

# APPELL SERIES OVER FINITE FIELDS AND GAUSSIAN HYPERGEOMETRIC SERIES

MOHIT TRIPATHI



DEPARTMENT OF MATHEMATICS  
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI  
GUWAHATI - 781039, INDIA

JANUARY 2021



# Appell series over finite fields and Gaussian hypergeometric series

by

Mohit Tripathi

Roll No. 156123023

Department of Mathematics

*submitted in fulfillment of the requirements  
of the degree of Doctor of Philosophy*

*to the*



Indian Institute of Technology Guwahati  
Guwahati - 781039, India

January 2021





*This work is dedicated*

*to*

*My Parents*

*for*

*encouraging me to chase my dreams!*



## Certificate

This is to certify that the thesis entitled **Appell series over finite fields and Gaussian hypergeometric series** submitted by **Mr. Mohit Tripathi** to the **Indian Institute of Technology Guwahati**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Date: January 20, 2021

Guwahati, India

**Dr. Rupam Barman**

**Associate Professor**

**Department of Mathematics**

**Indian Institute of Technology Guwahati**



# Acknowledgements

First and foremost, I would like to praise and thank God, the almighty, who has granted countless blessings, knowledge, and opportunity to me, so that I have been finally able to accomplish the thesis. Apart from the efforts of me, the success of this thesis depends largely on the encouragement and guidelines of many others. I take this opportunity to express my gratitude to the people who have been instrumental in the successful completion of this thesis.

First and foremost, I feel it a great privilege to express my deepest and the most sincere gratitude to my supervisor *Dr. Rupam Barman* for his invaluable guidance, enduring support and constant encouragement throughout this work. I am much indebted to him for spending his precious time into reading this thesis and providing the valuable suggestions and advice which helped to make this thesis worthwhile. Apart from academics, he taught me many things that helped me broaden my perspective in life. I am very grateful to have him as my advisor.

I want to convey my sincere gratitude to my doctoral committee of members *Prof. Anupam Saikia, Prof. K.V. Krishna* and *Dr. Vinay Wagh* for reviewing my research work periodically and giving valuable suggestions for the improvements of the same. I would like to express my deepest gratitude to *Prof. S. N. Bora, Dr. A. K. Chakrabarty* and *Dr. J. Swain* for there support and priceless advices and enormous help during my M.Sc and PhD.

I sincerely acknowledge Indian Institute of Technology, Guwahati for providing me various facilities necessary to carry out my research. I would also like to thank all my teachers and non-academic staffs of the department for their assistance in various ways during my research period.

I take this opportunity to express my heartfelt gratitude to my teachers *Dr. Vinay Saxena, Dr. N. K. Shukla, Shri Chaturbhuj Acharya Ji* and *Shri Kamlesh Pandey Sir* whose teachings proved to be building blocks of my life.

I would like to take the pleasure of thanking *Prof. Ken Ono* for his continuous support, inspiration, and helpful suggestions.

I wish to express my warmest gratitude to all my co-researchers at IIT Guwahati who were always there for me and helped me in various ways. I am also thankful to my friends Ashutosh, Prabhat and Sujeet for all their support.

Finally; I thank my family; my mother *Smt. Saroj Tripathi* and my father *Dr. Laxmi Kant Tripathi*, my sisters *Vibha* and *Pragya (Dullu)* for being most important part of this journey of my life, for being a constant source of moral support and joy which motivated me to keep going at difficult times, for always believing in me, encouraging me and for everything. I am also grateful to my grandfather *Late Shri Ram Achayaber Tripathi* for his support and inspiration during this journey.

At last, Immeasurable appreciations and deepest gratitude to all those people who contributed in many ways to the successful completion of my PhD and make it an unforgettable experience for me.

Date: January 20, 2021

Guwahati, India

Mohit Tripathi

# Abstract

In this thesis we study classical hypergeometric series and Appell series over finite fields, and find finite field analogues of several product and summation formulas satisfied by the classical hypergeometric series. Hypergeometric functions over finite field are known as *Gaussian hypergeometric series*. As an application of the product and summation formulas, we deduce several special values of  ${}_2F_1$ ,  ${}_3F_2$  and  ${}_4F_3$ -Gaussian hypergeometric series. Some of our special values of Gaussian hypergeometric series are evaluated at general arguments of the parameters.

Recently, finite field analogues of Appell series  $F_1$ ,  $F_2$  and  $F_3$  are introduced and their relations with certain Gaussian hypergeometric series are established. Integral representations of  $F_1$ ,  $F_2$  and  $F_3$  are used while defining their finite field analogues. However, integral representations of  $F_4$  are more complicated than the integral representations of  $F_1$ ,  $F_2$  and  $F_3$ . Therefore, it is not straightforward to find an appropriate finite field analogue of  $F_4$  using its integral representations. To overcome this problem, we define finite field analogues of classical Appell series  $F_1$ ,  $F_2$  and  $F_3$  using purely Gauss sums, and this allows us to define a finite field analogue of the Appell series  $F_4$ . We then establish finite field analogues of classical identities satisfied by the Appell series and hypergeometric series. As applications, we find several transformation formulas satisfied by the Gaussian hypergeometric series. For example, we

express a  ${}_4F_3$ -Gaussian hypergeometric series as a sum of two  ${}_2F_1$ -Gaussian hypergeometric series. We also express  ${}_4F_3$ -Gaussian hypergeometric series as a product of two  ${}_2F_1$ -Gaussian hypergeometric series.

Product formulas for Gaussian hypergeometric series have many significant applications. We find finite field analogues of certain product formulas satisfied by the classical hypergeometric series. We express product of two  ${}_2F_1$ -Gaussian hypergeometric series as  ${}_4F_3$ - and  ${}_3F_2$ -Gaussian hypergeometric series. We use properties of Gauss and Jacobi sums and our works on finite field Appell series to deduce these product formulas satisfied by the Gaussian hypergeometric series. We then use these transformations to evaluate explicitly some special values of  ${}_4F_3$ - and  ${}_3F_2$ -Gaussian hypergeometric series. By counting points on CM elliptic curves over finite fields, Ono found certain special values of  ${}_2F_1$ - and  ${}_3F_2$ -Gaussian hypergeometric series containing trivial and quadratic characters as parameters. Later, Evans and Greene found special values of certain  ${}_3F_2$ -Gaussian hypergeometric series containing arbitrary characters as parameters from where some of the values obtained by Ono follow as special cases. We show that some of the results of Evans and Greene follow from our product formulas including a finite field analogue of the classical Clausen's identity.

# Contents

<b>Certificate</b>	<b>i</b>
<b>Acknowledgements</b>	<b>iii</b>
<b>Abstract</b>	<b>v</b>
<b>Introduction</b>	<b>1</b>
<b>1 Preliminaries</b>	<b>7</b>
1.1 Classical hypergeometric and Appell series . . . . .	8
1.2 Multiplicative characters on finite fields . . . . .	9
1.3 Hypergeometric functions over finite fields . . . . .	12
1.3.1 Hypergeometric functions defined by Greene . . . . .	15
1.3.2 Hypergeometric functions defined by McCarthy . . . . .	18
1.3.3 Hypergeometric functions defined by Fuselier et. al. . . . .	21
<b>2 Finite field analogues of classical hypergeometric identities</b>	<b>23</b>
2.1 Introduction and statement of results . . . . .	23
2.2 Proof of Theorem 2.1 . . . . .	26
2.3 Proof of Theorem 2.2 . . . . .	31
2.4 Proof of Theorem 2.3 . . . . .	34

<b>3</b>	<b>Appell series <math>F_1, F_2</math> and <math>F_3</math> over finite fields</b>	<b>41</b>
3.1	Introduction . . . . .	41
3.2	Appell series $F_1^*, F_2^*$ and $F_3^*$ . . . . .	43
3.3	Main results . . . . .	44
3.4	Relationships of $F_1^*, F_2^*$ and $F_3^*$ with other variants . . . . .	51
3.5	Some applications . . . . .	56
<b>4</b>	<b>Appell series <math>F_4</math> over finite fields</b>	<b>61</b>
4.1	Introduction . . . . .	61
4.2	Main results . . . . .	62
4.3	Proof of Theorem 4.1 . . . . .	65
4.4	Proof of Theorem 4.2 . . . . .	67
4.5	Proof of Theorem 4.3 . . . . .	73
<b>5</b>	<b>Finite field analogues of identities satisfied by Appell series</b>	<b>85</b>
5.1	Introduction . . . . .	85
5.2	Statement of the main results . . . . .	86
5.3	Proof of Theorem 5.1 . . . . .	89
5.4	Proof of Theorem 5.2 . . . . .	97
5.5	Proof of Theorem 5.3 . . . . .	102
5.6	Proof of Theorem 5.4 . . . . .	106
5.7	Proof of Theorem 5.5 . . . . .	109
<b>6</b>	<b>Summation and product identities for Gaussian hypergeometric series</b>	<b>113</b>
6.1	Introduction . . . . .	113
6.2	Product formulas for Gaussian hypergeometric series . . . . .	115
6.3	Proof of Theorem 6.1 . . . . .	121
6.4	Proof of Theorem 6.2 . . . . .	125

6.5	Proof of Theorem 6.3 . . . . .	127
6.6	Proof of Theorem 6.4 . . . . .	128
6.7	Proof of Theorem 6.5 . . . . .	131
6.8	Proof of Theorem 6.6 . . . . .	133
6.9	A summation formula for Gaussian hypergeometric series . . . . .	135
<b>7</b>	<b>Special values of Gaussian hypergeometric series</b>	<b>141</b>
7.1	Introduction . . . . .	141
7.2	Special values of ${}_2F_1$ -Gaussian hypergeometric series . . . . .	143
7.3	Special values of ${}_3F_2$ -Gaussian hypergeometric series . . . . .	154
7.4	Special values of ${}_4F_3$ -Gaussian hypergeometric series . . . . .	166
	<b>Bibliography</b>	<b>171</b>
	<b>Publications</b>	<b>177</b>



# Introduction

In 1812, Gauss introduced  ${}_2F_1$ -classical hypergeometric series, see for example [27]. Hypergeometric series have great significance in many research areas such as algebraic varieties, differential equations, modular forms etc. For example, there are very powerful results expressing periods of abelian varieties such as elliptic curves, certain K3 surfaces and Calabi-Yau manifolds in terms of hypergeometric series.

In 1980s, Greene [31] introduced a finite field, character sum analogue of classical hypergeometric series that satisfies summation and transformation properties similar to those satisfied by the classical hypergeometric series. Finite field hypergeometric functions are also well-known as *Gaussian hypergeometric series*. Finite field hypergeometric functions are known to be related to various arithmetic objects. Some of the biggest motivations for studying finite field hypergeometric functions have been their connections with Fourier coefficients and eigenvalues of modular forms and with counting points on certain kinds of algebraic varieties. Their links to Fourier coefficients and eigenvalues of modular forms are established by many authors (see for example [2, 16, 22, 24, 26, 42, 49]). It is well-known that finite field hypergeometric functions can be used to count points on varieties over finite fields (see for example [10, 11, 24, 29, 28, 40, 41, 50, 53, 55]). Interestingly, several supercongruence conjectures connecting truncated classical hypergeometric sums to  $p$ -adic gamma function as well as Fourier coefficients of modular forms are proved

using finite field hypergeometric functions (see for example [1, 2, 26, 39, 46, 48, 49]). In view of such important applications, there has been growing interest in studying finite field analogues of results involving classical hypergeometric series. In this thesis we prove finite field analogues of several transformations satisfied by the classical hypergeometric series by evaluating certain character sums involving Gauss and Jacobi sums.

In recent years, there has been a study for Appell series over finite fields (see for example [33, 35, 43]). To define finite field analogues of the classical Appell series  $F_1, F_2$  and  $F_3$ , integral representations of these Appell series are used. However, integral representations of  $F_4$  are more complicated than the integral representations of  $F_1, F_2$  and  $F_3$ . Therefore, it is not straightforward to find an appropriate finite field analogue of  $F_4$  using its integral representations. Classical hypergeometric and Appell series are defined using rising factorial, and it is well-known that the rising factorials can be expressed as quotients of classical gamma function. Since the Gauss sum is a finite field analogue of the gamma function and Appell series are defined using products of rising factorials, it seems to be more appropriate to define finite field analogues of Appell series using Gauss sums. In this thesis we define four functions  $F_1^*, F_2^*, F_3^*$  and  $F_4^*$  as finite field analogues of Appell series  $F_1, F_2, F_3$  and  $F_4$ , respectively using purely Gauss sums. We establish these functions as finite field analogues of Appell series by proving results over finite fields analogous to those satisfied by the classical Appell series. We find finite field analogues of several classical identities connecting  ${}_2F_1$ -,  ${}_3F_2$ - and  ${}_4F_3$ -classical hypergeometric series to classical Appell series.

Greene [31] found several transformation formulas satisfied by the Gaussian hypergeometric series analogous to those satisfied by the classical hypergeometric series. Since then many mathematicians have obtained finite field analogues of transformation and summation identities satisfied by the classical hypergeometric series

(see for example [17, 18, 20, 21, 25, 45]). In [18, 19], Evans and Greene expressed  ${}_3F_2$ -hypergeometric series as a product of  ${}_2F_1$ -hypergeometric series over finite fields from where they deduced certain special values of  ${}_3F_2$ -hypergeometric series including a finite field analogue of the Clausen's identity. Recently, assuming the conjecture of van Geemen and van Straten, McCarthy and Papanicolas [47] related the eigenvalue of the Hecke operator of index  $p$  of a Siegel eigenform of degree 2 and level 8 to a  ${}_4F_3$ -Gaussian hypergeometric series. An identity expressing a  ${}_4F_3$ -Gaussian hypergeometric series as a product of  ${}_2F_1$ - and  ${}_3F_2$ -Gaussian hypergeometric series played a crucial role in their proof. In view of such important applications, in this thesis we have found finite field analogues of several product formulas satisfied by the classical hypergeometric series. We also show that many interesting results proved by Evans, Greene, and Ono follow from our product formulas including a finite field analogue of the Clausen's classical identity. We further express a  ${}_4F_3$ -Gaussian hypergeometric series as a sum of two  ${}_2F_1$ -Gaussian hypergeometric series using our works on finite field Appell series.

Finding special values of Gaussian hypergeometric series is an important and interesting problem. Special values of Gaussian hypergeometric series play an important role in solving many old conjectures and supercongruences. Many special values of  ${}_2F_1$ - and  ${}_3F_2$ -Gaussian hypergeometric series are obtained by using different techniques (see for example [2, 9, 10, 19, 31, 32, 37, 50, 52]). Finding values of Gaussian hypergeometric series containing arbitrary characters at specific values of the argument is a difficult problem. In this thesis, we have used our new transformation formulas to find several special values of Gaussian hypergeometric series. We have used our product formulas to find special values of  ${}_4F_3$ - and  ${}_3F_2$ -hypergeometric series at general values of the argument.

### Organization of the Thesis

We present the entire work of this thesis in seven chapters as described below.

- Chapter 1: Preliminaries

- Chapter 2: Finite field analogues of classical hypergeometric identities
- Chapter 3: Appell series  $F_1$ ,  $F_2$  and  $F_3$  over finite fields
- Chapter 4: Appell series  $F_4$  over finite fields
- Chapter 5: Finite field analogues of identities satisfied by Appell series
- Chapter 6: Summation and product identities for Gaussian hypergeometric series
- Chapter 7: Special values of Gaussian hypergeometric series

In Chapter 1 we introduce multiplicative characters on finite fields and define some important character sums, namely Gauss and Jacobi sums. We then introduce hypergeometric functions over finite fields, which are also famously known as *Gaussian hypergeometric series*. We also recall some basic results on character sums.

In Chapter 2 we prove finite field analogues of certain transformations satisfied by the classical hypergeometric series. Using properties of Gauss and Jacobi sums we evaluate certain character sums to establish these transformations. We then use these transformations to evaluate explicitly several special values of  ${}_2F_1$ -hypergeometric series over finite fields in Chapter 7. One of the finite field analogues of algebraic hypergeometric identities given by Fuselier, Long, Ramakrishna, Swisher, and Tu [25] also follows from one of our transformation formulas.

In Chapter 3 we define three functions  $F_1^*$ ,  $F_2^*$  and  $F_3^*$  as finite field analogues of Appell series  $F_1$ ,  $F_2$  and  $F_3$ , respectively using purely Gauss sums. We establish relations among  $F_1^*$ ,  $F_2^*$  and  $F_3^*$  analogous to those satisfied by the classical Appell series. Recently, several people have defined finite field analogues of Appell series using integral representations of Appell series [35, 33, 43]. We show that our functions  $F_1^*$ ,  $F_2^*$  and  $F_3^*$  are closely related to those functions.

Integral representations of the classical Appell series  $F_4$  are more complicated

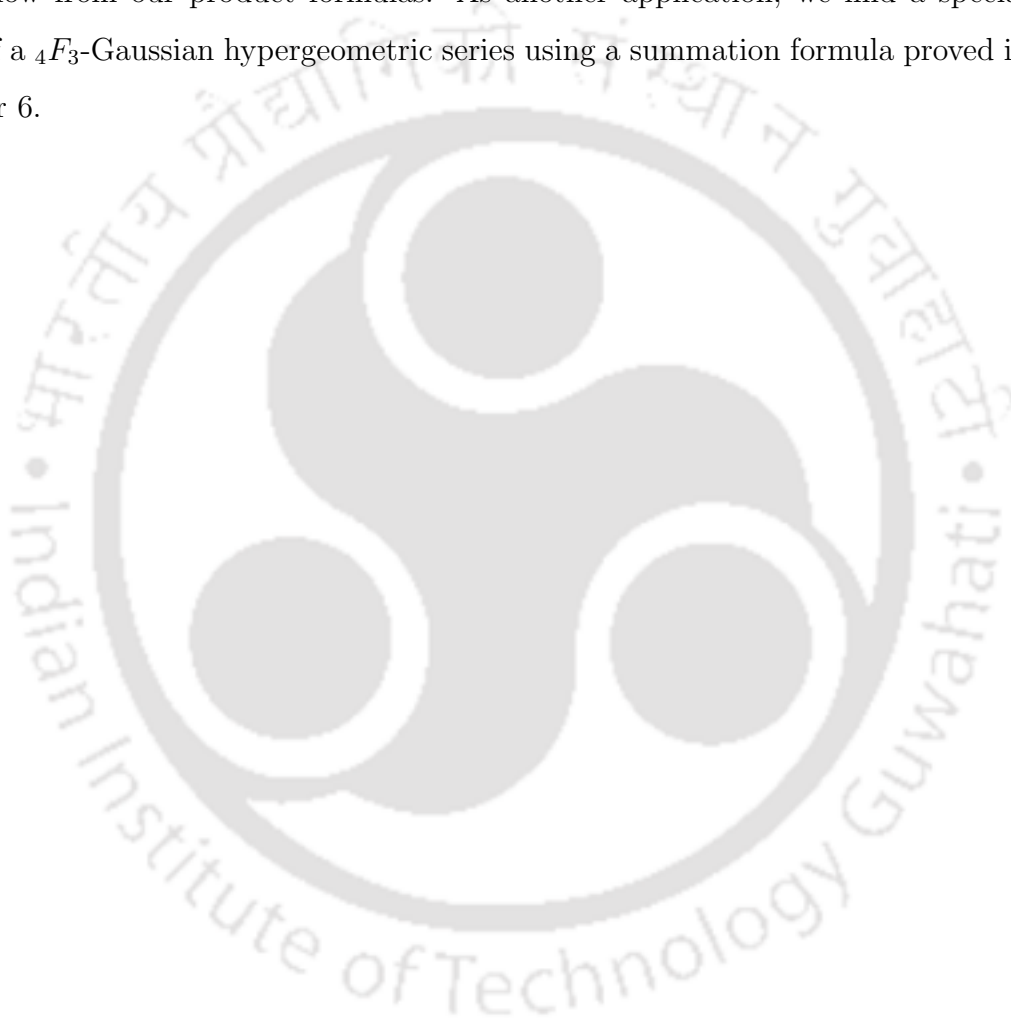
than the integral representations of  $F_1, F_2$  and  $F_3$ . Therefore, it is not straightforward to find an appropriate finite field analogue of  $F_4$  using integral representations of  $F_4$ . In Chapter 4 we define a function  $F_4^*$  as a finite field analogue of  $F_4$  using Gauss sums. We establish identities for  $F_4^*$  analogous to those satisfied by the classical Appell series  $F_4$ .

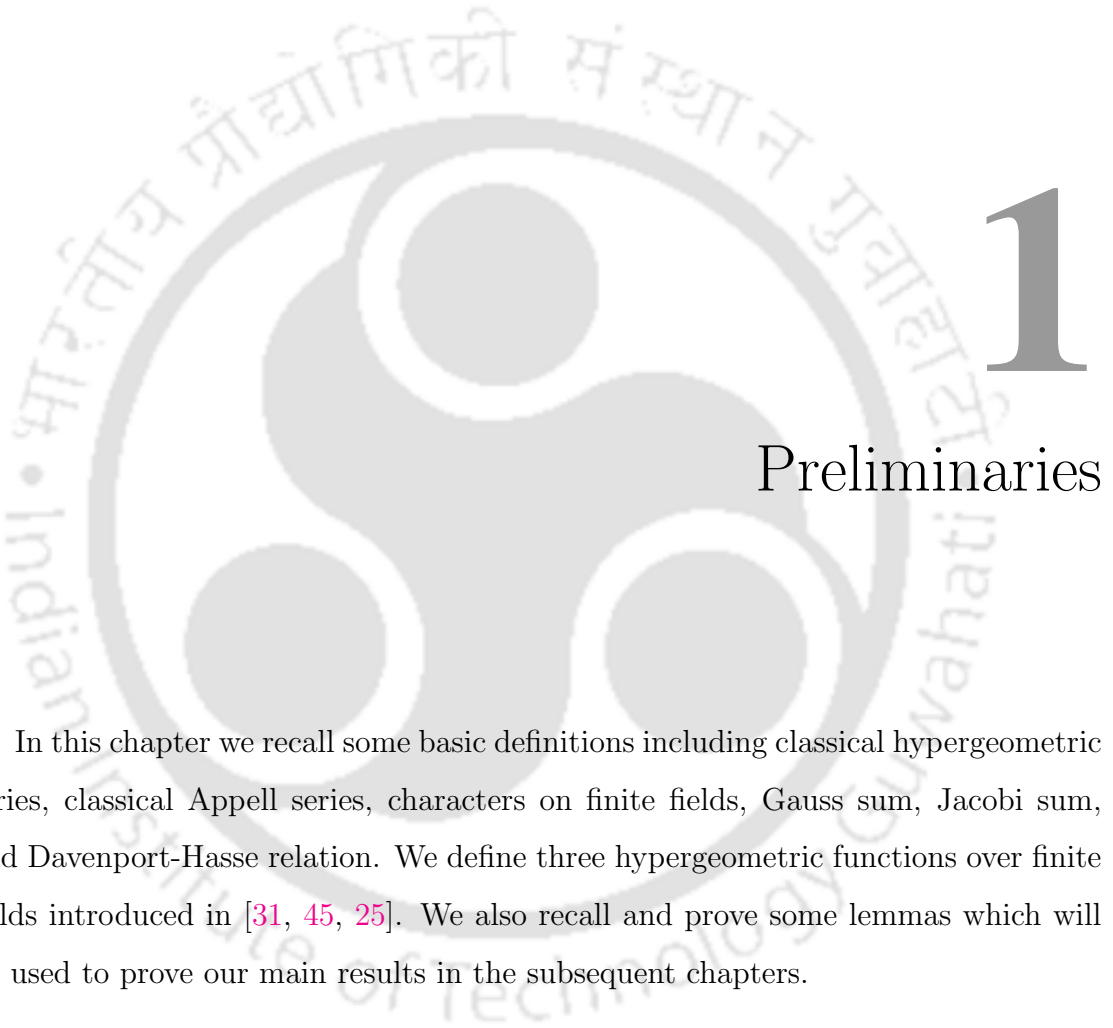
In Chapter 5 we find finite field analogues of certain identities satisfied by the classical  ${}_4F_3$ -hypergeometric series and Appell series. These identities have many interesting applications which are discussed in Chapter 6 and Chapter 7.

In Chapter 6 we find finite field analogues of certain product formulas satisfied by the classical hypergeometric series. We express product of two  ${}_2F_1$ -Gaussian hypergeometric series as  ${}_4F_3$ - and  ${}_3F_2$ -Gaussian hypergeometric series. We use properties of Gauss and Jacobi sums and our works on finite field Appell series to deduce these product formulas satisfied by the Gaussian hypergeometric series. As an application of the main results of Chapter 5, we find new summation and product formulas satisfied by the Gaussian hypergeometric series. For example, we express a  ${}_4F_3$ -Gaussian hypergeometric series as a sum of two  ${}_2F_1$ -Gaussian hypergeometric series. We find two identities expressing  ${}_4F_3$ -Gaussian hypergeometric series as a product of two  ${}_2F_1$ -Gaussian hypergeometric series. We also show that a finite field analogue of the classical Clausen's identity proved by Evans and Greene [18] follows from our product formulas.

In Chapter 7 we find several special values of Gaussian hypergeometric series. We use the transformations proved in Chapter 2 to evaluate explicitly several special values of  ${}_2F_1$ - hypergeometric series over finite fields. Certain special values of  ${}_2F_1$ - hypergeometric series over finite fields containing trivial and quadratic characters obtained by Ono follow from our special values of  ${}_2F_1$ - hypergeometric series containing arbitrary characters as parameters. We use the transformations proved in Chapter 6 to evaluate explicitly some special values of  ${}_4F_3$ - and  ${}_3F_2$ -Gaussian hypergeometric series. By counting points on CM elliptic curves over finite fields,

Ono found certain special values of  ${}_2F_1$ - and  ${}_3F_2$ -Gaussian hypergeometric series containing trivial and quadratic characters as parameters. Later, Evans and Greene [19, 18] found special values of certain  ${}_3F_2$ -Gaussian hypergeometric series containing arbitrary characters as parameters from where some of the values obtained by Ono follow as special cases. We show that some of the results of Evans and Greene also follow from our product formulas. As another application, we find a special value of a  ${}_4F_3$ -Gaussian hypergeometric series using a summation formula proved in Chapter 6.





# 1

## Preliminaries

In this chapter we recall some basic definitions including classical hypergeometric series, classical Appell series, characters on finite fields, Gauss sum, Jacobi sum, and Davenport-Hasse relation. We define three hypergeometric functions over finite fields introduced in [31, 45, 25]. We also recall and prove some lemmas which will be used to prove our main results in the subsequent chapters.

## 1.1 Classical hypergeometric and Appell series

Classical hypergeometric series have been studied for ages. Throughout the years, mathematicians established many nice relations between classical hypergeometric series and different mathematical objects. Ramanujan studied classical hypergeometric series rigorously and found many connections of classical hypergeometric series with other mathematical objects.

For a complex number  $a$  and a non-negative integer  $k$ , let  $(a)_k$  denote the rising factorial defined by

$$(a)_0 := 1 \quad \text{and} \quad (a)_k := a(a+1)(a+2)\cdots(a+k-1) \quad \text{for} \quad k > 0.$$

If  $\Gamma(x)$  denotes the Gamma function, then we have  $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ . For a non-negative integer  $n$ , and  $a_i, b_i \in \mathbb{C}$  with  $b_i \notin \{\dots, -3, -2, -1, 0\}$ , the (generalized) classical hypergeometric series  ${}_{n+1}F_n$  is defined by

$${}_{n+1}F_n \left( \begin{matrix} a_1, & a_2, & \dots, & a_{n+1} \\ & b_1, & \dots, & b_n \end{matrix} \middle| x \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{n+1})_k}{(b_1)_k \cdots (b_n)_k} \cdot \frac{x^k}{k!}, \quad (1.1)$$

which converges absolutely for  $|x| < 1$ . More details on hypergeometric series can be found in the books by Andrews, Askey and Roy [3], Bailey [6] and Slater [54].

If we consider the product of two  ${}_2F_1$ -hypergeometric series, we obtain a double series. Among them Appell's hypergeometric series of two variables are the most important ones, namely [6]

$$F_1(a; b, b'; c; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m+n} (b)_m (b')_n}{m! n! (c)_{m+n}} x^m y^n, \quad |x| < 1, |y| < 1; \quad (1.2)$$

$$F_2(a; b, b'; c, c'; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m+n} (b)_m (b')_n}{m! n! (c)_m (c')_n} x^m y^n, \quad |x| + |y| < 1; \quad (1.3)$$

$$F_3(a, a'; b, b'; c; x, y) = \sum_{m, n \geq 0} \frac{(a)_m (a')_n (b)_m (b')_n}{m! n! (c)_{m+n}} x^m y^n, \quad |x| < 1, |y| < 1; \quad (1.4)$$

$$F_4(a; b; c, c'; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m+n} (b)_{m+n}}{m! n! (c)_m (c')_n} x^m y^n, |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1. \quad (1.5)$$

There are many beautiful identities connecting Appell series to classical hypergeometric series. We will study finite field analogues of some of these identities in the subsequent chapters.

## 1.2 Multiplicative characters on finite fields

Let  $p$  be an odd prime and  $\mathbb{F}_q$  be the field with  $q = p^r$  elements. A *multiplicative character* on  $\mathbb{F}_q$  is a group homomorphism  $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$ . The set  $\widehat{\mathbb{F}_q^\times}$  consisting of all the multiplicative characters on  $\mathbb{F}_q^\times$  is a cyclic group of order  $q - 1$  under the product:  $(\chi\psi)(x) = \chi(x)\psi(x)$ . We denote by  $\bar{\chi}$  the inverse of  $\chi$ . Clearly  $\chi(x)$  is a  $(q - 1)$ -th root of unity for every  $x \neq 0$  and  $\bar{\chi}(x) = \overline{\chi(x)}$ . Here  $\overline{\chi(x)}$  is the conjugate of the complex number  $\chi(x)$ . We extend each character  $\chi \in \widehat{\mathbb{F}_q^\times}$  to all of  $\mathbb{F}_q$  by setting  $\chi(0) := 0$  including the trivial character  $\varepsilon$ . We also denote the quadratic character by  $\varphi$ .

The following lemma gives *orthogonality relations* for multiplicative characters.

**Lemma 1.1.** ([44, Chapter 5]). *We have*

$$(1) \sum_{x \in \mathbb{F}_q} \chi(x) = \begin{cases} q - 1 & \text{if } \chi = \varepsilon; \\ 0 & \text{if } \chi \neq \varepsilon. \end{cases}$$

$$(2) \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi(x) = \begin{cases} q - 1 & \text{if } x = 1; \\ 0 & \text{if } x \neq 1. \end{cases}$$

We next recall some properties of Gauss and Jacobi sums. For further details, see for example [13, 36, 44]. Let  $\zeta_p$  be a fixed primitive  $p$ -th root of unity in  $\mathbb{C}$ . The trace map  $\text{tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$  is given by

$$\text{tr}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{r-1}}.$$

Then the additive character  $\theta : \mathbb{F}_q \rightarrow \mathbb{C}$  is defined by

$$\theta(\alpha) = \zeta_p^{\text{tr}(\alpha)}.$$

It is easy to see that  $\theta(a + b) = \theta(a)\theta(b)$  and

$$\sum_{x \in \mathbb{F}_q} \theta(x) = 0. \quad (1.6)$$

**Definition 1.1** (Gauss sum). For  $\chi \in \widehat{\mathbb{F}_q^\times}$ , the Gauss sum is defined by

$$g(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x)\theta(x).$$

If  $\zeta_{q-1}$  is a primitive  $(q - 1)$ -th root of unity in  $\mathbb{C}$ , then  $g(\chi)$  lies in  $\mathbb{Q}(\zeta_p, \zeta_{q-1})$ . Using (1.6) it is easy to show that  $g(\varepsilon) = -1$ .

Let  $\delta$  denote the function on multiplicative characters defined by

$$\delta(A) = \begin{cases} 1, & \text{if } A \text{ is the trivial character;} \\ 0, & \text{otherwise.} \end{cases}$$

We also denote by  $\delta$  the function defined on  $\mathbb{F}_q$  by

$$\delta(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \neq 0. \end{cases}$$

We now state two important properties of Gauss sums.

**Lemma 1.2.** ([31, (1.12)]) For  $\chi \in \widehat{\mathbb{F}_q^\times}$  we have

$$g(\chi)g(\bar{\chi}) = q\chi(-1) - (q - 1)\delta(\chi).$$

**Lemma 1.3.** ([25, (17)]) For  $\chi \in \widehat{\mathbb{F}_q^\times}$  we have

$$\frac{1}{g(\overline{\chi})} = \frac{\chi(-1)g(\chi)}{q} - \frac{(q-1)}{q}\delta(\chi).$$

Another important product formula for Gauss sums is the Hasse-Davenport formula.

**Theorem 1.4.** [13, Davenport-Hasse Relation, Theorem 11.3.5] Let  $\chi$  be a character of order  $m$  on  $\mathbb{F}_q^\times$ , for some positive integer  $m$ . For character  $A$  on  $\mathbb{F}_q^\times$  we have

$$\prod_{i=0}^{m-1} g(\chi^i A) = -g(A^m)\overline{A^m}(m) \prod_{i=0}^{m-1} g(\chi^i) = g(A^m)\overline{A^m}(m) \prod_{i=1}^{m-1} g(\chi^i).$$

We use Davenport-Hasse relation for  $m = 2, 3, 4$ . When  $m = 2$ , we have the following identity.

**Lemma 1.5.** For  $A \in \widehat{\mathbb{F}_q^\times}$ , we have

$$g(A)g(\varphi A) = g(A^2)g(\varphi)\overline{A}(4).$$

For  $m = 3$ , we have the following lemma.

**Lemma 1.6.** Let  $\chi_3$  be character of order 3. Then for  $A \in \widehat{\mathbb{F}_q^\times}$ , we have

$$g(A)g(\chi_3 A)g(\chi_3^2 A) = g(A^3)g(\chi_3)g(\chi_3^2)\overline{A}(27).$$

For  $m = 4$ , we have the following lemma.

**Lemma 1.7.** Let  $\chi_4$  be character of order 4. Then for  $A \in \widehat{\mathbb{F}_q^\times}$ , we have

$$g(A)g(\chi_4 A)g(\varphi A)g(\chi_4^3 A) = g(A^4)g(\chi_4)g(\varphi)g(\chi_4^3)\overline{A}(256).$$

**Lemma 1.8.** ([31, (1.8)], [45, Thm 2.2]). For  $A, B, C, D \in \widehat{\mathbb{F}_q^\times}$  we have

$$\begin{aligned} & \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} g(A\chi)g(B\chi)g(C\bar{\chi})g(D\bar{\chi}) \\ &= \frac{g(AC)g(AD)g(BC)g(BD)}{g(ABCD)} + q(q-1)AB(-1)\delta(ABCD). \end{aligned}$$

We next define Jacobi sum.

**Definition 1.2** (Jacobi sum). For multiplicative characters  $A$  and  $B$  on  $\mathbb{F}_q$ , the Jacobi sum is defined by

$$J(A, B) := \sum_{x \in \mathbb{F}_q} A(x)B(1-x).$$

The following lemma gives a relation between Gauss and Jacobi sums.

**Lemma 1.9.** [31, (1.14)] For  $A, B \in \widehat{\mathbb{F}_q^\times}$  we have

$$J(A, B) = \frac{g(A)g(B)}{g(AB)} + (q-1)B(-1)\delta(AB).$$

### 1.3 Hypergeometric functions over finite fields

In 1980s, Greene [31] introduced a finite field, character sum analogue of classical hypergeometric series that satisfies summation and transformation properties similar to those satisfied by the classical hypergeometric series. Finite field hypergeometric functions are also well-known as *Gaussian hypergeometric series*. There are other finite field analogues of the classical hypergeometric series. For example, see [38, 45, 25]. In this thesis, we will use finite field hypergeometric functions introduced in [31, 45, 25].

It is easy to see that any function  $f : \mathbb{F}_q \rightarrow \mathbb{C}$  has a unique representation

$$f(x) = f(0) \cdot \delta(x) + \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} f_\chi \cdot \chi(x), \quad (1.7)$$

where

$$f_\chi = \frac{1}{q-1} \sum_{x \in \mathbb{F}_q} f(x) \cdot \bar{\chi}(x).$$

A character sum analogue for the classical binomial theorem follows easily from (1.7) with  $f(x) = A(1+x)$ , where  $A$  is a multiplicative character on  $\mathbb{F}_q$ .

**Theorem 1.10.** [31, Theorem 2.3] *For any  $A \in \widehat{\mathbb{F}_q^\times}$  and  $x \in \mathbb{F}_q$ , we have*

$$A(1+x) = \delta(x) + \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} J(A, \bar{\chi}) \chi(-x).$$

It follows from Theorem 1.10 that the finite field analogue for the binomial coefficient is the Jacobi sum.

**Definition 1.3.** *For multiplicative characters  $A$  and  $B$  on  $\mathbb{F}_q$ , the binomial coefficient  $\binom{A}{B}$  is defined by*

$$\binom{A}{B} := \frac{B(-1)}{q} J(A, \bar{B}) = \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x) \bar{B}(1-x). \quad (1.8)$$

In terms of binomial coefficients we now state two important variants of the binomial theorem [31, (2.10) & (2.11)]. For any  $A, B \in \widehat{\mathbb{F}_q^\times}$  and  $x \in \mathbb{F}_q$  we have

$$\bar{A}(1-x) = \delta(x) + \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \chi(x) \quad (1.9)$$

and

$$\overline{B}(x)\overline{A}B(1-x) = \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{B\chi} \chi(x). \quad (1.10)$$

We now prove another variant of the binomial theorem.

**Lemma 1.11.** *Let  $A \in \widehat{\mathbb{F}_q^\times}$  and  $x \in \mathbb{F}_q$ . For  $x \neq 0, 1$  we have*

$$\overline{A}(1-x) = \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(\overline{\chi})}{g(A)} \chi(-x).$$

*Proof.* From (1.8) and (1.9) we have

$$\overline{A}(1-x) = \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \chi(x) = \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} J(A\chi, \overline{\chi}) \chi(-x).$$

Lemma 1.9 yields

$$\begin{aligned} \overline{A}(1-x) &= \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(\overline{\chi})}{g(A)} \chi(-x) + \delta(A) \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi(x) \\ &= \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(\overline{\chi})}{g(A)} \chi(-x). \end{aligned}$$

Here, we use the fact that  $\sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi(x) = 0$  when  $x \neq 1$ . ■

Binomial coefficients satisfy many interesting properties. For example, we list the following from [31]:

$$\binom{A}{B} = \binom{A}{A\overline{B}}; \quad (1.11)$$

$$\binom{A}{B} = B(-1) \binom{B\overline{A}}{B}; \quad (1.12)$$

$$\binom{A}{B} = AB(-1) \binom{\overline{B}}{\overline{A}}; \quad (1.13)$$

$$\binom{A}{\varepsilon} = \binom{A}{A} = \frac{-1}{q} + \frac{q-1}{q} \delta(A); \quad (1.14)$$

$$\binom{\varepsilon}{A} = -\frac{A(-1)}{q} + \frac{q-1}{q} \delta(A); \quad (1.15)$$

and

$$\binom{A}{B} \binom{C}{A} = \binom{C}{B} \binom{C\overline{B}}{A\overline{B}} - \frac{q-1}{q^2} B(-1) \delta(A) + \frac{q-1}{q^2} AB(-1) \delta(B\overline{C}). \quad (1.16)$$

Using Lemma 1.9, we can re-write the binomial coefficient in terms of Gauss sums as given in the following lemma.

**Lemma 1.12.** *If  $A, B \in \widehat{\mathbb{F}}_q^\times$  then we have*

$$\binom{A}{B} = \frac{B(-1)g(A)g(\overline{B})}{qg(A\overline{B})} + \frac{q-1}{q} \delta(A\overline{B}).$$

### 1.3.1 Hypergeometric functions defined by Greene

Classical hypergeometric series have nice integral representations. For example, the following is an integral representation of the  ${}_2F_1$ -hypergeometric series [6, p. 4]:

$${}_2F_1 \left( \begin{matrix} a, & b \\ & c \end{matrix} \middle| z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^b (1-t)^{c-b} (1-tx)^{-a} \frac{dt}{t(1-t)}. \quad (1.17)$$

Using the integral representation (1.17), Greene [31] defined a finite field analogue of the classical  ${}_2F_1$ -hypergeometric series as follows.

**Definition 1.4.** [31, Definition 3.5] *For  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  and  $x \in \mathbb{F}_q$ , Greene's  ${}_2F_1$ -finite field hypergeometric function is defined as*

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) = \varepsilon(x) \frac{BC(-1)}{q} \sum_{y \in \mathbb{F}_q} B(y) \overline{BC} (1-y) \overline{A} (1-xy). \quad (1.18)$$

Greene expressed the above  ${}_2F_1$ -finite field hypergeometric function in terms of binomial coefficients.

**Theorem 1.13.** [31, Theorem 3.6] *We have*

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) = \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \binom{B\chi}{C\chi} \chi(x). \quad (1.19)$$

In general, for positive integer  $n$ , Greene defined the  ${}_{n+1}F_n$ - finite field hypergeometric function over  $\mathbb{F}_q$  as given below.

**Definition 1.5.** [31, Definition 3.10] *For  $A_0, \dots, A_n, B_1, \dots, B_n \in \widehat{\mathbb{F}_q^\times}$  and  $x \in \mathbb{F}_q$ ,*

$${}_{n+1}F_n \left( \begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \middle| x \right) = \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A_0\chi}{\chi} \binom{A_1\chi}{B_1\chi} \dots \binom{A_n\chi}{B_n\chi} \chi(x). \quad (1.20)$$

We now recall four transformation formulas of Greene.

**Theorem 1.14.** [31, Theorem 4.4] *For  $A, B, C \in \widehat{\mathbb{F}_q^\times}$  and  $x \in \mathbb{F}_q$ ,*

$$\begin{aligned} (i) \quad {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) &= A(-1) \cdot {}_2F_1 \left( \begin{matrix} A, & B \\ & ABC\bar{C} \end{matrix} \middle| 1-x \right) \\ &\quad + A(-1) \binom{B}{AC} \delta(1-x) - \binom{B}{C} \delta(x), \\ (ii) \quad {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) &= C(-1)\bar{A}(1-x) \cdot {}_2F_1 \left( \begin{matrix} A, & C\bar{B} \\ & C \end{matrix} \middle| \frac{x}{x-1} \right) \\ &\quad + A(-1) \binom{B}{AC} \delta(1-x), \\ (iii) \quad {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) &= \bar{B}(1-x) \cdot {}_2F_1 \left( \begin{matrix} C\bar{A}, & B \\ & C \end{matrix} \middle| \frac{x}{x-1} \right) \\ &\quad + A(-1) \binom{B}{AC} \delta(1-x), \end{aligned}$$

$$(iv) \quad {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) = C(-1)C\overline{A}\overline{B}(1-x) \cdot {}_2F_1 \left( \begin{matrix} C\overline{A}, & C\overline{B} \\ & C \end{matrix} \middle| x \right) \\ + A(-1) \left( \frac{B}{AC} \right) \delta(1-x).$$

Using the above transformations, we have the following formula.

**Lemma 1.15.** For  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  and  $x \in \mathbb{F}_q$  such that  $x \neq 0, 1$ , we have

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) = ABC(-1)\overline{A}(x) {}_2F_1 \left( \begin{matrix} A, & A\overline{C} \\ & A\overline{B} \end{matrix} \middle| \frac{1}{x} \right).$$

*Proof.* Using Theorem 1.14 (i) and (ii) we have

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) = BC(-1)\overline{A}(x) {}_2F_1 \left( \begin{matrix} A, & A\overline{C} \\ & A\overline{B} \end{matrix} \middle| \frac{x-1}{x} \right). \quad (1.21)$$

Again using Theorem 1.14 (i) in (1.21) we complete the proof.  $\blacksquare$

We now state three special values of Gaussian hypergeometric series obtained by Greene.

**Lemma 1.16.** [31, Theorem 4.9] Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$ . Then we have

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| 1 \right) = A(-1) \left( \frac{B}{AC} \right).$$

**Lemma 1.17.** [31, Theorem 4.37] Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$ . Then we have

$${}_3F_2 \left( \begin{matrix} A, & B, & C \\ & C\overline{A}, & C\overline{B} \end{matrix} \middle| 1 \right) \\ = \frac{q-1}{q^2} BC(-1)\delta(A) - \frac{q-1}{q^2} AC(-1)\delta(B) - \frac{q-1}{q^2} A(-1)\delta(C\overline{A}\overline{B})$$

$$+ AB(-1) \begin{cases} 0, & \text{if } C \neq \square; \\ \binom{D}{A} \binom{B\bar{D}}{AB\bar{D}} + (\varphi^D)_A (\varphi^{B\bar{D}})_{\varphi AB\bar{D}}, & \text{if } C = D^2. \end{cases}$$

**Lemma 1.18.** [31, (4.26)] *Let  $A, B, C, D, E \in \widehat{\mathbb{F}}_q^\times$ . Then we have*

$${}_3F_2 \left( \begin{matrix} A, & B, & D \\ & C, & E \end{matrix} \middle| 1 \right) = AB(-1) \cdot {}_3F_2 \left( \begin{matrix} C\bar{A}, & C\bar{B}, & D \\ & C, & \overline{ABCE} \end{matrix} \middle| 1 \right).$$

### 1.3.2 Hypergeometric functions defined by McCarthy

Greene's function is essentially defined using Jacobi sums, and hence often it is necessary to impose conditions on the parameters to relate the Jacobi sums to the required product of Gauss sums. In [45], McCarthy defined another finite field analogue of hypergeometric series purely in terms of Gauss sums.

**Definition 1.6.** *For  $A_0, \dots, A_n, B_1, \dots, B_n \in \widehat{\mathbb{F}}_q^\times$ , the McCarthy's finite field hypergeometric function  ${}_{n+1}F_n^*$  is defined by*

$$\begin{aligned} & {}_{n+1}F_n \left( \begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \middle| x \right)^* \\ &= \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}}_q^\times} \prod_{i=0}^n \frac{g(A_i \chi)}{g(A_i)} \prod_{j=1}^n \frac{g(B_j \chi)}{g(B_j)} g(\bar{\chi}) \chi(-1)^{n+1} \chi(x). \end{aligned} \quad (1.22)$$

The following proposition relates the two finite field hypergeometric functions defined by Greene in [31] and McCarthy in [45] under certain conditions on the parameters.

**Proposition 1.19.** [45, Proposition 2.5] *If  $A_0 \neq \varepsilon$  and  $A_i \neq B_i$  for  $1 \leq i \leq n$ , then for  $x \in \mathbb{F}_q$  we have*

$${}_{n+1}F_n \left( \begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \middle| x \right)^*$$

$$= \left[ \prod_{i=1}^n \binom{A_i}{B_i}^{-1} \right] {}_{n+1}F_n \left( \begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \middle| x \right).$$

We will also need the following two special cases which are not included in Proposition 1.19. The proofs of these cases are easy and follow directly by using (1.8), (1.9), (1.10), Lemma 1.2, Lemma 1.9 and the fact that  $g(\varepsilon) = -1$ .

Case 1. Let  $A_0 = \varepsilon$  and  $A_1 \neq B_1$ . Then we have

$${}_2F_1 \left( \begin{matrix} \varepsilon, & A_1 \\ & B_1 \end{matrix} \middle| x \right)^* = \varepsilon(x) - \overline{B_1}(x) \binom{A_1}{B_1}^{-1} \overline{A_1} B_1 (1-x). \quad (1.23)$$

Case 2: Let  $A_0 \neq \varepsilon$ ,  $A_1 = B_1 \neq \varepsilon$ . Then for  $x \neq 0$  we have

$${}_2F_1 \left( \begin{matrix} A_0, & A_1 \\ & A_1 \end{matrix} \middle| x \right)^* = -\overline{A_1}(x) \binom{A_0 \overline{A_1}}{\overline{A_1}} + \overline{A_0}(1-x). \quad (1.24)$$

Using (1.22) and Lemma 1.8 we have the following lemma.

**Lemma 1.20.** For  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  we have

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| 1 \right)^* = \frac{g(\overline{AC})g(\overline{BC})}{g(\overline{C})g(\overline{ABC})} + \frac{q(q-1)AB(-1)}{g(A)g(B)g(\overline{C})} \delta(\overline{ABC}).$$

**Lemma 1.21.** Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B \neq \varepsilon$  and  $B \neq C$ . For  $x \in \mathbb{F}_q$  such that  $x \neq 1$  we have

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right)^* = \overline{A}(1-x) {}_2F_1 \left( \begin{matrix} A, & \overline{BC} \\ & C \end{matrix} \middle| -\frac{x}{1-x} \right)^*.$$

*Proof.* By Proposition 1.19 we have

$$\begin{aligned} {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right)^* &= \begin{pmatrix} B \\ C \end{pmatrix}^{-1} {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) \\ &= \begin{pmatrix} B \\ C \end{pmatrix}^{-1} C(-1)\bar{A}(1-x) {}_2F_1 \left( \begin{matrix} A, & \bar{B}C \\ & C \end{matrix} \middle| -\frac{x}{1-x} \right) \\ &= \bar{A}(1-x) {}_2F_1 \left( \begin{matrix} A, & \bar{B}C \\ & C \end{matrix} \middle| -\frac{x}{1-x} \right)^*. \end{aligned}$$

The above equalities follow from Theorem 1.14 (ii), (1.12) and Proposition 1.19. ■

**Lemma 1.22.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B \neq \varepsilon$  and  $B \neq C, A \neq C$ . For  $x \in \mathbb{F}_q$  such that  $x \neq 1$  we have*

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right)^* = \overline{ABC}(1-x) {}_2F_1 \left( \begin{matrix} C\bar{A}, & \bar{B}C \\ & C \end{matrix} \middle| x \right)^*.$$

*Proof.* By Proposition 1.19 we have

$$\begin{aligned} {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right)^* &= \begin{pmatrix} B \\ C \end{pmatrix}^{-1} {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) \\ &= \begin{pmatrix} B \\ C \end{pmatrix}^{-1} C(-1)\overline{ABC}(1-x) {}_2F_1 \left( \begin{matrix} C\bar{A}, & C\bar{B} \\ & C \end{matrix} \middle| x \right) \\ &= \overline{ABC}(1-x) {}_2F_1 \left( \begin{matrix} C\bar{A}, & C\bar{B} \\ & C \end{matrix} \middle| x \right)^*. \end{aligned}$$

We note that the above equalities follow from Theorem 1.14 (iv), (1.12) and Proposition 1.19. ■

**Lemma 1.23.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$ . If  $A \neq \varepsilon, B \neq C$  and  $A \neq C$ , then for  $y \neq 1$  we*

have

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| y \right)^* = \bar{B}(1-y) {}_2F_1 \left( \begin{matrix} C\bar{A}, & B \\ & C \end{matrix} \middle| \frac{y}{y-1} \right)^*.$$

*Proof.* The result readily follows from Proposition 1.19 and Theorem 1.14 (iii). ■

**Lemma 1.24.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$ . If  $A \neq \varepsilon$ ,  $B \neq C$  and  $A \neq C$ , then for  $y \neq 1, 0$  we have*

$${}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| y \right)^* = A(-1) \binom{B}{C}^{-1} \binom{B}{ABC} {}_2F_1 \left( \begin{matrix} A, & B \\ & ABC \end{matrix} \middle| 1-y \right)^*.$$

*Proof.* The result readily follows from Proposition 1.19 and Theorem 1.14 (i). ■

From [31, Corollary 3.16 (iii)], we have

$${}_2F_1 \left( \begin{matrix} A, & B \\ & A \end{matrix} \middle| x \right) = \binom{B}{A} \varepsilon(x) \bar{B}(1-x) - \frac{1}{q} \bar{A}(-x) + \frac{q-1}{q} A(-1) \delta(1-x) \delta(B).$$

Using Proposition 1.19 and the above identity, we have the following lemma.

**Lemma 1.25.** *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$  be such that  $A \neq \varepsilon$  and  $A \neq B$ . For  $x \in \mathbb{F}_q$  such that  $x \neq 1$  we have*

$${}_2F_1 \left( \begin{matrix} A, & B \\ & A \end{matrix} \middle| x \right)^* = \varepsilon(x) \bar{B}(1-x) - \frac{1}{q} \binom{B}{A}^{-1} \bar{A}(-x).$$

### 1.3.3 Hypergeometric functions defined by Fuselier et. al.

In a recent paper [25], Fuselier et al. introduce another version of hypergeometric series over finite fields in a manner that is parallel to that of the classical

hypergeometric series by considering period functions for hypergeometric type algebraic varieties over finite fields. For multiplicative characters  $A, B, C$ , their  ${}_2\mathbb{F}_1$ -hypergeometric series is given by

$${}_2\mathbb{F}_1 \left[ \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right] := \frac{1}{J(B, \overline{BC})} {}_2\mathbb{P}_1 \left[ \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right], \quad (1.25)$$

where

$${}_2\mathbb{P}_1 \left[ \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right] := \frac{q^2}{(q-1)} BC(-1) \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \binom{B\chi}{C\chi} \chi(x) + \delta(x) J(B, \overline{BC}).$$

The relationship between the above finite field hypergeometric series and the Greene's hypergeometric series is the following:

$${}_2\mathbb{F}_1 \left[ \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right] = \frac{qBC(-1)}{J(B, \overline{BC})} {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| x \right) + \delta(x). \quad (1.26)$$

We note that, since we have used the definition of the binomial coefficient given by Greene, the above definition of  ${}_2\mathbb{P}_1$  series differs from its original definition given in [25] by a factor of  $q^2$ .

# 2

## Finite field analogues of classical hypergeometric identities

### 2.1 Introduction and statement of results

After introducing hypergeometric series over finite fields, Greene [31] found several transformation formulas satisfied by the finite field hypergeometric series analogous to those satisfied by the classical hypergeometric series. Since then many mathematicians have obtained finite field analogues of transformation and summation identities satisfied by the classical hypergeometric series (see for example

---

<sup>1</sup>The contents of this chapter are under review.

[31, 45, 18, 20, 17, 21]). In [25], Fuselier et. al. gave a systematic way to obtain certain types of hypergeometric transformation and evaluation formulas over finite fields and interpret them geometrically using a Galois representation perspective. As an application, they obtained a few finite field analogues of algebraic hypergeometric identities, quadratic and higher transformation formulas, and certain evaluation formulas. For example, one of the finite field analogues of algebraic hypergeometric identities is the following [25, Theorem 8.11]: For  $A \neq \varepsilon, \varphi$  and  $x \neq 0$ ,

$${}_2F_1 \left[ \begin{matrix} A, & A\varphi \\ & \varphi \end{matrix} \middle| x \right] = \left( \frac{1 + \varphi(x)}{2} \right) \left( \overline{A^2}(1 + \sqrt{x}) + \overline{A^2}(1 - \sqrt{x}) \right) \quad (2.1)$$

The above result is a finite field analogue of the following identity [54, (1.5.19)] satisfied by the classical hypergeometric series:

$${}_2F_1 \left( \begin{matrix} a, & a + \frac{1}{2} \\ & \frac{1}{2} \end{matrix} \middle| x \right) = \frac{1}{2} \left( (1 + \sqrt{x})^{-2a} + (1 - \sqrt{x})^{-2a} \right).$$

A more general identity [15, p. 112] is the following:

$${}_2F_1 \left( \begin{matrix} a, & a + \frac{1}{2} \\ & c \end{matrix} \middle| x \right) = (1 + \sqrt{x})^{-2a} {}_2F_1 \left( \begin{matrix} 2a, & c - \frac{1}{2} \\ & 2c - 1 \end{matrix} \middle| 2\sqrt{x}(1 + \sqrt{x})^{-1} \right).$$

In the following theorem, we establish a finite field analogue of the above identity.

**Theorem 2.1.** *Let  $A, C \in \widehat{\mathbb{F}}_q^\times$  with  $A \neq \varepsilon, \varphi$ . If  $x \in \mathbb{F}_q$ ,  $x \neq 0, -1$ , then we have*

$$\begin{aligned} \overline{A^2}(1+x) {}_2F_1 \left( \begin{matrix} A^2, & C\varphi \\ & C^2 \end{matrix} \middle| \frac{2x}{1+x} \right) &= \frac{\overline{C}(-4)g(C\varphi)g(A\overline{C}\varphi)}{g(\varphi)g(A\varphi)} {}_2F_1 \left( \begin{matrix} A, & \varphi A \\ & C \end{matrix} \middle| x^2 \right) \\ &\quad - \frac{(q-1)\overline{C}(-4)g(C\varphi)g(A\overline{C}\varphi)\overline{A}(1-x^2)}{qg(\varphi)g(A\varphi)} \delta(A\overline{C}\varphi). \end{aligned}$$

We next consider the following identity [5, (6.1)] satisfied by the classical hypergeometric series:

$$\begin{aligned} & {}_3F_2 \left( \begin{matrix} 2a - 1, & a + \frac{1}{2}, & a - b - \frac{1}{2} \\ & a + b + \frac{1}{2}, & a - \frac{1}{2} \end{matrix} \middle| x \right) \\ &= (1 - x)^{2b-2a} {}_3F_2 \left( \begin{matrix} 2b - 1, & b + \frac{1}{2}, & b - a - \frac{1}{2} \\ & a + b + \frac{1}{2}, & b - \frac{1}{2} \end{matrix} \middle| x \right). \end{aligned}$$

In the following theorem we establish a finite field analogue of the above identity.

**Theorem 2.2.** *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2, B^2, \overline{AB}\varphi \neq \varepsilon$ . For  $x \in \mathbb{F}_q$  we have*

$$\begin{aligned} & \overline{A^2}B^2(1-x) {}_3F_2 \left( \begin{matrix} B^2, & \varphi B, & B\overline{A}\varphi \\ & AB\varphi, & B\varphi \end{matrix} \middle| x \right) \\ &= \frac{AB\varphi(-1)g(A^2)g(\overline{B^2})}{g^2(\overline{AB}\varphi)} \left( \frac{\overline{A^2}}{\overline{A}} \right) \left( \frac{\overline{B^2}}{\overline{B}} \right)^{-1} {}_3F_2 \left( \begin{matrix} A^2, & \varphi A, & A\overline{B}\varphi \\ & AB\varphi, & A\varphi \end{matrix} \middle| x \right) \\ & - \frac{A(-1)\overline{B}\varphi(x)\overline{A^2}B^2(1-x)}{q} \left( \frac{B\varphi}{\overline{B}\varphi} \right) + \frac{g(A^2)g(\overline{B^2})}{qg^2(\overline{AB}\varphi)} \left( \frac{\overline{A^2}}{\overline{A}} \right) \left( \frac{\overline{B^2}}{\overline{B}} \right)^{-1} \overline{A}\varphi(-x) \left( \frac{A\varphi}{\overline{A}\varphi} \right) \\ & - \left( \frac{\overline{A^2}}{\overline{A}} \right) \left( \frac{A^2}{B^2} \right) \delta(1-x). \end{aligned}$$

In the following identity [15, p. 111], sum of two classical  ${}_2F_1$ -hypergeometric series is expressed as a  ${}_2F_1$ -hypergeometric series:

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} 2a, & 2b \\ & a + b + \frac{1}{2} \end{matrix} \middle| \frac{1 + \sqrt{x}}{2} \right) + {}_2F_1 \left( \begin{matrix} 2a, & 2b \\ & a + b + \frac{1}{2} \end{matrix} \middle| \frac{1 - \sqrt{x}}{2} \right) \\ &= 2 \frac{\Gamma(\frac{1}{2})\Gamma(a + b + \frac{1}{2})}{\Gamma(a + \frac{1}{2})\Gamma(b + \frac{1}{2})} {}_2F_1 \left( \begin{matrix} a, & b \\ & \frac{1}{2} \end{matrix} \middle| x \right). \end{aligned}$$

In the following theorem we establish a finite field analogue of the above identity.

**Theorem 2.3.** *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2, A, B\varphi \neq \varepsilon$ . Then, for  $x \in \mathbb{F}_q$  with*

$x \neq 0, \pm 1$ , we have

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \mid \frac{1+x}{2} \right) + {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \mid \frac{1-x}{2} \right) \\ &= \frac{2AB(-4)\varphi(-1)g(A\bar{B}\varphi)g(A)g(B\varphi)}{qg(A^2)} {}_2F_1 \left( \begin{matrix} A, & B \\ & \varphi \end{matrix} \mid x^2 \right). \end{aligned}$$

## 2.2 Proof of Theorem 2.1

*Proof.* Let

$$L := \overline{A^2}(1+x) {}_2F_1 \left( \begin{matrix} A^2, & C\varphi \\ & C^2 \end{matrix} \mid \frac{2x}{1+x} \right).$$

Using (1.19) and (1.9) we have

$$L = \frac{q^2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{A^2\chi}{\chi} \binom{C\varphi\chi}{C^2\chi} \binom{A^2\chi\lambda}{\lambda} \chi(2x)\lambda(-x).$$

If we apply the change of variables  $\chi \mapsto \bar{\chi}\bar{\lambda}$  we have

$$L = \frac{q^2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{A^2\bar{\chi}\bar{\lambda}}{\bar{\chi}\bar{\lambda}} \binom{C\varphi\bar{\chi}\bar{\lambda}}{C^2\bar{\chi}\bar{\lambda}} \binom{A^2\bar{\chi}}{\bar{\lambda}} \bar{\chi}(2x)\bar{\lambda}(-2).$$

We apply (1.13) on  $\binom{A^2\bar{\chi}\bar{\lambda}}{\bar{\chi}\bar{\lambda}}$  and  $\binom{C\varphi\bar{\chi}\bar{\lambda}}{C^2\bar{\chi}\bar{\lambda}}$ , and then (1.12) on  $\binom{A^2\bar{\chi}}{\bar{\lambda}}$  to obtain

$$L = \frac{q^2 C\varphi(-1)}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{\overline{A^2}\chi\lambda}{\lambda} \binom{\chi\lambda}{\overline{A^2}\chi\lambda} \binom{\overline{C^2}\chi\lambda}{\overline{C}\varphi\chi\lambda} \bar{\chi}(2x)\bar{\lambda}(2).$$

Using (1.16) on  $\binom{\overline{A^2\chi\lambda}}{\lambda} \binom{\chi\lambda}{\overline{A^2\chi\lambda}}$  we have

$$\begin{aligned} L &= \frac{q^2 C \varphi(-1)}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \left[ \binom{\chi\lambda}{\lambda} \binom{\chi}{\overline{A^2\chi}} - \frac{q-1}{q^2} \lambda(-1) \delta(\overline{A^2\chi\lambda}) + \frac{q-1}{q^2} \chi(-1) \delta(\overline{\chi}) \right] \\ &\quad \times \binom{\overline{C^2\chi\lambda}}{\overline{C\varphi\chi\lambda}} \overline{\chi}(2x) \overline{\lambda}(2) \\ &= \frac{q^2 C \varphi(-1)}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{\chi\lambda}{\lambda} \binom{\chi}{\overline{A^2\chi}} \binom{\overline{C^2\chi\lambda}}{\overline{C\varphi\chi\lambda}} \overline{\chi}(2x) \overline{\lambda}(2) - I_1 + I_2, \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} I_1 &= \frac{C \varphi(-1)}{(q-1)} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{\overline{C^2\chi\lambda}}{\overline{C\varphi\chi\lambda}} \overline{\chi}(2x) \overline{\lambda}(-2) \delta(\overline{A^2\chi\lambda}), \\ I_2 &= \frac{C \varphi(-1)}{(q-1)} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{\overline{C^2\chi\lambda}}{\overline{C\varphi\chi\lambda}} \overline{\chi}(-2x) \overline{\lambda}(2) \delta(\overline{\chi}). \end{aligned}$$

The term  $I_1$  is nonzero only when  $\chi\lambda = A^2$ . Now putting  $\lambda = A^2\overline{\chi}$ , and then using Lemma 1.1 and the fact that  $x \neq -1$  we find that  $I_1 = 0$ . The term  $I_2$  is nonzero only when  $\chi = \varepsilon$ , and hence using (1.10) we obtain

$$I_2 = \frac{\varphi(-1) \overline{C}(-4)}{q}. \quad (2.3)$$

Using (1.19) and the fact that  $I_1 = 0$  in (2.2) we have

$$\begin{aligned} L &= \frac{q C \varphi(-1)}{(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} {}_2F_1 \left( \chi, \frac{\overline{C^2\chi}}{\overline{C\varphi\chi}} \mid \frac{1}{2} \right) \binom{\chi}{\overline{A^2\chi}} \overline{\chi}(2x) + I_2 \\ &= \frac{q \varphi(-1) \overline{C}(-4)}{(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} {}_2F_1 \left( \overline{C\varphi}, \frac{\overline{C^2\chi}}{\overline{C\varphi\chi}} \mid -1 \right) \binom{\chi}{\overline{A^2\chi}} \overline{\chi}(x) + I_2. \end{aligned}$$

The last equality is obtained by applying part (iii) of Theorem 1.14. Now using Greene's definition (1.18), we have

$$\begin{aligned} L &= \frac{q\varphi(-1)\overline{C}(-4)}{(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \left[ \frac{C\varphi(-1)}{q} \sum_{y \in \mathbb{F}_q} \overline{C^2}\chi(y)C\varphi(1-y^2) \right] \left( \frac{\chi}{A^2\chi} \right) \overline{\chi}(x) + I_2 \\ &= \frac{q\overline{C}(4)}{(q-1)^2} \sum_{\chi, \gamma \in \widehat{\mathbb{F}_q^\times}} \left( \frac{\chi}{A^2\chi} \right) \binom{\overline{C}\varphi\gamma}{\gamma} \overline{\chi}(x) \sum_{y \in \mathbb{F}_q} \overline{C^2}\chi\gamma^2(y) + I_2. \end{aligned}$$

The last equality is obtained by using (1.9). The inner most sum is nonzero only when  $\chi = C^2\overline{\gamma^2}$ , and hence putting  $\chi = C^2\overline{\gamma^2}$  we have

$$L = \frac{q\overline{C}(4)}{(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \left( \frac{C^2\overline{\gamma^2}}{A^2C^2\overline{\gamma^2}} \right) \binom{\overline{C}\varphi\gamma}{\gamma} \overline{C^2}\gamma^2(x) + I_2.$$

Applying (1.13) and (1.11) on  $\binom{\overline{C}\varphi\gamma}{\gamma}$  we have

$$\begin{aligned} L &= \frac{q\varphi(-1)\overline{C}(-4)}{(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \left( \frac{\overline{\gamma}}{\overline{C}\varphi} \right) \left( \frac{C^2\overline{\gamma^2}}{A^2C^2\overline{\gamma^2}} \right) \overline{C}\gamma(x^2) + I_2 \\ &= \frac{q\varphi(-1)\overline{C}(-4)}{(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \left( \frac{\gamma}{\overline{C}\varphi} \right) \left( \frac{C^2\overline{\gamma^2}}{A^2C^2\overline{\gamma^2}} \right) \overline{C}\gamma(x^2) + I_2. \end{aligned}$$

Now using Lemma 1.12 and the fact that  $A^2 \neq \varepsilon$ , we have

$$\begin{aligned} L &= \frac{q\varphi(-1)\overline{C}(-4)}{(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \left[ \left( \frac{C\varphi(-1)g(\gamma)g(C\varphi)}{qg(C\varphi\gamma)} + \frac{q-1}{q}\delta(C\varphi\gamma) \right) \right. \\ &\quad \left. \times \frac{g(A^2\overline{C^2}\overline{\gamma^2})g(C^2\overline{\gamma^2})}{qg(A^2)} \right] \overline{C}\gamma(x^2) + I_2 \\ &= \frac{\overline{C}(4)}{q(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \frac{g(\gamma)g(C\varphi)g(A^2\overline{C^2}\overline{\gamma^2})g(C^2\overline{\gamma^2})\overline{C}\gamma(x^2)}{g(C\varphi\gamma)g(A^2)} + I_2 + I_3, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} I_3 &= \frac{\varphi(-1)\overline{C}(-4)}{q} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \frac{g(C^2\gamma^2)g(A^2\overline{C^2\gamma^2})\overline{C}\gamma(x^2)}{g(A^2)} \delta(C\varphi\gamma) \\ &= -\frac{\varphi(-1)\overline{C}(-4)}{q}. \end{aligned} \quad (2.5)$$

The last equality is obtained by putting  $\gamma = \overline{C}\varphi$  and then using the fact that  $g(\varepsilon) = -1$ . Using Lemma 1.5 on  $g(A^2\overline{C^2\gamma^2})$ ,  $g(C^2\gamma^2)$  and  $g(A^2)$ , we have

$$L = \frac{\overline{C}(4)g(C\varphi)}{q(q-1)g(\varphi)g(A\varphi)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \frac{g(\gamma)g(A\overline{C}\gamma)g(\varphi A\overline{C}\gamma)g(C\gamma)\overline{C}\gamma(x^2)}{g(A)} + I_2 + I_3. \quad (2.6)$$

If we apply the change of variables  $\gamma \mapsto \overline{C}\gamma$ , we have

$$\begin{aligned} L &= \frac{\overline{C}(4)g(C\varphi)}{q(q-1)g(\varphi)g(A\varphi)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \frac{g(\overline{C}\gamma)g(A\gamma)g(\varphi A\gamma)g(\overline{\gamma})\gamma(x^2)}{g(A)} + I_2 + I_3 \\ &= \frac{q\overline{C}(-4)g(C\varphi)g(A\overline{C}\varphi)}{(q-1)g(\varphi)g(A\varphi)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \left[ \left( \frac{\gamma(-1)g(A\gamma)g(\overline{\gamma})}{qg(A)} \right) \right. \\ &\quad \left. \times \left( \frac{C\gamma(-1)g(\varphi A\gamma)g\overline{C}\gamma}{qg(A\overline{C}\varphi)} \right) \right] \gamma(x^2) + I_2 + I_3. \end{aligned} \quad (2.7)$$

The last equality is obtained by multiplying both numerator and denominator by  $qg(A\varphi\overline{C})$  and then rearranging the terms. Now using Lemma 1.12 and the fact that  $A \neq \varepsilon$ , we have

$$\begin{aligned} L &= \frac{q\overline{C}(-4)g(C\varphi)g(A\overline{C}\varphi)}{(q-1)g(\varphi)g(A\varphi)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \binom{A\gamma}{\gamma} \left[ \binom{\varphi A\gamma}{C\gamma} - \frac{q-1}{q} \delta(A\overline{C}\varphi) \right] \gamma(x^2) + I_2 + I_3 \\ &= \frac{q\overline{C}(-4)g(C\varphi)g(A\overline{C}\varphi)}{(q-1)g(\varphi)g(A\varphi)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \binom{A\gamma}{\gamma} \binom{\varphi A\gamma}{C\gamma} \gamma(x^2) - I_4 + I_2 + I_3, \end{aligned} \quad (2.8)$$

where

$$I_4 = \frac{\overline{C}(-4)g(C\varphi)g(A\overline{C}\varphi)}{g(\varphi)g(A\varphi)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \binom{A\gamma}{\gamma} \gamma(x^2) \delta(A\overline{C}\varphi).$$

Using (1.9) and the fact that  $x \neq 0$ , we have

$$I_4 = \frac{(q-1)\overline{C}(-4)g(C\varphi)g(A\overline{C}\varphi)\overline{A}(1-x^2)\delta(A\overline{C}\varphi)}{qg(\varphi)g(A\varphi)}. \quad (2.9)$$

Finally employing (1.19), (2.3), (2.5) and (2.9) into (2.8), we complete the proof of the theorem.  $\blacksquare$

In the following corollary we show that the identity (2.1) of Fuselier et. al. readily follows from Theorem 2.1 by taking  $C = \varphi$ .

**Corollary 2.2.1.** *Let  $A \in \widehat{\mathbb{F}_q^\times}$  be such that  $A \neq \varepsilon, \varphi$ . For  $x \in \mathbb{F}_q^\times$ , we have*

$${}_2F_1 \left[ \begin{matrix} A, & A\varphi \\ & \varphi \end{matrix} \middle| x \right] = \left( \frac{1+\varphi(x)}{2} \right) \left( \overline{A^2}(1+\sqrt{x}) + \overline{A^2}(1-\sqrt{x}) \right).$$

*Proof.* If  $x$  is not a square, then the identity holds trivially (see for example [25, Theorem 8.11]). If  $x \neq 0$  is a square in  $\mathbb{F}_q$ , we deduce the identity from Theorem 2.1. We first take  $C = \varphi$  in Theorem 2.1 and then use the fact that  $g(\varepsilon) = -1$  to obtain

$$\overline{A^2}(1+x) {}_2F_1 \left( \begin{matrix} A^2, & \varepsilon \\ \varepsilon & \frac{2x}{1+x} \end{matrix} \middle| x \right) = -\frac{\varphi(-1)g(A)}{g(\varphi)g(A\varphi)} {}_2F_1 \left( \begin{matrix} A, & \varphi A \\ & \varphi \end{matrix} \middle| x^2 \right). \quad (2.10)$$

Again from [31, Corollary 3.16 (iv)], we have

$${}_2F_1 \left( \begin{matrix} A^2, & \varepsilon \\ & \varepsilon \end{matrix} \middle| x \right) = -\frac{1}{q} \overline{A^2}(1-x) + \binom{\varepsilon}{A^2}. \quad (2.11)$$

Now, employing (2.11) into (2.10), and then using (1.15), (1.26), Lemma 1.9 and Lemma 1.2, we readily deduce the required identity. ■

## 2.3 Proof of Theorem 2.2

*Proof.* If  $x = 0$  then the result is trivially true. Therefore, we assume that  $x \neq 0$ .

Using (1.20) and (1.9), we have

$$\begin{aligned} L &:= \overline{A^2}B^2(1-x) {}_3F_2 \left( \begin{matrix} B^2, & \varphi B, & B\overline{A}\varphi \\ & AB\varphi, & B\varphi \end{matrix} \middle| x \right) \\ &= \frac{q^2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{B^2\chi}{\chi} \binom{B\varphi\chi}{AB\varphi\chi} \binom{B\overline{A}\varphi\chi}{B\varphi\chi} \binom{A^2\overline{B^2}\lambda}{\lambda} \chi\lambda(x). \end{aligned}$$

The change of variables  $\lambda \mapsto \lambda\overline{\chi}$  yields

$$\begin{aligned} L &= \frac{q^2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{B^2\chi}{\chi} \binom{B\varphi\chi}{AB\varphi\chi} \binom{B\overline{A}\varphi\chi}{B\varphi\chi} \binom{A^2\overline{B^2}\lambda\overline{\chi}}{\lambda\overline{\chi}} \lambda(x) \\ &= \frac{q^2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \left[ \binom{B\overline{A}\varphi\chi}{AB\varphi\chi} \binom{\overline{A^2}}{\overline{A}} - \frac{q-1}{q^2} AB\varphi\chi(-1)\delta(B\varphi\chi) \right] \\ &\quad \times \binom{B^2\chi}{\chi} \binom{A^2\overline{B^2}\lambda\overline{\chi}}{\lambda\overline{\chi}} \lambda(x). \end{aligned}$$

The last equality is obtained by using (1.16) on  $\binom{B\varphi\chi}{AB\varphi\chi} \binom{B\overline{A}\varphi\chi}{B\varphi\chi}$  and the fact that  $A^2 \neq \varepsilon$ . We now have

$$L = \frac{q^2}{(q-1)^2} \binom{\overline{A^2}}{\overline{A}} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{B^2\chi}{\chi} \binom{B\overline{A}\varphi\chi}{AB\varphi\chi} \binom{A^2\overline{B^2}\lambda\overline{\chi}}{\lambda\overline{\chi}} \lambda(x) - I_1, \quad (2.12)$$

where

$$I_1 = \frac{AB\varphi(-1)}{(q-1)} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{B^2\chi}{\chi} \binom{A^2\overline{B^2}\lambda\overline{\chi}}{\lambda\overline{\chi}} \chi(-1)\lambda(x)\delta(B\varphi\chi).$$

The term  $I_1$  is nonzero only when  $\chi = \varphi\overline{B}$ . After putting  $\chi = \varphi\overline{B}$ , and then using (1.10) we obtain

$$I_1 = \frac{A(-1)}{q} \left( \frac{B\varphi}{\overline{B\varphi}} \right) \overline{B}\varphi(x)\overline{A^2}B^2(1-x). \quad (2.13)$$

Applying (1.13) on  $\binom{A^2\overline{B^2}\lambda\overline{\chi}}{\lambda\overline{\chi}}$  and then using (1.20), (2.12) yields

$$L = \frac{q}{(q-1)} \left( \frac{\overline{A^2}}{\overline{A}} \right) \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} {}_3F_2 \left( \begin{matrix} B^2, & B\varphi\overline{A}, & \overline{\lambda} \\ & AB\varphi, & \overline{A^2}\lambda\overline{B^2} \end{matrix} \mid 1 \right) \lambda(x) - I_1. \quad (2.14)$$

For  $A, B, C, D \in \widehat{\mathbb{F}_q^\times}$ , from [31, Theorem 4.35] we have

$${}_3F_2 \left( \begin{matrix} A, & B, & C \\ & D, & ABC\overline{D} \end{matrix} \mid 1 \right) = BC(-1) \binom{C}{D\overline{A}} \binom{B}{D\overline{C}} - \frac{1}{q} BD(-1) \binom{D\overline{B}}{A}. \quad (2.15)$$

Using (2.15) in (2.14), we obtain

$$\begin{aligned} L &= \frac{q}{(q-1)} \left( \frac{\overline{A^2}}{\overline{A}} \right) \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \left[ B\varphi\overline{A}\lambda(-1) \binom{\overline{\lambda}}{A\varphi\overline{B}} \binom{B\varphi\overline{A}}{AB\varphi\lambda} - \frac{1}{q} \binom{A^2}{B^2} \right] \lambda(x) - I_1 \\ &= \frac{q}{(q-1)} \left( \frac{\overline{A^2}}{\overline{A}} \right) BA\varphi(-1) \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \binom{\overline{\lambda}}{A\varphi\overline{B}} \binom{B\varphi\overline{A}}{AB\varphi\lambda} \lambda(-x) - I_2 - I_1, \end{aligned} \quad (2.16)$$

where

$$\begin{aligned} I_2 &= \frac{1}{(q-1)} \binom{\overline{A^2}}{\overline{A}} \binom{A^2}{B^2} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \lambda(x) \\ &= \binom{\overline{A^2}}{\overline{A}} \binom{A^2}{B^2} \delta(1-x). \end{aligned} \quad (2.17)$$

The last equality is obtained by using Lemma 1.1. If we apply (1.11) on  $\binom{\overline{\lambda}}{A\varphi\overline{B}}$  and (1.12) on  $\binom{B\varphi\overline{A}}{AB\varphi\lambda}$ , then (2.16) reduces to

$$L = \frac{q}{(q-1)} \binom{\overline{A^2}}{\overline{A}} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \binom{\overline{\lambda}}{\overline{AB\varphi\lambda}} \binom{A^2\lambda}{AB\varphi\lambda} \lambda(x) - I_2 - I_1.$$

Now using Lemma 1.12 and the fact that  $\overline{AB\varphi} \neq \varepsilon$ , we have

$$L = \frac{1}{q(q-1)g^2(\overline{AB\varphi})} \binom{\overline{A^2}}{\overline{A}} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} g(A^2\lambda)g(\overline{\lambda})g(\overline{AB\varphi\lambda})g(\overline{AB\varphi\lambda})\lambda(x) - I_2 - I_1. \quad (2.18)$$

Multiplying both numerator and denominator by  $qg(A^2)g(\overline{B^2})$  and then rearranging the terms, we obtain

$$\begin{aligned} L &= \frac{qAB\varphi(-1)g(A^2)g(\overline{B^2})}{(q-1)g^2(\overline{AB\varphi})} \binom{\overline{A^2}}{\overline{A}} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \left[ \left( \frac{\lambda(-1)g(A^2\lambda)g(\overline{\lambda})}{qg(A^2)} \right) \right. \\ &\quad \left. \times \left( \frac{AB\varphi\lambda(-1)g(\overline{AB\varphi\lambda})g(\varphi\overline{AB\lambda})}{qg(\overline{B^2})} \right) \right] \lambda(x) - I_2 - I_1 \\ &= \frac{qAB\varphi(-1)g(A^2)g(\overline{B^2})}{(q-1)g^2(\overline{AB\varphi})} \binom{\overline{A^2}}{\overline{A}} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \binom{A^2\lambda}{\lambda} \binom{\overline{AB\varphi\lambda}}{\overline{AB\varphi\lambda}} \lambda(x) - I_2 - I_1. \end{aligned} \quad (2.19)$$

The last equality is obtained by using Lemma 1.12 and the fact that  $A^2 \neq \varepsilon \neq \overline{B^2}$ .

Using (1.16) on  $\binom{A\varphi\lambda}{AB\varphi\lambda} \binom{A\overline{B}\varphi\lambda}{A\varphi\lambda}$  and the fact that  $B^2 \neq \varepsilon$ , we have

$$\binom{A\overline{B}\varphi\lambda}{AB\varphi\lambda} = \left(\frac{\overline{B^2}}{B}\right)^{-1} \binom{A\varphi\lambda}{AB\varphi\lambda} \binom{A\overline{B}\varphi\lambda}{A\varphi\lambda} + \frac{q-1}{q^2} AB\varphi\lambda(-1) \left(\frac{\overline{B^2}}{B}\right)^{-1} \delta(A\varphi\lambda). \quad (2.20)$$

Now using (2.20) in (2.19), we find that

$$L = \frac{qAB\varphi(-1)g(A^2)g(\overline{B^2})}{(q-1)g^2(A\overline{B}\varphi)} \left(\frac{A^2}{A}\right) \left(\frac{\overline{B^2}}{B}\right)^{-1} \times \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \binom{A^2\lambda}{\lambda} \binom{A\varphi\lambda}{AB\varphi\lambda} \binom{A\varphi\overline{B}\lambda}{A\varphi\lambda} \lambda(x) + I_3 - I_2 - I_1, \quad (2.21)$$

where

$$\begin{aligned} I_3 &= \frac{g(A^2)g(\overline{B^2})}{qg^2(A\overline{B}\varphi)} \left(\frac{A^2}{A}\right) \left(\frac{\overline{B^2}}{B}\right)^{-1} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \binom{A^2\lambda}{\lambda} \lambda(-x) \delta(A\varphi\lambda) \\ &= \frac{g(A^2)g(\overline{B^2})}{qg^2(A\overline{B}\varphi)} \left(\frac{A^2}{A}\right) \left(\frac{\overline{B^2}}{B}\right)^{-1} \overline{A}\varphi(-x) \binom{A\varphi}{\overline{A}\varphi}. \end{aligned} \quad (2.22)$$

The last equality is obtained by putting  $\lambda = \overline{A}\varphi$ . Finally employing (2.13), (2.17), (2.22) and (1.20) into (2.21), we complete the proof of the theorem.  $\blacksquare$

## 2.4 Proof of Theorem 2.3

*Proof.* Let

$$L := {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \middle| \frac{1+x}{2} \right) + {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \middle| \frac{1-x}{2} \right).$$

Using Theorem 1.14 (i), we have

$$L = 2 \cdot {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ AB\varphi & \end{matrix} \middle| \frac{1+x}{2} \right).$$

Using (1.19) and (1.9), we have

$$\begin{aligned} L &= \frac{2q^2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{A^2\chi}{\chi} \binom{B^2\chi}{AB\varphi\chi} \binom{\bar{\chi}\lambda}{\lambda} \lambda(-x)\bar{\chi}(2) \\ &= \frac{2q^2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \binom{A^2\chi\lambda}{\chi\lambda} \binom{B^2\chi\lambda}{AB\varphi\chi\lambda} \binom{\bar{\chi}}{\lambda} \lambda\left(\frac{-x}{2}\right)\bar{\chi}(2). \end{aligned}$$

The last equality is obtained by applying the change of variables  $\chi \mapsto \chi\lambda$ . Now using Lemma 1.12 and the fact that  $A^2 \neq \varepsilon$ , we have

$$\begin{aligned} L &= \frac{2q^2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \left[ \frac{\lambda\chi(-1)g(A^2\lambda\chi)g(\bar{\lambda}\chi)}{qg(A^2)} \binom{B^2\chi\lambda}{AB\varphi\chi\lambda} \right. \\ &\quad \left. \times \left( \frac{\lambda(-1)g(\bar{\lambda})g(\bar{\chi})}{qg(\bar{\chi}\lambda)} + \frac{q-1}{q}\delta(\bar{\chi}\lambda) \right) \right] \lambda\left(\frac{-x}{2}\right)\bar{\chi}(2) \\ &= \frac{2}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2\lambda\chi)g(\bar{\chi})g(\bar{\lambda})\bar{\chi}(-2)\lambda\left(\frac{-x}{2}\right)}{g(A^2)} \binom{B^2\chi\lambda}{AB\varphi\chi\lambda} + I_1, \end{aligned} \quad (2.23)$$

where

$$I_1 = \frac{2}{(q-1)} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2\lambda\chi)g(\bar{\chi}\lambda)\bar{\chi}(-2)\lambda\left(\frac{x}{2}\right)}{g(A^2)} \binom{B^2\chi\lambda}{AB\varphi\chi\lambda} \delta(\bar{\chi}\lambda).$$

The term  $I_1$  is nonzero only when  $\chi = \bar{\lambda}$ . As before, we find that

$$I_1 = -2 \binom{B^2}{AB\varphi} \delta(1+x).$$

Since  $x \neq -1$ , we have  $I_1 = 0$ . Now multiplying both numerator and denominator by  $qg(A^2\lambda)$  in (2.23), and then rearranging the terms and using the fact that  $I_1 = 0$ , we obtain

$$L = \frac{2q}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \frac{\chi(-1)g(A^2\lambda\chi)g(\bar{\chi})}{qg(A^2\lambda)} \frac{g(A^2\lambda)g(\bar{\lambda})\bar{\chi}(2)\lambda(\frac{-x}{2})}{g(A^2)} \begin{pmatrix} B^2\chi\lambda \\ AB\varphi\chi\lambda \end{pmatrix}.$$

Using Lemma 1.12, we have

$$\begin{aligned} L &= \frac{2q}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \left[ \begin{pmatrix} A^2\chi\lambda \\ \chi \end{pmatrix} - \frac{q-1}{q}\delta(A^2\lambda) \right] \begin{pmatrix} B^2\chi\lambda \\ AB\varphi\chi\lambda \end{pmatrix} \frac{g(A^2\lambda)g(\bar{\lambda})\bar{\chi}(2)\lambda(\frac{-x}{2})}{g(A^2)} \\ &= \frac{2q}{(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} A^2\chi\lambda \\ \chi \end{pmatrix} \begin{pmatrix} B^2\chi\lambda \\ AB\varphi\chi\lambda \end{pmatrix} \frac{g(A^2\lambda)g(\bar{\lambda})\bar{\chi}(2)\lambda(\frac{-x}{2})}{g(A^2)} - I_2, \end{aligned} \quad (2.24)$$

where

$$\begin{aligned} I_2 &= \frac{2}{(q-1)} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} B^2\chi\lambda \\ AB\varphi\chi\lambda \end{pmatrix} \frac{g(A^2\lambda)g(\bar{\lambda})\bar{\chi}(2)\lambda(\frac{-x}{2})}{g(A^2)} \delta(A^2\lambda) \\ &= -\frac{2\bar{A}(x^2)B(4)}{q}. \end{aligned} \quad (2.25)$$

The last equality is obtained by putting  $\lambda = \bar{A}^2$ , and then using (1.10) and  $g(\varepsilon) = -1$ . Applying (1.19) in (2.24), and then Theorem 1.14 (iii) we have

$$\begin{aligned} L &= \frac{2}{(q-1)} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} {}_2F_1 \left( \begin{matrix} A^2\lambda, & B^2\lambda \\ & AB\varphi\lambda \end{matrix} \middle| \frac{1}{2} \right) \frac{g(A^2\lambda)g(\bar{\lambda})\lambda(\frac{-x}{2})}{g(A^2)} - I_2 \\ &= \frac{2B(4)}{(q-1)} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} {}_2F_1 \left( \begin{matrix} \bar{A}B\varphi, & B^2\lambda \\ & AB\varphi\lambda \end{matrix} \middle| -1 \right) \frac{g(A^2\lambda)g(\bar{\lambda})\lambda(-x)}{g(A^2)} - I_2. \end{aligned}$$

Now using Greene's definition (1.18), we have

$$\begin{aligned} L &= \frac{2B(4)AB\varphi(-1)}{q(q-1)} \sum_{\lambda \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2\lambda)g(\bar{\lambda})\lambda(-x)}{g(A^2)} \sum_{y \in \mathbb{F}_q} B^2\lambda(y)A\bar{B}\varphi(1-y^2) - I_2 \\ &= \frac{2B(4)AB\varphi(-1)}{(q-1)^2} \sum_{\lambda, \gamma \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} \bar{A}B\varphi\gamma \\ \gamma \end{pmatrix} \frac{g(A^2\lambda)g(\bar{\lambda})\lambda(-x)}{g(A^2)} \sum_{y \in \mathbb{F}_q} B^2\lambda\gamma^2(y) - I_2. \end{aligned}$$

The last equality is obtained by using (1.9). The inner most sum is nonzero only when  $\lambda = \overline{B^2\gamma^2}$ , and hence we have

$$L = \frac{2B(4)AB\varphi(-1)}{(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} \bar{A}B\varphi\gamma \\ \gamma \end{pmatrix} \frac{g(A^2\overline{B^2\gamma^2})g(B^2\gamma^2)\overline{B}\gamma(x^2)}{g(A^2)} - I_2.$$

Applying (1.13) and (1.11) on  $\begin{pmatrix} \bar{A}B\varphi\gamma \\ \gamma \end{pmatrix}$  we have

$$\begin{aligned} L &= \frac{2B(4)}{(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} \bar{\gamma} \\ \bar{A}B\varphi \end{pmatrix} \frac{g(A^2\overline{B^2\gamma^2})g(B^2\gamma^2)\overline{B}\gamma(x^2)}{g(A^2)} - I_2 \\ &= \frac{2B(4)}{(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} \gamma \\ \bar{A}B\varphi \end{pmatrix} \frac{g(A^2\overline{B^2\gamma^2})g(B^2\overline{\gamma^2})\overline{B}\gamma(x^2)}{g(A^2)} - I_2. \end{aligned}$$

Using Lemma 1.12 we have

$$\begin{aligned} L &= \frac{2B(4)}{(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \left[ \frac{AB\varphi(-1)g(\gamma)g(A\bar{B}\varphi)}{qg(A\bar{B}\varphi\gamma)} + \frac{q-1}{q}\delta(A\bar{B}\varphi\gamma) \right] \\ &\quad \times \frac{g(A^2\overline{B^2\gamma^2})g(B^2\overline{\gamma^2})\overline{B}\gamma^2(x)}{g(A^2)} - I_2 \\ &= \frac{2B(4)AB\varphi(-1)}{q(q-1)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \frac{g(\gamma)g(A\bar{B}\varphi)g(A^2\overline{B^2\gamma^2})g(B^2\overline{\gamma^2})\overline{B}\gamma(x^2)}{g(A\bar{B}\varphi\gamma)g(A^2)} + I_3 - I_2, \quad (2.26) \end{aligned}$$

where

$$\begin{aligned} I_3 &= \frac{2B(4)}{q} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2 \overline{B^2} \gamma^2) g(B^2 \overline{\gamma^2}) \overline{B^2} \gamma^2(x) \delta(A \overline{B} \varphi \gamma)}{g(A^2)} \\ &= \frac{-2B(4) \overline{A}(x^2)}{q}. \end{aligned} \quad (2.27)$$

The last equality is obtained by putting  $\gamma = \overline{A}B\varphi$  and using the fact that  $g(\varepsilon) = -1$ . Using Lemma 1.5 on  $g(A^2 \overline{B^2} \gamma^2)$  and  $g(B^2 \overline{\gamma^2})$ , and then using the fact that  $g(\varphi)g(\varphi) = q\varphi(-1)$ , we have

$$\begin{aligned} L &= \frac{2AB(-4)}{q^2(q-1)g(A^2)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} g(\gamma)g(A\overline{B}\varphi)g(A\overline{B}\gamma)g(B\overline{\gamma})g(\varphi B\overline{\gamma})\overline{B^2} \gamma^2(x) + I_3 - I_2 \\ &= \frac{2AB(-4)g(A\overline{B}\varphi)}{q^2(q-1)g(A^2)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} g(B\gamma)g(A\gamma)g(\overline{\gamma})g(\varphi\overline{\gamma})\gamma(x^2) + I_3 - I_2. \end{aligned}$$

The last equality is obtained by applying  $\gamma \mapsto \gamma B$ . Now multiplying both numerator and denominator by  $g(A)g(B\varphi)$ , and then rearranging the terms we obtain

$$\begin{aligned} L &= \frac{2AB(-4)\varphi(-1)g(A\overline{B}\varphi)g(A)g(B\varphi)}{(q-1)g(A^2)} \\ &\quad \times \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \left[ \frac{\gamma(-1)g(A\gamma)g(\overline{\gamma})}{qg(A)} \right] \left[ \frac{\gamma\varphi(-1)g(B\gamma)g(\varphi\overline{\gamma})}{qg(B\varphi)} \right] \gamma(x^2) + I_3 - I_2 \\ &= \frac{2AB(-4)\varphi(-1)g(A\overline{B}\varphi)g(A)g(B\varphi)}{(q-1)g(A^2)} \sum_{\gamma \in \widehat{\mathbb{F}_q^\times}} \binom{A\gamma}{\gamma} \binom{B\gamma}{\varphi\gamma} \gamma(x^2) + I_3 - I_2. \end{aligned} \quad (2.28)$$

The last equality is obtained by using Lemma 1.12 and the fact that  $A \neq \varepsilon \neq B\varphi$ . Finally employing (2.25), (2.27) and (1.19) into (2.28), we complete the proof of the theorem.  $\blacksquare$

The following corollaries of Theorem 2.3 will be used to evaluate certain special values of Gaussian hypergeometric series in Chapter 7.

**Corollary 2.4.1.** *Let  $A, B \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^2, A, B\varphi \neq \varepsilon$ . Then, for  $x \in \mathbb{F}_q$  with  $x \neq 0, \pm 1$ , we have*

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \middle| \frac{1+x}{2} \right) \\ &= \frac{AB(-4)\varphi(-1)g(A\overline{B}\varphi)g(A)g(B\varphi)}{qg(A^2)} {}_2F_1 \left( \begin{matrix} A, & B \\ & \varphi \end{matrix} \middle| x^2 \right). \end{aligned}$$

*Proof.* Using Theorem 1.14 (i) in Theorem 2.3, we complete the proof of the corollary. ■

**Corollary 2.4.2.** *Let  $A, B \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^2, A, B\varphi \neq \varepsilon$ . Then, for  $x \in \mathbb{F}_q$  with  $x \neq 0, \pm 1$ , we have*

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \middle| \frac{1-x}{2} \right) \\ &= \frac{A(4)B(-4)\varphi(-1)g(A\overline{B}\varphi)g(A)g(B\varphi)}{qg(A^2)} {}_2F_1 \left( \begin{matrix} A, & B \\ & AB\varphi \end{matrix} \middle| 1-x^2 \right) \end{aligned}$$

The above result is a finite field analogue of the following identity [15, p. 111]:

$${}_2F_1 \left( \begin{matrix} a, & b \\ & a+b+\frac{1}{2} \end{matrix} \middle| x \right) = {}_2F_1 \left( \begin{matrix} 2a, & 2b \\ & a+b+\frac{1}{2} \end{matrix} \middle| \frac{1-\sqrt{1-x}}{2} \right).$$

*Proof.* Using Theorem 1.14 (i) in Theorem 2.3, we complete the proof of the corollary. ■

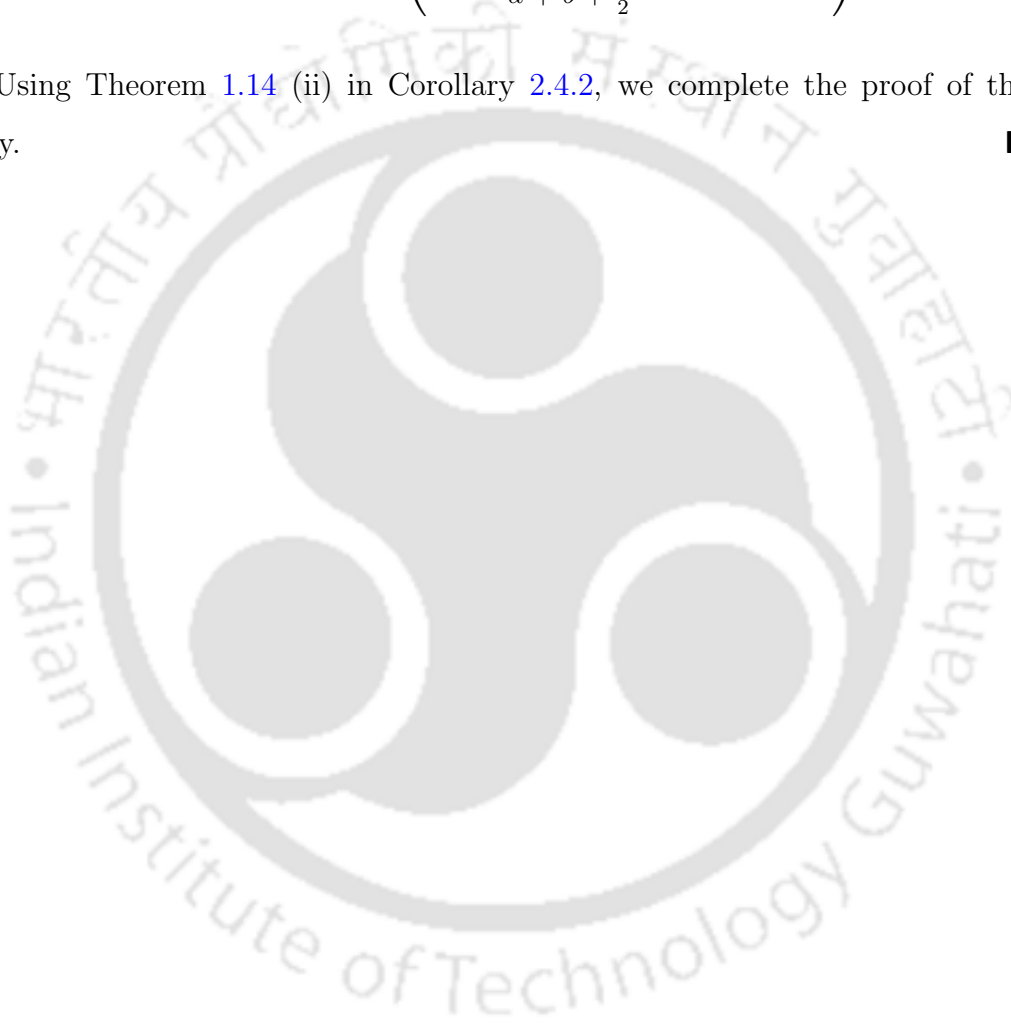
**Corollary 2.4.3.** *Let  $A, B \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^2, A, B\varphi \neq \varepsilon$ . Then, for  $x \in \mathbb{F}_q$  with  $x \neq 0, \pm 1$ , we have*

$${}_2F_1 \left( \begin{matrix} A, & B \\ & AB\varphi \end{matrix} \middle| 1-x^2 \right) = \frac{qA(-1)g(A^2)\overline{A^2}(1+x)}{B(4)g(A\overline{B}\varphi)g(A)g(B\varphi)} {}_2F_1 \left( \begin{matrix} A^2, & \overline{AB}\varphi \\ & AB\varphi \end{matrix} \middle| \frac{x-1}{x+1} \right).$$

The above result is a finite field analogue of the following identity [15, p. 111]:

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} a, & b \\ & a + b + \frac{1}{2} \end{matrix} \middle| x \right) \\ &= \left( \frac{1 + \sqrt{1-x}}{2} \right)^{-2a} {}_2F_1 \left( \begin{matrix} 2a, & a - b + \frac{1}{2} \\ & a + b + \frac{1}{2} \end{matrix} \middle| \frac{\sqrt{1-x}-1}{\sqrt{1-x}+1} \right). \end{aligned}$$

*Proof.* Using Theorem 1.14 (ii) in Corollary 2.4.2, we complete the proof of the corollary. ■



# 3

## Appell series $F_1$ , $F_2$ and $F_3$ over finite fields

### 3.1 Introduction

In recent times, there has been a study on Appell series over finite fields. Appell series  $F_1, F_2$  and  $F_3$  have integral representations. Using an integral representation of  $F_1$ , Li et. al. [43] defined a finite field analogue of  $F_1$  as follows. Let  $A, A', B, B', C, C'$  be multiplicative characters on  $\mathbb{F}_q$ . For  $x, y \in \mathbb{F}_q$ , the finite field

---

<sup>1</sup>The contents of this chapter have been published in *Int. J. Number Theory* (2020).

Appell series  $F_1^q$  is defined by

$$\begin{aligned} F_1^q(A; B, B'; C; x, y) \\ = \varepsilon(xy)AC(-1) \sum_{u \in \mathbb{F}_q} A(u)\overline{AC}(1-u)\overline{B}(1-ux)\overline{B}'(1-uy). \end{aligned} \quad (3.1)$$

Similarly, using an integral representation of  $F_2$ , He et. al. [35] defined the following function as a finite field analogue of  $F_2$ .

$$\begin{aligned} F_2^q(A; B, B'; C, C'; x, y) \\ = \varepsilon(xy)BB'CC'(-1) \sum_{u, v \in \mathbb{F}_q} B(u)B'(v)\overline{BC}(1-u)\overline{B}'C'(1-v)\overline{A}(1-ux-vy). \end{aligned} \quad (3.2)$$

Also, using an integral representation of  $F_3$ , He [33] defined the following function as a finite field analogue of  $F_3$ .

$$\begin{aligned} F_3^q(A, A'; B, B'; C; x, y) \\ = \varepsilon(xy)BB'(-1) \sum_{u, v \in \mathbb{F}_q} B(u)B'(v)C\overline{BB}'(1-u-v)\overline{A}(1-ux)\overline{A}'(1-vy). \end{aligned} \quad (3.3)$$

In [43], Li et. al. expressed the finite field Appell series  $F_1^q$  in terms of Greene's  ${}_2F_1$ -finite field hypergeometric series. They derived some reduction formulas for  $F_1^q$ . For instance, they proved that

$$F_1^q(A; B, \varepsilon; C; x, y) = \varepsilon(y) {}_2F_1 \left( \begin{matrix} B, & A \\ & C \end{matrix} \middle| x \right) - \varepsilon(x)\overline{AC}(1-y)\overline{BC}(y)\overline{B}(y-x).$$

The above formula can be considered as a finite field analogue of the following identity satisfied by the Appell series  $F_1$  and classical hypergeometric series.

$$F_1(a; b, 0; c; x, y) = {}_2F_1 \left( \begin{matrix} a, & b \\ & c \end{matrix} \middle| x \right).$$

Further, they derived certain transformation formulas for the finite field Appell series  $F_1^q$  analogous to those satisfied by the classical Appell series.

In [35], He et. al. expressed the finite field Appell series  $F_2^q$  in terms of Greene's  ${}_3F_2$ -finite field hypergeometric series, and derived some reduction and transformation formulas for  $F_2^q$  analogous to those satisfied by the classical Appell series. In [33], He did a similar study on the finite field Appell series  $F_3^q$ .

### 3.2 Appell series $F_1^*$ , $F_2^*$ and $F_3^*$

The finite field analogues of Appell series introduced in [43, 35, 33] used integral representations of  $F_1, F_2$  and  $F_3$ . They replaced the integrals with character sums suitably. However, the integral representations of the Appell series  $F_4$  are more complicated than  $F_1, F_2$  and  $F_3$ . Therefore, it is not straightforward to find an appropriate finite field analogue of  $F_4$  using the double integral representation. Classical hypergeometric series is defined using rising factorials which are related to Gamma function, and the Gauss sum is the finite field analogue of the Gamma function. Since Greene's function is defined using Jacobi sums, often it is necessary to impose conditions on the parameters to relate the Jacobi sums to the required product of Gauss sums. However, McCarthy's function does not need such conditions. With this motivation, we define finite field analogues of Appell series purely in terms of Gauss sums. The classical Appell series are defined after multiplying two  ${}_2F_1$ -hypergeometric series and then arranging the products of the rising factorials in some order. Here, we consider products of McCarthy's  ${}_2F_1$ -hypergeometric functions and follow analogously as in the case of Appell series. This leads to the definitions of three functions, denoted by  $F_1^*$ ,  $F_2^*$  and  $F_3^*$ , respectively and we establish them to be finite field analogues of classical Appell series. Let  $A, A', B, B', C, C'$

be multiplicative characters on  $\mathbb{F}_q$ . For  $x, y \in \mathbb{F}_q$ , we define

$$\begin{aligned} & F_1(A; B, B'; C; x, y)^* \\ &= \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi\psi)g(B\chi)g(B'\psi)g(\overline{C\chi\psi})g(\overline{\chi})g(\overline{\psi})}{g(A)g(B)g(B')g(\overline{C})} \chi(x)\psi(y); \end{aligned} \quad (3.4)$$

$$\begin{aligned} & F_2(A; B, B'; C, C'; x, y)^* \\ &= \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi\psi)g(B\chi)g(B'\psi)g(\overline{C\chi})g(\overline{C'\psi})g(\overline{\chi})g(\overline{\psi})}{g(A)g(B)g(B')g(\overline{C})g(\overline{C'})} \chi(x)\psi(y); \end{aligned} \quad (3.5)$$

$$\begin{aligned} & F_3(A, A'; B, B'; C; x, y)^* \\ &= \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(A'\psi)g(B\chi)g(B'\psi)g(\overline{C\chi\psi})g(\overline{\chi})g(\overline{\psi})}{g(A)g(A')g(B)g(B')g(\overline{C})} \chi(x)\psi(y). \end{aligned} \quad (3.6)$$

We will define a finite field analogue of the Appell series  $F_4$  in the next chapter.

### 3.3 Main results

In this section, we establish  $F_1^*$ ,  $F_2^*$  and  $F_3^*$  as finite field analogues of Appell series by proving results over finite fields analogous to those satisfied by the classical Appell series. We focus on finite field analogues of classical results concerning Appell series which are not discussed in [43, 35, 33]. Classical Appell series satisfy many transformation and reduction identities. There are beautiful relations connecting different types of Appell series. For example, we consider the following identities

$$\begin{aligned} F_1(a; b, b'; c; x, y) &= (1-y)^{-b'} F_3(a, c-a; b, b'; c; x, \frac{y}{y-1}), \\ F_1(b; a-b', b'; c; x, x(1-y)) &= (1-y)^{-b'} F_2(a; b, b'; c, a; x, \frac{-y}{1-y}). \end{aligned}$$

Finite field analogues of these results are not yet obtained. In the following two theorems we derive transformations analogous to the above two classical identities.

**Theorem 3.1.** Let  $A, C, B, B' \in \widehat{\mathbb{F}}_q^\times$  be such that  $A \neq C$ ,  $\bar{A} \neq \bar{C}B'$  and  $B' \neq \varepsilon$ . If  $x, y \in \mathbb{F}_q$  and  $y \neq 1$ , then

$$F_1(A; B, B'; C; x, y)^* = \bar{B}'(1-y)F_3(A, \bar{A}C; B, B'; C; x, \frac{y}{y-1})^* + \frac{g(B\bar{A})g(A\bar{C})\bar{A}(x)A\bar{C}(y)\bar{A}B'C(1-y)}{q\binom{B'}{AC}g(B)g(\bar{C})}.$$

**Theorem 3.2.** Let  $A, B, B', C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A \neq B'$  and  $A, B, B' \neq \varepsilon$ . If  $x, y \in \mathbb{F}_q^\times$  and  $y \neq 1$ , then

$$F_2(A; B, B'; C, A; x, \frac{-y}{1-y})^* = B'(1-y)F_1(B; \bar{A}B', B'; C; x, x(1-y))^* - \frac{\bar{A}(\frac{y}{1-y})}{q\binom{B'}{A}}.$$

In the proofs we use properties of finite field hypergeometric series introduced by Greene and McCarthy.

*Proof of Theorem 3.1.* If  $xy = 0$ , then the result is trivial. So, we take both  $x$  and  $y$  are non-zero. Let  $I = F_1(A; B, B'; C; x, y)^*$ . Then by definition we have

$$I = \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}}_q^\times} \frac{g(A\chi\psi)g(B\chi)g(B'\psi)g(\bar{C}\chi\psi)g(\bar{\chi})g(\bar{\psi})}{g(A)g(B)g(B')g(\bar{C})} \chi(x)\psi(y).$$

Multiplying both numerator and denominator by  $g(A\chi)g(\bar{C}\bar{\chi})$ , and then rearranging the terms we have

$$\begin{aligned} I &= \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}}_q^\times} \frac{g(A\chi)g(B\chi)g(\bar{C}\bar{\chi})g(\bar{\chi})\chi(x)}{g(A)g(B)g(\bar{C})} \frac{g(A\chi\psi)g(B'\psi)g(\bar{C}\chi\psi)g(\bar{\psi})\psi(y)}{g(A\chi)g(B')g(\bar{C}\bar{\chi})} \\ &= \frac{1}{(q-1)} \sum_{\chi \in \widehat{\mathbb{F}}_q^\times} \frac{g(A\chi)g(B\chi)g(\bar{C}\bar{\chi})g(\bar{\chi})\chi(x)}{g(A)g(B)g(\bar{C})} {}_2F_1 \left( \begin{matrix} A\chi, & B' \\ & C\chi \end{matrix} \middle| y \right)^* \end{aligned}$$

$$= \frac{1}{(q-1)} \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi \neq \bar{A}, \bar{B}'\bar{C}}} \frac{g(A\chi)g(B\chi)g(\bar{C}\chi)g(\bar{\chi})\chi(x)}{g(A)g(B)g(\bar{C})} {}_2F_1 \left( \begin{matrix} A\chi, & B' \\ & C\chi \end{matrix} \middle| y \right)^* + \alpha + \beta, \quad (3.7)$$

where

$$\alpha = \frac{1}{q-1} \frac{g(\varepsilon)g(\bar{B}\bar{A})g(\bar{A}\bar{C})g(A)\bar{A}(x)}{g(A)g(B)g(\bar{C})} {}_2F_1 \left( \begin{matrix} \varepsilon, & B' \\ & \bar{A}\bar{C} \end{matrix} \middle| y \right)^*$$

and

$$\beta = \frac{g(\bar{A}\bar{B}'\bar{C})g(\bar{B}\bar{B}'\bar{C})g(\bar{B}')g(\bar{B}'\bar{C})\bar{B}'\bar{C}(x)}{(q-1)g(A)g(B)g(\bar{C})} {}_2F_1 \left( \begin{matrix} \bar{A}\bar{B}'\bar{C}, & B' \\ & B' \end{matrix} \middle| y \right)^*.$$

Using (1.23) we obtain

$$\alpha = -\frac{g(\bar{B}\bar{A})g(\bar{A}\bar{C})\bar{A}(x)}{(q-1)g(B)g(\bar{C})} + \frac{g(\bar{B}\bar{A})g(\bar{A}\bar{C})\bar{A}(x)\bar{A}\bar{C}(y)\bar{A}\bar{B}'\bar{C}(1-y)}{(q-1)g(B)g(\bar{C})\left(\frac{B'}{\bar{A}\bar{C}}\right)}. \quad (3.8)$$

Also, using (1.24) we obtain

$$\beta = -\frac{g(\bar{A}\bar{B}'\bar{C})g(\bar{B}\bar{B}'\bar{C})g(\bar{B}')g(\bar{B}'\bar{C})\bar{B}'\bar{C}(x)\bar{B}'(y)\left(\frac{\bar{A}\bar{C}}{\bar{B}'}\right)}{(q-1)g(A)g(B)g(\bar{C})} + \frac{g(\bar{A}\bar{B}'\bar{C})g(\bar{B}\bar{B}'\bar{C})g(\bar{B}')g(\bar{B}'\bar{C})\bar{B}'\bar{C}(x)\bar{A}\bar{B}'\bar{C}(1-y)}{(q-1)g(A)g(B)g(\bar{C})}. \quad (3.9)$$

We now apply Lemma 1.23 to obtain

$$I = \frac{\bar{B}'(1-y)}{(q-1)} \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi \neq \bar{A}, \bar{B}'\bar{C}}} \left[ \frac{g(A\chi)g(B\chi)g(\bar{C}\chi)g(\bar{\chi})\chi(x)}{g(A)g(B)g(\bar{C})} \times {}_2F_1 \left( \begin{matrix} \bar{C}\bar{A}, & B' \\ & C\chi \end{matrix} \middle| \frac{y}{y-1} \right)^* \right] + \alpha + \beta. \quad (3.10)$$

If we apply Lemma 1.25 and the fact that  $g(\varepsilon) = -1$ , then the term under summation in (3.10) for  $\chi = \bar{A}$  becomes

$$\begin{aligned} & \frac{g(\varepsilon)g(B\bar{A})g(A\bar{C})\bar{A}(x)}{g(B)g(\bar{C})} {}_2F_1 \left( \begin{matrix} C\bar{A}, & B' \\ & C\bar{A} \end{matrix} \middle| \frac{y}{y-1} \right)^* \\ &= -\frac{B'(1-y)g(B\bar{A})g(A\bar{C})\bar{A}(x)}{g(B)g(\bar{C})} + \frac{1}{q} \frac{g(B\bar{A})g(A\bar{C})\bar{A}(x)A\bar{C}(\frac{y}{1-y})}{g(B)g(\bar{C})\binom{B'}{C\bar{A}}}. \end{aligned} \quad (3.11)$$

Similarly, if we apply (1.24), then the term under summation in (3.10) for  $\chi = B'\bar{C}$  becomes

$$\begin{aligned} & \frac{g(AB'\bar{C})g(BB'\bar{C})g(\bar{B}')g(\bar{B}'C)B'\bar{C}(x)}{g(A)g(B)g(\bar{C})} {}_2F_1 \left( \begin{matrix} C\bar{A}, & B' \\ & B' \end{matrix} \middle| \frac{y}{y-1} \right)^* \\ &= -\frac{B'(1-y)\bar{B}'(y)g(AB'\bar{C})g(BB'\bar{C})g(\bar{B}')g(\bar{B}'C)B'\bar{C}(x)\binom{A\bar{C}}{B'}}{g(A)g(B)g(\bar{C})} \\ &+ \frac{\bar{A}C(1-y)g(AB'\bar{C})g(BB'\bar{C})g(\bar{B}')g(\bar{B}'C)B'\bar{C}(x)}{g(A)g(B)g(\bar{C})}. \end{aligned} \quad (3.12)$$

Putting (3.11) and (3.12) into (3.10) we obtain

$$\begin{aligned} I &= \frac{\bar{B}'(1-y)}{(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(B\chi)g(\bar{C}\chi)g(\bar{\chi})\chi(x)}{g(A)g(B)g(\bar{C})} {}_2F_1 \left( \begin{matrix} C\bar{A}, & B' \\ & C\chi \end{matrix} \middle| \frac{y}{y-1} \right)^* \\ &+ \frac{g(B\bar{A})g(A\bar{C})\bar{A}(x)}{(q-1)g(B)g(\bar{C})} - \frac{\bar{B}'(1-y)g(B\bar{A})g(A\bar{C})\bar{A}(x)A\bar{C}(\frac{y}{1-y})}{q(q-1)g(B)g(\bar{C})\binom{B'}{C\bar{A}}} \\ &+ \frac{\bar{B}'(y)g(AB'\bar{C})g(BB'\bar{C})g(\bar{B}')g(\bar{B}'C)B'\bar{C}(x)\binom{A\bar{C}}{B'}}{(q-1)g(A)g(B)g(\bar{C})} \\ &- \frac{\bar{A}\bar{B}'C(1-y)g(AB'\bar{C})g(BB'\bar{C})g(\bar{B}')g(\bar{B}'C)B'\bar{C}(x)}{(q-1)g(A)g(B)g(\bar{C})} + \alpha + \beta. \end{aligned} \quad (3.13)$$

Finally, substituting (3.8) and (3.9) into (3.13), and then applying (1.22) we deduce the result. This completes the proof of the theorem.  $\blacksquare$

*Proof of Theorem 3.2.* Let  $I = F_2(A; B, B'; C, A; x, \frac{-y}{1-y})^*$ . Then by definition we have

$$\begin{aligned}
I &= \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi\psi)g(B\chi)g(B'\psi)g(\overline{C\chi})g(\overline{A\psi})g(\overline{\chi})g(\overline{\psi})}{g(A)g(B)g(B')g(\overline{C})g(\overline{A})} \chi(x)\psi\left(\frac{-y}{1-y}\right) \\
&= \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(B\chi)g(\overline{C\chi})g(\overline{\chi})\chi(x)}{g(A)g(B)g(\overline{C})} \\
&\quad \times \frac{g(A\chi\psi)g(B'\psi)g(\overline{A\psi})g(\overline{\psi})}{g(A\chi)g(B')g(\overline{A})} \psi\left(\frac{-y}{1-y}\right) \\
&= \frac{1}{(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(B\chi)g(\overline{C\chi})g(\overline{\chi})\chi(x)}{g(A)g(B)g(\overline{C})} {}_2F_1\left(\begin{matrix} A\chi, & B' \\ & A \end{matrix} \middle| \frac{-y}{1-y}\right)^* \\
&= \frac{1}{(q-1)} \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi \neq \varepsilon, \overline{A}}} \frac{g(A\chi)g(B\chi)g(\overline{C\chi})g(\overline{\chi})\chi(x)}{g(A)g(B)g(\overline{C})} {}_2F_1\left(\begin{matrix} A\chi, & B' \\ & A \end{matrix} \middle| \frac{-y}{1-y}\right)^* \\
&\quad + \alpha_1 + \alpha_2,
\end{aligned}$$

where  $\alpha_1 = \frac{g(\varepsilon)}{q-1} {}_2F_1\left(\begin{matrix} A, & B' \\ & A \end{matrix} \middle| \frac{-y}{1-y}\right)^*$  and

$$\alpha_2 = \frac{1}{q-1} \frac{g(\varepsilon)g(B\overline{A})g(A)g(\overline{A\overline{C}})\overline{A}(x)}{g(A)g(B)g(\overline{C})} {}_2F_1\left(\begin{matrix} \varepsilon, & B' \\ & A \end{matrix} \middle| \frac{-y}{1-y}\right)^*. \text{ Using Lemma 1.25}$$

and the fact that  $g(\varepsilon) = -1$  we have

$$\alpha_1 = -\frac{B'(1-y)}{q-1} + \frac{\overline{A}\left(\frac{y}{1-y}\right)}{q(q-1)\binom{B'}{A}}. \quad (3.14)$$

Also, using (1.23) we have

$$\alpha_2 = -\frac{g(B\overline{A})g(\overline{A\overline{C}})\overline{A}(x)}{(q-1)g(B)g(\overline{C})} + \frac{g(B\overline{A})g(\overline{A\overline{C}})\overline{A}(-xy)}{(q-1)g(B)g(\overline{C})} B'(1-y) \binom{B'}{A}^{-1}. \quad (3.15)$$

From Lemma 1.23 we have

$$I = \frac{B'(1-y)}{q-1} \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi \neq \varepsilon, \bar{A}}} \frac{g(A\chi)g(B\chi)g(\overline{C\chi})g(\bar{\chi})\chi(x)}{g(A)g(B)g(\overline{C})} {}_2F_1 \left( \begin{matrix} \bar{\chi}, & B' \\ & A \end{matrix} \middle| y \right)^* + \alpha_1 + \alpha_2. \quad (3.16)$$

Now, if we apply Lemma 1.24 in (3.16) then we have

$$I = \frac{B'(1-y)}{(q-1)\binom{B'}{A}} \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi \neq \varepsilon, \bar{A}}} \left[ \frac{g(A\chi)g(B\chi)g(\overline{C\chi})g(\bar{\chi})\chi(-x)}{g(A)g(B)g(\overline{C})} \left( \begin{matrix} B' \\ \bar{\chi}B'\bar{A} \end{matrix} \right) \right. \\ \left. \times {}_2F_1 \left( \begin{matrix} \bar{\chi}, & B' \\ & \bar{\chi}B'\bar{A} \end{matrix} \middle| 1-y \right)^* \right] + \alpha_1 + \alpha_2.$$

Using (1.8) and then Lemma 1.9 we obtain

$$I = \frac{AB'(-1)B'(1-y)}{q(q-1)\binom{B'}{A}} \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi \neq \varepsilon, \bar{A}}} \left[ \frac{g(B\chi)g(\overline{C\chi})g(\bar{\chi})\chi(x)}{g(A)g(B)g(\overline{C})} g(B')g(A\bar{B}'\chi) \right. \\ \left. \times {}_2F_1 \left( \begin{matrix} \bar{\chi}, & B' \\ & \bar{\chi}B'\bar{A} \end{matrix} \middle| 1-y \right)^* \right] + \alpha_1 + \alpha_2. \quad (3.17)$$

Now, adding and subtracting the terms for  $\chi = \varepsilon, \bar{A}$  in (3.17) we have

$$I = \frac{AB'(-1)B'(1-y)}{q(q-1)\binom{B'}{A}} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \left[ \frac{g(B\chi)g(\overline{C\chi})g(\bar{\chi})\chi(x)}{g(A)g(B)g(\overline{C})} g(B')g(A\bar{B}'\chi) \right. \\ \left. \times {}_2F_1 \left( \begin{matrix} \bar{\chi}, & B' \\ & \bar{\chi}B'\bar{A} \end{matrix} \middle| 1-y \right)^* \right] + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad (3.18)$$

where  $\alpha_3 = -\frac{AB'(-1)B'(1-y)}{q(q-1)\binom{B'}{A}} \frac{g(\varepsilon)g(B')g(A\bar{B}')}{g(A)} {}_2F_1 \left( \begin{matrix} \varepsilon, & B' \\ & B'\bar{A} \end{matrix} \middle| 1-y \right)^*$  and

$$\alpha_4 = -\frac{AB'(-1)B'(1-y)}{q(q-1)\binom{B'}{A}} \frac{g(B\bar{A})g(A\bar{C})\bar{A}(x)}{g(B)g(\bar{C})} g(B')g(\bar{B}') {}_2F_1 \left( \begin{matrix} A, & B' \\ & B' \end{matrix} \middle| 1-y \right)^*.$$

Using Lemma 1.9, (1.8), (1.11) and then (1.23) we obtain

$$\alpha_3 = \frac{B'(1-y)}{q-1} - \frac{\bar{A}(\frac{y}{1-y})}{(q-1)\binom{B'}{A}}. \quad (3.19)$$

Also, applying Lemma 1.2 and then (1.24) we obtain

$$\alpha_4 = \frac{g(B\bar{A})g(A\bar{C})\bar{A}(x)}{(q-1)g(B)g(\bar{C})} - \frac{B'(1-y)g(B\bar{A})g(A\bar{C})\bar{A}(-xy)}{(q-1)g(B)g(\bar{C})\binom{B'}{A}}. \quad (3.20)$$

Now, substituting the values of  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  into (3.18) we have

$$\begin{aligned} I &= \frac{AB'(-1)B'(1-y)}{q(q-1)\binom{B'}{A}} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \left[ \frac{g(B\chi)g(\bar{C}\chi)g(\bar{\chi})\chi(x)}{g(A)g(B)g(\bar{C})} g(B')g(A\bar{B}'\chi) \right. \\ &\quad \left. \times {}_2F_1 \left( \begin{matrix} \bar{\chi}, & B' \\ & \bar{\chi}B'\bar{A} \end{matrix} \middle| 1-y \right)^* \right] - \frac{\bar{A}(\frac{y}{1-y})}{q\binom{B'}{A}} \\ &= \frac{B'(1-y)AB'(-1)}{q(q-1)^2} \binom{B'}{A}^{-1} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \left[ \frac{g(B\chi)g(\bar{C}\chi)g(\bar{\chi}\psi)g(B'\psi)g(\bar{\psi})g(\chi\bar{B}'A\psi)}{g(A)g(B)g(\bar{C})} \right. \\ &\quad \left. \times \chi(x)\psi(1-y) \right] - \frac{\bar{A}(\frac{y}{1-y})}{q\binom{B'}{A}}. \end{aligned}$$

Transforming  $\chi \mapsto \chi\psi$ , and then using Lemma 1.9, (1.8), (1.13) and (1.11) we have

$$\begin{aligned} I &= \frac{B'(1-y)AB'(-1)}{q(q-1)^2} \binom{B'}{A}^{-1} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \left[ \frac{g(B\chi\psi)g(\bar{C}\chi\psi)g(\bar{\chi})g(B'\psi)g(\bar{\psi})g(A\bar{B}'\chi)}{g(B)g(A\bar{B}')g(B')g(\bar{C})} \right. \\ &\quad \left. \times \frac{g(A\bar{B}')g(B')}{g(A)} \chi\psi(x)\psi(1-y) \right] - \frac{\bar{A}(\frac{y}{1-y})}{q\binom{B'}{A}} \\ &= B'(1-y)F_1(B; A\bar{B}', B'; C; x, x(1-y))^* - \frac{\bar{A}(\frac{y}{1-y})}{q\binom{B'}{A}}, \end{aligned}$$

which completes the proof of the theorem. ■

### 3.4 Relationships of $F_1^*$ , $F_2^*$ and $F_3^*$ with other variants

As we mentioned earlier, the finite field Appell series  $F_1^q$ ,  $F_2^q$  and  $F_3^q$  are defined using integral representations of Appell series in [43, 35, 33]. In the following theorem, we prove that the above functions  $F_1^*$ ,  $F_2^*$  and  $F_3^*$  are closely related to the finite field Appell series  $F_1^q$ ,  $F_2^q$  and  $F_3^q$ , respectively. In [43, 35, 33] it is shown that the finite field Appell series  $F_1^q$ ,  $F_2^q$  and  $F_3^q$  are related to certain binomial coefficients of multiplicative characters. Using those relations and some properties of Jacobi and Gauss sums, we obtain the following relations between  $F_i^q$  and  $F_i^*$  for  $i = 1, 2, 3$ .

**Theorem 3.3.** *Let  $A, A', B, B', C, C' \in \widehat{\mathbb{F}_q^\times}$  and  $x, y \in \mathbb{F}_q$ .*

1. *If  $A \neq C$  and  $B, B' \neq \varepsilon$ , then*

$$F_1(A; B, B'; C; x, y)^* = \frac{1}{q} \binom{A}{C}^{-1} F_1^q(A; B, B'; C; x, y).$$

2. *If  $A \neq \varepsilon$ ,  $B \neq C$  and  $B' \neq C'$ , then*

$$F_2(A; B, B'; C, C'; x, y)^* = \frac{1}{q^2} \binom{B}{C}^{-1} \binom{B'}{C'}^{-1} F_2^q(A; B, B'; C, C'; x, y).$$

3. *If  $A, A' \neq \varepsilon$ ,  $B \neq \overline{B'}$  and  $\overline{B} \neq B'\overline{C}$ , then*

$$F_3(A, A'; B, B'; C; x, y)^* = \frac{B'(-1)}{q^2} \binom{C\overline{B}B'}{C}^{-1} \binom{B}{\overline{B'}}^{-1} F_3^q(A, A'; B, B'; C; x, y).$$

**Remark 3.4.1.** *In the above theorem the conditions on the parameters are required to simplify certain character sums. However, if we relax one or more conditions the evaluation of the character sums are not straightforward, and more terms will appear. For example, if we allow  $B' = C'$  in the 2nd expression, then under the conditions  $B \neq C$ ,  $A \neq B'$  and  $A, B, B' \neq \varepsilon$ , we have the following relation between*

$F_2^q$  and  $F_2^*$ :

$$F_2(A; B, B'; C, B'; x, y)^* = -\frac{1}{q} \binom{B}{C}^{-1} F_2^q(A; B, B'; C, B'; x, y) + (q-1)\overline{B'}(y) \binom{A\overline{B'}}{A} {}_2F_1 \left( \begin{matrix} A\overline{B'}, B \\ C \end{matrix} \middle| x \right)^*. \quad (3.21)$$

Before we present the proof, we note that the binomial coefficient  $\binom{A}{B}$  defined in [43] is  $q$  times the binomial coefficient defined by Greene. Since we are using the Greene's definition of binomial coefficients, we adjust with an appropriate factor of power of  $q$  whenever we recall a result of [43, 35, 33].

*Proof of Theorem 3.3.* The result is trivial if  $xy = 0$ . So, we assume that both  $x$  and  $y$  are non-zero. By definition we have

$$F_1(A; B, B'; C; x, y)^* = \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi\psi)g(B\chi)g(B'\psi)g(\overline{C\chi\psi})g(\overline{\chi})g(\overline{\psi})\chi(x)\psi(y)}{g(A)g(B)g(B')g(\overline{C})}.$$

We multiply the numerator and denominator by  $g(A\overline{C})$  and then use Lemma 1.9 to obtain

$$F_1(A; B, B'; C; x, y)^* = \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{J(A\chi\psi, \overline{C\chi\psi})J(B\chi, \overline{\chi})J(B'\psi, \overline{\psi})\chi(x)\psi(y)}{J(A, \overline{C})} = \binom{A}{C}^{-1} \frac{q^2}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi\psi}{C\chi\psi} \binom{B\chi}{\chi} \binom{B'\psi}{\psi} \chi(x)\psi(y). \quad (3.22)$$

From [43, Theorem 1.3], we have

$$F_1^q(A; B, B'; C; x, y) = \frac{q^3}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi\psi}{C\chi\psi} \binom{B\chi}{\chi} \binom{B'\psi}{\psi} \chi(x)\psi(y). \quad (3.23)$$

Now, from (3.22) and (3.23), we readily obtain the first identity of the theorem.

We now prove the second identity. By definition, we have

$$\begin{aligned} & F_2(A; B, B'; C, C'; x, y)^* \\ &= \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi\psi)g(B\chi)g(B'\psi)g(\overline{C}\chi)g(\overline{C'}\psi)g(\overline{\chi})g(\overline{\psi})\chi(x)\psi(y)}{g(A)g(B)g(B')g(\overline{C})g(\overline{C'})}. \end{aligned}$$

We multiply the numerator and denominator by  $g(\overline{BC})g(\overline{B'C'})g(A\chi)$ , and then rearrange the terms to obtain

$$\begin{aligned} & F_2(A; B, B'; C, C'; x, y)^* \\ &= \frac{g(\overline{BC})g(\overline{B'C'})}{(q-1)^2 g(B)g(B')g(\overline{C})g(\overline{C'})} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(\overline{\chi})}{g(A)} \frac{g(A\chi\psi)g(\overline{\psi})}{g(A\chi)} \\ & \times \frac{g(B\chi)g(\overline{C}\chi)}{g(\overline{BC})} \frac{g(B'\psi)g(\overline{C'}\psi)}{g(\overline{B'C'})} \chi(x)\psi(y) \\ &= \frac{q^2}{(q-1)^2 \binom{B}{C} \binom{B'}{C'}} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \binom{A\chi\psi}{\psi} \binom{B\chi}{C\chi} \binom{B'\psi}{C'\psi} \chi(x)\psi(y) \\ & + \frac{C(-1)\overline{A}(x)\overline{C'}(y)\overline{B'}C'(1-y)J(\overline{AB}, \overline{AC})}{q^2 \binom{B}{C} \binom{B'}{C'}}. \end{aligned} \quad (3.24)$$

The last equality is obtained using Lemma 1.9, (1.10), and properties of binomial coefficients. From [35, Theorem 1.3], we have

$$F_2^q(A; B, B'; C, C'; x, y) = \frac{q^4}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \binom{A\chi\psi}{\psi} \binom{B\chi}{C\chi} \binom{B'\psi}{C'\psi} \chi(x)\psi(y)$$

$$\begin{aligned}
& + q\bar{A}(-x)\bar{C}'(y)\bar{B}'C'(1-y)\binom{\bar{A}B}{B\bar{C}} \\
& = \frac{q^4}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \binom{A\chi\psi}{\psi} \binom{B\chi}{C\chi} \binom{B'\psi}{C'\psi} \chi(x)\psi(y) \\
& + C(-1)\bar{A}(x)\bar{C}'(y)\bar{B}'C'(1-y)J(\bar{A}B, \bar{A}\bar{C}). \tag{3.25}
\end{aligned}$$

We obtain the last equality from (1.11) and (1.8). Combining (3.24) and (3.25) we complete the proof of the second identity. Finally, we consider

$$\begin{aligned}
& F_3(A, A'; B, B'; C; x, y)^* \\
& = \frac{1}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(A'\psi)g(B\chi)g(B'\psi)g(\bar{C}\chi\psi)g(\bar{\chi})g(\bar{\psi})\chi(x)\psi(y)}{g(A)g(A')g(B)g(B')g(\bar{C})}.
\end{aligned}$$

Multiplying both numerator and denominator by  $g(C\bar{B}B')g(\bar{C}B'\bar{\chi})g(\bar{B}\bar{\chi})$  we have

$$\begin{aligned}
& F_3(A, A'; B, B'; C; x, y)^* \\
& = \frac{1}{(q-1)^2 g(C\bar{B}B')g(B)g(B')g(\bar{C})} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(\bar{\chi})}{g(A)} \frac{g(A'\psi)g(\bar{\psi})}{g(A')} \\
& \times \frac{g(C\bar{B}B')g(\bar{C}B'\bar{\chi})}{g(\bar{B}\bar{\chi})} \frac{g(\bar{C}\chi\psi)g(B'\psi)}{g(\bar{C}B'\bar{\chi})} g(B\chi)g(\bar{B}\bar{\chi})\chi(x)\psi(y). \tag{3.26}
\end{aligned}$$

Lemma 1.2 yields  $g(BB')g(\bar{B}\bar{B}') = qBB'(-1)$ . Using this value and Lemma 1.9 we obtain

$$\begin{aligned}
\frac{1}{g(C\bar{B}B')g(B)g(B')g(\bar{C})} & = \frac{g(BB')g(\bar{B}\bar{B}')}{qBB'(-1)g(C\bar{B}B')g(B)g(B')g(\bar{C})} \\
& = \frac{BB'(-1)}{q} J(C\bar{B}B', \bar{C})^{-1} J(B, B')^{-1}. \tag{3.27}
\end{aligned}$$

From (3.26) and (3.27) we have

$$F_3(A, A'; B, B'; C; x, y)^*$$

$$\begin{aligned}
&= \frac{BB'(-1)}{q(q-1)^2} J(C\overline{BB'}, \overline{C})^{-1} J(B, B')^{-1} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(\overline{\chi})}{g(A)} \frac{g(A'\psi)g(\overline{\psi})}{g(A')} \\
&\times \frac{g(C\overline{BB'})g(\overline{CB'\overline{\chi}})}{g(\overline{B\chi})} \frac{g(\overline{C\chi\psi})g(B'\psi)}{g(\overline{CB'\overline{\chi}})} g(B\chi)g(\overline{B\chi})\chi(x)\psi(y).
\end{aligned}$$

Now, applying Lemma 1.9, Lemma 1.2 and the fact that  $\overline{B} \neq B'\overline{C}$  we deduce that

$$\begin{aligned}
&F_3(A, A'; B, B'; C; x, y)^* \\
&= \frac{BB'(-1)}{q(q-1)^2} J(C\overline{BB'}, \overline{C})^{-1} J(B, B')^{-1} \\
&\times \left[ qB(-1) \sum_{\chi, \psi} J(A\chi, \overline{\chi}) J(A'\psi, \overline{\psi}) J(\overline{BB'}C, B'\overline{C}\overline{\chi}) J(\overline{C\chi\psi}, B'\psi) \chi(-x)\psi(y) \right. \\
&\quad - q(q-1)B'\overline{C}(x)B\overline{C}(-1) J(\overline{BB'}C, \varepsilon) J(AB'\overline{C}, \overline{B'}C) \sum_{\psi} J(A'\psi, \overline{\psi}) \psi(-y) \\
&\quad \left. - (q-1)\overline{B}(x) J(A\overline{B}, B) (BB'C(-1) + J(\overline{BB'}C, BB'\overline{C})) \right. \\
&\quad \left. \times \sum_{\psi} J(A'\psi, \overline{\psi}) J(B\overline{C}\overline{\psi}, B'\psi) \psi(y) \right]. \tag{3.28}
\end{aligned}$$

Employing (1.8), (1.14) and the fact that  $\overline{B} \neq B'\overline{C}$  we have

$$J(\overline{BB'}C, BB'\overline{C}) = -BB'C(-1).$$

Hence, (3.28) reduces to

$$\begin{aligned}
&F_3(A, A'; B, B'; C; x, y)^* \\
&= \frac{BB'(-1)}{q(q-1)^2} J(C\overline{BB'}, \overline{C})^{-1} J(B, B')^{-1} \\
&\times \left[ qB(-1) \sum_{\chi, \psi} J(A\chi, \overline{\chi}) J(A'\psi, \overline{\psi}) J(\overline{BB'}C, B'\overline{C}\overline{\chi}) J(\overline{C\chi\psi}, B'\psi) \chi(-x)\psi(y) \right. \\
&\quad \left. - q(q-1)B'\overline{C}(x)B\overline{C}(-1) J(\overline{BB'}C, \varepsilon) J(AB'\overline{C}, \overline{B'}C) \sum_{\psi} J(A'\psi, \overline{\psi}) \psi(-y) \right]. \tag{3.29}
\end{aligned}$$

Now, applying (1.8), (1.9) and properties of binomials in (3.29) we deduce that

$$F_3(A, A'; B, B'; C; x, y)^* = \frac{B'\overline{C}(x) \binom{\overline{A}}{B'\overline{C}} \overline{A}'(1-y)}{q \binom{C\overline{B}B'}{C} \binom{B}{B'}} + \frac{B'(-1)q^2}{(q-1)^2 \binom{C\overline{B}B'}{C} \binom{B}{B'}} \\ \times \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \binom{\overline{A}}{\chi} \binom{\overline{A}'}{\psi} \binom{C\overline{B}B'}{C\overline{B}'\chi} \binom{C\overline{B}'\chi}{C\chi\psi} \chi(x)\psi(y). \quad (3.30)$$

From [33, Theorem 2.1] we have

$$F_3^q(A, A'; B, B'; C; x, y) = \frac{q^4}{(q-1)^2} \sum_{\chi, \psi \in \widehat{\mathbb{F}_q^\times}} \binom{\overline{A}}{\chi} \binom{\overline{A}'}{\psi} \binom{C\overline{B}B'}{C\overline{B}'\chi} \binom{C\overline{B}'\chi}{C\chi\psi} \chi(x)\psi(y) \\ + q \cdot B'(-1)B'\overline{C}(x) \binom{\overline{A}}{B'\overline{C}} \overline{A}'(1-y). \quad (3.31)$$

Combining (3.30) and (3.31) we deduce the result. This completes the proof of the theorem.  $\blacksquare$

### 3.5 Some applications

We next find a finite field analogue of a classical identity as an application of Theorem 3.1 and Theorem 3.3. The following identity from [6, p. 80] connects the classical Appell series  $F_3$  and a classical hypergeometric series.

$$F_3(a, c-a; b, c-b; c; x, \frac{y}{y-1}) = (1-x)^{-a}(1-y)^{c-b} {}_2F_1 \left( \begin{matrix} a, & c-b \\ & c \end{matrix} \middle| \frac{y-x}{1-x} \right). \quad (3.32)$$

In the following theorem we give a finite field analogue of (3.32).

**Theorem 3.4.** *Let  $A, B, C \in \widehat{\mathbb{F}_q^\times}$  be such that  $B \neq \varepsilon, A \neq B, A \neq C$  and  $B \neq C$ .*

For  $y \neq 1$ , we have

$$\begin{aligned} & \varepsilon(x-y)F_3(A, C\bar{A}; B, \bar{B}C; C; x, \frac{y}{y-1})^* \\ &= \varepsilon(xy)\bar{A}(1-x)\bar{B}C(1-y)_2F_1\left(A, \bar{B}C \mid \frac{y-x}{1-x}\right)^* \\ & - \varepsilon(x-y)\frac{\bar{B}(-x)\bar{B}C(-y)\bar{B}C(1-y)}{q\binom{A}{C}} \\ & - \varepsilon(x-y)\frac{g(\bar{A}B)g(\bar{A}C)\bar{A}(x)A\bar{C}(y)\bar{A}C(1-y)}{q\binom{\bar{B}C}{\bar{A}C}g(B)g(\bar{C})}. \end{aligned}$$

*Proof.* Putting  $B' = \bar{B}C$  in [43, Corollary 3.2] we have

$$\begin{aligned} \varepsilon(x-y)F_1^q(A; B, \bar{B}C; C; x, y) &= q\varepsilon(xy)\bar{A}(1-x)_2F_1\left(\bar{B}C, A \mid \frac{y-x}{1-x}\right) \\ & - \varepsilon(y-x)\bar{B}(-x)\bar{B}C(-y). \end{aligned} \quad (3.33)$$

The first identity of Theorem 3.3 for  $B' = \bar{B}C$  and (3.33) yield

$$\begin{aligned} \varepsilon(x-y)F_1(A; B, \bar{B}C; C; x, y)^* &= \binom{A}{C}^{-1} \varepsilon(xy)\bar{A}(1-x)_2F_1\left(\bar{B}C, A \mid \frac{y-x}{1-x}\right) \\ & - \frac{1}{q}\varepsilon(y-x)\binom{A}{C}^{-1} \bar{B}(-x)\bar{B}C(-y). \end{aligned} \quad (3.34)$$

Now, the required identity readily follows from Theorem 3.1 and Proposition 1.19. ■

Using the relations given in Theorem 3.3 one can easily restate the results of He and Li in terms of  $F_1^*$ ,  $F_2^*$  and  $F_3^*$ . For example, we restate two of their results in the following corollary.

**Corollary 3.5.1.** *Let  $A, A', B, B', C, C' \in \widehat{\mathbb{F}}_q^\times$  and  $x, y \in \mathbb{F}_q$ .*

1. If  $A \neq C$  and  $A, B, B' \neq \varepsilon$ , then

$$F_1(A; B, B'; C; x, y)^* = \overline{B}(1-x)\overline{B'}(1-y)F_1(\overline{AC}; B, B'; C; \frac{x}{x-1}, \frac{y}{y-1})^*.$$

2. If  $A \neq \varepsilon$ ,  $B \neq C$  and  $B' \neq C'$ , then

$$\begin{aligned} F_2(A; B, B'; C, C'; x, y)^* &= \overline{A}(1-x)F_2(A; \overline{BC}, B'; C, C'; \frac{-x}{1-x}, \frac{y}{1-x})^* \\ &= \overline{A}(1-y)F_2(A; B, \overline{B'C'}; C, C'; \frac{x}{1-y}, \frac{-y}{1-y})^* \\ &= \overline{A}(1-x-y)F_2(A; \overline{BC}, \overline{B'C'}; C, C'; \frac{-x}{1-x-y}, \frac{-y}{1-x-y})^*, \end{aligned}$$

where the first identity holds for  $x \neq 1$  and  $B \neq \varepsilon$ , the second identity holds for  $y \neq 1$  and  $B' \neq \varepsilon$  and the third identity holds for  $x+y \neq 1$  and  $B, B' \neq \varepsilon$ .

The Appell series  $F_2$  satisfies the following transformation identity [6, p. 79].

$$F_2(a; b, b'; c, c'; x, y) = (1-y)^{-a} {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| \frac{x}{1-y} \right). \quad (3.35)$$

Using (3.21) and [35, Theorem 3.3] we deduce the following corollary which is a finite field analogue of (3.35).

**Corollary 3.5.2.** Let  $A, B, B', C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B, B' \neq \varepsilon$ ,  $A \neq B'$  and  $B \neq C$ . Then for  $y \neq 0, 1$ , we have

$$\begin{aligned} F_2(A; B, B'; C, B'; x, y)^* &= \overline{A}(1-y) {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| \frac{x}{1-y} \right)^* \\ &\quad - \overline{B'}(y) \left( \frac{A\overline{B'}}{\overline{B'}} \right) {}_2F_1 \left( \begin{matrix} A\overline{B'}, & B \\ & C \end{matrix} \middle| x \right)^*. \end{aligned}$$

We note that we need to express Theorem 3.3 of [35] in terms of McCarthy's finite field hypergeometric series using Proposition 1.19 to apply (3.21).





# 4

## Appell series $F_4$ over finite fields

### 4.1 Introduction

In this chapter we define a finite field analogue of the Appell series  $F_4$ . The finite field analogues of the Appell series  $F_1, F_2$  and  $F_3$  were introduced in [43, 35, 33] by using their integral representations. The following is an integral representation of the Appell series  $F_4$  (see for example [14, (68)]):

$$F_4(a; b; c, c'; x(1-y), y(1-x)) \\ = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)} \int_0^1 \int_0^1 [u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}]$$

<sup>1</sup>The contents of this chapter have been published in *Research in Number Theory* (2018).

$$\times (1 - ux)^{a-c-c'+1}(1 - vy)^{b-c-c'+1}(1 - ux - vy)^{c+c'-a-b-1}]dudv. \quad (4.1)$$

The above integral representation of  $F_4$  is more complicated than the integral representations of  $F_1, F_2$  and  $F_3$ . Therefore, it is not straightforward to find an appropriate finite field analogue of  $F_4$  using the double integral representation (4.1). To overcome this problem, in Chapter 3, we introduced finite field analogues of Appell series purely in terms of Gauss sums. The advantage of defining finite field analogues of Appell series using Gauss sums is that we can now define a finite field analogue of  $F_4$  as follows:

$$\begin{aligned} & F_4(A; B; C, C'; x, y)^* \\ &= \frac{1}{(q-1)^2} \sum_{\psi, \chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi\psi)g(B\chi\psi)g(\overline{C}\psi)g(\overline{C'}\chi)g(\overline{\psi})g(\overline{\chi})}{g(A)g(B)g(\overline{C})g(\overline{C'})} \psi(x)\chi(y). \end{aligned} \quad (4.2)$$

## 4.2 Main results

We now establish the function  $F_4^*$  as a finite field analogue of the Appell series  $F_4$  by proving results over finite fields analogous to classical results satisfied by  $F_4$ . For example, we prove the following result.

**Theorem 4.1.** *Let  $A, B, C, C' \in \widehat{\mathbb{F}_q^\times}$ . For  $x, y \in \mathbb{F}_q$  such that  $x, y \neq 1$  we have*

$$\begin{aligned} & \overline{A}(1-x)\overline{B}(1-y)F_4\left(A; B; C, C'; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right)^* \\ &= \frac{1}{(q-1)^2} \sum_{\psi, \chi \in \widehat{\mathbb{F}_q^\times}} {}_2F_1\left(\overline{\chi}, \begin{matrix} A\psi \\ C' \end{matrix} \middle| 1\right)^* {}_2F_1\left(\overline{\psi}, \begin{matrix} B\chi \\ C \end{matrix} \middle| 1\right)^* \\ & \quad \times \frac{g(A\psi)g(B\chi)g(\overline{\chi})g(\overline{\psi})}{g(A)g(B)} \psi(-x)\chi(-y). \end{aligned}$$

The above result is a finite field analogue of the following identity [51, p. 267]

satisfied by the Appell series  $F_4$ :

$$\begin{aligned} & (1-x)^{-a}(1-y)^{-b}F_4\left(a; b; c, c'; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right) \\ &= \sum_{n,k=0}^{\infty} {}_2F_1\left(\begin{matrix} -n, & a+k \\ & c' \end{matrix} \middle| 1\right) {}_2F_1\left(\begin{matrix} -k, & b+n \\ & c \end{matrix} \middle| 1\right) \frac{(a)_k(b)_n}{k!n!} x^k y^n. \end{aligned}$$

We now state a result where the classical Appell series  $F_4$  is expressed as a product of two  ${}_2F_1$ -classical hypergeometric series. For example, see [6, p. 81] and [51, Theorem 84, p. 269]. If neither  $c$  nor  $(1-c+a+b)$  is zero or a negative integer, then

$$\begin{aligned} & F_4\left(a; b; c, 1-c+a+b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right) \\ &= {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| \frac{-x}{1-x}\right) {}_2F_1\left(\begin{matrix} a, & b \\ & 1-c+a+b \end{matrix} \middle| \frac{-y}{1-y}\right). \end{aligned} \quad (4.3)$$

We prove the following result which is a finite field analogue of (4.3).

**Theorem 4.2.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B \neq \varepsilon$ , and  $B \neq C$ . For  $x, y \in \mathbb{F}_q$  such that  $x, y \neq 1$  we have*

$$\begin{aligned} & F_4\left(A; B; C, ABC\bar{C}; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right)^* \\ &= {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \middle| -\frac{x}{1-x}\right)^* {}_2F_1\left(\begin{matrix} A, & B \\ & ABC\bar{C} \end{matrix} \middle| -\frac{y}{1-y}\right)^* \\ &+ \frac{q(q-1)\overline{BC}(y)\overline{C}(xy)A(1-x)B(1-y)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \delta(\overline{AC}) \\ &- \frac{q^2 AC(-1)\overline{BC}(y)A(1-x)B(1-y)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \delta(1-xy). \end{aligned}$$

In addition, if  $xy \neq 1$  and  $A \neq C$ , then we have

$$\begin{aligned} & F_4 \left( A; B; C, ABC\bar{C}; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right)^* \\ &= {}_2F_1 \left( A, B \mid -\frac{x}{1-x} \right)^* {}_2F_1 \left( A, B \mid -\frac{y}{1-y} \right)^*. \end{aligned}$$

We next consider the following identity from [4, (4.1)] connecting the classical Appell series  $F_4$  and  $F_1$ .

$$\begin{aligned} & F_4 \left( a; b; c, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right) \\ &= (1-x)^a (1-y)^a F_1(a; c-b; 1+a-c; c; x, xy). \end{aligned} \quad (4.4)$$

In the following theorem we give a finite field analogue of (4.4).

**Theorem 4.3.** Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $B \neq \varepsilon$  and  $A \neq B \neq C \neq A$ . For  $x, y \in \mathbb{F}_q^\times$  such that  $x, y \neq 1$  we have

$$\begin{aligned} & F_4 \left( A; B; C, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right)^* \\ &= A((1-x)(1-y)) F_1(A; \bar{B}C, \bar{A}C; C; x, xy)^* - \frac{g(B)g(\bar{A}\bar{B})}{q \cdot g(A)} \bar{B}(y) B((1-x)(1-y)). \end{aligned}$$

**Remark 4.2.1.** In Theorem 3.2 and Theorem 3.1 we proved that  $F_1^*$  is related to  $F_2^*$  and  $F_3^*$ , respectively. With the help of Theorem 4.3 we readily obtain relations connecting  $F_4^*$  to  $F_2^*$  and  $F_3^*$ , respectively. For example, let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $B \neq \varepsilon$ ,  $A \neq B \neq C \neq A$  and  $A^2 \neq C^2$ . If  $x, y \in \mathbb{F}_q^\times$  satisfy  $x, y, xy \neq 1$  then we have the following relation connecting  $F_4^*$  and  $F_3^*$ .

$$\begin{aligned} & F_4 \left( A; B; C, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right)^* \\ &= A((1-x)(1-y)) \bar{A}C(1-xy) F_3(A, \bar{A}C; \bar{B}C, \bar{A}C; C; x, \frac{xy}{xy-1})^* \end{aligned}$$

$$\begin{aligned}
& + \frac{A((1-x)(1-y)) \overline{A^2} C^2 (1-xy) g(\overline{ABC}) g(\overline{AC}) \overline{A}(x) \overline{AC}(xy)}{q \binom{AC}{AC} g(\overline{BC}) g(\overline{C})} \\
& - \frac{g(B) g(\overline{AB})}{q \cdot g(A)} \overline{B}(y) B((1-x)(1-y)).
\end{aligned}$$

In two recent papers, He [34] and Frechette, Swisher and Tu [23] have studied finite field analogues of Appell-Lauricella hypergeometric series. They give some beautiful transformation formulas satisfied by the finite field Appell-Lauricella hypergeometric series. From Theorem 4.3, Theorem 3.3 (1) and [43, Theorem 3.2, Eqn (3.7)], we also obtain the following transformation satisfied by the Appell series  $F_4^*$ . Let  $A, B, C \in \widehat{\mathbb{F}_q^\times}$  be such that  $\overline{ABC}, B \neq \varepsilon$  and  $A \neq B \neq C \neq A$ . For  $x, y \in \mathbb{F}_q^\times$  such that  $x, y \neq 1$  we have

$$\begin{aligned}
& F_4 \left( A; B; C, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right)^* \\
& = A((1-x)(1-y)) \overline{A}(y) F_4 \left( A; \overline{AB}; C, \overline{AB}; \frac{x}{y}, \frac{(1-y)(x-1)}{y} \right)^*.
\end{aligned}$$

Frechette, Swisher and Tu [23] also prove that their finite field Appell-Lauricella hypergeometric functions are related to the number of points on the generalized Picard curves over  $\mathbb{F}_p$ . From this one easily obtains relation between the finite field Appell series  $F_1$  and the number of points on the generalized Picard curves over  $\mathbb{F}_p$ . We have shown that the finite field Appell series  $F_1, F_1^*$  and  $F_4^*$  are related. Thus, one can express the number of  $\mathbb{F}_p$ -points on the generalized Picard curves in terms of the finite field Appell series  $F_4^*$ .

### 4.3 Proof of Theorem 4.1

We write  $\sum_x$  to denote the sum over all multiplicative characters on  $\mathbb{F}_q$ .

*Proof.* The result holds trivially if either  $x = 0$  or  $y = 0$ . Therefore, we assume that

both  $x$  and  $y$  are nonzero. From (4.2) and then using Lemma 1.11 we have

$$\begin{aligned} L &:= \overline{A}(1-x)\overline{B}(1-y)F_4\left(A; B; C, C'; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right)^* \\ &= \frac{1}{(q-1)^2} \sum_{\psi, \chi} \frac{g(A\chi\psi)g(B\chi\psi)g(\overline{C\psi})g(\overline{C'\chi})g(\overline{\psi})g(\overline{\chi})}{g(A)g(B)g(\overline{C})g(\overline{C'})} \psi(-x)\chi(-y) \\ &\quad \times \overline{A\chi\psi}(1-x)\overline{B\chi\psi}(1-y) \\ &= \frac{1}{(q-1)^4} \sum_{\psi, \chi, \eta, \lambda} \frac{g(A\chi\psi\eta)g(B\chi\psi\lambda)g(\overline{C\psi\eta})g(\overline{C'\chi})g(\overline{\psi})g(\overline{\chi})g(\overline{\eta})g(\overline{\lambda})}{g(A)g(B)g(\overline{C})g(\overline{C'})} \psi\eta(-x)\chi\lambda(-y). \end{aligned}$$

The change of variables  $\psi \mapsto \psi\overline{\eta}$  yields

$$L = \frac{1}{(q-1)^4} \sum_{\psi, \chi, \eta, \lambda} \frac{g(A\chi\psi)g(B\chi\psi\lambda\overline{\eta})g(\overline{C\psi\eta})g(\overline{C'\chi})g(\overline{\psi\eta})g(\overline{\chi})g(\overline{\eta})g(\overline{\lambda})}{g(A)g(B)g(\overline{C})g(\overline{C'})} \psi(-x)\chi\lambda(-y).$$

Similarly, the change of variables  $\eta \mapsto \psi\overline{\eta}$  and (1.22) yield

$$\begin{aligned} L &= \frac{1}{(q-1)^3} \sum_{\psi, \chi, \lambda} \frac{g(A\chi\psi)g(B\chi\lambda)g(\overline{C'\chi})g(\overline{\chi})g(\overline{\lambda})g(\overline{\psi})}{g(A)g(B)g(\overline{C'})} \psi(-x)\chi\lambda(-y) \\ &\quad \times {}_2F_1\left(\begin{matrix} \overline{\psi}, & B\chi\lambda \\ & C \end{matrix} \middle| 1\right)^*. \end{aligned} \quad (4.5)$$

If we apply the change of variables  $\chi \mapsto \chi\overline{\lambda}$  then (4.5) reduces to

$$\begin{aligned} L &= \frac{1}{(q-1)^3} \sum_{\psi, \chi, \lambda} \frac{g(A\chi\overline{\lambda}\psi)g(B\chi)g(\overline{C'\chi\lambda})g(\overline{\chi\lambda})g(\overline{\lambda})g(\overline{\psi})}{g(A)g(B)g(\overline{C'})} \psi(-x)\chi(-y) \\ &\quad \times {}_2F_1\left(\begin{matrix} \overline{\psi}, & B\chi \\ & C \end{matrix} \middle| 1\right)^*. \end{aligned} \quad (4.6)$$

Finally, if we apply the change of variables  $\lambda \mapsto \chi\overline{\lambda}$  then (4.6) reduces to

$$L = \frac{1}{(q-1)^3} \sum_{\psi, \chi, \lambda} \frac{g(A\lambda\psi)g(B\chi)g(\overline{C'\lambda})g(\overline{\chi\lambda})g(\overline{\lambda})g(\overline{\psi})}{g(A)g(B)g(\overline{C'})} \psi(-x)\chi(-y)$$

$$\begin{aligned}
& \times {}_2F_1 \left( \begin{matrix} \bar{\psi}, & B\chi \\ & C \end{matrix} \mid 1 \right)^* \\
& = \frac{1}{(q-1)^2} \sum_{\psi, \chi} {}_2F_1 \left( \begin{matrix} \bar{\chi}, & A\psi \\ & C' \end{matrix} \mid 1 \right)^* {}_2F_1 \left( \begin{matrix} \bar{\psi}, & B\chi \\ & C \end{matrix} \mid 1 \right)^* \\
& \quad \times \frac{g(A\psi)g(B\chi)g(\bar{\chi})g(\bar{\psi})}{g(A)g(B)} \psi(-x)\chi(-y).
\end{aligned}$$

This completes the proof of the theorem. ■

#### 4.4 Proof of Theorem 4.2

*Proof.* If  $xy = 0$ , then the result is trivial. So, we take both  $x$  and  $y$  are nonzero.

Now, from Theorem 4.1 we have

$$\begin{aligned}
L & := \bar{A}(1-x)\bar{B}(1-y)F_4 \left( A; B; C, ABC\bar{C}; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right)^* \\
& = \frac{1}{(q-1)^2} \sum_{\psi, \chi} {}_2F_1 \left( \begin{matrix} \bar{\chi}, & A\psi \\ & ABC\bar{C} \end{matrix} \mid 1 \right)^* {}_2F_1 \left( \begin{matrix} \bar{\psi}, & B\chi \\ & C \end{matrix} \mid 1 \right)^* \\
& \quad \times \frac{g(A\psi)g(B\chi)g(\bar{\chi})g(\bar{\psi})}{g(A)g(B)} \psi(-x)\chi(-y).
\end{aligned}$$

Lemma 1.20 yields

$$\begin{aligned}
L & = \frac{1}{(q-1)^2} \sum_{\psi, \chi} \left( \frac{g(\overline{ABC\bar{\chi}})g(\overline{BC\psi})}{g(\overline{ABC})g(\overline{C\psi B\chi})} + \frac{q(q-1)A\psi\chi(-1)\delta(C\psi\overline{B\chi})}{g(\bar{\chi})g(A\psi)g(\overline{ABC})} \right) \\
& \quad \times \left( \frac{g(\overline{C\psi})g(\overline{BC\chi})}{g(\overline{C})g(\overline{B\chi C\psi})} + \frac{q(q-1)B\psi\chi(-1)\delta(B\chi\overline{C\psi})}{g(\bar{\psi})g(B\chi)g(\overline{C})} \right) \\
& \quad \times \frac{g(A\psi)g(B\chi)g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(-y)}{g(A)g(B)} \\
& = \frac{1}{(q-1)^2} \sum_{\psi, \chi} \frac{g(\overline{ABC\bar{\chi}})g(\overline{BC\psi})g(\overline{C\psi})g(\overline{BC\chi})g(A\psi)g(B\chi)}{g(A)g(B)g(\overline{C})g(\overline{ABC})g(\overline{C\psi B\chi})g(\overline{B\chi C\psi})} \\
& \quad \times g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(-y) + \alpha_1 + \alpha_2 + \alpha_3,
\end{aligned} \tag{4.7}$$

where

$$\alpha_1 = A(-1) \frac{q}{q-1} \sum_{\psi, \chi} \frac{g(\overline{C\psi})g(\overline{BC\chi})g(B\chi)g(\overline{\psi})}{g(A)g(B)g(\overline{C})g(\overline{ABC})g(\overline{BC\psi\chi})} \chi(y)\psi(x)\delta(\overline{BC\chi\psi}),$$

$$\alpha_2 = B(-1) \frac{q}{q-1} \sum_{\psi, \chi} \frac{g(\overline{ABC\chi})g(\overline{BC\psi})g(A\psi)g(\overline{\chi})}{g(A)g(B)g(\overline{ABC})g(\overline{BC\chi\psi})g(\overline{C})} \chi(y)\psi(x)\delta(\overline{BC\psi\chi}),$$

$$\alpha_3 = q^2 AB(-1) \sum_{\psi, \chi} \frac{\psi(-x)\chi(-y)\delta(\overline{BC\chi\psi})\delta(\overline{BC\psi\chi})}{g(A)g(B)g(\overline{ABC})g(\overline{C})}.$$

The above terms are nonzero only when  $\overline{\chi\psi} = \overline{BC}$ . So, after putting  $\overline{\chi} = \overline{BC\psi}$  and using the fact that  $g(\varepsilon) = -1$ , we obtain

$$\alpha_1 = -A(-1) \frac{q}{q-1} \sum_{\psi} \frac{g(\overline{C\psi})g(C\psi)g(\overline{\psi})g(\psi)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \overline{BC}(y)\psi(xy), \quad (4.8)$$

$$\alpha_2 = -B(-1) \frac{q}{q-1} \sum_{\psi} \frac{g(\overline{A\psi})g(A\psi)g(\overline{BC\psi})g(\overline{BC\psi})}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \overline{BC}(y)\psi(xy), \quad (4.9)$$

$$\alpha_3 = \frac{q^2 AB(-1)\overline{BC}(-y)}{g(A)g(B)g(\overline{ABC})g(\overline{C})} \sum_{\psi} \psi(xy).$$

In case of  $\alpha_3$ , Lemma 1.1 yields

$$\alpha_3 = \frac{q^2(q-1)AC(-1)\overline{BC}(y)}{g(A)g(B)g(\overline{ABC})g(\overline{C})} \delta(1-xy). \quad (4.10)$$

Using Lemma 1.2 we have

$$\alpha_1 = -\frac{A(-1)q}{q-1} \sum_{\psi} \frac{(qC\psi(-1) - (q-1)\delta(C\psi))(q\psi(-1) - (q-1)\delta(\psi))}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \overline{BC}(y)\psi(xy).$$

Now, using Lemma 1.1 we have

$$\begin{aligned} \alpha_1 &= -\frac{q^3 AC(-1)\overline{BC}(y)\delta(1-xy)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} + \frac{q^2 AC(-1)\overline{BC}(y)\overline{C}(xy)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \\ &+ \frac{q^2 AC(-1)\overline{BC}(y)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} - \frac{q(q-1)A(-1)\overline{BC}(y)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \delta(C). \end{aligned} \quad (4.11)$$

From (4.7) we have

$$L = \frac{1}{(q-1)^2} \sum_{\substack{\psi, \chi \\ \bar{\chi}\psi \neq \overline{BC}}} \frac{g(\overline{ABC}\bar{\chi})g(\overline{BC}\psi)g(\overline{C}\bar{\psi})g(\overline{BC}\chi)g(A\psi)g(B\chi)}{g(A)g(B)g(\overline{C})g(\overline{ABC})g(C\psi\overline{B}\chi)g(B\chi\overline{C}\psi)} \\ \times g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(-y) + \beta + \alpha_1 + \alpha_2 + \alpha_3,$$

where

$$\beta = \frac{BC(-1)}{(q-1)^2} \sum_{\psi} \frac{g(\overline{A}\bar{\psi})g(A\psi)g(\overline{BC}\psi)g(\overline{BC}\bar{\psi})g(\overline{C}\bar{\psi})g(C\psi)g(\psi)g(\bar{\psi})}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \overline{BC}(y)\psi(xy).$$

Using Lemma 1.2 on  $g(C\psi\overline{B}\chi)g(B\chi\overline{C}\psi)$  we have

$$L = \frac{BC(-1)}{q(q-1)^2} \sum_{\substack{\psi, \chi \\ \bar{\chi}\psi \neq \overline{BC}}} \frac{g(\overline{ABC}\bar{\chi})g(\overline{BC}\psi)g(\overline{C}\bar{\psi})g(\overline{BC}\chi)g(A\psi)g(B\chi)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \\ \times g(\bar{\chi})g(\bar{\psi})\psi(x)\chi(y) + \beta + \alpha_1 + \alpha_2 + \alpha_3 \\ = \frac{BC(-1)}{q(q-1)^2} \sum_{\psi, \chi} \frac{g(\overline{ABC}\bar{\chi})g(\overline{BC}\psi)g(\overline{C}\bar{\psi})g(\overline{BC}\chi)g(A\psi)g(B\chi)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \quad (4.12) \\ \times g(\bar{\chi})g(\bar{\psi})\psi(x)\chi(y) + \frac{q-1}{q}\beta + \alpha_1 + \alpha_2 + \alpha_3.$$

Employing Lemma 1.2 on  $g(C\psi)g(\overline{C}\bar{\psi})$  and  $g(\psi)g(\bar{\psi})$  we have

$$\beta = \frac{q^2 B(-1)}{(q-1)^2} \sum_{\psi} \frac{g(\overline{A}\bar{\psi})g(A\psi)g(\overline{BC}\bar{\psi})g(\overline{BC}\psi)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \overline{BC}(y)\psi(xy) - \beta_1 - \beta_2 + \beta_3, \quad (4.13)$$

where

$$\beta_1 = \frac{qBC(-1)}{q-1} \sum_{\psi} \frac{g(\overline{A}\bar{\psi})g(A\psi)g(\overline{BC}\bar{\psi})g(\overline{BC}\psi)\overline{BC}(y)\psi(xy)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \psi(-1)\delta(C\psi),$$

$$\beta_2 = \frac{qBC(-1)}{q-1} \sum_{\psi} \frac{g(\overline{A\psi})g(A\psi)g(\overline{BC\psi})g(\overline{BC\psi})\overline{BC}(y)\psi(xy)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} C\psi(-1)\delta(\psi),$$

$$\beta_3 = BC(-1) \sum_{\psi} \frac{g(\overline{A\psi})g(A\psi)g(\overline{BC\psi})g(\overline{BC\psi})\overline{BC}(y)\psi(xy)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \delta(C\psi)\delta(\psi).$$

Since  $\beta_1$  is nonzero only when  $\psi = \overline{C}$ , after putting  $\psi = \overline{C}$  we obtain

$$\beta_1 = \frac{qB(-1)g(\overline{B})g(B)g(\overline{AC})g(\overline{AC})\overline{BC}(y)\overline{C}(xy)}{(q-1)g(A)g(B)g(\overline{C})g(\overline{ABC})}. \quad (4.14)$$

Using Lemma 1.2 and the fact that  $B \neq \varepsilon$ , we have

$$\beta_1 = \frac{q^3AC(-1)\overline{BC}(y)\overline{C}(xy)}{(q-1)g(A)g(B)g(\overline{C})g(\overline{ABC})} - \frac{q^2\overline{BC}(y)\overline{C}(xy)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \delta(\overline{AC}). \quad (4.15)$$

Similarly,  $\beta_2$  is nonzero only when  $\psi = \varepsilon$ , and hence after putting  $\psi = \varepsilon$  we obtain

$$\begin{aligned} \beta_2 &= \frac{qB(-1)g(\overline{BC})g(\overline{BC})g(A)g(\overline{A})\overline{BC}(y)}{(q-1)g(A)g(B)g(\overline{C})g(\overline{ABC})} \\ &= \frac{q^3AC(-1)\overline{BC}(y)}{(q-1)g(A)g(B)g(\overline{C})g(\overline{ABC})}. \end{aligned} \quad (4.16)$$

We note that the last equality is obtained using Lemma 1.2 and the fact that  $A \neq \varepsilon$  and  $B \neq C$ . Similarly,  $\beta_3$  is nonzero only when  $\psi = \varepsilon$ , and hence after putting  $\psi = \varepsilon$  we obtain

$$\begin{aligned} \beta_3 &= \frac{BC(-1)g(\overline{A})g(A)g(\overline{BC})g(\overline{BC})\overline{BC}(y)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \delta(C) \\ &= \frac{q^2A(-1)\overline{BC}(y)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \delta(C). \end{aligned} \quad (4.17)$$

We note that the last equality is obtained using Lemma 1.2 and the fact that  $A \neq \varepsilon$  and  $B \neq C$ . Putting (4.13) and (4.9) into (4.12) we obtain

$$L = \frac{BC(-1)}{q(q-1)^2} \sum_{\psi, \chi} \frac{g(\overline{ABC\chi})g(\overline{BC\psi})g(\overline{C\psi})g(\overline{BC\chi})g(A\psi)g(B\chi)}{g(A)g(B)g(\overline{C})g(\overline{ABC})}$$

$$\times g(\bar{\chi})g(\bar{\psi})\psi(x)\chi(y) - \frac{q-1}{q}\beta_1 - \frac{q-1}{q}\beta_2 + \frac{q-1}{q}\beta_3 + \alpha_1 + \alpha_3.$$

Multiplying both numerator and denominator by  $g(B\bar{C})g(\bar{B}C)$  and rearranging the terms we have

$$L = \frac{BC(-1)}{q(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(\bar{B}C\psi)g(\bar{C}\bar{\psi})g(B\chi)g(B\bar{C}\chi)g(\bar{A}BC\bar{\chi})g(B\bar{C})g(\bar{B}C)}{g(A)g(\bar{B}C)g(\bar{C})g(B)g(B\bar{C})g(\bar{A}BC)} \\ \times g(\bar{\chi})g(\bar{\psi})\psi(x)\chi(y) - \frac{q-1}{q}\beta_1 - \frac{q-1}{q}\beta_2 + \frac{q-1}{q}\beta_3 + \alpha_1 + \alpha_3.$$

Using Lemma 1.2 and the fact that  $B \neq C$  we have

$$L = \frac{1}{(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(\bar{B}C\psi)g(\bar{C}\bar{\psi})g(B\chi)g(B\bar{C}\chi)g(\bar{A}BC\bar{\chi})}{g(A)g(\bar{B}C)g(\bar{C})g(B)g(B\bar{C})g(\bar{A}BC)} \\ \times g(\bar{\chi})g(\bar{\psi})\psi(x)\chi(y) - \frac{q-1}{q}\beta_1 - \frac{q-1}{q}\beta_2 + \frac{q-1}{q}\beta_3 + \alpha_1 + \alpha_3.$$

Now, (1.22) yields

$$L = {}_2F_1 \left( \begin{matrix} A, & \bar{B}C \\ & C \end{matrix} \middle| x \right)^* {}_2F_1 \left( \begin{matrix} B, & B\bar{C} \\ & ABC \end{matrix} \middle| y \right)^* - \frac{q-1}{q}\beta_1 - \frac{q-1}{q}\beta_2 \\ + \frac{q-1}{q}\beta_3 + \alpha_1 + \alpha_3. \quad (4.18)$$

Using Lemma 1.21 and  $x, y \neq 1$ , we have

$$L = \bar{A}(1-x)\bar{B}(1-y) {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| \frac{-x}{1-x} \right)^* {}_2F_1 \left( \begin{matrix} B, & A \\ & ABC \end{matrix} \middle| \frac{-y}{1-y} \right)^* \\ - \frac{q-1}{q}\beta_1 - \frac{q-1}{q}\beta_2 + \frac{q-1}{q}\beta_3 + \alpha_1 + \alpha_3. \quad (4.19)$$

Applying (4.10), (4.11), (4.15), (4.16) and (4.17) into (4.19) we have

$$L = \bar{A}(1-x)\bar{B}(1-y) {}_2F_1 \left( \begin{matrix} A, & B \\ & C \end{matrix} \middle| \frac{-x}{1-x} \right)^* {}_2F_1 \left( \begin{matrix} B, & A \\ & ABC \end{matrix} \middle| \frac{-y}{1-y} \right)^* \\ + \frac{q(q-1)\bar{B}C(y)\bar{C}(xy)}{g(A)g(B)g(\bar{C})g(\overline{ABC})} \delta(\bar{AC}) - \frac{q^2 AC(-1)\bar{B}C(y)\delta(1-xy)}{g(A)g(B)g(\bar{C})g(\overline{ABC})}.$$

Finally, multiplying both sides by  $A(1-x)B(1-y)$ , we complete the proof of the theorem.  $\blacksquare$

**Corollary 4.4.1.** *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B \neq \varepsilon$ , and  $A \neq B$ . For  $x, y \in \mathbb{F}_q^\times$  such that  $x, y, xy \neq 1$  we have*

$$F_4 \left( A; B; A, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right)^* \\ = B(1-x)A(1-y) - \frac{\binom{B}{A}^{-1} A((1-x)(1-y)) \bar{A}(x)}{q} \\ - \frac{\binom{A}{B}^{-1} B((1-x)(1-y)) \bar{B}(y)}{q} + \bar{A} \left( \frac{x}{x-1} \right) \bar{B} \left( \frac{y}{y-1} \right).$$

*Proof.* If we put  $A = C$  in Theorem 4.2, and then use Lemma 1.25, (1.8), Lemma 1.9 and Lemma 1.2, we readily deduce the desired identity.  $\blacksquare$

The above result is a finite field analogue of the following identity [4, (4.2)] satisfied by the appell series  $F_4$ :

$$F_4 \left( a; b; a, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right) \\ = (1-xy)^{-1}(1-x)^b(1-y)^a.$$

## 4.5 Proof of Theorem 4.3

*Proof.* From Theorem 4.1 we have

$$\begin{aligned} L &:= \bar{A}(1-x)\bar{B}(1-y)F_4\left(A; B; C, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right)^* \\ &= \frac{1}{(q-1)^2} \sum_{\psi, \chi \in \widehat{\mathbb{F}_q^\times}} {}_2F_1\left(\begin{matrix} \bar{\chi}, & A\psi \\ & B \end{matrix} \middle| 1\right)^* {}_2F_1\left(\begin{matrix} \bar{\psi}, & B\chi \\ & C \end{matrix} \middle| 1\right)^* \\ &\quad \times \frac{g(A\psi)g(B\chi)g(\bar{\chi})g(\bar{\psi})}{g(A)g(B)}\psi(-x)\chi(-y). \end{aligned}$$

Using Lemma 1.20 we have

$$\begin{aligned} L &= \frac{1}{(q-1)^2} \sum_{\psi, \chi} \left( \frac{g(\bar{B}\chi)g(A\bar{B}\psi)}{g(\bar{B})g(A\psi\bar{B}\chi)} + \frac{q(q-1)A\psi\chi(-1)\delta(A\psi\bar{B}\chi)}{g(\bar{\chi})g(A\psi)g(\bar{B})} \right) \\ &\quad \times \left( \frac{g(\bar{C}\psi)g(B\bar{C}\chi)}{g(\bar{C})g(B\chi\bar{C}\psi)} + \frac{q(q-1)B\psi\chi(-1)\delta(B\chi\bar{C}\psi)}{g(\bar{\psi})g(B\chi)g(\bar{C})} \right) \frac{g(A\psi)g(B\chi)g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(-y)}{g(A)g(B)} \\ &= \frac{1}{(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(A\bar{B}\psi)g(\bar{C}\psi)g(B\chi)g(B\bar{C}\chi)g(\bar{B}\chi)}{g(A)g(B)g(\bar{C})g(\bar{B})g(A\bar{B}\chi\psi)g(B\bar{C}\psi\chi)} \\ &\quad \times g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(-y) + \alpha + \beta + \gamma, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \frac{qA(-1)}{q-1} \sum_{\psi, \chi} \frac{g(\bar{C}\psi)g(B\bar{C}\chi)g(B\chi)g(\bar{\psi})}{g(A)g(B)g(\bar{C})g(\bar{B})g(B\chi\bar{C}\psi)}\psi(x)\chi(y)\delta(A\bar{B}\chi\psi), \\ \beta &= \frac{qB(-1)}{q-1} \sum_{\psi, \chi} \frac{g(\bar{B}\chi)g(A\bar{B}\psi)g(A\psi)g(\bar{\chi})}{g(A)g(B)g(\bar{C})g(\bar{B})g(A\psi\bar{B}\chi)}\psi(x)\chi(y)\delta(B\chi\bar{C}\psi), \\ \gamma &= q^2AB(-1) \sum_{\psi, \chi} \frac{\psi(-x)\chi(-y)\delta(A\psi\bar{B}\chi)\delta(B\chi\bar{C}\psi)}{g(A)g(B)g(\bar{B})g(\bar{C})}. \end{aligned}$$

The term  $\alpha$  is nonzero only when  $\bar{\chi}\psi = \bar{A}B$ . After putting  $\bar{\chi} = \bar{A}\bar{\psi}B$  we obtain

$$\alpha = \frac{qA(-1)}{q-1} \sum_{\psi} \frac{g(\bar{C}\bar{\psi})g(\bar{A}\bar{C}\psi)g(A\psi)g(\bar{\psi})}{g(A)g(B)g(\bar{C})g(\bar{B})g(\bar{A}\bar{C})} \bar{A}\bar{B}(y)\psi(xy).$$

Using Lemma 1.2 and the fact that  $B \neq \varepsilon$  we have

$$\alpha = \frac{AB(-1)}{q-1} \sum_{\psi} \frac{g(\bar{C}\bar{\psi})g(\bar{A}\bar{C}\psi)g(A\psi)g(\bar{\psi})}{g(A)g(\bar{C})g(\bar{A}\bar{C})} \bar{A}\bar{B}(y)\psi(xy).$$

Now, multiplying both numerator and denominator by  $g(\bar{A}\bar{C})$  and then using Lemma 1.2 and the fact that  $A \neq C$  we have

$$\alpha = \frac{BC(-1)g(\bar{A}\bar{C})}{q(q-1)} \sum_{\psi} \frac{g(\bar{C}\bar{\psi})g(\bar{A}\bar{C}\psi)g(A\psi)g(\bar{\psi})}{g(A)g(\bar{C})} \bar{A}\bar{B}(y)\psi(xy). \quad (4.20)$$

The term  $\beta$  is nonzero only when  $\bar{\psi}\chi = \bar{B}C$ . After putting  $\chi = \bar{B}C\psi$  we obtain

$$\beta = \frac{qB(-1)}{q-1} \sum_{\psi} \frac{g(\bar{C}\bar{\psi})g(\bar{A}\bar{B}\psi)g(A\psi)g(\bar{B}C\bar{\psi})}{g(A)g(B)g(\bar{C})g(\bar{B})g(\bar{A}\bar{C})} \bar{B}C(y)\psi(xy).$$

Using Lemma 1.2 and the fact that  $B \neq \varepsilon$  we have

$$\beta = \frac{1}{q-1} \sum_{\psi} \frac{g(\bar{C}\bar{\psi})g(\bar{A}\bar{B}\psi)g(A\psi)g(\bar{B}C\bar{\psi})}{g(A)g(\bar{C})g(\bar{A}\bar{C})} \bar{B}C(y)\psi(xy).$$

Multiplying both numerator and denominator by  $g(\bar{A}\bar{C})$  and then using Lemma 1.2 and the fact that  $A \neq C$  we have

$$\beta = \frac{AC(-1)g(\bar{A}\bar{C})}{q(q-1)} \sum_{\psi} \frac{g(\bar{C}\bar{\psi})g(\bar{A}\bar{B}\psi)g(A\psi)g(\bar{B}C\bar{\psi})}{g(A)g(\bar{C})} \bar{B}C(y)\psi(xy). \quad (4.21)$$

By the fact that  $A \neq C$  we obtain  $\gamma = 0$ . Using Lemma 1.2 on  $g(B)g(\bar{B})$  and the fact that  $B \neq \varepsilon$  we have

$$L = \frac{B(-1)}{q(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(A\bar{B}\psi)g(\bar{C}\bar{\psi})g(B\chi)g(\bar{B}\bar{C}\chi)g(\bar{B}\bar{\chi})}{g(A)g(\bar{C})g(A\bar{B}\bar{\chi}\psi)g(\bar{B}\bar{C}\bar{\psi}\chi)} \\ \times g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(-y) + \alpha + \beta.$$

Again, using Lemma 1.2 on  $g(B\chi)g(\bar{B}\bar{\chi})$  we have

$$L = \frac{1}{(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(A\bar{B}\psi)g(\bar{C}\bar{\psi})g(\bar{B}\bar{C}\chi)}{g(A)g(\bar{C})g(A\bar{B}\bar{\chi}\psi)g(\bar{B}\bar{C}\bar{\psi}\chi)} \\ \times g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(y) - \alpha_1 + \alpha + \beta,$$

where

$$\alpha_1 = \frac{B(-1)}{q(q-1)} \sum_{\psi, \chi} \frac{g(A\psi)g(A\bar{B}\psi)g(\bar{C}\bar{\psi})g(\bar{B}\bar{C}\chi)}{g(A)g(\bar{C})g(A\bar{B}\bar{\chi}\psi)g(\bar{B}\bar{C}\bar{\psi}\chi)} g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(-y)\delta(B\chi).$$

The term  $\alpha_1$  is nonzero only when  $\chi = \bar{B}$ , and hence after putting  $\chi = \bar{B}$  we obtain

$$\alpha_1 = \frac{\bar{B}(y)g(B)}{q(q-1)g(A)} \sum_{\psi} g(A\bar{B}\psi)g(\bar{\psi})\psi(-x).$$

Multiplying both numerator and denominator by  $g(A\bar{B})$  and then using Lemma 1.11 we have

$$\alpha_1 = \frac{g(A\bar{B})g(B)}{qg(A)} \bar{B}(y)\bar{A}\bar{B}(1-x). \quad (4.22)$$

Now, multiplying both numerator and denominator by  $g(\bar{A}\bar{B}\bar{\chi}\bar{\psi})g(\bar{B}\bar{C}\bar{\psi}\bar{\chi})$  we have

$$L = \frac{1}{(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(A\bar{B}\psi)g(\bar{C}\bar{\psi})g(\bar{B}\bar{C}\chi)g(\bar{A}\bar{B}\bar{\chi}\bar{\psi})g(\bar{B}\bar{C}\bar{\psi}\bar{\chi})}{g(A)g(\bar{C})g(A\bar{B}\bar{\chi}\psi)g(\bar{A}\bar{B}\bar{\chi}\bar{\psi})g(\bar{B}\bar{C}\bar{\psi}\chi)g(\bar{B}\bar{C}\bar{\psi}\bar{\chi})}$$

$$\begin{aligned}
& \times g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(y) - \alpha_1 + \alpha + \beta \\
= & \frac{1}{(q-1)^2} \sum_{\substack{\psi, \chi \\ \bar{\chi}\psi \neq \bar{A}B, \bar{B}\bar{C}}} \frac{g(A\psi)g(\bar{A}\bar{B}\psi)g(\bar{C}\bar{\psi})g(\bar{B}\bar{C}\chi)g(\bar{A}\bar{B}\chi\bar{\psi})g(\bar{B}\bar{C}\psi\bar{\chi})}{g(A)g(\bar{C})g(\bar{A}\bar{B}\chi\psi)g(\bar{A}\bar{B}\chi\bar{\psi})g(\bar{B}\bar{C}\psi\chi)g(\bar{B}\bar{C}\psi\bar{\chi})} \\
& \times g(\bar{\chi})g(\bar{\psi})\psi(-x)\chi(y) + \alpha_2 + \alpha_3 - \alpha_1 + \alpha + \beta,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_2 &= \frac{1}{(q-1)^2} \sum_{\substack{\psi \\ \bar{\chi} = \bar{A}\bar{\psi}B}} \frac{g(A\psi)g(\bar{A}\bar{B}\psi)g(\bar{C}\bar{\psi})g(\bar{A}\bar{C}\psi)g(\varepsilon)g(\bar{A}\bar{C})g(\bar{A}\bar{B}\bar{\psi})}{g(A)g(\bar{C})g(\varepsilon)g(\bar{A}\bar{C})g(\varepsilon)g(\bar{A}\bar{C})} \\
& \quad \times g(\bar{\psi})\bar{A}\bar{B}(y)\psi(-xy), \\
\alpha_3 &= \frac{1}{(q-1)^2} \sum_{\substack{\psi \\ \bar{\chi} = \bar{B}\bar{C}\bar{\psi}}} \frac{g(A\psi)g(\bar{A}\bar{B}\psi)g(\bar{C}\bar{\psi})g(\psi)g(\bar{A}\bar{C})g(\varepsilon)g(\bar{B}\bar{C}\bar{\psi})}{g(A)g(\bar{C})g(\bar{A}\bar{C})g(\varepsilon)g(\bar{A}\bar{C})g(\varepsilon)} \\
& \quad \times g(\bar{\psi})\bar{B}\bar{C}(y)\psi(-xy).
\end{aligned}$$

Using Lemma 1.2 on  $g(\bar{A}\bar{C})g(\bar{A}\bar{C})$  and the fact that  $g(\varepsilon) = -1$  and  $A \neq C$  we have

$$\begin{aligned}
\alpha_2 &= -\frac{AC(-1)}{q(q-1)^2} \sum_{\substack{\psi \\ \bar{\chi} = \bar{A}\bar{\psi}B}} \frac{g(A\psi)g(\bar{A}\bar{B}\psi)g(\bar{C}\bar{\psi})g(\bar{A}\bar{C}\psi)g(\bar{A}\bar{C})g(\bar{A}\bar{B}\bar{\psi})}{g(A)g(\bar{C})} \\
& \quad \times g(\bar{\psi})\bar{A}\bar{B}(y)\psi(-xy). \tag{4.23}
\end{aligned}$$

Again, using Lemma 1.2 on  $g(\bar{A}\bar{B}\psi)g(\bar{A}\bar{\psi}B)$  we have

$$\alpha_2 = -\frac{BC(-1)}{(q-1)^2} \sum_{\psi} \frac{g(A\psi)g(\bar{C}\bar{\psi})g(\bar{A}\bar{C}\psi)g(\bar{A}\bar{C})g(\bar{\psi})}{g(A)g(\bar{C})} \bar{A}\bar{B}(y)\psi(xy) + I_1, \tag{4.24}$$

where

$$I_1 = \frac{AC(-1)}{q(q-1)} \sum_{\psi} \frac{g(A\psi)g(\bar{C}\bar{\psi})g(\bar{A}\bar{C}\psi)g(\bar{A}\bar{C})g(\bar{\psi})}{g(A)g(\bar{C})} \bar{A}\bar{B}(y)\psi(-xy)\delta(\bar{A}\bar{B}\psi).$$

The term  $I_1$  is nonzero only when  $\psi = \overline{AB}$ . After putting  $\psi = \overline{AB}$  we obtain

$$I_1 = \frac{BC(-1)g(B)g(\overline{ABC})g(\overline{BC})g(\overline{AC})g(\overline{AB})\overline{AB}(x)}{q(q-1)g(A)g(\overline{C})}. \quad (4.25)$$

Using Lemma 1.2 on  $g(\overline{AC})g(\overline{AC})$  and the fact that  $g(\varepsilon) = -1$  and  $A \neq C$  we have

$$\alpha_3 = -\frac{AC(-1)}{q(q-1)^2} \sum_{\substack{\psi \\ \overline{\chi} = \overline{BC\psi}}} \frac{g(A\psi)g(\overline{AB\psi})g(\overline{C\psi})g(\psi)g(\overline{AC})g(\overline{BC\psi})}{g(A)g(\overline{C})} \quad (4.26)$$

$$\times g(\overline{\psi})\overline{BC}(y)\psi(-xy).$$

Employing Lemma 1.2 on  $g(\psi)g(\overline{\psi})$  we have

$$\alpha_3 = -\frac{AC(-1)}{(q-1)^2} \sum_{\psi} \frac{g(A\psi)g(\overline{AB\psi})g(\overline{C\psi})g(\overline{AC})g(\overline{BC\psi})}{g(A)g(\overline{C})} \overline{BC}(y)\psi(xy) + I_2, \quad (4.27)$$

where

$$I_2 = \frac{AC(-1)}{q(q-1)} \sum_{\psi} \frac{g(A\psi)g(\overline{AB\psi})g(\overline{C\psi})g(\overline{AC})g(\overline{BC\psi})}{g(A)g(\overline{C})} \overline{BC}(y)\psi(-xy)\delta(\psi).$$

The term  $I_2$  is nonzero only when  $\psi = \varepsilon$ , and so after putting  $\psi = \varepsilon$  we obtain

$$I_2 = \frac{AC(-1)g(\overline{AB})g(\overline{BC})g(\overline{AC})\overline{BC}(y)}{q(q-1)}. \quad (4.28)$$

Using Lemma 1.2 on  $g(\overline{AB\chi\psi})g(\overline{AB\chi\psi})$  and  $g(\overline{BC\psi\chi})g(\overline{BC\psi\chi})$  we have

$$L = \frac{AC(-1)}{q^2(q-1)^2} \sum_{\substack{\psi, \chi \\ \overline{\chi\psi} \neq \overline{AB}, \overline{BC}}} \frac{g(A\psi)g(\overline{AB\psi})g(\overline{C\psi})g(\overline{BC\chi})g(\overline{AB\chi\psi})g(\overline{BC\psi\chi})}{g(A)g(\overline{C})} \quad (4.29)$$

$$\times g(\overline{\chi})g(\overline{\psi})\psi(-x)\chi(y) + \alpha_2 + \alpha_3 - \alpha_1 + \alpha + \beta.$$

The term under summation for  $\bar{\chi} = \overline{A\psi}B$  in (4.29) is equal to

$$-\frac{AC(-1)}{q^2(q-1)^2} \sum_{\substack{\psi \\ \bar{\chi}=\overline{A\psi}B}} \frac{g(A\psi)g(\overline{A\psi})g(\overline{C\psi})g(\overline{AC\psi})g(\overline{AC})g(\overline{A\psi})}{g(A)g(\overline{C})} \quad (4.30)$$

$$\times g(\overline{\psi})\overline{AB}(y)\psi(-xy).$$

Hence, applying (4.23) and (4.30) into (4.29), we obtain

$$L = \frac{AC(-1)}{q^2(q-1)^2} \sum_{\substack{\psi, \chi \\ \bar{\chi}\psi \neq \overline{BC}}} \frac{g(A\psi)g(\overline{A\psi})g(\overline{C\psi})g(\overline{BC\chi})g(\overline{AB\chi\psi})g(\overline{BC\psi\bar{\chi}})}{g(A)g(\overline{C})} \quad (4.31)$$

$$\times g(\bar{\chi})g(\overline{\psi})\psi(-x)\chi(y) + \frac{q-1}{q}\alpha_2 + \alpha_3 - \alpha_1 + \alpha + \beta.$$

Similarly, the term under summation for  $\bar{\chi} = \overline{C\psi}B$  in (4.31) is equal to

$$-\frac{AC(-1)}{q^2(q-1)^2} \sum_{\substack{\psi \\ \bar{\chi}=\overline{BC\psi}}} \frac{g(A\psi)g(\overline{A\psi})g(\overline{C\psi})g(\psi)g(\overline{AC})g(\overline{BC\psi})}{g(A)g(\overline{C})} \quad (4.32)$$

$$\times g(\overline{\psi})\overline{BC}(y)\psi(-xy).$$

Then applying (4.26) and (4.32) into (4.31) we obtain

$$L = \frac{AC(-1)}{q^2(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(\overline{A\psi})g(\overline{C\psi})g(\overline{BC\chi})g(\overline{AB\chi\psi})g(\overline{BC\psi\bar{\chi}})}{g(A)g(\overline{C})} \quad (4.33)$$

$$\times g(\bar{\chi})g(\overline{\psi})\psi(-x)\chi(y) + \frac{q-1}{q}\alpha_2 + \frac{q-1}{q}\alpha_3 - \alpha_1 + \alpha + \beta.$$

Applying (4.20), (4.24) and (4.21), (4.27) into (4.33) we obtain

$$L = \frac{AC(-1)}{q^2(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(\overline{A\psi})g(\overline{C\psi})g(\overline{BC\chi})g(\overline{AB\chi\psi})g(\overline{BC\psi\bar{\chi}})}{g(A)g(\overline{C})} \quad (4.34)$$

$$\times g(\bar{\chi})g(\overline{\psi})\psi(-x)\chi(y) + \frac{q-1}{q}I_1 + \frac{q-1}{q}I_2 - \alpha_1.$$

Now, multiplying both numerator and denominator by  $g(\overline{AB\psi})g(\overline{BC\psi})g(\overline{BC})$  and then rearranging the terms, we obtain

$$L = \frac{AC(-1)}{q^2(q-1)^2} \sum_{\psi, \chi} \frac{g(\overline{A\psi B\chi})g(\overline{BC\chi})g(\overline{B\chi C\psi})g(\overline{\chi})g(A\psi)g(\overline{C\psi})g(\overline{AB\psi})}{g(\overline{AB\psi})g(\overline{BC})g(\overline{BC\psi})g(A)g(\overline{C})} \\ \times g(\overline{AB\psi})g(\overline{BC\psi})g(\overline{BC})g(\overline{\psi})\psi(-x)\chi(y) + \frac{q-1}{q}I_1 + \frac{q-1}{q}I_2 - \alpha_1.$$

Using (1.22) we have

$$L = \frac{AC(-1)}{q^2(q-1)} \sum_{\psi} \frac{g(A\psi)g(\overline{C\psi})g(\overline{AB\psi})g(\overline{AB\psi})g(\overline{BC\psi})g(\overline{BC})g(\overline{\psi})}{g(A)g(\overline{C})} \psi(-x) \\ \times {}_2F_1 \left( \begin{matrix} \overline{AB\psi}, & \overline{BC} \\ & \overline{BC\psi} \end{matrix} \middle| y \right)^* + \frac{q-1}{q}I_1 + \frac{q-1}{q}I_2 - \alpha_1 \\ = \frac{AC(-1)}{q^2(q-1)} \sum_{\substack{\psi \\ \psi \neq \varepsilon, \overline{AB}}} \frac{g(A\psi)g(\overline{C\psi})g(\overline{AB\psi})g(\overline{AB\psi})g(\overline{BC\psi})g(\overline{BC})g(\overline{\psi})}{g(A)g(\overline{C})} \psi(-x) \\ \times {}_2F_1 \left( \begin{matrix} \overline{AB\psi}, & \overline{BC} \\ & \overline{BC\psi} \end{matrix} \middle| y \right)^* + \beta_1 + \beta_2 + \frac{q-1}{q}I_1 + \frac{q-1}{q}I_2 - \alpha_1,$$

where

$$\beta_1 = \frac{AC(-1)g(\overline{AB})g(\overline{AB})g(\overline{BC})g(\overline{BC})g(\varepsilon)}{q^2(q-1)} {}_2F_1 \left( \begin{matrix} \overline{AB}, & \overline{BC} \\ & \overline{BC} \end{matrix} \middle| y \right)^*, \\ \beta_2 = \frac{AC(-1)g(B)g(\overline{ABC})g(\varepsilon)^2g(\overline{AC})g(\overline{BC})g(\overline{AB})\overline{AB}(-x)}{q^2(q-1)g(A)g(\overline{C})} {}_2F_1 \left( \begin{matrix} \varepsilon, & \overline{BC} \\ & \overline{AC} \end{matrix} \middle| y \right)^*.$$

Using Lemma 1.2 and the fact that  $A \neq B, B \neq C$  and  $g(\varepsilon) = -1$  we obtain

$$\beta_1 = -\frac{1}{(q-1)} {}_2F_1 \left( \begin{matrix} \overline{AB}, & \overline{BC} \\ & \overline{BC} \end{matrix} \middle| y \right)^*.$$

Using (1.24) we have

$$\beta_1 = \left( \frac{\overline{AC}}{\overline{BC}} \right) \frac{\overline{BC}(y)}{(q-1)} - \frac{\overline{AB}(1-y)}{q-1}. \quad (4.35)$$

Using (1.23) and the fact that  $g(\varepsilon) = -1$  we have

$$\begin{aligned} \beta_2 &= \frac{BC(-1)g(B)g(\overline{ABC})g(\overline{AC})g(\overline{AB})g(\overline{BC})\overline{AB}(x)}{q^2(q-1)g(A)g(\overline{C})} \\ &\quad - \left( \frac{\overline{BC}}{\overline{AC}} \right)^{-1} \frac{BC(-1)g(B)g(\overline{ABC})g(\overline{AC})g(\overline{AB})g(\overline{BC})\overline{AB}(x)\overline{AC}(y)\overline{AB}(1-y)}{q^2(q-1)g(A)g(\overline{C})}. \end{aligned}$$

Applying (1.12) on the second term we obtain

$$\begin{aligned} \beta_2 &= \frac{BC(-1)g(B)g(\overline{ABC})g(\overline{AC})g(\overline{AB})g(\overline{BC})\overline{AB}(x)}{q^2(q-1)g(A)g(\overline{C})} \quad (4.36) \\ &\quad - \left( \frac{\overline{AB}}{\overline{AC}} \right)^{-1} \frac{AB(-1)g(B)g(\overline{ABC})g(\overline{AC})g(\overline{AB})g(\overline{BC})\overline{AB}(x)\overline{AC}(y)\overline{AB}(1-y)}{q^2(q-1)g(A)g(\overline{C})}. \end{aligned}$$

Lemma 1.22 yields

$$\begin{aligned} L &= \frac{AC(-1)}{q^2(q-1)} \sum_{\substack{\psi \\ \psi \neq \varepsilon, \overline{AB}}} \frac{g(A\psi)g(\overline{C}\psi)g(\overline{AB}\psi)g(\overline{AB}\psi)g(\overline{BC}\psi)g(\overline{BC})g(\overline{\psi})}{g(A)g(\overline{C})} \psi(-x) \\ &\quad \times \overline{AB}(1-y) {}_2F_1 \left( \begin{matrix} \overline{AC}, & \overline{\psi} \\ & \overline{BC}\psi \end{matrix} \middle| y \right)^* + \beta_1 + \beta_2 + \frac{q-1}{q} I_1 + \frac{q-1}{q} I_2 - \alpha_1. \end{aligned} \quad (4.37)$$

Using Lemma 1.2 on  $g(\overline{AB}\psi)g(\overline{AB}\psi)$  in (4.37) and the fact that  $\psi \neq \overline{AB}$  we have

$$\begin{aligned} L &= \frac{BC(-1)}{q(q-1)} \sum_{\substack{\psi \\ \psi \neq \varepsilon, \overline{AB}}} \frac{g(A\psi)g(\overline{C}\psi)g(\overline{BC}\psi)g(\overline{BC})g(\overline{\psi})}{g(A)g(\overline{C})} \psi(x) \quad (4.38) \\ &\quad \times \overline{AB}(1-y) {}_2F_1 \left( \begin{matrix} \overline{AC}, & \overline{\psi} \\ & \overline{BC}\psi \end{matrix} \middle| y \right)^* + \beta_1 + \beta_2 + \frac{q-1}{q} I_1 + \frac{q-1}{q} I_2 - \alpha_1. \end{aligned}$$

If we consider the term under summation for  $\psi = \varepsilon$ , and then using Lemma 1.2, (1.23) and the fact that  $g(\varepsilon) = -1$  we obtain

$$I_3 = -\frac{A\bar{B}(1-y)}{q-1} + \left(\frac{A\bar{C}}{B\bar{C}}\right)^{-1} \frac{\bar{B}C(y)}{q-1}. \quad (4.39)$$

Now, using (4.35) and (4.39) we have

$$\begin{aligned} \beta_1 - I_3 &= \left(\frac{A\bar{C}}{B\bar{C}}\right) \frac{\bar{B}C(y)}{(q-1)} - \left(\frac{A\bar{C}}{B\bar{C}}\right)^{-1} \frac{\bar{B}C(y)}{q-1} \\ &= -\frac{qAC(-1)\bar{B}C(y)}{g(\bar{A}B)g(A\bar{C})g(B\bar{C})}. \end{aligned}$$

The last equality is obtained by using (1.8), Lemma 1.9 and Lemma 1.2. If we multiply both numerator and denominator by  $g(B\bar{C})g(A\bar{B})g(\bar{A}C)$  and then use Lemma 1.2 we have

$$\beta_1 - I_3 = -\frac{AC(-1)g(\bar{C}B)g(A\bar{B})g(\bar{A}C)\bar{B}C(y)}{q^2}. \quad (4.40)$$

Considering the term under summation for  $\psi = \bar{A}B$  in (4.38), and then using Lemma 1.25 we have

$$\begin{aligned} I_4 &= BC(-1) \frac{g(B)g(A\bar{B}\bar{C})g(\bar{A}C)g(A\bar{B})g(B\bar{C})\bar{A}B(x)}{q(q-1)g(A)g(\bar{C})} \\ &\quad - AB(-1) \left(\frac{A\bar{B}}{A\bar{C}}\right)^{-1} \frac{g(B)g(A\bar{B}\bar{C})g(\bar{A}C)g(A\bar{B})g(B\bar{C})\bar{A}B(x)A\bar{B}(1-y)\bar{A}C(y)}{q^2(q-1)g(A)g(\bar{C})}. \end{aligned} \quad (4.41)$$

Now, using (4.36) and (4.41) we have

$$\beta_2 - I_4 = -\frac{BC(-1)g(B)g(A\bar{B}\bar{C})g(\bar{A}C)g(A\bar{B})g(B\bar{C})\bar{A}B(x)}{q^2g(A)g(\bar{C})}. \quad (4.42)$$

Multiplying both numerator and denominator by  $g(\overline{BC})$  and then using Lemma 1.2 on  $g(\overline{BC})g(\overline{BC})$  and the fact that  $B \neq C$ , we have

$$L = \frac{1}{(q-1)} \sum_{\psi} \frac{g(A\psi)g(\overline{C\psi})g(\overline{BC\psi})g(\overline{\psi})}{g(A)g(\overline{C})g(\overline{BC})} \psi(x) \quad (4.43)$$

$$\times A\overline{B}(1-y) {}_2F_1 \left( \begin{matrix} A\overline{C}, & \overline{\psi} \\ & \overline{BC\psi} \end{matrix} \middle| y \right)^* + \beta_1 + \beta_2 - I_3 - I_4 + \frac{q-1}{q} I_1 + \frac{q-1}{q} I_2 - \alpha_1.$$

Employing (4.25), (4.28), (4.42) and (4.40) into (4.43) we obtain

$$L = \frac{1}{(q-1)} \sum_{\psi} \frac{g(A\psi)g(\overline{C\psi})g(\overline{BC\psi})g(\overline{\psi})}{g(A)g(\overline{C})g(\overline{BC})} \psi(x)$$

$$\times A\overline{B}(1-y) {}_2F_1 \left( \begin{matrix} A\overline{C}, & \overline{\psi} \\ & \overline{BC\psi} \end{matrix} \middle| y \right)^* - \alpha_1.$$

Using (1.22) and then rearranging the terms we have

$$L = \frac{A\overline{B}(1-y)}{(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi)g(\overline{C\psi})g(A\overline{C}\chi)g(\overline{BC\psi\chi})g(\overline{\psi\chi})g(\overline{\chi})}{g(A)g(\overline{BC})g(A\overline{C})g(\overline{C})} \psi(x)\chi(y) - \alpha_1.$$

If we apply the change of variables  $\psi \mapsto \psi\chi$  and then rearranging the terms, we have

$$L = \frac{A\overline{B}(1-y)}{(q-1)^2} \sum_{\psi, \chi} \frac{g(A\psi\chi)g(\overline{BC\psi})g(A\overline{C}\chi)g(\overline{C\psi\chi})g(\overline{\psi})g(\overline{\chi})}{g(A)g(\overline{BC})g(A\overline{C})g(\overline{C})} \psi(x)\chi(xy) - \alpha_1.$$

Finally, using (3.4) and (4.22) we obtain

$$L = A\overline{B}(1-y) F_1(A; \overline{BC}, A\overline{C}; C; x, xy)^* - \frac{g(B)g(A\overline{B})\overline{B}(y)\overline{AB}(1-x)}{q \cdot g(A)}.$$

Multiplying both side by  $A(1-x)B(1-y)$ , we complete the proof of the theorem.  $\blacksquare$

**Corollary 4.5.1.** *Let  $A, B \in \widehat{\mathbb{F}_q^\times}$  be such that  $B, A\overline{B}^2 \neq \varepsilon$ ,  $A \neq B$ . For  $x, y \in \mathbb{F}_q^\times$*

such that  $x, y \neq 1$  we have

$$\begin{aligned} & F_4 \left( A; B; A\bar{B}, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right)^* \\ &= A(1-y) {}_2F_1 \left( \begin{matrix} B, & A \\ & A\bar{B} \end{matrix} \middle| \frac{xy-x}{1-x} \right)^* - \frac{\bar{A}B^2(-x)\bar{B}(-xy)A((1-x)(1-y))}{q} \left( \frac{A}{A\bar{B}} \right)^{-1} \\ & \quad - \frac{g(B)g(A\bar{B})}{q.g(A)} \bar{B}(y)B((1-x)(1-y)). \end{aligned}$$

*Proof.* Putting  $C = A\bar{B}$  in Theorem 4.3 we have

$$\begin{aligned} L &:= F_4 \left( A; B; A\bar{B}, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right)^* \\ &= A((1-x)(1-y)) F_1(A; \bar{A}B^2, B; A\bar{B}; x, xy)^* - \alpha, \end{aligned}$$

where  $\alpha = \frac{g(B)g(A\bar{B})}{q.g(A)} \bar{B}(y)B((1-x)(1-y))$ .

Multiplying both sides of the above identity by  $\varepsilon(x-xy)$ , and using the first part of Theorem 3.3 we have

$$\begin{aligned} \varepsilon(x-xy)L &= \frac{\varepsilon(x-xy)A((1-x)(1-y))}{q} \left( \frac{A}{A\bar{B}} \right)^{-1} F_1(A; \bar{A}B^2, B; A\bar{B}; x, xy) \\ & \quad - \varepsilon(x-xy)\alpha. \end{aligned}$$

Using [43, Corollary 3.1] and Proposition 1.19 we have

$$\begin{aligned} \varepsilon(x-xy)L &= \varepsilon(x^2y)A(1-y) {}_2F_1 \left( \begin{matrix} B, & A \\ & A\bar{B} \end{matrix} \middle| \frac{xy-x}{1-x} \right)^* \\ & \quad - \frac{\varepsilon(xy-x)\bar{A}B^2(-x)\bar{B}(-xy)A((1-x)(1-y))}{q} \left( \frac{A}{A\bar{B}} \right)^{-1} \\ & \quad - \varepsilon(x-xy)\alpha. \end{aligned}$$

Finally, putting the value of  $\alpha$  and using the fact that  $\varepsilon(x-xy) = \varepsilon(x^2y) = 1$  we

complete the proof of the corollary. ■

The above result is a finite field analogue of the following identity [4, (4.4)] satisfied by the appell series  $F_4$ :

$$\begin{aligned} & F_4 \left( a; b; 1+a-b, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right) \\ &= (1-y)^a {}_2F_1 \left( \begin{matrix} a, & b \\ 1+a-b & \end{matrix} \middle| \frac{xy-x}{1-x} \right). \end{aligned}$$



# 5

## Finite field analogues of identities satisfied by Appell series

### 5.1 Introduction

Finite field analogues of identities connecting Appell series to classical  ${}_2F_1$ - and  ${}_3F_2$ -hypergeometric series are already known (see for example [33, 35, 43]). There are many beautiful identities connecting Appell series to classical  ${}_4F_3$ -hypergeometric series, and no finite field analogues of such classical identities are obtained till date. In this chapter, we establish finite field analogue of five identities connecting Appell

---

<sup>1</sup>The contents of this chapter are under review.

series to classical  ${}_4F_3$ -hypergeometric series. There are many important applications of the finite field analogues of these classical identities, and we shall discuss all these applications in Chapter 6 and Chapter 7.

## 5.2 Statement of the main results

In the following identity the classical Appell series  $F_2$  is expressed as a  ${}_4F_3$ -classical hypergeometric series (see for example [8, (4.4)]).

$$F_2(a; b, c; 2b, 2c; x, -x) = {}_4F_3 \left( \begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & \frac{b+c}{2}, & \frac{b+c+1}{2} \\ b + \frac{1}{2}, & c + \frac{1}{2}, & b + c \end{matrix} \mid x^2 \right). \quad (5.1)$$

In the following result we prove a finite field analogue of (5.1).

**Theorem 5.1.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B^4C^4, B^2\overline{C^2}, \overline{AB^2} \neq \varepsilon$ . For  $x \in \mathbb{F}_q$  such that  $x \neq 0$  we have*

$$\begin{aligned} & F_2(A^2; B^2, C^2; B^4, C^4; x, -x)^* \\ &= \frac{q^2 \overline{ABC}(4)g(A)g(\overline{AB^2})g(\overline{BC}\varphi)g(\overline{BC}\varphi)}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} {}_4F_3 \left( \begin{matrix} A, & A\varphi, & BC, & BC\varphi \\ B^2\varphi, & C^2\varphi, & B^2C^2 \end{matrix} \mid x^2 \right) \\ & - \left[ \frac{g(B^2C^2)g(\overline{C^2})}{qg(B^2)} + 1 \right] \frac{(q-1)g(B^2C^2)g(\overline{B^4C^4})}{g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \delta(B^2) \\ & + \frac{q(q-1)(\delta(B^2) - \delta(C^2))g(\overline{B^2C^2})}{g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} {}_2F_1 \left( \begin{matrix} A^2, & B^2C^2 \\ B^4C^4 \end{matrix} \mid x \right) \\ & - \frac{(q-1)^2(\delta(B^2) - \delta(C^2))g(\overline{B^2C^2})}{g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \delta(A^2)B^2C^2 \left( \frac{1-x}{x^2} \right) \\ & + \frac{q(q-1)g(A)g(\overline{B^2}A)g^2(\overline{BC}\varphi)}{\overline{ABC}(4)g(A^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} {}_4F_3 \left( \begin{matrix} A, & A\varphi, & BC, & BC\varphi \\ B^2\varphi, & B^2C^2\varphi, & B^2C^2 \end{matrix} \mid x^2 \right) \delta(B^2). \end{aligned}$$

**Remark 5.2.1.** *Suppose that  $B^2, C^2 \neq \varepsilon$ . Then  $\delta(B^2) = \delta(C^2) = 0$ . Hence, we*

have the following corollary of Theorem 5.1 which is an exact finite field analogue of (5.1). We have proved Theorem 5.1 without the conditions  $B^2, C^2 \neq \varepsilon$  so that we can derive certain applications of the theorem by allowing  $B, C \in \{\varepsilon, \varphi\}$ .

**Corollary 5.2.1.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B^2, C^2, B^4C^4, B^2\overline{C^2}, A\overline{B^2} \neq \varepsilon$ . For  $x \in \mathbb{F}_q$ , we have*

$$\begin{aligned} & F_2(A^2; B^2, C^2; B^4, C^4; x, -x)^* \\ &= \frac{q^2 \overline{ABC}(4)g(A)g(\overline{AB^2})g(\overline{BC}\varphi)g(\overline{BC}\varphi)}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} {}_4F_3 \left( \begin{matrix} A, & A\varphi, & BC, & BC\varphi \\ & B^2\varphi, & C^2\varphi, & B^2C^2 \end{matrix} \middle| x^2 \right). \end{aligned}$$

The following is another identity [8, (4.5)] satisfied by the Appell series  $F_2$  and a  ${}_4F_3$ -classical hypergeometric series.

$$F_2(a; b, b; c, c; x, -x) = {}_4F_3 \left( \begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & b, & c-b \\ & c, & \frac{c}{2}, & \frac{c+1}{2} \end{matrix} \middle| x^2 \right). \quad (5.2)$$

In the following result we prove a finite field analogue of (5.2).

**Theorem 5.2.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B, \overline{BC^2}, B^2\overline{C^2}, A\overline{C^2}\varphi \neq \varepsilon$ . For  $x \in \mathbb{F}_q$ , we have*

$$\begin{aligned} & F_2(A^2; B, B; C^2, C^2; x, -x)^* \\ &= \frac{qg(\overline{BC^2})g(A)g(A\varphi\overline{C^2})g(\overline{BC})g(\overline{BC}\varphi)}{B(-1)\overline{AC}(4)g(A^2)g(B)g^2(\overline{C^2})} {}_4F_3 \left( \begin{matrix} A, & A\varphi, & B, & \overline{BC^2} \\ & C^2, & C, & C\varphi \end{matrix} \middle| x^2 \right). \end{aligned}$$

In the following identity the classical Appell series  $F_3$  is expressed as a  ${}_4F_3$ -classical hypergeometric series (see for example [8, (4.3)]).

$$F_3(a, a; b, b; c; x, -x) = {}_4F_3 \left( \begin{matrix} \frac{a+b}{2}, & \frac{a+b+1}{2}, & a, & b \\ & a+b, & \frac{c}{2}, & \frac{c+1}{2} \end{matrix} \middle| x^2 \right). \quad (5.3)$$

In the following result we prove a finite field analogue of (5.3).

**Theorem 5.3.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2, B^2, A^2B^2, A^2\overline{C}, B^2\overline{C}\varphi \neq \varepsilon$ . For  $x \in \mathbb{F}_q$ , we have*

$$F_3(A^2, A^2; B^2, B^2; C^2; x, x)^* \\ = \frac{qABC\overline{C}(4)g(AB)g(\overline{AB}\varphi)g(A^2\overline{C})g(B^2\overline{C}\varphi)}{g(A^2)g(B^2)g(\overline{C}^2)} {}_4F_3 \left( \begin{matrix} AB, & AB\varphi, & A^2, & B^2 \\ & A^2B^2, & C, & C\varphi \end{matrix} \mid x^2 \right).$$

In the following identity the classical Appell series  $F_4$  is expressed as a  ${}_4F_3$ -classical hypergeometric series (see for example [56, (127)]).

$$F_4(a; b; c, c; x, -x) = {}_4F_3 \left( \begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & \frac{b}{2}, & \frac{b+1}{2} \\ & c, & \frac{c}{2}, & \frac{c+1}{2} \end{matrix} \mid -4x^2 \right). \quad (5.4)$$

We prove the following result which is a finite field analogue of (5.4).

**Theorem 5.4.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, A\overline{C}^2\varphi, B\overline{C} \neq \varepsilon$ . For  $x \in \mathbb{F}_q$ , we have*

$$F_4(A^2; B^2; C^2, C^2; x, -x)^* \\ = \frac{q^2ABC\overline{C}(4)g(A)g(A\overline{C}^2\varphi)g^2(B\overline{C})}{g(A^2)g(B^2)g^2(\overline{C}^2)g(\varphi)} {}_4F_3 \left( \begin{matrix} A, & A\varphi, & B, & B\varphi \\ & C^2, & C, & C\varphi \end{matrix} \mid -4x^2 \right).$$

The following is another identity satisfied by the classical Appell series  $F_4$  and a  ${}_4F_3$ -classical hypergeometric series (see for example [56, (126)]).

$$F_4(a; b; d, e; x, x) = {}_4F_3 \left( \begin{matrix} a, & b, & \frac{d+e}{2}, & \frac{d+e-1}{2} \\ & d, & e, & d+e-1 \end{matrix} \mid 4x \right). \quad (5.5)$$

We prove the following result which is a finite field analogue of (5.5).

**Theorem 5.5.** Let  $A, B, D, E \in \widehat{\mathbb{F}_q^\times}$  be such that  $A, D^2E^2, \overline{D^2E^2}, \overline{BD^2} \neq \varepsilon$ . For  $x \in \mathbb{F}_q$ , we have

$$\begin{aligned} & F_4(A; B; D^2, E^2; x, x)^* \\ &= \frac{q^2 DE(4)g(\overline{BD^2})g(\overline{DE})g(\overline{DE}\varphi)}{(q-1)g(\varphi)g(B)g(\overline{D^2})g(\overline{E^2})} {}_4F_3 \left( \begin{matrix} A, & B, & DE, & DE\varphi \\ & D^2, & E^2, & D^2E^2 \end{matrix} \middle| 4x \right). \end{aligned}$$

**Notation 5.2.1.** In the rest of this chapter,  $\chi, \lambda, \eta$  denote multiplicative characters on  $\mathbb{F}_q$ . We write  $\sum_{\chi, \lambda}$  to denote the sum which is taken over all multiplicative characters  $\chi$  and  $\lambda$ .

### 5.3 Proof of Theorem 5.1

*Proof.* Here  $x \neq 0$ . From (3.5) we have

$$\begin{aligned} L &:= F_2(A^2; B^2, C^2; B^4, C^4; x, -x)^* \\ &= \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A^2\chi\lambda)g(B^2\chi)g(C^2\lambda)g(\overline{B^4\chi})g(\overline{C^4\lambda})g(\overline{\chi})g(\overline{\lambda})}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \chi(x)\lambda(-x). \end{aligned}$$

The change of variables  $\chi \mapsto \chi\overline{\lambda}$  yields

$$L = \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A^2\chi)g(B^2\chi\overline{\lambda})g(C^2\lambda)g(\overline{B^4\chi\lambda})g(\overline{C^4\lambda})g(\overline{\chi\lambda})g(\overline{\lambda})}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \chi(x)\lambda(-1).$$

Multiplying both numerator and denominator by  $q^3\chi(-1)g(\overline{\chi})g(B^2C^2\chi)g(\overline{B^4C^4\chi})$  and then rearranging the terms, we obtain

$$\begin{aligned} L &= \frac{q^3}{(q-1)^2} \sum_{\chi, \lambda} \left[ \frac{\lambda(-1)g(\overline{\chi\lambda})g(\overline{\lambda})}{qg(\overline{\chi})} \right] \left[ \frac{\chi\lambda(-1)g(C^2\lambda)g(B^2\chi\overline{\lambda})}{qg(B^2C^2\chi)} \right] \\ &\quad \times \left[ \frac{\lambda(-1)g(\overline{B^4\chi\lambda})g(\overline{C^4\lambda})}{qg(\overline{B^4C^4\chi})} \right] G_1(\chi)\chi(-x), \end{aligned}$$

where

$$G_1(\chi) = \frac{g(A^2\chi)g(B^2C^2\chi)g(\overline{B^4C^4\chi})g(\overline{\chi})}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})}.$$

Now, using Lemma 1.12 we have

$$\begin{aligned} L &= \frac{q^3}{(q-1)^2} \sum_{\chi, \lambda} \left[ \binom{\overline{\chi}\lambda}{\lambda} - \frac{q-1}{q} \delta(\overline{\chi}) \right] \left[ \binom{C^2\lambda}{B^2\overline{\chi}\lambda} - \frac{q-1}{q} \delta(B^2C^2\chi) \right] \\ &\quad \times \left[ \binom{\overline{B^4\chi}\lambda}{C^4\lambda} - \frac{q-1}{q} \delta(\overline{B^4C^4\chi}) \right] G_1(\chi)\chi(-x) \\ &= \frac{q^3}{(q-1)^2} \sum_{\chi, \lambda} \binom{\overline{\chi}\lambda}{\lambda} \binom{C^2\lambda}{B^2\overline{\chi}\lambda} \binom{\overline{B^4\chi}\lambda}{C^4\lambda} G_1(\chi)\chi(-x) - I_1 - I_2 - I_3. \end{aligned} \quad (5.6)$$

The last equality is obtained by using the fact that  $B^4C^4 \neq \varepsilon$  which yields

$$\delta(\overline{\chi})\delta(B^2C^2\chi) = \delta(\overline{\chi})\delta(\overline{B^4C^4\chi}) = \delta(B^2C^2\chi)\delta(\overline{B^4C^4\chi}) = 0$$

for all  $\chi$ . Also,

$$\begin{aligned} I_1 &= \frac{q^2}{q-1} \sum_{\chi, \lambda} \binom{C^2\lambda}{B^2\overline{\chi}\lambda} \binom{\overline{B^4\chi}\lambda}{C^4\lambda} G_1(\chi)\chi(-x)\delta(\overline{\chi}); \\ I_2 &= \frac{q^2}{q-1} \sum_{\chi, \lambda} \binom{\overline{\chi}\lambda}{\lambda} \binom{\overline{B^4\chi}\lambda}{C^4\lambda} G_1(\chi)\chi(-x)\delta(B^2C^2\chi); \\ I_3 &= \frac{q^2}{q-1} \sum_{\chi, \lambda} \binom{\overline{\chi}\lambda}{\lambda} \binom{C^2\lambda}{B^2\overline{\chi}\lambda} G_1(\chi)\chi(-x)\delta(\overline{B^4C^4\chi}). \end{aligned}$$

Since  $I_1$  is nonzero only when  $\chi = \varepsilon$ , so putting  $\chi = \varepsilon$  and then using Lemma 1.12 with the given condition  $B^4C^4 \neq \varepsilon$  and Lemma 1.8, we deduce that

$$\begin{aligned} I_1 &= \frac{q^2}{q-1} \sum_{\lambda} \binom{C^2\lambda}{B^2\lambda} \binom{\overline{B^4}\lambda}{C^4\lambda} G_1(\varepsilon) \\ &= \frac{G_1(\varepsilon)}{(q-1)g(B^2C^2)g(\overline{B^4C^4})} \sum_{\lambda} g(C^2\lambda)g(\overline{B^4}\lambda)g(B^2\overline{\lambda})g(\overline{C^4\lambda}) \end{aligned}$$

$$\begin{aligned}
&= \frac{g(\overline{B^2})g(\overline{C^2})}{g(\overline{B^2C^2})}G_1(\varepsilon) \\
&= \frac{g(B^2C^2)g(\overline{C^2})}{g(B^2)}G_1(\varepsilon) - \frac{q-1}{q} \frac{g(B^2C^2)g(\overline{C^2})}{g(B^2)}G_1(\varepsilon)\delta(B^2).
\end{aligned}$$

The last equality is obtained by multiplying both numerator and denominator by  $g(B^2)g(B^2C^2)$  and then using Lemma 1.2 with  $B^2C^2 \neq \varepsilon$ . Similarly,  $I_2$  is nonzero only when  $\chi = \overline{B^2C^2}$ . So, putting  $\chi = \overline{B^2C^2}$  and then using (1.19), Lemma 1.16 and (1.14) with  $B^2\overline{C^2} \neq \varepsilon$ , we obtain

$$\begin{aligned}
I_2 &= \frac{q^2}{q-1} \sum_{\lambda} \binom{B^2C^2\lambda}{\lambda} \binom{\overline{B^2C^2}\lambda}{C^4\lambda} G_1(\overline{B^2C^2})\overline{B^2C^2}(x) \\
&= q \cdot {}_2F_1 \left( \begin{matrix} B^2C^2, \overline{B^2C^2} \\ C^4 \end{matrix} \middle| 1 \right) G_1(\overline{B^2C^2})\overline{B^2C^2}(x) \\
&= q \binom{\overline{B^2C^2}}{\overline{B^2C^2}} G_1(\overline{B^2C^2})\overline{B^2C^2}(x) \\
&= -G_1(\overline{B^2C^2})\overline{B^2C^2}(x).
\end{aligned}$$

Again,  $I_3$  is nonzero only when  $\chi = \overline{B^4C^4}$ , and so putting  $\chi = \overline{B^4C^4}$  and then using  $g(\varepsilon) = -1$ , we find that

$$\begin{aligned}
I_3 &= -\frac{q^2}{q-1} \sum_{\lambda} \binom{B^4C^4\lambda}{\lambda} \binom{C^2\lambda}{B^2C^4\lambda} \frac{g(A^2\overline{B^4C^4})g(\overline{B^2C^2})g(B^4C^4)\overline{B^4C^4}(x)}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \\
&= -\frac{qg(A^2\overline{B^4C^4})g(\overline{B^2C^2})g(B^4C^4)\overline{B^2C^2}(x^2)}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \binom{C^2}{\overline{B^2}}.
\end{aligned}$$

The last equality is obtained by using (1.19) and Lemma 1.16. We now use Lemma 1.12 with  $B^2C^2 \neq \varepsilon$ , and replace the Gauss sums  $g(A^2\overline{B^4C^4})$ ,  $g(\overline{B^2C^2})$  and  $g(B^4C^4)$  by employing Lemma 1.5 and then use Lemma 1.2 to obtain

$$I_3 = -\frac{ABC(4)g(\overline{AB^2C^2})g(\overline{AB^2C^2}\varphi)g(\overline{BC})g(\overline{BC}\varphi)g(B^2C^2\varphi)\overline{B^2C^2}(x^2)}{q\varphi(-1)g(\varphi)g(A^2)g(\overline{B^4})g(\overline{C^4})}$$

$$= -\frac{ABC(4)g(\overline{AB^2C^2})g(\overline{AB^2C^2}\varphi)g(\overline{BC})g(\overline{BC}\varphi)g(\varphi)g(B^2C^2\varphi)\overline{B^2C^2}(x^2)}{q^2g(A^2)g(\overline{B^4})g(\overline{C^4})}.$$

We note that the last equality is obtained by using Lemma 1.2 after multiplying both numerator and denominator by  $g(\varphi)$ . Using (1.20) and then Lemma 1.18 in (5.6), we obtain

$$\begin{aligned} L &= \frac{q^2}{q-1} \sum_x {}_3F_2 \left( \begin{matrix} \bar{\chi}, & C^2, & \overline{B^4\chi} \\ & \overline{B^2\chi}, & C^4 \end{matrix} \middle| 1 \right) G_1(\chi)\chi(-x) - I_1 - I_2 - I_3 \\ &= \frac{q^2}{q-1} \sum_x {}_3F_2 \left( \begin{matrix} \overline{B^2}, & \overline{B^2C^2\chi}, & \overline{B^4\chi} \\ & \overline{B^2\chi}, & C^2\overline{B^2} \end{matrix} \middle| 1 \right) G_1(\chi)\chi(x) - I_1 - I_2 - I_3. \end{aligned}$$

Employing the change of variables  $\chi \mapsto \chi\overline{B^4}$  and then Lemma 1.17, we have

$$\begin{aligned} L &= \frac{q^2}{q-1} \sum_x {}_3F_2 \left( \begin{matrix} \overline{B^2}, & \overline{B^2C^2\chi}, & \bar{\chi} \\ & \overline{B^2\chi}, & C^2\overline{B^2} \end{matrix} \middle| 1 \right) G_1(\overline{B^4\chi})\overline{B^4\chi}(x) - I_1 - I_2 - I_3 \\ &= J_1 - I_1 - I_2 - I_3 - I_4 + \frac{q^2}{q-1} \sum_x G_1(\overline{B^4\chi})\overline{B^4\chi}(-x) \\ &\quad \times \begin{cases} 0, & \text{if } \bar{\chi} \neq \square; \\ \left(\frac{\eta}{\overline{B^2}}\right)\left(\frac{B^2C^2\eta}{\overline{C^2\eta}}\right) + \left(\frac{\eta\varphi}{\overline{B^2}}\right)\left(\frac{\varphi B^2C^2\eta}{\varphi\overline{C^2\eta}}\right), & \text{if } \bar{\chi} = \eta^2. \end{cases} \end{aligned}$$

Here,

$$J_1 = (\delta(B^2) - \delta(C^2)) \sum_x G_1(\overline{B^4\chi})\overline{B^4\chi}(x)$$

and

$$I_4 = \sum_x G_1(\overline{B^4\chi})\overline{B^4\chi}(-x)\delta(B^2C^2\chi).$$

Since  $\delta(B^2\overline{C^2\chi})$  is nonzero only when  $\chi = B^2\overline{C^2}$ , so we have

$$I_4 = G_1(\overline{B^2C^2})\overline{B^2C^2}(x) = -I_2.$$

Also,

$$\begin{aligned} J_1 &= (\delta(B^2) - \delta(C^2)) \sum_x G_1(\overline{B^4\chi})\overline{B^4\chi}(x) \\ &= (\delta(B^2) - \delta(C^2)) \sum_x G_1(\chi)\chi(x) \\ &= (\delta(B^2) - \delta(C^2)) \sum_x \frac{g(A^2\chi)g(B^2C^2\chi)g(\overline{B^4C^4\chi})g(\overline{\chi})}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \chi(x) \\ &= \frac{q^2g(\overline{B^2C^2})(\delta(B^2) - \delta(C^2))}{g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \sum_x \left[ \binom{A^2\chi}{\chi} - \frac{q-1}{q}\delta(A^2) \right] \binom{B^2C^2\chi}{B^4C^4\chi} \chi(x) \\ &= \frac{q^2g(\overline{B^2C^2})(\delta(B^2) - \delta(C^2))}{g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \sum_x \binom{A^2\chi}{\chi} \binom{B^2C^2\chi}{B^4C^4\chi} \chi(x) \\ &\quad - \frac{q(q-1)g(\overline{B^2C^2})(\delta(B^2) - \delta(C^2))\delta(A^2)}{g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \sum_x \binom{B^2C^2\chi}{B^4C^4\chi} \chi(x) \\ &= \frac{q(q-1)g(\overline{B^2C^2})(\delta(B^2) - \delta(C^2))}{g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} {}_2F_1 \left( \begin{matrix} A^2, & B^2C^2 \\ & B^4C^4 \end{matrix} \middle| x \right) \\ &\quad - \frac{(q-1)^2g(\overline{B^2C^2})(\delta(B^2) - \delta(C^2))}{g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} B^2C^2 \left( \frac{1-x}{x^2} \right) \delta(A^2). \end{aligned}$$

In the evaluation of  $J_1$  we have used Lemma 1.12 with  $B^2C^2 \neq \varepsilon$  and then (1.19) and (1.10), respectively.

We next evaluate the last term of  $L$ . If we put  $\overline{\chi} = \eta^2$  and take the sum over all  $\eta \in \widehat{\mathbb{F}_q^\times}$ , then we will count the squares of  $\widehat{\mathbb{F}_q^\times}$  twice. Hence, we have

$$\begin{aligned} L &= \frac{q^2}{2(q-1)} \sum_\eta \left[ \binom{\eta}{\overline{B^2}} \binom{B^2\overline{C^2}\eta}{\overline{C^2}\eta} + \binom{\varphi\eta}{\overline{B^2}} \binom{B^2\overline{C^2}\varphi\eta}{\overline{C^2}\varphi\eta} \right] G_1(\overline{B^4\eta^2})\overline{B^4\eta^2}(x) \\ &\quad - I_1 - I_3 + J_1. \end{aligned} \tag{5.7}$$

If we apply the change of variables  $\eta \mapsto \varphi\eta$  in the sum containing  $\left(\frac{\varphi\eta}{B^2}\right)\left(\frac{B^2\overline{C^2}\varphi\eta}{\overline{C^2}\varphi\eta}\right)$ , then (5.7) yields

$$L = \frac{q^2}{q-1} \sum_{\eta} \left(\frac{\eta}{B^2}\right) \left(\frac{B^2\overline{C^2}\eta}{\overline{C^2}\eta}\right) G_1(\overline{B^4\eta^2})\overline{B^4\eta^2}(x) - I_1 - I_3 + J_1.$$

The change of variables  $\eta \mapsto \overline{B^2}\eta$  yields

$$L = \frac{q^2}{q-1} \sum_{\eta} \left(\frac{\overline{B^2}\eta}{B^2}\right) \left(\frac{\overline{C^2}\eta}{\overline{B^2}\overline{C^2}\eta}\right) G_1(\eta^2)\eta(x^2) - I_1 - I_3 + J_1.$$

Lemma 1.12 yields

$$\begin{aligned} L &= \frac{q^2}{q-1} \sum_{\eta} \left[ \frac{g(\overline{B^2}\eta)g(B^2)}{qg(\overline{\eta})} + \frac{q-1}{q}\delta(\overline{\eta}) \right] \left[ \frac{\eta(-1)g(\overline{C^2}\eta)g(B^2C^2\eta)}{qg(B^2)} + \frac{q-1}{q}\delta(B^2) \right] \\ &\quad \times G_1(\eta^2)\eta(x^2) - I_1 - I_3 + J_1 \\ &= \frac{1}{q-1} \sum_{\eta} \frac{g(A^2\eta^2)g(B^2C^2\eta^2)g(\overline{B^4C^4}\eta^2)g(\overline{B^2}\eta)g(\overline{C^2}\eta)g(B^2C^2\eta)g(\overline{\eta^2})}{g(\overline{\eta})g(A^2)g(B^2)g(C^2)g(B^4)g(C^4)} \eta(-x^2) \\ &\quad - I_1 - I_3 + I_5 + J_1 + J_2 + J_3, \end{aligned}$$

where

$$I_5 = \sum_{\eta} \frac{\eta(-1)g(\overline{C^2}\eta)g(B^2C^2\eta)}{g(B^2)} G_1(\eta^2)\eta(x^2)\delta(\eta) = \frac{g(B^2C^2)g(\overline{C^2})}{g(B^2)} G_1(\varepsilon).$$

Also,

$$J_2 = (q-1)\delta(B^2) \sum_{\eta} G_1(\eta^2)\eta(x^2)\delta(\eta) = (q-1)G_1(\varepsilon)\delta(B^2)$$

and

$$\begin{aligned} J_3 &= \delta(B^2) \sum_{\eta} \frac{g(\overline{B^2\eta})g(B^2)}{g(\overline{\eta})} G_1(\eta^2)\eta(x^2) \\ &= \delta(B^2) \sum_{\eta} \frac{g(A^2\eta^2)g(B^2C^2\eta^2)g(\overline{B^4C^4\eta^2})g(\overline{B^2\eta})g(\overline{\eta^2})}{g(\overline{\eta})g(A^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \eta(x^2). \end{aligned}$$

We now replace the Gauss sums  $g(A^2\eta^2)$ ,  $g(B^2C^2\eta^2)$ ,  $g(\overline{\eta^2})$ , and  $g(\overline{B^4C^4\eta^2})$  by using Lemma 1.5 to obtain

$$\begin{aligned} J_3 &= \delta(B^2) \sum_{\eta} \left[ \frac{g(A\eta)g(A\varphi\eta)}{g(\varphi)A\eta(4)} \right] \left[ \frac{g(BC\eta)g(BC\varphi\eta)}{g(\varphi)BC\eta(4)} \right] \left[ \frac{g(\overline{B^2C^2\eta})g(\overline{B^2C^2\varphi\eta})}{g(\varphi)B^2C^2\eta(4)} \right] \\ &\quad \times \left[ \frac{g(\overline{\eta})g(\overline{\varphi\eta})}{g(\varphi)\eta(4)} \right] \frac{g(\overline{B^2\eta})}{g(\overline{\eta})g(A^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \eta(x^2). \end{aligned}$$

Using the fact that  $g(\varphi)^2 = q\varphi(-1)$  and then applying the change of variables  $\eta \mapsto \eta\varphi$  we have

$$\begin{aligned} J_3 &= \frac{A\overline{B}(4)}{q^2C(4)} \sum_{\eta} \frac{g(A\eta)g(A\varphi\eta)g(BC\eta)g(BC\varphi\eta)g(\overline{B^2C^2\eta})g(\overline{B^2C^2\varphi\eta})g(\overline{B^2\varphi\eta})g(\overline{\eta})}{g(A^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \\ &\quad \times \eta(x^2)\delta(B^2). \end{aligned}$$

Multiplying both numerator and denominator by  $q^2g(A)g(\overline{AB^2})g^2(\overline{BC}\varphi)$  and then rearranging the terms, we obtain

$$\begin{aligned} J_3 &= \frac{q^2A\overline{B}C(4)g(A)g(\overline{AB^2})g^2(\overline{BC}\varphi)}{g(A^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \sum_{\eta} \left[ \frac{g(A\eta)g(\overline{\eta})}{q\eta(-1)g(A)} \right] \left[ \frac{g(A\varphi\eta)g(\overline{B^2\varphi\eta})}{q\varphi\eta(-1)g(\overline{B^2}A)} \right] \\ &\quad \times \left[ \frac{\varphi\eta(-1)g(BC\eta)g(\overline{B^2C^2\varphi\eta})}{qg(\overline{BC}\varphi)} \right] \left[ \frac{\eta(-1)g(BC\varphi\eta)g(\overline{B^2C^2\eta})}{qg(\overline{BC}\varphi)} \right] \eta(x^2)\delta(B^2). \end{aligned}$$

Using Lemma 1.12 with  $A, \overline{AB^2}, B^2C^2 \neq \varepsilon$  and then (1.20), we finally obtain

$$J_3 = \frac{q(q-1)g(A)g(\overline{B^2A})g^2(\overline{BC}\varphi)}{\overline{ABC}(4)g(A^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} {}_4F_3 \left( \begin{matrix} A, A\varphi, BC, BC\varphi \\ B^2\varphi, B^2C^2\varphi, B^2C^2 \end{matrix} \middle| x^2 \right) \delta(B^2).$$

This completes the evaluation of all the extra terms present in  $L$ . We now evaluate the main term of  $L$ . Replacing the Gauss sums  $g(A^2\eta^2)$ ,  $g(B^2C^2\eta^2)$ ,  $g(\overline{\eta^2})$ , and  $g(\overline{B^4C^4\eta^2})$  by employing Lemma 1.5 we obtain

$$\begin{aligned} L &= \frac{1}{q-1} \sum_{\eta} \left[ \frac{g(A\eta)g(A\varphi\eta)}{g(\varphi)\overline{A}\eta(4)} \right] \left[ \frac{g(BC\eta)g(BC\varphi\eta)}{g(\varphi)\overline{BC}\eta(4)} \right] \left[ \frac{g(\overline{B^2C^2}\eta)g(\overline{B^2C^2}\varphi\eta)}{g(\varphi)\overline{B^2C^2}\eta(4)} \right] \\ &\quad \times \left[ \frac{g(\overline{\eta})g(\overline{\varphi\eta})}{g(\varphi)\overline{\eta}(4)} \right] \frac{g(\overline{B^2}\eta)g(\overline{C^2}\eta)g(B^2C^2\eta)}{g(\overline{\eta})g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \eta(-x^2) \\ &\quad - I_1 - I_3 + I_5 + J_1 + J_2 + J_3. \end{aligned}$$

Employing Lemma 1.2 on  $g(\varphi)g(\varphi)$  and  $g(\overline{B^2C^2}\eta)g(B^2C^2\eta)$  we obtain

$$\begin{aligned} L &= \frac{\overline{ABC}(4)}{q(q-1)} \sum_{\eta} \frac{g(A\eta)g(A\varphi\eta)g(BC\eta)g(BC\varphi\eta)g(\overline{B^2C^2}\varphi\eta)g(\overline{B^2}\eta)g(\overline{C^2}\eta)g(\overline{\varphi\eta})}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \eta(x^2) \\ &\quad - I_1 - I_3 + I_5 + J_1 + J_2 + J_3 - I_6, \end{aligned}$$

where

$$\begin{aligned} I_6 &= \frac{\overline{ABC}(4)}{q^2} \sum_{\eta} \frac{g(A\eta)g(A\varphi\eta)g(BC\eta)g(BC\varphi\eta)g(\overline{B^2C^2}\varphi\eta)g(\overline{B^2}\eta)g(\overline{C^2}\eta)g(\overline{\varphi\eta})}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \\ &\quad \times \eta(-x^2)\delta(\overline{B^2C^2}\eta) \\ &= \frac{\overline{ABC}(4)g(\overline{AB^2C^2})g(\overline{AB^2C^2}\varphi)g(\overline{BC})g(\overline{BC}\varphi)g(\varphi)g(B^2C^2\varphi)\overline{B^2C^2}(x^2)}{q^2g(A^2)g(\overline{B^4})g(\overline{C^4})} \\ &= -I_3. \end{aligned}$$

In the evaluation of  $I_6$  we have used the fact that  $\delta(\overline{B^2C^2}\eta) \neq 0$  only when  $\eta = \overline{B^2C^2}$ .

Now the change of variables  $\eta \mapsto \eta\varphi$  yields

$$L = \sum_{\eta} \frac{g(A\eta)g(A\varphi\eta)g(BC\eta)g(BC\varphi\eta)g(\overline{B^2C^2\eta})g(\overline{B^2\varphi\eta})g(\overline{C^2\varphi\eta})g(\overline{\eta})\eta(x^2)}{q(q-1)\overline{ABC}(4)g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \\ - I_1 + I_5 + J_1 + J_2 + J_3.$$

Multiplying both numerator and denominator by  $q^3g(A)g(\overline{B^2A})g(\overline{BC\varphi})g(\overline{BC\varphi})$  and then rearranging the terms, we obtain

$$L = \frac{q^3\overline{ABC}(4)g(A)g(\overline{B^2A})g(\overline{BC\varphi})g(\overline{BC\varphi})}{(q-1)g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} \sum_{\eta} \left[ \frac{\eta(-1)g(A\eta)g(\overline{\eta})}{qg(A)} \right] \\ \times \left[ \frac{\varphi\eta(-1)g(A\varphi\eta)g(\overline{B^2\varphi\eta})}{qg(\overline{B^2A})} \right] \left[ \frac{\varphi\eta(-1)g(BC\eta)g(\overline{C^2\varphi\eta})}{qg(\overline{BC\varphi})} \right] \\ \times \left[ \frac{\eta(-1)g(BC\varphi\eta)g(\overline{B^2C^2\eta})}{qg(\overline{BC\varphi})} \right] \eta(x^2) - I_1 + I_5 + J_1 + J_2 + J_3.$$

Since  $A, \overline{AB^2}, \overline{BC\varphi}, \overline{BC\varphi} \neq \varepsilon$ , so using Lemma 1.12 and (1.20) we have

$$L = \frac{q^2\overline{ABC}(4)g(A)g(\overline{B^2A})g(\overline{BC\varphi})g(\overline{BC\varphi})}{g(A^2)g(B^2)g(C^2)g(\overline{B^4})g(\overline{C^4})} {}_4F_3 \left( \begin{matrix} A, & A\varphi, & BC, & BC\varphi \\ & B^2\varphi, & C^2\varphi, & B^2C^2 \end{matrix} \middle| x^2 \right) \\ - I_1 + I_5 + J_1 + J_2 + J_3.$$

Putting the values of  $I_1, I_5, J_1, J_2, J_3$ , and  $G_1(\varepsilon)$  with the fact that  $g(\varepsilon) = -1$ , we complete the proof of the theorem.  $\blacksquare$

## 5.4 Proof of Theorem 5.2

*Proof.* The result holds trivially for  $x = 0$ . Therefore, we assume that  $x \neq 0$ . From (3.5) we have

$$L := F_2(A^2; B, B; C^2, C^2; x, -x)^*$$

$$= \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A^2\chi\lambda)g(B\chi)g(B\lambda)g(\overline{C^2\chi})g(\overline{C^2\lambda})g(\overline{\chi})g(\overline{\lambda})}{g(A^2)g^2(B)g^2(\overline{C^2})} \chi(x)\lambda(-x).$$

The change of variables  $\chi \mapsto \chi\overline{\lambda}$  yields

$$L = \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A^2\chi)g(B\chi\overline{\lambda})g(B\lambda)g(\overline{C^2\chi\lambda})g(\overline{C^2\lambda})g(\overline{\chi\lambda})g(\overline{\lambda})}{g(A^2)g^2(B)g^2(\overline{C^2})} \chi(x)\lambda(-1).$$

Multiplying both numerator and denominator by  $g(\overline{C^2\chi})g(\overline{BC^2})$  and then rearranging the terms, we obtain

$$L = \frac{q^3}{(q-1)^2} \sum_{\chi, \lambda} \left[ \frac{\lambda(-1)g(B\lambda)g(\overline{\lambda})}{qg(B)} \right] \left[ \frac{B\chi\lambda(-1)g(\overline{C^2\chi\lambda})g(B\chi\overline{\lambda})}{qg(\overline{BC^2})} \right] \\ \times \left[ \frac{\lambda(-1)g(\overline{\chi\lambda})g(\overline{C^2\lambda})}{qg(\overline{C^2\chi})} \right] G_2(\chi)\chi(-x),$$

where

$$G_2(\chi) = \frac{B(-1)g(\overline{BC^2})}{g(A^2)g(B)g^2(\overline{C^2})} g(A^2\chi)g(\overline{C^2\chi}).$$

Employing Lemma 1.12 and the fact that  $B, \overline{BC^2} \neq \varepsilon$  we have

$$L = \frac{q^3}{(q-1)^2} \sum_{\chi, \lambda} \binom{B\lambda}{\lambda} \binom{\overline{C^2\chi\lambda}}{\overline{B\chi\lambda}} \left[ \binom{\overline{\chi\lambda}}{\overline{C^2\lambda}} - \frac{q-1}{q} \delta(\overline{C^2\chi}) \right] G_2(\chi)\chi(-x) \\ = \frac{q^3}{(q-1)^2} \sum_{\chi, \lambda} \binom{B\lambda}{\lambda} \binom{\overline{C^2\chi\lambda}}{\overline{B\chi\lambda}} \binom{\overline{\chi\lambda}}{\overline{C^2\lambda}} G_2(\chi)\chi(-x) - I_1, \quad (5.8)$$

where

$$I_1 = \frac{q^2}{q-1} \sum_{\chi, \lambda} \binom{B\lambda}{\lambda} \binom{\overline{C^2\chi\lambda}}{\overline{B\chi\lambda}} G_2(\chi)\chi(-x)\delta(\overline{C^2\chi}).$$

Since  $I_1$  is nonzero only when  $\chi = \overline{C^2}$ , so putting  $\chi = \overline{C^2}$  we obtain

$$I_1 = \frac{q^2}{q-1} \sum_{\lambda} \binom{B\lambda}{\lambda} \binom{\lambda}{\overline{B}C^2\lambda} G_2(\overline{C^2})\overline{C^2}(x).$$

Using (1.19), Lemma 1.16, (1.15), and the fact that  $\overline{B^2}C^2 \neq \varepsilon$ , we have

$$\begin{aligned} I_1 &= qG_2(\overline{C^2})\overline{C^2}(x) {}_2F_1 \left( \begin{matrix} B, & \varepsilon \\ & \overline{B}C^2 \end{matrix} \mid 1 \right) \\ &= qG_2(\overline{C^2})\overline{C^2}(x) B(-1) \binom{\varepsilon}{\overline{B^2}C^2} \\ &= -B(-1)G_2(\overline{C^2})\overline{C^2}(x). \end{aligned}$$

Now using (1.20) in (5.8), we obtain

$$L = \frac{q^2}{q-1} \sum_{\chi} {}_3F_2 \left( \begin{matrix} B, & \overline{C^2}\chi, & \overline{\chi} \\ & \overline{B}\chi, & C^2 \end{matrix} \mid 1 \right) G_2(\chi)\chi(-x) - I_1.$$

Since  $B, \overline{B}C^2 \neq \varepsilon$  and  $\delta(\overline{C^2}\chi) = 0$  when  $\chi \neq \square$ , so employing Lemma 1.17 we find that  ${}_3F_2 \left( \begin{matrix} B, & \overline{C^2}\chi, & \overline{\chi} \\ & \overline{B}\chi, & C^2 \end{matrix} \mid 1 \right)$  is nonzero only when  $\chi$  is a square. If we put  $\overline{\chi} = \eta^2$  and take the sum over all  $\eta \in \widehat{\mathbb{F}}_q^\times$ , then we will count the squares of  $\widehat{\mathbb{F}}_q^\times$  twice. Hence, using Lemma 1.17 we find that

$$\begin{aligned} L &= \frac{q^2 B(-1)}{2(q-1)} \sum_{\eta} \left[ \binom{\eta}{B} \binom{\overline{C^2}\eta}{\overline{B}C^2\eta} + \binom{\eta\varphi}{B} \binom{\overline{C^2}\eta\varphi}{\overline{B}C^2\eta\varphi} \right] G_2(\overline{\eta^2})\overline{\eta^2}(x) - I_1 - I_2 \\ &= \frac{q^2 B(-1)}{q-1} \sum_{\eta} \binom{\eta}{B} \binom{\overline{C^2}\eta}{\overline{B}C^2\eta} G_2(\overline{\eta^2})\overline{\eta^2}(x) - I_1 - I_2, \end{aligned}$$

where

$$I_2 = \frac{B(-1)}{2} \sum_{\eta} G_2(\overline{\eta^2})\overline{\eta^2}(x) \delta(\overline{C^2}\eta^2) = B(-1)G_2(\overline{C^2})\overline{C^2}(x) = -I_1.$$

In the evaluation of  $I_2$  we have used the fact that  $\delta(\overline{C^2\eta^2}) \neq 0$  only when  $\eta = C, C\varphi$ .

Now employing Lemma 1.12 with  $B \neq \varepsilon$  we deduce that

$$\begin{aligned} L &= \frac{q^2 B(-1)}{q-1} \sum_{\eta} \left[ \frac{B(-1)g(\eta)g(\overline{B})}{qg(\overline{B}\eta)} + \frac{q-1}{q} \delta(\overline{B}\eta) \right] \\ &\quad \times \left[ \frac{B\eta(-1)g(\overline{C^2\eta})g(C^2\overline{B}\eta)}{qg(\overline{B})} \right] G_2(\overline{\eta^2})\overline{\eta}(x^2) \\ &= \frac{B(-1)}{q-1} \sum_{\eta} \frac{g(\overline{C^2\eta})g(C^2\overline{B}\eta)g(\eta)}{g(\overline{B}\eta)} G_2(\overline{\eta^2})\overline{\eta}(-x^2) + I_3, \end{aligned}$$

where

$$\begin{aligned} I_3 &= \sum_{\eta} \frac{g(\overline{C^2\eta})g(C^2\overline{B}\eta)}{g(\overline{B})} G_2(\overline{\eta^2})\overline{\eta}(-x^2)\delta(\overline{B}\eta) \\ &= \frac{g(\overline{BC^2})g(\overline{B^2C^2})}{g(\overline{B})} G_2(\overline{B^2})\overline{B}(-x^2). \end{aligned}$$

The last equality is obtained by putting  $\eta = B$ . Employing Lemma 1.3 on  $g(\overline{B}\eta)$  we have

$$\begin{aligned} L &= \frac{B(-1)}{q-1} \sum_{\eta} g(\overline{C^2\eta})g(C^2\overline{B}\eta)g(\eta) \left[ \frac{B\eta(-1)g(\overline{B}\eta)}{q} - \frac{q-1}{q} \delta(\overline{B}\eta) \right] G_2(\overline{\eta^2})\overline{\eta}(-x^2) + I_3 \\ &= \frac{1}{q(q-1)} \sum_{\eta} g(\overline{C^2\eta})g(C^2\overline{B}\eta)g(\eta)g(\overline{B}\eta)G_2(\overline{\eta^2})\overline{\eta}(x^2) + I_3 - I_4, \end{aligned}$$

where

$$\begin{aligned} I_4 &= \frac{B(-1)}{q} \sum_{\eta} g(\overline{C^2\eta})g(C^2\overline{B}\eta)g(\eta)G_2(\overline{\eta^2})\overline{\eta}(-x^2)\delta(\overline{B}\eta) \\ &= \frac{B(-1)g(\overline{B})}{q} g(\overline{BC^2})g(\overline{B^2C^2})G_2(\overline{B^2})\overline{B}(-x^2) \\ &= \frac{g(\overline{BC^2})g(\overline{B^2C^2})}{g(\overline{B})} G_2(\overline{B^2})\overline{B}(-x^2) \\ &= I_3. \end{aligned}$$

We evaluate  $I_4$  by putting  $\eta = B$ , and then use Lemma 1.2 with  $B \neq \varepsilon$ . Now the change of variables  $\eta \mapsto \bar{\eta}$  yields

$$\begin{aligned} L &= \frac{1}{q(q-1)} \sum_{\eta} g(\overline{C^2\eta})g(\overline{BC^2\eta})g(\bar{\eta})g(B\eta)G_2(\eta^2)\eta(x^2) \\ &= \frac{B(-1)g(\overline{BC^2})}{q(q-1)g(A^2)g(B)g^2(\overline{C^2})} \sum_{\eta} g(A^2\eta^2)g(\overline{C^2\eta^2})g(\overline{C^2\eta})g(\overline{BC^2\eta})g(\bar{\eta})g(B\eta)\eta(x^2). \end{aligned}$$

Replacing the Gauss sums  $g(A^2\eta^2)$  and  $g(\overline{C^2\eta^2})$  by using Lemma 1.5 we deduce that

$$\begin{aligned} L &= \frac{B(-1)g(\overline{BC^2})}{q(q-1)g(A^2)g(B)g^2(\overline{C^2})} \sum_{\eta} \left[ \frac{g(A\eta)g(A\varphi\eta)}{g(\varphi)\overline{A\eta}(4)} \right] \left[ \frac{g(\overline{C\eta})g(\overline{C\eta\varphi})}{g(\varphi)\overline{C\eta}(4)} \right] \\ &\quad \times g(\overline{C^2\eta})g(\overline{C^2\overline{B\eta}})g(B\eta)g(\bar{\eta})\eta(x^2) \\ &= \frac{B\varphi(-1)\overline{AC}(4)g(\overline{BC^2})}{q^2(q-1)g(A^2)g(B)g^2(\overline{C^2})} \\ &\quad \times \sum_{\eta} g(A\eta)g(A\varphi\eta)g(\overline{C\eta})g(\overline{C\eta\varphi})g(\overline{C^2\eta})g(\overline{C^2\overline{B\eta}})g(B\eta)g(\bar{\eta})\eta(x^2). \end{aligned}$$

The last equality is obtained by using the fact that  $g(\varphi)^2 = q\varphi(-1)$ . Multiplying both numerator and denominator by  $q^2g(A)g(A\varphi\overline{C^2})g(\overline{BC})g(\overline{BC\varphi})$  and then rearranging the terms, we obtain

$$\begin{aligned} L &= \frac{q^2B(-1)\overline{AC}(4)g(\overline{BC^2})g(A)g(A\varphi\overline{C^2})g(\overline{BC})g(\overline{BC\varphi})}{(q-1)g(A^2)g(B)g^2(\overline{C^2})} \sum_{\eta} \left[ \frac{\eta(-1)g(A\eta)g(\bar{\eta})}{qg(A)} \right] \\ &\quad \times \left[ \frac{C^2\eta(-1)g(A\varphi\eta)g(\overline{C^2\eta})}{qg(A\varphi\overline{C^2})} \right] \left[ \frac{g(B\eta)g(\overline{C\eta})}{qC\eta(-1)g(\overline{BC})} \right] \left[ \frac{g(\overline{BC^2\eta})g(\overline{C\eta\varphi})}{qC\varphi\eta(-1)g(\overline{BC\varphi})} \right] \eta(x^2). \end{aligned}$$

Using Lemma 1.12 along with  $A \neq \varepsilon$ ,  $\overline{AC^2} \neq \varphi$ ,  $B^2\overline{C^2} \neq \varepsilon$  and then (1.20), we obtain

$$L = \frac{qB(-1)\overline{AC}(4)g(\overline{BC^2})g(A)g(A\varphi\overline{C^2})g(\overline{BC})g(\overline{BC\varphi})}{g(A^2)g(B)g^2(\overline{C^2})}$$

$$\times {}_4F_3 \left( \begin{matrix} A, A\varphi, B, \overline{BC^2} \\ C^2, C, C\varphi \end{matrix} \mid x^2 \right).$$

This completes the proof of the theorem. ■

## 5.5 Proof of Theorem 5.3

*Proof.* The result holds trivially for  $x = 0$ . Therefore, we assume that  $x \neq 0$ . From (3.6) we have

$$\begin{aligned} L &:= F_3(A^2, A^2; B^2, B^2; C^2; x, -x)^* \\ &= \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A^2\chi)g(A^2\lambda)g(B^2\chi)g(B^2\lambda)g(\overline{C^2\chi\lambda})g(\overline{\chi})g(\overline{\lambda})}{g^2(A^2)g^2(B^2)g(\overline{C^2})} \chi(x)\lambda(-x). \end{aligned}$$

The change of variables  $\chi \mapsto \chi\overline{\lambda}$  yields

$$L = \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A^2\chi\overline{\lambda})g(A^2\lambda)g(B^2\chi\overline{\lambda})g(B^2\lambda)g(\overline{C^2\chi})g(\overline{\chi\lambda})g(\overline{\lambda})}{g^2(A^2)g^2(B^2)g(\overline{C^2})} \chi(x)\lambda(-1).$$

Multiplying both numerator and denominator by  $q^3g(A^2B^2\chi)$  and then rearranging the terms, we obtain

$$\begin{aligned} L &= \frac{q^3}{(q-1)^2} \sum_{\chi, \lambda} \left[ \frac{\lambda(-1)g(A^2\lambda)g(\overline{\lambda})}{qg(A^2)} \right] \left[ \frac{\chi\lambda(-1)g(\overline{\chi\lambda})g(B^2\chi\overline{\lambda})}{qg(B^2)} \right] \\ &\quad \times \left[ \frac{\chi\lambda(-1)g(B^2\lambda)g(A^2\chi\overline{\lambda})}{qg(A^2B^2\chi)} \right] G_3(\chi)\chi(x), \end{aligned}$$

where

$$G_3(\chi) = \frac{g(A^2B^2\chi)g(\overline{C^2\chi})}{g(A^2)g(B^2)g(\overline{C^2})}.$$

Employing Lemma 1.12 with the given condition  $A^2, B^2 \neq \varepsilon$  we have

$$\begin{aligned} L &= \frac{q^3}{(q-1)^2} \sum_{x,\lambda} \binom{A^2\lambda}{\lambda} \binom{\bar{\chi}\lambda}{B^2\chi\lambda} \left[ \binom{B^2\lambda}{A^2\chi\lambda} - \frac{q-1}{q} \delta(A^2B^2\chi) \right] G_3(\chi)\chi(x) \\ &= \frac{q^3}{(q-1)^2} \sum_{x,\lambda} \binom{A^2\lambda}{\lambda} \binom{B^2\lambda}{A^2\chi\lambda} \binom{\bar{\chi}\lambda}{B^2\chi\lambda} G_3(\chi)\chi(x) - I_1, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{q^2}{q-1} \sum_{x,\lambda} \binom{A^2\lambda}{\lambda} \binom{\bar{\chi}\lambda}{B^2\chi\lambda} G_3(\chi)\chi(x)\delta(A^2B^2\chi) \\ &= \frac{q^2}{q-1} \sum_{\lambda} \binom{A^2\lambda}{\lambda} \binom{A^2B^2\lambda}{A^2\lambda} G_3(\overline{A^2B^2})\overline{A^2B^2}(x) \\ &= q \cdot {}_2F_1 \left( \begin{matrix} A^2, & A^2B^2 \\ & A^2 \end{matrix} \middle| 1 \right) G_3(\overline{A^2B^2})\overline{A^2B^2}(x) \\ &= q \binom{A^2B^2}{\varepsilon} G_3(\overline{A^2B^2})\overline{A^2B^2}(x) \\ &= -G_3(\overline{A^2B^2})\overline{A^2B^2}(x). \end{aligned}$$

We have evaluated  $I_1$  using the fact that  $\delta(A^2B^2\chi) \neq 0$  only when  $\chi = \overline{A^2B^2}$ , and then we use (1.19), Lemma 1.16 and (1.14) with the given condition  $A^2B^2 \neq \varepsilon$ .

Now (1.20) yields

$$L = \frac{q^2}{q-1} \sum_x {}_3F_2 \left( \begin{matrix} A^2, & B^2, & \bar{\chi} \\ & A^2\chi, & B^2\chi \end{matrix} \middle| 1 \right) G_3(\chi)\chi(x) - I_1. \quad (5.9)$$

Since  $A^2, B^2 \neq \varepsilon$  and  $\delta(\overline{A^2B^2}\chi) = 0$  when  $\chi \neq \square$ , so employing Lemma 1.17 we find that  ${}_3F_2 \left( \begin{matrix} A^2, & B^2, & \bar{\chi} \\ & A^2\chi, & B^2\chi \end{matrix} \middle| 1 \right)$  is nonzero only when  $\chi$  is a square. If we put  $\bar{\chi} = \eta^2$  and take the sum over all  $\eta \in \widehat{\mathbb{F}}_q^\times$ , then we will count the squares of  $\widehat{\mathbb{F}}_q^\times$  twice. Hence, using Lemma 1.17 and following similar steps as shown in the proofs

of Theorem 5.1 and Theorem 5.2, we have

$$L = \frac{q^2}{q-1} \sum_{\eta} \binom{\eta}{A^2} \binom{B^2\bar{\eta}}{A^2B^2\bar{\eta}} G_3(\bar{\eta}^2)\bar{\eta}(x^2) - I_1 - I_2, \tag{5.10}$$

where

$$I_2 = \frac{1}{2} \sum_{\eta} G_3(\bar{\eta}^2)\bar{\eta}^2(x)\delta(\overline{A^2B^2\eta^2}) = G_3(\overline{A^2B^2})\overline{A^2B^2}(x) = -I_1.$$

Employing Lemma 1.12 in (5.10) and then using  $A^2 \neq \varepsilon$  we find that

$$\begin{aligned} L &= \frac{q^2}{q-1} \sum_{\eta} \left[ \frac{g(\eta)g(\overline{A^2})}{qg(\overline{A^2\eta})} + \frac{q-1}{q}\delta(\overline{A^2\eta}) \right] \left[ \frac{\eta(-1)g(B^2\bar{\eta})g(\overline{A^2B^2\eta})}{qg(\overline{A^2})} \right] G_3(\bar{\eta}^2)\bar{\eta}(x^2) \\ &= \frac{1}{q-1} \sum_{\eta} \frac{g(\eta)g(B^2\bar{\eta})g(\overline{A^2B^2\eta})}{g(\overline{A^2\eta})} G_3(\bar{\eta}^2)\bar{\eta}(-x^2) + I_3, \end{aligned}$$

where

$$\begin{aligned} I_3 &= \sum_{\eta} \frac{g(B^2\bar{\eta})g(\overline{A^2B^2\eta})}{g(\overline{A^2})} G_3(\bar{\eta}^2)\bar{\eta}(-x^2)\delta(\overline{A^2\eta}) \\ &= \frac{g(\overline{A^2B^2})g(\overline{B^2})\overline{A^4}(x)}{g(\overline{A^2})} G_3(\overline{A^4}). \end{aligned} \tag{5.11}$$

Replacing  $g(\overline{A^2\eta})$  by employing Lemma 1.3 we find that

$$\begin{aligned} L &= \frac{1}{q-1} \sum_{\eta} g(\eta)g(B^2\bar{\eta})g(\overline{A^2B^2\eta}) \left[ \frac{\eta(-1)g(A^2\bar{\eta})}{q} - \frac{q-1}{q}\delta(A^2\bar{\eta}) \right] G_3(\bar{\eta}^2)\bar{\eta}(-x^2) + I_3 \\ &= \frac{1}{q(q-1)} \sum_{\eta} g(\eta)g(B^2\bar{\eta})g(\overline{A^2B^2\eta})g(A^2\bar{\eta})G_3(\bar{\eta}^2)\bar{\eta}(x^2) + I_3 - I_4, \end{aligned}$$

where

$$I_4 = \frac{1}{q} \sum_{\eta} g(\eta)g(B^2\bar{\eta})g(\overline{A^2B^2\eta})G_3(\bar{\eta}^2)\bar{\eta}(-x^2)\delta(A^2\bar{\eta})$$

$$\begin{aligned}
&= \frac{1}{q} g(A^2) g(\overline{A^2 B^2}) g(\overline{B^2}) G_3(\overline{A^4}) \overline{A^4}(x) \\
&= I_3.
\end{aligned}$$

We obtain  $I_4 = I_3$  by using  $g(A^2)g(\overline{A^2}) = q$ . The change of variables  $\eta \mapsto \bar{\eta}$  yields

$$L = \frac{1}{q(q-1)} \sum_{\eta} g(\bar{\eta}) g(B^2 \eta) g(\overline{A^2 B^2 \eta}) g(A^2 \eta) G_3(\eta^2) \eta(x^2).$$

We use Lemma 1.5 to replace the Gauss sums  $g(A^2 B^2 \eta^2)$  and  $g(\overline{C^2 \eta^2})$  present in the term  $G_3(\eta^2)$  to obtain

$$\begin{aligned}
L &= \frac{1}{q(q-1)g(A^2)g(B^2)g(\overline{C^2})} \sum_{\eta} \left[ \frac{g(AB\eta)g(AB\varphi\eta)}{g(\varphi)\overline{AB\eta}(4)} \right] \left[ \frac{g(\overline{C\eta})g(\overline{C\eta}\varphi)}{g(\varphi)C\eta(4)} \right] \\
&\quad \times g(A^2 \eta) g(B^2 \eta) g(\overline{A^2 B^2 \eta}) g(\bar{\eta}) \eta(x^2) \\
&= \frac{\varphi(-1)ABC\overline{C}(4)}{q^2(q-1)g(A^2)g(B^2)g(\overline{C^2})} \\
&\quad \times \sum_{\eta} g(AB\eta)g(AB\varphi\eta)g(\overline{C\eta})g(\overline{C\eta}\varphi)g(A^2 \eta)g(B^2 \eta)g(\overline{A^2 B^2 \eta})g(\bar{\eta})\eta(x^2).
\end{aligned}$$

The last equality is obtained by using the fact that  $g(\varphi)^2 = q\varphi(-1)$ . Now multiplying both numerator and denominator by  $q^2 g(AB)g(\overline{AB}\varphi)g(A^2\overline{C})g(B^2\overline{C}\varphi)$  and then rearranging the terms, we obtain

$$\begin{aligned}
L &= \frac{q^2 ABC\overline{C}(4)g(AB)g(\overline{AB}\varphi)g(A^2\overline{C})g(B^2\overline{C}\varphi)}{(q-1)g(A^2)g(B^2)g(\overline{C^2})} \sum_{\eta} \left[ \frac{\eta(-1)g(AB\eta)g(\bar{\eta})}{qg(AB)} \right] \\
&\quad \times \left[ \frac{g(AB\varphi\eta)g(\overline{A^2 B^2 \eta})}{q\eta(-1)g(\overline{AB}\varphi)} \right] \left[ \frac{g(A^2 \eta)g(\overline{C\eta})}{qC\eta(-1)g(A^2\overline{C})} \right] \left[ \frac{C\varphi\eta(-1)g(B^2 \eta)g(\overline{C\eta}\varphi)}{qg(B^2\overline{C}\varphi)} \right] \eta(x^2).
\end{aligned}$$

Finally, employing Lemma 1.12 along with the given conditions  $A^2 B^2, A^2 \overline{C}, B^2 \overline{C} \varphi \neq \varepsilon$  and then (1.20), we complete the proof of the theorem.  $\blacksquare$

## 5.6 Proof of Theorem 5.4

*Proof.* The result holds trivially for  $x = 0$ . Therefore, we assume that  $x \neq 0$ . From (4.2) we have

$$\begin{aligned} L &:= F_4(A^2; B^2; C^2, C^2; x, -x)^* \\ &= \frac{1}{(q-1)^2} \sum_{x, \lambda} \frac{g(A^2 \chi \lambda) g(B^2 \chi \lambda) g(\overline{C^2 \chi}) g(\overline{C^2 \lambda}) g(\overline{\chi}) g(\overline{\lambda})}{g(A^2) g(B^2) g^2(\overline{C^2})} \chi(x) \lambda(-x). \end{aligned}$$

The change of variables  $\chi \mapsto \chi \overline{\lambda}$  yields

$$L = \frac{1}{(q-1)^2} \sum_{x, \lambda} \frac{g(A^2 \chi) g(B^2 \chi) g(\overline{C^2 \chi \lambda}) g(\overline{C^2 \lambda}) g(\overline{\chi \lambda}) g(\overline{\lambda})}{g(A^2) g(B^2) g^2(\overline{C^2})} \chi(x) \lambda(-1).$$

Multiplying both numerator and denominator by  $q^2 g^2(\overline{C^2 \chi})$ , and then rearranging the terms and using Lemma 1.12, we obtain

$$\begin{aligned} L &= \frac{q^2}{(q-1)^2} \sum_{x, \lambda} \left[ \frac{\lambda(-1) g(\overline{C^2 \chi \lambda}) g(\overline{\lambda})}{q g(\overline{C^2 \chi})} \right] \left[ \frac{\lambda(-1) g(\overline{\chi \lambda}) g(\overline{C^2 \lambda})}{q g(\overline{C^2 \chi})} \right] G_4(\chi) \chi(x) \lambda(-1) \\ &= \frac{q^2}{(q-1)^2} \sum_{x, \lambda} \left[ \binom{\overline{C^2 \chi \lambda}}{\lambda} - \frac{q-1}{q} \delta(\overline{C^2 \chi}) \right] \left[ \binom{\overline{\chi \lambda}}{C^2 \lambda} - \frac{q-1}{q} \delta(\overline{C^2 \chi}) \right] G_4(\chi) \chi(x) \lambda(-1) \\ &= \frac{q^2}{(q-1)^2} \sum_{x, \lambda} \binom{\overline{C^2 \chi \lambda}}{\lambda} \binom{\overline{\chi \lambda}}{C^2 \lambda} G_4(\chi) \chi(x) \lambda(-1) - I_1 - I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} G_4(\chi) &= \frac{g(A^2 \chi) g(B^2 \chi) g^2(\overline{C^2 \chi})}{g(A^2) g(B^2) g^2(\overline{C^2})}; \\ I_1 &= \frac{q}{q-1} \sum_{x, \lambda} \binom{\overline{C^2 \chi \lambda}}{\lambda} G_4(\chi) \chi(x) \lambda(-1) \delta(\overline{C^2 \chi}); \\ I_2 &= \frac{q}{q-1} \sum_{x, \lambda} \binom{\overline{\chi \lambda}}{C^2 \lambda} G_4(\chi) \chi(x) \lambda(-1) \delta(\overline{C^2 \chi}); \end{aligned}$$

$$I_3 = \sum_{\chi, \lambda} G_4(\chi)\chi(x)\lambda(-1)\delta^2(\overline{C^2}\chi).$$

Since  $I_1$  is nonzero only when  $\chi = \overline{C^2}$ , so after putting  $\chi = \overline{C^2}$  and then using (1.14) and Lemma 1.1 we obtain

$$\begin{aligned} I_1 &= \frac{q}{q-1} \sum_{\lambda} \binom{\lambda}{\lambda} G_4(\overline{C^2})\overline{C^2}(x)\lambda(-1) \\ &= \frac{q}{q-1} \sum_{\lambda} \left[ \frac{-1}{q} + \frac{q-1}{q}\delta(\lambda) \right] G_4(\overline{C^2})\overline{C^2}(x)\lambda(-1) \\ &= -\frac{G_4(\overline{C^2})\overline{C^2}(x)}{q-1} \sum_{\lambda} \lambda(-1) + G_4(\overline{C^2})\overline{C^2}(x) \sum_{\lambda} \lambda(-1)\delta(\lambda) \\ &= G_4(\overline{C^2})\overline{C^2}(x). \end{aligned}$$

Following similar steps as shown in the evaluation of  $I_1$ , we find that

$$\begin{aligned} I_2 &= G_4(\overline{C^2})\overline{C^2}(x) = I_1; \\ I_3 &= G_4(\overline{C^2})\overline{C^2}(x) \sum_{\lambda} \lambda(-1) = 0. \end{aligned}$$

Now (1.19) yields

$$L = \frac{q}{q-1} \sum_{\chi} {}_2F_1 \left( \begin{matrix} \overline{C^2}\chi, & \bar{\chi} \\ & C^2 \end{matrix} \middle| -1 \right) G_4(\chi)\chi(x) - 2I_1.$$

Using Greene's definition (1.18), we have

$$\begin{aligned} L &= \frac{q}{q-1} \sum_{\chi} \left[ \frac{C^2\chi(-1)}{q} \sum_{y \in \mathbb{F}_q} \bar{\chi}(y)C^2\chi(1-y^2) \right] G_4(\chi)\chi(x) - 2I_1 \\ &= \frac{q}{(q-1)^2} \sum_{\chi, \eta} \binom{\overline{C^2}\chi\eta}{\eta} G_4(\chi)\chi(-x) \sum_{y \in \mathbb{F}_q} \eta^2 \bar{\chi}(y) - 2I_1. \end{aligned}$$

The last equality is obtained by using (1.9). The inner most sum is nonzero only when  $\chi = \eta^2$ , and hence putting  $\chi = \eta^2$  and using Lemma 1.1 we have

$$L = \frac{q}{q-1} \sum_{\eta} \binom{\overline{C^2\eta}}{\eta} G_4(\eta^2)\eta(x^2) - 2I_1.$$

Lemma 1.12 yields

$$\begin{aligned} L &= \frac{q}{q-1} \sum_{\eta} \left[ \frac{\eta(-1)g(\overline{C^2\eta})g(\overline{\eta})}{qg(\overline{C^2\eta^2})} + \frac{q-1}{q}\delta(\overline{C^2\eta^2}) \right] G_4(\eta^2)\eta(x^2) - 2I_1 \\ &= \frac{1}{q-1} \sum_{\eta} \frac{g(\overline{C^2\eta})g(\overline{\eta})}{g(\overline{C^2\eta^2})} G_4(\eta^2)\eta(-x^2) - 2I_1 + I_3 \\ &= \frac{1}{q-1} \sum_{\eta} \frac{g(A^2\eta^2)g(B^2\eta^2)g(\overline{C^2\eta^2})g(\overline{C^2\eta})g(\overline{\eta})}{g(A^2)g(B^2)g^2(\overline{C^2})} \eta(-x^2) - 2I_1 + I_3, \end{aligned}$$

where

$$I_3 = \sum_{\eta} G_4(\eta^2)\eta(x^2)\delta(\overline{C^2\eta^2}) = 2G_4(\overline{C^2})\overline{C^2}(x) = 2I_1.$$

In the evaluation of  $I_3$  we have used the fact that  $\delta(\overline{C^2\eta^2}) \neq 0$  only when  $\eta = \overline{C}, \overline{C}\varphi$ . We now use Lemma 1.5 to replace the Gauss sums  $g(A^2\eta^2)$ ,  $g(B^2\eta^2)$ , and  $g(\overline{C^2\eta^2})$  to obtain

$$\begin{aligned} L &= \frac{1}{q-1} \sum_{\eta} \left[ \frac{g(A\eta)g(A\eta\varphi)}{g(\varphi)\overline{A\eta}(4)} \right] \left[ \frac{g(B\eta)g(B\eta\varphi)}{g(\varphi)\overline{B\eta}(4)} \right] \left[ \frac{g(\overline{C\eta})g(\overline{C\eta\varphi})}{g(\varphi)\overline{C\eta}(4)} \right] \\ &\quad \times \frac{g(\overline{C^2\eta})g(\overline{\eta})}{g(A^2)g(B^2)g^2(\overline{C^2})} \eta(-x^2) \\ &= \frac{A\overline{B}\overline{C}(4)}{q(q-1)} \sum_{\eta} \frac{g(A\eta)g(A\eta\varphi)g(B\eta)g(B\eta\varphi)g(\overline{C\eta})g(\overline{C\eta\varphi})g(\overline{C^2\eta})g(\overline{\eta})}{\varphi(-1)g(\varphi)g(A^2)g(B^2)g^2(\overline{C^2})} \eta(-4x^2). \end{aligned}$$

We note that the last equality is obtain by using  $g(\varphi)^2 = q\varphi(-1)$ . Now multiplying both numerator and denominator by  $q^3g(A)g(\overline{C^2}A\varphi)g^2(\overline{BC})$  and then rearranging

the terms, we obtain

$$L = \frac{q^3 ABC(4)g(A)g(\overline{C^2}A\varphi)g^2(\overline{BC})}{(q-1)g(\varphi)g(A^2)g(B^2)g^2(\overline{C^2})} \sum_{\eta} \left[ \frac{g(A\eta)g(\overline{\eta})}{q\eta(-1)g(A)} \right] \left[ \frac{g(A\varphi\eta)g(\overline{C^2}\eta)}{q\eta(-1)g(\overline{AC^2}\varphi)} \right] \\ \times \left[ \frac{C\eta(-1)g(B\eta)g(\overline{C}\eta)}{qg(\overline{BC})} \right] \left[ \frac{C\varphi\eta(-1)g(B\varphi\eta)g(\overline{C}\varphi\eta)}{qg(\overline{BC})} \right] \eta(-4x^2).$$

Finally, using Lemma 1.12 with the given conditions  $A, \overline{AC^2}\varphi, \overline{BC} \neq \varepsilon$  and (1.20), we complete the proof.  $\blacksquare$

## 5.7 Proof of Theorem 5.5

*Proof.* The result holds trivially for  $x = 0$ . Therefore, we assume that  $x \neq 0$ . From (4.2) we have

$$L := F_4(A; B; D^2, E^2; x, x)^* \\ = \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A\chi\lambda)g(B\chi\lambda)g(\overline{D^2}\chi)g(\overline{E^2}\lambda)g(\overline{\chi})g(\overline{\lambda})}{g(A)g(B)g(\overline{D^2})g(\overline{E^2})} \chi(x)\lambda(x).$$

The change of variables  $\chi \mapsto \chi\overline{\lambda}$  yields

$$L = \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A\chi)g(B\chi)g(\overline{D^2}\chi\lambda)g(\overline{E^2}\lambda)g(\overline{\chi}\lambda)g(\overline{\lambda})}{g(A)g(B)g(\overline{D^2})g(\overline{E^2})} \chi(x) \\ = \frac{1}{(q-1)^2} \sum_{\chi} \frac{g(A\chi)g(B\chi)\chi(x)}{g(A)g(B)g(\overline{D^2})g(\overline{E^2})} \sum_{\lambda} g(\overline{D^2}\chi\lambda)g(\overline{\chi}\lambda)g(\overline{E^2}\lambda)g(\overline{\lambda}).$$

Using Lemma 1.8 we have

$$L = \frac{1}{q-1} \sum_{\chi} \frac{g(A\chi)g(B\chi)g(\overline{D^2E^2}\chi)g(\overline{D^2}\chi)g(\overline{E^2}\chi)g(\overline{\chi})}{g(\overline{D^2E^2}\chi^2)g(A)g(B)g(\overline{D^2})g(\overline{E^2})} \chi(x) + I_1,$$

where

$$I_1 = q \sum_x \frac{g(A\chi)g(B\chi)\chi(x)}{g(A)g(B)g(\overline{D^2})g(\overline{E^2})} \delta(\overline{D^2E^2\chi^2}).$$

Since  $I_1$  is nonzero only when  $\chi = \overline{DE}, \overline{DE}\varphi$ , so we have

$$I_1 = q \frac{g(\overline{ADE})g(\overline{BDE})\overline{DE}(x) + g(\overline{ADE}\varphi)g(\overline{BDE}\varphi)\overline{DE}\varphi(x)}{g(A)g(B)g(\overline{D^2})g(\overline{E^2})}. \tag{5.12}$$

Using Lemma 1.3 on  $g(\overline{D^2E^2\chi^2})$  we find that

$$L = \sum_x \frac{g(A\chi)g(B\chi)g(\overline{D^2E^2\chi})g(\overline{D^2\chi})g(\overline{E^2\chi})g(\overline{D^2E^2\chi^2})g(\overline{\chi})}{q(q-1)g(A)g(B)g(\overline{D^2})g(\overline{E^2})} \chi(x) + I_1 - I_2,$$

where

$$I_2 = \frac{1}{q} \sum_x \frac{g(A\chi)g(B\chi)g(\overline{D^2E^2\chi})g(\overline{D^2\chi})g(\overline{E^2\chi})g(\overline{\chi})}{g(A)g(B)g(\overline{D^2})g(\overline{E^2})} \chi(x) \delta(\overline{D^2E^2\chi^2}).$$

Similarly,  $I_2$  is nonzero only when  $\chi = \overline{DE}$  and  $\overline{DE}\varphi$ , and hence

$$I_2 = \frac{g(\overline{ADE})g(\overline{BDE})g(\overline{DE})g(\overline{DE})g(\overline{ED})g(\overline{DE})\overline{DE}(x)}{qg(A)g(B)g(\overline{D^2})g(\overline{E^2})} + \frac{g(\overline{ADE}\varphi)g(\overline{BDE}\varphi)g(\overline{DE}\varphi)g(\overline{ED}\varphi)g(\overline{DE}\varphi)g(\overline{DE}\varphi)\overline{DE}\varphi(x)}{qg(A)g(B)g(\overline{D^2})g(\overline{E^2})}. \tag{5.13}$$

Replacing  $g(\overline{DE})g(\overline{DE})$ ,  $g(\overline{DE})g(\overline{ED})$ ,  $g(\overline{DE}\varphi)g(\overline{DE}\varphi)$  and  $g(\overline{ED}\varphi)g(\overline{DE}\varphi)$  by using Lemma 1.2 with the given conditions  $\overline{D^2E^2}, \overline{D^2E^2} \neq \varepsilon$ , we have

$$I_2 = q \frac{g(\overline{ADE})g(\overline{BDE})\overline{DE}(x) + g(\overline{ADE}\varphi)g(\overline{BDE}\varphi)\overline{DE}\varphi(x)}{g(A)g(B)g(\overline{D^2})g(\overline{E^2})} = I_1.$$

Replacing the Gauss sum  $g(D^2E^2\chi^2)$  by using Lemma 1.5 we obtain

$$L = \frac{DE(4)}{q(q-1)} \sum_{\chi} \frac{g(A\chi)g(B\chi)g(\overline{D^2E^2\chi})g(\overline{D^2\chi})g(\overline{E^2\chi})g(DE\chi)g(DE\varphi\chi)g(\overline{\chi})}{g(\varphi)g(A)g(B)g(\overline{D^2})g(\overline{E^2})} \chi(4x).$$

Multiplying both numerator and denominator by  $q^3g(\overline{BD^2})g(\overline{DE})g(\overline{DE}\varphi)$  and then rearranging the terms, we obtain

$$\begin{aligned} L &= \frac{q^3DE(4)g(\overline{BD^2})g(\overline{DE})g(\overline{DE}\varphi)}{(q-1)g(\varphi)g(B)g(\overline{D^2})g(\overline{E^2})} \sum_{\chi} \left[ \frac{g(A\chi)g(\overline{\chi})}{q\chi(-1)g(A)} \right] \left[ \frac{g(B\chi)g(\overline{D^2\chi})}{q\chi(-1)g(\overline{BD^2})} \right] \\ &\quad \times \left[ \frac{\chi(-1)g(DE\chi)g(\overline{E^2\chi})}{qg(\overline{DE})} \right] \left[ \frac{\chi(-1)g(DE\varphi\chi)g(\overline{D^2E^2\chi})}{qg(\overline{DE}\varphi)} \right] \chi(4x). \end{aligned}$$

Finally, using Lemma 1.12 with the given conditions  $A, D^2E^2, \overline{D^2E^2}, \overline{BD^2} \neq \varepsilon$  and then (1.20), we complete the proof of the theorem. ■



# 6

## Summation and product identities for Gaussian hypergeometric series

### 6.1 Introduction

Greene [31] found several transformation formulas satisfied by the Gaussian hypergeometric series analogous to those satisfied by the classical hypergeometric series. Since then many mathematicians have obtained finite field analogues of transformation and summation identities satisfied by the classical hypergeometric

---

<sup>1</sup>Some parts of this chapter are published in *Research in Number Theory* (2020) and some parts are under review.

series (see for example [17, 18, 20, 21, 25, 45]). Finite field hypergeometric series are known to be related to various arithmetic objects. Some of the biggest motivations for studying finite field hypergeometric functions have been their connections with Fourier coefficients and eigenvalues of modular forms and with counting points on certain kinds of algebraic varieties. Assuming the conjecture of van Geemen and van Straten, McCarthy and Papanicolas [47] related the eigenvalue of the Hecke operator of index  $p$  of a Siegel eigenform of degree 2 and level 8 to  ${}_4F_3 \left( \begin{matrix} \varphi, \varphi, \varphi, \varphi \\ \varepsilon, \varepsilon, \varepsilon \end{matrix} \middle| -1 \right)$ . The following identity played a crucial role in their proof:

$${}_4F_3 \left( \begin{matrix} \varphi, \varphi, \varphi, \varphi \\ \varepsilon, \varepsilon, \varepsilon \end{matrix} \middle| -1 \right) = {}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| -1 \right) \cdot {}_3F_2 \left( \begin{matrix} \chi_4, \varphi, \varphi \\ \varepsilon, \varepsilon \end{matrix} \middle| 1 \right).$$

In [18, 19], Evans and Greene expressed  ${}_3F_2$ -hypergeometric series as a product of  ${}_2F_1$ -hypergeometric series over finite fields from where they deduced certain special values of  ${}_3F_2$ -hypergeometric series including a finite field analogue of the Clausen's identity. In this chapter, we prove finite field analogues of certain product formulas satisfied by the classical hypergeometric series. As an application of the main results of Chapter 5, we find new summation and product formulas satisfied by the finite field hypergeometric functions. For example, we express a  ${}_4F_3$ -finite field hypergeometric function as a sum of two  ${}_2F_1$ -finite field hypergeometric functions. We also find two identities expressing  ${}_4F_3$ -finite field hypergeometric functions as a product of two  ${}_2F_1$ -finite field hypergeometric functions.

## 6.2 Product formulas for Gaussian hypergeometric series

In the following theorem, we express a  ${}_4F_3$ -hypergeometric series as a product of two  ${}_2F_1$ -hypergeometric series over finite fields.

**Theorem 6.1.** *Let  $A, B, C \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^2, B^2 \neq \varepsilon$ ,  $A^2 \neq C$ , and  $B^2 \neq C$ . For  $x \neq 1$ , we have*

$$\begin{aligned}
& {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & C \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & A^2 B^2 \bar{C} \end{matrix} \middle| x \right) \\
&= \frac{qAB(4)g(\bar{A}^2)g(AB\bar{C})g(\overline{ABC}\varphi)}{g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)g(\varphi)} {}_4F_3 \left( \begin{matrix} A^2, & B^2, & AB, & AB\varphi \\ & A^2B^2, & C, & A^2B^2\bar{C} \end{matrix} \middle| 4x(1-x) \right) \\
&\quad - \frac{(q-1)AB(4)g(\bar{A}^2)g(AB\bar{C})g(\overline{ABC}\varphi)}{g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)g(\varphi)} \left[ {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB\varphi \\ & A^2B^2, & A^2B^2\bar{C} \end{matrix} \middle| 4x(1-x) \right) \right. \\
&\quad \times \delta(\overline{ABC}) + {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB \\ & A^2B^2, & C \end{matrix} \middle| 4x(1-x) \right) \delta(\overline{ABC}\varphi) \left. \right] \\
&\quad + \frac{q\bar{C}A^2(1-x)C\bar{B}^2(x)}{g(A^2)g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)} \delta \left( \frac{1-2x}{(1-x)^2} \right) - \frac{(q-1)g(\bar{A}B)g(\overline{AB})\overline{AB}(x-x^2)}{q^2g(A^2)g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)} \\
&\quad \times [(q-1)\delta(AB)\delta(\overline{ABC}) - qAB(-1)\delta(\overline{ABC}) - qABC(-1)\delta(AB)] \\
&\quad - \frac{(q-1)g(\overline{AB}\varphi)g(\overline{AB}\varphi)\overline{AB}\varphi(x-x^2)}{q^2g(A^2)g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)} [(q-1)\delta(\overline{ABC}\varphi)\delta(\overline{AB}\varphi) \\
&\quad - qAB\varphi(-1)\delta(\overline{ABC}\varphi) - qABC\varphi(-1)\delta(\overline{AB}\varphi)].
\end{aligned}$$

We show that many interesting results proved by Evans, Greene, and Ono follow from the above transformation including a finite field analogue of the Clausen's classical identity. We have stated Theorem 6.1 with minimum conditions on the parameters so that certain known results can be deduced, and therefore there are some extra terms in the formula. The extra terms will disappear if we put some additional

conditions on the parameters. For example, we have the following corollary.

**Corollary 6.2.1.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2, B^2, A^2B^2, A^2B^2\overline{C^2} \neq \varepsilon$ ,  $A^2 \neq C$ , and  $B^2 \neq C$ . For  $x \neq 1, \frac{1}{2}$ , we have*

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & C \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & A^2B^2\overline{C} \end{matrix} \middle| x \right) \\ &= \frac{qAB(4)g(\overline{A^2})g(\overline{ABC})g(\overline{ABC}\varphi)}{g(\overline{B^2})g(\overline{B^2\overline{C}})g(\overline{A^2C})g(\varphi)} {}_4F_3 \left( \begin{matrix} A^2, & B^2, & AB, & AB\varphi \\ & A^2B^2, & C, & A^2B^2\overline{C} \end{matrix} \middle| 4x(1-x) \right). \end{aligned}$$

If we apply Proposition 1.19 to Corollary 6.2.1, we obtain the following identity satisfied by the McCarthy's finite field hypergeometric series.

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & C \end{matrix} \middle| x \right)^* {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & A^2B^2\overline{C} \end{matrix} \middle| x \right)^* \\ &= {}_4F_3 \left( \begin{matrix} A^2, & B^2, & AB, & AB\varphi \\ & A^2B^2, & C, & A^2B^2\overline{C} \end{matrix} \middle| 4x(1-x) \right)^*. \end{aligned}$$

The above identity is a finite field analogue of the following identity [7, (6.1)] satisfied by the classical hypergeometric series:

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} \alpha, & \beta \\ & \gamma \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} \alpha, & \beta \\ & \alpha + \beta - \gamma \end{matrix} \middle| x \right) \\ &= {}_4F_3 \left( \begin{matrix} \alpha, & \beta, & \frac{1}{2}(\alpha + \beta), & \frac{1}{2}(\alpha + \beta + 1) \\ & \alpha + \beta, & \gamma, & \alpha + \beta - \gamma + 1 \end{matrix} \middle| 4x(1-x) \right). \end{aligned}$$

The following transformation satisfied by the classical hypergeometric series is equivalent to the Clausen's identity [7].

$${}_2F_1 \left( \begin{matrix} \alpha, & \beta \\ & \alpha + \beta + \frac{1}{2} \end{matrix} \middle| 4x(1-x) \right)^2$$

$$= {}_3F_2 \left( \begin{matrix} 2\alpha, & 2\beta, & \alpha + \beta \\ & 2\alpha + 2\beta, & \alpha + \beta + \frac{1}{2} \end{matrix} \mid 4x(1-x) \right). \quad (6.1)$$

From Theorem 6.1, we prove the following result which is a finite field analogue of (6.1).

**Theorem 6.2.** *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2, B^2, A\overline{B}\varphi, AB, AB\varphi \neq \varepsilon$ . For  $x \neq 1, \frac{1}{2}$ , we have*

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A, & B \\ & AB\varphi \end{matrix} \mid 4x(1-x) \right)^2 = \frac{AB(4)g(B)^2g(A\varphi)^2}{qg(A^2)g(B^2)} \\ & \times {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB \\ & A^2B^2, & AB\varphi \end{matrix} \mid 4x(1-x) \right) + \frac{g(B)^2g(A\varphi)^2A\overline{B}\varphi(x-x^2)}{q^2g(A^2)g(B^2)}. \end{aligned}$$

We note that a finite field analogue of the Clausen's identity was also obtained by Evans and Greene [18, Thm 1.5]. Theorem 6.2 can also be deduced from [18, Thm 1.5] by taking  $S = B, C = AB\varphi$ , and then employing Lemma 1.9 and Lemma 1.2.

The following identity expresses a  ${}_4F_3$  classical hypergeometric series as a product of two  ${}_2F_1$  classical hypergeometric series [7, (7.4)].

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} \alpha, & \beta \\ & \gamma \end{matrix} \mid x \right) {}_2F_1 \left( \begin{matrix} \gamma - \beta, & 1 - \beta \\ & \alpha - \beta - 1 \end{matrix} \mid x \right) \\ & = (1-x)^{\beta-\alpha-\gamma} {}_4F_3 \left( \begin{matrix} \alpha, & \gamma - \beta, & \frac{1}{2}(\alpha + \gamma - \beta), & \frac{1}{2}(\alpha + \gamma - \beta + 1) \\ \alpha + \gamma - \beta, & \gamma, & \alpha - \beta + 1 \end{matrix} \mid \frac{-4x}{(1-x)^2} \right). \end{aligned} \quad (6.2)$$

In the following theorem, we prove a finite field analogue of (6.2).

**Theorem 6.3.** *Let  $A, D, E \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2, E^2, A^2\overline{D^2E^2}, A^2D^2\overline{E^2} \neq \varepsilon, A^2 \neq$*

$D^2$ , and  $D^2 \neq E^2$ . For  $z \neq 1$ , we have

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & E^2 \\ & D^2 \end{matrix} \middle| z \right) {}_2F_1 \left( \begin{matrix} D^2 \overline{E^2}, & \overline{E^2} \\ & A^2 \overline{E^2} \end{matrix} \middle| z \right) \\ &= \frac{E^2(z)}{q} \delta(1-z^2) + \frac{ADE(4)\overline{A^2 D^2 E^2}(1-z)g(\overline{AED})g(\overline{AED}\varphi)}{g(\varphi)} \\ & \quad \times {}_4F_3 \left( \begin{matrix} A^2, & D^2 \overline{E^2}, & ADE, & ADE\varphi \\ & A^2 D^2 \overline{E^2}, & D^2, & A^2 \overline{E^2} \end{matrix} \middle| \frac{-4z}{(1-z)^2} \right). \end{aligned}$$

If we assume  $z^2 \neq 1$  in Theorem 6.3, then Proposition 1.19 yields

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & E^2 \\ & D^2 \end{matrix} \middle| z \right)^* {}_2F_1 \left( \begin{matrix} D^2 \overline{E^2}, & \overline{E^2} \\ & A^2 \overline{E^2} \end{matrix} \middle| z \right)^* \\ &= \overline{A^2 D^2 E^2} (1-z) {}_4F_3 \left( \begin{matrix} A^2, & D^2 \overline{E^2}, & ADE, & ADE\varphi \\ & A^2 D^2 \overline{E^2}, & D^2, & A^2 \overline{E^2} \end{matrix} \middle| \frac{-4z}{(1-z)^2} \right)^*, \end{aligned}$$

which is an exact finite field analogue of (6.2).

The following is another product formula satisfied by the classical hypergeometric series [7, (6.3)].

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} \alpha, & \beta \\ & \gamma \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} \alpha, & \gamma - \beta \\ & \gamma \end{matrix} \middle| x \right) \\ &= (1-x)^{-\alpha} {}_4F_3 \left( \begin{matrix} \alpha, & \beta, & \gamma - \alpha, & \gamma - \beta \\ & \gamma, & \frac{1}{2}\gamma, & \frac{\gamma+1}{2} \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right). \end{aligned} \quad (6.3)$$

We prove the following result which is a finite field analogue of (6.3).

**Theorem 6.4.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B, C^2 \neq \varepsilon$  and  $A, B \neq C^2$ . For*

$x \neq 1$ , we have

$$\begin{aligned}
& {}_2F_1 \left( \begin{matrix} A, B \\ C^2 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A, C^2 \bar{B} \\ C^2 \end{matrix} \middle| x \right) = \frac{qAB(-1)\bar{A}^2B(1-x)\bar{C}^2(x)}{g(A)g(\bar{B})g(\bar{A}C^2)g(\bar{B}C^2)} \delta \left( \frac{x-2}{x-1} \right) \\
& + \frac{q\bar{A}(1-x)g(\bar{A}C)g(\bar{B}C\varphi)}{\varphi(-1)C(4)g(\varphi)g(\bar{A}C^2)g(\bar{B})} {}_4F_3 \left( \begin{matrix} A, B, \bar{A}C^2, \bar{B}C^2 \\ C^2, C, C\varphi \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right) \\
& + \frac{(q-1)\bar{A}(1-x)g(\bar{A}C)g(\bar{B}C\varphi)}{\varphi(-1)C(4)g(\varphi)g(\bar{A}C^2)g(\bar{B})} \left[ \frac{q-1}{q} {}_2F_1 \left( \begin{matrix} A, B \\ C^2 \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right) \delta(\bar{A}C)\delta(\bar{B}C\varphi) \right. \\
& \left. - {}_3F_2 \left( \begin{matrix} A, B, \bar{B}C^2 \\ C^2, C\varphi \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right) \delta(\bar{A}C) - {}_3F_2 \left( \begin{matrix} A, B, \bar{A}C^2 \\ C^2, C \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right) \delta(\bar{B}C\varphi) \right] \\
& - \frac{(q-1)\bar{A}(1-x)\bar{C}(x^2)C(1-x)}{qg(A)g(\bar{B})g(\bar{B}C^2)g(\bar{A}C^2)} [(q-1)\delta(\bar{A}C)\delta(\bar{B}C) - qBC(-1)\delta(\bar{A}C) \\
& - qAC(-1)\delta(\bar{B}C) + (q-1)\varphi(1-x)\delta(\bar{A}C\varphi)\delta(\bar{B}C\varphi) - qBC(-1)\varphi(x-1)\delta(\bar{A}C\varphi) \\
& - qAC(-1)\varphi(x-1)\delta(\bar{B}C\varphi)].
\end{aligned}$$

If we put some additional conditions on the parameters in Theorem 6.4, we readily obtain the following identity.

**Corollary 6.2.2.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B, C^2, A^2\bar{C}^2, B^2\bar{C}^2 \neq \varepsilon$  and  $A, B \neq C^2$ . For  $x \neq 1$ , we have*

$$\begin{aligned}
& {}_2F_1 \left( \begin{matrix} A, B \\ C^2 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A, C^2 \bar{B} \\ C^2 \end{matrix} \middle| x \right) = \frac{qAB(-1)\bar{A}^2B(1-x)\bar{C}^2(x)}{g(A)g(\bar{B})g(\bar{A}C^2)g(\bar{B}C^2)} \delta \left( \frac{x-2}{x-1} \right) \\
& + \frac{q\varphi(-1)\bar{C}(4)\bar{A}(1-x)g(\bar{A}C)g(\bar{B}C\varphi)}{g(\varphi)g(\bar{A}C^2)g(\bar{B})} {}_4F_3 \left( \begin{matrix} A, B, \bar{A}C^2, \bar{B}C^2 \\ C^2, C, C\varphi \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right).
\end{aligned}$$

If we assume  $x \neq 2$  in Corollary 6.2.2, then Proposition 1.19 yields

$${}_2F_1 \left( \begin{matrix} A, B \\ C^2 \end{matrix} \middle| x \right)^* {}_2F_1 \left( \begin{matrix} A, C^2 \bar{B} \\ C^2 \end{matrix} \middle| x \right)^*$$

$$= \overline{A}(1-x) {}_4F_3 \left( \begin{matrix} A, B, \overline{AC}^2, \overline{BC}^2 \\ C^2, C, C\varphi \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right)^*$$

which is an exact finite field analogue of (6.3).

In the following theorem, we prove a product formula expressing a  ${}_4F_3$ -Gaussian hypergeometric series as a product of two  ${}_2F_1$ -Gaussian hypergeometric series employing Theorem 5.4.

**Theorem 6.5.** *Let  $A, B \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^2, B^2, \overline{AB}, \overline{AB}\varphi \neq \varepsilon$ . Suppose that  $AB$  is a square character. Then, for  $z \in \mathbb{F}_q$  satisfying  $z \neq \pm 1$ , we have*

$$\begin{aligned} & {}_4F_3 \left( \begin{matrix} A, A\varphi, B, B\varphi \\ AB, D, D\varphi \end{matrix} \middle| -\frac{4z^2}{(1-z^2)^2} \right) \\ &= \frac{\overline{D}(4)g(\overline{AB})^2g(A^2)g(\varphi)}{g(A)g(B^2)g(\overline{B}\varphi)g(\overline{BD})^2} {}_2F_1 \left( \begin{matrix} A^2, B^2 \\ AB \end{matrix} \middle| -\frac{z}{1-z} \right) {}_2F_1 \left( \begin{matrix} A^2, B^2 \\ AB \end{matrix} \middle| \frac{z}{1+z} \right) \\ &\quad - \frac{AB(-1)\overline{AB}(z)A^2(1+z)B^2(1-z)g(\varphi)}{D(4)g(A)g(\overline{B}\varphi)g(\overline{BD})^2} \delta(1+z^2), \end{aligned}$$

where  $D^2 = AB$ .

We note that there are two product identities in Theorem 6.5 each corresponding to a square root of  $AB$ . In the following theorem, we prove another product formula employing Theorem 5.5.

**Theorem 6.6.** *Let  $A, B, D \in \widehat{\mathbb{F}_q^\times}$  be such that  $A, B, A^2B^2, \overline{AD}^2, \overline{BD}^2, \overline{ABD}^4, \overline{ABD}^4\varphi \neq \varepsilon$ . Suppose that  $AB$  is a square character. Then, for  $z \in \mathbb{F}_q$  satisfying  $z \neq 1$ , we have*

$$\begin{aligned} & {}_4F_3 \left( \begin{matrix} A, B, G, G\varphi \\ D^2, \overline{ABD}^2, AB \end{matrix} \middle| 4z(1-z) \right) \\ &= \frac{(q-1)A(-1)g(\overline{B})g(\overline{AD}^2)g(\varphi)}{qG(4)g(D^2\overline{G})g(\overline{G}\varphi)} {}_2F_1 \left( \begin{matrix} A, B \\ D^2 \end{matrix} \middle| z \right) {}_2F_1 \left( \begin{matrix} A, B \\ \overline{ABD}^2 \end{matrix} \middle| z \right) \end{aligned}$$

$$- \frac{(q-1)AB(-1)\overline{AD}^2(1-z)\overline{BD}^2(z)g(\varphi)}{G(4)g(A)g(\overline{BD}^2)g(D^2\overline{G})g(\overline{G}\varphi)} \delta\left(\frac{1-2z}{(1-z)^2}\right),$$

where  $G^2 = AB$ .

Again in Theorem 6.6, there are two product identities. We get a product formula corresponding to each square root of  $AB$ .

### 6.3 Proof of Theorem 6.1

In the following lemma, we re-write Theorem 4.2 in terms of Greene's finite field hypergeometric series.

**Lemma 6.7.** *Let  $A, B, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, B \neq \varepsilon$ ,  $B \neq C$ , and  $A \neq C$ . For  $z, w \in \mathbb{F}_q$  such that  $z, w \neq 1$  we have*

$$\begin{aligned} & {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \middle| z\right) {}_2F_1\left(\begin{matrix} A, & B \\ & ABC\overline{C} \end{matrix} \middle| w\right) \\ &= \frac{A(-1)g(B)g(\overline{C})g(\overline{ABC})}{qg(\overline{B})g(\overline{BC})g(\overline{AC})} F_4\left(A; B; C, ABC\overline{C}; z(1-w), w(1-z)\right)^* \\ & \quad + \frac{qB(-1)\overline{A}(1-z)\overline{BC}(w)\overline{C}(1-w)}{g(A)g(\overline{B})g(\overline{BC})g(\overline{AC})} \delta\left(\frac{1-w-z}{(1-z)(1-w)}\right). \end{aligned}$$

*Proof.* Using Proposition 1.19 in Theorem 4.2 we have

$$\begin{aligned} & {}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \middle| -\frac{x}{1-x}\right) {}_2F_1\left(\begin{matrix} A, & B \\ & ABC\overline{C} \end{matrix} \middle| -\frac{y}{1-y}\right) \\ &= \begin{pmatrix} B \\ C \end{pmatrix} \begin{pmatrix} B \\ ABC\overline{C} \end{pmatrix} \times \left[ F_4\left(A; B; C, ABC\overline{C}; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right)^* \right. \\ & \quad \left. + \frac{q^2 AC(-1)\overline{BC}(y)A(1-x)B(1-y)}{g(A)g(B)g(\overline{C})g(\overline{ABC})} \delta(1-xy) \right]. \quad (6.4) \end{aligned}$$

Applying Lemma 1.12 and Lemma 1.2, and then putting  $z = \frac{-x}{1-x}$  and  $w = \frac{-y}{1-y}$  in (6.4), we complete the proof. ■

We now present a proof of our main product formula Theorem 6.1.

*Proof of Theorem 6.1.* The result is trivially true if  $x = 0$ . So, we assume that  $x \neq 0$ . Let

$$L := {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & C \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & A^2 B^2 \bar{C} \end{matrix} \middle| x \right).$$

Using Lemma 6.7, we have

$$L = \frac{g(B^2)g(\bar{C})g(\overline{A^2 B^2 C})}{qg(\bar{B}^2)g(B^2 \bar{C})g(\overline{A^2 C})} F_4(A^2; B^2; C, A^2 B^2 \bar{C}; x(1-x), x(1-x))^* + I_1, \quad (6.5)$$

where

$$I_1 = \frac{q\bar{C}A^2(1-x)C\bar{B}^2(x)}{g(A^2)g(\bar{B}^2)g(B^2 \bar{C})g(\overline{A^2 C})} \delta \left( \frac{1-2x}{(1-x)^2} \right). \quad (6.6)$$

Now employing (4.2) into (6.5) we have

$$L = \frac{1}{q(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2 \chi \lambda)g(B^2 \chi \lambda)g(\bar{C} \chi)g(\overline{A^2 B^2 C \bar{\lambda}})g(\bar{\lambda})g(\bar{\chi})}{g(A^2)g(\bar{B}^2)g(B^2 \bar{C})g(\overline{A^2 C})} \chi \lambda (x - x^2) + I_1.$$

The change of variables  $\chi \mapsto \chi \bar{\lambda}$  yield

$$L = \frac{1}{q(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2 \chi)g(B^2 \chi)g(\bar{C} \chi \lambda)g(\overline{A^2 B^2 C \bar{\lambda}})g(\bar{\lambda})g(\bar{\chi} \lambda)}{g(A^2)g(\bar{B}^2)g(B^2 \bar{C})g(\overline{A^2 C})} \chi (x - x^2) + I_1.$$

Using Lemma 1.8, we have

$$L = \frac{1}{q(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2 \chi)g(B^2 \chi)g(\overline{A^2 B^2 \chi})g(\overline{A^2 B^2 C \bar{\chi}})g(\bar{C} \chi)g(\bar{\chi})}{g(A^2)g(\bar{B}^2)g(B^2 \bar{C})g(\overline{A^2 C})g(\overline{A^2 B^2 \chi^2})} \chi (x - x^2) + I_1 + I_2, \quad (6.7)$$

where

$$\begin{aligned} I_2 &= C(-1) \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2\chi)g(B^2\chi)\chi(x-x^2)}{g(A^2)g(\overline{B^2})g(\overline{B^2C})g(\overline{A^2C})} \delta(\overline{A^2B^2\chi^2}) \\ &= C(-1) \left[ \frac{g(\overline{AB})g(\overline{AB})\overline{AB}(x-x^2) + g(\overline{AB}\varphi)g(\overline{AB}\varphi)\overline{AB}\varphi(x-x^2)}{g(A^2)g(\overline{B^2})g(\overline{B^2C})g(\overline{A^2C})} \right]. \end{aligned} \quad (6.8)$$

The last equality is obtained by putting  $\chi = \overline{AB}, \overline{AB}\varphi$ . Now using Lemma 1.3 in (6.7), we have

$$\begin{aligned} L &= \frac{1}{q^2(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2\chi)g(B^2\chi)g(\overline{C\chi})g(\overline{A^2B^2\chi})g(\overline{A^2B^2C\chi})}{g(A^2)g(\overline{B^2})g(\overline{B^2C})g(\overline{A^2C})} \\ &\quad \times g(A^2B^2\chi^2)g(\overline{\chi})\chi(x-x^2) + I_1 + I_2 - I_3, \end{aligned} \quad (6.9)$$

where

$$\begin{aligned} I_3 &= \frac{1}{q^2} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2\chi)g(B^2\chi)g(\overline{C\chi})g(\overline{A^2B^2\chi})g(\overline{A^2B^2C\chi})g(\overline{\chi})\chi(x-x^2)}{g(A^2)g(\overline{B^2})g(\overline{B^2C})g(\overline{A^2C})} \delta(A^2B^2\chi^2) \\ &= I_2 + \frac{(q-1)g(\overline{AB})g(\overline{AB})\overline{AB}(x-x^2)}{q^2g(A^2)g(\overline{B^2})g(\overline{B^2C})g(\overline{A^2C})} [\{(q-1)\delta(AB) - qAB(-1)\}\delta(\overline{ABC}) \\ &\quad - qABC(-1)\delta(AB)] + \frac{(q-1)g(\overline{AB}\varphi)g(\overline{AB}\varphi)\overline{AB}\varphi(x-x^2)}{q^2g(A^2)g(\overline{B^2})g(\overline{B^2C})g(\overline{A^2C})} \\ &\quad \times [(q-1)\delta(\overline{ABC}\varphi)\delta(\overline{AB}\varphi) - qAB\varphi(-1)\delta(\overline{ABC}\varphi) - qABC\varphi(-1)\delta(\overline{AB}\varphi)]. \end{aligned} \quad (6.10)$$

The last equality is obtained by putting  $\chi = \overline{AB}, \overline{AB}\varphi$  and using Lemma 1.2 on  $g(AB)g(\overline{AB}), g(\overline{ABC})g(\overline{ABC}), g(\overline{AB}\varphi)g(\overline{AB}\varphi)$  and  $g(\overline{ABC}\varphi)g(\overline{ABC}\varphi)$ . Now using Lemma 1.5 on  $g(A^2B^2\chi^2)$ , (6.9) reduces to

$$L = \frac{AB(4)}{q^2(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A^2\chi)g(B^2\chi)g(\overline{C\chi})g(\overline{A^2B^2\chi})g(\overline{A^2B^2C\chi})g(\overline{AB}\chi)}{g(A^2)g(\overline{B^2})g(\overline{B^2C})g(\overline{A^2C})g(\varphi)}$$

$$\times g(AB\varphi\chi)g(\bar{\chi})\chi(4x - 4x^2) + I_1 + I_2 - I_3.$$

Multiplying both numerator and denominator by  $q^4g(\bar{A}^2)g(ABC\bar{C})g(\overline{ABC}\varphi)$  and then rearranging the terms we have

$$\begin{aligned} L &= \frac{q^2AB(4)g(\bar{A}^2)g(ABC\bar{C})g(\overline{ABC}\varphi)}{(q-1)g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)g(\varphi)} \\ &\times \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \left( \frac{g(A^2\chi)g(\bar{\chi})\chi(-1)}{qg(\bar{A}^2)} \times \frac{g(B^2\chi)g(\bar{A}^2B^2\chi)\chi(-1)}{qg(\bar{A}^2)} \times \frac{g(AB\chi)g(\bar{C}\chi)C\chi(-1)}{qg(ABC\bar{C})} \right. \\ &\left. \times \frac{g(AB\varphi\chi)g(\bar{A}^2B^2C\bar{\chi})C\chi(-1)}{qg(\overline{ABC}\varphi)} \chi(4x - 4x^2) \right) + I_1 + I_2 - I_3. \end{aligned}$$

Lemma 1.12 yields

$$\begin{aligned} L &= \frac{q^2AB(4)g(\bar{A}^2)g(ABC\bar{C})g(\overline{ABC}\varphi)}{(q-1)g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)g(\varphi)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A^2\chi}{\chi} \left( \binom{AB\chi}{C\chi} - \frac{q-1}{q}\delta(ABC\bar{C}) \right) \\ &\times \binom{B^2\chi}{A^2B^2\chi} \left( \binom{AB\varphi\chi}{A^2B^2C\chi} - \frac{q-1}{q}\delta(\overline{ABC}\varphi) \right) \chi(4x - 4x^2) + I_1 + I_2 - I_3. \end{aligned}$$

Using (1.20) we have

$$\begin{aligned} L &= \frac{qAB(4)g(\bar{A}^2)g(ABC\bar{C})g(\overline{ABC}\varphi)}{g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)g(\varphi)} {}_4F_3 \left( \begin{matrix} A^2, & B^2, & AB, & AB\varphi \\ & A^2B^2, & C, & A^2B^2\bar{C} \end{matrix} \middle| 4x(1-x) \right) \\ &\quad - \frac{(q-1)AB(4)g(\bar{A}^2)g(ABC\bar{C})g(\overline{ABC}\varphi)}{g(\bar{B}^2)g(B^2\bar{C})g(\bar{A}^2C)g(\varphi)} \\ &\quad \times \left[ {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB\varphi \\ & A^2B^2, & A^2B^2\bar{C} \end{matrix} \middle| 4x(1-x) \right) \delta(ABC\bar{C}) \right. \\ &\quad \left. + {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB \\ & A^2B^2, & C \end{matrix} \middle| 4x(1-x) \right) \delta(\overline{ABC}\varphi) \right] + I_1 + I_2 - I_3. \end{aligned} \tag{6.11}$$

Finally employing (6.6), (6.8), and (6.10) into (6.11), we complete the proof of the theorem.  $\blacksquare$

## 6.4 Proof of Theorem 6.2

We now state a special case of Theorem 6.1 which will be used to prove Theorem 6.2 and to derive certain special values.

**Corollary 6.4.1.** *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2, B^2, A\bar{B}\varphi \neq \varepsilon$ . For  $x \neq 1$  we have*

$$\begin{aligned} {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \middle| x \right)^2 &= \frac{qAB(4)g(\overline{A^2})}{g(\overline{B^2})g(\overline{AB\varphi})^2} {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB \\ & A^2B^2, & AB\varphi \end{matrix} \middle| 4x(1-x) \right) \\ &+ \frac{g(AB\varphi)g(\overline{AB\varphi})g(\overline{AB\varphi})}{q^2g(\overline{B^2})g(A^2)g(\overline{AB\varphi})} \overline{AB\varphi}(x-x^2) + \frac{qA^3\overline{B\varphi}(1-x)\overline{AB\varphi}(x)}{g(A^2)g(\overline{B^2})g(\overline{AB\varphi})^2} \delta \left( \frac{1-2x}{(x-1)^2} \right) \\ &+ \frac{(q-1)\varphi(-1)g(\overline{AB})g(\overline{AB})\overline{AB}(x-x^2)}{qg(A^2)g(\overline{B^2})g(\overline{AB\varphi})^2} \delta(AB) \\ &+ \frac{(q-1)g(\overline{AB\varphi})\overline{AB\varphi}(x-x^2)}{q^2g(A^2)g(\overline{B^2})g(\overline{AB\varphi})} [\delta(AB\varphi) + qAB\varphi(-1)]. \end{aligned}$$

*Proof.* The result is trivially true if  $x = 0$ . So, let  $x \neq 0$ . Putting  $C = AB\varphi$  in Theorem 6.1 and using the fact that  $g(\varepsilon) = -1$  we have

$$\begin{aligned} {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \middle| x \right)^2 &= -\frac{qAB(4)g(\overline{A^2})}{g(\overline{B^2})g(\overline{AB\varphi})^2} {}_4F_3 \left( \begin{matrix} A^2, & B^2, & AB, & AB\varphi \\ & A^2B^2, & AB\varphi, & AB\varphi \end{matrix} \middle| 4x(1-x) \right) \\ &+ \frac{(q-1)AB(4)g(\overline{A^2})}{g(\overline{B^2})g(\overline{AB\varphi})^2} {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB \\ & A^2B^2, & AB\varphi \end{matrix} \middle| 4x(1-x) \right) \\ &+ \frac{qA^3\overline{B\varphi}(1-x)\overline{AB\varphi}(x)}{g(A^2)g(\overline{B^2})g(\overline{AB\varphi})^2} \delta \left( \frac{1-2x}{(1-x)^2} \right) + \frac{(q-1)g(\overline{AB})g(\overline{AB})\overline{AB}(x-x^2)}{q\varphi(-1)g(A^2)g(\overline{B^2})g(\overline{AB\varphi})^2} \delta(AB) \end{aligned}$$

$$+ \frac{(q-1)g(\overline{AB\varphi})\overline{AB\varphi}(x-x^2)}{q^2g(A^2)g(\overline{B^2})g(\overline{AB\varphi})} [\delta(AB\varphi) + qAB\varphi(-1)]. \quad (6.12)$$

Using (1.20) and (1.14) we have

$$\begin{aligned} & - \frac{qAB(4)g(\overline{A^2})}{g(\overline{B^2})g(\overline{AB\varphi})^2} {}_4F_3 \left( \begin{matrix} A^2, & B^2, & AB, & AB\varphi \\ & A^2B^2, & AB\varphi, & AB\varphi \end{matrix} \middle| 4x(1-x) \right) \\ & = \frac{AB(4)g(\overline{A^2})}{g(\overline{B^2})g(\overline{AB\varphi})^2} {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB \\ & A^2B^2, & AB\varphi \end{matrix} \middle| 4x(1-x) \right) - I_1, \end{aligned} \quad (6.13)$$

where

$$\begin{aligned} I_1 & = \frac{qAB(4)g(\overline{A^2})}{g(\overline{B^2})g(\overline{AB\varphi})^2} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} A^2\chi \\ \chi \end{pmatrix} \begin{pmatrix} B^2\chi \\ A^2B^2\chi \end{pmatrix} \begin{pmatrix} AB\chi \\ AB\varphi\chi \end{pmatrix} \chi(4x-4x^2)\delta(AB\varphi\chi) \\ & = - \frac{g(AB\varphi)g(\overline{AB\varphi})g(\overline{AB\varphi})}{q^2g(\overline{B^2})g(A^2)g(\overline{AB\varphi})} \overline{AB\varphi}(x-x^2). \end{aligned} \quad (6.14)$$

The last equality is obtained by putting  $\chi = \overline{AB\varphi}$ , and then using Lemma 1.12 and  $g(\varepsilon) = -1$ . Finally, combining (6.12), (6.13) and (6.14), we complete the proof. ■

*Proof of Theorem 6.2.* From [30, (4.33)], we have

$${}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \middle| x \right) = \frac{B(-1)g(B^2)g(\overline{AB\varphi})}{g(B)g(A\varphi)} {}_2F_1 \left( \begin{matrix} A, & B \\ & AB\varphi \end{matrix} \middle| 4x(1-x) \right). \quad (6.15)$$

Using the given conditions  $x \neq 1, \frac{1}{2}$  and  $AB, AB\varphi \neq \varepsilon$ , Corollary 6.4.1 yields

$${}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB\varphi \end{matrix} \middle| x \right)^2 = \frac{qAB(4)g(\overline{A^2})}{g(\overline{B^2})g(\overline{AB\varphi})^2} {}_3F_2 \left( \begin{matrix} A^2, & B^2, & AB \\ & A^2B^2, & AB\varphi \end{matrix} \middle| 4x(1-x) \right)$$

$$+ \frac{g(AB\varphi)g(\overline{AB}\varphi)g(\overline{AB}\varphi)}{q^2g(\overline{B^2})g(A^2)g(\overline{AB}\varphi)}\overline{AB}\varphi(x-x^2) + \frac{(q-1)g(\overline{AB}\varphi)\overline{AB}\varphi(x-x^2)}{qg(A^2)g(\overline{B^2})g(\overline{AB}\varphi)}AB\varphi(-1). \quad (6.16)$$

Combining (6.15) and (6.16), and then employing Lemma 1.2 we complete the proof. ■

## 6.5 Proof of Theorem 6.3

*Proof.* The result is trivially true if  $z = 0$ . Let  $x \neq 0$ . Putting  $C = D^2$  and  $B = D\overline{E}$  in Theorem 6.1, we have

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & D^2\overline{E^2} \\ & D^2 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A^2, & D^2\overline{E^2} \\ & A^2\overline{E^2} \end{matrix} \middle| x \right) \\ &= {}_4F_3 \left( \begin{matrix} A^2, & D^2\overline{E^2}, & ADE\overline{E}, & ADE\overline{E}\varphi \\ & A^2D^2\overline{E^2}, & D^2, & A^2\overline{E^2} \end{matrix} \middle| 4x(1-x) \right) \\ &\times \frac{qADE\overline{E}(4)g(\overline{A^2})g(\overline{AED})g(\overline{AED}\varphi)}{g(\overline{D^2E^2})g(\overline{E^2})g(\overline{A^2D^2})g(\varphi)} + \frac{qA^2\overline{D^2}(1-x)E^2(x)}{g(A^2)g(\overline{D^2E^2})g(\overline{E^2})g(\overline{A^2D^2})} \delta \left( \frac{1-2x}{(x-1)^2} \right). \end{aligned} \quad (6.17)$$

Using Proposition 1.19 we find that

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & D^2\overline{E^2} \\ & D^2 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A^2, & D^2\overline{E^2} \\ & A^2\overline{E^2} \end{matrix} \middle| x \right) \\ &= \left( \frac{D^2\overline{E^2}}{A^2\overline{E^2}} \right) \left( \frac{A^2}{A^2\overline{E^2}} \right)^{-1} {}_2F_1 \left( \begin{matrix} A^2, & D^2\overline{E^2} \\ & D^2 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} D^2\overline{E^2}, & A^2 \\ & A^2\overline{E^2} \end{matrix} \middle| x \right) \\ &= \frac{g(D^2\overline{E^2})g(E^2)}{g(A^2)g(\overline{A^2D^2})} \overline{A^2D^2}E^2(1-x) {}_2F_1 \left( \begin{matrix} A^2, & E^2 \\ & D^2 \end{matrix} \middle| \frac{x}{x-1} \right) \\ &\times {}_2F_1 \left( \begin{matrix} D^2\overline{E^2}, & \overline{E^2} \\ & A^2\overline{E^2} \end{matrix} \middle| \frac{x}{x-1} \right). \end{aligned} \quad (6.18)$$

The last equality is obtained by using Lemma 1.12 and Theorem 1.14 (ii). Now using (6.18) in (6.17) and Lemma 1.2 we have

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A^2, & E^2 \\ & D^2 \end{matrix} \middle| \frac{x}{x-1} \right) {}_2F_1 \left( \begin{matrix} D^2 \overline{E^2}, & \overline{E^2} \\ & A^2 \overline{E^2} \end{matrix} \middle| \frac{x}{x-1} \right) \\ &= \frac{ADE \overline{(4)A^2 D^2 \overline{E^2}} (1-x) g(A \overline{E} \overline{D}) g(\overline{A} E D \varphi)}{g(\varphi)} \\ &\times {}_4F_3 \left( \begin{matrix} A^2, & D^2 \overline{E^2}, & ADE \overline{E}, & ADE \overline{E} \varphi \\ & A^2 D^2 \overline{E^2}, & D^2, & A^2 \overline{E^2} \end{matrix} \middle| 4x(1-x) \right) \\ &+ \frac{\overline{E^2} (1-x) E^2(x)}{q} \delta \left( \frac{1-2x}{(x-1)^2} \right). \end{aligned}$$

Finally, putting  $z = \frac{x}{x-1}$ , we complete the proof of the theorem. ■

## 6.6 Proof of Theorem 6.4

*Proof.* The result is trivially true if  $x = 0$ . So, let  $x \neq 0$ . Let

$$L := {}_2F_1 \left( \begin{matrix} A, & B \\ & C^2 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A, & C^2 \overline{B} \\ & C^2 \end{matrix} \middle| x \right).$$

Using Theorem 1.14 (i) and (ii) we have

$$L = A(-1) \overline{A} (1-x) {}_2F_1 \left( \begin{matrix} A, & B \\ & C^2 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A, & B \\ & ABC^2 \end{matrix} \middle| \frac{1}{1-x} \right). \quad (6.19)$$

Employing Lemma 6.7 into (6.19) yields

$$L = \frac{\overline{A} (1-x) g(B) g(\overline{C^2}) g(\overline{ABC^2})}{q g(\overline{B}) g(BC^2) g(\overline{AC^2})} F_4 \left( A; B; C^2, ABC^2; \frac{x^2}{x-1}, 1 \right)^* + I_1, \quad (6.20)$$

where

$$I_1 = \frac{qAB(-1)\overline{A^2}B(1-x)\overline{C^2}(x)}{g(A)g(\overline{B})g(\overline{AC^2})g(\overline{BC^2})} \delta\left(\frac{x-2}{x-1}\right). \quad (6.21)$$

Using (4.2) in (6.20) we obtain

$$L = \frac{\overline{A}(1-x)}{q(q-1)^2} \sum_{\chi, \lambda \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi\lambda)g(B\chi\lambda)g(\overline{C^2}\chi)g(\overline{ABC^2}\lambda)g(\overline{\lambda})g(\overline{\chi})}{g(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})} \chi\left(\frac{x^2}{x-1}\right) + I_1.$$

Using Lemma 1.8 yields

$$L = \frac{\overline{A}(1-x)}{q(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(B\chi)g(\overline{C^2}\chi)g(\overline{AC^2}\chi)g(\overline{BC^2}\chi)g(\overline{\chi})}{g(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})g(C^2\chi^2)} \chi\left(\frac{x^2}{x-1}\right) + I_1 + I_2, \quad (6.22)$$

where

$$\begin{aligned} I_2 &= AB(-1)\overline{A}(1-x) \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(\overline{C^2}\chi)g(\overline{\chi})}{g(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})} \chi\left(\frac{x^2}{x-1}\right) \delta(C^2\chi^2) \\ &= \frac{qAB(-1)\overline{A}(1-x)\overline{C^2}(x)C(1-x)}{g(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})} [\varphi(1-x) + 1]. \end{aligned} \quad (6.23)$$

The last equality is obtained by putting  $\chi = \overline{C}, \overline{C}\varphi$ , and then using Lemma 1.2 and the fact that  $C^2 \neq \varepsilon$ . Now using Lemma 1.3 in (6.22) we have

$$\begin{aligned} L &= \frac{\overline{A}(1-x)}{q^2(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(B\chi)g(\overline{C^2}\chi)g(\overline{AC^2}\chi)g(\overline{BC^2}\chi)g(\overline{C^2}\chi^2)g(\overline{\chi})}{g(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})} \chi\left(\frac{x^2}{x-1}\right) \\ &\quad + I_1 + I_2 - I_3, \end{aligned} \quad (6.24)$$

where

$$\begin{aligned}
 I_3 &= \frac{\overline{A}(1-x)}{q^2} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(B\chi)g(\overline{C^2\chi})g(\overline{AC^2\chi})g(\overline{BC^2\chi})g(\overline{\chi})}{g(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})} \chi \left( \frac{x^2}{x-1} \right) \delta(\overline{C^2\chi^2}) \\
 &= \frac{\overline{A}(1-x)\overline{C}(x^2)C(x-1)}{q^2g(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})} [g(\overline{AC})g(\overline{BC})g(\overline{C})g(\overline{AC})g(\overline{BC})g(C) \\
 &\quad + g(\overline{AC}\varphi)g(\overline{BC}\varphi)g(\overline{C}\varphi)g(\overline{AC}\varphi)g(\overline{BC}\varphi)g(C\varphi)\varphi(x-1)].
 \end{aligned}$$

The last equality is obtained by putting  $\chi = \overline{C}, \overline{C}\varphi$ . Using Lemma 1.2 on  $g(\overline{AC})g(\overline{AC}), g(\overline{BC})g(\overline{BC}), g(\overline{AC}\varphi)g(\overline{AC}\varphi), g(\overline{BC}\varphi)g(\overline{BC}\varphi), g(C)g(\overline{C})$  and  $g(C\varphi)g(\overline{C}\varphi)$  with the fact that  $C^2 \neq \varepsilon$ , we have

$$\begin{aligned}
 I_3 &= I_2 + \frac{(q-1)\overline{A}(1-x)\overline{C}(x^2)C(1-x)}{qg(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})} [(q-1)\delta(\overline{AC})\delta(\overline{BC}) - qBC(-1)\delta(\overline{AC}) \\
 &\quad - qBC(-1)\varphi(x-1)\delta(\overline{AC}\varphi) + (q-1)\varphi(1-x)\delta(\overline{AC}\varphi)\delta(\overline{BC}\varphi) \\
 &\quad - qAC(-1)\delta(\overline{BC}) - qAC(-1)\varphi(x-1)\delta(\overline{BC}\varphi)]. \tag{6.25}
 \end{aligned}$$

Now using Lemma 1.5 in (6.24) we have

$$\begin{aligned}
 L &= \frac{\overline{A}(1-x)\overline{C}(4)}{q^2(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(B\chi)g(\overline{C^2\chi})g(\overline{AC^2\chi})g(\overline{BC^2\chi})g(\overline{C\chi})g(\varphi\overline{C\chi})g(\overline{\chi})}{g(A)g(\overline{B})g(\overline{BC^2})g(\overline{AC^2})g(\varphi)} \\
 &\quad \times \chi \left( \frac{x^2}{4(x-1)} \right) + I_1 + I_2 - I_3.
 \end{aligned}$$

Multiplying both numerator and denominator by  $q^4g(\overline{AC})g(\overline{BC}\varphi)\varphi(-1)$  and then rearranging the terms we have

$$\begin{aligned}
 L &= \frac{q^2\varphi(-1)\overline{C}(4)\overline{A}(1-x)g(\overline{AC})g(\overline{BC}\varphi)}{(q-1)g(\varphi)g(\overline{AC^2})g(\overline{B})} \\
 &\quad \times \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \left( \frac{g(A\chi)g(\overline{\chi})\chi(-1)}{qg(A)} \right) \left( \frac{g(B\chi)g(\overline{C^2\chi})\chi(-1)}{qg(\overline{BC^2})} \right) \left( \frac{g(\overline{AC^2\chi})g(\overline{C\chi})C\chi(-1)}{qg(\overline{AC})} \right)
 \end{aligned}$$

$$\times \left( \frac{g(\overline{BC}^2\chi)g(\overline{\varphi C\chi})\varphi C\chi(-1)}{qg(\overline{BC}\varphi)} \right) \chi \left( \frac{x^2}{4x-4} \right) + I_1 + I_2 - I_3. \quad (6.26)$$

Using Lemma 1.12 and the fact that  $A, \overline{BC}^2 \neq \varepsilon$  in (6.26) we have

$$\begin{aligned} L &= \frac{q^2\varphi(-1)\overline{C}(4)\overline{A}(1-x)g(\overline{AC})g(\overline{BC}\varphi)}{(q-1)g(\varphi)g(\overline{AC}^2)g(\overline{B})} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \left[ \binom{\overline{AC}^2\chi}{C\chi} - \frac{q-1}{q}\delta(\overline{AC}) \right] \\ &\times \binom{B\chi}{C^2\chi} \left[ \binom{\overline{BC}^2\chi}{C\varphi\chi} - \frac{q-1}{q}\delta(\overline{BC}\varphi) \right] \chi \left( \frac{x^2}{4x-4} \right) + I_1 + I_2 - I_3. \end{aligned}$$

Employing (1.20) yields

$$\begin{aligned} L &= \frac{q\varphi(-1)\overline{C}(4)\overline{A}(1-x)g(\overline{AC})g(\overline{BC}\varphi)}{g(\varphi)g(\overline{AC}^2)g(\overline{B})} {}_4F_3 \left( \begin{matrix} A, B, \overline{AC}^2, \overline{BC}^2 \\ C^2, C, C\varphi \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right) \\ &+ \frac{(q-1)\varphi(-1)\overline{C}(4)\overline{A}(1-x)g(\overline{AC})g(\overline{BC}\varphi)}{g(\varphi)g(\overline{AC}^2)g(\overline{B})} \left[ \frac{q-1}{q} {}_2F_1 \left( \begin{matrix} A, B \\ C^2 \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right) \right. \\ &\times \delta(\overline{AC})\delta(\overline{BC}\varphi) - {}_3F_2 \left( \begin{matrix} A, B, \overline{BC}^2 \\ C^2, C\varphi \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right) \delta(\overline{AC}) \\ &\left. - {}_3F_2 \left( \begin{matrix} A, B, \overline{AC}^2 \\ C^2, C \end{matrix} \middle| \frac{-x^2}{4(1-x)} \right) \delta(\overline{BC}\varphi) \right] + I_1 + I_2 - I_3. \quad (6.27) \end{aligned}$$

Finally combining (6.21), (6.23), (6.25) and (6.27), we complete the proof.  $\blacksquare$

## 6.7 Proof of Theorem 6.5

*Proof.* Let  $D$  be a square root of  $AB$ . Then the other root is  $D\varphi$ . Putting  $C = D$  and  $x = \frac{z}{1-z^2}$  in Theorem 5.4, for  $A, \overline{B}\varphi, \overline{AB}, \overline{AB}\varphi \neq \varepsilon$ , we have

$$F_4(A^2; B^2; AB, AB; \frac{z}{1-z^2}, -\frac{z}{1-z^2})^*$$

$$= \frac{q^2 D(4)g(A)g(\overline{B}\varphi)g(B\overline{D})^2}{g(A^2)g(B^2)g^2(\overline{AB})g(\varphi)} {}_4F_3 \left( \begin{matrix} A, A\varphi, B, B\varphi \\ AB, D, D\varphi \end{matrix} \middle| -\frac{4z^2}{(1-z^2)^2} \right). \quad (6.28)$$

Putting  $A = A^2$ ,  $B = B^2$ ,  $C = AB$ ,  $x = -z$ , and  $y = z$  in Theorem 4.2, for  $A^2, B^2, \overline{AB} \neq \varepsilon$ , we have

$$\begin{aligned} & F_4(A^2; B^2; AB, AB; \frac{z}{1-z^2}, -\frac{z}{1-z^2})^* \\ &= {}_2F_1 \left( \begin{matrix} A^2, B^2 \\ AB \end{matrix} \middle| \frac{z}{1+z} \right)^* {}_2F_1 \left( \begin{matrix} A^2, B^2 \\ AB \end{matrix} \middle| -\frac{z}{1-z} \right)^* \\ & - \frac{q^2 AB(-1)\overline{AB}(z)A^2(1+z)B^2(1-z)}{g(A^2)g(B^2)g(\overline{AB})^2} \delta(1+z^2) \\ &= \frac{q^2 g(\overline{AB})^2}{g(B^2)^2 g(\overline{AB})^2} {}_2F_1 \left( \begin{matrix} A^2, B^2 \\ AB \end{matrix} \middle| \frac{z}{1+z} \right) {}_2F_1 \left( \begin{matrix} A^2, B^2 \\ AB \end{matrix} \middle| -\frac{z}{1-z} \right) \\ & - \frac{q^2 AB(-1)\overline{AB}(z)A^2(1+z)B^2(1-z)}{g(A^2)g(B^2)g(\overline{AB})^2} \delta(1+z^2). \end{aligned} \quad (6.29)$$

The last equality is obtained by employing Proposition 1.19 with the given conditions  $A^2 \neq \varepsilon$  and  $A \neq B$ , and then Lemma 1.12. Now (6.28) and (6.29) yield

$$\begin{aligned} & {}_4F_3 \left( \begin{matrix} A, A\varphi, B, B\varphi \\ AB, D, D\varphi \end{matrix} \middle| -\frac{4z^2}{(1-z^2)^2} \right) \\ &= \frac{\overline{D}(4)g^2(\overline{AB})g(A^2)g(\varphi)}{g(A)g(B^2)g(\overline{B}\varphi)g(B\overline{D})^2} {}_2F_1 \left( \begin{matrix} A^2, B^2 \\ AB \end{matrix} \middle| -\frac{z}{1-z} \right) {}_2F_1 \left( \begin{matrix} A^2, B^2 \\ AB \end{matrix} \middle| \frac{z}{1+z} \right) \\ & - \frac{AB(-1)\overline{AB}(z)A^2(1+z)B^2(1-z)g(\varphi)}{D(4)g(A)g(\overline{B}\varphi)g(B\overline{D})^2} \delta(1+z^2). \end{aligned} \quad (6.30)$$

To complete the proof of the theorem, we need to prove (6.30) for the other root  $D\varphi$ . That is, we need to prove that

$$\begin{aligned}
& {}_4F_3 \left( \begin{matrix} A, & A\varphi, & B, & B\varphi \\ & AB, & D\varphi, & D \end{matrix} \middle| -\frac{4z^2}{(1-z^2)^2} \right) \\
&= \frac{\overline{D}(4)g^2(\overline{AB})g(A^2)g(\varphi)}{g(A)g(B^2)g(\overline{B\varphi})g(\overline{BD\varphi})^2} {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB \end{matrix} \middle| -\frac{z}{1-z} \right) {}_2F_1 \left( \begin{matrix} A^2, & B^2 \\ & AB \end{matrix} \middle| \frac{z}{1+z} \right) \\
&- \frac{AB(-1)A\overline{B}(z)A^2(1+z)B^2(1-z)g(\varphi)}{D(4)g(A)g(\overline{B\varphi})g(\overline{BD\varphi})^2} \delta(1+z^2). \tag{6.31}
\end{aligned}$$

Using (1.20) and Lemma 1.12 with the facts that  $\overline{AB}, \overline{AB\varphi} \neq \varepsilon$  we have

$$\begin{aligned}
& {}_4F_3 \left( \begin{matrix} A, & A\varphi, & B, & B\varphi \\ & AB, & D\varphi, & D \end{matrix} \middle| -\frac{4z^2}{(1-z^2)^2} \right) \\
&= \frac{g(\overline{BD})^2}{g(\overline{BD\varphi})^2} {}_4F_3 \left( \begin{matrix} A, & A\varphi, & B, & B\varphi \\ & AB, & D, & D\varphi \end{matrix} \middle| -\frac{4z^2}{(1-z^2)^2} \right). \tag{6.32}
\end{aligned}$$

Multiplying (6.30) by  $\frac{g(\overline{BD})^2}{g(\overline{BD\varphi})^2}$  and then employing (6.32) we arrive at (6.31). This completes the proof of the theorem. ■

## 6.8 Proof of Theorem 6.6

*Proof.* Let  $G$  be a square root of  $AB$ . Then the other root is  $G\varphi$ . Putting  $E = \overline{DG}$  and  $x = z(1-z)$  in Theorem 5.5, for  $A, AB, \overline{BD^2}, \overline{ABD^4} \neq \varepsilon$ , we have

$$\begin{aligned}
F_4(A; B; D^2, \overline{ABD^2}; z(1-z), z(1-z))^* &= \frac{q^2 G(4)g(\overline{D^2B})g(D^2\overline{G})g(\overline{G\varphi})}{(q-1)g(\varphi)g(B)g(\overline{D^2})g(\overline{ABD^2})} \\
&\times {}_4F_3 \left( \begin{matrix} A, & B, & G, & G\varphi \\ & D^2, & \overline{ABD^2}, & AB \end{matrix} \middle| 4z(1-z) \right). \tag{6.33}
\end{aligned}$$

Putting  $C = D^2$  and  $w = z$  in Lemma 6.7, for  $A, B, \overline{AD^2}, \overline{BD^2} \neq \varepsilon$ , we have

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} A, & B \\ & D^2 \end{matrix} \middle| z \right) {}_2F_1 \left( \begin{matrix} A, & B \\ & \overline{ABD^2} \end{matrix} \middle| z \right) \\ &= \frac{A(-1)g(B)g(\overline{D^2})g(\overline{ABD^2})}{qg(\overline{B})g(\overline{BD^2})g(\overline{AD^2})} F_4 \left( A; B; D^2, \overline{ABD^2}; z(1-z), z(1-z) \right)^* \\ &+ \frac{qB(-1)\overline{A}(1-z)\overline{BD^2}(z)\overline{D^2}(1-z)}{g(A)g(\overline{B})g(\overline{BD^2})g(\overline{AD^2})} \delta \left( \frac{1-2z}{(1-z)^2} \right). \end{aligned} \quad (6.34)$$

Now (6.33) and (6.34) yield

$$\begin{aligned} & {}_4F_3 \left( \begin{matrix} A, & B, & G, & G\varphi \\ & D^2, & \overline{ABD^2}, & \overline{AB} \end{matrix} \middle| 4z(1-z) \right) \\ &= \frac{(q-1)A(-1)g(\overline{B})g(\overline{AD^2})g(\varphi)}{qG(4)g(D^2\overline{G})g(\overline{G\varphi})} {}_2F_1 \left( \begin{matrix} A, & B \\ & D^2 \end{matrix} \middle| z \right) {}_2F_1 \left( \begin{matrix} A, & B \\ & \overline{ABD^2} \end{matrix} \middle| z \right) \\ &- \frac{(q-1)AB(-1)\overline{AD^2}(1-z)\overline{BD^2}(z)g(\varphi)}{G(4)g(A)g(\overline{BD^2})g(D^2\overline{G})g(\overline{G\varphi})} \delta \left( \frac{1-2z}{(1-z)^2} \right). \end{aligned} \quad (6.35)$$

To complete the proof of the theorem, we need to prove (6.35) for the other root  $G\varphi$ . That is, we need to prove that

$$\begin{aligned} & {}_4F_3 \left( \begin{matrix} A, & B, & G\varphi, & G \\ & D^2, & \overline{ABD^2}, & \overline{AB} \end{matrix} \middle| 4z(1-z) \right) \\ &= \frac{(q-1)A(-1)g(\overline{B})g(\overline{AD^2})g(\varphi)}{qG(4)g(D^2\overline{G\varphi})g(\overline{G})} {}_2F_1 \left( \begin{matrix} A, & B \\ & D^2 \end{matrix} \middle| z \right) {}_2F_1 \left( \begin{matrix} A, & B \\ & \overline{ABD^2} \end{matrix} \middle| z \right) \\ &- \frac{(q-1)AB(-1)\overline{AD^2}(1-z)\overline{BD^2}(z)g(\varphi)}{G(4)g(A)g(\overline{BD^2})g(D^2\overline{G\varphi})g(\overline{G})} \delta \left( \frac{1-2z}{(1-z)^2} \right). \end{aligned} \quad (6.36)$$

Now using (1.20) and Lemma 1.12 with the facts that  $A^2B^2, \overline{D^4}AB, \overline{D^4}AB\varphi, \neq \varepsilon$  we have

$$\begin{aligned}
 & {}_4F_3 \left( \begin{matrix} A, & B, & G\varphi, & G \\ & D^2, & AB\overline{D^2}, & AB \end{matrix} \mid 4z(1-z) \right) \\
 &= \frac{g(D^2\overline{G})g(\overline{G}\varphi)}{g(D^2\overline{G}\varphi)g(\overline{G})} {}_4F_3 \left( \begin{matrix} A, & B, & G, & G\varphi \\ & D^2, & AB\overline{D^2}, & AB \end{matrix} \mid 4z(1-z) \right). \tag{6.37}
 \end{aligned}$$

Multiplying (6.35) by  $\frac{g(D^2\overline{G})g(\overline{G}\varphi)}{g(D^2\overline{G}\varphi)g(\overline{G})}$ , and then employing (6.37) we arrive at (6.36). This completes the proof of the theorem. ■

## 6.9 A summation formula for Gaussian hypergeometric series

In the following theorem, we express a  ${}_4F_3$ -Gaussian hypergeometric series as a sum of two  ${}_2F_1$ -Gaussian hypergeometric series by employing Theorem 5.1.

**Theorem 6.8.** *Let  $A, C \in \widehat{\mathbb{F}}_q^\times$  be such that  $A, C^4 \neq \varepsilon$ . For  $x \in \mathbb{F}_q$ , we have*

$$\begin{aligned}
 & {}_4F_3 \left( \begin{matrix} A, & A\varphi, & C\varphi, & C \\ & \varphi, & C^2\varphi, & C^2 \end{matrix} \mid x^2 \right) \\
 &= \frac{\overline{AC}(4)g(\overline{C^2})g(A^2)}{g(A)^2g(\overline{C})^2} \left[ {}_2F_1 \left( \begin{matrix} A^2, & C^2 \\ & C^4 \end{matrix} \mid x \right) + {}_2F_1 \left( \begin{matrix} A^2, & C^2 \\ & C^4 \end{matrix} \mid -x \right) \right] \\
 &+ \frac{(q-1)\overline{C^4}(x)g(\overline{C^2})g(A^2)}{\overline{AC}(4)g(A)^2g(\overline{C})^2} \left[ \frac{q-1}{q}C^2(1-x) - \overline{A^2}C^2(1-x) - \frac{1}{q}C^2(1+x) \right] \delta(A^2) \\
 &+ \frac{\overline{AC}(4)g(A^2)}{g(A)^2g(\overline{C})^2} \left[ g(C^2\overline{A^2})g(\overline{C^4}A^2)\overline{A^2}(x) - qg(\overline{C^2}) \left( \frac{C^2}{\overline{A^2}C^4} \right) \right] \delta(1-x).
 \end{aligned}$$

By putting some additional conditions, we can remove the extra terms, and hence we get the following corollary.

**Corollary 6.9.1.** Let  $A, C \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^2, C^4 \neq \varepsilon$ . For  $x \in \mathbb{F}_q, x \neq 1$ , we have

$$\begin{aligned} & {}_4F_3 \left( \begin{matrix} A, A\varphi, C\varphi, C \\ \varphi, C^2\varphi, C^2 \end{matrix} \middle| x^2 \right) \\ &= \frac{\overline{AC}(4)g(\overline{C^2})g(A^2)}{g(A)^2g(\overline{C})^2} \left[ {}_2F_1 \left( \begin{matrix} A^2, C^2 \\ C^4 \end{matrix} \middle| x \right) + {}_2F_1 \left( \begin{matrix} A^2, C^2 \\ C^4 \end{matrix} \middle| -x \right) \right]. \end{aligned}$$

*Proof of Theorem 6.8.* The result hold trivially for  $x = 0$ . Therefore, we assume that  $x \neq 0$ . If we take  $B = \varphi$ , then the hypothesis of Theorem 5.1 is true for all characters  $A$  and  $C$  such that  $A, C^4 \neq \varepsilon$ . So, putting  $B = \varphi$  in Theorem 5.1 and then using Lemma 1.2 with  $C^2 \neq \varepsilon$  and putting the value  $g(\varepsilon) = -1$  we find that

$$\begin{aligned} F_2(A^2; \varepsilon, C^2; \varepsilon, C^4; x, -x)^* &= \frac{qA\overline{C}(4)g^2(A)g^2(\overline{C})}{g(A^2)g(C^2)g(\overline{C^4})} {}_4F_3 \left( \begin{matrix} A, A\varphi, C\varphi, C \\ \varphi, C^2\varphi, C^2 \end{matrix} \middle| x^2 \right) \\ &+ \frac{q(q-1)g(\overline{C^2})}{g(C^2)g(\overline{C^4})} {}_2F_1 \left( \begin{matrix} A^2, C^2 \\ C^4 \end{matrix} \middle| x \right) - \frac{(q-1)^2g(\overline{C^2})}{g(C^2)g(\overline{C^4})} C^2 \left( \frac{1-x}{x^2} \right) \delta(A^2). \end{aligned} \quad (6.38)$$

Now using (3.5) we have

$$\begin{aligned} L &:= F_2(A^2; \varepsilon, C^2; \varepsilon, C^4; x, -x)^* \\ &= \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A^2\chi\lambda)g(\chi)g(C^2\lambda)g(\overline{\chi})g(\overline{C^4\lambda})g(\overline{\chi})g(\overline{\lambda})}{g(A^2)g(\varepsilon)g(C^2)g(\varepsilon)g(\overline{C^4})} \chi(x)\lambda(-x). \end{aligned}$$

Employing Lemma 1.2 on  $g(\chi)g(\overline{\chi})$  and then using  $g(\varepsilon) = -1$  we find that

$$L = \frac{q}{(q-1)^2} \sum_{\chi, \lambda} \frac{g(A^2\chi\lambda)g(C^2\lambda)g(\overline{C^4\lambda})g(\overline{\chi})g(\overline{\lambda})}{g(A^2)g(C^2)g(\overline{C^4})} \chi(-x)\lambda(-x) - I_1,$$

where

$$I_1 = \frac{1}{q-1} \sum_{\chi, \lambda} \frac{g(A^2 \chi \lambda) g(C^2 \lambda) g(\overline{C^4 \lambda}) g(\overline{\chi}) g(\overline{\lambda})}{g(A^2) g(C^2) g(\overline{C^4})} \chi(x) \lambda(-x) \delta(\chi).$$

Since  $I_1$  is nonzero only when  $\chi = \varepsilon$ , so putting  $\chi = \varepsilon$  and then using  $g(\varepsilon) = -1$  we obtain

$$I_1 = -\frac{1}{q-1} \sum_{\lambda} \frac{g(A^2 \lambda) g(C^2 \lambda) g(\overline{C^4 \lambda}) g(\overline{\lambda})}{g(A^2) g(C^2) g(\overline{C^4})} \lambda(-x).$$

Multiplying both numerator and denominator by  $q^2 g(\overline{C^2})$  and then rearranging the terms, we obtain

$$I_1 = -\frac{q^2 g(\overline{C^2})}{(q-1) g(C^2) g(\overline{C^4})} \sum_{\lambda} \left[ \frac{\lambda(-1) g(A^2 \lambda) g(\overline{\lambda})}{q g(A^2)} \right] \left[ \frac{\lambda(-1) g(C^2 \lambda) g(\overline{C^4 \lambda})}{q g(\overline{C^2})} \right] \lambda(-x).$$

Employing Lemma 1.12 with  $C^2 \neq \varepsilon$  and then (1.19) and (1.10) yield

$$\begin{aligned} I_1 &= -\frac{q^2 g(\overline{C^2})}{(q-1) g(C^2) g(\overline{C^4})} \sum_{\lambda} \binom{A^2 \lambda}{\lambda} \binom{C^2 \lambda}{C^4 \lambda} \lambda(-x) + \frac{q g(\overline{C^2}) \delta(A^2)}{g(C^2) g(\overline{C^4})} \sum_{\lambda} \binom{C^2 \lambda}{C^4 \lambda} \lambda(-x) \\ &= -\frac{q g(\overline{C^2})}{g(C^2) g(\overline{C^4})} {}_2F_1 \left( \begin{matrix} A^2, C^2 \\ C^4 \end{matrix} \middle| -x \right) + \frac{(q-1) g(\overline{C^2})}{g(C^2) g(\overline{C^4})} C^2 \left( \frac{1+x}{x^2} \right) \delta(A^2). \end{aligned} \quad (6.39)$$

Now multiplying both numerator and denominator by  $q g(A^2 \lambda)$  and then using Lemma 1.12, we obtain

$$\begin{aligned} L &= \frac{q^2}{(q-1)^2} \sum_{\chi, \lambda} \left[ \frac{\chi(-1) g(A^2 \chi \lambda) g(\overline{\chi})}{q g(A^2 \lambda)} \right] \frac{g(A^2 \lambda) g(C^2 \lambda) g(\overline{C^4 \lambda}) g(\overline{\lambda})}{g(A^2) g(C^2) g(\overline{C^4})} \chi(x) \lambda(-x) - I_1 \\ &= \frac{q^2}{(q-1)^2} \sum_{\chi, \lambda} \binom{A^2 \lambda \chi}{\chi} \frac{g(A^2 \lambda) g(C^2 \lambda) g(\overline{C^4 \lambda}) g(\overline{\lambda})}{g(A^2) g(C^2) g(\overline{C^4})} \chi(x) \lambda(-x) - J_1 - I_1, \end{aligned} \quad (6.40)$$

where

$$\begin{aligned} J_1 &= \frac{q}{q-1} \sum_{\lambda} \frac{g(A^2\lambda)g(C^2\lambda)g(\overline{C^4\lambda})g(\overline{\lambda})}{g(A^2)g(C^2)g(\overline{C^4})} \lambda(-x)\delta(A^2\lambda) \sum_x \chi(x) \\ &= -\frac{qg(C^2\overline{A^2})g(\overline{C^4A^2})}{g(C^2)g(\overline{C^4})} \overline{A^2}(x)\delta(1-x). \end{aligned} \quad (6.41)$$

The last equality is obtained by using the fact that  $\delta(A^2\lambda) \neq 0$  only when  $\lambda = \overline{A^2}$ , Lemma 1.1 and then putting the value  $g(\varepsilon) = -1$ . If we multiply both numerator and denominator by  $q^2g(\overline{C^2})$  and use (1.9), then (6.40) yields

$$\begin{aligned} L &= \frac{q^3\overline{A^2}(1-x)g(\overline{C^2})}{(q-1)g(C^2)g(\overline{C^4})} \sum_{\lambda} \left[ \frac{\lambda(-1)g(A^2\lambda)g(\overline{\lambda})}{qg(A^2)} \right] \left[ \frac{\lambda(-1)g(C^2\lambda)g(\overline{C^4\lambda})}{qg(\overline{C^2})} \right] \lambda\left(\frac{x}{x-1}\right) \\ &\quad - J_1 - I_1. \end{aligned}$$

We note that the above expression is well-defined even for  $x = 1$  due to the presence of the factor  $\overline{A^2}(1-x)$ . That is, the main term of  $L$  is equal to 0 when  $x = 1$ . Now Lemma 1.12 and the given condition  $C^2 \neq \varepsilon$  yield

$$L = \frac{q^3\overline{A^2}(1-x)g(\overline{C^2})}{(q-1)g(C^2)g(\overline{C^4})} \sum_{\lambda} \binom{A^2\lambda}{\lambda} \binom{C^2\lambda}{C^4\lambda} \lambda\left(\frac{x}{x-1}\right) - J_1 - I_1 - I_2, \quad (6.42)$$

where

$$\begin{aligned} I_2 &= \frac{q^2\overline{A^2}(1-x)g(\overline{C^2})}{g(C^2)g(\overline{C^4})} \sum_{\lambda} \binom{C^2\lambda}{C^4\lambda} \lambda\left(\frac{x}{x-1}\right) \delta(A^2) \\ &= \frac{q(q-1)\overline{A^2}(1-x)g(\overline{C^2})}{g(C^2)g(\overline{C^4})} C^2 \left(\frac{x-1}{x^2}\right) \delta(A^2). \end{aligned} \quad (6.43)$$

The last equality is obtained by using (1.10). Now combining (6.43), (6.41), (6.39)

and (6.42), and then employing (1.19) we find that

$$\begin{aligned}
 L = & \frac{q^2 \overline{A^2} (1-x) g(\overline{C^2})}{g(C^2) g(\overline{C^4})} {}_2F_1 \left( \begin{matrix} A^2, & C^2 \\ & C^4 \end{matrix} \middle| \frac{x}{x-1} \right) \\
 & + \frac{qg(\overline{C^2})}{g(C^2)g(\overline{C^4})} {}_2F_1 \left( \begin{matrix} A^2, & C^2 \\ & C^4 \end{matrix} \middle| -x \right) - \frac{(q-1)g(\overline{C^2})}{g(C^2)g(\overline{C^4})} C^2 \left( \frac{1+x}{x^2} \right) \delta(A^2) \\
 & - \frac{q(q-1)\overline{A^2}(1-x)g(\overline{C^2})}{g(C^2)g(\overline{C^4})} C^2 \left( \frac{x-1}{x^2} \right) \delta(A^2) + \frac{q\overline{A^2}(x)g(\overline{C^4}A^2)g(C^2\overline{A^2})}{g(C^2)g(\overline{C^4})} \delta(1-x).
 \end{aligned} \tag{6.44}$$

From Theorem 1.14 (ii) we have

$$\overline{A^2}(1-x) {}_2F_1 \left( \begin{matrix} A^2, & C^2 \\ & C^4 \end{matrix} \middle| \frac{x}{x-1} \right) = {}_2F_1 \left( \begin{matrix} A^2, & C^2 \\ & C^4 \end{matrix} \middle| x \right) - \left( \frac{C^2}{A^2 C^4} \right) \delta(1-x). \tag{6.45}$$

Finally, employing (6.45) into (6.44), and then putting this value of  $L$  in (6.38) we complete the proof of the theorem. ■



# 7

## Special values of Gaussian hypergeometric series

### 7.1 Introduction

Finding special values of Gaussian hypergeometric series is an important and interesting problem. Special values of Gaussian hypergeometric series play an important role in solving many old conjectures and supercongruences. After introducing Gaussian hypergeometric series in [31], Greene deduced certain special values of  ${}_2F_1$  and  ${}_3F_2$ -hypergeometric series using symmetric and transformation formulas of

---

<sup>1</sup>Some parts of this chapter are published in *Research in Number Theory* (2020) and some parts are under review.

Gaussian hypergeometric series. Since then many special values of hypergeometric functions over finite fields are obtained using different techniques (see for example [2, 9, 10, 12, 19, 32, 50, 52]). Most of the known results give values of  ${}_2F_1$  and  ${}_3F_2$ -hypergeometric series containing trivial, quadratic and quartic characters as parameters. Finding values of Gaussian hypergeometric series containing arbitrary characters at specific values of the argument is a difficult problem. Only a few such values are known till date (see for example [9, 19, 31]). In this chapter, we have used our new transformation formulas to find several special values of  ${}_2F_1$ ,  ${}_3F_2$  and  ${}_4F_3$ -Gaussian hypergeometric series. We first state some results of Greene on special values of Gaussian hypergeometric series.

**Lemma 7.1.** ([31, (4.11)]) *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$ . Then we have*

$${}_2F_1 \left( \begin{matrix} A, & B \\ & \bar{A}B \end{matrix} \middle| -1 \right) = \begin{cases} 0, & \text{if } B \neq \square; \\ \binom{C}{A} + (\varphi_A^C), & \text{if } B = C^2. \end{cases}$$

**Lemma 7.2.** ([31, (4.14)]) *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$ . Then we have*

$${}_2F_1 \left( \begin{matrix} A, & B \\ & A^2 \end{matrix} \middle| 2 \right) = A(-1) \begin{cases} 0, & \text{if } B \neq \square; \\ \binom{C}{A} + (\varphi_A^C), & \text{if } B = C^2. \end{cases}$$

**Lemma 7.3.** ([31, (4.15)]) *Let  $A, B \in \widehat{\mathbb{F}}_q^\times$ . Then we have*

$${}_2F_1 \left( \begin{matrix} A, & \bar{A} \\ & \bar{A}B \end{matrix} \middle| \frac{1}{2} \right) = A(-2) \begin{cases} 0, & \text{if } B \neq \square; \\ \binom{C}{A} + (\varphi_A^C), & \text{if } B = C^2. \end{cases}$$

## 7.2 Special values of ${}_2F_1$ -Gaussian hypergeometric series

In the following theorems, we find special values of  ${}_2F_1$ -hypergeometric series containing arbitrary characters at certain arguments.

**Theorem 7.4.** *Let  $q = p^r$ ,  $p > 2$  a prime with  $q \equiv 1$  or  $7 \pmod{8}$ . Let  $A \in \widehat{\mathbb{F}}_q^\times$  be such that  $A \neq \varepsilon, \varphi$ . Then we have*

$${}_2F_1 \left( \begin{matrix} A^2, & A^2\varphi \\ & A^4 \end{matrix} \middle| \frac{\pm 2\sqrt{2}}{1 \pm \sqrt{2}} \right) = \frac{A^2(1 \pm \sqrt{2})\overline{A}(-16)g(A^2\varphi)g(\overline{A}\varphi)}{g(\varphi)g(A\varphi)} \times \begin{cases} 0, & \text{if } A\varphi \neq \square; \\ \binom{\gamma}{A} + \binom{\gamma\varphi}{A}, & \text{if } A\varphi = \gamma^2. \end{cases}$$

*Proof.* It is known that 2 is a square in  $\mathbb{F}_q$  if and only if  $q \equiv 1$  or  $7 \pmod{8}$ . Putting  $C = A^2$  and  $x = \pm\sqrt{2}$  in Theorem 2.1 we find that

$$\overline{A^2}(1 \pm \sqrt{2}){}_2F_1 \left( \begin{matrix} A^2, & A^2\varphi \\ & A^4 \end{matrix} \middle| \frac{\pm 2\sqrt{2}}{1 \pm \sqrt{2}} \right) = \frac{\overline{A}(16)g(A^2\varphi)g(\overline{A}\varphi)}{g(\varphi)g(A\varphi)} {}_2F_1 \left( \begin{matrix} A, & A\varphi \\ & A^2 \end{matrix} \middle| 2 \right). \quad (7.1)$$

Now employing Lemma 7.2 into (7.1), we complete the proof of the theorem.  $\blacksquare$

**Theorem 7.5.** *Let  $q = p^r$ ,  $p > 2$  a prime with  $q \equiv 1 \pmod{8}$ . Let  $\chi_4$  be a character of order 4. Then for  $B \in \widehat{\mathbb{F}}_q^\times$  we have*

$${}_2F_1 \left( \begin{matrix} \varphi, & B\chi_4 \\ & \varphi B^2 \end{matrix} \middle| \frac{2}{1 \pm \sqrt{2}} \right) = -\frac{(q-1)\varphi(2 \pm \sqrt{2})\chi_4(-8)\overline{B}(-4)g(\chi_4 B)g(\overline{B})}{qg(\varphi)g(\overline{\chi_4})}\delta(\overline{B}) \\ + \frac{\varphi(2 \pm \sqrt{2})\chi_4(8)\overline{B}(-4)g(\chi_4 B)g(\overline{B})}{g(\varphi)g(\overline{\chi_4})} \times \begin{cases} 0, & \text{if } B \neq \square; \\ \binom{\gamma}{\chi_4} + \binom{\gamma\varphi}{\chi_4}, & \text{if } B = \gamma^2. \end{cases}$$

*Proof.* Putting  $C = \chi_4^3 B$ ,  $A = \chi_4$  and  $x = \pm \frac{1}{\sqrt{2}}$  in Theorem 2.1 we find that

$$\begin{aligned} \varphi(2 \pm \sqrt{2}) {}_2F_1 \left( \begin{matrix} \varphi, & B\chi_4 \\ & \varphi B^2 \end{matrix} \middle| \frac{2}{1 \pm \sqrt{2}} \right) &= \frac{\chi_4 \bar{B}(-4)g(\chi_4 B)g(\bar{B})} {g(\varphi)g(\bar{\chi}_4)} {}_2F_1 \left( \begin{matrix} \chi_4, & \bar{\chi}_4 \\ & \bar{\chi}_4 B \end{matrix} \middle| \frac{1}{2} \right) \\ &\quad - \frac{(q-1)\chi_4 \bar{B}(-4)g(\chi_4 B)g(\bar{B})\chi_4(2)} {qg(\varphi)g(\bar{\chi}_4)} \delta(\bar{B}). \end{aligned} \quad (7.2)$$

Now using Lemma 7.3 in (7.2), we complete the proof of the theorem.  $\blacksquare$

**Theorem 7.6.** *Let  $\chi_3$  be a character of order 3. Let  $S$  be a character on  $\mathbb{F}_q$  whose order is strictly greater than 3. If  $S$  is a square, then we have*

$$\begin{aligned} {}_2F_1 \left( \begin{matrix} \bar{S}^3, & \bar{S}^2\varphi \\ & \bar{S}^4 \end{matrix} \middle| \frac{4}{2 \pm \sqrt{3}} \right) &= \frac{S^3(\sqrt{3})S(16)g(\bar{S}^2\varphi)g(\sqrt{S})} {S^3(\sqrt{3} \pm 2)g(\varphi)g(\sqrt{S^3})} \\ &\times \begin{cases} 0, & \text{if } q \equiv 11 \pmod{12}; \\ \frac{S(8)\bar{S}(27)J(\sqrt{S}, \sqrt{S^3}\varphi)} {J(\varphi, S)} \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right], & \text{if } q \equiv 1 \pmod{12}. \end{cases} \end{aligned}$$

*Proof.* It is known that 3 is a square in  $\mathbb{F}_q$  if and only if  $q \equiv 1$  or  $11 \pmod{12}$ . Let  $A^2 = \bar{S}^3$ . Then,  $A = \sqrt{\bar{S}^3}$  or  $A = \sqrt{\bar{S}^3}\varphi$ . Putting  $C = \bar{S}^2$ ,  $A = \sqrt{\bar{S}^3}$  and  $x = \frac{\pm 2}{\sqrt{3}}$  in Theorem 2.1, we deduce that

$$\begin{aligned} S^3 \left( \frac{\sqrt{3} \pm 2}{\sqrt{3}} \right) {}_2F_1 \left( \begin{matrix} \bar{S}^3, & \bar{S}^2\varphi \\ & \bar{S}^4 \end{matrix} \middle| \frac{\pm 4}{\sqrt{3} \pm 2} \right) \\ &= \frac{S(16)g(\bar{S}^2\varphi)g(\sqrt{S}\varphi)} {g(\varphi)g(\sqrt{S^3}\varphi)} {}_2F_1 \left( \begin{matrix} \sqrt{\bar{S}^3}, & \sqrt{\bar{S}^3}\varphi \\ & \bar{S}^2 \end{matrix} \middle| \frac{4}{3} \right) \\ &= \frac{S(16)g(\bar{S}^2\varphi)g(\sqrt{S}\varphi)} {g(\varphi)g(\sqrt{S^3}\varphi)} \left( \frac{\varphi\sqrt{\bar{S}^3}}{\bar{S}^2} \right) {}_2F_1 \left( \begin{matrix} \sqrt{\bar{S}^3}\varphi, & \sqrt{\bar{S}^3} \\ & \bar{S}^2 \end{matrix} \middle| \frac{4}{3} \right)^* \\ &= \frac{S(16)g(\bar{S}^2\varphi)g(\sqrt{S}\varphi)} {g(\varphi)g(\sqrt{S^3}\varphi)} \left( \frac{\varphi\sqrt{\bar{S}^3}}{\bar{S}^2} \right) \left( \frac{\sqrt{\bar{S}^3}}{\bar{S}^2} \right)^{-1} {}_2F_1 \left( \begin{matrix} \sqrt{\bar{S}^3}\varphi, & \sqrt{\bar{S}^3} \\ & \bar{S}^2 \end{matrix} \middle| \frac{4}{3} \right). \end{aligned}$$

The last two equalities are obtained by using Proposition 1.19. Now using Lemma 1.12 and the fact that  $S^2\sqrt{S^3}\varphi \neq \varepsilon \neq S^2\sqrt{S^3}$ , we have

$$\begin{aligned} & S^3 \left( \frac{\sqrt{3} \pm 2}{\sqrt{3}} \right) {}_2F_1 \left( \begin{matrix} \overline{S^3}, \overline{S^2}\varphi \\ \overline{S^4} \end{matrix} \middle| \frac{\pm 4}{\sqrt{3} \pm 2} \right) \\ &= \frac{S(16)g(\overline{S^2}\varphi)g(\sqrt{S})}{g(\varphi)g(\sqrt{S^3})} {}_2F_1 \left( \begin{matrix} \sqrt{S^3}\varphi, \sqrt{S^3} \\ \overline{S^2} \end{matrix} \middle| \frac{4}{3} \right). \end{aligned} \quad (7.3)$$

If we take the other square root  $A = \sqrt{S^3}\varphi$ , then putting  $C = \overline{S^2}$ ,  $A = \sqrt{S^3}\varphi$  and  $x = \frac{\pm 2}{\sqrt{3}}$  in Theorem 2.1, we readily obtain (7.3). Now, if  $S$  is a square and its order is not equal to 3, then from [9, Theorem 1.8 (i)] we have

$${}_2F_1 \left( \begin{matrix} \sqrt{S^3}\varphi, \sqrt{S^3} \\ \overline{S^2} \end{matrix} \middle| \frac{4}{3} \right) = \begin{cases} 0, & \text{if } q \equiv 2 \pmod{3}; \\ S \binom{8}{27} \frac{J(\sqrt{S}, \sqrt{S^3}\varphi)}{J(\varphi, S)} \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right] & \text{if } q \equiv 1 \pmod{3}. \end{cases} \quad (7.4)$$

Using (7.4) into (7.3) we complete the proof of the theorem.  $\blacksquare$

**Theorem 7.7.** *Let  $q = p^r$ ,  $p > 2$  a prime with  $q \equiv 1 \pmod{8}$ . Let  $\chi_4$  be a character of order 4. If  $B \in \widehat{\mathbb{F}_q^\times}$  and  $B \neq \varphi$ , then we have*

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} \varphi, B^2 \\ \overline{\chi_4}B \end{matrix} \middle| \frac{1 \pm \sqrt{2}}{2} \right) \\ &= \frac{\chi_4(4)B(-4)g(\overline{\chi_4}B)g(\chi_4)g(B\varphi)}{qg(\varphi)} \begin{cases} 0, & \text{if } B \neq \square; \\ \binom{\gamma}{\chi_4} + \binom{\gamma\varphi}{\chi_4}, & \text{if } B = \gamma^2. \end{cases} \end{aligned}$$

*Proof.* Putting  $x = \pm\sqrt{2}$  and  $A = \chi_4$  in Corollary 2.4.1, we have

$${}_2F_1 \left( \begin{matrix} \varphi, & B^2 \\ & \chi_4 B \varphi \end{matrix} \middle| \frac{1 \pm \sqrt{2}}{2} \right) = \frac{\chi_4 B(-4)g(\chi_4 \bar{B}\varphi)g(\chi_4)g(B\varphi)}{qg(\varphi)} {}_2F_1 \left( \begin{matrix} \chi_4, & B \\ & \varphi \end{matrix} \middle| 2 \right). \quad (7.5)$$

Now using Lemma 7.2 in (7.5), we complete the proof of the theorem.  $\blacksquare$

In [10, Theorem 1.4], Barman and Kalita obtained four special values of  ${}_2F_1$  hypergeometric series containing trivial and quartic characters at the arguments  $1/9, 8/9, -1/8$  and  $-8$ , respectively. For example, one of the values is the following:

$${}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \middle| \frac{1}{9} \right) = \chi_4(-1)\varphi(3) \left[ \binom{\chi_4}{\varphi} + \binom{\chi_4^3}{\varphi} \right]. \quad (7.6)$$

In the following theorem we evaluate these values for any character  $A \neq \varepsilon, \varphi$ .

**Theorem 7.8.** *Let  $A \in \widehat{\mathbb{F}}_q^\times$  be such that  $A \neq \varepsilon, \varphi$ . Then we have*

$$\begin{aligned} (a) \quad {}_2F_1 \left( \begin{matrix} A, & A\varphi \\ & \bar{A}^2\varphi \end{matrix} \middle| \frac{1}{9} \right) &= \frac{q\bar{A}(256)A(9)g(A^2)}{g(A)g(\bar{A}^2)g(A^3)} \left[ \binom{\bar{A}}{A^2} + \binom{\varphi\bar{A}}{A^2} \right] \\ &\quad + \frac{(q-1)\varphi(-1)\bar{A}(32)A(9)g(A^2)}{g(\varphi)g(A)g(A\varphi)} \delta(A^3). \\ (b) \quad {}_2F_1 \left( \begin{matrix} A, & A\varphi \\ & A^4 \end{matrix} \middle| \frac{8}{9} \right) &= \frac{q\bar{A}(-256)A(9)g(A^2)}{g(A)g(\bar{A}^2)g(A^3)} \left[ \binom{\bar{A}}{A^2} + \binom{\varphi\bar{A}}{A^2} \right] \\ &\quad + \frac{(q-1)A\varphi(-1)\bar{A}(32)A(9)g(A^2)}{g(\varphi)g(A)g(A\varphi)} \delta(A^3). \\ (c) \quad {}_2F_1 \left( \begin{matrix} A, & \bar{A}^3 \\ & \bar{A}^2\varphi \end{matrix} \middle| \frac{-1}{8} \right) &= \frac{q\varphi(-1)\bar{A}(32)g(A^2)}{g(A)g(\bar{A}^2)g(A^3)} \left[ \binom{\bar{A}}{A^2} + \binom{\varphi\bar{A}}{A^2} \right] \\ &\quad + \frac{(q-1)\bar{A}(4)g(A^2)}{g(\varphi)g(A)g(A\varphi)} \delta(A^3). \end{aligned}$$

$$(d) \quad {}_2F_1 \left( \begin{matrix} A, & A^3\varphi \\ & A^4 \end{matrix} \middle| -8 \right) = \frac{q\bar{A}(-256)g(A^2)}{g(A)g(\bar{A}^2)g(A^3)} \left[ \binom{\bar{A}}{A^2} + \binom{\varphi\bar{A}}{A^2} \right] \\ + \frac{(q-1)A\varphi(-1)\bar{A}(32)g(A^2)}{g(\varphi)g(A)g(A\varphi)} \delta(A^3).$$

*Proof.* Putting  $C = \bar{A}^2\varphi$ ,  $x = -\frac{1}{3}$  in Theorem 2.1, and then using Lemma 1.2 and Lemma 1.5 on  $g(A\varphi)$ , we deduce that

$${}_2F_1 \left( \begin{matrix} A, & A\varphi \\ & \bar{A}^2\varphi \end{matrix} \middle| \frac{1}{9} \right) = \frac{q\bar{A}(256)A(9)g(A^2)}{g(A)g(\bar{A}^2)g(A^3)} {}_2F_1 \left( \begin{matrix} A^2, & \bar{A}^2 \\ & A^4 \end{matrix} \middle| -1 \right) \\ + \frac{(q-1)\varphi(-1)\bar{A}(32)A(9)g(A^2)}{g(A)g(\varphi)g(A\varphi)} \delta(A^3). \quad (7.7)$$

Now using Lemma 7.1 in (7.7), we complete the proof of part (a). Using Theorem 1.14 (i) and (ii) in (a), we obtain the required special values given in (b) and (c), respectively. Finally, using Theorem 1.14 (ii) in (b), we complete the proof of (d).  $\blacksquare$

**Corollary 7.2.1.** *Let  $A \in \widehat{\mathbb{F}}_q^\times$  be such that  $A \neq \varepsilon, \varphi$ . Then we have*

$$(a) \quad {}_2F_1 \left( \begin{matrix} A, & \bar{A}^3 \\ & \varphi \end{matrix} \middle| \frac{9}{8} \right) = \frac{q\varphi(-1)\bar{A}(-32)g(A^2)}{g(A)g(\bar{A}^2)g(A^3)} \left[ \binom{\bar{A}}{A^2} + \binom{\varphi\bar{A}}{A^2} \right] \\ + \frac{(q-1)\bar{A}(-4)g(A^2)}{g(\varphi)g(A)g(A\varphi)} \delta(A^3). \\ (b) \quad {}_2F_1 \left( \begin{matrix} A, & A^3\varphi \\ & \varphi \end{matrix} \middle| 9 \right) = \frac{q\bar{A}(256)g(A^2)}{g(A)g(\bar{A}^2)g(A^3)} \left[ \binom{\bar{A}}{A^2} + \binom{\varphi\bar{A}}{A^2} \right] \\ + \frac{(q-1)\varphi(-1)\bar{A}(32)g(A^2)}{g(\varphi)g(A)g(A\varphi)} \delta(A^3).$$

*Proof.* Putting  $x = \frac{9}{8}$  in Theorem 1.14 (i), we have

$${}_2F_1 \left( A, \frac{A^3}{\varphi} \mid \frac{9}{8} \right) = A(-1) {}_2F_1 \left( A, \frac{A^3}{A^2\varphi} \mid \frac{-1}{8} \right). \quad (7.8)$$

Now using Theorem 7.8 (c) in (7.8), we complete the proof of part (a).

Again, putting  $x = 9$  in Theorem 1.14 (i), we have

$${}_2F_1 \left( A, A^3\varphi \mid 9 \right) = A(-1) {}_2F_1 \left( A, \frac{A^3\varphi}{A^4} \mid -8 \right). \quad (7.9)$$

Now using Theorem 7.8 (d) in (7.9), we complete the proof of part (b). ■

**Remark 7.2.1.** We readily obtain Theorem 1.4 of [10] by taking  $A = \chi_4$  in Theorem 7.8. We also note that the extra terms containing  $\delta(A^3)$  in Theorem 7.8 and Corollary 7.2.1 will disappear if  $A$  has order greater than 3.

We next evaluate special values of  ${}_2F_1$  hypergeometric series containing characters of specific orders, namely characters of orders dividing 8. Let  $\chi_8$  be a character of order 8. In the following two theorems, we find special values of  ${}_2F_1$  hypergeometric series containing  $\varepsilon, \varphi$  and  $\chi_8$  at a few specific arguments.

**Theorem 7.9.** Let  $q = p^r$ ,  $p > 2$  a prime with  $q \equiv 1 \pmod{8}$ . Then we have

$$(a) \quad {}_2F_1 \left( \chi_8, \frac{\chi_8^3}{\varphi} \mid \frac{49}{81} \right) = \chi_8(-1)\varphi(3) \left[ \binom{\chi_8^2}{\varphi} + \binom{\chi_8^6}{\varphi} \right].$$

$$(b) \quad {}_2F_1 \left( \varphi, \frac{\varphi}{\varepsilon} \mid \frac{3 \mp 2\sqrt{2}}{3 \pm 2\sqrt{2}} \right) = \varphi \left( \frac{6\sqrt{2} \pm 3}{4\sqrt{2} \pm 6} \right) \varphi(6 \pm 12\sqrt{2}) \left[ \binom{\chi_4}{\varphi} + \binom{\chi_4^3}{\varphi} \right].$$

*Proof.* Putting  $x = \frac{-7}{9}$  and  $A = \chi_8, B = \chi_8^3$  in Corollary 2.4.1, we have

$${}_2F_1 \left( \chi_8^2, \frac{\chi_8^6}{\varepsilon} \mid \frac{1}{9} \right) = \frac{g(\chi_8)g(\chi_8^3\varphi)}{q} {}_2F_1 \left( \chi_8, \frac{\chi_8^3}{\varphi} \mid \frac{49}{81} \right). \quad (7.10)$$

Now using (7.6) in (7.10), and then Lemma 1.2 we complete the proof of (a). Again, putting  $x = \frac{-2\sqrt{2}\pm 3}{6\sqrt{2}\pm 3}$  and  $A = \chi_4, C = \varepsilon$  in Theorem 2.1, we have

$$\varphi \left( \frac{4\sqrt{2}\pm 6}{6\sqrt{2}\pm 3} \right) {}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{3\mp 2\sqrt{2}}{3\pm 2\sqrt{2}} \right) = {}_2F_1 \left( \begin{matrix} \chi_4, \chi_4^3 \\ \varphi \end{matrix} \middle| \left( \frac{-2\sqrt{2}\pm 3}{6\sqrt{2}\pm 3} \right)^2 \right). \quad (7.11)$$

For  $q \equiv 1 \pmod{8}$ , from [12, Theorem 1.8] we have

$${}_2F_1 \left( \begin{matrix} \chi_4, \chi_4^3 \\ \varepsilon \end{matrix} \middle| \left( \frac{-2\sqrt{2}\pm 3}{6\sqrt{2}\pm 3} \right)^2 \right) = \varphi(6 \pm 12\sqrt{2}) \left[ \begin{matrix} \chi_4 \\ \varphi \end{matrix} + \begin{matrix} \chi_4^3 \\ \varphi \end{matrix} \right]. \quad (7.12)$$

Now using (7.12) in (7.11), we complete the proof of part (b). ■

**Theorem 7.10.** *Let  $q = p^r$ ,  $p > 2$  a prime with  $q \equiv 1 \pmod{12}$ . Then we have*

$${}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{\sqrt{3}\mp 2}{\sqrt{3}\pm 2} \right) = \varphi \left( \frac{6 \pm \sqrt{3}}{4 \pm 2\sqrt{3}} \right) \varphi \left( \frac{8 \pm 5\sqrt{3}}{12 \pm 6\sqrt{3}} \right) \left[ \begin{matrix} \varphi \\ \chi_3 \end{matrix} + \begin{matrix} \varphi \\ \chi_3^2 \end{matrix} \right],$$

where  $\chi_3$  is a character of order 3.

*Proof.* Taking  $C = \varepsilon$ ,  $A = \chi_4$  and  $x = \frac{-2\pm\sqrt{3}}{6\pm\sqrt{3}}$  in Theorem 2.1, we have

$$\varphi \left( \frac{4 \pm 2\sqrt{3}}{6 \pm \sqrt{3}} \right) {}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{\sqrt{3}\mp 2}{\sqrt{3}\pm 2} \right) = {}_2F_1 \left( \begin{matrix} \chi_4, \chi_4^3 \\ \varepsilon \end{matrix} \middle| \left( \frac{-2 \pm \sqrt{3}}{6 \pm \sqrt{3}} \right)^2 \right). \quad (7.13)$$

For  $q \equiv 1 \pmod{12}$ , from [12, Theorem 1.8] we have

$${}_2F_1 \left( \begin{matrix} \chi_4, \chi_4^3 \\ \varepsilon \end{matrix} \middle| \left( \frac{-2 \pm \sqrt{3}}{6 \pm \sqrt{3}} \right)^2 \right) = \varphi \left( \frac{8 \pm 5\sqrt{3}}{12 \pm 6\sqrt{3}} \right) \left[ \begin{matrix} \varphi \\ \chi_3 \end{matrix} + \begin{matrix} \varphi \\ \chi_3^2 \end{matrix} \right]. \quad (7.14)$$

Now using (7.14) in (7.13), we complete the proof. ■

In the following theorem, we find special values of  ${}_2F_1$  hypergeometric series containing characters of order 4 at a few specific values of  $x$ .

**Theorem 7.11.** *Let  $q = p^r$ ,  $p > 3$  be prime. Then we have*

$$(a) \quad {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4 \\ \varphi \end{matrix} \middle| \left( \frac{6\sqrt{2} \mp 3}{2\sqrt{2} \pm 3} \right)^2 \right) = \chi_4(-1)\varphi(3 \pm 2\sqrt{2}) \left[ \binom{\chi_4}{\varphi} + \binom{\chi_4^3}{\varphi} \right],$$

*if  $q \equiv 1 \pmod{8}$ .*

$$(b) \quad {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4 \\ \varphi \end{matrix} \middle| \left( \frac{6 \mp \sqrt{3}}{2 \pm \sqrt{3}} \right)^2 \right) = \chi_4(-1)\varphi(3 \pm 2\sqrt{3}) \left[ \binom{\varphi}{\chi_3} + \binom{\varphi}{\chi_3^2} \right],$$

*if  $q \equiv 1 \pmod{12}$ .*

*Proof.* Taking  $A = \chi_4 = B$  and  $x = \frac{6\sqrt{2} \mp 3}{2\sqrt{2} \pm 3}$  in Corollary 2.4.1, we have

$${}_2F_1 \left( \begin{matrix} \varphi, & \varphi \\ \varepsilon \end{matrix} \middle| \frac{4\sqrt{2}}{2\sqrt{2} \pm 3} \right) = \frac{g(\chi_4)g(\chi_4\varphi)}{q} {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4 \\ \varphi \end{matrix} \middle| \left( \frac{6\sqrt{2} \mp 3}{2\sqrt{2} \pm 3} \right)^2 \right). \quad (7.15)$$

For  $q \equiv 1 \pmod{8}$ , from [37, Theorem 1.1 (a)] we have

$${}_2F_1 \left( \begin{matrix} \varphi, & \varphi \\ \varepsilon \end{matrix} \middle| \frac{4\sqrt{2}}{2\sqrt{2} \pm 3} \right) = \varphi(3 \pm 2\sqrt{2}) \left[ \binom{\chi_4}{\varphi} + \binom{\chi_4^3}{\varphi} \right]. \quad (7.16)$$

Now employing (7.16) into (7.15) and the fact that  $g(\chi_4)g(\chi_4\varphi) = q\chi_4(-1)$ , we complete the proof.

Again, taking  $A = \chi_4 = B$  and  $x = \frac{6 \mp \sqrt{3}}{2 \pm \sqrt{3}}$  in Corollary 2.4.1, we have

$${}_2F_1 \left( \begin{matrix} \varphi, & \varphi \\ \varepsilon \end{matrix} \middle| \frac{4}{2 \pm \sqrt{3}} \right) = \frac{g(\chi_4)g(\chi_4\varphi)}{q} {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4 \\ \varphi \end{matrix} \middle| \left( \frac{6 \mp \sqrt{3}}{2 \pm \sqrt{3}} \right)^2 \right). \quad (7.17)$$

From [37, Theorem 1.1 (b)] we have

$${}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{4}{2 \pm \sqrt{3}} \right) = \begin{cases} 0, & \text{if } q \equiv 11 \pmod{12}; \\ \varphi(3 \pm 2\sqrt{3}) \left[ \binom{\varphi}{x_3} + \binom{\varphi}{x_3^2} \right] & \text{if } q \equiv 1 \pmod{12}. \end{cases} \quad (7.18)$$

Now using (7.18) in (7.17) we complete the proof.  $\blacksquare$

In [50], Ono explicitly evaluated special values of  ${}_2F_1$  and  ${}_3F_2$  hypergeometric series containing trivial and quadratic characters by counting points on elliptic curves with complex multiplication over  $\mathbb{F}_p$ . By relating the values of  ${}_3F_2$  hypergeometric series obtained by Ono to squares of certain  ${}_2F_1$  hypergeometric series, Kalita [37] evaluated special values of squares of certain  ${}_2F_1$  hypergeometric series containing trivial and quadratic characters. For example, recall the following result from [37].

**Theorem 7.12.** ([37, Theorem 1.2]) *Let  $p$  be an odd prime. Then*

$$\begin{aligned} (i) \quad & {}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{2}{1 \pm \sqrt{2}} \right)^2 = \begin{cases} 0, & \text{if } p \equiv 7 \pmod{8}; \\ \frac{4x^2}{p^2}, & \text{if } x^2 + 2y^2 = p \equiv 1 \pmod{8}. \end{cases} \\ (ii) \quad & {}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{2}{1 \pm \sqrt{-3}} \right)^2 = \frac{4x^2}{p^2}, \text{ if } p \equiv 1, 7 \pmod{12}, \text{ and } x^2 + 3y^2 = p. \\ (iii) \quad & {}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{16}{8 \pm 3\sqrt{7}} \right)^2 = \begin{cases} 0, & \text{if } p \equiv 3, 19, 27 \pmod{28} \text{ and } p \neq 3; \\ \frac{4x^2}{p^2}, & \text{if } x^2 + 7y^2 = p \equiv 1, 9, 25 \pmod{28}. \end{cases} \\ (iv) \quad & {}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{2}{1 \pm 3\sqrt{-7}} \right)^2 = \frac{4x^2}{p^2}, \text{ if } x^2 + 7y^2 = p \equiv 1, 9, 11, 15, 23, 25 \pmod{28}. \end{aligned}$$

As a consequence of the above result, we evaluate the following special values of squares of certain  ${}_2F_1$  hypergeometric series containing characters of order 4.

**Theorem 7.13.** *Let  $p$  be an odd prime. Then*

$$(a) {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \middle| \frac{-1}{3} \right)^2 = \frac{4x^2}{p^2}, \text{ if } p \equiv 1 \pmod{12} \text{ and } x^2 + 3y^2 = p.$$

$$(b) {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \middle| \frac{64}{63} \right)^2 = \frac{4x^2}{p^2}, \text{ if } x^2 + 7y^2 = p \equiv 1, 9, 25 \pmod{28}.$$

$$(c) {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \middle| \frac{-1}{63} \right)^2 = \frac{4x^2}{p^2}, \text{ if } x^2 + 7y^2 = p \equiv 1, 9, 25 \pmod{28}.$$

*Proof.* (a) It is known that  $-3$  is a square modulo  $p$  if and only if  $p \equiv 1$  or  $7 \pmod{12}$ . Now, putting  $A = \chi_4$ ,  $C = \varepsilon$  and  $x = \frac{1}{\sqrt{-3}}$  in Theorem 2.1, we find that

$$\varphi \left( 1 + \frac{1}{\sqrt{-3}} \right) {}_2F_1 \left( \begin{matrix} \varphi, & \varphi \\ & \varepsilon \end{matrix} \middle| \frac{2}{1 + \sqrt{-3}} \right) = {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \middle| \frac{-1}{3} \right). \quad (7.19)$$

Squaring both sides of (7.19), and then using Theorem 7.12 (ii), we complete the proof of part (a).

(b) It is known that  $7$  is a square modulo  $p$  if and only if  $p \equiv 1, 3, 9, 19, 25$  or  $27 \pmod{28}$ . We put  $A = \chi_4$ ,  $C = \varepsilon$  and  $x = \frac{8}{3\sqrt{7}}$  in Theorem 2.1 to obtain

$$\varphi \left( 1 + \frac{8}{3\sqrt{7}} \right) {}_2F_1 \left( \begin{matrix} \varphi, & \varphi \\ & \varepsilon \end{matrix} \middle| \frac{16}{8 + 3\sqrt{7}} \right) = {}_2F_1 \left( \begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \middle| \frac{64}{63} \right). \quad (7.20)$$

As before, squaring both sides of (7.20) and then using Theorem 7.12 (iii), we complete the proof of (b).

(c) It is known that  $-7$  is a square modulo  $p$  if and only if  $p \equiv 1, 9, 11, 15, 23$  or  $25$

(mod 28). Now, taking  $A = \chi_4$ ,  $C = \varepsilon$  and  $x = \frac{1}{3\sqrt{-7}}$  in Theorem 2.1, we have

$$\varphi\left(1 + \frac{1}{3\sqrt{-7}}\right) {}_2F_1\left(\begin{matrix} \varphi, & \varphi \\ & \varepsilon \end{matrix} \middle| \frac{2}{1 + 3\sqrt{-7}}\right) = {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \middle| \frac{-1}{63}\right). \quad (7.21)$$

Squaring both sides of (7.21) and then using Theorem 7.12 (iv), we complete the proof of (c). This completes the proof of the theorem.  $\blacksquare$

**Theorem 7.14.** *Let  $p$  be an odd prime. Then*

$$\begin{aligned} (a) \quad & {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4 \\ & \varphi \end{matrix} \middle| \left(\frac{3 \mp \sqrt{2}}{1 \pm \sqrt{2}}\right)^2\right)^2 = \frac{4x^2}{p^2}, \text{ if } x^2 + 2y^2 = p \equiv 1 \pmod{8}. \\ (b) \quad & {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4 \\ & \varphi \end{matrix} \middle| \left(\frac{3 \mp \sqrt{-3}}{1 \pm \sqrt{-3}}\right)^2\right)^2 = \frac{4x^2}{p^2}, \text{ if } p \equiv 1 \pmod{12} \text{ and } x^2 + 3y^2 = p. \\ (c) \quad & {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4 \\ & \varphi \end{matrix} \middle| \left(\frac{24 \mp 3\sqrt{7}}{8 \pm 3\sqrt{7}}\right)^2\right)^2 = \frac{4x^2}{p^2}, \text{ if } x^2 + 7y^2 = p \equiv 1, 9, 25 \pmod{28}. \\ (d) \quad & {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4 \\ & \varphi \end{matrix} \middle| \left(\frac{3 \mp 3\sqrt{-7}}{1 \pm 3\sqrt{-7}}\right)^2\right)^2 = \frac{4x^2}{p^2}, \text{ if } x^2 + 7y^2 = p \equiv 1, 9, 25 \pmod{28}. \end{aligned}$$

*Proof.* Taking  $A = B = \chi_4$  and  $x = \frac{3 \mp \sqrt{2}}{1 \pm \sqrt{2}}$  in Corollary 2.4.1, we have

$${}_2F_1\left(\begin{matrix} \varphi, & \varphi \\ & \varepsilon \end{matrix} \middle| \frac{2}{1 \pm \sqrt{2}}\right) = \chi_4(-1) {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4 \\ & \varphi \end{matrix} \middle| \left(\frac{3 \mp \sqrt{2}}{1 \pm \sqrt{2}}\right)^2\right). \quad (7.22)$$

Now using Theorem 7.12 (i) in (7.22), we complete the proof of (a). We next put  $A = B = \chi_4$  and  $x = \frac{3 \mp \sqrt{-3}}{1 \pm \sqrt{-3}}$  in Corollary 2.4.1 to obtain

$${}_2F_1\left(\begin{matrix} \varphi, & \varphi \\ & \varepsilon \end{matrix} \middle| \frac{2}{1 \pm \sqrt{-3}}\right) = \chi_4(-1) {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4 \\ & \varphi \end{matrix} \middle| \left(\frac{3 \mp \sqrt{-3}}{1 \pm \sqrt{-3}}\right)^2\right). \quad (7.23)$$

Now employing Theorem 7.12 (ii) in (7.23), we complete the proof of (b). We again put  $A = B = \chi_4$  and  $x = \frac{24 \mp 3\sqrt{7}}{8 \pm 3\sqrt{7}}$  in Corollary 2.4.1 to get

$${}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{16}{8 \pm 3\sqrt{7}} \right) = \chi_4(-1) {}_2F_1 \left( \begin{matrix} \chi_4, \chi_4 \\ \varphi \end{matrix} \middle| \left( \frac{24 \mp 3\sqrt{7}}{8 \pm 3\sqrt{7}} \right)^2 \right). \quad (7.24)$$

Now using Theorem 7.12 (iii) in (7.24), we complete the proof of (c). Finally, putting  $A = B = \chi_4$  and  $x = \frac{3 \mp 3\sqrt{-7}}{1 \pm 3\sqrt{-7}}$  in Corollary 2.4.1, we have

$${}_2F_1 \left( \begin{matrix} \varphi, \varphi \\ \varepsilon \end{matrix} \middle| \frac{2}{1 \pm 3\sqrt{-7}} \right) = \chi_4(-1) {}_2F_1 \left( \begin{matrix} \chi_4, \chi_4 \\ \varphi \end{matrix} \middle| \left( \frac{3 \mp 3\sqrt{-7}}{1 \pm 3\sqrt{-7}} \right)^2 \right). \quad (7.25)$$

Now using Theorem 7.12 (iv) in (7.25), we complete the proof (d). This completes the proof of the theorem.  $\blacksquare$

**Remark 7.2.2.** *In view of Theorem 1.14, there are certain more special values of  ${}_2F_1$  hypergeometric series similar to the above theorems at different arguments.*

### 7.3 Special values of ${}_3F_2$ -Gaussian hypergeometric series

We first recall a result from [25].

**Lemma 7.15.** [25, Thm 8.11] *Let  $A \in \widehat{\mathbb{F}_q^\times}$  be such that  $A \neq \varepsilon, \varphi$ . For  $x \in \mathbb{F}_q^\times$ , we have*

$${}_2\mathbb{F}_1 \left[ \begin{matrix} A, A\varphi \\ \varphi \end{matrix} \middle| x \right] = \left( \frac{1 + \varphi(x)}{2} \right) \left( \overline{A^2}(1 + \sqrt{x}) + \overline{A^2}(1 - \sqrt{x}) \right). \quad (7.26)$$

We remark that the formula (7.26) is well-defined and the value of the hypergeometric series will be equal to 0 if  $x$  is not a square.

To deduce the special values obtained by Evans and Greene from our product formulas proved in Chapter 6, we need to use the fact that  $A(-1) = -1$  if  $A$  is a non-square character. In the following two lemmas, we prove this fact.

**Lemma 7.16.** *Let  $A \in \widehat{\mathbb{F}_q^\times}$  be of order  $m > 1$ . Then  $A(-1) = -1$  if and only if  $m$  is even and  $\frac{q-1}{m}$  is odd.*

*Proof.* Let  $g$  be a generator of the cyclic group  $\mathbb{F}_q^\times$ . Since  $m$  is the order of the character  $A$ , therefore  $A(g) = \zeta$ , a primitive  $m$ -th root of unity. We have  $A(-1) = A(g^{\frac{q-1}{2}}) = \zeta^{\frac{q-1}{2}}$ . Suppose that  $A(-1) = -1$ . Then  $(-1)^m = A^m(-1) = 1$ , and hence  $m$  is even. Also,  $\zeta^{\frac{q-1}{2}} = A(-1) = -1 = \zeta^{\frac{m}{2}}$ . This gives  $\frac{q-1}{2} \equiv \frac{m}{2} \pmod{m}$ , and hence  $\frac{q-1}{m} \equiv 1 \pmod{2}$  or equivalently  $\frac{q-1}{m}$  is odd. Conversely, if  $m$  is even and  $\frac{q-1}{m}$  is odd then  $\frac{q-1}{2} \equiv \frac{m}{2} \pmod{m}$ . Hence,  $-1 = \zeta^{\frac{m}{2}} = \zeta^{\frac{q-1}{2}}$ . This implies that  $A(-1) = -1$ . ■

**Lemma 7.17.** *If  $A \in \widehat{\mathbb{F}_q^\times}$  is not a square, then  $A(-1) = -1$ .*

*Proof.* Let  $m$  be the order of the character  $A$ . Then  $G = \langle A \rangle$  is a cyclic subgroup of  $\widehat{\mathbb{F}_q^\times}$  of order  $m$ . Since  $A$  is not a square character, so  $A^2$  is not a generator of  $G$ . This implies that  $\gcd(2, m) = 2$ , that is  $m$  is even. We next prove that  $\frac{q-1}{m}$  is odd. Otherwise,  $\widehat{\mathbb{F}_q^\times}$  will have an element of order  $2m$ , say  $B$ . Then we must have  $\langle A \rangle = \langle B^2 \rangle$ . This is a contradiction to the fact that  $A$  is not a square. Hence  $\frac{q-1}{m}$  is odd. Using Lemma 7.16 we complete the proof of the lemma. ■

In [50], Ono found several special values of  ${}_2F_1$ - and  ${}_3F_2$ -Gaussian hypergeometric series containing trivial and quadratic characters as parameters by counting points on CM elliptic curves. We find the following special value which generalizes a result of Ono.

**Theorem 7.18.** *Let  $A \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^2, A^6 \neq \varepsilon$ . Then we have*

$${}_3F_2 \left( \begin{matrix} A^2, & A^6, & A^4\varphi \\ & A^8, & A^4 \end{matrix} \middle| -8 \right) = \frac{\overline{A}(256)g(A^2)^2g(\overline{A^6})}{qg(\overline{A^2})} \left[ \binom{A^3}{A^2} + \binom{A^3\varphi}{A^2} \right]^2$$

$$-\frac{\overline{A}(4096)}{q} - \frac{q-1}{q^3} \overline{A}(4096) \varphi(2) g(\overline{A^2} \varphi) g(A^2 \varphi) \delta(A^4 \varphi).$$

*Proof.* Putting  $x = -1$ ,  $B = A^3 \varphi$  in Corollary 6.4.1 and then using Lemma 1.2 on  $g(A^2)g(\overline{A^2})$  and  $g(A^4)g(\overline{A^4})$  we have

$$\begin{aligned} {}_3F_2 \left( \begin{matrix} A^2, & A^6, & A^4 \varphi \\ & A^8, & A^4 \end{matrix} \middle| -8 \right) &= \frac{\overline{A}(256)g(A^2)^2g(\overline{A^6})}{qg(\overline{A^2})} {}_2F_1 \left( \begin{matrix} A^2, & A^6 \\ & A^4 \end{matrix} \middle| -1 \right)^2 \\ &- \frac{\overline{A}(4096)}{q} - \frac{(q-1)}{q^3} \varphi(2) \overline{A}(4096) g(\overline{A^2} \varphi) g(A^2 \varphi) \delta(A^4 \varphi). \end{aligned} \tag{7.27}$$

We complete the proof by combining Lemma 7.1 and (7.27). ■

Putting  $A = \chi_4$  in Theorem 7.18 we readily obtain the following special value obtained by Ono [50] when  $q \equiv 1 \pmod{4}$ .

**Corollary 7.3.1.** *Let  $q \equiv 1 \pmod{4}$ . We have*

$${}_3F_2 \left( \begin{matrix} \varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon \end{matrix} \middle| -8 \right) = \left[ \binom{\chi_4}{\varphi} + \binom{\overline{\chi_4}}{\varphi} \right]^2 - \frac{1}{q}.$$

Using our product formulas, we next find special values of  ${}_3F_2$ -hypergeometric series at  $x = -1$  and  $x = \frac{1}{4}$ , respectively. We note that these two results were also proved by Evans and Greene, see for example [19, Thm 1.3 & 1.4].

**Theorem 7.19.** *Suppose that  $C$  is a multiplicative character whose order is not equal to 1, 2, 4. Then for  $q \equiv 1 \pmod{8}$  we have*

$${}_3F_2 \left( \begin{matrix} \varphi, & C^2 \varphi, & C \varphi \\ & C^2, & C \end{matrix} \middle| -1 \right) = \begin{cases} \frac{1}{q}, & \text{if } C\chi_4 \neq \square ; \\ \frac{1}{q} + \frac{2}{q^2} \operatorname{Re}(J(D, \varphi)J(\overline{D}\chi_4, \varphi)), & \text{if } C\chi_4 = D^2. \end{cases}$$

*Proof.* Putting  $A = \chi_4$  and  $x = \frac{1+\sqrt{2}}{2}$  in Corollary 6.4.1 and then using Lemma 1.2

we have

$$\begin{aligned} & {}_3F_2 \left( \begin{matrix} \varphi, & B^2, & B\chi_4 \\ & B^2\varphi, & B\bar{\chi}_4 \end{matrix} \middle| -1 \right) \\ &= \frac{\overline{B\chi_4}(4)g(\overline{B^2})g(B\chi_4)^2}{qg(\varphi)} {}_2F_1 \left( \begin{matrix} \varphi, & B^2 \\ \overline{\chi_4}B & \end{matrix} \middle| \frac{1+\sqrt{2}}{2} \right)^2 - \frac{B\chi_4(-1)}{q}, \end{aligned} \quad (7.28)$$

where  $B \neq \varepsilon, \varphi, \chi_4, \bar{\chi}_4$ . From Theorem 7.7, we have

$$\begin{aligned} & {}_2F_1 \left( \begin{matrix} \varphi, & B^2 \\ \overline{\chi_4}B & \end{matrix} \middle| \frac{1 \pm \sqrt{2}}{2} \right) \\ &= \frac{\chi_4(4)B(-4)g(\overline{\chi_4}B)g(\chi_4)g(B\varphi)}{qg(\varphi)} \begin{cases} 0, & \text{if } B \neq \square; \\ \binom{D}{\chi_4} + \binom{D\varphi}{\chi_4}, & \text{if } B = D^2. \end{cases} \end{aligned} \quad (7.29)$$

Since  $q \equiv 1 \pmod{8}$ , we have  $\chi_4(4) = \varphi(2) = 1$ . Now, combining (7.29), (7.28), and Lemma 1.2 we find that

$$\begin{aligned} M &:= {}_3F_2 \left( \begin{matrix} \varphi, & B^2, & B\chi_4 \\ & B^2\varphi, & B\bar{\chi}_4 \end{matrix} \middle| -1 \right) \\ &= -\frac{B(-1)}{q} + \frac{B(4)}{q^2g(\varphi)}g(\chi_4)^2g(B\varphi)^2g(\overline{B^2}) \begin{cases} 0, & \text{if } B \neq \square; \\ \left[ \binom{D}{\chi_4} + \binom{D\varphi}{\chi_4} \right]^2, & \text{if } B = D^2. \end{cases} \end{aligned} \quad (7.30)$$

When  $B = D^2$  we have

$$\begin{aligned} M &= -\frac{1}{q} + \frac{D^2(4)}{q^2g(\varphi)}g(\chi_4)^2g(D^2\varphi)^2g(\overline{D^4}) \left[ \binom{D}{\chi_4} + \binom{D\varphi}{\chi_4} \right]^2 \\ &= -\frac{1}{q} + \frac{g(\chi_4)g(D\chi_4)g(D\bar{\chi}_4)g(\overline{D})g(\overline{D}\varphi)}{q^2g(\bar{\chi}_4)} \left[ \binom{D}{\chi_4} + \binom{D\varphi}{\chi_4} \right]^2. \end{aligned}$$

The last equality is obtained by using Lemma 1.5 on  $g(D^2\varphi)$  and Lemma 1.7 on

$g(\overline{D^4})$ . Employing Lemma 1.12 and Lemma 1.2 we find that

$$M = \frac{D(-1)g(D)g(\overline{D}\varphi)g(D\chi_4)}{q^2g(D\overline{\chi_4})} + \frac{D(-1)g(\overline{D})g(D\varphi)g(D\overline{\chi_4})}{q^2g(D\chi_4)} + \frac{1}{q}. \quad (7.31)$$

Now using Lemma 1.9 and Lemma 1.2 we have

$$J(D, \varphi)J(\overline{D}\chi_4, \varphi) = \frac{qg(D)g(\overline{D}\chi_4)}{g(D\varphi)g(D\overline{\chi_4})}. \quad (7.32)$$

Using  $\overline{g(A)} = A(-1)g(\overline{A})$ , (7.32) yields

$$\overline{J(D, \varphi)J(\overline{D}\chi_4, \varphi)} = \frac{qg(\overline{D})g(D\overline{\chi_4})}{g(\overline{D}\varphi)g(D\chi_4)}. \quad (7.33)$$

Combining (7.33), (7.32), (7.31) and Lemma 1.2 we find that

$$M = \frac{1}{q} + \frac{2}{q^2} \operatorname{Re}(J(D, \varphi)J(\overline{D}\chi_4, \varphi)), \quad (7.34)$$

where  $B = D^2$ . By Lemma 7.17 we have  $B(-1) = -1$  if  $B$  is not a square. Now, using (7.34) in (7.30) we have

$${}_3F_2 \left( \begin{matrix} \varphi, & B^2, & B\chi_4 \\ & B^2\varphi, & B\overline{\chi_4} \end{matrix} \middle| -1 \right) = \begin{cases} \frac{1}{q}, & \text{if } B \neq \square; \\ \frac{1}{q} + \frac{2}{q^2} \operatorname{Re}(J(D, \varphi)J(\overline{D}\chi_4, \varphi)), & \text{if } B = D^2. \end{cases}$$

Clearly,  $B \neq \varepsilon, \varphi, \chi_4, \overline{\chi_4}$  if and only if  $B\overline{\chi_4} \neq \varepsilon, \varphi, \chi_4, \overline{\chi_4}$ . We complete the proof of the theorem by putting  $B = C\chi_4$ . ■

**Theorem 7.20.** *Suppose that  $C$  is a multiplicative character which is a square and its order is strictly greater than 4. Then we have*

$${}_3F_2 \left( \begin{matrix} \overline{C}, & C^3, & C \\ & C^2, & C\varphi \end{matrix} \middle| \frac{1}{4} \right) = \begin{cases} -\frac{C(4)}{q}, & \text{if } q \equiv 11 \pmod{12}; \\ \frac{C(4)}{q} [q + 2\operatorname{Re}(J(C, \chi_3)J(\overline{C}, \chi_3))], & \text{if } q \equiv 1 \pmod{12}. \end{cases}$$

*Proof.* Let  $S$  be a multiplicative character which is a square, and let its order be strictly greater than 4. Putting  $A = \sqrt{S^3}$ ,  $B = \sqrt{S}$  and  $x = \frac{2+\sqrt{3}}{4}$  in Corollary 6.4.1 and then using Lemma 1.2 we have

$${}_3F_2 \left( \begin{matrix} \overline{S^3}, S, \overline{S} \\ \overline{S^2}, \overline{S}\varphi \end{matrix} \middle| \frac{1}{4} \right) = \frac{S(4)g(\overline{S})g(S^2\varphi)^2}{qg(S^3)} {}_2F_1 \left( \begin{matrix} \overline{S^3}, S \\ \overline{S}\varphi \end{matrix} \middle| \frac{2+\sqrt{3}}{4} \right)^2 - \frac{\overline{S}(4)}{q}. \quad (7.35)$$

From Theorem 7.6, we have

$${}_2F_1 \left( \begin{matrix} \overline{S^3}, \overline{S^2}\varphi \\ \overline{S^4} \end{matrix} \middle| \frac{4}{2 \pm \sqrt{3}} \right) = \frac{S^3(\sqrt{3})S(16)g(\overline{S^2}\varphi)g(\sqrt{S})}{S^3(\sqrt{3} \pm 2)g(\varphi)g(\sqrt{S^3})} \\ \times \begin{cases} 0, & \text{if } q \equiv 11 \pmod{12}; \\ \frac{S(8)\overline{S}(27)J(\sqrt{S}, \sqrt{S^3}\varphi)}{J(\varphi, S)} \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right], & \text{if } q \equiv 1 \pmod{12}. \end{cases} \quad (7.36)$$

Employing Lemma 1.15 into (7.36), and then using Lemma 1.2, we deduce from (7.35) that

$$M := {}_3F_2 \left( \begin{matrix} \overline{S^3}, S, \overline{S} \\ \overline{S^2}, \overline{S}\varphi \end{matrix} \middle| \frac{1}{4} \right) = -\frac{\overline{S}(4)}{q} + \frac{\varphi(-1)g(\overline{S})g(\sqrt{S})^2 J(\sqrt{S}, \sqrt{S^3}\varphi)^2}{S(27)\overline{S}(16)g(S^3)g(\sqrt{S^3})^2 J(\varphi, S)^2} \\ \times \begin{cases} 0, & \text{if } q \equiv 11 \pmod{12}; \\ \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right]^2, & \text{if } q \equiv 1 \pmod{12}. \end{cases} \quad (7.37)$$

Using Lemma 1.9 and Lemma 1.2 in (7.37), for  $q \equiv 1 \pmod{12}$ , we have

$$M = -\frac{\overline{S}(4)}{q} + \frac{qS(16)\overline{S}(27)g(\overline{S})g(\sqrt{S^3}\varphi)^2}{g(S^3)g(\sqrt{S^3})^2 g(S)^2} \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right]^2.$$

Using Lemma 1.5 on  $g(\sqrt{S^3})$ , and then employing Lemma 1.2 we find that

$$M = -\frac{\bar{S}(4)}{q} + \frac{q\bar{S}(4)\bar{S}(27)g(\bar{S})}{g(\bar{S}^3)g(S)^2} \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right]^2.$$

Lemma 1.12, Lemma 1.6, and Lemma 1.2 yield

$$M = \frac{\bar{S}(4)}{q^2} \left[ q + \frac{qg(\chi_3^2)^2}{g(\bar{S}\chi_3^2)g(S\chi_3^2)} + \frac{qg(\chi_3)^2}{g(\bar{S}\chi_3)g(S\chi_3)} \right]. \quad (7.38)$$

Using Lemma 1.9 and Lemma 1.2 we have

$$J(S, \chi_3)J(\bar{S}, \chi_3) = \frac{qg(\chi_3)^2}{g(S\chi_3)g(\bar{S}\chi_3)}. \quad (7.39)$$

Using  $\overline{g(A)} = A(-1)g(\bar{A})$ , (7.39) yields

$$\overline{J(S, \chi_3)J(\bar{S}, \chi_3)} = \frac{qg(\chi_3^2)^2}{g(\bar{S}\chi_3^2)g(S\chi_3^2)}. \quad (7.40)$$

Now, employing (7.40), (7.39) and (7.38) into (7.37) we have

$$\begin{aligned} & {}_3F_2 \left( \begin{matrix} \bar{S}^3, & S, & \bar{S} \\ \bar{S}^2, & \bar{S}\varphi & \end{matrix} \middle| \frac{1}{4} \right) \\ &= \begin{cases} \frac{\bar{S}(4)}{q}, & \text{if } q \equiv 11 \pmod{12}; \\ \frac{\bar{S}(4)}{q} [q + 2\operatorname{Re}(J(S, \chi_3)J(\bar{S}, \chi_3))], & \text{if } q \equiv 1 \pmod{12}. \end{cases} \end{aligned} \quad (7.41)$$

Using (1.20), Lemma 1.12, and Lemma 1.2 we find that

$${}_3F_2 \left( \begin{matrix} \bar{S}^3, & S, & \bar{S} \\ \bar{S}^2, & \bar{S}\varphi & \end{matrix} \middle| \frac{1}{4} \right) = {}_3F_2 \left( \begin{matrix} S, & \bar{S}^3, & \bar{S} \\ \bar{S}^2, & \bar{S}\varphi & \end{matrix} \middle| \frac{1}{4} \right). \quad (7.42)$$

Combining (7.41) and (7.42), and then putting  $S = \bar{C}$  we complete the proof of the theorem. ■

**Remark 7.3.1.** By Lemma 7.16 we have  $\varphi(-1) = -1$  if  $q \equiv 11 \pmod{12}$  and  $\varphi(-1) = 1$  if  $q \equiv 1 \pmod{12}$ . To deduce Theorem 7.20 from [19, Theorem 1.3], we need to use the values of  $\varphi(-1)$  accordingly.

In the following theorem, we find values of  ${}_3F_2$ -hypergeometric series at  $x = -8$ .

**Theorem 7.21.** Let  $A \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2, A^6 \neq \varepsilon$  and  $A^4 \neq \varphi$ . Then we have

$$\begin{aligned} {}_3F_2 \left( \begin{matrix} \overline{A^2}, A^2, \varphi \\ A^4, \overline{A^4} \end{matrix} \middle| -8 \right) &= \frac{g(\varphi)}{g(\overline{A^4})g(A^4\varphi)} \left( \frac{\overline{A^2}}{A^4} \right)^{-1} \left[ \left( \frac{\overline{A}}{A^2} \right) + \left( \frac{\varphi\overline{A}}{A^2} \right) \right] \\ &\times \left[ \left( \frac{A}{\overline{A^2}} \right) + \left( \frac{\varphi A}{\overline{A^2}} \right) \right] + \frac{q-1}{q} \left( \frac{\overline{A^2}}{A^4} \right)^{-1} {}_3F_2 \left( \begin{matrix} A^2, \overline{A^2}, \varphi \\ \varepsilon, \overline{A^4} \end{matrix} \middle| -8 \right) \delta(\overline{A^4}) \\ &- \frac{1}{q^2} \left( \frac{\varphi}{\overline{A^4}} \right) \left( \frac{\overline{A^2}}{A^4} \right)^{-1} - \frac{(q-1)g(\varphi)}{q^3g(\overline{A^4})g(A^4\varphi)} \left( \frac{\overline{A^2}}{A^4} \right)^{-1} \left[ \delta(\overline{A^4}) + q \right]. \end{aligned}$$

*Proof.* Putting  $B = \overline{A}, C = A^4$  and  $x = 2$  in Theorem 6.1 we have

$$\begin{aligned} &{}_2F_1 \left( \begin{matrix} A^2, \overline{A^2} \\ A^4 \end{matrix} \middle| 2 \right) {}_2F_1 \left( \begin{matrix} A^2, \overline{A^2} \\ \overline{A^4} \end{matrix} \middle| 2 \right) \\ &= \frac{qg(\overline{A^2})g(\overline{A^4})g(A^4\varphi)}{g(A^2)^2g(\overline{A^6})g(\varphi)} {}_4F_3 \left( \begin{matrix} A^2, \overline{A^2}, \varepsilon, \varphi \\ \varepsilon, A^4, \overline{A^4} \end{matrix} \middle| -8 \right) \\ &- \frac{(q-1)g(\overline{A^2})g(\overline{A^4})g(A^4\varphi)}{g(A^2)^2g(\overline{A^6})g(\varphi)} {}_3F_2 \left( \begin{matrix} A^2, \overline{A^2}, \varphi \\ \varepsilon, \overline{A^4} \end{matrix} \middle| -8 \right) \delta(\overline{A^4}) \\ &+ \frac{(q-1)g(\overline{A^2})}{q^2g(A^2)^2g(\overline{A^6})} \left[ \delta(\overline{A^4}) + q \right]. \end{aligned} \tag{7.43}$$

From (1.20), (1.14) and using (1.16) on  $\binom{A^2\chi}{\chi} \binom{\chi}{A^4\chi}$  with the fact that  $A^2 \neq \varepsilon$ , we obtain

$${}_4F_3 \left( \begin{matrix} A^2, \overline{A^2}, \varepsilon, \varphi \\ \varepsilon, A^4, \overline{A^4} \end{matrix} \middle| -8 \right) = \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}}_q^\times} \binom{A^2\chi}{\chi} \binom{\overline{A^2}\chi}{\chi} \binom{\chi}{A^4\chi} \binom{\varphi\chi}{\overline{A^4}\chi} \chi^{(-8)}$$

$$= \left(\frac{\overline{A^2}}{A^4}\right) {}_3F_2 \left( \begin{matrix} \overline{A^2}, A^2, \varphi \\ A^4, \overline{A^4} \end{matrix} \mid -8 \right) + \frac{1}{q^2} \left(\frac{\varphi}{\overline{A^4}}\right). \quad (7.44)$$

Using Proposition 1.19 and the fact that  $A^2 \neq \varepsilon$  we have

$$\begin{aligned} {}_2F_1 \left( \begin{matrix} A^2, \overline{A^2} \\ \overline{A^4} \end{matrix} \mid 2 \right) &= \left(\frac{\overline{A^2}}{A^4}\right) \left(\frac{A^2}{\overline{A^4}}\right)^{-1} {}_2F_1 \left( \begin{matrix} \overline{A^2}, A^2 \\ \overline{A^4} \end{matrix} \mid 2 \right) \\ &= \frac{g(\overline{A^2})g(A^6)}{g(A^2)^2} {}_2F_1 \left( \begin{matrix} \overline{A^2}, A^2 \\ \overline{A^4} \end{matrix} \mid 2 \right). \end{aligned} \quad (7.45)$$

The last equality is obtained by using Lemma 1.12. Finally, employing (7.45), (7.44), Lemma 1.2 and Lemma 7.2 into (7.43) we complete the proof.  $\blacksquare$

We remark that Corollary 7.3.1 also follows from Theorem 7.21 by taking  $A = \chi_4$ . We also show that the following result of Evans and Greene [18, Thm 1.9] follows from Theorem 7.21.

**Theorem 7.22.** *Suppose that  $A$  is a multiplicative character whose order is not equal to 1, 2, 3, 4, 6, 8. Then we have*

$${}_3F_2 \left( \begin{matrix} \varphi, A^2, \overline{A^2} \\ A^4, \overline{A^4} \end{matrix} \mid -8 \right) = \frac{1}{q} + \frac{\overline{A^2}(4)J(\overline{A^2}, A^6)}{q^2 J(A^2, A^2)} [J(A^2, A)^2 + J(A^2, A\varphi)^2].$$

*Proof.* Using Lemma 1.12 and Lemma 1.2 in Theorem 7.21 we have

$$\begin{aligned} &{}_3F_2 \left( \begin{matrix} \overline{A^2}, A^2, \varphi \\ A^4, \overline{A^4} \end{matrix} \mid -8 \right) \\ &= \frac{g(\varphi)g(A^2)}{qg(\overline{A^2})g(A^4\varphi)} + \frac{g(A)g(A^2)g(\varphi)g(\overline{A}\varphi)}{qg(\overline{A^2})g(A^3)g(\varphi\overline{A^3})g(A^4\varphi)} + \frac{g(\overline{A})g(A^2)g(A\varphi)g(\varphi)}{qg(\overline{A^2})g(\overline{A^3})g(\varphi\overline{A^3})g(A^4\varphi)}. \end{aligned} \quad (7.46)$$

Now (1.20) and Lemma 1.12 yield

$${}_3F_2 \left( \begin{matrix} \overline{A^2}, & A^2, & \varphi \\ & A^4, & \overline{A^4} \end{matrix} \mid -8 \right) = \frac{g(\varphi)g(A^2)}{g(\overline{A^2})g(A^4\varphi)} {}_3F_2 \left( \begin{matrix} \varphi, & A^2, & \overline{A^2} \\ & A^4, & \overline{A^4} \end{matrix} \mid -8 \right). \quad (7.47)$$

Combining (7.47) and (7.46) we have

$${}_3F_2 \left( \begin{matrix} \varphi, & A^2, & \overline{A^2} \\ & A^4, & \overline{A^4} \end{matrix} \mid -8 \right) = \frac{1}{q} + \frac{g(A)g(\overline{A}\varphi)}{qg(A^3)g(\varphi\overline{A^3})} + \frac{g(\overline{A})g(A\varphi)}{qg(\overline{A^3})g(\varphi A^3)}. \quad (7.48)$$

Using Lemma 1.9 we find that

$$\begin{aligned} & \frac{\overline{A^2}(4)J(\overline{A^2}, A^6)}{q^2J(A^2, A^2)} [J(A^2, A)^2 + J(A^2, A\varphi)^2] \\ &= \frac{\overline{A^2}(4)}{q^2} \left[ \frac{g(A)^2g(A^6)g(\overline{A^2})}{g(A^3)^2} + \frac{g(A\varphi)^2g(A^6)g(\overline{A^2})}{g(A^3\varphi)^2} \right] \\ &= \frac{g(A)g(\overline{A}\varphi)}{qg(A^3)g(\varphi\overline{A^3})} + \frac{g(\overline{A})g(A\varphi)}{qg(\overline{A^3})g(\varphi A^3)}. \end{aligned} \quad (7.49)$$

The last equality is obtained by using Lemma 1.5 on  $g(\overline{A^2})$  and  $g(A^6)$ , and then we use Lemma 1.2. We now complete the proof by combining (7.49) and (7.48). ■

In the following theorem we find values of  ${}_3F_2$ -hypergeometric series at  $x = 4$ .

**Theorem 7.23.** *Suppose that  $S$  is a multiplicative character which is a square and its order is strictly greater than 4. Then we have*

$$\begin{aligned} & {}_3F_2 \left( \begin{matrix} \overline{S^3}, & \overline{S}, & \overline{S^2}\varphi \\ & \overline{S^4}, & \overline{S^2} \end{matrix} \mid 4 \right) = -\frac{\varphi(-1)S(16)}{q} \\ & + \frac{S(16)\overline{S}(27)J(\overline{S}, \overline{S})}{J(\overline{S^3}, S)} \begin{cases} 0, & \text{if } q \equiv 11 \pmod{12}; \\ \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right]^2, & \text{if } q \equiv 1 \pmod{12}. \end{cases} \end{aligned}$$

We note that the above result was also proved by Barman and Kalita by counting

points on certain algebraic curves over finite fields, see for example [9, Thm 1.7]. Here we present a different proof using our product formulas.

*Proof of Theorem 7.23.* Given that  $S$  is a character which is a square and its order is strictly greater than 4. Putting  $A = \overline{S^3}$ ,  $B = \overline{S^2\varphi}$ ,  $C = \overline{S^2}$  and  $x = \frac{4}{2+\sqrt{3}}$  in Theorem 6.4 and using the fact that  $g(\varepsilon) = -1$  we have

$$\begin{aligned} {}_2F_1 \left( \begin{matrix} \overline{S^3}, \overline{S^2\varphi} \\ \overline{S^4} \end{matrix} \middle| \frac{4}{2+\sqrt{3}} \right)^2 &= \frac{(q-1)S(256)S(\sqrt{3}-2)\overline{S^5}(2+\sqrt{3})}{g(\overline{S^3})g(S^2\varphi)^2g(\overline{S})} \\ &+ \frac{(q-1)\varphi(-1)S(16)S^3(\sqrt{3}-2)\overline{S^3}(\sqrt{3}+2)g(S)}{g(\varphi)g(\overline{S})g(S^2\varphi)} {}_3F_2 \left( \begin{matrix} \overline{S^3}, \overline{S^2\varphi}, \overline{S} \\ \overline{S^4}, \overline{S^2} \end{matrix} \middle| 4 \right) \\ &- \frac{q\varphi(-1)S(16)S^3(\sqrt{3}-2)\overline{S^3}(\sqrt{3}+2)g(S)}{g(\varphi)g(\overline{S})g(S^2\varphi)} {}_4F_3 \left( \begin{matrix} \overline{S^3}, \overline{S^2\varphi}, \overline{S}, \overline{S^2\varphi} \\ \overline{S^4}, \overline{S^2}, \overline{S^2\varphi} \end{matrix} \middle| 4 \right). \end{aligned} \quad (7.50)$$

Now using (1.20) and (1.14) we have

$${}_4F_3 \left( \begin{matrix} \overline{S^3}, \overline{S^2\varphi}, \overline{S}, \overline{S^2\varphi} \\ \overline{S^4}, \overline{S^2}, \overline{S^2\varphi} \end{matrix} \middle| 4 \right) = -\frac{1}{q} {}_3F_2 \left( \begin{matrix} \overline{S^3}, \overline{S^2\varphi}, \overline{S} \\ \overline{S^4}, \overline{S^2} \end{matrix} \middle| 4 \right) + I_1, \quad (7.51)$$

where

$$\begin{aligned} I_1 &= \sum_{\chi} \binom{\overline{S^3}\chi}{\chi} \binom{\overline{S^2\varphi}\chi}{\overline{S^4}\chi} \binom{\overline{S}\chi}{\overline{S^2}\chi} \chi(4)\delta(\overline{S^2\varphi}\chi) \\ &= -\frac{\varphi(-1)S(16)}{q} \binom{\overline{S}\varphi}{S^2\varphi} \binom{S\varphi}{\varphi} = -\frac{S(16)g(\varphi)g(\overline{S^2\varphi})}{q^2g(S)g(\overline{S^3})} = -\frac{\varphi(-1)S(16)g(\varphi)}{qg(S)g(\overline{S^3})g(S^2\varphi)}. \end{aligned} \quad (7.52)$$

The above equality is obtained by putting  $\chi = S^2\varphi$  and then using (1.15), Lemma 1.12, and Lemma 1.2. By combining (7.50), (7.51), (7.52), Lemma 1.2 and Theorem

7.6, we have

$$\begin{aligned}
& {}_3F_2 \left( \begin{matrix} \overline{S^3}, \overline{S^2}\varphi, \overline{S} \\ \overline{S^4}, \overline{S^2} \end{matrix} \middle| 4 \right) \\
&= -\frac{\varphi(-1)S(16)g(\varphi)}{g(\overline{S^3})g(S^2\varphi)g(S)} + \frac{\varphi(-1)g(\overline{S})g(\varphi)g(\overline{S^2}\varphi)g(\sqrt{S})^2 J(\sqrt{S}, \sqrt{S^3}\varphi)^2}{q\overline{S}(1024)S(27)g(S)g(\sqrt{\overline{S^3}})^2 J(\varphi, S)^2} \\
&\times \begin{cases} 0, & \text{if } q \equiv 11 \pmod{12}; \\ \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right]^2, & \text{if } q \equiv 1 \pmod{12}. \end{cases} \tag{7.53}
\end{aligned}$$

Lemma 1.9 and Lemma 1.2 yield

$$\begin{aligned}
& \frac{\varphi(-1)S(1024)\overline{S}(27)g(\overline{S})g(\varphi)g(\overline{S^2}\varphi)g(\sqrt{S})^2 J(\sqrt{S}, \sqrt{S^3}\varphi)^2}{qg(S)g(\sqrt{\overline{S^3}})^2 J(\varphi, S)^2} \\
&= \frac{q\varphi(-1)S(1024)\overline{S}(27)g(\overline{S})g(\overline{S^2}\varphi)g(\sqrt{S^3}\varphi)^2}{g(S)^3 g(\sqrt{\overline{S^3}})^2 g(\varphi)} \\
&= \frac{\varphi(-1)S(16)\overline{S}(27)g(\overline{S})g(\overline{S^2}\varphi)g(S^3)^2 g(\varphi)}{qg(S)^3}. \tag{7.54}
\end{aligned}$$

The last equality is obtained by using Lemma 1.5 on  $g(\sqrt{S^3}\varphi)$  and then using Lemma 1.2. Combining (7.54) and (7.53) we have

$$\begin{aligned}
& {}_3F_2 \left( \begin{matrix} \overline{S^3}, \overline{S^2}\varphi, \overline{S} \\ \overline{S^4}, \overline{S^2} \end{matrix} \middle| 4 \right) = -\frac{\varphi(-1)S(16)g(\varphi)}{g(\overline{S^3})g(S^2\varphi)g(S)} + \frac{\varphi(-1)g(\overline{S})g(\overline{S^2}\varphi)g(S^3)^2 g(\varphi)}{qS(27)\overline{S}(16)g(S)^3} \\
&\times \begin{cases} 0, & \text{if } q \equiv 11 \pmod{12}; \\ \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right]^2, & \text{if } q \equiv 1 \pmod{12}. \end{cases} \tag{7.55}
\end{aligned}$$

Lemma 1.12 and (1.20) yield

$${}_3F_2 \left( \begin{matrix} \overline{S^3}, \overline{S^2}\varphi, \overline{S} \\ \overline{S^4}, \overline{S^2} \end{matrix} \middle| 4 \right) = \frac{g(S^3)g(\varphi)}{g(S^2\varphi)g(S)} {}_3F_2 \left( \begin{matrix} \overline{S^3}, \overline{S}, \overline{S^2}\varphi \\ \overline{S^4}, \overline{S^2} \end{matrix} \middle| 4 \right). \tag{7.56}$$

Using (7.56) in (7.55) and then using Lemma 1.2 we have

$$\begin{aligned}
 {}_3F_2 \left( \begin{matrix} \overline{S^3}, & \overline{S}, & \overline{S^2}\varphi \\ & \overline{S^4}, & \overline{S^2} \end{matrix} \mid 4 \right) &= -\frac{\varphi(-1)S(16)}{q} \\
 + \frac{S(16)\overline{S}(27)g(\overline{S})g(S^3)}{g(S)^2} &\begin{cases} 0, & \text{if } q \equiv 11 \pmod{12}; \\ \left[ \binom{S}{\chi_3} + \binom{S}{\chi_3^2} \right]^2, & \text{if } q \equiv 1 \pmod{12}. \end{cases} \quad (7.57)
 \end{aligned}$$

Using Lemma 1.9 and Lemma 1.2 we have

$$\frac{J(\overline{S}, \overline{S})}{J(\overline{S^3}, S)} = \frac{g(\overline{S})g(S^3)}{g(S)^2}. \quad (7.58)$$

Finally, combining (7.58) and (7.57) we complete the proof.  $\blacksquare$

## 7.4 Special values of ${}_4F_3$ -Gaussian hypergeometric series

Using our product formulas we now find special values of  ${}_4F_3$ -Gaussian hypergeometric series. In the following theorem, we find special values of  ${}_4F_3$ -hypergeometric series at general values of the argument.

**Theorem 7.24.** *Let  $q \equiv 1 \pmod{4}$ . Let  $A \in \widehat{\mathbb{F}}_q^\times$  be such that  $A^2 \notin \{\varepsilon, \varphi, \chi_4, \overline{\chi_4}\}$ . For  $x \neq 0, 1$ , we have*

$$\begin{aligned}
 (i) \quad {}_4F_3 \left( \begin{matrix} A^2, & A^2\varphi, & A^2\chi_4, & A^2\overline{\chi_4} \\ & A^4\varphi, & A^4, & \varphi \end{matrix} \mid 4x(1-x) \right) &= \frac{\overline{A^4}\varphi(2)}{g(\varphi)g(A^2\chi_4)g(\overline{A^2}\chi_4)} \\
 &\times \left( \frac{1+\varphi(1-x)}{2} \right) \left( \frac{1+\varphi(x)}{2} \right) \left( \overline{A^4}(1+\sqrt{1-x}) + \overline{A^4}(1-\sqrt{1-x}) \right) \\
 &\times \left( \overline{A^4}(1+\sqrt{x}) + \overline{A^4}(1-\sqrt{x}) \right) - \frac{A^2\varphi(x)\overline{A^4}\varphi(2)\overline{A^6}(x-1)g(\varphi)}{qg(A^2\chi_4)g(\overline{A^2}\chi_4)} \delta \left( \frac{1-2x}{(1-x)^2} \right),
 \end{aligned}$$

$$\begin{aligned}
(ii) \quad & {}_4F_3 \left( \begin{matrix} A^2, & A^2\varphi, & A^2\chi_4, & A^2\overline{\chi}_4 \\ & A^4\varphi, & A^4, & \varphi \end{matrix} \middle| \frac{-4x}{(1-x)^2} \right) = \frac{\overline{A^4}\varphi(2)}{g(\varphi)g(A^2\chi_4)g(\overline{A^2}\chi_4)} \\
& \times \frac{(1+\varphi(1-x))(1+\varphi(x^2-x))}{4} \left( \overline{A^4} \left( 1 + \frac{1}{\sqrt{1-x}} \right) + \overline{A^4} \left( 1 - \frac{1}{\sqrt{1-x}} \right) \right) \\
& \times \left( \overline{A^4} \left( 1 + \sqrt{\frac{x}{x-1}} \right) + \overline{A^4} \left( 1 - \sqrt{\frac{x}{x-1}} \right) \right) \\
& - \frac{A^2\varphi(x)\overline{A^4}\varphi(2)A^4\varphi(x-1)g(\varphi)}{qg(A^2\chi_4)g(\overline{A^2}\chi_4)} \delta(1-x^2).
\end{aligned}$$

We note that the above formulas are well-defined. Since  $q \equiv 1 \pmod{4}$ ,  $x-1$  is a square if and only if  $1-x$  is a square. In (i), if  $x$  or  $1-x$  is not a square, then the term containing the product  $(1+\varphi(x))(1+\varphi(1-x))$  will disappear. In (ii), if  $x$  or  $1-x$  is not a square, then the term containing the product  $(1+\varphi(x^2-x))(1+\varphi(1-x))$  will disappear.

*Proof.* Putting  $B = A\chi_4, C = A^4$  in Theorem 6.1 and then using Lemma 1.2 we have

$$\begin{aligned}
& {}_4F_3 \left( \begin{matrix} A^2, & A^2\varphi, & A^2\chi_4, & A^2\overline{\chi}_4 \\ & A^4\varphi, & A^4, & \varphi \end{matrix} \middle| 4x(1-x) \right) \\
& = \frac{\overline{A^2}\chi_4(4)g(A^2)g(\varphi)g(\overline{A^2}\varphi)^2}{qg(\overline{A^2})g(A^2\chi_4)g(\overline{A^2}\chi_4)} {}_2F_1 \left( \begin{matrix} A^2, & A^2\varphi \\ & A^4 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A^2, & A^2\varphi \\ & \varphi \end{matrix} \middle| x \right) \\
& - \frac{A^2\varphi(x)\overline{A^2}\chi_4(4)\overline{A^6}(1-x)g(\varphi)}{qg(A^2\chi_4)g(\overline{A^2}\chi_4)} \delta \left( \frac{1-2x}{(1-x)^2} \right). \tag{7.59}
\end{aligned}$$

Using Theorem 1.14 (i) we have

$$\begin{aligned}
& {}_2F_1 \left( \begin{matrix} A^2, & A^2\varphi \\ & A^4 \end{matrix} \middle| x \right) {}_2F_1 \left( \begin{matrix} A^2, & A^2\varphi \\ & \varphi \end{matrix} \middle| x \right) \\
& = {}_2F_1 \left( \begin{matrix} A^2, & A^2\varphi \\ & \varphi \end{matrix} \middle| (1-x) \right) {}_2F_1 \left( \begin{matrix} A^2, & A^2\varphi \\ & \varphi \end{matrix} \middle| x \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{J(A^2\varphi, \overline{A^2})^2}{q^2} \left( \frac{1 + \varphi(1-x)}{2} \right) \\
&\times \left( \frac{1 + \varphi(x)}{2} \right) \left( \overline{A^4}(1 + \sqrt{1-x}) + \overline{A^4}(1 - \sqrt{1-x}) \right) \left( \overline{A^4}(1 + \sqrt{x}) + \overline{A^4}(1 - \sqrt{x}) \right).
\end{aligned} \tag{7.60}$$

The last equality obtained by using (1.26) and Lemma 7.15. Finally, using (7.60), (7.59), Lemma 1.9 and Lemma 1.2 we complete the proof of (i). Replacing  $x$  by  $\frac{x}{x-1}$  in Theorem 7.24 (i), we complete the proof of (ii).  $\blacksquare$

Putting  $x = \frac{1}{2}$  in Theorem 7.24 (i) we find the following special value of a  ${}_4F_3$ -Gaussian hypergeometric series.

**Corollary 7.4.1.** *Let  $q \equiv 1 \pmod{4}$ . Let  $A \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^2 \notin \{\varepsilon, \varphi, \chi_4, \overline{\chi_4}\}$ . We have*

$$\begin{aligned}
&{}_4F_3 \left( \begin{matrix} A^2, & A^2\varphi, & A^2\chi_4, & A^2\overline{\chi_4} \\ A^4\varphi, & A^4, & \varphi \end{matrix} \middle| 1 \right) = -\frac{g(\varphi)}{qg(A^2\chi_4)g(\overline{A^2\chi_4})} \\
&+ \begin{cases} \frac{1}{g(\varphi)g(A^2\chi_4)g(\overline{A^2\chi_4})} \left[ 2 + \overline{A^8} \left( 1 + \sqrt{2} \right) + \overline{A^8} \left( 1 - \sqrt{2} \right) \right], & \text{if } q \equiv 1 \pmod{8}; \\ 0, & \text{if } q \equiv 5 \pmod{8}. \end{cases}
\end{aligned}$$

The summation identity proved in Theorem 6.8 can be used to find special values of  ${}_4F_3$ -hypergeometric series by using known values of  ${}_2F_1$ -hypergeometric series. For example, in the following theorem, we find a special value of a  ${}_4F_3$ -hypergeometric series at 1 for any character  $A$  satisfying  $A^4 \neq \varepsilon$ .

**Theorem 7.25.** *Let  $A \in \widehat{\mathbb{F}_q^\times}$  be such that  $A^4 \neq \varepsilon$ . We have*

$$\begin{aligned}
&{}_4F_3 \left( \begin{matrix} A, & A\varphi, & \overline{A}\varphi, & \overline{A} \\ \varphi, & \overline{A^2}\varphi, & \overline{A^2} \end{matrix} \middle| 1 \right) \\
&= \frac{\overline{A}(16)g(A^2)g(A^6)g(\overline{A^4})}{qg(A)^4} + \frac{\overline{A}(16)g(A^2)^2}{g(A)^4} \left[ \binom{\overline{A}}{A^2} + \binom{\overline{A}\varphi}{A^2} \right].
\end{aligned}$$

*Proof.* Putting  $C = \bar{A}$  and  $x = 1$  in Theorem 6.8 we have

$${}_4F_3 \left( \begin{matrix} A, A\varphi, \bar{A}\varphi, \bar{A} \\ \varphi, \bar{A}^2\varphi, \bar{A}^2 \end{matrix} \middle| 1 \right) = \frac{\bar{A}(16)g(A^2)}{g^4(A)} \left[ g(A^6)g(\bar{A}^4) - qg(A^2) \left( \frac{\bar{A}^2}{\bar{A}^6} \right) \right] \\ + \frac{\bar{A}(16)g^2(A^2)}{g^4(A)} \left[ {}_2F_1 \left( \begin{matrix} A^2, \bar{A}^2 \\ \bar{A}^4 \end{matrix} \middle| 1 \right) + {}_2F_1 \left( \begin{matrix} A^2, \bar{A}^2 \\ \bar{A}^4 \end{matrix} \middle| -1 \right) \right].$$

Now using Lemma 1.16 and Lemma 7.1 we have

$${}_4F_3 \left( \begin{matrix} A, A\varphi, \bar{A}\varphi, \bar{A} \\ \varphi, \bar{A}^2\varphi, \bar{A}^2 \end{matrix} \middle| 1 \right) = \frac{\bar{A}(16)g(A^2)g(A^6)g(\bar{A}^4)}{g^4(A)} \\ + \frac{(1-q)\bar{A}(16)g^2(A^2)}{g^4(A)} \left( \frac{\bar{A}^2}{\bar{A}^6} \right) + \frac{\bar{A}(16)g^2(A^2)}{g^4(A)} \left[ \left( \frac{\bar{A}}{A^2} \right) + \left( \frac{\bar{A}\varphi}{A^2} \right) \right]. \quad (7.61)$$

Using Lemma 1.12 on  $\left( \frac{\bar{A}^2}{\bar{A}^6} \right)$  with  $A^4 \neq \varepsilon$  and Lemma 1.2 on  $g(A^2)g(\bar{A}^2)$  we have

$$\frac{(1-q)\bar{A}(16)g^2(A^2)}{g^4(A)} \left( \frac{\bar{A}^2}{\bar{A}^6} \right) = \frac{(1-q)\bar{A}(16)g(A^2)g(A^6)}{g^4(A)g(A^4)} \\ = \frac{(1-q)\bar{A}(16)g(A^2)g(A^6)g(\bar{A}^4)}{qg^4(A)}. \quad (7.62)$$

The last equality is obtained by replacing  $g(A^4)$  with  $g(\bar{A}^4)$  by using Lemma 1.3. Finally, employing (7.62) into (7.61) we obtain the required value of the hypergeometric function. ■



## Bibliography

- [1] Scott Ahlgren. Gaussian hypergeometric series and combinatorial congruences. In *Symbolic computation, number theory, special functions, physics and combinatorics (Gainesville, FL, 1999)*, volume 4 of *Dev. Math.*, pages 1–12. Kluwer Acad. Publ., Dordrecht, 2001.
- [2] Scott Ahlgren and Ken Ono. A Gaussian hypergeometric series evaluation and Apéry number congruences. *J. Reine Angew. Math.*, 518:187–212, 2000.
- [3] George E. Andrews, Richard Askey, and Ranjan Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.
- [4] Wilfrid N. Bailey. Some infinite integrals involving bessel functions. *Proc. London Mathematical Society*, 40.
- [5] Wilfrid N. Bailey. Products of generalized hypergeometric series. *Proc. London Mathematical Society*, 28:242–254, 1928.
- [6] Wilfrid N. Bailey. *Generalized hypergeometric series*. Cambridge University Press, 1935.
- [7] Wilfrid N. Bailey. Some theorem concerning products of hypergeometric series. *Proc. London Mathematical Society*, 38:377–384, 1935.

- [8] Wilfrid N. Bailey. On the sum of a terminating  ${}_3F_2(1)$ . *Quart. J. Math. Oxford Ser. (2)*, 4:237–240, 1953.
- [9] Rupam Barman and Gautam Kalita. Certain values of Gaussian hypergeometric series and a family of algebraic curves. *Int. J. Number Theory*, 8(4):945–961, 2012.
- [10] Rupam Barman and Gautam Kalita. Elliptic curves and special values of Gaussian hypergeometric series. *J. Number Theory*, 133(9):3099–3111, 2013.
- [11] Rupam Barman and Gautam Kalita. Hypergeometric functions over  $f_q$  and traces of frobenius for elliptic curves. *Proc. Amer. Math. Soc.*, 141(10):3403–3410, 2013.
- [12] Rupam Barman and Neelam Saikia. Certain character sums and hypergeometric series. *Pacific J. Math.*, 295(2):271–289, 2018.
- [13] Bruce C. Berndt, Ronald J. Evans, and Kenneth S. Williams. *Gauss and Jacobi sums*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1998. A Wiley-Interscience Publication.
- [14] Joseph L. Burchnell and Theodore W. Chaundy. Expansions of Appell’s double hypergeometric functions. *Quart. J. Math. Oxford Ser.*, 11:249–270, 1940.
- [15] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vol. I*. Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981.
- [16] Ron Evans. Hypergeometric  ${}_3F_2(1/4)$  evaluations over finite fields and Hecke eigenforms. *Proc. Amer. Math. Soc.*, 138(2):517–531, 2010.
- [17] Ron Evans. Some mixed character sum identities of Katz. *J. Number Theory*, 179:17–32, 2017.

- [18] Ron Evans and John Greene. Clausen's theorem and hypergeometric functions over finite fields. *Finite Fields Appl.*, 15(1):97–109, 2009.
- [19] Ron Evans and John Greene. Evaluations of hypergeometric functions over finite fields. *Hiroshima Math. J.*, 39(2):217–235, 2009.
- [20] Ron Evans and John Greene. A quadratic hypergeometric  ${}_2F_1$  transformation over finite fields. *Proc. Amer. Math. Soc.*, 145(3):1071–1076, 2017.
- [21] Ron Evans and John Greene. Some mixed character sum identities of Katz II. *Res. Number Theory*, 3:Paper No. 8, 14, 2017.
- [22] Sharon Frechette, Ken Ono, and Matthew Papanikolas. Gaussian hypergeometric functions and traces of Hecke operators. *Int. Math. Res. Not.*, 2004(60):3233–3262, 2004.
- [23] Sharon Frechette, Holly Swisher, and Fang-Ting Tu. A cubic transformation formula for Appell-Lauricella hypergeometric functions over finite fields. *Res. Number Theory*, 4(2):Paper No. 27, 27, 2018.
- [24] Jenny G. Fuselier. Hypergeometric functions over  $\mathbb{F}_p$  and relations to elliptic curves and modular forms. *Proc. Amer. Math. Soc.*, 138(1):109–123, 2010.
- [25] Jenny G. Fuselier, Ling. Long, Holly Ramakrishna, Ravi. Swisher, and Fang-Ting Tu. Hypergeometric functions over finite fields. *Memoirs of the AMS*, 2019.
- [26] Jenny G. Fuselier and Dermot McCarthy. Hypergeometric type identities in the  $p$ -adic setting and modular forms. *Proc. Amer. Math. Soc.*, 144(4):1493–1508, 2016.
- [27] Carl F. Gauss. Disquisitiones generales circa seriem infinitam. *Comm. Soc. Reg. Gött. II*, 3:123–162, 1882.

- [28] Heidi Goodson. A complete hypergeometric point count formula for Dwork hypersurfaces. *J. Number Theory*, 179:142–171, 2017.
- [29] Heidi Goodson. Hypergeometric functions and relations to Dwork hypersurfaces. *Int. J. Number Theory*, 13(2):439–485, 2017.
- [30] John Greene. *Character sum analogues for hypergeometric and generalized hypergeometric functions over finite fields*. ProQuest LLC, Ann Arbor, MI, 1984. Thesis (Ph.D.)—University of Minnesota.
- [31] John Greene. Hypergeometric functions over finite fields. *Trans. Amer. Math. Soc.*, 301(1):77–101, 1987.
- [32] John Greene and Dennis W. Stanton. A character sum evaluation and Gaussian hypergeometric series. *J. Number Theory*, 23(1):136–148, 1986.
- [33] Bing He. A finite field analogue for appell series  $f_3$ . *arXiv: 1704.03509v1*, 2017.
- [34] Bing He. A lauricella hypergeometric series over finite fields. *arXiv: 1610.04473v3*, 2017.
- [35] Bing He, Long Li, and Ruiming Zhang. An Appell series over finite fields. *Finite Fields Appl.*, 48:289–305, 2017.
- [36] Kenneth Ireland and Michael Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
- [37] Gautam Kalita. Values of Gaussian hypergeometric series and their connections to algebraic curves. *Int. J. Number Theory*, 14(1):1–18, 2018.
- [38] Nicholas M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1990.

- [39] Timothy Kilbourn. An extension of the Apéry number supercongruence. *Acta Arith.*, 123(4):335–348, 2006.
- [40] Masao Koike. Hypergeometric series over finite fields and Apéry numbers. *Hiroshima Math. J.*, 22(3):461–467, 1992.
- [41] Catherine Lennon. Gaussian hypergeometric evaluations of traces of Frobenius for elliptic curves. *Proc. Amer. Math. Soc.*, 139(6):1931–1938, 2011.
- [42] Catherine Lennon. Trace formulas for Hecke operators, Gaussian hypergeometric functions, and the modularity of a threefold. *J. Number Theory*, 131(12):2320–2351, 2011.
- [43] Long Li, Xin Li, and Rui Mao. Appell series  $\mathbb{F}_1$  over finite fields. *Int. J. Number Theory*, 14(3):727–738, 2018.
- [44] Rudolf Lidl and Harald Niederreiter. *Finite fields*, volume 20 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1997. With a foreword by P. M. Cohn.
- [45] Dermot McCarthy. Transformations of well-poised hypergeometric functions over finite fields. *Finite Fields Appl.*, 18(6):1133–1147, 2012.
- [46] Dermot McCarthy and Robert Osburn. A  $p$ -adic analogue of a formula of Ramanujan. *Arch. Math. (Basel)*, 91(6):492–504, 2008.
- [47] Dermot McCarthy and Matthew A. Papanikolas. A finite field hypergeometric function associated to eigenvalues of a Siegel eigenform. *Int. J. Number Theory*, 11(8):2431–2450, 2015.
- [48] Eric Mortenson. A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function. *J. Number Theory*, 99(1):139–147, 2003.

- [49] Eric Mortenson. Supercongruences for truncated  ${}_{n+1}F_n$  hypergeometric series with applications to certain weight three newforms. *Proc. Amer. Math. Soc.*, 133(2):321–330, 2005.
- [50] Ken Ono. Values of Gaussian hypergeometric series. *Trans. Amer. Math. Soc.*, 350(3):1205–1223, 1998.
- [51] Earl D. Rainville. *Special functions*. The Macmillan Co., New York, 1960.
- [52] Mohammad Sadek, Nermine El-Sissi, Arman Shamsi Zargar, and Naser Zamani. Evaluation of Gaussian hypergeometric series using Huff’s models of elliptic curves. *Ramanujan J.*, 48(2):357–368, 2019.
- [53] Adriana Salerno. Counting points over finite fields and hypergeometric functions. *Funct. Approx. Comment. Math.*, 49(1):137–157, 2013.
- [54] Lucy Joan Slater. *Generalized hypergeometric functions*. Cambridge University Press, Cambridge, 1966.
- [55] M. Valentina Vega. Hypergeometric functions over finite fields and their relations to algebraic curves. *Int. J. Number Theory*, 7(8):2171–2195, 2011.
- [56] Raimundas Vidūnas. Specialization of Appell’s functions to univariate hypergeometric functions. *J. Math. Anal. Appl.*, 355(1):145–163, 2009.

## Publications

### Publications from Thesis work

1. M. Tripathi and R. Barman, *A finite field analogue of the Appell series  $F_4$* , Research in Number Theory 4 (2018), Article no. 35, 23 pages.
2. M. Tripathi, N. Saikia and R. Barman, *Appell's hypergeometric series over finite fields*, International Journal of Number Theory 16 (2020), no. 4, 673–692.
3. M. Tripathi and R. Barman, *Certain product formulas and values of Gaussian hypergeometric series*, Research in Number Theory 6 (2020), Article no. 26, 29 pages.
4. M. Tripathi and R. Barman, *Certain transformations and special values of hypergeometric functions over finite fields*, Submitted for publication.
5. M. Tripathi and R. Barman, *Appell series over finite fields and Gaussian hypergeometric series*, Submitted for publication.