / products. It turns out that any tree with property (SR) is the corona of some tree with K_1. The fourth part of the thesis is in this direction. We first supply a family of bipartite graphs with property (R) using the corona of a bipartite graph and a single vertex. Note that in general a graph with property (R) need not be a corona of two graphs and need not even be bipartite. We characterize all trees with property (SR) and show that such a tree is the corona of some tree and an isolated vertex. We give a combinatorial description of the inverse of the adjacency matrix of a bipartite graph with a unique perfect matching which generalizes a known result of Buckley, Doty and Harary [14], Godsil [30] and Pavlikova [62]. We obtain a crucial result about the behaviour of the smallest (ordinary) eigenvalue of a nonsingular tree under a particular graph operation, defined by Xu in [69]. This relation and the earlier combinatorial description of the inverse are then used to settle the relatively harder task of characterizing all trees with property (R). We show that it is exactly the class of all trees with property (SR). Further we also give some more equivalent characterizations of this class.

The thesis is organized as follows. Section 1.2 of this chapter contains some known results on algebraic connectivity and the characteristic set which will be used in the later part of the thesis.

In chapter 2, we characterize graphs in which spectral integral variation occurs in one place where the changed eigenvalue is the algebraic connectivity. This chapter contains a complete characterization of trees which have 1 as the third smallest Laplacian eigenvalue. A characterization of trees in which spectral integral variation in one place occurs by adding edge, where the changed eigenvalue is the third smallest Laplacian eigenvalue is also contained here.

In Chapter 3, the focus is on the Laplacian spectra of graphs obtained by means of some operations on graphs. Results describing the Laplacian spectrum of the graph $G[F, u_1, \ldots, u_k, H_v]$ are supplied here.

Chapter 4 contains an investigation on the Laplacian spectra of graphs obtained by the four graph products: cartesian product, categorical product, strong product and lexicographic product. Some results on algebraic connectivity and characteristic set are also supplied here.

In Chapter 5, we investigate both the ordinary and the Laplacian spectrum of corona of two graphs. The main results here are: a complete characterization ordinary spectrum of $F \circ H$ when H is regular and F is connected, a complete characterization Laplacian spectrum of $F \circ H$ when F is connected, a construction of graphs with nonisomorphic Perron branches, and a construction of infinitely many pairs of nonisomorphic graphs which are simultaneously adjacency and Laplacian cospectral.

In Chapter 6, we introduce the concept of property (R) and property (SR) for the adjacency matrix of a graph. The highlights here are: the existence of a class of graphs with property (R), the characterization of trees with property (SR), a combinatorial description of the inverse of the adjacency matrix of a bipartite graph with a unique perfect matching, and the characterization of trees with property (R).

1.2 Preliminaries : the characteristic set

Let G be a connected graph. Define a relation R on the edge set of G as: e_1Re_2 if and only if either $e_1 = e_2$ or there is a simple cycle containing both of them. Then R is an equivalence relation. Let $E_1 \bigcup E_2 \bigcup \cdots \bigcup E_k$ be the decomposition of the edge set into equivalence classes. The subgraphs $G_i, i = 1, \ldots, k$ of G consisting of all edges in E_i and all vertices adjacent to them is called a *block* of G. A vertex v is called a *point of articulation* if v is common to more than one block. Note that for a tree each edge is a block and each internal vertex is a point of articulation. Let Y is a Fiedler vector. A component H of G is said to be *zero* (*positive, negative*) if $Y(v) = 0\,(Y(v) > 0, Y(v) < 0)), \forall\, v \in H$. The component H is said to be *nonzero* if it is not a zero component. By a *nonzero* (*zero, negative, positive*) vertex of G we mean a vertex of G such that $Y(v) \neq 0$ ($Y(v) = 0,\ Y(v) < 0,\ Y(v) > 0$, respectively). A real $n \times n$ matrix A is called an *M-matrix* if there exists a nonnegative matrix B with maximal eigenvalue r such that $A = cI - B$, where $c \geq r$. (See Berman and Plemmons [10], Horn and Johnson [42] and Minc [56]).

A general version of the following result can be found in Fiedler [27] which reveals a nice relationship between a Fiedler vector and the graph structure.

Theorem 1.2.1 (Fiedler [27, Theorem (3,12)]) *Let G be a connected graph and Y a Fiedler vector. Then exactly one of the following statements holds:*

(a) *There exists a unique block B_0 of G with both positive and negative vertices. Each other blocks have either all positive, or all negative, or only zero vertices.*

(b) *No block of G contains positive and negative values simultaneously. In this case there exists a single vertex v with $Y(v) = 0$ which has a neighbor u with $Y(u) \neq 0$. This vertex is a point of articulation.*

The following result is essentially contained in [27].

Proposition 1.2.2 (Fiedler [27]) *Let G be a connected graph and Y a Fiedler vector. Then exactly one of the following statements holds:*
Case 1. $C(G, Y) = \{v\}$, where v is a point of articulation,
Case 2. Or Case 1 does not hold and there is a unique block B (called characteristic block) of G which contains all the characteristic vertices and edges.

In the case when G is a tree, the above results are strengthened by Fiedler in [27] and discussed in greater details by Grone and Merris in [35, 53]. In [47], Kirkland and Fallat have proved the following result for the general case.

Proposition 1.2.3 (Kirkland and Fallat [47, Corollary 2.1]) *Let G be connected. If Case 1 of Proposition 1.2.2 holds, then for any Fiedler vector Z of G, $C(G, Z) = \{v\}$. If Case 2 holds, then for any Fiedler vector Z of G the characteristic block of G is B.*

Recall that if Case 1 of Proposition 1.2.3, then G is called a Type I graph. In Section 5.3 of Chapter 5, we supply a class of Type I graphs with nonisomorphic Perron branches.

Several authors have investigated the elements of $C(G, Y)$ and their location (see, for example, Kirkland and Fallat [47], Kirkland and Neumann [48] and Kirkland, Neumann and Shader [49]). However, the concept of a characteristic set, which takes a unified view of the characteristic vertices and characteristic edges, has been introduced by Bapat and Pati in [3]. The relation between the characteristic set and nonnegative matrix theory is exploited.

The following result discloses a nice relationship between the algebraic connectivity and some principal submatrices of $L(G)$ and has many applications.

Lemma 1.2.4 (Bapat and Pati [3, Lemma 6]) *Let G be a connected simple graph and $a(G)$ the algebraic connectivity. Let W be a set of vertices of G such that $G \setminus W$ is disconnected. Let G_1, G_2 be two components of $G \setminus W$ and let $\hat{L}_1$, $\hat{L}_2$ be the principal submatrices of L corresponding to G_1, G_2, respectively. Suppose $\tau(\hat{L}_1) \leq \tau(\hat{L}_2)$. Then either $\tau(\hat{L}_2) > a(G)$ or $\tau(\hat{L}_1) = \tau(\hat{L}_2) = a(G)$.*

As an immediate corollary we have the following result.

Corollary 1.2.5 *Let G is a connected graph with the algebraic connectivity $a(G)$. Let W be a set of vertices of G such that $G \setminus W$ is disconnected with at least $m \geq 2$ components. $G_1, G_2, \ldots, G_m$ and $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_m$ be the corresponding principal submatrices of $L(G)$. If $\tau(\hat{L}_i) = a(G)$, for some $i = 1, 2, \ldots, m$, then $\tau(\hat{L}_i) \geq a(G)$, for all i.*

Let G be a connected graph. Let Y be a Fiedler vector of G such that $W = C(G, Y)$ contains vertices only. It is well-known that G has at least two Perron branches at W (see [3]). Suppose that G has t Perron branches at W. It is also known that (Kirkland and Fallat [47]) in case $W = \{u\}$, the multiplicity of the algebraic connectivity is exactly $t - 1$ (see Bapat, Kirkland and Pati [4] for a result which is true for a much wider class of matrices).

In general when there is no restriction on the cardinality of W the following can be said. Note that the following result has already been proved in the thesis of Dr. S. Pati. For the proof one can also see the article [8], by Barik and Pati.

Lemma 1.2.6 (Barik and Pati [8, Lemma 8]) *Let G be a connected graph. Let Y be a Fiedler vector of $L(G)$. Suppose that $W = C(G, Y)$ contains vertices only. Suppose that G has t (≥ 2) Perron branches $G_1, G_2, \ldots, G_t$ at W. Then the multiplicity of $a(G)$ is at least $t - 1$.*

A natural question is whether the multiplicity can actually be higher. A simple example confirms this fact affirmatively.

Example 1.2.1 Consider the cycle $G = C_4$ on $\{1, 2, 3, 4\}$.

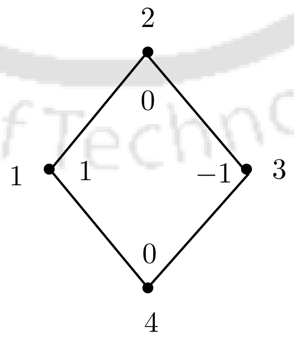


Figure 1.1: A description of a Fiedler vector of C_4 (note that the numbers outside of the graph are vertex labels and the numbers inside of the graph are entries of the Fiedler vector).

It is easy to see that $Y = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$, is a Fiedler vector of G. In this case $W = C(G, Y) = \{2, 4\}$. There are only two Perron branches $\{1\}$ and $\{3\}$ of G at W. By Lemma

1.2.6 the multiplicity of the algebraic connectivity is at least one. One can check that the multiplicity of the algebraic connectivity is two.

In [4], the authors have shown that when $W = \{u\}$ one can get $t-1$ linearly independent Fiedler vectors using the positive eigenvectors corresponding to the Perron branches of G at W. A similar result holds in the general case.

Theorem 1.2.7 *Let G be a connected graph and Y be a Fiedler vector of G. Suppose that $W = C(G,Y)$ contains vertices only and $G_1, G_2, \ldots, G_t,\ t \geq 2$ be the Perron branches of G at W. Let $\hat{L}_i$ be the principal submatrix of $L(G)$ corresponding to G_i, $i = 1, 2, \ldots, t$. Then the following statements are true.*

(i) *Each $G_i,\ i \in \{1, \ldots, t\}$ is either positive or negative or zero and $\tau(\hat{L}_i) = a(G)$.*

(ii) *We can get $t-1$ linearly independent Fiedler vectors $X_1, X_2, \ldots, X_{t-1}$ of $L(G)$ such that for each vector X_j, $X_j(w) = 0$, for all $w \in W$ and exactly one of the components G_i, $i \in \{1, \ldots, t\}$ is positive and exactly one of the components G_i, $i \in \{1, \ldots, t\}$ is negative with respect to X_j.*

(iii) *For each of the above Fiedler vectors X_i, $C(G, X_i) = W$.*

Proof. Item (i) follows easily, because there is a nonzero branch, say H of G at W. If H contains a negative vertex and a positive vertex then H is bound to contain either a characteristic edge or a characteristic vertex, which is not possible by the hypothesis. Suppose that H contains a positive vertex, thus the subvector $Y(H)$ is nonnegative. Since $L(G)Y = a(G)Y$ it follows that $\hat{L}(H)Y(H) = a(G)Y(H)$. As $\hat{L}(H)$ is a nonsingular symmetric M-matrix, its inverse is positive. It follows that $Y(H)$ is positive and $\tau(\hat{L}(H)) = a(G)$. Thus H is a Perron branch at W. If G_i is any other Perron branch at W such that $\tau(\hat{L}_i) < a(G)$, then considering H, G_i and applying Lemma 1.2.4, we get $a(G) < \tau(\hat{L}(H))$, which is a contradiction.

(ii) Now we construct a Fiedler vector of the form described in the statement. Consider the positive eigenvectors Z_1 and Z_2 of $\hat{L}(G_1)$ and $\hat{L}(G_2)$. Let

$$k = \frac{\sum\limits_{v \in G_1} Z_1(v)}{\sum\limits_{v \in G_2} Z_2(v)}.$$

Observe that in the above definition of k we use the fact that the entries of Z_2 agree in sign.

With a permutation similarity operation we can write

$$L(G) = \begin{bmatrix} \hat{L}_1 & 0 & L_{13} \\ 0 & \hat{L}_2 & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}.$$

Let

$$X_1 = \begin{bmatrix} Z_1 \\ -kZ_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

One can see that

$$\mathbb{1}^T X_1 = 0 \text{ and}$$
$$X_1^T L(G) X_1 = a(G) X_1^T X_1. \tag{1.1}$$

Recall that $\mathbb{1}$ is the eigenvector of $L(G)$ corresponding to the smallest eigenvalue 0 and $L(G)$ is a symmetric matrix. Thus from the above equation one can easily show that $L(G)X_1 = a(G)X_1$. One can construct other Fiedler vectors by considering the positive eigenvectors Z_1, Z_i of $\hat{L}(G_1)$, $\hat{L}(G_i)$, case by case where, $3 \leq i \leq t$. It is clear that these Fiedler vectors are linearly independent.

To prove (iii), we claim that each vertex $w \in W$, is adjacent to at least one vertex of each of $G_1, \ldots G_t$.

Suppose that w is not adjacent to any vertex of G_1 (say). Since $w \in W = C(G, Y)$, a characteristic vertex, there exist two components, say G_2, G_3 such that w is adjacent to a vertex from each of G_2, G_3. Let $W' = W \backslash \{w\}$. Notice that $G \backslash W'$ has at least two components, one is G_1, the other one is H which contains G_2, G_3, w (this one may actually contain some more vertices).

With some permutation similarity operations we can write

$$\hat{L}(H) - \tau(\hat{L}(G_2))I = \left[\begin{array}{c|c} \hat{L}(G_2) - \tau(\hat{L}(G_2))I & C \\ \hline C^t & D - \tau(\hat{L}(G_2))I \end{array}\right].$$

Note that since w is adjacent to G_2, we have $C \neq 0$. Thus $[\hat{L}(G_2) - \tau(\hat{L}(G_2))I]Z_2 = 0$ and $C^t Z_2 \neq 0$ (as $C \leq 0$), where Z_2 is the positive eigenvector of $\hat{L}(G_2)$. An application of Corollary 5 of [3] tells that $\tau(\hat{L}(H)) < \tau(\hat{L}(G_2)) = \tau(\hat{L}(G_1))$.

Now, applying Lemma 1.2.4, we see that $a(G) < \tau(\hat{L}(G_1))$, which is a contradiction. Thus the claim is justified and the proof is complete in view of the fact that for each of these Fiedler vectors X_i, G_1 is always positive and G_{i+1} is negative. ■

Remark 1.2.8 *The technique used in* (ii) *is taken from H. van der Holst* [67].

Next we give a class of graphs satisfying the conditions of Lemma 1.2.6, for which the multiplicity is exactly $t-1$.

This construction is easy in view of Theorem 3.1.1 of Chapter 3. Given any natural number $t \geq 2$, consider the join G of a graph H_1 of order k with H_2 which consists of t isolated vertices. We observe that $S(G) = \{0, k, k, \ldots, k, t + \lambda_2(H_1), \ldots, t + \lambda_k(H_1), n = k + t\}$. We see that the multiplicity of $a(G)$ is exactly $t-1$ if $t + \lambda_2(H_1) > k$. Note that if we choose H_1 to be complete then $\lambda_2(H_1) = k$, and thus the requirement is satisfied. Thus the class $G = K_k \vee H_2$, where H_2 is a graph of $t(\geq 2)$ isolated vertices $\{1, \ldots, t\}$ has multiplicity exactly $t-1$. Note that any Fiedler vector Y of G is a linear combination of

$$e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_t,$$

and thus $C(G, Y) =$ vertices of K_k. For the case $k = 1$, the graph G becomes a star on $t+1$ vertices.

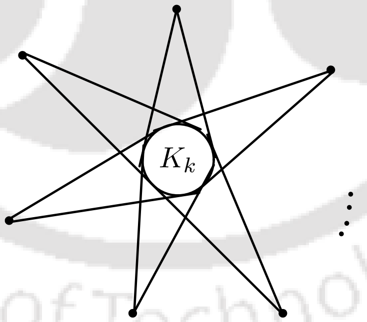


Figure 1.2: $K_k \vee K_t^c$, $k \geq 1$, satisfies the conditions of Lemma 1.2.6, and the multiplicity of the algebraic connectivity is exactly $t-1$.

In view of the above discussion it is clear that for $t \geq 2$, the complete bipartite graph $K_{t,t}$ is an example which satisfies the condition of Lemma 1.2.6, but the multiplicity of the algebraic connectivity is $2t-2$,

Another interesting question can be asked at this point. Suppose that G satisfies the condition of Lemma 1.2.6. We know the following. If the multiplicity of $a(G)$ is exactly $t-1$ then for each Fiedler vector Y we have $C(G,Y)$ consists of vertices only (follows from Theorem 1.2.7). Is the converse true? That is, if we know that for each Fiedler vector Y of G, the set $C(G,Y)$ consists of vertices only, does that force the multiplicity of $a(G)$ to be $t-1$?

The following result answers this question affirmatively.

Theorem 1.2.9 *Let G be a connected graph and Y be a Fiedler vector with $W = C(G,Y)$ consisting of vertices only. Suppose that there are t (≥ 2) Perron branches $G_1, \dots, G_t$ of G at W. Then the following statements are equivalent.*

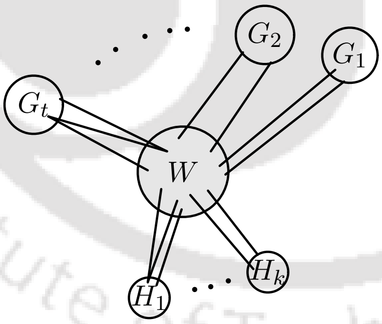
(i) *The multiplicity of $a(G)$ is $t-1$.*

(ii) *For each Fiedler vector Z the set $C(G,Z) = W$.*

(iii) *For each Fiedler vector Z the set $C(G,Z)$ consists of vertices only.*

Proof: In view of the foregone discussions we show only (iii) $\Rightarrow$ (i).

Suppose that (iii) holds and (i) does not. We recall (Theorem 1.2.7) that there are $t-1$ linearly independent Fiedler vectors $X_1, \dots, X_{t-1}$, where for $i = 1, \dots, t-1$, X_i is positive on G_1, negative on G_{i+1} and zero elsewhere. Let $H_1, \dots, H_k$ be the branches of G at W which are not Perron branches.



Let Z be a Fiedler vector of G which is not a linear combination of these X_is. Thus the vector $U = Z + \sum_{i=1}^{t-1} \alpha_i X_i$ is also a Fiedler vector of G. Note that when α_is are large positive numbers, U makes G_1 positive and each of $G_2, \dots, G_t$ negative. By the hypothesis $C(G,U)$ contains vertices only. Thus $C(G,U) \subset W \bigcup \bigcup_{i=1}^{k} H_i$. By Theorem 1.2.7 we know that each $w \in W$ is adjacent to at least one vertex of each of $G_1, \dots, G_t$. Thus $W \subset C(G,U)$.

Note that $G_1, \dots, G_t, H_1, \dots, H_k$ are the components of $G \backslash W$, and $U(W) = 0$. If $U(H_1) \neq 0$, it follows that $\hat{L}(H_1)U(H_1) = a(G)U(H_1)$. Thus $\tau(\hat{L}(H_1)) \leq a(G)$, which is a contradiction to the fact that H_1 is not a Perron branch. Thus for each $i = 1, \dots, k$ we have $U(H_i) = 0$.

Since $G_1, \ldots, G_t$, are Perron branches, and $\hat{L}(G_i)$ is inverse-positive and $\hat{L}(G_i)U(G_i) = a(G)U(G_i)$, we see by applying Perron-Frobenius theory that $U(G_i)$ is the Perron vector for $(\hat{L}(G_i))^{-1}$.

It now follows that for the Fiedler vector Z either $Z(G_i)$ is zero or is the Perron vector for $(\hat{L}(G_i))^{-1}$. Thus $C(G, Z) = W$.

To show that Z is a linear combination of X_is, assume that Z is nonzero on $G_1, \ldots, G_s, 2 \leq s \leq t$. Consider the vector $Z_1 = Z - \gamma_1 X_1$, where the number γ_1, is so chosen that $Z_1(u_2) = 0$, for some $u_2 \in G_2$. If $Z_1 = 0$, we are done, otherwise Z_1 is also a Fiedler vector. Since $Z_1(u_2) = 0$, it follows that $Z_1(G_2) = 0$. Thus for Z_1 at most $s - 1$ components out of $G_1, G_3 \ldots, G_s$ are nonzero.

We repeat the above procedure at most $s-1$ times to obtain $Z_{s-1} = Z - \gamma_1 X_1 - \gamma_2 X_2 - \cdots - \gamma_{s-1} X_{s-1}$, for some real numbers γ_i, $i = 1, \cdots, s-1$ such that the components $G_2, G_3, \ldots, G_s$ are zero components for Z_{s-1}. Note that for Z_{s-1} the vertices in G_1 agree in sign. Since each Fiedler vector must have a positive vertex and a negative vertex, it follows that $Z_{s-1} = 0$. Hence the proof is complete. ■

Chapter 2

Spectral integral variations in graphs

2.1 Preliminaries

Let $G = (V, E)$ be a general graph with vertex set $V = \{1, 2, \dots, n\}$. Denote by $G + e$ the general graph obtained from G by adding an edge or a loop e. Using *Courant-Weyl inequalities* (Theorem 2.1 in [17],) we have the following theorem which is also stated in [21, 57].

Theorem 2.1.1 (Fan [21, Theorem 1]; Mohar [57, Theorem 3.2]) *Let G be a general graph of order n. Let $G+e$ be the graph obtained by adding the edge or loop e in G. Then the eigenvalues of $L(G)$ interlace those of $L(G+e)$, that is,*

$$0 = \lambda_1(G) = \lambda_1(G+e) \leq \lambda_2(G) \leq \lambda_2(G+e) \leq \cdots \leq \lambda_n(G) \leq \lambda_n(G+e).$$

Since the sum of the eigenvalues of a matrix A is the trace of A, we have

$$\sum_{i=1}^{n} (\lambda_i(G+e) - \lambda_i(G)) = 2 \text{ (or 1) if } e \text{ is an edge (or a loop).}$$

Thus at least one inequality in Theorem 2.1.1 must be strict.

Given a vertex v of a graph G, let $N(v)$ denote the set of all neighbors of v in G. For a simple graph G with two nonadjacent vertices i and j, W. So [65] showed that by adding the edge e one eigenvalue of $L(G)$ increases by 2 (and $n-1$ eigenvalues remain unchanged) if and only if $N(i) = N(j)$. Fan [21] introduced and studied spectral integral variation in general graphs by adding an edge or a loop. It follows from Theorem 2.1.1 that if the spectral integral variation of a general graph occurs by adding an edge or a loop, then it must occur either in one place or in two places.

Definition 2.1.1 Let G be a general graph of order n and e be any edge or loop. We say that the spectral integral variation of G occurs in one place by adding e if exactly one eigenvalue of $L(G)$ increases by 2 (or 1) when the edge (or loop) e is added to G. We say that the spectral integral variation of G occur in two places by adding e (here e is an edge) if exactly two eigenvalues of $L(G)$ increase by 1 each when e is added to G.

Here we give examples of two connected graphs where adding an edge results in spectral integral variation in one and two places.

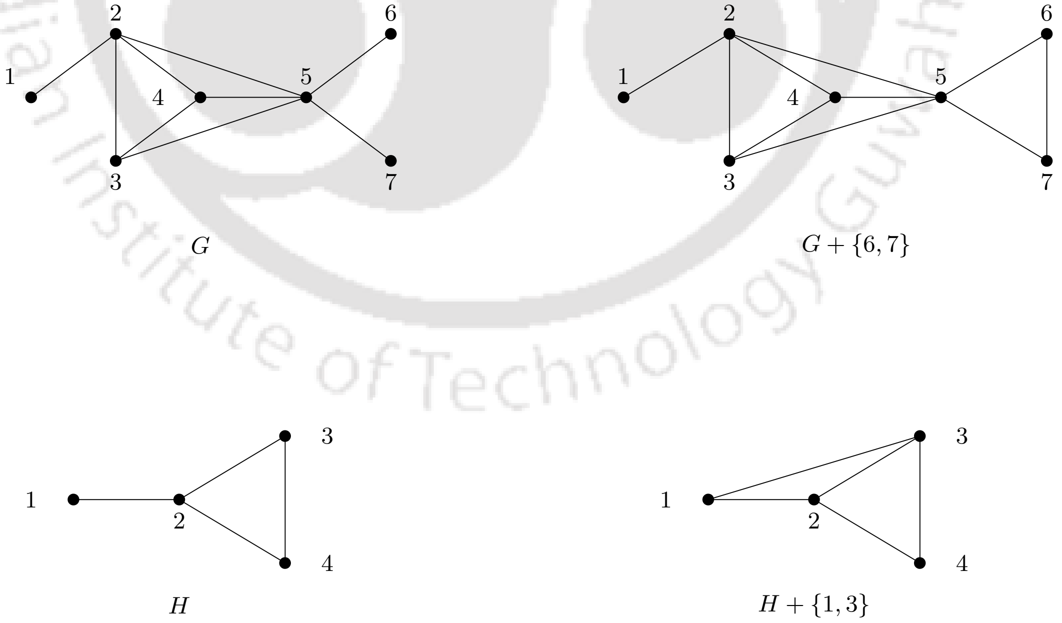
Example 2.1.1 Consider the graphs G and H in Figure 2.1. It can be verified that

$$S(G) = (0, 0.7035, 1, 1.3427, 4, 4.8813, 6.0725)$$

and

$$S(G + \{6, 7\}) = (0, 0.7035, 1.3427, 3, 4, 4.8813, 6.0725).$$

Thus by adding the edge $\{6, 7\}$ spectral integral variation of G occurs in one place where the third smallest eigenvalue, $\lambda_3(G)$ increases from 1 to 3.



G $\qquad$ $G + \{6, 7\}$

H $\qquad$ $H + \{1, 3\}$

Figure 2.1: Spectral integral variation in graphs

Note that for the graph H we have

$$S(H) = (0, 1, 3, 4) \qquad \text{and} \qquad S(H + \{1, 3\}) = (0, 2, 4, 4).$$

Here spectral integral variation of H occurs in two places while adding the edge $\{1,3\}$, where both $a(H)$ and $\lambda_3(H)$ increase by 1 each.

The following lemma gives a sufficient condition for the occurrence of spectral integral variation in a graph G while adding an edge.

Lemma 2.1.2 (Fan [21, Lemma 1]) *Let G be a connected graph of order n, e be an edge not in G, and $S(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. If the spectral integral variation of G occurs in one place by adding an edge e, then*

$$S(G+e) = \Big(S(G) \setminus \{\lambda_k\}\Big) \cup \{\lambda_k + 2\}, \text{ for some } k \in \{1, 2, \ldots, n\}.$$

By $\hat{e}_i$ we denote the vector with ith entry equal to 1 and all other entries 0. A vector $x \in R^n$ is called a *Faria vector*, if $x = \hat{e}_i - \hat{e}_j$, for some $i \neq j$.

The following result gives necessary and sufficient conditions for the occurrence of spectral integral variation of G in one place while adding a new edge.

Theorem 2.1.3 (Fan [21]) *Let $G = (V, E)$ be a connected graph of order n, $e = \{i, j\}$, $i \neq j$, be an edge not in G. Then the following conditions are equivalent.*

1. *The spectral integral variation of G occurs in one place by adding e.*
2. *One eigenvector of $L(G)$ is the Faria vector of which the ith and jth entries are nonzero.*
3. *$N(i) = N(j)$.*

As an immediate corollary we have the following result.

Corollary 2.1.4 *Let $G = (V, E)$ be a connected graph of order n and let i, j be two nonadjacent vertices of G. If the spectral integral variation G occurs in one place by adding edge between i and j, then the Faria vector with the ith and jth entry nonzero is an eigenvector corresponding to the changed eigenvalue λ_k(say) of $L(G)$ and $\lambda_k = d_i$, the degree of the ith vertex.*

Kirkland [46] has characterized all graphs in which spectral integral variation occurs in two places by adding an edge e and discussed the case where one of the changed eigenvalue is the algebraic connectivity. He has characterized the graphs in which spectral integral variation occurs in two places, with the algebraic connectivity increasing from 1 to 2.

A natural question at this point is the following: *can we study the spectral integral variation in case of adjacency matrices*? In [60], Pan, Fan and Li have introduced and studied spectral integral variation considering adjacency matrices in place of Laplacian matrices. Since the trace of the adjacency matrix of a graph is 0, it follows that by adding an edge the adjacency spectral variation occurs at least in two places. If an edge is added between two isolated vertices, then adjacency spectral integral variation occurs in two places, with one zero ordinary eigenvalue increases by 1, and another zero ordinary eigenvalue decreases by 1. For the connected case, it has been proved in [60] that one cannot construct a new adjacency integral connected graph with order $n \geq 3$ from a known one by adding an edge. In [60], the authors have proved that the spectrum of $A(G+e)$ is different from that of $A(G)$ only in two places with one eigenvalue increases by m and another eigenvalue decreases by m, where $m > 0$ is a rational number, if and only if G is an empty graph with order 2. Thus the spectral rational variation is also not possible for a connected graph with order $n \geq 3$. To characterize the case that adjacency spectral irrational variation occurs just in two places is still an open problem.

2.2 Spectral integral variation of graphs with a(G) increasing by 2

In this section we shall characterize the graphs G in which addition of a new edge would result in spectral integral variation in just one place where the eigenvalue that gets changed is $a(G)$. Fan [21] was the first to point out the following result. (Strictly speaking, the article contains a general version of this result.)

Corollary 2.2.1 *Let G_1, G_2 be two disjoint graphs. Then the spectral integral variation of $G_1 \cup G_2$ occurs in one place by adding an edge $e = \{u, v\}$, $u \in G_1$, $v \in G_2$ if and only if both u, v are isolated vertices.*

In Corollary 2.2.1, notice that the changed eigenvalue is the algebraic connectivity. Thus it is natural to ask for characterization of connected graphs in which spectral integral variation occurs in one place by adding an edge between two nonadjacent vertices where the changed eigenvalue is the algebraic connectivity.

The following example shows that such variations do not depend on the multiplicity of the algebraic connectivity.

Example 2.2.1 See the graph G in Figure 2.2. We have, $S(G) = (0, 2, 2, 4)$ and $S(G_1) = S(G + \{2, 4\}) = (0, 2, 4, 4)$. Note that the algebraic connectivity of G has multiplicity 2 and spectral integral variation of G occurs in one place by adding the edge $\{2,4\}$ where the changed eigenvalue is the algebraic connectivity.

But $a(G_1)$ has multiplicity one and $S(G_1+\{1,3\}) = (0,4,4,4)$. Thus, spectral integral variation of G_1 occurs in one place by adding the edge $\{1,3\}$ where the changed eigenvalue is the algebraic connectivity.

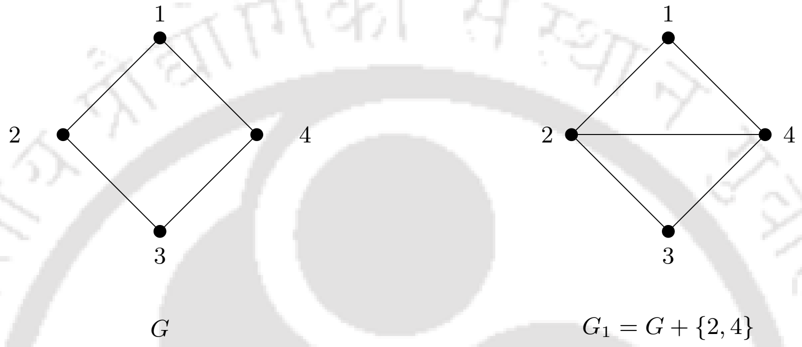


Figure 2.2: Spectral integral variation of G occurring in one place where $a(G)$ changes, and $a(G)$ has multiplicity more than one.

The following is an useful observation.

Lemma 2.2.2 *Suppose that G is a connected graph of order $n \geq 4$ with a simple algebraic connectivity and the Fiedler vector $Y = \hat{e}_n - \hat{e}_{n-1}$. Assume that the edges $\{1,2\}, \{n-1,n\} \notin G$. Let $H = G + \{1,2\}$. Then $a(H) = a(G)$ and it is a simple eigenvalue of $L(H)$ with Y as the Fiedler vector. Further the spectral integral variation of both G and H occur in one place by adding the edge $\{n-1,n\}$ where the changed eigenvalue is the algebraic connectivity.*

Proof: By Theorem 2.1.1 we have

$$a(H) \geq a(G).$$

Observe that the vector Y is an eigenvector of $L(H)$ corresponding to the eigenvalue $a(G)$. Since $a(G) > 0$, and $\tau(L(H)) = 0$, it follows that $a(G) = a(H)$ and Y is the corresponding eigenvector. Again by Theorem 2.1.1, we have

$$a(H) = \lambda_2(G) < \lambda_3(G) \leq \lambda_3(H),$$

which implies that $a(H)$ is simple.

Let $G' = G + \{n-1, n\}$. Let $\lambda \neq a(G)$ be an eigenvalue of $L(G)$ and Z be a corresponding eigenvector. Thus $Y^t Z = 0$, that is $Z(n-1) = Z(n)$. Thus

$$[L(G') - L(G)]Z \;=\; (\hat{e}_{n-1} - \hat{e}_n)(\hat{e}_{n-1} - \hat{e}_n)^t Z \;=\; 0.$$

So

$$L(G')Z \;=\; L(G)Z \;=\; \lambda Z.$$

We see that spectral integral variation occurs in one place by adding the edge $\{n-1, n\}$ to G where the changed eigenvalue is $a(G)$. Proof of spectral integral variation of H occurring in one place by adding the edge $\{n-1, n\}$ where the changed eigenvalue is $a(H)$ is similar. ▪

The following is one of our main results that characterizes all graphs with simple algebraic connectivity in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the algebraic connectivity. Note that, by K_n we denote the complete graph of order n.

Theorem 2.2.3 *Let G be a connected graph with vertices $\{1, 2, \ldots, n\}$ and a simple algebraic connectivity. Then spectral integral variation occurs in one place by adding a new edge $\{i, j\}$ where the changed eigenvalue is $a(G)$ if and only if $G = K_n \setminus \{i, j\}$.*

Proof: If $G = K_n \setminus \{i, j\}$, then the spectrum of $L(G)$ and $L(G + \{i, j\})$ are

$$S(G) = (0, n-2, n, \ldots, n) \quad \text{and} \quad S(G + \{i, j\}) = (0, n, n, \ldots, n),$$

respectively. Hence spectral integral variation occurs in one place and the changed eigenvalue is $a(G)$.

Conversely, assume that $a(G)$ has multiplicity one and spectral integral variation of G occurs in one place by adding the edge $\{i, j\}$ where the changed eigenvalue is $a(G)$. It follows that if $S(G) = (0 = \lambda_1, \lambda_2 = a(G), \lambda_3, \ldots, \lambda_n)$ then $S(G + \{i, j\}) = (S(G) \setminus \{\lambda_2\}) \cup \{\lambda_2 + 2\}$. By Theorem 2.1.3 and the corollary following that we have $N(i) = N(j)$, $\lambda_2 = |N(i)| = |N(j)|$ and the Faria vector $Y = \hat{e}_i - \hat{e}_j$ is a Fiedler vector of G.

Let $|N(i)| = k$. Clearly $C(G, Y) = N(i)$, the characteristic set of G. Note that $G \setminus C(G, Y)$ is disconnected with at least two components $\{i\}$ and $\{j\}$. Let $G_1, G_2, \ldots, G_m$ be the $m (\geq 2)$ components of $G \setminus C(G, Y)$, where $G_1 = \{i\}$ and $G_2 = \{j\}$. Let $\hat{L}(G_t)$ be the principal submatrix of $L(G)$ corresponding to G_t and $L(G_t)$ be the Laplacian matrix of the graph G_t, for $t = 1, 2, \ldots, m$. After a permutation similarity operation we have

$$L(G) = \left[\begin{array}{cccc|ccc}
\hat{L}(G_1) & 0 & \cdots & 0 & & & \\
0 & \hat{L}(G_2) & \cdots & 0 & & & \\
\vdots & \vdots & \ddots & \vdots & C_1 & \cdots & C_k \\
0 & 0 & \cdots & \hat{L}(G_m) & & & \\
\hline
 & & C_1^t & & d_1 & \cdots & * \\
 & & \vdots & & * & \ddots & * \\
 & & C_k^t & & * & \cdots & d_k
\end{array}\right].$$

By Corollary 1.2.5 (Chapter 1), $\tau(\hat{L}(G_t)) \geq k$, for all $t = 1, 2, \ldots, m$.

We claim that for each t, every vertex of G_t is adjacent to each of the k vertices of $C(G, Y)$ in G. To see the claim suppose that there is a component, say G_3, and a vertex $v \in G_3$ such that v is not adjacent to all vertices of $C(G, Y)$ in G. Thus

$$\hat{L}(G_3) < kI + L(G_3) \Rightarrow \mathbb{1}^t \hat{L}(G_3) \mathbb{1} < \mathbb{1}^t [kI + L(G_3)] \mathbb{1} = |G_3| k,$$

where $|G_3|$ denotes the number of vertices in the graph G_3. Thus by Rayleigh-Ritz theorem [42], we have $\tau(\hat{L}(G_3)) < k$, which is a contradiction to Lemma 1.2.4. So the claim is justified.

Since each vertex of G_t, $t = 1, 2, \ldots, m$, is adjacent to every vertex in $C(G, Y)$, we see that

$$\hat{L}(G_t) = kI + L(G_t).$$

Hence $\tau(\hat{L}(G_t)) = k$, for all $t = 1, 2, \ldots, m$. Thus each of the m components $G_1, G_2, \ldots, G_m$ are Perron branches of G at $C(G, Y)$. By Lemma 1.2.6, the multiplicity of $\lambda_2 = a(G)$ is at least $m - 1$. But from the hypothesis the multiplicity of λ_2 is 1. So $m - 1 \leq 1$. But we know that $m \geq 2$ and thus $m = 2$. Thus $G_1 = \{i\}$ and $G_2 = \{j\}$ are the only two components of $G \setminus C(G, Y)$. Hence $G = G^* \vee (\{i\} + \{j\})$ where G^* is the subgraph of G induced by $C(G, Y)$. We recall that $C(G, Y) = N(i)$, and notice that $|C(G, Y)| = n - 2$. It is known that any eigenvalue of the Laplacian matrix of size n does not exceed n (see [52]).

Now suppose that G^* is not complete. Thus $n \geq 4$. Let $E(G^*) = K_{n-2} \setminus \{e_1, e_2, \ldots, e_p\}$, for some $p \geq 1$, where K_{n-2} is the complete graph with vertex set $\{1, 2, \ldots, n\} \setminus \{i, j\}$ and $\{e_1 = \{i', j'\}, e_2, \ldots, e_p\} \subset E(K_{n-2})$. Let H be the graph obtained by adding edges $e_2, \ldots, e_p$ to G. Repeated application of Lemma 2.2.2 reveals that the eigenvalue $a(H) = n - 2$ is simple. But the vector $Y' = \hat{e}_{i'} - \hat{e}_{j'}$ is also an eigenvector corresponding to the eigenvalue $n - 2$, which is a contradiction. ∎

The following is an immediate corollary.

Corollary 2.2.4 *Let G be a connected graph on vertices $\{1, 2, \ldots, n\}$ such that 1 and 2 are nonadjacent in G. Suppose also that $a(G + \{1, 2\}) - a(G) = 2$. Then $a(G)$ is simple, $a(G) = n - 2, G + \{1, 2\} = K_n$.*

Proof: By Theorem 2.1.1 and the discussion following, we have that $\lambda_i(G + \{i, j\}) = \lambda_i(G)$, for all $i \neq 2$. Thus spectral integral variation occurs in one place where the changed eigenvalue is $a(G)$. Again by Theorem 2.1.1,

$$a(G + \{i, j\}) \leq \lambda_3(G),$$

so that $a(G)$ is simple. The rest follows from Theorem 2.2.3. ■

Now we consider the case where the multiplicity of $a(G)$ is at least 2 and spectral integral variation of G occurs in one place by adding edge between i and j where the changed eigenvalue is $a(G)$.

Theorem 2.2.5 *Let G be a connected graph with vertices $\{1, 2, \ldots, n\}$ and i, j be two nonadjacent vertices in G. Then spectral integral variation of G occurs in one place by adding edge between i and j where $a(G)$ is the changed eigenvalue if and only if $G = G^* \vee (G_1 + \{i\} + \{j\})$, where G^* is a graph of order k(say), $1 \leq k \leq n - 2$, with $a(G^*) \geq 2k - n$ and G_1 is any graph on $n - k - 2$ vertices.*

Proof: Let $G = G^* \vee (G_1 + \{i\} + \{j\})$, where G^* is a graph of order k, $1 \leq k \leq n - 2$, with $a(G^*) \geq 2k - n$ and G_1 is any graph on $n - k - 2$ vertices. Let

$S(G^*) = (0 = \lambda_1(G^*), \lambda_2(G^*), \ldots, \lambda_k(G^*))$ and $S(G_1) = (0 = \lambda_1(G_1), \lambda_2(G_1), \ldots, \lambda_{n-k-2}(G_1))$.

Then by Theorem 3.1.1 (Chapter 3),

$$S(G) = (0, \lambda_2(G^*) + n - k, \ldots, \lambda_k(G^*) + n - k, \lambda_2(G_1) + k, \ldots, \lambda_{n-k-2}(G_1) + k, k, k, n).$$

Since $a(G^*) \geq 2k - n$, that is, $a(G^*) + n - k \geq k$, we have $a(G) = k$. Now

$$G + \{i, j\} = G^* \vee (G_1 + K_2),$$

where K_2 is the complete graph with vertex set $\{i, j\}$ and

$$S(G+\{i, j\}) = (0, \lambda_2(G^*)+n-k, \ldots, \lambda_k(G^*)+n-k, \lambda_2(G_1)+k, \ldots, \lambda_{n-k-2}(G_1)+k, k, k+2, n).$$

Hence spectral integral variation of G occurs in one place by adding the edge between i and j where $a(G)$ is the changed eigenvalue.

Conversely, assume that spectral integral variation of G occurs in one place by adding edge between i and j where the changed eigenvalue is $a(G)$. We proceed in a similar way as in Theorem 2.2.3 to see that $G = G^* \vee (G_1 + \{i\} + \{j\})$, where G^* is the graph induced by the set of vertices $N(i) = C(G, Y)$, where $Y = \hat{e}_i - \hat{e}_j$ is the Faria-Fiedler vector of G and $a(G) = k$. Thus

$$S(G) = (0, \lambda_2(G^*) + n - k, \ldots, \lambda_k(G^*) + n - k, \lambda_2(G_1) + k, \ldots, \lambda_{n-k-2}(G_1) + k, k, k, n).$$

It follows that $\lambda_2(G^*) + n - k \geq k$, that is, $\lambda_2(G^*) \geq 2k - n$. Thus the proof is complete. ■

Let us denote by $K_{m,n}$ the complete bipartite graph with parts of size m and n. Now consider the following example.

Example 2.2.2 Consider graph $K_{4,3}$ on parts $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7\}$ (see Figure 2.3). We know that $S(K_{4,3}) = (0, 3, 3, 3, 4, 4, 7)$. Using Theorem 2.1.3, we have

$$S(K_{4,3} + \{5, 6\}) = (0, 3, 3, 3, 4, 6, 7) \quad \text{and} \quad S(K_{4,3} + \{1, 2\}) = (0, 3, 3, 4, 4, 5, 7).$$

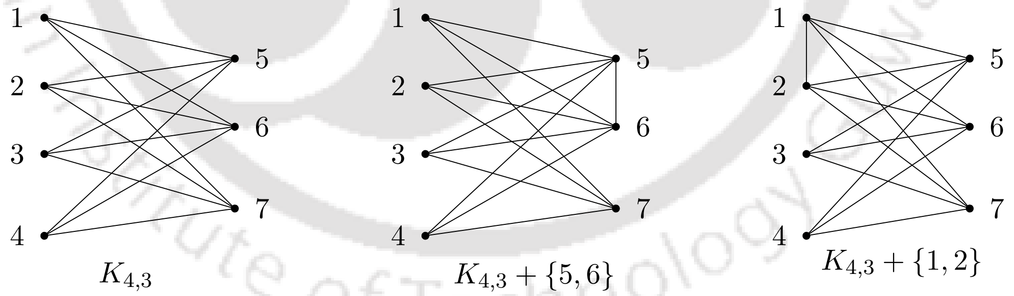


Figure 2.3: Graphs of $K_{4,3}$, $K_{4,3} + \{5, 6\}$ and $K_{4,3} + \{1, 2\}$.

Thus by adding an edge between 1 and 2 the spectral integral variation occurs in one place where the changed eigenvalue is the algebraic connectivity. But if we add an edge between 5 and 6 (the vertices which belong to the part B), then the spectral integral variation occurs in one place, but the algebraic connectivity does not change.

In general we have the following result as an application Theorem 2.2.5.

Corollary 2.2.6 *Let $2 \leq m < n$, and $K_{m,n}$ be the complete bipartite graph on parts A of m vertices and B of n vertices. Let $i, j \in A$ and $i', j' \in B$. Then the following statements hold.*

(i) *Spectral integral variation occurs in one place by adding the edge $\{i', j'\}$ where the changed eigenvalue is the algebraic connectivity.*

(ii) *Spectral integral variation does not occur in one place where the changed eigenvalue is the algebraic connectivity by adding the edge $\{i, j\}$.*

Proof: Follows immediately from Theorem 2.2.5. ■

Remark 2.2.7 *In Corollary 2.2.6, if $m = n$, then by adding either the edge $\{i, j\}$ or $\{i', j'\}$ the spectral integral variation occurs in one place where the changed eigenvalue is the algebraic connectivity.*

The following result describes a way to construct a graph G which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the algebraic connectivity.

Corollary 2.2.8 *Let G^* be a graph on k vertices and G_1 be a graph on $m, (m \geq k)$ vertices with at least two isolated vertices i, j. Put $G = G^* \vee G_1$. Then in G, spectral integral variation occurs in one place by adding the edge $\{i, j\}$, where the changed eigenvalue is the algebraic connectivity.*

Proof: Note that $|V(G)| = k + m$ and $a(G^*) \geq 0 \geq 2k - (k + m)$. Following Theorem 2.2.5 we see that G has the required structure so that addition of the edge $\{i, j\}$ results in spectral integral variation where the changed eigenvalue is $a(G)$. ■

2.3 Spectral Integral Variation in one place with $a(G)$ unchanged

After the characterization of graphs in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the algebraic connectivity it is natural to ask the following two questions:

(1) Characterize graphs G in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the Laplacian spectral radius, $\mu(G)$.

(2) Characterize graphs G in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the third smallest Laplacian eigenvalue, $\lambda_3(G)$.

It is not so difficult to answer (1). To do this we need the following result of Grone and Merris [34].

Lemma 2.3.1 *[34] Let G be a connected graph with order n. Then*

$$\mu(G) \geq \Delta(G) + 1,$$

with equality if and only if $\Delta(G) = n - 1$, where $\Delta(G)$ denotes the maximum degree of a vertex in G.

The following result shows that the spectral integral variation of a connected graph G does not occur in one place by adding edge between i and j where the changed eigenvalue is the laplacian spectral radius.

Lemma 2.3.2 *Let G be a connected graph with vertex set $\{1, 2, \ldots, n\}$. Let i, j be any two nonadjacent vertices in G. Then spectral integral variation of G does not occur in one place by adding edge between i and j where the changed eigenvalue is $\mu(G)$.*

Proof: On the contrary suppose that spectral integral variation of G occurs in one place by adding an edge where the changed eigenvalue is $\mu(G)$. Then using Theorem 2.1.3 and the corollary following that we have $N(i) = N(j)$, $\mu(G) = |N(i)| = |N(j)|$. From Lemma 2.3.1, $\mu(G) \geq \Delta(G) + 1$. This implies that $|N_i| > \Delta(G)$, which gives a contradiction. ∎

The following is an immediate corollary.

Corollary 2.3.3 *Let G be a graph with vertex set $\{1, 2, \ldots, n\}$. Let i, j be two vertices in G such that $N(i) = N(j) \neq \emptyset$. Then $\mu(G) = \mu(G + \{i, j\})$, that is, the Laplacian spectral radius does not change by adding an edge between i and j.*

Proof: Consider the component C of G containing the vertices i and j. The conclusion is now immediate from Lemma 2.3.2. ∎

The characterization of graphs in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the third smallest Laplacian eigenvalue is a rather difficult task. Thus we restrict our search to trees with such a variation. The following result gives an equivalent condition for a tree in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the kth smallest Laplacian eigenvalue.

Proposition 2.3.4 *Let i and j be two nonadjacent vertices in a tree T. Then the following two statements are equivalent.*

(i) *The spectral integral variation of T occurs in one place by adding an edge between i and j, where the changed eigenvalue is the kth smallest Laplacian eigenvalue.*

(ii) *The kth smallest Laplacian eigenvalue of T equals to 1. The vertices i and j are two pendant vertices (vertices of degree one) with the same neighbour.*

Proof: (i)⇒(ii). By Theorem 2.1.3, we see that $N(i) = N(j) \neq \emptyset$ and by Corollary 2.1.4, the changed eigenvalue is $|N(i)|$. If possible let u and v be two vertices in $N(i) = N(j)$. Then $[i, u, j, v, i]$ is a cycle. Hence $|N(i)| = |N(j)| = 1$, that is, the kth smallest laplacian eigenvalue (changed eigenvalue) is 1.

(ii)⇒(i). By Theorem 2.1.3, we see that the spectral integral variation in T occurs by adding the edge $\{i, j\}$. By Corollary 2.1.4, the changed eigenvalue is 1. Thus (i) follows. ■

In view of the previous result, our task becomes to characterize trees with 1 as the third smallest Laplacian eigenvalue and with two pendant vertices having a common neighbour.

Let G be a graph on n vertices. Let $\hat{G}$ be the graph obtained from G by adding a new pendant vertex to some vertex of G. Then using the Courant-Weyl inequalities [17, 42], we have the following result, which we shall use later.

Lemma 2.3.5 (Grone, Merris and Sunder [33, Corollary 4.2]) *Let G be a graph on n vertices. Let $\hat{G}$ be the graph obtained from G by adding a new pendant vertex to some vertex of G. Then*

$$\lambda_1(\hat{G}) \leq \lambda_1(G) \leq \lambda_2(\hat{G}) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(\hat{G}) \leq \lambda_n(G) \leq \lambda_{n+1}(\hat{G}),$$

where $\lambda_i(G)$ and $\lambda_i(\hat{G})$ are the ith smallest eigenvalues of $L(G)$ and $L(\hat{G})$, respectively.

The following is a well-known result that describes the Laplacian spectrum of a path (see Fiedler [25] and Grone, Merris, and Sunder [33]).

Theorem 2.3.6 (Fiedler [25]) *Let P_n be the path of order n. Then*

$$S(P_n) = \left(0,\ 2\left(1 - \cos\frac{\pi}{n}\right), 2\left(1 - \cos\frac{2\pi}{n}\right), \ldots, 2\left(1 - \cos\frac{(n-1)\pi}{n}\right)\right).$$

The following result characterizes all trees with algebraic connectivity 1.

Corollary 2.3.7 *The star on $n \geq 3$ vertices is the only tree which can have algebraic connectivity 1.*

Proof: Let T be a tree which is not a star. So T has a diameter of at least 3. Think of T being generated from P_4, a path of length 4, by adding pendant vertices one after another. Thus by Lemma 2.3.5 and Theorem 2.3.6, we have

$$\lambda_2(T) \leq \lambda_2(P_4) < 1.$$

The conclusion follows. ■

From Theorem 2.2.5 we have that: *star is the only tree for which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the algebraic connectivity.* Thus in this case, the class of trees T with $a(T) = 1$ is precisely the class of trees in which spectral integral variation occurs in one place by adding an edge, where the changed eigenvalue is $a(T)$.

2.4 Trees with third smallest Laplacian eigenvalue 1

In this section we characterize all trees with third smallest Laplacian eigenvalue 1. Henceforth the multiplicity of the algebraic connectivity is assumed to be one (that is, $a(T)$ is simple), unless stated otherwise.

The multiplicity of the Laplacian eigenvalue 1 of a tree has already been studied by Faria [24] and Grone, Merris and Sunder [33]. Let T be a tree with pendant vertices $v_1, v_2, \ldots, v_k, k \geq 2$, adjacent to the vertex v. Define vectors $Y_i, \quad i = 1, \ldots, k-1$ as $Y_i(v_1) = 1, \; Y_i(v_{i+1}) = -1, \; Y(w) = 0$, for all other $w \in T$. Then it can be verified that the nonzero vectors Y_is are linearly independent eigenvectors of T corresponding to the Laplacian eigenvalue 1. It follows that

$$p(T) - q(T) \leq m_T(1), \tag{2.1}$$

where $p(T)$ is the number of pendant vertices in T and $q(T)$ is the number of *quasipendant* vertices (vertices adjacent to a pendant vertex) and $m_T(1)$ is the multiplicity of 1 as an eigenvalue of $L(T)$. Note that in the case of a star $\mathbf{S}_{n+1}$ on $n+1$ vertices, the above inequality becomes equality.

Naturally, one wonders whether it is possible for a tree with $p(T) = q(T)$ to have a Laplacian eigenvalue 1. The path P_6, with 1 as the third smallest Laplacian eigenvalue is the most suitable answer. In fact, there is a larger class of paths with the property.

Proposition 2.4.1 *The paths P_{3k}, $k > 1$ are precisely the paths with $p(T) = q(T)$ and which have 1 as a Laplacian eigenvalue.*

Proof: It follows from Theorem 2.3.6 that $1 \in S(P_n)$ if and only if $\frac{k\pi}{n} = \frac{\pi}{3}$ for some $k > 1$. That is, if and only if $n = 3k, k > 1$. The conclusion follows. ■

The class of trees with $p(T) = q(T)$ and with a Laplacian eigenvalue 1 is a much larger, see Barik, Lal and Pati [6].

Let T be a tree and v be a vertex. Let B be a branch of T at v. Let $\hat{L}(B)$ be the principal submatrix of $L(T)$ corresponding to B. The matrix $\hat{L}(B)$ is invertible (see Kirkland, Neumann and Shader [49]) and the inverse $G = (g_{ij})$ is described as follows: let i, j be vertices in B. Then g_{ij} = number of edges in common between the two paths $P_{v,i}$ and $P_{v,j}$ obtained by joining the vertex v to the vertices i and j, respectively. Observe that $\hat{L}(B)^{-1}$ is always entrywise positive.

Note that when B_1 and B_2 are two branches of T at v such that $\hat{L}(B_1)^{-1}$ is entrywise dominated by a proper principal submatrix of $\hat{L}(B_2)^{-1}$, then using Perron-Frobenius theory, we see $\rho(\hat{L}(B_1)^{-1}) < \rho(\hat{L}(B_2)^{-1})$, where $\rho(A)$ denotes the spectral radius of A, that is, $\tau(\hat{L}(B_1)) > \tau(\hat{L}(B_2))$, where $\tau(A)$ denotes the smallest eigenvalue of A.

A general version of the following useful result can be found in Pati [61].

Lemma 2.4.2 (Pati [61, Lemma 2.6]) *Let T be a tree. Let v be a vertex of T such that $T - v$ is disconnected with at least 3 branches. Let T_1, T_2, T_3 be three of the branches of $T - v$ and $\hat{L}(T_1), \hat{L}(T_2), \hat{L}(T_3)$ be the corresponding principal submatrices of $L(T)$. Suppose that $\tau(\hat{L}(T_1)) \leq \tau(\hat{L}(T_2)) \leq \tau(\hat{L}(T_3))$. Then either $\tau(\hat{L}(T_3)) > \lambda_3$ or $\tau(\hat{L}(T_2)) = \tau(\hat{L}(T_3)) = \lambda_3$. Thus, it is always true that $\tau(\hat{L}(T_3)) \geq \lambda_3$.*

The following result puts a restriction on the diameter of a tree with 1 as the third smallest Laplacian eigenvalue.

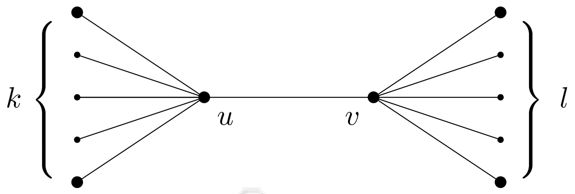
Lemma 2.4.3 *Let T be a tree with diameter at least 6. Then $\lambda_3(T) < 1$.*

Proof: For the path P_7 on 7 vertices, $\lambda_3(P_7) = 2\left(1 - \cos\frac{2\pi}{7}\right) < 1$. It follows from Lemma 2.3.5 that $\lambda_3(T) \leq \lambda_3(P_7) < 1$. ■

In view of Lemma 2.4.3, a tree with $\lambda_2 < 1$ and $\lambda_3 = 1$ must have diameter $3, 4$, or 5. Below we show that all trees with diameter 3, except P_4, have the third smallest Laplacian eigenvalue 1. Note that these trees do not have algebraic connectivity 1.

Theorem 2.4.4 *Any tree T with diameter 3, except P_4 has $\lambda_3 = 1$.*

Proof: Recall that $L(P_4)$ doesn't have eigenvalue 1. Any tree T different from P_4 with diameter 3 has the form as in the following figure,



where $k \geq l \geq 1$ and $k > 1$. As $p(T) - q(T) = n - 4 (\geq 1)$, using (3.5), we have $m_T(1) \geq n - 4$. We know $0 \in S(T)$ and $a(T) < 1$. As $\lambda_3(P_4) > 1$, by the repeated application of Lemma 2.3.5, we have $\lambda_n(T) > 1$ and $\lambda_{n-1}(T) > 1$. And since $n \geq 5, n - 1 \neq 3$. Thus $\lambda_3(T) = 1$. ■

Next, we talk about trees of diameter 4. The following result characterizes all trees of diameter 4, with 1 as the third smallest Laplacian eigenvalue.

Theorem 2.4.5 *Let T be any tree of diameter 4 (we assume $P = [1, 2, 3, 4, 5]$ corresponds to its diameter). Then $\lambda_3(T) = 1$ if and only if deg(3) $= 2$, and $p(T) - q(T) > 0$.*

Proof: Let T be a tree on $n (n \geq 6)$ vertices with diameter 4 such that $p(T) - q(T) > 0$ and $deg(3) = 2$. Thus T has the form as shown in Figure 2.4, where $k \geq l \geq 1$ and $k > 1$. As $p(T) - q(T) = n - 5 (\geq 1)$, using (3.5), we have $m_T(1) \geq n - 5$.

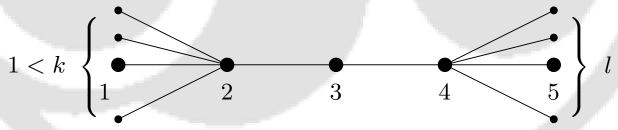
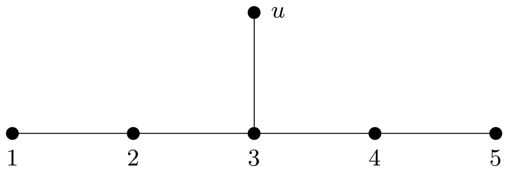


Figure 2.4: Tree of diameter 4 with $p(T) - q(T) > 0$

We know $0 \in S(T)$ and $a(T) < 1$. As $\lambda_3(P_5) > 1$, by the repeated application of Lemma 2.3.5, we have $\lambda_n(T) > 1$, $\lambda_{n-1}(T) > 1$ and $\lambda_{n-2}(T) > 1$. Thus $\lambda_3(T) = 1$.

Now, consider the tree $\bar{T}$ obtained from the path $P_5 = [1, 2, 3, 4, 5]$ by adding a pendant vertex u (say) at 3.



$\bar{T}$

Then $\bar{T}-3$ has 3 branches B_1, B_2, B_3, where B_1 contains vertices $1, 2$, B_2 contains vertices $4, 5$ and B_3 contains u. Observe that $\tau(\bar{L}(B_i)) < 1$, for $i = 1, 2$ and $\tau(\bar{L}(B_3)) = 1$. Thus using Lemma 2.4.2, $\lambda_3(\bar{T}) < 1$. Let T be a tree satisfying the statement of the theorem with $deg(3) \geq 3$. Then T is either equal to the tree $\hat{T}$ or is obtained from $\bar{T}$ by adding pendant vertices successively. Thus using Lemma 2.3.5, $\lambda_3(T) < 1$.

Any tree T of diameter 4 which is not of the form described in the statement is either isomorphic to $\bar{T}$ or can be obtained from $\bar{T}$ by adding pendant vertices. Thus by using Lemma 2.3.5, $\lambda_3(T) < 1$. ■

Our next result characterizes all trees of diameter 5 and with 1 as the third smallest Laplacian eigenvalue.

Theorem 2.4.6 *Let T be a tree of diameter 5. Then $\lambda_3(T) = 1$ if an only if T is obtained from a path $P_6 = [1, 2, 3, 4, 5, 6]$ by adding $l \geq 0$ new pendant vertices at 5 and $k \geq l$, new pendant vertices at 2.*

Proof: Let T be a tree on $n (n \geq 6)$ vertices as described in the statement. Thus T has the form as shown in Figure 2.5, where $l \geq 0$ and $k \geq l$.

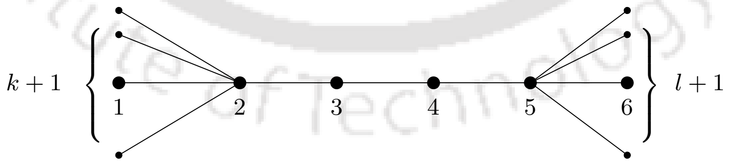


Figure 2.5: Structure of T

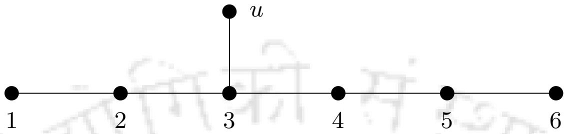
Using (3.5) and the discussion before that $m_T(1) \geq n-6$, and we have $n-6$ linearly independent eigenvectors of $L(T)$ corresponding to the eigenvalue 1. Observe that

$$\begin{bmatrix} 0 & -(k+1) & -(k+1) & 0 & 1 & \dots & 1 & \frac{-(k+1)}{l+1} & \dots & \frac{-(k+1)}{l+1} \end{bmatrix}^T$$

is also an eigenvector of $L(T)$ corresponding the eigenvalue 1 and is linearly independent of the other $n-6$ eigenvectors corresponding to 1. Thus $m_T(1) \geq n-5$. We know $0 \in S(T)$

and $a(T) < 1$. Also since $\lambda_4(P_6) > 1$, by the repeated application of Lemma 2.3.5 we have $\lambda_i(T) > 1$, for $i = n, n-1, n-2$. Thus $\lambda_3(T) = 1$.

Now consider a tree $\hat{T}$ obtained from the path $P_6 = [1,2,3,4,5,6]$ by adding a pendant vertex u (say) at 3.



Then $\hat{T} - 3$ has 3 branches B_1, B_2, B_3, where B_1 contains vertices $1, 2$, B_2 contains vertices $4, 5, 6$ and B_3 contains u. Notice that $\tau(\hat{L}(B_i)) < 1$, for $i = 1, 2$. Also $\tau(\hat{L}(B_3)) = 1$. Thus using Lemma 2.4.2, $\lambda_3(\hat{T}) < 1$. Any tree T of diameter 5 which is not of the form described in the statement is either isomorphic to $\hat{T}$ or can be obtained from $\hat{T}$ by adding pendant vertices. Thus by using Lemma 2.3.5, $\lambda_3(T) < 1$. ■

We summarize our discussions to state the main result of this section, which characterizes all trees with 1 as the third smallest Laplacian eigenvalue.

Theorem 2.4.7 *Let T be a tree on n vertices. Then $\lambda_3(T) = 1$ if and only if T is one of the following types:*

(i) *T is any tree with diameter 3, other than P_4. (Thus $p(T) - q(T) > 0$.)*

(ii) *T is a tree with diameter 4 (assume the path on $1,2,3,4,5$ corresponds to its diameter) such that $deg(3) = 2$ and $p(T) - q(T) > 0$.*

(iii) *T is obtained from the path on $1,2,3,4,5,6$ by adding $l \geq 0$ new pendant vertices at 5 and $k\ (\geq l)$ new pendant vertices at 2.*

Thus the only tree with $p(T) = q(T)$ and $\lambda_3(T) = 1$ is P_6.

2.5 Spectral integral variation of trees with λ_3 increasing by 2

Using the results proved in Section 2.4, here we characterize all trees in which spectral integral variation occurs in one place by adding an edge where the changed eigenvalue is the third smallest Laplacian eigenvalue. We require a notation before that. Let $s > 1, k > 1, l > 0$ be integers. By $T_{s,l,k}$, denote the tree obtained by taking the path $P_s = [1, 2, \ldots, s]$ and adding l pendant vertices at 1 and k pendant vertices at s.

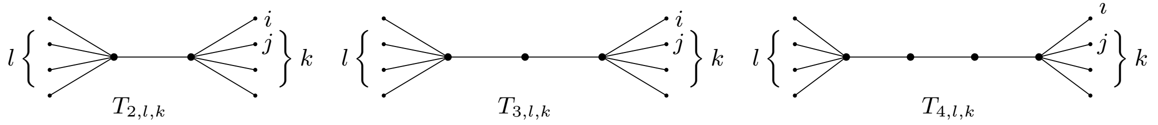


Figure 2.6: $T_{2,l,k}, T_{3,l,k}$ and $T_{4,l,k}$

Theorem 2.5.1 *Let i and j be two non adjacent vertices in a tree T of order n. Then spectral integral variation of T occurs in one place by adding an edge between i and j where the changed eigenvalue is the third smallest Laplacian eigenvalue if and only if T is one of $T_{2,l,k}$, $T_{3,l,k}$ or $T_{4,l,k}$ and i, j are two pendant vertices adjacent to the same vertex.* (*See Figure* 2.6)

Proof: Suppose T is one of $T_{2,l,k}$, $T_{3,l,k}$ or $T_{4,l,k}$ and i, j are two pendant vertices adjacent to the same vertex. From Theorem 2.4.7, we have $\lambda_3(T) = 1 = |N(i)| = |N(j)|$. Thus using Theorem 2.1.3, spectral integral variation of T occurs in one place by adding an edge between i and j where the Laplacian eigenvalue 1 changes to 3.

Conversely, let the spectral integral variation of T occurs in one place by adding an edge between i and j where the changed eigenvalue is λ_3. Thus by Theorem 2.1.3 and discussion following that we have $N(i) = N(j)$ and $\lambda_3 = |N(i)| = |N(j)|$.

If $|N(i)| = |N(j)| \geq 2$, then we will get a cycle in T, which is not possible. So $|N(i)| = |N(j)| = 1$ and i, j are pendant vertices adjacent to the same vertex in T. Hence $\lambda_3 = 1$. Now, using Theorem 2.4.7, T is a tree of the desired form. ■

As an immediate corollary we have the following result which characterizes all trees in which λ_3 has multiplicity one and spectral integral variation occurs in one place by adding an edge where λ_3 increases by 2.

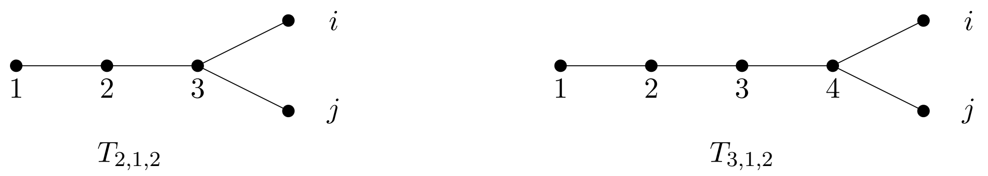


Figure 2.7: $T_{2,1,2}, T_{3,1,2}$

Corollary 2.5.2 *Let T be a tree with vertex set $\{1, 2, \ldots, n\}$ and i, j be two non adjacent vertices in T. Let the third smallest Laplacian eigenvalue have multiplicity one. Then spectral integral variation of T occurs in one place by adding an edge between i and j where the changed eigenvalue is the third smallest Laplacian eigenvalue if and only if T is a tree obtained from a path of length two or three by adding two pendant vertices i and j to one end (that is, T is either $T_{2,1,2}$ or $T_{3,1,2}$ and i, j are two pendant vertices adjacent to the same vertex). (See Figure 2.7)*

Proof: Follows immediately from Theorem 2.5.1. ∎

Chapter 3

Graph operations and the Laplacian spectra

3.1 Preliminaries

In this chapter we investigate the Laplacian spectra of graphs which are obtained by using some operations on other graphs. The Laplacian spectra of graphs obtained by operations like, complement, disjoint union, and join are well studied. We start by stating some of the results concerning to these graph operations, partly for the purpose of motivation and partly due to the frequent use of these results in the thesis. The proofs of these results can be found in [17, 25, 51, 57].

(a) **Complement.** Let G^c denote the complement of a graph G. Observe that, if G has n vertices, then

$$L(G) + L(G^c) = nI - J = L(K_n).$$

If X is any nonzero vector such that $X \perp \mathbb{1}$, then we have

$$L(G)X + L(G^c)X = nX.$$

In particular, $X \perp \mathbb{1}$ is an eigenvector of $L(G)$ if and only if X is an eigenvector of $L(G^c)$. It follows that

$$\lambda_2(G^c) = n - \lambda_n(G),\ \lambda_3(G^c) = n - \lambda_{n-1}(G),\ \ldots,\ \lambda_n(G^c) = n - \lambda_2(G). \tag{3.1}$$

Thus, a graph and its complement have the same set of eigenvectors. Furthermore, $\lambda_n(G) \leq n$ with equality if and only if $a(G^c) = 0$, if and only if G^c is disconnected.

We note that

$$a(G^c) = n - \mu(G). \tag{3.2}$$

(b) **Disjoint union.** The *disjoint union* $F + H$ of two graphs F and H is the graph $\Big(V(F) \cup V(H), E(F) \cup E(H)\Big)$, with the assumption that $V(F) \cap V(H) = \emptyset$. The Laplacian matrix of $F + H$ may be viewed as

$$L(F + H) = \begin{bmatrix} L(F) & 0 \\ 0 & L(H) \end{bmatrix}.$$

It follows that the Laplacian eigenvalues of $F + H$ are precisely the Laplacian eigenvalues of F and H, including multiplicities. We note here that $a(F + H)$ is always 0 as $F + H$ is disconnected.

(c) **Join.** The *join* $F \vee H$ of two graphs F and H is defined to be the graph $(F^c + H^c)^c$. In other words $F \vee H$ is the graph obtained from $F + H$ by inserting edges from each vertex of F to every vertex of H. Suppose that the orders of F and H are m and n, respectively. Observe that the Laplacian matrix of $F \vee H$ may be viewed as

$$L(F \vee H) = \begin{bmatrix} nI + L(F) & -J \\ -J^T & mI + L(H) \end{bmatrix}.$$

If $X \perp \mathbb{1}$ is any eigenvector of $L(F)$ corresponding to an eigenvalue λ_i, $i > 1$, then we have that

$$\begin{bmatrix} X^T & \mathbf{0}^T \end{bmatrix} L(F \vee H) \begin{bmatrix} X \\ \mathbf{0} \end{bmatrix} = X^T \Big[nI + L(F)\Big] X = n + \lambda_i.$$

In a similar way we see that $m + \mu_i$ is also an eigenvalue of $L(F \vee H)$, for each eigenvalue μ_i, $i > 1$, of $L(H)$. As 0 is an eigenvalue of $L(F \vee H)$ and the trace is the sum of the eigenvalues, we conclude that $m + n$ is also an eigenvalue of $L(F \vee H)$. Thus we have the following result from Merris [51, Theorem 2.1].

Theorem 3.1.1 Merris [51] *Let F and H be two graphs on m and n vertices, respectively. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ be the eigenvalues of $L(F)$ and $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ be the eigenvalues of $L(H)$. Then the eigenvalues of $L(F \vee H)$ are*

$$0,\ m + n,\ \lambda_2 + n, \lambda_3 + n, \ldots, \lambda_m + n,\ \mu_2 + m, \mu_3 + m, \ldots, \mu_n + m.$$

The eigenvalue $m+n$ *of* $L(F \vee H)$ *corresponds to an eigenvector* Y *where*

$$Y(v) = \begin{cases} -n & \text{if } v \in F, \\ m & \text{if } v \in H. \end{cases}$$

It follows from the above theorem that

$$a(F \vee H) = min(a(F) + n, a(H) + m). \tag{3.3}$$

Note that if a graph G of order n is the join of two graphs, then by Theorem 3.1.1, $\lambda_n(L(G)) = n$. Conversely, if $\lambda_n(L(G)) = n$ then by (3.1), we have that $\lambda_2(L(G^c)) = 0$, that is G^c is disconnected. Hence G must be a join of two graphs. This observation has been noted in Godsil [31].

3.2 Graphs with pockets

In this section we define a new graph operation and discuss the Laplacian eigenvalues of the resulting graphs. We have seen that the stars form an important family of trees. A *generalized star* is a tree T having at most one vertex of degree greater than 2. Notice that the class of generalized stars contains the class of stars. Let T be a generalized star with a vertex w such that $T - w$ is a disjoint union of paths $T_1, \ldots, T_r$. Then we refer to w as a *central vertex* of T. Note that if T has a vertex w of degree at least 3 then the central vertex is w. If T_i is a path of order p_i, $i = 1, \ldots, r$, then the generalized star T is denoted by $\mathbf{S}[p_1, p_2, \ldots p_r]$. If $p_1 = p_2 = \cdots = p_r = l$, then we denote the generalized star by $\mathbf{S}_{r,l}$. Note that, a path is also a generalized star; in this case, any vertex is considered to be a central vertex. To know some more background of the generalized stars see the paper of Johnson, Duarte and Saiago [44]. Let us consider an example here.

Figure 3.1: Generalized stars

Example 3.2.1 The trees $\mathbf{S}[2,3,4,4]$ and $\mathbf{S}_{5,3}$ in Figure 3.1 are generalized stars with central vertices w and $\bar{w}$, respectively.

Observe that the graph $\mathbf{S}[2,3,4,4]$ may be viewed as the result of of attaching a new path of length 2 to the central vertex of $\mathbf{S}[3,4,4]$. It may also be viewed as the result of attaching a new path of length 1 to each of the pendant vertices of $\mathbf{S}[1,2,3,3]$. This motivates us to bring in the following definition.

Definition 3.2.1 Let F, H_v be two connected graphs, v be a specified vertex of H_v and $u_1, \ldots, u_k$ be some vertices in F. Let $G = G[F, u_1, \ldots, u_k, H_v]$ be the graph obtained by taking one copy of F and k copies of H_v, and then attaching the ith copy of H_v to the vertex u_i, $i = 1, \ldots, k$, identifying u_i with the vertex v of the ith copy. Then the copies of the graph H_v that are attached to the vertices u_i, $i = 1, \ldots, k$ are referred to as *pockets*, and we describe G as a *graph with k pockets* .

Example 3.2.2 Consider the graphs F and H_v as shown in Figure 3.2. Then the graph $G[F, u_1, u_2, u_3, H_v]$ with 3 pockets is shown in the same figure.

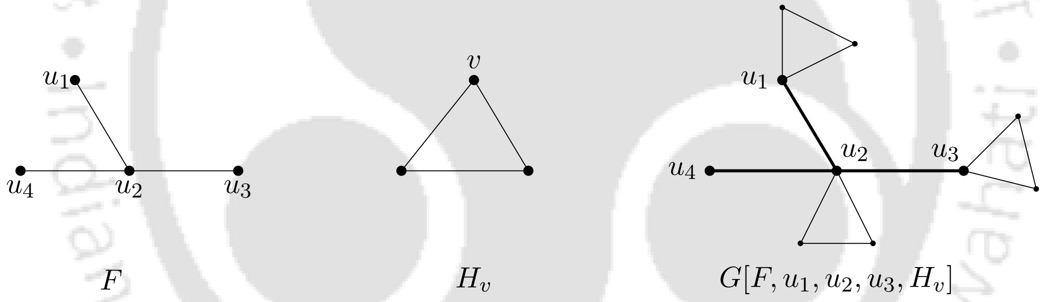


Figure 3.2: Graphs of F, H_v and $G[F, u_1, u_2, u_3, H_v]$.

It is natural to wonder how far the Laplacian spectrum of F and H_v are reflected in the Laplacian spectrum of $G[F, u_1, \ldots, u_k, H_v]$. Due to the very general nature of this operation it is difficult to give a complete description of the Laplacian spectrum of $G[F, u_1, \ldots, u_k, H_v]$ using the Laplacian spectra of F and H_v. Nevertheless some of the Laplacian eigenvalues of $G[F, u_1, \ldots, u_k, H_v]$ can always be described by the Laplacian eigenvalues of F and H_v and in some particular cases we can describe the complete Laplacian spectrum of $G[F, u_1, \ldots, u_k, H_v]$.

The study of graphs with pockets has already been done in some special cases, see, for example, Molitierno and Neumann [59]. Let $G = G[r, K_m]$ be the graph obtained by attaching a copy of K_m to each pendant vertex of the star $\mathbf{S}_{r+1}$ on $r+1$ vertices, by identifying each pendant vertex with a vertex v of K_m. Molitierno and Neumann [59] have described the complete Laplacian spectrum of $G[r, K_m]$.

Theorem 3.2.1 [59] *Let $G = G[r, K_m]$ be the graph obtained by attaching a copy of K_m to each vertex of degree* 1 *of $\mathbf{S}_{r+1}$, each at a common vertex v (say) of K_m. Then*

(i) $0 \in S(G)$ *with multiplicity* 1,

(ii) $m \in S(G)$ *with multiplicity* $(m-2)r$,

(iii) $\dfrac{m+1 \pm \sqrt{(m+1)^2-4}}{2} \in S(G)$ *with multiplicity* $r-1$ *and*

(iv) $\dfrac{r+m+1 \pm \sqrt{(r+m+1)^2-4(rm+1)}}{2} \in S(G)$ *with multiplicity* 1.

If $R = [r_{ij}]$ and S are two matrices then the *tensor product* of R and S is defined to be the partitioned matrix $[r_{ij}S]$ and is denoted by $R \otimes S$. We shall prove some results to generalize Theorem 3.2.1. We take a generalized star $\mathbf{S}_{r,l}$ and a graph H_v of order m $(m > 1)$, with a specified vertex v of degree $m-1$. Let $G[r, l, H_v]$ be the graph obtained by attaching a copy of H_v to each but the central vertex of $S_{r,l}$, each at the vertex v of H_v. In this case except $2l$ eigenvalues, we describe all other eigenvalues and eigenvectors of $L(G[r, l, H_v])$ using the eigenvalues and eigenvectors of $L(\mathbf{S}_{r,l})$ and $L(H_v)$, respectively. To do this we need the following lemma.

Lemma 3.2.2 *Let $\mathbf{S}_{r,l}$ be a generalized star on $rl+1$ vertices with central vertex w. Suppose that $S(\mathbf{S}_{r,l}) = (0 = \delta_1, \delta_2, \ldots, \delta_{rl+1})$. Then*

$$\begin{aligned} &\delta_2 = \delta_3 = \cdots = \delta_r, \\ &\delta_{r+2} = \delta_{r+3} = \cdots = \delta_{2r}, \\ &\vdots \\ &\delta_{(l-1)r+2} = \delta_{(l-1)r+3} = \cdots = \delta_{lr}, \end{aligned}$$

and the eigenvectors of $L(\mathbf{S}_{r,l})$ corresponding to these eigenvalues have entry 0 *corresponding to the vertex w.*

Proof: Let P_l be the path of order l. Assume that w is the first vertex of $\mathbf{S}_{r,l}$. The square matrix with (i,i)th entry equal to one and all other entries zero is denoted by $\hat{E}_i$. With a permutation similarity operation we can write

$$L(\mathbf{S}_{r,l}) = \left(\begin{array}{c|ccc} r & -\hat{e}_1^T & \cdots & -\hat{e}_1^T \\ \hline -\hat{e}_1 & & & \\ \vdots & & I \otimes (L(P_l) + \hat{E}_1) & \\ -\hat{e}_1 & & & \end{array}\right).$$

Suppose that the eigenvalues of the matrix $L(P_l) + \hat{E}_1$ are $\beta_1, \beta_2, \ldots, \beta_l$. Since $I \otimes (L(P_l) + \hat{E}_1)$ is a principal submatrix of $L(\mathbf{S}_{r,l})$, by the interlacing properties of symmetric matrices, we have

$$\delta_{pr+2} = \delta_{pr+3} = \cdots = \delta_{pr+r} = \beta_{p+1}, \text{ for } p = 0, 1, 2, \ldots, l-1.$$

Suppose that $Y_1, Y_2, \ldots, Y_l$ are eigenvectors of $L(P_l) + \hat{E}_1$ corresponding to the eigenvalues $\beta_1, \beta_2, \ldots, \beta_l$, respectively. Then observe that for $i = 1, 2, \ldots, l$,

$$\begin{pmatrix} 0 \\ Y_i \\ -Y_i \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} 0 \\ Y_i \\ \mathbf{0} \\ -Y_i \\ \vdots \\ \mathbf{0} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ Y_i \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ -Y_i \end{pmatrix},$$

are $r-1$ linearly independent eigenvectors corresponding to the eigenvalue β_i of $L(\mathbf{S}_{r,l})$ having entry 0 corresponding to the vertex w. Hence the proof. ■

As an immediate corollary we have the following result.

Corollary 3.2.3 *Let $\mathbf{S}_{r,l}$ be a generalized star on $rl+1$ vertices, $r \geq 2$, with central vertex w. Then $a(\mathbf{S}_{r,l})$ is of multiplicity $r-1$ and is independent of r.*

Proof: Let $S(\mathbf{S}_{r,l}) = (0 = \delta_1, \delta_2, \ldots, \delta_{rl+1})$. Let Y be a Fiedler vector of $\mathbf{S}_{r,l}$. From Lemma 3.2.2, we have

$$C(G, Y) = \{w\}.$$

Thus applying Theorem 1.2.9 of Chapter 1, multiplicity of $a(\mathbf{S}_{r,l})$ is $r-1$. ■

The following is one of our main results of this section.

Theorem 3.2.4 *Let $\mathbf{S}_{r,l}$ be a generalized star on $rl+1$ vertices with central vertex w. Let H_v be a graph of order m $(m > 1)$, with a specified vertex v of degree $m-1$. Let $G = G[r, l, H_v]$ be the graph obtained by attaching a copy of H_v to each but the central vertex of $\mathbf{S}_{r,l}$, each at the vertex v of H_v. Suppose that $S(H_v) = (0 = \gamma_1, \gamma_2, \ldots, \gamma_m)$ and $S(\mathbf{S}_{r,l}) = (0 = \delta_1, \delta_2, \ldots, \delta_{rl+1})$. Then*

(i) $0 \in S(G)$ *with multiplicity* 1,

(ii) $\gamma_j \in S(G)$ *with multiplicity* rl *for* $j = 2, \ldots, m-1$ *and*

(iii) $\dfrac{\delta_i + m \pm \sqrt{(\delta_i+m)^2 - 4\delta_i}}{2} \in S(G)$ *with multiplicity* $r-1$ *for* $i = 2, r+2, \dots, (l-1)r+2$.

Proof: Since the vertex v is of degree $m-1$, H_v can be written as $H_v = \{v\} \vee H$, where H is the graph obtained from H_v after deleting the vertex v and the edges incident to it. Thus the Laplacian matrix of G is

$$L(G) = \left[\begin{array}{c|c} L(\mathbf{S}_{r,l}) + \left[\begin{array}{c|c} 0 & \mathbf{0}^T \\ \hline \mathbf{0} & (m-1)I \end{array}\right] & \mathbb{1}^T \otimes \begin{bmatrix} \mathbf{0}^T \\ -I \end{bmatrix} \\ \hline \mathbb{1} \otimes \begin{bmatrix} \mathbf{0}^T \\ -I \end{bmatrix}^T & (L(H)+I) \otimes I \end{array}\right].$$

Suppose that $\mathbb{1} = Y_1, Y_2, \dots, Y_m$ are eigenvectors of $L(H_v)$ corresponding to the eigenvalues $\gamma_1, \gamma_2, \dots, \gamma_m$, respectively. Since $H_v = \{v\} \vee H$, using Theorem 3.1.1, we have $\gamma_m = m$ and $Y_m = \mathbb{1} - m\,\hat{e}_1$ (assuming v as the first vertex of H_v). Thus,

$$Y_j(v) = 0, \text{ for } j = 2, 3, \dots, m-1.$$

Hence for $j = 2, 3, \dots, m-1$,

$$\begin{pmatrix} 0 \\ Y_j \otimes \hat{e}_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ Y_j \otimes \hat{e}_{rl} \end{pmatrix}$$

are rl linearly independent eigenvectors corresponding to the eigenvalue γ_j of $L(G)$.

Let $\eta_1 = \dfrac{\delta_i + m + \sqrt{(\delta_i+m)^2 - 4\delta_i}}{2}$ and $\eta_2 = \dfrac{\delta_i + m - \sqrt{(\delta_i+m)^2 - 4\delta_i}}{2}$.

Suppose that $Z_1^i, Z_2^i, \dots, Z_{r-1}^i$, are $r-1$ linearly independent eigenvectors of $L(\mathbf{S}_{r,l})$ corresponding to the eigenvalue δ_i, where $i = 2, r+2, \dots, (l-1)r+2$. Now Lemma 3.2.2 guarantees that

$$Z_t^i(w) = 0, \text{ for } t = 1, 2, \dots, r-1.$$

Let for $t = 1, 2, \ldots, r-1$, $\overline{Z^i}_t$ be the vector obtained from Z_t^i by deleting the entry corresponding to the vertex w of $\mathbf{S}_{r,l}$. Then observe that for $t = 1, 2, \ldots, r-1$,

$$\begin{pmatrix} Z_t^i \\ \frac{1}{1-\eta_s}\overline{Z^i}_t \\ \vdots \\ \frac{1}{1-\eta_s}\overline{Z^i}_t \end{pmatrix}$$

are $r-1$ linearly independent eigenvectors corresponding to the eigenvalue η_s of $L(G)$, for $s = 1, 2$. Also 0 is an eigenvalue of $L(G)$ afforded by the eigenvector $\mathbb{1}$. Hence the proof. ■

Example 3.2.3 The tree G in Figure 3.3 is obtained by attaching a copy of K_2 to each but the central vertex of $\mathbf{S}_{4,2}$, each at one end vertex of K_2. We know $S(K_2) = (0, 2)$. Using MATLAB,

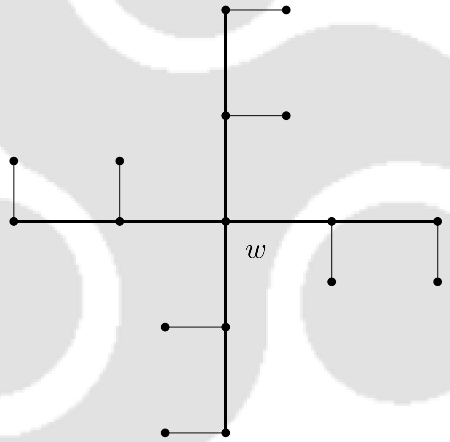


Figure 3.3: $G = G[4, 2, K_2]$

we obtain

$$S(\mathbf{S}_{4,2}) = (0,\ 0.3820,\ 0.3820,\ 0.3820,\ 1.6972,\ 2.6180,\ 2.6180,\ 2.6180,\ 5.3028)$$

and

$$S(G) = (0,\ 0.1729,\ 0.1729,\ 0.1729,\ \underline{0.5737},\ 0.6617,\ 0.6617,\ 0.6617,\ \underline{1.7875},\ 2.2091,\ 2.2091,$$

$$2.2091,\ \underline{2.8769},\ 3.9563,\ 3.9563,\ 3.9563,\ \underline{5.7619}).$$

Note that except four Laplacian eigenvalues (which are underlined) all other Laplacian eigenvalues of G can be obtained from the Laplacian eigenvalues of K_2 and $\mathbf{S}_{4,2}$, using Theorem 3.2.4.

Now taking $l = 1$, $\mathbf{S}_{r,l}$ becomes the star $\mathbf{S}_{r+1}$ and we obtain the following result as a corollary of Theorem 3.2.4, whose proof is much similar to that of Theorem 3.2.4.

Corollary 3.2.5 *Let H_v be a graph of order m $(m > 1)$, with a specified vertex v of degree $m-1$. Let $G = G[r, H_v]$ be the graph obtained by attaching a copy of H_v to each vertex of degree 1 of $\mathbf{S}_{r+1}$, each at the vertex v of H_v. Suppose that $S(H_v) = (0 = \gamma_1, \gamma_2, \ldots, \gamma_m)$. Then*

(i) $0 \in S(G)$,

(ii) $\gamma_j \in S(G)$ *with multiplicity r for $j = 2, \ldots, m-1$,*

(iii) $\dfrac{m+1 \pm \sqrt{(m+1)^2 - 4}}{2} \in S(G)$ *with multiplicity $r-1$ and*

(iv) $\dfrac{r+m+1 \pm \sqrt{(r+m+1)^2 - 4(rm+1)}}{2} \in S(G)$ *with multiplicity* 1.

Notice that Corollary 3.2.5 describes the complete Laplacian spectrum of $G = G[r, H_v]$ using the Laplacian spectrum of H_v, where H_v is any graph of order m with a specified vertex v of degree $m-1$ and generalizes Theorem 3.2.1.

Finally we consider a more general case. We take any two graphs F and H_v of orders n and m $(m > 1)$, respectively. Let v be a specified vertex of H_v and $u_1, \ldots, u_k$ be any k $(1 \le k \le n)$ arbitrarily chosen vertices in F. Let $G = G[F, u_1, \ldots, u_k, H_v]$ be the graph as defined in Definition 3.2.1. Then except $n+k$ eigenvalues, we describe all other eigenvalues of $L(G)$ using the eigenvalues of $L(F)$ and $L(H_v)$, when v has degree $m-1$. Moreover, we show that the other $n+k$ eigenvalues of $L(G)$ are independent of the graph H_v. This is shown in the following result.

Theorem 3.2.6 *Let F and H_v be graphs of orders n and m $(m > 1)$, respectively. Let v be a specified vertex in H_v of degree $m-1$ and $u_1, \ldots, u_k \in F$. Let $G = G[F, u_1, \ldots, u_k, H_v]$ be the graph as defined in Definition 3.2.1. Suppose that $S(H_v) = (0 = \gamma_1, \gamma_2, \ldots, \gamma_m)$. Then $\gamma_2, \ldots, \gamma_{m-1}$ are eigenvalues of $L(G)$, each of multiplicity k, and the other $n+k$ eigenvalues of $L(G)$ are independent of the graph H_v.*

Proof: Since v is of degree $m-1$, H_v can be written as $H_v = \{v\} \vee H$, where H is the graph obtained from H_v after deleting the vertex v and the edges incident to it. Let $\mathbf{S}_m$, be the star of order m with v as the central vertex. Without loss of generality let us assume that $u_1, \ldots, u_k$ are the first k vertices of F. Thus with a permutation similarity operation we can write

$$L(G[F,u_1,\ldots,u_k,\mathbf{S}_m]) = \left[\begin{array}{c||c} L(F)+(m-1)\left[\begin{array}{c|c} I & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] & \left[\begin{array}{c} I\otimes -\mathbb{1}^T \\ \hline \mathbf{0} \end{array}\right] \\ \hline \left[\begin{array}{c|c} I\otimes -\mathbb{1} & \mathbf{0} \end{array}\right] & I \end{array}\right].$$

Observe that the following are $(m-2)k$ linearly independent eigenvectors of $L(G[F,u_1,\ldots,u_k,\mathbf{S}_m])$ corresponding to the eigenvalue 1.

$$U_1^i = \left(\begin{array}{c} \mathbf{0} \\ \hline \hat{e}_1-\hat{e}_i \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{array}\right), U_2^i = \left(\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \\ \hat{e}_1-\hat{e}_i \\ \vdots \\ \mathbf{0} \end{array}\right), \ldots, U_k^i = \left(\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \hat{e}_1-\hat{e}_i \end{array}\right),$$

for $i = 2,3,\ldots,m-1$.

Let λ be an eigenvalue of $L(G[F,u_1,\ldots,u_k,\mathbf{S}_m])$ with an eigenvector

$$X = \left(\begin{array}{c} \hat{X} \\ \hline Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{array}\right)$$

orthogonal to the eigenvectors $U_1^i, U_2^i, \ldots, U_k^i$, for $i = 2,3,\ldots,m-1$. Thus $Y_j = \kappa_j \mathbb{1}$, for some scalar κ_j, $j = 1,2,\ldots,k$.

Also $L(G[F,u_1,\ldots,u_k,\mathbf{S}_m])X = \lambda X$ implies that

$$L(F)\hat{X} + (m-1)\begin{pmatrix} \hat{X}(1) \\ \hat{X}(2) \\ \vdots \\ \hat{X}(k) \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} \Sigma_{i=1}^{m-1} Y_1(i) \\ \Sigma_{i=1}^{m-1} Y_2(i) \\ \vdots \\ \Sigma_{i=1}^{m-1} Y_k(i) \\ \mathbf{0} \end{pmatrix} = \lambda \hat{X} \tag{3.4}$$

and

$$\begin{pmatrix} -\hat{X}(1)\mathbb{1} \\ -\hat{X}(2)\mathbb{1} \\ \vdots \\ -\hat{X}(k)\mathbb{1} \end{pmatrix} + \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{pmatrix} = \lambda \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{pmatrix}. \tag{3.5}$$

Note that,

$$L(G[F,u_1,\ldots,u_k,H_v]) = \left[\begin{array}{c|c} L(F)+(m-1)\left[\begin{array}{c|c} I & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0}\end{array}\right] & \left[\begin{array}{c} I\otimes -\mathbb{1}^T \\ \hline \mathbf{0}\end{array}\right] \\ \hline \left[\begin{array}{c|c} I\otimes -\mathbb{1} & \mathbf{0}\end{array}\right] & I\otimes (L(H)+I)\end{array}\right].$$

Using Equations 3.4 and 3.5, we have

$$L(G[F,u_1,\ldots,u_k,H_v])X$$

$$= \left(\begin{array}{c} L(F)\hat{X}+(m-1)\begin{pmatrix}\hat{X}(1)\\ \hat{X}(2)\\ \vdots\\ \hat{X}(k)\\ \mathbf{0}\end{pmatrix} - \begin{pmatrix}\Sigma_{i=1}^{m-1}Y_1(i)\\ \Sigma_{i=1}^{m-1}Y_2(i)\\ \vdots\\ \Sigma_{i=1}^{m-1}Y_k(i)\\ \mathbf{0}\end{pmatrix} \\ \hline \begin{pmatrix}-\hat{X}(1)\mathbb{1}\\ -\hat{X}(2)\mathbb{1}\\ \vdots\\ -\hat{X}(k)\mathbb{1}\end{pmatrix} + \begin{pmatrix}L(H)Y_1+Y_1\\ L(H)Y_2+Y_2\\ \vdots\\ L(H)Y_k+Y_k\end{pmatrix}\end{array}\right) = \lambda\begin{pmatrix}\hat{X}\\ Y_1\\ Y_2\\ \vdots\\ Y_k\end{pmatrix},$$

as $L(H)Y_j = L(H)\kappa_j\mathbb{1} = 0$, for $j = 1,2,\ldots,k$.

Thus, if λ is an eigenvalue of $L(G[F,u_1,\ldots,u_k,\mathbf{S}_m])$ afforded by an eigenvector X orthogonal to the eigenvectors $U_1^i, U_2^i, \ldots, U_k^i$, for $i = 2,3,\ldots,m-1$, then λ is also an eigenvalue of $L(G[F,u_1,\ldots,u_k,H_v])$ afforded by the same eigenvector X, for any graph H_v with a specified vertex v of degree $m-1$.

Also note that $\gamma_2,\gamma_3,\ldots,\gamma_{m-1}$ are the remaining $(m-2)$ eigenvalues of $L(G[F,u_1,\ldots,u_k,H_v])$, each of multiplicity k afforded by the eigenvectors

$$\begin{pmatrix}\mathbf{0}\\ \hat{e}_t\otimes Z_2\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \hat{e}_t\otimes Z_3\end{pmatrix}, \ldots, \begin{pmatrix}\mathbf{0}\\ \hat{e}_t\otimes Z_{m-1}\end{pmatrix}, \text{ for } t = 1,2,\ldots,k,$$

respectively, where $Z_2, Z_3, \ldots, Z_{m-1}$ are eigenvectors of $L(H)$ corresponding to the eigenvalues $\gamma_2-1, \gamma_3-1, \ldots, \gamma_{m-1}-1$, respectively. Hence the proof. ■

Theorem 3.2.6 is illustrated by the following example.

Example 3.2.4 Consider the graphs F, H_v of order $4, 5$, respectively in Figure 3.4. The vertex v of H_v has degree 4. $G[F, u_1, H_v]$ is the graph obtained by attaching one copy of H_v to the vertex u_1 of F at v. $G[F, u_1, \mathbf{S}_5]$ is the graph obtained by attaching one copy of $\mathbf{S}_5$ to the same vertex of F at the central vertex of the star $\mathbf{S}_5$. It can be checked that $S(F) = (0, 2, 2, 4)$ and $S(H_v) = (0, 1.5858, 3, 4.4142, 5)$. Also

$$S(G[F, u_1, \mathbf{S}_5]) = (0, 0.7029, 1, 1, 1, 2, 3.2132, 7.0839)$$

and

$$S(G[F, u_1, H_v]) = (0, 0.7029, 1.5858, 2, 3, 3.2132, 4.4142, 7.0839).$$

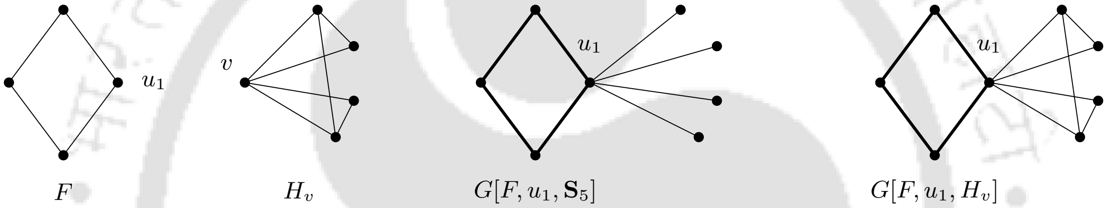


Figure 3.4: Graphs of $F, H_v, G[F, u_1, \mathbf{S}_5]$ and $G[F, u_1, H_v]$.

Notice that $S(G[F, u_1, H_v])$ can be obtained from $S(G[F, u_1, \mathbf{S}_5])$ and $S(H_v)$ as described in Theorem 3.2.6.

3.3 Some applications

In Example 3.2.4, notice that after replacing $\mathbf{S}_5$ by the graph H_v, the Laplacian spectral radius of the resulting graph does not change. The following result tells that this is also true in general.

Proposition 3.3.1 *Let F and H_v be graphs of orders n and m $(m > 1)$, respectively. Let v be a specified vertex in H_v of degree $m - 1$ and $u_1, \ldots, u_k \in F$. Let $G = G[F, u_1, \ldots, u_k, H_v]$ be the graph as defined in Definition* 3.2.1. *Then $\mu(G)$, the Laplacian spectral radius of G is independent of the graph H_v.*

Proof: Notice that $G = G[F, u_1, \ldots, u_k, H_v]$ contains a vertex of degree at least $m + 1$. Thus by using Lemma 2.3.1 of Chapter 2,

$$\mu(G) \geq m + 2 > \gamma_{m-1}.$$

This implies that $\mu(G)$ is one of the $n+k$ eigenvalues of $L(G)$ that are independent of the graph H_v. Hence the proof. ■

As an immediate corollary we have the following result which has already been proved by Guo in [36].

Corollary 3.3.2 *Let G be a graph of order n. Let $v_1, v_2, \ldots, v_p$ be p pendant vertices of G adjacent to a common vertex v. Suppose that G^* be the graph obtained from G by adding t $(1 \leq t \leq \frac{p(p-1)}{2})$ edges arbitrarily, among the vertices $v_1, v_2, \ldots, v_p$. Then $\mu(G) = \mu(G^*)$.*

To get a more general result on Laplacian spectral radius we define the following graph operation. Let F and H be two graphs on disjoint sets of m and n vertices, respectively. Define $F^k[H]$ as the graph on $m+n$ vertices obtained by joining some $k, 1 \leq k \leq m$ vertices of F to each and every vertex of H. Notice that this operation is a generalization of the operation 'join'. Thus we call it as *generalized join.*

Example 3.3.1 Consider the graphs F and $H = K_2$ in Figure 3.5. The graph $F^3[H]$ obtained by joining the 3 vertices, u_1, u_2, u_3 of F to each and every vertex of H is also shown in the same figure.

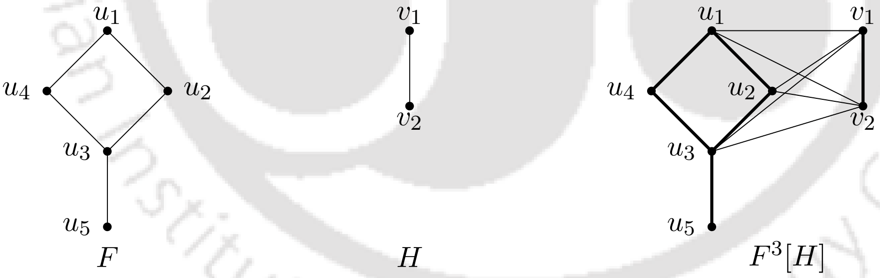


Figure 3.5: Generalized join of two graphs

Note 3.3.3 *If $|V(F)| = m$, $|V(H)| = n$, $|E(F)| = q_1$, and $|E(H)| = q_2$, then*

$$|V(F^k[H])| = m+n, \quad \text{and} \quad |E(F^k[H])| = q_1 + q_2 + kn.$$

Theorem 3.3.4 *Let F and H be graphs of orders m and n, respectively. Let $G = F^k[H]$ be the graph obtained by joining some $k, 1 \leq k \leq m$ vertices of F to each vertex of H. Suppose that $S(H) = (0 = \gamma_1, \gamma_2, \ldots, \gamma_n)$. Then $\gamma_2 + k, \ldots, \gamma_n + k$ are eigenvalues of $L(G)$, each with multiplicity 1, and the other $m+1$ eigenvalues of $L(G)$ are independent of the graph H.*

Proof: The proof is much similar to that of Theorem 3.2.6. ∎

Proposition 3.3.5 *Let F be a graph of order m and H be a graph of order $n, n > 1$. Let $G = F^k[H]$ be the graph obtained by joining some $k, 1 \leq k \leq m$ vertices of F to each vertex of H. Then $\mu(G)$ is independent of the graph H.*

Proof: Notice that $F^k[H]$ contains $K_{k,n}$ as a subgraph. Thus $\mu(G) \geq n + k$. Thus $\mu(G)$ is either $n + k$ or one of the $m + 1$ eigenvalues of $L(G)$ that are independent of the graph H. Hence the proof. ∎

The following result generalizes Corollary 3.3.2 and Corollary 2.3.3 of Chapter 2.

Corollary 3.3.6 *Let G be a graph of order m. Let $v_1, v_2, \ldots, v_p$ be p vertices in G such that $N(v_1) = N(v_2) = \cdots = N(v_p) \neq \phi$. Let G^* be the graph obtained from G by adding $t, 1 \leq t \leq \frac{p(p-1)}{2}$ edges arbitrarily, among the vertices $v_1, v_2, \ldots, v_p$. Then $\mu(G) = \mu(G^*)$.*

Proof: Proof follows immediately from Proposition 3.3.5. ∎

Chapter 4

Graph products and the resulting Laplacian spectra

4.1 Introduction

Let F and H be two graphs with disjoint vertex sets $\{u_1, \dots, u_{p_1}\}$ and $\{v_1, \dots, v_{p_2}\}$, respectively. A *graph product* G of F and H is a new graph, whose vertex set $V(G) = V(F) \times V(H)$, the cartesian product of $V(F)$ and $V(H)$. The adjacency of two distinct vertices $(u_i, v_j), (u_r, v_s) \in V(G)$ is determined entirely by the *adjacency* (*or equality or non adjacency*) of u_i and u_r in F and that of v_j and v_s in H. Thus we may define the vertices (u_i, v_j) and (u_r, v_s) to be adjacent if some of the following 8 conditions are satisfied.

1. $\{u_i, u_r\} \in E(F)$ and $v_j = v_s$.
2. $\{u_i, u_r\} \notin E(F)$ and $v_j = v_s$,
3. $\{u_i, u_r\} \in E(F)$ and $\{v_j, v_s\} \in E(H)$,
4. $\{u_i, u_r\} \notin E(F)$ and $\{v_j, v_s\} \in E(H)$,
5. $\{u_i, u_r\} \in E(F)$ and $\{v_j, v_s\} \notin E(H)$,
6. $\{u_i, u_r\} \notin E(F)$ and $\{v_j, v_s\} \notin E(H)$,
7. $u_i = u_r$ and $\{v_j, v_s\} \in E(H)$ and
8. $u_i = u_r$ and $\{v_j, v_s\} \notin E(H)$.

And in these 8 cases the adjacency of (u_i, v_j) and (u_r, v_s) can be defined to be either in product graph of F and H or in its complement. Thus $2^8 = 256$ different types of graph products can be defined. The graphs obtained by taking products of two graphs are called as *product graphs*, and the two graphs are called as *factors*. The most commonly used graph products, given by conditions sufficient and necessary for adjacency, are listed in the following table.

Graph product name	Symbol	Definition
Graph cartesian product	$F\Box H$	$\Big(u_i = u_r \text{ and } \{v_j, v_s\} \in E(H)\Big)$ or $\Big(\{u_i, u_r\} \in E(F) \text{ and } v_j = v_s\Big)$
Graph categorical product	$F \times H$	$\{u_i, u_r\} \in E(F)$ and $\{v_j, v_s\} \in E(H)$
Graph strong product	$F\boxtimes H$	$\Big(u_i = u_r \text{ and } \{v_j, v_s\} \in E(H)\Big)$ or $\Big(\{u_i, u_r\} \in E(F) \text{ and } v_j = v_s\Big)$ or $\Big(\{u_i, u_r\} \in E(F) \text{ and } \{v_j, v_s\} \in E(H)\Big)$
Graph lexicographic product	$F[H]$	$\Big(\{u_i, u_r\} \in E(F)\Big)$ or $\Big(u_i = u_r \text{ and } \{v_j, v_s\} \in E(H)\Big)$

Note that the terminology is not quite standardized, for example the graph lexicographic product is called as the composition in Harary [39].

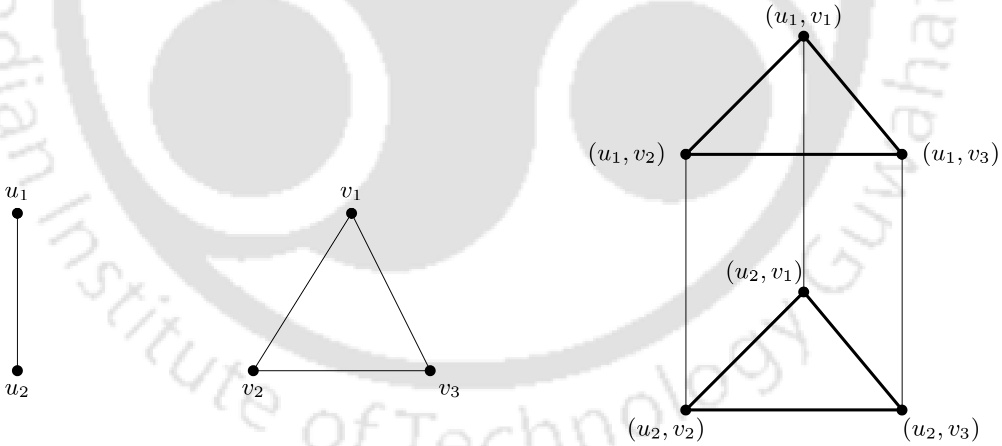
The four products listed above are known as the standard graph products and have been studied by many graph theorist. Imrich and Klavžar [43] have studied the graph products and their structural properties more deeply and noticed that the four standard products are the most relevant ones and the other products can be neglected. Note that, the four standard products have a common property that: if we take the product of two simple graphs, then we will get a simple graph. Also, all the four products are associative and except the lexicographic product all the other three products are commutative.

Like the graph operations, graph products are also used in constructing many important classes of graphs. Any graph invariant can be studied on graph products (see [43]). The standard problem is that describing some properties of a graph invariant of a product graph when we know the corresponding invariants of the factors. Investigation of the spectra/Laplacian spectra of product graphs is also an interesting topic for researchers. Results describing the adjacency matrix and its spectra (called ordinary spectra) of the product graphs can be found in Cvetković, Doob and Sachs [17] and Imrich and Klavžar [43]. Results describing the Laplacian eigenvalues and eigenvectors of cartesian product of graphs have already been derived (see Mohar [57]). In this chapter, we focus on the other three standard products, describe the Laplacian matrix of product graphs in terms of the Laplacian matrices of the factors and investigate the Laplacian spectra of these product graphs. We also supply some new results relating to the algebraic connectivity of the cartesian product graph.

4.2 Graph cartesian product

The cartesian product $F \Box H$ of two graphs F and H is a graph with vertex set $V(F) \times V(H)$ where the adjacency of vertices is determined by the following rule: (u_i, v_j) and (u_r, v_s) are adjacent if **either** $\Big(u_i = u_r \text{ and } \{v_j, v_s\} \in E(H)\Big)$ **or** $\Big(\{u_i, u_r\} \in E(F) \text{ and } v_j = v_s\Big)$. One may also view $F \Box H$ as the graph obtained from F by replacing each of its vertices with a copy of H and each of its edges with $|V(H)|$ edges joining corresponding vertices of H in the two copies. It is known that the graph $F \Box H$ is isomorphic to the graph $F \Box H$.

Example 4.2.1 Let $F = P_2$ and $H = C_3$, the cycle of order 3. The graph $F \Box H$, obtained by taking cartesian product of F and H, is shown in Figure 4.1.



F H $F \Box H$

Figure 4.1: Cartesian product

Note 4.2.1 *If* $|V(F)| = p_1$, $|V(H)| = p_2$, $|E(F)| = q_1$, *and* $|E(H)| = q_2$, *then*

$$|V(F \Box H)| = p_1 p_2, \qquad \textit{and} \qquad |E(F \Box H)| = p_1 q_2 + p_2 q_1.$$

Cartesian product has been widely investigated and is arguably the most interesting one among all products (see Mohar [57]). For example, by taking the n-fold cartesian product of

K_2 one can get the *n-cube*, Q_n on 2^n vertices. It is known that $F \Box H$ is connected if and only if both the graphs F and H are connected (see [43]). Fiedler [25] has observed that

$$L(F \Box H) = L(F) \otimes I + I \otimes L(H). \tag{4.1}$$

The next result which follows from (4.1), has been noted in that article (see also Merris [52]).

Theorem 4.2.2 (Fiedler [25]; Merris [52]) *Let F and H be graphs with*

$$S(F) = (\lambda_1, \ldots, \lambda_{p_1}) \quad \textit{and} \quad S(H) = (\mu_1, \ldots, \mu_{p_2}).$$

Then the eigenvalues of $L(F \Box H)$ are

$$\lambda_i + \mu_j, \quad 1 \leq i \leq p_1,\, 1 \leq k \leq p_2.$$

Moreover, if X_i is an eigenvector of $L(F)$ affording λ_i and Y_j is an eigenvector of $L(H)$ affording μ_j, then $X_i \otimes Y_j$ is an eigenvector of $L(F \Box H)$ affording $\lambda_i + \mu_j$. In particular

$$a(F \Box H) = min(a(F), a(H)).$$

The next result which follows from Theorem 4.2.2, describes the characteristic set of the cartesian product. It has some beautiful applications. If X is a Fiedler vector of a graph F, then by $\tilde{V}(F, X)$ we denote the set of vertices involved in $C(F, X)$ (these may be characteristic vertices or end vertices of characteristic edges) and by $\tilde{E}(F, X)$ we denote the set of edges involved in $C(F, X)$ (these are the characteristic edges). Let $\tilde{C}(F, X)$ denote the graph with vertex set $\tilde{V}(F, X)$ and edge set $\tilde{E}(F, X)$.

Theorem 4.2.3 *Let F and H be graphs on vertices $\{u_1, u_2, \ldots, u_{p_1}\}$ and $\{v_1, v_2, \ldots, v_{p_2}\}$, respectively. Let F^* be the graph obtained from F by deleting all edges and H^* be the graph obtained from H by deleting all edges. Then the following statements hold.*

(a) *If $a(F) = a(H)$, then $a(F \Box H) = a(F)$ and the multiplicity of $a(F \Box H)$ is the sum of the multiplicities of $a(F)$ and $a(H)$. If X and Y are Fiedler vectors of F and H, respectively, then $\mathbb{1} \otimes Y$ and $X \otimes \mathbb{1}$ are Fiedler vectors of $F \Box H$. Furthermore,*

$$\tilde{C}(F \Box H, X \otimes \mathbb{1}) = \tilde{C}(F, X) \Box H^*.$$

Similarly

$$\tilde{C}(F \Box H, \mathbb{1} \otimes Y) = F^* \Box \tilde{C}(H, Y).$$

(b) *If* $a(F) < a(H)$, *then* $a(F\square H) = a(F)$ *and the multiplicity of* $a(F\square H)$ *is same as the multiplicity of* $a(F)$. *If* X *is a Fiedler vector of* F, *then* $X \otimes \mathbb{1}$ *is a Fiedler vector of* $F\square H$. *Furthermore,*

$$\tilde{C}(F\square H, X \otimes \mathbb{1}) = \tilde{C}(F, X)\square H^*.$$

Proof: (a). If $a(F) = a(H)$, then using Theorem 4.2.2, we have $a(F\square H) = a(F)$ and the multiplicity of $a(F\square H)$ is the sum of the multiplicities of $a(F)$ and $a(H)$. Also it follows from Theorem 4.2.2 that if X and Y are Fiedler vectors of F and H, respectively, then $\mathbb{1} \otimes Y$ and $X \otimes \mathbb{1}$ are Fiedler vectors of $F\square H$. Thus $u_i \in \tilde{C}(F, X)$ is a characteristic vertex if and only if $(u_i, v_j) \in \tilde{C}(F\square H, X \otimes \mathbb{1})$, $j = 1, 2, \ldots, p_2$, are characteristic vertices. Similarly, $u_i \in \tilde{C}(F, X)$ is an end vertex of a characteristic edge if and only if $(u_i, v_j) \in \tilde{C}(F\square H, X\otimes\mathbb{1})$, $j = 1, 2, \ldots, p_2$, are end vertices of characteristic edges.

Also $\{u_i, u_k\} \in \tilde{C}(F, X)$ if and only if $\Big\{(u_i, v_j), (u_k, v_j)\Big\} \in \tilde{C}(F\square H, X\otimes\mathbb{1})$, $j = 1, 2, \ldots, p_2$. Thus, it follows that

$$\tilde{C}\Big(F\square H, X \otimes \mathbb{1}\Big) = \tilde{C}(F, X)\square H^*.$$

The other equality may be proved similarly. Proof of (b) is similar to (a). ■

We now give an illustration of Theorem 4.2.3 (a).

Example 4.2.2 Consider the path P_3 with vertex set $\{u_1, u_2, u_3\}$ and the cycle C_6 with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Note that $S(P_3) = (0, 1, 3)$ and $X = (-1, 0, 1)^T$ is a Fiedler vector of P_3. Hence $\tilde{C}(P_3, X)$ is the graph with vertex set $\{u_2\}$ and no edges. Note also that $S(C_6) = (0, 1, 1, 3, 3, 4)$. Two linearly independent Fiedler vectors of C_6 are given by $Y_1 = (0, 1, 1, 0, -1, -1)^T$ and $Y_2 = (1, 1, 0, -1, -1, 0)^T$. Hence

$$\tilde{C}(C_6, Y_1) = (\{u_1, u_4\}, \emptyset)\,, \quad \text{and} \quad \tilde{C}(C_6, Y_2) = (\{u_3, u_6\}, \emptyset)\,.$$

By Theorem 4.2.3, we see that $a(P_3\square C_6) = 1$ and the multiplicity of $a(P_3\square C_6)$ is 3. By the

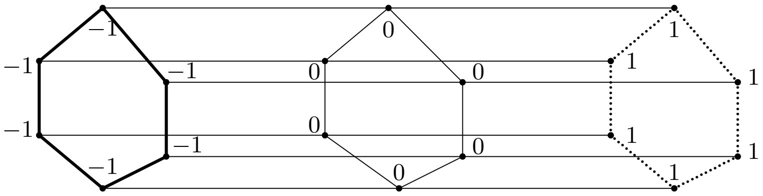


Figure 4.2: $C(P_3\square C_6, X \otimes \mathbb{1})$

same theorem the vectors $X \otimes \mathbb{1}$, $\mathbb{1} \otimes Y_1$ and $\mathbb{1} \otimes Y_2$ are three linearly independent Fiedler vectors of $P_3 \Box C_6$. Observe that

$$X \otimes \mathbb{1} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T,$$

so that $\tilde{C}(P_3 \Box C_6, X \otimes \mathbb{1})$ is the graph with vertices $\{(u_2, v_i) : i = 1, \ldots, 6\}$ and no edges. It can be seen (in Figure 4.2) that

$$\tilde{C}(P_3 \Box C_6, X \otimes \mathbb{1}) = \tilde{C}(P_3, X) \Box C_6^*.$$

Similarly, the vector

$$\mathbb{1} \otimes Y_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}^T,$$

so that

$$\tilde{C}(P_3 \Box C_6, \mathbb{1} \otimes Y_1) = P_3^* \Box \tilde{C}(C_6, Y_1),$$

which is shown in Figure 4.3. The characteristic set $C(P_3 \Box C_6, \mathbb{1} \otimes Y_2)$ has a similar description.

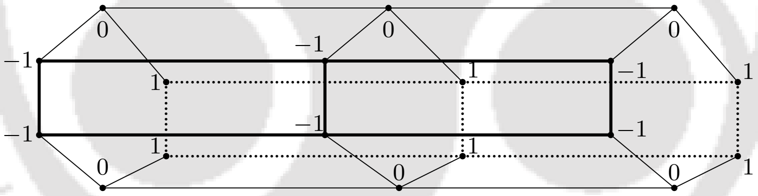


Figure 4.3: $C(P_3 \Box C_6, \mathbb{1} \otimes Y_1)$

The occurrence of Theorem 4.2.3 (b) is illustrated in Example 4.2.2.

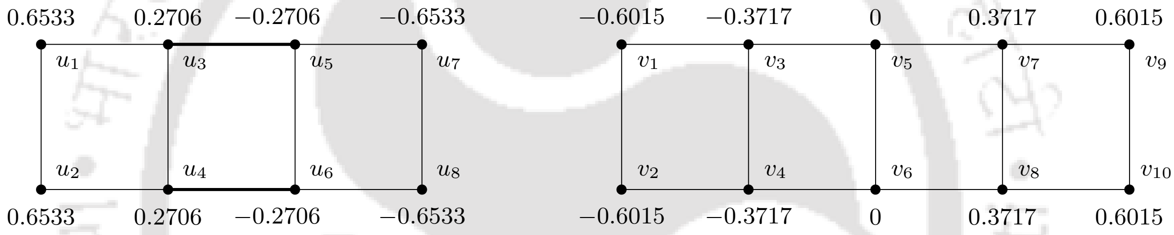
As a straightforward application of Theorem 4.2.3 we have the following result describing the case of simple algebraic connectivity of cartesian product.

Corollary 4.2.4 *Let F and H be two graphs. Then $a(F \Box H)$ is simple if and only if* **either** *$a(F) < a(H)$ and $a(F)$ is simple* **or** *$a(H) < a(F)$ and $a(H)$ is simple. In the former case the characteristic set of $F \Box H$ is the cartesian product of the characteristic set of F with the graph H. In the latter case the characteristic set of $F \Box H$ is the cartesian product of the graph F with the characteristic set of H.*

Proof: Proof follows immediately from Theorem 4.2.3. ∎

As an immediate application we see that the ladder of length at least 2 has a simple algebraic connectivity. Furthermore, we have a description of the Fiedler vector and the characteristic set of this graph (see Corollary 4.2.5). Let us give an example for motivation.

Example 4.2.3 Consider the two ladders of length 3 and 4 in Figure 4.4. Note that $a(G_1) = 0.5858 = a(P_4)$, and $Z_1 = (0.6533, 0.6533, 0.2706, 0.2706, -0.2706, -0.2706, -0.6533, -0.6533)^T$ is a Fiedler vector vector of G_1. Hence $C(G_1, Z_1) = \Big\{\{u_3, u_5\}, \{u_4, u_6\}\Big\}$. Also $a(G_2) = 0.382 = a(P_5)$, and $Z_2 = (-0.6015, -0.6015, -0.3717, -0.3717, 0, 0, 0.3717, 0.3717, 0.6015, 0.6015)^T$ is a Fiedler vector of G_2. Thus, $C(G_2, Z_2) = \{v_5, v_6\}$. A complete description of these two Fiedler vectors is given in Figure 4.4.



$G_1 = P_4 \Box P_2$ $G_2 = P_5 \Box P_2$

Figure 4.4:

Corollary 4.2.5 *Let* $G = P_{k+1} \Box P_2$, *be a ladder of length* k $(k \geq 2)$ *and vertex set* $\{1, 2, \ldots, 2k\}$. *Then*

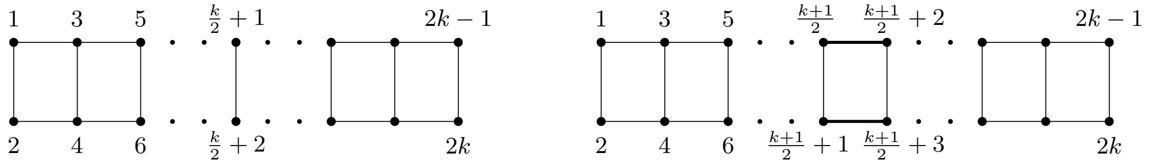
(i) $a(G)$ *is simple.*

(ii) *If* X *is a Fiedler vector of* P_{k+1}, *then* $X \otimes \mathbb{1}$ *a Fiedler vector of* G.

(iii) *If* k *is even, then the characteristic set of* G *consists of two zero vertices,* $\frac{k}{2} + 1, \frac{k}{2} + 2$ *only.*

(iv) *If* k *is odd, then the characteristic set of* G *consists of exactly two edges,* $\{\frac{k+1}{2}, \frac{k+1}{2} + 2\}, \{\frac{k+1}{2} + 1, \frac{k+1}{2} + 3\}$.

Proof: Since $a(P_{k+1}) < a(P_2)$ for $k \geq 3$, using Corollary 4.2.4 we have that $a(G) = a(P_{k+1})$ and that $a(G)$ is simple. Proofs of (ii), (iii) and (iv) also follow from Corollary 4.2.4 easily. ∎



k is even $\qquad$ k is odd

Figure 4.5: Ladder of length k

Consider the graph G_2 in Figure 4.4. Observe that the entries of the Fiedler vector have certain monotonicity property. For example, the entries increase along $[v_6, v_8, v_{10}]$ and decrease along $[v_6, v_4, v_2]$. In general a Fiedler vector of $T \Box H$ may not have such a property, where T is a tree and H is any graph. The following result gives a sufficient condition for the Fiedler vector of $T \Box H$ to have the monotonicity property. Let H be a subgraph of a connected graph G and $v \in G$ be a vertex. Then by $d(H, v)$ we denote the minimum length of a path from a vertex of H to v. In particular, $d(H, v) = 0$ if and only if $v \in H$.

Theorem 4.2.6 *Let T be a tree with simple algebraic connectivity and H be a graph such that $a(H) > a(T)$. Let Z be a Fiedler vector of $G = T \Box H$ and $\mathcal{C} = C(G, Z)$. Let $P = [v = v_0, v_1, \ldots, v_k]$ be a path in G such that $d(\mathcal{C}, v_{i+1}) > d(\mathcal{C}, v_i)$, $i = 0, 1, \ldots, k-1$ with $d(\mathcal{C}, v) = 0$. Then the entries of Z monotonically increase (monotonically decrease / identically zero) along P if v is positive (negative / zero).*

Proof: Let T be on vertices $u_1, \ldots, u_n$ and H be on vertices $x_1, \ldots, x_m$. Let Y be a Fiedler vector of T. It is known that $|C(T, Y)| = 1$ and $C(T, Y)$ is either an edge or a singleton vertex. Assume first that $C(T, Y) = \{\{u_1, u_2\}\}$ and $Y(u_1) > 0$. By Theorem 4.2.3, the vector $Z = Y \otimes \mathbb{1}$ is a Fiedler vector of G and

$$\mathcal{C} = C(G, Z) = \Big\{\{(u_1, x_i), (u_2, x_i)\} \,:\, i = 1, \ldots, m\Big\}.$$

Thus the point v is a point from the set $\{(u_1, x_i), (u_2, x_i)\}$, say $v = v_0 = (u_1, x_5)$. Since v_i satisfy $d(\mathcal{C}, v_{i+1}) > d(\mathcal{C}, v_i)$, it follows that $v_i = (u_{r_i}, x_5)$, where $P' = [u_1, u_{r_1}, u_{r_2}, \ldots, u_{r_k}]$ is a path in T which starts at u_1 and does not pass through u_2. Thus by Fiedler's monotonicity theorem the entries of Y increase along P'. But by Theorem 4.2.3 we know that $Z(v) = Y(u_1)$ and $Z(v_i) = Y(u_{r_i})$, for $i = 1, \ldots, k$. Thus we are done in this case.

The proof is similar in the case when $C(T, Y)$ is a single vertex. ■

As another application we have the following result showing the construction of an infinite class of graphs with nonisomorphic Perron branches, whose characteristic set consists of vertices only.

Theorem 4.2.7 *Let T be a Type I tree with nonisomorphic Perron branches and H be any graph such that $a(T) < a(H)$. Let $G_1 = T\Box H$ and for $k = 2, 3, \ldots$ define $G_k = G_{k-1}\Box H$. Then the class consisting of G_k, $k = 1, 2, \ldots$ is an infinite class of graphs with nonisomorphic Perron branches, whose characteristic set consists of vertices only.*

Proof: Let Y be a Fiedler vector of T and $C(T, Y) = \{v\}$. Let $G_1 = T\Box H$. Since $a(T) < a(H)$, using Theorem 4.2.3, $a(G_1) = a(T)$ and $Y_1 = Y \otimes \mathbb{1}$ is a Fiedler vector of G_1. Thus $C(G_1, Y_1)$ consists of vertices only. Note that G_1 is a graph with nonisomorphic Perron branches.

Put $G_2 = G_1\Box H$. Now as $a(G_1) = a(T) < a(H)$, again by using Theorem 4.2.3, $a(G_2) = a(T)$ and $Y_2 = Y_1 \otimes \mathbb{1}$ is a Fiedler vector of G. Thus $C(G_2, Y_2)$ consists of vertices only and G_2 is also a graph with nonisomorphic perron branches.

Now for $k = 2, 3, \ldots$ define $G_k = G_{k-1}\Box H$ and $Y_k = Y_{k-1} \otimes \mathbb{1}$. Then in a similar argument as above we can prove that for each k, $k = 1, 2, \ldots$, $C(G_k, Y_k)$ consists of vertices only and G_k is a graph with nonisomorphic perron branches.

Thus using Theorem 1.2.1 we have the result. ■

Our next result is a direct application of Theorem 4.2.2. It describes the construction of a new Laplacian integral graph from the known ones.

Corollary 4.2.8 *Let F and H be two graphs on disjoint vertex sets. Then $F\Box H$ is Laplacian integral if and only if both F and H are Laplacian integral.*

Proof: If F and H are Laplacian integral, then by Theorem 4.2.2, $F\Box H$ is also Laplacian integral.

Conversely, suppose $F\Box H$ is Laplacian integral, but F is not Laplacian integral. Let $\lambda \in S(F)$ and λ is not an integer. Now using Theorem 4.2.2, $\lambda \in S(F\Box H)$, implies λ is an integer, which is a contradiction. Hence the proof is complete. ■

4.3 Graph categorical product

The categorical product $F \times H$ of two graphs F and H is the product of F and H where the adjacency is determined by the following rule: (u_i, v_j) and (u_r, v_s) are adjacent in $F \times H$ if $\Big(\{u_i, u_r\} \in E(F)\Big)$ **and** $\Big(\{v_j, v_s\} \in E(H)\Big)$.

Note that an edge $e = \{u_i, u_r\} \in F$ and an edge $e' = \{v_j, v_s\} \in H$ give rise to two edges $\Big\{(u_i, v_j), (u_r, v_s)\Big\}$, $\Big\{(u_i, v_s), (u_r, v_j)\Big\} \in F \times H$. Thus $|E(F \times H)| = 2|E(F)||E(H)|$.

Example 4.3.1 Let $F = P_2$ and $H = P_4$. The graph $F \times H$, obtained by taking categorical product of F and H, is shown in Figure 4.6.

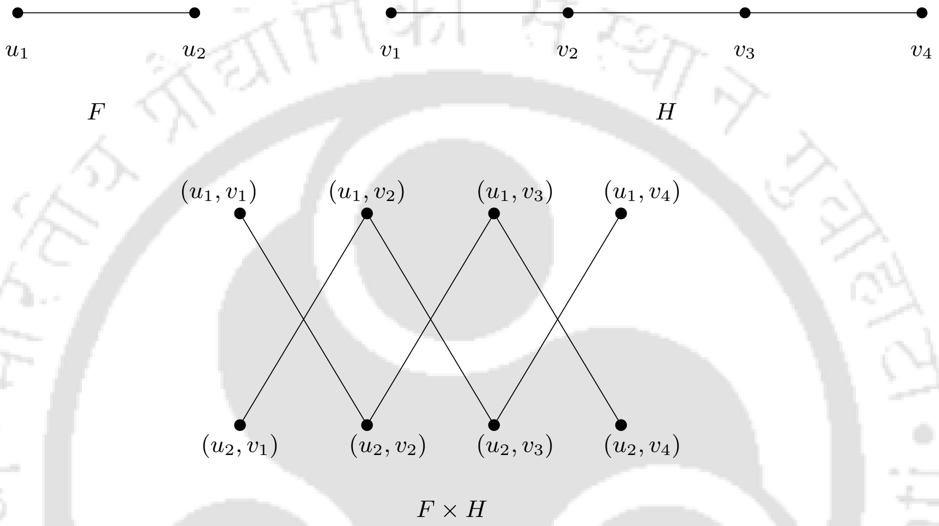


Figure 4.6: Categorical product

The following result describes the Laplacian matrix of $F \times H$ in terms of the Laplacian matrices of F and H.

Proposition 4.3.1 *Let F and H be two graphs on p_1 and p_2 vertices, respectively. Then the Laplacian matrix of $F \times H$ is:*

$$L(F \times H) = D(F) \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H). \tag{4.2}$$

Proof: Let F be on vertices $\{u_1, u_2, \ldots, u_{p_1}\}$ and H be on vertices $\{v_1, v_2, \ldots, v_{p_2}\}$. Hence $F \times H$ is a graph on vertices $\{(u_i, v_j) \mid i = 1, 2, \ldots, p_1;\ \ j = 1, 2, \ldots, p_2\}$. From the definition of graph categorical product, the adjacency matrix of $F \times H$ is $A(F \times H) = [a_{pqrs}]$, where for $p = 1, 2, \ldots, p_1,\ q = 1, 2, \ldots, p_2,\ r = 1, 2, \ldots, p_1$ and $s = 1, 2, \ldots, p_2$,

$$a_{pqrs} = \begin{cases} 1, & \text{if } \{u_p, u_r\} \in F \text{ and } \{v_q, v_s\} \in H, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $A(F \times H)$ is the partitioned matrix

$$\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1p_1} \\ B_{21} & B_{22} & \cdots & B_{2p_1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{p_11} & B_{p_12} & \cdots & B_{p_1p_1} \end{bmatrix},$$

where for $i = 1, 2, \ldots, p_1$ and $j = 1, 2, \ldots, p_1$,

$$B_{ij} = \begin{cases} A(H), & \text{if } \{u_i, u_j\} \in F, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Note that in the above matrix the ordering of the rows and columns is done as

$$(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_{p_2}), \ldots, (u_{p_1}, v_1), (u_{p_1}, v_2), \ldots, (u_{p_1}, v_{p_2}).$$

It follows that

$$A(F \times H) = A(F) \otimes A(H) \quad \text{and} \quad D(F \times H) = D(F) \otimes D(H).$$

Thus,

$$\begin{aligned} L(F \times H) &= D(F) \otimes D(H) - A(F) \otimes A(H) \\ &= D(F) \otimes D(H) - (D(F) - L(F)) \otimes (D(H) - L(H)) \\ &= D(F) \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H), \end{aligned}$$

as required. ■

Recall that the cartesian product of two connected graphs is always a connected graph. But this is not true for the categorical product. One can see Example 4.3.1. Let F and H be two connected graphs. The following result gives a necessary and sufficient condition for $F \times H$ to be connected.

Theorem 4.3.2 (Imrich and Klavžar [43]) *Let F and H be graphs with at least one edge. Then $F \times H$ is connected if and only if both F and H are connected and at least one of them is non-bipartite. Further more, if both F and H are connected and bipartite, then $F \times H$ has exactly two connected components.*

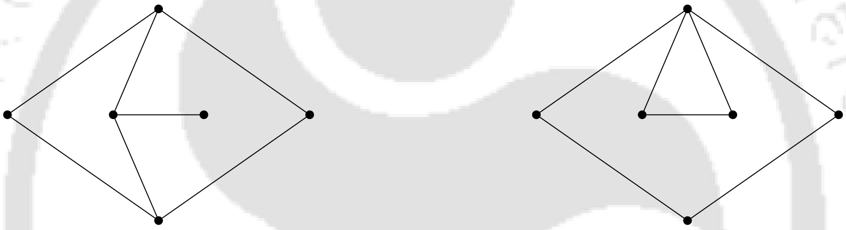
The following is an immediate corollary.

Corollary 4.3.3 *Let F and H be connected graphs of order p_1 and p_2, respectively. Then $a(F \times H) = 0$ if and only if both F and H are bipartite.*

Proof: Proof is immediate from Theorem 4.3.2. ■

Let us consider the following pair of graphs taken from van Dam and Haemers [66]. Graphs F_1 and F_2 shown in Figure 4.7 are nonisomorphic, as one is bipartite and the other is not. They have the same Laplacian spectrum, as their Laplacian matrices have the same characteristic polynomial

$$\mathbf{C}(F_1; x) = \mathbf{C}(F_2; x) = x^6 - 14x^5 + 73x^4 - 176x^3 + 192x^2 - 72x.$$



F_1 F_2

Figure 4.7: A pair of Laplacian cospectral graphs

In view of the previous corollary we see that $a(F_1 \times P_2) = 0$ and $a(F_2 \times P_2) \neq 0$. Thus we see that the complete characterization of the Laplacian spectrum of $F \times H$ is not possible using only the Laplacian spectra of F and H, in general. The following result describes a case where it can be.

Theorem 4.3.4 *Let F and H be connected regular graphs on p_1 and p_2 vertices and of regularities r $(1 \leq r \leq p_1 - 1)$ and s $(1 \leq s \leq p_2 - 1)$, respectively. Suppose that $S(F) = (\lambda_1, \lambda_2, \ldots, \lambda_{p_1})$ and $S(H) = (\mu_1, \mu_2, \ldots, \mu_{p_2})$. Then*

$$r\mu_j + \lambda_i s - \lambda_i \mu_j \in S(F \times H), \quad \textit{for} \quad i = 1, \ldots, p_1, \ \ j = 1, \ldots, p_2.$$

Proof: Suppose that X_i and Y_j are eigenvectors of $L(F)$ and $L(H)$, affording the eigenvalues λ_i and μ_j, respectively. That is, $L(F)X_i = \lambda_i X_i$, $X_i \neq 0$, and $L(H)Y_j = \mu_j Y_j$, $Y_j \neq 0$. Thus using Proposition 4.3.1, we have

$$L(F \times H)(X_i \otimes Y_j) \quad = \quad \Big(D(F) \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H)\Big)(X_i \otimes Y_j)$$

$$\begin{aligned} &= (rI \otimes L(H))(X_i \otimes Y_j) + (L(F) \otimes sI)(X_i \otimes Y_j) \\ &\qquad\qquad - (L(F) \otimes L(H))(X_i \otimes Y_j) \\ &= rX_i \otimes \mu_j Y_j + \lambda_i X_i \otimes sY_j - \lambda_i X_i \otimes \mu_j Y_j \\ &= (r\mu_j + \lambda_i s - \lambda_i \mu_j)(X_i \otimes Y_j). \end{aligned}$$

Hence the proof is complete. ■

In the case when at least one of the two graphs F and H is regular one can describe some of the Laplacian eigenvalues of $F \times H$ using that of F and H. The following result tells that.

Proposition 4.3.5 *Let F be a regular graph on m vertices with regularity r and H be any graph on n vertices. Suppose that $S(H) = (\mu_1, \ldots, \mu_n)$. Then*

$$r\mu_j \in S(F \times H) \text{ for } j = 1, 2 \ldots, n.$$

Proof: Let Y_j be an eigenvector of $L(H)$ corresponding to the eigenvalue μ_j for $j = 1, 2 \ldots, n$. By using Proposition 4.3.1

$$\begin{aligned} L(F \times H) &= D(F) \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H) \\ &= rI_m \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H). \end{aligned}$$

Thus $r\mu_j$ is an eigenvalue of $L(F \times H)$ afforded by the eigenvector $\mathbb{1} \otimes Y_j$ for $j = 1, \ldots, n$. ■

The following is an immediate corollary.

Corollary 4.3.6 *Let F be a regular graph on m vertices with regularity r and H be any graph on n vertices. If $F \times H$ is Laplacian integral, then H is Laplacian integral.*

Proof: Proof follows directly from Proposition 4.3.5. ■

In general the categorical product of two Laplacian integral graphs is not necessarily Laplacian integral. For example, consider the graphs $F = K_3$ and $H = P_3$ (See Figure 4.8). We know that $S(K_3) = (0, 3, 3)$ and $S(P_3) = (0, 1, 3)$. Thus both F and H are Laplacian integral. But it can be checked that $S(F \times H) = (0, 1.2679, 1.2679, 2, 2, 2, 4.7321, 4.7321, 6)$.

The following result shows that $K_n \times P_3$, $n \geq 3$, is not Laplacian integral though K_n and P_3 are Laplacian integral graphs. Notice that in this result we have determined the complete Laplacian spectrum of $K_n \times P_3$, though P_3 is not regular. Note also that the graphs $K_n \times P_3$ have all but two Laplacian eigenvalues integral.

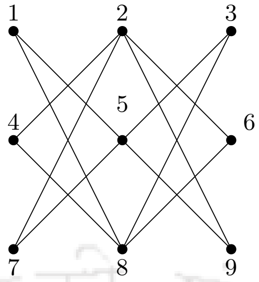


Figure 4.8: The graph of $K_3 \times P_3$

Theorem 4.3.7 *Let* $F = K_n$, $n \geq 3$ *and* $H = P_3$. *Then*

(i) $0 \in S(K_n \times P_3)$ *with multiplicity* 1,

(ii) $3n - 3 \in S(K_n \times P_3)$ *with multiplicity* 1,

(iii) $\dfrac{3n-3+\sqrt{n^2-2n+9}}{2} \in S(K_n \times P_3)$ *with multiplicity* $n-1$,

(iv) $\dfrac{3n-3-\sqrt{n^2-2n+9}}{2} \in S(K_n \times P_3)$ *with multiplicity* $n-1$, *and*

(v) $n - 1 \in S(K_n \times P_3)$ *with multiplicity* n.

In particular, $F \times H$ *is not Laplacian integral.*

Proof: Let $G = K_n \times P_3$, $n \geq 3$. Note that $S(K_n) = (0, n, \ldots, n)$, $S(P_3) = (0, 1, 3)$ and G is of regularity $n - 1$. Thus, applying Proposition 4.3.5, 0, $n - 1$ and $3n - 3 \in S(G)$. By using Proposition 4.3.1, we have

$$L(G) = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 & \hat{L}_2 & \cdots & \hat{L}_2 \\ \hat{L}_2 & \hat{L}_1 & \hat{L}_2 & \cdots & \hat{L}_2 \\ \hat{L}_2 & \hat{L}_2 & \hat{L}_1 & \cdots & \hat{L}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{L}_2 & \hat{L}_2 & \cdots & \hat{L}_2 & \hat{L}_1 \end{bmatrix},$$

where

$$\hat{L}_1 = \begin{bmatrix} n-1 & 0 & 0 \\ 0 & 2n-2 & 0 \\ 0 & 0 & n-1 \end{bmatrix} \text{ and } \hat{L}_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Observe that if λ is an eigenvalue of $\hat{L}_1 + \hat{L}_2$ afforded by the eigenvector X, then

$$\begin{bmatrix} X \\ X \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} X \\ \mathbf{0} \\ X \\ \vdots \\ \mathbf{0} \end{bmatrix}, \cdots, \begin{bmatrix} X \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ X \end{bmatrix}$$

are $n-1$ linearly independent eigenvectors corresponding to the eigenvalue λ of $L(G)$.

By simple calculation it can be obtained that the eigenvalues of $\hat{L}_1 + \hat{L}_2$ are

$$\eta_1 = \frac{3n-3+\sqrt{n^2-2n+9}}{2}, \quad \eta_2 = \frac{3n-3-\sqrt{n^2-2n+9}}{2} \text{ and } \eta_3 = n-1.$$

It is easy to show that $\sqrt{n^2-2n+9}$ is not an integer. Suppose that $\sqrt{n^2-2n+9} = k$, where k is an integer. Thus, $n^2-2n+9 = k^2$, which implies that $(k+n-1)[k-(n-1)] = 8$. Thus we have the following two cases:

(i) $k+n-1 = 4$ and $k-(n-1) = 2$

(ii) $k+n-1 = 8$ and $k-(n-1) = 1$.

If case (i) holds, then we have $k = 3$ and $n = 2$, which is not possible as $n \geq 3$. If case (ii) holds, then $k = 4.5 = n$, which is also not possible.

Thus $\sqrt{n^2-2n+9}$ is not an integer. And hence both η_1 and η_2 are not integers. ■

It is natural to wonder whether we can have classes of graphs on which Laplacian integrality of F and H is equivalent to Laplacian integrality of $F \times H$. The following result supplies such a class.

Corollary 4.3.8 *Let F and H be two regular graphs of regularity r and s, respectively. Then $F \times H$ is Laplacian integral if and only if both F and H are Laplacian integral .*

Proof: Follows easily from Theorem 4.3.4. ■

Another interesting problem is the following. Suppose that the graphs F and H are Laplacian integral. Can we give a necessary and sufficient condition on F and H so that $F \times H$ is Laplacian integral. By Corollary 4.3.8, the condition that F and H are regular is clearly sufficient. It is not necessary as can be seen in the following example.

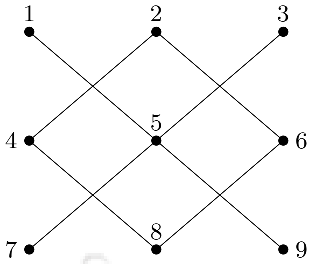


Figure 4.9: Categorical product of P_3 with itself

Example 4.3.2 Let us take $F = H = P_3$. Then both F and H are nonregular Laplacian integral graphs. The product $F \times H$ is shown in Figure 4.9. Observe that $F \times H = C_4 + \mathbf{S}_5$ (see Figure 4.9). Thus $S(F \times H) = (0, 0, 1, 1, 1, 2, 2, 4, 5)$ and hence $F \times H$ is also Laplacian integral.

Let F and H be two graphs and $G = F \times H$. The following result gives an upper bound for the algebraic connectivity of G.

Proposition 4.3.9 *Let F and H be two graphs and $G = F \times H$. Then*

$$a(G) \leq \Delta(F)\Delta(H) + \Big(\Delta(F) - \mu(F)\Big)\Big(\mu(H) - \Delta(H)\Big). \tag{4.3}$$

Proof: By Proposition 4.3.1, we know that $L(G) = D(F) \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H)$. Let X and Y be eigenvectors of $L(F)$ and $L(H)$ corresponding to the eigenvalues $\mu(F)$ and $\mu(H)$, respectively. Let $Z = X \times Y$. Note that $Z \perp \mathbb{1}$. Thus

$$\begin{aligned}
&a(G) \leq Z^T L(G) Z \\
&= Z^T(D(F) \otimes L(H))Z + Z^T(L(F) \otimes D(H))Z - Z^T(L(F) \otimes L(H))Z \\
&\leq \Delta(F)\mu(H) + \mu(F)\Delta(H) - \mu(F)\mu(H) \\
&= \Delta(F)\Delta(H) + (\Delta(F) - \mu(F))(\mu(H) - \Delta(H)).
\end{aligned}$$

Hence the proof. ■

The following well-known [25] result describes the Laplacian spectrum of a cycle and has been used later.

Theorem 4.3.10 (Fiedler [25]) *Let C_n be the cycle of order n. Then*

$$S(C_n) = \left(0,\ 2\left(1 - \cos\frac{2(n-1)\pi}{n}\right), 2\left(1 - \cos\frac{2(n-2)\pi}{n}\right), \ldots, 2\left(1 - \cos\frac{4\pi}{n}\right)\right).$$

The bound given in Proposition 4.3.9 is tight, as the equality holds in the case where $F = C_{2m}$ and $H = C_{2n}$, $m, n \geq 2$. Indeed, as F and H are bipartite, we have by Theorem 4.3.2, $a(F \times H) = 0$. On the other hand it is well known that $\mu(C_n) = 4$, for $n \geq 4$ (comes from Theorem 4.3.10). Thus

$$\Delta(F)\Delta(H) + \Big(\Delta(F) - \mu(F)\Big)\Big(\mu(H) - \Delta(H)\Big) = 4 + (-2)(-2) = 0.$$

A natural problem is to characterise the class of graphs attaining the bound in Proposition 4.3.9. The following result supplies an answer in a special case.

Theorem 4.3.11 *Let $F = C_{2n}$, $n \geq 2$, and H be bipartite connected graphs. Then the equality holds in* (4.3) *if and only if H is regular.*

Proof: As F and H are bipartite $a(F \times H) = 0$. Thus the equality holds in (4.3) if and only if $\mu(H) = 2\Delta(H)$. Suppose first that $\mu(H) = 2\Delta(H)$. Anderson and Morley [2] has proved that

$$\mu(G') \leq \max\Big\{d(u) + d(v) \, : \, \{u, v\} \in E(G')\Big\},$$

for any graph G' and the equality holds if and only if G' is a bipartite semiregular graph. As

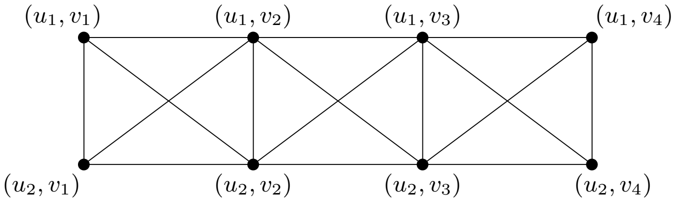
$$\mu(H) \leq \max\Big\{d(u) + d(v) \, : \, \{u, v\} \in E(H)\Big\} \leq 2\Delta(H) = \mu(H),$$

we see that H is semiregular. Further, as $d(u) + d(v) = 2\Delta(H)$ for some edge $\{u, v\} \in E(H)$, we see that H is regular. Conversely, if H is regular, then again by the result of Anderson and Morley $\mu(H) = 2\Delta(H)$ and thus the equality in (4.3) holds. ■

4.4 Graph strong product

The strong product $F \boxtimes H$ of two graphs F and H is a graph where the adjacency is determined by the following rule: (u_i, v_j) and (u_r, v_s) are adjacent in $F \boxtimes H$ if **either** $\Big(u_i = u_r$ and $\{v_j, v_s\} \in E(H)\Big)$ **or** $\Big(\{u_i, u_r\} \in E(F)$ and $v_j = v_s\Big)$ **or** $\Big(\{u_i, u_r\} \in E(F)$ and $\{v_j, v_s\} \in E(H)\Big)$.

Example 4.4.1 Let $F = P_2$, the path of order 2 and $H = P_4$, the path of order 4. The graph $F \boxtimes H$, obtained by taking strong product of F and H, is shown in Figure 4.10.



$F \boxtimes H$

Figure 4.10: Strong product

Observe that $A(F \boxtimes H) = A(F \Box H) + A(\times H)$ and hence $E(F \boxtimes H)) = E(F \Box H) + E(F \times H)$. Thus, if $|V(F)| = p_1$, $|V(H)| = p_2$, $|E(F)| = q_1$ and $|E(H)| = q_2$, then $|E(F \boxtimes H)| = p_1 q_2 + p_2 q_1 + 2q_1 q_2$. Thus we have the following result.

Proposition 4.4.1 *Let F and H be two graphs. Then*

$$L(F \boxtimes H) = L(F \Box H) + L(F \times H). \tag{4.4}$$

Proof: The proof follows from the definition of strong product, cartesian product, categorical product, Equation (4.1) and Proposition 4.3.1. ■

In general, the complete description of the Laplacian spectrum of $F \boxtimes H$ cannot be obtained only from the Laplacian spectra of F and H. This is clear from the following example.

Example 4.4.2 Consider the pair of graphs F_1 and F_2 shown in Figure 4.7. It has already been mentioned that they are nonisomorphic and they have the same Laplacian spectrum. It is easy to check that

$$S(F_1 \boxtimes P_2) = (0, 1.5279, 4, 6, 6, 6, 6, 6, 6, 6, 10, 10.4721)$$

and

$$S(F_2 \boxtimes P_2) = (0, 1.5279, 4, 4, 6, 6, 6, 6, 8, 8, 8, 10.4721).$$

Thus $F_1 \boxtimes P_2$ and $F_2 \boxtimes P_2$ have different Laplacian spectra.

The following result describes the complete Laplacian spectrum of $F \boxtimes H$ using the Laplacian spectra of F and H in the case both F and G are regular.

Theorem 4.4.2 *Let F and H be connected regular graphs on p_1 and p_2 vertices and of regularities r and s, respectively. Suppose that $S(F) = (\lambda_1, \lambda_2, \ldots, \lambda_{p_1})$ and $S(H) = (\mu_1, \mu_2, \ldots, \mu_{p_2})$. Then*

$$\lambda_i(1+s) + \mu_j(1+r) - \lambda_i \mu_j \in S(F \boxtimes H), \quad \textit{for} \quad i = 1, \ldots, p_1, \ j = 1, \ldots, p_2.$$

Proof: Let X_i and Y_j be eigenvectors of $L(F)$ and $L(H)$, affording the eigenvalues λ_i and μ_j, respectively. That is, $L(F)X_i = \lambda_i X_i$, $X_i \neq 0$, and $L(H)Y_j = \mu Y_j$, $Y_j \neq 0$. Thus using Proposition 4.4.1, Equation 4.1 and Theorem 4.3.4, we have

$$\begin{aligned} L(F\boxtimes H)(X_i \otimes Y_j) &= L(F\square H)(X_i \otimes Y_j) + L(F \times H)(X_i \otimes Y_j) \\ &= (\lambda_i + \mu_j)(X_i \otimes Y_j) + (r\mu_j + \lambda_i s - \lambda_i \mu_j)(X_i \otimes Y_j) \\ &= (\lambda_i + \mu_j + r\mu_j + \lambda_i s - \lambda_i \mu_j)(X_i \otimes Y_j). \end{aligned}$$

Hence the proof. ■

The following is an immediate consequence of Theorem 4.4.2.

Corollary 4.4.3 *Let F and H are two regular graphs of regularity r and s, respectively. Then $F\boxtimes H$ is Laplacian integral if and only if both F and H are Laplacian integral.*

Proof: Follows easily from Theorem 4.4.2. ■

But in general the strong product of two Laplacian integral graphs does not give a Laplacian integral graph. For example take the Laplacian integral graph P_3. We know that $S(P_3) = (0, 1, 3)$ and hence P_3 is Laplacian integral. It can be proved by a similar argument as in Theorem 4.3.7 that the graph $P_3\boxtimes P_3$ is not Laplacian integral.

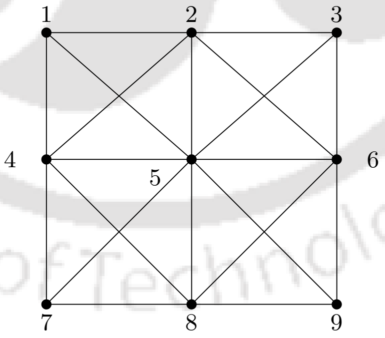


Figure 4.11: Strong product of P_3 with itself

Theorem 4.4.4 *$P_3\boxtimes P_3$ is not Laplacian integral.*

Proof: Observe that

$$L(G) = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 & \mathbf{0} \\ \hat{L}_2 & \hat{L}_3 & \hat{L}_2 \\ \mathbf{0} & \hat{L}_2 & \hat{L}_1 \end{bmatrix},$$

where

$$\hat{L}_1 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \hat{L}_2 = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \text{ and } \hat{L}_3 = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 5 \end{bmatrix}.$$

Thus if λ is an eigenvalue of $\hat{L}_1$ with corresponding eigenvector X, then λ is an eigenvalue of $L(G)$ with corresponding eigenvector $\begin{bmatrix} X \\ \mathbf{0} \\ -X \end{bmatrix}$. Let

$$B_1 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Note thatB_1, B_2 and $\hat{L}_1$ are M-matrices and B_2^{-1} is entrywise dominated by $\hat{L}_1^{-1}$ and $\hat{L}_1^{-1}$ is entrywise dominated by B_1^{-1}. Thus, using Perron-Frobenius theory

$$\tau(B_1) < \tau(\hat{L}_1) < \tau(B_2).$$

But $\tau(B_1) = 2$ and $\tau(B_2) = 3$. Thus $2 < \tau(\hat{L}_1) < 3$ and so $\tau(\hat{L}_1)$ is not an integer. Since $\tau(\hat{L}_1)$ is also an eigenvalue of $L(G)$, G is not Laplacian integral. ■

4.5 Graph lexicographic product

Let F and H be two graphs on vertex sets $\{u_1, u_2, \ldots, u_{p_1}\}$ and $\{v_1, v_2, \ldots, v_{p_2}\}$, respectively. The lexicographic product $F[H]$ of F and H is the product of F and H where the adjacency is determined by the following rule: (u_i, v_j) and (u_r, v_s) are adjacent in $F[H]$ if **either** $\Big(\{u_i, u_j\} \in E(F)\Big)$ **or** $\Big(u_i = u_r \text{ and } \{v_j, v_s\} \in E(H)\Big)$. Note that, the lexicographic product of F and H can be obtained from F by substituting a copy H_{u_i} (say), of H for every vertex u_i of F and by joining all vertices of H_{u_i} with all vertices of H_{u_r} if $\{u_i, u_r\} \in E(F)$. Thus it is also known as composition or substitution (see [38, 41]).

Note 4.5.1 *If* $|V(F)| = p_1$, $|V(H)| = p_2$, $|E(F)| = q_1$, *and* $|E(H)| = q_2$, *then*

$$|V(F[H])| = p_1p_2 = |V(H[F])|, \;\; |E(F[H])| = p_1q_2 + p_2^2q_1 \;\; \textit{and} \;\; |E(H[F])| = p_2q_1 + p_1^2q_2.$$

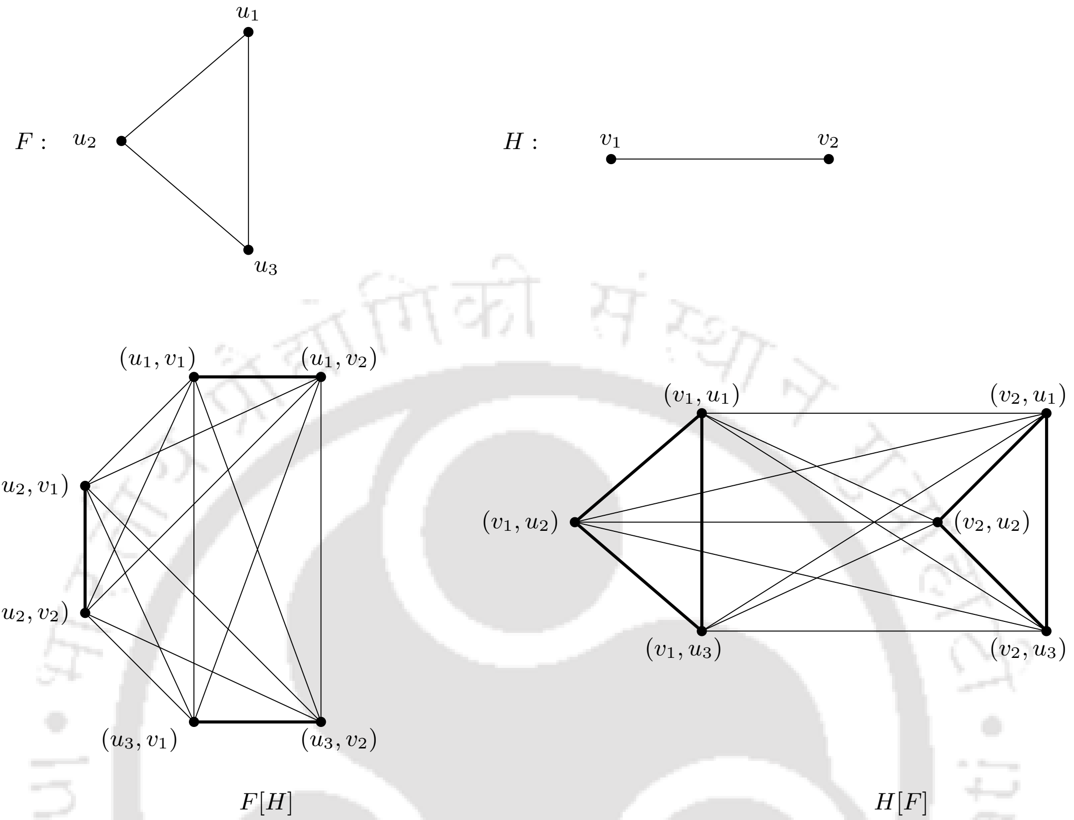


Figure 4.12: Lexicographic product

Example 4.5.1 Let $F = C_3$, the cycle of order 3 and $H = K_2$. The graph $F[H]$, obtained by taking lexicographic product of F and H, is shown in Figure 4.12. Notice that $F[H] \cong H[F] \cong K_6$.

From the definition of lexicographic product it follows that: Let F and H be two nontrivial graphs with at least two vertices. Then $F[H]$ is connected if and only if F is connected. Thus $F[H] \ncong H[F]$ whenever one of F or H is disconnected. Even in the case both the factors are connected, it need not commute. For example, Consider $F = K_2$ and $H = P_3$. $F[H]$ has 13 edges but $H[F]$ has only 11 edges. In [43], Imrich and Klavžar have characterized the cases when the lexicographic product commutes.

The following result describes the Laplacian matrix of $F[H]$ using the Laplacian matrices of F and H.

Proposition 4.5.2 *Let F and H be graphs on p_1 and p_2 vertices, respectively. Then*

$$L(F[H]) = L(F \Box H) + L(F) \otimes (J - I) + D(F) \otimes L(K_{p_2}). \tag{4.5}$$

Proof: Let F be on vertices $\{u_1, u_2, \ldots, u_{p_1}\}$ and H be on vertices $\{v_1, v_2, \ldots, v_{p_2}\}$. Thus $F[H]$ is a graph on vertices $\{(u_i, v_j) \mid i = 1, 2, \ldots, p_1,\ j = 1, 2, \ldots, p_2\}$. From the definition of graph lexicographic product, the adjacency matrix of $F[H]$ is $A(F[H]) = [a_{pqrs}]$, where for $p = 1, 2, \ldots, p_1$, $q = 1, 2, \ldots, p_2$, $r = 1, 2, \ldots, p_1$ and $s = 1, 2, \ldots, p_2$,

$$a_{pqrs} = \begin{cases} 1, & \text{if } p = r,\ q \neq s \text{ and } \{v_q, v_s\} \in E(H), \\ 0, & \text{if } p = r,\ q \neq s \text{ and } \{v_q, v_s\} \notin E(H), \\ 1, & \text{if } p \neq r \text{ and } \{u_p, u_r\} \in E(F), \\ 0, & \text{if } p \neq r \text{ and } \{u_p, u_r\} \notin E(F), \\ 0, & \text{otherwise.} \end{cases}$$

Thus $A(F[H])$ is the partitioned matrix

$$\begin{bmatrix} A(H) & B_{12} & \cdots & B_{1p_1} \\ B_{21} & A(H) & \cdots & B_{2p_1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{p_1 1} & B_{p_1 2} & \cdots & A(H) \end{bmatrix},$$

where for $i = 1, 2, \ldots, p_1$, $j = 1, 2, \ldots, p_1$ and $i \neq j$,

$$B_{ij} = \begin{cases} J, & \text{if } \{u_i, u_j\} \in E(F), \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Note that the ordering of the rows and columns of the matrix $A(F[H])$ is done as

$$(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_{p_2}), \ldots, (u_{p_1}, v_1), (u_{p_1}, v_2), \ldots, (u_{p_1}, v_{p_2}).$$

It follows that

$$A(F[H]) = I \otimes A(H) + A(F) \otimes J \text{ and } D(F[H]) = I \otimes D(H) + D(F) \otimes p_2 I.$$

Thus,

$$\begin{aligned} L(F[H]) &= I \otimes D(H) + D(F) \otimes p_2 I - I \otimes A(H) - A(F) \otimes J \\ &= I \otimes D(H) + D(F) \otimes p_2 I + I \otimes \Big(L(H) - D(H)\Big) + \Big(L(F) - D(F)\Big) \otimes J \\ &= I \otimes L(H) + L(F) \otimes J + D(F) \otimes (p_2 I - J) \\ &= L(F \Box H) + L(F) \otimes (J - I) + D(F) \otimes L(K_{p_2}). \end{aligned}$$

∎

Our next result describes the complete Laplacian spectrum of $F[H]$ using the Laplacian spectra of F and H.

Theorem 4.5.3 *Let F be a connected graph of order p_1 and H be any graph of order p_2. Suppose that $S(F) = (\lambda_1, \lambda_2, \ldots, \lambda_{p_1})$ and $S(H) = (\mu_1, \mu_2, \ldots, \mu_{p_2})$. Then*

(i) $\lambda_i p_2 \in S(F[H])$ *for* $i = 1, \ldots, p_1$,

(ii) $\mu_j + d(v_i)p_2 \in S(F[H])$ *for* $i = 1, 2, \ldots, p_1$ *and* $j = 2, \ldots, p_2$.

Thus

$$a(F[H]) = min\Big\{a(H) + \delta(F)p_2,\ a(F)p_2\Big\},$$

where $\delta(G)$ denotes the minimum degree of a vertex in G.

Proof: Using Proposition 4.5.2

$$\begin{aligned} L(F[H]) &= L(F\Box H) + L(F)\otimes(J-I) + D(F)\otimes L(K_{p_2}) \\ &= L(F)\otimes I + I\otimes L(H) + L(F)\otimes J - L(F)\otimes I + D(F)\otimes L(K_{p_2}) \\ &= I\otimes L(H) + L(F)\otimes J + D(F)\otimes L(K_{p_2}). \end{aligned}$$

Let X_i and Y_j be eigenvectors of $L(F)$ and $L(H)$ corresponding to the eigenvalues λ_i and μ_j, respectively. Observe that for $i = 1, 2, \ldots, p_1$,

$$\begin{aligned} L(F[H])(X_i\otimes \mathbb{1}) &= \Big(I\otimes L(H) + L(F)\otimes J + D(F)\otimes L(K_{p_2})\Big)(X_i\otimes \mathbb{1}) \\ &= \lambda_i p_2(X_i\otimes \mathbb{1}). \end{aligned}$$

Thus

$$\lambda_i p_2 \in S(F[H]) \text{ for } i = 1, \ldots, p_1.$$

Now for $i = 1, 2, \ldots, p_1$ and $j = 2, \ldots, p_2$,

$$\begin{aligned} L(F[H])(\hat{e}_i\otimes Y_j) &= \Big(I\otimes L(H) + L(F)\otimes J + D(F)\otimes L(K_{p_2})\Big)(\hat{e}_i\otimes Y_j) \\ &= \Big(\mu_j + d(v_i)p_2\Big)(\hat{e}_i\otimes Y_j). \end{aligned}$$

Thus

$$\mu_j + d(v_i)p_2 \in S(F[H]), \quad \text{for} \quad i = 1, \ldots, p_1 \quad \text{and} \quad j = 2, \ldots, p_2.$$

■

The following result comes immediately from Theorem 4.5.3, using which we can construct new Laplacian integral graphs from the known ones.

Corollary 4.5.4 *Let F and H be two graphs of order p_1 and p_2, respectively. Then $F[H]$ is Laplacian integral if and only if both F and H are Laplacian integral.*

Proof: Proof is immediate from Theorem 4.5.3. ■

It follows from Theorem 4.5.3 that $a(F[H]) = min\Big\{a(H) + \delta(F)p_2,\ a(F)p_2\Big\}$. The following result tells when each of the cases occur.

Corollary 4.5.5 *Let F be a connected graph of order p_1 and H be any graph of order p_2. If $F \neq K_{p_1}$, then*

$$a(F[H]) = a(F)p_2.$$

Further, if $F = K_{p_1}$, then $a(F[H]) = a(H) + (p_1 - 1)p_2$.

Proof: It is known (see Fiedler [25] and Merris [52]) that if $F \neq K_{p_1}$, then $a(F) < \delta(F)$. Thus if F is not a complete graph, then $a(F[H]) = a(F)p_2$.

If $F = K_{p_1}$, then

$$\begin{aligned} a(H) + \delta(F)p_2 &= a(H) + (p_1 - 1)p_2 \\ &= \Big(a(H) - p_2\Big) + p_1p_2 \\ &= \Big(a(H) - p_2\Big) + a(F)p_2 \\ &\leq a(F)p_2. \end{aligned}$$

Thus by Theorem 4.5.3, $a(F[H]) = a(H) + \delta(F)p_2 = a(H) + (p_1 - 1)p_2$. ■

Chapter 5

Spectra and Laplacian spectra of coronas

5.1 Introduction

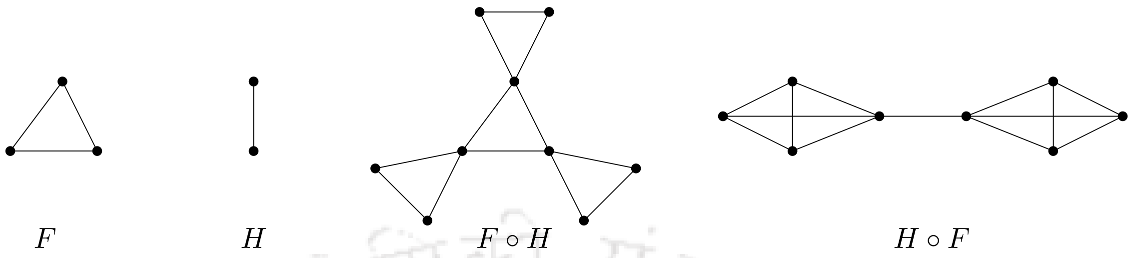
In this chapter, we discuss on the spectra and Laplacian spectra of coronas, with some applications. The *corona* operation is defined by Frucht and Harary in the following way:

Definition 5.1.1 (Frucht and Harary [29]; Harary [39]) Let F and H be two graphs on disjoint sets of n and m vertices, respectively. The *corona* $F \circ H$ of F and H is defined as the graph obtained by taking one copy of F and n copies of H, and then joining the ith vertex of F to every vertex in the ith copy of H.

Note that the corona $F \circ H$ has $n(m+1)$ vertices and $|E(F)| + n(|E(H)| + m)$ edges.

Example 5.1.1 Let $F = C_3$, the cycle of order 3 and $H = K_2$. The two different coronas $F \circ H$ and $H \circ F$ are shown in Figure 5.1.

Observe that the corona operation is a particular case of the operation defined in Definition 3.2.1 of Chapter 3. Let F be a graph on vertices $1, 2, \ldots, n$ and H be any graph. Take a new vertex v and add edges between v and each vertex of H to obtain H_v. Then $G = G[F, 1, \ldots, n, H_v]$, the graph with n pockets is nothing but the corona $F \circ H$.



F H $F \circ H$ $H \circ F$

Figure 5.1: Coronas

Let F be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and H be a graph of order $m \geq 1$. Suppose that $G = F \circ H$. Thus the adjacency matrix of G is

$$A(G) = \left[\begin{array}{c|ccc} A(F) & I_n & \cdots & I_n \\ \hline I_n & & & \\ \vdots & & A(H) \otimes I_n & \\ I_n & & & \end{array}\right],$$

where $A(F)$ and $A(H)$ are the adjacency matrices of the graphs F and H, respectively.

Let $L(F)$ and $L(H)$ be the Laplacian matrices of the graphs F and H, respectively. Thus the Laplacian matrix of G is

$$L(G) = \left[\begin{array}{c|ccc} L(F) + mI_n & -I_n & \cdots & -I_n \\ \hline -I_n & & & \\ \vdots & & (L(H) + I_m) \otimes I_n & \\ -I_n & & & \end{array}\right].$$

In [29, 39], the authors have described some important properties of the coronas. In section 5.2, we study the spectral properties of the coronas. In Section 5.3, we prove some structural results on the Fiedler vectors of the coronas and offer an application. Further, we construct infinitely many pairs of nonisomorphic graphs with the same spectrum and the same Laplacian spectrum using corona operation.

5.2 Spectra and Laplacian spectra of $F \circ H$

Let F be a connected graph with vertex set $V = \{1, 2, \ldots, n\}$ and H be a regular graph of order m and regularity r, $r \leq m - 1$. The following result gives a complete characterization of the eigenvalues and the eigenvectors of $A(F \circ H)$.

Theorem 5.2.1 *Let F be a graph of order n, H be an r-regular graph of order m and $G = F \circ H$. Let $\sigma(F) = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\sigma(H) = (\eta_1, \eta_2, \ldots, \eta_m = r)$. Then*

(i) $\dfrac{\mu_i + r \pm \sqrt{(r-\mu_i)^2 + 4m}}{2} \in \sigma(G)$ *with multiplicity* 1 *for* $i = 1, \ldots, n$ *and*

(ii) $\eta_j \in \sigma(G)$ *with multiplicity* n *for* $j = 1, \ldots, m-1$.

Proof: Let $X_1, \ldots, X_n$ be the orthonormal eigenvectors of $A(F)$ corresponding to the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$, respectively. For $i = 1, \ldots, n$, let

$$\lambda_i = \frac{\mu_i + r + \sqrt{(r-\mu_i)^2 + 4m}}{2} \quad \text{and} \quad \hat{\lambda}_i = \frac{\mu_i + r - \sqrt{(r-\mu_i)^2 + 4m}}{2}.$$

Note that $\dfrac{\mu_i + r + \sqrt{(r-\mu_i)^2 + 4m}}{2} = r$ implies $m = 0$, so that $\lambda_i, \hat{\lambda}_i$ is never r.

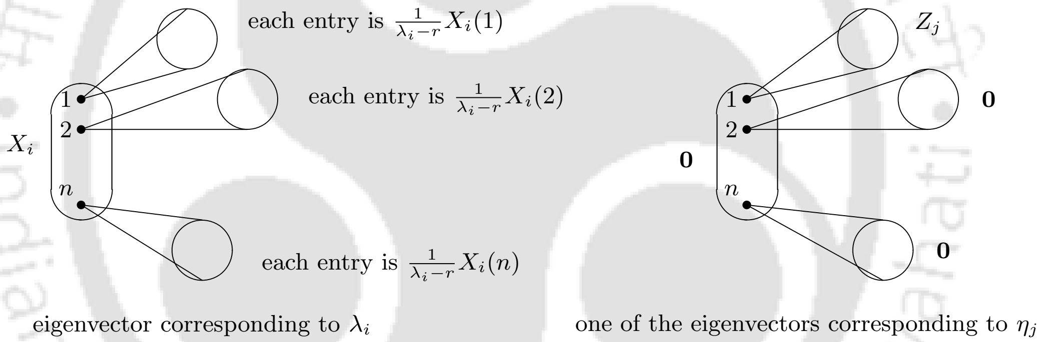


Figure 5.2: Description of eigenvectors

Observe that λ_i and $\hat{\lambda}_i$ are eigenvalues of $A(G)$ corresponding to the eigenvectors

$$\begin{pmatrix} X_i \\ \frac{1}{\lambda_i - r}X_i \\ \vdots \\ \frac{1}{\lambda_i - r}X_i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_i \\ \frac{1}{\hat{\lambda}_i - r}X_i \\ \vdots \\ \frac{1}{\hat{\lambda}_i - r}X_i \end{pmatrix},$$

respectively (see figure 5.2, picture on left).

Further, for $1 \leq j \leq m-1$, let Z_j be the eigenvector corresponding to the eigenvalue η_j of $A(H)$. Then for $i = 1, \ldots, n$, we have (see figure 5.2, picture on right, for $i = 1$)

$$A(G)\begin{pmatrix} \mathbf{0} \\ Z_j \otimes \hat{e}_i \end{pmatrix} = \eta_j \begin{pmatrix} \mathbf{0} \\ Z_j \otimes \hat{e}_i \end{pmatrix}.$$

In the previous equation we use that H is r-regular and hence $Z_j \perp \mathbb{1}$, for $j = 1, 2, \ldots, m-1$. Hence the proof. ■

The following result gives the spectral radius of $F \circ H$.

Corollary 5.2.2 *Let F be a graph of order n and H be an r-regular graph of order m and $G = F \circ H$. Then*

$$\rho(G) = \frac{\rho(F) + r + \sqrt{(r - \rho(F))^2 + 4m}}{2}.$$

Proof: Using Theorem 5.2.1 it is sufficient to show that $\dfrac{\rho(F) + r + \sqrt{(r - \rho(F))^2 + 4m}}{2} \geq r$. But $\dfrac{\rho(F) + r + \sqrt{(r - \rho(F))^2 + 4m}}{2} < r$ implies that $m < 0$, which is a contradiction. Thus the proof is complete. ■

Next we talk about the Laplacian spectrum of $F \circ H$. The following result describes the Laplacian eigenvalues and Laplacian eigenvectors of $F \circ H$ using the Laplacian eigenvalues and Laplacian eigenvectors of F and H.

Theorem 5.2.3 *Let F and H be any two graphs, not necessarily regular and $G = F \circ H$. Suppose that $S(F) = (0 = \nu_1, \nu_2, \ldots, \nu_n)$ and $S(H) = (0 = \delta_1, \delta_2, \ldots, \delta_m)$. Then*

(i) $\dfrac{\nu_i + m + 1 \pm \sqrt{(\nu_i + m + 1)^2 - 4\nu_i}}{2} \in S(G)$ *with multiplicity* 1 *for* $i = 1, \ldots, n$, *and*

(ii) $\delta_j + 1 \in S(G)$ *with multiplicity n for $j = 2, \ldots, m$.*

Proof: Suppose that $\mathbb{1} = Y_1, Y_2, \ldots, Y_n$, are the eigenvectors of $L(F)$ corresponding to the eigenvalues $0 = \nu_1, \nu_2, \ldots, \nu_n$, respectively. For $i = 1, \ldots, n$, let

$$\gamma_i = \frac{\nu_i + m + 1 + \sqrt{(\nu_i + m + 1)^2 - 4\nu_i}}{2} = \frac{\nu_i + m + 1 + \sqrt{(\nu_i + m - 1)^2 + 4m}}{2},$$

$$\hat{\gamma}_i = \frac{\nu_i + m + 1 - \sqrt{(\nu_i + m + 1)^2 - 4\nu_i}}{2} = \frac{\nu_i + m + 1 - \sqrt{(\nu_i + m - 1)^2 + 4m}}{2}.$$

Notice that $\dfrac{\nu_i + m + 1 \pm \sqrt{(\nu_i + m - 1)^2 + 4m}}{2} = 1$ implies $m = 0$, so that $\gamma_i, \hat{\gamma}_i$ are never 1.

Observe that γ_i and $\hat{\gamma}_i$ are eigenvalues of $L(G)$ afforded by the eigenvectors

$$\begin{pmatrix} Y_i \\ \frac{1}{1-\gamma_i} Y_i \\ \vdots \\ \frac{1}{1-\gamma_i} Y_i \end{pmatrix} \text{ and } \begin{pmatrix} Y_i \\ \frac{1}{1-\hat{\gamma}_i} Y_i \\ \vdots \\ \frac{1}{1-\hat{\gamma}_i} Y_i \end{pmatrix},$$

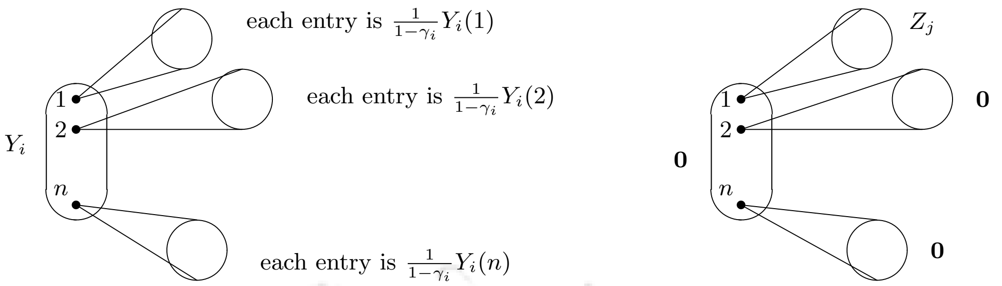


Figure 5.3: Description of eigenvectors

respectively (see figure 5.3, picture on the left).

Also if the eigenvalues $\delta_1(=0), \delta_2, \ldots, \delta_{m-1}, \delta_m$ of $L(H)$ are afforded by the eigenvectors $Z_1, Z_2, \ldots, Z_m$, respectively, then for $j = 2, \ldots, m$,

$$\begin{pmatrix} \mathbf{0} \\ Z_j \otimes \hat{e}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ Z_j \otimes \hat{e}_2 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{0} \\ Z_j \otimes \hat{e}_n \end{pmatrix} \text{(see figure 5.3, picture on the right)}$$

are the n linearly independent eigenvectors corresponding to the eigenvalue $\delta_j + 1$ of $L(G)$. Hence the proof is complete. ■

The following is an immediate corollary.

Corollary 5.2.4 *Let F and H be any two graphs, not necessarily regular and $G = F \circ H$. Then the following statements hold:*

(i) $1 \notin S(G)$ *if and only if H is connected.*

(ii) $m + 1 \in S(G)$ *always.*

(iii) *If $n > 1$, then* $a(G) = \frac{a(F)+m+1-\sqrt{(a(F)+m+1)^2-4a(F)}}{2} < 1$.

Proof: Item (i) and (ii) are routine. Item (iii) follows from the fact that

$$\frac{\nu_i + m + 1 - \sqrt{(\nu_i + m - 1)^2 + 4m}}{2} < 1$$

and if $\nu_i \leq \nu_j$, then

$$\frac{\nu_i + m + 1 - \sqrt{(\nu_i + m - 1)^2 + 4m}}{2} \leq \frac{\nu_j + m + 1 - \sqrt{(\nu_j + m - 1)^2 + 4m}}{2}.$$

Hence the proof is complete. ■

5.3 Some applications

5.3.1 Construction of Type I graphs with nonisomorphic Perron branches

Let Y be a Fiedler vector of $G = F \circ H$. The following is an easy consequence of Theorem 5.2.3 which describes the characteristic set of $F \circ H$.

Corollary 5.3.1 *Let F be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and H be any graph of order m and $G = F \circ H$. Then exactly one of the following statements holds.*
Case 1. For some Fiedler vector Y of F, $C(F, Y) = \{v\}$. Then for each Fiedler vector Z of G we have $C(G, Z) = \{v\}$.
Case 2. For some Fiedler vector Y of F there is a unique characteristic block B_1. Then for each Fiedler vector Z of G the characteristic block of G is also B_1.

Proof: We know

$$a(G) = \frac{a(F) + m + 1 - \sqrt{(a(F) + m + 1)^2 - 4a(F)}}{2} < 1,$$

and the vector

$$\begin{pmatrix} Y \\ \frac{1}{1-a(G)}Y \\ \vdots \\ \frac{1}{1-a(G)}Y \end{pmatrix}$$

is a Fiedler vector of G, where Y is a Fiedler vector of the graph F. Hence the proof is complete. ■

It is easy to construct a Type I tree with isomorphic perron branches. The question of constructing Type I trees with nonisomorphic Perron branches has been asked by Kirkland in [45]. The answer to this question is "yes", by the following example of a tree T, (see figure 5.4) taken from Grone and Merris [35] which is of Type I and has nonisomorphic Perron branches.

An infinite class of Type I trees with nonisomorphic Perron branches has been constructed in Kirkland [45]. The following result follows directly from the previous result and helps in the constructions of Type I graphs with nonisomorphic Perron branches.

Corollary 5.3.2 *Let $F = T$ be a tree and H be any graph. Then the characteristic set $C(G, Y)$ of $G = T \circ H$ (with respect to any Fiedler vector Y) is completely determined by the nature*

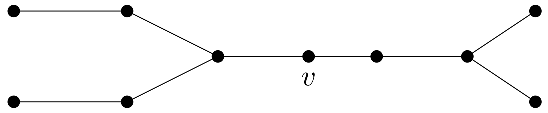


Figure 5.4: A Type I tree with characteristic vertex v

of T. *The set* $C(G, Y)$ *always has only one element, either a vertex or an edge. Further* $C(G, Y) = C(G, Z) = C(T, X)$, *where* Z, X *are any Fiedler vectors of* G, T, *respectively.*

In particular if T *is Type I with characteristic vertex* v *then* G *is Type I with characteristic vertex* v.

In view of this result, all we need is to take a Type I tree T on more than 2 vertices with nonisomorphic Perron branches and any graph H. Then $G = T \circ H$ is an example of a Type I graph with nonisomorphic Perron branches. Note that $G \circ H'$ is also an example, for any other graph H'. In this way we can construct an infinite class of Type I graphs with nonisomorphic Perron branches.

In particular, considering the tree T in Figure 5.4, which is known to be Type I with nonisomorphic Perron branches, and taking H to be an isolated vertex, we see that $T, T \circ H, (T \circ H) \circ H, \ldots$ gives us an infinite class of Type I trees with nonisomorphic Perron branches (which is different from the one constructed in Kirkland [45]).

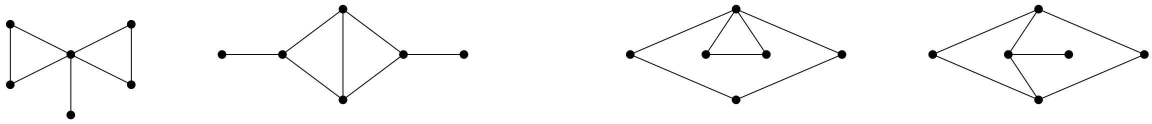
5.3.2 Cospectral and Laplacian cospectral graphs

Here we discuss another application of the results which are proved earlier. Two graphs F and H are called *cospectral* if the spectrum of $A(F)$ and $A(H)$ are the same. Two graphs are called *Laplacian cospectral* if $L(F)$ and $L(H)$ have the same spectrum. Note that if two graphs are isomorphic then they are cospectral and Laplacian cospectral. Here we discuss about the pairs of graphs which are not isomorphic but cospectral and Laplacian cospectral. Here we give examples of one pair of cospectral graphs and another pair of Laplacian cospectral graphs.

Example 5.3.1 Consider the pair of graphs F_1 and H_1 in Figure 5.5.

One can check that

$$\sigma(F_1) = \sigma(H_1) = (-1.9032, -1, -1, .1939, 1, 2.7093).$$



F_1 H_1 F_2 H_2

Figure 5.5: F_1 and H_1 are cospectral and F_2 and H_2 are Laplacian cospectral.

Thus F_1 and H_1 are cospectral. Note that they are not Laplacian cospectral as

$$S(F_1) = (0, 1, 1, 3, 3, 6) \text{ and } S(H_1) = (0, .5858, 1.2679, 3.4142, 4, 4.7321).$$

On the other hand the graphs F_2 and H_2 are Laplacian cospectral as

$$S(F_2) = S(H_2) = (0, 0.7639, 2, 3, 3, 5.2361)$$

but not cospectral. $\sigma(F_2) = (-2.1912, -1, -0.5767, 0, 1.2644, 2.5035)$ and

$$\sigma(H_2) = (-2.5243, -0.7923, 0, 0, 0.7923, 2.5243).$$

In the above example the graphs F_1 and H_1 are cospectral but not Laplacian cospectral whereas F_2 and H_2 are Laplacian cospectral but not cospectral. There are also examples of pairs of graphs which are both cospectral and Laplacian cospectral (See van Dam and Haemers [66]). Note that if two graphs are regular and cospectral then they are also Laplacian cospectral. See, for example, the graphs F and H in Figure 5.6 which are taken from [66].

F H

Figure 5.6: A pair of cospectral regular graphs.

This topic has been an area of interest for many researchers. We refer the reader to [66] and the references therein to know more. Here we construct infinite pairs of nonisomorphic graphs which are cospectral and Laplacian cospectral.

Let F and H be two nonisomorphic cospectral and Laplacian cospectral graphs (take for example the two graphs given in Figure 5.6). Let B be the graph of an isolated vertex. Let $F_1 = F \circ B$ and $H_1 = H \circ B$. Now for $i = 2, \ldots$ define $F_i = F_{i-1} \circ B$ and $H_i = H_{i-1} \circ B$. By Theorems 5.2.1, 5.2.3, we see that the spectrum and the Laplacian spectrum of F_1 and H_1 is completely determined by the spectrum of F and H and they are the same. Use of induction leads us to the following conclusion.

Corollary 5.3.3 *Let F_i and H_i be graphs as defined above, for $i \in \mathbb{N}$. Then for each i the graphs F_i, H_i are nonisomorphic cospectral and Laplacian cospectral nonregular graphs.*

Remark 5.3.4 *In the above construction we have taken B as the graph of an isolated vertex. Thus the resulting graphs have pendant vertices. If we replace B by any other graph then we can get infinite pairs of nonisomorphic nonregular graphs without any pendant vertex which are cospectral and Laplacian cospectral.*

Chapter 6

Graphs with a reciprocal eigenvalue property

6.1 Introduction

Let G be a graph with vertex set $\{1, 2, \ldots, n\}$. Throughout this chapter the spectrum of G is defined as

$$\sigma(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)),$$

where $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ are the eigenvalues of $A(G)$. As $A(G)$ is a nonnegative matrix, $\lambda_n(G)$ is nonnegative and it equals the *spectral radius* of G which will be denoted here by $\rho(G)$. If G is connected then $A(G)$ is irreducible, thus using Perron-Frobenius theorem the spectral radius of G is of multiplicity one to which there corresponds a positive eigenvector, called the *Perron vector*. A graph G is said to be *singular* if $A(G)$ is singular. A graph G which is not singular is called *nonsingular*.

It is well-known that (see Cvetković, Doob and Sach [17]) a graph G is bipartite if and only if the negative of each eigenvalue of G is also an eigenvalue of G. We ask a question which is quite similar: *Characterize the graphs which satisfy the property that the reciprocal of each eigenvalue of* G *is also an eigenvalue of* G.

We say that *a graph* G *has property* (R) if $\frac{1}{\lambda}$ is an eigenvalue of G whenever λ is an eigenvalue of G. Further, if λ and $\frac{1}{\lambda}$ have the same multiplicity, for each eigenvalue λ then we say that the *graph has property* (SR).

Example 6.1.1 The graphs G_1 and G_2 in Figure 6.1 satisfy property (SR).

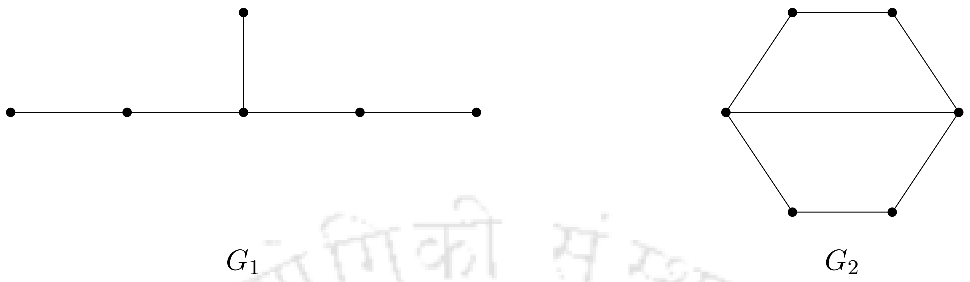


Figure 6.1: Two graphs with property (SR)

The eigenvalues of G_1 are

$$1, -1, \frac{\sqrt{2}+\sqrt{6}}{2}, \frac{\sqrt{2}-\sqrt{6}}{2}, \frac{-\sqrt{2}+\sqrt{6}}{2}, \frac{-\sqrt{2}-\sqrt{6}}{2},$$

and the eigenvalues of $A(G_2)$ are $1, -1, \pm 1+\sqrt{2}, \pm 1-\sqrt{2}$.

A graph G is said to have a *perfect matching* if there exists a spanning forest whose components are solely paths on two vertices. A graph in general can have more than one perfect matching. It follows from Cvetkov́ic, Doob and Sachs [17, Theorem 1.3 and Proposition 1.1] that when a tree has a perfect matching, it is unique and such trees are precisely the trees which are nonsingular. In general any graph with a unique perfect matching is nonsingular.

In the next section, we supply a class of graphs satisfying property (R), using the corona of a bipartite graph and a single vertex. We characterize the trees satisfying property (SR). We show that this is the class of trees on $2n$ vertices with n matchings which are leaves. We supply suitable examples to show that a graph with property (R) is not necessarily the corona of two graphs and is not necessarily bipartite.

6.2 Trees with property (SR)

It is obvious that a graph with property (R) or property (SR) must be nonsingular. The following result which can be found in [17] helps us to proceed further.

Lemma 6.2.1 (Cvetkov́ic, Doob and Sachs [17]) *Let G be a graph with vertex set $\{1, 2, \ldots, n\}$. If two or more pendant vertices have a common neighbor, then G is singular.*

Let us first investigate graphs with property (R). The first examples are paths P_2 and P_4. For P_2 the eigenvalues are $1, -1$ where as for P_4 the eigenvalues are $\frac{1\pm\sqrt{5}}{2}, \frac{-1\pm\sqrt{5}}{2}$. A careful examination of P_4 leads us to the following result which gives a class of bipartite graphs with property (R).

Lemma 6.2.2 *Let G_1 be any graph and G be obtained by adding a new pendant vertex to every vertex of G_1. Then λ is an eigenvalue of G if and only if $\frac{-1}{\lambda}$ is an eigenvalue of G. Further, if G_1 is bipartite then G has property (R).*

Proof: Let G_1 be on n vertices. It is clear that $G = G_1 \circ K_1$. Thus $A(G) = \left[\begin{array}{c|c} A(G_1) & I_n \\ \hline I_n & \mathbf{0} \end{array}\right]$. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of $A(G_1)$ corresponding to the eigenvectors $x_1, \ldots, x_n$, respectively, where the set $\{x_1, \ldots, x_n\}$ is orthonormal. Then the vectors

$$\begin{bmatrix} x_1 \\ \frac{2}{\mu_1+\sqrt{\mu_1^2+4}}x_1 \end{bmatrix}, \begin{bmatrix} x_1 \\ \frac{2}{\mu_1-\sqrt{\mu_1^2+4}}x_1 \end{bmatrix}, \ldots, \begin{bmatrix} x_n \\ \frac{2}{\mu_n+\sqrt{\mu_n^2+4}}x_n \end{bmatrix}, \begin{bmatrix} x_n \\ \frac{2}{\mu_n-\sqrt{\mu_n^2+4}}x_n \end{bmatrix}$$

are all eigenvectors of $A(G)$ corresponding to the eigenvalues

$$\frac{\mu_1 + \sqrt{\mu_1^2+4}}{2}, \frac{\mu_1 - \sqrt{\mu_1^2+4}}{2}, \ldots, \frac{\mu_n + \sqrt{\mu_n^2+4}}{2}, \frac{\mu_n - \sqrt{\mu_n^2+4}}{2},$$

respectively.

We observe that $\dfrac{\mu_i + \sqrt{\mu_i^2+4}}{2} \, \dfrac{\mu_i - \sqrt{\mu_i^2+4}}{2} = -1$ and the first conclusion follows. Note that if G_1 is bipartite then G is bipartite. Thus if $\lambda \in \sigma(G)$ then by the above $\frac{-1}{\lambda} \in \sigma(G)$ and as G is bipartite $\frac{1}{\lambda} \in \sigma(G)$. ■

The following is an immediate corollary.

Corollary 6.2.3 *Let $G = G_1 \circ K_1$. Then*

a. *G is nonsingular and the determinant of $A(G) = (-1)^n$, where n is the number of vertices in G_1.*

b. *There are n positive and n negative eigenvalues of G. If λ_i are the positive eigenvalues of G then $\sum\limits_{i=1}^{n} \lambda_i = \sum\limits_{i=1}^{n} \frac{1}{\lambda_i}$.*

c. *$\rho(G) = \frac{\rho(G_1)+\sqrt{\rho(G_1)^2+4}}{2}$, where $\rho(H)$ is the spectral radius of a graph H.*

It is natural to ask whether the converse of Lemma 6.2.2 is true, that is, if G is any graph which has property (R), is it necessarily the corona of a bipartite graph and K_1? The answer is no, in general, as can be seen from the following example.

Example 6.2.1 The graph G in Figure 6.2 satisfies property (R). The eigenvalues of G are

$$1, 1, \frac{-3 \pm \sqrt{5}}{2}, \frac{1+\sqrt{33}+\sqrt{18+2\sqrt{33}}}{4}, \frac{1+\sqrt{33}-\sqrt{18+2\sqrt{33}}}{4},$$
$$\frac{1-\sqrt{33}+\sqrt{18-2\sqrt{33}}}{4}, \frac{1-\sqrt{33}-\sqrt{18-2\sqrt{33}}}{4}.$$

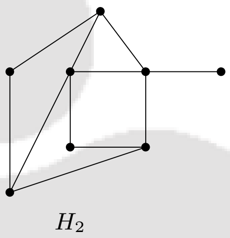


Figure 6.2: A graph with property (R) which is not a corona

Notice that G is not bipartite. We can argue that G is not the corona of two graphs. Suppose that $G = G_1 \circ G_2$. Thus $8 = |V(G)| = (|V(G_2)| + 1)|V(G_1)|$. Note that $|V(G_2)|$ cannot be 1 because in that case G should have 4 pendant vertices. If $|V(G_2)| \geq 2$ then G cannot have a pendant vertex. Thus G is not a corona.

One can see that it is difficult to characterize graphs with property (R). Thus one may ask for a characterization of all such trees. Here we have two immediate questions.

1. Characterize all trees with property (R).
2. Characterize all trees with property (SR).

In this section we supply an answer to question 2. It turns out that any tree with property (SR) is of the form $T \circ K_1$, for some tree T. Such trees are called *corona trees*. To do this we need the following lemma.

Lemma 6.2.4 (Brualdi and Ryser [13]; Cvetković, Doob and Sachs [17]) *Let* $P(T;x) = x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_{n-2}x^2 + C_{n-1}x + C_n$ *be the characteristic polynomial of a tree* T *on* n *vertices. Then* $C_{2i+1} = 0$*, and*

$$C_{2i} = (-1)^i(\textit{the number of pairwise disjoint edge subsets of size } i).$$

The following result is an useful observation.

Lemma 6.2.5 *Let G be a graph on n vertices with property (SR) and $P(G;x) = \Sigma_{i=0}^{n} a_i(G)x^{n-i}$ be the characteristic polynomial of $A(G)$. Then $|a_i(G)| = |a_{n-i}(G)|$, for $i = 0, 1, \ldots, n$.*

Proof: Since G satisfies property (SR), G is nonsingular. Moreover $P(G;x)$ and $x^n P(G; \frac{1}{x})$ have the same roots. Since $P(G;x)$ is monic and the leading coefficient of $x^n P(G; \frac{1}{x})$ is ± 1, it follows that $P(G;x) = \pm x^n P(G; \frac{1}{x})$ and the conclusion follows. ■

The following result characterizes all trees with property (SR).

Theorem 6.2.6 *Let T be a tree on n vertices. Then T has property (SR) if and only if $T = T_1 \circ K_1$, for some tree T_1.*

Proof: We prove the only if part here and the if part follows from Lemma 6.2.2. Let T have property (SR). Then $n = 2k$, for some k. If $k = 1, 2$, then the only nonsingular trees of order $2k$ are the paths which are $K_1 \circ K_1, K_2 \circ K_1$. Assume that $k \geq 3$. Further, T has a perfect matching. Let $\mathcal{M} = \{f_i = \{u_i, v_i\}, i = 1, \ldots k\}$ be those edges of T (see Figure 6.3). Note here that if we put back the remaining $k-1$ edges we get the figure of T.

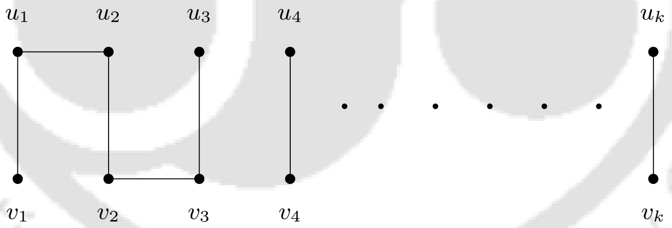


Figure 6.3:

We claim that for each edge f_i at least one of u_i, v_i is of degree 1 in T. Suppose it is not the case. Thus there is an edge, say f_2, such that u_2, v_2 both have degrees greater than 1, say $\{u_1, u_2\}, \{v_2, v_3\}$ are present.

Let

$$P(x) = x^{2k} + C_1 x^{2k-1} + C_2 x^{2k-2} + \cdots + C_{2k-2}x^2 + C_{2k-1}x + C_{2k}$$

be the characteristic polynomial of the tree T. By Lemma 6.2.5, $|C_2| = |C_{2k-2}|$.

But by Lemma 6.2.4, $|C_2| =$ the number of edges in $T = 2k-1$ and $|C_{2k-2}| =$ the number of pairwise disjoint edge subsets of size $k-1$. Hence the number of pairwise disjoint edge subsets of size $k-1$ is $2k-1$.

There are k pairwise edge disjoint subsets of size $k-1$ of T of the form

$$\{f_1,\dots,f_k\}\setminus\{f_1\},\{f_1,\dots,f_k\}\setminus\{f_2\},\dots,\{f_1,\dots,f_k\}\setminus\{f_k\}.$$

Any edge e of T which is not in $\mathcal{M}$ is incident to exactly two edges in $\mathcal{M}$, say f_i, f_j and gives us a pairwise edge disjoint subset of size $k-1$ of T of the form

$$\{e\}\cup\mathcal{M}\setminus\{f_i,f_j\}.$$

We will have $k-1$ such pairwise edge disjoint subsets of size $k-1$ of T.

Further, the set $\{\{u_1,u_2\},\{v_2,v_3\},f_4,\dots f_k\}$ is also a pairwise edge disjoint subset of size $k-1$ of T. Thus the number of pairwise edge disjoint subsets of size $k-1$ of T exceeds $2k-1$, which is a contradiction and the claim is justified. Assume that the k pendant vertices of T are $\{u_1,\dots,u_k\}$ and let T_1 be the subtree of T induced by $\{v_1,\dots,v_k\}$. Then $T=T_1\circ K_1$ and the proof is complete. ■

To characterize trees with property (R), we need some more results. In section 6.3, we give a combinatorial description of the inverse of the adjacency matrix of a bipartite graph with a unique perfect matching. And using that in section 6.4, we characterize the trees satisfying property (R).

6.3 Bipartite graphs with a unique perfect matching

The following definition of an *alternating path* is taken from Buckley, Doty, and Harary [14, p.156]:

Definition 6.3.1 Let G be a graph with a unique perfect matching. A path $P(i,j)=[i=i_1,i_2,\dots,i_{2k}=j]$ from vertex i to vertex j in G is said to be an *alternating path* if the edges $\{i_1,i_2\},\{i_3,i_4\},\dots,\{i_{2k-1},i_{2k}\}$ are edges in the perfect matching. A single edge $\{i,j\}$ from the perfect matching is also considered as an alternating path.

For a graph with a unique perfect matching, there can be more than one alternating paths between a pair of vertices. If T is a nonsingular tree and i,j are two vertices, a necessary condition for the existence of an alternating path between i and j is that the distance between i and j is odd. It can easily be seen that this is not sufficient, take for example, $P_4\circ K_1$. Naturally one wonders, given two vertices i,j in a nonsingular tree, how can one find out whether there is an alternating path between i and j, without finding the perfect matching explicitly. It is easy to see that the path $P(i,j) = [i = i_1, i_2, \ldots, i_{2k} = j]$ is alternating if and only if each connected component of $T \setminus \{i_1, \ldots, i_{2k}\}$ has even number of vertices. Let us ask the following question. *Given a nonsingular tree, what is the total number of alternating paths in it?* We will give an answer to this which requires the description of the inverse of the adjacency matrix. A combinatorial description of the inverse of the adjacency matrix of a nonsingular tree has been given in Buckley, Doty, and Harary [14, Theorem 3] and in Pavlikova [62, Theorem 1]. A description of the inverse of the adjacency matrix of a bipartite graph without a cycle of length $4m$ is given in Cvetković, Doob and Sachs [17, Exercise 8, p. 258]. We supply a simple combinatorial description of the inverse of a bipartite graph with a unique perfect matching from which the description of the inverse of a nonsingular tree follows as a corollary. It follows that the total number of alternating paths in a nonsingular tree is the total number of nonzero entries in A^{-1} and the number attains its minimum at the value $n-1$ if and only if the tree is a corona tree.

Let G be a bipartite graph with a unique perfect matching $\mathcal{M}$ and let $\mathcal{P}_G$ denote the collection of all alternating paths in G. Note that if $P(i,j)$ is an alternating path between vertices i and j, then the number of edges in $P(i,j)$ which are not in $\mathcal{M}$ is $\frac{|P(i,j)|-1}{2}$, where $|P(i,j)|$ is the number edges in $P(i,j)$.

The following result gives a description of the inverse of the adjacency matrix of a bipartite connected graph with a unique perfect matching.

Lemma 6.3.1 *Let G be a bipartite connected graph on n vertices with a unique perfect matching $\mathcal{M}$ and let A be its adjacency matrix. Let $B = [b_{i,j}] \in \underline{R}^{n,n}$, where*

$$b_{i,j} = \begin{cases} \sum\limits_{P(i,j) \in \mathcal{P}_G} (-1)^{\frac{|P(i,j)|-1}{2}}, & \textit{if at least one } P(i,j) \in \mathcal{P}_G \textit{ can be found,} \\ 0, & \textit{otherwise.} \end{cases} \tag{6.1}$$

Then $B = A^{-1}$.

Proof: The (i,j)th entry of AB is given by

$$(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} = \sum_{k;\, \{k,i\} \in E(G)} b_{k,j}. \tag{6.2}$$

Thus for each $i = 1, 2, \ldots, n$,

$$(AB)_{i,i} = \sum_{k;\, \{k,i\} \in E(G)} b_{k,i} = 1,$$

as there exist exactly one vertex, say i', such that $\{i', i\} \in \mathcal{M}$.

Now let i, j be two distinct vertices in G. Suppose that for each vertex v adjacent to i, there is no alternating path $P(v, j)$ between v and j so that, by 6.1, $b_{v,j} = 0$. Then from 6.2 we have that $(AB)_{i,j} = 0$.

Assume now that there is a vertex $v \neq i'$ adjacent to i such that $P(v, j) = [v = x_1, x_2, \ldots, x_m = j]$ is an alternating path. In this case $P' = [i', i, P(v, j)] = [i', i, v, x_2, \ldots, x_m = j]$ is also alternating. We claim that this is also a path. If not then, as $P(v, j)$ is a path, there are two cases to consider:

Case I. Some $x_l = i'$ and $x_{l+1} = i$. In this case we have an even cycle $[i, v = x_1, x_2, \ldots, x_l, x_{l+1} = i]$ of which every alternate edge is in the matching. Hence the cycle is a simple one. Clearly we cannot have a unique matching in this case.

Case II. Some $x_{l+1} = i$ and $x_l \neq i'$. In this case we have an odd cycle $[i, v = x_1, \ldots, x_l, x_{l+1} = i]$. This is not possible as G is bipartite.

Thus the claim is justified.

Conversely, if there is an alternate path $P(i', j)$ from i' to j, it must have the form $[i', i, v = x_1, \ldots, j]$. Thus there must exist a vertex $v \neq i'$ adjacent to i such that an alternating path from v to j exists.

Let $S = \{v_1, v_2, \ldots v_r\}$, with $v_l \neq i'$, be the vertices adjacent to i such that there are alternating paths from v_l to j. We have already seen that the alternating paths from i' to j are precisely of the form $[i', i, P(v_l, j)]$, where $P(v_l, j)$ is an alternating path from v_l to j. Hence

$$(AB)_{i,j} \;=\; b_{i',j} + \sum_{v_s \in S} b_{v_s,j} \;=\; \sum_{v_s \in S} -b_{v_s,j} + \sum_{v_s \in S} b_{v_s,j} \;=\; 0$$

and the proof is done. ■

As a corollary, we have the following result that gives the combinatorial description of the inverse of a nonsingular tree.

Corollary 6.3.2 (Pavlikova [62, Theorem 1]; Buckley, Doty, and Harary [14, Theorem 3]) *Let T be a nonsingular tree on n vertices and A be its adjacency matrix. Let $B = [b_{i,j}]_n$, where*

$$b_{i,j} \;=\; \begin{cases} (-1)^{\frac{|P(i,j)|-1}{2}}, & \text{if } P(i,j) \in \mathcal{P}_T, \\ 0, & \text{otherwise.} \end{cases}$$

Then $B = A^{-1}$.

Next, we show that if T is a nonsingular tree, then A^{-1} is similar to the adjacency matrix of some graph via a diagonal matrix of ±1's. Note that the following result has already been observed by Godsil [30]. Here we give a different proof. We use the term *non-matching edge* to mean an edge of T which is not in $\mathcal{M}$.

Lemma 6.3.3 *Let T be a nonsingular tree on vertices $1, \ldots, n$. Let 1 be a pendant vertex in T. Define the entries of the vector $\tilde{d}$ via $\tilde{d}_i = (-1)^{n_i}$, where n_i is the number of non-matching edges in $P(1, i)$, for $i = 1, \ldots, n$. Then $\mathrm{diag}(\tilde{d})A^{-1}\mathrm{diag}(\tilde{d})$ is the adjacency matrix of some graph.*

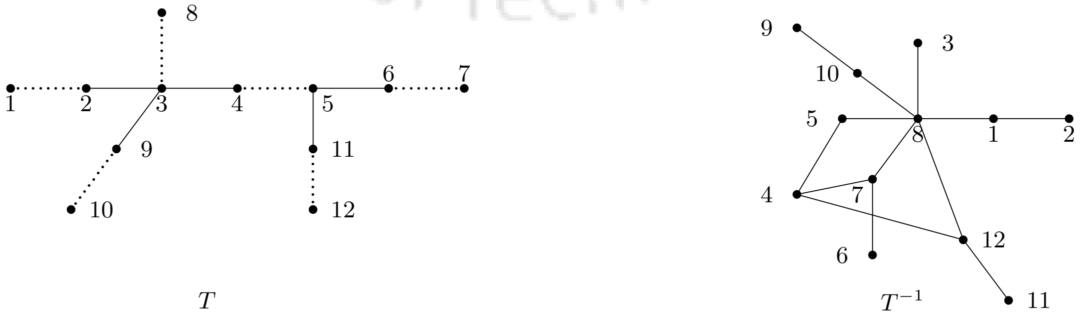
Proof: Let $D = \mathrm{diag}(\tilde{d}), B = A^{-1}$, and $F = DBD$. Thus $f_{i,j} = \tilde{d}_i\tilde{d}_j b_{i,j}$ and, since D is invertible, we observe that $f_{i,j} = 0$ if and only if $b_{i,j} = 0$.

Suppose that $b_{i,j} \neq 0$. It follows from the properties of B found in Lemma 6.3.1 that the path $P(i, j)$ is an alternating path. Let k be the vertex on $P(i, j)$ which is nearest to 1. Then $n_i + n_j = 2n_k + r$, where $r = \frac{d(i,j)-1}{2}$ is the number of non-matching edges in $P(i, j)$. Thus $\tilde{d}_i\tilde{d}_j b_{i,j} = (-1)^{n_i}(-1)^{n_j}(-1)^r = (-1)^{2(n_k+r)} = 1$. As F is symmetric, the proof is complete. ■

Remark 6.3.4 *In the above proof as $D = D^{-1}$, it follows that A^{-1} is similar to the adjacency matrix of some graph and, furthermore, that $f_{i,j} = |b_{i,j}|$.*

Henceforth, for a nonsingular tree T, we use the notation T^{-1} for the graph as indicated in Lemma 6.3.3.

Example 6.3.1 In Figure 6.4, a nonsingular tree and its inverse graph are given. The dotted lines represent the edges in $\mathcal{M}$.



T T^{-1}

Figure 6.4: A nonsingular tree and its inverse

Remark 6.3.5 *Let T be a nonsingular tree. It is clear that $V(T) = V(T^{-1})$ and $\{i, j\} \in T^{-1}$ if and only if $P(i, j)$ is an alternating path in T. Thus $|E(T^{-1})| = |\mathcal{P}_T|$. From Example* 6.3.1 *we see that T^{-1} may, indeed, not be a tree. It can easily be seen that any alternating path of length at least* 5 *in T creates a cycle in T^{-1}. Since $A(T^{-1})$ is similar to the matrix $A(T)^{-1}$, we further see that $A(T^{-1})$ and $A(T)^{-1}$ have the same eigenvalues. Thus λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}. In particular,*

$$\rho(T) \;=\; \frac{1}{\tau(T^{-1})} \text{ and } \tau(T) \;=\; \frac{1}{\rho(T^{-1})}.$$

Thus invertible trees have interesting spectral and combinatorial properties and these we study here. We begin with the following observation:

Lemma 6.3.6 *Let T be a nonsingular tree. Then T^{-1} is connected.*

Proof: This follows easily from Lewin and Neumann [50, Lemma 0] which states that if the (j, i)th cofactor of the matrix $B = (b_{i,j}) \in \underline{C}^{n,n}$ is nonzero, then in the directed graph of B there is a path from vertex i to vertex j. Now suppose that B is invertible, its directed graph is strictly connected, and $C = (c_{i,j}) = B^{-1}$. Let i and j be arbitrary vertices in the directed graph of C. As B has a connected directed graph, there exists indices $i_1, \ldots, i_r$ such that $b_{i,i_1}, b_{i_1,i_2}, \ldots, b_{i_r,j} \neq 0$. But then the $(i_1, i), (i_2, i_1), \ldots, (j, i_r)$ cofactors of the matrix C are all nonzero. Hence in the directed graph of C there is a path from vertex i to vertex i_1, and so on. Finally, there is a path from vertex i_r to vertex j. To complete the proof let $B = A(T)$ and $C = (A(T))^{-1}$. ■

Corollary 6.3.7 *Let T be a nonsingular tree on n vertices. Then the smallest positive eigenvalue of T, $\tau(T)$ is simple. Moreover, to $\tau(T)$ there corresponds an eigenvector whose entries are all nonzero.*

Proof: Since T^{-1} is a connected graph, $A(T^{-1})$ is irreducible. Thus using Perron–Frobenius Theorem $\rho(T^{-1})$ is a simple eigenvalue to which there corresponds a positive eigenvector, say y. By Remark 6.3.5, we first immediately have that $\tau(T) = \frac{1}{\rho(T^{-1})}$.

Next observe that in the above if y is the Perron vector of T^{-1}, then taking D as defined in Lemma 6.3.3 and noting that $D = D^{-1}$, we see that have $A(T)(Dy) = \tau(T)Dy$. Thus the eigenvector corresponding to the smallest positive eigenvalue of a nonsingular tree has no zero entries and the signs of the entries are determined by D. ■

The following is an immediate consequence for which the proof is omitted.

Corollary 6.3.8 *Let T be a nonsingular tree on n vertices. Then*

$$|\mathcal{P}_T| \ = \ |E(T^{-1})| \ \geq \ n-1.$$

Lemma 6.3.9 *Let T be a nonsingular tree on $2n$ vertices. Then the graph T^{-1} is bipartite.*

Proof: We know that T is bipartite. Let (A, B) be the bi–partition of the vertex set of T, where $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{M} = \{\{u_i, v_i\}, i = 1, 2, \ldots, n\}$ be the perfect matching in T. We wish to show that (A, B) is also a bi–partition of T^{-1}.

Suppose that $\{u_1, u_2\} \in T^{-1}$. Then $P(u_1, u_2)$ is an alternating path in T. As this path has odd length, we see that u_1 and u_2 cannot be in the same part of the bipartition (A, B) of T and we obtain a contradiction. ■

Note that the results on connectedness of T^{-1} (Lemma 6.3.6) and the graph T^{-1} is bipartite (Lemma 6.3.9) are already pointed out by Godsil in [30]. Here we have given different proofs.

Another interesting set of observations is as follows:

Theorem 6.3.10 *Let T be a nonsingular tree on $2n$ vertices. Then the following are equivalent:*

(i) *$|\mathcal{P}_T|$ is at a minimum possible value and equals $2n - 1$.*

(ii) *T^{-1} is a tree.*

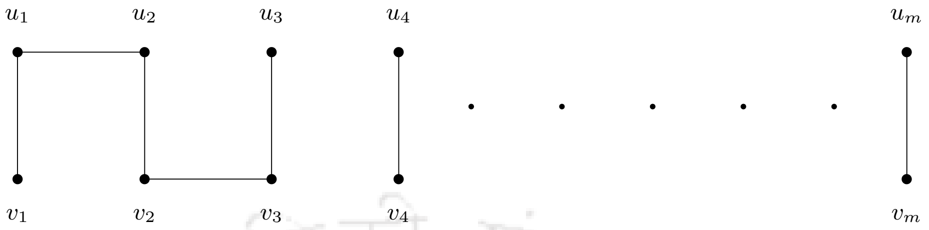
(iii) *T is a corona tree.*

(iv) *T^{-1} is isomorphic to T.*

Proof: (i)⇒(ii). Assume that $|\mathcal{P}_T| = 2n - 1$. As $|\mathcal{P}_T| = |E(T^{-1})|$ and T^{-1} is connected, it follows that T^{-1} is a tree.

(ii)⇒(iii). Assume that T^{-1} is a tree. Thus T^{-1} does not have a cycle implying that T cannot have alternating paths of length more than 3. Without loss, assume that T has at least 6 vertices. Let $\mathcal{M} = \{\{u_i, v_i\} : i = 1, \ldots, n\}$ be the perfect matching in T.

We first claim that one endvertex of each matching edge is a pendant vertex in T. To see that this is true suppose, by way of contradiction, that there is a matching edge, say, $\{u_2, v_2\}$ such that u_2 is adjacent to some vertex other than v_2, say u_1, and that v_2 is adjacent to some vertex other than u_2, say, v_3.



Then $[v_1, u_1, u_2, v_2, v_3, u_3]$ is an alternating path of length 5 in T. In this case T^{-1} should contain a cycle which cannot be the case. Hence T is obtained by adding pendant vertices to each vertex of some other tree which means that T is a corona tree.

(iii)⇒(iv). Suppose that T is a corona tree. Let $\mathcal{M} = \{\{u_i, v_i\} : i = 1, \ldots n\}$ be the perfect matching in T and assume that the vertices u_i are the pendant vertices. Define $f : V(T) \to V(T^{-1})$ as $f(u_i) = v_i$ and $f(v_i) = u_i$. Using the description of T^{-1} we see that, for each matching edge $\{u_i, v_i\} \in T$, the edge $\{f(u_i), f(v_i)\} = \{v_i, u_i\} \in T^{-1}$ because $\{u_i, v_i\}$ is an alternating path in T. For any non–matching edge $\{v_i, v_j\} \in T$, we see that the path $[u_i, v_i, v_j, u_j]$ is an alternating path in T. Hence $\{f(v_i), f(v_j)\} = \{u_i, u_j\} \in T^{-1}$. Thus $\{x, y\} \in T$ implying that $\{f(x), f(y)\} \in T^{-1}$.

Conversely, suppose $\{f(x), f(y)\} \in T^{-1}$ implying that there is an alternating path between $f(x)$ and $f(y)$ in T. As T is a corona tree with the above description, it follows that either (i) $f(x) = u_i$ and $f(y) = v_i$, for some i, or (ii) $f(x) = v_i$ and $f(y) = u_i$, for some i, or (iii) $f(x) = u_i$ and $f(y) = u_j$, for some $i \neq j$. In the case of (i) we see that $x = v_i$ and $y = u_i$ and we know that $\{x, y\} \in T$. Case (ii) is treated in a similar to case (i). In case of (iii) the alternating path between $f(x) = u_i$ and $f(y) = u_j$ in T is none other than $[u_i, v_i, v_j, u_j]$. This means that the vertices v_i and v_j are adjacent in T. Thus $\{x, y\} = \{v_i, v_j\} \in T$ and we see that $\{f(x), f(y)\} \in T^{-1}$ implying that $\{x, y\} \in T$.

(iv) ⇒(i). Suppose that T^{-1} is isomorphic to T so that $|\mathcal{P}_T| = |E(T^{-1})| = |E(T)| = 2n - 1$ and the proof is complete. ■

Remark 6.3.11 *Many of the results presented in this section has been independently proved by Akabari and Kirkland [1], though the proof techniques used by them are different from our techniques. The author became aware of this from the reports of the thesis examiners.*

6.4 Trees with property(R)

In section 6.2 it has been shown that a tree has property (SR) if and only if it is a corona tree. In this section, we shall further add to this equivalence by showing that a tree has property (SR) if and only if it has property (R). We approach this task by examining the behavior of the spectral radius of certain graphs under certain graph operations.

For the purpose of developing the results of this section we need the following notions from Xu [69]. First, let G be a graph, then an *edge-cut* is the set of all edges having one endvertex in some proper vertex subset S and another endvertex in $V(G) \setminus S$. A *cut-edge* (or *bridge*) is an edge-cut consisting of a single edge. We use these notions in the following definition:

Definition 6.4.1 Let G be a graph on n vertices. Let $e = \{u, v\}$ be a cut-edge of G such that each of the vertices u and v has degree at least two. Denote by $G[u, v]$ the graph obtained from G by deleting the edge e, identifying the vertices u and v (suppose that the new vertex is still denoted by u), and then attaching a new pendant vertex v at u. See in picture below.

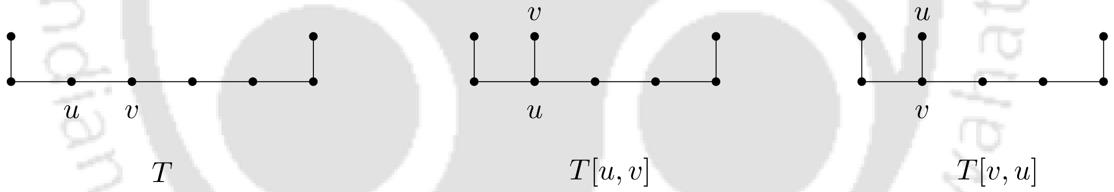


Figure 6.5: T, $T[u, v]$ and $T[v, u]$.

The behavior of the spectral radius under the above operation has been studied by Xu in [69], (see also [37, Lemma 2.7]) where the following result can be found:

Lemma 6.4.1 *Let* $\{u, v\}$ *be an edge of a tree* T *such that each of the vertices* u *and* v *has degree at least two. Then* $\rho(T[u, v]) > \rho(T)$.

The following result is important to the development of the results in this section:

Lemma 6.4.2 *Let* T *be a nonsingular tree on* n *vertices with a perfect matching* $\mathcal{M}$ *and suppose that* $\{1, 2\}$ *is a matching edge in* T *such that both the vertices* 1 *and* 2 *have a degree of at least* 2. *If* $\rho(T[1, 2]^{-1}) > 2$ *then* $\tau(T[1, 2]) > \tau(T)$.

Proof: Consider the tree T. Let

$$F_1 = \{x \in V(T) : \text{the path joining } x \text{ and } 1 \text{ does not contain } 2\}$$

and

$$F_2 = \{z \in V(T) : \text{the path joining } z \text{ and } 2 \text{ does not contain } 1\}.$$

Note that F_1 and F_2 are disjoint and $F_1 \cup F_2 \cup \{1,2\} = V(T)$. Note also that if $x \in F_1$ and $z \in F_2$, then $P(x,1) \notin \mathcal{P}_T$ and $P(z,2) \notin \mathcal{P}_T$.

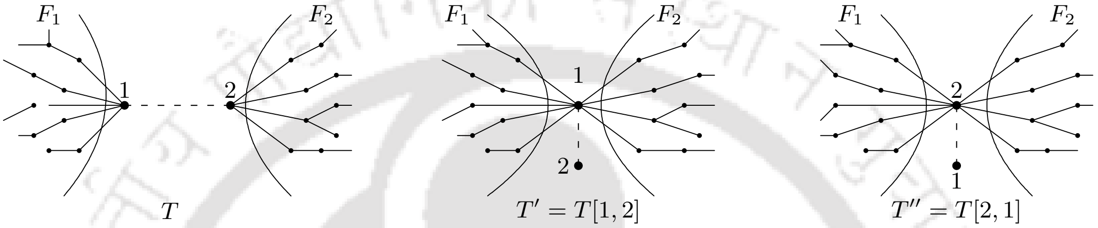


Figure 6.6:

Let $i_1, i_2, \ldots, i_k$ be the vertices of T such that $i_t \in F_1$ and $P(i_t, 2) \in \mathcal{P}_T$, for $t = 1, 2, \ldots, k$. Let $j_1, j_2, \ldots, j_p$ be the vertices of T such that $j_l \in F_2$ and $P(j_l, 1) \in \mathcal{P}_T$, for $l = 1, 2, \ldots, p$. Then $P(i_t, j_l) \in \mathcal{P}_T$, for each $t = 1, 2, \ldots, k$, and for each $l = 1, 2, \ldots, p$. Thus $\{j_l, 1\}, \{i_t, 2\}, \{j_l, i_t\} \in T^{-1}$, for each $l = 1, 2, \ldots, p$, and each $t = 1, 2, \ldots, k$.

Let $T' = T[1,2]$ and $T'' = T[2,1]$. Consider the tree T'. Observe that $P(i_t, 2), P(j_l, 2) \in \mathcal{P}_{T'}$, for $t = 1, 2, \ldots, k$, and for $l = 1, 2, \ldots, p$. But $P(j_l, 1)$, $P(i_t, 1)$, $P(j_l, i_t) \notin \mathcal{P}_{T'}$, for $l = 1, 2, \ldots, p$, and for $t = 1, 2, \ldots, k$. Thus $\{j_l, 2\}$, $\{i_t, 2\} \in T'^{-1}$ and $\{j_l, 1\}, \{i_t, 1\}, \{j_l, i_t\} \notin T'^{-1}$, for $l = 1, 2, \ldots, p$ and for $t = 1, 2, \ldots, k$.

Similarly $\{j_l, 1\}, \{i_t, 1\} \in T''^{-1}$, and $\{j_l, 2\}, \{i_t, 2\}, \{j_l, i_t\} \notin T''^{-1}$, for $l = 1, 2, \ldots, p$, and for $t = 1, 2, \ldots, k$.

We have thus obtained that:

$$A(T'^{-1}) - A(T^{-1}) = \begin{array}{c} 1 \\ 2 \\ i_t \\ j_l \\ \\ \\ \end{array} \left[\begin{array}{cc|c|c|c|c} 0 & 0 & \mathbf{0}^t & -\mathbb{1}^t & \mathbf{0}^t & \mathbf{0}^t \\ 0 & 0 & \mathbf{0}^t & \mathbb{1}^t & \mathbf{0}^t & \mathbf{0}^t \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & -J & \mathbf{0} & \mathbf{0} \\ \hline -\mathbb{1} & \mathbb{1} & -J & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

and

$$A(T''^{-1}) - A(T^{-1}) = \begin{array}{c|cc|c|c|c|c|} & & & & & & \\ 1 & 0 & 0 & \mathbb{1}^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{0}^t \\ 2 & 0 & 0 & -\mathbb{1}^t & \mathbf{0}^t & \mathbf{0}^t & \mathbf{0}^t \\ \hline i_t & \mathbb{1} & -\mathbb{1} & \mathbf{0} & -J & \mathbf{0} & \mathbf{0} \\ \hline j_l & \mathbf{0} & \mathbf{0} & -J & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} .$$

Let y' and y'' be the unit Perron vectors of $A(T'^{-1})$ and $A(T''^{-1})$, respectively. Note that T'^{-1} and T''^{-1} are identical (only the labels of vertices 1 and 2 are interchanged). Thus $y'(1) = y''(2), y'(2) = y''(1)$, and $y'(i) = y''(i)$, for all $i = 3, 4, \ldots, n$. Furthermore,

$$\begin{aligned} & y'^t[A(T'^{-1}) - A(T^{-1})]y' \\ & \quad = -2y'(1)\sum_{l=1}^{p} y'(j_l) + 2y'(2)\sum_{l=1}^{p} y'(j_l) - 2\sum_{t=1}^{k} y'(i_t)\sum_{l=1}^{p} y'(j_l). \end{aligned}$$

Thus

$$\rho(T'^{-1}) - y'^t A(T^{-1})y' = 2\left[y'(2) - y'(1) - \sum_{t=1}^{k} y'(i_t)\right]\sum_{l=1}^{p} y'(j_l). \tag{6.3}$$

Similarly

$$\begin{aligned} \rho(T''^{-1}) - y''^t A(T^{-1})]y'' &= 2\left[y''(1) - y''(2) - \sum_{l=1}^{p} y''(j_l)\right]\sum_{t=1}^{k} y''(i_t) \\ &= 2\left[y'(2) - y'(1) - \sum_{l=1}^{p} y'(j_l)\right]\sum_{t=1}^{k} y'(i_t). \end{aligned} \tag{6.4}$$

Assume now that $\rho' = \rho(T'^{-1}) \geq 2$. Using eigenvalue–eigenvector relation for $\rho' = \rho(T'^{-1})$ we have that

$$y'(1) + \sum_{t=1}^{k} y'(i_t) + \sum_{l=1}^{p} y'(j_l) = \rho' y'(2) \geq 2y'(2).$$

But as y' is entrywise positive we must have that

$$\text{either} \quad y'(1) + \sum_{t=1}^{k} y'(i_t) > y'(2) \quad \text{or} \quad y'(1) + \sum_{l=1}^{p} y'(j_l) > y'(2).$$

On using equations (6.3) and (6.4), we get that

$$\text{either} \quad \rho(T'^{-1}) < y'^t A(T^{-1})y' \quad \text{or} \quad \rho(T''^{-1}) < y''^t A(T^{-1})]y''.$$

Noting $\rho(T'^{-1}) = \rho(T''^{-1})$ and using the properties of the spectral radius we have that

$$\rho(T^{-1}) \;>\; \rho(T'^{-1}) \Rightarrow \tau(T) \;<\; \tau(T').$$

■

As a consequence we get the following useful result.

Lemma 6.4.3 *Suppose that G_1 is a connected graph with $n \geq 4$ vertices and $G = G_1 \circ K_1$. Then $\rho(G) > 2$.*

Proof: By Corollary 5.2.2,

$$\rho(G) \;=\; \frac{\rho(G_1) + \sqrt{\rho(G_1)^2 + 4}}{2}.$$

But $\rho(G_1) \geq \rho(T_{G_1})$, where T_{G_1} is a spanning tree of G_1, as the adjacency matrix of G_1 is irreducible and dominates the adjacency matrix of T_{G_1}. Also since T_{G_1} contains a subtree on 4 vertices T_4, by Cauchy interlacing theorem $\rho(T_{G_1}) \geq \rho(T_4)$, where T_4 is any tree on 4 vertices. Now it is known that there are only two trees on 4 vertices and for both $\rho(T_4) \geq \frac{1+\sqrt{5}}{2}$. Thus

$$\rho(G) \;\geq\; \frac{\frac{1+\sqrt{5}}{2} + \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2 + 4}}{2} \;>\; \frac{\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 + 4}}{2} \;=\; 2.$$

■

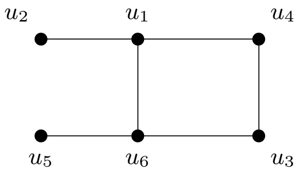
To prove the main result of this section we require one more technical lemma.

Lemma 6.4.4 *Let T be a nonsingular tree on $n \geq 8$ vertices with a perfect matching $\mathcal{M}$. Suppose further that T is not a corona tree. Then T has two vertices i and j, each of degree at least two, such that $\{i, j\} \in \mathcal{M}$ and $\rho(T[i, j]^{-1}) \geq 2$.*

Proof: Let T be a tree as in the statement. Since T is not a corona tree, it has a matching edge, say $\{3, 4\}$, whose endvertices are not pendant vertices. Thus it has a path of length 5, say, $[1, 2, 3, 4, 5, 6]$ such that the edges $\{1, 2\}, \{3, 4\}, \{5, 6\} \in \mathcal{M}$. Note that $T' = T[3, 4]$ is either a corona tree or not.

Case I. T' is a corona tree. It follows by Lemma 6.4.3, that $\rho(T') > 2$. But as T' is a corona tree, T' is isomorphic to T'^{-1} and so, in this case, we are done.

Case II. T' is not a corona tree. In this case, as we already know that T' is invertible and we conclude that T' has an alternating path of length 5, say $[u_1, u_2, u_3, u_4, u_5, u_6]$. In this case T'^{-1} must have the graph below as an induced subgraph:



Thus $A(T'^{-1})$ has $A(C_4)$ as a principal submatrix, where C_4 denotes the cycle on 4 vertices. By the Cauchy interlacing theorem, it follows that $\rho(T'^{-1}) > \rho(C_4) = 2$. ■

We are now ready to prove our main result of this section.

Theorem 6.4.5 *Let T be a nonsingular tree on $n = 2m$ vertices. Then the following are equivalent.*

(i) *T has property (SR).*

(ii) *T has property (R).*

(iii) *T is a corona tree.*

(iv) *$|\mathcal{P}_T| = 2m - 1$.*

(v) *T^{-1} is a tree.*

(vi) *T^{-1} is isomorphic to T.*

Proof: The equivalence of (i) and (iii) has been proved in section 6.2. Thus, in view of Theorem 6.3.10, it suffices to prove the equivalence of (i) and (ii).

That (i) implies (ii) is immediate by definition. Suppose then that (ii) holds, but (iii) does not. As T is invertible and not a corona, T has an alternating path of length 5. Note that if T is a path, so that $A(T)$ is a tridiagonal matrix with zeros on the main diagonal and 1's in the sub– and super–diagonal. It is well known, see for example Gregory and Karney [32, p.10], that the eigenvalues of T are given by $2\cos(\frac{k\pi}{2m+1})$, $k = 1, \ldots, 2m$, and so $\rho(T) < 2$ and $\tau(T) < \frac{1}{2}$. It follows that T cannot have property (R). Thus T has a vertex of degree at least 3 showing that T must have at least 8 vertices. On applying Lemma 6.4.4, we obtain that there is a matching edge $\{u, v\}$ in T such that $T_1 = T[u, v]$ and such that $\rho(T_1^{-1}) \geq 2$. But then, from Lemma 6.4.2, we see that $\tau(T_1) \geq \tau(T)$. Lemma 6.4.1 now yields that $\rho(T_1) > \rho(T)$.

Continuing, if T_1 is not a corona tree, we can repeat the above process to get T_2 and so on. Since at each such reduction the new tree has at least one more pendant vertex, the process must stop when we can not obtain any further matching edge with endvertices having degrees of at least 2, that is, when the tree thus obtained, say T_k, is the corona tree. Then

$$\tau(T_k) \;\geq\; \tau(T) \;\text{ and }\; \rho(T_k) > \;\rho(T). \tag{6.5}$$

It has been proved that if G is a bipartite graph, then $G \circ K_1$ has property (R). Now T_k being a corona tree, has property (R). Thus from the left inequality in 6.5 we have that

$$\rho(T) \;=\; \frac{1}{\tau(T)} \;\leq\; \frac{1}{\tau(T_k)} \;=\; \rho(T_k),$$

a contradiction to the right inequality in 6.5. ∎

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