DOCTORAL THESIS

Thermodynamic and fluid interpretations of gravitational field equations: General relativity and beyond

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Declaration by the student

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I hereby declare that works presented in the thesis entitled "**Thermodynamic and fluid interpretations of gravitational field equations: General relativity and beyond**" has been carried out by me under the supervision of Dr. Bibhas Ranjan Majhi at the Department of Physics, Indian Institute of Technology Guwahati, India. The thesis has not been submitted anywhere else for any degree. Works presented in the thesis are all my own unless referenced to the contrary in the thesis.

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Certificate from the Supervisor

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It is certified that the work contained in the thesis entitled "**Thermodynamic and fluid interpretations of gravitational field equations: General relativity and beyond**" by Mr. Sumit Dey (Roll No - 186121030), a Ph.D. student in the Department of Physics, Indian Institute of Technology Guwahati is carried out under my supervision and has not been submitted elsewhere for the award of any other degree.

Bibhasranjan Majhi.

Dr. Bibhas Ranjan Majhi

Date: May 26, 2023

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Disclaimer

The bibliography included in this thesis is, by no means complete but contains the ones which are consulted thoroughly by me. I apologize for inadvertently missing out some of the research papers, review articles and other scientific documents pertaining to the focus of this thesis which should also have been cited.

- Sumit Dey

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Abstract

The intriguing connections between gravity and thermodynamics have been a long standing subject of study. The conventional laws of black hole mechanics have often provided deep insights into the nature of gravity. The dynamics involving black holes can be recast in a form analogous to the laws of thermodynamics. The universality of a black hole, whose features are completely independent of the collapsing matter has been suggested to be the thermodynamic limit of the underlying microscopic (quantum) degrees of freedom. The classical and semi-classical aspects of black hole mechanics provide a window into quantum gravity. As a result, the study of the geometrical features of a black hole has been a major research theme. However, it has been established that the connections between gravity and thermodynamics are not just only restricted to black holes. In fact, any given event of a spacetime has a well defined temperature and entropy assigned to it by the so called Rindler observers.

It was shown by Jacobson, that one can derive the Einstein field equations from an equilibrium thermodynamic relationship implemented for local causal horizons established at a given point in spacetime. Later, it was shown by Padmanabhan that the Einstein field equations projected on a general null surface assume a thermodynamic interpretation analogous to the first law of thermodynamics for a certain physical process. Damour also showed that the field equations of general relativity on a null surface assume the dynamics which look structurally equivalent to Navier-Stokes equations. These strong interconnections between the gravitational field equations and the laws of thermodynamics and fluid flow form the basis of the emergent nature of gravity. Under this view, the gravitational dynamics are a result of the coarse-graining of the underlying microscopic degrees of freedom.

In this thesis, we explore further the picture of this emergence of the gravitational field equations from a classical stand-point and test its validity to theories beyond general relativity.

In part one of this thesis, we develop in detail the geometrical construction of a general integrable null hypersurface in the Riemann-Cartan spacetime. The Riemann-Cartan spacetime is a generalization to the usual (pseudo)-Riemannian spacetime (equipped with the Levi-Civita connection) in the sense of allowing non-trivial torsion in it. We develop in detail the evolution equations of certain geometric data established on the null surface. In part two of the thesis, we try to interpret the physical nature of the gravitational field equations on the null surface in the light of the evolution equations constructed in part one. Our first study is the general case of gravitational theories described on spacetimes equipped with the Levi-Civita connection. We show in a covariant fashion

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that the field equations on the null surface under the process of virtual displacement take up a thermodynamic structure without taking recourse to any explicit coordinate system adapted to the null surface. Next, we take the specific case of scalar-tensor theory and show such a thermodynamic interpretation of the field equations allow us to shed some light on the issue of the physical (in)equivalences between the Einstein and Jordan frame. We also provide a proof of the zeroth law for Killing horizons in the scalar-tensor theory. Next, we take the explicit case of Einstein-Cartan gravity and show similar thermodynamic interpretation exists for the gravitational field equations on the null surface. We also study the dynamics of a geometrical data called the Há́jiček 1-form on the null surface in Einstein-Cartan gravity and show that under suitable conditions, it looks like a Cosserat fluid. This strengthens the analogy of the horizon or null surface dynamics to that of a viscous fluid flow, for theories even beyond general relativity. Finally, we conclude the thesis with a brief discussion of the conclusions and potential future directions.

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when Paddy in this lecture course arrived at the Einstein's equations.

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List of Publications

1. **Effective metric in fluid-gravity duality through parallel transport: a proposal**

   Shounak De, **Sumit Dey** and Bibhas Ranjan Majhi

   Phys. Rev. D **99** *no.12, 124024 (2019)* [arXiv: 1901.05735 [hep-th]

2. **General framework to study the extremal phase transition of black holes**

   Krishnakanta Bhattacharya, **Sumit Dey** , Bibhas Ranjan Majhi and Saurav Samanta

   Phys. Rev. D **99***, no.12, 124047 (2019)* [arXiv: 1903.03434 [gr-qc]].

3. **Gravity dual of Navier-Stokes equation in a rotating frame through parallel transport**

   Sumit Dey, Shounak De and Bibhas Ranjan Majhi

   Phys. Rev. D **102***, no.6, 064003 (2020)* [arXiv: 2002.06801 [hep-th]].

4*. **Covariant approach to the thermodynamic structure of a generic null surface**

   Sumit Dey and Bibhas Ranjan Majhi

   Phys. Rev. D **102***, no.12, 124044 (2020)* [arXiv: 2009.08221 [gr-qc]].

5*. **Thermodynamic structure of a generic null surface in scalar-tensor theory and the zeroth law**

   Sumit Dey, Krishnakanta Bhattacharya and Bibhas Ranjan Majhi

   Phys. Rev. D **104***, no.12, 124038 (2021)* [arXiv:2105.07787 [gr-qc]].

   Sumit Dey

6*. **Kinematics and dynamics of null hypersurfaces in the Einstein-Cartan spacetime and related thermodynamic interpretation**

   Sumit Dey and Bibhas Ranjan Majhi

   Phys. Rev. D **105***, no.6, 064047 (2022)* [arXiv:2201.01131 [gr-qc]].

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7*. **Possible fluid interpretation and tidal force equation on a generic null hypersurface in Einstein-Cartan theory**

Sumit Dey and Bibhas Ranjan Majhi

Phys. Rev. D **106**, *no.10, 104005 (2022)* [arXiv:2206.11875 [gr-qc]].

Note: ⋆ marked publications are included in the thesis.

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c	Speed of light	page 3
$h = 2\pi\hbar$	Planck constant	page 3

Notations and conventions

We adopt the metric signature $(-,+,+,+)$ and work in $d = 4$ spacetime dimensions. We will use the geometrized unit system where c, $\hbar$ and G have been set to one. The bulk spacetime indices are designated by the lowercase Latin letters $a, b, \cdots$. The spatial coordinates on a time constant slice are designated by the Greek letters $\mu, \nu, \cdots$. The coordinates on the null hypersurface $\mathcal{H}$ are designated by Greek letters, with a tilde on them, $\tilde{\mu}, \tilde{\nu}, \cdots$. The spatial coordinates on the two dimensional spatial cross section S_t of $\mathcal{H}$ are designated by the uppercase Latin letters $A, B, \cdots$. All kinematical and dynamical quantities associated with $\mathcal{H}$ in the Riemann-Cartan spacetime will be designated with a hat on them. The equivalent quantities in the usual spacetime without torsion will be unhatted.

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Dedicated to my family . . .

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Chapter 1

Introduction

General relativity has been one of the most singular intellectual pursuits into the mysteries of the universe and cosmos with wide ranging consequences, implications and deep (open) questions into the nature of physical reality. One of the most fantastic predictions of general relativity is that of a spacetime with a black hole. Colloquially speaking, the black hole is a region from which not even light can escape. The *event horizon* denotes the boundary of the black hole region. The very definition of a black hole in a given spacetime implies the notion of *inescapability*. Inescapability requires the notion of a *boundary*, which obviously is a *one-way* membrane. A given boundary in a spacetime can be of three types depending upon the nature of its normal vector, *viz*, timelike, spacelike and null. The timelike surface is two-way membrane and hence cannot the support the notion of inescapability. Spacelike and null surfaces are however one-way membranes. Before laying out the formal definition of a black hole, let us discuss some of its properties. The event horizon is a null surface of codimension one. It is ruled by null geodesics, that once enter the horizon can never leave it. The event horizon as a null structure is smooth; it does not have any caustics (the point where the null geodesics converge). The event horizon is a causal null boundary. No signal originating within the black hole region can affect any observer outside the event horizon, thus being causally disconnected from the rest of the spacetime. The definition of an event horizon is a global concept. It requires a complete knowledge of the spacetime in order to demarcate the event horizon. In order to decide upon whether any particle is within the event horizon, one needs to know beforehand its complete evolution into the future in order to decide whether it can escape and reach some observer stationed at infinity (in the spacetime) or remain completely unobserved. A spacetime containing an isolated single black hole is not invariant under time reversal operation. A radially infalling causal geodesic falling into the black hole region can never retrace its path back. A black hole portrays universality. The final state of a black hole system that is generated from gravitational collapse is independent of the inherent details of the collapsing matter. That is, all black holes in the universe can be defined by a set of three parameters which are its mass, charge and angular momentum.

Let us now state the definition of a black hole region and the event horizon. We will

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assume that the spacetime manifold $(\mathcal{M}, g)$ is time orientable and has a conformal completion at null infinity. The spacetime hence possesses a future null infinity $\mathcal{J}^+$ which is assumed to be complete. Then the black hole region $\mathcal{B}_{BH}$ denotes the set of points not in the causal past $J^-(\mathcal{J}^+)$ of the future null infinity [1–3],

$$\mathcal{B}_{BH} \equiv \mathcal{M} \setminus (J^-(\mathcal{J}^+) \cap \mathcal{M}) . \tag{1.1}$$

This definition shows that no future directed causal curve originating from a point in the black hole region $\mathcal{B}_{BH}$ can reach the future null infinity. Not even light can escape from the black hole region. The boundary of the black hole region is usually defined as the event horizon $\mathcal{H}_{BH}$,

$$\mathcal{H}_{BH} \equiv \partial\mathcal{B}_{BH} . \tag{1.2}$$

Via a series of great intellectual endeavors, it has been shown that there exists a very strong parallel between the laws of classical thermodynamics and that of black hole mechanics. These connections came to life following the seminal works of Bekenstein, Hawking and others [4–8].

Bekenstein came to the conclusion that if the second law of thermodynamics is not to be violated then an event horizon should be assigned an entropy and temperature. This can be understood from the following thought experiment. Suppose some 'hot' matter (hence containing large entropy) fell into the event horizon, then all traces of its information is lost to an observer stationed outside the horizon. This would apparently reduce the total entropy of the universe. Hence in order to 'save' the second law, Bekenstein conjectured that an event horizon should be endowed with an entropy. Via the area law [9], the entropy of an event horizon was postulated to be ,

$$\frac{S_{BH}}{k_B} = \hat{\eta}\frac{A_{BH}}{l_P^2} , \tag{1.3}$$

where $\hat{\eta}$ is a proportionality constant. The area A_{BH} of the event horizon is normalized to the Planck area $(l_P^2) = G\hbar/c^3$ due to dimensional reasons. In the above relation, we have restored the natural units. The entropy of the event horizon and the outside system *i.e.* $S_{BH} + S_{\text{out}}$ must never decrease under any process. They should satisfy the generalized second law. While studying quasi-static processes involving two nearby Kerr black holes, Bekenstein [10] showed that they satisfy the following relationship under an infinitesimal transformation ,

$$dM = \frac{\kappa}{8\pi G} dA_{BH} + \Omega_{BH} dJ , \tag{1.4}$$

where M, J and Ω_{BH} are the mass, angular momentum and angular velocity respectively of the Kerr black hole. Hence it was conjectured that if the above relation was to be interpreted as the first law of black hole mechanics implying energy conservation, then the temperature T of the event horizon must be of the form ,

$$k_B T = \frac{\hbar}{8\pi c}\frac{\kappa}{\hat{\eta}}\,. \tag{1.5}$$

Hence, it was assumed that the temperature of the event horizon is proportional to its surface gravity κ. However, black holes being classical geometric objects should not radiate. This can also be seen from the relation (1.5), in the classical limit of $\hbar \to 0$ where the temperature of the black hole is $T \to 0$, thus reproducing the correct classical temperature for the black hole. Classically, a black hole should be a zero temperature object and therefore cannot radiate. The fact that via (1.5), a non-radiating classical black hole is assigned a temperature, produces a contradiction to the second law of thermodynamics. This can be understood from a simple thought experiment. We consider a black hole inside a box that provides a thermal bath at temperature T_{out}. Suppose an amount of radiation ΔE falls into the black hole, thus reducing the entropy of the box by an amount $\Delta S_{\text{out}} = -\Delta E/T_{\text{out}}$. The entropy of the black hole should increase by the amount $\Delta S_{BH} = \Delta E/T$. Hence the entropy variation of the total system is,

$$\Delta S_{\text{total}} = \Delta E\left(\frac{1}{T} - \frac{1}{T_{\text{out}}}\right)\,. \tag{1.6}$$

If the temperature of the box is less than that of the black hole *i.e.* $T_{\text{out}} < T$, then the total entropy variation ΔS_{total} is negative, which violates the generalized second law. Hence, in order to get away from this paradox, black holes must radiate.

This was indeed shown by Hawking in 1975 [11]. Hawking showed in his brilliant work while investigating quantum fields in a black hole spacetime, that there arises a thermal (Hawking) radiation flux from the black hole. The thermal spectrum of such radiation coming from the black hole corresponds to a temperature of $T = \hbar\kappa/2\pi c$. Before this seminal work, Hawking himself was unconvinced of the underlying deep connection between black hole mechanics and the laws of thermodynamics. Hawking's calculations fixed the value of the proportionality constant $\hat{\eta}$ introduced by Bekenstein to be $\hat{\eta} = 1/4$. This implies that the entropy of the black hole is given by,

$$\frac{S_{Bh}}{k_B} = \frac{1}{4}\frac{A_{BH}}{l_P^2}\,. \tag{1.7}$$

This convinced Hawking that the analogy of assigning a temperature and entropy to an event horizon is of fundamental essence. Hawking's work of deriving the black hole radiation is based on semi-classical quantum analysis, by carefully analyzing the notion of vacuum in curved spacetime. The vacuum of quantum field theory in curved spacetime is observer dependent. The vacuum of zero-particle initial state in the black hole background will be populated with particles (field excitations) with respect to an observer at the future null infinity $\mathcal{J}^+$. Thus the observer at $\mathcal{J}^+$ will observe a thermal distribution corresponding to the temperature $\hbar\kappa/2\pi c$ for the initial vacuum state at the past null infinity $\mathcal{J}^-$. Instead of following Hawking's original derivation, we can show the emergence of temperature in a heuristic sense that will hide much of the rigorous mathematics. If in the Euclidean formulation, the partition function and the Green functions of a statistical mechanical system are periodic in the imaginary time with a period $\beta = 1/T$, then such a system is in thermal equilibrium at the temperature T.

Let us consider a static spherically symmetric black hole spacetime and study its near-horizon (but outside) geometry. The metric of such a spacetime is given by,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2 \,. \tag{1.8}$$

The horizon $r = r_h$ of such a spacetime is given by $f(r_h) = 0$. The surface gravity is then given by $\kappa = 1/2f'(r_h)$. Introducing the proper distance ρ from horizon via $d\rho = dr/\sqrt{f(r)}$, it can be shown that the near-horizon expansion of the function $f(r)$ goes as,

$$f(r) = f'(r_h)(r - r_h) + \mathcal{O}((r - r_h)^2) = \kappa^2\rho^2 + \mathcal{O}(\rho^4) \,. \tag{1.9}$$

The near-horizon metric then follows as,

$$ds^2 = -\kappa^2\rho^2 dt^2 + d\rho^2 + r_h d\Omega_2^2 = -\rho^2 d\eta^2 + d\rho^2 + r_h d\Omega_2^2 \,, \tag{1.10}$$

where we have used $\eta = \kappa t$. In order to describe a quantum field theory (QFT) at finite temperature, we need to analytically continue to the Euclidean signature *i.e.* $t \to -i\tau$. If the imaginary time in the Euclidean continuation of the QFT is periodic, then it implies that the QFT is at a finite temperature. Since the black hole is a thermal system, we periodically identify τ to have a period of $\beta = 1/T$ *i.e.* $\tau \sim \tau + \beta$. Here, we have set the Boltzmann and the Planck constants to one. We now consider the analytic continuation of the near-horizon geometry of the static spherically symmetric spacetime to the Euclidean signature. This gives us, near the horizon,

$$ds_E^2 = \kappa^2\rho^2 d\tau^2 + d\rho^2 + r_h d\Omega_2^2 = \rho^2 d\theta^2 + d\rho^2 + r_h d\Omega_2^2 \,, \tag{1.11}$$

where we have set $\theta = \kappa\tau$. The first two terms in the Euclidean continuation of the near-horizon metric (1.11) is simply the plane $\mathbb{R}^2$ described in polar coordinates. The Lorentzian metric (1.10) is regular at $r = r_h$, which is mapped to $\rho = 0$ in the Euclidean metric. Hence, in order to avoid the conical singularity (in the Euclidean metric) at $\rho = 0$, it implies that θ must be periodic with a period of 2π. This then implies that the imaginary time must be periodic in $2\pi/\kappa$ *i.e.* $\tau \sim \tau + 2\pi/\kappa$. We notice that the time t is the proper time corresponding to observers at spatial infinity ($r \to \infty$). Hence such observers would feel that the black horizon is at a temperature $T = \kappa/2\pi$.

1.1 Overview of black-hole thermodynamics

Let us now state the four laws of black hole mechanics.

- **Zeroth law** : *For a black hole in equilibrium, the surface gravity is constant.*
  The zeroth law of conventional (mechanical) thermodynamics defines the notion of a quantity called temperature which is constant for two systems in thermal equilibrium. However, for black hole mechanics, the notion of equilibrium is different. We do not speak of two black hole systems being in equilibrium with each other. Rather, the notion of a black hole event horizon being in equilibrium necessitates it being a Killing horizon $\mathcal{H}^{(K)}$. The Killing horizon, being a null hypersurface is generated by a Killing vector field that becomes null on $\mathcal{H}^{(K)}$. The non-affinity parameter associated with this Killing field is denoted as κ. For a stationary spacetime admitting a non-degenerate ($\kappa \neq 0$) Killing horizon $\mathcal{H}^{(K)}$, the non-affinity parameter is called the surface gravity. Generally speaking, for the case of general relativity, if the matter and the non-gravitational fields satisfy the null dominant energy condition, then the non-affinity parameter is constant over the Killing horizon [5, 12]. Without the assumption of energy conditions, and hence independent of the field equations, the zeroth law has been proved for an event horizon in static or axisymmetric stationary spacetimes (with $t - \phi$ reflection symmetry) [13].

- **First law**: We assume a stationary black hole in general relativity, with mass M, horizon surface area A and angular momentum J. We consider a quasi-static perturbation process that takes the black hole to a further stationary state with the global parameters of the mass, horizon area and angular momentum to be $(M + \delta M)$, $(A + \delta A)$ and $(J + \delta J)$ respectively.
  For such changes in the global properties of the black hole, the first law is simply an identity of the form,

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_{BH}\delta J \, . \tag{1.12}$$

  The black hole's surface gravity is κ and its rotation velocity is Ω_{BH}. The perturbation process is described via a stress energy tensor $T^{(m)}_{ab}$ and all the variations are taken to be of the first order. Initially, the proof was provided for Kerr black holes by Bekenstein [10]. Later, the first law was extended to any diffeomorphism invariant gravitational theory by Wald [14]. The analogy with the conventional first law of thermodynamics (stating the conservation of energy) under a quasi-static reversible mechanical process of the form $dE = TdS - pdV$ is quite evident, if we identify the mass, area and surface gravity of the black hole with the thermodynamic entities of energy, entropy and temperature. That is $M \leftrightarrow E$, $A/4 \leftrightarrow S$ and $\kappa/2\pi \leftrightarrow T$. Under this analogy, the term $\Omega_{BH}dJ$ would represent the work done on the black hole under the quasi-static perturbation process.

- **Second law**: *For any classical process in an asymptotically flat spacetime satisfying the Einstein field equations, if the Ricci tensor R_{ab} satisfies the null energy condition i.e. $R_{ab}l^a l^b \geq 0$ for any null vector l^a on the black hole horizon, then its surface area can never decrease i.e.,*

$$\delta A \geq 0\,. \tag{1.13}$$

The second law follows from the area theorem by Hawking [9]. The area theorem follows from the focusing theorem and the fact that the family of null geodesics l^a that rule the event horizon $\mathcal{H}_{BH}$, when followed into the future have no end points [15]. Any null vector can enter into the event horizon and become a part of the null generators of the horizon; however once they enter, they can never leave $\mathcal{H}_{BH}$. This means that the event horizon is geodesically complete and has no caustics. The area theorem then guarantees, that if $R_{ab}l^a l^b \geq 0$, the expansion scalar θ_l of the future pointing null generators l^a of the event horizon is positive or zero,

$$\theta_l \geq 0\,. \tag{1.14}$$

If the initial expansion of the null geodesics forming the horizon were negative (and the null energy condition holds), then the focusing theorem states that these null generators will form a caustic ($\theta_l \to -\infty$) in a finite (affine) time. This comes in contradiction with the assumption that the (smooth) event horizon is ruled via null geodesics that by definition do not form a caustic. Hence the expansion scalar θ_l is everywhere positive on the event horizon. The surface area of an event horizon always increases.

- **Third law**: *There does not exist any physical process(es) that can reduce the surface gravity of a black hole to zero within a finite advanced time.*
  This theorem requires that the stress energy tensor is bounded and the weak energy condition is satisfied. Extremal black holes have by definition zero surface gravity. This third law then states that no black hole can become extremal in a finite advanced time. This case of extremal black holes contradicts the Nernst formulation of standard third law of thermodynamics. The Nernst formulation states that at zero temperature, the entropy is constant and does not depend on the macroscopic

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variables of the system. Together with Planck's postulate, this asserts that the entropy at zero temperature is zero. The extremal black holes with zero surface gravity still have finite surface area. The area of extremal black holes depends on the mass and hence the entropy is non-zero for zero surface gravity. However, the third law of black holes is not violated in the unattainability formulation, which states there does not exit a finite number of thermodynamical processes that makes the system go to absolute zero; absolute zero can only be reached asymptotically. The third law of black hole mechanics was proved by Israel [16].

These four laws of black hole mechanics present an astonishing similarity with the well known laws of thermodynamics. The laws of classical thermodynamics are a set of precise well defined constraint relations for a macroscopic system under thermal interactions. The laws of thermodynamics as such do not probe into the microscopic details of the system even though they strongly hint towards it. Similarly, the laws of black hole mechanics probe into the global properties of a black hole spacetime and are a set of precise relations for processes involving the black hole. As such they are of paramount importance for a theory of quantum gravity. In fact, the Bekenstein-Hawking entropy for a black hole provides one of the very few test beds for quantum gravity. We can have an idea of the entropy and the number of microstates for a black hole by considering as concrete example the case of Schwarzschild black hole in general relativity. For a one solar mass black hole in this case, the entropy $S_{BH} = A_{BH}k_Bc^3/4\hbar G$ at the event horizon $r_h = 2GM/c^2$ gives the numerical value of the order $10^{54} J/K$. The number of microstates $\mathcal{N}$ associated with this entropy is given by $\mathcal{N} = \exp(S_{BH}/k_B)$ is of the order of $\exp(10^{77})$. This is astronomical by any standards and hence the black hole is a highly entropic object. The problem of quantum gravity is to identify the microscopic degrees of freedom that account for the entropy of the black hole.

1.2 Local definitions of temperature and entropy

In the previous section, we saw that the laws of black hole thermodynamics allow us to assign thermodynamic notions of entropy, temperature and energy to black hole event horizons. Since any theory of gravity is built upon the principle of general covariance, there exists the democracy of observers. We should also in principle consider the notion of local horizons (Rindler horizons) that are purely due to the state of motion of observers. In fact, the notion of the horizon as being a one-way membrane is *observer-dependent*. For example, in the Schwarzschild spacetime, radially infalling observers (plunging into the $r < 2M$) region do not perceive the event horizon. Hence, if we are to attribute thermodynamic interpretation to the dynamics of gravity with respect to (w.r.t.) a notion

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of a horizon, we should not prefer one set of observers over the other. The only requirement is that the observer should perceive the horizon. Hence a static observer in the Schwarzschild spacetime observing the dynamics of the event horizon should not be considered more important than say any locally accelerating observing perceiving the dynamics of his Rindler horizon. We should not be making distinctions over the notions of horizons as being 'absolute' (say event horizons) or 'observer-dependent' (the Rindler horizons) and treat all horizons on the same footing [17, 18]. Whatever interpretation an observer can assign to the dynamics of the horizon, is only via the variables accessible to him or her.

The teleological feature of event horizons also presents a conceptual difficulty. Such horizons having a global definition, requires us to have the complete knowledge of the spacetime in order to pin point the event horizon. As a result, it might be convenient to work with *local* or *quasi-local* definitions of horizons. Given any point P of the spacetime, we can have a local Rindler horizon corresponding to an accelerated Rindler observer. A family of Rindler observers can have such a local causal horizon formed by the boundary of the union of the past light cones constructed along their trajectories. Such a local Rindler horizon can be generated by an (approximate) timelike boost Killing vector $\vec{\xi}$ whose norm vanishes on the Rindler horizon. The Rindler horizon, as a result is a null surface. So whatever notion of horizon we work with, say global or local, in essence the horizon is one-way causal null surface that separates the region of spacetime into 'inside' and 'outside'. Intuitively, this allows the notion of entropy to be assigned to the null horizon.

In the previous section, we saw that the notion of (Hawking) temperature assigned to a global event horizon was a consequence of the (Hawking) radiation flux from the black hole. However, from the observation by Unruh [19], we know that given any point P of the spacetime, an accelerated Rindler observer assigns a temperature called the Unruh temperature to the local Rindler horizon. Even in the flat Minkowski spacetime, the Rindler horizon provides a thermal bath at the Unruh temperature for the accelerated observer. In fact, given any spacetime endowed with a horizon, a local patch of the null surface may be approximated by a Rindler horizon. Such Rindler observers will assign a local temperature to the horizon in a well defined way even if there exists no radiation flux from the horizon. This can again be seen quite heuristically via the Euclidean form of the Rindler metric. In fact, the near-horizon geometry of the static spherically symmetric metric used in (1.10) is precisely that of the Rindler metric for the Minkowski spacetime. Let us discuss the interpretation of temperature assigned to the Rindler horizon by accelerated observers. The Rindler metric in $1+1$ dimensions is given by,

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \,. \tag{1.15}$$

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The local proper time for accelerated observers moving along constant ρ is given via $dt_{\text{loc}}^2 = \rho^2 d\eta^2$. Analytic continuation to the imaginary time via $\eta \to -i\theta$ gives us the Rindler metric in the Euclidean signature,

$$ds_E^2 = \rho^2 d\theta^2 + d\rho^2 \,, \tag{1.16}$$

with $d\tau_{\text{loc}}^2 = \rho^2 d\theta^2$. Again since the Rindler metric is just the flat regular Minkowski metric, in order to avoid the conical singularity at $\rho = 0$, the imaginary time θ must be periodic in 2π. This implies that τ_{loc} must also be periodic with a period of $2\pi/\rho$. Since t_{loc} is the proper time measured by an accelerated Rindler observer moving along $\rho =$ constant, such observer would measure the Rindler horizon to be at a temperature of $T = 1/(2\pi\rho)$. The constant acceleration of such an observer is given by $a = 1/\rho$. This tells us that in the Minkowski spacetime, owing to the state of constant acceleration, Rindler observers will measure the temperature of the Rindler horizon to be at the Unruh temperature $T = \frac{a}{2\pi}$.

The Unruh effect has its origins in the fact that the vacuum state for an inertial observer is not the same as that for an accelerated Rindler observer. The positive frequency modes (and hence the creation and annihilation operators) used by the Rindler observers is quite different from the Minkowski observer; as a result of which they perceive different vacua even though both of them are living in the same flat spacetime. The positive frequency modes used by the accelerated observer is related to the positive and negative frequency modes of the inertial observer via the Bogoliubov transformations. The Bogoliubov transformations mix both the creation and annihilation operators, resulting that one observer's vacuum will contain particles for the other. The concept of vacuum is left invariant under Lorentz transformations, not under general coordinate transformations. In fact, using the simple case of a massless scalar field as a probe, it can be shown that the expectation value of the particle number operator (for the Rindler frame) on the Minkowski vacuum $|0_M\rangle$ gives exactly a Planckian thermal radiation spectrum corresponding to the Unruh temperature $T = a/2\pi$. This means that for the Rindler observer, the Minkowski vacuum is filled with particles (field excitations) and the Rindler horizon provides a thermal bath at the Unruh temperature. There exists a strong interconnection between the Unruh and the Hawking effect. Taking the example of the static spherically symmetric spacetime, we had showed that its near-horizon geometry is well approximated by the Rindler spacetime. In fact, the proper acceleration of static observers near the horizon increases as one approaches the horizon, effectively taking infinite acceleration to stay static on the horizon. We could be of the viewpoint, that the radiation measured by such a static observer near enough to the horizon is the Unruh radiation as opposed to some black-hole radiation. Using the redshift factor on the Unruh radiation for such near-horizon static observers to stationary observers at spatial infinity allows us to recover the Hawking radiation and hence the Hawking temperature.

Having defined the notion of a local temperature assigned to any given event of the spacetime by an accelerated observer who perceives the Rindler horizon, let us come to the notion of an entropy that can defined locally. The Rindler horizon, as a null surface is a Killing horizon. The global definition of an event horizon (1.2) does not presume any symmetries of the spacetime. However, now we will consider the case of Killing horizons. We will assume that the spacetime manifold $(\mathcal{M}, g)$ allows for a 1-parameter group of isometries whose orbits form the timelike Killing vector field $\vec{\xi}$ such that $\pounds_{\xi} g_{ab} = 0$. The Killing horizon is a null surface $\mathcal{H}^{(K)}$ such that the Killing field $\vec{\xi}$ becomes null on $\mathcal{H}^{(K)}$. Not all event horizons are Killing horizons. In fact, the strong rigidity theorem [6, 20, 21] in $d = 4$ dimensions asserts that for a stationary spacetime, the event horizon is a Killing horizon. For an (constant) accelerated observer in the $(1 + 1)$ Minkowski spacetime, the Rindler horizon is the plane $x = |t|$. The Rindler observer moving along the $\rho =$ constant trajectory (1.15) defines a timelike Killing vector field in the temporal direction. Such a vector field $\vec{\xi} = x\vec{\partial}_t + t\vec{\partial}_x$ is the generator of Lorentz boosts in the $(t - x)$ plane. This vector field becomes null on the Rindler horizon $x = |t|$. Hence, we see that the Rindler horizon is a Killing horizon.

We will now discuss the notion of entropy of a Killing horizon as that of the *Noether charge* associated with such a horizon. For such a discussion, we will consider a general class of gravity theories whose gravitational Lagrangians $\mathcal{L}_{\rm grav}(g^{ab}, R^a{}_{bcd})$ depend on the metric and the curvature tensor, but not on the covariant derivatives of the latter. Specifically, we have,

$$16\pi \mathcal{A}_{\rm grav} = \int_{\mathcal{V}} d^4x \sqrt{-g}\, \mathcal{L}_{\rm grav}(g^{ab}, R^a{}_{bcd}) \,, \tag{1.17}$$

where, $\mathcal{V}$ denotes the region of integration in the spacetime manifold. It can be shown [22, 23] that the variation of this gravitational action yields the generic structure of the form,

$$16\pi\, \delta \mathcal{A}_{\rm grav} = \int_{\mathcal{V}} d^4x \sqrt{-g} \left(\mathcal{E}_{ab} \delta g^{ab} + \nabla_a \delta v^a \right) , \tag{1.18}$$

where $\nabla_a \delta v^a$ generates a surface term. The form of the symmetric tensor $\mathcal{E}_{ab}$ is given by,

$$\mathcal{E}_{ab} = \frac{\partial \mathcal{L}_{\rm grav}}{\partial g^{ab}}_{|R^a{}_{bcd}} - \frac{1}{2} g_{ab}\, \mathcal{L}_{\rm grav} - 2\nabla^i \nabla^j P_{aijb} \,, \tag{1.19}$$

where P_{abcd} has been defined as

$$P_a{}^{bcd} \equiv \frac{\partial \mathcal{L}_{\rm grav}}{\partial R^a{}_{bcd}}_{|g_{ab}} \,. \tag{1.20}$$

P_{abcd} has all the algebraic symmetries of the curvature tensor. The boundary term is given by,

$$\delta v^i = 2P_a{}^{bic}\delta\Gamma^a{}_{bc} + 2\left(\nabla_j P_k{}^{ji}{}_l\right)\delta g^{kl} \,. \tag{1.21}$$

Corresponding to the matter action, we have by definition,

$$\delta \mathcal{A}_{\text{matter}} \equiv -\frac{1}{2}\int_{\mathcal{V}} d^4x\sqrt{-g}\,T^{(m)}_{ab}\,\delta g^{ab} \,, \tag{1.22}$$

where $T^{(m)}_{ab}$ denotes the energy-momentum tensor. Extremizing the gravitational and the matter action w.r.t the inverse metric tensor, yields,

$$\frac{\delta\left(\mathcal{A}_{\text{grav}} + \mathcal{A}_{\text{matter}}\right)}{\delta g^{ab}} = 0 \implies \mathcal{E}_{ab} = 8\pi T^{(m)}_{ab} \,. \tag{1.23}$$

Without imposing the field equations, one can arrive at a *generalized Bianchi identity* [22–24] of the form,

$$\nabla_i \mathcal{E}^{ij} = 0 \,. \tag{1.24}$$

This *off-shell* generalized Bianchi identity owes its origin to the general covariance of the gravitational theory of the form (1.17). The fact that the gravitational action $\mathcal{A}_{\text{grav}}$ is invariant under infinitesimal diffeomorphism of the form $x^i \to x^i + \xi^i(x)$ leads to an off-shell conserved current $(J_{\text{off}})^i$. Such a conserved current can be shown to have the form,

$$(J_{\text{off}})^i = 2\mathcal{E}^i{}_j\xi^j + \mathcal{L}_{\text{grav}}\,\xi^i + \delta_{\vec{\xi}}v^i \,. \tag{1.25}$$

In the above, for now, we have $\xi^i(x)$ being a general vector field and not the Killing field. The boundary term $\delta_{\vec{\xi}}v^i$ is evaluated on (1.21) with the variations of the metric and the Christoffel connection as,

$$\begin{aligned} \delta_{\vec{\xi}}g^{ab} &= -\pounds_{\vec{\xi}}g^{ab} = \nabla^a\xi^b + \nabla^b\xi^a \,. \\ \delta_{\vec{\xi}}\Gamma^a{}_{bc} &= R^a{}_{(bc)d}\xi^d - \nabla_{(b}\nabla_{c)}\xi^a \,. \end{aligned} \tag{1.26}$$

This off-shell current is conserved albeit the gravitational field equations and is crucially dependent on the generalized Bianchi identity,

$$\nabla_i (J_{\text{off}})^i = 0 \,. \tag{1.27}$$

Via further manipulations on the gravitational Lagrangian of the form (1.17), it can shown [23], that the off-shell conserved current takes the specific form of

$$(J_{\text{off}})^i = 2\nabla_j\left(P^{iabj} + P^{ibaj}\right)\nabla_a\xi_b - 4\left(\nabla_j\nabla_k P^{ijkb}\right)\xi_b - 2P^{baic}\nabla_c\nabla_a\xi_b \,. \tag{1.28}$$

Conservation of such a current implies the possibility of an anti-symmetric tensor field $(J_{\text{off}})^{ij}$ from which it can be derived,

$$(J_{\text{off}})^i = \nabla_j (J_{\text{off}})^{ij} \,. \tag{1.29}$$

Such a (non-unique) anti-symmetric tensor field for the Lagrangian $\mathcal{L}_{\text{grav}}(g^{ab}, R^a{}_{bcd})$ takes the form of,

$$(J_{\text{off}})^{ij} = 2P^{ijab}\nabla_a\xi_b - 4\left(\nabla_a P^{ijab}\right)\xi_b \,. \tag{1.30}$$

The fact that we have a conserved current associated with the diffeomorphism invariance of the gravitational Lagrangian leads us to define a conserved *Noether charge* Q_{Noether}.

For the specific case, when the infinitesimal diffeomorphism of the form $x^a \to x^a + \xi^a(x)$ is generated by a Killing vector field $\vec{\xi}$, the boundary term $\delta_{\vec{\xi}} v^i$ vanishes exactly. This is because both the metric and the Christoffel symbols are left invariant under the infinitesimal diffeomorphism generated by the Killing vector field. Under such a specific choice of the diffeomorphism, we have,

$$(J_{\text{off}})^i_{\text{KV}} = 2\mathcal{E}^i{}_j\xi^j(x) + \mathcal{L}_{\text{grav}}\,\xi^i \,. \tag{1.31}$$

Consider a Killing horizon $\mathcal{H}^{(K)}$ with respect to a Killing vector field $\vec{\xi}$ such that it coincides with the non-affinely parametrized null generators $\vec{l}$ of $\mathcal{H}^{(K)}$ defined via,

$$\vec{\xi}\cdot\vec{\xi} \overset{\mathcal{H}^{(K)}}{=} 0 \quad \text{and} \quad \vec{\xi} \overset{\mathcal{H}^{(K)}}{=} \vec{l} \,. \tag{1.32}$$

We can compute the Noether charge as the flux of the off-shell Noether current (1.28) over a Cauchy hypersurface $\mathcal{C}$, such that the closed boundary $\partial\mathcal{C}$ of the Cauchy slice is $\partial\mathcal{C} = \mathcal{J} \cup S_\infty$,

$$\begin{aligned} Q_{\text{Noether}} \equiv \int_{\mathcal{C}} d\Sigma_a\,(J_{\text{off}})^a &= \frac{1}{2}\oint_{\partial\mathcal{C}} dS_{ab}\,(J_{\text{off}})^{ab} \\ &= -\frac{1}{2}\int_{\mathcal{J}} dS_{ab}\,(J_{\text{off}})^{ab} + \frac{1}{2}\int_{\mathcal{S}_\infty} dS_{ab}\,(J_{\text{off}})^{ab} \,. \end{aligned} \tag{1.33}$$

In the above, $\mathcal{J}$, is a closed 2-dimensional spatial sub-manifold of the Killing horizon $\mathcal{H}^{(K)}$ and $\mathcal{S}_\infty$ is the boundary of the Cauchy slice at spatial infinity. dS_{ab} is the directed surface element on these 2-boundaries. Computation of the surface integral in (1.33) at the spatial infinity S_∞ gives the numerical value of the ADM Hamiltonian when the Killing vector field is appropriately chosen to correspond to time translation and rotations. Computation of the Noether charge at the Hilling horizon gives,

$$Q^{\mathcal{H}^{(K)}}_{\text{Noether}} \equiv Q_{\mathcal{H}^{(K)}} \equiv \frac{1}{2}\int_{\mathcal{J}} dS_{ab}(J_{\text{off}})^{ab} \,. \tag{1.34}$$

Let us now, as an example, consider the Einstein-Hilbert action. The off-shell anti-symmetric Noether tensor for this gravitational theory is given by,

$$(J^{\text{EH}}_{\text{off}})^{ij} = \frac{1}{16\pi}\left(\nabla^i\xi^j - \nabla^j\xi^i\right) . \tag{1.35}$$

Computing the Noether charge on a Killing horizon for the Einstein-Hilbert case with the directed 2-dimensional surface element [25] on $\mathcal{J}$ given by $dS_{ab} = d^2\Theta\sqrt{q}(l_a k_b - k_a l_b)$ turns out to be,

$$\begin{aligned} Q^{\text{EH}}_{\mathcal{H}^{(K)}} &\overset{\mathcal{H}^{(K)}}{=} \frac{1}{2}\int_{\mathcal{J}} d^2\Theta\sqrt{q}(l_a k_b - k_a l_b)\left[\frac{1}{16\pi}(\nabla^a\xi^b - \nabla^b\xi^a)\right] \\ &= -\int_{\mathcal{J}} d^2\Theta\frac{\sqrt{q}}{4}\left(\frac{\kappa}{2\pi}\right) . \end{aligned} \tag{1.36}$$

In the above, we used the fact that $\vec{k}$ is the auxiliary null vector field on $\mathcal{H}^{(K)}$ defined via the conditions $\vec{k}\cdot\vec{k} \overset{\mathcal{H}^{(K)}}{=} 0$ and $\vec{k}\cdot\vec{l} \overset{\mathcal{H}^{(K)}}{=} -1$. The surface gravity κ of the Killing horizon is given by $\xi^j\nabla_j\xi^i \overset{\mathcal{H}^{(K)}}{=} l^j\nabla_j l^i \overset{\mathcal{H}^{(K)}}{=} \kappa\xi^i \overset{\mathcal{H}^{(K)}}{=} \kappa l^i$. The induced metric on $\mathcal{J}$ is given by $q_{ab} \overset{\mathcal{H}^{(K)}}{=} g_{ab} + l_a k_b + k_a l_b$ and $\Theta^A = (\Theta^2, \Theta^3)$ are the intrinsic coordinates on $\mathcal{J}$. Provided that the null dominant energy condition holds, the zeroth law states that the surface gravity κ is constant over $\mathcal{H}^{(K)}$ in general relativity. In that case, we have for the Einstein-Hilbert theory via (1.36),

$$Q^{\text{EH}}_{\mathcal{H}^{(K)}} = -\left(\frac{\kappa}{2\pi}\right)\frac{A_{\mathcal{H}^{(K)}}}{4} = -T\, S_{\text{Bekenstein}} \,, \tag{1.37}$$

where $T = \kappa/2\pi$ is the Hawking temperature and $S_{\text{Bekenstein}} = A_{\mathcal{H}^{(K)}}/4$ is the Bekenstein-Hawking entropy of the Killing horizon.

Motivated by this example, it is natural to suggest that the entropy of a Killing horizon for the general class of gravitational theories given by $\mathcal{L}_{\text{grav}}(g^{ab}, R^a{}_{bcd})$ is [26, 27],

$$S_{\text{Noether}} = -\frac{2\pi}{\kappa}\, Q_{\mathcal{H}^{(K)}} = -\left(\frac{2\pi}{\kappa}\right)\frac{1}{2}\int_{\mathcal{J}} dS_{ab}(J_{\text{off}})^{ab} \,. \tag{1.38}$$

This motivation was fortified by the works of Wald [28, 29] which considered a Killing horizon with a bifurcation 2-surface $\mathcal{B}$, on which the Killing vector vanishes. It was shown, via the covariant phase space formalism [30–32], under linear perturbations, that the following result holds [28, 29, 33],

$$0 = \underbrace{\frac{1}{2}\left(\delta \int_{\mathcal{B}} dS_{ab}\,(J_{\text{off}})^{ab}\right)}_{-(\kappa/2\pi)\delta S_{\text{Noether}}} + \underbrace{\frac{1}{2}\delta\left(\int_{S_\infty} dS_{ab}\,(J_{\text{off}})^{ab}\right) + \int_{S_\infty} dS_{ab}\,\xi^{[a}\delta v^{b]}}_{\delta E - \Omega_{\mathcal{H}^{(K)}}\delta J}\,. \tag{1.39}$$

The above relation can be identified analogous to the first law of thermodynamics under linear perturbations of the background stationary Killing horizon $\mathcal{H}^{(K)}$. The spatial infinity S_∞ computations provide the "work done" terms $\delta E - \Omega_{\mathcal{H}^{(K)}}\delta J$, where E and J are the energy and angular momentum respectively. If the relation (1.39) is indeed to be interpreted as a thermodynamic identity, then the entropy of the Killing horizon must necessarily be of the form (1.38). That is, for any diffeomorphism invariant gravity theory, the black hole entropy is related to the Noether charge associated with the isometry of the Killing field $\vec{\xi}$ that generates the horizon. For the case of Lovelock gravity [33], the Noether entropy evaluated on a stationary event horizon turns out to be the Wald entropy given as,

$$S_{\text{Wald}} = -2\pi \int_{\mathcal{B}} d^2\Theta \sqrt{q}\,\epsilon_{ab}\epsilon_{cd}P^{abcd}\,, \tag{1.40}$$

where $\mathcal{B}$ is the bifurcation 2-surface and ϵ_{ab} is the bi-normal to it. It has been shown [34] that provided that $\mathcal{B}$ is a regular surface, the computation of the Wald entropy remains unaffected if computed on an arbitrary spatial cross-section $\mathcal{J}$ of the stationary event horizon.

This analysis shows that entropy can be assigned to a Rindler horizon which is a Killing horizon via the Noether prescription. Since a Rindler horizon can be constructed at any point in the spacetime, there exists observers in the spacetime who can attribute both temperature T, entropy density s and hence heat density Ts associated with each event P of a given spacetime. Spacetime is hence 'hot'.

1.3 Local versions of thermodynamic laws and the emergent paradigm

Since, locally both temperature and entropy can be assigned to a given event in the spacetime, we could expect for local thermodynamic laws to be valid. In essence the notion of *Clausius* type thermodynamical identities that can be expressed in the form $\delta Q = T\delta S$ can be established locally. Such an identity can be thought to be a re-expression of the first law. In essence such a Clausius type identity can again be interpreted as an energy conservation or entropy balance equation. Let us try to motivate this locally for any given event in a spacetime. The same idea can be applied to global event horizons. We saw that the local observers on account of their acceleration perceive a Rindler horizon. When some amount of matter crosses across the Rindler horizon, all the information pertaining to the matter is lost for the Rindler observer. Hence for such an observer, the entropy corresponding to the matter is lost. The phenomenon of matter flow crossing the Rindler horizon constitutes a heat flux δQ crossing across the horizon. In order to compensate for the entropy loss, the Rindler observer has to assign an entropy increase $\delta S = \delta Q/T$ to the horizon, where T is the Unruh temperature of the Rindler horizon, thus implying an entropy balance equation. It can also be interpreted via $\delta Q = T\delta S$ as a energy conservation equation that is valid locally for any given point of the spacetime, thus quantifying a version of the first law. This form of the energy conservation or entropy balance can be applied to any event P of the spacetime for any diffeomorphism invariant theory of gravity. The reason behind this is the structure of the off-shell Noether current (1.31) for an (approximate) Killing vector field $\vec{\xi}$ generating the Killing (Rindler) horizon. We will provide a much more rigorous justification of such a local version of the Clausius identity in the next section when we discuss the physical process first law in the next section.

In fact, the deep connection between the gravitational field equations and such versions of Clausius or thermodynamical relations established locally at any given point P of the spacetime gave rise to idea that perhaps the dynamics of gravity is *emergent*. This idea has been supported by the following observations :

- It was shown by Jacobson [35], that the Einstein field equations can be 'derived' from a local equilibrium constitutive relation $\delta Q = T\delta S$ called the Clausius identity applied to local causal horizon (null surface) constructed at any event of the spacetime.

- It was shown by Padmanabhan [36], that the Einstein field equations under a certain physical process, when projected onto a generic null surface, takes the form of a thermodynamic identity analogous to the first law of thermodynamics.

We see that null surfaces are quite universal in such analysis. Null surfaces in a spacetime can act as a horizon to special class of observers. These observers can attribute both notions of temperature and entropy density to these null surfaces locally. Thus each point in the spacetime can be assigned an observer-dependent heat density Ts. The fact that spacetime is hot, means that it has a micro-structure or that there are *atoms of spacetime*.

Aside from these thermodynamical implications of the gravitational field equations another conclusive evidence came from the work of Damour [37]. It was shown that there exists a correspondence between the field equations of general relativity when projected onto a generic null hypersurface and equations of fluid-dynamics. The horizon dynamics has an interpretation of that of a viscous fluid membrane. It has also been shown that the Navier-Stokes fluid equations can be derived from an action principle [38]. When such an action is extremized with respect to null vectors in the spacetime, the field equations

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are precisely the Navier-Stokes equations. These results point to the deep interplay of gravitational field equations with the laws of thermodynamics and fluid flow and hence point to the notion of gravity being *emergent*. These evidences point towards the *emergent paradigm of gravity* which propounds the view that the statistical mechanics of the atoms of spacetime gives rise to the gravitational field equations [39, 40]. A few other observations also hint towards the emergence of the gravitational field equations. It has been shown [41, 42] that when gravitational and matter heat density constructed for every null surface in the spacetime is maximized, then it leads to the relevant gravitational field equations. So the field equations of gravity can also be derived from a thermodynamic variational principle. Such a thermodynamic extremum principle can be related to the zero point length of spacetime arising at the Planck scales [43, 44]. This then identifies the characteristics of the underlying microscopic theory, such that coarse-graining of these degrees of freedom lead to the gravitational field equations in the long wavelength limit. The emergent paradigm of gravity provides direct thermodynamical interpretation to quantities built out of the metric and the Christoffel connection in terms of temperature and entropy [45, 46]. The Noether current and hence the charge associated to the diffeomorphism invariance of the gravitational action under a time evolution vector field can also be provided a thermodynamic connotation [46]. In fact, all these vivid thermodynamical connections are not just only for Einstein gravity, but can be extended to Lanczos-Lovelock theories of gravity where the horizon entropy is not proportional to the surface area as opposed to general relativity. This suggests that the emergent paradigm transcends general relativity and is hinting to something deeper within the structures of gravity and the nature of spacetime. Obviously, the notion of emergence that we speak of here is that of the field equations and not of the emergence of manifold structure of the spacetime [47].

In the next sections, we will elaborate about the physical process first law and Jacobson's analysis of deriving the Einstein field equations from the Clausius identity. We will also discuss briefly Padmanabhan's approach of interpreting the Einstein field equations as a thermodynamic identity as well as Damour's work of relating the Einstein field equations to fluid behavior.

1.4 The physical process first law of black hole mechanics

We could ask the question as to whether such Clausius type thermodynamic identity of the form $T\delta S = \delta Q$ exists for local definitions of horizons say the Rindler horizons. This would be in accordance with the democracy of the observers who on account of their motion perceive a local causal horizon. This is where the physical process first law (PPFL) comes into play. The PPFL does not require the spacetime to be asymptotically flat and hence applies to a wide class of horizons. The PPFL is local and hence can be

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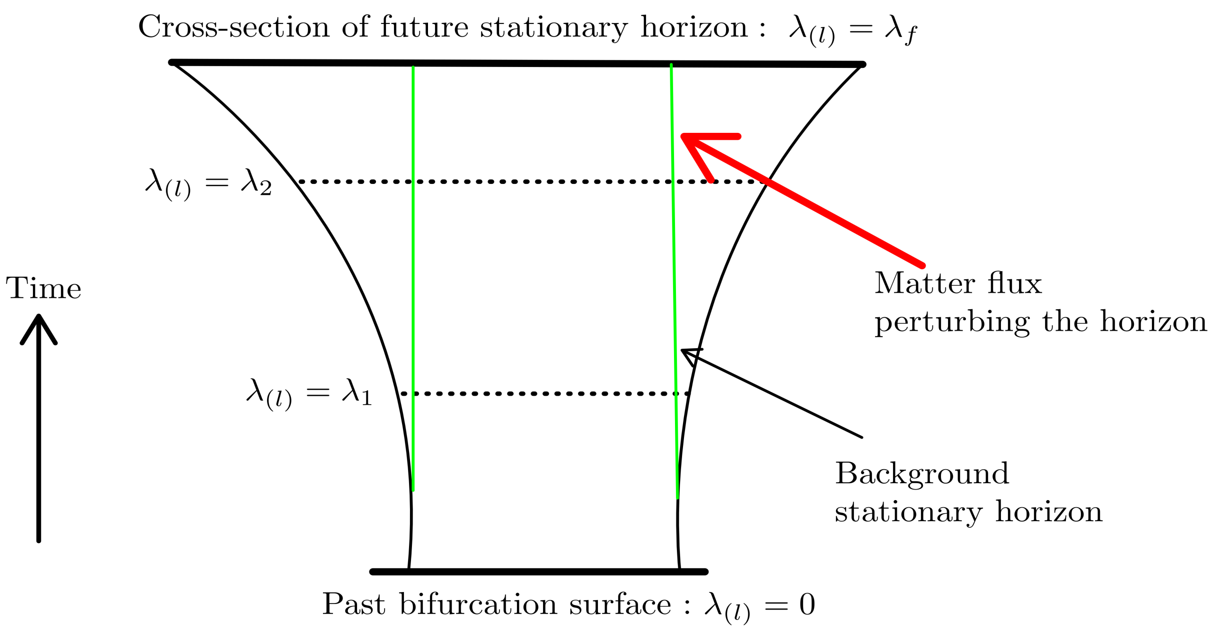

FIGURE 1.1: The unperturbed evolution of the stationary background Killing horizon is shown by the green curve. The evolution of the dynamical horizon under the perturbation is shown by the black curve.

applied to local Rindler horizons constructed at any event in the spacetime [48–50]. The basic requirement for the PPFL is that there exists a Killing horizon $\mathcal{H}^{(K)}$ generated by a timelike Killing vector field (whose norm vanishes on $\mathcal{H}^{(K)}$) and that the spacetime can be extended to the past such that the Killing horizon possesses a regular bifurcation 2-surface. For the PPFL, unlike the equilibrium case, we truly consider a dynamical scenario in which the background stationary Killing horizon is perturbed by a matter flux crossing across $\mathcal{H}^{(K)}$. The PPFL then relates the entropy variation for the dynamical horizon to the matter flux.

In the construction of the PPFL, the null generators $\vec{l}$ of $\mathcal{H}^{(K)}$ will be affinely parametrized with the affine parameter value $\bar{\lambda}_{(l)}$. The Killing field $\vec{\xi}$, which is null on the horizon is non-affinely parametrized with the Killing parameter v. As a result, we have,

$$l^a = \frac{dx^a}{d\bar{\lambda}_{(l)}}; \quad l^j \nabla_j l^i = 0; \quad \xi^i = \frac{dx^i}{dv}; \quad \xi^j \nabla_j \xi^i \overset{\mathcal{H}^{(K)}}{=} \kappa \xi^i \,, \tag{1.41}$$

with κ being the surface gravity of the Killing horizon. In general, for such a choice of parametrization, it can be shown that,

$$\xi^i \overset{\mathcal{H}^{(K)}}{=} \kappa \bar{\lambda}_{(l)} l^a \,. \tag{1.42}$$

The bifurcation 2-surface of the Killing horizon is assumed to be located at $\bar{\lambda}_{(l)} = 0$. Given a spatial 2-dimensional cross-section defined via $\bar{\lambda}_{(l)} =$ constant on the future

Killing horizon ($\bar{\lambda}_{(l)} > 0$), there exists two orthogonal null directions $\vec{l}$ and $\vec{k}$ to it. The auxiliary null vector field $\vec{k}$ is as usual defined via $\vec{k} \cdot \vec{k} = 0$ and $\vec{k} \cdot \vec{l} = -1$. The induced metric on a transverse 2-dimensional cross-section $\mathcal{J}_{\mathcal{H}^{(K)}}$ is given via $q_{ab} = g_{ab} + l_a k_b + k_a l_b$. The expansion scalar and the shear corresponding to the null generators $\vec{l}$ are given via θ_l and ${}^{(l)}\sigma_{ab}$. Similarly, the expansion scalar and the shear tensor corresponding to the auxiliary field $\vec{k}$ are given by θ_k and ${}^{(k)}\sigma_{ab}$. The null Raychaudhuri equation (NRE) corresponding to the affinely parametrized null geodesics $\vec{l}$ is given by,

$$\frac{d\theta_l}{d\bar{\lambda}_{(l)}} = -\frac{1}{2}\theta_l^2 - {}^{(l)}\sigma_{ab}{}^{(l)}\sigma^{ab} - R_{ab}l^a l^b \,. \tag{1.43}$$

Let us now study the dynamics of the gravitational field equations under the perturbation of the stationary Killing horizon $\mathcal{H}^{(K)}$ via the influx of the matter energy momentum tensor across $\mathcal{H}^{(K)}$. The strength of the perturbation is given by a parameter ϵ. As an assumption, it can be demanded that on the bifurcation surface $\mathcal{B}$ of the background stationary Killing horizon, the expansions and the shears of both $\vec{l}$ and $\vec{k}$ vanish [51]. This can be demanded since the horizon $\mathcal{H}^{(K)}$ at the bifurcation 2-surface starts out as a stationary background configuration before the perturbation kicks in. The expansion θ_l and the shear ${}^{(l)}\sigma_{ab}$ on the future dynamical event horizon $\mathcal{H}^{(K)}_{(+)}$ is of the order $\mathcal{O}(\epsilon)$. The expansion θ_k and the shear ${}^{(k)}\sigma_{ab}$ are zeroth order on the future dynamical horizon $\mathcal{H}^{(K)}_{(+)}$ and hence of order $\mathcal{O}(\epsilon)$ on the bifurcation surface of the dynamical horizon. We would be considering a general covariant diffeomorphism invariant theory of gravity under this dynamical perturbation process by the matter flux. Obviously, the Bekenstein-Hawking entropy density for the horizon will not work since we are considering theories beyond general relativity. Moreover, since the process is dynamical, and the horizon under the perturbation is non-stationary, the Wald entropy density turns out to be ambiguous. Whatever be the case, the horizon entropy for such a dynamical horizon can be expressed in the form,

$$S_{\text{dynamical}} \equiv \frac{1}{4}\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}(1+\bar{\rho}) \,, \tag{1.44}$$

where $\sqrt{q}\bar{\rho}$ is some locally constructed entropy density involving higher curvature contributions. In the limit when the dynamical horizon goes over to the stationary background Killing horizon, (1.44) should give us the Wald entropy. Setting $\bar{\rho} = 0$ under stationarity implies neglecting the higher curvature contributions and we get the Bekenstein-Hawking entropy. For such a choice of general diffeomorphism invariant theory of gravity, the field equations take the form of $G_{ab} + H_{ab} = 8\pi T^{(m)}_{ab}$, where H_{ab} involves higher curvature contributions and is a measure of the deviation from general relativity.

Under the process of perturbation, let us compute the change in the entropy by integrating from the transverse cross-section $\mathcal{J}^1_{\mathcal{H}^{(K)}}$ situated at the affine parameter value $\bar{\lambda}_{(l)} = \lambda_1$ to $\mathcal{J}^2_{\mathcal{H}^{(K)}}$ situated at $\bar{\lambda}_{(l)} = \lambda_2$.

$$\delta S_{\text{dynamical}} = \frac{1}{4}\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)} \frac{d}{d\bar{\lambda}_{(l)}}\Big(\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}(1+\bar{\rho})\Big) = \frac{1}{4}\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)} \int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\,\bar{\zeta}_{(l)}\,, \tag{1.45}$$

where $\bar{\zeta}_{(l)}$ has been defined as $\bar{\zeta}_{(l)} \equiv \theta_l(1+\bar{\rho}) + d\bar{\rho}/d\bar{\lambda}_{(l)}$. In the above, we have used the relation for the expansion scalar for the affinely parametrized null generators as $\theta_l = \frac{1}{\sqrt{q}}\frac{d}{d\bar{\lambda}_{(l)}}\sqrt{q}$. We integrate by parts the relation (1.45) to get,

$$\begin{aligned}\delta S_{\text{dynamical}} = &\frac{1}{4}\Big(\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\bar{\lambda}_{(l)}\,\bar{\zeta}_{(l)}\Big)\Big|_{\lambda_1}^{\lambda_2} \\ &-\frac{1}{4}\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)} \int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\Big(\theta_l\bar{\zeta}_{(l)} + \frac{d\bar{\zeta}_{(l)}}{d\bar{\lambda}_{(l)}}\Big)\,.\end{aligned} \tag{1.46}$$

Upon using the NRE for the affinely parametrized null geodesic generators *i.e.* (1.43) in (1.46), we obtain for the variation of the entropy of the horizon as,

$$\begin{aligned}\delta S_{\text{dynamical}} = &\frac{1}{4}\Big(\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\bar{\lambda}_{(l)}\,\bar{\zeta}_{(l)}\Big)\Big|_{\lambda_1}^{\lambda_2} \\ -&\frac{1}{4}\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)} \int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\,\Big[(1+\bar{\rho})\frac{\theta_l^2}{2} + 2\theta_l\frac{d\bar{\rho}}{d\bar{\lambda}_{(l)}} - (1+\bar{\rho})\,{}^{(l)}\sigma^2 \\ &- (1+\bar{\rho})R_{ab}l^a l^b + \frac{d^2\bar{\rho}}{d\bar{\lambda}^2_{(l)}}\Big]\,.\end{aligned} \tag{1.47}$$

Next, upon using the field equations for the given diffeomorphism invariant theory of gravity, we obtain,

$$\begin{aligned}\delta S_{\text{dynamical}} = &\frac{1}{4}\Big(\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\bar{\lambda}_{(l)}\,\bar{\zeta}_{(l)}\Big)\Big|_{\lambda_1}^{\lambda_2} + 2\pi\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)} \int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\,T^{(m)}_{ab}l^a l^b \\ &+\frac{1}{4}\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)} \int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\,\Big(-(1+\bar{\rho})\frac{\theta_l^2}{2} + (1+\bar{\rho})\,{}^{(l)}\sigma^2\Big) \\ &-\frac{1}{4}\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)} \int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\Big[\frac{d^2\bar{\rho}}{d\bar{\lambda}^2_{(l)}} + 2\theta_l\frac{d\bar{\rho}}{d\bar{\lambda}_{(l)}} - \bar{\rho}R_{ab}l^a l^b + H_{ab}l^a l^b\Big]\,.\end{aligned} \tag{1.48}$$

The above expression for the variation of the entropy under the evolution of the horizon from the cross-section $\bar{\lambda}_{(l)} = \lambda_1$ to $\bar{\lambda}_{(l)} = \lambda_2$ is quite general and has not taken into

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account any order of perturbation under the dynamical evolution of the horizon. We will do that now. We see that on the future horizon, θ_l^2 and ${}^{(l)}\sigma^2$ are of the order $\mathcal{O}(\epsilon^2)$ in the strength of the perturbation. Hence to the first order, the change in the entropy associated with the evolution of the horizon, under the perturbation is given as,

$$\delta S^{(1)}_{\text{dynamical}} = = \frac{1}{4}\Big(\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\bar{\lambda}_{(l)}\,\bar{\zeta}_{(l)}\Big)|_{\lambda_1}^{\lambda_2} + 2\pi\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)}\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\,T^{(m)}_{ab}l^a l^b$$
$$-\frac{1}{4}\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)}\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\Big[\frac{d^2\bar{\rho}}{d\bar{\lambda}^2_{(l)}} + 2\theta_l\frac{d\bar{\rho}}{d\bar{\lambda}_{(l)}} - \bar{\rho}R_{ab}l^a l^b + H_{ab}l^a l^b\Big]\,. \tag{1.49}$$

It can easily be verified that for Einstein gravity, the second line in the first order variation of the entropy (1.49) vanishes identically by setting $\bar{\rho}$ and H_{ab} to zero. Moreover, it can be shown [51] that for Einstein gravity with $f(R)$ corrections, the second line in the R.H.S of (1.49) again vanishes exactly for the choice of $\bar{\rho} = \alpha f'(R)$. Taking cue from this, it is postulated that for any diffeomorphism invariant theory of gravity,

$$\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)}\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\Big[\frac{d^2\bar{\rho}}{d\bar{\lambda}^2_{(l)}} + 2\theta_l\frac{d\bar{\rho}}{d\bar{\lambda}_{(l)}} - \bar{\rho}R_{ab}l^a l^b + H_{ab}l^a l^b\Big] = \mathcal{O}(\epsilon^2)\,, \tag{1.50}$$

in the order of perturbation. Hence, the first order variation of the entropy of the horizon under the perturbation can be shown to be,

$$\delta S^{(1)}_{\text{dynamical}} = = \frac{1}{4}\Big(\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\bar{\lambda}_{(l)}\,\bar{\zeta}_{(l)}\Big)|_{\lambda_1}^{\lambda_2} + 2\pi\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}_{(l)}\bar{\lambda}_{(l)}\int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta\sqrt{q}\,T^{(m)}_{ab}l^a l^b\,. \tag{1.51}$$

The first term in the R.H.S of Eq (1.51) is a surface term. At this point we make a further assumption. Remember that we had already assumed for the existence of a regular bifurcation 2-surface for the Killing horizon stationed at $\bar{\lambda}_{(l)} = 0$ which corresponds to infinite past of the (non-affine) Killing parameter v since $\bar{\lambda}_{(l)} = \frac{1}{\kappa}\exp[\kappa v]$. The final assumption that we make is that the horizon is stable under perturbations. That is, we must wait long enough to the asymptotic future so that after the perturbation has acted out, the horizon finally settles to an equilibrium state where its expansion and shear vanish. This assumption has been motivated by the cosmic censorship conjecture [52, 53]. While considering the first order variation of the entropy for the dynamical horizon, we have neglected contributions which were of the second order in the strength of perturbation. This means that we are essentially considering small perturbations in the future horizon $\mathcal{H}^{(K)}_{(+)}$ that do not diverge under the evolution and remain small throughout. This ensures that the Killing horizon remains a smooth structure throughout its evolution and there does not form any caustics. This assumption remains parallel with a conventional thermodynamic system where if we wait long enough, with all the dissipation having acted out, the system finally reaches thermal equilibrium. Under this assumption, the first boundary terms drop out since the integral is then evaluated at $\bar{\lambda}_{(l)} = \lambda_1 = 0$ bifurcation slice $\mathcal{B}$ and final asymptotic stationary slice $\bar{\lambda}_{(l)} = \lambda_2 \to \lambda_f$. So finally we have,

$$\delta S_{\text{dynamical}} = 2\pi \int_{\bar{\lambda}_{(l)}=0}^{\bar{\lambda}_{(l)}=\lambda_f} d\bar{\lambda}_{(l)} \bar{\lambda}_{(l)} \int_{\mathcal{J}_{\mathcal{H}^{(K)}}} d^2\Theta \sqrt{q}\, T^{(m)}_{ab} l^a l^b \,. \tag{1.52}$$

The directed surface element [25] on the Killing horizon is $d\Sigma_a = -d\bar{\lambda}_{(l)} l_a d^2\Theta\sqrt{q}$. We use the relation $\vec{\xi} \overset{\mathcal{H}^{(K)}}{=} \kappa\bar{\lambda}_{(l)}\vec{l}$ for the background Killing horizon $\mathcal{H}^{(K)}$. Since, we have assumed that $\mathcal{H}^{(K)}$ for stationary spacetime possesses a bifurcation 2-surface, the zeroth law holds and κ remains constant over the background Killing horizon. So finally, this yields,

$$\frac{\kappa}{2\pi}\delta S_{\text{dynamical}} = -\int_{\mathcal{H}^{(K)}} d\Sigma^a T^{(m)}_{ab} \xi^b \,. \tag{1.53}$$

The above equation (1.53) defines the PPFL. If matter accretes into the Killing horizon, resulting in a truly dynamical process, then the PPFL relates the flux of the energy-momentum tensor across the dynamical horizon to its entropy change. The temperature associated with this process is the Hawking temperature of the background Killing horizon $T = \kappa/2\pi$. If the matter fields satisfy the null energy condition ($T^{(m)}_{ab} l^a l^b \geq 0$), then (1.52) verifies that the entropy change under the process is positive. If we identify the matter flux as some *heat exchanged* δQ across the horizon, we then have under the PPFL, a kind of local Clausius identity interpretation,

$$T\delta S_{\text{dynamical}} = \delta Q \,, \tag{1.54}$$

where we define

$$\delta Q \equiv -\int_{\mathcal{H}^{(K)}} d\Sigma^a T^{(m)}_{ab} \xi^b \,. \tag{1.55}$$

Getting back to (1.51), we might not have thrown away the boundary terms. We can still provide a local thermodynamic interpretation to the PPFL provided we interpret the boundary terms as a change of energy δE associated with the horizon membrane [54]. We would then have,

$$T\delta S_{\text{dynamical}} = \delta E + \delta Q \,. \tag{1.56}$$

Finally, the PPFL is local and hence can be used by Rindler observers who can construct

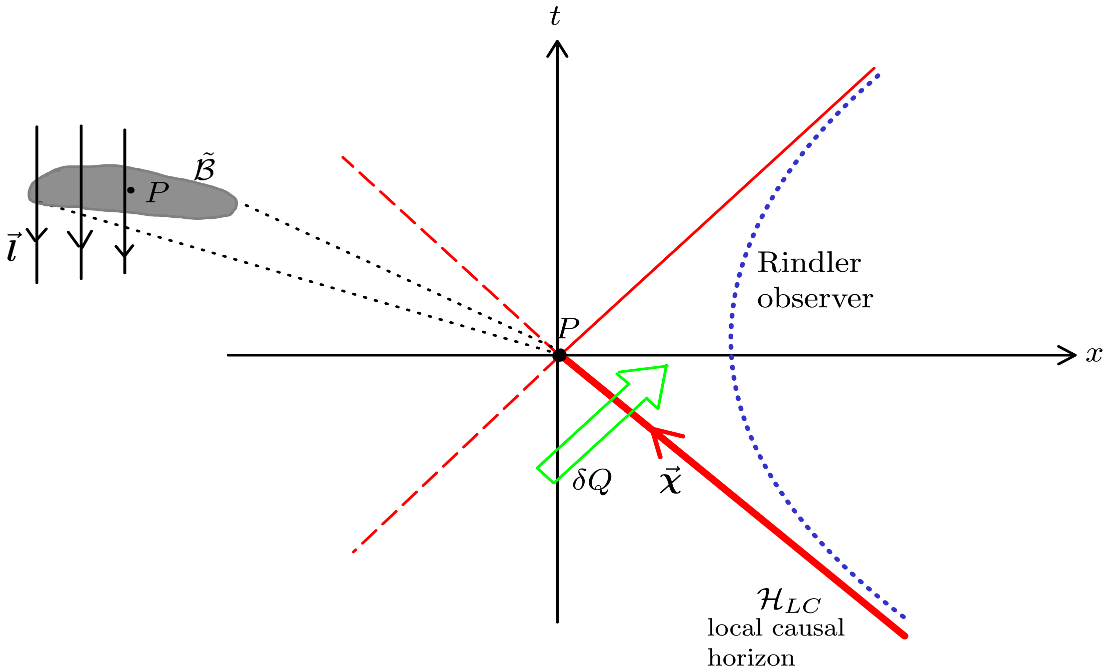

FIGURE 1.2: The Rindler horizon is $x = |t|$. The local causal horizon $\mathcal{H}_{LC}$ ($x = -t$) acts as a diathermic wall for Rindler observers.

at any given point P of the spacetime a Rindler (Killing) horizon provided with a bifurcation surface $\mathcal{B}$. For such observers, the local Clausius identity of the form (1.54) holds under matter flux across the horizon. This will provide us some motivation to consider Jacobson's procedure [35] of deriving the Einstein field equation from a local Clausius identity applied to a given event in the spacetime.

1.5 "Deriving" Einstein field equations from a thermodynamic identity: ála Jacobson

In this section we will follow through Jacobson's analysis [35] of obtaining the Einstein field equations from an equilibrium thermodynamic constitutive relationship valid for a local causal horizon established on a given point in the spacetime. Let us discuss the construction for Jacobson's analysis. Given any event P in the spacetime manifold $(\mathcal{M}, g)$, we can consider a local patch of spacelike 2-surface $\tilde{\mathcal{B}}$ which contains the point P. Given this surface, one side of the boundary of the causal past of all points lying on $\tilde{\mathcal{B}}$ can define a local causal horizon at P.

We can construct a local inertial frame (LIF) around the point P depending on the curvature scale in the neighborhood of the point. If one selects a region with an intrinsic length scale l such that $l << \mathcal{R}(P)^{-1/2}$, where $\mathcal{R}(P)$ is the radius of curvature of the spacetime at the point P, then the metric in the neighborhood of P can be approximated by the Minkowski metric. Using the Lorentz invariance of the spacetime in the region of size l about P, we can construct a LIF for the local patch $\tilde{\mathcal{B}}$. Using Riemann normal coordinates $\{x^i\}$ in this patch, we set the point P to be $x^i = 0$. Having constructed this LIF, we could accelerate along a preferred spatial axis (say x-axis, implying a boost in the $t-x$ plane) with an acceleration κ. For such an accelerated observer, we can introduce the local Rindler frame (LRF) by the usual Rindler transformations. This in principle can always be done in a region $l << \mathcal{R}(P)^{-1/2}$ which is satisfied provided $\kappa^{-1} << \mathcal{R}(P)^{-1/2}$. A large enough value of the acceleration κ satisfies such a condition. The LRF construction allows for a family of uniformly accelerated observers who perceive the plane $x = \pm t$ as the local Rindler horizon. Now that we had previously designated one side of the boundary of the causal past of the set of points lying in $\tilde{\mathcal{B}}$ as the local causal horizon, it implies that the past Rindler horizon $x = -t$ coincides with the local causal horizon $\mathcal{H}_{LC}$. Hence we see that in Jacobson's construction the null generators $\vec{l}$ of the local causal horizon are past pointing. Corresponding to translations in the Rindler time, we can have an (approximate) future pointing boost Killing vector field $\vec{\xi}$ generating an (approximate) Killing horizon such that the norm of $\vec{\xi}$ vanishes on the Killing horizon. We can consider the local causal horizon $\mathcal{H}_{LC}$ to be a part of this (approximate) Killing horizon. The non-affine parameter for $\vec{\xi}$ is v satisfying $\xi^i \nabla_i v = 1$. Since the null generators (in Jacobson's construction) are affinely parametrized (with the parameter value $\bar{\lambda}_{(l)}$), we have $\vec{\xi} = -\bar{\lambda}_{(l)}\kappa\vec{l}$ and $\bar{\lambda}_{(l)} = -\frac{1}{\kappa}\exp[-\kappa v]$. The 2-surface $\tilde{\mathcal{B}}$ is situated at the value $\bar{\lambda}_{(l)} = 0$ which corresponds to infinite Killing parameter v. The approximate Killing vector within its region of validity satisfies,

$$\nabla_{(a}\xi_{b)} = 0 \quad \text{and} \quad \nabla_a \nabla_b \xi_c = R_{dabc}\xi^d \,. \tag{1.57}$$

We finally comment that the above construction of a local Rindler horizon at a given spacetime event is independent of the gravitational field equations and only depends on the principle of equivalence.

Generally, null surfaces in any given spacetime act as one-way membranes. In fact, locally any null surface can be approximated by a Rindler horizon generated by an Killing vector field. Given any event P, let an accelerated observer consider an amount of matter flux across his Rindler horizon. Once the matter crosses over the Rindler horizon, all information pertaining to it is lost for the observer. This would imply a decrease in the total entropy for the outside accelerated observer who perceives the Rindler horizon. This would then imply a violation of the sacrosanct second law of thermodynamics. In order to rescue this principle, the outside observer must assign an entropy to the Rindler horizon when the matter flux crosses across the horizon. Corresponding to the

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temperature T of the horizon (as perceived by the observer), if the matter flux is δQ, then $\delta S = \delta Q/T$ should be the entropy increase for the Rindler horizon. Jacobson's idea was to "invert" such a local thermodynamical relationship established for causal horizons $\mathcal{H}_{LC}$ constructed at P to evolve out the Einstein field equations.

At this point, we will move away from Jacobson's original construction using local causal horizons. Rather, we will consider a general integrable null surface $\mathcal{H}$ and a special coordinate system called the Gaussian null coordinate (GNC) system adapted to it. We do this because we will also consider Padmanabhan's approach of interpreting the gravitational field equations on a null surface as a thermodynamic identity [36]. Padmanabhan's program was generalized to the case of an integrable null surface $\mathcal{H}$ in the spacetime and specifically used the GNC system. Hence, in order to elucidate the structural similarities and distinctions between Jacobson's and Padmanabhan's analysis, we will proceed with a general null surface $\mathcal{H}$ and the GNC providing a coordinate chart for the spacetime in the neighborhood of $\mathcal{H}$.

Let us begin briefly discussing about the GNC system. The details of it can be found in [55–58]. In this coordinate system $x^a = (u, r, x^2, x^3)$, the generic integrable null surface is defined by $r = 0$. The non-affine parameter along the null generators $\vec{l}$ of such a null surface is the parameter u. The null-generators in such a construction is assumed to be future pointing. The spatial transverse coordinates $x^A = (x^2, x^3)$ of the null surface are carried along the integral curves of $\vec{l}$. As a result, the null surface is parameterized by (u, x^2, x^3). To move away from the null surface into the neighborhood of the spacetime, we deploy the auxiliary null vector field $\vec{k}$. In the GNC construction, the auxiliary null vector is assumed to be tangent to affine geodesics with the affine parameter $-r$. The spacetime line interval in the vicinity of the null surface is given by,

$$ds^2 = -2r\alpha du^2 + 2dudr - 2r\beta_A du dx^A + q_{AB} dx^A dx^B \, . \tag{1.58}$$

The six independent functions α, β_A and q_{AB} precisely account for the six independent degrees of freedom in the metric defined up to the freedom of general coordinate transformations. Let us list the components of the null generator and the auxiliary null vector $\vec{k} = -\frac{\vec{\partial}}{\partial r}$ on the null surface at $r = 0$,

$$l^i = (1, 0, 0, 0) \, ; \quad l_i = (0, 1, 0, 0) \, ; \quad k^i = (0, -1, 0, 0) \, ; \quad k_i = (-1, 0, 0, 0) \, . \tag{1.59}$$

The time development vector $\vec{\xi} = \frac{\vec{\partial}}{\partial u}$ in this coordinate system has the interpretation of being the Killing vector field for the Schwarzschild and the Rindler metric when they are written in the GNC form. Hence local Rindler observers who identify the null surface locally as their Rindler horizon, assign the vector field $\vec{\xi} = \frac{\vec{\partial}}{\partial u}$ to be their local time direction. The norm of the vector $\vec{\xi}$ is given by $\vec{\xi} \cdot \vec{\xi} = g_{uu} = -2r\alpha$. This shows that time development vector is timelike for $r > 0$ and spacelike for $r < 0$ provided α is chosen positive. $\vec{\xi}$ goes to null on $r = 0$. Hence the null surface $r = 0$ partitions the spacetime into an inner and outer region. The value of the affine parameter r on the null surface is set to zero with the auxiliary null vector pointing towards the 'ingoing' direction. Any dynamics of the null surface will be studied by observers in the outside $r > 0$ region.

In order to evolve out the connection between the gravitational field equations and thermodynamics we consider the vector field $G^a{}_b\xi^b$. This vector field is quite important since it is related to the off-shell Noether current (1.25) or the gravitational momentum [59, 60]. It will turn out that two projection components of the vector field $G^a{}_b\xi^b$ on the null surface *viz* $G_{ab}l^al^b$ and $G_{ab}k^al^b$ have been used by Jacobson and Padmanabhan respectively in their analysis.

Let us begin with the projection component (on the null surface) used by Jacobson *i.e.* the $G_{ab}l^al^b$. In the GNC coordinates, using (1.59), we find that,

$$G_{ab}l^al^b = G^{ab}l_al_b = R^{ab}l_al_b = R_{ab}l^al^b = R_{uu} = R^{rr}\,. \tag{1.60}$$

Using the GNC metric and the Christoffel connection components [57], it can be shown that,

$$R_{uu} = R^{rr} = G_{ab}l^al^b = G^{ab}l_al_b = \alpha\partial_u(\ln\sqrt{q}) - \partial_u^2(\ln\sqrt{q}) - \frac{1}{4}q^{AC}q^{BD}(\partial_u q_{AB})(\partial_u q_{CD})\,. \tag{1.61}$$

The second fundamental form $\Theta_{ab} = q_a^i q_b^j \nabla_i l_j$ [61] or the deformation rate tensor $\chi_{ab} = q_a^i q_b^j \pounds_{\vec{l}} q_{ij}$ [61] of the null surface (for spacetimes equipped without torsion) is a completely spatial tensor,

$$\Theta_{AB} = \chi_{AB} = \frac{1}{2}\partial_u q_{AB}\,, \tag{1.62}$$

with all the rest of the components being zero. The trace of the second fundamental form gives the expansion scalar θ_l for the null generators $\vec{l}$,

$$\theta_l = q^{ab}\nabla_a l_b = \partial_u(\ln\sqrt{q})\,. \tag{1.63}$$

Using the relations (1.62) and (1.63) in (1.61), we have,

$$R_{uu} = R^{rr} = \alpha\theta_l - \partial_u\theta_l - \Theta^{AB}\Theta_{AB}\,. \tag{1.64}$$

Since $R_{uu} = R_{ab}l^a l^b$, (1.64) is nothing but the null Raychaudhuri equation written out in the GNC, expressible as,

$$\partial_u \theta_l = \alpha\theta_l - \Theta^{AB}\Theta_{AB} - R_{ab}l^a l^b \; , \tag{1.65}$$

which expresses the rate of change of the expansion scalar θ_l along the null generators $\vec{l}$ quantified by the term $\partial_u \theta_l$. The term $\alpha\theta_l$ appears since we have chosen a non-affine parameter u associated with the null generators. This can be seen from the fact that the time development vector $\vec{\xi}$ (which becomes null on the $\mathcal{H}$) satisfies,

$$\lim_{r\to 0} \xi^j \nabla_j \xi^i \overset{\mathcal{H}}{=} \alpha(u, r=0, x^A)\xi^i \; . \tag{1.66}$$

This shows that the non-affinity parameter associated with the (non-affinely) parameterized null generators of $\mathcal{H}$ is $\alpha_{|r=0}$. Hence we can identity $\alpha_{|r=0}$ to be the surface gravity of the Killing horizon that becomes synonymous with a local patch of the null surface $\mathcal{H}$.

However, in accordance with Jacobson's analysis and the PPFL, we would like to have the null generators of $\mathcal{H}$ being affinely parameterized and hence satisfying $l^j\nabla_j l^i = 0$. This puts a restriction on the choice of the function α. Specifically, we see that for affinely parameterized null geodesics, the non-affinity parameter should go to zero on the $r = 0$ surface. Hence, we must have $\alpha(u, r = 0, x^A) = 0$. This can be implemented by the suitable choice $\alpha(u, r.x^A) = r\gamma(u, x^A)$, where $\gamma(u, x^A)$ is a smooth function in the neighborhood of the null surface. Hence the GNC metric in the vicinity of the null surface under affine parameterization is,

$$ds^2 = -2r^2\gamma du^2 + 2dudr - 2r\beta_A dudx^A + q_{AB}dx^A dx^B \; . \tag{1.67}$$

Under this choice, the structure of the NRE given via (1.65) becomes,

$$\partial_u \theta_l = -\Theta_{AB}\Theta^{AB} - R_{ab}l^a l^b \; . \tag{1.68}$$

Since, the null surface is integrable and the spacetime does not have torsion, the second fundamental form Θ_{AB} is completely spatial (orthogonal to both $\vec{l}$ and $\vec{k}$) and symmetric. This implies that the anti-symmetric vorticity tensor for the null surface is zero.

Here, we come to the first assumption by Jacobson. The null surface is assumed to be in equilibrium. This states that the expansion θ_l and the shear ${}^{(l)}\sigma_{ab}$ of the null surface is zero. Let us interpret this. We are given an initial configuration of the null surface $\mathcal{H}$ on which the expansion and the shear vanish. This null surface is then subjected to a matter flux crossing it. This matter flux perturbs the null surface along its null generators. That is to say the initial spacelike cross-section defined by $(r = 0)$ and $(u = 0)$ is now perturbed to another spacelike cross-section defined by $(r = 0)$ and $(u = \delta u)$ under the matter flux. However, as usual, we will assume that the perturbation is weak enough so that there does not form any caustics and the regular integrable structure of the null surface remains. Under this perturbation process, the initial expansion and shear of the horizon cross-section vanish (assumption), however their derivatives (along the null generators) do not. Under this assumption, we then have the following structure of the NRE,

$$\partial_u \theta_l = -R_{ab} l^a l^b \, . \tag{1.69}$$

We can integrate the above equation forward with the initial condition that at $u = 0$, the expansion scalar θ_l vanishes. This gives us (within the small perturbation regime),

$$\theta_l = -(R_{ab} l^a l^b) u \, . \tag{1.70}$$

The above solution ensures the analyticity in the expansion scalar avoiding the formation of caustics. We remind that the null generators in the GNC construction is future pointing and u hence increases towards the future. Let us then compute the variation δA of the cross-sectional area $\mathcal{J}$ of the null surface under the perturbation process. Using the fact that the expansion scalar is given by $\theta_l = \frac{1}{\sqrt{q}} \partial_u \sqrt{q}$, we have,

$$\begin{aligned} \delta A &= \int_{u=0}^{u=\delta u} du \frac{d}{du} \int_{\mathcal{J}} d^2x \sqrt{q} = \int_{u=0}^{u=\delta u} du \int_{\mathcal{J}} d^2x \sqrt{q}\, \theta_l \\ &= -\int_{u=0}^{u=\delta u} du\, u \int_{\mathcal{J}} d^2x \sqrt{q}\, R_{ab} l^a l^b \, . \end{aligned} \tag{1.71}$$

We would now require the second assumption by Jacobson. It is assumed that the variation of the entropy under the perturbation due to the matter flux across the null surface is proportional to the area change,

$$\delta S = \eta \delta A = -\eta \int_{u=0}^{u=\delta u} du\, u \int_{\mathcal{J}} d^2x \sqrt{q}\, R_{ab} l^a l^b \, . \tag{1.72}$$

We will now choose observers in the vicinity of the null surface, who on account of their motion will perceive a local patch of the null surface as their Rindler horizon. Such observers would be able to assign an acceleration temperature to the null surface. The acceleration of the Rindler observer is inversely proportional to the proper distance from the horizon. As such, we will consider highly accelerated observers such that they are very close to the null surface, effectively going to the infinite acceleration limit on the null surface. We will consider those observers moving along the field $\tilde{\chi}^i$ defined by,

$$\tilde{\chi}^i = (u, -r, 0, 0) \, , \tag{1.73}$$

in the affinely parameterized GNC metric (1.67). The unit normalized four-velocity χ^i of these observers is given by $\chi^i = N\tilde{\chi}^i$. We remind that the null geodesic generators of $\mathcal{H}$ in the GNC metric (1.67) is given by $\frac{dx^a}{du} = (1,0,0,0)$ with u being the affine parameter and $l^i\nabla_i l^j = 0$ w.r.t the metric (1.67). On the null surface, we have,

$$\lim_{r\to 0} \tilde{\chi}^i = u(1,0,0,0) = ul^i \,. \tag{1.74}$$

The acceleration of the observers moving along $\chi^i = N\tilde{\chi}^i$ is N and hence the Unruh temperature associated with such observers to the patch of the null surface is given by $T = N/2\pi$. The physical process involved in this construction is that the matter flux crossing across the horizon results in an entropy change of the null surface which is related to the area variation of $\mathcal{H}$ along its null generators. The heat energy δQ (1.55) due to matter flux associated with the energy momentum tensor $T^{(m)}_{ab}$ is given by,

$$dQ = -\int_{\mathcal{H}} d\Sigma^a T^{(m)}_{ab}\chi^b = N\int_{u=0}^{u=\delta u} du\, u\int_{\mathcal{J}} d^2x\sqrt{q}T^{(m)}_{ab}l^a l^b \,. \tag{1.75}$$

In the above, we have used the fact that the directed surface element on the null surface is given by $d\Sigma_a = -l_a du d^2x\sqrt{q}$ and that on $\mathcal{H}$, $\chi^a = Nul^a$ (w.r.t the affinely parameterized GNC metric).

The final assumption in Jacobson's analysis is to equate the heat flux δQ with $T\delta S$ *i.e.* the entropy variation of the null surface is precisely due to the matter flux,

$$N\int_{u=0}^{u=\delta u} du\, u\int_{\mathcal{J}} d^2x\sqrt{q}T^{(m)}_{ab}l^a l^b = -\frac{N}{2\pi}\eta\int_{u=0}^{u=\delta u} du\, u\int_{\mathcal{J}} d^2x\sqrt{q}\, R_{ab}l^a l^b \,. \tag{1.76}$$

This gives us,

$$R_{ab}l^a l^b = -\frac{2\pi}{\eta}T^{(m)}_{ab}l^a l^b \,. \tag{1.77}$$

which is precisely what Jacobson had obtained except for the minus sign. This could be interpreted as follows. If the null energy condition on $T^{(m)}_{ab}$ is satisfied, then the L.H.S of (1.76) is positive. However the entropy variation term on the R.H.S of (1.76) turns out to be a negative quantity. We had seen while dealing with our analysis of the PPFL, that if the matter flux into the horizon is positive, then the entropy variation associated with the horizon is also positive. The issue arises due to a subtle difference in the constructions of Jacobson and the GNC method. As mentioned in the beginning of this section, the (affinely parameterized) null geodesic generators of the local causal horizon is past pointing, whereas the (approximate) Killing vector field is future directed. However, in the (affine) GNC construction, both the time development vector field and the null generators are future pointing. If we take this factor into account and consider past pointing null generators of $\mathcal{H}$ in the affinely parametrized GNC construction, then we have got to change the directed surface element of $\mathcal{H}$ to $d\Sigma_a = l_a du d^2x\sqrt{q}$. This then gives us for the perturbation process,

$$N\int_{u=0}^{u=\delta u} du\, u \int_{\mathcal{J}} d^2x\sqrt{q}T^{(m)}_{ab}l^a l^b = \frac{N}{2\pi}\eta\int_{u=0}^{u=\delta u} du\, u \int_{\mathcal{J}} d^2x\sqrt{q}\, R_{ab}l^a l^b\,. \tag{1.78}$$

We finally have Jacobson's result,

$$R_{ab}l^a l^b = \frac{2\pi}{\eta}T^{(m)}_{ab}l^a l^b\,. \tag{1.79}$$

The above relation (1.79) implies,

$$R_{ab} + \psi g_{ab} = \frac{2\pi}{\eta}T^{(m)}_{ab}\,, \tag{1.80}$$

where ψ is an undetermined integration constant. Upon using the local conservation of the energy momentum tensor *i.e.* $\nabla^b T^{(m)}_{ab} = 0$ and the Bianchi identity $\nabla_b R^b{}_a = 1/2\nabla_a R$, we obtain by using the derivative operator ∇^b on both sides of (1.80),

$$\nabla_a\left(\frac{1}{2}R + \psi\right) = 0\,. \tag{1.81}$$

This gives us that $\psi = -1/2R + \Lambda$, where Λ is an arbitrary integration constant. Thus we obtain for the specific choice of $\eta = 1/4$, the Einstein field equation,

$$R_{ab} - \frac{1}{2}g_{ab}R - \Lambda g_{ab} = 8\pi T^{(m)}_{ab}\,. \tag{1.82}$$

In fact, Jacobson's analysis has been extended to the non-equilibrium setting [62] to obtain the field equations for general relativity or their higher order theories via a local thermodynamic constitutive relation. For such an analysis, a clear distinction needs to be made between the reversible and the irreversible parts of the entropy generation terms. Specific cases for $f(R)$ [63] and Einstein-Cartan gravity [64] have also been performed. There have been attempts to generalize Jacobson's procedure to higher curvature theories. Attempts have been made for more general diffeomorphism-invariant theories [65–68]. One major issue has been the fact that since the planar Rindler horizon corresponding to the (planar) boost vector $\vec{\xi} = x\vec{\partial}_t + t\vec{\partial}_x$ say, in the $(t-x)$ plane has no closed spacelike cross-sections, there exists extra boundary contributions to the variation of the entropy (associated to the horizon under the heat flux) due to the edges of the planar section of the Rindler horizon.

Recently, it has been shown [69], that both the Einstein field equations and the field equations for general diffeomorphism-invariant theories of gravity could be derived in

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the spirit of Jacobson's analysis by considering a *stretched future light cone*. The stretched future light cone is a timelike codimension 1 surface generated by a radial boost vector rather than a planar boost vector. This results in a spherical Rindler horizon associated to these radially accelerating observers. As such, even in the Minkowski spacetime itself, these radial boosts do not represent true isometries. They naturally fail to satisfy the Killing identities. The authors of [69] assume local holography for the stretched light cones. For Einstein gravity, the entropy associated with these stretched future light cones is the Bekenstein-Hawking entropy whereas for general diffeomorphism invariant theories of gravity, the entropy associated is the Wald entropy. The advantage of the spherical Rindler horizon is that they possess closed spherical cross-sections. The authors of [69] show that (in the computation for the variation in the entropy) most of the terms resulting from the failure of the Killing identities integrate to zero on the sphere. The remaining contribution precisely cancels out the change of the entropy that results from the natural expansion of the (approximate) spherical Rindler hyperboloid. What remains after all such cancellations is what the authors in [69] call the *reversible* part of the entropy variation. Employing the Clausius identity and equating this reversible change ($T\Delta S_{\text{rev}}$) to the heat flux associated to the stress-energy tensor, the authors of [69] have been able to obtain both the Einstein field equations as well as those for general diffeomorphism invariant theories of gravity.

Before concluding this section, let us talk about an alternative to the perturbation process. Instead of physically dropping matter across the horizon, one can think of a *virtual displacement* of the horizon along its null generators. Such a virtual displacement will engulf matter from the vicinity of the horizon thus decreasing the entropy (for outside observers). However, such a virtual displacement should be infinitesimal since the information and hence the entropy of the matter is not lost till it gets eaten up by the horizon. This engulfing process should be of the matter infinitesimally close by the horizon. This is because observers will not account for any loss of entropy till the moment the matter stays in the outside region of the horizon and is 'observed' by the observers. Once the matter infinitesimally close (to the horizon) gets engulfed by the horizon under the virtual displacement process, the observer will account for a loss of entropy. Hence, in order to preserve entropy balance, the observer must assign an entropy increase to the horizon under this virtual displacement process. This interpretation will now allow us to transition to Padmanabhan's analysis of regarding the gravitational field equations as a thermodynamic identity. Such interpretation would be brought about by a virtual displacement of $\mathcal{H}$; but along a different direction.

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1.6 Padmanabhan's thermodynamic interpretation of gravitational field equations

The analysis by Padmanabhan [36] is in the opposite direction to that of Jacobson. For the case of general relativity, it was shown by Padmanabhan that the Einstein field equations have a thermodynamic interpretation when projected onto a null surface. Initially the result was shown for the case of static and spherically symmetric [70–74] black hole event horizons. For such symmetric cases of the horizon the ${T^{(m)r}}_r$ component of the stress energy tensor has the interpretation of being the radial transverse pressure P across the horizon [74, 75]. It was shown that a certain component of the Einstein field equations of the form ${G^r}_r = 8\pi {T^{(m)r}}_r$ for a virtual displacement of the horizon in the radial direction assumed a structure of the form $T\delta_\lambda S = \delta_\lambda E + P\delta_\lambda V$. The variations in the quantities are for an affine parameter λ. Padmanabhan was later able to generalize this result (valid for static and spherically symmetric spacetimes) to any arbitrary null surface in a given spacetime without imposing any additional symmetries [36]. We will now analyze Padmanabhan's general result again via the GNC system.

Considering the vector field ${G^a}_b\xi^b$, Padmanabhan took the other projection component $G_{ab}k^a l^b$ on the null surface. As mention in [36], this component is more suitable for the thermodynamic interpretation since it represents the component of the vector field ${G^a}_b l^b$ along the null surface. Hence, inherently, $G_{ab}k^a l^b$ is a better suited projection component. Using the Einstein field equations, we have on the null surface,

$$G_{ab}k^a l^b = {G^r}_r = 8\pi T^{(m)}_{ab} k^a l^b = -8\pi {T^{(m)u}}_u = -8\pi {T^{(m)r}}_r \,. \tag{1.83}$$

It can be shown in [36] that the component ${G^r}_r$ in the GNC system is,

$${G^r}_r = \alpha\partial_r \ln\sqrt{q} - \frac{1}{2}\,{}^2R + \frac{1}{\sqrt{q}}\partial_r\partial_u\sqrt{q} + \frac{1}{4}\beta_A\beta^A + \frac{1}{2\sqrt{q}}\partial_A(\sqrt{q}\beta^A)\,, \tag{1.84}$$

where ${}^2R$ is the 2-dimensional Ricci scalar of the spacelike cross-section $r = 0$ and $u =$ constant. Now using the fact that the expansion scalar corresponding to the auxiliary null vector field is given by $\theta_k = -\partial_r \ln\sqrt{q}$, we have,

$$\frac{1}{\sqrt{q}}\partial_r\partial_u\sqrt{q} = -\partial_u\theta_l - \theta_l\theta_k\,. \tag{1.85}$$

Using the above relation (1.85) in (1.84), we have,

$$\partial_u\theta_k = -\alpha\theta_k - \frac{1}{2}\,{}^2R - \theta_l\theta_k + \frac{1}{4}\beta_A\beta^A + \frac{1}{2\sqrt{q}}\partial_A(\sqrt{q}\beta^A) - {G^r}_r\,. \tag{1.86}$$

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Since $\partial_u \theta_k$ quantifies the evolution rate of the expansion scalar (for $\vec{k}$) along the null generators parameterized by u, the relation (1.86) is a special case of the cross-focusing equations [76] derived in the Gaussian null coordinates. In a later chapter, we will derive a covariant version of this evolution equation and interpret it as the null Raychaudhuri equation for the ingoing expansion scalar θ_k.

We will subject the null hypersurface to the process of virtual displacement along the auxiliary null vector field $\vec{k}$. This provides an infinitesimal displacement that perturbs the cross-section of the horizon initially at $r = 0$ to $r = \delta r$. Coupled with the Einstein field equation, the relation (1.84) under this virtual displacement process takes the structure of,

$$\int_{\mathcal{J}} d^2x \, T\delta_r s = \delta_r E + F\delta r \, . \tag{1.87}$$

Here, $\delta_r s$ is the variation of the Bekenstein-Hawking entropy density as the null hypersurface is perturbed along $\vec{k}$. The temperature associated to the null surface is $T = \alpha(u, r = 0, x^A)/2\pi$. The variation of the energy under this process is $\delta_r E$. The work done under the virtual displacement is $F\delta r$, where F is the integral of $T^{(m)}_{ab} k^a l^b$ over the null surface. Initially, the generic thermodynamic identity (1.87) was provided for spacetimes with a high degree of symmetry [71, 73] which allowed the thermodynamic interpretation as,

$$T\delta_r S = \delta_r E + F\delta r \, . \tag{1.88}$$

The result (1.87) was also generalized to Lanczos-Lovelock theories of gravity [77–79]. We not go further here into the details and interpretation of the results since we will revisit this entire program in chapter 4. This is because, the GNC construction makes the thermodynamic interpretation dependent on a specific choice of coordinate system adapted to the null surface. In chapter 4, we will provide a coordinate-independent generalization of the thermodynamic interpretation of the gravitational field equations under the virtual displacement. Our analysis will remove the requirement of some restrictions that are inherent in the GNC construction. There, we will make the equivalence between our coordinate-independent interpretation and the one achieved via the GNC construction.

Before we end this section, let us reiterate some structural differences between Jacobson's and Padmanabhan's approach. For Jacobson, the matter flux perturbs the null surface along the null generators or equivalently, the virtual displacement occurs along $\vec{l}$. The entropy variation occurs because the cross-sectional area changes along the null generators. For this, the relevant projection component to use is $G_{ab} l^a l^b$ and hence the matter flux component is $T^{(m)}_{ab} l^a l^b$. For Padmanabhan, the virtual displacement is along $\vec{k}$. The entropy change occurs due to the variation of the null surface along the transverse field $\vec{k}$. The relevant projection component to use is $G_{ab} k^a l^b$. However, it has been claimed

[36] that the more natural projection component to use to bring the thermodynamic interpretation is $G_{ab}k^a l^b$. This is evident from from the approach of Padmanabhan, who does not use any restriction of equilibrium condition on the null surface to bring about the thermodynamic interpretation.

1.7 Fluid interpretation of the field equations: ála Damour

Let us finally come to the other spatial projection $G^a{}_b q_{ac} l^b$ (onto the closed cross-section $\mathcal{J}$) of the vector field $G^a{}_b \xi^b$ on the null surface $\mathcal{H}$. This component is related to the dynamics of the Hájiček 1-form [61, 80] associated with the null surface when the gravitational field equations are used. In fact, it was shown by Damour [37, 81] that when the Einstein field equations are projected onto the black hole event horizon via the component $G_{ab}l^a q^b{}_c$, the resulting dynamics look very similar to the Navier-Stokes equation. This result can be generalized to any arbitrary null surface in the spacetime. The dynamics of the Háȷ́iček 1-form when viewed through a suitable coordinate system adapted to the null surface, gives rise to the Damour-Navier-Stokes equation. It was later shown by Padmanabhan [82] that when the Damour-Navier-Stokes dynamics is viewed with respect an inertial reference frame, it looks exactly like a Navier-Stokes fluid. This shows that for an outside observer, a black hole horizon, or for that matter any null surface (whose local patch can be approximated by a Rindler horizon) behaves like a viscous fluid membrane. This forms the basis of the membrane paradigm [83] which establishes a connection between black hole dynamics and non-relativistic mechanical fluids. The membrane paradigm considers a stretched membrane with a fictitious fluid residing on it. The dynamics of the black hole horizon perturbed by external fields mirrors the response of this fictitious fluid under interaction with such fields. The viscous fluid then possesses a stress-tensor that sources the gravitational field. For the event horizon in general relativity, the shear viscosity for this membrane fluid is $\eta = 1/16\pi$. Dividing it with the Bekenstein-Hawking entropy density $s = 1/4$, we obtain a dimensionless number $\eta/s = 1/4\pi$. To arrive at this result in the membrane paradigm, only the knowledge of the dynamics of the horizon and its associated thermodynamic quantities is sufficient. However, the same ratio was also obtained under string theoretic considerations [84–86]. It was suggested that this ratio, called the KSS bound is the universal lower bound for all materials.

We will not discuss further the fluid interpretation in this section. This is because we will be devoted to this question largely in chapter 7, however for Einstein-Cartan gravity [87]. For the case of general relativity the interested reader may look at [61, 80, 82].

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1.8 Objective and chapter-wise outline of the thesis

It is the objective of this thesis to explore further these connections between gravitational dynamics and thermodynamics and fluid behavior. In fact, the thermodynamic identity attested to the gravitational field equations in the works of Padmanabhan has been explicitly dependent on a choice of coordinate system adapted to the generic null surface or for spacetimes with a high degree of symmetry. As a result of this, the thermodynamic entities are coordinate dependent. We will try to attest a *covariant* thermodynamic interpretation to the relevant gravitational field equations with respect to the generic integrable null surface for any gravitational theory formulated on spacetimes equipped with the unique Levi-Civita connection. If the emergent paradigm of gravity is indeed correct, then it should transcend to theories beyond Einstein gravity. It is in this respect, that we also consider Einstein-Cartan (EC) gravity [87, 88] and study in detail whether the relevant gravitational field equations lend themselves a thermodynamic and fluid interpretation.

The entire thesis consists of two parts excluding introduction and conclusion. The first chapter introduces and motivates some well known results of black hole physics. We also briefly discussed the physical interpretations brought to the gravitational field equations in the context of Einstein gravity following the works of Jacobson, Padmanabhan and Damour. We wrap off this chapter by briefly introducing the reader to the emergent paradigm of gravity. We then head over to part I of this thesis which includes two chapters. Part I of the thesis is devoted in its entirety to a detailed study and exploration of the geometry, both intrinsic and extrinsic, of a generic integrable null surface in the Riemann-Cartan (RC) spacetime. The RC spacetime is a Lorentzian manifold provided with a metric-compatible connection that is independent of the metric and hence supports non-trivial torsion in it. We develop in depth various geometric/kinematical quantities associated with the null surface in the spacetimes with torsion as these were absent in the literature . We also study the relevant evolution laws of these kinematical quantities since they would be indispensable in our goal of attesting the gravitational field equations the required thermodynamic and fluid interpretation. As a result part I forms the geometrical background of this thesis, that introduces the relevant geometrical and mathematical tools. Part II of the thesis is where we put in the physics. This part deals completely in analyzing the relevant evolution equations with respect to the chosen gravity theory. In part II, chapter 4 deals with attesting a covariant thermodynamic interpretation to the gravitational field equations for any gravity theory formulated on spacetimes provided with the Levi-Civita connection. Chapter 5 is devoted to elucidating the physical (in)equivalences between the Einstein and Jordan frame for scalar tensor (ST) theory of gravity on the basis of this covariant thermodynamic interpretation. Chapters 6 and 7 are devoted completely to establishing the thermodynamic and fluid interpretation

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to the field equations of Einstein-Cartan gravity. We present a brief conclusion and future directions for our findings in chapter 8.

The contents of this thesis is based on the works [89–92]. We summarize the contents of the following chapters for the convenience of the reader.

Chapter 2:

Following the objective of this thesis, we begin with its central tool: a generic integrable null hypersurface $\mathcal{H}$. Since we aim to explore the thermodynamic and fluid interpretations of the gravitational field equations to theories beyond general relativity, with a specific eye to Einstein-Cartan gravity, we begin with the in-depth construction of a null hypersurface in the Riemann-Cartan spacetime. The RC spacetime forms the geometric backdrop for the EC gravity theory, where the metric and the connection (metric-compatible) are treated as independent geometric entities. We begin this chapter with a brief introduction to the geometrical properties of the RC spacetime with non-trivial torsion present in it. We then describe in detail the geometric construction of an integrable null hypersurface in the RC spacetime. Hypersurface orthogonality leads us to the fact that the null generators forming the null surface in the RC spacetime are null geodesics; yet they are not auto-parallel. To specify the null surface, we describe in detail both its *intrinsic* and *extrinsic* geometry. Following a $1+3$ decomposition of the null surface, we allow for the construction of a unique transverse auxiliary null vector field. This allows us to define a unique projection tensor or induced metric (which encodes the intrinsic geometry) onto a spacelike submanifold of our null surface. For the extrinsic geometry, we describe in detail, the construction of the Weingarten map (the shape operator) and the second fundamental form restricted to the null surface. Viewing the null surface as an embedded submanifold in the ambient RC spacetime, we then describe various kinematical quantities associated with $\mathcal{H}$. Among them, we describe in detail, the extended Weingarten map and the extended second fundamental form. We follow this up with the description of the rotation and the Hajicek 1-forms. We also describe in detail notions of the deformation rate tensor, the transversal deformation rate tensor and the projected deviation tensor. These kinematical quantities form a part of the geometric data on the integrable surface and quantity its extrinsic geometry. We will be interested in the evolution dynamics of these geometric data for our purposes of providing physical interpretation to the gravitational field equations.

Chapter 3:

In this chapter, we focus exclusively on the dynamical evolution of some of the kinematical quantities introduced in the previous chapter. We will consider the evolution dynamics along the null generators of our integrable null surface. This is because, in the geometrical construction of $\mathcal{H}$, the null generators are related to the time evolution vector field. We begin with the evolution of the *outgoing* expansion scalar corresponding to the null generators of $\mathcal{H}$ in the RC spacetime. In this general analysis, we end up with the generalization of the null Raychaudhuri equation corresponding to the outgoing expansion scalar in the RC spacetime. Next, we analyze the evolution dynamics of the *ingoing* expansion scalar corresponding to the transverse auxiliary null vector field. However, such an analysis has been done under the *geodesic constraint*. The geodesic constraint forces the null generators to be both geodesics as well as auto-parallel null curves in the RC spacetime. We call the resulting dynamics as the NRE corresponding to the ingoing expansion scalar. Finally, we look at the evolution (along the null generators) of the Hajicek 1-form. This analysis yields for us the generalization to the Hajicek equation for the null surface in the RC spacetime. The evolution equations so obtained are completely geometric in structure, in the sense that the gravitational field equations have not been used. It is only when we encode the dynamics of the relevant theory of gravity into these evolution equations can we interpret them of having thermodynamic and fluid interpretation. This will be the objective in the later chapters. The resulting evolution equations corresponding to the above three cases for spacetimes equipped with the usual Levi-Civita connection have also been studied by setting the torsion tensor to zero.

Chapter 4:

Having laid out the evolution equations of relevant kinematical quantities (of $\mathcal{H}$) we focus exclusively in this chapter, on spacetimes provided with the unique Levi-Civita connection and hence gravity theories constructed in such spacetimes thereof. We begin with the NRE corresponding to the ingoing expansion scalar. Such an evolution equation was a crucial input by Jacobson, when he "derived" the Einstein field equations from the equilibrium Clausius identity applied to local Rindler horizons. Here, we interpret the gravitational dynamics corresponding to the NRE under a certain *virtual displacement* process. The virtual displacement in this case perturbs the null surface along the null generators. This allows us to interpret the gravitational dynamics corresponding to the NRE of the outgoing expansion scalar as a thermodynamic identity in a covariant fashion. It is noticed that in the special case of a stationary hole hole system, the expression of the energy is related to the well known Komar energy. We also see from this analysis that the integrated form of the resulting thermodynamic identity leads to a generalized form of the Smarr formula. Next, we analyze the gravitational dynamics corresponding to the NRE of the ingoing expansion scalar defined with respect to the auxiliary null vector field. The dynamics has been studied under the process of a virtual displacement which perturbs the integrable null surface along the auxiliary null vector field. This yields a covariant interpretation of the resulting gravitational dynamics. Contrary to earlier approaches, all the relevant thermodynamic quantities turn out to covariant and foliation-independent. As such the results of our analysis can be implemented on any coordinate system adapted to the null surface. Moreover, our thermodynamic analysis is independent of the gravitational theory in the sense that only requirement is that the gravitational theory needs to described on a spacetime equipped with the Levi-Civita connection. The equivalence of our covariant thermodynamic interpretation with previous coordinate dependent thermodynamic interpretation to the resulting gravitational dynamics in the specific case of Einstein gravity has also been established.

Chapter 5:

The issue of physical equivalences or in-equivalences between the Einstein and Jordan frame in the scalar-tensor theory of gravity has been a long standing issue as of such. In this chapter, we have tried to throw some light on this issue by studying the thermodynamic interpretation of the gravitational field equations via the NRE (corresponding to the ingoing expansion scalar) in both the respective frames under the physical process of virtual displacement. We show that field equations of gravity acquire a form quite identical in structure to the first law of thermodynamics when projected (in a certain way) on a generic null surface constructed in the scalar tensor theory. We then show, explicitly the equivalences of thermodynamic quantities like temperature, entropy, energy and work done in the Einstein and Jordan frame. This is consistent with previous findings of a Killing horizon. The entire analysis is based on a covariant geometrical construction of the general null hypersurface. We also provide a concrete proof of the zeroth law in the scalar-tensor theory for the null surface generated by a Killing vector field.

Chapter 6:

If this *emergent paradigm of gravity* is indeed correct, then it should hopefully transcend to theories beyond Einstein gravity. It is in this respect that we consider the case of Einstein-Cartan gravity. In this chapter we again investigate whether the gravitational field equations in EC theory lend themselves a thermodynamic interpretation. We hence begin this chapter with a brief review of the gravitational field equations for the EC theory. Considering a general integrable null surface in the RC spacetime, we use the evolution dynamics of the ingoing expansion scalar corresponding to the auxiliary null vector field under the *geodesic constraint*. We then consider the physical process of virtual displacement that perturbs the null surface along the auxiliary null field. Employing the EC gravity field equations to the resulting dynamical evolution law (for the ingoing expansion scalar) allows us to interpret it in a form analogous to the first law of thermodynamics. As opposed to the Einstein gravity case, we observe here that the variation of the entropy density under the virtual displacement process has a non-zero contribution due to non-trivial torsion current flowing along the null generators. We also see that the energy and work function terms are suitably modified by the inclusion of torsion in the gravity theory. We also consider the case when the torsion tensor is completely antisymmetric under which the geodesic constraint is trivially satisfied. We also discuss the notion of *equilibrium* for the null surface under this virtual displacement process for the EC gravity case.

Chapter 7:

In this chapter, we concern ourselves with providing a fluid-dynamic interpretation to the Einstein-Cartan gravitational field equations. To this effect, we consider the evolution dynamics of the Hajicek 1-form of an integrable null surface in the EC theory. Employing the relevant field equations and the *geodesic constraint*, we propose a possible fluid interpretation of this evolution equation by connecting it to the *Cosserat* generalization of the the Navier-Stokes fluid. This has been done by analyzing the dynamics of the Hajicek 1-form in a set of coordinates adapted to the null surface and then going to a local inertial frame. This shows that the horizon or the null surface also in the case of EC theory behaves as a viscous membrane whose dynamics are synonymous to that a null Cosserat fluid. An analogous viewpoint can also be built under the motive that the usual material derivative for fluids should be replaced by the Lie derivative. Finally, we also derive the tidal force equation for the null surface dynamics in the Einstein-Cartan theory.

Chapter 8:

We dedicate this final chapter of our thesis for the possible extensions of the results discussed in the thesis as well as a list of new problems that could be explored in the future. We have also highlighted several important conclusions related to our work.

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Part I

A generic null surface: The geometrical playground

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Chapter 2

A generic null hypersurface: Geometry and kinematics

2.1 Introduction and motivation

We saw that the notion of inescapability requires the notion of a *boundary*. A boundary obviously is a one-way membrane. Spacelike and null surfaces are one-way membranes. It turns out that the black-hole boundary, also called the *event horizon* is a codimension one null surface [2, 93–95]. Hence in order to answer questions about black holes the study of null surfaces becomes essential.

Null surfaces are ubiquitous in the study of general relativity. They arise not just only for global event horizons, but also *quasi-local* definitions of horizons [96–102]. They are essential ingredients in the study of causality and the singularity theorems [3, 103–110]. Null surfaces hold a very special status in the arena of spacetime both in terms of its rich mathematical and physical interpretations. The gravitational field equations have distinguished meanings on the null surface. In essence, both the constraint and the evolution set of field equations on the null surface give rise to very important physical interpretations. The study of null surfaces is essential in the context of black hole thermodynamics as well as providing thermodynamic status to the gravitational field equations [4–8, 11, 14, 28, 35, 36, 41, 45, 46, 59, 71, 72, 111–124]. The thermodynamic description of gravity is not just limited to the special case of event horizons, but rather for any arbitrary null hypersurface in the spacetime. Thermodynamics essentially points to the statistical mechanics of "microstates" residing on the null surface. Hence for possible routes to quantization of gravity, null surfaces become important.

A partial aim of the thesis is to provide a self-contained description of the geometry of a general integrable null surface in the Riemann-Cartan spacetime. The complete geometrical description of null surfaces in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ provided with the Levi-Civita connection $\boldsymbol{\nabla}$ has already been presented in detail [61, 125]. The RC spacetime is a particular example of non-Riemannian spacetime. The use of non-Riemannian spacetimes become important while describing test particles with spin (intrinsic degree

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of freedom). The spin can couple to the non-Riemannian parameters of the spacetime. Gauge theories of gravity require the use of non-Riemannian geometries while gauging the spacetime symmetries [126, 127]. The RC spacetime arises while localizing or gauging the Poincaré symmetry which leads to the Poincaré gauge theory of gravity (PGT) [126, 128, 129]. The simplest of the PGT is the Einstein-Cartan spacetime which is constructed on the arena of the RC spacetime.

Let us briefly review the contents of this chapter. We begin with a brief review of the RC spacetime and lay out its necessary geometrical properties in Sec. 2.2. Next, in Sec. (2.3) we begin with the geometric construction of a generic integrable null hypersurface $\mathcal{H}$ in the RC spacetime. Here, we describe in detail both the intrinsic and the extrinsic geometry of our null surface. We then proceed to describe in Sec. (2.4) various kinematical quantities of $\mathcal{H}$ that will be of much relevance to us in our analysis. They are the extended Weingarten map, the extended second fundamental form, the rotation and the Hájiček 1-forms, the deformation and the transversal deformation rate tensors and the projected deviation tensor. Various of these kinematical quantities will have much relevance and specific roles in the fluid and thermodynamic interpretation of the gravitational field equations. Finally we conclude this chapter with a very brief discussion about a few coordinate systems that can be adapted to a general null surface in Sec. 2.5.

2.2 The Riemann-Cartan spacetime: The requisite geometric background

Keeping the future in mind, we would like to consider the construction of a general integrable null hypersurface in the RC spacetime. The RC spacetime is a generalization of the Riemannian spacetime in the sense that it allows for torsion to be present in the spacetime. The entire geometric construction of a null hypersurface in the Riemannian spacetime is already present in the literature [61]. However, such an analysis has been absent for spacetimes having a non-trivial torsion present in it. It is an objective of this thesis to fill this present gap and study in depth the resulting kinematics and dynamics of a null hypersurface in the RC spacetime.

The RC spacetime manifold $\mathcal{M}$, endowed with the metric g and designated as $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ is provided with a metric-compatible affine connection $\hat{\boldsymbol{\nabla}}$,

$$\hat{\nabla}_a g_{bc} = 0\,. \tag{2.1}$$

2.2.1 Geometrical properties of $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$:

Let us now very briefly review the geometrical properties of such a spacetime. For details refer to [87]. Just to set the convention straight, we define the covariant derivative of a

(r,s) rank tensor $T^{a_1\cdots a_r}{}_{b_1\cdots b_s}$ to be,

$$\hat{\nabla}_a T^{a_1\cdots a_r}{}_{b_1\cdots b_s} \equiv \partial_a T^{a_1\cdots a_r}{}_{b_1\cdots b_s} + \hat{\Gamma}^{a_1}{}_{ai_1} T^{i_1\cdots a_r}{}_{b_1\cdots b_s} + \cdots + \hat{\Gamma}^{a_r}{}_{ai_r} T^{a_1\cdots i_r}{}_{b_1\cdots b_s} - \hat{\Gamma}^{j_1}{}_{ab_1} T^{a_1\cdots a_r}{}_{j_1\cdots b_s} \cdots - \hat{\Gamma}^{j_s}{}_{ab_s} T^{a_1\cdots a_r}{}_{b_1\cdots j_s}\,, \tag{2.2}$$

where notice that the differentiating index "a" sits at the first position in the subscript indices of affine connections. We will follow this convention in the subsequent analysis. The torsion tensor is basically defined as,

$$T^a{}_{bc} \equiv \hat{\Gamma}^a{}_{bc} - \hat{\Gamma}^a{}_{cb}\,, \tag{2.3}$$

implying that the torsion tensor is antisymmetric in the last two indices. This general metric-compatible affine connection coefficient is related to the symmetrical Levi-Civita connection coefficient (the Christoffel symbol) $\Gamma^a{}_{bc}$ via the contorsion tensor $K^a{}_{bc}$,

$$\hat{\Gamma}^a{}_{bc} = \Gamma^a_{bc} + K^a{}_{bc}\,. \tag{2.4}$$

The contorsion tensor can be expressed in terms of the torsion tensor as,

$$K^a{}_{bc} = \frac{1}{2}\left(T^a{}_{bc} + T_b{}^a{}_c + T_c{}^a{}_b\right). \tag{2.5}$$

The above relation can be obtained by the familiar trick of setting $(-\hat{\nabla}_a g_{bc} + \hat{\nabla}_b g_{ca} + \hat{\nabla}_c g_{ab}) = 0$ via (2.1) and then using (2.2). It is quite easy to verify (according to the convention followed in (2.3)) that the contorsion tensor K_{abc} is antisymmetric in the first and last indices. Here, we use the metric tensor g_{ab} to raise and lower all spacetime indices of a tensorial quantity. Let us now look at the trace of the torsion tensor defined via contracting the first and the third index. Following properties can then be easily deduced from the definition,

$$T_b \equiv g^{ac} T_{abc} = T^a{}_{ba} = -T^a{}_{ab}\,;$$
$$K^a{}_{ab} = -T_b; \quad K^a{}_{ba} = 0; \quad K_b{}^a{}_a = T_b\,. \tag{2.6}$$

Another quantity of interest that comes into play is the modified torsion tensor $S^a{}_{bc}$ defined as,

$$S^a{}_{bc} \equiv T^a{}_{bc} + \delta^a{}_b T_c - \delta^a{}_c T_b\,, \tag{2.7}$$

which like the torsion tensor is antisymmetric in the last two indices.

In general, the existence of torsion in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ can be characterized by the fact that the action of the commutator of the covariant derivatives $\hat{\boldsymbol{\nabla}}$ on any scalar

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field does not vanish,

$$[\hat{\nabla}_a, \hat{\nabla}_b]\Phi = -T^d_{ab}(\hat{\nabla}_d\Phi) \,. \tag{2.8}$$

The corresponding action on contravariant and covariant vectors are summarized below,

$$[\hat{\nabla}_a, \hat{\nabla}_b]A^i = \hat{R}^i{}_{kab}A^k - T^d_{ab}(\hat{\nabla}_d A^i) \,; \tag{2.9}$$
$$[\hat{\nabla}_a, \hat{\nabla}_b]\omega_c = -\hat{R}^d{}_{cab}\omega_d - T^d_{ab}(\hat{\nabla}_d\omega_c) \,. \tag{2.10}$$

The Riemann tensor in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ follows the usual definition in terms of the affine connection $\hat{\Gamma}^a{}_{bc}$ and as per our convention is,

$$\hat{R}^a{}_{bcd} \equiv \partial_c\hat{\Gamma}^a{}_{db} - \partial_d\hat{\Gamma}^a{}_{cb} + \hat{\Gamma}^a{}_{ci}\hat{\Gamma}^i{}_{db} - \hat{\Gamma}^a{}_{di}\hat{\Gamma}^i{}_{cb} \,. \tag{2.11}$$

The following symmetries of the Riemann tensor are then quite evident,

$$\hat{R}_{abcd} = -\hat{R}_{abdc} \quad \text{and} \quad \hat{R}_{abcd} = -\hat{R}_{bacd} \,. \tag{2.12}$$

However, the usual symmetry under pairwise exchange of the indices does not follow through over here, as $\hat{R}_{cdab} = \hat{R}_{abcd} + \hat{\mathcal{Q}}_{abcd}$ where,

$$\begin{aligned}\hat{\mathcal{Q}}_{abcd} = -\frac{3}{2}\Big(&\hat{\nabla}_{[b}T_{|a|cd]} - \hat{\nabla}_{[a}T_{|b|cd]} - \hat{\nabla}_{[d}T_{|c|ab]} + \hat{\nabla}_{[c}T_{|d|ab]} \\ &+ T_{ae[b}T^e{}_{cd]} - T_{be[a}T^e{}_{cd]} - T_{ce[d}T^e{}_{ab]} + T_{de[c}T^e{}_{ab]}\Big) \,. \end{aligned} \tag{2.13}$$

In the last equation, $||$ indicates the enclosed index barred from antisymmetrization. Similarly, the usual first and the second Bianchi identities do not follow:

$$\hat{R}^d{}_{[cab]} = \hat{\nabla}_{[a}T^d{}_{bc]} - T^f{}_{[ab}T^d{}_{c]f} \,; \tag{2.14}$$
$$\hat{\nabla}_{[a}\hat{R}^f{}_{|d|bc]} = -T^k{}_{[ab}\hat{R}^f{}_{|dk|c]} \,. \tag{2.15}$$

Here, we have used the convention that,

$$\begin{aligned} A_{[i_1\cdots i_n]} &= \frac{1}{n!}\sum_\sigma(-1)^{\epsilon_\sigma}A_{i_{\sigma(1)}\cdots i_{\sigma(n)}} \,, \\ A_{(i_1\cdots i_n)} &= \frac{1}{n!}\sum_\sigma A_{i_{\sigma(1)}\cdots i_{\sigma(n)}} \,, \end{aligned} \tag{2.16}$$

where the summation is over all possible permutations $\{\sigma\}$ of the set $\{1, 2, \cdots, n\}$ and

$$\begin{aligned} \epsilon_\sigma &= 0, \quad \text{when } \sigma \text{ is an even permutation of } \{1, 2, \cdots, n\} \\ \epsilon_\sigma &= 1, \quad \text{when } \sigma \text{ is an odd permutation of } \{1, 2, \cdots, n\} \,. \end{aligned} \tag{2.17}$$

The Ricci tensor is no longer symmetric owing to the presence of torsion,

$$\hat{R}_{[ab]} = -\frac{1}{2}(\hat{\nabla}_i + T_i)S^i{}_{ab} \,. \tag{2.18}$$

In analogy with the Einstein tensor of the usual Riemannian geometry, we introduce the tensor $\hat{G}^a{}_b \equiv \hat{R}^a{}_b - \frac{1}{2}\delta^a{}_b\hat{R}$ in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. As anticipated, due to the presence of torsion, the tensor $\hat{G}^a{}_b$ fails to be divergenceless,

$$\hat{\nabla}_a \hat{G}^a{}_b = -T^k{}_{ab}\hat{R}^a{}_k + \frac{1}{2}T^{kad}\hat{R}_{adkb} \,. \tag{2.19}$$

The Lie derivative of the metric tensor of $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ along a given vector field $\boldsymbol{v}$ is,

$$\pounds_v g_{ab} = \hat{\nabla}_a v_b + \hat{\nabla}_b v_a + (T_{acb} + T_{bca})v^c \,. \tag{2.20}$$

Now, we consider the decomposition of curvature tensor $\hat{R}_{abcd}$ in the RC spacetime into a Riemannian part (R_{abcd}) and an extra part that involves the torsion contributions. It can be shown that,

$$\hat{R}^a{}_{bcd} = R^a{}_{bcd} + (\hat{\nabla}_c K^a{}_{db} - \hat{\nabla}_d K^a{}_{cb}) + T^i{}_{cd}K^a{}_{ib} + (K^i{}_{cb}K^a{}_{di} - K^i{}_{db}K^a{}_{ci}) \,. \tag{2.21}$$

Similarly, taking the contraction to yield the Ricci tensor, it can be easily verified that,

$$\hat{R}_{ab} = R_{ab} + \hat{\nabla}_i K^i{}_{ba} + \hat{\nabla}_b T_a + T^i{}_{jb}K^j{}_{ia} + K^i{}_{ja}K^j{}_{bi} + T_i K^i{}_{ba} \,. \tag{2.22}$$

Let us just finally remind ourselves that all the above mentioned results valid in the RC spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ go over to usual familiar results in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ provided the Levi-Civita connection $\boldsymbol{\nabla}$ once the torsion tensor has been set to zero. Again, for details of these discussions, see [87].

2.3 Geometric construction of the null surface in the RC spacetime

In this introduction to the structure of a generic null hypersurface in the RC spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ and its associated kinematics and dynamics (to be introduced in the next chapter), we will stick to the notations and formalism introduced in [61], which provided the formulation for the Riemannian spacetime. In fact part of our objective is to see what modifications do the kinematics and dynamics of a general null surface incur provided our ambient spacetime has non-trivial torsion present in it. It is to this respect that we adopt the formalism introduced in [61] and follow the notions. Even though we will

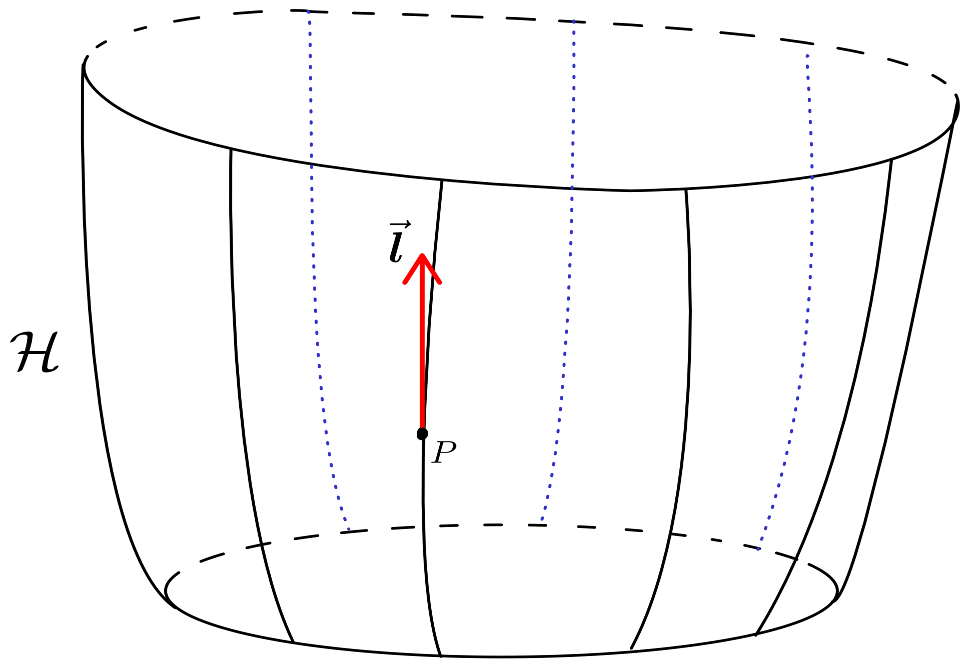

FIGURE 2.1: A null hypersurface $\mathcal{H}$ of codimension 1 generated by the null vector $\vec{l}$ (shown in red).

work in $d = 4$ spacetime dimensions, generalizations to higher spacetime dimensions follows quite naturally.

A null integrable hypersurface $\mathcal{H}$ is basically a surface of codimension one such that the induced metric $\boldsymbol{q}$ on it is degenerate. The null hypersurface is characterized by its induced metric and the extrinsic curvature (viewed as an embedding in the ambient spacetime). The fact that the induced metric on the null surface in degenerate means the existence of a vector field $\vec{l}$ in the tangent bundle $\mathcal{T}(\mathcal{H})$ of $\mathcal{H}$ that is orthogonal to any vector field defined on $\mathcal{H}$. The induced metric $\boldsymbol{q}$ is $(0,2)$ rank tensor defined on the cotangent space $\mathcal{T}^*_P(\mathcal{H})$ at any point P of $\mathcal{H}$ *i.e.* $\boldsymbol{q} \in \mathcal{T}^*_P(\mathcal{H}) \otimes \mathcal{T}^*_P(\mathcal{H})$. For any vector $\vec{v}$ lying in the tangent space of $\mathcal{H}$, we have,

$$\forall \vec{v} \in \mathcal{T}_P(\mathcal{H}), \qquad \boldsymbol{q}(\vec{l},\vec{v}) = 0\,. \tag{2.23}$$

The vector field $\vec{l}$ is null w.r.t the ambient spacetime metric $\boldsymbol{g}$ *i.e.* $\boldsymbol{g}(\vec{l},\vec{l}) = \vec{l}\cdot\vec{l} = 0$. Both the intrinsic and the extrinsic properties of the null surface $\mathcal{H}$ need to be defined properly in the RC spacetime.

We consider the existence of a generic null hypersurface $\mathcal{H}$ in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, defined via the scalar field $u(x^a) = 0$. The surface $\mathcal{H}$ is integrable into a hypersurface-orthogonal null surface such that its null normal 1-form $\underline{\boldsymbol{l}}$ is given by,

$$l_a = -e^{\rho}\partial_a u = -e^{\rho}\hat{\nabla}_a u\,, \tag{2.24}$$

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where ρ is some smooth scalar field on $\mathcal{H}$. The co-efficient e^{ρ} that relates the null normal l_a with the gradient of the scalar field $\partial_a u$ is chosen to be negative such that the null normal vector $\vec{l}$ is future pointing. This can be done by a suitable choice of the scalar field $u(x^a)$. Notice that the null normal cannot be provided a unique normalization on account of the fact that $\vec{l} \cdot \vec{l} = 0$. We notice that as of yet, our construction permits the support of $\vec{l}$ to be constrained only on $\mathcal{H}$. Hence the notion of taking derivatives of the null normal vector $\vec{l}$ away from the null surface is not well defined. In order to have well defined operations valid in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\mathbf{\nabla}})$ like the covariant derivative $\hat{\mathbf{\nabla}}$, we cannot be only constrained on the single null hypersurface $u(x^a) = 0$. The support of the null vector field $\vec{l}$ needs to be extended from the null surface to at least in its vicinity. To remedy this issue, we can consider the existence of an auxiliary null foliation in the neighborhood of our null hypersurface $\mathcal{H}$. Following Carter [130], this is facilitated by considering not just a single hypersurface $u = 0$, but by rather a family of null hypersurfaces $u(x^a) = c$, where c is a constant. Hence the spacetime is foliated by a family of null hypersurfaces $\mathcal{H}_u$ out which our particular chosen surface $\mathcal{H}_{u=0} = \mathcal{H}$ is just an element with $c = 0$. This null foliation of $(\mathcal{M}, \mathbf{g}, \hat{\mathbf{\nabla}})$ in the neighborhood of $\mathcal{H}$ extends the validity of the scalar field ρ and hence $\vec{l}$ to not just on $\mathcal{H}$, but to an open neighborhood of $(\mathcal{M}, \mathbf{g}, \hat{\mathbf{\nabla}})$ about $\mathcal{H}$. This finally allows us to perform operations that are valid on the ambient spacetime rather than on the hypersurface. Even though such a foliation is non-unique, yet the geometrical quantities of interest that will be introduced and evaluated on our chosen hypersurface $\mathcal{H}$ does not depend on the choice of foliation.

Next, we proceed to a discussion of the Frobenius identity on our integrable null surface. The fact that we have been able to write our null normal in the form of (2.24) means that the exterior derivative of the null normal to the hypersurface satisfies,

$$\boldsymbol{d}\underline{\boldsymbol{l}} = \boldsymbol{d}\rho \wedge \underline{\boldsymbol{l}} \, . \tag{2.25}$$

This represents the Frobenius theorem in its dual formulation [2]. The Frobenius identity quantifies the fact that the hyperplane, normal to $\underline{\boldsymbol{l}}$, is integrable into our null hypersurface $\mathcal{H}$ and is hence hypersurface-orthogonal. As a consequence of the dual formulation of the Frobenius identity we have,

$$\underline{\boldsymbol{l}} \wedge \boldsymbol{d}\underline{\boldsymbol{l}} = \underline{\boldsymbol{l}} \wedge \boldsymbol{d}\rho \wedge \underline{\boldsymbol{l}} = -\boldsymbol{d}\rho \wedge \underline{\boldsymbol{l}} \wedge \underline{\boldsymbol{l}} = 0 \, . \tag{2.26}$$

In the index notation, the above implies,

$$l_{[a}\partial_b l_{c]} = 0 \, . \tag{2.27}$$

Converting to the spacetime covariant derivatives, the above formula translates to,

$$\hat{\omega}_{abc} \equiv l_{[a}\hat{\nabla}_b l_{c]} = -l_a\left(T^d{}_{bc}\, l_d\right) - l_b\left(T^d{}_{ca}\, l_d\right) - l_c\left(T^d{}_{ab} l_d\right) . \tag{2.28}$$

Thus we see that due to the presence of torsion in the spacetime, hypersurface orthogonality does not imply a zero twist $\hat{\omega}_{abc}$. Again going back to Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the twist tensor is zero as the torsion tensor vanishes,

$$\omega_{abc} \overset{(\mathcal{M},\mathbf{g},\boldsymbol{\nabla})}{=} l_{[a}\nabla_b l_{c]} = 0 . \tag{2.29}$$

2.3.1 Hypersurface-orthogonal null geodesic congruence

As a consequence of the Frobenius identity (2.25), we have,

$$\partial_a l_b - \partial_b l_a = \left(\hat{\nabla}_a l_b - \hat{\nabla}_b l_a\right) + T^c{}_{ab}\, l_c = \left(\nabla_a l_b - \nabla_b l_a\right) = (\partial_a \rho) l_b - (\partial_b \rho) l_a . \tag{2.30}$$

Contracting the above formula with l^a, we obtain,

$$l^a \hat{\nabla}_a l_b + T_{cab}\, l^a l^c = l^a \nabla_a l_b = (l^a \partial_a \rho) l_b . \tag{2.31}$$

The operator ∇ is the covariant derivative of the spacetime taken w.r.t the Levi-Civita connection. Defining the directional rate of change of the scalar field $\rho(x^a)$ along the null generators l^a to be κ, *i.e.* $l^a\partial_a\rho = \kappa$ and using the antisymmetry of the torsion tensor in its last two indices (2.3), we hence have,

$$l^a \hat{\nabla}_a l_b - T_{abc}\, l^a l^c = l^a \nabla_a l_b = \kappa l_b . \tag{2.32}$$

The above equation indicates that even though the vector field l^a is the null generator of $\mathcal{H}$, yet it is not an auto-parallel vector field *i.e.* l^a does not satisfy the parallel transport equation w.r.t to the spacetime connection $\hat{\nabla}$,

$$l^a \hat{\nabla}_a l_b = \kappa l_b + T_{abc} l^a l^c = \kappa l_b + \mathbb{T}_b , \tag{2.33}$$

where $\mathbb{T}_b \equiv T_{abc}\, l^a l^c$. Notice that even though $\vec{l}$ is not an auto-parallel vector field in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, yet $\vec{l}$ is extremal in the sense that they are null geodesic curves of extremal length. This is because the notion of extremal curves is defined only w.r.t. the Levi-Civita connection. It is quite clear that the null normal vector field $\vec{l}$ is both a geodesic congruence as well as auto-parallel in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$.

It maybe noted that if we impose the constraint,

$$\mathbb{T}_b \equiv T_{abc}\, l^a l^c = 0 , \tag{2.34}$$

in the Riemann-Cartan spacetime, we then have, via (2.33), that the null generators are parallel-transported along themselves with κ being the non-affinity parameter,

$$l^a\hat{\nabla}_a l_b \overset{\mathbb{T}_b=0}{=} \kappa l_b \,. \tag{2.35}$$

The condition (2.34) will be termed as the *geodesic constraint*. Application of the geodesic constraint implies,

$$l^a\hat{\nabla}_a l_b = l^a\nabla_a l_b - K^c{}_{ab}\, l^a l_c \overset{\mathbb{T}_b=0}{=} l^a\nabla_a l_b = \kappa l_b \,, \tag{2.36}$$

thus verifying the fact that if the null generators $\vec{l}$ of $\mathcal{H}$ satisfy the parallel-transport equation w.r.t. the connection $\hat{\boldsymbol{\nabla}}$, then they also satisfy the geodesic equation w.r.t. the Levi-Civita connection $\boldsymbol{\nabla}$. The above geodesic constraint (2.34) represents the vanishing of the torsion current $T_{abc}l^a l^c$. In the context of Killing horizons established in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ it has been shown in [64] that the above condition of the vanishing of such a torsion current is necessary to establish the zeroth law of black hole mechanics. This allows a notion of equilibrium to be defined for such a horizon. We will have much to say about this later in chapter 6 when we begin discussing about our thermodynamic interpretation alluded to the gravitational dynamics.

2.3.2 Intrinsic geometry of the null surface

Before proceeding to define the intrinsic geometry of $\mathcal{H}$ via the induced metric $\boldsymbol{q}$, expressed through the ambient spacetime metric $\boldsymbol{g}$, we are faced with two issues. Firstly, we have no unique normalization for our null generators $\vec{l}$. Secondly, because of the unique structure of our null hypersurface $\mathcal{H}$, we do not have any notion of a vector that is transverse to $\mathcal{H}$. Hence we do not have any well defined projection tensor onto the null surface. To remedy these issues we have to add some extra structure onto $\mathcal{H}$. This we do via a $3+1$ induced foliation of our null family of hypersurfaces $u(x^a) = c$. For this, we consider in our spacetime at least in the vicinity of the null family of hypersurfaces $\mathcal{H}_u$, a stack of spacelike hypersurfaces, defined via $\Sigma_t \equiv t(x^a) =$ constant slices. The timelike normal to these spacelike slices is defined via,

$$n_a = -N\partial_a t = -N\hat{\nabla}_a t \,. \tag{2.37}$$

The timelike normal is normalized *i.e.* $n_a n^a = -1$ and the proportionality factor N is called the lapse function. The orthogonal projection tensor onto the surfaces Σ_t defined as,

$$\gamma^a{}_b = \delta^a{}_b + n^a n_b \,, \tag{2.38}$$

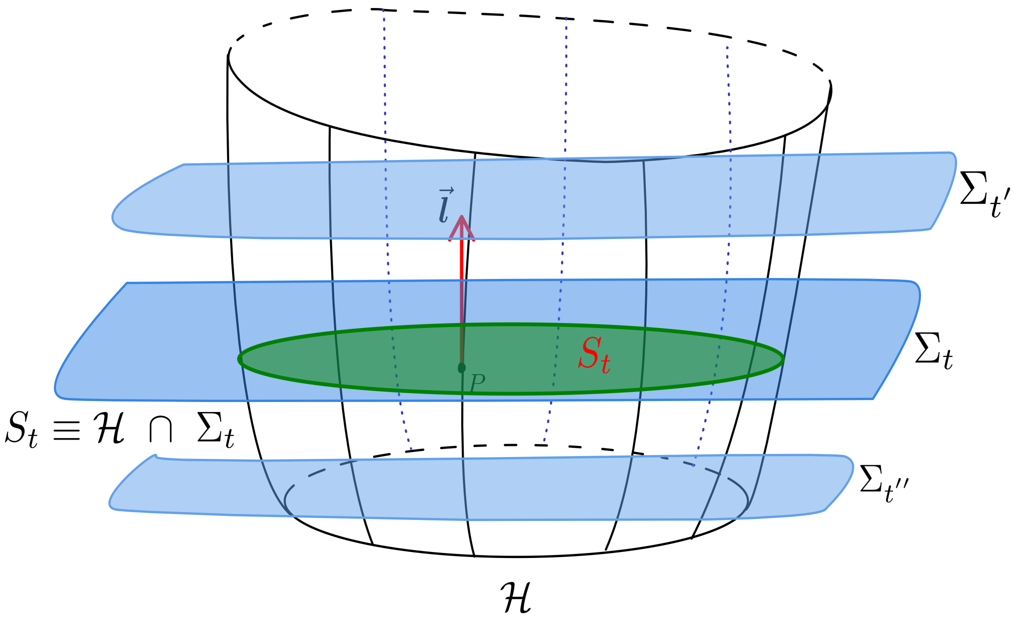

FIGURE 2.2: The null surface being intersected by a family of spacelike hypersurfaces Σ_t. The intersection of $\mathcal{H}$ and a given Σ_t is the spatial cross-section S_t of codimension 2.

projects any vector in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ onto the surface Σ_t. Now with the help of this spacelike foliation we can introduce a chosen coordinate system for $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. Let the independent coordinates parametrizing the $t(x^a) = \kappa$ (constant) surface be $x^\mu = (x^1, x^2, x^3)$. This allows a locally well defined coordinate system to be established in an open neighborhood of $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ given via $x^a = (t, x^\mu)$. The coordinate time vector $\vec{\boldsymbol{t}}$ along its flow basically joins the points having the same spatial coordinates x^μ for the different time slices and is defined as,

$$\vec{\boldsymbol{t}} \equiv \frac{\vec{\partial}}{\partial t} \quad \text{and} \quad t^a \partial_a t = 1 \,. \tag{2.39}$$

This allows an orthogonal decomposition of the time vector as follows,

$$t^a = N n^a + \beta^a \,. \tag{2.40}$$

The vector β^a is basically the projection of the time vector onto the $t(x^a) = \kappa$ slice and is known as the shift vector.

Now, we finally come to the topic of foliating our family of null hypersurfaces by this stack of spacelike hypersurfaces. The spacelike slices Σ_t cut our generic null surface $\mathcal{H}$ into a stack of 2-dimensional cross-sections S_t defined as,

$$S_t \equiv \mathcal{H} \cap \Sigma_t \,. \tag{2.41}$$

As we can visualize, the family of these transverse spacelike 2-dimensional cross-sections

S_t provide a foliation of $\mathcal{H}$. The value of the scalar field $t(x^a)$ = constant can be chosen as a parameter along each null generator l^a of $\mathcal{H}$. With this (in general) non-affine parametrization t, we can in essence normalize [61] the null normal by demanding that,

$$l^a = \frac{dx^a}{dt} \,. \tag{2.42}$$

Geometrically this means that the 2-surfaces S_t can be Lie-dragged along the null generators l^a thus forming our null hypersurface $\mathcal{H}$. It is in this sense that the null vector $\vec{l}$ is called the null generator of $\mathcal{H}$. Thus following (2.41), the transverse 2-surfaces S_t can be defined as level sets of the scalar field $u(x^a) = 0$ such that it is the set of all the points P,

$$S_t \equiv \{P \in \mathcal{M},\ \ P \in S_t : u(P) = 0 \ \text{ and } \ t(P) = t\} \,. \tag{2.43}$$

Hence, with this induced $3+1$ foliation of $\mathcal{H}$, we have been able to fix a unique non-affine parametrization of our null generators $\vec{l}$. As mentioned earlier, in order to define a projector onto the tangent space $\mathcal{T}_P(\mathcal{H})$ of $\mathcal{H}$, we need to have a direction that is transverse to the hypersurface. There exists no unique transverse direction to $\mathcal{H}$. The auxiliary null vector $\vec{k}_{(\mathrm{au})} \in \mathcal{T}_P(\mathcal{M})$ defined via the relations,

$$\vec{k}_{(\mathrm{au})} \cdot \vec{k}_{(\mathrm{au})} = 0 \ \text{ and } \ \vec{k}_{(\mathrm{au})} \cdot \vec{l} \equiv -1 \,, \tag{2.44}$$

defines a notion of a vector field transverse to the null surface $\mathcal{H}$. However such a vector is non-unique. We will see that the $3+1$ induced foliation of $\mathcal{H}$ actually resolves this issue of non-uniqueness in the auxiliary null vector field definition.

Let us consider a unit spacelike vector field $\vec{s}$ belonging to the tangent space of Σ_t and which is directed outward from the transverse 2-surface S_t (S_t is considered as an embedding of codimension one in the spacelike surface Σ_t). This vector field follows the properties given as,

$$\begin{gathered} \vec{s} \cdot \vec{s} = 1, \quad \vec{n} \cdot \vec{s} = 0, \quad s^a \partial_a u < 0 \\ \forall \vec{v} \in \mathcal{T}_P(\Sigma_t),\ \ \vec{v} \in \mathcal{T}_P(S_t) \quad \Longleftrightarrow \quad \vec{s} \cdot \vec{v} = 0 \,. \end{gathered} \tag{2.45}$$

Then it can be quite easily shown [61] that, the $3+1$ decomposition of the null generator is,

$$\vec{l} = N(\vec{n} + \vec{s}) \,. \tag{2.46}$$

Obviously it can be seen that the projection of the null generator onto the spacelike slices Σ_t given by $\gamma^a_{\ b} l^b = N s^a$ with $N > 0$ points in the exterior direction w.r.t. S_t. It is in this respect that the null generators are "outgoing" w.r.t. the 2-surface S_t. Using (2.24) and

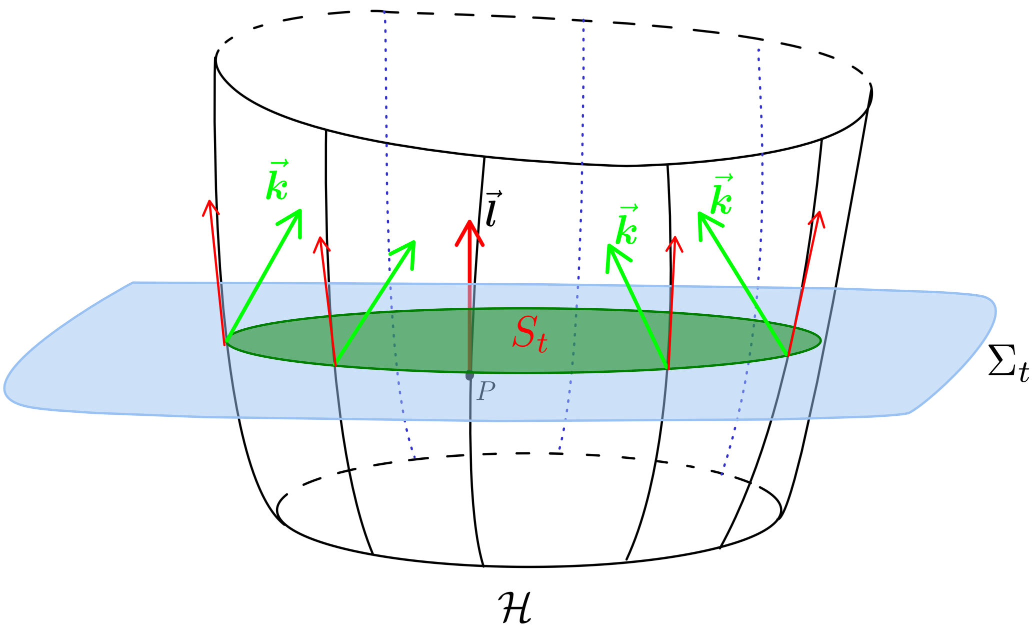

FIGURE 2.3: The auxiliary null vector field (shown in neon green) represents a null vector transverse to $\mathcal{H}$. The codimension 2-spacelike surface S_t is orthogonal to the plane containing $\vec{l}$ and $\vec{k}$.

(2.37), it can very easily be shown that,

$$s_a = \left(\frac{1}{N}\right) l_a - n_a \;=\; N\partial_a t - \left(\frac{e^\rho}{N}\right)\partial_a u \;=\; N\partial_a t + M\partial_a u\,, \tag{2.47}$$

where $M = -e^\rho/N$. Now once we have provided a null normal n_a to the spacelike slices Σ_t and the unit outward spacelike vector field s_a to the transverse cross-sections S_t, we can define an orthogonal projection tensor q_{ab} onto the spacelike 2-surface S_t as,

$$q_{ab} = g_{ab} + n_a n_b - s_a s_b\,. \tag{2.48}$$

Now it can be very easily verified that $q^i{}_k q^k{}_j = q^i{}_j$ and that $q^i{}_j v^j$ is the projected part of the vector v^i onto the 2-surface S_t, hence proving that q_{ab} is indeed an orthogonal projection tensor onto S_t.

Now let us come to the discussion of the construction of a unique ingoing transverse auxiliary null vector field to $\mathcal{H}$. Any null vector pointing in the direction $(\vec{n} - \vec{s})$ points in the ingoing direction w.r.t. S_t. We can normalize this ingoing auxiliary null field [61] by taking it as,

$$\vec{k} = \frac{1}{2N}(\vec{n} - \vec{s})\,. \tag{2.49}$$

The normalization has been chosen so that $\vec{k}$ is consistent with a unique definition of the auxiliary null vector field thanks to the foliation of the null surface $\mathcal{H}$ by the stack of spacelike slices Σ_t,

$$\vec{l}\cdot\vec{k} = -1, \quad \vec{k}\cdot\vec{k} = 0 \text{ and } \vec{k}\cdot\vec{e_A} = 0\,, \tag{2.50}$$

where $\{\vec{e_A}\}$ denotes the set of basis vectors lying on the tangent space $\mathcal{T}_P(S_t)$ of S_t. By using the Eqs. (2.49), (2.47) and (2.37) we have,

$$k_a = -\partial_a t - \left(\frac{M}{2N}\right)\partial_a u\,. \tag{2.51}$$

The unique orthogonal projection tensor onto the transverse 2-surface S_t can also be defined in terms of the null normal l_a and the auxiliary null normal k_a as,

$$q_{ab} = g_{ab} + l_a k_b + k_a l_b\,. \tag{2.52}$$

Its trivial to verify that $q^a{}_b l^b = 0$ and $q^a{}_b k^b = 0$. Finally, owing to the induced $(3+1)$ foliation of $\mathcal{H}$ by the stack of spacelike Σ_t, we have at our disposal both a unique normalized null normal $\vec{l}$ (parameterized by the non-affine parameter t) and a unique auxiliary transverse null vector field $\vec{k}$.

The above construction associated with $\mathcal{H}$ allows us to consider the following unique projection tensor onto the tangent space of $\mathcal{H}$,

$$\Pi^a{}_b = \delta^a{}_b + k^a l_b = q^a{}_b - l^a k_b\,. \tag{2.53}$$

This projection tensor basically projects any vector belonging to $\mathcal{T}_P(\mathcal{M})$ to only the part belonging to $\mathcal{T}_P(\mathcal{H})$. The projection tensor satisfies the following properties as can be easily verified along with the fact that $\Pi^i{}_k\Pi^k{}_j = \Pi^i{}_j$,

$$\begin{aligned}&\Pi^a{}_b l^b = l^a, \quad \Pi^a{}_b k^b = 0,\\ &\Pi^a{}_b l_a = 0 \text{ and } \Pi^a{}_b k_a = k_b\,.\end{aligned} \tag{2.54}$$

However, even though $\Pi^i{}_j$ acts as a good projection tensor onto $\mathcal{T}_P(\mathcal{H})$, it is not a symmetric object which is definitely what we expect of an induced metric.

To understand the nature of the induced metric onto the null hypersurface $\mathcal{H}$, let us consider the following choice of basis $\{\vec{e}_i\}$ on $\mathcal{T}_P(\mathcal{M})$,

$$\vec{e}_i = (\vec{k}, \vec{l}, \vec{e_A})\,. \tag{2.55}$$

Of this, the tangent basis vectors $\{\vec{e}_{\tilde{\mu}}\}$ on $\mathcal{T}_P(\mathcal{H})$ are,

$$\vec{e}_{\tilde{\mu}} = (\vec{l}, \vec{e_A})\,. \tag{2.56}$$

It can easily be verified that the cotangent basis $\{\underline{e}^i\}$ of $\mathcal{T}_P^*(\mathcal{M})$ satisfying $\langle \underline{e}^i, \vec{e}_j \rangle = \delta^i{}_j$ is,

$$\underline{e}^i = (-\underline{l}, -\underline{k}, \underline{e}^A) \tag{2.57}$$

Obviously we have,

$$q^i{}_j l^j = 0; \quad q^i{}_j k^j = 0; \quad \text{and} \quad q^i{}_j (\vec{e}_A)^j = (\vec{e}_A)^i \tag{2.58}$$

Thus as seen from Eq. (2.58), we see that $q^i{}_j$ projects spacetime vectors onto the tangent subspace of $\mathcal{T}_P(\mathcal{M})$ spanned by the basis vectors $\{\vec{e_A}\}$. The metric induced onto the null hypersurface $\mathcal{H}$ spanned by the basis vector $\vec{e}_{\tilde{\mu}} = (\vec{l}, \vec{e}_A)$ is defined as,

$$h_{\tilde{\mu}\tilde{\nu}} = \boldsymbol{g}(\vec{e}_{\tilde{\mu}}, \vec{e}_{\tilde{\nu}}) = g_{ij}(\vec{e}_{\tilde{\mu}})^i (\vec{e}_{\tilde{\nu}})^j \,. \tag{2.59}$$

Obviously, $h_{11} = g_{ij} l^i l^j = 0$ and $h_{1A} = g_{ij} l^i (\vec{e}_A)^j = 0$. The transverse components of the induced metric on $\mathcal{H}$ are given by,

$$h_{AB} = g_{ij}(\vec{e}_A)^i (\vec{e}_B)^j = (q_{ij} - l_i k_j - k_i l_j)(\vec{e}_A)^i (\vec{e}_B)^j = q_{ij}(\vec{e}_A)^i (\vec{e}_B)^j = q_{AB} \,. \tag{2.60}$$

Hence we notice that entire information of the induced metric $h_{\tilde{\mu}\tilde{\nu}}$ is captured by only the components $h_{AB} = q_{AB}$, where $q_{AB} = \boldsymbol{g}(\vec{e}_A, \vec{e}_B) = g_{ij}(\vec{e}_A)^i (\vec{e}_B)^j$ is the metric induced on the 2-dimensional spatial cross-section S_t orthogonal to both $\vec{l}$ and $\vec{k}$. Hence the induced metric onto the null surface $\mathcal{H}$ is given by (in the basis $(\vec{k}, \vec{l}, \vec{e}_A)$),

$$h_{\tilde{\mu}\tilde{\nu}} = \begin{pmatrix} 0 & 0 \\ 0 & q_{AB} \end{pmatrix} \tag{2.61}$$

Since the entire information of the induced metric $h_{\tilde{\mu}\tilde{\nu}}$ is contained in q_{AB}, we will from now on call the restriction of the ambient metric $\boldsymbol{g}$ onto $\mathcal{H}$ *i.e.* $\boldsymbol{g}_{|\mathcal{H}} \equiv \boldsymbol{q}$. Eq. (2.61) also reveals that the metric $\boldsymbol{g}$ restricted to $\mathcal{H}$ has the signature $(0, +, +)$ and hence the induced metric onto $\mathcal{H}$ as a hypersurface of codimension one is indeed degenerate.

2.3.3 Extrinsic geometry of the null hypersurface

Let us now come to the discussion of the extrinsic geometry of the null surface $\mathcal{H}$ as viewed as an embedding in the RC spacetime. For any hypersurface of codimension one embedded in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, the extrinsic curvature captures the notion of bending of the hypersurface. The extrinsic curvature is quantified by the Weingarten map (also known as the shape operator). The Weingarten map at a point P on the hypersurface measures how its normal changes as we move along a vector on the tangent space established on the hypersurface at P. Let us now focus on our null surface $\mathcal{H}$. For any

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vector $\vec{v} \in T_P(\mathcal{H})$, we have the definition of the Weingarten map ${}^{\mathcal{H}}\hat{\Upsilon}^a{}_b$ as follows,

$$ {}^{\mathcal{H}}\hat{\Upsilon}^a{}_b v^b \equiv \hat{\boldsymbol{\nabla}}_v \, l^a = v^i \hat{\nabla}_i l^a \, . \tag{2.62} $$

Now, since the notion of the Weingarten map involves taking the covariant derivative of the null generator along a tangent vector to $\mathcal{H}$, the quantity ${}^{\mathcal{H}}\hat{\Upsilon}^a{}_b v^b$ is independent of the null foliation. The quantity ${}^{\mathcal{H}}\hat{\Upsilon}^a{}_b v^b$ again itself lies on the tangent space of $\mathcal{H}$ established on P as verified via the following,

$$ l_a({}^{\mathcal{H}}\hat{\Upsilon}^a{}_b v^b) = l_a \hat{\boldsymbol{\nabla}}_v \, l^a = l_a v^b \hat{\nabla}_b l^a = 0 \implies ({}^{\mathcal{H}}\hat{\Upsilon}^a{}_b v^b)\vec{e}_a \in \mathcal{T}_P(\mathcal{H}) \, . \tag{2.63} $$

In Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ with the Levi-Civita connection, the Weingarten map ${}^{\mathcal{H}}\Upsilon^a{}_b$ is self-adjoint. However, the presence of non-trivial torsion in the Riemann-Cartan spacetime forces the Weingarten map to not be self-adjoint. This can be shown along the following lines. Consider two vectors $\vec{u}$ and $\vec{v}$ established on the tangent space of $\mathcal{H}$ at the point P *i.e.* $l_a u^a = 0 = l_a v^a$. Then obviously the Lie commutator of these two vectors again lies on the tangent space of $\mathcal{H}$ at P *i.e.* $[\vec{u}, \vec{v}] \in \mathcal{T}_P(\mathcal{H})$ as can be verified by showing that,

$$ \begin{aligned} l_a[\vec{u}, \vec{v}]^a &= l_a u^b \hat{\nabla}_b v^a - l_a v^b \hat{\nabla}_b u^a - l_a T^a_{\ bc} \, u^b v^c \\ &= u^b v^a \left(\hat{\nabla}_a l_b - \hat{\nabla}_b l_a \right) - T^a_{\ bc} \, l_a u^b v^c \\ &= u^b v^a \left((\partial_a \rho) \, l_b - (\partial_b \rho) \, l_a - T^c_{\ ab} \, l_c \right) - T^a_{\ bc} \, l_a u^b v^c \\ &= -l_c T^c_{\ ba} \, u^a v^b - l_c T^c{}_{ab} u^a v^b = 0 \, . \end{aligned} \tag{2.64} $$

Getting back to our reasoning and using the definition of the torsion tensor we have the following,

$$ \begin{aligned} u_a \, ({}^{\mathcal{H}}\hat{\Upsilon}^a{}_b v^b) &= u_a (v^b \hat{\nabla}_b l^a) = -l_a v^b \hat{\nabla}_b u^a \\ &= -l_a u^b \hat{\nabla}_b v^a + l_a [\vec{u}, \vec{v}]^a + T_{abc} l^a u^b v^c \\ &= v_a (u^b \hat{\nabla}_b l^a) + T_{abc} l^a u^b v^c = v_a \, ({}^{\mathcal{H}}\hat{\Upsilon}^a{}_b u^b) + T_{abc} l^a u^b v^c \, . \end{aligned} \tag{2.65} $$

This shows that the presence of non-zero torsion in the Riemann-Cartan spacetime accounts for the fact that $u_a \, {}^{\mathcal{H}}\hat{\Upsilon}^a{}_b v^b \neq v_a \, {}^{\mathcal{H}}\hat{\Upsilon}^a{}_b u^b$ and hence the Weingarten map ${}^{\mathcal{H}}\hat{\Upsilon}^a{}_b$ is in general not self-adjoint.

Let us then move on to the concept of the second fundamental form of $\mathcal{H}$. The second fundamental form ${}^{\mathcal{H}}\hat{\Theta}_{ab}$ is a second rank $(0, 2)$ tensor belonging to the cotangent space of the hypersurface $\mathcal{H}$ defined as the following. Consider any two vectors $\vec{u}$ and $\vec{v}$ belonging to the tangent space of $\mathcal{H}$, then,

$$ {}^{\mathcal{H}}\hat{\Theta}_{ab}u^a v^b \equiv u_a \left({}^{\mathcal{H}}\hat{\Upsilon}^a{}_b v^b\right) \implies {}^{\mathcal{H}}\hat{\Theta}_{ab}\underline{e}^a \otimes \underline{e}^b \in \mathcal{T}^*_P(\mathcal{H}) \otimes \mathcal{T}^*_P(\mathcal{H}) \,. \tag{2.66} $$

Notice that since the Weingarten map in $(\mathcal{M}, \mathbf{g}, \hat{\nabla})$ is not self-adjoint, the second fundamental form is not symmetric in its two indices. It can very easily be verified that,

$$ {}^{\mathcal{H}}\hat{\Theta}_{ab}v^a u^b = {}^{\mathcal{H}}\hat{\Theta}_{ab}u^a v^b + T_{abc}l^a v^b u^c \,. \tag{2.67} $$

The above same analysis in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \nabla)$ will allow us to conclude that the second fundamental form ${}^{\mathcal{H}}\Theta_{ab}$ on $\mathcal{H}$ is symmetric,

$$ {}^{\mathcal{H}}\hat{\Theta}_{ab} \overset{(\mathcal{M},\mathbf{g},\nabla)}{\underline{=}} {}^{\mathcal{H}}\hat{\Theta}_{ba} \,. \tag{2.68} $$

2.4 Kinematics of the null hypersurface $\mathcal{H}$: viewed from the ambient spacetime

Following [61], what we imply by kinematical quantities are all those geometrical entities that have first order derivatives of the null vector fields $\vec{l}$ and $\vec{k}$, their associated 1-forms $\underline{\boldsymbol{l}}$ and $\underline{\boldsymbol{k}}$ and the metric fields $\boldsymbol{g}$ and $\boldsymbol{q}$ as well. By first order derivatives, we mean spacetime covariant derivatives $\hat{\nabla}$ as well as the Lie derivatives along $\vec{l}$ and $\vec{k}$. The extension of such kinematical quantities (to be described in detail in this section) to case of the Riemann-Cartan spacetime $(\mathcal{M}, \mathbf{g}, \hat{\nabla})$ is quite necessary for our analysis. In doing so, we will keep track of the modifications that arise in these kinematical quantities when we consider torsion in the spacetime.

2.4.1 The extended Weingarten map and extended second fundamental form

Previously we have defined our Weingarten map or the shape operator corresponding to vectors constrained only on the hypersurface $\mathcal{H}$. Now, as we have a unique projection tensor $\Pi^i{}_j$ onto the tangent space of $\mathcal{H}$ (because of the induced $3+1$ foliation of $\mathcal{H}$), we can extend the definition of the Weingarten map to vectors living in the tangent space $\mathcal{T}_P(\mathcal{M})$. The extended Weingarten map $\hat{\Upsilon}^a{}_b$ is defined for vectors living in $\mathcal{T}_P(\mathcal{M})$ as,

$$ \hat{\Upsilon}^a{}_b v^b \equiv {}^{\mathcal{H}}\hat{\Upsilon}^a{}_b \left(\Pi^b{}_c v^c\right) = \hat{\boldsymbol{\nabla}}_{\boldsymbol{\Pi}(\vec{v})} l^a = (\Pi^b{}_c v^c)\hat{\nabla}_b l^a = \left(\delta^b{}_c + k^b l_c\right) v^c \hat{\nabla}_b l^a \,, \tag{2.69} $$

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where contrary to the earlier section, now $\vec{v} \in \mathcal{T}_P(\mathcal{M})$. From the above, it follows that,

$$\hat{\Upsilon}^a{}_b = \hat{\nabla}_b l^a + (k^c \hat{\nabla}_c l^a) l_b \,. \tag{2.70}$$

Then one finds that,

$$\hat{\Upsilon}^a{}_b l^b = l^b \hat{\nabla}_b l^a = \kappa l^a + T_b{}^a{}_c\, l^b l^c = \kappa l^a + \mathbb{T}^a \,, \tag{2.71}$$

$$\hat{\Upsilon}^a{}_b k^b = 0 \,, \tag{2.72}$$

where $\mathbb{T}^a = T_b{}^a{}_c\, l^b l^c$. It is worthwhile to notice that the action of the extended Weingarten map onto any spacetime vector $\vec{v} \in \mathcal{T}_P(\mathcal{M})$ is to effectively map it to another vector belonging to the tangent space of $\mathcal{H}$. This can quite easily seen by,

$$l_a \hat{\Upsilon}^a{}_b v^b = l_a \hat{\boldsymbol{\nabla}}_{\boldsymbol{\Pi}(\vec{v})} l^a = \frac{1}{2} \Pi^a{}_b v^b \hat{\nabla}_a (l_d l^d) = 0 \,. \tag{2.73}$$

This clearly shows that $\hat{\Upsilon}^a{}_b v^b$ belongs to the tangent space $\mathcal{T}_P(\mathcal{H})$ for any vector $\vec{v} \in \mathcal{T}_P(\mathcal{M})$. For this reason the action of the extended Weingarten map onto any spacetime vector is the same as its action on the projected part of the vector onto the tangent space of $\mathcal{H}$ *i.e.*

$$\hat{\Upsilon}^a{}_b (\Pi^b{}_c v^c) = \hat{\Upsilon}^a{}_b v^b \,. \tag{2.74}$$

Notice, that as evident from Eq. (2.71), $\vec{l}$ is not an eigenvector of the extended Weingarten map $\hat{\Upsilon}^a{}_b$ exclusively due to the presence of torsion in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. However, in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ without any intrinsic torsion, the null generators $\vec{l}$ become the eigenvector for the extended Weingarten map $\Upsilon^a{}_b$ with the eigenvalue of the non-affinity parameter κ,

$$\Upsilon^a{}_b l^b \overset{(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})}{=} \kappa l^a \,. \tag{2.75}$$

In the same vein, we can extend the notion of the second fundamental form ${}^{\mathcal{H}}\hat{\Theta}_{ab}$ to its action over vectors living in $\mathcal{T}_P(\mathcal{M})$ rather than $\mathcal{T}_P(\mathcal{H})$. Hence for any two vectors $(\vec{u}, \vec{v}) \in \mathcal{T}_P(\mathcal{M}) \times \mathcal{T}_P(\mathcal{M})$ we define the extended second fundamental form $\hat{\Theta}_{ab}$ as the following,

$$\begin{aligned}
\hat{\Theta}_{ab} u^a v^b &\equiv {}^{\mathcal{H}}\hat{\Theta}_{ab} \Big(\Pi^a{}_c u^c \, \Pi^b{}_d v^d \Big) = (\Pi^a{}_c u^c)(\Pi^b{}_d v^d) \hat{\nabla}_b l_a \\
&= \Big((q^a{}_c - l^a k_c) u^c \Big) \Big((q^b{}_d - l^b k_d) v^d \Big) \hat{\nabla}_b l_a \\
&= \Big(q^a{}_c q^b{}_d u^c v^d \hat{\nabla}_b l_a \Big) - (k_d v^d) q^a{}_c u^c \Big(\kappa l_a + \mathbb{T}_a \Big) + (k_c u^c)(k_d v^d) l^a \Big(\kappa l_a + \mathbb{T}_a \Big) \\
&= \Big(q^c{}_a q^d{}_b \hat{\nabla}_d l_c - q^c{}_a k_b \mathbb{T}_c \Big) u^a v^b \,.
\end{aligned} \tag{2.76}$$

This naturally allows us to define the extended second fundamental form as a bilinear,

$$\hat{\Theta}_{ab} = \left(q^c{}_a q^d{}_b \hat{\nabla}_d l_c\right) - \left(q^c{}_a k_b \mathbb{T}_c\right) . \tag{2.77}$$

In the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, we naturally have,

$$\Theta_{ab} \overset{(\mathcal{M},\mathbf{g},\boldsymbol{\nabla})}{=} q^c{}_a q^d{}_b \nabla_d l_c . \tag{2.78}$$

Naturally by inspection it can be observed that,

$$\hat{\Theta}_{ab} l^a = 0, \;\; \hat{\Theta}_{ab} l^b = q^c{}_a \mathbb{T}_c, \;\; \hat{\Theta}_{ab} k^a = 0 \;\; \text{and} \;\; \hat{\Theta}_{ab} k^b = 0 . \tag{2.79}$$

$$\Theta_{ab} l^a = 0, \;\; \Theta_{ab} l^b = 0, \;\; 0\Theta_{ab} k^a = 0 \;\; \Theta_{ab} k^b = 0 . \tag{2.80}$$

The above Eq. (2.79) very clearly shows that in the presence of torsion in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, the extended second fundamental form $\hat{\Theta}_{ab}$ (as opposed to Θ_{ab})is not a second rank tensor lying only in the space $\mathcal{T}^*(S_t) \otimes \mathcal{T}^*(S_t)$; rather it lies in the space $\mathcal{T}^*(\mathcal{H}) \otimes \mathcal{T}^*(\mathcal{H})$. We can compute the trace of this extended second fundamental form,

$$\begin{aligned} \overset{(e)}{\hat{\theta}_l} &\equiv g^{ab} \hat{\Theta}_{ab} = g^{ab} \left[\left(q^c{}_a q^d{}_b \hat{\nabla}_d l_c\right) - \left(q^c{}_a k_b \mathbb{T}_c\right)\right] = q^{cd} \hat{\nabla}_d l_c \\ &= (g^{cd} + l^c k^d + k^c l^d) \hat{\nabla}_d l_c = \hat{\nabla}_a l^a - \kappa + k_a \mathbb{T}^a . \end{aligned} \tag{2.81}$$

The trace of the second fundamental form in the Riemannian spacetime follows,

$$\overset{(e)}{\theta_l} \equiv g^{ab} \Theta_{ab} = \nabla_a l^a - \kappa . \tag{2.82}$$

Imposing the geodesic constraint *i.e.* $\mathbb{T}^a = 0$, we notice that the extended second fundamental form lies in the orthogonal transverse space (to $\vec{l}$ and $\vec{k}$) $\mathcal{T}^*(S_t) \otimes \mathcal{T}^*(S_t)$ and hence is orthogonal to both $\vec{l}$ and $\vec{k}$. Note that again due to the presence of torsion in the spacetime, the extended second fundamental form is not symmetric in its indices *i.e.* $\hat{\Theta}_{ab} \neq \hat{\Theta}_{ba}$. For the specific case of the geodesic constraint $\mathbb{T}_a = 0$, the extended second fundamental form being a bilinear established on the 2-dimensional transverse space orthogonal to both $\vec{l}$ and $\vec{k}$ can be provided an irreducible decomposition into a symmetric trace part $\overset{(e)}{\hat{\theta}_l}$, a traceless symmetric part ${}^{(l,e)}\hat{\sigma}_{ab}$ and an antisymmetric traceless part ${}^{(l,e)}\hat{\omega}_{ab}$,

$$\hat{\Theta}_{ab} = \frac{1}{2} q_{ab} \overset{(e)}{\hat{\theta}_l} + {}^{(l,e)}\hat{\sigma}_{ab} + {}^{(l,e)}\hat{\omega}_{ab} . \tag{2.83}$$

$$\overset{(e)}{\hat{\theta}_l} \overset{\mathbb{T}^a=0}{=} (\hat{\nabla}_a l^a - \kappa), \quad {}^{(l,e)}\hat{\sigma}_{ab} \overset{\mathbb{T}^a=0}{=} \hat{\Theta}_{(ab)} - \frac{1}{2} q_{ab} \overset{(e)}{\hat{\theta}_l} \;\; \text{and} \;\; {}^{(l,e)}\hat{\omega}_{ab} \overset{\mathbb{T}^a=0}{=} \hat{\Theta}_{[ab]} . \tag{2.84}$$

In the same manner, the extended second fundamental form Θ_{ab} of $\mathcal{H}$ in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ can be decomposed as,

$$\Theta_{ab} = \frac{1}{2} q_{ab} \overset{(e)}{\theta_{l}} + {}^{(l,e)}\sigma_{ab} \,. \tag{2.85}$$

$$\overset{(e)}{\theta_{l}} = (\nabla_a l^a - \kappa) \text{and} \quad {}^{(l,e)}\sigma_{ab} = \Theta_{(ab)} - \frac{1}{2} q_{ab} \overset{(e)}{\theta_{l}} \quad . \tag{2.86}$$

Note that the trace of the extended second fundamental form $\overset{(e)}{\hat{\theta}_{l}}$ is not to be designated as the expansion scalar corresponding to the null congruence $\vec{l}$ in the RC spacetime since it does not quantify the fractional rate of change of the area element $\sqrt{q}$ of S_t along $\vec{l}$. We will soon develop the proper notion of an expansion scalar corresponding to null congruences in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ for our purpose.

2.4.2 The Rotation 1-form and the Hájíček 1-form

A quantity of great interest and utility for practical calculations dealing with this analysis is the spacetime covariant derivative of the null normal l_a. Now having foliated the spacetime in the neighborhood of our generic null surface $\mathcal{H}$ by a family of null hypersurfaces and the slicing of this null family by a stack of spacelike slices Σ_t, we have now a well defined notion of $\hat{\nabla}_a l_b$. According to (2.76), we have for any $(\vec{u}, \vec{v}) \in \mathcal{T}_P(\mathcal{M}) \times \mathcal{T}_P(\mathcal{M})$,

$$\begin{aligned}
\hat{\Theta}_{ab} u^a v^b &= (\Pi^a{}_c u^c) \hat{\boldsymbol{\nabla}}_{\boldsymbol{\Pi}(\vec{v})} l_a = \Big(u^a + (l_c u^c) k^a \Big) \hat{\boldsymbol{\nabla}}_{\boldsymbol{\Pi}(\vec{v})} l_a \\
&= u^a (\Pi^b{}_d v^d) \hat{\nabla}_b l_a + (l_c u^c) k^a \hat{\boldsymbol{\nabla}}_{\boldsymbol{\Pi}(\vec{v})} l_a \\
&= u^a \Big(v^b + (l_d v^d) k^b \Big) \hat{\nabla}_b l_a + (l_c u^c) k_a \Big(\chi^a{}_d v^d \Big) \\
&= (\hat{\nabla}_b l_a) u^a v^b + u^a (l_d v^d)(k^b \hat{\nabla}_b l_a) + (l_a u^a) k_b \Big(\chi^b{}_d v^d \Big) \,.
\end{aligned} \tag{2.87}$$

The rotation 1-form $\underset{\sim}{\hat{\omega}} \in \mathcal{T}^*_P(\mathcal{M})$ is defined in such a way that its action on any vector $\vec{f} \in \mathcal{T}_P(\mathcal{M})$ is given via,

$$\hat{\omega}_a f^a \equiv -k_a \, (\hat{\Upsilon}^a{}_b f^b) \,. \tag{2.88}$$

Then using (2.88) in (2.87), we obtain,

$$\hat{\Theta}_{ab} u^a v^b = (\hat{\nabla}_b l_a) u^a v^b + (l_b k^c \hat{\nabla}_c l_a) u^a v^b - (l_a u^a)(\hat{\omega}_b v^b) \,. \tag{2.89}$$

Now, since u^a and v^a are arbitrary, one finds after rearranging,

$$\hat{\nabla}_a l_b = \hat{\Theta}_{ba} + \hat{\omega}_a l_b - l_a (\hat{\boldsymbol{\nabla}}_{\vec{k}} l_b) \,. \tag{2.90}$$

The corresponding analogue of Eq. (2.90) in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ is given by,

$$\nabla_a l_b \overset{(\mathcal{M},\mathbf{g},\boldsymbol{\nabla})}{=} \Theta_{ba} + \omega_a l_b - l_a(\boldsymbol{\nabla}_{\vec{k}} l_b) \,. \tag{2.91}$$

The above expansions Eq. (2.90) and Eq. (2.91) of the spacetime covariant derivative of the null normal are going to be of significant practical interest to us.

Let us look at some properties of the rotation 1- form in the RC spacetime. Using the basic definition (2.88), it can very easily verified that,

$$\hat{\omega}_a k^a = 0 \text{ and } \hat{\omega}_a l^a = \kappa - k_a \mathbb{T}^a \,. \tag{2.92}$$

Combining (2.70) and (2.90), we have,

$$\hat{\Upsilon}^a{}_b = \hat{\Theta}^a{}_b + \hat{\omega}_b l^a \,. \tag{2.93}$$

Let us now proceed to obtain an expression of the rotation 1-form. We begin by noticing that for any vector $\vec{f} \in \mathcal{T}_P(\mathcal{M})$, we have,

$$\begin{aligned} \hat{\omega}_a f^a &= -k_a \hat{\Upsilon}^a{}_b f^b = -k_a \Pi^b{}_d f^d \hat{\nabla}_b l^a = -k_a \Big((\delta^b_d + k^b l_d) f^d \Big) \hat{\nabla}_b l^a \\ &= -k_a \Big(f^b + k^b (l_d f^d) \Big) \hat{\nabla}_b l^a = -(k_b \hat{\nabla}_a l^b) f^a - \Big(k_b (k^c \hat{\nabla}_c l^b) l_a \Big) f^a \,. \end{aligned} \tag{2.94}$$

This allows us to have,

$$\hat{\omega}_a = -(k_b \hat{\nabla}_a l^b) - l_a k_b (\hat{\boldsymbol{\nabla}}_{\vec{k}} l^b) \,. \tag{2.95}$$

The corresponding analogue of the rotation 1-form $\underline{\omega}$ in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ is given by,

$$\omega_a = -(k_b \nabla_a l^b) - l_a k_b (\boldsymbol{\nabla}_{\vec{k}} l^b) \,. \tag{2.96}$$

The above equation Eq. (2.95) stands as a working definition of the rotation 1-form. However, we will soon come up with other expressions of this rotation 1-form that will be useful later on.

The Hájiček 1-form $\underline{\hat{\Omega}} \in \mathcal{T}^*_P(S_t)$ is defined as the projection of the rotation 1-form $\underline{\hat{\omega}}$ onto the cotangent space of S_t.

$$\hat{\Omega}_a \equiv q^b{}_a \hat{\omega}_b \,. \tag{2.97}$$

Following the above definition, we have for any vector $\vec{v} \in \mathcal{T}_P(\mathcal{M})$,

$$\hat{\Omega}_a v^a \equiv (q^b{}_a \hat{\omega}_b) v^a = \hat{\omega}^a (q^a{}_b v^b) \,. \tag{2.98}$$

Using the fact that $q^a{}_b = \Pi^a{}_b + l^a k_b$, we have,

$$\hat{\Omega}_a v^a = \hat{\omega}_a\Big((\Pi^a{}_b + l^a k_b)v^b\Big) = \hat{\omega}_a(\Pi^a{}_b v^b) + (\hat{\omega}_a l^a)k_b v^b$$
$$= \hat{\omega}_a v^a + (\kappa - k_a \mathbb{T}^a)(k_b v^b)\ . \tag{2.99}$$

In the above, we have used the fact that $\hat{\omega}_a(\Pi^a{}_b v^b) = \hat{\omega}_a v^a$ precisely via the property of the extended Weingarten map as shown in (2.74). This allows us to have a relationship between the rotation 1-form and the Hájiček 1-form as,

$$\hat{\omega}_a = \hat{\Omega}_a - \kappa k_a + (k_b \mathbb{T}^b)k_a\ . \tag{2.100}$$

The analogue of the above expression in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ is,

$$\omega_a \overset{(\mathcal{M},\mathbf{g},\boldsymbol{\nabla})}{=} \Omega_a - \kappa k_a\ . \tag{2.101}$$

From the dual formulation of the Frobenius theorem for the null normal $\underline{\boldsymbol{l}}$, we have upon using (2.90),

$$\boldsymbol{d}\underline{\boldsymbol{l}} = \boldsymbol{d}\rho \wedge \underline{\boldsymbol{l}} = (\partial_a l_b - \partial_b l_a)\underline{\boldsymbol{e}}^a \otimes \underline{\boldsymbol{e}}^b$$
$$= \Big(\hat{\nabla}_a l_b - \hat{\nabla}_b l_a + T_{cab} l^c\Big)\underline{\boldsymbol{e}}^a \otimes \underline{\boldsymbol{e}}^b$$
$$= \Big(\hat{\Theta}_{ba} - \hat{\Theta}_{ab} + \hat{\omega}_a l_b - \hat{\omega}_b l_a + (\hat{\boldsymbol{\nabla}}_{\vec{k}} l_a)l_b - (\hat{\boldsymbol{\nabla}}_{\vec{k}} l_b)l_a + T_{cab} l^c\Big)\underline{\boldsymbol{e}}^a \otimes \underline{\boldsymbol{e}}^b\ . \tag{2.102}$$

Using (2.77), we have the antisymmetric part of the extended second fundamental form as,

$$\hat{\Theta}_{ba} - \hat{\Theta}_{ab} = q^c{}_a q^d{}_b\Big((\hat{\nabla}_c l_d - \hat{\nabla}_d l_c) + (k_b q^c{}_a - k_a q^c{}_b)\mathbb{T}_c\Big)\ . \tag{2.103}$$

Using the fact that on account of the dual formulation of the Frobenius theorem *i.e.* $(\partial_a l_b - \partial_b l_a) = (\partial_a \rho)l_b - (\partial_b \rho)l_a$, we have,

$$\hat{\Theta}_{ba} - \hat{\Theta}_{ab} = q^c{}_a q^d{}_b T_{fdc}\, l^f + (k_b q^c{}_a - k_a q^c{}_b)\mathbb{T}_c\ . \tag{2.104}$$

This allows us to have,

$$\boldsymbol{d}\underline{\boldsymbol{l}} = \Big(q^c{}_a q^d{}_b T_{fdc}\, l^f + (k_b q^c{}_a - k_a q^c{}_b)\mathbb{T}_c + T_{cab} l^c + \hat{\omega}_a l_b - \hat{\omega}_b l_a$$
$$+ (\hat{\boldsymbol{\nabla}}_{\vec{k}} l_a)l_b - (\hat{\boldsymbol{\nabla}}_{\vec{k}} l_b)l_a\Big)\underline{\boldsymbol{e}}^a \otimes \underline{\boldsymbol{e}}^b\ . \tag{2.105}$$

Using the definition of the projection tensor $q^a{}_b = \delta^a{}_b + l^a k_b + k^a l_b$ and few lines of simple algebra it can be shown that,

$$q^c{}_a q^d{}_b T_{fdc}\, l^f + (k_b q^c{}_a - k_a q^c{}_b)\mathbb{T}_c + T_{cab} l^c = (T_{cda} l^c k^d) l_b - (T_{cdb} l^c k^d) l_a \,. \tag{2.106}$$

This finally results in the fact that,

$$\boldsymbol{d}\underline{\boldsymbol{l}} = \Big(\hat{\omega}_a l_b - \hat{\omega}_b l_a + (\hat{\boldsymbol{\nabla}}_{\vec{k}} l_a) l_b - (\hat{\boldsymbol{\nabla}}_{\vec{k}} l_b) l_a + (T_{cda} l^c k^d) l_b - (T_{cdb} l^c k^d) l_a\Big) \underline{\boldsymbol{e}}^a \otimes \underline{\boldsymbol{e}}^b \,. \tag{2.107}$$

Provided we define a 1-form $\underline{\boldsymbol{\mathfrak{T}}} \equiv (T_{cda} l^c k^d) \underline{\boldsymbol{e}}^a$, we can succinctly via (2.107) express the exterior derivative of the null normal as,

$$\boldsymbol{d}\underline{\boldsymbol{l}} = \underline{\hat{\boldsymbol{\omega}}} \wedge \underline{\boldsymbol{l}} + (\hat{\boldsymbol{\nabla}}_{\vec{k}} \underline{\boldsymbol{l}}) \wedge \underline{\boldsymbol{l}} + \underline{\boldsymbol{\mathfrak{T}}} \wedge \underline{\boldsymbol{l}} = \Big(\underline{\hat{\boldsymbol{\omega}}} + (\hat{\boldsymbol{\nabla}}_{\vec{k}} \underline{\boldsymbol{l}}) + \underline{\boldsymbol{\mathfrak{T}}}\Big) \wedge \underline{\boldsymbol{l}} \,. \tag{2.108}$$

The comparison of the above relation with $\boldsymbol{d}\underline{\boldsymbol{l}} = \boldsymbol{d}\rho \wedge \underline{\boldsymbol{l}}$, provides a relationship between the scalar field ρ and the rotation 1-form, via the equation,

$$\partial_a \rho = \hat{\omega}_a + \hat{\boldsymbol{\nabla}}_{\vec{k}} l_a + T_{cda} l^c k^d \,. \tag{2.109}$$

Let us proceed to derive another expression of the rotation 1-form. For that, notice that the exterior derivative of the auxiliary null normal $\underline{\boldsymbol{k}}$ via (2.51) can be simply expressed as,

$$\boldsymbol{d}\underline{\boldsymbol{k}} = \frac{1}{2N^2}\, \boldsymbol{d}\Big(\ln(\frac{N}{M})\Big) \wedge \underline{\boldsymbol{l}} \,. \tag{2.110}$$

Since the auxiliary null normal $\underline{\boldsymbol{k}}$ does not satisfy the dual formulation of the Frobenius theorem it can be interpreted that the hyperplane orthogonal to the auxiliary null normal is not integrable. From the definition (2.88) of the rotation 1-form, we have for any vector $\vec{v} \in \mathcal{T}_P(\mathcal{M})$,

$$\begin{aligned}\hat{\omega}_a v^a &= -k_a(\hat{\Upsilon}^a{}_b v^b) = -k_a({}^{\mathcal{H}}\hat{\Upsilon}^a{}_b \Pi^b{}_c v^c) = -k_a \hat{\boldsymbol{\nabla}}_{\boldsymbol{\Pi}(\vec{v})} l^a \\ &= -k_a(\Pi^b{}_c v^c)\hat{\nabla}_b l^a = l^a(\Pi^b{}_c v^c)(\hat{\nabla}_b k_a) \,. \end{aligned} \tag{2.111}$$

From the relation of the exterior derivative of the auxiliary null normal, we have from (2.110),

$$\begin{aligned} \partial_a k_b - \partial_b k_a &= \Big(\hat{\nabla}_a k_b - \hat{\nabla}_b k_a + T^c{}_{ab} k_c\Big) = \frac{1}{2N^2}\Big(\partial_a \ln(\frac{N}{M})\, l_b - \partial_b \ln(\frac{N}{M})\, l_a\Big) \,, \\ \implies \hat{\nabla}_b k_a &= \hat{\nabla}_a k_b + T^c{}_{ab} k_c - \frac{1}{2N^2}\Big(\partial_a \ln(\frac{N}{M})\, l_b - \partial_b \ln(\frac{N}{M})\, l_a\Big) \,. \end{aligned} \tag{2.112}$$

Employing (2.112) in (2.111), we end up having,

$$\begin{aligned}
\hat{\omega}_a v^a &= l^a \Pi^b{}_c v^c \Big[\hat{\nabla}_a k_b + T^c{}_{ab} k_c - \frac{1}{2N^2}\Big(\partial_a \ln(\frac{N}{M})\, l_b - \partial_b \ln(\frac{N}{M})\, l_a\Big)\Big] \\
&= l^a \Pi^b{}_c v^c \hat{\nabla}_a k_b + l^a \Pi^b{}_d v^d T^c_{ab} k_c \\
&= l^a (\delta^b{}_c + k^b l_c) v^c \hat{\nabla}_a k_b + l^a (q^b{}_d - l^b k_d) v^d T^c{}_{ab} k_c \\
&= (l^b \hat{\nabla}_b k_a) v^a + T_{cdb} k^c l^d q^d{}_a v^a \,.
\end{aligned} \tag{2.113}$$

Thus we finally retrieve an alternative and useful expression of the rotation 1-form $\underline{\boldsymbol{\omega}}$ as,

$$\hat{\omega}_a = (l^b \hat{\nabla}_b k_a) + T_{bcd} k^b l^c q^d{}_a \,. \tag{2.114}$$

The corresponding analogue of the rotation 1-form in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ is given by,

$$\omega_a = l^b \nabla_b k_a \,. \tag{2.115}$$

The above formula Eq. (2.114) allows us very easily to verify the relations (2.92).

2.4.3 The deformation rate tensor

The fact that the null generators $\vec{l}$ are parameterized by the (non-affine) time parameter t, (given by Eq. (2.42)), makes it a convenient vector field along which the evolution of geometrical quantities associated with $\mathcal{H}$ can be considered. The deformation rate tensor is one such quantity. The deformation rate tensor $\hat{\chi}_{ab}$ essentially quantifies the rate at which the metric $\boldsymbol{q}$ of the 2-surface S_t changes as we evolve along the null generators $\vec{l}$. Following [61], the deformation rate tensor is defined as,

$$\hat{\chi}_{ab} \equiv \frac{1}{2} q^i{}_a q^j{}_b \pounds_{l} q_{ij} \,. \tag{2.116}$$

Using the definition (2.20) of the Lie derivative of the metric tensor g_{ab} along the null generators $\vec{l}$, it is quite easy to notice that,

$$\hat{\chi}_{ab} = \frac{1}{2} q^c{}_a q^d{}_b \Big(\hat{\nabla}_c l_d + \hat{\nabla}_d l_c + (T_{cfd} + T_{dfc}) l^f \Big) \,. \tag{2.117}$$

Then using (2.77), the deformation rate tensor can be also be expressed as,

$$\hat{\chi}_{ab} = \frac{1}{2}\Big(\hat{\Theta}_{ab} + \hat{\Theta}_{ba}\Big) + \frac{1}{2}\Big(q^c{}_a k_b + q^c{}_b k_a\Big)\mathbb{T}_c + \frac{1}{2} q^c{}_a q^d{}_b \Big(T_{cfd} + T_{dfc}\Big) l^f \,. \tag{2.118}$$

For the spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the deformation rate tensor (in the absence of torsion) follows,

$$\chi_{ab} \overset{(\mathcal{M},\mathbf{g},\boldsymbol{\nabla})}{\underline{\underline{\quad}}} \Theta_{ab} \,. \tag{2.119}$$

A point worthwhile to notice is that in the presence of torsion in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, the extended second fundamental form $\hat{\Theta}_{ab}$ and the deformation rate tensor $\hat{\chi}_{ab}$ are not equivalent. Contrary to the extended second fundamental form $\hat{\Theta}_{ab}$, in general the deformation rate tensor is by definition both symmetric and orthogonally transverse to the space spanned by $\vec{\boldsymbol{l}}$ and $\vec{\boldsymbol{k}}$,

$$\hat{\chi}_{ab} l^a = 0 = \hat{\chi}_{ab} l^b \text{ and } \hat{\chi}_{ab} k^a = 0 = \hat{\chi}_{ab} k^b \,. \tag{2.120}$$

We can in fact perform an irreducible decomposition of the deformation rate tensor in terms a symmetric trace part and a traceless symmetric part,

$$\hat{\chi}_{ab} = \frac{1}{2} q_{ab} \overset{(d)}{\hat{\theta}_l} + {}^{(l,d)}\hat{\sigma}_{ab} \,, \tag{2.121}$$

where $\overset{(d)}{\hat{\theta}_l}$ is the outgoing expansion scalar and ${}^{(l,d)}\hat{\sigma}_{ab}$ is the traceless shear tensor corresponding to the null congruence $\vec{\boldsymbol{l}}$. The reason as to why $\overset{(d)}{\hat{\theta}_l}$ is called the expansion scalar is because it represents the fractional rate of change of the area element of the transverse spacelike 2-surface S_t as we move along the null generators $\vec{\boldsymbol{l}}$. We will prove this result shortly in Sec. 2.5.1 (see Eq. (2.160)). The trace of the deformation rate tensor is given by,

$$\begin{aligned} \overset{(d)}{\hat{\theta}_l} &= g^{ab} \hat{\chi}_{ab} = \frac{1}{2} q^{cd} \pounds_{\boldsymbol{l}} q_{cd} = \frac{1}{2} q^{cd} \pounds_{\boldsymbol{l}} g_{cd} \\ &= \frac{1}{2} q^{cd} \left(\hat{\nabla}_c l_d + \hat{\nabla}_d l_c + (T_{cfd} + T_{dfc}) l^f \right) \\ &= \frac{1}{2} (g^{cd} + l^c k^d + k^c l^d) \left(\hat{\nabla}_c l_d + \hat{\nabla}_d l_c + (T_{cfd} + T_{dfc}) l^f \right) \\ &= \hat{\nabla}_a l^a + T_a l^a - \kappa = \nabla_a l^a - \kappa \,. \end{aligned} \tag{2.122}$$

In the above, we have used Eq. (2.116) and Eq. (2.117). Similarly, in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the irreducible decomposition of the deformation rate tensor follows as,

$$\chi_{ab} \overset{(\mathcal{M},\mathbf{g},\boldsymbol{\nabla})}{\underline{\underline{\quad}}} \frac{1}{2} \overset{(d)}{\theta_l} + {}^{(l,d)}\sigma_{ab} \,. \tag{2.123}$$

With the induced metric $\boldsymbol{q}$ on the integrable null surface $\mathcal{H}$ (defined by the scalar function $u(x^a) = 0$ in both $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ and $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$) we expect that the fractional rate of change of the area element on S_t along the null generators $\vec{\boldsymbol{l}}$ be the same in both $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ and

$(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$. This is quantified by the fact that,

$$\overset{(d)}{\hat{\theta}_{l}} = \overset{(d)}{\theta_{l}} = \hat{\nabla}_a l^a + T_a l^a - \kappa = \nabla_a l^a - \kappa \,. \tag{2.124}$$

In the spirit of Eq. (2.90) we will find it beneficial to expand the spacetime covariant derivative of the null normal l_a in terms of the deformation rate tensor. To that extent, using Eq. (2.118) and Eq. (2.104), we can provide a relationship between the deformation rate tensor and the extended second fundamental form,

$$\hat{\chi}_{ab} = \hat{\Theta}_{ba} + k_a q^c{}_b \mathbb{T}_c + q^c{}_a q^d{}_b K_{fcd} l^f \,. \tag{2.125}$$

Finally, upon using Eq. (2.90), we get our desired expression relating the covariant derivative of the null normal l_a with the deformation rate tensor,

$$\hat{\nabla}_a l_b = \hat{\chi}_{ab} + \hat{\omega}_a l_b - l_a (k^i \hat{\nabla}_i l_b) - k_a q^c{}_b \mathbb{T}_c - q^c{}_a q^d{}_b K_{fcd} l^f \,. \tag{2.126}$$

2.4.4 Transversal deformation rate tensor

Much like the deformation rate tensor we can also look for the projection (onto the transverse spacelike surface S_t) of the Lie derivative of the metric q_{ab}, however now in a direction transverse to $\mathcal{H}$. This transverse direction is provided by the auxiliary null vector field $\vec{k}$. We define the transversal deformation rate tensor as,

$$\hat{\Xi}_{ab} \equiv \frac{1}{2} q^c{}_a q^d{}_b \, \pounds_{\boldsymbol{k}} q_{cd} \,. \tag{2.127}$$

Using the properties of the Lie derivative of the metric g_{ab} and the fact that $q_{ab} l^a = q_{ab} k^b = 0$, the transversal deformation rate tensor can be expressed as,

$$\hat{\Xi}_{ab} = \frac{1}{2} q^c{}_a q^d{}_b \pounds_{k} g_{cd} = \frac{1}{2} q^c{}_a q^d{}_b \Big(\hat{\nabla}_c k_d + \hat{\nabla}_d k_c + (T_{cfd} + T_{dfc}) k^f \Big) \,. \tag{2.128}$$

From (2.110), in the index notation, we have,

$$(\partial_a k_b - \partial_b k_a) = (\hat{\nabla}_a k_b - \hat{\nabla}_b k_a + T_{cab} k^c) = \frac{1}{2N^2} \partial_a (\ln \frac{N}{M}) l_b - \frac{1}{2N^2} \partial_b (\ln \frac{N}{M}) l_a \,. \tag{2.129}$$

Using (2.129) for the covariant derivative of the auxiliary field in (2.128), we have,

$$\begin{aligned} \hat{\Xi}_{ab} &= \frac{1}{2} q^c{}_a q^c{}_b \Big[2\hat{\nabla}_d k_c + \frac{1}{2N^2} \partial_a (\ln \frac{N}{M}) l_b - \frac{1}{2N^2} \partial_b (\ln \frac{N}{M}) l_a \\ &\qquad - T_{fcd} k^f + T_{cfd} k^f + T_{dfc} k^f \Big] \,, \\ &= q^c{}_a q^d{}_b \Big(\hat{\nabla}_d k_c + K_{fdc} k^f \Big) \,. \end{aligned} \tag{2.130}$$

Obviously, in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the transversal deformation rate tensor assumes the simple form,

$$\Xi_{ab} = q^c{}_a q^d{}_b (\nabla_d k_c) \ . \tag{2.131}$$

As evident from its definition (2.127), the transversal deformation rate tensor is symmetric and orthogonal to the space spanned by the vectors $\vec{l}$ and $\vec{k}$ *i.e.*

$$\hat{\Xi}_{ab} l^a \ = \ 0 \ = \ \hat{\Xi}_{ab} l^b \text{ and } \hat{\Xi}_{ab} k^a \ = \ 0 \ = \ \hat{\Xi}_{ab} k^b \ . \tag{2.132}$$

Just like for the case of the null generators, it would be of great practical interest for us to calculate the spacetime covariant derivative of the auxiliary null vector field $\vec{k}$. This quantity is again a well defined quantity, thanks to the null foliation of the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ in the neighborhood of $\mathcal{H}$. For this, we will just have to manipulate the part $q^c{}_a q^b{}_d \hat{\nabla}_d k_c$ in the expression (2.130) for $\hat{\Xi}_{ab}$:

$$\begin{aligned}
q^c{}_a q^d{}_b \hat{\nabla}_d k_c &= (\delta^c{}_a + l^c k_a + k^c l_a)(\delta^d{}_b + l^d k_b + k^d l_b)\hat{\nabla}_d k_c \\
&= (\delta^c{}_a + l^c k_a)\Big(\hat{\nabla}_b k_c + k_b (l^d \hat{\nabla}_d k_c) + l_b (\kappa^d \hat{\nabla}_d k_c)\Big) \\
&= \hat{\nabla}_b k_a + k_b (l^d \hat{\nabla}_d k_a) + l_b (k^d \hat{\nabla}_d k_a) + k_a (l^c \hat{\nabla}_b k_c) \\
&\quad\ + k_a k_b l^c (l^d \hat{\nabla}_d k_c) + k_a l_b l^c (k^d \hat{\nabla}_d k_c) \\
&= \hat{\nabla}_b k_a + k_b \Big[\hat{\omega}_a - T_{cdf} k^c l^d q^f{}_a\Big] + l_b (k^d \hat{\nabla}_d k_a) - k_a k^c (\hat{\nabla}_b l_c) \\
&\qquad\qquad - k_a k_b k^c (l^d \hat{\nabla}_d l_c) + k_a l_b l^c (k^d \hat{\nabla}_d k_c) \\
&= \hat{\nabla}_b k_a + k_b \hat{\omega}_a + l_b (\hat{\boldsymbol{\nabla}}_{\vec{k}} k_a) - k_a k^c \Big(\hat{\Theta}_{cb} + \hat{\omega}_b l_c - l_b \hat{\boldsymbol{\nabla}}_{\vec{k}} l_c\Big) \\
&\qquad - k_a k_b k^c (\kappa l_c + \mathbb{T}_c) - k_a l_b k^c (k^d \hat{\nabla}_d l_c) - k_b T_{cdf} k^c l^d q^f{}_a \\
&= \hat{\nabla}_b k_a + k_b \hat{\omega}_a + l_b (\hat{\boldsymbol{\nabla}}_{\vec{k}} k_a) + k_a \hat{\omega}_b \\
&\quad + \kappa k_a k_b - k_a k_b k_c \mathbb{T}^c - k_b T_{cdf} k^c l^d q^f{}_a \\
&= \hat{\nabla}_b k_a + k_b \hat{\omega}_a + l_b (\hat{\boldsymbol{\nabla}}_{\vec{k}} k_a) + k_a \hat{\Omega}_b - k_b T_{cdf} k^c l^d q^f{}_a \ .
\end{aligned} \tag{2.133}$$

For the above result, we have used Eq. (2.114) in the fourth line, Eq. (2.90) in the fifth line and Eq. (2.100) in the seventh line. Putting the value of $q^c{}_a q^b{}_d \hat{\nabla}_d k_c$ as obtained in Eq. (2.133) into Eq. (2.130) and rearranging, we obtain our desired quantity *i.e.* the covariant derivative of the auxiliary null normal in terms of the transversal deformation rate tensor,

$$\hat{\nabla}_a k_b = \hat{\Xi}_{ab} - \hat{\Omega}_a k_b - k_a \hat{\omega}_b - l_a (k^i \hat{\nabla}_i k_b) + k_a T_{cdf} k^c l^d q^f{}_a - q^c{}_a q^d{}_b K_{fcd} k^f \ . \tag{2.134}$$

For the spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the above expansion for the covariant derivative of the auxiliary null normal reduces to,

$$\nabla_a k_b = \Xi_{ab} - \Omega_a k_b - k_a \omega_b - l_a (k^i \nabla_i k_b) \, . \tag{2.135}$$

In the same way, the transversal deformation rate tensor $\hat{\Xi}_{ab}$ can be decomposed into a symmetric trace part and a symmetric traceless part,

$$\hat{\Xi}_{ab} = \frac{1}{2} q_{ab} \overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} + {}^{(\boldsymbol{k},d)}\hat{\sigma}_{ab} \, . \tag{2.136}$$

The trace of the transversal deformation rate tensor is the expansion scalar corresponding to the auxiliary $\vec{\boldsymbol{k}}$ congruence. Again, as we will show later that the trace is truly the expansion scalar in the sense that it quantifies the fractional rate of change of the area element of S_t as we move along $\vec{\boldsymbol{k}}$. The ingoing expansion scalar corresponding to the auxiliary null vector field, via Eq. (2.130) is then given by,

$$\begin{aligned}
\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} &= g^{ab} \hat{\Xi}_{ab} = q^{cd} (\hat{\nabla}_d k_c + K_{fdc} k^f) \\
&= q^{cd} \hat{\nabla}_d k_c + \frac{1}{2} (g^{cd} + l^c k^d + k^c l^d) \Big(T_{dfc} + T_{cfd} \Big) k^f \\
&= (q^{cd} \hat{\nabla}_d k_c) + T_a k^a - T_{dcf} k^d l^c k^f \, .
\end{aligned} \tag{2.137}$$

Analogously, we have the following relations valid for $\mathcal{H}$ in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$,

$$\Xi_{ab} = \frac{1}{2} q_{ab} \overset{(d)}{\theta_{\boldsymbol{k}}} + {}^{(\boldsymbol{k},d)}\sigma_{ab} \quad \text{and} \quad \overset{(d)}{\theta_{\boldsymbol{k}}} \equiv \theta_{\boldsymbol{k}} = q^{cd} \nabla_d k_c \, . \tag{2.138}$$

Also as a matter of notation in the Riemannian spacetime, we define ${}^{(\boldsymbol{k},d)}\sigma_{ab} \equiv {}^{(\boldsymbol{k})}\sigma_{ab}$.

2.4.5 The projected deviation tensor

Till now, we have discussed about three kinematical second rank tensors that are of interest to us. They are the extended second fundamental form $\hat{\Theta}_{ab}$, the deformation rate tensor $\hat{\chi}_{ab}$ and the transversal deformation rate tensor $\hat{\Xi}_{ab}$. As we shall see later, all these quantities will be very relevant in the analysis of providing our advertised thermodynamic interpretation as applied to $\mathcal{H}$. There also exists another second rank tensor called the deviation tensor that we wont require for our thermodynamic interpretation. However, for the sake of completeness, we provide a discussion of such a kinematical quantity in this section. Shortly afterwards, in the next chapter, we provide a detailed derivation of the null Raychaudhuri equation corresponding to the expansion scalar of the null generators $\vec{\boldsymbol{l}}$ without the assumption of the geodesic constraint (2.34). To corroborate our results with the existing literature [64] we would require the construction of the deviation tensor and its relation to the extended second fundamental form. In this section, we go ahead with the discussion of the kinematics of the null hypersurface by describing the deviation tensor or effectively its projected part. On the outset, let us mention that the discussion of the deviation tensor does not require a foliation of $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ by a family of null hypersurfaces and their subsequent slicing by a stack of spacelike surfaces. All we actually require is a congruence of null trajectories (not necessarily geodesic or auto-parallel) and build upon the premise that the deviation vector field $\vec{\boldsymbol{\eta}}$ is Lie-transported along the null congruence $\vec{\boldsymbol{l}}$. The deviation vector essentially measures the deviation between two neighboring null trajectories. The above condition means,

$$[\vec{\boldsymbol{\eta}}, \vec{\boldsymbol{l}}] = 0 \,. \tag{2.139}$$

In index notation this translates to,

$$\begin{aligned} l^a \partial_i \eta^a - \eta^i \partial_i l^a &= 0 \,, \\ l^i \hat{\nabla}_i \eta^a &= \eta^i \left(\hat{\nabla}_i l^a + T^a_{ji}\, l^j \right) \equiv \eta^i \bar{\mathcal{B}}^a_{\ i} \,. \end{aligned} \tag{2.140}$$

Here $\bar{\mathcal{B}}^a_{\ i}$ is called the deviation tensor which measures the failure of the deviation vector to be parallel-transported along the null congruence. This is given by,

$$\bar{\mathcal{B}}_{ai} = \hat{\nabla}_i l_a + T_{aji} l^j \,. \tag{2.141}$$

The auxiliary null vector field k^a to such a null congruence is as usual defined via the relations $l_a k^a = -1$ and $k^a k_a = 0$, with the projection tensor onto a 2-dimensional spacelike cross-section of the null congruence being $q_{ab} = g_{ab} + l_a k_b + k_a l_b$. We can easily verify that the deviation tensor is not orthogonal to both l^a and k^a as seen from,

$$\begin{aligned} &\bar{\mathcal{B}}_{ab} l^a = -\mathbb{T}_b, \quad \bar{\mathcal{B}}_{ab} l^b = \kappa l_a + \mathbb{T}_a \,, \\ &\bar{\mathcal{B}}_{ab} k^a = -l^a \hat{\nabla}_b k_a + T_{aib} k^a l^i \quad \text{and} \quad \bar{\mathcal{B}}_{ab} k^b = k^b \hat{\nabla}_b l_a + T_{aib} l^i k^b \,. \end{aligned} \tag{2.142}$$

We will project the deviation tensor onto the transverse spacelike 2-surface of the null congruence to define the projected deviation tensor $\hat{\mathcal{B}}_{ab}$,

$$\hat{\mathcal{B}}_{ab} \equiv q^c_{\ a} q^d_{\ b} \bar{\mathcal{B}}_{cd} \,. \tag{2.143}$$

Opening up the projection tensors $q^a_{\ b} = (\delta^a_{\ b} + l^a k_b + k^a l_b)$ in (2.143), a few lines of simple algebra lead us to,

$$\hat{\mathcal{B}}_{ab} = \bar{\mathcal{B}}_{ab} + l_b k^d \bar{\mathcal{B}}_{ad} + l_a k^c \bar{\mathcal{B}}_{cb} + l_a l_b k^c k^d \bar{\mathcal{B}}_{cd} + \mathbb{T}_a k_b - \mathbb{T}_b k_a + (l_a k_b - l_b k_a)\kappa_c \mathbb{T}^c \,. \tag{2.144}$$

Next, using the relation (2.141) and (2.114) in the above, we obtain,

$$\begin{aligned}\hat{\mathcal{B}}_{ab} = \hat{\nabla}_b l_a + T_{akb} l^k + (\mathbb{T}_a k_b - \mathbb{T}_b k_a) + (l_a k_b - k_a l_b)\mathbb{T}^c k_c - l_a \hat{\omega}_b + l_b (k^i \hat{\nabla}_i l_a) \\ + (T_{akj} l^k k^j) l_b + l_a T_{ikb} k^i l^k + l_a l_b (T_{ikj} k^i l^k k^j) \, . \end{aligned} \tag{2.145}$$

The above relation relates the spacetime covariant derivative of the null normal with the projected deviation tensor. Finally using (2.90) in (2.145), we end up deriving a relationship between the projected deviation tensor and the extended second fundamental form,

$$\begin{aligned}\hat{\Theta}_{ab} = \hat{\mathcal{B}}_{ab} - T_{akb} l^k - (\mathbb{T}_a k_b - k_a \mathbb{T}_b) - (l_a k_b - k_a l_b)\mathbb{T}^c k_c \\ + (T_{aij} k^i l^j) l_b - l_a T_{pqr} k^p l^q q^r{}_b \, . \end{aligned} \tag{2.146}$$

We can again perform an irreducible decomposition of the projected deviation tensor into a symmetric trace part, a symmetric traceless part and an antisymmetric part,

$$\hat{\mathcal{B}}_{ab} = \frac{1}{2} q_{ab} \overset{(B)}{\hat{\theta}_l} + {}^{(l,B)}\hat{\sigma}_{ab} + {}^{(l,B)}\hat{\omega}_{ab} \, . \tag{2.147}$$

Computing the trace of the projected deviation tensor we obtain,

$$\begin{aligned}\overset{(B)}{\hat{\theta}_l} &= g^{ab} \hat{\mathcal{B}}_{ab} = q^{ab} \hat{\mathcal{B}}_{ab} \\ &= (g^{ab} + l^a k^b k^a l^b) \left[\hat{\nabla}_b l_a + T_{acb} l^c \right] = \hat{\nabla}_a l^a - \kappa + T_a l^a \, . \end{aligned} \tag{2.148}$$

We notice that $\overset{(B)}{\hat{\theta}_l} = \overset{(d)}{\hat{\theta}_l}$ and hence the trace of the projected deviation tensor also quantifies the expansion scalar of the null congruence.

We end our discussion on these kinematical quantities by mentioning that in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the extended second fundamental form, the deformation rate tensor and the projected deviation tensor are all equivalent and spatial (orthogonal to $\vec{l}$ and $\vec{k}$),

$$\Theta_{ab} = \chi_{ab} = \mathcal{B}_{ab} \overset{(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})}{\equiv} q^i{}_a q^j{}_b \nabla_j l_i \, . \tag{2.149}$$

Hence the corresponding traces of these spatial tensors are the same,

$$\overset{(e)}{\theta_l} = \overset{(d)}{\theta_l} = \overset{(B)}{\theta_l} \overset{(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})}{\equiv} \nabla_a l^a - \kappa \equiv \theta_l \, . \tag{2.150}$$

The corresponding shear tensors are also the same,

$${}^{(l,e)}\sigma_{ab} = {}^{(l,d)}\sigma_{ab} = {}^{(l,B)}\sigma_{ab} \equiv {}^{(l)}\sigma_{ab} \, . \tag{2.151}$$

2.5 Notions of coordinate systems adapted to the null surface

Let us allude to in this section, a few coordinate systems that are adapted to the null surface $\mathcal{H}$. We will first describe a coordinate system in the spirit of $3+1$ foliation of the spacetime and how such a coordinate system can be adapted to $\mathcal{H}$. Then we will list out a few other coordinate systems that are used in the context of null surfaces without going into any details.

Now that we foliated our chosen null surface $\mathcal{H}$ by a family of spacelike hypersurfaces Σ_t (defined by t = constant slices), we can in essence set up a system of coordinates on the null surface. We can construct a coordinate system $x^i = (t, x^\mu)$ in the spirit of the $3+1$ decomposition of the given spacetime in the neighborhood of $\mathcal{H}$. Here t is the coordinate associated with the time development/evolution vector $\vec{\boldsymbol{t}} = \frac{\vec{\partial}}{\partial t} = (1,0,0,0)$ and $x^\mu = (x^1, x^2, x^3)$ are the spacelike coordinates on the t = constant slice. The time evolution vector field can then be decomposed in terms of the normal $\vec{\boldsymbol{n}}$ to Σ_t and a spatial shift vector $\vec{\boldsymbol{\beta}} \in \mathcal{T}_P(\Sigma_t)$ as,

$$\vec{\boldsymbol{t}} = N\,\vec{\boldsymbol{n}} + \vec{\boldsymbol{\beta}}\,, \tag{2.152}$$

where N represents the usual lapse function. Viewing the spacelike manifold $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$ as an embedding in the spatial slice Σ_t, we can decompose the shift vector $\vec{\boldsymbol{\beta}}$ as follows,

$$\vec{\boldsymbol{\beta}} = b\vec{\boldsymbol{s}} - \vec{\boldsymbol{V}}\,, \tag{2.153}$$

where $b = \vec{\boldsymbol{s}} \cdot \vec{\boldsymbol{\beta}}$ and $\vec{\boldsymbol{V}}$ is a spatial vector lying in the tangent space of S_t (and hence $\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{s}} = 0$). $\vec{\boldsymbol{V}}$ is usually denoted as the surface velocity of the null surface $\mathcal{H}$. The time development vector basically connects neighboring slices of Σ_t and Σ_{t+dt} with the same spatial coordinates. We now describe the notion of a coordinate system *stationary* w.r.t $\mathcal{H}$ [61]. If in this coordinate system (x^i), the equation of $\mathcal{H}$ does not depend on the coordinate t and only involves the spatial coordinates x^μ, then (x^i) is a coordinate system that is stationary w.r.t $\mathcal{H}$. This basically means that the location of $\mathcal{H}$ is fixed via the spatial coordinates by some scalar function say $f(x^1, x^2, x^3) = 0$. For such a stationary coordinate system adapted to $\mathcal{H}$, it can be shown that [61],

$$\vec{\boldsymbol{l}} \overset{\mathcal{H}}{=} \vec{\boldsymbol{t}} + \vec{\boldsymbol{V}}\,. \tag{2.154}$$

One specific choice would be to set $f(x^1, x^2, x^3) = x^1 = 0$. So with this choice, $x^1 = 0$ defines the transverse two-dimensional cross-section S_t on Σ_t. With this, the coordinates then on S_t surface are $x^A = (x^2, x^3)$ and the coordinate basis vectors of $T_p(S_t)$ are $\vec{\boldsymbol{e}}_A = \vec{\boldsymbol{\partial}}_A = (\vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = (\vec{\boldsymbol{\partial}}_2, \vec{\boldsymbol{\partial}}_3)$. The coordinate time evolution vector $\vec{\boldsymbol{t}} = \vec{\boldsymbol{\partial}}_t$ connects same spatial points along neighboring Σ_t hypersurfaces. Hence the coordinates defined on the null surface are $x^{\tilde{\mu}} = (t, x^A)$ with the coordinate basis vectors on $T_p(\mathcal{H})$ being $\vec{e}_{\tilde{\mu}} = (\vec{t}, \vec{e}_A) = (\vec{\partial}_t, \vec{\partial}_A)$. It can be shown that the metric of the null surface $\mathcal{H}$ w.r.t the coordinate system $x^{\tilde{\mu}} = (t, x^A)$ is given by [61, 82],

$$ds^2_{|\mathcal{H}} = q_{\tilde{\mu}\tilde{\nu}} dx^{\tilde{\mu}} dx^{\tilde{\nu}} = q_{AB}(dx^A - V^A dt)(dx^B - V^B dt) \,. \tag{2.155}$$

Another popular choice of coordinate system describing the spacetime in the vicinity of $\mathcal{H}$ is the Gaussian null coordinates that is named in analogy with the Gaussian normal coordinates [25, 131]. We described this system briefly in section 1.5.

Another choice of providing a coordinate grid to the spacetime in the vicinity of $\mathcal{H}$ is using the so called double null foliation. Here we begin with a spatial surface of codimension two (in our case this would correspond to the surface S_t). We consider two scalar fields such the intersection of the level sets or surfaces (corresponding to the scalar fields) defines our codimension two surface. Both the null normal one form $\underline{\boldsymbol{l}}$ and the auxiliary null normal one form $\underline{\boldsymbol{k}}$ can be expanded as linear combinations of the gradient one forms of the two scalar fields. One distinguishing feature in such a construction is that both $\vec{\boldsymbol{l}}$ and $\vec{\boldsymbol{k}}$ are in general not hypersurface-orthogonal. For the construction and details of the double null foliation the reader may wish to consult [132–134].

2.5.1 Proof of the expansion scalars as being the rate of change of area element on S_t

We conclude this chapter on the discussion of the geometry and kinematics of our integrable null surface $\mathcal{H}$ by convincing ourselves that $\overset{(d)}{\hat{\theta}_k}$ and $\overset{(d)}{\hat{\theta}_l}$ represent the expansion scalars of the auxiliary null vector field $\vec{\boldsymbol{k}}$ and the null generators $\vec{\boldsymbol{l}}$ respectively. Let us represent the (in general) non-affine parameter along the auxiliary null vector field to be $\lambda_{(k)}$. Hence we have $k^a = -\frac{dx^a}{d\lambda_{(k)}}$. The crucial negative sign is because of the fact that the auxiliary null field $\vec{\boldsymbol{k}}$ is ingoing as opposed to the null generators $\vec{\boldsymbol{l}}$ which are outgoing. On account of the determinant of the transverse metric of the 2-surface S_t being a scalar density, we have,

$$\frac{d\sqrt{q}}{d\lambda_{(k)}} = \frac{1}{2}\sqrt{q} q^{AB} \frac{d}{d\lambda_{(k)}} q_{AB} = -\frac{1}{2}\sqrt{q} q^{AB} k^i \hat{\nabla}_i \Big(g_{ab} (\vec{e}_A)^a (\vec{e}_A)^b \Big).$$

Therefore one finds,

$$\begin{aligned} -\frac{1}{\sqrt{q}} \frac{d}{d\lambda_{(k)}} \sqrt{q} &= \frac{1}{2} q^{AB} g_{ab} (\vec{e}_B)^b (k^i \hat{\nabla}_i (\vec{e}_A)^a) + \frac{1}{2} q^{AB} g_{ab} (\vec{e}_A)^a (k^i \hat{\nabla}_i (\vec{e}_B)^b) \\ &= q^{AB} g_{ab} (\vec{e}_B)^b (k^i \hat{\nabla}_i (\vec{e}_A)^a) \,. \end{aligned} \tag{2.156}$$

Under the construction of the null hypersurface $\mathcal{H}$, the basis vectors $\{\vec{e}_A\}$ of the tangent space established on the 2-surface S_t are Lie-transported along the auxiliary null field *i.e.* $[\vec{k}, \vec{e}_A] = 0$. This results in,

$$k^i \hat{\nabla}_i (\vec{e}_A)^a = (\vec{e}_A)^a \hat{\nabla}_i k^a + T^a_{\ bc} k^b (\vec{e}_A)^c \ . \tag{2.157}$$

Using Eq. (7.47) in Eq. (7.46), we obtain,

$$\begin{aligned} -\frac{1}{\sqrt{q}} \frac{d}{d\lambda_{(k)}} \sqrt{q} &= q^{AB} g_{ab} (\vec{e}_B)^b \Big((\vec{e}_A)^i (\hat{\nabla}_i k^a) + T^a_{\ dc} k^d \, (\vec{e}_A)^c \Big) \\ &= q^{ab} \hat{\nabla}_a k_b + q^{ab} T_{acb} k^b \ . \end{aligned} \tag{2.158}$$

In the above, we have used the fact that $q^{AB} (\vec{e}_A)^a (\vec{e}_B)^b = q^{ab}$. Upon using the relation (2.134) in Eq. (7.48), we obtain,

$$-\frac{1}{\sqrt{q}} \frac{d}{d\lambda_{(k)}} \sqrt{q} = q^{ab} \hat{\Xi}_{ab} - q^{cd} K_{fcd} k^f + q^{ab} T_{acb} k^c = \overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} \tag{2.159}$$

Hence we indeed verify that the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}$ represents the fractional rate of change of the 2-surface area element $\sqrt{q}$ along the auxiliary null vector field $\vec{k}$. Strictly along the lines of the previous analysis, its quite straightforward to establish that,

$$\frac{1}{\sqrt{q}} \frac{d}{d\lambda_{(l)}} \sqrt{q} = q^{ab} \hat{\chi}_{ab} = \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} \ , \tag{2.160}$$

where $\lambda_{(l)} = t$ represents the non-affine parameter for the outgoing null generators $\vec{l}$.

2.6 Summary and discussions

Our goal in this chapter was to define in a concrete sense the construction of a general integrable null hypersurface in the RC spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. The RC spacetime allows for non-trivial torsion to be present associated with its connection $\hat{\boldsymbol{\nabla}}$. We then studied the geometric construction of such a generic null surface $\mathcal{H}$ in this spacetime. That is, we studied in depth both the intrinsic and extrinsic geometry of $\mathcal{H}$ in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. Even though the in-depth study of null surfaces have been done in Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, such a coherent presentation, we believe was lacking in the literature for the RC spacetime. Let us mention at the outset that the RC spacetime is not completely general. It is constrained by the fact that the linear connection $\hat{\boldsymbol{\nabla}}$ is metric-compatible. It gives rise to those class of gravitational theories (called the metric-affine theories) where the non-metricity tensor has been set to zero. One of the main aims of this thesis is to study the thermodynamic and fluid dynamic interpretation of the gravitational field equations of the Einstein-Cartan theory w.r.t a generic null surface. The EC theory is a sub-class of such metric affine theories. For that purpose it is quite imperative that we study quite in its generality the geometry of a null surface in the RC spacetime.

With a view towards this goal, we began in this chapter with a very brief review about the geometrical properties of the RC spacetime. We then discussed about the construction of a null surface of codimension one generated by a hypersurface-orthogonal null congruence $\vec{l}$. We then studied in depth both the intrinsic and extrinsic geometry of such an integrable surface in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. We saw that the induced metric on the null surface is degenerate. In order to discuss concretely the kinematics of the null surface as viewed from the ambient spacetime, we considered a null foliation of the spacetime in the neighborhood of $\mathcal{H}$ by a family of null hypersurfaces. This allowed us to define the notion of a derivative of the null generators along a direction away from the surface. We then proceeded to study the extended Weingarten map (also known as the shape operator) as well as the extended second fundamental form. We then discussed about the rotation 1-form and the Hájíček 1-form. Next, we considered the kinematical evolution of the induced metric $\boldsymbol{q}$ along the null generators $\vec{l}$ and the auxiliary null vector field $\vec{k}$. These led us to the concepts of the deformation rate and the transversal deformation rate tensors. Finally we discussed about the projected (onto the 2-surface S_t) deviation tensor. Such construction of the null surface $\mathcal{H}$ and its defined kinematical quantities will be useful in the later chapters where we discuss the physical interpretations of the dynamics of the relevant gravitational theory with respect to the null surface. The few salient features that separate a generic hypersurface-orthogonal null surface $\mathcal{H}$ in RC spacetime from the one in say Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, are the following.

- Since the ambient connection ($\hat{\boldsymbol{\nabla}}$) is not torsion-free, we see that the connection $\hat{\boldsymbol{\mathcal{D}}}$ compatible with the spatial 2-metric $\boldsymbol{q}$ of the null surface is also non-unique and not torsion-free. The submanifold on the spatial 2-dimensional cross-section $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$ is provided with a metric-compatible linear connection $\hat{\boldsymbol{\mathcal{D}}}$ *i.e.* $\hat{\mathcal{D}}_a q_{bc} = 0$. We will have much more to say on this when we study the dynamics of $\mathcal{H}$ in the next chapter.

- Even in spite of the assumption of hypersurface-orthogonality of $\mathcal{H}$ in the RC spacetime, we found that the twist vector does not vanish.

- The relevant kinematical quantities are modified due to the presence of torsion. We saw that the Weingarten map is not self-adjoint for $\mathcal{H}$ in the RC spacetime. This leads to the fact that the extended second fundamental form (not symmetric) has only $\vec{k}$ as the degeneracy direction and not $\vec{l}$. The extended second fundamental form, the deformation rate tensor (symmetric) and the projected deviation tensor obviously are not equivalent and agree only when the torsion vanishes.

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Chapter 3

Dynamics of the generic null hypersurface

3.1 Introduction and motivation

In the last chapter, we have constructed the relevant kinematical quantities of interest for our analysis. Now, we shall proceed to discuss the evolution dynamics of these kinematical variables along the null generators $\vec{l}$ of $\mathcal{H}$. These will become the building blocks for our later chapters. Lets us remind ourselves that by "kinematics", we mean those quantities obtainable as the first order spacetime derivative of the null vector fields $\vec{l}$ and $\vec{k}$, the null 1-forms $\underline{l}$ and $\underline{k}$ as well as the metric field $\boldsymbol{g}$ and $\boldsymbol{q}$. Following [61] by "dynamics", we mean the quantities involving first derivatives of kinematical objects along a specified vector field in the spacetime. The null generator $\vec{l}$ is related to the time evolution vector field $\vec{t}$. For our consideration we would specifically call the first derivative (Lie or covariant) of kinematical variables along $\vec{l}$ to be dynamical quantities of interest. There can be evolution equations of kinematical variables along the auxiliary null vector field $\vec{k}$. However, with the hindsight of interpreting the gravitational field equations w.r.t $\mathcal{H}$ as having a fluid/thermodynamic nature, we would only require for our purposes, the evolution equations along $\vec{l}$. Now, as far Einstein's theory of gravity is concerned, the field equations expressed w.r.t a generic null hypersurface $\mathcal{H}$ give rise to important physical interpretations. To look at this, consider the expansion of the vector field $G^a{}_b l^b$ in the basis $(\vec{l}, \vec{k}, \vec{e}_A)$,

$$G^a{}_b l^b = \phi_1 l^a + \phi_2 k^a + \phi^A (\vec{e}_A)^a \,, \tag{3.1}$$

with $\phi_1 = -G_{ab}k^a l^b$, $\phi_2 = -G_{ab}l^a l^b$ and $\phi_A = (G_{ab}l^b q^a{}_c)(\vec{e}_A)^c$. These projection components have important thermodynamic and fluid-dynamic interpretation as far as Einstein gravity is concerned. It is well known that $G_{ab}l^a l^b$ arises in the null Raychaudhuri equation that governs the dynamical evolution of the outgoing expansion scalar along $\vec{l}$ [25, 61]. The projection component $G_{ab}l^b q^a{}_c$ arises in the context of the Damour-Navier-Stokes equation that describes the evolution of the Hájíček 1-form along $\vec{l}$ [61, 76, 80, 82]. The DNS equation is structurally similar to the NS equation. In fact, it has been pointed out that in a boosted inertial reference frame, the DNS equation reduces to the NS equation [82]. Finally, the projection component $G_{ab}k^a l^b$ in the Riemannian spacetime is related to the dynamics of the ingoing expansion scalar $\overset{(d)}{\theta}_{\boldsymbol{k}}$ along $\vec{\boldsymbol{l}}$. For Einstein gravity as well as Lanczos-Lovelock models of gravity this dynamical evolution law leads to a thermodynamic identity for a well defined physical process involving $\mathcal{H}$ [36]. This thermodynamic identity is structurally similar to the first law of (conventional) thermodynamics. Such evolution equations for $\overset{(d)}{\theta}_{\boldsymbol{l}}$ and $\overset{(d)}{\theta}_{\boldsymbol{k}}$ form a part of the so called focusing and cross-focusing equations [76, 135]. The focusing and cross-focusing equations and their relation to thermodynamics for local definitions of horizons have been studied in [135].

Our objective in the later chapters is to provide a thermodynamic and fluid-dynamic interpretation to the gravitational field equations corresponding to the RC spacetime. For that purpose, we would need the evolution equations corresponding to the projection components $\hat{G}_{ab}l^a l^b$, $\hat{G}_{ab}k^a l^b$ and $\hat{G}_{ab}l^a q^b{}_c$, where $\hat{G}_{ab}$ is the analogue of the Einstein tensor in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. Motivated by the results explicitly established for the case of Einstein gravity, we aim to see what exactly are the corresponding dynamical evolution equations in the RC spacetime. However, our consideration of the dynamics will be focused entirely on the null case. The NRE corresponding to the evolution of the outgoing expansion scalar $\overset{(d)}{\hat{\theta}}_{\boldsymbol{l}}$ (along $\vec{\boldsymbol{l}}$) will be related to the projection component $\hat{G}_{ab}l^a l^b$ and have already been derived in the literature [64, 136, 137]. However, for the sake of completeness, in Sec. 3.2, we provide a detailed derivation of the NRE for the outgoing expansion scalar $\overset{(d)}{\hat{\theta}}_{\boldsymbol{l}}$ within the formalism and convention that we follow. The NRE and the tidal force equation [61] (which concerns the geodesic deviation equation between two neighboring null congruences) are a part of the optical scalar equations [136, 138] obtained within the Newman-Penrose formalism [139, 140]. The Raychaudhuri equation and other optical scalar equations for timelike congruences in the case of RC spacetime has been derived in [137, 141, 142]. We also derive the dynamical evolution equations of the ingoing expansion scalar $\overset{(d)}{\hat{\theta}}_{\boldsymbol{k}}$ and the Hájiček 1-form in Sec. 3.3 and Sec. 3.4 and show they are determined by the projection components $\hat{G}_{ab}k^a l^b$ and $\hat{G}_{ab}l^a q^b{}_c$ respectively.

3.2 The dynamics for $\hat{G}_{ab}l^a l^b$: The null Raychaudhuri equation for $\mathcal{H}$ in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$

We begin with the dynamical evolution equation corresponding to the projection component $\hat{G}_{ab}l^b l^b$. Doing this, we arrive at the NRE for our generic null hypersurface in the RC spacetime. The Raychaudhuri equations (for both timelike and null congruences) find diverse applications in physics and hence hold enormous significance. The Raychaudhuri equations at their core are geometric relations involving the dynamics of *flows* and hence find applications beyond theories of gravity. For a very pedagogical introduction to the Raychaudhuri equations and its diverse applications to various areas in physics, the reader may wish to consult [143].

In order to arrive at the NRE, we can reap the benefit of the induced foliation of our spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ in the neighbourhood of $\mathcal{H}$ by the null family of hypersurfaces. The NRE determines the dynamics of the outgoing expansion scalar $\overset{(d)}{\hat{\theta}}_l$ along the null generator $\vec{l}$. To arrive at the NRE we start from the null Codacci identity established on $\mathcal{H}$,

$$\hat{\nabla}_a(\hat{\nabla}_b l^a) - \hat{\nabla}_b(\hat{\nabla}_a l^a) = \hat{R}_{ab}l^a - T^d_{\ ab}(\hat{\nabla}_d l^a)\,. \tag{3.2}$$

Making judicious use of Eq. (2.90) on both sides of (3.2), we arrive at the following evolution equation,

$$\begin{aligned}(\hat{\nabla}_a T_b{}^a{}_c)l^b l^c + (T_{abc} + T_{cba} + T_{bac})\hat{\Theta}^{ab}l^c + (k_a \mathbb{T}^a)(k_b \mathbb{T}^b) - \hat{\Theta}_{ab}\hat{\Theta}^{ba} \\ +\kappa \overset{(d)}{\hat{\theta}}_l - l^b \hat{\nabla}_b \overset{(d)}{\hat{\theta}}_l + l^b l^a(\hat{\nabla}_b T_a) + T^a \mathbb{T}_a = \hat{R}_{ab}l^a l^b\,.\end{aligned} \tag{3.3}$$

The proof of the above relation (3.3) has been shown in the appendix 3.6. Notice that (3.3) relates the evolution $l^b \hat{\nabla}_b \overset{(d)}{\hat{\theta}}_l$ of the outgoing expansion scalar $\overset{(d)}{\hat{\theta}}_l$ to $\hat{R}_{ab}l^a l^b$ and hence $\hat{G}_{ab}l^a l^b$ (since $\hat{G}_{ab} = \hat{R}_{ab} - 1/2\, g_{ab}\hat{R}$).

However, we should be careful not to assign (3.3) as being interpreted as the NRE. This is because we notice the trace of the extended second fundamental form $\hat{\Theta}_{ab}$ is not the true outgoing expansion scalar $\overset{(d)}{\hat{\theta}}_l$. It would be better to rewrite any $\hat{\Theta}_{ab}$ occurring in (3.3) by the corresponding projected deviation tensor $\hat{\mathcal{B}}_{ab}$ or the deformation rate tensor $\hat{\chi}_{ab}$, since the trace of both of these tensors gives the true outgoing expansion scalar $\overset{(d)}{\hat{\theta}}_l$. This can be done by the virtue of (2.146) or (2.125). However here, we will progress with the projected deviation tensor in order to corroborate our results with [64]. Using (2.146) through some moderate but straightforward algebra, it can be shown that,

$$\begin{aligned}\hat{\Theta}_{ab}\hat{\Theta}^{ba} = \hat{\mathcal{B}}_{ab}\hat{\mathcal{B}}^{ba} - 2\hat{\mathcal{B}}^{ab}l^c T_{bca} + (T_{aib}l^i)(T^{bka}l_k) + (\mathbb{T}_a k^a)(\mathbb{T}_b k^b) \\ -2\mathbb{T}^a T_{aib}l^i k^b\,.\end{aligned} \tag{3.4}$$

Using the symmetries of the torsion tensor, similar straightforward algebra shows that,

$$\begin{aligned}l^c(T_{abc} + T_{cba} + T_{bac})\hat{\Theta}^{ab} &= -\hat{\mathcal{B}}^{ab}l^c(T_{acb} + T_{bca} + T_{cab}) \\ + T^{akb}l_k l^c(T_{acb} + T_{bca} + T_{cab}) &+ 2T_{cab}l^c \mathbb{T}^a k^b - 2(\mathbb{T}_a k^a)(\mathbb{T}_b k^b) - 2T_{ijb}k^i l^j \mathbb{T}^b\,.\end{aligned} \tag{3.5}$$

Combining the last two relations (3.4) and (3.5), we have after some simplification,

$$l^c(T_{abc}+T_{cba}+T_{bac})\hat{\Theta}^{ab}-\hat{\Theta}_{ab}\hat{\Theta}^{ba}=-\hat{\mathcal{B}}^{ab}(T_{acb}-T_{bca}+T_{cab})l^c$$
$$+T^a{}_i{}^b l^i l^j(T_{ajb}+T_{jab})-\hat{\mathcal{B}}^{ab}\hat{\mathcal{B}}_{ba}-3(\mathbb{T}_a k^a)(\mathbb{T}_b k^b)-2\mathbb{T}^a T_{abc}k^b l^c$$
$$-2\mathbb{T}^c T_{abc}k^a l^b+2\mathbb{T}^b T_{cba}l^c k^a\,. \tag{3.6}$$

Putting (3.6) in (3.3), we obtain as a result,

$$l^a\hat{\nabla}_a\overset{(d)}{\hat{\theta}_l}=-\hat{R}_{ab}l^a l^b+(\hat{\nabla}^a T_{bac})l^b l^c+\kappa\overset{(d)}{\hat{\theta}_l}+l^b l^a(\hat{\nabla}_b T_a)+T^a\mathbb{T}_a$$
$$-\hat{\mathcal{B}}^{ab}(T_{acb}-T_{bca}+T_{cab})l^c+l^i l^j T^a{}_i{}^b(T_{ajb}+T_{jab})-\hat{\mathcal{B}}^{ab}\hat{\mathcal{B}}_{ba}$$
$$+\Big[-2(\mathbb{T}_a k^a)(\mathbb{T}_b k^b)-2\mathbb{T}^a k^b l^c(T_{abc}+T_{bca}+T_{cba})\Big]\,. \tag{3.7}$$

As usual, using (2.147) and using the fact that $\hat{R}_{ab}l^a l^b=\hat{G}_{ab}l^a l^b$, we have finally the NRE corresponding to a hypersurface-orthogonal null congruence,

$$l^a\hat{\nabla}_a\overset{(d)}{\hat{\theta}_l}=-\hat{G}_{ab}l^a l^b+(\hat{\nabla}^a T_{bac})l^b l^c+\kappa\overset{(d)}{\hat{\theta}_l}+l^b l^a(\hat{\nabla}_b T_a)+T^a\mathbb{T}_a$$
$$-{}^{(l,B)}\omega^{ab}(T_{acb}-T_{bca}+T_{cab})l^c+l^i l^j T^a{}_i{}^b(T_{ajb}+T_{jab})$$
$$-\frac{1}{2}(\overset{(d)}{\hat{\theta}_l})^2-{}^{(l,B)}\hat{\sigma}_{ab}{}^{(l,B)}\sigma^{ab}+{}^{(l,B)}\hat{\omega}_{ab}{}^{(l,B)}\omega^{ab}$$
$$+\Big[-2(\mathbb{T}_a k^a)(\mathbb{T}_b k^b)-2\mathbb{T}^a k^b l^c(T_{abc}+T_{bca}+T_{cba})\Big]\,. \tag{3.8}$$

In the Riemannian spacetime $(\mathcal{M},\mathbf{g},\boldsymbol{\nabla})$, the above evolution equation (3.8) reduces to,

$$l^a\nabla_a\theta_l=-G_{ab}l^a l^b+\kappa\theta_l-\frac{1}{2}(\theta_l)^2-{}^{(l)}\sigma_{ab}{}^{(l)}\sigma^{ab}\,. \tag{3.9}$$

This is the usual NRE for the null generators $\vec{l}$ in the Riemannian spacetime. The above Eq. (3.8) represents the evolution of the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_l}$ along the null congruence $\vec{l}$. Notice, that as of yet we have not imposed the geodesic constraint and hence even though the congruence $\vec{l}$ is geodesic, it not auto-parallel. The present Eq. (3.8) should be matched with Eq. (66) of [64]. Eq. (66) of [64] has been written down for an affinely parametrized null congruence ($\kappa=0$). Our Eq. (3.8) matches exactly with Eq. (66) of [64] except for the last terms in the squared parentheses (*i.e.* the terms containing $\mathbb{T}_a$). Even though the authors of [64] claim that their Eq. (66) represents the NRE in its full generality, we believe that they have missed the terms in the square parentheses. Notice that the NRE (3.8) in this generality depends upon the auxiliary null vector field $\vec{k}$. This is somewhat a rather peculiar feature. This is because, for our construction the auxiliary null vector is uniquely defined. The auxiliary null field is transverse to the null generator as well as being orthogonal to the 2-dimensional sub-space S_t. The dynamical equation (3.8) necessarily implies that the evolution of the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_l}$ along the null generator $\vec{l}$ actually encodes information of a direction that is transverse to the null generators and orthogonal to the spacelike submanifold of $\mathcal{H}$. It is quite an instructive exercise to verify that if we decompose (3.8) into its pure Riemannian parts and the pure torsion terms on both sides of the equation, we end up having the NRE for the outgoing expansion scalar $\overset{(d)}{\theta_l} = \theta_l$ of $\mathcal{H}$ established in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ (provided with the Levi-Civita connection $\boldsymbol{\nabla}$). The explicit details of this has been relegated to appendix 3.7.

Finally, if we want to consider a system of hypersurface-orthogonal auto-parallel geodesic null congruence generating $\mathcal{H}$, then we have to impose the geodesic constraint $\mathbb{T}_a = 0$ in (3.8). For this particular case then, we have,

$$
\begin{aligned}
l^a \hat{\nabla}_a \overset{(d)}{\hat{\theta}_l} = & -\hat{R}_{ab} l^a l^b + \kappa \overset{(d)}{\hat{\theta}_l} - \frac{1}{2} (\overset{(d)}{\hat{\theta}_l})^2 - {}^{(l,B)}\hat{\sigma}_{ab} {}^{(l,B)}\sigma^{ab} + {}^{(l,B)}\hat{\omega}_{ab} {}^{(l,B)}\omega^{ab} \\
& - {}^{(l,B)}\omega^{ab} (T_{acb} - T_{bca} + T_{cab}) l^c + l^i l^j T^a{}_i{}^b (T_{ajb} + T_{jab}) \\
& + (\hat{\nabla}^a T_{bac}) l^b l^c + l^b l^a (\hat{\nabla}_b T_a) \, . \qquad (3.10)
\end{aligned}
$$

The above equation under the geodesic constraint determines the evolution of the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_l}$ along $\vec{l}$ and contains explicitly no signature of the auxiliary null field $\vec{k}$.

3.3 The dynamics related to $\hat{G}_{ab}k^a l^b$

In this section, we will focus on the dynamical evolution equation related to the projection component $\hat{G}_{ab}k^a l^b$. In doing so, we will first derive the evolution equation along $\vec{l}$ of the transversal deformation rate tensor $\hat{\Xi}_{ab}$. Then we will proceed to find the evolution rate of the expansion scalar $\overset{(d)}{\hat{\theta}_k}$ along the null generators. We will see in a later chapter, that this will then provide our starting point towards a thermodynamic interpretation that can be attributed to the relevant projection component. At the outset, let us establish the fact that the dynamics of $\hat{G}_{ab}l^a k^b$ will be studied entirely keeping the geodesic constraint Eq. (2.34) in mind. We will use the fact that $\mathbb{T}^a = 0$ in all subsequent analysis. Under the geodesic constraint, as we have discussed at the end of section 2.4.1, that the null generators of $\mathcal{H}$ are both auto-parallel and geodesic. To proceed, we start with the Ricci identity established on the manifold of the transverse spacelike 2-surface $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$, where $\hat{\boldsymbol{\mathcal{D}}}$ is the spatial covariant derivative compatible with $\boldsymbol{q}$ *i.e.* $\hat{\mathcal{D}}_a q_{bc} = 0$. By definition, for any spatial vector $\vec{v} \in \mathcal{T}_P(S_t)$, we have

$$\hat{\mathcal{D}}_a v^b \equiv q^i{}_a q^b{}_j \hat{\nabla}_i v^j \text{ and } \hat{\mathcal{D}}_a v_b \equiv q^i{}_a q^j{}_b \hat{\nabla}_a v_b \,, \tag{3.11}$$

and its obvious generalization to tensor fields on S_t follows. So for the spatial vector $\vec{v}$,

$$[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] v^a = {}^{(2)}\hat{R}_{ab}\, v^a - {}^{(2)}T^d{}_{ab} \hat{\mathcal{D}}_d v^a \,, \tag{3.12}$$

where, ${}^{(2)}\hat{R}_{ab}$ and ${}^{(2)}T^a{}_{bc}$ are the 2-dimensional Ricci and torsion tensors established respectively on $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$. Let us first take on the left-hand side (L.H.S) of (3.12). We show (via a detailed calculation in Appendix 3.8) that,

$$\begin{aligned}
[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] v^a &= \hat{\mathcal{D}}_a(\hat{\mathcal{D}}_b v^a) - \hat{\mathcal{D}}_b(\hat{\mathcal{D}}_a v^a) \\
&= \Big[(\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} - q^{cd} T_{cfd} l^f) q^m_b - \hat{\chi}^m_b + q^c{}_b q^{dm} K_{fcd} l^f \Big] (k_s \hat{\nabla}_m v^s) \\
&+ \Big[(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{cd} T_{cfd} k^f) q^m_b - \hat{\Xi}^m_b + q^c{}_b q^{dm} K_{fcd} k^f \Big] (l_s \hat{\nabla}_m v^s) \\
&+ q^m_b q^l{}_s q^p{}_t \hat{R}^s{}_{plm} v^t + q^c{}_b q^d{}_s q^m_f T^f{}_{cd} (\hat{\nabla}_m v^s) \,.
\end{aligned} \tag{3.13}$$

Looking at the R.H.S of (3.12) we have on account of ${}^{(2)}T^a{}_{bc}$ being a tensor defined on the transverse 2-dimensional space S_t,

$$\begin{aligned}
&{}^{(2)}\hat{R}_{ab}\, v^a - {}^{(2)}T^d{}_{ab} \hat{\mathcal{D}}_d v^a = {}^{(2)}\hat{R}_{ab} v^a + {}^{(2)}T^d{}_{ba} q^m_d q^a{}_s (\hat{\nabla}_m v^s) \\
&= {}^{(2)}\hat{R}_{ab}\, v^a + q^c{}_b q^d{}_a {}^{(2)}T^f{}_{cd} q^m_f q^a{}_s (\hat{\nabla}_m v^s) = {}^{(2)}\hat{R}_{ab}\, v^a + q^c{}_b q^d{}_s q^m_f {}^{(2)}T^f{}_{cd} (\hat{\nabla}_m v^s) \,.
\end{aligned} \tag{3.14}$$

Incidentally, it can be proven that the complete projection of the spacetime torsion onto the 2-dimensional transverse space S_t is equivalent to the intrinsic torsion established in $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$ *i.e.*

$$q^c{}_b q^d{}_s q^m_f T^f{}_{cd} = q^c{}_b q^d{}_s q^m_f {}^{(2)}T^f_{cd} \,. \tag{3.15}$$

For the detailed derivation of this, see Appendix 3.9. Upon equating the L.H.S and the R.H.S of (3.12) via the relations (3.13) and (3.14) respectively, and using (3.15), we end up having,

$$
\begin{aligned}
&-\underbrace{\left[(\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} - q^{cd}T_{cfd}l^f)q^m_b - \hat{\chi}^m_b + q^c{}_b q^{dm} K_{fcd} l^f\right](v^a \hat{\nabla}_m k_a)}_{\text{expression 1}} \\
&-\underbrace{\left[(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{cd}T_{cfd}k^f)q^m_b - \hat{\Xi}^m_b + q^c{}_b q^{dm} K_{fcd} k^f\right](v^a \hat{\nabla}_m l_a)}_{\text{expression 2}} \\
&+ q^m_b q^l{}_s q^p{}_a \hat{R}^s{}_{plm} v^a = {}^{(2)}\hat{R}_{ab} v^a
\end{aligned} \tag{3.16}
$$

Let us focus on expression 1. Again as usual, using (2.134) for the covariant derivative of the auxiliary null normal and simplifying, we obtain,

$$
\begin{aligned}
\text{expression 1} = v^a \Big[&\hat{\chi}^m_b\, \hat{\Xi}_{ma} - \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} - q^{cd}T_{cfd}l^f\Big)\hat{\Xi}_{ba} - q^c{}_b \hat{\Xi}^d{}_a (K_{fcd}l^f) \\
&- \hat{\chi}^c{}_b q^d{}_a (K_{fcd}k^f) + \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} - q^{cd}T_{cfd}l^f\Big) q^c{}_b q^d{}_a (K_{fcd}k^f) \\
&+ q^c{}_b q^{di} q^j{}_a (K_{fcd}l^f)(K_{hij}k^h)\Big] .
\end{aligned} \tag{3.17}
$$

Analogously, for expression 2, we use (2.126) and simplify to yield,

$$
\begin{aligned}
\text{expression 2} = v^a \Big[&\hat{\Xi}^m_b\, \hat{\chi}_{ma} - \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{cd}T_{cfd}k^f\Big)\hat{\chi}_{ba} - q^c{}_b \hat{\chi}^d{}_a (K_{fcd}k^f) \\
&- \hat{\Xi}^i{}_b q^j{}_a (K_{hij}l^h) + \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{cd}T_{cfd}k^f\Big) q^i{}_b q^j{}_a (K_{hij}l^h) \\
&+ q^c{}_b q^{di} q^j{}_a (K_{fcd}k^f)(K_{hij}l^h)\Big] .
\end{aligned} \tag{3.18}
$$

Adding up both the expressions (3.17) and (3.18) in (3.16), we end up having the expression of the 2-dimensional Ricci tensor in terms of the 4-dimensional spacetime quantities (since v^a is arbitrary),

$$
\begin{aligned}
{}^{(2)}\hat{R}_{ab} = {}& q^m_b q^l{}_s q^p{}_a\, \hat{R}^s{}_{plm} \\
&+ \Big[\Big(\hat{\chi}^m_b \hat{\Xi}_{ma} + \hat{\Xi}^m_b \hat{\chi}_{ma}\Big) - \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} - q^{ij}T_{ihj}l^h\Big)\Big(\hat{\Xi}_{ba} - q^c{}_b q^d{}_a K_{fcd}k^f\Big) \\
&- \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big)\Big(\hat{\chi}_{ba} - q^c{}_b q^d{}_a K_{fcd}l^f\Big) - \Big(q^c{}_b \hat{\Xi}^d{}_a + q^d{}_a \hat{\Xi}^c{}_b\Big) K_{fcd}l^f \\
&- \Big(q^c{}_b \hat{\chi}^d{}_a + q^d{}_a \hat{\chi}^c{}_b\Big) K_{fcd}k^f + \Big(q^c{}_b q^{di} q^j{}_a + q^i{}_b q^{jc} q^d{}_a\Big)(K_{fcd}l^f)(K_{hij}k^h)\Big] .
\end{aligned} \tag{3.19}
$$

Now, we focus on the term $q^m_{\ b} q^l_{\ s} q^p_{\ a}\ \hat{R}^s_{\ plm}$ of (3.19). After some routine computations (see Appendix 3.10), it can be shown that,

$$
\begin{aligned}
q^m_{\ b} q^l_{\ s} q^p_{\ a}\ \hat{R}^s_{\ plm} &= q^m_{\ b} q^p_{\ a} \hat{\mathcal{Q}}_{rmsp} k^r l^s + q^m_{\ b} q^p_{\ a} \hat{R}_{pm} + 2\hat{\Omega}_a \hat{\Omega}_b - (\hat{\Omega}_a \hat{\mathscr{P}}_b + \hat{\mathscr{P}}_a \hat{\Omega}_b) \\
&- q^m_{\ b} q^p_{\ a} l^r \hat{\nabla}_r \Big[2\hat{\Xi}_{mp} - \Big(q^i_{\ m} q^j_{\ p} + q^i_{\ p} q^j_{\ m} \Big) (K_{hij} k^h) \Big] \\
&+ q^m_{\ b} q^p_{\ a} \Big[\Big(\hat{\nabla}_m \hat{\omega}_p + \hat{\nabla}_p \hat{\omega}_m \Big) - \Big(\hat{\nabla}_m \hat{\mathscr{P}}_p + \hat{\nabla}_p \hat{\mathscr{P}}_m \Big) \Big] \\
&- \Big(\hat{\chi}^r_{\ b} \hat{\Xi}_{ra} + \hat{\chi}^r_{\ a} \hat{\Xi}_{rb} \Big) + \Big(q^j_{\ a} \hat{\chi}^i_{\ b} + q^j_{\ b} \hat{\chi}^i_{\ a} \Big) (K_{hij} k^h) + \Big(q^c_{\ a} \hat{\Xi}^d_{\ b} + q^c_{\ b} \hat{\Xi}^d_{\ a} \Big) (K_{fcd} l^f) \\
&- \Big(\hat{\Xi}^c_{\ a} q^d_{\ b} + \hat{\Xi}^c_{\ b} q^d_{\ a} \Big) (T_{cfd} l^f) - \Big[q^c_{\ b} q^j_{\ a} q^{di} + q^c_{\ a} q^j_{\ b} q^{di} \Big] (K_{fcd} l^f)(K_{hij} k^h) \\
&+ \Big[q^j_{\ a} q^d_{\ b} q^{ci} + q^j_{\ b} q^d_{\ a} q^{ci} \Big] (T_{cfd} l^f)(K_{hij} k^h)\,.
\end{aligned}
\tag{3.20}
$$

In the above, we have defined the quantity $\hat{\mathscr{P}}_a$ accordingly as,

$$
\hat{\mathscr{P}}_a \equiv T_{bcd} k^b l^c q^d_{\ a}\,. \tag{3.21}
$$

Obviously, the quantity $\hat{\mathscr{P}}_a$ is orthogonal to both l^a and k^a and hence is defined on the 2-surface S_t. Converting the spacetime covariant derivatives present in the above relation (3.20) to Lie derivatives along the null generator $\vec{l}$, we obtain from the above,

$$
\begin{aligned}
q^m_{\ b} q^l_{\ s} q^p_{\ a}\ \hat{R}^s_{\ plm} &= q^m_{\ b} q^p_{\ a} \hat{\mathcal{Q}}_{rmsp} k^r l^s + q^m_{\ b} q^p_{\ a} \hat{R}_{pm} + 2\hat{\Omega}_a \hat{\Omega}_b - (\hat{\Omega}_a \hat{\mathscr{P}}_b + \hat{\mathscr{P}}_a \hat{\Omega}_b) \\
&- q^m_{\ b} q^p_{\ a} \pounds_l \Big[2\hat{\Xi}_{mp} - \Big(q^i_{\ m} q^j_{\ p} + q^i_{\ p} q^j_{\ m} \Big) (K_{hij} k^h) \Big] + \Big(\hat{\chi}_{ar} \hat{\Xi}^r_{\ b} + \hat{\chi}_{br} \hat{\Xi}^r_{\ a} \Big) \\
&+ \hat{\mathcal{D}}_a (\hat{\Omega}_b - \hat{\mathscr{P}}_b) + \hat{\mathcal{D}}_b (\hat{\Omega}_a - \hat{\mathscr{P}}_a) - 2\kappa \hat{\Xi}_{ab} + \Big(\hat{\Xi}^c_{\ a} q^d_{\ b} + \hat{\Xi}^c_{\ b} q^d_{\ a} \Big) (T_{cfd} l^f) \\
&- \Big(q^c_{\ a} \hat{\Xi}^d_{\ b} + q^c_{\ b} \hat{\Xi}^d_{\ a} \Big) (K_{fcd} l^f) + \Big[q^i_{\ a} (\kappa q^j_{\ b} - \hat{\chi}^j_{\ b}) + q^i_{\ b} (\kappa q^j_{\ a} - \hat{\chi}^j_{\ a}) \Big] (K_{hij} k^h) \\
&- \Big[\Big(q^i_{\ a} q^d_{\ b} + q^i_{\ b} q^d_{\ a} \Big) q^{jc} \Big] (T_{cfd} l^f)(K_{hij} k^h) \\
&+ \Big[\Big(q^c_{\ a} q^i_{\ b} + q^c_{\ b} q^i_{\ a} \Big) q^{jd} \Big] (K_{fcd} l^f)(K_{hij} k^h)\,.
\end{aligned}
\tag{3.22}
$$

We relegate the details of the computation to Appendix 3.10. Now all that remains to do is to put Eq. (3.22) into Eq. (3.19) to get to the evolution equation of the transversal deformation rate tensor,

$$
\begin{aligned}
& q^m{}_b q^p{}_a \pounds_{\boldsymbol{l}} \hat{\Xi}_{mp} - \frac{1}{2} q^m{}_b q^p{}_a \pounds_{\boldsymbol{l}} \Big[\Big(q^i{}_m q^j{}_p + q^i{}_p q^j{}_m \Big) (K_{hij} k^h) \Big] = -\frac{1}{2} {}^{(2)}\hat{R}_{ab} \\
& + \frac{1}{2} q^m{}_b q^p{}_a \hat{\mathcal{Q}}_{rmsp} k^r l^s + \frac{1}{2} q^m{}_b q^p{}_a \hat{R}_{pm} + \Big(\hat{\chi}_{ar} \hat{\Xi}^r{}_b + \hat{\chi}_{br} \hat{\Xi}^r{}_a \Big) \\
& + \frac{1}{2} \Big(\hat{\mathcal{D}}_a (\hat{\Omega}_b - \hat{\mathscr{P}}_b) + \hat{\mathcal{D}}_b (\hat{\Omega}_a - \hat{\mathscr{P}}_a) \Big) - \Big(\kappa + \frac{\overset{(d)}{\hat{\theta}}_{\boldsymbol{l}}}{2} \Big) \hat{\Xi}_{ab} \\
& - \frac{\overset{(d)}{\hat{\theta}}_{\boldsymbol{k}}}{2} \hat{\chi}_{ab} + \hat{\Omega}_a \hat{\Omega}_b - \frac{1}{2} (\hat{\Omega}_a \hat{\mathscr{P}}_b + \hat{\Omega}_b \hat{\mathscr{P}}_a) + \frac{1}{2} \Big[(\hat{\Xi}^c{}_a q^d{}_b + \hat{\Xi}^c{}_b q^d{}_a) + q^{cd} \hat{\Xi}_{ab} \Big] (T_{cfd} l^f) \\
& + \frac{1}{2} \Big[\overset{(d)}{\hat{\theta}}_{\boldsymbol{k}} \, q^c{}_b q^d{}_a - q^c{}_a \hat{\Xi}^d{}_b - q^c{}_b \hat{\Xi}^d{}_a - q^c{}_b \hat{\Xi}^d{}_a - q^d{}_a \hat{\Xi}^c{}_b \Big] (K_{fcd} l^f) \\
& + \frac{1}{2} \Big[q^i{}_a \Big(\kappa q^j{}_b - \hat{\chi}^j{}_b \Big) + q^i{}_b \Big(\kappa q^j{}_a - \hat{\chi}^j{}_a \Big) \\
& \qquad + \overset{(d)}{\hat{\theta}}_{\boldsymbol{l}} \, q^i{}_b q^j{}_a - \hat{\chi}^i{}_b q^j{}_a - \hat{\chi}^j{}_a q^i{}_b + q^{ij} \hat{\chi}_{ab} \Big] (K_{hij} k^h) \\
& - \frac{1}{2} \Big[\Big(q^i{}_a q^d{}_b + q^i{}_b q^d{}_a \Big) q^{jc} + q^{cd} q^i{}_b q^j{}_a \Big] (T_{cfd} l^f)(K_{hij} k^h) \\
& - \frac{1}{2} \Big(q^{ij} q^c{}_b q^d{}_a \Big) (T_{ihj} k^h)(K_{fcd} l^f) \\
& + \frac{1}{2} \Big[\Big(q^c{}_a q^i{}_b + q^c{}_b q^i{}_a \Big) q^{jd} + q^c{}_b q^{di} q^j{}_a + q^i{}_b q^{jc} q^d{}_a \Big] (K_{fcd} l^f)(K_{hij} k^h) \, . \qquad (3.23)
\end{aligned}
$$

So we have arrived at the evolution equation for the transversal deformation rate tensor in the metric-compatible general affine spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ under the geodesic constraint $(\mathbb{T}^a = 0)$ *i.e.* the null geodesic generators of $\mathcal{H}$ are parallel-transported along themselves. In the absence of torsion in the spacetime this equation matches with Eq. (6.43) of [61].

Now, that we have derived the evolution equation of the transversal deformation rate tensor, we proceed to study the evolution of the expansion scalar $\overset{(d)}{\hat{\theta}}_{\boldsymbol{k}}$. This would enable us to provide the thermodynamic relation established on $\mathcal{H}$ for the gravitational dynamics in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. To get to the evolution equation for $\overset{(d)}{\hat{\theta}}_{\boldsymbol{k}}$, all we need to is to take the trace of the evolution equation of the transversal deformation rate tensor (3.23). However this can be avoided to the benefit of a shorter route that involves taking the trace of (3.19) and

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(3.20) and then combining them. The result of the trace computation leads us to,

$$
\begin{aligned}
-\kappa\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) = \Big[\frac{1}{2}{}^{(2)}\hat{R} + l^r\hat{\nabla}_r\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a \\
-\hat{\mathcal{D}}_a(\hat{\Omega}^a - \hat{\mathscr{P}}^a) + \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}q^{cd} - \hat{\Xi}^{cd}\Big)(T_{cfd}l^f) \\
-\Big(q^{dj}q^{ci} - q^{cd}q^{ij}\Big)(T_{cbd})(K_{aij})k^a l^b\Big] \\
-\Big[\hat{G}_{ab} + (\hat{\nabla}_a T_b - \hat{\nabla}_b T_a) + (\hat{\nabla}_i + T_i)T^i{}_{ab}\Big]k^a l^b \,. \qquad (3.24)
\end{aligned}
$$

In the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the above dynamical evolution equation reduces to,

$$
-\kappa\theta_{\boldsymbol{k}} = \Big[\frac{1}{2}{}^{(2)}R + l^r\nabla_r\theta_{\boldsymbol{k}} - \Omega_a\Omega^a - \mathcal{D}_a\Omega^a + \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}}\Big] - G_{ab}k^a l^b \,. \qquad (3.25)
$$

The above equation (3.25) becomes a special case of the cross-focusing equation [76]. All the relevant steps involved in the computation for the trace leading up to (3.24) have been shown in Appendix 3.11. The above geometrical relation involves the directional derivative of the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}$ along the null generator $\vec{l}$ being related to the component $\hat{G}_{ab}k^a l^b$. It is in this sense that the above equation can be referred to as the NRE for $\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}$.

3.4 The dynamics corresponding to $\hat{G}_{ab}l^a q^b{}_c$

In this section, we will describe the dynamical evolution law of a kinematical quantity that is related to the projection component $\hat{G}_{ab}l^a q^b{}_c$. The kinematical quantity of interest to us, whose dynamics we study here is the the Hájíček 1-form. We will indeed see that dynamical evolution of $\hat{\Omega}_a$ along $\vec{l}$ is related to the projection component $\hat{G}_{ab}l^a q^b{}_c$. To arrive at this, let us begin with the null Ricci identity established for l^a,

$$
\Big[\hat{\nabla}_b, \hat{\nabla}_a\Big]l^b = \hat{R}_{ca}l^c - T^i{}_{ba}(\hat{\nabla}_i l^b) \,. \qquad (3.26)
$$

We will be interested in the projection of the above dynamical equation on the transverse two-surface S_t of $\mathcal{H}$,

$$
\Big[\hat{\nabla}_b, \hat{\nabla}_a\Big]l^b q^a{}_t = \hat{R}_{ab}l^a q^b{}_t - T^i{}_{ba}(\hat{\nabla}_i l^b)q^a{}_t \,. \qquad (3.27)
$$

From here along, our procedure will follow [61]. In order to remove the clutter of indices, let us define the following spatial tensor,

$$\tilde{t}_{ij} \equiv q_j{}^r q_i{}^s K_{trs} l^t \,. \tag{3.28}$$

Let us also define the following quantity,

$$\hat{\Phi}^b{}_a \equiv \hat{\chi}^b{}_a - q^c{}_a q^{db} K_{fcd} l^f = \hat{\chi}^b{}_a - \tilde{t}^b{}_a \,. \tag{3.29}$$

$\hat{\Phi}^b{}_a$ is a completely spatial $(1,1)$ tensor that is orthogonal to both the $\vec{l}$ and $\vec{k}$ directions. It is easily verified that,

$$\hat{\Phi}_{ab} - \hat{\Phi}_{ba} = 2\hat{\Phi}_{[ab]} = q^c{}_a q^d{}_b T_{fcd} l^f \,. \tag{3.30}$$

It can be shown that the (Lie) evolution of the Hájíček 1-form $\underline{\hat{\Omega}}$ along the null generator $\vec{l}$ is indeed related to the quantity $\hat{R}_{ab}l^a q^b{}_t = \hat{G}_{ab}l^a q^b{}_t$ via the following relationship,

$$\begin{aligned}
& q^a{}_t \pounds_l \hat{\Omega}_a + \hat{\mathcal{D}}_b \hat{\Phi}^b{}_t - \hat{\chi}_{bt} \mathscr{P}^b + q^c{}_t q^d{}_b (K_{fcd} l^f) \mathscr{P}^b - \hat{\Xi}_{tb} \mathbb{T}^b \\
& + \hat{\Omega}_t \Big(\overset{(d)}{\hat{\theta}_l} + k_b \mathbb{T}^b - T_b l^b \Big) - \hat{\mathcal{D}}_t \Big(\overset{(d)}{\hat{\theta}_l} + \kappa - T_b l^b \Big) + q^c{}_t q^d{}_b (K_{fdc} k^f) \mathbb{T}^b \\
& = \hat{G}_{ab} l^a q^b{}_t - T_{iba} \hat{\chi}^{bi} q^a{}_t + q^a{}_t (T_{iba} k^i q^{cb} \mathbb{T}_c) + q^a{}_t (q^{ci} q^{db} T_{iba} (K_{fcd} l^f)) \,.
\end{aligned} \tag{3.31}$$

The explicit proof of the above equation (3.31) has been provided in Appendix 3.12. We now consider the completely spatial tensor $\hat{\Phi}_{bt}$ and perform an irreducible decomposition of it by breaking it up into a trace part $(\overset{*}{\hat{\theta}})$, a symmetric traceless part $(\overset{*}{\hat{\sigma}}_{bt})$ and an anti-symmetric traceless part $(\overset{*}{\hat{\omega}}_{bt})$. The trace of this spatial tensor $\hat{\Phi}_{bt}$ is,

$$\begin{aligned}
\overset{*}{\hat{\theta}} = q^{bt} \hat{\Phi}_{bt} &= q^b{}t \Big(\hat{\chi}_{bt} - q^c{}_t q^d{}_b K_{fcd} l^f \Big) = \overset{(d)}{\hat{\theta}_l} - q^{dc} K_{fcd} l^f \\
&= \overset{(d)}{\hat{\theta}_l} - (g^{cd} + l^c k^d + k^c l^d) K_{fcd} l^f = \overset{(d)}{\hat{\theta}_l} - T_b l^b + \mathbb{T}_b k^b \,.
\end{aligned} \tag{3.32}$$

In arriving at the above result, we have used the antisymmetry of the contorsion tensor in the first and third indices along with the fact that $K_a{}^b{}_b = T_a$. The symmetric traceless part of the tensor $\hat{\Phi}_{bt}$ is ,

$$\overset{*}{\hat{\sigma}}_{bt} = \hat{\Phi}_{(bt)} - \frac{1}{2}q_{bt}\overset{*}{\hat{\theta}} = \left(\hat{\chi}_{(bt)} - q^c{}_t q^d{}_b K_{f(cd)} l^f\right) - \frac{1}{2}q_{bt}\left(\overset{(d)}{\hat{\theta}_l} - T_b l^b + \mathbb{T}_b k^b\right)$$
$$= \left(\hat{\chi}_{bt} - \frac{1}{2}q_{bt}\overset{(d)}{\hat{\theta}_l}\right) - \left(q^c{}_t q^d{}_b K_{f(cd)} l^f - \frac{1}{2}q_{bt}(T_b l^b - \mathbb{T}_b k^b)\right)$$
$$= {}^{(l,d)}\sigma_{bt} - \left(q^c{}_t q^d{}_b K_{f(cd)} l^f - \frac{1}{2}q_{bt}(T_b l^b - \mathbb{T}_b k^b)\right). \tag{3.33}$$

Similarly, the traceless anti-symmetric part of $\hat{\Phi}_{bt}$ is,

$$\overset{*}{\hat{\omega}}_{bt} = \hat{\Phi}_{[bt]} = \hat{\chi}_{[bt]} - \frac{1}{2}q^c{}_t q^d{}_b (K_{fcd} - K_{fdc}) l^f = \frac{1}{2}q^c{}_t q^d{}_b T_{fdc} l^f . \tag{3.34}$$

Using (3.32), (3.33) and (3.34), we obtain,

$$\hat{\Phi}_{bt} = \frac{1}{2}q_{bt}\left(\overset{(d)}{\hat{\theta}_l} - T_b l^b + \mathbb{T}_b k^b\right) + \overset{*}{\hat{\sigma}}_{bt} + \frac{1}{2}q^c{}_t q^d{}_b T_{fdc} l^f . \tag{3.35}$$

The above result allows us to have,

$$\hat{\mathcal{D}}_b \hat{\Phi}^b{}_t = \hat{\mathcal{D}}_t\left[\frac{1}{2}\left(\overset{(d)}{\hat{\theta}_l} - T_b l^b + \mathbb{T}_b k^b\right)\right] + \hat{\mathcal{D}}_b \overset{*}{\hat{\sigma}}{}^b{}_t + \hat{\mathcal{D}}^b\left(\frac{1}{2}q^c{}_t q^d{}_b T_{fdc} l^f\right). \tag{3.36}$$

Using (3.36) in (3.31) and further simplifying, we end up with,

$$q^a{}_t \pounds_l \hat{\Omega}_a + \hat{\Omega}_t\left(\overset{(d)}{\hat{\theta}_l} - T_b l^b + k_b \mathbb{T}^b\right) - \hat{\mathcal{D}}_t\left(\kappa + \frac{1}{2}(\overset{(d)}{\hat{\theta}_l} - T_b l^b - k_b \mathbb{T}^b)\right) + \hat{\mathcal{D}}_b \overset{*}{\hat{\sigma}}{}^b{}_t$$
$$+ \hat{\mathcal{D}}^b\left(\frac{1}{2}q^c{}_t q^d{}_b T_{fdc} l^f\right) - \hat{\chi}_{bt}\hat{\mathscr{P}}^b + q^c{}_t q^d{}_b (K_{fcd} l^f)\hat{\mathscr{P}}^b - \hat{\Xi}_{tb}\mathbb{T}^b + q^c{}_t q^d{}_b (K_{fdc} k^f)\mathbb{T}^b$$
$$= \hat{G}_{ab} l^a q^b{}_t - T_{iba}\hat{\chi}^{bi} q^a{}_t + q^a{}_t (T_{iba} k^i q^{cb}\mathbb{T}_c) + q^a{}_t (q^{ci} q^{db} T_{iba}(K_{fcd} l^f)) . \tag{3.37}$$

The above geometrical relationship established in the RC spacetime relates the (Lie)evolution of the Hájiček one-form $\hat{\Omega}_a$ along the null generators in its full generality with the projection component $\hat{R}_{ab} l^a q^b{}_t$ or $\hat{G}_{ab} l^a q^b{}_t$. Note that, in arriving at (3.37), we have not used the geodesic constraint condition $\mathbb{T}_b = 0$.

We can express the above relationship in a slightly different version by switching over to the extended second fundamental form $\hat{\Theta}_{bt}$. This we do by noticing via the relationship between the extended second fundamental form and the deformation rate tensor, *i.e.* (2.125). We see that upon using (2.125), we have,

$$-\hat{\Theta}_{ba}\hat{\mathscr{P}}^b q^a{}_t = -\hat{\chi}_{bt}\hat{\mathscr{P}}^b + q^c{}_t q^d{}_b (K_{fcd} l^f)\hat{\mathscr{P}}^b . \tag{3.38}$$

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Similarly the last three terms on the R.H.S of (3.37) can be combined via (2.125) to get,

$$
\begin{aligned}
&- T_{iba}\hat{\chi}^{bi}q^{a}{}_{t} + q^{a}{}_{t}(T_{iba}k^{i}q^{cb}\mathbb{T}_{c}) + q^{a}{}_{t}(q^{ci}q^{db}T_{iba}(K_{fcd}l^{f})) = \\
&- T_{iba}q^{a}{}_{t}\left(\hat{\chi}^{bi} - k^{i}q^{cb}\mathbb{T}_{c} - q^{ci}q^{db}K_{fcd}l^{f}\right) = -T_{iba}q^{a}{}_{t}\hat{\Theta}^{bi} .
\end{aligned}
\tag{3.39}
$$

Using (3.38) and (3.39) in (3.37), we express the Lie-evolution of the Hájíček one-form in a slightly different form,

$$
\begin{aligned}
&q^{a}{}_{t}\pounds_{l}\hat{\Omega}_{a} + \hat{\Omega}_{t}\Big(\overset{(d)}{\hat{\theta}_{l}} - T_{b}l^{b} + k_{b}\mathbb{T}^{b}\Big) - \hat{\mathcal{D}}_{t}\Big(\kappa + \frac{1}{2}(\overset{(d)}{\hat{\theta}_{l}} - T_{b}l^{b} - k_{b}\mathbb{T}^{b})\Big) + \hat{\mathcal{D}}_{b}\overset{*}{\hat{\sigma}}{}^{b}{}_{t} \\
&+ \hat{\mathcal{D}}^{b}\Big(\frac{1}{2}q^{c}{}_{t}q^{d}{}_{b}T_{fdc}l^{f}\Big) - \hat{\Theta}_{ba}\hat{\mathscr{P}}^{b}q^{a}{}_{t} - \hat{\Xi}_{tb}\mathbb{T}^{b} + q^{c}{}_{t}q^{d}{}_{b}(K_{fdc}k^{f})\mathbb{T}^{b} \\
&= \hat{G}_{ab}l^{a}q^{b}{}_{t} - T_{iba}\hat{\Theta}^{bi}q^{a}{}_{t} .
\end{aligned}
\tag{3.40}
$$

Obviously, in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the above evolution equation reduces to,

$$
q^{a}{}_{t}\pounds_{l}\Omega_{a} + \Omega_{t}\overset{(d)}{\theta_{l}} - \mathcal{D}_{t}\Big(\kappa + \frac{1}{2}(\overset{(d)}{\theta_{l}}\Big) + \mathcal{D}_{b}\ ({}^{(l,d)}\sigma^{b}{}_{t}) = G_{ab}l^{a}q^{b}{}_{t} . \tag{3.41}
$$

Note that, in contrast to the deformation rate tensor, the extended second fundamental form is not symmetric and completely spatial as evidenced from the relations (2.79) and (2.104). So in essence, via Eq. (3.40), we have have arrived at the dynamical evolution equation of the Hájíček 1-form along the null generators $\vec{l}$.

3.5 Summary and discussion

In this present chapter, we have gone ahead and discussed the relevant dynamics of some of the kinematical quantities introduced in the previous chapter. As mentioned before, the null generators $\vec{l}$ associated with the non-affine parameter t provides a natural vector field along which the evolution of kinematical quantities associated with $\mathcal{H}$ can be considered. The first such kinematical quantity is the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_{l}}$. We considered the evolution of $\overset{(d)}{\hat{\theta}_{l}}$ along the null vector field $\vec{l}$ and saw that its dynamical evolution is related to the projection component $\hat{G}_{ab}l^{a}l^{b}$. We have called this evolution equation to be the NRE in the RC spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. There exists one peculiar distinguishing feature of the NRE in the RC spacetime from the NRE in Riemannian spacetime. The evolution equation for $\overset{(d)}{\hat{\theta}_{l}}$ involves explicit dependence on the auxiliary null vector field $\vec{k}$ in the sense that the torsion tensor field is coupled to $\vec{k}$ as evident from Eq. (3.8). This feature is absent in the Riemannian spacetime as torsion vanishes there. However, if we impose the geodesic constraint, then such explicit dependence on $\vec{k}$ vanishes. The geodesic constraint forces the null geodesics to be auto-parallel in the RC spacetime. The geodesic constraint will be important in the notion of *equilibrium* for the null surface $\mathcal{H}$ in the RC spacetime. This will be elaborated further in chapter 6 when the discuss the thermodynamics associated with the gravitational field equations. Next, we considered the dynamical evolution of the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_k}$ along $\vec{l}$ and saw that it was related to the projection component $\hat{G}_{ab}k^a l^b$. We have called such an evolution equation Eq. (3.24) to be the NRE corresponding to the ingoing expansion scalar. However, the entire analysis was done under the geodesic constraint right from the start. Finally, we considered the evolution dynamics of the Hájiček 1-form $\underline{\hat{\boldsymbol{\Omega}}}$ along the null generators $\vec{l}$ and saw that it was related to the projection component $\hat{G}_{ab}l^a q^b{}_c$ as evident from Eq. (3.37).

In principle, we could also have have considered evolution dynamics of kinematical quantities along the auxiliary null field $\vec{k}$ or along an arbitrary vector field $\vec{X} \equiv \alpha\vec{l} + \beta\vec{k}$ $(\alpha, \beta > 0)$ lying in the orthogonal tangent space to $\mathcal{T}_P(S_t)$. However, for the objective of providing a thermodynamic and fluid-dynamic interpretation to the gravitational field equations in the RC spacetime, the ones that we have considered would suffice. For a study of the evolution equation of such kinematical quantities along the arbitrary vector field $\vec{X}$ in the GR spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the reader may wish to consult [76, 135]. It would quite an instructive exercise to see what modifications arise for the dynamical equations once we go over to the RC spacetime.

Now, the objective of the chapters 2 and 3 have been to present in its generality, the construction, geometry (both intrinsic and extrinsic), kinematics and dynamics of an integrable null hypersurface $\mathcal{H}$ in the RC spacetime. So in a sense, the first two chapters lay out the geometrical foundations with which we need to work on. Till this point, we have not used the relevant gravitational dynamics to be employed in the RC spacetime. This will be our focus in the subsequent chapters.

Appendices

3.6 Proof of Eq. (3.3)

We begin by manipulating the first term on the L.H.S of (3.2) using the relation (2.90) and (2.122),

$$\begin{aligned}
\hat{\nabla}_a(\hat{\nabla}_b l^a) &= \hat{\nabla}_a \hat{\Theta}^a{}_b + l^a \hat{\nabla}_a \hat{\omega}_b + \hat{\omega}_b (\hat{\nabla}_a l^a) - (\hat{\nabla}_a l_b)(k^i \hat{\nabla}_i l^a) - l_b \hat{\nabla}_a (k^i \hat{\nabla}_i l^a) \\
&= \hat{\nabla}_a \hat{\Theta}^a{}_b + l^a \hat{\nabla}_a \hat{\omega}_b + \hat{\omega}_b \Big(\overset{(d)}{\hat{\theta}_l} + \kappa - T_a l^a \Big) \\
&\quad - \Big(\hat{\Theta}_{ba} + \hat{\omega}_a l_b - l_a (k^j \hat{\nabla}_j l_b) \Big) (k^i \hat{\nabla}_i l^a) - l_b \hat{\nabla}_a (k^i \hat{\nabla}_i l^a) \\
&= \hat{\nabla}_a \hat{\Theta}^a{}_b + l^a \hat{\nabla}_a \hat{\omega}_b + \hat{\omega}_b \Big(\overset{(d)}{\hat{\theta}_l} + \kappa - T_a l^a \Big) - \hat{\Theta}_{ba} (k^i \hat{\nabla}_i l^a) \\
&\quad - l_b \Big(\hat{\omega}_a (k^i \hat{\nabla}_i l^a) - \hat{\nabla}_a (k^i \hat{\nabla}_i l^a) \Big) \, . \qquad (3.42)
\end{aligned}$$

Let us now proceed with the second term in the R.H.S of (3.2). Again using the relation (2.90) and (2.122), we can similarly show that,

$$\begin{aligned}
\hat{\nabla}_b(\hat{\nabla}_a l^a) &= \hat{\nabla}_b \Big(\overset{(d)}{\hat{\theta}_l} + \kappa - T_a l^a \Big) \\
&= \hat{\nabla}_b \Big(\overset{(d)}{\hat{\theta}_l} + \kappa \Big) - l^a (\hat{\nabla}_b T_a) - T^a \hat{\Theta}_{ab} - \hat{\omega}_b (T_a l^a) + l_b \Big(T_a (k^i \hat{\nabla}_i l^a) \Big) \, . \qquad (3.43)
\end{aligned}$$

Similarly, using (2.90) for the second term in the R.H.S of (3.2), we have,

$$T^d{}_{ab} (\hat{\nabla}_d l^a) = T_{dab} \hat{\Theta}^{ad} + T_{dab} \hat{\omega}^d l^a - T_{dab} l^d (k^j \hat{\nabla}_j l^a) \, . \qquad (3.44)$$

Finally, upon using the relations (3.42), (3.43) and (3.44) in the null Cadacci equation (3.2) and simplifying, we end up having,

$$\begin{aligned}
&\hat{\nabla}_a \hat{\Theta}^a{}_b + l^a \hat{\nabla}_a \hat{\omega}_b + \hat{\omega}_b \Big(\overset{(d)}{\hat{\theta}_l} + \kappa \Big) - \hat{\Theta}_{ba} (k^i \hat{\nabla}_i l^a) - \hat{\nabla}_b (\overset{(d)}{\hat{\theta}_l} + \kappa) + l^a (\hat{\nabla}_b T_a) \\
&+ T^a \hat{\Theta}_{ab} - l_b \Big[\hat{\omega}_a (k^i \hat{\nabla}_i l^a) - \hat{\nabla}_a (k^i \hat{\nabla}_i l^a) + T_a (k^i \hat{\nabla}_i l^a) \Big] = \hat{R}_{ab} l^a - T_{fab} \hat{\Theta}^{af} \\
&- T_{fab} \hat{\omega}^f l^a + T_{fab} l^f (k^j \hat{\nabla}_j l^a) \, . \qquad (3.45)
\end{aligned}$$

To this end, we simply need to contract the previous equation (3.45) with l^b. Upon using the following relations, $\hat{\omega}_a l^a = \kappa - k_a \mathbb{T}^a$, $\hat{\Theta}_{ba} l^b = 0$, $\hat{\Theta}_{ab} l^b = q^c{}_a \mathbb{T}_c$ and $T^a q^c{}_a \mathbb{T}_c = T^a \mathbb{T}_a +$

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$(T^a l_a)(k^b \mathbb{T}_b)$ and simplifying the above contracted (with l^b) relation, we have,

$$l^b \hat{\nabla}_a \hat{\Theta}^a{}_b + l^b (l^a \hat{\nabla}_a \hat{\omega}_b) + (\kappa - k_a \mathbb{T}^a)(\overset{(d)}{\hat{\theta}_l} + \kappa) - l^b \hat{\nabla}_b (\overset{(d)}{\hat{\theta}_l} + \kappa)$$
$$+ l^b l^a (\hat{\nabla}_b T_a) + T^a \mathbb{T}_a + (T^a l_a)(k^b \mathbb{T}_b) = \hat{R}_{ab} l^a l^b - T_{fab} \hat{\Theta}^{af} l^b + \mathbb{T}_a (k^j \hat{\nabla}_j l^a) \, . \tag{3.46}$$

Next, we manipulate the term $l^b \hat{\nabla}_a \hat{\Theta}^a{}_b$ in the L.H.S of (3.46). Using the fact that $\hat{\Theta}^a{}_b l^b = q^{ca} \mathbb{T}_c$, we have upon using (2.90),

$$\begin{aligned} l^b \hat{\nabla}_a \hat{\Theta}^a{}_b &= \hat{\nabla}_a (\hat{\Theta}^a{}_b l^b) - \hat{\Theta}^a{}_b (\hat{\nabla}_a l^b) \\ &= \hat{\nabla}_a (q^{ca} \mathbb{T}_c) - \hat{\Theta}^a{}_b \Big(\hat{\Theta}^b{}_a + \hat{\omega}_a l^b - l_a (k^j \hat{\nabla}_j l^b) \Big) \\ &= \hat{\nabla}_a \Big(\mathbb{T}^a + (k^c \mathbb{T}_c) l^a \Big) - \hat{\Theta}_{ab} \hat{\Theta}^{ba} - \hat{\omega}_a \hat{\Theta}^a{}_b l^b \\ &= \hat{\nabla}_a \mathbb{T}^a + (l^a \hat{\nabla}_a k_c) \mathbb{T}^c + (l^a \hat{\nabla}_a \mathbb{T}_c) k^c + (k^c \mathbb{T}_c)(\overset{(d)}{\hat{\theta}_l} + \kappa) - (T_a l^a)(k^c \mathbb{T}_c) \\ &\quad - \hat{\Theta}_{ab} \hat{\Theta}^{ba} - \hat{\omega}_a \mathbb{T}^a - (\kappa - k_a \mathbb{T}^a)(k^c \mathbb{T}_c) \, . \end{aligned} \tag{3.47}$$

Upon using the relation of the rotation 1-form (2.114) in the previous relation (3.47) and simplifying, we end up with,

$$\begin{aligned} l^b \hat{\nabla}_a \hat{\Theta}^a{}_b = (\hat{\nabla}_a \mathbb{T}^a) - (T_{abc} k^a l^b) \mathbb{T}^c + (l^a \hat{\nabla}_a \mathbb{T}_c) k^c + (k^c \mathbb{T}_c)(\overset{(d)}{\hat{\theta}_l} - T_a l^a) \\ + (k^a \mathbb{T}_a)(k^b \mathbb{T}_b) - \hat{\Theta}_{ab} \hat{\Theta}^{ba} \, . \end{aligned} \tag{3.48}$$

After this, we focus on the term $l^b (l^a \hat{\nabla}_a \hat{\omega}_b)$ in the L.H.S of (3.46) and use the fact that $l^b \hat{\omega}_b = \kappa - k^a \mathbb{T}_a$ along with the implementation of (2.114),

$$\begin{aligned} l^b (l^a \hat{\nabla}_a \hat{\omega}_b) &= l^a \hat{\nabla}_a (\kappa - k_c \mathbb{T}^c) - \hat{\omega}_b (l^a \hat{\nabla}_a l^b) \\ &= l^a \hat{\nabla}_a \kappa - 2\hat{\omega}_a \mathbb{T}^a + (T_{abc} k^a l^b) \mathbb{T}^c - k^c (l^a \hat{\nabla}_a \mathbb{T}_c) - \kappa^2 + \kappa (k_a \mathbb{T}^a) \, . \end{aligned} \tag{3.49}$$

Adding (3.48) and (3.49) leads to upon simplification,

$$\begin{aligned} l^b \hat{\nabla}_a \hat{\Theta}^a{}_b + l^b l^a \hat{\nabla}_a \hat{\omega}_b = \hat{\nabla}_a \mathbb{T}^a + (k_b \mathbb{T}^b)(\overset{(d)}{\hat{\theta}_l} - T_a l^a) + (k_a \mathbb{T}^a)(k_b \mathbb{T}^b) - \hat{\Theta}_{ab} \hat{\Theta}^{ba} \\ + l^a \hat{\nabla}_a \kappa - 2\hat{\omega}_a \mathbb{T}^a - \kappa^2 + \kappa (k_a \mathbb{T}^a) \, . \end{aligned} \tag{3.50}$$

Putting the above relation (3.50) in (3.46) and simplifying, we get

$$\begin{aligned} &\hat{\nabla}_a \mathbb{T}^a + (k_a \mathbb{T}^a)(k_b \mathbb{T}^b) - \hat{\Theta}_{ab} \hat{\Theta}^{ba} - 2\hat{\omega}_a \mathbb{T}^a + \kappa \overset{(d)}{\hat{\theta}_l} - l^b \hat{\nabla}_b \overset{(d)}{\hat{\theta}_l} + l^b l^a (\hat{\nabla}_b T_a) + T^a \mathbb{T}_a \\ &= \hat{R}_{ab} l^a l^b - T_{fab} \hat{\Theta}^{af} l^b + \mathbb{T}_a (k^j \hat{\nabla}_j l^a) \, . \end{aligned} \tag{3.51}$$

After this, we consider a further manipulation of the term $\hat{\nabla}_a \mathbb{T}^a$ using the relation (2.90),

$$\begin{aligned}\hat{\nabla}_a \mathbb{T}^a &= \hat{\nabla}_a (T_b{}^a{}_c\, l^b l^c) = (\hat{\nabla}_a T_b{}^a{}_c) l^b l^c + T_b{}^a{}_c (\hat{\nabla}_a l^b) l^c + T_b{}^a{}_c l^b (\hat{\nabla}_a l^c) \\ &= (\hat{\nabla}_a T_b{}^a{}_c) l^b l^c + T_{bac} l^c \hat{\Theta}^{ba} + T_{bac} l^b \hat{\Theta}^{ca} + 2\mathbb{T}^a \hat{\omega}_a + \mathbb{T}_c (k^j \hat{\nabla}_j l^c)\,. \end{aligned} \tag{3.52}$$

Putting the above relation (3.52) in (3.51) and further simplifying leads us to Eq. (3.3).

3.7 Reduction of NRE in $(\mathcal{M}, \mathrm{g}, \hat{\boldsymbol{\nabla}})$ to NRE in the Riemannian spacetime

It seems quite obvious that if we separate out Eq. (3.8) into its purely Riemannian part and the part involving the torsion terms on both the L.H.S and R.H.S, then the torsion terms on both sides of the relation must cancel out, leaving aside the NRE for the null generators $\vec{l}$ in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$. For arriving at the result, we make the following observations based on the projected deviation tensor. It can quite simply be established that,

$$\begin{aligned}\hat{\mathcal{B}}_{ab} &= \mathcal{B}_{ab} + (K_{abc} - T_{abc}) l^c + \Big[l_a (K_{dbc} - T_{dbc}) k^d l^c + l_b (K_{adc} - T_{adc}) k^d l^c \Big] \\ &\quad + l_a l_b (K_{cdf} - T_{cdf}) k^c k^d l^f + (\mathbb{T}_a k_b - k_a \mathbb{T}_b) + (l_a k_b - k_a l_b)(k_i \mathbb{T}^i)\,, \end{aligned} \tag{3.53}$$

where $\mathcal{B}_{ab} = q_a^i q_b^j (\nabla_j l_i)$ is the projected deviation tensor as computed for the null congruence $\vec{l}$ in the spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ endowed with the Levi-Civita connection. Taking the trace of (3.53) on both sides leads to the fact that the outgoing expansion scalars for the Riemann-Cartan and the Riemannian versions are the same *i.e.*

$$\overset{(d)}{\hat{\theta}_l} = \theta_l = \hat{\nabla}_a l^a - \kappa + T_a l^a = \nabla_a l^a - \kappa\,. \tag{3.54}$$

Similarly, the shear tensors are related by,

$${}^{(l,B)}\hat{\sigma}_{ab} = {}^{(l,B)}\sigma_{ab} + l_a l_b (K_{cdf} - T_{cdf}) k^c k^d l^f\,, \tag{3.55}$$

where ${}^{(l,B)}\sigma_{ab} = \mathcal{B}_{(ab)} - \frac{1}{2} q_{ab} \theta_l$. For a hypersurface-orthogonal null congruence $\mathcal{H}$ generated by $\vec{l}$ in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, we have the antisymmetric part of the projected deviation tensor $\mathcal{B}_{ab}$ to be zero *i.e.* $\mathcal{B}_{[ab]} = 0$ [25]. Hence,

$$\begin{aligned}{}^{(l,B)}\hat{\omega}_{ab} = \hat{\mathcal{B}}_{[ab]} &= K_{acb} l^c + (l_a K_{dcb} - l_b K_{dca}) k^d l^c + (\mathbb{T}_a k_b - k_a \mathbb{T}_b) \\ &\quad + (l_a k_b - k_a l_b)(k_i \mathbb{T}^i)\,. \end{aligned} \tag{3.56}$$

Finally we would require the decomposition of the Ricci tensor in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ in terms of the pure Riemannian counterpart and pure torsion terms, *i.e.*,

$$\hat{R}_{ab} = R_{ab} + \hat{\nabla}_i K^i{}_{ba} + \hat{\nabla}_b T_a + T^i{}_{jb} K^j{}_{ia} + K^i{}_{ja} K^j{}_{bi} + T_i K^i{}_{ba} \,. \tag{3.57}$$

Putting (3.54), (3.55), (3.56) and (3.57) in (3.8), leads upon simplification to the well known NRE for θ_l,

$$l^a \nabla_a \theta_l = -R_{ab} l^a l^b + \kappa \theta_l - \frac{1}{2}\theta_l^2 - {}^{(l,B)}\sigma_{ab} \, {}^{(l,B)}\sigma^{ab} \,. \tag{3.58}$$

3.8 Derivation of the relation (3.13)

Let us manipulate the first term *i.e.* $\hat{\mathcal{D}}_a(\hat{\mathcal{D}}_b v^a)$ of (3.12),

$$\begin{aligned}
\hat{\mathcal{D}}_a(\hat{\mathcal{D}}_b v^a) &= q^i{}_b q^l{}_k \hat{\nabla}_l (q^m_i q^k{}_s \hat{\nabla}_m v^s) \\
&= q^i{}_b q^l{}_s \hat{\nabla}_l (\delta^m_i + l^m k_i + k^m l_i)(\hat{\nabla}_m v^s) \\
&+ q^m_b q^l{}_k \hat{\nabla}_l (\delta^k_s + l^k k_s + k^k l_s)(\hat{\nabla}_m v^s) + q^m_b q^l{}_s \hat{\nabla}_l \hat{\nabla}_m v^s \,. \\
&= q^i{}_b q^l{}_s l^m (\hat{\nabla}_l k_i)(\hat{\nabla}_m v^s) + q^i{}_b q^l{}_s k^m (\hat{\nabla}_l l_i)(\hat{\nabla}_m v^s) + q^m_b q^l{}_k k_s (\hat{\nabla}_l l^k)(\hat{\nabla}_m v^s) \\
&\quad + q^m_b q^l{}_k l_s (\hat{\nabla}_l k^k)(\hat{\nabla}_m v^s) + q^m_b q^l{}_s \hat{\nabla}_l \hat{\nabla}_m v^s \,.
\end{aligned} \tag{3.59}$$

Upon using (2.134) and (2.126) in (3.59), we have,

$$\begin{aligned}
\hat{\mathcal{D}}_a(\hat{\mathcal{D}}_b v^a) &= q^i{}_b q^l{}_s l^m \left[\hat{\Xi}_{li} - q^c{}_l q^d{}_i K_{fcd} k^f\right](\hat{\nabla}_m v^s) \\
&+ q^i{}_b q^l{}_s k^m \left[\hat{\chi}_{li} - q^c{}_l q^d{}_i K_{fcd} l^f\right](\hat{\nabla}_m v^s) \\
&+ q^m_b q^{lk} l_s \left[\hat{\Xi}_{lk} - q^c_l q^d{}_k K_{fcd} k^f\right](\hat{\nabla}_m v^s) + q^m_b q^{lk} k_s \left[\hat{\chi}_{lk} - q^c{}_l q^d{}_k K_{fcd} l^f\right](\hat{\nabla}_m v^s) \\
&+ q^m_b q^l{}_s \hat{\nabla}_l \hat{\nabla}_m v^s \,.
\end{aligned} \tag{3.60}$$

The above expression can be very easily expressed as,

$$\begin{aligned}
\hat{\mathcal{D}}_a(\hat{\mathcal{D}}_b v^a) &= \left[\hat{\Xi}_{sb} - q^c{}_s q^d{}_b K_{fcd} k^f\right](l^m \hat{\nabla}_m v^s) + \left[\hat{\chi}_{sb} - q^c{}_s q^d{}_b K_{fcd} l^f\right](k^m \hat{\nabla}_m v^s) \\
&\quad + \left[(\overset{(d)}{\hat{\theta}_k} - q^{cd} K_{fcd} k^f) q^m_b\right](l_s \hat{\nabla}_m v^s) + \left[(\overset{(d)}{\hat{\theta}_l} - q^{cd} K_{fcd} l^f) q^m_b\right](k_s \hat{\nabla}_m v^s) \\
&\quad + q^m_b q^l{}_s \hat{\nabla}_l \hat{\nabla}_m v^s \,.
\end{aligned} \tag{3.61}$$

Let us now deal with the second term $\hat{\mathcal{D}}_b(\hat{\mathcal{D}}_a v^a)$ of (3.12),

$$\hat{\mathcal{D}}_b(\hat{\mathcal{D}}_a v^a) = q^m_s q^r{}_b \hat{\nabla}_r (q^i{}_m + q^s{}_j \hat{\nabla}_i v^j) \,. \tag{3.62}$$

Using an exactly similar analysis as was done for $\hat{\mathcal{D}}_a(\hat{\mathcal{D}}_b v^a)$, it can be verified that,

$$\begin{aligned}\hat{\mathcal{D}}_b(\hat{\mathcal{D}}_a v^a) = &\left[\hat{\Xi}_{bs} - q^c{}_b q^d{}_s K_{fcd} k^f\right](l^m \hat{\nabla}_m v^s) + \left[\hat{\chi}_{bs} - q^c{}_b q^d{}_s K_{fcd} l^f\right](k^m \hat{\nabla}_m v^s) \\ &+ \left[\hat{\chi}^m_{\ b} - q^c{}_b q^{dm} K_{fcd} l^f\right](k_s \hat{\nabla}_m v^s) + \left[\hat{\Xi}^m_{\ b} - q^c{}_b q^{dm} K_{fcd} k^f\right](l_s \hat{\nabla}_m v^s) \\ &+ q^m{}_b q^l{}_s \hat{\nabla}_m \hat{\nabla}_l v^s \,. \end{aligned} \tag{3.63}$$

Before proceeding forward let us list a few results obtained from the properties of the torsion and contorsion tensor,

$$(q^c{}_b q^d{}_s K_{fcd} - q^d{}_s q^c{}_b K_{fdc}) k^f = q^c{}_b q^d{}_s (K_{fcd} - K_{fdc}) k^f = q^c{}_b q^d{}_s (T_{fcd} k^f) \,, \tag{3.64}$$

$$(q^c{}_b q^d{}_s K_{fcd} - q^d{}_s q^c{}_b K_{fdc}) l^f = q^c{}_b q^d{}_s (T_{fcd} l^f) \,. \tag{3.65}$$

Finally, subtracting (3.63) from (3.61) and utilizing the symmetry of the deformation rate and transversal deformation rate tensors along with, (3.64) and (3.65) we have our desired result (3.13).

3.9 Proof of Eqn. (3.15)

Due to the presence of torsion in the ambient spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, the submanifold $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$ is not torsion-free with an intrinsic ${}^{(2)}T^a{}_{bc}$ present in it. For any two vectors $(\vec{\boldsymbol{X}}, \vec{\boldsymbol{Y}}) \in \mathcal{T}_P(S_t) \otimes \mathcal{T}_P(S_t)$, we have from the definition of torsion as,

$${}^{(2)}\boldsymbol{T}(\vec{\boldsymbol{X}}, \vec{\boldsymbol{Y}}) = \hat{\mathcal{D}}_{\vec{\boldsymbol{X}}} \vec{\boldsymbol{Y}} - \hat{\mathcal{D}}_{\vec{\boldsymbol{Y}}} \vec{\boldsymbol{X}} - {}^{(2)}[\vec{\boldsymbol{X}}, \vec{\boldsymbol{Y}}] \,, \tag{3.66}$$

where ${}^{(2)}[\vec{\boldsymbol{X}}, \vec{\boldsymbol{Y}}] = {}^{(2)}\pounds_{\vec{\boldsymbol{X}}} \vec{\boldsymbol{Y}}$ is the intrinsic Lie bracket defined for the manifold $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$. In index notation, this translates to,

$${}^{(2)}T^a_{\ bc} X^b Y^c = X^i \hat{\mathcal{D}}_i Y^a - Y^i \hat{\mathcal{D}}_i X^a - {}^{(2)}\pounds_{\vec{\boldsymbol{X}}} Y^a \,. \tag{3.67}$$

Now, since the vectors $\vec{\boldsymbol{X}}$ and $\vec{\boldsymbol{Y}}$ lie in the tangent space established on S_t, the Lie bracket of these two vectors also belongs to $\mathcal{T}_P(S_t)$. Following from the Frobenius theorem [144], we have,

$${}^{(2)}[\vec{\boldsymbol{X}}, \vec{\boldsymbol{Y}}]^a = q^a{}_b [\vec{\boldsymbol{X}}, \vec{\boldsymbol{Y}}]^b \,. \tag{3.68}$$

Expanding the above relation (3.68), we have,

$$X^b \hat{\mathcal{D}}_b Y^a - Y^b \hat{\mathcal{D}}_b X^a - {}^{(2)}T^a_{\ bc} X^b Y^c = q^a{}_b \left[X^c \hat{\nabla}_c Y^b - Y^c \hat{\nabla}_c X^b - T^b_{\ cd} X^c Y^d\right] . \tag{3.69}$$

Using the fact that $\vec{\boldsymbol{X}}$ and $\vec{\boldsymbol{Y}}$ are spatial tangent vectors on S_t, we have as consequence, $q^a{}_b X^c \hat{\nabla}_c Y^b = X^c \hat{\mathcal{D}}_c Y^a$ and $q^a{}_b Y^c \hat{\nabla}_c X^b = \mathcal{Y}^c \hat{\mathcal{D}}_c X^a$. Using these relations in (3.69) and simplifying, we obtain the following,

$$ {}^{(2)}T^{a}{}_{bc}X^{b}Y^{c} = T^{a}{}_{bc}X^{b}Y^{c} + l^{a}k^{d}T_{dbc}X^{b}Y^{c} + k^{a}l^{d}T_{dbc}X^{b}Y^{c} \,. \tag{3.70} $$

From this, it follows that,

$$ {}^{(2)}T^{a}{}_{bc}q^{b}{}_{i}q^{c}{}_{j}X^{i}Y^{j} = T^{a}{}_{bc}q^{b}{}_{i}q^{c}{}_{j}X^{i}Y^{j} + l^{a}k^{d}T_{dbc}q^{b}{}_{i}q^{c}{}_{j}X^{i}Y^{j} + k^{a}l^{d}T_{dbc}q^{b}{}_{i}q^{c}{}_{j}X^{i}Y^{j} \,. \tag{3.71} $$

From the above, we can have,

$$ q^{f}{}_{a}\,{}^{(2)}T^{a}{}_{bc}q^{b}{}_{i}q^{c}{}_{j}X^{i}Y^{j} = q^{f}{}_{a}T^{a}{}_{bc}q^{b}{}_{i}q^{c}{}_{j}X^{i}Y^{j} \,. \tag{3.72} $$

Since all the indices of the 4-dimension torsion tensor have been projected onto the surface S_t we have finally the result,

$$ q^{c}{}_{b}q^{d}{}_{s}q^{m}{}_{f}T^{f}{}_{cd} = q^{c}{}_{b}q^{d}{}_{s}q^{m}{}_{f}\,{}^{(2)}T^{f}{}_{cd} = {}^{(2)}T^{m}{}_{bs} \,. \tag{3.73} $$

3.10 Derivation of the relations (3.20) and (3.22)

We begin by noticing that,

$$ q^{m}{}_{b}q^{l}{}_{s}q^{p}{}_{a}\,\hat{R}^{s}{}_{plm} = q^{m}{}_{b}q^{p}{}_{a}\left(\hat{R}_{pm} + \hat{R}_{sprm}l^{r}k^{s} + \hat{R}_{sprm}k^{r}l^{s}\right) \,. \tag{3.74} $$

Let us list a few relevant properties involving curvature tensors in metric affine spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. Even though the Riemann curvature tensor is antisymmetric in the first two and the last two indices, it is not symmetric under pairwise exchange,

$$ \begin{aligned} \hat{R}_{cdab} =& \hat{R}_{abcd} + \hat{\mathcal{Q}}_{abcd} \text{ , where} \\ & \hat{\mathcal{Q}}_{abcd} = -\frac{3}{2}\Big(\hat{\nabla}_{[b}T_{|a|cd]} - \hat{\nabla}_{[a}T_{|b|cd]} - \hat{\nabla}_{[d}T_{|c|ab]} + \hat{\nabla}_{[c}T_{|d|ab]} \\ & + T_{ae[b}T^{e}{}_{cd]} - T_{be[a}T^{e}{}_{cd]} - T_{ce[d}T^{e}{}_{ab]} + T_{de[c}T^{e}{}_{ab]}\Big) \,, \end{aligned} \tag{3.75} $$

where || indicates the enclosed index barred from antisymmetrization. Using the above property (3.75) in (3.74), we obtain,

$$ q^{m}{}_{b}q^{l}{}_{s}q^{p}{}_{a}\,\hat{R}^{s}{}_{plm} = q^{m}{}_{b}q^{p}{}_{a}\left(\hat{R}_{pm} + l^{r}(\hat{R}_{sprm}k^{s}) + l^{r}(\hat{R}_{smrp}k^{s}) + \hat{\mathcal{Q}}_{rmsp}k^{r}l^{s}\right) \,. \tag{3.76} $$

Using the Ricci identity (2.10) for the auxiliary vector field $\vec{k}$, we have,

$$
\begin{aligned}
& q^m{}_b q^l{}_s q^p{}_a \, \hat{R}^s{}_{plm} = q^m{}_b q^p{}_a \Big(\hat{R}_{pm} - l^r \hat{\nabla}_r \hat{\nabla}_m k_p + l^r \hat{\nabla}_m \hat{\nabla}_r k_p - l^r T^d{}_{rm} (\hat{\nabla}_d k_p) \\
& - l^r \hat{\nabla}_r \hat{\nabla}_p k_m + l^r \hat{\nabla}_p \hat{\nabla}_r k_m - l^r T^d{}_{rp} (\hat{\nabla}_d k_m) \Big) + q^m{}_b q^p{}_a \hat{\mathcal{Q}}_{rmsp} k^r l^s \\
& = q^m{}_b q^p{}_a \Big[\hat{R}_{pm} \underbrace{- l^r \hat{\nabla}_r (\hat{\nabla}_m k_p) - l^r \hat{\nabla}_r (\hat{\nabla}_p k_m)}_{A_1} \underbrace{+ \hat{\nabla}_m (l^r \hat{\nabla}_r k_p) + \hat{\nabla}_p (l^r \nabla_r k_m)}_{A_2} \\
& \underbrace{- (\hat{\nabla}_m l^r)(\hat{\nabla}_r k_p) - (\hat{\nabla}_p l^r)(\hat{\nabla}_r k_m)}_{A_3} \underbrace{- l^r T^d{}_{rm} (\hat{\nabla}_d k_p) - l^r T^d{}_{rp} (\hat{\nabla}_d k_m)}_{A_4} \Big] \\
& + q^m{}_b q^p{}_a \hat{\mathcal{Q}}_{rmsp} k^r l^s \, .
\end{aligned}
\tag{3.77}
$$

Let us build on this calculation step by step. We first evaluate the projection of the quantity A_1. Using (2.134) and the symmetry of the transversal deformation rate tensor we obtain,

$$
\begin{aligned}
& q^m{}_b q^p{}_a \Big(- l^r \hat{\nabla}_r (\hat{\nabla}_m k_p) - l^r \hat{\nabla}_r (\hat{\nabla}_p k_m) \Big) = \\
& - q^m{}_b q^p{}_a l^r \Big[2 \hat{\nabla}_r \hat{\Xi}_{mp} - \hat{\Omega}_m (\hat{\nabla}_r k_p) - \hat{\Omega}_p (\hat{\nabla}_r k_m) \\
& - (\hat{\nabla}_r k_m) \hat{\omega}_p - (\hat{\nabla}_r k_p) \hat{\omega}_m - (\hat{\nabla}_r l_m)(k^i \hat{\nabla}_i k_p) - (\hat{\nabla}_r l_p)(k^i \hat{\nabla}_i k_m) \\
& + (\hat{\nabla}_r k_m)(T_{dep} k^d l^e) + (\hat{\nabla}_r k_p)(T_{dem} k^d l^e) - \hat{\nabla}_r \Big((q^c{}_m q^d{}_p + q^c{}_p q^d{}_m) K_{fcd} k^f \Big) \Big]
\end{aligned}
\tag{3.78}
$$

Upon this present relation (3.78), we will use the auto-parallel equation under the geodesic constraint *i.e.* $l^b \hat{\nabla}_b l^a = \kappa l^a$ as well as (2.114). Let us further bring in the notation we introduced earlier in Eq. (3.21) to reduce the clutter of indices,

$$
\hat{\mathscr{P}}_a \equiv T_{bcd} k^b l^c q^d{}_a \, .
\tag{3.79}
$$

Finally simplifying (3.78), we have,

$$
\begin{aligned}
& q^m{}_b q^p{}_a \Big(- l^r \hat{\nabla}_r (\hat{\nabla}_m k_p) - l^r \hat{\nabla}_r (\hat{\nabla}_p k_m) \Big) = \\
& - q^m{}_b q^p{}_a l^r \hat{\nabla}_r \Big[2 \hat{\Xi}_{mp} - (q^c{}_m q^d{}_p + q^c{}_p q^d{}_m)(K_{fcd} k^f) \Big] + 4 \hat{\Omega}_a \hat{\Omega}_b - 3 \hat{\mathscr{P}}_a \hat{\Omega}_b \\
& - 3 \hat{\Omega}_a \hat{\mathscr{P}}_b + 2 \hat{\mathscr{P}}_a \hat{\mathscr{P}}_b \, .
\end{aligned}
\tag{3.80}
$$

Proceeding to the next term in (3.77), and using $l^r \hat{\nabla}_r k_p = \hat{\omega}_p - \hat{\mathscr{P}}_p$, we have,

$$
\begin{aligned}
& q^m{}_b q^p{}_a \Big(\hat{\nabla}_m (l^r \hat{\nabla}_r k_p) + \hat{\nabla}_p (l^r \hat{\nabla}_r k_m) \Big) = \\
& \qquad q^m{}_b q^p{}_a \Big((\hat{\nabla}_m \hat{\omega}_p + \hat{\nabla}_p \hat{\omega}_m) - (\hat{\nabla}_m \hat{\mathscr{P}}_p + \hat{\nabla}_p \hat{\mathscr{P}}_m) \Big) \, .
\end{aligned}
\tag{3.81}
$$

Proceeding on to the spatial projection of the term A_3, we again as usual use the relations (2.134) and (2.126) and simplify to have,

$$q^m_{\ b}q^p_{\ a}\Big(-(\hat{\nabla}_m l^r)(\hat{\nabla}_r k_p)-(\hat{\nabla}_p l^r)(\hat{\nabla}_r k_m)\Big)=-\hat{\chi}^r_{\ b}\hat{\Xi}_{ra}-\hat{\chi}^r_{\ a}\hat{\Xi}_{rb}-2\hat{\Omega}_a\hat{\Omega}_b$$
$$+(\hat{\Omega}_a\hat{\mathscr{P}}_b+\hat{\Omega}_b\hat{\mathscr{P}}_a)+\Big[q^j_{\ a}\hat{\chi}^i_{\ b}+q^j_{\ b}\hat{\chi}^i_{\ a}\Big](K_{hij}k^h)+\Big[q^c_{\ a}\hat{\Xi}^d_{\ b}+q^c_{\ b}\hat{\Xi}^d_{\ a}\Big](K_{fcd}l^f)$$
$$-\Big[q^c_{\ b}q^{di}q^j_{\ a}+q^c_{\ a}q^{di}q^j_{\ b}\Big](K_{fcd}l^f)(K_{hij}k^h)\,. \tag{3.82}$$

Similar analysis on the spatial projection of term A_4 leads us to,

$$q^m_{\ b}q^p_{\ a}\Big(-l^rT^d_{\ rm}(\hat{\nabla}_d k_p)-l^rT^d_{\ rp}(\hat{\nabla}_d k_m)\Big)=-\Big(\hat{\Xi}^c_{\ a}q^d_{\ b}+\hat{\Xi}^c_{\ b}q^d_{\ a}\Big)(T_{cfd}l^f)$$
$$-2\hat{\mathscr{P}}_a\hat{\mathscr{P}}_b+(\hat{\Omega}_a\hat{\mathscr{P}}_b+\hat{\Omega}_b\hat{\mathscr{P}}_a)+\Big(q^d_{\ b}q^j_{\ a}q^{ci}+q^d_{\ a}q^j_{\ b}q^{ci}\Big)(T_{cdf}l^f)(K_{hij}k^h)\,. \tag{3.83}$$

Adding up (3.80), (3.81) , (3.83) and (3.83) in (3.77) and proceeding to simplify, we have our result (3.20).

Notice that in (3.20), there exists the term $q^m_{\ b}q^p_{\ a}l^r\hat{\nabla}_r(2\hat{\Xi}_{mp})$. We will convert the covariant derivative into a Lie derivative term along the null generator to go ahead towards our construction of the evolution equation of the transversal deformation rate tensor. It is quite easy to show that,

$$\pounds_{\boldsymbol{l}}\hat{\Xi}_{mp}=l^i\partial_i\hat{\Xi}_{mp}+\hat{\Xi}_{mi}\partial_p l^i+\hat{\Xi}_{ip}\partial_m l^i$$
$$=l^r\hat{\nabla}_r\hat{\Xi}_{mp}+\hat{\Xi}^i_{\ m}(\hat{\nabla}_p l_i)+\hat{\Xi}^i_{\ p}(\hat{\nabla}_m l_i)+\hat{\Xi}^s_{\ p}(T_{sim}l^i)+\hat{\Xi}^s_{\ m}(T_{sip}l^i)\,. \tag{3.84}$$

Projecting the Lie derivative along the null generator $\vec{l}$ of the transversal deviation rate tensor $\pounds_{\boldsymbol{l}}\hat{\Xi}_{mp}$ on to the spatial cross-section S_t , we the use (2.126) and simplify to have,

$$-2q^m_{\ b}q^p_{\ a}l^r\hat{\nabla}_r(2\hat{\Xi}_{mp})=-2q^m_{\ b}q^p_{\ a}\pounds_{\boldsymbol{l}}\hat{\Xi}_{mp}+2\Big[\hat{\Xi}^c_{\ a}q^d_{\ b}+\hat{\Xi}^c_{\ b}q^d_{\ a}\Big](T_{cfd}l^f)$$
$$2\Big(\hat{\chi}_{ai}\hat{\Xi}^i_{\ b}+\hat{\chi}_{bi}\hat{\Xi}^i_{\ a}\Big)-2\Big[q^c_{\ a}\hat{\Xi}^d_{\ b}+q^c_{\ b}\hat{\Xi}^d_{\ a}\Big](K_{fcd}l^f)\,. \tag{3.85}$$

There also exists the term $q^m_{\ b}q^p_{\ a}l^r\Big[\hat{\nabla}_r\Big(q^i_{\ m}q^j_{\ p}+q^i_{\ p}q^j_{\ m}\Big)(K_{hij}k^h)\Big]$ in (3.20). In a similar fashion, following (3.84) and (3.85), we want to convert the covariant derivative into a Lie derivative term. After a few lines of simple algebra, we have,

$$q^m_{\ b}q^p_{\ a}l^r\Big[\hat{\nabla}_r\Big(q^i_{\ m}q^j_{\ p}+q^i_{\ p}q^j_{\ m}\Big)(K_{hij}k^h)\Big]=q^m_{\ b}q^p_{\ a}\pounds_{\boldsymbol{l}}\Big[\Big(q^i_{\ m}q^j_{\ p}+q^i_{\ p}q^j_{\ m}\Big)(K_{hij}k^h)\Big]$$
$$-\Big[\Big(q^j_{\ a}q^d_{\ b}+q^j_{\ b}q^d_{\ a}\Big)q^{ic}+\Big(q^i_{\ a}q^d_{\ b}+q^i_{\ b}q^d_{\ a}\Big)q^{jc}\Big](T_{cfd}l^f)(K_{hij}k^h)$$
$$-\Big[\hat{\chi}^i_{\ a}q^j_{\ b}+\hat{\chi}^i_{\ b}q^j_{\ a}+\hat{\chi}^j_{\ a}q^i_{\ b}+\hat{\chi}^j_{\ b}q^i_{\ a}\Big](K_{hij}k^h)$$
$$+\Big[\Big(q^c_{\ a}q^j_{\ b}+q^c_{\ b}q^j_{\ a}\Big)q^{id}+\Big(q^c_{\ a}q^i_{\ b}+q^c_{\ b}q^i_{\ a}\Big)q^{jd}\Big](K_{fcd}l^f)(K_{hij}k^h)\,. \tag{3.86}$$

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We have one more transformation to do. We consider the sixth term of the R.H.S. of Eq. (3.20) *i.e.* $q^m_{\ b} q^p_{\ a} \Big[\Big(\hat{\nabla}_m \hat{\omega}_p + \hat{\nabla}_p \hat{\omega}_m \Big) - \Big(\hat{\nabla}_m \hat{\mathscr{P}}_p + \hat{\nabla}_p \hat{\mathscr{P}}_m \Big) \Big]$. We aim to convert the space-time covariant derivatives into spatial derivatives. Notice that $\hat{\mathscr{P}}_a$ acts on the tangent space of S_t and hence is a spatial covector. Upon using the relation $\hat{\omega}_a = \hat{\Omega}_a - \kappa l_a$ under the geodesic constraint, we have,

$$q^m_{\ b} q^p_{\ a} \Big[\Big(\hat{\nabla}_m \hat{\omega}_p + \hat{\nabla}_p \hat{\omega}_m \Big) - \Big(\hat{\nabla}_m \hat{\mathscr{P}}_p + \hat{\nabla}_p \hat{\mathscr{P}}_m \Big) \Big] = \\ q^m_{\ b} q^p_{\ a} \Big[(\hat{\nabla}_m \hat{\Omega}_p + \hat{\nabla}_p \hat{\Omega}_m) - \kappa (\hat{\nabla}_m k_p) - \kappa (\hat{\nabla}_p k_m) \Big] - (\hat{\mathcal{D}}_a \hat{\mathscr{P}}_b + \hat{\mathcal{D}}_b \hat{\mathscr{P}}_a) \,. \tag{3.87}$$

Upon using Eq. (2.134) in the above Eq. (3.87), we have,

$$q^m_{\ b} q^p_{\ a} \Big[\Big(\hat{\nabla}_m \hat{\omega}_p + \hat{\nabla}_p \hat{\omega}_m \Big) - \Big(\hat{\nabla}_m \hat{\mathscr{P}}_p + \hat{\nabla}_p \hat{\mathscr{P}}_m \Big) \Big] = \\ \hat{\mathcal{D}}_a (\hat{\Omega}_b - \hat{\mathscr{P}}_b) + \hat{\mathcal{D}}_b (\hat{\Omega}_a - \hat{\mathscr{P}}_a) - 2\kappa \hat{\Xi}_{ab} + \kappa \Big(q^i_{\ b} q^j_{\ a} + q^i_{\ a} q^j_{\ b} \Big) (K_{hij} k^h) \,. \tag{3.88}$$

At the end of this, finally using (3.85), (3.86) and (3.88) in (3.20), we obtain, after some simplification, our desired result (3.22).

3.11 Derivation of the result (3.24)

Let us for the benefit of the reader list the individual traces of the terms in the R.H.S of Eq. (3.20).

(1) $g^{ab} q^m_{\ b} q^p_{\ a} \hat{\mathcal{Q}}_{rmsp} k^r l^s = q^{bd} \hat{\mathcal{Q}}_{abcd} k^a l^c$.

(2) $g^{ab} q^m_{\ b} q^p_{\ a} \hat{R}_{pm} = \hat{\mathcal{R}} + \Big(\hat{R}_{ab} l^a k^b + \hat{R}_{ab} k^a l^b \Big)$.

(3) $-g^{ab} q^m_{\ b} q^p_{\ a} l^r \hat{\nabla}_r \Big[2\hat{\Xi}_{mp} - \Big(q^i_{\ m} q^j_{\ p} + q^i_{\ p} q^j_{\ m} \Big) (K_{hij} k^h) \Big] = - 2 l^r \hat{\nabla}_r \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij} T_{ihj} k^h \Big)$.

(4) $g^{ab} (2\hat{\Omega}_a \hat{\Omega}^a) = 2\hat{\Omega}^a \hat{\Omega}_a$.

(5) $-g^{ab} (\hat{\Omega}_a \hat{\mathscr{P}}_b + \hat{\mathscr{P}}_a \hat{\Omega}_b) = -2\hat{\Omega}_a \hat{\mathscr{P}}^a$.

(6) $g^{ab} q^m_{\ b} q^p_{\ a} \Big[\Big(\hat{\nabla}_m \hat{\omega}_p + \hat{\nabla}_p \hat{\omega}_m \Big) - \Big(\hat{\nabla}_m \hat{\mathscr{P}}_p + \hat{\nabla}_p \hat{\mathscr{P}}_m \Big) \Big] = 2\hat{\mathcal{D}}_a (\hat{\Omega}^a - \hat{\mathscr{P}}^a) - 2\kappa \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij} T_{ihj} k^h \Big)$.

(7) $-g^{ab} \Big(\hat{\chi}^r_{\ b} \hat{\Xi}_{ra} + \hat{\chi}^r_{\ a} \hat{\Xi}_{rb} \Big) = -2\hat{\Xi}_{ab} \hat{\chi}^{ab}$.

(8) $g^{ab} \Big(q^j_{\ a} \hat{\chi}^i_{\ b} + q^j_{\ b} \hat{\chi}^i_{\ a} \Big) (K_{hij} k^h) = 2\hat{\chi}^{ij} T_{ihj} k^h$.

(9) $g^{ab}\Big(q^c{}_a\hat{\Xi}^d{}_b + q^c{}_b\hat{\Xi}^d{}_a\Big)(K_{fcd}l^f) = 2\hat{\Xi}^{cd}T_{cfd}l^f$.

(10) $-g^{ab}\Big(\hat{\Xi}^c{}_a q^d{}_b + \hat{\Xi}^c{}_b q^d{}_a\Big)(T_{cfd}l^f) = -2\hat{\Xi}^{cd}T_{cfd}l^f$.

(11) $-g^{ab}\Big[q^c{}_b q^j{}_a q^{di} + q^c{}_a q^j{}_b q^{di}\Big](K_{fcd}l^f)(K_{hij}k^h) = -2q^{cj}q^{di}(K_{fcd}l^f)(K_{hij}k^h)$.

(12) $g^{ab}\Big[q^j{}_a q^d{}_b q^{ci} + q^j{}_b q^d{}_a q^{ci}\Big](T_{cfd}l^f)(K_{hij}k^h) = 2q^{dj}q^{ci}(T_{cfd}l^f)(K_{hij}k^h)$.

Adding up the traces we have,

$$\begin{aligned}
&g^{ab}(q^m_b q^l{}_s q^p{}_a \hat{R}^s{}_{plm}) = q^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c + \hat{R} \\
&+\Big(\hat{R}_{ab}l^a k^b + \hat{R}_{ab}k^a l^b\Big) - 2l^r\hat{\nabla}_r\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) \\
&+2\hat{\Omega}^a\hat{\Omega}_a - 2\hat{\Omega}_a\hat{\mathscr{P}}^a + 2\hat{\mathcal{D}}_a(\hat{\Omega}^a - \hat{\mathscr{P}}^a) - 2\kappa\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) \\
&-2\hat{\Xi}_{ab}\hat{\chi}^{ab} + 2\hat{\chi}^{ij}T_{ihj}k^h - 2q^{cj}q^{di}(K_{fcd}l^f)(K_{hij}k^h) + 2q^{dj}q^{ci}(T_{cfd}l^f)(K_{hij}k^h)\,. \qquad (3.89)
\end{aligned}$$

Now we have to take the trace of (3.19) and put the value of $g^{ab}(q^m_b q^l{}_s q^p{}_a \hat{R}^s{}_{plm})$ from (3.89). That leads us to, upon simplification,

$$\begin{aligned}
&-\kappa\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) = \frac{1}{2}{}^{(2)}\hat{R} - \frac{1}{2}q^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c - \frac{1}{2}\hat{R} \\
&-\frac{1}{2}\Big(\hat{R}_{ab}l^a k^b + \hat{R}_{ab}k^a l^b\Big) - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a \\
&+l^r\hat{\nabla}_r\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \hat{\mathcal{D}}_a(\hat{\Omega}^a - \hat{\mathscr{P}}^a) + \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) \\
&-\Big[\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}q^{cd} - \hat{\Xi}^{cd}\Big](T_{cfd}l^f) - \Big[q^{dj}q^{ci} - q^{cd}q^{ij}\Big](T_{cfd}l^f)(K_{hij}k^h)\,. \qquad (3.90)
\end{aligned}$$

Onwards, using the symmetries of the tensor $\hat{\mathcal{Q}}_{abcd}$, we go on to compute the quantity $q^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c$. The tensor $\hat{\mathcal{Q}}_{abcd}$ like the curvature tensor is antisymmteric in the first and the second pair of indices. We have hence upon using the symmetries of $\hat{\mathcal{Q}}_{abcd}$,

$$q^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c = g^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c - \hat{\mathcal{Q}}_{abcd}k^a k^c l^b l^d\,. \qquad (3.91)$$

From (3.75), we see that all the individual terms inside the expression for $\hat{\mathcal{Q}}_{abcd}$ is anti-symmetric in either a and c or b and d. Hence $\hat{\mathcal{Q}}_{abcd}k^a k^c l^b l^d$ vanishes. Hence,

$$\begin{aligned}
q^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c &= -\frac{3}{2}g^{bd}k^a l^c\Big(\hat{\nabla}_{[b}T_{|a|cd]} - \hat{\nabla}_{[a}T_{|b|cd]} - \hat{\nabla}_{[d}T_{|c|ab]} + \hat{\nabla}_{[c}T_{|d|ab]} \\
&\quad + T_{ae[b}T^e{}_{cd]} - T_{be[a}T^e{}_{cd]} - T_{ce[d}T^e{}_{ab]} + T_{de[c}T^e{}_{ab]}\Big) \\
&= -\frac{3}{2}g^{bd}k^a l^c\Big(-\hat{\nabla}_{[a}T_{|b|cd]} + \hat{\nabla}_{[c}T_{|d|ab]} - T_{be[a}T^e{}_{cd]} + T_{de[c}T^e{}_{ab]}\Big)\,. \qquad (3.92)
\end{aligned}$$

Upon expanding the antisymmetric parts, we have,

$$
\begin{aligned}
-\frac{3}{2}g^{bd}k^a l^c\Big(-\hat{\nabla}_{[a}T_{|b|cd]}+\hat{\nabla}_{[c}T_{|d|ab]}\Big) &= k^a l^c\Big(\hat{\nabla}_a T_c-\hat{\nabla}_c T_a+\hat{\nabla}_d T^d_{ac}\Big)\\
-\frac{3}{2}g^{bd}k^a l^c\Big(-T_{be[a}T^e{}_{cd]}+T_{de[c}T^e{}_{ab]}\Big) &= k^a l^c\Big(g^{bd}(T_{bea}T^e{}_{cd}-T_{bec}T^e{}_{ad})+T_e T^e{}_{ac}\Big)\,. \qquad (3.93)
\end{aligned}
$$

Hence we have as a result,

$$
q^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c=\Big[\Big(\hat{\nabla}_a T_b-\hat{\nabla}_b T_a\Big)+\hat{\nabla}_i T^i{}_{ab}+\Big(T^i{}_{ea}T^e{}_{bi}-T^i{}_{eb}T^e{}_{ai}\Big)+T_i T^i{}_{ab}\Big]k^a l^b\,. \qquad (3.94)
$$

However, owing to the symmetry property of the torsion tensor, it is quite easy to see that,

$$
T^i{}_{ea}T^e{}_{bi}-T^i{}_{eb}T^e{}_{ai}=0\,. \qquad (3.95)
$$

Hence we have,

$$
q^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c=\Big[\Big(\hat{\nabla}_a T_b-\hat{\nabla}_b T_a\Big)+\left(\hat{\nabla}_i+T_i\right)T^i{}_{ab}\Big]k^a l^b\,. \qquad (3.96)
$$

Next, we will deal with the term $(\hat{R}_{ab}l^a k^b+\hat{R}_{ab}k^a l^b)=(\hat{R}_{ab}+\hat{R}_{ba})k^a l^b$ in (3.90). Following simply the definition of the Riemann-curvature tensor in the spacetime $(\mathcal{M},\mathbf{g},\hat{\boldsymbol{\nabla}})$, we have for the Ricci tensor,

$$
\hat{R}_{ba}=\hat{R}_{ab}+\Big(\hat{\nabla}_a T_b-\hat{\nabla}_b T_a\Big)+\Big(\hat{\nabla}_i+T_i\Big)T^i{}_{ab}\,. \qquad (3.97)
$$

Using the above relation (3.97) and the fact that $\hat{R}=-g_{ab}l^a k^b\hat{R}$, we obtain,

$$
\begin{aligned}
&-\frac{1}{2}\Big(\hat{R}_{ab}l^a k^b+\hat{R}_{ab}k^a l^b\Big)-\frac{1}{2}\hat{R}=\\
&-\Big[\hat{G}_{ab}+\frac{1}{2}(\hat{\nabla}_a T_b-\hat{\nabla}_b T_a)+\frac{1}{2}(\hat{\nabla}_i+T_i)T^i{}_{ab}\Big]k^a l^b\,, \qquad (3.98)
\end{aligned}
$$

where as usual we have $\hat{G}_{ab}=\hat{R}_{ab}-\frac{1}{2}g_{ab}\hat{R}$. Using the relations (3.98) and (3.96), we have,

$$
\begin{aligned}
-\frac{1}{2}q^{bd}\hat{\mathcal{Q}}_{abcd}k^a l^c-\frac{1}{2}\Big(\hat{R}_{ab}l^a k^b+\hat{R}_{ab}k^a l^b\Big)-\frac{1}{2}\hat{R} &= -\hat{G}_{ab}k^a l^b\\
&-\Big[(\hat{\nabla}_a T_b-\hat{\nabla}_b T_a)+(\hat{\nabla}_i+T_i)T^i{}_{ab}\Big]k^a l^b \qquad (3.99)
\end{aligned}
$$

Let us then rewrite (3.90) using (3.99) and hence end up with the relation (3.24).

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3.12 Derivation of the relation (3.31)

Let us begin with the first term within the parentheses in the L.H.S of (3.26). Using (2.126) in this term, one finds,

$$\hat{\nabla}_b(\hat{\nabla}_a l^b) = \hat{\nabla}_b\Big[\hat{\chi}_a^{\ b} + \hat{\omega}_a l^b - l_a(k^i\hat{\nabla}_i l^b) - k_a q^{cb}\mathbb{T}_c - q^c_{\ a}q^{db}K_{fcd}l^f\Big] . \tag{3.100}$$

Then, we make use of (2.134) and (2.122) in (3.100) for the spacetime covariant derivative of the ingoing auxiliary null vector field k^a and the covariant divergence of the null generator l^a respectively. Upon simplification, this leads to,

$$\begin{aligned}\hat{\nabla}_b(\hat{\nabla}_a l^b) &= \hat{\nabla}_b\hat{\chi}^b_{\ a} + l^b\hat{\nabla}_b\hat{\omega}_a + \hat{\omega}_a\Big(\overset{(d)}{\hat{\theta}_l} + \kappa - T_i l^i\Big) \\ &\quad - \hat{\chi}_{ab}(k^j\hat{\nabla}_j l^b) + q^c_{\ a}\mathbb{T}_c(k_b(k^j\hat{\nabla}_j l^b)) \\ &\quad + q^d_{\ a}q^c_{\ b}K_{fcd}l^f(k^j\hat{\nabla}_j l^b) - \hat{\Xi}_a^{\ c}\mathbb{T}_c + q^d_{\ a}q^{ci}\mathbb{T}_i K_{fcd}k^f - \hat{\nabla}_b(q^c_{\ a}q^{db}K_{fcd}l^f) \\ &\quad - l_a\Big(\hat{\omega}_b(k^j\hat{\nabla}_j l^b) + \hat{\nabla}_b(k^j\hat{\nabla}_j l^b)\Big) + k_a\Big(\hat{\Omega}^c\mathbb{T}_c - \hat{\nabla}_b(q^{cb}\mathbb{T}_c)\Big) .\end{aligned} \tag{3.101}$$

Similarly, taking on the second term in the commutator bracket of L.H.S of (3.26), we have upon using (2.122),

$$\begin{aligned}\hat{\nabla}_a(\hat{\nabla}_b l^b) &= \hat{\nabla}_a(\overset{(d)}{\hat{\theta}_l} + \kappa) - (\hat{\nabla}_a T_i)l^i - T^i\hat{\chi}_{ia} - \hat{\omega}_a(T^i l_i) + l_a\Big(T^i(k^j\hat{\nabla}_j l_i)\Big) \\ &\quad + k_a(q^c_{\ i}\mathbb{T}_c T^i) + q^c_{\ a}q^d_{\ b}T^b(K_{fcd}l^f) .\end{aligned} \tag{3.102}$$

Using the above results we compute the quantity $\Big[\hat{\nabla}_b, \hat{\nabla}_a\Big]l^b$. We then project it on the two-surface S_t and have the expression for the L.H.S of (3.27),

$$\begin{aligned}\Big[\hat{\nabla}_b, \hat{\nabla}_a\Big]l^b q^a_{\ t} &= q^a_{\ t}\hat{\nabla}_b\Big[\hat{\chi}^b_{\ a} - q^c_{\ a}q^{db}K_{fcd}l^f\Big] + q^a_{\ t}(l^b\hat{\nabla}_b\hat{\omega}_a) + \hat{\Omega}_a\Big(\overset{(d)}{\hat{\theta}_l} + \kappa\Big) \\ &- \hat{\chi}_{tb}(k^j\hat{\nabla}_j l^b) + q^c_{\ t}\mathbb{T}_c k_b(k^j\hat{\nabla}_j l^b) - \hat{\Xi}^c_{\ t}\mathbb{T}_c - \hat{\mathcal{D}}_t\Big(\overset{(d)}{\hat{\theta}_l} + \kappa\Big) + q^a_{\ t}(\hat{\nabla}_a T_i)l^i + \hat{\chi}_{ti}T^i \\ &+ q^c_{\ t}q^d_{\ b}\Big[(K_{fdc}l^f)(k^j\hat{\nabla}_j l^b) + (K_{fdc}k^f)\mathbb{T}^b - (K_{fcd}l^f)T^b\Big] .\end{aligned} \tag{3.103}$$

Let us now, focus on the term $q^a{}_t \hat{\nabla}_b \left[\hat{\chi}^b{}_a - q^c{}_a q^{db} K_{fcd} l^f\right]$ in R.H.S of (3.103) and try to manipulate it. We see then,

$$
\begin{aligned}
q^a{}_t \hat{\nabla}_b \left[\hat{\chi}^b{}_a - q^c{}_a q^{db} K_{fcd} l^f\right] &= q^a{}_t \hat{\nabla}_b \hat{\Phi}^b{}_a = q^a_t \delta^j{}_b \delta^k{}_a (\hat{\nabla}_j \hat{\Phi}^b{}_k) \\
&= q^a{}_t (q^j{}_b - l^j k_b - k^j l_b)(q^k{}_a - l^k k_a - k^k l_a) \hat{\nabla}_j \hat{\Phi}^b{}_k \\
&= q^j{}_b q^a{}_t q^k{}_a \hat{\nabla}_j \hat{\Phi}^b{}_k - q^k{}_t l^j k_b \hat{\nabla}_j \hat{\Phi}^b{}_k - q^k{}_t k^j l_b \hat{\nabla}_j \hat{\Phi}^b{}_k \\
&= \hat{\mathcal{D}}_b \hat{\Phi}^b{}_t + q^k{}_t \hat{\Phi}^b{}_k (l^j \hat{\nabla}_j k_b) + q^k{}_t \hat{\Phi}^b{}_k (k^j \hat{\nabla}_j l_b) \\
&= \hat{\mathcal{D}}_b \hat{\Phi}^b{}_t + q^k{}_t \hat{\Phi}^b{}_k \Big(\hat{\omega}_b - \hat{\mathscr{P}}_b\Big) + q^k{}_t \hat{\Phi}^b{}_k (k^j \hat{\nabla}_j l_b) \\
&= \hat{\mathcal{D}}_b \hat{\Phi}^b{}_t + \hat{\Phi}^b{}_t \Big(\hat{\Omega}_b - \hat{\mathscr{P}}_b\Big) + \hat{\chi}_{bt} (k^j \hat{\nabla}_j l^b) - q^c{}_t q^d{}_b (K_{fcd} l^f)(k^j \hat{\nabla}_j l^b) \,. \qquad (3.104)
\end{aligned}
$$

In the above, we use the definition Eq. (3.21) of the spatial covector $\hat{\mathscr{P}}_a$. In arriving at (3.104), we have used orthogonality relations of the spatial tensor $\hat{\Phi}^a{}_b$ (w.r.t $\vec{l}$ and $\vec{k}$) and (2.114). Similarly, we have used (2.97) in the above manipulation. Next, we focus on the term $q^a{}_t (l^b \hat{\nabla}_b \hat{\omega}_a)$ of (3.103). Using (2.100), we have,

$$
\begin{aligned}
q^a{}_t (l^b \hat{\nabla}_b \hat{\omega}_a) &= q^a{}_t l^b \hat{\nabla}_b \Big(\hat{\Omega}_a - \kappa k_a + (\boldsymbol{k} \cdot \mathbb{T}) k_a\Big) \\
&= q^a{}_t (l^b \hat{\nabla}_b \hat{\Omega}_a) - (\kappa - \vec{\boldsymbol{k}} \cdot \vec{\mathbb{T}}) q^b{}_t (\hat{\Omega}_b - \hat{\mathscr{P}}_b) \,. \qquad (3.105)
\end{aligned}
$$

Here, we have used the notation that $\vec{\boldsymbol{k}} \cdot \vec{\mathbb{T}} = k_b \mathbb{T}^b$. In the above, we have as usual used (2.114) and (2.97). Note that using (2.126) for the expansion of the spacetime covariant derivative of the null generator in terms of the deformation rate tensor, it can quite easily be shown that,

$$
q^a{}_t (l^b \hat{\nabla}_b \hat{\Omega}_a) = q^a{}_t \pounds_{\boldsymbol{l}} \hat{\Omega}_a - \hat{\chi}_{tb} \hat{\Omega}^b + q^c{}_t \hat{\Omega}^d K_{fcd} l^f - q^a{}_t T_{bca} \hat{\Omega}^b l^c \,. \qquad (3.106)
$$

Using this in (3.105), we finally obtain,

$$
q^a{}_t (l^b \hat{\nabla}_b \hat{\omega}_a) = q^a{}_t \pounds_{\boldsymbol{l}} \hat{\Omega}_a - \hat{\chi}_{tb} \hat{\Omega}^b - (\kappa - \vec{\boldsymbol{k}} \cdot \vec{\mathbb{T}})(\hat{\Omega}_t - \hat{\mathscr{P}}_t) + q^a{}_t \hat{\Omega}^b l^c (K_{cab} - T_{bca}) \,. \qquad (3.107)
$$

Upon using (3.107) and (3.104) in (3.103), we obtain, after a bit simplification,

$$
\begin{aligned}
\Big[\hat{\nabla}_b, \hat{\nabla}_a\Big] l^b q^a{}_t &= \hat{\mathcal{D}}_b \hat{\Phi}^b{}_t + \hat{\Phi}^b{}_t (\hat{\Omega}_b - \hat{\mathscr{P}}_b) + q^a{}_t \pounds_{\boldsymbol{l}} \hat{\Omega}_a + \hat{\chi}_{ti} (T^i - \hat{\Omega}^i) \\
&- \hat{\Xi}_{ti} \mathbb{T}^i - (\kappa - \vec{\boldsymbol{k}} \cdot \vec{\mathbb{T}})(\hat{\Omega}_t - \hat{\mathscr{P}}_t) + q^a{}_t \hat{\Omega}^b l^c (K_{cab} - T_{bca}) \\
&+ \hat{\Omega}_t (\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} + \kappa) + q^c{}_t \mathbb{T}_c k_b (k^j \hat{\nabla}_j l^b) - \hat{\mathcal{D}}_t (\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} + \kappa) + q^a{}_t (\hat{\nabla}_a T_i) l^i \\
&+ q^c{}_t q^d{}_b \Big[(T_{fdc} l^f)(k^j \hat{\nabla}_j l^b) + (K_{fdc} k^f) \mathbb{T}^b - (K_{fcd} l^f) T^b\Big] \,. \qquad (3.108)
\end{aligned}
$$

We manipulate further a few terms on the R.H.S of (3.108). We note that upon using the definition (3.29), we have,

$$\hat{\Phi}_{bt}(\hat{\Omega}^b - \hat{\mathscr{P}}^b) + \hat{\chi}_{bt}(T^b - \hat{\Omega}^b) = \hat{\chi}_{bt}(T^b - \hat{\mathscr{P}}^b) - q^c{}_t\hat{\Omega}^d(K_{fcd}l^f) + q^c{}_t\hat{\mathscr{P}}^d(K_{fcd}l^f) \, . \tag{3.109}$$

Next, we manipulate the term $q^a{}_t(\hat{\nabla}_a T_b)l^b$ in the R.H.S of (3.108) via using (2.126):

$$\begin{aligned} q^a{}_t(\hat{\nabla}_a T_b)l^b &= q^a{}_t\Big(\hat{\nabla}_a(T_b l^b) - T^b(\hat{\nabla}_a l_b)\Big) \\ &= \hat{\mathcal{D}}_t(T_b l^b) - \hat{\chi}_{ta}T^a - \hat{\Omega}_t(T_a l^a) + q^c{}_t q^d{}_b T^b(K_{fcd}l^f) \, . \end{aligned} \tag{3.110}$$

Finally using the relations (3.109) and (3.110) in (3.108), we obtain after simplification,

$$\begin{aligned} \Big[\hat{\nabla}_b, \hat{\nabla}_a\Big] l^b q^a{}_t &= \hat{\mathcal{D}}_b\hat{\Phi}^b{}_t + q^a{}_t \pounds_{\boldsymbol{l}}\hat{\Omega}_a - \hat{\chi}_{bt}\hat{\mathscr{P}}^b + q^c{}_t\hat{\mathscr{P}}^d(K_{fcd}l^f) - \hat{\Xi}_{ti}\mathbb{T}^i \\ & - (\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})\hat{\Omega}_t + (\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})\hat{\mathscr{P}}_t - q^a{}_t\hat{\Omega}^b l^c T_{bca} + \hat{\Omega}_t\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} + \kappa\Big) + q^c{}_t\mathbb{T}_c k_b(k^j\hat{\nabla}_j l^b) \\ & - \hat{\mathcal{D}}_t\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} + \kappa - T_b l^b\Big) - \hat{\Omega}_t(T_a l^a) + q^c{}_t q^d{}_b(T_{fdc}l^f)(k^j\hat{\nabla}_j l^b) + q^c{}_t q^d{}_b(K_{fdc}k^f)\mathbb{T}^b \, . \end{aligned} \tag{3.111}$$

Now, we bring our focus to the R.H.S of (3.27). As usual, we use (2.126) for the expansion of the covariant derivative of the null generator. This yields,

$$\begin{aligned} \hat{R}_{ab}l^a q^b{}_t - T^i{}_{ba}(\hat{\nabla}_i l^b)q^a{}_t &= \hat{R}_{ab}l^a q^b{}_t - T^i{}_{ba}\hat{\chi}^b{}_i q^a{}_t - (T_{iba}\hat{\omega}^i l^b)q^a{}_t \\ & + (T_{iba}l^i)(k^j\hat{\nabla}_j l^b)q^a{}_t + T_{iba}k^i q^{cb}\mathbb{T}_c q^a{}_t + T^i{}_{ba}q^c{}_i q^{db}(K_{fcd}l^f)q^a{}_t \, . \end{aligned} \tag{3.112}$$

We now express the term $(T_{iba}l^i)(k^j\hat{\nabla}_j l^b)q^a{}_t$ in the R.H.S of (3.112) in a different way:

$$\begin{aligned} (T_{iba}l^i)(k^j\hat{\nabla}_j l^b)q^a{}_t &= q^c_t\delta^d{}_b(T_{fdc}l^f)(k^j\hat{\nabla}_j l^b) \\ &= q^c{}_t(q^d{}_b - k^d l_b - l^d k_b)(T_{fdc}l^f)(k^j\hat{\nabla}_j l^b) \\ &= q^c{}_t q^d{}_b(T_{fdc}l^f)(k^j\hat{\nabla}_j l^b) + q^c{}_t k_b\mathbb{T}_c(k^j\hat{\nabla}_j l^b) \, . \end{aligned} \tag{3.113}$$

Upon using (3.113) in (3.112) and using (2.100), we obtain,

$$\begin{aligned} \hat{R}_{ab}l^a q^b{}_t - T^i{}_{ba}(\hat{\nabla}_i l^b)q^a{}_t &= \hat{R}_{ab}l^a q^b{}_t - T_{iba}\hat{\chi}^{bi}q^a{}_t - T_{iba}\hat{\Omega}^i l^b q^a{}_t + (\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})\hat{\mathscr{P}}_t \\ & + q^c{}_t q^d{}_b(T_{fdc}l^f)(k^j\hat{\nabla}_j l^b) + q^c{}_t k_b\mathbb{T}_c(k^j\hat{\nabla}_j l^b) \\ & + T_{iba}k^i q^{cb}\mathbb{T}_c q^a{}_t + q^a{}_t q^{ci}q^{db}T_{iba}K_{fcd}l^f \, . \end{aligned} \tag{3.114}$$

Now, all we got to do is to equate (3.111) and (3.114) via (3.27) and simplify. To this effect, we obtain (3.31).

Part II

Applications to theories of gravity

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Chapter 4

Covariant approach to the thermodynamic structure of a generic null surface in Riemannian spacetime

4.1 Introduction and motivation

Now, we begin our journey of interpreting the gravitational field equations w.r.t the generic null hypersurface $\mathcal{H}$. Our objective will be to see whether we can attest any thermodynamic or fluid dynamic interpretation to the gravitational dynamics. In the previous chapter, we have constructed the dynamical evolution equations of quantities described for $\mathcal{H}$ in the RC spacetime. We saw that such evolution equations reduced to the well known laws for the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ by setting the torsion to zero. Our first testbed will be studying the thermodynamic interpretation to the gravitational field equations for a spacetime without any torsion built into it and hence provided with the Levi-Civita connection $\boldsymbol{\nabla}$. We will not be focusing on the fluid-dynamic interpretation for Einstein gravity since this is already a standard result in the literature. Damour [37] (in the context of a black hole event horizon in Einstein gravity) showed that the particular projection, $G_{ab}l^a q^b{}_c$, on S_t leads to the Damour Navier-Stokes (DNS) equation which is structurally quite similar to the Navier-Stokes (NS) equation. However, the DNS equation can as well be obtained for any generic null hypersurface $\mathcal{H}$ in the spacetime [38, 61, 82] (see Eq. (6.14) of [61] for an excellent review).

The intriguing connection between gravitational dynamics explored on the black hole horizon and classical thermodynamics was laid bare in the seventies following the work of Bekenstein, Hawking and others [4–8] (for a review see [14, 112, 115]). This led to the development of the famous black hole mechanics which are a set of intricate equivalences. For every law of black hole mechanics, there exists a corresponding law of classical thermodynamics, thus allowing the black hole to be considered as a thermodynamic object. However these connections can very well be equivalently established not just only for

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black hole horizons, but rather on any arbitrary null hypersurface. This allows the attribution of thermodynamical quantities like temperature, entropy etc for any null surface [45].[1]

We saw in Chapter 3 that $G_{ab}l^a l^b$ is related to the null Raychaudhuri equation [25, 61]. The NRE equation is a purely geometrical relation which relates the evolution of the outgoing expansion scalar $\overset{(d)}{\theta_l}$ along the null generators $\vec{l}$ with $G_{ab}l^a l^b$. The NRE equation was used as a crucial input by Jacobson to derive the Einstein field equations from the Clausius identity $\delta S = \frac{\delta Q}{T}$, applied on the local Rindler horizons [35]. The Rindler horizons are assumed to be at thermodynamic equilibrium and δQ represents the matter energy flux traversing across the causal horizon which results in the change of entropy δS (known as Clausius entropy) associated with the horizon. The equilibrium condition requires the crucial restrictive assumption of the vanishing of the second fundamental form and the shear tensor on the null horizon. It is postulated that δS is proportional to the area change of the horizon. The above formalism was extended to the non equilibrium case, in the regime of which, the shear tensor and the expansion scalar on the null surface cannot be set to zero [62–64]. Gravitational equations for certain modified theories of gravity were also obtained from such similar thermodynamic considerations [63]. Later this concept of Clausius entropy was extended to arbitrary bifurcate null surface [148] and the Einstein equations were also derived for stretched light cone [69]. Moreover, Jacobson [149] derived the Einstein field equations as applied to local causal diamonds (constructed at any point in the spacetime) by extremizing the total entanglement entropy of the null horizon and the matter inside of it.

Next, we consider the other relevant projection component of the Einstein tensor. It was shown by Padmanabhan and his collaborators [36, 70, 73] that a certain projection of the Einstein field equations (via $G_{ab}l^a k^b$) yields a thermodynamic interpretation which is structurally similar to the first law of thermodynamics. The main difference between Padmanabhan's [36, 70, 73] and Jacobson's [35] approaches in order to relate thermodynamics is the choice of the component of the Einstein equation. For Jacobson, the relevant projection component is $G_{ab}l^a l^b$, whereas for Padmanabhan, the choice is $G_{ab}l^a k^b$. In fact, it is pointed out in [36] that the neater component to consider is $G_{ab}l^a k^b$ which produces the thermodynamic identity without any restrictive assumptions like the vanishing of the second fundamental form and the shear tensor on $\mathcal{H}$ (which was a crucial assumption in Jacobson's approach). The argument behind this is $G_{ab}l^a k^b$ picks out the component of $G^a_{\ b}l^b$ along $\vec{l}$, the null generators which are intrinsic to the null surface $\mathcal{H}$. Whereas the other one corresponds to that of $G^a_{\ b}l^b$ along $\vec{k}$ (see section 3 of [36] for more details). Padmanabhan's approach has been generalized to the case of Lanczos-Lovelock theories of gravity [77–79] as well. However, such a thermodynamic interpretation (following

[1] Although not fully understood, but there exists certain progress in this direction (see [145–147] for few instances).

Padmanabhan) has been done for a generic null hypersurface by invoking the adapted Gaussian null coordinates [55–58]. This makes the identified expression of the thermodynamics entities to be in "non covariant" form. In this chapter, we aim to investigate whether in a completely covariant fashion $G_{ab}l^a l^b$ and $G_{ab}l^a k^b$ can be provided any physical interpretation, without invoking any specific gravitational dynamics; *i.e.* solely based on the properties of the null surface $\mathcal{H}$. In our analysis, the NRE equation (for both the outgoing and ingoing expansion scalars) is the starting point in providing a physical interpretation for the two concerned projections. We will show that the underlying form of the gravitational dynamics of the background spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ is not "explicitly" necessary to provide such interpretation.

Let us briefly summarize the contents of this chapter. We begin in section 4.2 by elucidating the constructional differences between the GNC chart and the $3+1$ foliation of the null family $\mathcal{H}_u$ of $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ (that we introduced in chapter 2). In section 4.3, we analyze if $G_{ab}l^a l^b$ can be provided a physical/thermodynamic interpretation. We begin with the NRE equation for the outgoing expansion scalar θ_l and then integrate it on the transverse spacelike surface S_t. Performance of a *virtual displacement* along the null generators of $\mathcal{H}$ leads to a possibility of thermodynamic identity. We feel that this is not a surprising result at all as NRE equation has been used in the search of thermodynamics of horizons and therefore it possesses such an inherent structure. Still we present this study in order to provide a segue into our main topic of providing a physical interpretation to $G_{ab}l^a k^b$ in a covariant way. In going through the steps we shall observe a few interesting features of the approach which are probably not emphasized in literature. It is noticed that in the special case of a stationary black hole system, the expression of the energy is related to the well known Komar energy E_K (see Chapter 4, page number 149 of [25]). Moreover the integrated form of the thermodynamic identity leads to a generalized form of Smarr formula [150] which, as given in literature, is of the form $E_K = 2ST$. Here T is the temperature of the horizon (see [119, 151–153] for discussions related to this identity).

Next we concentrate our attention to $G_{ab}l^a k^b$ in Section 4.4. Invoking the NRE equation corresponding to the ingoing expansion scalar θ_k and integrating it on the transverse space S_t and allowing for the virtual displacement along $\vec{k}$, we arrive at a thermodynamic interpretation of $G_{ab}l^a k^b$ which is structurally equivalent to the first law of thermodynamics. The induced null foliation (though non-unique) of the spacetime manifold $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ (which we introduced in Sec. 2.3) allows us to have a completely covariant expression of the energy term. This is because the expression of the energy term contains geometrical quantities defined on the null surface $\mathcal{H}$. These geometrical quantities once defined on $\mathcal{H}$ will be independent of the foliation chosen. Here we provide our definition of the *"geometrical work function"* in order to make way for the thermodynamic identity independent of any theory of gravity established on the background spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$.

Previously equivalent thermodynamic interpretation (analogous to the first law of thermodynamics) have been provided in the Einstein-Hilbert [36] and Lanczos-Lovelock theory [79]. However there are certain important differences between the work in [36],[79] and ours. In [36] and [79], the derivations of the thermodynamic identity have been performed near a generic null hypersurface without any assumed symmetries of the spacetime. However the derivations have been performed w.r.t an adapted null coordinate system constructed in the neighborhood of the generic null surface $\mathcal{H}$ known as the GNC system. One noticeable feature of such a construction is that the expression of the energy is compatible to the GNC metric only. For the GNC construction, the auxiliary null vector field $\vec{k}$ is affinely parametrized and hypersurface-orthogonal. We in our case, however foliate the spacetime in the vicinity of the generic null surface $\mathcal{H}$ by a family or stack of null hypersurfaces. Then allowing for the $3+1$ induced foliation of the family of the null surfaces, we derive exactly the same structural thermodynamic identity in a completely coordinate-independent fashion. Our construction does not require $\vec{k}$ to be affinely parametrized and hypersurface-orthogonal. Another difference in our approach from that adopted in earlier ones is that our starting point is the NRE equation corresponding to θ_k, whereas no such equation has been explicitly used in these works. It may be pointed out that the work function (or pressure) in [36, 74, 79] has consistently been defined as $P = -T_{ab}l^a k^b$ *i.e.* owing entirely to the matter energy tensor. The entropy density has then been defined as the Bekenstein-Hawking entropy density for the Einstein-Hilbert case [36] and as the the Wald entropy density for the Lanczos-Lovelock models [79]. However in our interpretation, we have identified what we call as the *"geometrical work function" (or geometrical pressure)*, entirely from geometrical quantities. In analogy to the entanglement entropy, we call our identified entropy density as the "entanglement entropy density" since it depends on the geometry of the relevant surface. Under the umbrella of such an interpretation, we have aimed to provide the thermodynamic identification independent of any theory of gravity whose solution space is the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$.

For the reader, we summarize the structural and interpretational difference between the approach in [36, 79] and ours.

- The thermodynamic identity in [36, 79] and is brought through the GNC construction while ours is brought about through a $3+1$ foliation of the null family $\mathcal{H}_u$.

- The expression of the energy in the GNC is solely adapted to these coordinates. On the contrary, ours is in a covariant form and hence can be applied to any structure of the null surface.

- In [36, 79], the pressure or work function has been consistently defined w.r.t the matter energy tensor $P = -T_{ab}l^a k^b$, while we define a so called "geometrical work function".

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- The entropy density for the case of Einstein gravity is the Bekenstein-Hawking entropy density [36], while it is the Wald entropy density for the Lanczos-Lovelock models [79]. For our interpretation, the entropy density is consistently the entanglement entropy density irrespective of the theory of gravity.

In Sec. 4.5, we show that the covariant expression of the energy we derive entirely matches with the expression of the energy obtained via the GNC system in [36] for the Einstein-Hilbert case. We use the appendices to provide derivation of results used in this chapter.

4.2 $3+1$ induced foliation of $\mathcal{H}$ vs the GNC construction

As mentioned earlier, our covariant interpretation of thermodynamic identity will be brought about in the spirit of $3+1$ induced foliation of $\mathcal{H}$. To remind the reader, the $3+1$ foliation of $\mathcal{H}$ by spacelike slices just brings about the extra structure in $\mathcal{H}$ so that both the null generators and the auxiliary null vector field are fixed uniquely. However, as far as the previous literature is concerned, the relevant thermodynamic interpretation has been brought about in the GNC construction. So it is important to bring about structural differences between the systems. This discussion is important in order to identify the underlying constructional differences used in the thermodynamic interpretation from the earlier attempts which rely on the structure of GNC. In the GNC construction with the coordinates (u, r, x^A), the generic null hypersurface is stationed at $r = 0$. The null normal to the hypersurface $\mathcal{H}$ is defined as the gradient of the $r =$ constant surfaces *i.e.* $l_a = \partial_a r$ and are non-affinely parametrized satisfying $l^j \nabla_j l^i = \kappa l^i$. The auxiliary null vector field k^a in the GNC is by construction chosen to be along affinely parametrized null geodesics. That is, we move away from the null surface stationed at $r = 0$ along the ingoing null geodesic of k^a. In the GNC, the auxiliary null vector field $k^a = -(\partial/\partial r)^a$ has the affine parameter r and points along the direction of decreasing r (ingoing). It can also be seen that the null geodesic k_a is hypersurface-orthogonal to the $u =$ constant surfaces *i.e.* $k_a = -\partial_a u$. Hence we see that the coordinates adapted to the null surface $\mathcal{H}$ at $r = 0$ are (u, x^A). As a result of this adapted coordinatization (u, r, x^A) of $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ in the vicinity of $\mathcal{H}$, the thermodynamic interpretation of the gravitational dynamics via the projection component $G_{ab}k^a l^b$ comes explicitly as GNC dependent. However within the construction of induced $3+1$ foliation of $\mathcal{H}$, we do not impose any such specific coordinate system as *per se*. Within this scheme, as mentioned earlier, all the relevant geometrical quantities of interest will be foliation-independent and the thermodynamic identity will be completely coordinate-independent. In our construction, the auxiliary null vector field need not be along null geodesics as well as hypersurface-orthogonal. The auxiliary null vector field in our case is by construction an ingoing normal to the spacelike 2-surfaces S_t and hence extends out into the open neighborhood of $\mathcal{H}$. It can be shown via Eq. (2.110), that k_a satisfies the following relation,

$$\partial_a k_b - \partial_b k_a = \frac{1}{2N^2}\Big[\partial_a\Big(\ln(\frac{N}{M})\Big)l_b - \partial_b\Big(\ln(\frac{N}{M})\Big)l_a\Big]\,, \tag{4.1}$$

where $M = -\frac{e^\rho}{N}$. The relation (4.1) shows that $\vec{k}$ is not hypersurface-orthogonal (as on the right hand side (RHS) l_a appears). Hence k^a is not the generator of any null hypersurface or in other words, the hyperplane normal to k^a is not integrable into a smooth submanifold surface. It can also be shown (see [61]) that,

$$k^i\nabla_i k_a = -\frac{1}{2N^2}\Pi^i{}_a\partial_i \ln\Big(\frac{N}{M}\Big)\,, \tag{4.2}$$

where $\Pi^i{}_a = \delta^i{}_a + k^i l_a$ is the projection tensor onto $\mathcal{H}$ along k_i. The relation (4.2) essentially shows that the auxiliary null vector field k^a does not satisfy the geodesic equation as opposed to that in the GNC construction.

4.3 $G_{ab}l^a l^b$: route to a thermodynamic identity ?

The quantity $G_{ab}l^a l^b$ comes about in the NRE for the null generators of $\vec{l}$ of $\mathcal{H}$. Usually, NRE equation is used to explore the thermodynamical behavior of a black hole horizon. Both for proving the area increase theorem as well as finding the first law of black hole mechanics, the NRE plays a central role. It must be noted that in all these analyses, the NRE equation usually comes in the middle of the calculation and is always applied for the Killing vector which is null on the horizon. For instance, see the discussion around Eq. (8.168) to Eq. (8.173) of [95]. Moreover, in Jacobson's analysis [35], the NRE has been used to derive the Einstein's field equations from the Clausius relation (an equilibrium local constitutive relation). Therefore, apparently the NRE provides an inherent thermodynamical structure on the horizon w.r.t the gravitational dynamics. Although it is not explicitly mentioned in literature, but the way it has been used, one can immediately identify this property. In all these earlier analysis, the expressions for thermodynamical entities are taken as input at the very beginning and then finally the NRE is used to obtain the required conclusion. Also, as we mentioned earlier, the analysis has been strictly confined to the Killing horizon case (or asymptotically Killing).

Here we shall take the "opposite" route. We will begin with the NRE for an arbitrary null congruence $\vec{l}$ (say), not necessarily a Killing one. We will only assume that the null vector field $\vec{l}$ is hypersurface-orthogonal, generating an integrable surface $\mathcal{H}$. Interestingly, the integration of the NRE on the two-dimensional subspace S_t (which is orthogonal to $\vec{l}$), leads to a first law of thermodynamics like structure. This analysis has some noticeable features. First of all, this is valid for any arbitrary integrable null surface $\mathcal{H}$ and so the results are valid beyond Killing vector field-generated horizons. Secondly, a more general expression of gravitational energy can be obtained. Finally, this thermodynamic structure is the property of the gravitational dynamics w.r.t. the null surface embedded in the Riemannian spacetime, and is independent of the specifics of which gravitational theory we use.

Let us now start our calculation. The NRE [25, 61] for the outgoing expansion scalar θ_l in $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ as given by Eq. (3.9),

$$l^a \nabla_a \theta_l = \frac{d\theta_l}{d\lambda_{(l)}} = \kappa\theta_l - \frac{1}{2}\theta_l^2 - {}^{(l)}\sigma_{ab}{}^{(l)}\sigma^{ab} - G_{ab}l^a l^b \,, \tag{4.3}$$

where l^a, satisfying $l^j \nabla_j l^i = \kappa l^i$, is parametrized by non-affine parameter $\lambda_{(l)} = t$, *i.e.* $l^a = dx^a/d\lambda_{(l)} = dx^a/dt$. We now make a virtual displacement of the null hypersurface along its own generators by an amount $\delta\lambda_{(l)}$. We will discuss about the physical interpretation of the virtual displacement later in this chapter. In essence, the virtual displacement is a "physical process" that *virtually* shifts the two-dimensional cross-section S_t stationed at the non-affine parameter value $\lambda_{(l)} = t$ to say $\lambda_{(l)} + \delta\lambda_{(l)} = t + \delta\lambda_{(l)}$. This virtual displacement process shifts $\mathcal{H}$ along $\vec{l}$ by a parameter value $\delta\lambda_{(l)}$. Multiplying both sides of equation (4.3) by the transverse elementary area $dA = \sqrt{q}d^2x$ of S_t and $\delta\lambda_{(l)}$ and then integrating on the two-surface S_t, one finds,

$$\delta\lambda_{(l)} \int_{S_t} dA\, \kappa\theta_l \;=\; \delta\lambda_{(l)} \int_{S_t} dA \left[\frac{d\theta_l}{d\lambda_{(l)}} + \frac{\theta_l^2}{2} + {}^{(l)}\sigma_{ab}{}^{(l)}\sigma^{ab} + G_{ab}l^a l^b\right] . \tag{4.4}$$

Now since we know that

$$\theta_l = \frac{1}{\sqrt{q}}\frac{d\sqrt{q}}{d\lambda_{(l)}} \,, \tag{4.5}$$

the term on the left hand side (LHS) of (4.4) can be expressed as follows:

$$\frac{1}{8\pi}\delta\lambda_{(l)} \int_{S_t} dA\, \kappa\theta_l = \int_{S_t} d^2x \frac{\kappa}{2\pi}\delta\lambda_{(l)} \frac{d}{d\lambda_{(l)}}\left(\frac{1}{4}\sqrt{q}\right) = \int_{S_t} d^2x T \delta_{\lambda(l)} s \,. \tag{4.6}$$

In the above we introduced a factor $1/(8\pi)$. Here we have identified $T = \frac{\kappa}{2\pi}$ as the temperature of the null surface and is hence related to the non-affinity parameter κ of the null generators of the null hypersurface $\mathcal{H}$. We here also identify $s = \frac{\sqrt{q}}{4}$ as the entropy density of the null surface. This, in analogy to entanglement entropy, we may interpret as entanglement entropy density (more will be discussed on this analogy in the next section). In the same way, multiplying the R. HS of (4.4) by the numerical factor $(1/8\pi)$, the resultant equation can be interpreted as the following thermodynamic identity:

$$\int_{S_t} d^2x T \delta_{\lambda(l)} s = \delta_{\lambda(l)} E \,, \tag{4.7}$$

where E is the energy associated to the null surface, given by

$$E = \frac{1}{8\pi}\int d\lambda_{(l)} \int_{S_t} dA \left[\frac{d\theta_l}{d\lambda_{(l)}} + \frac{\theta_l^2}{2} + {}^{(l)}\sigma_{ab}{}^{(l)}\sigma^{ab} + G_{ab}l^a l^b\right] . \tag{4.8}$$

Note that in the whole analysis, we never used Einstein's field equations or any gravitational field equations (defined for $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$) and hence the result is very generic to any null surface and whatever the gravitational theory be. The only requirement is that background for the gravitational theory needs to be Riemannian.

We can however provide an alternative interpretation to the NRE (5.19) under the process of the virtual displacement $\delta\lambda_{(l)}$. We first multiply both sides of Eq. (5.19) by the transverse area element of the 2-surface S_t together with a multiplicative factor of $\frac{1}{8\pi}$ *i.e.* $\frac{1}{8\pi}dA = \frac{1}{8\pi}\sqrt{q}d^2x$ and the virtual displacement $\delta\lambda_{(l)}$. We then integrate the resulting equation over S_t,

$$\frac{1}{8\pi}\delta\lambda_{(l)}\int_{S_t} d^2x\sqrt{q}\,\kappa\theta_l = \frac{1}{8\pi}\delta\lambda_{(l)}\int_{S_t} d^2x\sqrt{q}\left(-\frac{\theta_l^2}{2} + {}^{(l)}\sigma_{ab}{}^{(l)}\sigma^{ab}\right) + \frac{1}{8\pi}\delta\lambda_{(l)}\int_{S_t} d^2x\sqrt{q}\left[\frac{d\theta_l}{d\lambda_{(l)}} + \theta_l^2 + G_{ab}l^a l^b\right] . \tag{4.9}$$

Following Eq. (4.6), the LHS can be identified as,

$$\frac{1}{8\pi}\delta\lambda_{(l)}\int_{S_t} d^2x\sqrt{q}\,\kappa\theta_l = \int_{S_t} d^2x T\delta_{\lambda(l)}s\,, \tag{4.10}$$

where $s = \frac{\sqrt{q}}{4}$ is the entropy density of the null hypersurface and will be identified as the entanglement entropy density. Looking at the first term of the RHS of (4.9), we find the integrand contains the well known dissipation term $\mathbb{D}$ corresponding to the null hypersurface,

$$\mathbb{D} = \Theta_{ab}\Theta^{ab} - \theta_l^2 = -\frac{1}{2}\theta_l^2 + {}^{(l)}\sigma_{ab}{}^{(l)}\sigma^{ab} . \tag{4.11}$$

The identification of the dissipation term basically comes from the $G_{ab}l^a q^b{}_c$ projection component, which results in the DNS equation. The physical interpretation of this viscous dissipation term and its observer dependence has been explored in [82]. Finally, if we identify the variation of the energy under $\delta\lambda_{(l)}$ virtual displacement as,

$$\delta_{\lambda(l)}E = \frac{1}{8\pi}\delta\lambda_{(l)}\int_{S_t} d^2x\sqrt{q}\left[\frac{d\theta_l}{d\lambda_{(l)}} + \theta_l^2 + G_{ab}l^a l^b\right] , \tag{4.12}$$

we can then identify (4.9) as ,

$$\int_{S_t} d^2x T\delta_{_{\lambda(l)}} s = \frac{1}{8\pi}\delta\lambda_{(l)} \int_{S_t} d^2x\sqrt{q}\,\mathbb{D} + \delta_{_{\lambda(l)}} E\,. \tag{4.13}$$

The first term on the RHS of (4.13) is identified as the heat generation part under the virtual displacement $\delta\lambda_{(l)}$ process due to irreversible viscous dissipation effects of the null surface $\mathcal{H}$,

$$\delta_{_{\lambda(l)}} Q_{\rm dis} \equiv \frac{1}{8\pi}\delta\lambda_{(l)} \int_{S_t} d^2x\sqrt{q}\,\mathbb{D}\,. \tag{4.14}$$

A very interesting feature of this dissipation term $\mathbb{D}$ is that it comprises entirely of geometrical quantities. Hence it can be thought that the dissipative sector of the heat generation basically arises due to gravitational energy fluxes through the null surface. A similar interpretation was previously put forth in [63]. The authors of [63] show via an analogy between the horizon null congruence and a continuous medium that the dissipative heat generation is purely due to gravitational effects. Our expression of the internal heat production (4.14) for non-affine generators l^a matches with Eq. (47) of [63] for non-affine Killing vector field approximately generating the Killing horizon, provided we set θ_l to zero in our case. The authors in [63] constructed a local Rindler wedge in the neighborhood of a spacetime point P. The Rindler horizon is approximately generated by the Killing vector field ξ^a which satisfies $\xi^a\nabla_a\xi^b = \kappa\xi^b$. The Killing vector field is tangent to the affinely parametrized null congruence l^a of the Rindler horizon. The viscous dissipative part of the heat generation term or *uncompensated heat* in Eq. (47) of [63] contains the norm $||\hat{\sigma}||$, where $\hat{\sigma}_{ab}$ is the shear tensor corresponding to the Killing congruence. However in our construction, the arbitrary null surface is generated by the non-affinely parameterized l^a field and hence all geometric quantities in our expression (4.14) of the dissipative heat generation pertains to the l^a congruence itself. Under such an identification, we have from Eq. (4.13),

$$\int_{S_t} d^2x T\delta_{_{\lambda(l)}} s = \delta_{_{\lambda(l)}} Q_{\rm dis} + \delta_{_{\lambda(l)}} E\,. \tag{4.15}$$

We now carry over this analysis to the special case of a *stationary* black hole system, for example the Kerr spacetime. In that case, the non-affinity parameter κ is independent of the transverse coordinates $\{x^A\}$ and hence can be taken outside the integral in equation (4.6). This allows us to interpret equation (4.7) in the more familiar setting of the thermodynamic identity (first law of thermodynamics),

$$T\delta_{_{\lambda(l)}} S = \delta_{_{\lambda(l)}} E\,, \tag{4.16}$$

where $S = \frac{A}{4}$ is the entropy of the null hypersurface and hence proportional to its area A. Let us now investigate the expression (4.8) of the energy E for the stationary black hole case. In this case $\mathcal{H}$ is the event horizon of the black hole and l^a is the timelike Killing vector which is null on $\mathcal{H}$. Therefore, we denote l^a as $l^a \equiv \xi^a = (1, 0, 0, \Omega_H)$ where Ω_H is the angular velocity of the black hole. Then, the first term on the RHS of (4.8) can be evaluated as follows. Integration over $\lambda_{(l)}$ yields $\int_{S_t} dA\, \theta_l \Big|_{\lambda_{(l)}=1}^{\lambda_{(l)}=2}$ and since the value of θ_l vanishes for the stationary horizon, this term will not contribute. This obviously entails the fact, that the initial and the final states of the horizon are stationary and hence in equilibrium after the process of virtual displacement has been applied. In this case, for the "quasi-static physical process" the next two terms of RHS can be neglected on S_t compared to the other terms (known as equilibrium or near-equilibrium situation [154, 155]). Therefore the energy expression (4.8), for this special case reduces to

$$E = \frac{1}{8\pi} \int d\lambda_{(l)} \int_{S_t} dA\, G_{ab}\xi^a \xi^b \,. \tag{4.17}$$

Also analogously we investigate the second alternative interpretation provided via Eq. (4.15), through the introduction of the viscous dissipative part of the heat generation under the "quasi-static physical process" for the stationary black hole system. However for the stationary black hole case, the virtual displacement is through a quasi-equilibrium process. That is, initially the 2-surface S_t is at equilibrium at $\lambda_{(l)} = t$ and then via a quasi-static process it is displaced to the stationary equilibrium state at $\lambda_{(l)} + \delta\lambda_{(l)} = t + \delta\lambda_{(l)}$. As a result of this quasi-static virtual displacement process, the dissipative heat generation $\delta_{\lambda(l)} Q_{\text{dis}}$ is basically zero. Under such a process for the stationary black hole, we hence have (4.16). The energy expression for the stationary black hole case in this alternative interpretation still remains the same and is given by (4.17).

This expression of the energy (4.17) is known to be proportional to the Komar expression for conserved quantity (see [25] for details on Komar conserved quantity), calculated on the horizon. The volume element on $\mathcal{H}$ is $d\Sigma^a = -\xi^a d\lambda_{(l)} dA$. Also, we can express $R_{ab}\xi^b$ as $R_{ab}\xi^b = [\nabla_b, \nabla_a]\xi^b = (1/2)\nabla^b(\nabla_a\xi_b - \nabla_b\xi_a) \equiv (1/2)\nabla^b J_{ab}$, where in the last step of the Killing equation $\nabla_a\xi_b + \nabla_b\xi_a = 0$ has been used. Note that J_{ab} can be identified as the Noether potential for Einstein gravity. Using all these in (4.17) one obtains

$$E = -\frac{1}{16\pi} \int_{\mathcal{H}} d\Sigma_a \nabla_b J^{ab} = -\frac{1}{32\pi} \oint_{S_t} d\Sigma_{ab} J^{ab} \,. \tag{4.18}$$

In the last equality, the Stoke's law for integration has been applied. Now the Komar conserved quantity is defined as [25]

$$E_K = -\frac{1}{16\pi} \oint_{S_t} d\Sigma^{ab} J_{ab} \,, \tag{4.19}$$

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which for Killing vector corresponding to time translational symmetry, gives mass term M_H at the horizon while that for Killing vector corresponding to azimuthal symmetry leads to $-2\Omega_H J_H$ where J_H is the angular momentum at S_t. Therefore (4.19) for stationary black hole, like Kerr metric, yields $E_K = M_H - 2\Omega_H J_H$. Comparison of (4.18) and (4.19) yields $E = (1/2)E_K$.

Next we can integrate Eq. (4.16) over $\lambda_{(l)}$. Since for stationary background $l^a = \xi^a$ and $\xi^a \nabla_a \kappa = 0$, we have

$$\delta_{\lambda(l)} T = \delta\lambda_{(l)} \frac{dT}{d\lambda_{(l)}} = \delta\lambda_{(l)} \xi^a \nabla_a T = 0\,. \tag{4.20}$$

Therefore integration of (4.16) yields $E = TS$ and since $E = (1/2)E_K$, in terms of Komar conserved quantity on horizon, one finds

$$E_K = 2TS\,. \tag{4.21}$$

This has already been shown in literature [119, 151–153] to be the general form of the famous Smarr formula [150] (for a particular dynamical black hole, similar relations have also been achieved [156]). For instance, in the case of Kerr, the above leads to our well known Smarr expression $M_H = 2\Omega_H J_H + (\kappa A/4\pi)$. This shows that the integrated form of NRE (incorporating the relevant gravitational dynamics) on the stationary horizon along the Killing vector is the Smarr relation.

4.4 $G_{ab}l^ak^b$, the favourable candidate for thermodynamic interpretation: a covariant approach

We are now in a position to hit the better candidate among the different projections of G_{ab}; *i.e.* $G_{ab}l^ak^b$ (the logic for better choice has been discussed in [36]) which serves as a thermodynamic identity. The approach will be similar to the earlier section and hence the outcome will be covariant in nature. We start with the following evolution equation of the transversal deformation rate tensor Ξ_{ab} along the null generators l^a of the hypersurface $\mathcal{H}$:

$$\begin{aligned} q^i_a q^j_b \pounds_l \Xi_{ij} &= \frac{1}{2}(\mathcal{D}_a\Omega_b + \mathcal{D}_b\Omega_a) + \Omega_a\Omega_b - \frac{1}{2}{}^2R_{ab} + \frac{1}{2}q^i_a q^j_b R_{ij} - \Big(\kappa + \frac{\theta_l}{2}\Big)\Xi_{ab} \\ &- \frac{\theta_k}{2}\Theta_{ab} + \Theta_{ai}\Xi^i_b + \Xi_{ai}\Theta^i_b\,. \end{aligned} \tag{4.22}$$

The detailed derivation of this is given in [61]. Analogously, Eq. (4.22) follows directly from Eq. (3.23) by setting all the torsion terms to zero. Taking trace on both sides of Eq.

(4.22) we obtain the following identity:

$$-\kappa\theta_{\boldsymbol{k}} \;=\; \Big(-\mathcal{D}_a\Omega^a - \Omega_a\Omega^a + \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} + l^i\nabla_i\theta_{\boldsymbol{k}} + \tfrac{1}{2}{}^2R \Big) - G_{ij}l^ik^k \,. \tag{4.23}$$

The above relation is same as Eq. (3.25) which relates the evolution of the ingoing expansion scalar $\theta_{\boldsymbol{k}}$ along $\vec{\boldsymbol{l}}$ to the projection component $G_{ab}l^ak^b$. It is in this sense we call the above equation, the NRE for $\theta_{\boldsymbol{k}}$ in the Riemannian spacetime.

Below, we shall show, taking inspiration from the earlier section, the NRE (4.23) via the virtual displacement along $\vec{\boldsymbol{k}}$ can also be provided an interpretation as a thermodynamic relation on the 2-surface S_t, without explicitly invoking the specifics of the underlying gravitational dynamics.

The auxiliary null vector field k^i is parameterized as $k^i = -(dx^i/d\lambda_{(k)})$, where $\lambda_{(k)}$ is the parameter along the k^i field. Note that negative sign is chosen here in the parametrization. The reason is as follows. Usually the null vector l^a is chosen to be outgoing and so its 'radial' component increases along integral curves of $\vec{\boldsymbol{l}}$. Whereas the auxiliary vector k^a is regarded as ingoing one and hence the 'radial' component along integral curves of $\vec{\boldsymbol{k}}$ decreases along this field. Now here we are interested in the thermodynamic interpretation of (4.23) when provided with a virtual displacement along $\vec{\boldsymbol{k}}$. In this case, to identify the relevant thermodynamic entities like entropy, energy, *etc.* in its usual meaning, it is required to define change of x^a along k^a as positive one. Therefore we have the coordinate variation under the virtual displacement $\delta\lambda_{(k)}$ as $\delta x^a = -k^a\delta\lambda_{(k)}$. The physical interpretation of this displacement has been explained in [36] and we briefly mention it here. Let us consider two null surfaces under the null based foliation of the spacetime by the family $\mathcal{H}_u$. The null surfaces have to be the solutions to the specific theory of gravity that we are considering *implicitly*. Let us suppose that the null surfaces are stationed at $\lambda_{(k)} = 0$ and $\lambda_{(k)} = \delta\lambda_{(k)}$. The virtual displacement $\delta\lambda_{(k)}$ is essentially a process that lets us shift from one solution of the hypersurface to the other since k^a is an ingoing vector field. Then the expansion of the congruence of the auxiliary null vector field in terms of rate of area element of S_t is given as,

$$\begin{aligned}\theta_{\boldsymbol{k}} \;&=\; q^{ij}(\nabla_i k_j) = \frac{1}{2}q^{ij}\pounds_k q_{ij} = \frac{1}{\sqrt{q}}\pounds_k\sqrt{q}\\ &=\; -\frac{1}{\sqrt{q}}\frac{d}{d\lambda_{(k)}}\sqrt{q}\,.\end{aligned} \tag{4.24}$$

The details of this relation has been sketched in Sec. 2.5.1. Now we multiply both sides of (4.23) with $\delta\lambda_{(k)}$ and the elemental area $\sqrt{q}d^2x$ on the 2-surface S_t along with an overall factor of $\frac{1}{8\pi}$. Then the integration over the 2-surface yields,

$$-\delta\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\frac{\kappa}{2\pi}\frac{1}{4}\theta_{\boldsymbol{k}} = \delta\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi}\left[\frac{1}{2}{}^2R + l^i\nabla_i\theta_{\boldsymbol{k}} - \Omega_a\Omega^a - \mathcal{D}_A\Omega^A\right] - \delta\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi}\, G_{ij}l^ik^j\,. \tag{4.25}$$

The LHS of above equation (4.25) can be rewritten in the following form:

$$-\delta\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\frac{\kappa}{2\pi}\frac{1}{4}\theta_{\boldsymbol{k}} = \int_{S_t} d^2x\frac{\kappa}{2\pi}\delta\lambda_{(k)}\frac{d}{d\lambda_{(k)}}\left(\frac{1}{4}\sqrt{q}\right) = \int_{S_t} d^2x T\delta_{\lambda(k)}s\,, \tag{4.26}$$

where we associate the temperature T of the null surface S_t as being related to the non-affinity parameter via $T = (\kappa/2\pi)$. The entropy density s of the null surface is identified to be $s = (\sqrt{q}/4)$. We identify this entropy density defined on the null hypersurface to be the entanglement entropy density. We will have more to say on the nature of this entropy density shortly.

Now focusing on the first term on the RHS of (4.25), we identify it as the variation of the energy associated with the null surface S_t under the virtual displacement $\delta\lambda_{(k)}$, *i.e.*

$$\delta_{\lambda(k)}E \equiv \delta\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi}\left[\frac{1}{2}\,{}^2R + l^i\nabla_i\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} - \Omega_a\Omega^a - \mathcal{D}_A\Omega^A\right]. \tag{4.27}$$

Performing an indefinite integration over $\lambda_{(k)}$ allows us to have an expression for the energy associated with the 2-surface S_t,

$$E = \int d\lambda_{(k)}\int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi}\left[\frac{1}{2}\,{}^2R + l^i\nabla_i\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} - \Omega_a\Omega^a - \mathcal{D}_A\Omega^A\right]. \tag{4.28}$$

Before proceeding, we note that the expression of the energy as obtained in (4.28) is reminiscent of the Hawking-Hayward energy definition [157, 158]. Our aim here is to show that the analogous NRE for the ingoing expansion scalar $\theta_{\boldsymbol{k}}$ (4.23) has a thermodynamic interpretation under the process of the virtual displacement $\delta\lambda_{(k)}$. That is, we proceed towards interpreting (4.25) as a thermodynamic identity. To this end we have identified the LHS of (4.25) as $T\delta_{\lambda(k)}s$ integrated on the transverse spatial 2-surface S_t (of $\mathcal{H}$) and the first term on the RHS of (4.25) as the variation of the energy of the null surface under the virtual displacement. The thermodynamic identity would be complete if we are allowed to interpret the second term of of (4.25) as the virtual work done under the displacement of the null surface S_t by $\delta\lambda_{(k)}$. Allowing ourselves the liberty, we identify the "geometric work function" associated with the virtual displacement $\delta\lambda_{(k)}$ as $P \equiv -1/(8\pi)G_{ij}l^ik^j$.

Following this, we have,

$$-\delta\lambda_{(k)}\int_{S_t}d^2x\sqrt{q}\frac{1}{8\pi}\,G_{ij}l^ik^j=\delta\lambda_{(k)}\int_{S_t}d^2x\sqrt{q}P\equiv F\delta\lambda_{(k)}\,, \tag{4.29}$$

where F is the integral of the work function over the transverse space S_t of the null surface and $F\delta\lambda_{(k)}$ is to be interpreted as the virtual work done under $\delta\lambda_{(k)}$. Combining (4.26), (4.27) and (4.29), we see that (4.25) can be succinctly formulated as,

$$\int_{S_t}d^2xT\delta_{\lambda(k)}s=\delta_{\lambda(k)}E+F\delta\lambda_{(k)}\,. \tag{4.30}$$

We remind that this interpretation holds for all the variations that are consistent with the virtual displacement. That is to physically interpret this, let us say that our virtual displacement is a "physical" process that virtually shifts our null surface $\mathcal{H}$ from say $\lambda_{(k)}=0$ to $\lambda_{(k)}=\delta\lambda_{(k)}$. Under such a virtual variation process, energy flows through the null surface $\mathcal{H}$. The energy is given by $\delta_{\lambda(k)}E$. The energy then contributes to the heat energy $\int_{S_t}d^2xT\delta_{\lambda(k)}s$ and the virtual work done $F\delta\lambda_{(k)}$ under this virtual displacement process.

We further note that the expression of the energy (4.28) can be rewritten as,

$$E=\frac{1}{2}\int d\lambda_{(k)}\left(\frac{\chi}{2}\right)+\frac{1}{8\pi}\int d\lambda_{(k)}\int_{S_t}d^2x\sqrt{q}\left[l^i\nabla_i\theta_k+\theta_l\theta_k-\Omega_a\Omega^a-\mathcal{D}_A\Omega^A\right]\,, \tag{4.31}$$

having noted that χ represents the following integral over S_t (a 2 dimensional manifold) defined as,

$$\chi=\frac{1}{4\pi}\int_{S_t}d^2x\sqrt{q}\,{}^2R\,. \tag{4.32}$$

If the transverse space S_t of the null surface is compact then, χ is precisely equal to the Euler characteristic of the S_t; if not, then χ is defined via the integral (4.32). For example if the topology of $\mathcal{H}$ is $\mathbb{R}\times\mathbb{S}^2$, then S_t is the compact surface $\mathbb{S}^2$ and hence χ then represents the Euler characteristics of the sphere.

Now, we restrict to the special case where the non-affinity parameter κ and hence the temperature T associated with the null surface S_t is independent of the transverse coordinates of S_t (for example a stationary black hole system). In that sense T can be taken outside the integral, and then identifying the total change of the entropy S of the null surface under the virtual displacement as $\delta_{\lambda(k)}S=\int_{S_t}d^2x\delta_{\lambda(k)}s$, we have further simplification of (4.30),

$$T\delta_{\lambda(k)}S=\delta_{\lambda(k)}E+F\delta\lambda_{(k)}\,. \tag{4.33}$$

Before proceeding ahead towards showing the equivalence of our approach with previous results and interpretations let us mention the differences instead first. We iterate that our analysis is independent of any gravitational theory as *per se*. For example we have not invoked the Einstein's field equations or any other gravitational field equations for that matter in our interpretation of the NRE (for θ_k) as providing a thermodynamic identity. Whatever be the gravitational theory, all we require is that the corresponding solution space is $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ provided with the Levi-Civita connection. This is in contrast to previous results, which have been specifically formulated for explicit theories of gravity [36, 74, 79]. This is precisely the reason as to why we define a gravitational/geometric work function $P = -1/(8\pi)G_{ij}l^i k^j$ as opposed to $P = -T^{(m)}_{ab}l^a k^b$ as is done in previous works [36, 74, 79] which identify the work function or pressure entirely in terms of the matter energy momentum tensor. This also entails as to why irrespective of any given gravitational theory, we have identified the entropy density of the null hypersurface under the virtual displacement process to be the entanglement entropy density. The observer under such a virtual displacement process is the null observer moving along the integral curves of the null generators l^a. We assume that our generic null hypersurface $\mathcal{H}$ actually partitions the spacetime into timelike and spacelike regions. Then the quantum fields living in spatial slices on both these two sides can be entangled. The degrees of freedom (dof) of the quantum fields in the spacelike acausal region is not accessible to a timelike observer in the timelike causal region. The timelike observer then calculates the reduced density matrix by tracing out the dof of the quantum fields on the acausal side. The entanglement entropy is then defined as the Von Neumann entropy of this reduced density matrix. By introducing a momentum cut-off the entanglement entropy is shown to be proportional to the area of the null surface $\mathcal{H}$. Since the entropy density introduced in our case is proportional to $\sqrt{q}$, with an analogy to entanglement entropy, we propose that this as entanglement entropy density as measured by the null observer, moving along l^a. In this regard we mention that a similar concept has been taken by Jacobson [35, 62, 149] at the very beginning of his analysis in order to obtain the Einstein's equation by extremizing the entropy of the Rindler horizon as well as for a causal diamond. This also makes connections with the Einstein gravity which we will elaborate further in the next section. For the case of general relativity, under the process of virtual displacement along $\vec{k}$, the entropy density has been shown to be precisely the Bekenstein-Hawking entropy density $s = \sqrt{q}/4$. The pressure identified for this virtual displacement in GR is $P = -(1/8\pi)G_{ab}l^a l^b = -T^{(m)}_{ab}l^a l^b$. For static spherically symmetric spacetimes in general relativity $-T^{(m)}_{ab}l^a l^b = T^{(m)r}{}_r$ which has the interpretation of being the transverse radial pressure. We will have more to say on these connections to previous results in the next section.

Let us mention again, that in our analysis leading towards the thermodynamic interpretation (under the virtual displacement $\delta\lambda_{(k)}$), (4.30) is independent of any coordinate system as opposed to [36, 79], which produces the equivalent thermodynamic identity, but under the null adapted GNC coordinates. A specific requirement under the GNC system is the fact that there is only one null hypersurface stationed at the position $r = 0$.

This null hypersurface partitions the spacetime between timelike and spacelike regions. However in our case, we have foliated the spacetime in the neighborhood of $\mathcal{H}$ by a family of null hypersurfaces $\mathcal{H}_u$ and have focused on producing the the thermodynamic identity on any one of them, say $\mathcal{H}_{u=0} = \mathcal{H}$. A specific advantage of such a foliation is that all the relevant geometrical quantities that can be defined on the null surface (for example expansion scalar, second fundamental form etc) are independent of the foliation. Another requirement specific to the GNC analysis is that the auxiliary null vector field is affinely parametrized *i.e.* $k_a = -\partial_a u$ or in other words k^a is hypersurface-orthogonal to $u =$ constant surfaces. This, we believe is a certain restriction on the analysis. We can however do away with such a restriction. Under the system of the foliation of spacetime introduced in 2.3, we do not require k^a to be affinely parametrized and hypersurface-orthogonal. In fact, under this general structure, the hyperplane normal to k^a cannot be integrated into some integrable surface. As a result of such a null foliation of the spacetime in the vicinity of $\mathcal{H}$, our interpretation of the energy and the work function pertain entirely to geometric quantities defined in the spacetime manifold. The way we have made a distinction between the energy term and the work function is to identify that the energy expression (4.28) contains terms that are defined geometrically for $\mathcal{H}$ or for the transverse 2-surface S_t along with a term involving the directional derivative (along $\vec{l}$) of such a geometrical quantity defined on $\mathcal{H}$. However the work function (4.29) contains terms that are defined for the entire spacetime manifold. In doing so, we have obtained a covariant (but foliation based) expression of the energy of the null surface. Previous expositions [36, 79] into the energy term under the purview of providing a thermodynamic interpretation have however come under the context of an adapted coordinate system w.r.t a fiduciary null surface *i.e.* the GNC construction. As a result, previous such descriptions of the energy have been coordinate dependent.

As a mathematical curiosity, the above relation (4.23) can also be derived in an alternative method following [74]. In [74], the thermodynamic identity analogous to equation (4.33) was shown for static null horizons. We generalize the results to any arbitrary null hypersurface via the use of two following relations. The first one relates the Ricci tensor (${}^2R_{ab}$) of the 2-dimensional transverse Riemannian manifold (S_t, q) with the 4-dimensional Riemann curvature tensor of the spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, the second fundamental form Θ_{ab} of the null hypersurface $\mathcal{H}$ and the transversal deformation rate Ξ_{ab} (see [61]),

$$ {}^2R_{ab} = q_a^{\ i} q_b^{\ j} q_c^{\ k} R^c_{\ jki} - \Xi_{ab}\theta_{\boldsymbol{l}} - \Theta_{ab}\theta_{\boldsymbol{k}} + \Theta_a^{\ c}\Xi_{cb} + \Xi_a^{\ c}\Theta_{cb} \,. \tag{4.34} $$

Taking the trace of the above equation, we obtain,

$$ {}^2R = q^{ij} q^{ck} R_{cjki} - 2\theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} + 2\Theta^{ab}\Xi_{ab} \,. \tag{4.35} $$

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Using the definition of the orthogonal projection tensor onto the 2-surface S_t as $q^{ab} = g^{ab} + l^a k^b + k^a l^b$, we have,

$$q^{ij} q^{ck} R_{cjki} = R + 4R_{ij} l^i k^j + 2R_{abcd} l^a k^b k^c l^d \,. \tag{4.36}$$

Similarly using the irreducible decomposition of both the second fundamental form of $\mathcal{H}$ and the traversal deformation tensor of the 2-surface S_t *i.e.* $\Theta_{ab} = \frac{1}{2}\theta_{\boldsymbol{l}}\, q_{ab} + {}^{(\boldsymbol{l})}\sigma_{ab}$ and $\Xi_{ab} = \frac{1}{2}\theta_{\boldsymbol{k}}\, q_{ab} + {}^{(\boldsymbol{k})}\sigma_{ab}$, we get,

$$2\Theta_{ab}\Xi^{ab} = \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} + 2{}^{(\boldsymbol{l})}\sigma_{ab}{}^{(\boldsymbol{k})}\sigma^{ab} \,. \tag{4.37}$$

Upon using the equations (4.36) and (4.37) in equation (4.35), we obtain as a result,

$$R = {}^2R - 4R_{ij} l^i k^j - 2R_{abcd} l^a k^b k^c l^d + \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} - 2{}^{(\boldsymbol{l})}\sigma_{ab}{}^{(\boldsymbol{k})}\sigma^{ab} \,. \tag{4.38}$$

The second relation that will be put to use is,

$$R_{abcd} l^a k^b k^c l^d = \mathcal{D}_a \Omega^a + \Omega_a \Omega^a - \kappa\theta_{\boldsymbol{k}} - l^i \nabla_i \theta_{\boldsymbol{k}} - \frac{1}{2}\theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} - {}^{(\boldsymbol{l})}\sigma_{ab}{}^{(\boldsymbol{k})}\sigma^{ab} - R_{ab} l^a k^b \,. \tag{4.39}$$

A detailed derivation of the above result is provided in appendix 4.7. In fact the relation (4.39) can be regarded as a generalization to Eq. (5) of [74] (which is valid only for static null horizons) to any arbitrary null surface. Now simply the use of equation (4.39) in equation (4.38) leads us to equation (4.23).

In the next section, we show the equivalence of the energy term (4.28) and the gravitational/geometric work function term with those obtained in [36] via the GNC construction under the purview of Einstein gravity.

4.5 Connection with existing results

In the previous section 4.4 we landed ourselves with a covariant expression of the energy (4.28) associated with the null surface $\mathcal{H}$ resulting from a virtual displacement $\delta\lambda_{(k)}$ along the ingoing auxiliary null field $\vec{k}$. We now aim to compute this expression of the energy in the GNC system. To this end, we mention that the metric expressed in the GNC (u, r, x^A) reads,

$$ds^2 = -2r\alpha du^2 + 2dudr - 2r\beta_A dudx^A + q_{AB} dx^A dx^B \,, \tag{4.40}$$

where the six independent parameters $(\alpha, \beta_A, q_{AB})$ are dependent on the coordinates (u, r, x^A). The null hypersurface in this system is stationed at $r = 0$. The relevant inverse metric as well as its Christoffel connection coefficients have been calculated in [57]. The components of the null normal and the auxiliary null normal in this coordinate system are,

$$\begin{aligned} l_a &= (0,1,0,0) \qquad k_a = (-1,0,0,0) \\ l^a &= (1, 2r\alpha + r^2\beta^2, r\beta^A) \quad k^a = (0,-1,0,0)\,. \end{aligned} \tag{4.41}$$

Before proceeding, we now invoke the Einstein's field equations and note that the work function previously defined as $P = -1/(8\pi)G_{ij}l^i k^j$ when evaluated on the null hypersurface $r = 0$, yields $P = -1/(8\pi)G_{ij}l^i k^j = (-T^{(m)}_{ij}l^i k^j) = (-T^{(m)ij}l_i k_j) = -T^{(m)a}{}_b l_a k^b = T^{(m)ur} = T^{(m)r}{}_r = T^{(m)}_{ur} = T^{(m)u}{}_u$. In static spherically symmetric spacetimes $T^{(m)r}{}_r$ has the interpretation of being the radial or the normal pressure [74, 75]. Hence the integral of the work function $F = \int_{S_t} d^2x\sqrt{q}P$ in the static spherically symmetric case is $F = \int_{S_t} d^2x\sqrt{q}P = \int_{S_t} d^2x\sqrt{q}T^{(m)r}{}_r$, which is to be interpreted as the average normal force on the spatial two-dimensional cross-section S_t.

We note that all the quantities in the integrand of the expression of energy (4.28) are to be evaluated on the null hypersurface *i.e.* at $r = 0$. Looking at the term $\theta_{\boldsymbol{l}}$, we obtain,

$$\theta_{\boldsymbol{l}} = q^{ab}\nabla_a l_b = -q^{AB}\Gamma^r_{AB}\,. \tag{4.42}$$

The value of Γ^r_{AB} ,

$$\begin{aligned} \Gamma^r_{AB} = &-\frac{1}{2}\partial_u q_{AB} - \frac{1}{2}(r^2\beta^2 + 2r\alpha)\partial_r q_{AB} \\ &+ \frac{1}{4}r\Big(\mathcal{D}_A\beta_B + \mathcal{D}_B\beta_A\Big)\,. \end{aligned} \tag{4.43}$$

Evaluating $\theta_{\boldsymbol{l}}$ on the null hypersurface,

$$\theta_{\boldsymbol{l}|r=0} = -q^{AB}\Gamma^r_{AB|_{r=0}} = \frac{1}{2}q^{AB}\partial_u q_{AB} = \frac{1}{\sqrt{q}}\Big(\partial_u\sqrt{q}\Big) = \partial_u\Big(\ln\sqrt{q}\Big)\,. \tag{4.44}$$

Looking at the computation of $\theta_{\boldsymbol{k}}$, we have,

$$\theta_{\boldsymbol{k}} = q^{ab}\nabla_a k_b = q^{AB}\Gamma^u_{AB}\,. \tag{4.45}$$

The value of Γ^u_{AB} is $-\frac{1}{2}\partial_r q_{AB}$. Evaluating $\theta_{\boldsymbol{k}}$ on the null hypersurface,

$$\theta_{\boldsymbol{k}|_{r=0}} = q^{AB}\Gamma^u_{AB|_{r=0}} = -\frac{1}{\sqrt{q}}\Big(\partial_r\sqrt{q}\Big)\,. \tag{4.46}$$

Computation of $l^i\nabla_i\theta_{\boldsymbol{k}}$ on the null hypersurface, with the components of l^i given in (4.41), yields,

$$l^i\nabla_i\theta_{\boldsymbol{k}|_{r=0}} = -\partial_u\Big(\frac{1}{\sqrt{q}}\partial_r\sqrt{q}\Big)\,. \tag{4.47}$$

This allows us to have,

$$\theta_l \theta_{\mathbf{k}|_{r=0}} + l^i \nabla_i \theta_{\mathbf{k}|_{r=0}} = -\frac{1}{\sqrt{q}} \partial_r \partial_u \sqrt{q}\,. \tag{4.48}$$

Next, noting that $\Omega_a = \omega_a + \kappa k_a$, we have $\Omega_a \Omega^a = \omega_a \omega^a$, where ω_a refers to the rotation one form defined in the manifold. For the evaluation of $\omega_a = l^i \nabla_i k_a$, with l^i and k_a provided from (4.41), we have,

$$\omega^u = 0\,, \quad \omega_r = 0\,. \tag{4.49}$$

As a consequence of this, we have $\omega_a \omega^a = \omega_A \omega^A$. The relevant quantities are,

$$\begin{aligned} \omega_A &= \frac{1}{2}\beta_A + \frac{1}{2} r\Big(\partial_r \beta_A\Big) - \frac{1}{2} r \beta^B \Big(\partial_r q_{AB}\Big) \\ \omega^A &= q^{AC}\Big[\frac{1}{2}\beta_C + \frac{1}{2} r\Big(\partial_r \beta_C\Big) - \frac{1}{2} r \beta^B \Big(\partial_r q_{BC}\Big)\Big] \end{aligned} \tag{4.50}$$

Evaluation of $\Omega_a \Omega^a$ on the null hypersurface at $r = 0$ yields,

$$\Omega_a \Omega^a_{|r=0} = \omega_A \omega^A_{|r=0} = \frac{1}{4}\beta_A \beta^A\,. \tag{4.51}$$

Finally we are left with the evaluation of $\mathcal{D}_A \Omega^A$ on the null hypersurface. The calculations follow as,

$$\Omega^A = q^{AB}\omega_B = q^{AB}\Big(\frac{1}{2}\beta_B + \frac{1}{2} r\Big(\partial_r \beta_B\Big) - \frac{1}{2} r \beta^D \Big(\partial_r q_{BD}\Big)\Big) \tag{4.52}$$

$$\mathcal{D}_A \Omega^A = \frac{1}{\sqrt{q}}\partial_A\Big[\sqrt{q}\Big(\frac{1}{2}\beta^A + \frac{1}{2} r q^{AB}\Big(\partial_r \beta_B\Big) - \frac{1}{2} r q^{AB}\beta^D\Big(\partial_r q_{BD}\Big)\Big)\Big] \tag{4.53}$$

$$\mathcal{D}_A \Omega^A_{|r=0} = \frac{1}{2}\frac{1}{\sqrt{q}}\partial_A(\sqrt{q}\beta^A)\,. \tag{4.54}$$

In the GNC coordinates, the virtual displacement $\delta x^a = \frac{dx^a}{d\lambda_{(k)}}\delta\lambda = -k^a \delta\lambda = (0, \delta\lambda = \delta r, 0, 0)$. Finally putting the values of the relevant quantities obtained in (4.48), (4.51) and (4.54) into the expression of the energy in (4.28), we obtain,

$$\begin{aligned} E =& \frac{1}{2}\int dr\Big(\frac{\chi}{2}\Big) \\ &- \frac{1}{8\pi}\int dr\Big[\int_{S_t} d^2x\sqrt{q}\Big(\frac{1}{\sqrt{q}}\partial_r\partial_u\sqrt{q} + \frac{1}{4}\beta_A\beta^A\Big) + \frac{1}{2}\frac{1}{\sqrt{q}}\partial_A(\sqrt{q}\beta^A)\Big]\,. \end{aligned} \tag{4.55}$$

We note that,

$$\int_{r=1}^{r=2} dr \int_{S_t} d^2x\sqrt{q}\Big(\frac{1}{\sqrt{q}}\partial_r\partial_u\sqrt{q}\Big) = \int_{S_t} d^2x\partial_u\sqrt{q}|_{r=1}^{r=2}\,. \tag{4.56}$$

With this, the energy of the null hypersurface manifests as,

$$\begin{aligned} E =&\frac{1}{2}\int dr\Big(\frac{\chi}{2}\Big) \\ &-\frac{1}{8\pi}\int_{S_t} d^2x\partial_u\sqrt{q} - \frac{1}{16\pi}\int dr\int_{S_t} d^2x\sqrt{q}\Big[\frac{1}{2}\beta_A\beta^A + \frac{1}{\sqrt{q}}\partial_A(\sqrt{q}\beta^A)\Big] \end{aligned} \tag{4.57}$$

We find that the expression of the energy obtained in the GNC (4.57) via the covariant form of the expression of the energy (4.28) matches with Eq. (53) in [36]. Provided the 2-dimension surface S_t is compact, the above expression can be simplified to,

$$E = \frac{1}{2}\int dr\Big(\frac{\chi}{2}\Big) - \frac{1}{8\pi}\int_{S_t} d^2x\partial_u\sqrt{q} - \frac{1}{16\pi}\int dr\int_{S_t} d^2x\sqrt{q}\Big[\frac{1}{2}\beta_A\beta^A\Big]\,. \tag{4.58}$$

The reason as to why (4.58) is called the energy term is because it provides the expression of the energy in quite a few well known cases. For a review of these specific cases, the reader may wish to see [36]. As an example, for the Schwarzschild metric, the energy term (4.58) reduces to the mass.

4.6 Summary and discussions

Our first ground of application was to study the null surface in the Riemannian spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ provided with the Levi-Civita connection. Whatever be the gravitational theory whose solution space is given by the spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$, we tried interpreting whether a thermodynamic identity can be provided covariantly to the dynamics corresponding to $G_{ab}l^a l^b$ and $G_{ab}l^a k^b$ on the integrable null surface $\mathcal{H}$.

For the dynamics pertaining to $G_{ab}l^a l^b$, we started with the NRE for the outgoing expansion scalar θ_l (4.3), and then provided a virtual displacement $\delta\lambda_{(l)}$. We then integrated the resulting equation onto the transverse spacelike 2-surface S_t and obtained relevant thermodynamical structures (4.7) and (4.15). We have provided two alternative interpretations of the resulting thermodynamic identity. In the first interpretation (4.7), we identified that under the virtual displacement process $\delta\lambda_{(l)}$, an amount of energy $\delta_{\lambda(l)}E$ sweeps across through the null surface $\mathcal{H}$. The expression of the energy is provided in (4.8). This energy flow results in the heat exchanged as a result of the entropy variation of the null surface. The temperature of $\mathcal{H}$ is associated with the non-affinity parameter κ of the null generators and the entropy density is proportional to $\sqrt{q}$ of the area element of S_t. In our second interpretation (4.15), we have identified the energy variation $\delta_{\lambda(l)}E$

and the irreversible heat $\delta_{\lambda(l)} Q_{\text{dis}}$ that flows (as a result of the virtual displacement process) across through the null surface $\mathcal{H}$. These quantities are given by (4.12) and (4.14) respectively. This results in the heat energy generation (4.10) due to the variation of the entropy density of the null surface identified as $\delta_{\lambda(l)} s$, where s is again postulated to be proportional to $\sqrt{q}$ of the area element of S_t. The irreversible heat generation (4.14) is due to the viscous dissipative effects present in the null surface. We also identified that this dissipative heat generation must be entirely due to geometric fluxes since the dissipation term contains only geometrical quantities established on $\mathcal{H}$. Finally, we showed for the explicit case of a stationary black hole system, that the integrated form of the NRE (for $\theta_{\boldsymbol{l}}$) over the virtual displacement produces the generalized Smarr formula (4.21). We also showed that the energy term (in both our interpretations) is proportional to the Komar energy term (4.18) for this special case.

Next we focused on the more relevant projection component $G_{ab} l^a k^b$ and the dynamical evolution (for $\theta_{\boldsymbol{k}}$ along $\vec{\boldsymbol{l}}$) related to it for providing the thermodynamic interpretation in a covariant fashion. In literature, previous works [36, 70, 73] had solidified the fact that the gravitational field equations via $G_{ab} l^a k^b$ can be provided a thermodynamical relationship which is structurally quite similar to the first law of thermodynamics. However they had been proposed for event horizons in spacetimes with a high degree of symmetry or for a general null surface, but in an adapted coordinate system called the GNC system. This results in the expression of the energy being either very simplified or dependent on the GNC coordinates. Here, we have tried to show, in quite a very general setting whether a similar interpretation can be provided without the need of adapting any coordinate system w.r.t the null surface or imposing any symmetries in the spacetime. In our approach, we started out with the NRE (for $\theta_{\boldsymbol{k}}$) (4.23) and provided a virtual displacement $\delta\lambda_{(k)}$. We then integrated the resulting equation over the 2-surface S_t. This procedure allowed us to obtain our required thermodynamic interpretation in a covariant fashion. However our proposed interpretation does have major differences with the previous approaches. We have not invoked in our analysis any specific gravitational field equations and hence proposed that our interpretation is not specific to any particular theory of gravity. This required us to propose that the entropy density associated with the null surface under the virtual displacement process is actually the entanglement entropy density assigned to $\mathcal{H}$ by a null observer moving along the integral curves of the null generators l^a. This is because in our case the entropy density is actually proportional to $\sqrt{q}$ of our area element on the 2-surface S_t. The temperature is again found to be proportional to the non-affinity parameter κ associated with the null generators l^a. To have a consistent thermodynamic interpretation irrespective of any particular theory of gravity we have defined a so called geometric/gravitational pressure $P = -1/(8\pi) G_{ij} l^i k^j$. This is in contrast with the earlier methods defining the pressure entirely through the matter energy tensor. Moreover, the identified energy here is in covariant form and so can be applied to any metric adapted to the null surface. This added advantage of our formalism can be very useful for further progress of this field.

We have indeed claimed to use the notion of entanglement entropy associated with the generic surface in question while studying the thermodynamic interpretation of the relevant gravitational field equations under the approach of Padmanabhan. This was done precisely, so that we could attest a coordinate-independent notion of thermodynamic interpretation of the gravitational dynamics irrespective of the gravity theory that is being considered. This interpretation is surely different from Padmanabhan's viewpoint where exclusively for Einstein gravity, the entropy is the Bekenstein-Hawking entropy and for Lanczos-Lovelock theory, the entropy is the Wald entropy. The pressure term following Padmanabhan's works have been precisely due to the stress-energy tensor. We have have however identified a geometric pressure to attest to the thermodynamics of the gravitational field equations irrespective of the theory of gravity. Hence for theories other than Einstein gravity, the notions of pressure between Padmanabhan's and our work differ. Our notion of using the entanglement entropy for the null surface coincides with the work of [64]. However, we have no rigorous justification of its use as of now. The use of the entanglement entropy allows the thermodynamic analogy of the gravitational field equations to be set in irrespective of the gravity theory.

It would certainly be worthwhile to use the notion of entanglement entropy as regards to a causal diamond instead of a single null surface. In fact, the derivations of the field equations for Einstein gravity and for generic diffeomorphism invariant theories of gravity following the approach of Jacobson has already been done in [69]. On the other hand, to our knowledge, Padmanabhan's analysis using causal diamonds have not been done previously. The notion of entanglement entropy can then be naturally used for causal diamonds based on spacelike cross-sections. We look forward to examine this issue in the near future.

Appendices

4.7 Derivation of Eq. (4.39)

Before delving into the derivation, we note two relations involving the covariant derivatives of the null normals l^a and k^a, which we had derived earlier in Eq. (2.91) and Eq. (2.135) and which we are going to put to heavy usage. We list them here again for better reference of the reader.

$$\nabla_a l_b = \Theta_{ab} + \omega_a l_b - l_a(k^i \nabla_i l_b) , \tag{4.59}$$

and

$$\nabla_a k_b = \Xi_{ab} - \Omega_a k_b - k_a \omega_b - l_a(k^i \nabla_i k_b) . \tag{4.60}$$

We start with the Ricci identity for the null normals l^a and k^a,

$$l^a(\nabla_a \nabla_b k_c) = l^a(\nabla_b \nabla_a k_c) - R_{abfc} l^a k^f . \tag{4.61}$$

We focus on the LHS of Eq. (4.61). Upon using Eq. (4.60), we obtain,

$$\begin{aligned} l^a(\nabla_a \nabla_b k_c) = l^a(\nabla_a \Xi_{bc}) - (l^a \nabla_a \Omega_b) k_c - \Omega_b(l^a \nabla_a k_c) - (l^a \nabla_a k_b)\omega_c \\ - k_b(l^a \nabla_a \omega_c) - (l^a \nabla_a l_b)(k^i \nabla_i k_c) - l_b l^a \nabla_a(k^i \nabla_i k_c). \end{aligned} \tag{4.62}$$

Upon using the relations $l^a \nabla_a l_b = \kappa l_b$, $l^a \nabla_a k_b = \omega_b$ and $\omega_a = \Omega_a - \kappa k_a$, we contract the above Eq. (4.62) with $k^b l^c$ to have,

$$l^a(\nabla_a \nabla_b k_c) k^b l^c = -\Omega_b \Omega^b + \kappa l^c k^i(\nabla_i k_c) + l^c l^a \nabla_a(k^i \nabla_i k_c) . \tag{4.63}$$

We now focus on the first term of the RHS of Eq. (4.61) *i.e.* $l^a \nabla_b(\nabla_a k_c)$. Again upon using Eq. (4.60), we have,

$$l^a \nabla_b(\nabla_a k_c) = \; -\Xi_{ac}(\nabla_b l^a) + \Omega_a(\nabla_b l^a) k_c - (l^a \nabla_b k_a)\omega_c + \nabla_b \omega_c . \tag{4.64}$$

Contracting the above Eq. (4.64) with $k^b l^c$ and using the fact that $\Omega_a l^a = 0$, $\Omega_a k^a = 0$ and $\omega_a l^a = \kappa$ we obtain,

$$\begin{aligned} l^a(\nabla_b \nabla_a k_c) k^b l^c &= -\Omega_a(\nabla_b l^a) k^b - \kappa k^b l^a(\nabla_b k_a) + k^b l^c(\nabla_b \omega_c) \\ &= l^a k^b(\nabla_b \Omega_a) - \kappa l^c k^i(\nabla_i k_c) + k^b l^c(\nabla_b \omega_c) . \end{aligned} \tag{4.65}$$

We focus on the first term of the RHS of Eq. (4.65) *i.e.* $l^a k^b(\nabla_b\Omega_a)$,

$$l^a k^b(\nabla_b\Omega_a) = (q^{ab} - g^{ab} - k^a l^b)(\nabla_b\Omega_a)$$
$$= q^{ab}\left(\delta^i_b\delta^k_a(\nabla_i\Omega_k)\right) - (\nabla_b\Omega^b) + l^b\Omega^a(\nabla_b k_a)\,. \tag{4.66}$$

Upon using the completeness relation $\delta^a_b = q^a_b - l_b k^a - l^a k_b$, we have after some simple algebra,

$$l^a k^b(\nabla_b\Omega_a) = q^{ab}\left(\mathcal{D}_b\Omega_a\right) - (\nabla_a\Omega^a) + \Omega_a\Omega^a\,. \tag{4.67}$$

Putting Eq. (4.67) into Eq. (4.65), we obtain,

$$l^a(\nabla_b\nabla_a k_c)k^b l^c = q^{ab}\left(\mathcal{D}_b\Omega_a\right) - (\nabla_a\Omega^a) + \Omega_a\Omega^a - \kappa l^c k^i(\nabla_i k_c)$$
$$+k^b l^c(\nabla_b\omega_c)\,. \tag{4.68}$$

We now contract the Ricci identity *i.e.* Eq. (4.61) on both sides with $k^b l^c$. Following this we use the relations (4.63) and (4.68) onto the contracted Ricci identity to obtain,

$$R_{abcd}l^a k^b k^c l^d = q^{ab}\left(\mathcal{D}_b\Omega_a\right) - (\nabla_a\Omega^a) + 2\Omega_a\Omega^a - 2\kappa l^c k^i(\nabla_i k_c)$$
$$+ k^b l^c(\nabla_b\omega_c) - l^c l^a\nabla_a(k^i\nabla_i k_c)\,. \tag{4.69}$$

The term $-2\kappa l^c k^i(\nabla_i k_c)$ can further be manipulated as,

$$-2\kappa l^c k^i(\nabla_i k_c) = -2\kappa(q^{ci} - g^{ci} - k^c l^i)(\nabla_i k_c) = -2\kappa\theta_{\mathbf{k}} + 2\kappa(\nabla_a k^a)\,. \tag{4.70}$$

Putting Eq. (4.70) in Eq. (4.69) we obtain as a result,

$$R_{abcd}l^a k^b k^c l^d = \mathcal{D}_b\Omega^b - (\nabla_a\Omega^a) + 2\Omega_a\Omega^a - 2\kappa\theta_{\mathbf{k}} + 2\kappa(\nabla_a k^a)$$
$$+ k^b l^c(\nabla_b\omega_c) - l^c l^a\nabla_a(k^i\nabla_i k_c)\,. \tag{4.71}$$

Following this result, we focus on the last term on the RHS of Eq. (4.71) *i.e.* $l^c l^a\nabla_a(k^i\nabla_i k_c)$ and manipulate it in the following sense,

$$l^c l^a\nabla_a(k^i\nabla_i k_c) = l^c\omega^i(\nabla_i k_c) + l^c l^a k^i\nabla_a(\nabla_i k_c)\,.$$

Upon using Eq. (4.60), we have,

$$l^c l^a\nabla_a(k^i\nabla_i k_c) = \Omega_i\Omega^i - \kappa l^c(k^i\nabla_i k_c) + l^c l^a k^i\nabla_a(\nabla_i k_c)\,. \tag{4.72}$$

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We proceed to manipulate the last term on the RHS of Eq. (4.72) with the help of Eq. (4.60),

$$\begin{aligned} l^c l^a k^i \nabla_a(\nabla_i k_c) &= l^a(q^{ic} - g^{ic} - l^i k^c)\nabla_a(\nabla_i k_c) \\ &= q^{ic} l^a \nabla_a \Xi_{ic} - l^a q^{ic}\Omega_i(\nabla_a k_c) - l^a q^{ic}(\nabla_a k_i)\omega_c \\ &- l^a \nabla_a(\nabla_i k^i) - l^c k^a(\nabla_c \omega_a)\,. \end{aligned} \tag{4.73}$$

Putting Eq. (4.73) in Eq. (4.72) we obtain along with the use of Eq. (4.60),

$$\begin{aligned} l^c l^a \nabla_a(k^i \nabla_i k_c) &= \Omega_i \Omega^i - \kappa l^c(k^f \nabla_f k_c) + q^{ic}(l^a \nabla_a \Xi_{ic}) \\ &- 2 l^a \Omega^c(\nabla_a k_c) - l^a \nabla_a(\nabla_i k^i) - l^c k^a(\nabla_c \omega_a) \\ &= -\Omega_i \Omega^i - \kappa l^c(k^f \nabla_f k_c) + q^{ic}(l^a \nabla_a \Xi_{ic}) \\ &- l^a \nabla_a(\nabla_i k^i) - l^c k^a(\nabla_c \omega_a)\,. \end{aligned} \tag{4.74}$$

Putting the value of $l^c l^a \nabla_a(k^i \nabla_i k_c)$ from Eq. (4.74) in Eq. (4.71), we obtain,

$$\begin{aligned} R_{abcd} l^a k^b k^c l^d &= \mathcal{D}_b \Omega^b - (\nabla_a \Omega^a) + 3\Omega_a \Omega^a - \kappa l^c k^i(\nabla_i k_c) - q^{ab}(l^i \nabla_i \Xi_{ab}) \\ &+ l^a \nabla_a(\nabla_i k^i) + (l^c k^b + k^c l^b)(\nabla_c \omega_b) \\ &= \mathcal{D}_a \Omega^a - (\nabla_a \Omega^a) + 3\Omega_a \Omega^a - \kappa l^c k^i(\nabla_i k_c) - q^{ab}(l^i \nabla_i \Xi_{ab}) \\ &+ l^a \nabla_a(\nabla_i k^i) + (q^{cb} - g^{cb})(\nabla_c \omega_b)\,. \end{aligned} \tag{4.75}$$

Expanding the above result and after a few lines of simple manipulations we obtain,

$$\begin{aligned} R_{abcd} l^a k^b k^c l^d &= \mathcal{D}_a \Omega^a - (\nabla_a \Omega^a) + 3\Omega_a \Omega^a - \kappa l^c k^i(\nabla_i k_c) - q^{ab}(l^i \nabla_i \Xi_{ab}) \\ &+ q^{ab}(\nabla_a \omega_b) - (\nabla_a l^i)(\nabla_i k^a) - R_{ab} l^a k^b\,. \end{aligned} \tag{4.76}$$

To this end, we focus at the $-q^{ab}(l^i \nabla_i \Xi_{ab})$ term and using the fact,

$$\pounds_{\boldsymbol{l}} \Xi_{ab} = l^i \nabla_i \Xi_{ab} + \Xi_{ai}(\nabla_b l^i) + \Xi_{ib}(\nabla_a l^i)$$

along with Eq. (4.59), we have,

$$-q^{ab}(l^i \nabla_i \Xi_{ab}) = -q^{ab} \pounds_{\boldsymbol{l}} \Xi_{ab} + 2\Xi_{ab}\Theta^{ab} = -q^{ab}\pounds_{\boldsymbol{l}} \Xi_{ab} + \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} + 2\,{}^{(\boldsymbol{l})}\sigma_{ab}\,{}^{(\boldsymbol{k})}\sigma^{ab}\,. \tag{4.77}$$

Upon using the irreducible decomposition of the transversal deformation rate tensor Ξ_{ab}, it is fairly straightforward to show that,

$$q^{ab}\pounds_{\boldsymbol{l}} \Xi_{ab} = \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} + 2\,{}^{(\boldsymbol{l})}\sigma_{ab}\,{}^{(\boldsymbol{k})}\sigma^{ab} + l^i \nabla_i \theta_{\boldsymbol{k}}\,. \tag{4.78}$$

Using Eq. (4.78) in Eq. (4.77), we obtain,

$$-q^{ab}(l^i\nabla_i\Xi_{ab}) = -l^i\nabla_i\theta_{\boldsymbol{k}} \,. \tag{4.79}$$

Upon using Eq. (4.79) and the relation $-\kappa l^c k^i(\nabla_i k_c) = -\kappa\theta_{\boldsymbol{k}} + \kappa(\nabla_a k^a)$ in Eq. (4.76), we have as a result,

$$\begin{aligned} R_{abcd}l^a k^b k^c l^d = {} & \mathcal{D}_a\Omega^a - (\nabla_a\Omega^a) + 3\Omega_a\Omega^a - \kappa\theta_{\boldsymbol{k}} + \kappa(\nabla_a k^a) - l^i\nabla_i\theta_{\boldsymbol{k}} \\ & + q^{ab}(\nabla_a\omega_b) - (\nabla_a l^i)(\nabla_i k^a) - R_{ab}l^a k^b \,. \end{aligned} \tag{4.80}$$

Let us now take a look at the term $(\nabla_a l^i)(\nabla_i k^a)$. Using the relations (4.59) and (4.60), it can manipulated quite simply to be,

$$(\nabla_a l^i)(\nabla_i k^a) = \Theta_{ab}\Xi^{ba} + \Omega_a\Omega^a - k^a\nabla_a\kappa + l^a k^b(\nabla_b\omega_a) \,. \tag{4.81}$$

Looking at the last term on the RHS of Eq. (4.81), *i.e.* $l^a k^b(\nabla_b\omega_a)$, we obtain in the process of manipulation as

$$l^a k^b(\nabla_b\omega_a) = \left(q^{ab} - g^{ab} - l^b k^a\right)(\nabla_b\omega_a) = q^{ab}(\nabla_b\omega_a) - (\nabla_b\omega^b) + \Omega_a\Omega^a \,. \tag{4.82}$$

Equating Eq. (4.82) in Eq. (4.81), we have,

$$(\nabla_a l^i)(\nabla_i k^a) - q^{ab}(\nabla_b\omega_a) = \Theta_{ab}\Xi^{ba} + 2\Omega_a\Omega^a - k^a\nabla_a\kappa - (\nabla_b\omega^b) \,. \tag{4.83}$$

Looking at Eq. (4.80) we manipulate the terms $(\nabla_a\Omega^a) - \kappa(\nabla_a k^a)$ using the relation $\omega_a = \Omega_a - \kappa k_a$,

$$(\nabla_a\Omega^a) - \kappa(\nabla_a k^a) = (\nabla_a\omega^a) + (k^a\nabla_a\kappa) \,. \tag{4.84}$$

To this end, we obtain, using Eq. (4.84) and Eq. (4.83),

$$\begin{aligned} & -(\nabla_a\Omega^a) + \kappa(\nabla_a k^a) + q^{ab}(\nabla_a\omega_b) - (\nabla_a l^i)(\nabla_i k^a) = -\Theta_{ab}\Xi^{ab} - 2\Omega_a\Omega^a \\ & = -\frac{1}{2}\theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} - {}^{(\boldsymbol{l})}\sigma_{ab}{}^{(\boldsymbol{k})}\sigma^{ab} - 2\Omega_a\Omega^a \,. \end{aligned} \tag{4.85}$$

Finally, putting Eq. (4.85) in Eq. (4.80) we obtain our desired result,

$$R_{abcd}l^a k^b k^c l^d = \mathcal{D}_a\Omega^a + \Omega_a\Omega^a - \kappa\theta_{\boldsymbol{k}} - l^i\nabla_i\theta_{\boldsymbol{k}} - \frac{1}{2}\theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}} - {}^{(\boldsymbol{l})}\sigma_{ab}{}^{(\boldsymbol{k})}\sigma^{ab} - R_{ab}l^a k^b \,. \tag{4.86}$$

Chapter 5

Thermodynamic structure of a generic null surface and the zeroth law in scalar-tensor theory

5.1 Introduction and Motivation

Now, that we advertised in the previous chapter, we can attribute a given thermodynamic interpretation to a particular dynamics of an integrable null hypersurface $\mathcal{H}$ in a background Riemannian spacetime equipped with a Levi-Civita connection, whatever the gravitational theory be. Our thermodynamic interpretation was brought about without any particular reference to a given coordinate system and we systematically obtained covariant expressions of relevant thermodynamic quantities. We checked that for the case of Einstein gravity, our results match exactly with well known previous results that were done either in spacetimes with high degree of symmetry or had used an explicit coordinate chart (GNC). In this chapter, we will still remain within the realm of the ambient spacetime provided with the Levi-Civita connection ($\boldsymbol{\nabla}$), however the gravitational dynamics (and its subsequent connection to thermodynamics) to be studied here is that of scalar-tensor theory.

In spite of the enormous success of GR, there are several reasons which imply that the actual behavior of gravity might deviate significantly from Einstein GR in the strong gravity region, where GR is not experimentally well-tested. However, in pursuit of developing a more accurate theory of gravity, one has to remember that Einstein's theory of gravity cannot be ruled out completely because of its sheer success against the observational tests and also due to its infallible predictions: such as the presence of black holes, gravitational waves *etc.*, existence of which were proved later. Therefore, a more accurate version of the theory of gravitation is more likely to be a modified version of Einstein GR instead of being a radically new theory [159, 160]. The ST theory is one of the most popular among the modified theories of gravity for various reasons [161–166]. Unlike Einstein gravity, the dynamical variable in this theory is not only the metric tensor, but also the

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scalar degrees of freedom are accounted for this theory in the form of "non-minimal coupling" between the scalar field and the curvature. This theory is described in two different frames which raises several issues in the literature, particularly on the equivalence of the physical results described in these two frames [167–191]. The original frame, where the non-minimal coupling is present, is known as the Jordan frame. With the help of a conformal transformation, the non-minimal coupling can be removed and the theory can be expressed equivalently in the Einstein frame. In that case, the curvature and scalar field are separated out and the scalar field behaves like an external source. Now the issues of the two frames are the following: the apparent mathematical equivalence between the two frames via the conformal transformation raises the question on whether the two frames are physically equivalent [168–172, 174, 181–184, 188–191] or one of the two frames is more physical than the other one [167]. The behavior of energy and other conserved charges under conformal transformations for ST as well as higher curvature theories of gravity have been studied in [177]. In [177], the authors show that such charges are invariant under conformal transformations provided the conformal factor goes over to unity at infinity. However, a proper thermodynamic description was detailed out in [188–191]. In these works [188–191], the authors have shown that the thermodynamic first law can be obtained in the two different frames using the existing well-defined formalisms of general relativity. In addition, earlier works show that the thermodynamic parameters are exactly equivalent in two different frames. This provides considerable improvement of the previous work [174], in which the equivalence of the thermodynamic parameters are subject to a few assumptions such as the asymptotic flatness of the spacetime.

The works stated above [188–191] (describing thermodynamics laws in the two frames of the scalar-tensor theory) are done in the context of the black hole horizon. As mentioned in the previous chapter, in Einstein GR, it is known for a long time that the thermodynamic structure of general relativity is present in any arbitrary null surface [36, 45, 59, 72, 92, 146, 147] and is not restricted to the black hole horizon. In fact, the thermodynamics of a null surface is very significant in the context of "emergent gravity" paradigm, which was first predicted by Sakharov [192] and later the idea was resurrected by Jacobson by establishing the fact that the Einstein's equation can be obtained as an equation of state from the Clausius relation on a local Rindler horizon [35]. On the other hand, Padmanabhan and his group established the fact that the governing dynamical equations in GR (such as the Einstein's equation) has a thermodynamic structure on the horizon (see the review [72]). In particular, we are driven by the fact that the Einstein's equation, when suitably projected on a null surface, takes the form of a thermodynamic identity [36] (Interestingly, this has been successfully extended to any order Lanczos-Lovelock gravity as well [59]). Therefore, within ST theory, one needs to check the possibility for developing the first law of thermodynamics like structure for an arbitrary integrable null-surface $\mathcal{H}$. This will provide the generality and robustness of the earlier claim on obtaining the

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thermodynamic structure in this theory. In the process of obtaining the first law for a generic null surface in ST theory, we need to identify certain terms as the temperature to draw the analogy between the gravitational thermodynamics and the conventional thermodynamics. To claim the analogous expression of temperature as the physical thermodynamic quantity, we need to investigate as to whether the expression is consistent with other thermodynamic laws, such as the zeroth law. Now, the idea of temperature becomes meaningful only in the equilibrium thermodynamic system; in gravity this is analogous to the Killing horizon.[1] So far we know, the zeroth law has not been explored rigorously in ST theory. Therefore, we need to check whether the identified temperature satisfies the zeroth law for the Killing horizon, which is a special category of null-surface and represents the equilibrium thermodynamic system. In summary, the motivation of the present work is straightforward; *i.e.* obtaining the first law for a generic null-surface and proving the zeroth law for the Killing horizon.[2] Thus, the present work is motivated to fill the gaps in the literature and to establish the thermodynamics of the scalar-tensor gravity in a more concrete manner.

To obtain the thermodynamic interpretation of the gravitational field equations as applied to the generic integrable null hypersurface in the ST theory w.r.t. both the Einstein and Jordan frame, we would again employ the same strategy of analyzing the gravitational dynamics via the NRE (for θ_k) Eq. (3.25) under the virtual displacement process as detailed out in the previous chapter. We adopt this method [92] in the context of scalar-tensor gravity and show that the same method works well to obtain the thermodynamic first law in the two frames. In addition, we also prove that the thermodynamic parameters in the two frames are equivalent, as it has been suggested earlier for the stationary black hole horizon (*i.e.* the Killing horizon) [189]. However, as we discuss later, obtaining thermodynamic law in the Jordan frame is quite non-trivial as compared to the Einstein frame, where the latter case is very much similar to that of the Einstein gravity. Thereafter, we prove the zeroth law for the Killing horizon. To our knowledge, the zeroth law has not been studied extensively for the scalar-tensor theory of gravity. However, there exists some comments in the literature stating that for any sensible definition of zeroth law in ST gravity, the scalar field is required to be constant on the horizon [169]. In our analysis, we show that imposition of such a strong restriction is not required. Instead, what it only requires is that the scalar field needs to be Lie-transported along the direction of the Killing vector *i.e.* the scalar field is required to be independent of only one coordinate, which is along the direction of the generator of the horizon surface.

Let us also mention that the gravitational dynamics of the ST theory as seen in the

[1]It is the horizon, which behaves like a thermodynamic object in black hole thermodynamics and the Killing horizon corresponds to a stationary black hole horizon [6].

[2]Note that the area increase theorem (*i.e.* the second law of black hole thermodynamics) has already been proved in the ST theory [189].

dynamical evolution of the Hájíček 1-form (along $\vec{l}$) under the consideration of the projection component $G_{ab}l^a q^b_{\ c}$ and its subsequent characterization as a relevant DNS equation (for both the frames) has been extensively studied in [190]. Hence in this chapter, we fill focus exclusively on thermodynamic interpretation of the ST gravitational dynamics incorporated by the projection component $G_{ab}l^a k^b$ and see whether its possible to shed some light on the physical (in)equivalences between the frames purely standing on the thermodynamic viewpoint.

Let us give an overview of this chapter. In Sec. 5.2 we begin with a very brief review of the action and field dynamics in the Einstein and Jordan frames. Next we proceed in Sec. 5.3 to our essential study of the covariant formulation of the thermodynamic identity established on a generic null hypersurface in the two frames. We go ahead in Sec. 5.3.1 towards our construction of the thermodynamic identity in both frames. This allows us then to attribute the equivalence of thermodynamic variables in the two frames. Finally, in order to provide a concrete interpretation of the notion of temperature in the two frames, we establish the proof of the zeroth law in Sec. 5.4. This proof has been performed in two different ways as applied to Killing horizons in the two frames.

5.2 Actions and equations of motion in the two frames : A brief review

Among the modified theories of gravity, the ST theory is a much viable and discussed one. In the original *Jordan frame*, the scalar field ϕ is non-minimally coupled to the Ricci scalar R. The total action for the ST theory in the Jordan frame $(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)$ is given by,

$$\mathcal{A}^{(ST)} = \int_V d^4x\sqrt{-g}\frac{1}{16\pi}\Big(\phi R - \frac{\omega(\phi)}{\phi}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\Big) + \mathcal{A}^{(m)}\,, \tag{5.1}$$

where $\omega(\phi)$ is known as the Brans-Dicke parameter, which is kept as a variable of the scalar-field ϕ. When $\omega(\phi)$ is considered as the constant parameter, the scalar-tensor theory reduces to the Brans-Dicke theory. Also, $V(\phi)$ corresponds to the arbitrary scalar-field potential and $\mathcal{A}^{(m)} = \int_V d^4x\sqrt{-g}\mathcal{L}^{(m)}$ is the ordinary matter action (ordinary in the sense that the matter fields are not coupled to the scalar field ϕ). The resulting field equation of g^{ab} corresponding to the action (5.1) with a suitable Gibbons-Hawking-York (GHY) surface term is [188, 189]

$$\begin{aligned} E_{ab} &= \frac{1}{16\pi}\Big[\phi G_{ab} + \frac{\omega}{2\phi}g_{ab}\nabla^i\phi\nabla_i\phi - \frac{\omega}{\phi}\nabla_a\phi\nabla_b\phi + \frac{V}{2}g_{ab} - \nabla_a\nabla_b\phi + g_{ab}\nabla_i\nabla^i\phi\Big] \\ &= \frac{1}{2}T^{(m)}_{ab}\,, \end{aligned} \tag{5.2}$$

where $T^{(m)}_{ab} = (-2/(\mathcal{M},g))\partial((\mathcal{M},g)\mathcal{L}^{(m)})/\partial g^{ab}$ represents the matter energy momentum tensor corresponding to $\mathcal{A}^{(m)}$.

In the Einstein frame $(\mathcal{M},\tilde{g},\tilde{\nabla},\tilde{\phi})$ (where $\tilde{\nabla}$ is the Levi-Civita connection associated with the metric $\tilde{g}$), we can remove the non-minimal coupling by the following set of conformal transformations on the metric and rescaling of the scalar field respectively,

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \tag{5.3}$$

$$d\tilde{\phi} = \sqrt{\frac{2\omega(\phi)+3}{16\pi}}\frac{d\phi}{\phi}\,, \tag{5.4}$$

where $\Omega^2 = \phi$ along with the condition that $\phi > 0$. The related field equation in Einstein frame turns out to be [188, 189]

$$\tilde{E}_{ab} = \frac{\tilde{G}_{ab}}{16\pi} - \frac{1}{2}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} + \frac{1}{4}\tilde{g}_{ab}\tilde{\nabla}^i\tilde{\phi}\tilde{\nabla}_i\tilde{\phi} + \frac{1}{2}\tilde{g}_{ab}U(\tilde{\phi}) = \frac{1}{2}\tilde{T}^{(m)}_{ab}\,, \tag{5.5}$$

where $U(\tilde{\phi}) = V(\phi)/(16\pi\phi^2)$ and $\tilde{T}^{(m)}_{ab} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial(\sqrt{-\tilde{g}}L^{(m)})}{\partial\tilde{g}^{ab}} = \frac{1}{\phi}T^{(m)}_{ab}$ represents the matter energy momentum tensor corresponding to matter action in the Einstein frame. The gravitational field equation in the Einstein frame (5.5) can be expressed in the similar form of Einstein's equation as $\tilde{G}_{ab} = 8\pi(\tilde{T}^{(\tilde{\phi})}_{ab} + \tilde{T}^{(m)}_{ab})$ where,

$$\tilde{T}^{(\tilde{\phi})}_{ab} = \tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{2}\tilde{g}_{ab}\tilde{\nabla}^i\tilde{\phi}\tilde{\nabla}_i\tilde{\phi} - \tilde{g}_{ab}U(\tilde{\phi})\,. \tag{5.6}$$

Throughout this paper we will follow the notation as presented in this section, where the tilde variables are reserved for the Einstein frame and the untilde ones are for the Jordan frame.

5.3 Covariant thermodynamic description on a generic null surface: equivalence between Jordan and Einstein frames

As usual for the coordinate-independent characterization of the desired thermodynamics, we will foliate the spacetime (provided with the Levi-Civita connection) in the neighborhood of our general null surface by a family of null surfaces. We would then as usual slice this null family by a stack of spacelike slices Σ_t. The intersection of the spacelike slice Σ_t with our null surface is the codimension two spacelike cross-section S_t. As usual with the convention introduced in the previous section $\vec{\tilde{l}}$ and $\vec{\tilde{k}}$ will denote the null generators and the auxiliary null vector field respectively for our null surface $\tilde{\mathcal{H}}$ in the Einstein frame. The induced metric $\tilde{\boldsymbol{q}}$ onto the submanifold $(\tilde{S}_t, \tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\mathcal{D}}})$ provided with a torsion-free metric-compatible connection $\tilde{\boldsymbol{\mathcal{D}}}$ ($\tilde{\mathcal{D}}_a\tilde{q}_{bc} = 0$) is given as,

$$\tilde{q}_{ab} = \tilde{g}_{ab} + \tilde{l}_a\tilde{k}_b + \tilde{k}_a\tilde{l}_b \, . \tag{5.7}$$

The same considerations hold for our null surface $\mathcal{H}$ in the Jordan frame. With these prerequisites, we now move on to discuss the procedure to obtain the thermodynamic law for a general null hypersurface in the two frames of the scalar-tensor theory.

5.3.1 Thermodynamic first law of a generic null surface in scalar-tensor gravity

In the previous chapter, we introduced a new covariant description of the thermodynamics alluded to specific dynamics (w.r.t. a generic null surface) of any given gravitational theory that properly reproduces the previous coordinate dependent results in the case of Einstein gravity. In the present section, we want to check whether this new formulation works well in ST gravity *i.e.*, whether we can obtain a similar thermodynamic identity in both the frames from the recent approach, as prescribed in [92]. In this analysis the starting point was Eq. (3.25) which governs the evolution of the ingoing expansion scalar $\theta_{\boldsymbol{k}}$ along the null generators. We rewrite the evolution equation here, for the convenience of the reader.

$$-\kappa\theta_{\boldsymbol{k}} = \Big[\frac{1}{2}{}^{(2)}R + l^r\nabla_r\theta_{\boldsymbol{k}} - \Omega_a\Omega^a - \mathcal{D}_a\Omega^a + \theta_{\boldsymbol{l}}\theta_{\boldsymbol{k}}\Big] - G_{ab}k^a l^b \, , \tag{5.8}$$

where $\mathcal{D}_a$ is the torsion-free unique covariant derivative operator defined on the manifold $(S_t, \boldsymbol{q})$ and ${}^{(2)}R$ denotes the Ricci scalar associated with the operator $\mathcal{D}_a$. This identity does not take into account any information of the particulars of the dynamics of gravitational field and hence one can use it in any theory of gravity. Below, in this identity the explicit form of field equations for g_{ab} in ST theory will be used in order to investigate our goal.

5.3.1.1 Einstein frame

We start our analysis in Einstein frame as the situation is simpler in this frame. The non-minimal coupling no longer exists in this frame and, the scalar field appears like the external field. In the Einstein frame $(\mathcal{M}, \tilde{g}, \tilde{\boldsymbol{\nabla}}, \tilde{\phi})$, we assume the existence of a generic null hypersurface $\tilde{\mathcal{H}}$. Let us briefly describe the nature of such a null surface (for details see [61]) in the Einstein frame which is designated by the constant value of the scalar field $u(x^a)$. The null normal $\tilde{l}_a$ to $\tilde{\mathcal{H}}$ is given by $\tilde{l}_a = -e^{\tilde{\rho}}\tilde{\nabla}_a u$, with $\tilde{\rho}$ being a scalar function on $\tilde{\mathcal{H}}$. The integrable null surface $\tilde{\mathcal{H}}$ is generated by null generators $\tilde{\boldsymbol{l}}$ satisfying the geodesic equation $\tilde{l}^a\tilde{\nabla}_a\tilde{l}^b = \tilde{\kappa}\tilde{l}^b$. The integrability of the null surface is defined by the Frobenius's theorem, which in its dual formulation [2] reads,

$$\tilde{\nabla}_a \tilde{l}_b - \tilde{\nabla}_b \tilde{l}_a = (\tilde{\nabla}_a \tilde{\rho}) \tilde{l}_b - (\tilde{\nabla}_b \tilde{\rho}) \tilde{l}_a \, . \tag{5.9}$$

The non-affinity parameter of the null generators assumes the value $\tilde{\kappa} = \tilde{l}^a \tilde{\nabla}_a \tilde{\rho}$. The transverse 2-dimensional spacelike submanifold of this null hypersurface is designated by $(\tilde{S}_t, \tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\mathcal{D}}})$. In this frame, since every quantity is represented by tilde variable, we express the identity (5.8) in the following form:

$$-\tilde{\kappa}\tilde{\theta}_{\tilde{\boldsymbol{k}}} = -\mathcal{D}_a \tilde{\Omega}^a - \tilde{\Omega}_a \tilde{\Omega}^a + \tilde{\theta}_{\tilde{\boldsymbol{l}}} \tilde{\theta}_{\tilde{\boldsymbol{k}}} + \tilde{l}^i \tilde{\nabla}_i \tilde{\theta}_{\tilde{\boldsymbol{k}}} + \frac{1}{2} \, {}^{(2)}\tilde{R} - \tilde{G}_{ab} \tilde{l}^a \tilde{k}^b \, . \tag{5.10}$$

From the above equation (5.10), one can obtain the thermodynamic first law considering the virtual displacement process along the auxiliary null vector field $\vec{\tilde{\boldsymbol{k}}}$. We just reiterate the idea which goes as follows. We consider the auxiliary null vector field as being parametrized by $\lambda_{(\tilde{k})}$, which means $\tilde{k}^i = -dx^i/d\lambda_{(\tilde{k})}$. The reason for the negative sign has been explained in the previous chapter. Furthermore, we consider that a set of two null surfaces are located at $\lambda_{(\tilde{k})} = 0$ and at $\lambda_{(\tilde{k})} = \delta\lambda_{(\tilde{k})}$. A virtual displacement $\delta\lambda_{(\tilde{k})}$ implies a shift from one solution of null hypersurface to the other. Then the coordinate variation under the mentioned virtual displacement is given as $\delta x^i = -\tilde{k}^i \delta\lambda_{(\tilde{k})}$. Next, we multiply both sides of the Eq. (5.10) with $\delta\lambda_{(\tilde{k})}$ (along with an overall factor $1/8\pi$) and integrate it over the two-surface $\tilde{S}_t$, which yields

$$\begin{aligned}
&- \int_{\tilde{S}_t} d^2x \sqrt{\tilde{q}} \, \delta\lambda_{(\tilde{k})} \frac{\tilde{\kappa}}{2\pi} \frac{1}{4} \tilde{\theta}_{\tilde{\boldsymbol{k}}} \\
&= \int_{\tilde{S}_t} d^2x \sqrt{\tilde{q}} \, \delta\lambda_{(\tilde{k})} \frac{1}{8\pi} \Big[\frac{1}{2} {}^2\tilde{R} + \tilde{l}^i \tilde{\nabla}_i \tilde{\theta}_{\tilde{\boldsymbol{k}}} + \tilde{\theta}_{\tilde{\boldsymbol{l}}} \tilde{\theta}_{\tilde{\boldsymbol{k}}} - \tilde{\Omega}_a \tilde{\Omega}^a - \tilde{\mathcal{D}}_a \tilde{\Omega}^a \Big] \\
&- \int_{\tilde{S}_t} d^2x \sqrt{\tilde{q}} \, \delta\lambda_{(\tilde{k})} \Big[\tilde{T}^{(\tilde{\phi})}_{ab} + \tilde{T}^{(m)}_{ab} \Big] \tilde{l}^a \tilde{k}^b \, .
\end{aligned} \tag{5.11}$$

In the above we have used the gravitational field equation of the Einstein frame (5.5). The above equation (5.11) can be given the interpretation analogous to the first law of thermodynamics as applied to the null surface via,

$$\int_{\tilde{S}_t} d^2x \tilde{T} \delta_{\lambda(\tilde{k})} \tilde{s} = \delta_{\lambda(\tilde{k})} \tilde{E} + \tilde{F} \delta\lambda_{(\tilde{k})} \, , \tag{5.12}$$

where, the thermodynamic parameters are identified as the following. We identify the temperature as $\tilde{T} = \tilde{\kappa}/2\pi$, the entropy density s is identified as $s = \sqrt{\tilde{q}}/4$ and, the change of entropy density (s) due to the virtual displacement is denoted by $\delta_{\lambda(\tilde{k})} \tilde{s}$, which is given as

$$\delta_{\lambda(\tilde{k})}\tilde{s} = \frac{d\tilde{s}}{d\lambda_{(\tilde{k})}}\delta\lambda_{(\tilde{k})} = \frac{1}{4}\frac{d\sqrt{\tilde{q}}}{d\lambda_{(\tilde{k})}}\delta\lambda_{(\tilde{k})} = -\frac{1}{4}\sqrt{\tilde{q}}\,\tilde{\theta}_{\tilde{k}}\delta\lambda_{(\tilde{k})}\,. \tag{5.13}$$

While obtaining the last step in the above relation (5.13), we have used (see [25, 61])

$$\tilde{\theta}_{\tilde{k}} = -\frac{1}{\sqrt{\tilde{q}}}\frac{d\sqrt{\tilde{q}}}{d\lambda_{(\tilde{k})}}\,. \tag{5.14}$$

The total entropy in Einstein frame is given as

$$\tilde{S} = \int_{\tilde{S}_t} d^2x\tilde{s} = \frac{1}{4}\int_{\tilde{S}_t}\sqrt{\tilde{q}}d^2x\,, \tag{5.15}$$

which is consistent with the area law of the entropy. The variation of energy $\tilde{E}$ due to the virtual displacement (in (5.12)) is given as

$$\delta_{\lambda(\tilde{k})}\tilde{E} = \frac{1}{8\pi}\int_{\tilde{S}_t} d^2x\sqrt{\tilde{q}}\,\delta\lambda_{(\tilde{k})}\Big[\frac{1}{2}{}^2\tilde{R} + \tilde{l}^i\tilde{\nabla}_i\tilde{\theta}_{\tilde{k}} + \tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} - \tilde{\Omega}_a\tilde{\Omega}^a - \tilde{\mathcal{D}}_a\tilde{\Omega}^a\Big]\,. \tag{5.16}$$

An indefinite integration over $\lambda_{(\tilde{k})}$ provides the expression of energy associated with the two-surface S_t, which is given as

$$\tilde{E} = \frac{1}{8\pi}\int\int_{\tilde{S}_t} d^2x\sqrt{\tilde{q}}\,d\lambda_{(\tilde{k})}\Big[\frac{1}{2}{}^2\tilde{R} + \tilde{l}^i\tilde{\nabla}_i\tilde{\theta}_{\tilde{k}} + \tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} - \tilde{\Omega}_a\tilde{\Omega}^a - \tilde{\mathcal{D}}_a\tilde{\Omega}^a\Big]\,. \tag{5.17}$$

The above expression has been identified as the energy term inspired by the fact that it reduces to expressions of that for well known spacetimes as explained in section 4.5. For example, it has been shown in [92] that specifically for Einstein gravity the covariant expression of the energy term matches with the expression of the energy expressed in the GNC system [36]. For example, the covariant energy term for the Schwarzschild metric (in Einstein gravity) reduces to the mass Finally, we identify the pressure ($\tilde{P}$) as $\tilde{P} = -(\tilde{T}^{(\tilde{\phi})}_{ab} + \tilde{T}^{(m)}_{ab})\tilde{l}^a\tilde{k}^b$ in the similar way as it has been identified in [36, 74, 79]. The total work due to the virtual displacement $\delta\lambda_{(\tilde{k})}$ is given as

$$\tilde{W} = \tilde{F}\delta\lambda_{(\tilde{k})} = \int_{\tilde{S}_t} d^2x\sqrt{\tilde{q}}\,\delta\lambda_{(\tilde{k})}\tilde{P} = -\int_{\tilde{S}_t} d^2x\sqrt{\tilde{q}}\,\delta\lambda_{(\tilde{k})}(\tilde{T}^{(\tilde{\phi})}_{ab} + \tilde{T}^{(m)}_{ab})\tilde{l}^a\tilde{k}^b\,. \tag{5.18}$$

Here $\tilde{F}$ is the integral of the pressure over the two-surface S_t and hence can be given the interpretation of the generalized force conjugate to the virtual displacement $\delta\lambda_{(\tilde{k})}$. The null hypersurface $\tilde{\mathcal{H}}$ is obviously considered to be a solution of the field equations in $(\mathcal{M}, \tilde{g}, \tilde{\boldsymbol{\nabla}}, \tilde{\phi})$. As a result of this virtual displacement process, an amount of energy $\delta_{\lambda(\tilde{k})}\tilde{E}$ flows across the null hypersurface. Part of this energy contributes in the entropy generation term $\int_{\tilde{S}_t} d^2x \tilde{T} \delta_{\lambda(\tilde{k})} \tilde{s}$ and the other contributes to the virtual work done $\tilde{F}\delta\lambda_{(\tilde{k})}$. Let us note, before proceeding next, that all the relevant quantities (geometrical, physical and thermodynamical) in the Jordan frame will be denoted without the use of any tilde as opposed to the Einstein frame.

5.3.1.2 Jordan frame

We now proceed to obtain the thermodynamic law in the Jordan frame taking hints from the analysis in the Einstein frame. The dynamics of the ingoing expansion scalar $\theta_{\boldsymbol{k}}$ along the null generators $\vec{l}$ in the Jordan frame is given by (5.8). We see that the evolution equations for $\tilde{\theta}_{\tilde{\boldsymbol{k}}}$ in the Einstein frame and $\theta_{\boldsymbol{k}}$ in the Jordan frame *i.e.* the null Raychaudhuri equations (for the ingoing expansion scalars) are form-invariant under conformal transformations, *viz* Eq (5.10) and Eq (5.8). This is to be anticipated since they are evolution equations valid as geometrical identities on any arbitrary null hypersurface. Conformal transformations after all do not alter the causal structure of null hypersurfaces. In fact, it can also be proven that the geodesic equation for the null generators as well as the null Raychaudhuri equations (for the outgoing expansion scalars $\tilde{\theta}_{\tilde{\boldsymbol{l}}}$ and $\theta_{\boldsymbol{l}}$) remain form-invariant under conformal transformations for a generic null hypersurface in the Einstein and Jordan frames. Simply multiplying equation (5.8) by $\delta\lambda_{(k)}/8\pi$ and integrating over the two-surface S_t does not lead to the correct expression of thermodynamic law and identification of thermodynamic quantities. The reason is the following. It has been found in the earlier works [188–190] for a Killing horizon that the thermodynamic quantities are equivalent in the two frames. Therefore we expect our present thermodynamic quantities, defined on a generic null surface, must be equivalent in the two frames at least when the null normal becomes the symmetry generator of a Killing horizon. Let us now check whether this is the case. If we multiply Eq. (5.8) by $\delta\lambda_{(k)}/8\pi$ and integrate over two-surface, the term on left hand side then yields $-\frac{1}{8\pi}\int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)}\kappa\theta_{\boldsymbol{k}}$ which by the earlier argument can be expressed as

$$-\frac{1}{8\pi}\int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)}\kappa\theta_{\boldsymbol{k}} = \int_{S_t} d^2x \frac{\kappa}{2\pi}\delta_{\lambda}\left(\frac{\sqrt{q}}{4}\right) . \tag{5.19}$$

Now if the null surface is a Killing horizon, then κ is constant on S_t (we will explicitly prove this later in section 5.4). In this case, the above is expressed as $(\kappa/2\pi)\delta_{\lambda}(A/4)$, from which one can identify temperature and entropy as $\kappa/2\pi$ and $A/4$ respectively. But this is in conflict with the earlier result [188, 189] since this is not equivalent to its counter part in Einstein frame. For Killing horizon we know that $\tilde{T} = T$ and $\tilde{S} = \tilde{A}/4 = S = \phi A/4$. But this is not what we are obtaining from the above. Hence the above simple extension of the approach will not be consistent to known cases.

The remedy can be found from the earlier work [190]. From the analysis of fluid-gravity correspondence in scalar-tensor gravity [190], it is known that the the parameters of the Einstein frame (such as $\tilde{\kappa}$, $\tilde{\Omega}^a$, $\tilde{\theta}_{(\tilde{l})}$ *etc.*) are used in the Jordan frame as well to obtain the equivalent framework, where the physical parameters of the Einstein frame becomes equivalent to the same in the Jordan frame. Here we adopt the same method. Therefore, we plan to obtain the dynamics associated with the projection component $G_{ab}l^a k^b$ in terms of the parameters of Einstein frame (such as $\tilde{\kappa}$, $\tilde{\Omega}^a$, $\tilde{\theta}_{\tilde{l}}$, $\tilde{\theta}_{\tilde{k}}$ *etc.*) and in terms of the the covariant derivative operator and the null vectors of the Jordan frame (*i.e.* ∇_i, l^a, k^a *etc.*). The desired relation can be obtained either from (5.10) or from (5.8) as (5.10) and (5.8) are equivalent under the conformal transformation (see the Appendix 5.6). For simplicity, here we obtain it from eq. (5.10). In the Appendix 5.7 we obtain the same equation from eq. (5.8).

Firstly, we show how the different quantities in one frame are connected to the same in the other frame. From the conformal transformation relation (5.3), we obtain that the null vectors change between the two frames in the following manner [190]

$$\begin{aligned} \tilde{l}^a &= l^a, \qquad \tilde{l}_a = \phi l_a \\ \tilde{k}^a &= \frac{1}{\phi}k^a, \qquad \tilde{k}_a = k_a \,. \end{aligned} \tag{5.20}$$

Let us take note of the nature of the null hypersurface in the Jordan frame. Its important to stress that we are not considering a different null surface $\mathcal{H}$ in the Jordan frame. The hypersurface is still defined by $u(x^a) = 0$ in the Jordan spacetime as well. To establish this fact, it is sufficient to prove that $\mathcal{H}$ still represents an integrable hypersurface generated by $\vec{l}$ under the conformal transformation. Taking cue from (5.20) and (5.9), its quite easy to show that,

$$\tilde{\nabla}_a\tilde{l}_b - \tilde{\nabla}_b\tilde{l}_a = (\tilde{\nabla}_a\tilde{\rho})\tilde{l}_b - (\tilde{\nabla}_b\tilde{\rho})\tilde{l}_a = \phi(\nabla_a l_b - \nabla_b l_a) + (\nabla_a\phi)l_b - (\nabla_b\phi)l_a \,. \tag{5.21}$$

This implies,

$$(\nabla_a l_b - \nabla_b l_a) = (\partial_a\tilde{\rho} - \nabla_a \ln\phi)l_b - (\partial_b\tilde{\rho} - \nabla_a \ln\phi)l_a = (\partial_a\rho)l_b - (\partial_b\rho)l_a \,, \tag{5.22}$$

with the scalar function ρ on $\mathcal{H}$ defined by $\tilde{\rho} = \rho + \ln\phi + \text{constant}$. The relation (5.22) guarantees the hypersurface orthogonality of the null surface $\mathcal{H}$ generated by $\vec{l}$ defined via $l_a = -e^{\rho}\nabla_a u$. The non-affinity parameter of the null generators of $\mathcal{H}$ are defined via $\kappa = l^a\nabla_a\rho$.

From the relation (5.20), we find,

$$\tilde{\theta}_{\tilde{l}} = \theta_l + l^i \nabla_i \ln\phi\,,$$
$$\tilde{\theta}_{\tilde{k}} = \frac{1}{\phi}\Big[\theta_k + k^i \nabla_i \ln\phi\Big],$$
$$\tilde{\kappa} = \kappa + l^i \nabla_i \ln\phi\,,$$
$$\tilde{\omega}_a = \omega_a + \frac{1}{2}\Big[l_a k^i \nabla_i \ln\phi + \nabla_a \ln\phi - k_a l^i \nabla_i \ln\phi\Big]\,,$$
$$\tilde{\Omega}_a = \Omega_a + \frac{1}{2} q^b{}_a \nabla_b \ln\phi\,. \tag{5.23}$$

Now, we start from the relation (5.10) and change the covariant derivative operator of the Einstein frame (*i.e.* $\tilde{\nabla}_i$) to the covariant derivative of the Jordan frame (*i.e.* ∇_i). Also, the null vectors of the Einstein frame ($\tilde{l}^i$ and $\tilde{k}^i$) are transformed to the null vectors of the Jordan frame using eq. (5.20). However, we keep the kinematical parameters (such as $\tilde{\kappa}$, $\tilde{\Omega}^a$, $\tilde{\theta}_{\tilde{l}}$, $\tilde{\theta}_{\tilde{k}}$ *etc.*) unchanged. In addition, the Ricci tensor, the intrinsic scalar curvature of the whole manifold and the same of the two-surface are expressed in terms of their Jordan frame's counterpart. With these goals in our mind, we obtain

$$\tilde{\mathcal{D}}_a \tilde{\Omega}^a = \mathcal{D}_a \tilde{\Omega}^a + \tilde{\Omega}^i \nabla_i (\ln\phi)\,. \tag{5.24}$$

The intrinsic scalar curvature of the two-surface transforms as [2]

$${}^{(2)}\tilde{R} = \frac{{}^{(2)}R}{\phi} - \frac{1}{\phi}\mathcal{D}^i \mathcal{D}_i (\ln\phi)\,. \tag{5.25}$$

The quantity $\tilde{G}_{ab}\tilde{l}^a\tilde{k}^b$ changes under conformal transformation as

$$\begin{aligned}\tilde{G}_{ab}\tilde{l}^a\tilde{k}^b \;=\; & \frac{G_{ab}l^a k^b}{\phi} + \frac{3}{2\phi} l^a k^b \nabla_a(\ln\phi)\nabla_b(\ln\phi) - \frac{l^a k^b}{\phi^2}\nabla_a\nabla_b\phi - \frac{1}{\phi^2}\nabla^i\nabla_i\phi \\ & + \frac{3}{4\phi}\nabla_i(\ln\phi)\nabla^i(\ln\phi)\,. \end{aligned} \tag{5.26}$$

Using the transformation relations (5.24), (5.25) and (5.26) in (5.10) one obtains the desired dynamics related to $G_{ab}l^a k^b$ projection component in the Jordan frame, which is given as

$$\begin{aligned}-\tilde{\kappa}\tilde{\theta}_{\tilde{k}} =& -\mathcal{D}_a\tilde{\Omega}^a - \tilde{\Omega}^i\nabla_i(\ln\phi) - \tilde{\Omega}_a\tilde{\Omega}^a + \tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} + l^i\nabla_i\tilde{\theta}_{\tilde{k}} + \frac{1}{2\phi}\,{}^{(2)}R \\ & -\frac{1}{2\phi}\mathcal{D}^i\mathcal{D}_i(\ln\phi) - \Big(\frac{G_{ab}l^a k^b}{\phi} + \frac{3}{2\phi} l^a k^b\nabla_a(\ln\phi)\nabla_b(\ln\phi) - \frac{l^a k^b}{\phi^2}\nabla_a\nabla_b\phi \\ & -\frac{1}{\phi^2}\nabla^i\nabla_i\phi + \frac{3}{4\phi}\nabla_i(\ln\phi)\nabla^i(\ln\phi)\Big)\,. \end{aligned} \tag{5.27}$$

To interpret the above relation (5.27) as the thermodynamic identity, we firstly use the field equation in the Jordan frame (*i.e.* eq. (5.2)) in (5.27), which yields upon multiplication by the scalar field ϕ on both sides as,

$$
\begin{aligned}
&-\phi\tilde{\kappa}\tilde{\theta}_{\tilde{k}} = \\
&-\phi\mathcal{D}_a\tilde{\Omega}^a - \tilde{\Omega}^i\nabla_i(\ln\phi) - \phi\tilde{\Omega}_a\tilde{\Omega}^a + \phi\tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} + \phi l^i\nabla_i\tilde{\theta}_{\tilde{k}} + \frac{1}{2}\,{}^{(2)}R - \frac{1}{2}\mathcal{D}^i\mathcal{D}_i(\ln\phi) \\
&- l^a k^b\Big[\Big(\frac{2\omega+3}{2}\Big)\Big\{\nabla_a(\ln\phi)\nabla_b(\ln\phi) - \frac{1}{2}g_{ab}\nabla^i(\ln\phi)\nabla_i(\ln\phi)\Big\} - \frac{V}{2\phi}g_{ab}\Big] \\
&- \frac{8\pi}{\phi}T^{(m)}_{ab}l^a k^b\,. \qquad (5.28)
\end{aligned}
$$

The terms inside the square bracket of (5.28) can be identified as the quantity $8\pi\tilde{T}^{(\tilde{\phi})}_{ab}$ as computed in the Jordan frame. Note that the same energy-momentum tensor for ϕ field was also obtained in [190] when the dynamics (of the Háj́iček 1-form) associated with the projection component $G_{ab}l^a q^b_c$ was interpreted as Damour-Navier-Stokes equation in Jordan frame. Also, we know that the energy-momentum tensor of the external matter fields are connected in the two frames as $\tilde{T}^{(m)}_{ab} = T^{(m)}_{ab}/\phi$. We now follow the same procedure as that of the Einstein frame to obtain the first law of thermodynamics. In the Einstein frame we considered the virtual displacement of the null hypersurface from $\lambda_{(\tilde{k})} = 0$ to $\lambda_{(\tilde{k})} = \delta\lambda_{(\tilde{k})}$ *i.e* by an amount of $\delta\lambda_{(\tilde{k})}$. We obviously expect this numerical value of the displacement to remain the same when we consider an analogous virtual displacement in the Jordan frame. We have the relation,

$$
\delta x^a = -\tilde{k}^a\delta\lambda_{(\tilde{k})} = -\frac{k^a}{\phi}\delta\lambda_{(\tilde{k})} = -k^a\delta\lambda_k\,. \qquad (5.29)
$$

This above relation allows us to interpret $\delta\lambda_{(\tilde{k})} = \phi\,\delta\lambda_k$. This can also be understood by the following way. We know $\tilde{k}^a = -dx^a/d\lambda_{\tilde{k}}$ and $k^a = -dx^a/d\lambda_k$ and as $\tilde{k}^a = k^a/\phi$, we must have $\delta\lambda_{(\tilde{k})} = \phi\delta\lambda_{(k)}$. Hence multiplying the relation (5.28) with $\delta\lambda_{(\tilde{k})}/8\pi = \phi\,\delta\lambda_{(k)}/8\pi$ and integrating it over the transverse 2-surface S_t with the integration measure $\sqrt{q}$, we have,

$$
\begin{aligned}
&-\int_{S_t} d^2x\phi\sqrt{q}\frac{\tilde{\kappa}}{2\pi}\frac{1}{4}\tilde{\theta}_{\tilde{k}}\,\delta\lambda_{(\tilde{k})} = -\int_{S_t} d^2x\sqrt{q}\Big[\tilde{T}^{(\tilde{\phi})}_{ab} + \frac{T^{(m)}_{ab}}{\phi}\Big]l^a k^b\,\delta\lambda_{(\tilde{k})} \\
&+\int_{S_t} d^2x\sqrt{q}\frac{\phi}{8\pi}\Big[\frac{1}{2\phi}{}^{(2)}R + l^i\nabla_i\tilde{\theta}_{\tilde{k}} + \tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} - \tilde{\Omega}_a\tilde{\Omega}^a - \mathcal{D}_A\tilde{\Omega}^A - \tilde{\Omega}^i\nabla_i(\ln\phi) \\
&-\frac{1}{2\phi}\mathcal{D}^i\mathcal{D}_i(\ln\phi)\Big]\,\delta\lambda_{(\tilde{k})}\,. \qquad (5.30)
\end{aligned}
$$

This allows us to have,

$$
\begin{aligned}
&-\int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)}\phi^2\frac{\tilde{\kappa}}{2\pi}\frac{1}{4}\tilde{\theta}_{\tilde{k}} = -\int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)}\phi\Big[\tilde{T}^{(\tilde{\phi})}_{ab} + \frac{T^{(m)}_{ab}}{\phi}\Big]l^a k^b \\
&+\int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)}\frac{\phi^2}{8\pi}\Big[\frac{1}{2\phi}{}^{(2)}R + l^i\nabla_i\tilde{\theta}_{\tilde{k}} + \tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} - \tilde{\Omega}_a\tilde{\Omega}^a - \mathcal{D}_A\tilde{\Omega}^A \\
&-\tilde{\Omega}^i\nabla_i(\ln\phi) - \frac{1}{2\phi}\mathcal{D}^i\mathcal{D}_i(\ln\phi)\Big]\,. \qquad (5.31)
\end{aligned}
$$

As earlier, the above equation (5.31) can be interpreted as the first law of the gravitational dynamics w.r.t the null-surface $\mathcal{H}$ in the Jordan frame, which is given as

$$
\int_{S_t} d^2x T\delta_{\lambda(k)} s = \delta_{\lambda(k)} E + F\delta\lambda_{(k)}\,. \qquad (5.32)
$$

For the moment we do not give the covariantly identified thermodynamical quantities in the Jordan frame. This will be given in the next discussion where we will show their equivalence with those in Einstein frame.

5.3.1.3 Thermodynamic equivalence in two frames

In the following, it will be shown that we not only obtain the first law of thermodynamics in the two frames, but also the fact that the thermodynamic parameters are equivalent in the two frames. Firstly, we identify the temperature in the Jordan frame as $T = \tilde{\kappa}/2\pi$. This is equivalent to the temperature $\tilde{T}$ in the Einstein frame. Here, the entropy density (s) in the Jordan frame is defined as $s = \sqrt{q}\phi/4$. Therefore,

$$
\begin{aligned}
\delta_{\lambda(k)}s &= \frac{ds}{d\lambda_{(k)}}\delta\lambda_{(k)} = \frac{\delta\lambda_{(k)}}{4}\Big(\phi\frac{d\sqrt{q}}{d\lambda_{(k)}} + \sqrt{q}\frac{d\phi}{d\lambda_{(k)}}\Big) \\
&= -\delta\lambda_{(k)}\frac{\sqrt{q}\phi}{4}\Big(\theta_{\boldsymbol{k}} + k^i\nabla_i(\ln\phi)\Big) = -\frac{1}{4}\phi^2\sqrt{q}\tilde{\theta}_{\tilde{k}}\delta\lambda_{(k)} = -\frac{1}{4}\sqrt{\tilde{q}}\,\tilde{\theta}_{\tilde{k}}\delta\lambda_{(\tilde{k})} \\
&= \delta_{\lambda(\tilde{k})}\tilde{s}\,, \qquad (5.33)
\end{aligned}
$$

where we have used

$$
\theta_{\boldsymbol{k}} = -\frac{1}{\sqrt{q}}\frac{d\sqrt{q}}{d\lambda_{(k)}}\,. \qquad (5.34)
$$

The total entropy in the Jordan frame is defined in the similar way as that of the Einstein frame, which is given as

$$
S = \int_{S_t} s d^2x = \int_{S_t}\phi\frac{\sqrt{q}}{4}d^2x = \int_{\tilde{S}_t}\frac{\sqrt{\tilde{q}}}{4}d^2x = \tilde{S}\,. \qquad (5.35)
$$

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Here, we have used the fact that $\sqrt{\tilde{q}} = \phi\sqrt{q}$. Therefore, we obtain that the entropy density and the entropy in the two frames are equivalent. Also, let us note that the usual area law of entropy is not valid in the Jordan frame. But, the obtained expression of entropy is consistent with earlier observation [176].

The variation of the energy in Jordan frame due to the virtual displacement is given as

$$\delta_{\lambda(k)}E = \frac{1}{8\pi}\int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)}\phi^2\Big[\frac{1}{2\phi}{}^{(2)}R + l^i\nabla_i\tilde{\theta}_{\tilde{k}} + \tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} - \tilde{\Omega}_a\tilde{\Omega}^a - \mathcal{D}_A\tilde{\Omega}^A - \tilde{\Omega}^i\nabla_i(\ln\phi) - \frac{1}{2\phi}\mathcal{D}^i\mathcal{D}_i(\ln\phi)\Big]\,. \tag{5.36}$$

The expression of energy associated with the two-surface S_t is identified as

$$E = \frac{1}{8\pi}\int\int_{S_t} d^2x\sqrt{q}\,d\lambda_{(k)}\phi^2\Big[\frac{1}{2\phi}{}^{(2)}R + l^i\nabla_i\tilde{\theta}_{\tilde{k}} + \tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} - \tilde{\Omega}_a\tilde{\Omega}^a - \mathcal{D}_A\tilde{\Omega}^A - \tilde{\Omega}^i\nabla_i(\ln\phi) - \frac{1}{2\phi}\mathcal{D}^i\mathcal{D}_i(\ln\phi)\Big]\,. \tag{5.37}$$

Its quite evident using (5.24) and (5.25) that the expressions for the variation of the energy (5.36) and the energy (5.37) in the Jordan frame are equivalent to the ones established in the Einstein frame, *viz* (5.16) and (5.17) respectively. Hence we have established the fact that the energy terms are equivalent in both the frames under the process of virtual displacement.

The work done under the virtual displacement process in the Jordan frame is identified as

$$W = -\int_{S_t} d^2x\,\sqrt{q}\,\delta\lambda_{(k)}\phi\Big(\tilde{T}^{(\tilde{\phi})}_{ab} + \frac{T^{(m)}_{ab}}{\phi}\Big)l^a k^b = F\delta\lambda_{(k)}\,. \tag{5.38}$$

Using the relevant transformations *i.e* $k^a = \phi\tilde{k}^a$, $\tilde{T}^{(m)}_{ab} = \frac{1}{\phi}T^{(m)}_{ab}$, $\delta\lambda_{(k)} = \delta\lambda_{(\tilde{k})}/\phi$ and $\sqrt{\tilde{q}} = \phi\sqrt{q}$ we obtain,

$$\begin{aligned} F\delta\lambda_{(k)} &= -\int_{S_t} d^2x\,\sqrt{q}\,\delta\lambda_{(k)}\phi\Big(\tilde{T}^{(\tilde{\phi})}_{ab} + \frac{T^{(m)}_{ab}}{\phi}\Big)l^a k^b \\ &= -\int_{\tilde{S}_t} d^2x\sqrt{\tilde{q}}\,\delta\lambda_{(\tilde{k})}\Big(\tilde{T}^{(\tilde{\phi})}_{ab} + \tilde{T}^{(m)}_{ab}\Big)\tilde{l}^a\tilde{k}^b = \tilde{F}\delta\lambda_{(\tilde{k})}\,. \end{aligned} \tag{5.39}$$

Hence we see that the work done under the virtual displacement process is equivalent in both the Jordan and the Einstein frames. Even though the work function turns out to be equivalent, the pressure terms in the respective frames are not synonymous under our interpretation. We identify the pressure (P) in the Jordan frame as,

$$P = -\left(\phi\tilde{T}^{(\tilde{\phi})}_{ab} + T^{(m)}_{ab}\right) l^a k^b \,. \tag{5.40}$$

Obviously, the pressure functions in the two frames are not equivalent *i.e.* $\tilde{P} \neq P$. Our identification of the pressure term comes from the fact that the force conjugate to the virtual displacement $\delta\lambda_{(k)}$ in the Jordan frame is given as the integral of the pressure term over the 2-surface S_t,

$$F = \int_{S_t} d^2x \sqrt{q} P \,. \tag{5.41}$$

So far we have seen that, like Einstein gravity, the ST theory has also similar thermodynamic structure on a generic null surface. We found the thermodynamic quantities on both the frames and constructed them in such a way that they are equivalent. It must be mentioned that this identification of quantities is purely analogy. A comparison with the familiar thermodynamics yields such interpretations. But it may happen that the aforesaid null surface may not be describing an equilibrium system and therefore defining the geometric quantities in terms of thermodynamic entities runs into trouble. Hence the discussion till now has been based on a formal analogy between gravitational equations and conventional thermodynamic identities. On the contrary if the manifold has a Killing horizon present in it (which represents a stationary solution of the gravity theory) then, in the light of constancy of surface gravity on the horizon and area (more generally entropy) increase theorem, the thermodynamic interpretation is much more logical. Having said that, we mention that the entropy increase theorem for a Killing horizon in the ST theory has been discussed in literature [189]. But constancy of surface gravity on the equilibrium Killing horizon in this theory, as far we aware of, has not been proven explicitly. Of course, there is a mention in literature that for the zeroth law to hold, the scalar field ϕ must be constant on the horizon, *i.e.* it must not only be independent of the coordinate along the null generator, but also of coordinates on S_t. In our point of view the latter restriction is very strong. Therefore we aim to look into this issue here. We will posit the existence of a black-hole spacetime. The Killing vector field is only timelike in some open region of the manifold *i.e.* outside a compact region. We mean that only this open region of the spacetime is stationary. The vanishing norm of the Killing vector field determines the position of the Killing horizon. We will see in the next discussion that the existence of a timelike Killing vector field in the stationary region of the spacetime and the scalar field ϕ being Lie-transported along it are enough to prove the constancy of surface gravity on the horizon. Therefore to obtain the zeroth law in general, ϕ can be a function of coordinates on S_t.

5.4 Study of the zeroth law in both the frames

Having stated our motivation, we are now going to prove the zeroth law (in other words, constancy of surface gravity on the Killing horizon) in this section. As far as the literature is concerned, the proof of the zeroth law crucially depends on the assumptions in the theory. The assumptions constrain the generality of the proof in turn. As far as we know of, the zeroth law has been proven under three specific assumptions.

- Use of the gravitational field equations along with the assumption that the non-gravitational and matter fields satisfy the Null Dominant Energy Condition (NDEC): This approach does not assume any extra symmetries of the spacetime other than the existence of a Killing vector field. This has been explicitly proven for the case of Einstein gravity [2] and Lanczos-Lovelock gravity [193]. Our proof of the constancy of the surface gravity in this section for the case of ST gravity rests upon this assumption.

- Assumption of the existence of bifurcate Killing horizons without the need of any gravitational field equations [25]: This however is restrictive since not all Killing horizons admit a bifurcation 2-surface.

- Assumptions of extra symmetries in the spacetime without the need of any field equations: This has been explicitly shown in the case of static and circular (stationary axisymmetric with t-ϕ reflection symmetry) spacetimes admitting the Killing horizon [13, 94, 194]. We present a proof (which we hope will add to the existing literature) of the zeroth law for static spacetimes in Appendix 5.8.

Our analysis will be done both in Einstein and Jordan frames. In order to do that we start by constructing the background requisites.

Let us posit the existence of a Killing vector $\vec{\tilde{\chi}}$ in the Einstein frame $(\mathcal{M}, \tilde{g}, \tilde{\nabla}, \tilde{\phi})$ defined via,

$$\pounds_{\tilde{\chi}}\, \tilde{g}_{ab} \overset{(\mathcal{M},\tilde{g},\tilde{\nabla},\tilde{\phi})}{=} 0\,. \tag{5.42}$$

Using the above fact and $\tilde{g}_{ab} = \phi g_{ab}$, we have,

$$\pounds_{\tilde{\chi}}\, g_{ab} \overset{(\mathcal{M},g,\nabla,\phi)}{=} -\frac{1}{\phi}\Big(\pounds_{\tilde{\chi}}\, \phi\Big) g_{ab}\,. \tag{5.43}$$

This shows that provided $\pounds_{\tilde{\chi}}\, \phi \overset{(\mathcal{M},g,\nabla,\phi)}{\neq} 0$, the vector field $\vec{\tilde{\chi}}$ becomes the conformal Killing vector field in the Jordan spacetime $(\mathcal{M}, g, \nabla, \phi)$. However, provided we impose the constraint,

$$\pounds_{\tilde{\chi}}\, \phi \overset{(\mathcal{M},g,\nabla,\phi)}{=} 0\,, \tag{5.44}$$

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we observe that $\vec{\tilde{\chi}}$ is also the Killing vector field in the Jordan frame as well. As a matter of field renaming (as per our conventions) we can define the generator of this Killing symmetry in the Jordan spacetime $(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)$ to be $\vec{\chi}$ and hence $\vec{\tilde{\chi}}$ and $\vec{\chi}$ coincide in $(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)$ *i.e.*,

$$\tilde{\chi}^a \overset{(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)}{=} \chi^a \,. \tag{5.45}$$

The above relation has been followed from [175] and has also been imposed in [174]. Obviously, we notice that the contravariant components of the Killing vectors match in the two frames, whereas the covariant vectors are related by the conformal factor. Hence the constraint (5.44) translates to the condition,

$$\chi^a \nabla_a \phi \overset{(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)}{=} 0 \,. \tag{5.46}$$

We now show what the condition (5.46) implies in the Einstein frame. In fact, taking help of the rescaling of the scalar field ϕ (5.4) we can show that,

$$\tilde{\chi}^a \tilde{\nabla}_a \tilde{\phi} = \tilde{\chi}^a \partial_a \tilde{\phi} = \sqrt{\frac{(2\omega(\phi)+3)}{16\pi}}\, \frac{1}{\phi}\, \chi^a \nabla_a \phi \,. \tag{5.47}$$

The above relation clearly implies that setting the constraint $\chi^a \nabla_a \phi \overset{(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)}{=} 0$ in the Jordan frame results in an analogous constraint in the Einstein frame, *i.e.*,

$$\tilde{\chi}^a \tilde{\nabla}_a \tilde{\phi} \overset{(\mathcal{M}, \tilde{g}, \tilde{\boldsymbol{\nabla}}, \tilde{\phi})}{=} 0 \,. \tag{5.48}$$

Having established the connections between the constraints (5.46) and (5.48) in the two frames, we now switch our attention to Killing horizons established in the two respective spacetimes. We reiterate that the Einstein frame $(\mathcal{M}, \tilde{g}, \tilde{\boldsymbol{\nabla}}, \tilde{\phi})$ and the Jordan frame $(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)$ admit the Killing vector fields $\vec{\tilde{\chi}}$ and $\vec{\chi}$ respectively upon which we have assumed the existence of the constraints (5.46) and (5.48).

A Killing horizon $\tilde{\mathcal{H}}^{(K)}$ in the Einstein frame $(\mathcal{M}, \tilde{g}, \tilde{\boldsymbol{\nabla}}, \tilde{\phi})$ admitting the Killing vector field $\vec{\tilde{\chi}}$ is by definition a null hypersurface of co-dimension one such that $\vec{\tilde{\chi}}$ is the normal vector to $\tilde{\mathcal{H}}^{(K)}$ and hence coincides with the null generators of $\tilde{\mathcal{H}}^{(K)}$. Under the assumption of the constraint (5.46) and ϕ being finite on horizon, we necessarily see that the Killing horizon $\tilde{\mathcal{H}}^{(K)}$ under the conformal transformation (5.3) and scalar field re-scaling (5.4) is mapped to a Killing horizon $\mathcal{H}^{(K)}$ in the Jordan frame $(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)$. The null generators of $\mathcal{H}^{(K)}$ coincide with the Killing field $\vec{\chi}$ on $\mathcal{H}^{(K)}$. Furthermore, we assume that the respective Killing horizons have (transverse to the null generators) spacelike codimension two cross-sections that are closed manifolds. The null generators satisfy the pregeodesic equation on their respective Killing horizons,

$$\tilde{\chi}^b \tilde{\nabla}_b \tilde{\chi}^a \stackrel{\tilde{\mathcal{H}}^{(K)}}{=} \tilde{\kappa} \tilde{\chi}^a \tag{5.49}$$

and

$$\chi^b \nabla_b \chi^a \stackrel{\mathcal{H}^{(K)}}{=} \kappa \chi^a \,, \tag{5.50}$$

where $\tilde{\kappa}$ and κ are the non-affinity parameters and in this context of Killing horizons are the surface gravities associated with the null generators $\vec{\tilde{\chi}}$ and $\vec{\chi}$ of $\tilde{\mathcal{H}}^{(K)}$ and $\mathcal{H}^{(K)}$ respectively. It is worth noticing from (5.23) and under the constraint (5.44) imposed on the scalar field that $\tilde{\kappa}$ and κ are same.

We now shift our attention towards the consideration of the zeroth law of black hole mechanics as applied to the Killing horizons in the two frames. Our proof towards the constancy of the surface gravity in the Killing horizon will demand the dynamical content of the theory, in the sense that we will use the gravitational field equations. The dynamical field equations come into play along with the assumption of some energy conditions. For our case, we will assume that the NDEC holds. We will prove the zeroth law as applied to the Killing horizons in both the frames in two different ways.

5.4.1 Approach I

For the first approach we basically follow [2]. We observe that the relations (5.49) and (5.50) are applicable only on the respective Killing horizons. Hence directly applying the derivative operator ∇_a onto such relations that are only valid on the Killing horizon leads us to problems. In order to prove the constancy of the surface gravity we basically need to take its directional derivative along a vector/tensor field that lies in the tangent plane of the Killing horizon. The Killing horizon being a null surface, makes it impossible to have a well defined projection tensor onto it using only the metric and the null normal. However we can look at the tensor field $\epsilon^{abcd}\chi_a$ which is tangent to the Killing horizon as evident from the fact that $\epsilon^{abcd}\chi_a\chi_b = 0$. Here ϵ_{abcd} is the spacetime volume form. Hence we can apply the derivative operator $\epsilon^{abcd}\chi_a\nabla_b$ as applied to relations that are valid only on the Killing horizon. Taking the completely antisymmetric nature of the volume form, we may as well take the derivative operator $\chi_{[a}\nabla_{b]}$ as applied to relations valid only on the Killing horizon. For any Killing horizon generated by $\vec{\chi}$, we have the relation [2] ,

$$\chi_{[a}\nabla_{b]}\kappa \stackrel{\mathcal{H}^{(K)}}{=} -\chi_{[a}R_{b]}{}^f \chi_f \,, \tag{5.51}$$

where R_{ab} stands for the Ricci tensor.

5.4.1.1 Einstein frame

Let us now begin our analysis in the Einstein frame $(\mathcal{M}, \tilde{g}, \tilde{\boldsymbol{\nabla}}, \tilde{\phi})$ as applied to the Killing horizon $\tilde{\mathcal{H}}^{(K)}$ generated by $\vec{\tilde{\chi}}$. The resulting equation concerning the change of the surface gravity $\tilde{\kappa}$ along any direction tangent to the Killing horizon $\mathcal{H}^{(K)}$is given by,

$$\tilde{\chi}_{[a}\tilde{\nabla}_{b]}\tilde{\kappa} \stackrel{\tilde{\mathcal{H}}^{(K)}}{=} -\tilde{\chi}_{[a}\tilde{R}_{b]}{}^{f}\tilde{\chi}_{f} \, . \tag{5.52}$$

Using the field equations (5.5), we compute the R.H.S of (5.52) which leads us to,

$$\begin{aligned}\tilde{\chi}_{[a}\tilde{R}_{b]}{}^{f}\tilde{\chi}_{f} \;&= 16\pi\Big[\tfrac{1}{2}\tilde{\chi}_{[a}\tilde{T}^{(m)f}_{b]}\tilde{\chi}_{f} + \tfrac{1}{2}\tilde{\chi}_{[a}\tilde{\nabla}_{b]}\tilde{\phi}\;(\tilde{\nabla}^{f}\tilde{\phi}\tilde{\chi}_{f}) - \tfrac{1}{4}\tilde{\chi}_{[a}\delta_{b]}^{f}\tilde{\chi}_{f}\;(\tilde{\nabla}^{i}\tilde{\phi}\tilde{\nabla}_{i}\tilde{\phi}) \\ &\qquad -\tfrac{1}{2}\tilde{\chi}_{[a}\delta_{b]}^{f}\tilde{\chi}_{f}\;U(\tilde{\phi}) + \tfrac{1}{32\pi}\tilde{\chi}_{[a}\delta_{b]}^{f}\tilde{\chi}_{f}\;\tilde{R}\Big] \, .\end{aligned} \tag{5.53}$$

Using the constraint (5.48) as applied to the Einstein frame, the above relation simplifies, which allows us to express (5.52) as,

$$\tilde{\chi}_{[a}\tilde{\nabla}_{b]}\tilde{\kappa} \stackrel{\tilde{\mathcal{H}}^{(K)}}{=} -\tilde{\chi}_{[a}\tilde{R}_{b]}{}^{f}\tilde{\chi}_{f} \stackrel{\tilde{\mathcal{H}}^{(K)}}{=} -8\pi\tilde{\chi}_{[a}\tilde{T}^{(m)f}_{b]}\tilde{\chi}_{f} \, . \tag{5.54}$$

Next, we notice that projecting the field equations (5.5) along the null generators of $\tilde{\mathcal{H}}^{(K)}$ we have,

$$\begin{aligned}\tilde{E}_{ab}\tilde{\chi}^{a}\tilde{\chi}^{b} \;&= \tfrac{1}{16\pi}\tilde{G}_{ab}\tilde{\chi}^{a}\tilde{\chi}^{b} - \tfrac{1}{2}(\tilde{\chi}^{a}\tilde{\nabla}_{a}\tilde{\phi})\;(\tilde{\chi}^{b}\tilde{\nabla}_{b}\tilde{\phi}) + \tfrac{1}{4}\vec{\tilde{\chi}}^{2}\;\tilde{\nabla}^{i}\tilde{\phi}\tilde{\nabla}_{i}\tilde{\phi} + \tfrac{1}{2}\vec{\tilde{\chi}}^{2}U(\tilde{\phi}) \, , \\ &= \tfrac{1}{2}\tilde{T}^{(m)}_{ab}\tilde{\chi}^{a}\tilde{\chi}^{b}\end{aligned} \tag{5.55}$$

where $\vec{\tilde{\chi}}^{2}$ stands for $\vec{\tilde{\chi}}\cdot\vec{\tilde{\chi}} = \tilde{g}_{ab}\tilde{\chi}^{a}\tilde{\chi}^{b}$. Employing the constraint as applied in the Einstein frame (5.48) and the fact that $\tilde{\chi}$ is null on the Killing horizon $\tilde{\mathcal{H}}^{(K)}$, we obtain,

$$\tilde{R}_{ab}\tilde{\chi}^{a}\tilde{\chi}^{b} \stackrel{\tilde{\mathcal{H}}^{(K)}}{=} 8\pi\tilde{T}^{(m)}_{ab}\tilde{\chi}^{a}\tilde{\chi}^{b} \, . \tag{5.56}$$

As mentioned earlier, we assume that our Killing horizon $\tilde{\mathcal{H}}^{(K)}$ is a null hypersurface provided with the topology $\tilde{\mathcal{H}}^{(K)} \simeq \mathbb{R}\times\tilde{\mathcal{J}}$, where the spacelike cross-section $\tilde{\mathcal{J}}$ is a closed 2 dimensional manifold (this is similar to the S_t describing the cross-section of our earlier generic null surface and being a closed submanifold). The induced metric onto the cross-section $\tilde{\mathcal{J}}$ is designated as $\tilde{q}_{ab}$. The null generator $\vec{\tilde{\chi}}$ satisfies (5.42), which implies that $\vec{\tilde{\chi}}$ is a symmetry generator of $\tilde{\mathcal{H}}^{(K)}$. Now since $\tilde{q}_{ab}$ is the metric induced by $\tilde{g}_{ab}$ on $\tilde{\mathcal{J}}$ and the fact that the basis vectors on $\tilde{\mathcal{J}}$ are Lie-transported along the null generators, we have the second fundamental form $\tilde{\Theta}_{ab}$ of $\tilde{\mathcal{H}}^{(K)}$ (which coincides with the deformation rate tensor for the integrable null hypersurface in the absence of torsion) [61] vanishing identically,

$$\tilde{\Theta}_{ab} = \frac{1}{2}\tilde{q}^{c}{}_{a}\tilde{q}^{d}{}_{b}\,\pounds_{\tilde{\chi}}\tilde{q}_{cd} \stackrel{\tilde{\mathcal{H}}^{(K)}}{=} 0 \, . \tag{5.57}$$

The irreducible decomposition of the deformation tensor

$$\tilde{\Theta}_{ab} = \frac{1}{2}\tilde{q}_{ab}\,\tilde{\theta}_{(\tilde{\chi})} + \tilde{\sigma}_{ab}\,, \tag{5.58}$$

where $\tilde{\theta}_{(\tilde{\chi})}$ denotes the expansion scalar corresponding to the null generator $\vec{\tilde{\chi}}$ and $\tilde{\sigma}_{ab}$ the shear tensor necessitates the fact that,

$$\tilde{\theta}_{(\tilde{\chi})} \overset{\tilde{\mathcal{H}}^{(K)}}{=} 0 \quad \text{and} \quad \tilde{\sigma}_{ab} \overset{\tilde{\mathcal{H}}^{(K)}}{=} 0\,. \tag{5.59}$$

This is precisely because the cross-section $\tilde{\mathcal{J}}$ is spacelike in nature. Now we can use the NRE as applied on $\tilde{\mathcal{H}}^{(K)}$ to find the value of $\tilde{R}_{ab}\tilde{\chi}^a\tilde{\chi}^b$. The NRE reads as,

$$\tilde{\chi}^a\tilde{\nabla}_a\tilde{\theta}_{(\tilde{\chi})} - \tilde{\kappa}\tilde{\theta}_{(\tilde{\chi})} + \frac{1}{2}\tilde{\theta}^2_{(\tilde{\chi})} + \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} \overset{\tilde{\mathcal{H}}^{(K)}}{=} -\tilde{R}_{ab}\tilde{\chi}^a\tilde{\chi}^b\,. \tag{5.60}$$

As applied to the specific Killing Horizon $\tilde{\mathcal{H}}^{(K)}$, where we established that the expansion scalar and the shear tensor for $\vec{\tilde{\chi}}$ vanish, the NRE implies,

$$\tilde{R}_{ab}\tilde{\chi}^a\tilde{\chi}^b \overset{\tilde{\mathcal{H}}^{(K)}}{=} 0\,. \tag{5.61}$$

This entails the fact from (5.56) that,

$$\tilde{T}^{(m)}_{ab}\tilde{\chi}^a\tilde{\chi}^b \overset{\tilde{\mathcal{H}}^{(K)}}{=} 0\,. \tag{5.62}$$

From the above relation we can conclude that the vector field $\tilde{T}^{(m)a}{}_b\,\tilde{\chi}^b$ lies on the tangent plane of the Killing horizon and hence is either null (collinear to $\vec{\tilde{\chi}}$) or spacelike (in the tangent plane of $\tilde{\mathcal{J}}$). To proceed ahead, we will make the assumption that the matter and the non-gravitational fields in $(\mathcal{M}, \tilde{\mathbf{g}}, \tilde{\boldsymbol{\nabla}}, \tilde{\phi})$ satisfy the Null Dominant Energy Condition (NDEC). The NDEC states that the vector field $\tilde{W}^a$ defined as,

$$\tilde{W}^a \equiv -\tilde{T}^{(m)a}{}_b\tilde{\chi}^b \tag{5.63}$$

is future directed causal (null or timelike) for the future directed null generator $\vec{\tilde{\chi}}$ of $\tilde{\mathcal{H}}^{(K)}$. However, we have already shown that on $\tilde{\mathcal{H}}^{(K)}$, the vector field $\tilde{T}^{(m)a}{}_b\,\tilde{\chi}^b$ can either be null or spacelike. Hence the NDEC forces $\tilde{T}^{(m)a}{}_b\,\tilde{\chi}^b$ to be null on the Killing horizon and hence collinear to the null generators,

$$-\tilde{T}^{(m)a}{}_b\,\tilde{\chi}^b \overset{\tilde{\mathcal{H}}^{(K)}}{=} \tilde{\alpha}\tilde{\chi}^a\,, \tag{5.64}$$

where $\tilde{\alpha}$ is some proportionality factor. Using the above relation in (5.54), we finally end up with the establishment of the zeroth law as applied on the Killing horizon in

$(\mathcal{M}, \tilde{g}, \tilde{\nabla}, \tilde{\phi})$,

$$\tilde{\chi}_{[a}\tilde{\nabla}_{b]}\tilde{\kappa} \stackrel{\tilde{\mathcal{H}}^{(K)}}{=} 0\,. \tag{5.65}$$

This basically shows the constancy of the surface gravity over the entire Killing horizon $\tilde{\mathcal{H}}^{(K)}$ established in the Einstein frame $(\mathcal{M}, \tilde{g}, \tilde{\nabla}, \tilde{\phi})$ by the null generators $\tilde{\chi}$.

5.4.1.2 Jordan frame

Now, we proceed towards the establishment of the zeroth law in the Jordan frame $(\mathcal{M}, g, \nabla, \phi)$. As again, we reiterate that under the constraint (5.46), the Einstein frame $(\mathcal{M}, \tilde{g}, \tilde{\nabla}, \tilde{\phi})$ with the Killing vector $(\vec{\tilde{\chi}})$ is mapped (under the conformal transformation of the metric and the scaling of the scalar field) to the Jordan frame $(\mathcal{M}, g, \nabla, \phi)$ with the Killing vector field $\vec{\chi}$. We further posit the existence of a Killing horizon $\mathcal{H}^{(K)}$ in the Jordan frame where its null generators $\vec{l}$ coincide with the Killing vector $\vec{\chi}$,

$$\vec{l} \stackrel{\mathcal{H}^{(K)}}{=} \vec{\chi}\,. \tag{5.66}$$

The topology of the Killing horizon in the Jordan frame should remain the same, in the sense that the spacelike cross-section of codimension two of the null surface is assumed to be a closed manifold. The same analysis towards the fact that the Killing horizon $\mathcal{H}^{(K)}$ is a non-expanding horizon follows. The null Killing vector $\vec{\chi}$ is a symmetry generator of the Killing Horizon $\mathcal{H}^{(K)}$,

$$\pounds_{\chi} g_{ab} \stackrel{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.67}$$

This again implies that the deformation rate tensor and the second fundamental tensor corresponding to $\mathcal{H}^{(K)}$ vanishes. So does the expansion scalar and the shear tensor corresponding to the null generator $\vec{\chi}$. Again, application of the NRE for $\vec{\chi}$, leads us to the fact that,

$$R_{ab}\chi^a\chi^b \stackrel{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.68}$$

As applied to the Killing horizon $\mathcal{H}^{(K)}$, analogous relation holds regarding the directional derivative of the surface gravity along any vector field tangent to the null surface (5.51). Its is quite easy to verify that the R.H.S of (5.51) upon application of the field equations in the Jordan frame leads us to,

$$\begin{aligned} &-\chi_{[a}R_{b]}{}^{f}\chi_f \stackrel{(\mathcal{M},g,\nabla,\phi)}{=} \\ &-\frac{1}{\phi}\Big[8\pi\chi_{[a}T^{(m)f}_{b]}\chi_f - \frac{\omega}{2\phi}\chi_{[a}\delta^{f}_{b]}\chi_f\,(\nabla_i\phi\nabla^i\phi) + \frac{\omega}{\phi}\chi_{[a}\nabla_{b]}\phi\,(\chi^f\nabla_f\phi) \\ &-\frac{1}{2}\chi_{[a}\delta^{f}_{b]}\chi_f\,V(\phi) + (\chi_{[a}\nabla_{b]}\nabla^f\phi)\chi_f - \chi_{[a}\delta^{f}_{b]}\chi_f(\nabla_i\nabla^i\phi)\Big] \\ &-\frac{1}{2}\chi_{[a}\delta^{f}_{b]}\chi_f\,R\,. \end{aligned} \tag{5.69}$$

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Using the constraint as applied in the Jordan frame (5.46) and simplifying the above result, we have then for (5.51),

$$\chi_{[a}\nabla_{b]}\kappa \overset{\mathcal{H}^{(K)}}{=} -\frac{1}{\phi}\Big(8\pi\chi_{[a}T^{(m)f}_{b]}\chi_f + \chi^f\left(\chi_{[a}\nabla_{b]}\nabla_f\,\phi\right)\Big)\,. \tag{5.70}$$

Next, we have the fact that,

$$\pounds_\chi(\nabla_a\phi) \overset{(\mathcal{M},g_{ab},\phi)}{=} 0\,. \tag{5.71}$$

This is again to be expected since the scalar field ϕ is Lie-transported along $\vec{\chi}$ as evident under the constraint (5.46) and therefore the quantity $(\nabla_a\phi)$ is expected to satisfy the symmetry of the spacetime. However we give a brief sketch of its proof in Appendix 5.9. Using (5.71), we can verify that,

$$\chi^f\left(\chi_{[a}\nabla_{b]}\nabla_f\,\phi\right) \overset{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.72}$$

A detailed outlined proof of this is given in Appendix 5.9. So finally, we obtain from (5.70) and (5.72),

$$\chi_{[a}\nabla_{b]}\kappa \overset{\mathcal{H}^{(K)}}{=} -\frac{1}{\phi}8\pi\chi_{[a}T^{(m)f}_{b]}\chi_f\,. \tag{5.73}$$

Next, we proceed to calculate $T^{(m)}_{ab}\chi^a\chi^b$ on the Killing horizon. Using the field equations of motion (5.2), we can show that,

$$\begin{aligned} E_{ab}\chi^a\chi^b \overset{(\mathcal{M},g,\nabla,\phi)}{=} & \tfrac{1}{16\pi}\Big[\phi G_{ab}\chi^a\chi^b + \tfrac{\omega}{2\phi}\vec{\chi}^2\left(\nabla_i\phi\nabla^i\phi\right) - \tfrac{\omega}{\phi}(\chi^a\nabla_a\phi)\,(\chi^b\nabla_b\phi) \\ & + \tfrac{1}{2}\vec{\chi}^2V(\phi) - \chi^a\chi^b\nabla_a\nabla_b\phi + \vec{\chi}^2\left(\nabla_i\nabla^i\phi\right)\Big] = \tfrac{1}{2}T^{(m)}_{ab}\chi^a\chi^b\,. \end{aligned} \tag{5.74}$$

On the Killing horizon $\mathcal{H}^{(K)}$, $\vec{\chi}$ is null and the projection component $R_{ab}\chi^a\chi^b$ vanishes (5.68). Upon using the constraint relation (5.46), we obtain from (5.74),

$$-\chi^a\chi^b\nabla_a\nabla_b\phi \overset{\mathcal{H}^{(K)}}{=} 8\pi T^{(m)}_{ab}\chi^a\chi^b\,. \tag{5.75}$$

Using the relation (5.71), it can be easily shown that $\chi^a\chi^b\nabla_a\nabla_b\phi$ vanishes on $\mathcal{H}^{(K)}$,

$$\chi^a\chi^b\nabla_a\nabla_b\phi = \chi^a\left(\pounds_\chi(\nabla_a\phi) - \nabla_b\phi\nabla_a\chi^b\right) \overset{\mathcal{H}^{(K)}}{=} -\kappa\chi^b\nabla_b\phi \overset{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.76}$$

Hence this allows us to finally conclude that,

$$T^{(m)}_{ab}\chi^a\chi^b \overset{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.77}$$

The above relation relation implies as usual that the vector field $T^{(m)a}_{\;\;b}\chi^b$ lies on the tangent space of the Killing horizon $\mathcal{H}^{(K)}$ and hence is either null or spacelike. From the invariance of the matter action under conformal transformations,

$$\tilde{\mathcal{A}}^{(m)} = \int d^4x\sqrt{-\tilde{g}}\tilde{\mathcal{L}}^{(m)} = \int d^4x\sqrt{-g}\mathcal{L}^{(m)} = \mathcal{A}^{(m)}\,, \tag{5.78}$$

we necessarily have the following relation between the matter (and non-gravitational) Lagrangians between the Einstein and the Jordan frames, under the conformal transformation rule (5.3),

$$\tilde{\mathcal{L}}^{(m)} = \Omega^{-4}\mathcal{L}^{(m)}\,. \tag{5.79}$$

From the definition of the matter energy momentum tensor,

$$\tilde{T}^{(m)}_{ab} = \frac{2}{\sqrt{-\tilde{g}}}\frac{\delta}{\delta\tilde{g}^{ab}}\left(\sqrt{-\tilde{g}}\,\tilde{\mathcal{L}}^{(m)}\right) = \Omega^{-4}\frac{\partial g^{cd}}{\partial\tilde{g}^{ab}}\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{cd}}\left(\sqrt{-g}\,\mathcal{L}^{(m)}\right), \tag{5.80}$$

we have the following relations between the matter energy momentum tensors in the two conformal frames,

$$\tilde{T}^{(m)}_{ab} = \Omega^{-2}T^{(m)}_{ab}\,, \qquad \tilde{T}^{(m)a}_{\;\;b} = \Omega^{-4}T^{(m)a}_{\;\;b}\,, \qquad \tilde{T}^{(m)ab} = \Omega^{-6}T^{(m)ab} \tag{5.81}$$

Now, since $\Omega^2 = \phi$ is a strictly positive function of the spacetime coordinates, we conclude via (5.81) that if the NDEC holds in the Einstein frame, then it must also necessarily hold in the Jordan frame. The vector field W^a defined as,

$$W^a \equiv -T^{(m)a}_{\;\;b}\chi^b\,, \tag{5.82}$$

is future directed timelike or null for any future directed null vector field $\vec{\chi}$. But as again, we have previously shown that $T^{(m)a}_{\;\;b}$ can either be null or spacelike. Hence the NDEC as applied to $\mathcal{H}^{(K)}$ forces $T^{(m)a}_{\;\;b}$ to be null on the Killing horizon and hence is collinear to its null generators,

$$-T^{(m)a}_{\;\;b}\chi^b \overset{\mathcal{H}^{(K)}}{=} \alpha\chi^a\,, \tag{5.83}$$

where α is some proportionality factor. Finally using the above relation in (5.73), we get to our desired goal,

$$\chi_{[a}\nabla_{b]}\kappa \overset{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.84}$$

So we have essentially established the constancy of the surface gravity κ over the Killing horizon $\mathcal{H}^{(K)}$ *i.e.* the zeroth law holds for the Killing horizon established in the Jordan frame under the constraint (5.46).

5.4.2 Approach II

Now we give a different proof of the zeroth law in the two frames considered. However, this proof also relies upon the dynamical content of the theory in the sense that the field equations are used under the fact that the NDEC holds in both the frames. The method we follow is adopted from [1]. Let us begin with very generic considerations in the sense that suppose our spacetime $(\mathcal{M}, g)$ admits a Killing vector field $\vec{\chi}$. The vector field $\vec{\chi}$ then generates the Killing horizon in the given spacetime, in the sense that $\vec{\chi}$ coincides with the null generators of the Killing horizon. The surface gravity κ of the Killing horizon $\mathcal{H}$ is defined as,

$$\kappa^2 \overset{\mathcal{H}}{=} -\frac{1}{2}\nabla_a\chi_b\nabla^a\chi^b \,. \tag{5.85}$$

We can show, without using the gravitational field equations, that κ is constant along the null generators. After all, this is to be expected since $\vec{\chi}$ is the symmetry generator of the horizon $\mathcal{H}$. Taking directional derivative of the above equation (5.85) along the null generators $\vec{\chi}$, we have,

$$2\kappa(\chi^i\nabla_i\kappa) \overset{\mathcal{H}}{=} -(\nabla^a\chi^b)(\chi^i\nabla_i\nabla_a\chi_b) \overset{\mathcal{H}}{=} -(\nabla^a\chi^b)R_{baid}\chi^i\chi^d \overset{\mathcal{H}}{=} 0\,. \tag{5.86}$$

Since κ is non-zero on the horizon (non-degenerate), we necessarily have,

$$(\chi^i\nabla_i\kappa) \overset{\mathcal{H}}{=} 0\,. \tag{5.87}$$

As a result, once we have established the fact that we have respective Killing horizons in the two frames, we should be content in proving the constancy of the surface gravity only along the spacelike directions of the submanifold $(\mathcal{J}, \boldsymbol{q})$. This is exactly the point where we will require the respective field equations in the two frames. As before, we assume the Killing horizon $\mathcal{H}$ has a topology of $\mathbb{R}\times\mathcal{J}$, where $\mathcal{J}$ is a spacelike closed manifold transverse to the null generators. We can establish the relation [1],

$$\mathcal{D}_c\kappa \overset{\mathcal{H}}{=} -R_{ab}\chi^a q^b{}_c\,, \tag{5.88}$$

where $\mathcal{D}_c$ denotes the spatial covariant derivative w.r.t the spacelike manifold $(\mathcal{J}, \boldsymbol{q})$ and $q^a{}_b = \delta^a{}_b + \chi^a k_b + k^a\chi_b$ denotes the induced metric on $\mathcal{J}$ with k^a being the auxiliary null vector transverse to $\mathcal{H}$.

5.4.2.1 Einstein frame

We now follow up with this in the Einstein frame $(\mathcal{M}, \tilde{g}, \tilde{\boldsymbol{\nabla}}, \tilde{\phi})$ where we have for the Killing horizon $\tilde{\mathcal{H}}^{(K)}$ generated by $\vec{\tilde{\chi}}$ (having the spacelike cross-section $(\tilde{\mathcal{J}}, \tilde{\boldsymbol{q}}, \tilde{\mathcal{D}})$),

$$\tilde{\mathcal{D}}_c\tilde{\kappa} \overset{\tilde{\mathcal{H}}^{(K)}}{=} -\tilde{R}_{ab}\tilde{\chi}^a\tilde{q}^b{}_c\,. \tag{5.89}$$

Using the field equations in the Einstein frame (5.5) its quite easy to show that,

$$\tilde{R}_{ab}\tilde{\chi}^a\tilde{q}^b{}_c = 16\pi\left(\frac{1}{2}\tilde{T}^{(m)}_{ab}\tilde{\chi}^a\tilde{q}^b_c + \frac{1}{2}(\tilde{\chi}^a\tilde{\nabla}_a\tilde{\phi})\,\tilde{\mathcal{D}}_c\tilde{\phi}\right). \tag{5.90}$$

Use of the constraint relation (5.48) allows us to have,

$$\tilde{\mathcal{D}}_c\tilde{\kappa} \overset{\tilde{\mathcal{H}}^{(K)}}{=} -8\pi\tilde{T}^{(m)}_{ab}\tilde{\chi}^a\tilde{q}^b{}_c\,. \tag{5.91}$$

Invoking the validity of the NDEC as applied to the Einstein frame, we have,

$$-\tilde{T}^{(m)}_{ab}\tilde{\chi}^a \overset{\tilde{\mathcal{H}}^{(K)}}{=} \tilde{\beta}\tilde{\chi}_b\,, \tag{5.92}$$

where $\tilde{\beta}$ is some proportionality factor. This further allows us to conclude that the R.H.S of (5.91) on the Killing horizon $\tilde{\mathcal{H}}^{(K)}$ is,

$$-8\pi\tilde{T}^{(m)}_{ab}\tilde{\chi}^a\tilde{q}^b{}_c \overset{\tilde{\mathcal{H}}^{(K)}}{=} -8\pi\tilde{\beta}\,\tilde{\chi}_b\tilde{q}^b{}_c = 0\,. \tag{5.93}$$

The last part comes from the fact that the null generator of $\tilde{\mathcal{H}}^{(K)}$ is orthogonal to the spacelike cross-section $\tilde{\mathcal{J}}$. So in essence, we have finally showed that in the Einstein frame, the zeroth law holds,

$$\tilde{\mathcal{D}}_c\tilde{\kappa} \overset{\tilde{\mathcal{H}}^{(K)}}{=} 0\,. \tag{5.94}$$

5.4.2.2 Jordan frame

We now proceed towards the Jordan frame where we have the relation established on the Killing horizon $\mathcal{H}^{(K)}$ (with the spacelike cross-section $(\mathcal{J}, \boldsymbol{q}, \mathcal{D})$),

$$\mathcal{D}_c\kappa \overset{\mathcal{H}^{(K)}}{=} -R_{ab}\chi^a q^b{}_c\,. \tag{5.95}$$

Again, using the field equations (5.2) for the Jordan frame and the constraint (5.46) it is quite easy to show that,

$$R_{ab}\chi^a q^b{}_c = \frac{1}{\phi}\left(8\pi T^{(m)}_{ab}\chi^a q^b{}_c + \chi^a q^b{}_c\nabla_a\nabla_b\phi\right). \tag{5.96}$$

The quantity $\chi^a q^b{}_c\nabla_a\nabla_b\phi$ vanishes on the Killing horizon $\mathcal{H}^{(K)}$,

$$\chi^a q^b{}_c\nabla_a\nabla_b\phi \overset{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.97}$$

This has has been shown in Appendix 5.10. This allows us again to have,

$$\mathcal{D}_c\kappa \overset{\mathcal{H}^{(K)}}{=} -\frac{1}{\phi}8\pi T^{(m)}_{ab}\chi^a q^b{}_c\,. \tag{5.98}$$

Similar validity of the NDEC in the Jordan frame allows us to establish the fact that the R.H.S of (5.98) vanishes on $\mathcal{H}^{(K)}$. Hence we finally establish the zeroth law as well in the Jordan frame.

$$\mathcal{D}_c\kappa \overset{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.99}$$

Since temperature is proportional to surface gravity, the above analysis shows that the temperature is constant over the horizon. This we have shown separately in both the frames.

Note that the zeroth law has been proved here for a Killing horizon. The Rigidity theorem [6, 20, 21] for the case of general relativity ensures that (under certain assumptions) that the event horizon of a stationary black hole is a Killing horizon. Till now, it is not known whether the rigidity theorem also holds for scalar tensor gravity theory. Therefore the present analysis is only valid for a Killing horizon only.

5.5 Conclusion

There has been much debate about the physical (in)equivalence of the Jordan and the Einstein frame and the question still remains as to what can be considered "more" physical than the other. Any establishment of (in)equivalences of physical and thermodynamical quantities can only help us to address such long-standing issues. Our present work has been focused in this particular direction aimed at the thermodynamic aspects of the gravitational theories in the two frames. In the earlier works, it had been shown in the context of Killing horizons present in the spacetime that the thermodynamic parameters are equivalent in the two frames. However, the presence of the Killing horizon imposes symmetry requirements on the spacetime. Moreover, since the Killing horizon describes a stationary equilibrium black hole system, the equivalence of such thermodynamic parameters is restricted only to equilibrium processes. However, it has been established that at least for Einstein gravity and the Lanczos-Lovelock gravity the gravitational field equations expressed in the neighborhood of a generic null hypersurface assumes a thermodynamic interpretation in analogy with the first law of thermodynamics. The presence of this generic null surface does not ask for any symmetry requirements on the spacetime. The resulting thermodynamical interpretation given under the context of virtual displacement of the null hypersurface incorporates both equilibrium as well as non-equilibrium processes. That is such an interpretation is capable of handing internal entropy generation due to dissipation or viscous effects under the process of virtual displacement [92].

We have shown that such an equivalence of the relevant thermodynamic parameters also exists in the case of ST theory of gravity. For this, we used the gravitational dynamics associated with the projection component $G_{ab}l^a k^b$ onto a generic null hypersurface established in both the Einstein and the Jordan frames. Through the process of virtual displacements, we connected the dynamics related to the component $G_{ab}l^a k^b$ to the relevant thermodynamic identities (established on the null hypersurface) in both the frames in a completely covariant fashion. We stress again that such an identity has been interpreted not via any coordinate system adapted to the null hypersurface (say the Gaussian null coordinate system). Our analysis has been done completely in a covariant fashion which allows us to provide covariant expressions of the relevant thermodynamical parameters, which can then be adapted to any coordinate system of the person's choice describing the spacetime in the neighborhood of the null surface. This allowed us to interpret from the analogous thermodynamical first law established in both the frames, that the quantities like temperature, entropy density, energy and the work function are equivalent in both the frames. Finally, this nicely ties in with another interpretation provided under the umbrella of the projection component $G_{ab}l^a q^b{}_c$, the dynamics associated to which leads to the Damour-Navier-Stokes equation. The equivalence of the relevant fluid variables (of the DNS equation) in the two frames had previously been established. Thus such fluid variables and thermodynamic parameters operate on an equal footing when the two frames are considered. This we hope lends much ground to the issue about the physical equivalences between the two frames.

Let us reinstate the fact that the above thermodynamical interpretation (using the field equations in the two frames) had been drawn based on analogy with conventional thermodynamics. This however does not allow concrete physical interpretation of the thermodynamic variables, especially the temperature. In conventional thermodynamics, the temperature is essentially an intensive variable whose constancy defines the notion of thermal equilibrium between two thermal systems under contact. This is essentially the statement of the zeroth law of thermodynamics. However, for gravitational dynamics, there is actually no notion of two black hole systems being in thermal equilibrium with each other. The zeroth law of black hole mechanics says that a black hole system in thermal equilibrium must be by definition a Killing horizon (defining a stationary black hole system) over which its surface gravity is constant. This constancy of the surface gravity allows us then to give a concrete identification and interpretation of the temperature associated with any generic null hypersurface. So our analysis would be quite well rounded if we could prove the zeroth law as established in Killing horizons in the two frames. The zeroth law for Scalar-Tensor theory had been established in the literature under the constraint that the scalar field needed to be constant over the Killing horizon. However, we believe that this is a bit too restrictive. In second part of our analysis, we showed that in order for the zeroth law to hold in both the frames, the only requirement we demand of the scalar field is for it to respect the symmetry of the given spacetime. That is, we only demanded that the scalar field is Lie-transported along the symmetry generator of the spacetime. This implies that on the Killing horizon the scalar field is independent of the coordinate along the null symmetry generator, but can very well depend on the angular/transverse coordinates. We did not put any extra symmetries on the spacetime other than to impose that the matter and non-gravitational fields in the two frames satisfy the NDEC.

Finally, we believe that our results based on the thermodynamic identity valid on any generic null hypersurface and the proof of the zeroth law in the two frames provide some clarifications into questions regarding their physical equivalences (or in-equivalences for that matter). It is worthwhile to mention that at the classical level a certain class of $f(R)$ gravity can be cast in the form of ST theories (as not possible in general; for instance see [195] and references therein). The thermodynamic structure of such $f(R)$ theories can be discussed along the present line of thought. We hope that our analysis will help shed more light onto the nature of physics in both the Einstein and the Jordan frame.

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Appendices

5.6 Equivalence of Eq. (5.10) and Eq. (5.8) under the conformal transformation

Using eq. (5.23), one obtains the following relations

$$\tilde{\mathcal{D}}_a\tilde{\Omega}^a = \frac{1}{\phi}\mathcal{D}_a\Omega^a + \frac{1}{2\phi}\left[\theta_l k^i + \theta_k l^i\right]\nabla_i(\ln\phi) + \frac{q^{ab}}{2\phi}\nabla_a\nabla_b(\ln\phi)\,, \tag{5.100}$$

$$\tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} = \frac{1}{\phi}\left[\theta_l\theta_k + \left(\theta_l k^i + \theta_k l^i\right)\nabla_i(\ln\phi) + l^i k^j\nabla_i(\ln\phi)\nabla_j(\ln\phi)\right], \tag{5.101}$$

$$\begin{aligned}\tilde{l}^i\tilde{\nabla}_i\tilde{\theta}_{\tilde{k}} =& \frac{1}{\phi}l^i\nabla_i\theta_k + \frac{1}{\phi}\left[\Omega^i - \kappa k^i - \theta_k l^i\right]\nabla_i(\ln\phi) - \frac{l^i k^j}{\phi}\nabla_i(\ln\phi)\nabla_j(\ln\phi)\\ &+ \frac{l^i k^j}{\phi}\nabla_i\nabla_j(\ln\phi)\,.\end{aligned} \tag{5.102}$$

Using (5.24), (5.25), (5.26), (5.100), (5.101) and (5.102) in (5.10) one obtains (5.8).

5.7 Obtaining Eq. (5.27) starting from Eq. (5.8)

We start from (5.8) and write each parameters of the Jordan frame (such as θ_l, θ_k, κ, Ω^a *etc.*), in terms of the parameters of the Einstein frame (such as $\tilde{\theta}_{\tilde{l}}$, $\tilde{\theta}_{\tilde{k}}$, $\tilde{\kappa}$, $\tilde{\Omega}^a$ *etc.*) using Eq. (5.23). We consider term-by-term of Eq. (5.8) and obtain the following relations for each term

$$\kappa\theta_k = \phi\tilde{\kappa}\tilde{\theta}_{\tilde{k}} - \tilde{\kappa}k^i\nabla_i(\ln\phi) - \phi\tilde{\theta}_{\tilde{k}}l^i\nabla_i(\ln\phi) + l^i k^j\nabla_i(\ln\phi)\nabla_j(\ln\phi)\,. \tag{5.103}$$

$$\mathcal{D}_a\Omega^a = \phi\mathcal{D}_a\tilde{\Omega}^a + \tilde{\Omega}^a\mathcal{D}_a\phi - \frac{1}{2}\mathcal{D}^a\mathcal{D}_a(\ln\phi)\,. \tag{5.104}$$

$$\theta_l\theta_k = \phi\tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} - \phi\tilde{\theta}_{\tilde{k}}l^i\nabla_i(\ln\phi) - \tilde{\theta}_{\tilde{l}}k^i\nabla_i(\ln\phi) + l^i k^j\nabla_i(\ln\phi)\nabla_j(\ln\phi)\,. \tag{5.105}$$

Now, we know that $\mathcal{D}^a\mathcal{D}_a(\ln\phi) = q^i_j\nabla_i(q^{jk}\nabla_k(\ln\phi))$, from which it can be obtained that

$$\mathcal{D}^a\mathcal{D}_a(\ln\phi) = q^{ij}\nabla_i\nabla_j(\ln\phi) + [\tilde{\theta}_{\tilde{l}}k^i + \tilde{\theta}_{\tilde{k}}l^i]\nabla_i(\ln\phi)\,. \tag{5.106}$$

Writing θ_l and θ_k in terms of $\tilde{\theta}_{\tilde{l}}$ and $\tilde{\theta}_{\tilde{k}}$, one further obtains

$$\mathcal{D}^a\mathcal{D}_a(\ln\phi) = q^{ij}\nabla_i\nabla_j(\ln\phi) + \phi\tilde{\theta}_{\tilde{k}}l^i\nabla_i(\ln\phi) + \tilde{\theta}_{\tilde{l}}k^i\nabla_i(\ln\phi) - 2l^ik^j\nabla_i(\ln\phi)\nabla_j(\ln\phi)\,. \tag{5.107}$$

Using (5.107) in (5.105), one obtains

$$\theta_l\theta_k = \phi\tilde{\theta}_{\tilde{l}}\tilde{\theta}_{\tilde{k}} - \mathcal{D}^a\mathcal{D}_a(\ln\phi) + q^{ij}\nabla_i\nabla_j(\ln\phi) - l^ik^j\nabla_i(\ln\phi)\nabla_j(\ln\phi)\,. \tag{5.108}$$

Writing Ω^a in terms of $\tilde{\Omega}^a$ we obtain

$$\Omega^a\Omega_a = \phi\tilde{\Omega}^a\tilde{\Omega}_a - \phi\tilde{\Omega}^i\nabla_i(\ln\phi) + \frac{1}{4}q^{ij}\nabla_i(\ln\phi)\nabla_j(\ln\phi)\,. \tag{5.109}$$

Finally we obtain

$$l^i\nabla_i\theta_k = \phi l^i\nabla_i\tilde{\theta}_{\tilde{k}} + \phi\tilde{\theta}_{\tilde{k}}l^i\nabla_i(\ln\phi) - \phi\tilde{\Omega}^i\nabla_i(\ln\phi) + \frac{1}{2}q^{ij}\nabla_i(\ln\phi)\nabla_j(\ln\phi) + \phi\tilde{\kappa}\tilde{k}^i\nabla_i(\ln\phi) - l^ik^j\nabla_i(\ln\phi)\nabla_j(\ln\phi) - l^ik^j\nabla_i\nabla_j(\ln\phi)\,. \tag{5.110}$$

Now, using (5.103), (5.104), (5.108), (5.109) and (5.110) in Eq. (5.8), one obtains Eq. (5.27).

5.8 Proof the zeroth law for static spacetimes

We assume the spacetime $(\mathcal{M}, g, \boldsymbol{\nabla})$ to be static *i.e.* both stationary (admitting a Killing vector field $\vec{\chi}$) and the Killing vector field $\vec{\chi}$ as being hypersurface-orthogonal. The hypersurface orthogonality condition implies that over the manifold we have,

$$\chi_{[c}\nabla_b\chi_{a]} = \chi_c\nabla_b\chi_a + \chi_b\nabla_a\chi_c + \chi_a\nabla_c\chi_b = 0\,. \tag{5.111}$$

The above relation is valid on the manifold and not just only on the Killing horizon $\mathcal{H}^{(K)}$ in the given spacetime. Hence we can safely take the derivative of the above relation:

$$\nabla^a\Big[\chi_c\nabla_b\chi_a + \chi_b\nabla_a\chi_c + \chi_a\nabla_c\chi_b\Big] = 0\,. \tag{5.112}$$

Using the Killing equation $\nabla_a\chi_b + \nabla_b\chi_a = 0$, its quite easy to show that the above relation reduces to,

$$\chi_c\nabla_a\nabla_b\chi^a + \chi_b\Box\chi_c + \chi^a\nabla_a\nabla_c\chi_b = 0\,. \tag{5.113}$$

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We now define the following quantity P_a to be,

$$P_a = R^f{}_a \chi_f \, . \tag{5.114}$$

Simple manipulations allow us to have,

$$\begin{aligned} P_a = R^f{}_a \chi_f = [\nabla_b, \nabla_a]\chi^b \\ = \nabla_b \nabla_a \chi^b - \nabla_a \nabla_b \chi^b = \nabla_b \nabla_a \chi^b = -\nabla_b \nabla^b \chi_a = -\Box \chi_a \quad . \end{aligned} \tag{5.115}$$

Use of Eq. (5.115) in Eq. (5.113) yields,

$$\chi_c P_b - \chi_b P_c = -\chi^a \nabla_a \nabla_c \chi_b \, . \tag{5.116}$$

Since $\vec{\chi}$ is a symmetry generator of the spacetime, any tensor constructed out of $\vec{\chi}$ and g_{ab} will also respect the spacetime symmetry. Such a tensor field is the quantity $T_{cb} = \nabla_c \chi_b$. Explicitly, this means that,

$$\pounds_\chi T_{cb} = 0 \, . \tag{5.117}$$

This follows that,

$$\begin{aligned} \chi^a \nabla_a \nabla_c \chi_b + \nabla_a \chi_b (\nabla_c \chi^a) + \nabla_c \chi_a (\nabla_b \chi^a) = 0 \, , \\ \chi^a \nabla_a \nabla_c \chi_b = 0 \, . \end{aligned} \tag{5.118}$$

Use of Eq. (5.118) in Eq. (5.116) implies,

$$\chi_c P_b - \chi_b P_c = 0 \, . \tag{5.119}$$

Now, as applied onto the Killing horizon $\mathcal{H}^{(K)}$, we have the relation (5.51), upon which using Eq. (5.119) leads to,

$$\begin{aligned} \chi_d \, \chi_{[a} \nabla_{b]} \kappa &\overset{\mathcal{H}^{(K)}}{=} -\chi_d \chi_f R^f{}_{[b} \chi_{a]} \\ &\overset{\mathcal{H}^{(K)}}{=} -\frac{\chi_d}{2} \left(\chi_f R^f{}_b \chi_a - \chi_f R^f{}_a \chi_b \right) \\ &\overset{\mathcal{H}^{(K)}}{=} -\frac{\chi_d}{2} (P_b \chi_a - P_a \chi_b) = 0 \, . \end{aligned} \tag{5.120}$$

Since we assume our Killing horizon to be non-degenerate, we essentially have $\chi_{[a} \nabla_{b]} \kappa \overset{\mathcal{H}^{(K)}}{=} 0$. This essentially proves the zeroth law for static spacetimes admitting a Killing horizon without the need of the dynamical field equations.

5.9 Proof of the relation (5.72)

We begin by showing that the Lie derivative of $\nabla_a\phi$ along the null generator of $\vec{\chi}$ vanishes over $(\mathcal{M}, g, \boldsymbol{\nabla}, \phi)$ using the constraint (5.46),

$$\begin{aligned}\pounds_\chi(\nabla_a\phi) =& \chi^f\nabla_a\nabla_f\phi + (\nabla_f\phi)(\nabla_a\chi^f)\\ =& \nabla_a(\chi^f\nabla_f\phi) - (\nabla_a\chi^f)(\nabla_f\phi) + (\nabla_f\phi)(\nabla_a\chi^f) = 0\,.\end{aligned} \tag{5.121}$$

We have then,

$$\chi^f(\chi_{[a}\nabla_{b]}\nabla_f\,\phi) = \frac{1}{2}\Big(\chi^f\chi_a\nabla_b\nabla_f\phi - \chi^f\chi_b\nabla_a\nabla_f\phi\Big)\,. \tag{5.122}$$

Using the first line of (5.121) we have $\chi_b\chi^f\nabla_f\nabla_a\phi = -\chi_b(\nabla_c\phi)(\nabla_a\chi^c)$. Putting this in the second term of above relation, we have,

$$\begin{aligned}\chi^f(\chi_{[a}\nabla_{b]}\nabla_f\,\phi) =& \frac{1}{2}\Big(\chi_a\nabla_b(\chi^f\nabla_f\phi) - \chi_a(\nabla_b\chi^f)(\nabla_f\phi) + \chi_b(\nabla_f\phi)(\nabla_a\chi^f)\Big)\\ =& \frac{1}{2}\nabla^f\phi\Big(\chi_a\nabla_f\chi_b + \chi_b\nabla_a\chi_f\Big)\,.\end{aligned} \tag{5.123}$$

Let us then invoke the hypersurface orthogonality of the integrable null hypersurface $\mathcal{H}^{(K)}$ generated by the null vector field $\vec{\chi}$ in the absence of torsion,

$$\chi_a\nabla_f\chi_b + \chi_f\nabla_b\chi_a + \chi_b\nabla_a\chi_f \overset{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.124}$$

Using the relation (5.124), we have,

$$\chi^f(\chi_{[a}\nabla_{b]}\nabla_f\,\phi) \overset{\mathcal{H}^{(K)}}{=} -\frac{1}{2}\Big(\chi^f\nabla_f\phi(\nabla_b\chi_a)\Big) \overset{\mathcal{H}^{(K)}}{=} 0\,. \tag{5.125}$$

This proves our desired relation.

5.10 Proof of the relation (5.97)

Next we proceed to give a proof of (5.97). Using the relation (5.121), we have on the Killing horizon $\mathcal{H}^{(K)}$,

$$\begin{aligned}\chi^a q^b{}_c\nabla_a\nabla_b\phi =& -q^b{}_c(\nabla_a\phi)(\nabla_b\chi^a) = -(\delta^b{}_c + \chi^b k_c + k^b\chi_c)(\nabla_a\phi)(\nabla_b\chi^a)\\ =& -\nabla^a\phi\Big(\nabla_c\chi_a + \kappa k_c\chi_a + k^b\chi_c\nabla_b\chi_a\Big)\,.\end{aligned} \tag{5.126}$$

From the hypersurface orthogonality condition of the Killing horizon (5.124), we have,

$$\begin{aligned}\chi^a q^b{}_c \nabla_a \nabla_b \phi &\overset{\mathcal{H}^{(K)}}{=} -\nabla^a \phi \Big(\nabla_c \chi_a + \kappa k_c \chi_a - k_b (\chi_b \nabla_a \chi_c + \chi_a \nabla_c \chi_b) \Big) \\ &\overset{\mathcal{H}^{(K)}}{=} -\nabla^a \phi \Big(\nabla_c \chi_a + \kappa k_c \chi_a + \nabla_a \chi_c - (k^b \nabla_c \chi_b) \chi_a \Big) \,. \end{aligned} \tag{5.127}$$

Use of the fact that $\vec{\chi}$ is a symmetry generator of $\mathcal{H}^{(K)}$ and the constraint condition (5.46), allows us to have,

$$\chi^a q^b{}_c \nabla_a \nabla_b \phi \overset{\mathcal{H}^{(K)}}{=} 0 \,. \tag{5.128}$$

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Chapter 6

Thermodynamic structure of Einstein-Cartan gravity via a generic null hypersurface

6.1 Introduction and Motivation

In the past two chapters, we have laid out the completely co-ordinate independent methodology of interpreting the particular dynamics of *any* gravity theory whose solution space is the spacetime manifold endowed with a Levi-Civita connection $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ with that of having a given thermodynamic interpretation. For this we needed to project the relevant dynamics onto a generic integrable null surface $\mathcal{H}$. We also saw how this methodology can be applied to the case of ST theory to elucidate the (in)equivalences between the Einstein and the Jordan frame. Next, to move ahead with the discussion, as mentioned in Sec. 4.1, the field equations of Einstein gravity render themselves a fluid dynamic interpretation on a generic null surface. Moreover, it was shown by Jacobson [35] that the Einstein field equations could be "derived" from an equilibrium local thermodynamic constitutive relation (Clausius identity) applied to approximate Rindler horizons constructed at any point of the spacetime (endowed with the Levi-Civita connection).

These vivid connections of gravitational dynamics with thermodynamics and fluid equation established in the context of generic null hypersurfaces form the motivation for thinking about gravity as an *"emergent phenomenon"*.[1] In a paradigm shift, the "emergent gravity" program considers gravitational dynamics to be not fundamental. Rather it considers gravity to be *emergent* from fundamental degrees of freedom associated with the gravitational field [45, 46, 151, 196–198]. That is, gravity and its dynamics emerge much like thermodynamics of matter arises as an effective theory from the statistical mechanics of its constituent atoms.

Under this point of view, if gravity is indeed emergent (as seen especially for Einstein gravity), then the connections between gravitational dynamics and thermodynamics

[1] Emergent nature of gravity was initially introduced in 1967 by Sakharov [192].

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should indeed transcend to other theories of gravity. Here, in our case we take the example of Einstein-Cartan theory [199]. The EC theory is built in the geometrical backdrop of the Riemann-Cartan (RC) spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. The EC theory is a natural extension of Einstein gravity obtained by including the intrinsic spin of the particle(s) in the geometrization of spacetime [88]. The presence of intrinsic spin causes non-zero torsion in the spacetime geometry and the relevant gravitational field equations are the Einstein-Cartan-Kibble-Sciama (ECKS) equations [200–204] (for textbook expositions and reviews see [87, 129, 205–208]). The presence of spin allows for a non-zero spin angular momentum tensor in addition to the energy-momentum tensor. In the macroscopic classical domain, the spin degrees of freedom cancel out due to their dipole nature and hence dynamics of macroscopic bodies are characterized by the energy-momentum tensor alone. However, in the microscopic regime, one cannot ignore the spin angular momentum which actually "sources" torsion as being a geometric field in the spacetime in addition to the metric tensor.

In this chapter, we aim to address exclusively the question whether the projection component $\hat{G}_{ab}k^a l^b$ on $\mathcal{H}$ in the EC theory can be provided a thermodynamic interpretation in a completely covariant way. Here, $\hat{G}_{ab}$ is the analogue (not symmetric) of the Einstein tensor in the RC spacetime. So in this chapter, we are effectively moving to a gravity theory whose solution space is not endowed with the Levi-Civita connection $\boldsymbol{\nabla}$. Let us pause to mention that the component $\hat{G}_{ab}l^a l^b$ is related to the NRE determining the dynamical evolution of the outgoing expansion scalar. The corresponding NRE in the RC spacetime has been derived in Eq. (3.8). The evolution equations, corresponding to the expansion, shear and vorticity, for congruences of both timelike as well as null curves in spacetimes with torsion (under different assumptions on the nature of the torsion) have been provided in [64, 137, 209, 210]. The thermodynamic interpretation for $\hat{G}_{ab}l^a l^b$ has been provided in [64] and hence will not be pursued here. In order to provide a coordinate-independent thermodynamic interpretation to the gravitational field equations in the EC theory via the projection component $\hat{G}_{ab}k^a l^b$ (onto $\mathcal{H}$), we would as usual require the dynamical evolution of the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_k}$ along the null generators $\vec{l}$. This has been explicitly derived in Eq. (3.24). However, there is crucial restriction. The NRE for the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_k}$ *i.e.* Eq. (3.24) has been derived under the *geodesic constraint* Eq. (2.34). We will see that with the help of the notion of virtual displacement [2] applied to this evolution equation and the relevant field equations, we provide a covariant thermodynamic interpretation to $\hat{G}_{ab}k^a l^b$. In doing so, we will be able to access how the thermodynamic parameters and their interpretations are affected by the inclusion of torsion under the geodesic constraint.

[2]In this context, the concept of virtual displacement was initially introduced in [211].

The organization of this chapter is as follows. In Sec. 6.2, we very briefly discuss about ECKS gravitational field equations in the EC theory. In Sec. 6.3, we begin our in-depth study of the thermodynamic interpretation provided to $\hat{G}_{ab}k^a l^b$ and also discuss some special cases. Finally, we conclude in Sec. 6.4.

6.2 A brief review of the gravitational field equations for the EC theory

The gravitational action in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ will henceforth be referred to as the Einstein-Cartan action $\mathcal{A}_{\rm EC}$. For details refer to [87, 207, 212]. In this theory both the metric and the torsion tensors are treated as independent dynamical variables. The total action for the theory is,

$$\mathcal{A}_{\rm tot} = \mathcal{A}_{\rm EC} + \mathcal{A}_{\rm m} = \frac{1}{16\pi}\int_{\mathcal{V}} d^4x\sqrt{-g}\hat{R} + \mathcal{A}_{\rm m}\,, \tag{6.1}$$

where, $\mathcal{A}_{\rm m}$ is the corresponding matter or non-gravitational action. Obviously, the above action is extremized by varying w.r.t. both the metric and the torsion (preferably here the contorsion tensor) to yield the field equations. The Einstein-Cartan-Sciama-Kibble field equation (by varying w.r.t. the metric) is [87, 207, 212],

$$\hat{G}_{ab} + \frac{1}{2}(\hat{\nabla}_c + T_c)\Big(-S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c\Big) = 8\pi T^{(m)}_{ab}\,, \tag{6.2}$$

where $T^{(m)}_{ab}$ is the matter stress energy-momentum tensor. The field equation obtained by extremizing the total action w.r.t. the contorsion tensor is,

$$S^a{}_{bc} = 8\pi\tau^a{}_{bc}\,, \tag{6.3}$$

where $\tau^a{}_{bc}$ is the spin angular momentum tensor. Hence given a matter Lagrangian depending upon the metric, the matter field and its first derivative, the variation of the matter action is given as,

$$\delta\mathcal{A}_{\rm m} \equiv -\frac{1}{2}\int_{\mathcal{V}} d^4x\sqrt{-g}\Big[T^{(m)}_{ab}\delta g^{ab} + \tau^{b}{}_{a}{}^{c}\,\delta K^a{}_{bc}\Big]\,. \tag{6.4}$$

This indicates that the matter energy-momentum tensor $T^{(m)}_{ab}$ is symmetric whereas the spin angular momentum tensor $\tau^a{}_{bc}$ is antisymmetric in the last two indices. In anticipation of the result we are trying to achieve, let us state the following identity,

$$\Big(\hat{\nabla}_a T_b - \hat{\nabla}_b T_a\Big) + \Big(\hat{\nabla}_i + T_i\Big)T^i{}_{ab} = \Big(\hat{\nabla}_i + T_i\Big)S^i{}_{ab}\,. \tag{6.5}$$

The above result can quite easily be verified by using the definition of the modified torsion tensor (2.7). Upon using (6.5) in (6.2), we obtain,

$$\hat{G}_{ab} + \left(\hat{\nabla}_a T_b - \hat{\nabla}_b T_a\right) + \left(\hat{\nabla}_i + T_i\right) T^i{}_{ab} = 8\pi T^{(m)}_{ab} + \frac{1}{2}(\hat{\nabla}_i + T_i)\left[3S^i{}_{ab} + S_a{}^i{}_b + S_b{}^i{}_a\right]. \tag{6.6}$$

Using (6.3), the last term on the right hand side (R.H.S) of the above equation can be expressed in terms of the spin angular momentum tensor. This form of the gravitational field equation (6.6) will be used later in our analysis.

6.3 Thermodynamic interpretation provided to the NRE (3.24) via virtual displacement $\delta\lambda_{(k)}$

In Eq. (3.24), we had derived the evolution dynamics of the expansion scalar of the ingoing auxiliary null vector field $\vec{k}$ along the null generators $\vec{l}$ and showed that as a purely geometric relationship, it is related to the projection component $\hat{G}_{ab}k^a l^b$. Let us try to motivate the reason of this particular choice of arranging the terms in Eq. (3.24). Notice that all the terms in the first squared parentheses for the R.H.S of Eq. (3.24) except for $l^r\hat{\nabla}_r\left(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\right)$ contains geometrical/kinematical quantities that are defined on the transverse 2-surface S_t or the null surface $\mathcal{H}$. The terms in the second squared parentheses for the R.H.S of (3.24) involves rather quantities defined for the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ and are not restricted to S_t or $\mathcal{H}$.

Till this point, we have not used the dynamics of the gravitational field equations. We will now use the ECKS field equation corresponding to the metric tensor. We will rather use the form given in (6.6). Use of this in (3.24), we have,

$$\begin{aligned}
-\kappa\left(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\right) = &\left[\frac{1}{2}{}^{(2)}\hat{R} + l^r\hat{\nabla}_r\left(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\right) - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a\right. \\
&- \hat{\mathcal{D}}_a(\hat{\Omega}^a - \hat{\mathscr{P}}^a) + \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}\left(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\right) - \left(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} q^{cd} - \hat{\Xi}^{cd}\right)(T_{cfd}l^f) \\
&\left. - \left(q^{dj}q^{ci} - q^{cd}q^{ij}\right)(T_{cbd})(K_{aij})k^a l^b\right] \\
&- \left[8\pi T^{(m)}_{ab} + \frac{1}{2}(\hat{\nabla}_i + T_i)\left(3S^i{}_{ab} + S_a{}^i{}_b + S_b{}^i{}_a\right)\right]k^a l^b .
\end{aligned} \tag{6.7}$$

Before proceeding to interpret (6.7) as a thermodynamic identity established on the generic null surface $\mathcal{H}$, it is necessary to convince ourselves that $\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}$ and $\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}$ indeed represent the expansion scalars of the auxiliary null field $\vec{k}$ and the null generators $\vec{l}$ respectively. This we have already done in Sec. 2.5.1 arriving at Eqs (2.159) and (2.160).

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Now, we come to the point where we discuss the physical process under which a thermodynamic interpretation can be alluded to. The physical process is a virtual displacement $\delta\lambda_{(k)}$ along the auxiliary null vector field. The notion of virtual displacement has been adopted from the analysis in [36]. The virtual displacement basically shifts our null hypersurface $\mathcal{H}$ along $\vec{k}$. Consider the foliation of $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ in the neighborhood of $\mathcal{H}$ by the null family $\mathcal{H}_u$. Let us suppose that $\mathcal{H}$ is stationed at the value of $\lambda_{(k)} = 0$ and in the null family, there exists another surface at the value of $\lambda_{(k)} = \delta\lambda_{(k)}$. Of course, both of these null surfaces are solutions of the Einstein-Cartan spacetime. The virtual displacement is the physical process that shifts us from the null surface at $\lambda_{(k)} = 0$ to $\lambda_{(k)} = \delta\lambda_{(k)}$. Let us multiply then both sides of Eq. (6.7) with $\delta\lambda_{(k)}$ along with a multiplicative factor of $\frac{1}{8\pi}$. We integrate the resulting equation on the 2-dimensional spacelike cross-section S_t of $\mathcal{H}$. This results in,

$$
\begin{aligned}
&-\int_{S_t} d^2x\sqrt{q}\Big[\frac{\kappa}{2\pi}\Big(\frac{1}{4}\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - \frac{1}{4}q^{ij}T_{ihj}k^h\Big)\Big]\delta\lambda_{(k)} \\
&= \int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi}\Big[\frac{1}{2}{}^{(2)}\hat{R} + l^r\hat{\nabla}_r\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a \\
&- \hat{\mathcal{D}}_a\Big(\hat{\Omega}^a - \hat{\mathscr{P}}^a\Big) + \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}q^{cd} - \hat{\Xi}^{cd}\Big)(T_{fd}l^f) \\
&- \Big(q^{dj}q^{ci} - q^{cd}q^{ij}\Big)(T_{cbd})(K_{aij})k^a l^b\Big]\delta\lambda_{(k)} \\
&- \int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi}\Big[8\pi T^{(m)}_{ab} + \frac{1}{2}(\hat{\nabla}_i + T_i)\Big(3S^i{}_{ab} + S_a{}^i{}_b + S_b{}^i{}_a\Big)\Big]k^a l^b\delta\lambda_{(k)}\,. \qquad (6.8)
\end{aligned}
$$

Let us now focus on the term in the L.H.S of Eq. (6.8). We can rewrite it as,

$$
\begin{aligned}
&-\int_{S_t} d^2x\sqrt{q}\Big[\frac{\kappa}{2\pi}\Big(\frac{1}{4}\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - \frac{1}{4}q^{ij}T_{ihj}k^h\Big)\Big]\delta\lambda_{(k)} \\
&= \int_{S_t} d^2x\frac{\kappa}{2\pi}\Big[\sqrt{q}\frac{1}{\sqrt{q}}\frac{d}{d\lambda_{(k)}}\Big(\frac{\sqrt{q}}{4}\Big) + \sqrt{q}\frac{1}{4}q^{ij}T_{ihj}k^h\Big]\delta\lambda_{(k)} \\
&= \int_{S_t} d^2x\frac{\kappa}{2\pi}\Big[\delta\lambda_{(k)}\frac{d}{d\lambda_{(k)}}\Big(\frac{\sqrt{q}}{4}\Big) + \delta\lambda_{(k)}\Big(\frac{\sqrt{q}}{4}q^{ij}T_{ihj}k^h\Big)\Big] \\
&= \int_{S_t} d^2x\, T\Big(\delta_{\lambda_{(k)}}s_{\text{null}} + \delta_{\lambda_{(k)}}s_{\text{tor}}\Big)\,. \qquad (6.9)
\end{aligned}
$$

Here, we identify the temperature associated with the null surface under the process of virtual displacement $\delta\lambda_{(k)}$ to be $T = (\kappa/2\pi)$. We postulate that the variation of the total entropy density occurs from two contributions. First is the entropy generation term of the null surface itself. The entropy density of the null surface $\mathcal{H}$ is proportional to the area element $\sqrt{q}$ of the 2-surface S_t *i.e.* $s_{\text{null}} = \sqrt{q}/4$.[3] This part of the entropy generation is purely due to the variation of cross-sectional transverse area elements S_t as we move in the transverse $\vec{k}$ direction under the virtual displacement. But this not the end of the story. Due to the presence of non-trivial torsion in the spacetime, there happens to be another entropy generation term $\delta_{\lambda_{(k)}} s_{\text{tor}}$. In order to have an understanding for the source of it, let us consider the following particular torsion current $T_i{}^h{}_j q^{ij}$. Obviously the quantity $q^{ij}T_{ihj}k^h$ is negative the component of this torsion current along the null generators $\vec{l}$. We here define the entropy variation under the virtual displacement $\delta\lambda_{(k)}$ due to presence of this non-trivial torsion current $T_i{}^h{}_j q^{ij}$ to be $\delta_{\lambda_{(k)}} s_{\text{tor}}$:

$$\delta_{\lambda_{(k)}} s_{\text{tor}} = \delta\lambda_{(k)}\left(\frac{\sqrt{q}}{4} q^{ij}T_{ihj}k^h\right) . \tag{6.10}$$

Thus we see that there exists two causes of entropy generation under the virtual displacement $\delta\lambda_{(k)}$. One arises primarily due to the variation of the transverse cross-sectional area element S_t. The other arises due to a non-trivial torsion current flowing along the null generators. We will have something more to say on this at the end of this section.

Having done this, let us now look at the first term in the R.H.S of (6.8). We identify this term to be the variation of energy $\delta_{\lambda_{(k)}} E$ associated with the physical process of virtual displacement $\delta\lambda_{(k)}$,

$$\begin{aligned}\delta_{\lambda_{(k)}} E = \int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)}\frac{1}{8\pi}\Big[&\frac{1}{2}{}^{(2)}\hat{R} + l^r\hat{\nabla}_r\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a \\ &- \hat{\mathcal{D}}_a\Big(\hat{\Omega}^a - \hat{\mathscr{P}}^a\Big) + \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} q^{cd} - \hat{\Xi}^{cd}\Big)(T_{cfd}l^f) \\ &- \Big(q^{dj}q^{ci} - q^{cd}q^{ij}\Big)(T_{cbd})(K_{aij})k^a l^b\Big] .\end{aligned} \tag{6.11}$$

We can in principle perform an integration over the non-affine parameter $\lambda_{(k)}$ of the auxiliary null field to provide an expression of the energy associated with the null surface $\mathcal{H}$,

$$\begin{aligned}E = \int d\lambda_{(k)} \int_{S_t} d^2x\sqrt{q}\frac{1}{8\pi}\Big[&\frac{1}{2}{}^{(2)}\hat{R} + l^r\hat{\nabla}_r\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a \\ &- \hat{\mathcal{D}}_a\Big(\hat{\Omega}^a - \hat{\mathscr{P}}^a\Big) + \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big) - \Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} q^{cd} - \hat{\Xi}^{cd}\Big)(T_{cfd}l^f) \\ &- \Big(q^{dj}q^{ci} - q^{cd}q^{ij}\Big)(T_{cbd})(K_{aij})k^a l^b\Big] .\end{aligned} \tag{6.12}$$

Let us reiterate that our aim is to provide a thermodynamic interpretation to the NRE

[3] Same identification of entropy has been done in [213] through Noether prescription on a Killing horizon.

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(corresponding to the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_k}$) in analogy with the first law of thermodynamics. That would be complete, if we have the liberty to interpret the following expression to be the pressure term P,

$$P \equiv -\frac{1}{8\pi}\left[8\pi T^{(m)}_{ab} + \frac{1}{2}(\hat{\nabla}_i + T_i)\left(3S^i{}_{ab} + S_a{}^i{}_b + S_b{}^i{}_a\right)\right]k^a l^b\,. \tag{6.13}$$

The force F conjugate to the physical process of virtual displacement $\delta\lambda_{(k)}$ is simply then the integral of the pressure term over the transverse surface S_t,

$$F = \int_{S_t} d^2x\sqrt{q}\,P\,. \tag{6.14}$$

Now once this interpretation is allowed (we will try to justify this shortly), the process of virtual displacement of the null surface $\mathcal{H}$ along the auxiliary null field described via (6.8) can be succinctly restated as,

$$\int_{S_t} d^2x\; T\left(\delta_{\lambda_{(k)}} s_{\text{null}} + \delta_{\lambda_{(k)}} s_{\text{tor}}\right) = \delta_{\lambda_{(k)}} E + F\delta\lambda_{(k)}\,. \tag{6.15}$$

The above interpretation is made possible only under a virtual displacement of the null hypersurface $\mathcal{H}$ in the auxiliary null field $\vec{k}$ direction. The virtual displacement is to be thought of as a physical process that "virtually" shifts the position of $\mathcal{H}$ from stationed at $\lambda_{(k)} = 0$ to the position at say $\lambda_{(k)} = \delta\lambda_{(k)}$. The virtual work done under this process is $F\delta\lambda_{(k)}$. As a result of this, an amount of energy $\delta_{\lambda_{(k)}} E$ sweeps through the null surface. The corresponding change in the heat energy is $\int_{S_t} d^2x T(\delta_{\lambda_{(k)}} s_{\text{null}} + \delta_{\lambda_{(k)}} s_{\text{tor}})$.

Let us now describe the motivation behind the pressure term (6.13). The pressure term contains the term $-T^{(m)}_{ab}k^a l^b$. In the case of Einstein and Lanczos-Lovelock gravity, this particular term has been consistently identified as the pressure under the process of virtual displacement [36, 74, 79]. For static spherically symmetric spacetimes, this particular term has the value $-T^{(m)}_{ab}k^a l^b = T^{(m)r}{}_r$, which has the interpretation of being the radial or the normal pressure[74, 75]. However, when dealing with the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, we see that there are necessarily extra terms in the pressure. Notice that there are quadratic terms involving the torsion [and hence the modified torsion which can then be related to the spin angular momentum tensor via the field equation (6.3)]. For example consider the following term in the pressure,

$$-\frac{1}{8\pi}\frac{1}{2}T_i\left(3S^i{}_{ab} + S_a{}^i{}_b + S_b{}^i{}_a\right)k^a l^b = -8\pi\frac{1}{4}g^{ac}\tau_{aci}\left(3\tau^i{}_{ab} + \tau_a{}^i{}_b + \tau_b{}^i{}_a\right)k^a l^b\,. \tag{6.16}$$

In arriving at the above relation, we have used (6.3) and the fact that $T_i = \frac{1}{2}g^{ac}S_{aci} = \frac{1}{2}S^a{}_{ai}$. Such quadratic terms in the spin tensor actually represent spin-spin contact interaction and hence produce a correction to the matter energy-momentum tensor [87]. Our definition of the pressure involves such spin-spin interaction terms in addition to the matter energy-momentum tensor. In addition to the energy-momentum tensor and the spin-spin contact interaction terms we also have a derivative of modified torsion tensors in the pressure term *i.e.* $-\frac{1}{8\pi}\frac{1}{2}\hat{\nabla}_i(3S^i{}_{ab}+S_a{}^i{}_b+S_b{}^i{}_a)k^a l^b = -\frac{1}{2}\hat{\nabla}_i(\tau^i{}_{ab}+\tau_a{}^i{}_b+\tau_b{}^i{}_a)k^a l^b$. In Chapter 4, while analyzing the thermodynamic interpretation provided to a generic null surface (under virtual displacement) in Riemannian spacetimes without any torsion, we described the notion of a "gravitational pressure" defined as $P = -\frac{1}{8\pi}G_{ab}k^a l^b$. Hence for a generic null surface in Einstein gravity $P = -\frac{1}{8\pi}G_{ab}k^a l^b = -T^{(m)}_{ab}k^a l^b$ *i.e.* the field equation gives rise to the pressure term. In the same spirit, we identify the pressure as $-\frac{1}{8\pi}[\hat{G}_{ab}+(\hat{\nabla}_a T_b-\hat{\nabla}_b T_a)+(\hat{\nabla}_i+T_i)T^i{}_{ab}]k^a l^b$. Once the ECKS equation (6.6) is used on this it actually reduces to the value of the pressure (6.13). In addition to the above motivation, there lies another reason behind the (not so obvious) definition of energy term (6.11) and the work function (6.13) under the virtual displacement. As already mentioned previously, we have partitioned the NRE (for $\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}$) (3.24) in such a way, that the energy contribution arises entirely from geometrical quantities defined on the transverse submanifold S_t or on the null surface $\mathcal{H}$ (in addition to the scalar field term $l^r\hat{\nabla}_r\Big(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - q^{ij}T_{ihj}k^h\Big)$). Contrary to this, the pressure term (leading to the work function) is entirely from quantities defined in the manifold $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. In fact, it has been explicitly shown [92] that at least for Einstein gravity (with zero torsion), the covariant expression of the energy (6.12) reduces to expressions of energy for well known spacetimes. For example, the computation of (6.12) for the usual Schwarzschild metric gives us the mass term. It is in this spirit, that the natural generalization of energy term for a generic $\mathcal{H}$ in EC gravity theory follows.

6.3.1 Case of completely antisymmetric torsion:

Let us now come to the important specific case of the torsion being completely antisymmetric. Applications of completely antisymmetric torsion tensor have been discussed in string and superstring theories [214]. For the case of a string-inspired gravitational theory, the Kalb-Ramond field is identified with a completely antisymmetric torsion background [215]. In the case of completely antisymmetric torsion, the expressions in our thermodynamic analysis simplify significantly. Firstly, the geodesic constraint (2.34) no longer needs to be assumed but rather is a consequence of total antisymmetry of the torsion tensor. Let us focus on the NRE corresponding to the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}$ *i.e.* (6.7).

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The expression simplifies to,

$$-\kappa\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} = \Big[\frac{1}{2}{}^{(2)}\hat{R} + l^r\hat{\nabla}_r\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a - \hat{\mathcal{D}}_a(\hat{\Omega}^a - \hat{\mathscr{P}}^a)$$
$$+ \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - \frac{1}{2}q^{dj}q^{ci}S_{cbd}S_{aij}k^a l^b\Big] - \Big[8\pi T^{(m)}_{ab} + \frac{1}{2}\hat{\nabla}_i\Big(3S^i{}_{ab}\Big)\Big]k^a l^b\,. \tag{6.17}$$

In the above, we have used the fact that for completely antisymmetric torsion, $S_{abc} = T_{abc}$ and $K_{abc} = \frac{1}{2}T_{abc}$. Proceeding ahead with the process of virtual displacement, we can attest the thermodynamic interpretation to this specific case as well. The heat energy associated with the process now is,

$$\int_{S_t} d^2x T\delta_{\lambda_{(k)}} s_{\text{null}}\,, \tag{6.18}$$

where $s_{\text{null}} = \frac{\sqrt{q}}{4}$. Obviously, the entropy generation term $\delta_{\lambda_{(k)}} s_{\text{tor}}$ due to the torsion current component $q^{ij}T_{ihj}k^h$ flowing along the null generators $\vec{\boldsymbol{l}}$ of $\mathcal{H}$ is zero owing to the total antisymmetry of torsion. Hence under the virtual displacement process the only change in the entropy density occurs via the change in the transverse area element $\sqrt{q}$ of $\mathcal{H}$. The amount of energy flow along the null hypersurface under such considerations is,

$$\delta_{\lambda_{(k)}} E = \int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)}\frac{1}{8\pi}\Big[\frac{1}{2}{}^{(2)}\hat{R} + l^r\hat{\nabla}_r\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a$$
$$- \hat{\mathcal{D}}_a\Big(\hat{\Omega}^a - \hat{\mathscr{P}}^a\Big) + \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - \frac{1}{2}q^{dj}q^{ci}S_{cbd}S_{aij}k^a l^b\Big]\,. \tag{6.19}$$

The corresponding identification of the pressure term under such a process in the case of totally antisymmetric torsion tensor is,

$$P = -\frac{1}{8\pi}\Big[8\pi T^{(m)}_{ab} + \frac{1}{2}\hat{\nabla}_i\Big(3S^i{}_{ab}\Big)\Big]k^a l^b\,. \tag{6.20}$$

6.3.2 Notion of *equilibrium* for the null surface $\mathcal{H}$ in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$

Now, let us discuss the case of an equilibrium null hypersurface $\mathcal{H}_{\text{eq}}$, *i.e.* we want a truly stationary description of our null surface/horizon. First of all, we would require our theory to have non-propagating torsion. The Einstein-Cartan action $\mathcal{A}_{\text{EC}}$ is given by (6.1),

$$\mathcal{A}_{\text{EC}} = \frac{1}{16\pi}\int_{\mathcal{V}}\sqrt{-g}\,\hat{R} = \frac{1}{16\pi}\int_{\mathcal{V}}\sqrt{-g}\Big[R + 2\nabla_i T^i - T^a T_a + K^{imj}K_{mij}\Big]\,. \tag{6.21}$$

We see that this gravitational action does not contain second derivatives of the torsion term. Hence, in such theories, the torsion field itself does not propagate. This can also be seen from the ECKS field equations Eq. (6.3) which shows that the source of torsion is the spin angular momentum density tensor. However this relation is purely algebraic and involves no second order derivatives of the torsion term. The EC theory is such an example of a gravitational theory with non-propagating torsion. However, the torsion can indirectly propagate through some other field with which it is coupled. For instance, here the torsion is carried by the propagation of g_{ab}. In principle, whatever be the case, for a truly stationary description of our null hypersurface, we would require any torsion current flowing along the null surface $\mathcal{H}$ to be zero. The first among such a non-trivial torsion current is $T_{abc}l^a l^c$. Setting this to zero implies our geodesic constraint (2.34). In fact, when considering the case of a Killing horizon (a stationary equilibrium description of the horizon), such a torsion current needs to be eliminated for removing inequivalent definitions of surface gravity [64]. The second among such torsion current that we need to consider for our purposes is $T_{ihj}q^{ij}$. The component of this torsion current flowing along the null surface (*i.e.* along the null generators $\vec{l}$ is precisely $q^{ij}T_{ihj}k^h$. We should demand for this component to vanish in order to have a stationary description of the null surface/horizon. In fact, the authors of [64] have shown that only the geodesic constraint (2.34) is required to prove the zeroth law of black-hole mechanics for a Killing horizon established in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. They do not demand specifically the requirement that $q^{ij}T_{ihj}k^h$ be zero as well for the Killing horizon. In order to prove the zeroth law, the authors consider the specific case of a Killing horizon having a bifurcation 2-surface. However, not all stationary horizons have a bifurcation 2-surface. Here we postulate that for a true stationary and hence equilibrium notion of a Killing horizon, we require both the conditions $T_{abc}l^a l^c = 0$ and $q^{ij}T_{ijh}k^h = 0$ to be simultaneously implemented. These two constraints represent our equilibrium conditions. For such a Killing horizon established in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, the surface gravity and hence the temperature is constant over the horizon. Moreover the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}$ of the null generators vanish by definition for a Killing horizon. Now if we perform a virtual displacement process for such a Killing horizon, then the thermodynamic interpretation becomes quite clear. The variation of the entropy due to the torsion term *i.e.* $\delta_{\lambda_{(k)}} s_{\text{tor}}$ is by default zero under our equilibrium conditions. Since the temperature is constant over the Killing horizon, while considering Eq. (6.15), we can take T outside the integral. We then identify the total change of the entropy S_{null} of the null surface (Killing horizon) to be $\delta_{\lambda_{(k)}} S_{\text{null}} = \int_{S_t} d^2x \; \delta_{\lambda_{(k)}} s_{\text{null}}$. We then finally have the thermodynamic interpretation established on the Killing horizon in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ under the virtual displacement $\delta\lambda_{(k)}$ to be,

$$T\delta_{\lambda_{(k)}} S_{\text{null}} = \delta_{\lambda_{(k)}} E + F\delta\lambda_{(k)} \,. \tag{6.22}$$

The variation of the energy term is,

$$\delta_{\lambda_{(k)}} E = \int_{S_t} d^2x\sqrt{q}\,\delta\lambda_{(k)} \frac{1}{8\pi}\Big[\frac{1}{2}{}^{(2)}\hat{R} + l^r\hat{\nabla}_r\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} - \hat{\Omega}_a\hat{\Omega}^a + \hat{\Omega}_a\hat{\mathscr{P}}^a$$
$$- \hat{\mathcal{D}}_a\left(\hat{\Omega}^a - \hat{\mathscr{P}}^a\right) - \left(\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}} q^{cd} - \hat{\Xi}^{cd}\right)(T_{cfd}l^f) - q^{dj}q^{ci}(T_{cbd})(K_{aij})k^a l^b\Big] . \tag{6.23}$$

Similarly the pressure term for the virtual displacement of the Killing horizon is,

$$P \equiv -\frac{1}{8\pi}\left[8\pi T^{(m)}_{ab} + \frac{1}{2}(\hat{\nabla}_i + T_i)\left(3S^i{}_{ab} + S_a{}^i{}_b + S_b{}^i{}_a\right)\right]k^a l^b . \tag{6.24}$$

Now, having discussed the physical interpretation of the thermodynamic identity as applied to a generic hypersurface-orthogonal null surface (satisfying the geodesic constraint) in the spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ and its relevant specifications to the case of completely antisymmetric torsion and the equilibrium case, we delve a little bit more into the possible origins of the total entropy variation term. In this regard it helps to compare our results with the interpretation provided in [64]. In this paper, the authors in the context of a local causal horizon established in the Riemann-Cartan spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ assume an area-entropy law, where they propose that the variation of the entropy is proportional to the variation of the horizon cross-section [see Eq. (69) of [64]]. However, as we have seen in our case, w.r.t (6.9), that the total variation of the entropy density is due to the sum of two contributions. One is due to the variation of the null surface/horizon cross-section under the virtual displacement $\delta\lambda_{(k)}$. The other is the entropy generation term (6.10) due to the non-zero torsion current $q^{ij}T_{ihj}$. One can surely think as to why there is no such entropy generation term due to the torsion current in the process involving the local causal horizons described in [64]. To our understanding, this stems from the difference in the processes involved. Even though there is no mention of a virtual displacement process in [64], the entropy variation in [64] as applied to local causal horizons is surely due to some physical process (here that represents a local constitutive relation of entropy balance law on the local causal horizon). This physical process (involving matter fluxes across the horizon) clearly perturbs the local causal horizon along its null generators $\vec{\boldsymbol{l}}$. As a result the NRE corresponding to the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_{\boldsymbol{l}}}$ has been used to compute the variation of the horizon cross-section. Clearly, whatever the case may be, under the process of varying the local causal horizons along their null generators, there does not arise the need for a torsion current of the type $q^{ij}T_{ihj}$. However, the process that we are considering virtually shifts our null surface along the transverse auxiliary null field $\vec{\boldsymbol{k}}$. The physical processes involved in both of these considerations are very different. For our case, as the NRE corresponding to the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_{\boldsymbol{k}}}$ suggests, we have to very well take into consideration the entropy generation term due to the torsion current $q^{ij}T_{ihj}$. Setting the component of this torsion current along the null generators to zero (along with the geodesic constraint), which we have seen, represents our notion of an equilibrium horizon.

6.4 Discussion and conclusion

The main aim of the present analysis was to investigate whether the thermodynamic interpretation of gravitational dynamics is possible in the presence of torsion in the spacetime. We found that a particular projection of the field equation of EC theory of gravity on a generic null surface indeed provides a thermodynamic structure. The idea was originally introduced in [36, 211] based on an infinitesimal virtual displacement along the auxiliary null vector field. Since the original analysis was a non-covariant one, we here followed the spirit of our earlier work [92] in order to provide a covariant formalism for a specific spacetime with torsion.

Having developed all the necessary geometrical tools in Chapters 2 and 3, we extended the formalism that we developed in Chapter 4 to the case of EC theory. We focused on one particular projection of ECKS equations on our generic null surface. Then following [36] we provided the process of virtual displacement of $\mathcal{H}$ in the transverse auxiliary null vector direction. This enabled us to interpret this evolution equation "similar" to the first law of thermodynamics in a covariant fashion. We saw the presence of a non-trivial torsion current $T_i{}^h{}_j q^{ij}$ leading to non-zero torsion current component $T_{ihj}q^{ij}k^h$ along the null generators. This led to an additional entropy generation term under the virtual displacement process. The amount of energy flow across the null surface under such process now contains additional terms depending on the non-trivial torsion tensor. Similarly, the pressure term is not defined only with respect to the matter energy-momentum tensor. It contains suitable spin-spin contact interaction terms as well as covariant derivatives of the torsion term. We mention that the present thermodynamic interpretation is strictly based on the geodesic constraint as our evolution equation was derived within this condition. All of the analysis, we saw, consequently reduces to the familiar form when we set the torsion to zero, *i.e.* say, for Einstein gravity [92]. The special case of the torsion field being completely antisymmetric and its consequences were also discussed. Finally, we commented upon the case of null surface $\mathcal{H}$ being in equilibrium in the EC gravity theory.

Let us at this point discuss our approach to the viewpoint of torsion. There have been predominantly two notions of torsion. It can be considered either as a geometric field or that of a background dynamical field. Here, in our analysis, we have leaned onto the geometrical perspective. This is quite evident in the way we factored the energy and work done term under the virtual displacement $\delta\lambda_{(k)}$. In the NRE (of the ingoing

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expansion scalar $\overset{(d)}{\hat{\theta}}_{\boldsymbol{k}}$) (3.24), we had factored out the energy terms on the basis that they contained terms purely defined on the two-surface S_t or the null surface $\mathcal{H}$ [along with the term $l^r \hat{\nabla}_r (\overset{(d)}{\hat{\theta}}_{\boldsymbol{k}} - q^{ij} T_{ihj} k^h)$]. The work function contained terms defined entirely on the four dimensional manifold $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$. This thermodynamic interpretation provided to $\hat{G}_{ab} k^a l^b$ through the virtual displacement essentially takes this viewpoint from the very beginning that torsion is a geometric field. It is only at the end once the dynamics of the EC theory has been established (letting the torsion be sourced by the spin angular momentum tensor (6.3)) that we can also interpret the work function or rather the pressure (6.13) in terms of the matter energy-momentum tensor $T^{(m)}_{ab}$ and the spin angular momentum tensor $\tau^a{}_{bc}$. However we can right away begin with the viewpoint of torsion being a dynamical background field. This viewpoint lets the torsion terms be a part of an effective stress-energy tensor T^{eff}_{ab}. This effective stress-energy tensor is related to the Einstein tensor of the spacetime provided with the Levi-Civita connection (see Eq. (2.5.10) of [87]),

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi (T^{(m)}_{ab} + U_{ab}) = 8\pi T^{\text{eff}}_{ab} \,, \tag{6.25}$$

where U_{ab} contains terms quadratic in $\tau^a{}_{bc}$ and hence represents spin-spin contact interaction terms. Obviously owing to the Bianchi identity, the effective stress-energy tensor is covariantly conserved with respect to the Levi-Civita connection $\boldsymbol{\nabla}$. We in our approach did not proceed with such consideration of an effective stress-energy tensor [for the spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$] and went purely by the geometrical interpretation. We could however have started with the dynamical equation for ingoing expansion scalar $\theta_{\boldsymbol{k}}$ for the spacetime $(\mathcal{M}, \mathbf{g}, \boldsymbol{\nabla})$ *i.e.* Eq. (3.25)

$$-\kappa \theta_{\boldsymbol{k}} = \Big(- \mathcal{D}_a \Omega^a - \Omega_a \Omega^a + \theta_{\boldsymbol{l}} \theta_{\boldsymbol{k}} + l^i \nabla_i \theta_{\boldsymbol{k}} + \frac{1}{2} {}^2 R \Big) - G_{ab} k^a l^b \,. \tag{6.26}$$

Using the fact that $\overset{(d)}{\hat{\theta}}_{\boldsymbol{k}} = \theta_{\boldsymbol{k}}$, we can certainly use the form of the ECKS field equation (6.25) in (6.26). Then the process of virtual displacement would on (6.26) have yielded for us different energy variation and work done terms. The pressure, as we can anticipate would depend on the matter energy tensor as well the spin-spin contact interaction terms. The variation of energy term would also be different from what had been obtained previously. Similarly looking on the L.H.S of (6.26), we see that the entropy generation term is purely due to the change in the cross-sectional area of the null hypersurface under the virtual displacement. Under this interpretation, there is no identification of an entropy generation term due to a non-zero torsion current. The question then naturally arises as to which interpretation for the torsion field is correct. Is it good to consider it a geometric field or would it be better if torsion acted as a background field? This dilemma was also addressed in [64] where the ECKS field equations were derived from a generalized Clausius identity $\delta Q = T(dS + dS_i)$ applied to a local Rindler horizon. In the paper [64], the authors discussed that the internal entropy production term dS_i followed quite naturally when torsion was considered as geometric field. However if torsion was proposed as a background dynamical field then such a term had to imposed by hand in an *ad-hoc* fashion to recover the EC equations. We also believe that our analysis is more structured towards interpretation of torsion being a geometric field. However, our stand on this issue is by no means definitive and remains open to further scrutiny and interpretation.

Chapter 7

Possible fluid interpretation and tidal force equation on a generic null hypersurface in Einstein-Cartan theory

7.1 Introduction and motivation

The fascinating connection between gravitational dynamics and fluid-dynamics has been a subject of great interest for long. One of the earliest works relating the dynamics of gravity and that of hydrodynamics appeared in the doctoral thesis of Damour [37], wherein there are suggestions of a connection between horizon and fluid dynamics. This work contains the evolution dynamics of a given geometric data on an arbitrary null surface, now known as the Damour-Navier-Stokes (DNS) equation. The same equation is also obtained in terms of coordinates adapted to a null surface [82] [61] by projecting the Einstein's equations of motion onto the null hypersurface (a similar analysis has also been done in [190] for scalar-tensor gravity theory to obtain DNS like equation). Moreover, a corresponding action formulation of the same has been greatly detailed in [38]. A connection in this regard has also been obtained in the membrane paradigm approach by Price and Thorne in [216]. The membrane paradigm was applied in [217] in the context of asymptotically AdS spacetimes to show the dynamics of the membrane being described by the incompressible NS equation. In [218], the authors have obtained an analogous DNS type equation for both future outer trapped horizons and dynamical horizons (which are spacelike). One peculiarity of the DNS equation as obtained on a null horizon is that the bulk viscosity of the horizon fluid is negative. This makes the null horizon fluid unfit to have a connection with ordinary fluids. However the authors of [218] show that the horizon fluid on both future outer trapped horizons and the dynamical horizons have a positive value of the bulk viscosity.

In the AdS/CFT context, it has been shown that the dissipative behavior of an AdS black hole agrees with the hydrodynamics of the holographically dual CFT. In this approach the NS equation together with its corrections arise under a gradient expansion of the Einstein's equations. This has been studied extensively and important works in this regard include [219–222]. More recently in a cut-off surface approach by Bredberg *et al.* [223], it has been shown by explicit construction that for every solution of the incompressible NS equation in $(p+1)$-dimensions, there is a uniquely associated dual solution of the vacuum Einstein equations in $(p+2)$-dimensions. The metric of [223] has been extended to all orders perturbatively via gradient expansion in [224], thus yielding higher order corrections to the NS equation as well as the incompressibility condition. In [225], the authors have generalized the cut-off surface approach by expounding on the dynamics of the dual field theory living on the boundary of AdS spacetime, provided the Dirichlet boundary conditions on the $r = r_c$ cut-off surface is ensured. The authors show that there exists a critical radius as we go towards the horizon, beyond which, a relativistic description of the fluid living on the cut-off surface is not valid because of the acausal propagation of sound modes. Allowing the non relativistic scaling, the authors retrieve that Ricci flat gravitational duals to the incompressible NS equations. In [226], the authors provide a general approach to fluid/gravity correspondence, where the base metric is no longer the flat Rindler metric, but rather a generic static metric. The spacetime is endowed with a general bulk stress energy tensor and an event horizon. This cut-off surface approach has been applied in various cases, see [227–229]. For example, it was extended for higher curvature gravity theories [230–234] as well as for the AdS [235, 236] and dS [237] gravity theories (for other theories, like black branes, see [238]). In [228], it was shown that an incompressible DNS-like equation can be obtained in the cut-off surface approach. In this case the obtained metric is a solution of Einstein's equations of motion in the presence of a particular type of matter. Also a corresponding relativistic situation has been discussed extensively in [239]. Symmetries of the vacuum Einstein equations have been exploited to develop a formalism for solution-generating transformations of the corresponding NS fluid duals in [240]. The fluid description on the Kerr horizon has also been explored in [241] (see [242] for the isolated horizon case). The correspondence has also been established for general rotating black holes yielding a Coriolis force term [243]. For extensive reviews of the fluid-gravity correspondence, refer to [72, 244, 245].

In this chapter, we will not deal with the fluid/gravity duality from the gauge-gravity context. Our analysis will be entirely within the purview of the membrane paradigm of gravity, where we will focus on the dynamics of the gravitational field equations on a generic null hypersurface. For this, we will have to focus on the projection component $\hat{G}_{ab}l^a q^b{}_c$. As mentioned in the earlier chapters, for the case of Einstein gravity, $G_{ab}l^a q^b{}_c$ leads to the Lie evolution of the Hájíček 1-form along the null generators which is then interpreted as the DNS equation [61, 82]. It also works in scalar-tensor theory of gravity as well [190]. In this chapter, we will focus exclusively whether $\hat{G}_{ab}l^a q^b{}_c$ would enable us to attribute to the ECKS field equations any fluid/elastic continuum model interpretation. In particular, under the geodesic constraint, we will be able to see when we write down the dynamics of the Hájíček 1-form in a coordinate system adapted to the null surface $\mathcal{H}$, the structure we get is quite similar to the Cosserat generalization of the NS fluid (under appropriate identifications of the fluid variables with the kinematical variables of $\mathcal{H}$ and the external force density). In fact, we will see that w.r.t a local inertial frame, the above dynamical equation will indeed reduce to the Cosserat fluid equation. The Cosserat fluid is a real world fluid dynamical system that incorporates intrinsic angular momentum via the rotational degrees of freedom for its constituent fluid particles.

Even though, much of our analysis is based on providing a possible fluid interpretation for the ECKS field equations, we also present the tidal force equation for the null surface $\mathcal{H}$ in its most generic sense. The only imposition, that we will make while arriving at the tidal force equation is that the geodesic null congruence generates an integrable hypersurface, *i.e.* it satisfies the Frobenius identity. In doing this analysis, we are again led to derive to the NRE (for $\overset{(d)}{\hat{\theta}_l}$) for the given integrable hypersurface $\mathcal{H}$ in the RC spacetime (however in a different form). The tidal equation and the NRE analyzed for the EC theory furnish a part of the optical scalar equations obtained under the Newman-Penrose formalism.

The organization of this chapter is as follows. In section 7.2, we recollect our dynamical evolution dynamics of the Hájíček 1-form related to the projection component $\hat{G}_{ab}l^a q^b{}_c$ and connect it with the gravitational dynamics via the ECKS field equations. In the next section 7.3, we try to argue whether the ECKS field equation w.r.t $\mathcal{H}$ via the component $\hat{G}_{ab}l^a q^b{}_c$ can be attributed any possible fluid dynamical/elastic theory interpretation. Here, we come to the conclusion that the resulting dynamics cannot be compared with the DNS fluid. The dynamics has rather an analogy with the Cosserat generalization of the NS fluid which we try to make more precise in a boosted local inertial frame. We do this to argue whether the dynamics of the "null fluid" established on $\mathcal{H}$ has connections/analogy with some real world fluid scenarios. For the sake of completeness, we also present in section 7.4 the tidal force equation governing a congruence of null geodesics in the RC spacetime. We then conclude in 7.5 and in the appendices provide detailed derivations of some of the expressions used in the text.

7.2 The relevant gravitational dynamics projected onto $\mathcal{H}$

In Chapter 3, we derived the evolution dynamics of the Hájíček 1-form along the null generators $\vec{l}$ in Eq. (3.40). Our objective now would be to use the ECKS field equations on this evolution equation and get to the extension of the Hájíček equation [61] in the case of the EC gravity theory. Using the ECKS field equations (6.2), we have,

$$\hat{G}_{ab}l^a q^b{}_t = 8\pi T^{(m)}_{ab} l^a q^b{}_t - \frac{1}{2}(\hat{\nabla}_c + T_c)\Big(- S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c\Big) l^a q^b{}_t \,. \tag{7.1}$$

Note that in (3.40) or (3.37), we have related the dynamical evolution (along $\vec{l}$) of $\hat{\underline{\Omega}}$ with $\hat{G}_{ab}l^a q^b{}_t$ instead of $\hat{G}_{ab}q^a{}_t l^b$ as anticipated in the transverse projection component of the vector field $\hat{G}^a{}_b l^b$. However, this is not a matter of concern, since at this point, we eventually use the field equations as seen in (7.1). Using (7.1) in (3.40), we have,

$$
\begin{aligned}
& q^a{}_t \pounds_l \hat{\Omega}_a + \hat{\Omega}_t\Big(\overset{(d)}{\hat{\theta}_l} - T_b l^b + k_b \mathbb{T}^b\Big) - \hat{\mathcal{D}}_t\Big(\kappa + \frac{1}{2}(\overset{(d)}{\hat{\theta}_l} - T_b l^b - k_b \mathbb{T}^b)\Big) + \hat{\mathcal{D}}_b \overset{*}{\hat{\sigma}}{}^b{}_t \\
& + \hat{\mathcal{D}}^b\Big(\frac{1}{2}q^c{}_t q^d{}_b T_{fdc} l^f\Big) - \hat{\Theta}_{ba}\mathscr{P}^b q^a{}_t - \hat{\Xi}_{tb}\mathbb{T}^b + q^c{}_t q^d{}_b (K_{fdc}k^f)\mathbb{T}^b \\
& = 8\pi T^{(m)}_{ab} l^a q^b{}_t - \frac{1}{2}(\hat{\nabla}_c + T_c)\Big(- S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c\Big) l^a q^b{}_t - T_{iba}\hat{\Theta}^{bi} q^a{}_t\,. \qquad (7.2)
\end{aligned}
$$

The above relation reduces to the equation (6.15) in [61] in the absence of torsion in the spacetime and then hence defines the Hájíček equation under the membrane paradigm. We notice that the above general dynamical evolution law of the Hájíček one-form $\hat{\Omega}_a$ *i.e.* (7.2) involves the transversal deformation rate tensor $\hat{\Xi}_{tb}$. The transversal deformation rate tensor in essence measures the projection of the Lie derivative of the transverse induced metric q_{ab} along the ingoing auxiliary null field $\vec{k}$. So the dynamics involving the evolution of the Hájíček 1-form *i.e.* (7.2) involves a part that contains the evolution of q_{ab} along $\vec{k}$. This is however in stark contrast to the evolution equation of the Hájíček 1-form in the Riemannian spacetime [61]. Such a dynamical evolution equation in the Riemannian spacetime involves the evolution of relevant kinematical quantities along the outgoing null generator $\vec{l}$ and none so in the direction of the ingoing field $\vec{k}$. Such terms that encode explicit information of the auxiliary null field $\vec{k}$ can also be seen from the corresponding NRE (for $\overset{(d)}{\hat{\theta}_l}$) in the RC spacetime. Without invoking the geodesic constraint, the most general form of the NRE has been derived in (3.8) . A different variant of the NRE in RC spacetime has been derived in the Appendix 7.7 (see Eq. (7.67)). It is quite clear that in the presence of torsion in the spacetime, the dynamical evolution of the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_l}$ along $\vec{l}$ encodes information about the uniquely defined ingoing $\vec{k}$ field. This is again in contrast with the Riemannian spacetime, where the NRE (for $\overset{(d)}{\hat{\theta}_l}$) carries no explicit terms involving the auxiliary $\vec{k}$ field. Coming back to the present analysis, in the presence of torsion in the spacetime, *i.e.* in the RC spacetime, the evolution equation of the Hájíček 1-form $\hat{\Omega}_a$ involves terms like $\hat{\Xi}_{tb}$. This is preferably due to the fact that in the RC spacetime, the null generators of our integrable null hypersurface $\mathcal{H}$ are not parallel-transported along themselves. However these null generators are themselves null geodesics w.r.t. the Levi-Civita connection $\boldsymbol{\nabla}$. However, if we impose the geodesic constraint *i.e.* $\mathbb{T}_b = T_{abc}l^a l^c = 0$, then we force the null generators to be simultaneously auto-parallel (w.r.t to the connection $\hat{\boldsymbol{\nabla}}$) and extremal length geodesics (w.r.t $\boldsymbol{\nabla}$). Hence, we notice that under the geodesic constraint, we remove in equations

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(3.37) and (3.40) any reference to evolution of kinematical/geometrical quantities along the auxiliary null field $\vec{k}$. As a side note, this feature was also shared by the NRE (for $\overset{(d)}{\hat{\theta}_l}$) in the RC spacetime, where the application of the geodesic constraint removed explicit references of evolution of kinematical quantities along $\vec{k}$. Thus, under the geodesic constraint, we have our relevant dynamical evolution laws for $\hat{\Omega}_a$ to be,

$$
\begin{aligned}
& q^a{}_t \pounds_l \hat{\Omega}_a + \hat{\Omega}_t \Big(\overset{(d)}{\hat{\theta}_l} - T_b l^b \Big) - \hat{\mathcal{D}}_t \Big(\kappa + \frac{1}{2} (\overset{(d)}{\hat{\theta}_l} - T_b l^b) \Big) + \hat{\mathcal{D}}_b \overset{*}{\hat{\sigma}}{}^b{}_t \\
& + \hat{\mathcal{D}}^b \Big(\frac{1}{2} q^c{}_t q^d{}_b T_{fdc} l^f \Big) - \hat{\Theta}_{ba} \mathscr{P}^b q^a{}_t = 8\pi T^{(m)}_{ab} l^a q^b{}_t \\
& - \frac{1}{2} (\hat{\nabla}_c + T_c) \Big(- S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c \Big) l^a q^b{}_t - T_{iba} \hat{\Theta}^{bi} q^a{}_t \, .
\end{aligned}
\tag{7.3}
$$

The above can be expressed in the following alternative structure as well:

$$
\begin{aligned}
& q^a{}_t \pounds_l \hat{\Omega}_a + \hat{\Omega}_t \Big(\overset{(d)}{\hat{\theta}_l} - T_b l^b \Big) - \hat{\mathcal{D}}_t \Big(\kappa + \frac{1}{2} (\overset{(d)}{\hat{\theta}_l} - T_b l^b) \Big) + \hat{\mathcal{D}}_b \overset{*}{\hat{\sigma}}{}^b{}_t \\
& + \hat{\mathcal{D}}^b \Big(\frac{1}{2} q^c{}_t q^d{}_b T_{fdc} l^f \Big) - \hat{\chi}_{bt} \hat{\mathscr{P}}^b + q^c{}_t q^d{}_b (K_{fcd} l^f) \hat{\mathscr{P}}^b \\
& = 8\pi T^{(m)}_{ab} l^a q^b{}_t - \frac{1}{2} (\hat{\nabla}_c + T_c) \Big(- S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c \Big) l^a q^b{}_t \\
& - T_{iba} \hat{\chi}^{bi} q^a{}_t + q^a{}_t (q^{ci} q^{db} T_{iba} (K_{fcd} l^f)) \, .
\end{aligned}
\tag{7.4}
$$

The above dynamical equations (7.3) and (7.4) are completely covariant relations written in any arbitrary coordinate system. It is this equation that in absence of torsion in the spacetime reduces to the Damour-Navier-Stokes (DNS) equation under a specific choice of coordinates adapted to the null surface $\mathcal{H}$. So, in view of this, let us discuss the structural aspects of this dynamical equation. Firstly, note that (7.3) contains the Lie derivative of the Hájiček 1-form along the null generator $\vec{l}$. In a coordinate system adapted to the null hypersurface $\mathcal{H}$, we have [61],

$$
\vec{l} \overset{\mathcal{H}}{=} \vec{t} + \vec{V} \, ,
\tag{7.5}
$$

where $\vec{t}$ is the time evolution vector field (essentially connecting similar spatial points on the $t =$ constant spacelike Σ_t slices foliating $\mathcal{H}$) and $\vec{V}$ is a spacelike vector field tangent to the two-surface S_t. For a coordinate system $(t, x^\mu = \{x^1, x^2, x^3\})$ adapted to the null hypersurface $\mathcal{H}$, its location is prescribed by say $x^1 = 1$ and hence the coordinates on the transverse 2-surface are prescribed by $(x^A = \{x^2, x^3\})$. For such an adapted coordinate system, we hence have $l^a \overset{\mathcal{H}}{=} t^a + V^A \partial_A$ and that $q^a{}_A = \delta^a{}_A$. For such a choice of coordinates, its quite easy to show that,

$$q^a{}_A \pounds_l \hat{\Omega}_a \overset{\mathcal{H}}{=} q^a{}_A \pounds_{t+V} \hat{\Omega}_a = \frac{\partial \hat{\Omega}_A}{\partial t} + V^B \hat{\mathcal{D}}_B \hat{\Omega}_A + \hat{\Omega}_B \hat{\mathcal{D}}_A V^B + {}^{(2)}T^B_{CA} V^C \hat{\Omega}_B \,, \tag{7.6}$$

where ${}^{(2)}T^A_{BC}$ is the induced torsion tensor on the transverse submanifold $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$. Inserting the above relation (7.6) into (7.3), we obtain,

$$\begin{aligned}
&\frac{\partial \hat{\Omega}_A}{\partial t} + V^B \hat{\mathcal{D}}_B \hat{\Omega}_A + \hat{\Omega}_B \hat{\mathcal{D}}_A V^B + {}^{(2)}T^B_{CA} V^C \hat{\Omega}_B + \hat{\Omega}_A (\overset{(d)}{\hat{\theta}_l} - T_b l^b) \\
&= 8\pi q^b{}_A T^{(m)}_{ab} l^a - \frac{1}{2}(\hat{\nabla}_c + T_c)\Big(- S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c \Big) l^a q^b{}_A + \hat{\Theta}_{ba} \mathscr{P}^b q^a{}_A \\
&- T_{iba} \hat{\Theta}^{bi} q^a_A + \hat{\mathcal{D}}_A \kappa - \hat{\mathcal{D}}_B \overset{*}{\hat{\sigma}}{}^B{}_A + \frac{1}{2} \hat{\mathcal{D}}_A (\overset{(d)}{\hat{\theta}_l} - T_b l^b) - \hat{\mathcal{D}}^B \Big(\frac{1}{2} q^c{}_A q^d{}_B T_{fdc} l^f \Big) \,.
\end{aligned} \tag{7.7}$$

The initial two terms in the L.H.S of (7.7) denote the material derivative of the Háj́iček 1-form $\hat{\Omega}_A$ along $\vec{V}$. In the context of Einstein gravity, as applied to a black-hole event horizon, Damour interpreted $\vec{V}$ as the surface velocity of the horizon. In this context, $\vec{V}$ is to be interpreted as the surface velocity of the null hypersurface $\mathcal{H}$ w.r.t. the adapted coordinates in the RC spacetime. The extra term $\hat{\Omega}_B \hat{\mathcal{D}}_A V^B$ involving the derivative of the velocity field V^A is present (as in the DNS case) along with an extra term ${}^{(2)}T^B_{CA} V^C \hat{\Omega}_B$. This is because the ambient spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ carries intrinsic torsion that induces a non-symmetric connection $\hat{\boldsymbol{\mathcal{D}}}$ and hence torsion on the submanifold $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$. It is worth mentioning that in analogy with the *membrane paradigm* approach, we can exchange the Lie derivative operator with the operator $\hat{\mathcal{D}}_{\bar{t}}$ [216] such that its operation on the Háj́iček 1-form is given as,

$$\hat{\mathcal{D}}_{\bar{t}} \hat{\Omega}_i \equiv q^a{}_i (l^j \hat{\nabla}_j \hat{\Omega}_a) \,, \tag{7.8}$$

which quantifies the projection (onto S_t) of the covariant derivative of the Háj́iček 1-form along $\vec{l}$. It can then quite easily be seen that,

$$q^a{}_i \pounds_l \hat{\Omega}_a = \hat{\mathcal{D}}_{\bar{t}} \hat{\Omega}_i + \hat{\Omega}_k \hat{\Theta}^k{}_j q^j{}_i + T_{kja} l^j \hat{\Omega}^k q^a{}_i \,. \tag{7.9}$$

Using (7.9) in (7.3), we can also interpret the dynamical evolution of the Háj́iček 1-form in the following way,

$$\begin{aligned}
&\hat{\mathcal{D}}_{\bar{t}} \hat{\Omega}_i + \hat{\Omega}_i \Big(\overset{(d)}{\hat{\theta}_l} - T_b l^b \Big) - \hat{\mathcal{D}}_i \Big(\kappa + \frac{1}{2} (\overset{(d)}{\hat{\theta}_l} - T_b l^b) \Big) + \hat{\mathcal{D}}_j \overset{*}{\hat{\sigma}}{}^j{}_i + \hat{\mathcal{D}}^j \Big(\frac{1}{2} q^c{}_i q^d{}_j T_{fdc} l^f \Big) \\
&+ (\hat{\Omega}_k - \mathscr{P}_k) \hat{\Theta}^k{}_i = 8\pi T^{(m)}_{jk} l^j q^k{}_i \\
&- \frac{1}{2} (\hat{\nabla}_j + T_j) \Big(- S^j{}_{mn} + S_{mn}{}^j + S_{nm}{}^j \Big) l^m q^n{}_i - T_{kja} (\hat{\Theta}^{jk} + l^j \hat{\Omega}^k) q^a{}_i \,.
\end{aligned} \tag{7.10}$$

In the absence of torsion in the spacetime, the above equation reduces to the "Hájíček equation" arising in the context of membrane paradigm approach [216]. Though we have these two notions of the derivative operator acting on $\hat{\Omega}_a$, the Lie derivative presents itself as a natural generalization to the material derivative in curved manifolds.

7.3 Possible connections with fluid dynamics/ generalized continuum mechanics

7.3.1 Projected ECKS field equations on $\mathcal{H}$ are not equivalent to DNS:

It is quite well known that for the case of vanishing torsion, the DNS equation describes the dynamics of a 2-dimensional null viscous fluid living on $\mathcal{H}$. In fact, it has also been shown that the DNS equation is exactly identical to the NS equation provided we view it in a boosted inertial frame [82]. Much in the same way, we would like to have an interpretation for the dynamical equations (7.3) or (7.7).

We notice in (7.3), the presence of the spatial covariant derivatives of $\overset{*}{\hat{\sigma}}_{ab}$ and $(\frac{1}{2}q^c{}_a q^d{}_b T_{fdc} l^f)$ which denote the traceless symmetric and antisymmetric parts of $\hat{\Phi}_{ab}$ respectively. With respect to the adapted coordinate system $(x^a = (t, x^\mu) = (x^0, x^\mu))$, it can be shown that, (for derivation, see Appendix 7.6)

$$\hat{\chi}_{AB} \stackrel{\mathcal{H}}{=} \frac{1}{2}\Big(\partial_t q_{AB} + \hat{\mathcal{D}}_A V_B + \hat{\mathcal{D}}_B V_A + ({}^{(2)}T_{ACB} + {}^{(2)}T_{BCA})V^C\Big) , \tag{7.11}$$

and

$$\hat{\Phi}_{AB} = \hat{\chi}_{AB} - \tilde{t}_{AB} \stackrel{\mathcal{H}}{=} \frac{1}{2}\Big(\partial_t q_{AB} + \hat{\mathcal{D}}_A V_B + \hat{\mathcal{D}}_B V_A - 2K_{0BA} - {}^{(2)}T_{DBA}V^D\Big) . \tag{7.12}$$

In the case of the geodesic constraint ($\mathbb{T}_a = 0$), it can easily be verified that $\hat{\Theta}_{ab} \stackrel{\mathbb{T}_a=0}{=} \hat{\chi}_{ab} - \tilde{t}_{ab} = \hat{\Phi}_{ab}$ *i.e.* the second fundamental form of the null hypersurface $\mathcal{H}$ coincides with the spatial tensor $\hat{\Phi}_{ab}$. Hence we have,

$$\begin{aligned} \hat{\Theta}_{BA} &\stackrel{\mathbb{T}_a=0}{=} \hat{\Phi}_{BA} = \hat{\chi}_{BA} - \tilde{t}_{BA} \\ &\stackrel{\mathcal{H}}{=} \frac{1}{2}\Big(\partial_t q_{AB} + \hat{\mathcal{D}}_A V_B + \hat{\mathcal{D}}_B V_A - 2K_{0AB} - {}^{(2)}T_{DAB}V^D\Big) . \end{aligned} \tag{7.13}$$

We do indeed see that w.r.t. the adapted coordinate system, $\hat{\Theta}_{BA}$ contains the term $\hat{\mathcal{D}}_A V_B + \hat{\mathcal{D}}_B V_A$. Let us mention the reason as to why we are looking at the spatial tensor $\hat{\Theta}_{BA}$. In the absence of torsion, for the case of Einstein gravity, the second fundamental form of $\mathcal{H}$ in $(\mathcal{M}, g, \boldsymbol{\nabla})$ is of the form,

$$\Theta_{BA} \overset{(\mathcal{M},g,\boldsymbol{\nabla})}{=} \frac{1}{2}(\partial_t q_{AB} + \mathcal{D}_A V_B + \mathcal{D}_B V_A) \, . \tag{7.14}$$

For the suitable choice case of an adapted coordinate system on the null surface $\mathcal{H}$, we can make the induced metric q_{AB} of S_t independent of the time evolution parameter t *i.e.* $\partial_t q_{AB} = 0$. Then, we notice that the second fundamental form of $\mathcal{H}$ in Einstein gravity has the same form as that of the stress tensor of a viscous fluid (having no internal angular momentum) with velocity V_A. The symmetric combination of the velocity gradient tensor *i.e.* $1/2(\mathcal{D}_A V_B + \mathcal{D}_B V_A)$ can as usual be broken down into a trace part and a traceless shear part. The trace part which contains the divergence of the velocity field V_A is necessarily interpreted as the expansion scalar corresponding to the fluid flow lines. In that same respect, the trace of the second fundamental form $\Theta_{AB} = \Theta_{BA}$ gives the true expansion scalar of the null surface $\mathcal{H}$. This is perhaps the central reason as to why (in the absence of torsion), (7.3) or (7.7) can be interpreted as the DNS equation or the NS equation (in a boosted inertial frame) [82]. The viscous stress tensor for a conventional two-dimensional NS fluid is necessarily of the form $2\eta\sigma_{AB} + \xi\delta_{AB}\theta$, where η and ξ stand for the shear and bulk viscosity coefficients respectively. For the NS fluid, the trace-free shear tensor σ_{AB} is built from the derivatives of the velocity field V_A. In the case of vanishing torsion tensor, the spatial tensors $\hat{\Theta}_{AB}$, $\hat{\Phi}_{AB}$ and $\hat{\chi}_{AB}$ all coincide.

Coming back to the EC theory, we indeed see via (7.13) that the second fundamental form of $\mathcal{H}$ as usual contains the term $\hat{\mathcal{D}}_A V_B + \hat{\mathcal{D}}_B V_A$. However, now for the generic RC spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, $\hat{\Theta}_{BA}$ has the extra terms of $\partial_t q_{AB}$ and $-2K_{0AB} - {}^{(2)}T_{DAB}V^D$ involving the contorsion and the two dimension torsion tensor. These terms have no direct interpretation of fluid variables. Via certain choices of the metric and the adapted coordinate system along with the freedom of rescaling $\vec{l}$, one can make q_{AB} independent of t. However, the term involving the contorsion and torsion tensor *i.e.* $-2K_{0AB} - {}^{(2)}T_{DAB}V^D$ cannot be set to zero for the generic RC spacetime. Moreover, now in the RC spacetime, the second fundamental form $\hat{\Theta}_{BA}$ has both symmetric as well as antisymmetric parts due to the presence of the torsion tensor in its definition (7.13). This is certainly a far cry from the usual spacetime without torsion. It is precisely for this reason that by a particular choice of the metric and an adapted coordinate system, we cannot set $\partial_t q_{AB} - 2K_{0AB} - {}^{(2)}T_{DAB}V^D = 0$. It will not be possible to compensate for the antisymmetric contribution. The presence of the antisymmetric part in the definition (7.13) of $\hat{\Theta}_{BA}$ is the reason as to why we have a term $\hat{\mathcal{D}}^b\left(\frac{1}{2}q^c{}_t q^d{}_b T_{fdc} l^f\right)$ involving the spatial derivative of an antisymmetric tensor. This shows that in no way, can (7.3) or (7.7) be interpreted as some kind of (modified) DNS equation in the RC spacetime. This is because the stress tensor for the DNS fluid is by default symmetric. The precise symmetry property of the

Cauchy stress tensor is intimately connected to the underlying assumption that the material points describing the continuum (fluid) system have no intrinsic angular momentum. But this does fly in the face of the assumption underlying the RC spacetime in the EC theory. The very source of torsion is the intrinsic spin-angular momentum tensor describing the intrinsic spin of particles in the geometrization of RC spacetime. Thus there is a non-zero contribution involving the spatial covariant derivative of a completely antisymmetric tensor in (7.3) and (7.7). It is for this reason, we can safely conclude that if we were to allude a fluid/elastic medium interpretation to the dynamics of our null hypersurface $\mathcal{H}$ in EC theory, its shear tensor would certainly not be symmetric in general. Hence it is quite certain, that the projected ECKS field equations on $\mathcal{H}$ (as observed via the projection component $\hat{G}_{ab}l^a q^b{}_c$) cannot be interpreted as the DNS (or NS) fluid equation.

7.3.2 Any real life analogies possible?

Perhaps the most that we can stretch our imagination to give a real life analogy with the dynamics of (7.3) or (7.7) is that of Cosserat [246] fluids. In fact, Cosserat theory describes a classical elastic continuum in which the material point bodies have translation as well as rotational degrees of freedom. Conventional Navier-Stokes dynamics does not incorporate any intrinsic length scales. Each point of the Cosserat medium can be visualized as an infinitesimal rigid body. There exists both stress and couple stress as responses to the translation and rotational degrees of freedom respectively.

Let us now briefly describe the Cosserat fluid in Cartesian coordinates y^μ. The particle velocity and spin are designated as $v_\mu(t, y^\nu)$ and $w_\mu(t, y^\nu)$ respectively. The velocity gradient tensor can as usual be broken into a symmetric and antisymmetric part,

$$\partial_\mu v_\nu = D_{\mu\nu} + W_{\mu\nu} \,, \tag{7.15}$$

where $D_{\mu\nu} = \frac{1}{2}(\partial_\mu v_\nu + \partial_\nu v_\mu)$ and $W_{\mu\nu} = \frac{1}{2}(\partial_\mu v_\nu - \partial_\nu v_\mu)$ represent strain rate (or classical deformation rate) tensor and the rotation rate (or vorticity) tensor respectively. The Cosserat generalization to the NS equations are then described as [246],

$$\rho\left(\partial_t v_\alpha + v^\mu \partial_\mu v_\alpha\right) = \partial_\alpha P + \gamma \partial^\mu(\partial_\mu v_\alpha + \partial_\alpha v_\mu) + \gamma_c \partial^\mu([\partial_\mu v_\alpha - \partial_\alpha v_\mu] - 2\epsilon_{\mu\alpha\rho} w^\rho) + f^{\text{ext}}_\alpha \,. \tag{7.16}$$

In the above Eq. (7.16), P stands for the fluid pressure and γ is identified as the classical macroscopic fluid viscosity and is related to the symmetric part of the shear tensor. The quantity γ_c represents an extra viscosity material parameter. The relative spin of the Cosserat fluid particle w.r.t the background vorticity is accounted for by the extra viscosity parameter γ_c [246]. f^{ext}_α represents the external volume or body force density that acts per unit volume of the fluid.

7.3.2.1 In coordinates adapted to $\mathcal{H}$

Let us rewrite (7.7) in the following suggestive way,

$$\begin{aligned}
&\partial_t\Big(\frac{-\hat{\Omega}_A}{8\pi}\Big) + V^B\hat{\mathcal{D}}_B\Big(\frac{-\hat{\Omega}_A}{8\pi}\Big) \\
&+ \Big(\frac{-\hat{\Omega}_B}{8\pi}\Big)\hat{\mathcal{D}}_A V^B + {}^{(2)}T^B_{CA}V^C\Big(\frac{-\hat{\Omega}_B}{8\pi}\Big) + \Big(\frac{-\hat{\Omega}_A}{8\pi}\Big)(\overset{(d)}{\hat{\theta}_l} - T_b l^b) = -q^b{}_A T^{(m)}_{ab} l^a \\
&+ \frac{1}{16\pi}(\hat{\nabla}_c + T_c)\Big(- S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c\Big) l^a q^b{}_A - \frac{1}{8\pi}\hat{\Theta}_{ba}\mathscr{P}^b q^a{}_A + \frac{1}{8\pi}T_{iba}\hat{\Theta}^{bi}q^a_A \\
&- \hat{\mathcal{D}}_A\Big(\frac{\kappa}{8\pi}\Big) + \hat{\mathcal{D}}^B\Big[\frac{1}{8\pi}\Big(\overset{*}{\hat{\sigma}}_{BA} - \frac{1}{2}q_{AB}(\overset{(d)}{\hat{\theta}_l} - T_b l^b)\Big)\Big] \\
&+ \hat{\mathcal{D}}^B\Big[\frac{1}{8\pi}\Big(- \frac{1}{2}q^d{}_B q^c{}_A T_{fcd} l^f\Big)\Big] \, . \qquad (7.17)
\end{aligned}$$

Now referring to (7.17), we make the following identification,

$$\begin{aligned}
f^{\text{ext}}_A \equiv &- T^{(m)}_{ab} q^b{}_A l^a + \frac{1}{16\pi}(\hat{\nabla}_c + T_c)\Big(- S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c\Big) l^a q^b{}_A \\
&- \frac{1}{8\pi}\Big(\hat{\Theta}_{ba}\mathscr{P}^b q^a{}_A - T_{iba}\hat{\Theta}^{bi}q^a_A\Big) \, , \qquad (7.18)
\end{aligned}$$

where f^{ext}_A represents the surface force density on the two-dimensional cross-section S_t. It is the momentum per unit area per unit coordinate time t. We notice that the definition of force density includes the projection of matter energy momentum tensor onto the two-surface S_t via $-T^{(m)}_{ab}q^b{}_A l^a$. However, it also includes contribution due to the torsion tensor as well. Specifically we note that in this identification that we have made for the force density, there is the existence of the terms $\hat{\Theta}_{ba}\mathscr{P}^b q^a{}_A$ and $-T_{iba}\hat{\Theta}^{bi}q^a_A$. This means that the torsion tensor coupled with the second fundamental form $\hat{\Theta}_{ab}$ provides a kind of a force density on the surface S_t. Once such identification has been made, we make comparisons between (7.17) and (7.16). This will allow us to make the necessary identification for the description of the two-dimensional viscous null fluid governed by (7.17). The surface momentum density π_A is related to the Hájiček 1-form as

$$\pi_A = -\frac{1}{8\pi}\hat{\Omega}_A \, . \qquad (7.19)$$

The velocity of our null fluid is V^A, whereas $\kappa/(8\pi)$ denotes the null fluid pressure. From the symmetric part of the stress tensor, we note that $1/(16\pi)$ is the shear viscosity coefficient, whereas $-1/(16\pi)$ represents the bulk viscosity coefficient. When comparing (7.16) and (7.17), we notice that the comparative terms (for the antisymmetric part of the stress tensor) are respectively, $\gamma_c\partial^\mu(\partial_\mu v_\alpha - \partial_\alpha v_\mu - 2\epsilon_{\mu\alpha\rho}w^\rho)$ and $\hat{\mathcal{D}}^B\Big[\frac{1}{8\pi}\Big(- \frac{1}{2}q^d{}_B q^c{}_A T_{fcd} l^f\Big)\Big]$. It is clear that on the side of the null fluid dynamics (7.17), there does not exist any term of the form $\hat{\mathcal{D}}^B(\hat{\mathcal{D}}_B V_A - \hat{\mathcal{D}}_A V_B)$. This is also evident from the expansion the second fundamental form $\hat{\Theta}_{AB}$ (7.13) which does not contain any antisymmetric combination of $\hat{\mathcal{D}}_B V_A$. Given (7.16), the generic stress for the Cosserat fluid has antisymmetric contribution. The antisymmetric part of the Cauchy stress tensor for the Cosserat fluid is $\gamma_c(\partial_\mu v_\alpha - \partial_\alpha v_\mu - 2\epsilon_{\mu\alpha\rho}w^\rho)$. So the null fluid that we are trying to describe via (7.17) is a special case of the Cosserat fluid in the sense that the antisymmetric part of its own stress tensor will have no contribution from $\hat{\mathcal{D}}_B V_A - \hat{\mathcal{D}}_A V_B$. Having decided on this issue, we now compare $2\gamma_c\partial^\mu(-\epsilon_{\mu\alpha\rho}w^\rho)$ and $\hat{\mathcal{D}}^B\Big[\frac{1}{8\pi}\Big(-\frac{1}{2}q^d{}_B q^c{}_A T_{fcd} l^f\Big)\Big]$. Its clear that the spin w^ρ of the Cosserat fluid particle is related to the torsion tensor on the null fluid side. This was perhaps anticipated since the origin of torsion is the spin angular momentum density in the matter action part. The extra material viscosity parameter γ_c for the null fluid is hence identified as $\gamma_c = 1/(16\pi)$. This establishes the connection that $\epsilon_{\mu\alpha\rho}w^\rho$ is related to $(1/2)T_{fAB}l^f$. In the characterization of the two-dimensional null fluid we have to take notice of the fact that there exists the extra factor of $\hat{\Omega}_B\hat{\mathcal{D}}_A V^B + {}^{(2)}T^B_{\;CA}V^C\hat{\Omega}_B$ along with the material derivative term in (7.7). The analog of the first extra term in the case of Einstein gravity *i.e.* $\Omega_B\mathcal{D}_A V^B$ was already present in the DNS equation.

In Einstein gravity case it has been observed that the extra factors can be removed if the analysis is done in locally constructed inertial frame related to the adapted coordinates [82]. Then the Háj́iček 1-form equation exactly maps to NS dynamics. Having this instance, next we want to proceed our analysis in the local inertial frame. This would make the analogy more transparent.

7.3.2.2 Analysis in a boosted local inertial frame constructed in adapted coordinates

In this section we want to look at the evolution equation of $\hat{\Omega}_A$ in the adapted coordinate system $x^i = (t, x^\mu)$ *i.e.* (7.7) from the perspective of a local inertial observer. However, before we go over to that particular analysis, we should comment on whether it is indeed possible to construct a local inertial frame (LIF) in the EC theory. The EC theory is provided with the non symmetric, however metric-compatible Cartan connection $\hat{\boldsymbol{\nabla}}$. In fact, it has been pointed out that the only connection (with torsion) that is compatible with the Einstein equivalence principle is the Cartan connection [64, 247, 248]. The equivalence principle in the context of EC theory means the existence of a unique local frame where all the components of the Cartan connection coefficients vanish. If we insist on only holonomic or coordinate bases, then the only possible way for the connection coefficients (in the coordinate bases) to vanish is to demand that the torsion tensor vanishes. However, one can introduce a local Lorentz or orthonormal tetrad basis at the point p in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ where we want to construct our local inertial frame. The local orthonormal tetrad basis are required to be locally Minkowskian at the point p. With respect to this anholonomic tetrad bases, it has been explicitly shown [64, 248], that it is indeed possible to set the Cartan connection coefficient (in the local orthonormal tetrad bases) to zero at the point

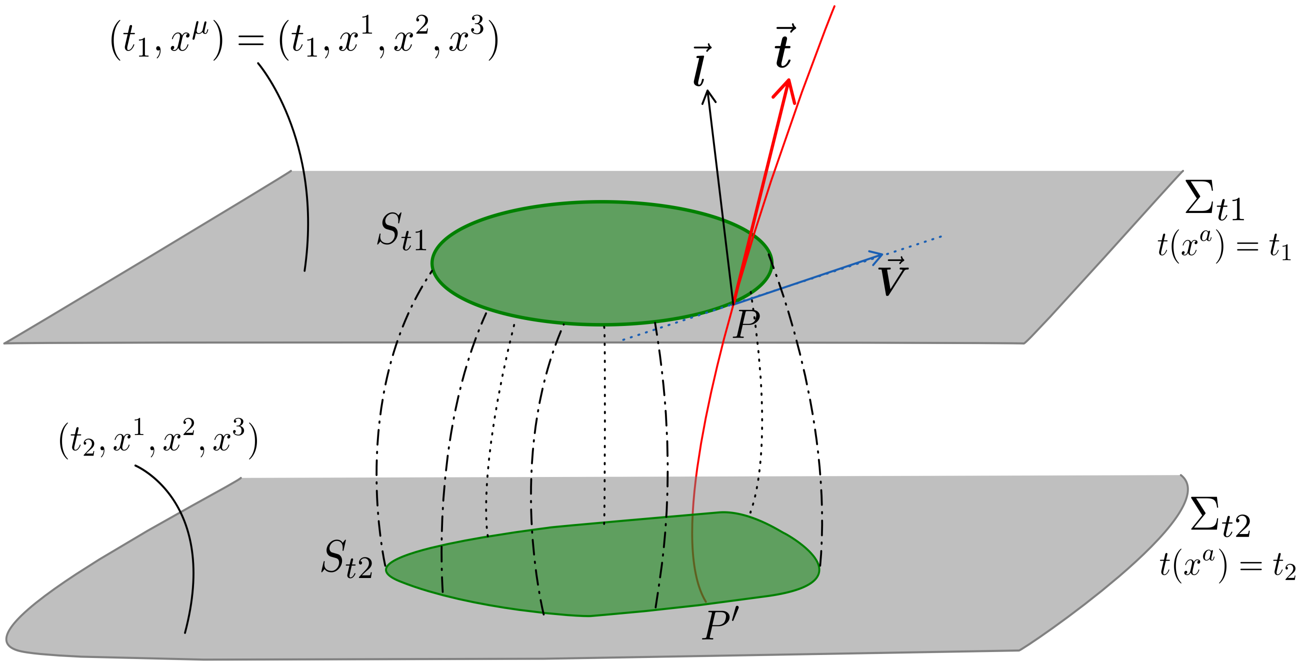

FIGURE 7.1: Construction of the adapted coordinate system w.r.t. $\mathcal{H}$. The surface velocity vector $\vec{V}$ is tangent to the cross-section S_t. The time evolution vector field $\vec{t}$ (shown in red) connects the same spatial points on Σ_t.

p, without demanding that the torsion tensor also vanishes identically. We will hence not repeat the argument here.

In order to study the dynamics of the null surface $\mathcal{H}$ via (7.7) in the adapted coordinate system by constructing a LIF at a point p on $\mathcal{H}$, we need to revisit both the intrinsic and the extrinsic geometry of the null surface which we have done in chapter 2. Let us again go back to the notion of the adapted coordinate system that we discussed in Sec. 2.5. The coordinates on S_t surface are $x^A = (x^2, x^3)$ and the coordinate basis vectors of $T_p(S_t)$ are $\vec{e}_A = \vec{\partial}_A = (\vec{e}_2, \vec{e}_3) = (\vec{\partial}_2, \vec{\partial}_3)$. The coordinate time evolution vector $\vec{t} = \vec{\partial}_t$ connects same spatial points along neighboring Σ_t hypersurfaces. We remind that the coordinates defined on the null surface are $x^{\tilde{\mu}} = (t, x^A)$. On the null surface $\mathcal{H}$, we have $\vec{l} \overset{\mathcal{H}}{=} \vec{t} + \vec{V}$, where $\vec{V} \in T_p(S_t)$. Hence w.r.t the coordinates on $\mathcal{H}$ we have $l^{\tilde{\mu}} \overset{\mathcal{H}}{=} (1, V^A)$. From (2.155), it can be easily verified that $l_{\tilde{\mu}} = q_{\tilde{\mu}\tilde{\nu}} l^{\tilde{\nu}} = (l_t, \{l_A\}) \overset{\mathcal{H}}{=} (0, 0, 0)$. The null generator $\vec{l}$ belongs the tangent space $T_p(\mathcal{H})$ of $\mathcal{H}$ and hence can be lowered w.r.t the induced metric $\boldsymbol{q}$.

In the discussion of the extrinsic geometry of the null surface $\mathcal{H}$ in Sec. 2.3.3, we had discussed in detail about the Weingarten map. The Weingarten map was defined for any vector field $\vec{v} \in T_p(\mathcal{H})$ as,

$$ {}^{\mathcal{H}}\hat{\Upsilon}^i{}_j v^j = \hat{\boldsymbol{\nabla}}_v l^i \,. \tag{7.20}$$

Since the vector field $\vec{v}$ is arbitrary, we have as a result, ${}^{\mathcal{H}}\hat{\Upsilon}^i{}_j = \hat{\nabla}_j l^i$. With respect to the coordinates on $\mathcal{H}$, we hence have, ${}^{\mathcal{H}}\hat{\Upsilon}^{\tilde{\alpha}}{}_{\tilde{\beta}} = \hat{\nabla}_{\tilde{\beta}} l^{\tilde{\alpha}}$. The second fundamental form ${}^{\mathcal{H}}\boldsymbol{\Theta}$

restricted to $\mathcal{H}$ is defined for arbitrary vectors $(\vec{u}, \vec{v}) \in T_p(\mathcal{H}) \times T_p(\mathcal{H})$ in the following way [61, 90],

$$^{\mathcal{H}}\Theta_{ij} u^i v^j \equiv u_i \, (^{\mathcal{H}}\hat{\Upsilon}^i{}_j v^j) \,. \tag{7.21}$$

Let us make the choice of the vectors $\vec{u}$ and $\vec{v}$ to be the basis vectors $\vec{e}_{\tilde{\mu}}$ and $\vec{e}_{\tilde{\nu}}$ respectively. Then, we have,

$$\begin{aligned} &^{\mathcal{H}}\Theta_{ij}(\vec{e}_{\tilde{\mu}})^i(\vec{e}_{\tilde{\nu}})^j = (\vec{e}_{\tilde{\mu}})_i \, ^{\mathcal{H}}\hat{\Upsilon}^i{}_j(\vec{e}_{\tilde{\nu}})^j = (\vec{e}_{\tilde{\mu}})_i(\hat{\nabla}_j l^i)(\vec{e}_{\tilde{\nu}})^j = (\vec{e}_{\tilde{\mu}})_i(\hat{\nabla}_j l^i)\delta^j{}_{\tilde{\nu}} \,; \\ &^{\mathcal{H}}\Theta_{ij}\delta^i_{\tilde{\mu}}\delta^j_{\tilde{\nu}} = (\vec{e}_{\tilde{\mu}})_i(\hat{\nabla}_{\tilde{\nu}} l^i) = -l_i\hat{\nabla}_{\tilde{\nu}}(\vec{e}_{\tilde{\mu}})^i = -l_i(\hat{\nabla}_{\tilde{\nu}}\delta^i{}_{\tilde{\mu}}) = -l_i\hat{\Gamma}^i{}_{\tilde{\nu}j}\delta^j{}_{\tilde{\mu}} \,; \\ &^{\mathcal{H}}\Theta_{\tilde{\mu}\tilde{\nu}} = -l_i\hat{\Gamma}^i{}_{\tilde{\nu}\tilde{\mu}} \,. \end{aligned} \tag{7.22}$$

So we see that the second fundamental form restricted to the null surface $\mathcal{H}$ is proportional to the connection coefficients of the RC spacetime. The non symmetric nature of $^{\mathcal{H}}\boldsymbol{\Theta}$ (as a $(0,2)$ tensor field) is also evident from the fact that the connection coefficients are not symmetric. In fact, restricted to the transverse space S_t, we see that $^{\mathcal{H}}\Theta_{AB} = -l_i\hat{\Gamma}^i{}_{BA}$. The above analysis leading to (7.22) can also be arrived from the relation (2.90). Expressing (2.90) w.r.t the coordinates $x^{\tilde{\mu}}$ on $\mathcal{H}$, we have,

$$\begin{aligned} \hat{\nabla}_{\tilde{\nu}} l_{\tilde{\mu}} &= \hat{\Theta}_{\tilde{\mu}\tilde{\nu}} + \hat{\omega}_{\tilde{\nu}} l_{\tilde{\mu}} - l_{\tilde{\nu}}(\hat{\nabla}_{\vec{k}} l_{\tilde{\mu}}) \\ \implies \hat{\Theta}_{\tilde{\mu}\tilde{\nu}} &= -\hat{\Gamma}^i{}_{\tilde{\nu}\tilde{\mu}} l_i \,. \end{aligned} \tag{7.23}$$

In the above, we have used the fact that $l_{\tilde{\mu}} \stackrel{\mathcal{H}}{=} (0,0,0)$.

Next, we come to analysis of the rotation 1-form, again restricted on $\mathcal{H}$. In order to facilitate the form of the rotation 1-form restricted to $\mathcal{H}$, we would require a knowledge of the uniquely defined auxiliary null vector field $\vec{k}$. The auxiliary null vector field is not defined on the tangent space of $\mathcal{H}$. Hence we would now require to choose a basis for $T_p(\mathcal{M})$. Let us define the outward pointing spacelike unit normal to the surface S_t by $\vec{s}$ [61]. Hence the basis vectors on $T_p(\Sigma_t)$ are $\vec{e}_\mu = (\vec{s}, \vec{e}_A)$. Extending the Σ_t surfaces along the time evolution vector field, we have that the basis for $T_p(\mathcal{M})$ is $\vec{e}_i = (\vec{t}, \vec{s}, \vec{e}_A)$. The covector k_i decomposed in such a basis is $k_i = (k_t, k_s, k_A)$. From the condition that $\vec{k} \cdot \vec{V} = 0$, we have $k_A = 0$. Similarly, the condition, $\vec{k} \cdot \vec{l} = -1$ leads us to have $k_t = -1$. Hence the expansion of the covector k_i in the basis $(\vec{t}, \vec{s}, \vec{e}_A)$ is $k_i = (-1, k_s, 0)$. From the working definition of the rotation 1-form (2.95), we consider the projection of $\hat{\Omega}_i$ onto the null surface $\mathcal{H}$. The projection tensor onto $\mathcal{H}$ is given by $\Pi^i{}_j = \delta^i{}_j + k^i l_j$ [61]. The projected part of the rotation 1-form onto $\mathcal{H}$ is defined by,

$$^{\mathcal{H}}\hat{\omega}_i \equiv \Pi^j{}_i\hat{\omega}_j = -\Pi^j{}_i(k_m\hat{\nabla}_j l^m) = -k_m\hat{\nabla}_i l^m + l_i k_m k^j\hat{\nabla}_j l^m \,. \tag{7.24}$$

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In the above, we have used the fact that $\Gamma^{j}{}_{i} l_j = 0$. We now specifically look at the components of ${}^{\mathcal{H}}\hat{\omega}_i$ in the coordinate basis of $T_p(\mathcal{H})$.

$$\begin{aligned}
{}^{\mathcal{H}}\hat{\omega}_{\tilde{\alpha}} &= -k_m(\hat{\nabla}_{\tilde{\alpha}} l^m) + l_{\tilde{\alpha}} k_m (k^j \hat{\nabla}_j l^m) = -k_m(\hat{\nabla}_{\tilde{\alpha}} l^m) \\
&\stackrel{\mathcal{H}}{=} -k_0 \hat{\nabla}_{\tilde{\alpha}} l^0 - k_1(\hat{\nabla}_{\tilde{\alpha}} l^1) - k_A(\hat{\nabla}_{\tilde{\alpha}} l^A) \stackrel{\mathcal{H}}{=} \hat{\nabla}_{\tilde{\alpha}} l^0 - k_1(\hat{\nabla}_{\tilde{\alpha}} l^1) \\
{}^{\mathcal{H}}\hat{\omega}_{\tilde{\alpha}} &\stackrel{\mathcal{H}}{=} \hat{\Gamma}^0{}_{\tilde{\alpha}j} l^j - \hat{\Gamma}^1{}_{\tilde{\alpha}j} l^j k_1 \,.
\end{aligned} \tag{7.25}$$

The projection of the rotation 1-form onto the 2-surface S_t is the Hájíček 1-form $\hat{\Omega}_A$. It is clear that ${}^{\mathcal{H}}\hat{\omega}_{\tilde{\alpha}} = ({}^{\mathcal{H}}\hat{\omega}_0, {}^{\mathcal{H}}\hat{\omega}_A) = ({}^{\mathcal{H}}\hat{\omega}_0, \hat{\Omega}_A)$. This allows us to identify,

$$\hat{\Omega}_A = {}^{\mathcal{H}}\hat{\omega}_A \stackrel{\mathcal{H}}{=} \hat{\Gamma}^0{}_{Aj} l^j - \hat{\Gamma}^1{}_{Aj} l^j k_1 \,. \tag{7.26}$$

So, again w.r.t the basis established on $\mathcal{H}$, we find that the rotation 1-form (restricted to $\mathcal{H}$) is proportional to the connection coefficients. Similar analysis can also be brought through a tetrad basis $(\vec{\boldsymbol{n}}, \vec{\boldsymbol{s}}, \vec{\boldsymbol{e}}_A)$ of $T_p(\mathcal{M})$, where $\vec{\boldsymbol{n}}$ denotes the timelike unit normal to the $t(x^i) =$ constant surface. It can be shown that the Hájíček 1-form w.r.t the above mentioned basis is associated to the tetrad connection coefficients [80] as $\hat{\Omega}_A = \hat{\Gamma}^1{}_{0A}$.

Having done this analysis, let us now look at the evolution equation for $\hat{\Omega}_A$ *i.e.* (7.17) in the adapted (or stationary) coordinates $x^i = (t, x^\mu)$ w.r.t $\mathcal{H}$. We propose to consider this expression around the given event point $p \in \mathcal{H}$ in a LIF. In the LIF, the connection coefficients will vanish, but not their derivatives. We will be working with a Lorentz boosted inertial frame given by the fact that the metric coefficients are constant. The metric in this Lorentz boosted LIF is diagonal in structure with $V^A \neq 0$. The physical interpretation of such a boosted inertial frame has been explained in detail in [82]. Under the consideration of such a boosted LIF, all the terms that are proportional to the connection coefficients vanish. Hence $\hat{\Omega}_A$ and $\hat{\Theta}_{BA}$ vanish in the LIF, however not their derivatives. Let us remember the fact that (7.17) has been derived under the geodesic constraint. Under this assumption, the trace of the second fundamental form $\hat{\Theta}_{BA}$ which is $(\overset{(d)}{\hat{\theta}_l} - T_b l^b)$, the shear tensor $\overset{*}{\hat{\sigma}}_{BA}$ and the antisymmetric part $\hat{\Theta}_{[BA]} = \frac{1}{2} q^c{}_A q^d{}_B T_{fdc} l^f = \frac{1}{2}[T_{0BA} + {}^2T_{DBA} V^D]$ all vanish in the boosted LIF. Since, we are working under the geodesic constraint, the second fundamental form is completely a spatial bilinear (given by the fact that $\hat{\Theta}_{ab} l^a = 0, \hat{\Theta}_{ab} l^b = 0, \hat{\Theta}_{ab} k^a = 0$ and $\hat{\Theta}_{ab} k^b = 0$). Let us look at the term $-\frac{1}{8\pi}\left(\hat{\Theta}_{ba} \mathscr{P}^b q^a{}_A - T_{iba} \hat{\Theta}^{bi} q^a_A\right)$ in the external force density term (7.18). The term within the parentheses (in the geodesic constraint) evaluates to $\left(\hat{\Theta}_{CB} \mathscr{P}^C q^B{}_A - {}^2T_{CBD} \hat{\Theta}^{BC} q^D{}_A\right)$ which naturally vanishes in the boosted LIF. Finally, under these considerations, let us consider the evolution equation

(7.17), which now becomes (remembering that in the adapted coordinates $q^a_A = \delta^a{}_B$),

$$\begin{aligned}
&\partial_t\Big(\frac{-\hat{\Omega}_A}{8\pi}\Big) + V^B\partial_B\Big(\frac{-\hat{\Omega}_A}{8\pi}\Big) = \\
&-\delta^b{}_A T^{(m)}_{ab} l^a + \frac{1}{16\pi}(\partial_c + T_c)\Big(-S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c\Big) l^a \delta^b{}_A \\
&-\partial_A\Big(\frac{\kappa}{8\pi}\Big) + \partial^B\Big[\frac{1}{8\pi}\Big(\overset{*}{\hat{\sigma}}_{BA} - \frac{1}{2}\delta_{AB}(\overset{(d)}{\hat{\theta}_l} - T_b l^b)\Big)\Big] + \partial^B\Big[\frac{1}{8\pi}\Big(-\frac{1}{2}\delta^d{}_B\delta^c{}_A T_{fcd} l^f\Big)\Big] \,. \qquad (7.27)
\end{aligned}$$

Even though the term $\hat{\Omega}_A(\overset{(d)}{\hat{\theta}_l} - T_b l^b)$ is zero in the LIF, it can be formally added so that its analogous structure with the Cosserat generalization to the NS fluid equation becomes evident. Now, in the boosted LIF, the equation (7.27) should be compared with the Cosserat fluid equation (7.16). The comparison shows that the dynamics of the Hájíček 1-form via the ECKS field equations looked through the adapted coordinate system and viewed in a LIF is structurally similar to that a two-dimensional null Cosserat fluid. Hence we say that in the EC theory, the generic null surface behaves as a viscous Cosserat fluid. Here, we have the exact material derivative of the momentum density $\pi_A = -\hat{\Omega}_A/8\pi$. The null fluid pressure is $\kappa/8\pi$ while the shear and extra material viscosity coefficients are $1/16\pi$. The bulk viscosity coefficient is $-1/16\pi$. The external force density is $f^{\text{ext}}_A \equiv -T^{(m)}_{ab}\delta^b{}_A l^a + \frac{1}{16\pi}(\partial_c + T_c)\Big(-S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c\Big) l^a \delta^b{}_A$ which quite naturally arises from the term $-1/8\pi\hat{G}_{ab} l^a q^b{}_A$, once the ECKS field equations are applied in the LIF. This completes our analysis of the evolution equation (7.7) through an adapted coordinate system w.r.t. $\mathcal{H}$ about the point $p \in \mathcal{H}$, where a boosted LIF has been constructed.

A few comments are in order. Let us discuss again about the stress tensor of the Cosserat null fluid that we have been considering. The analogy was complete under the consideration that the stress tensor $\mathcal{S}_{AB}$ for our two-dimensional null fluid describing the dynamics of $\mathcal{H}$ in the EC theory has the form $\mathcal{S}_{BA} = 2\eta\overset{*}{\hat{\sigma}}_{BA} + \xi\delta_{BA}(\overset{(d)}{\hat{\theta}_l} - T_b l^b) - 2\gamma_c(1/2)T_{fAB} l^f$. It is obviously necessary that the trace-free shear tensor is built from the derivative of the velocity field V_A *i.e.* we must have,

$$\hat{\mathcal{D}}^B\overset{*}{\hat{\sigma}}_{BA} = \frac{1}{2}\hat{\mathcal{D}}^B\Big[(\hat{\mathcal{D}}_B V_B + \hat{\mathcal{D}}_A V_B) - \frac{1}{2}q_{AB}(\hat{\mathcal{D}}_C V^C)\Big] \,. \qquad (7.28)$$

However, as evident from (7.13), that would be possible only if $\partial_t q_{AB} - 2K_{0(AB)} = \partial_t q_{AB} - K_{0AB} - K_{0BA}$ was equal to zero. Now, it might be that in some choice of the adapted coordinate system, we might be able to set $\partial_t q_{AB} = K_{0AB} + K_{0AB}$. However, the interpretation of $\partial_t q_{AB}$ in terms of fluid variables is under debate. This issue was also present in the DNS case [82]. In terms of the analogy, we pointed out that the momentum density of our null fluid in the EC theory is proportional to the Hájíček 1-form $\hat{\Omega}_a$ which a kinematical quantity. Its physical interpretation would be clearer if it was shown that the momentum density π_A is indeed proportional to the velocity of the null fluid V_A. For the case of Einstein gravity, via the use of a particular adapted coordinate system w.r.t $\mathcal{H}$, it was indeed shown that the Hájíček 1-form was proportional to V_A [38]. The same logic applies over here except that in the case of EC theory, we would have extra terms involving the two-dimensional torsion tensor on S_t. We notice, that for our null fluid, both the shear viscosity and the extra material viscosity coefficients are positive. However, the bulk viscosity coefficient is negative. This feature was also present for the case of Einstein gravity. This is in contrast to the real world scenario where viscosity coefficients are positive. In the case of hydrodynamics, a negative bulk viscosity would lead to local entropy decrease with time. Reasons for the negative sign in the bulk viscosity has been attributed to the teleological nature of the horizon [218, 249–251]. Finally, while considering the antisymmetric part of the stress tensor, we made the analogy that the quantity $\epsilon_{\mu\alpha\rho}w^{\rho}$ (7.16) is related to $(1/2)T_{fAB}l^{f}$ for our null fluid (7.17). One could have the viewpoint, that it might be better to relate T_{fAB} to the intrinsic spin angular momentum tensor for the EC theory via the use of $T_{abc} = S_{abc} + (1/2)(g_{ab}S_c - g_{ac}S_b)$ and $S_{abc} = 8\pi\tau_{abc}$, where $S_b = g^{ac}S_{abc}$. In that sense, perhaps the connection between the spin w^{μ} of the Cosserat fluid particle and the intrinsic spin density of our null fluid might have been more evident. This would have naturally resulted in a different material viscosity parameter γ_c. However, in dealing, with this analysis, we are of the viewpoint, that different kinematical quantities of $\mathcal{H}$ are provided fluid interpretation (like $-\hat{\Omega}_A/8\pi$ as momentum density, $\kappa/8\pi$ as the pressure etc). Similarly, here we adopt the geometrical viewpoint of torsion rather than that of a dynamical field. Such issues regarding whether torsion is to be interpreted as a geometrical or dynamical field has been explored in [64, 90].

7.3.2.3 Covariant generalization of Cosserat fluid

Finally, let us note that we have made a formal analogy between (7.7) and the Cosserat generalization to the NS fluid equation (7.16). However instead, we could have taken the viewpoint that the natural generalization of the material derivative (in real world fluids) to the case of (hypersurface) null fluids on a genuinely curved background is the notion of the Lie derivative. Under that viewpoint, all the extra terms *i.e.* $\hat{\Omega}_B\hat{\mathcal{D}}_A V^B + {}^{(2)}T^{B}_{CA}V^C\hat{\Omega}_B$ can be effectively incorporated in the Lie derivative term as evident from (7.6). As a result, we have from (7.6) and (7.7),

$$\begin{aligned}
& q^a{}_A \pounds_l \hat{\Omega}_a + \hat{\Omega}_A (\overset{(d)}{\hat{\theta}_l} - T_b l^b) \\
& = 8\pi q^b{}_A T^{(m)}_{ab} l^a - \frac{1}{2}(\hat{\nabla}_c + T_c)\Big(- S^c{}_{ab} + S_{ab}{}^c + S_{ba}{}^c\Big) l^a q^b{}_A + \hat{\Theta}_{ba} \mathscr{P}^b q^a{}_A \\
& - T_{iba} \hat{\Theta}^{bi} q^a_A + \hat{\mathcal{D}}_A \kappa - \hat{\mathcal{D}}_B \overset{*}{\hat{\sigma}}{}^B{}_A - \hat{\mathcal{D}}^B \Big(\frac{1}{2} q^c{}_A q^d{}_B T_{fdc} l^f \Big) + \frac{1}{2} \hat{\mathcal{D}}_A (\overset{(d)}{\hat{\theta}_l} - T_b l^b) \, . \qquad (7.29)
\end{aligned}$$

The above dynamical equation for our null fluid can then be compared with the Cosserat generalization of the NS equation (7.16) with the identification that the material derivative has now been replaced with the Lie derivative term (for the null fluid).

7.4 The tidal force equation in the RC spacetime $(\mathcal{M}, g, \hat{\nabla})$

We present the derivation of the tidal force equation in the RC spacetime for the sake of completeness. We will see that the tidal force equation is related to the evolution of the symmetric traceless shear tensor ${}^{(l,d)}\hat{\sigma}_{ab}$ along the null generators $\vec{l}$. As again, we begin with the Ricci identity,

$$\begin{aligned}
l^k \Big[\hat{\nabla}_k, \hat{\nabla}_j \Big] l_i &= l^k \Big(- \hat{R}^m{}_{ikj} l_m - T^m{}_{kj} \hat{\nabla}_m l_i \Big) \\
\implies l^k \hat{\nabla}_k \hat{\nabla}_j l_i &= l^k \hat{\nabla}_j (\hat{\nabla}_k l_i) - \hat{R}_{mikj} l^m l^k - T^m{}_{kj} l^k (\hat{\nabla}_m l_i) \, . \qquad (7.30)
\end{aligned}$$

Our analysis follows [61]. With respect to (3.29), we have the expansion for the covariant derivative of the null generators as,

$$\hat{\nabla}_a l_b = \hat{\Phi}_{ba} + \hat{\omega}_a l_b - l_a (k^i \hat{\nabla}_i l_b) - k_a q^c{}_b \mathbb{T}_c \, . \qquad (7.31)$$

This allows us to have,

$$
\begin{aligned}
& q^i{}_m q^j{}_n l^r \hat{\nabla}_r \left({}^{(l,d)}\sigma_{ij} - \tilde{t}_{ij} \right) + {}^{(l,d)}\sigma_m{}^i q_n^j T_{irj} l^r - \tilde{t}_m{}^i q_n^j T_{irj} l^r \\
& + \frac{1}{2} \overset{(d)}{\hat{\theta}_l} \left(2 {}^{(l,d)}\sigma_{mn} - \tilde{t}_{mn} - q^i{}_m q^j{}_n T_{jri} l^r + \frac{1}{2} q_{mn} (q^{rs} T_{rts} l^t) + \tilde{t}_{nm} \right) \\
& - {}^{(l,d)}\sigma_{mi} \tilde{t}^i{}_n - {}^{(l,d)}\sigma_{ni} \tilde{t}^i_m + q_{mn} ({}^{(l,d)}\sigma_{ij} \tilde{t}^{ij}) + \tilde{t}_{mi} \tilde{t}^i{}_n - \frac{1}{2} q_{mn} (\tilde{t}^{ij} \tilde{t}_{ji}) \\
& - (\kappa - \vec{k} \cdot \mathbb{T}) \left(\frac{1}{2} q_{mn} (q^{rs} T_{rts} l^t) + {}^{(l,d)}\sigma_{mn} - \tilde{t}_{mn} \right) + \frac{1}{2} q_{mn} \left(l^k \hat{\nabla}_k (q^{rs} T_{rts} l^t) \right) \\
& - \frac{1}{2} q_{mn} \left({}^{(l,d)}\sigma^{ij} T_{jri} l^r - \tilde{t}^{ij} T_{jri} l^r \right) + q^r{}_m \hat{\Omega}_n \mathbb{T}_r \\
& + \frac{1}{2} q_{mn} \hat{\mathcal{D}}_i (q^{ij} \mathbb{T}_j) - \frac{1}{2} q_{mn} (q^{ij} \hat{\Omega}_i \mathbb{T}_j) - \hat{\mathcal{D}}_n (q^r{}_m \mathbb{T}_r) \\
& + q^b{}_m q^d{}_n \left((\hat{\nabla}_c K_{adb} - \hat{\nabla}_d K_{acb}) + T^i{}_{cd} K_{aib} + (K^i{}_{cb} K_{adi} - K^i{}_{db} K_{aci}) \right) l^a l^c \\
& - \frac{1}{2} q_{mn} \left(\hat{\nabla}_i K^i{}_{ca} + \hat{\nabla}_c T_a + T^i{}_{jc} K^j{}_{ia} + K^i{}_{ja} K^j{}_{ci} + T_i K^i{}_{ca} \right) l^a l^c = \\
& - q^b{}_m q^d{}_n C_{abcd} l^a l^c \, .
\end{aligned} \tag{7.32}
$$

A detailed derivation of the above result has been shown in Appendix 7.7. Even though we are considering an integrable null hypersurface $\mathcal{H}$ generated by $\vec{l}$ in the RC spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, the null generators are themselves null geodesics in the sense that they satisfy $l^b \nabla_b l^a = \kappa l^a$. However, in this general setup, the null generators are not autoparallel along themselves. In the context of a congruence of geodesic null curves, the term $-q^b{}_m q^d{}_n C_{abcd} l^a l^c$ (present on the R.H.S of (7.32)) is related to the geodesic deviation equation [2, 95] between two null geodesics. The driving force behind the relative acceleration between two neighboring null geodesics is directly related to the term involving the Weyl tensor in the R.H.S of (7.32). It is in this respect that in literature, the above Eq. (7.32) is called the tidal force equation [216]. Our analysis also allows us to arrive at a different form of the NRE (as seen in (7.67)) corresponding to the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_l}$ as compared to the ones presented in [64, 90]. Let us mention that the tidal force equation for a null congruence has also been derived in [209] (see Eq. (63) of [209]). However, the traceless shear tensor considered in [209] is different from ${}^{(l,d)}\sigma_{mn}$ analyzed over here. In [209], the shear tensor considered is the traceless symmetric part of the projected deviation tensor [90] *i.e.* $\hat{\mathcal{B}}_{ab} = q_a^i q_b^j B_{ij}$, where $B_{ij} = \hat{\nabla}_j l_i + T_{iaj} l^a$. As evident, from its definition, the projected deviation tensor is not symmetric. Its decomposition involves a non trivial antisymmetric part. However, our analysis leading to (7.32) deals with the deformation rate tensor (2.116), which by definition is symmetric. It can be shown that the trace part of both the deformation rate and the projected deviation tensor are equivalent [90]. However, owing to the difference in their definitions, the symmetric traceless part of these two projected tensors ($\hat{\mathcal{B}}_{ab}$ and $\hat{\chi}_{ab}$) are different.

7.5 Discussions and comments

As pointed out in [36], in the case of Einstein gravity, the structure of Einstein field equations near an arbitrary null surface can be understood w.r.t the components of the vector field $G^a{}_b l^b$ on the null surface. The components turn out to have very precise physical and thermodynamical interpretations. These insights have paved way to the understanding that the gravitational field equations (at least in the context of Einstein gravity) is perhaps emergent from underlying degrees of freedom associated with the gravitational field. Here, we have studied and tested the claim of emergence of the relevant gravitational field equations of a different theory of gravity. Our object of study has been the EC theory, which is the simplest of all possible gravitational theories under the umbrella of Poincaré gauge theory (PGT) of gravity [88, 126, 128, 205, 252]. As again, to understand the structure of the ECKS gravitational field equations about $\mathcal{H}$, we consider the relevant projection components of $\hat{G}^a{}_b l^b$. They, as usual turn out to be $\hat{G}_{ab} l^a l^b$, $\hat{G}_{ab} k^a l^b$ and $\hat{G}_{ab} q^a{}_c l^b$. Naturally, the goal towards probing the emergent nature of the ECKS field equations would be to find out and analyze the physical underpinnings of these components. $\hat{G}_{ab} l^a l^b$ is related to the dynamical evolution rule (the NRE) of the outgoing expansion scalar for the geodesic null congruence generating the integrable null hypersurface $\mathcal{H}$. We mentioned that the EC theory is the only gravity theory with torsion that is compatible with the Einstein equivalence principle. This means that locally about a given event p of the spacetime, we can construct a local inertial frame, without requiring the torsion tensor to vanish [64]. This can be achieved by going to a local Lorentz (tetrad) basis as opposed to a holonomic coordinate basis. The Lorentz frames are introduced by requiring that the induced metric in such basis is locally Minkowskian. Hence depending upon the length scale set by the torsion and the curvature tensor, we can again in the EC theory choose a sufficiently small region about the event p, where the metric becomes locally Minkowskian.

With regards to the approach by Jacobson, we can now introduce approximate Rindler observers who will perceive the approximate Rindler horizon to be a thermal system at the Unruh temperature of $T = \frac{\hbar a}{2\pi}$. Employing the procedure of Jacobson, and assuming the notion of local holography, the authors of [64] have been able to derive the Einstein-Cartan field equations as an equation of state from a (modified) Clausius identity, by employing additional irreversible entropy generation terms. For the notion of local holography, the authors of [64] have assumed that the entropy of the approximate local Rindler horizon in the EC theory is proportional to its area. They argue that for a causal horizon, the origin of its entropy is due to vacuum entanglement between the quantum gravitational degrees of freedom across the horizon. They argue that the presence of torsion (in the EC theory at least) that is sourced by the matter distribution away from the horizon is not expected to affect the constant of proportionality of the entanglement entropy across the horizon.

Similarly, it was shown in chapter 6, that the component $\hat{G}_{ab}k^a l^b$ is related to the dynamical evolution law of the ingoing expansion scalar of the null geodesic congruence forming $\mathcal{H}$. Application of the process of virtual displacement of $\mathcal{H}$ along the auxiliary null vector field allowed the interpretation of the ECKS field equations (via the component $\hat{G}_{ab}k^a l^b$) as a structure analogous to the first law of thermodynamics. In this chapter we provided the physical interpretation of the component $\hat{G}_{ab}q^a{}_c l^b$.

We have shown that the transverse spatial component $\hat{G}_{ab}q^a{}_c l^b$ or rather $\hat{G}_{ab}l^a q^b{}_c$ is indeed related to the dynamical evolution rule of the Hájíček 1-form for $\mathcal{H}$. In the case of Einstein gravity, the correspondence of the dynamical evolution law for Ω_a in a set of coordinates adapted to $\mathcal{H}$ yields a structure called the DNS equation. The DNS equation for the null fluid on $\mathcal{H}$ is structurally very similar to the NS equation. The NS or the DNS equation is marked by the fact that its Cauchy stress tensor is symmetric by definition. This we expect, since the NS equation describes a fluid with no intrinsic angular momentum. This is turn, on the Einstein gravity side is guaranteed by the fact that there does not exist any contribution to the matter energy-momentum tensor due to the spin degrees of freedom in the microscopic domain. This, we surely cannot neglect in the EC theory. The source of torsion in the EC theory is precisely due to the intrinsic spin angular momentum tensor. The matter energy momentum sources the metric whereas torsion as a geometric field is sourced by the spin angular momentum density tensor. Hence, if it were indeed possible to interpret the evolution law for the Hájíček 1-form $\hat{\Omega}_a$ of $\mathcal{H}$ in terms of a fluid/elastic model theory, then that particular theory must have a stress tensor that has a non-zero antisymmetric contribution. This we explicitly see from the term $\hat{\mathcal{D}}^b\left(\frac{1}{2}q^c{}_t q^d{}_b T_{fdc} l^f\right)$ in (7.3). However, we reiterate that the evolution law (with which the fluid correspondence has to be made) has been considered under the geodesic constraint that forces the null generators of $\mathcal{H}$ to be simultaneously auto-parallel and geodesic.

We note that the presence of the antisymmetric contribution to the stress tensor in the case of EC theory hinted us to look at any real life scenario of fluids/elastic model systems with built in intrinsic angular momentum. One such hint came from [246, 253]. The Cosserat generalization to the NS equation describes a continuum fluid system in which the constituent material point bodies have translation as well as rotational degrees of freedom. Hence the Cosserat fluid does have an antisymmetric contribution to its stress tensor. Next, in order to furnish the analogy of the dynamical evolution equation of $\hat{\Omega}_a$ with that of fluid dynamical equation of the Cosserat fluid, we adopted two ways. In the first approach, we expressed the evolution equation of $\hat{\Omega}_a$ in terms of a coordinate system (t, x^μ) adapted to the null hypersurface $\mathcal{H}$ (7.7). Making suitable identification of the external body force density (7.18), we compared our resulting equation with the Cosserat fluid equation (7.16). The momentum density of our null fluid on $\mathcal{H}$ turned out to be $-\hat{\Omega}_A/(8\pi)$. The comparison allowed us to extract the viscosity coefficients with the shear and the extra material viscosity parameter γ_c turning out to be positive. The bulk viscosity coefficient of our null fluid as in the Einstein case remains negative. We then performed the analysis in a boosted LIF which made the analogy of our dynamical evolution equation of $\hat{\Omega}_A$ with the Cosserat fluid more evident. In the second approach, we made the identification that the material derivative of the Cosserat fluid (written in Cartesian coordinates in Galilean spacetime) should be replaced by the Lie derivative. This is on the basis that the Lie derivative of the momentum density should be the natural generalization of the material derivative in a genuinely curved background.

Having done this analysis, we should be cautious about the analogy brought in. There are certain issues regarding the fluid interpretation of the gravitational field equations which are in still under debate. Firstly, w.r.t (7.13), consider the quantity, $\partial_t q_{AB}$. It is for a convenient choice of the adapted coordinates w.r.t. $\mathcal{H}$, can we set $\partial_t q_{AB}$ to zero. However, the physical interpretation of this quantity in terms of fluid variables is still lacking. This was also the case for Einstein gravity. In addition, for the case of the ECKS theory, we understand, w.r.t. to (7.13), there is an extra contribution to the extended second fundamental form $\hat{\Theta}_{BA}$ written in the adapted set of coordinates (t, x^μ) *i.e.* $\frac{1}{2}(-2K_{0AB} - {}^{(2)}T_{DAB}V^D)$. The antisymmetric part of the second fundamental form *i.e.* $\hat{\Theta}_{[BA]} = 1/2(T_{0BA} + {}^{(2)}T_{DBA}V^D) = 1/2(q^d{}_B q^c{}_A)T_{fdc}l^f$ should be related to spin w^μ of the Cosserat fluid particle (7.16). However, the symmetric part of $\hat{\Theta}_{BA}$ barring aside the term $1/2(\hat{\mathcal{D}}_A V_B + \hat{\mathcal{D}}_B V_A)$ (analogous to the strain rate or classical deformation rate tensor $D_{\mu\nu}$ in (7.15)) *i.e.* the term $\partial_t q_{AB} - K_{0AB} - K_{0BA}$ does not have direct fluid interpretation. It is again only by a convenient choice of the adapted coordinates can we set this extra term to zero. Hence the correspondence of the dynamical evolution law for $\hat{\Omega}_a$ with the Cosserat fluid is in no way watertight. Further scrutiny and insight into the analogy is desired.

For a real world fluid, negative value of bulk viscosity coefficient would imply a dilation or contraction instability. This would translate to the fact, that the global null surface $\mathcal{H}$ is unstable under external perturbations. The fact, that the bulk viscosity coefficient is negative is in agreement with the fact that a generic hypersurface has the tendency to continually contract to expand. For the case of event horizon, the expansion vanishes under the equilibrium condition attained at far future and hence it stabilizes [218]. In any case, the event horizon is a global concept, that requires full knowledge of the spacetime or complete future predictability of any Cauchy surface. There are local concepts of horizons which are bereft of this teleological property. It has been shown that the generalized DNS equation applied to future outer trapping horizon and dynamical horizon in $(\mathcal{M}, g, \boldsymbol{\nabla})$ leads to a positive bulk viscosity coefficient [80, 218]. It is also desirable to see whether we encounter the same positive bulk viscosity coefficient for such local horizons in the RC spacetime.

This “completes” the physical interpretation for the trifecta of the relevant projection components of the vector field $\hat{G}^a{}_b l^b$ in the EC theory. The fact that the ECKS field equations expressed w.r.t to a generic null surface $\mathcal{H}$ has both thermodynamic (via $\hat{G}_{ab}l^a l^b$ and $\hat{G}_{ab}k^a l^b$) and fluid dynamic (via $\hat{G}_{ab}q^a{}_c l^b$) interpretation lends strength to the concept of the emergent paradigm of gravity.

For the sake of completeness, we also computed the tidal force equation for a null geodesic congruence in the RC spacetime. Here, we did not employ the geodesic constraint and kept the analysis general. The only assumption that went into the tidal force equation was that the geodesic null congruence forms an integrable hypersurface, *i.e.* the Frobenius identity is satisfied.

Appendices

7.6 Derivation of the relations (7.11) and (7.12)

We will now expand the deformation rate tensor $\hat{\chi}_{ij} = \frac{1}{2}q_i{}^m q_j{}^m \pounds_{\boldsymbol{l}} q_{mn}$ w.r.t to the coordinate system (t, x^μ) generated by the foliation of $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ by the stack of spacelike hypersurfaces Σ_t. Our analysis follows [61]. With respect to the $(3+1)$ foliation, we have in general that $\vec{\boldsymbol{l}} = \vec{\boldsymbol{t}} + \vec{\boldsymbol{V}} + (N-b)\vec{\boldsymbol{s}}$, where N is the lapse function and $\vec{\boldsymbol{s}}$ is the outward pointing spacelike unit normal to transverse cross-section S_t on Σ_t. The orthogonal decomposition of the time evolution vector field $\vec{\boldsymbol{t}}$ is given by $\vec{\boldsymbol{t}} = N\vec{\boldsymbol{n}} + \vec{\boldsymbol{\beta}}$. The spatial shift vector $\vec{\boldsymbol{\beta}}$ can again be provided an orthogonal decomposition on Σ_t via $\vec{\boldsymbol{\beta}} = b\vec{\boldsymbol{s}} - \vec{\boldsymbol{V}}$, where $\vec{\boldsymbol{V}}$ is a vector field established on the tangent space of S_t. As a consequence, we have,

$$\hat{\chi}_{ij} = \frac{1}{2}q_i{}^r q_j{}^s \left[\pounds_{\boldsymbol{t}} q_{rs} + \pounds_{\boldsymbol{V}} q_{rs} + \pounds_{(N-b)\boldsymbol{s}} q_{rs}\right] . \tag{7.33}$$

Expanding the term $\pounds_{\boldsymbol{V}} q_{rs}$ in terms of the spacetime covariant derivative, we have,

$$\pounds_{\boldsymbol{V}} q_{rs} = V^i \hat{\nabla}_i q_{rs} + q_{ri}(\hat{\nabla}_s V^i) + q_{is}(\hat{\nabla}_r V^i) + T^k_{\ ir} q_{ks} V^i + T^k_{\ is} q_{rk} V^i . \tag{7.34}$$

Same expansion for the term $\pounds_{(N-b)\boldsymbol{s}} q_{rs}$ leads us to,

$$\begin{aligned}\pounds_{(N-b)\boldsymbol{s}} q_{rs} = {} & (N-b)s^i \hat{\nabla}_i q_{rs} + q_{ri}\hat{\nabla}_s[(N-b)s^i] + q_{is}\hat{\nabla}_r[(N-b)s^i] \\ & + T^k_{\ ir} q_{ks}(N-b)s^i + T^k_{\ is} q_{rk}(N-b)s^i .\end{aligned} \tag{7.35}$$

Putting the relations (7.34) and (7.35) in (7.33), and using the fact that
$q_i{}^r q_j{}^s V^i \hat{\nabla}_i q_{rs} = 0$ and $(N-b) q_i{}^r q_j{}^s s^i \hat{\nabla}_i q_{rs} = 0$, we simplify the resulting expression for

$\hat{\chi}_{ij}$. to have,

$$\hat{\chi}_{ij} = \frac{1}{2}\Big[q_i{}^r q_j{}^s \pounds_{\boldsymbol{t}} q_{rs} + \hat{\mathcal{D}}_j V_i + \hat{\mathcal{D}}_i V_j + {}^{(2)}T_{jli}V^l + {}^{(2)}T_{ilj}V^l + (N-b)\Big(\hat{H}_{ij} + \hat{H}_{ji} + q_i{}^r q_j{}^k T_{klr}s^l + q_i{}^k q_j{}^r T_{klr}s^l\Big)\Big] . \tag{7.36}$$

In the above, we have used the following definitions. The spatial covariant derivative (compatible with the induced metric $\boldsymbol{q}$ of S_t) of any vector field $\vec{\boldsymbol{V}}$ lying in the tangent space of S_t is given by $\hat{\mathcal{D}}_i V_j \equiv q_i{}^r q_j{}^k (\hat{\nabla}_r V_k)$. The two-dimensional torsion tensor ${}^{(2)}T^a{}_{bc}$ as result of the induced connection $\hat{\boldsymbol{\mathcal{D}}}$ on the submanifold $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$ is given via the relation,

$$ {}^{(2)}T^a{}_{bc} = q^a{}_m q_b{}^d q_c{}^f T^m{}_{df} . \tag{7.37}$$

The proof of (7.37) has been presented in [90]. Similarly, the extrinsic curvature H_{ij} of the 2-surface S_t viewed as embedded hypersurface in the 3-dimensional spacelike surface Σ_t is given by,

$$H_{ij} = q_i{}^r q_j{}^k (\hat{\nabla}_r s_k) . \tag{7.38}$$

Now, for an adapted coordinate system w.r.t $\mathcal{H}$, we have $b \overset{\mathcal{H}}{=} N$ and $q^i_A = \delta^i{}_A$ [61]. Thus on the null hypersurface, we have $\vec{\boldsymbol{l}} \overset{\mathcal{H}}{=} \vec{\boldsymbol{t}} + \vec{\boldsymbol{V}}$. As a result, from (7.36), we have,

$$\hat{\chi}_{ij} \overset{\mathcal{H}}{=} \frac{1}{2}\Big[q_i{}^r q_j{}^s \pounds_{\boldsymbol{t}} q_{rs} + \hat{\mathcal{D}}_j V_i + \hat{\mathcal{D}}_i V_j + \Big({}^{(2)}T_{jli} + {}^{(2)}T_{ilj}\Big)V^l\Big] . \tag{7.39}$$

Similarly, the spatial tensor $\tilde{t}_{ij}$ can be expressed in the following way on the null surface $\mathcal{H}$,

$$\tilde{t}_{ij} = q_i{}^s q_j{}^r K_{trs} l^t \overset{\mathcal{H}}{=} q_i{}^s q_j{}^r K_{trs}(t^t + V^t) = (q_i{}^s q_j{}^r K_{mrs} t^m) + {}^2K_{tji}V^t . \tag{7.40}$$

Finally, the spatial tensor $\hat{\Phi}_{ij}$, turns out to be,

$$\begin{aligned} \hat{\Phi}_{ij} &= \hat{\chi}_{ij} - \tilde{t}_{ij} \\ &\overset{\mathcal{H}}{=} \frac{1}{2}\Big[q_i{}^r q_j{}^s \pounds_{\boldsymbol{t}} q_{rs} + \hat{\mathcal{D}}_j V_i + \hat{\mathcal{D}}_i V_j \\ &\quad + \Big({}^{(2)}T_{jli} + {}^{(2)}T_{ilj}\Big)V^l - 2q_i{}^s q_j{}^r K_{mrs} t^m - 2({}^2K_{lji})V^j\Big] . \end{aligned} \tag{7.41}$$

Using the fact that for the adapted coordinate system $q^i{}_A = \delta^i{}_A$, we have via (7.41),

$$\begin{aligned} \hat{\Phi}_{AB} &\overset{\mathcal{H}}{=} \frac{1}{2}\Big[\pounds_{\boldsymbol{t}} q_{AB} + \hat{\mathcal{D}}_A V_B + \hat{\mathcal{D}}_B V_A \\ &\quad + \Big({}^2T_{ADB} + {}^2T_{BDA}\Big)V^D - 2K_{0BA} - 2({}^2K_{DBA})V^D\Big] . \end{aligned} \tag{7.42}$$

Using the definition of contorsion tensor, we easily arrive at (7.12). Similar analysis on (7.39) leads us to (7.11).

7.7 Derivation of (7.32)

Let us manipulate the first term in the R.H.S of (7.30), by repeated use of (2.126). Upon using, (2.126) on the term $l^m\hat{\nabla}_j(\hat{\nabla}_m l_i)$, we have,

$$\begin{aligned} l^m\hat{\nabla}_j(\hat{\nabla}_m l_i) &= l^m\hat{\nabla}_j(\hat{\Phi}_{im}) + (l^m\hat{\nabla}_j\hat{\omega}_m)l_i + (\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})(\hat{\nabla}_j l_i) - (l^m\hat{\nabla}_j k_m)q^r{}_i\mathbb{T}_r \\ &+ \hat{\nabla}_j(q^r{}_i\mathbb{T}_r) - \hat{\Phi}_{im}(\hat{\nabla}_j l^m) + (l^m\hat{\nabla}_j\hat{\omega}_m)l_i + (\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})(\hat{\nabla}_j l_i) - l^m(\hat{\nabla}_j k_m)q^r{}_i\mathbb{T}_r \\ &+ \hat{\nabla}_j(q^r{}_i\mathbb{T}_r)\,. \end{aligned} \tag{7.43}$$

We now use the fact that the $(0,2)$ tensor $\hat{\Phi}_{ij}$ is a completely spatial tensor and hence, $l^m\hat{\nabla}_j(\hat{\Phi}_{im}) = -\hat{\Phi}_i{}^m(\hat{\nabla}_j l_m)$, upon which we again make use of (2.126). This leads us to,

$$\begin{aligned} l^m\hat{\nabla}_j(\hat{\nabla}_m l_i) &= -\hat{\Phi}_i{}^m\hat{\Phi}_{mj} + l_j\hat{\Phi}_{im}(k^r\hat{\nabla}_r l^m) + k_j q^s{}_m\mathbb{T}_s\hat{\Phi}_i{}^m - q^r{}_i\mathbb{T}_r\hat{\omega}_j \\ &- l_j\Big(q^r{}_i\mathbb{T}_r k^m(k^s\hat{\nabla}_s l_m)\Big) + l_i(l^m\hat{\nabla}_j\hat{\omega}_m) + (\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})\hat{\Phi}_{ij} + l_i(\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})\hat{\omega}_j \\ &- l_j\Big[(\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})(k^s\hat{\nabla}_s l_i)\Big] - k_j\Big[(\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})q^s{}_i\mathbb{T}_s\Big] + \hat{\nabla}_j(q^r{}_i\mathbb{T}_r)\,. \end{aligned} \tag{7.44}$$

Let us look at the third term $-T^m{}_{kj}l^k(\hat{\nabla}_m l_i)$ in the R.H.S of (7.30). Again, upon use of the relation (2.126), and the symmetry property of the torsion tensor, we have,

$$\begin{aligned} -T^m{}_{kj}l^k(\hat{\nabla}_m l_i) &= -\hat{\Phi}_{im}T^m{}_{kj}l^k - \mathbb{T}_j(k^s\hat{\nabla}_s l_i) + \hat{\mathscr{P}}_j q^r{}_i\mathbb{T}_r - (T_{mkj}\hat{\omega}^m l^k)l_i \\ &+ l_j(T_{mks}k^m l^k k^s)q^r{}_i\mathbb{T}_r\,. \end{aligned} \tag{7.45}$$

In principle, we would like to project the Eq. (7.30) onto the 2-surface S_t. As a consequence, we have,

$$\begin{aligned} q^i{}_m q^j{}_n l^k\hat{\nabla}_j(\hat{\nabla}_k l_i) &= -\hat{\Phi}_{mt}\hat{\Phi}^t{}_n - \hat{\Omega}_n(q^r{}_m\mathbb{T}_r) + (\kappa - \vec{\boldsymbol{k}}\cdot\vec{\mathbb{T}})\hat{\Phi}_{mn} \\ &+ q^i{}_m q^j{}_n\hat{\nabla}_j(q^r{}_i\mathbb{T}_r)\,. \end{aligned} \tag{7.46}$$

Similarly,

$$-q^i{}_m q^j{}_n T^r{}_{kj}l^k(\hat{\nabla}_r l_i) = -\hat{\Phi}_{mr}T^r{}_{kj}l^k q^j{}_n - (q^j{}_n\mathbb{T}_j)(q^i{}_m k^s\hat{\nabla}_s l_i) + \hat{\mathscr{P}}_n q^r{}_m\mathbb{T}_r\,. \tag{7.47}$$

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Thus upon using (7.46) and (7.47), we have,

$$q^i{}_m q^j{}_n \left(l^k \hat{\nabla}_j(\hat{\nabla}_k l_i) - \hat{R}_{mikj} l^m l^k - T^r{}_{kj} l^k (\hat{\nabla}_r l_i) \right) = -\hat{\Phi}_{mt}\hat{\Phi}^t{}_n - \hat{\Omega}_n (q^r{}_m \mathbb{T}_r)$$
$$+ (\kappa - \vec{\boldsymbol{k}} \cdot \vec{\mathbb{T}})\hat{\Phi}_{mn} + q^i{}_m q^j{}_n \hat{\nabla}_j (q^r{}_i \mathbb{T}_r) - q^i{}_m q^j{}_n \hat{R}_{risj} l^r l^s - \hat{\Phi}_{mr} T^r{}_{kj} l^k q^j{}_n$$
$$- (q^j{}_n \mathbb{T}_j)(q^i{}_m k^s \hat{\nabla}_s l_i) + \hat{\mathscr{P}}_n q^r{}_m \mathbb{T}_r \,. \tag{7.48}$$

Let us focus back on the term in the L.H.S of (7.30). As usual, use of (2.126) and (2.114) on the term $l^k \hat{\nabla}_k (\hat{\nabla}_j l_i)$ leads to,

$$l^k \hat{\nabla}_k (\hat{\nabla}_j l_i) = l^k \hat{\nabla}_k \hat{\Phi}_{ij} + l_i (l^k \hat{\nabla}_k \hat{\omega}_j) + \hat{\omega}_j (\kappa l_i + \mathbb{T}_i) - \kappa l_j (k^s \hat{\nabla}_s l_i) - \mathbb{T}_j (k^s \hat{\nabla}_s l_i)$$
$$- l_j \left(l^r \hat{\nabla}_r (k^s \hat{\nabla}_s l_i) \right) - (\hat{\omega}_j - \hat{\mathscr{P}}_j) q^r{}_i \mathbb{T}_r - k_j \left(l^s \hat{\nabla}_s (q^r{}_i \mathbb{T}_r) \right) . \tag{7.49}$$

Projection of the above Eq. (7.49) onto the 2-surface S_t leads to,

$$q^i{}_m q^j{}_n l^k \hat{\nabla}_k (\hat{\nabla}_j l_i) = q^i{}_m q^j{}_n (l^k \hat{\nabla}_k \hat{\Phi}_{ij}) - (q^j{}_n \mathbb{T}_j)(q^i{}_m k^s \hat{\nabla}_s l_i) + \hat{\mathscr{P}}_n (q^r{}_m \mathbb{T}_r) \,. \tag{7.50}$$

We equate the Eqs. (7.50) and (7.48) following (7.30). Upon using the fact that,

$$q^i{}_m q^j{}_n \hat{\nabla}_j (q^r{}_i \mathbb{T}_r) = \hat{\mathcal{D}}_n (q^r{}_m \mathbb{T}_r) \,, \tag{7.51}$$

we end up after some simplification, with the following result,

$$q^i{}_m q^j{}_n (l^k \hat{\nabla}_k \hat{\Phi}_{ij}) = -\hat{\Phi}_{mt}\hat{\Phi}^t{}_n + (\kappa - \vec{\boldsymbol{k}} \cdot \vec{\mathbb{T}})\hat{\Phi}_{mn} - q^i{}_m q^j{}_n \hat{R}_{risj} l^r l^s - \hat{\Omega}_n (q^r{}_m \mathbb{T}_r)$$
$$+ \hat{\mathcal{D}}_n (q^r{}_m \mathbb{T}_r) - \hat{\Phi}_{mr} T^r{}_{kj} l^k q^j{}_n \,. \tag{7.52}$$

Essentially, we would want to convert the covariant derivative of the spatial tensor $\hat{\Phi}_{ab}$ along the null generator into its Lie derivative counterpart. To that effect, in the RC spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, it is quite easy to establish that for any spatial tensor, we have,

$$\pounds_{\boldsymbol{l}} \hat{\Phi}_{ij} = l^r \hat{\nabla}_r \hat{\Phi}_{ij} + l^r \hat{\Phi}_{kj} T^k{}_{ri} + l^r \hat{\Phi}_{ik} T^k{}_{rj} + \hat{\Phi}_{kj} (\hat{\nabla}_i l^k) + \hat{\Phi}_{ik} (\hat{\nabla}_j l^k) \,. \tag{7.53}$$

This leads to,

$$q^i{}_m q^j{}_n \pounds_{\boldsymbol{l}} \hat{\Phi}_{ij} = q^i{}_m q^j{}_n (l^r \hat{\nabla}_r \hat{\Phi}_{ij}) + \hat{\Phi}_{kn} T^k{}_{ri} l^r q^i{}_m + \hat{\Phi}_{mk} T^k{}_{rj} l^r q^j{}_n$$
$$+ q^i{}_m \hat{\Phi}^k{}_n (\hat{\nabla}_i l_k) + q^j{}_n \hat{\Phi}_m{}^k (\hat{\nabla}_j l_k) \,. \tag{7.54}$$

Again, using the relation for the covariant derivative of the null generator *i.e.* (2.126) and the fact that $\hat{\Phi}_{ij}$ is a completely transverse spatial tensor, we can easily show that,

$$q^i{}_m \hat{\Phi}^k{}_n (\hat{\nabla}_i l_k) + q^j{}_n \hat{\Phi}_m{}^k (\hat{\nabla}_j l_k) = \hat{\Phi}_{im} \hat{\Phi}^i{}_n + \hat{\Phi}_{mi} \hat{\Phi}^i{}_n \,. \tag{7.55}$$

Using (7.55) in (7.54), we hence have,

$$
\begin{aligned}
q^i{}_m q^j{}_n (l^r \hat{\nabla}_r \hat{\Phi}_{ij}) = q^i{}_m q^j{}_n \pounds_l \hat{\Phi}_{ij} - \hat{\Phi}_{kn} T^k{}_{ri} l^r q^i{}_m - \hat{\Phi}_{mk} T^k{}_{rj} l^r q^j{}_n \\
- \hat{\Phi}_{im} \hat{\Phi}^i{}_n - \hat{\Phi}_{mi} \hat{\Phi}^i{}_n \, .
\end{aligned} \tag{7.56}
$$

Employing (7.56) in (7.52) and simplifying a bit, we end up having,

$$
\begin{aligned}
q^i{}_m q^j{}_n \pounds_l \hat{\Phi}_{ij} = \hat{\Phi}_{im} \hat{\Phi}^i{}_n + (\kappa - \vec{k} \cdot \vec{\mathbb{T}}) \hat{\Phi}_{mn} - q^i{}_m q^j{}_n \hat{R}_{risj} l^r l^s \\
+ \hat{\Phi}_{kn} T^k{}_{ri} l^r q^i{}_m - \hat{\Omega}_n (q^r{}_m \mathbb{T}_r) + \hat{\mathcal{D}}_n (q^r{}_m \mathbb{T}_r) \, .
\end{aligned} \tag{7.57}
$$

The above Eq. (7.57) defines the dynamical (Lie) evolution (along the null generator $\vec{l}$ of $\mathcal{H}$) of the spatial tensor $\hat{\Phi}_{ij}$ as projected on to the two-surface S_t. On taking the trace of (7.57), it can perhaps be anticipated that it leads to the dynamical evolution equation of the trace of $\hat{\Phi}_{ij}$ along the null generator. This we will proceed with and carry out explicitly. Let us then take the trace of the L.H.S of (7.57) *i.e.* $q^{ij} \pounds_l \hat{\Phi}_{ij}$. Let us note that the trace of the spatial tensor $\hat{\Phi}_{ij}$ is,

$$
g^{ij} \hat{\Phi}_{ij} = q^{ij} \hat{\Phi}_{ij} = q^{ij} (\hat{\chi}_{ij} - \tilde{t}_{ij}) = \overset{(d)}{\hat{\theta}_l} - q^{rs} K_{trs} l^t = \overset{(d)}{\hat{\theta}_l} - q^{rs} T_{rts} l^t \, . \tag{7.58}
$$

Computing the trace of the L.H.S of (7.57), we have,

$$
\begin{aligned}
q^{ij} \pounds_l \hat{\Phi}_{ij} &= q^{ij} l^r \hat{\nabla}_r \hat{\Phi}_{ij} + l^r \hat{\Phi}^{ji} T_{jri} + l^r \hat{\Phi}^{ji} T_{irj} + \hat{\Phi}^{ij} (\hat{\nabla}_j l_i) + \hat{\Phi}^{ji} (\hat{\nabla}_j l_i) \\
&= l^r \hat{\nabla}_r (g^{ij} \hat{\chi}_{ij}) + l^r (\hat{\chi}^{ji} - \tilde{t}^{ji})(T_{jri} + T_{irj}) + (\hat{\chi}^{ij} - \tilde{t}^{ij})(\hat{\chi}_{ij} - \tilde{t}_{ij}) \\
&\quad (\hat{\chi}^{ji} - \tilde{t}^{ji})(\hat{\chi}_{ij} - \tilde{t}_{ij}) \\
&= l^k \hat{\nabla}_k \Big(\overset{(d)}{\hat{\theta}_l} - q^{rs} K_{trs} l^t \Big) + 2 l^r \hat{\chi}^{ij} T_{irj} - l^r \tilde{t}^{ij} (T_{irj} + T_{jri}) + 2 \hat{\chi}^{ij} \hat{\chi}_{ij} \\
&\quad - \tilde{t}_{ij} (2 \hat{\chi}^{ij}) - (\tilde{t}_{ij} \hat{\chi}^{ij} + \tilde{t}_{ji} \hat{\chi}^{ji}) + \tilde{t}_{ij} (\tilde{t}^{ij} + \tilde{t}^{ji}) \, .
\end{aligned} \tag{7.59}
$$

Upon using the irreducible decomposition of the deformation rate tensor $\hat{\chi}_{ij}$ *i.e.* (2.121), the above relation (7.59) can be expanded to,

$$
\begin{aligned}
q^{ij} \pounds_l \hat{\Phi}_{ij} = l^k \hat{\nabla}_k \Big(\overset{(d)}{\hat{\theta}_l} - q^{rs} K_{trs} l^t \Big) + (l^t q^{rs} T_{rts}) \overset{(d)}{\hat{\theta}_l} + 2 l^r ({}^{(l,d)}\sigma^{ij} T_{irj}) \\
- l^r \tilde{t}^{ij} (T_{irj} + T_{jri}) + (\overset{(d)}{\hat{\theta}_l})^2 + 2 ({}^{(l,d)}\sigma_{ij})({}^{(l,d)}\sigma^{ij}) - 2 (q^{rs} K_{trs} l^t) \overset{(d)}{\hat{\theta}_l} \\
- 4 ({}^{(l,d)}\sigma_{ij}) \tilde{t}^{ij} + \tilde{t}_{ij} (\tilde{t}^{ij} + \tilde{t}^{ji}) \, .
\end{aligned} \tag{7.60}
$$

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Let us then compute the trace of the terms present in the R.H.S of (7.57). We have then,

$$q^{mn}\hat{\Phi}_{im}\hat{\Phi}^{i}{}_{m} = \frac{1}{2}(\overset{(d)}{\hat{\theta}_{l}})^2 + ({}^{(l,d)}\sigma_{ij})({}^{(l,d)}\sigma^{ij}) - (q^{rs}K_{trs}l^t)\overset{(d)}{\hat{\theta}_{l}} - 2({}^{(l,d)}\sigma_{ij})\tilde{t}^{ij} + \tilde{t}_{ij}\tilde{t}^{ij}\,; \tag{7.61}$$

$$q^{mn}(\kappa - \vec{k}\cdot\vec{\mathbb{T}})\hat{\Phi}_{mn} = (\kappa - \vec{k}\cdot\vec{\mathbb{T}})\left(\overset{(d)}{\hat{\theta}_{l}} - q^{rs}K_{trs}l^t\right)\,; \tag{7.62}$$

$$-q^{mn}q^{i}{}_{m}q^{j}{}_{n}\hat{R}_{risj}l^r l^s = -\hat{R}_{ij}l^i l^j\,; \tag{7.63}$$

$$q^{mn}\hat{\Phi}_{kn}T^{k}{}_{ri}l^r q^{i}{}_{m} = \frac{1}{2}(l^t q^{rs}T_{rts})\overset{(d)}{\hat{\theta}_{l}} + ({}^{(l,d)}\sigma^{ij})T_{irj}l^r - \tilde{t}^{ij}T_{irj}l^r\,; \tag{7.64}$$

$$-q^{mn}\hat{\Omega}_n(q^{r}{}_{m}\mathbb{T}_r) = -\hat{\Omega}_n(q^{rn}\mathbb{T}_r)\,; \tag{7.65}$$

$$q^{mn}\hat{\mathcal{D}}_n(q^{r}{}_{m}\mathbb{T}_r) = \hat{\mathcal{D}}_i(q^{ij}\mathbb{T}_j)\,. \tag{7.66}$$

We then add up (7.61), (7.62), (7.63), (7.64), (7.65) and (7.66) to obtain the trace of the R.H.S of (7.57). Equating the resultant relation with (7.60), we end up after some simplification,

$$\begin{aligned} l^k\hat{\nabla}_k\overset{(d)}{\hat{\theta}_{l}} = &-\frac{1}{2}(\overset{(d)}{\hat{\theta}_{l}})^2 - ({}^{(l,d)}\sigma_{ij})({}^{(l,d)}\sigma^{ij}) + (\kappa - \vec{k}\cdot\vec{\mathbb{T}})\left(\overset{(d)}{\hat{\theta}_{l}} - q^{rs}T_{rts}l^t\right) - \hat{R}_{ij}l^i l^j \\ &+ l^k\hat{\nabla}_k(q^{rs}T_{rts}l^t) + \frac{1}{2}(q^{rs}T_{rts}l^t)\overset{(d)}{\hat{\theta}_{l}} - ({}^{(l,d)}\sigma^{ij})T_{irj}l^r + \tilde{t}^{ij}T_{jri}l^r \\ &+ 2({}^{(l,d)}\sigma^{ij})\tilde{t}_{ij} - \tilde{t}^{ij}\tilde{t}_{ji} - q^{ij}\hat{\Omega}_i\mathbb{T}_j + \hat{\mathcal{D}}_j(q^{ij}\mathbb{T}_i)\,. \end{aligned} \tag{7.67}$$

The resulting equation provides the geometrical dynamics of the evolution of the expansion scalar $\overset{(d)}{\hat{\theta}_{l}}$ (corresponding to the outgoing null generators) along $\vec{l}$ and relates it to the quantity $\hat{R}_{ij}l^i l^j = \hat{G}_{ij}l^i l^j$. This equation (7.67) can hence be identified as the null Raychaudhuri equation corresponding to a congruence of hypersurface-orthogonal null generators $\vec{l}$. The above form of the NRE should be compared with Eq. (A18) of [90]. Let us mention a few structural differences between them. Here, the construction of the NRE has been done via the taking the irreducible decomposition of the deformation rate tensor $\hat{\chi}_{ij}$ *i.e.* (2.121). The deformation rate tensor by construction is a symmetric spatial tensor and hence ${}^{(l,d)}\sigma_{ij}$ represents its symmetric traceless part. As contrasted with Eq. A18 of [90], this form the NRE does not incorporate any antisymmetric rotation terms. Eq. A18 of [90] has been constructed with the dynamical variable $\hat{\mathcal{B}}_{ij}$ which is the projected deviation tensor. The irreducible decomposition of the the projected deviation tensor involves an antisymmetric traceless part ${}^{(l,B)}\hat{\omega}_{ab}$ and hence in principle ${}^{(l,d)}\hat{\sigma}_{ab} \neq {}^{(l,B)}\hat{\sigma}_{ab}$.

Let us get back to the dynamical equation involving the spatial tensor $\hat{\Phi}_{ij}$ *i.e.* (7.57). We focus on the term $q^i{}_m q^j{}_n \pounds_l \hat{\Phi}_{ij}$ in the L.H.S of (7.57). Using the definition of the deformation rate tensor $\hat{\chi}_{ij}$ (2.116) and its irreducible decomposition (2.121) its quite easy to show that,

$$\begin{aligned} q^i{}_m q^j{}_n \pounds_l \hat{\Phi}_{ij} = \frac{1}{2} q_{mn} \left(\overset{(d)}{\hat{\theta}_l} \right)^2 + \overset{(d)}{\hat{\theta}_l} \left({}^{(l,d)}\sigma_{mn} \right) + \frac{1}{2} q_{mn} \left(l^r \hat{\nabla}_r \overset{(d)}{\hat{\theta}_l} \right) \\ + q^i{}_m q^j{}_n \pounds_l \left({}^{(l,d)}\sigma_{ij} - \tilde{t}_{ij} \right) . \end{aligned} \tag{7.68}$$

Now, we incorporate the form of the NRE given in (7.67) to put down the value of $l^r \hat{\nabla}_r \overset{(d)}{\hat{\theta}_l}$. This leads us to,

$$\begin{aligned} q^i{}_m q^j{}_n \pounds_l \hat{\Phi}_{ij} &= q^i{}_m q^j{}_n \pounds_l \left({}^{(l,d)}\sigma_{ij} - \tilde{t}_{ij} \right) + \frac{1}{4} q_{mn} \left(\overset{(d)}{\hat{\theta}_l} \right)^2 + \overset{(d)}{\hat{\theta}_l} \left({}^{(l,d)}\sigma_{mn} \right) \\ &- \frac{1}{2} q_{mn} \left({}^{(l,d)}\sigma_{ij} {}^{(l,d)}\sigma^{ij} \right) + \frac{1}{2} q_{mn} (\kappa - \vec{k} \cdot \vec{\mathbb{T}}) \left(\overset{(d)}{\hat{\theta}_l} - q^{rs} T_{rts} l^t \right) - \frac{1}{2} q_{mn} \hat{R}_{ij} l^i l^j \\ &+ \frac{1}{2} q_{mn} \left(l^k \hat{\nabla}_k (q^{rs} T_{rts} l^t) \right) + \frac{1}{4} q_{mn} \left(\overset{(d)}{\hat{\theta}_l} (q^{rs} T_{rts} l^t) \right) - \frac{1}{2} q_{mn} \left({}^{(l,d)}\sigma^{ij} T_{irj} l^r \right) \\ &+ \frac{1}{2} q_{mn} \left(\tilde{t}^{ij} T_{jri} l^r \right) + q_{mn} \left({}^{(l,d)}\sigma^{ij} \tilde{t}_{ij} \right) - \frac{1}{2} q_{mn} \left(\tilde{t}^{ij} \tilde{t}_{ji} \right) - \frac{1}{2} q_{mn} \left(q^{ij} \hat{\Omega}_i \mathbb{T}_j \right) \\ &+ \frac{1}{2} q_{mn} \left(\hat{\mathcal{D}}_i (q^{ij} \mathbb{T}_j) \right) . \end{aligned} \tag{7.69}$$

Next, along the same lines as above, using (3.29) and (2.121), we can expand the R.H.S of (7.57). We have, as a result,

$$\begin{aligned} &\hat{\Phi}_{im} \hat{\Phi}^i{}_n + (\kappa - \vec{k} \cdot \vec{\mathbb{T}}) \hat{\Phi}_{mn} - q^i{}_m q^j{}_n \hat{R}_{risj} l^r l^s \\ &+ \hat{\Phi}_{kn} T^k{}_{ri} l^r q^i{}_m - \hat{\Omega}_n (q^r{}_m \mathbb{T}_r) + \hat{\mathcal{D}}_n (q^r{}_m \mathbb{T}_r) \\ &= \frac{1}{2} \overset{(d)}{\hat{\theta}_l} \left(q^i{}_m q^j{}_n T_{jri} l^r \right) + \left(q^i{}_m {}^{(l,d)}\sigma^j{}_n T_{jri} l^r \right) - \left(q^i{}_m \tilde{t}^j{}_n T_{jri} l^r \right) + \frac{1}{4} q_{mn} \left(\overset{(d)}{\hat{\theta}_l} \right)^2 \\ &+ \overset{(d)}{\hat{\theta}_l} \left({}^{(l,d)}\sigma_{mn} \right) - \frac{1}{2} \overset{(d)}{\hat{\theta}_l} \left(\tilde{t}_{mn} + \tilde{t}_{nm} \right) + {}^{(l,d)}\sigma_{im} {}^{(l,d)}\sigma^i{}_n - \left({}^{(l,d)}\sigma_{im} \tilde{t}^i{}_n + {}^{(l,d)}\sigma_{in} \tilde{t}^i{}_m \right) \\ &+ \tilde{t}_{im} \tilde{t}^i{}_n + \frac{1}{2} (\kappa - \vec{k} \cdot \vec{\mathbb{T}}) q_{mn} \overset{(d)}{\hat{\theta}_l} + (\kappa - \vec{k} \cdot \vec{\mathbb{T}}) {}^{(l,d)}\sigma_{mn} - (\kappa - \vec{k} \cdot \vec{\mathbb{T}}) \tilde{t}_{mn} \\ &- q^i{}_m q^j{}_n \hat{R}_{risj} l^r l^s - \hat{\Omega}_n q^r{}_m \mathbb{T}_r + \hat{\mathcal{D}}_n (q^r{}_m \mathbb{T}_r) . \end{aligned} \tag{7.70}$$

Now, all that we need to do is to invoke (7.57) and hence equate (7.69) and (7.70). After the necessary simplification, we end up with,

$$
\begin{aligned}
& q^i{}_m q^j{}_n \pounds_{\boldsymbol{l}} \left({}^{(l,d)}\sigma_{ij} - \tilde{t}_{ij} \right) = \\
& q_{mn} \left({}^{(l,d)}\sigma_{ij} {}^{(l,d)}\sigma^{ij} \right) + (\kappa - \vec{\boldsymbol{k}} \cdot \vec{\mathbb{T}}) \left(\frac{1}{2} q_{mn} (q^{rs} T_{rts} l^t) + {}^{(l,d)}\sigma_{mn} - \tilde{t}_{mn} \right) \\
& - \frac{1}{2} q_{mn} \left(l^k \hat{\nabla}_k (q^{rs} T_{rts} l^t) \right) + \frac{1}{2} \overset{(d)}{\hat{\theta}_{\boldsymbol{l}}} \left(q^i{}_m q^j{}_n T_{jri} l^r - \frac{1}{2} q_{mn} (q^{rs} T_{rts} l^t) - (\tilde{t}_{mn} + \tilde{t}_{nm}) \right) \\
& + \left(q^i{}_m {}^{(l,d)}\sigma^j{}_n T_{jri} l^r + \frac{1}{2} q_{mn} ({}^{(l,d)}\sigma^{ij} T_{jri} l^r) \right) - \left(q^i{}_m \tilde{t}^j{}_n T_{jri} l^r + \frac{1}{2} q_{mn} (\tilde{t}^{ij} T_{jri} l^r) \right) \\
& - \left({}^{(l,d)}\sigma_{im} \tilde{t}^i{}_n + {}^{(l,d)}\sigma_{in} \tilde{t}^i{}_m + q_{mn} ({}^{(l,d)}\sigma_{ij} \tilde{t}^{ij}) \right) + \left(\tilde{t}_{im} \tilde{t}^i{}_n + \frac{1}{2} q_{mn} (\tilde{t}^{ij} \tilde{t}_{ji}) \right) \\
& - \left(q^r{}_m \hat{\Omega}_n \mathbb{T}_r - \frac{1}{2} q_{mn} (q^{ij} \hat{\Omega}_i \mathbb{T}_j) \right) + \left(\hat{\mathcal{D}}_n (q^r{}_m \mathbb{T}_r) - \frac{1}{2} q_{mn} \hat{\mathcal{D}}_i (q^{ij} \mathbb{T}_j) \right) \\
& - q^b{}_m q^d{}_n \hat{R}_{abcd} l^a l^c + \frac{1}{2} q_{mn} \hat{R}_{ac} l^a l^c \,. \qquad (7.71)
\end{aligned}
$$

In the above, we have used the result, that for a spatial two-dimensional symmetric tensor, one has, ${}^{(l,d)}\sigma_{im} {}^{(l,d)}\sigma^i{}_n = \frac{1}{2} q_{mn} ({}^{(l,d)}\sigma_{ij} {}^{(l,d)}\sigma^{ij})$. Now, we would want to break the curvature tensor $\hat{R}_{abcd}$ in $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$ into the Riemannian part R_{abcd} and the torsion part and similarly for the Ricci tensor $\hat{R}_{ij}$. The decomposition of the Riemannian curvature tensor R_{abcd} reads as,

$$
R^a{}_{bcd} = C^a{}_{bcd} + \frac{1}{2} \left(R^a{}_c g_{bd} - R^a{}_d g_{bc} + R_{bd} \delta^a{}_c - R_{bc} \delta^a{}_d \right) + \frac{1}{6} R \left(g_{bc} \delta^a{}_d - g_{bd} \delta^a{}_c \right) , \qquad (7.72)
$$

where C_{abcd} is the traceless part of R_{abcd} called the Weyl tensor. Employing (2.21), (2.22) and (7.72), it is quite easy to verify that,

$$
\begin{aligned}
& - q^b{}_m q^d{}_n \hat{R}_{abcd} l^a l^c + \frac{1}{2} q_{mn} \hat{R}_{ac} l^a l^c = - q^b{}_m q^d{}_n C_{abcd} l^a l^c \\
& - q^b{}_m q^d{}_n \left((\hat{\nabla}_c K_{adb} - \hat{\nabla}_d K_{acb}) + T^i{}_{cd} K_{aib} + (K^i{}_{cb} K_{adi} - K^i{}_{db} K_{aci}) \right) l^a l^c \\
& + \frac{1}{2} q_{mn} \left(\hat{\nabla}_i K^i{}_{ca} + \hat{\nabla}_c T_a + T^i{}_{jc} K^j{}_{ia} + K^i{}_{ja} K^j{}_{ci} + T_i K^i{}_{ca} \right) l^a l^c \,. \qquad (7.73)
\end{aligned}
$$

Eq. (7.71) coupled with Eq. (7.73) would then define the dynamical evolution of the shear tensor ${}^{(l,d)}\sigma_{ij}$ corresponding to the deformation rate tensor. Our final goal of arriving at the tidal equation involves converting the projected (onto the two-surface S_t) Lie derivative (along $\vec{\boldsymbol{l}}$) of the quantity $({}^{(l,d)}\sigma_{ij} - \tilde{t}_{ij})$ into the covariant directional derivative counterpart. To that end, after a few trivial manipulations involving (2.126) and (2.121), it can be shown that,

$$q^i{}_m q^j{}_n \pounds_{\boldsymbol{l}} \left({}^{(\boldsymbol{l},d)}\sigma_{ij} - \tilde{t}_{ij} \right) = q^i{}_m q^j{}_n l^r \hat{\nabla}_r \left({}^{(\boldsymbol{l},d)}\sigma_{ij} - \tilde{t}_{ij} \right) + q^i{}_m \left({}^{(\boldsymbol{l},d)}\sigma^k{}_n - \tilde{t}^k{}_n \right) T_{kri} l^r$$
$$+ \left({}^{(\boldsymbol{l},d)}\sigma_m{}^k - \tilde{t}_m{}^k \right) q^j_n T_{krj} l^r + \overset{(d)}{\hat{\theta}}_{\boldsymbol{l}} \left({}^{(\boldsymbol{l},d)}\sigma_{mn} - \tilde{t}_{mn} \right) + 2\,{}^{(\boldsymbol{l},d)}\sigma_{mk}\,{}^{(\boldsymbol{l},d)}\sigma^k{}_n$$
$$- 2\,{}^{(\boldsymbol{l},d)}\sigma_{mk}\tilde{t}^k{}_n - {}^{(\boldsymbol{l},d)}\sigma^k{}_n(\tilde{t}_{mk} + \tilde{t}_{km}) + (\tilde{t}_{mk} + \tilde{t}_{km})\tilde{t}^k{}_n \,. \tag{7.74}$$

Finally, we equate (7.74) with (7.71) and use the relation (7.73). After some simplifications, we finally end up with (7.32).

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Chapter 8

Conclusions and outlook

8.1 Conclusions

Gravity has always been elusive in its road towards quantization. We still do not have a complete and consistent theory of quantum gravity. The laws of black hole mechanics, and the very strong thermodynamic implications of the gravitational field equations provide a strong framework that a given quantum theory of gravity should perhaps respect. The fact that event horizons have a temperature and entropy suggests one to look for the microscopic origins of black hole entropy. However, corresponding to any given event of a spacetime, an accelerated observer can assign a well defined entropy and temperature to its local causal Rindler horizon. Event horizons are in no way special. It has been shown following the works of Jacobson[35], that the Einstein field equations can be derived from a thermodynamic Clausius identity applied to such local Rindler horizons. In fact any general null surface $\mathcal{H}$ generated by a null vector field $\vec{l}$ can locally act as a Rindler horizon for a given set of accelerated observers. It has been shown that the gravitational field equations via the projection components $G_{ab}l^a l^b$, $G_{ab}k^a l^b$ and $G_{ab}q^a{}_c l^b$ have both thermodynamic and fluid interpretations. This suggests that perhaps the field equations have the same conceptual status as that of fluid dynamics. As an effective field theory, the field equations describe the long wavelength limit to the underlying statistical mechanics of the atoms of spacetime. The connection between the macroscopic continuum gravitational field equations and that of the microscopic quantum degrees of freedom is precisely via the 'descriptions' of temperature and entropy density assigned to an event of the spacetime. Various results [39, 40] suggest us to look more carefully at the 'emergence of gravity' in order to understand the gravitational dynamics more clearly. The central theme of this thesis has been to explore and fortify the thermodynamic and fluid interpretations of the gravitational field equations.

We see that the null hypersurface has been a central tool in our investigations into the physical interpretations of the gravitational field equations. To this effect, we wanted to describe in complete generality, the geometrical structure of an integrable null hypersurface $\mathcal{H}$ in the Riemann-Cartan spacetime. This we did in the second and the third

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chapters. The RC spacetime supports intrinsic torsion in it via the metric-compatible Cartan connection. Part of our aim in this thesis has been to explore the thermodynamic and fluid interpretations of the gravitational field equations of the Einstein-Cartan theory which is formulated in the geometric backdrop of the RC spacetime. The second and the third chapters introduced the necessary geometrical and mathematical tools into the description of an integrable null surface in the RC spacetime.

The findings and salient features of chapter 2 are summarized below.

- We saw that the null generators $\vec{l}$ of $\mathcal{H}$ in the RC spacetime are not parallel-transported along themselves; yet they are geodesic congruences (w.r.t. the Levi-Civita connection). It is only under the geodesic constraint (2.34), that we force the null generators to be both auto-parallel and geodesics.

- The null generator $\vec{l}$ is not an eigen-vector to the (extended) Weingarten map or the shape operator for the null surface. This translated to the fact that the (extended) second fundamental form is neither symmetric nor completely orthogonal to the plane containing $\vec{l}$ and $\vec{k}$. However, the imposition of the geodesic constraint makes it orthogonal to both $\vec{l}$ and $\vec{k}$. The extended second fundamental form is one among the relevant kinematical quantities that we described for null hypersurface.

- For the kinematical quantities, we also discussed three other important second rank tensor fields. They were the deformation rate tensor, the transversal deformation rate tensor and the projected deviation tensor. The deformation and the transversal deformation rate tensors were by construction symmetric and spatial (orthogonal to $\vec{l}$ and $\vec{k}$). The irreducible decompositions of these (spatial) tensors gave us various important notions of (outgoing and ingoing) expansion scalars and corresponding shear tensors.

- We also studied the rotation 1-form and its spatial projection, the Hájiček 1-form onto the submanifold. $(S_t, \boldsymbol{q}, \hat{\boldsymbol{\mathcal{D}}})$.

- All the relevant kinematics were studied under an null foliation of the RC spacetime in the neighborhood of $\mathcal{H}$ by a family of null surfaces.

Having developed the relevant kinematical quantities for the null surface $\mathcal{H}$ in the RC spacetime $(\mathcal{M}, \mathbf{g}, \hat{\boldsymbol{\nabla}})$, we needed to find their evolution dynamics along the null generators $\vec{l}$. This is because the null generators were related to the time evolution vector field for the spacetime. A detailed development of these evolution equations have been dealt with in chapter 3. At this very level, the evolution equations were completely geometrical in the sense that no gravitational field equations had been used. We saw that our evolution equations were related to the projection components of the vector field $\hat{G}^a{}_b l^b$ on the null surface. The vector field $\hat{G}^a{}_b l^b$ is very important in our analysis since it is related to the (off-shell) Noether current and gravitational momentum [59], both of which have thermodynamic interpretations. The important features in this chapter are :

- We studied in depth the null Raychaudhuri equation for the outgoing expansion scalar $\overset{(d)}{\hat{\theta}_l}$ of the null hypersurface $\mathcal{H}$ in the RC spacetime. The analysis had been performed in quite its generality without assuming the geodesic constraint. This evolution dynamics of $\overset{(d)}{\hat{\theta}_l}$ is related to the projection component $\hat{G}_{ab}l^a l^b$. Without the imposition of the geodesic constraint, we saw that the evolution dynamics (3.8) of $\overset{(d)}{\hat{\theta}_l}$ contained terms directly dependent on the auxiliary null field; a feature that was not shared in spacetimes without torsion.

- We studied in depth the evolution of the ingoing expansion scalar $\overset{(d)}{\hat{\theta}_k}$ along the null generators $\vec{l}$ under the geodesic constraint. This evolution was related to the projection component $\hat{G}_{ab}k^a l^b$ and forms a part of the cross-focusing equations [76].

- We finally studied the evolution law for the Hájíček 1-form along $\vec{l}$ both in completely generality and under the geodesic constraint. We saw that such an evolution is related to the projection component $\hat{G}_{ab}q^a{}_c l^b$.

Having developed in detail both the (intrinsic and extrinsic) geometry of the null surface $\mathcal{H}$ in the RC spacetime and its relevant associated dynamics, we tried to investigate whether the gravitational field equations have a thermodynamic and fluid interpretation. This is where we brought in the physics. We first wanted to remain exclusively in the realm of gravitational theories that were formulated in spacetimes equipped with the (unique) Levi-Civita connection. This we did in the fourth chapter 4, whose major findings are the following :

- We inverted Jacobson's procedure [35], by considering a virtual displacement of the null surface along its null generators $\vec{l}$. Under this process, the gravitational dynamics via the null Raychaudhuri equation (corresponding to θ_l) does indeed attain a thermodynamic interpretation. We provided two versions of this interpretation *viz* (4.7) and (4.13). We also showed that for a stationary Killing horizon in the spacetime, the integrated version of the NRE (for θ_l) under the virtual displacement produces the generalized Smarr formula.

- We then proceeded to the more 'relevant' projection component $G_{ab}k^a l^b$. For this we considered a process of virtual displacement that perturbs the null surface along the auxiliary null vector field $\vec{k}$. We then showed that any gravitational dynamics (for spacetimes with the Levi-Civita connection) via the the evolution equation for θ_k does indeed have a thermodynamic interpretation (4.30). Previous investigations into such analogous thermodynamic interpretations of the field equations had either been restricted to static and spherically symmetric spacetimes or had been dependent on a choice of adapted coordinate system (GNC system). Thanks to our null foliation, the thermodynamic interpretation attested to the gravitational field dynamics under the virtual displacement along $\vec{k}$ was completely independent of any choice of adapted coordinate system. The usual case of general relativity was also analyzed and mapped to the previous results under this formalism.

Having described the coordinate-independent thermodynamic interpretation brought to the gravitational field equations under a virtual displacement process along $\vec{k}$ for any gravity theory established in spacetimes with the Levi-Civita connection, we draw our attention to the specific case of scalar-tensor gravity. In particular, there arises the questions of physical equivalences or inequivalences between the the Einstein and the Jordan frame for this theory. We tried to shed some light on this issue via the thermodynamic interpretation of the field equations for ST gravity in chapter 5. The major findings of this chapter are :

- We saw that under the process of virtual displacement of the general null surface $\mathcal{H}$ along the auxiliary null field $\vec{k}$, the ST field equations assume the usual (coordinate-independent) thermodynamic interpretation in both the Einstein (5.12) and the Jordan frame (5.32). We explicitly showed the equivalences for the thermodynamic parameters of temperature, entropy, energy and the work done terms in both the frames. This result generalizes previous results that were known for the specific case of a Killing horizon.

- In order to make the interpretation more concrete and to establish the notion of equilibrium, we provided a general proof (for the first time) of the zeroth law for a Killing horizon in ST gravity. We did not use the assumption of a bifurcation 2-surface in the Killing horizon or any symmetries in the spacetime. Our general proof relied upon the null dominant energy condition being satisfied by the non-gravitational fields.

Having studied the specific examples of general relativity and scalar-tensor gravity in this thermodynamic formalism, we extended our view to a gravitational theory built in the backdrop of a spacetime with torsion present. Our objective in chapters six and seven was to investigate whether the thermodynamic and fluid interpretation of the field equations can be accounted for in the Einstein-Cartan theory. We began with the usual thermodynamic interpretation to the Einstein-Cartan-Kibble-Sciama field equations for the virtual displacement process accounted by $\hat{G}_{ab}k^a l^b$ in the sixth chapter 6. The salient features of this chapter are;

- We saw that under the virtual displacement process of $\mathcal{H}$ along $\vec{k}$, the ECKS field equations can again be consistently provided a covariant thermodynamic interpretation (6.15). However, the variation of the entropy term under this process no longer results only because of the change of (transverse) cross-sectional area (S_t). There arises additional entropy corrections due to torsion. Similarly, the variation of the energy and the work done terms are suitably modified by the inclusion of torsion in the theory.

- We studied the specific case of the thermodynamic interpretation for completely antisymmetric torsion. We also conjectured that for the notion of equilibrium to be defined for a Killing horizon in the EC theory, not only the geodesic constraint needs to be satisfied, but also there needs to be the vanishing of the torsion current of the form $q^{ij}T_{ihj}$.

The emergence of the ECKS field equations from a local thermodynamic constitutive relation applied to approximate Rindler (Killing) horizons constructed at any point in the spacetime had been previously shown in [64]. Hence the only interpretation left for the field equations was that of its fluid nature. In chapter 7, we tried to address this particular question. The main points of this chapter are;

- We analyzed the evolution dynamics of the Hájiček 1-form along $\vec{l}$ under the geodesic constraint. We then expressed the Hájiček equation in a coordinate system adapted to the null surface $\mathcal{H}$ and viewed the resultant dynamics from a local inertial frame. This showed the similarity of the null surface dynamics with that of a (null) Cosserat fluid.

- We extracted the transport coefficients for this null Cosserat fluid describing the dynamics of $\mathcal{H}$ in the EC theory. We saw that the Cauchy stress tensor for the null Cosserat fluid is no longer symmetrical. In fact, the antisymmetric part of the stress tensor is due to the torsion field which is sourced by the spin-angular momentum density in the matter sector of the EC theory.

- We also provided a covariant generalization to the Cosserat null fluid under the assumption that in genuine curved spacetime, the material derivative is replaced by the Lie derivative.

- We also derived the tidal force equation for the null congruences generating our integrable null surface.

We have seen that a running theme behind the thesis is that the gravitational field equations seem to suggest a very strong connection with thermodynamics or that of fluid flow. The conventional laws of black hole mechanics established on event horizons have stood as a strong testament to this fact. However, following the works of Jacobson, Padmanabhan and others, we have come to the conclusion that spacetime is indeed hot and indicates an inherent micro structure. Each point in spacetime according to a well defined observer has both entropy and temperature assigned to it. Such observers perceiving a local Rindler horizon can assign the gravitational dynamics both a thermodynamic and fluid interpretation. In the literature, the analysis had been constrained only to theories of gravity built in spacetimes provided with the Levi-Civita connection. In this thesis, we have formalized in a coherent fashion, a coordinate-independent interpretation to the thermodynamics and extended it to theories beyond general relativity. In particular our focus has been restricted to scalar-tensor and Einstein-Cartan gravity. We have consistently shown as well that the Einstein Cartan field equations assume a fluid interpretation as well.

Via these works in this thesis, we have hopefully strengthened the idea that perhaps gravitational dynamics is indeed emergent. We may need to give this alternative (thermodynamic) interpretation of the field equations its proper due. However, a lot needs to be done and justified. In the following, I provide some future directions in tandem with this approach that might be illuminating or interesting at best.

8.2 Scope for future works

From the thermodynamic perspective, we can look at the following :

8.2.1 Application to quasi-local definition of horizons

Global event horizons are teleological and highly non-local. However another characterization of black holes is of *quasi-local* nature. These are based on the notion of a trapped surface. The future directed null expansions from such a trapped surface is negative. The various such notions of quasi-local horizons are isolated horizons [99], apparent horizons [61], dynamical horizons [254] and slowly evolving horizons [255]. It can be worthwhile to establish and investigate the connections of the dynamics of the gravitational theory with respect to such quasi-local horizons. For example, for the apparent horizon, we need to define a congruence of null geodesics and its associated expansion θ_l. Considering a spacelike hypersurface Σ_t, there can be defined a local notion of a trapped surface. A trapped surface on Σ_t is a codimension 2-surface on Σ_t such that expansions of the both the outgoing and ingoing null congruences is less than zero for every point on the trapped surface i.e. $\theta_l \leq 0$ and $\theta_k \leq 0$. The subset of Σ_t that contains all the trapped surfaces is now essentially a codimension 1 region called the trapped region $\mathcal{T}$. The apparent horizon is then defined to be the boundary $\partial\mathcal{T}$ of such a trapped region. By definition the expansion scalar of the outgoing null congruence for the apparent horizon is zero i.e. $\theta_l = 0$. Hence apparent horizons are 2-dimensional surfaces associated with a given spacelike foliation of the spacetime. The world tube obtained by joining these apparent horizons is spacelike in general and not null. In a similar vein, there exists another quasi-local notion of a horizon called the trapping horizon [61]. The trapping horizon is a world tube constructed out of marginally trapped codimension 2 surfaces (where the expansion scalars for both the outgoing θ_l and ingoing θ_k null congruences vanish). Such trapping horizons are in general codimension one timelike hypersurfaces. Another associated quasi-local construct of the horizon which is spacelike is the dynamical horizon [97].

Both the intrinsic and the extrinsic geometry of such timelike/spacelike quasi-local horizons have been previously studied in the literature at least for spacetimes without torsion [61, 80, 256]. In fact, the "first law" of black hole thermodynamics has been studied for dynamical horizons [97]. Moreover, the fluid dynamical interpretation has been studied for these trapped and dynamical horizons resulting in a generalized Damour-Navier-Stokes equation [80]. An interesting feature that results from the fluid interpretation on these timelike horizons is that bulk viscosity is positive. This means that the fluid behavior of the horizon is stable under perturbations, which is quite distinct from that of null hypersurfaces. However, the methodology of Jacobson and Padmanabhan have not been largely applied to these non-null quasi-local horizons. These are interesting directions to look at. There are questions of observer dependence of the thermodynamics for these non-null surfaces. The null hypersurface is useful for thermodynamic analysis since the Rindler observers can perceive the null surface as their local Rindler horizon, thus associating notions of temperature and entropy density to the null surface. It is at present uncertain as to what classes of observers would be able to assign thermodynamic variables to these timelike/spacelike horizons.

Obviously, with the inclusion of torsion in the spacetime, the intrinsic and extrinsic geometry of such non-null surfaces would become rather more involved (however manageable); in this respect, non-null hypersurfaces are easier to handle than null surfaces. This is certainly an interesting direction to look at. Possibly, what are modifications to the fluid interpretation under the inclusion of torsion? What is the way ahead for Jacobson's and Padmanabhan's thermodynamic analysis still remains an open question for spacetimes with torsion. We could in principle study the notions of both equilibrium as well out-of-equilibrium thermodynamics since such quasi-local horizons can be adapted to such schemes. This is because isolated horizons correspond to equilibrium states and dynamical horizons refer to non-trivially evolving horizons and thus representing non-equilibrium states. The establishment of the black hole mechanics for different theories of gravity can be tested for such quasi-local horizons. These could have consequences and avenues in the field of quantum gravity and understanding the microscopic origins of gravity.

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8.2.2 Applications to Poincaré gauge theory

It would be worthwhile to investigate the above intriguing connection between gravitational dynamics and thermodynamics in the context of Poincaré gauge theory (PGT) [126]. The dynamical variables in PGT are the veilbeins and the Lorentz spin connection. Hence the study of the thermodynamic interpretation of the gravitational dynamics in terms of these dynamical variables is quite interesting. We could look at specific cases of PGT say the teleparallel gravity [252], coincident general relativity [257] *etc.*

8.2.3 Asymptotic symmetries of the spacetime

In the study of thermodynamics of black holes, the Noether charges and currents associated with the diffeomorphism invariance of the gravitational theory turns out to be very important. There exists a special class of symmetries called the *asymptotic symmetry* that forms a closed Virasoro algebra [258]. The entropy of the black hole is related to the central charge of this algebra. In the pioneering work by Bondi, Metzner and Sachs [259, 260], it was shown that the symmetry group corresponding to the isometries of an asymptotically flat spacetime at null infinity was infinite dimensional with the Poincaré symmetry as a subgroup. Considerable work has been devoted in specifying the appropriate boundary conditions on the gravitational field configuration. Similar asymptotic symmetry considerations has also been explored in the near-horizon analysis of a black hole for both extremal and non-extremal cases [147]. Most of the analysis in the literature has been done either for asymptotic null infinity or in the neighborhood of the event horizon. However we could have in principle considered any general physical boundary in the spacetime. Such a boundary could be either null or non-null in nature. It would be worthwhile to study the physical implications of the diffeomorphism symmetries associated with such a physical boundary for the gravitational theory. It would be interesting to study the asymptotic symmetries near such null or non-null hypersurfaces. For extremal null horizons, it has been shown [261] that the near-horizon symmetries are spontaneously broken. In the quantum theory, this leads to Goldstone modes that act as soft hairs and lead to a thermal nature of the horizon at the semi-classical level. This could have implications in the black hole information paradox.

For the fluid dynamic side, we could possibly look at the following :

8.2.4 In the context of gauge-gravity duality

The theory of *gauge-gravity duality* [220] predicts that the hydrodynamics of gauge theory can be effectively described by the long wavelength, long time limit dynamics of a black hole living in the bulk. In the fluid-gravity duality picture, the momentum constraint of the gravitational field equations (constraining the initial data on a timelike cut-off surface) is dual or equivalent to the NS fluid dynamics. It would be worthwhile to study the fluid sector in more details at the microscopic or the mesoscopic scales. This could reveal much about the microscopic details of the dual gravitational side.

8.2.5 Asymptotic symmetries of the spacetime

The asymptotic symmetries of the gravitational field in general relativity are related to the symmetries of a dual hypothetical fluid defined on the conformal boundary. When the spacetime is asymptotically anti-de-Sitter, then the boundary is timelike and the dual fluid is relativistic. When the spacetime is flat, the boundary is null and the associated dual fluid is Carrolian. Local transformations on the fluid side is dual to diffeomorphisms on the gravitational side. It would be interesting to see whether the dynamics of the null boundary (considering the evolution of various kinematical quantities) have any implications on the Carrolian fluid structure. The Bondi-Metzner-Sachs (BMS) type symmetry action has been studied on isolated horizons and the corresponding field theory interpretation has been provided in the membrane paradigm [262]. Supertranslations and superrotations as subset of the asymptotic symmetries that preserve the structure of the isolated horizons have been considered on a stretched membrane near the horizon. These symmetries when viewed from the non-relativistic field theory living on the membrane are spontaneously broken. It would be interesting to look at other quasi-local horizons in the spacetime and look at the physical interpretation of the corresponding asymptotic symmetries that preserve the horizon structure. The field theory in the membrane paradigm can also be analyzed for such other quasi-local horizons.

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Bibliography

1. Gourgoulhon, E. *Geometry and physics of black holes* https://relativite.obspm.fr/blackholes/bholes.pdf (Lecture notes: Draft, 2023).
2. Wald, R. M. *General relativity* (University of Chicago press, 2010).
3. Hawking, S. W. & Ellis, G. F. R. *The Large Scale Structure of Space-Time* ISBN: 978-0-521-20016-5, 978-0-521-09906-6, 978-0-511-82630-6, 978-0-521-09906-6 (Cambridge University Press, Feb. 2011).
4. Bekenstein, J. D. Black holes and entropy. *Phys. Rev. D* **7,** 2333–2346 (1973).
5. Bardeen, J. M., Carter, B. & Hawking, S. The Four laws of black hole mechanics. *Commun. Math. Phys.* **31,** 161–170 (1973).
6. Hawking, S. Black holes in general relativity. *Commun. Math. Phys.* **25,** 152–166 (1972).
7. Hawking, S. Black Holes and Thermodynamics. *Phys. Rev. D* **13,** 191–197 (1976).
8. Davies, P. Thermodynamics of Black Holes. *Proc. Roy. Soc. Lond. A* **A353,** 499–521 (1977).
9. Hawking, S. W. Gravitational Radiation from Colliding Black Holes. *Phys. Rev. Lett.* **26,** 1344–1346. https://link.aps.org/doi/10.1103/PhysRevLett.26.1344 (21 1971).
10. Bekenstein, J. D. Extraction of Energy and Charge from a Black Hole. *Phys. Rev. D* **7,** 949–953. https://link.aps.org/doi/10.1103/PhysRevD.7.949 (4 1973).
11. Hawking, S. W. Particle Creation by Black Holes. *Commun. Math. Phys.* **43** (eds Gibbons, G. W. & Hawking, S. W.) [Erratum: Commun.Math.Phys. 46, 206 (1976)], 199–220 (1975).
12. *Black holes. Les Astres occlus. Edited by C. DeWitt [and] B. S. DeWitt.* eng. ISBN: 0677156103 (Gordon and Breach, New York, 1973).
13. Carter, B. Republication of: Black hole equilibrium states Part II. General theory of stationary black hole states. *General Relativity and Gravitation* **42,** 653–744 (Mar. 2010).
14. Wald, R. M. The thermodynamics of black holes. *Living Rev. Rel.* **4,** 6. arXiv: gr-qc/9912119 (2001).

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15. Penrose, R. *Structure of space-time* in *Battelle Rencontres* (1968), 121–235.

16. Israel, W. Third Law of Black-Hole Dynamics: A Formulation and Proof. *Phys. Rev. Lett.* **57,** 397–399. https://link.aps.org/doi/10.1103/PhysRevLett.57.397 (4 1986).

17. Padmanabhan, T. Dark Energy and its Implications for Gravity. *Adv. Sci. Lett.* **2,** 174. arXiv: 0807.2356 [gr-qc] (2009).

18. Padmanabhan, T. Gravity as an emergent phenomenon: A Conceptual description. *AIP Conf. Proc.* **939** (ed Misra, A.) 114–123. arXiv: 0706.1654 [gr-qc] (2007).

19. Unruh, W. Notes on black hole evaporation. *Phys. Rev. D* **14,** 870 (1976).

20. Hollands, S., Ishibashi, A. & Wald, R. M. A Higher dimensional stationary rotating black hole must be axisymmetric. *Commun. Math. Phys.* **271,** 699–722. arXiv: gr-qc/0605106 (2007).

21. Moncrief, V. & Isenberg, J. Symmetries of Higher Dimensional Black Holes. *Class. Quant. Grav.* **25,** 195015. arXiv: 0805.1451 [gr-qc] (2008).

22. Padmanabhan, T. Some aspects of field equations in generalised theories of gravity. *Phys. Rev. D* **84,** 124041. arXiv: 1109.3846 [gr-qc] (2011).

23. Padmanabhan, T. *Gravitation: Foundations and frontiers* ISBN: 978-7-301-22787-9 (Cambridge University Press, Dec. 2014).

24. Kolekar, S. On the Bianchi identity in generalized theories of gravity. *Gen. Rel. Grav.* **54,** 92 (2022).

25. Poisson, E. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, U.K., Dec. 2009).

26. Wald, R. M. Black hole entropy is the Noether charge. *Phys. Rev. D* **48,** R3427–R3431. arXiv: gr-qc/9307038 (1993).

27. Iyer, V. & Wald, R. M. Some properties of Noether charge and a proposal for dynamical black hole entropy. *Phys. Rev. D* **50,** 846–864. arXiv: gr-qc/9403028 (1994).

28. Wald, R. M. Black hole entropy is the Noether charge. *Phys. Rev. D* **48,** R3427–R3431. https://link.aps.org/doi/10.1103/PhysRevD.48.R3427 (8 1993).

29. Iyer, V. & Wald, R. M. A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes. *Phys. Rev. D* **52,** 4430–4439. arXiv: gr-qc/9503052 (1995).

30. Harlow, D. & Wu, J.-Q. Covariant phase space with boundaries. *JHEP* **10,** 146. arXiv: 1906.08616 [hep-th] (2020).

31. Compère, G. & Fiorucci, A. Advanced Lectures on General Relativity. arXiv: 1801.07064 [hep-th] (Jan. 2018).

32. Grumiller, D. & Sheikh-Jabbari, M. M. *Black Hole Physics: From Collapse to Evaporation* ISBN: 978-3-031-10342-1, 978-3-031-10343-8 (Springer, Nov. 2022).

33. Padmanabhan, T. & Kothawala, D. Lanczos-Lovelock models of gravity. *Phys. Rept.* **531,** 115–171. arXiv: 1302.2151 [gr-qc] (2013).

34. Jacobson, T., Kang, G. & Myers, R. C. On black hole entropy. *Phys. Rev. D* **49,** 6587–6598. arXiv: gr-qc/9312023 (1994).

35. Jacobson, T. Thermodynamics of space-time: The Einstein equation of state. *Phys. Rev. Lett.* **75,** 1260–1263. arXiv: gr-qc/9504004 (1995).

36. Chakraborty, S., Parattu, K. & Padmanabhan, T. Gravitational field equations near an arbitrary null surface expressed as a thermodynamic identity. *JHEP* **10,** 097. arXiv: 1505.05297 [gr-qc] (2015).

37. Damour, T. *Quelques proprietes mecaniques, electromagnetiques, thermodynamiques et quantiques des trous noir* PhD thesis (Paris U., VI-VII, 1979).

38. Kolekar, S. & Padmanabhan, T. Action Principle for the Fluid-Gravity Correspondence and Emergent Gravity. *Phys. Rev. D* **85,** 024004. arXiv: 1109.5353 [gr-qc] (2012).

39. Padmanabhan, T. Emergent Gravity Paradigm: Recent Progress. *Mod. Phys. Lett. A* **30,** 1540007. arXiv: 1410.6285 [gr-qc] (2015).

40. Padmanabhan, T. Gravity and Quantum Theory: Domains of Conflict and Contact. *Int. J. Mod. Phys. D* **29,** 2030001. arXiv: 1909.02015 [gr-qc] (2019).

41. Padmanabhan, T. & Paranjape, A. Entropy of null surfaces and dynamics of spacetime. *Phys. Rev. D* **75,** 064004. arXiv: gr-qc/0701003 (2007).

42. Padmanabhan, T. Dark energy and gravity. *Gen. Rel. Grav.* **40,** 529–564. arXiv: 0705.2533 [gr-qc] (2008).

43. Kothawala, D. & Padmanabhan, T. Entropy density of spacetime as a relic from quantum gravity. *Phys. Rev. D* **90,** 124060. arXiv: 1405.4967 [gr-qc] (2014).

44. Kothawala, D. & Padmanabhan, T. Entropy density of spacetime from the zero point length. *Phys. Lett. B* **748,** 67–69. arXiv: 1408.3963 [gr-qc] (2015).

45. Parattu, K., Majhi, B. R. & Padmanabhan, T. Structure of the gravitational action and its relation with horizon thermodynamics and emergent gravity paradigm. *Phys. Rev. D* **87,** 124011. arXiv: 1303.1535 [gr-qc] (2013).

46. Padmanabhan, T. General Relativity from a Thermodynamic Perspective. *Gen. Rel. Grav.* **46,** 1673. arXiv: 1312.3253 [gr-qc] (2014).

47. Carlip, S. Challenges for Emergent Gravity. *Stud. Hist. Phil. Sci. B* **46,** 200–208. arXiv: 1207.2504 [gr-qc] (2014).

48. Jacobson, T. & Parentani, R. Horizon entropy. *Found. Phys.* **33,** 323–348. arXiv: gr-qc/0302099 (2003).

49. Amsel, A. J., Marolf, D. & Virmani, A. The Physical Process First Law for Bifurcate Killing Horizons. *Phys. Rev. D* **77,** 024011. arXiv: 0708.2738 [gr-qc] (2008).

50. Bhattacharjee, S. & Sarkar, S. Physical process first law and caustic avoidance for Rindler horizons. *Phys. Rev. D* **91,** 024024. arXiv: 1412.1287 [gr-qc] (2015).

51. Sarkar, S. Black Hole Thermodynamics: General Relativity and Beyond. *Gen. Rel. Grav.* **51,** 63. arXiv: 1905.04466 [hep-th] (2019).

52. Penrose, R. Gravitational collapse: The role of general relativity. *Riv. Nuovo Cim.* **1,** 252–276 (1969).

53. *Gravitational Collapse and Spacetime Singularities* (ed Joshi, P. S.) ISBN: 978-1-107-40536-3, 978-0-521-87104-4, 978-0-511-37283-4 (Cambridge University Press, Sept. 2012).

54. Mishra, A., Chakraborty, S., Ghosh, A. & Sarkar, S. On the physical process first law for dynamical black holes. *JHEP* **09,** 034. arXiv: 1709.08925 [gr-qc] (2018).

55. Friedrich, H., Racz, I. & Wald, R. M. On the rigidity theorem for space-times with a stationary event horizon or a compact Cauchy horizon. *Commun. Math. Phys.* **204,** 691–707. arXiv: gr-qc/9811021 (1999).

56. Morales, E. M. *On a second law of black hole mechanics in a higher derivative theory of gravity* 2008. https://www.theorie.physik.uni-goettingen.de/forschung/qft/theses/dipl/Morfa-Morales.pdf.

57. Parattu, K., Chakraborty, S., Majhi, B. R. & Padmanabhan, T. A Boundary Term for the Gravitational Action with Null Boundaries. *Gen. Rel. Grav.* **48,** 94. arXiv: 1501.01053 [gr-qc] (2016).

58. Racz, I. Stationary Black Holes as Holographs. *Class. Quant. Grav.* **24,** 5541–5572. arXiv: gr-qc/0701104 (2007).

59. Chakraborty, S. & Padmanabhan, T. Thermodynamical interpretation of the geometrical variables associated with null surfaces. *Phys. Rev. D* **92,** 104011. arXiv: 1508.04060 [gr-qc] (2015).

60. Padmanabhan, T. Momentum density of spacetime and the gravitational dynamics. *Gen. Rel. Grav.* **48,** 4. arXiv: 1506.03814 [gr-qc] (2016).

61. Gourgoulhon, E. & Jaramillo, J. L. A 3+1 perspective on null hypersurfaces and isolated horizons. *Phys. Rept.* **423,** 159–294. arXiv: gr-qc/0503113 (2006).

62. Eling, C., Guedens, R. & Jacobson, T. Non-equilibrium thermodynamics of spacetime. *Phys. Rev. Lett.* **96,** 121301. arXiv: gr-qc/0602001 (2006).

63. Chirco, G. & Liberati, S. Non-equilibrium Thermodynamics of Spacetime: The Role of Gravitational Dissipation. *Phys. Rev. D* **81,** 024016. arXiv: 0909.4194 [gr-qc] (2010).

64. Dey, R., Liberati, S. & Pranzetti, D. Spacetime thermodynamics in the presence of torsion. *Phys. Rev. D* **96,** 124032. arXiv: 1709.04031 [gr-qc] (2017).

65. Padmanabhan, T. Entropy density of spacetime and thermodynamic interpretation of field equations of gravity in any diffeomorphism invariant theory. arXiv: 0903.1254 [hep-th] (Mar. 2009).

66. Parikh, M. K. & Sarkar, S. Beyond the Einstein Equation of State: Wald Entropy and Thermodynamical Gravity. *Entropy* **18,** 119. arXiv: 0903.1176 [hep-th] (2016).

67. Brustein, R. & Hadad, M. The Einstein equations for generalized theories of gravity and the thermodynamic relation delta Q = T delta S are equivalent. *Phys. Rev. Lett.* **103.** [Erratum: Phys.Rev.Lett. 105, 239902 (2010)], 101301. arXiv: 0903.0823 [hep-th] (2009).

68. Guedens, R., Jacobson, T. & Sarkar, S. Horizon entropy and higher curvature equations of state. *Phys. Rev. D* **85,** 064017. arXiv: 1112.6215 [gr-qc] (2012).

69. Parikh, M. & Svesko, A. Einstein's equations from the stretched future light cone. *Phys. Rev. D* **98,** 026018. arXiv: 1712.08475 [hep-th] (2018).

70. Padmanabhan, T. Classical and quantum thermodynamics of horizons in spherically symmetric space-times. *Class. Quant. Grav.* **19,** 5387–5408. arXiv: gr-qc/0204019 (2002).

71. Padmanabhan, T. Gravity and the thermodynamics of horizons. *Phys. Rept.* **406,** 49–125. arXiv: gr-qc/0311036 (2005).

72. Padmanabhan, T. Thermodynamical Aspects of Gravity: New insights. *Rept. Prog. Phys.* **73,** 046901. arXiv: 0911.5004 [gr-qc] (2010).

73. Kothawala, D., Sarkar, S. & Padmanabhan, T. Einstein's equations as a thermodynamic identity: The Cases of stationary axisymmetric horizons and evolving spherically symmetric horizons. *Phys. Lett. B* **652,** 338–342. arXiv: gr-qc/0701002 (2007).

74. Kothawala, D. The thermodynamic structure of Einstein tensor. *Phys. Rev. D* **83,** 024026. arXiv: 1010.2207 [gr-qc] (2011).

75. Hayward, S. A. Unified first law of black hole dynamics and relativistic thermodynamics. *Class. Quant. Grav.* **15,** 3147–3162. arXiv: gr-qc/9710089 (1998).

76. Cao, L.-M. Deformation of Codimension-2 Surface and Horizon Thermodynamics. *JHEP* **03,** 112. arXiv: 1009.4540 [gr-qc] (2011).

77. Paranjape, A., Sarkar, S. & Padmanabhan, T. Thermodynamic route to field equations in Lanczos-Lovelock gravity. *Phys. Rev. D* **74,** 104015. arXiv: hep-th/0607240 (2006).

78. Kothawala, D. & Padmanabhan, T. Thermodynamic structure of Lanczos-Lovelock field equations from near-horizon symmetries. *Phys. Rev. D* **79,** 104020. arXiv: 0904.0215 [gr-qc] (2009).

79. Chakraborty, S. Lanczos-Lovelock gravity from a thermodynamic perspective. *JHEP* **08,** 029. arXiv: 1505.07272 [gr-qc] (2015).

80. Gourgoulhon, E. A Generalized Damour-Navier-Stokes equation applied to trapping horizons. *Phys. Rev. D* **72,** 104007. arXiv: gr-qc/0508003 (2005).

81. Damour, T. *Surface effects in black hole physics* https://www.ihes.fr/~damour/Articles/surfaceeffects.pdf (1982).

82. Padmanabhan, T. Entropy density of spacetime and the Navier-Stokes fluid dynamics of null surfaces. *Phys. Rev. D* **83,** 044048. arXiv: 1012.0119 [gr-qc] (2011).

83. Thorne, K. S., Price, R. H. & MacDonald, D. A. *Black holes: the membrane paradigm* (Yale University press, 1986).

84. Policastro, G., Son, D. T. & Starinets, A. O. The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma. *Phys. Rev. Lett.* **87,** 081601. arXiv: hep-th/0104066 (2001).

85. Kovtun, P., Son, D. T. & Starinets, A. O. Holography and hydrodynamics: Diffusion on stretched horizons. *JHEP* **10,** 064. arXiv: hep-th/0309213 (2003).

86. Dobado, A., Llanes-Estrada, F. J. & Rincon, J. M. T. The Status of the KSS bound and its possible violations: How perfect can a fluid be? *AIP Conf. Proc.* **1031** (eds Edelstein, J. D., Grandi, N., Nunez, C. A. & Schvellinger, M.) 221–231. arXiv: 0804.2601 [hep-ph] (2008).

87. Poplawski, N. J. Classical Physics: Spacetime and Fields. arXiv: 0911.0334 [gr-qc] (Nov. 2009).

88. Hehl, F. W., Von Der Heyde, P., Kerlick, G. D. & Nester, J. M. General Relativity with Spin and Torsion: Foundations and Prospects. *Rev. Mod. Phys.* **48,** 393–416 (1976).

89. Dey, S. & Majhi, B. R. Possible fluid interpretation and tidal force equation on a generic null hypersurface in Einstein-Cartan theory. *Phys. Rev. D* **106,** 104005. arXiv: 2206.11875 [gr-qc] (2022).

90. Dey, S. & Majhi, B. R. Kinematics and dynamics of null hypersurfaces in the Einstein-Cartan spacetime and related thermodynamic interpretation. *Phys. Rev. D* **105,** 064047. arXiv: 2201.01131 [gr-qc] (2022).

91. Dey, S., Bhattacharya, K. & Majhi, B. R. Thermodynamic structure of a generic null surface and the zeroth law in scalar-tensor theory. *Phys. Rev. D* **104,** 124038. arXiv: 2105.07787 [gr-qc] (2021).

92. Dey, S. & Majhi, B. R. Covariant approach to the thermodynamic structure of a generic null surface. *Phys. Rev. D* **102,** 124044. arXiv: 2009.08221 [gr-qc] (2020).

93. Carroll, S. M. *Spacetime and Geometry* ISBN: 978-0-8053-8732-2, 978-1-108-48839-6, 978-1-108-77555-7 (Cambridge University Press, July 2019).

94. Straumann, N. *General Relativity* (Springer, Dordrecht, 2013).

95. Padmanabhan, T. *Gravitation: foundations and frontiers* (Cambridge University Press, New York, United States of America, 2010).

96. Krishnan, B. in *Springer Handbook of Spacetime* (eds Ashtekar, A. & Petkov, V.) 527–555 (2014). arXiv: 1303.4635 [gr-qc].

97. Ashtekar, A. & Krishnan, B. Isolated and dynamical horizons and their applications. *Living Rev. Rel.* **7,** 10. arXiv: gr-qc/0407042 (2004).

98. Krishnan, B. Fundamental properties and applications of quasi-local black hole horizons. *Class. Quant. Grav.* **25** (eds Scott, S. M. & McClelland, D. E.) 114005. arXiv: 0712.1575 [gr-qc] (2008).

99. Ashtekar, A. *et al.* Isolated horizons and their applications. *Phys. Rev. Lett.* **85,** 3564–3567. arXiv: gr-qc/0006006 (2000).

100. Ashtekar, A., Beetle, C. & Lewandowski, J. Geometry of generic isolated horizons. *Classical and Quantum Gravity* **19,** 1195–1225. arXiv: gr-qc/0111067 [gr-qc] (Mar. 2002).

101. Ashtekar, A., Beetle, C. & Fairhurst, S. Isolated horizons: A Generalization of black hole mechanics. *Class. Quant. Grav.* **16,** L1–L7. arXiv: gr-qc/9812065 (1999).

102. Nielsen, A. B. *Physical Aspects of Quasi-Local Black Hole Horizons* in *International Journal of Modern Physics Conference Series* **7** (Jan. 2012), 67–83.

103. Penrose, R. Gravitational Collapse and Space-Time Singularities. *Phys. Rev. Lett.* **14,** 57–59. https://link.aps.org/doi/10.1103/PhysRevLett.14.57 (3 1965).

104. Hawking, S. W. & Penrose, R. The Singularities of gravitational collapse and cosmology. *Proc. Roy. Soc. Lond. A* **314,** 529–548 (1970).

105. Joshi, P. *Global Aspects in Gravitation and Cosmology* ISBN: 9780198500797. https://books.google.co.in/books?id=g0cIkAEACAAJ (Clarendon Press, 1996).

106. Witten, E. Light rays, singularities, and all that. *Rev. Mod. Phys.* **92,** 045004. https://link.aps.org/doi/10.1103/RevModPhys.92.045004 (4 2020).

107. Senovilla, J. M. M. Singularity Theorems and Their Consequences. *Gen. Rel. Grav.* **30,** 701. arXiv: 1801.04912 [gr-qc] (1998).

108. Senovilla, J. M. M. & Garfinkle, D. The 1965 Penrose singularity theorem. *Class. Quant. Grav.* **32,** 124008. arXiv: 1410.5226 [gr-qc] (2015).

109. Wall, A. C. The Generalized Second Law implies a Quantum Singularity Theorem. *Class. Quant. Grav.* **30.** [Erratum: Class.Quant.Grav. 30, 199501 (2013)], 165003. arXiv: 1010.5513 [gr-qc] (2013).

110. Modesto, L. Disappearance of the black hole singularity in loop quantum gravity. *Phys. Rev. D* **70,** 124009. https://link.aps.org/doi/10.1103/PhysRevD.70.124009 (12 2004).

111. Bekenstein, J. D. Generalized second law of thermodynamics in black-hole physics. *Phys. Rev. D* **9,** 3292–3300. https://link.aps.org/doi/10.1103/PhysRevD.9.3292 (12 1974).

112. Carlip, S. Black Hole Thermodynamics. *Int. J. Mod. Phys. D* **23,** 1430023. arXiv: 1410.1486 [gr-qc] (2014).

113. Jacobson, T. *Black hole thermodynamics today* in *8th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (MG 8)* (June 1997), 959–967. arXiv: gr-qc/9801015.

114. Wald, R. M. in (2020). arXiv: 1805.02302 [physics.hist-ph].

115. Wall, A. C. A Survey of Black Hole Thermodynamics. arXiv: 1804.10610 (gr-qc) (Apr. 2018).

116. Unruh, W. G. Notes on black-hole evaporation. *Physical Review D* **14,** 870 (1976).

117. Iyer, V. & Wald, R. M. Some properties of the Noether charge and a proposal for dynamical black hole entropy. *Phys. Rev. D* **50,** 846–864. https://link.aps.org/doi/10.1103/PhysRevD.50.846 (2 1994).

118. Paranjape, A., Sarkar, S. & Padmanabhan, T. Thermodynamic route to field equations in Lanczos-Lovelock gravity. *Phys. Rev. D* **74,** 104015. https://link.aps.org/doi/10.1103/PhysRevD.74.104015 (10 2006).

119. Padmanabhan, T. Gravitational entropy of static space-times and microscopic density of states. *Class. Quant. Grav.* **21,** 4485–4494. arXiv: gr-qc/0308070 (2004).

120. Bousso, R. The holographic principle. *Rev. Mod. Phys.* **74,** 825–874. https://link.aps.org/doi/10.1103/RevModPhys.74.825 (3 2002).

121. Ashtekar, A. & Krishnan, B. Dynamical Horizons: Energy, Angular Momentum, Fluxes, and Balance Laws. *Phys. Rev. Lett.* **89,** 261101. https://link.aps.org/doi/10.1103/PhysRevLett.89.261101 (26 2002).

122. Hayward, S. A. Energy Conservation for Dynamical Black Holes. *Phys. Rev. Lett.* **93,** 251101. https://link.aps.org/doi/10.1103/PhysRevLett.93.251101 (25 2004).

123. Verlinde, E. P. On the Origin of Gravity and the Laws of Newton. *JHEP* **04,** 029. arXiv: 1001.0785 [hep-th] (2011).

124. Parattu, K. Einstein Equations from/as Thermodynamics of Spacetime. *Fundam. Theor. Phys.* **187,** 339–352 (2017).

125. Jezierski, J., Kijowski, J. & Czuchry, E. Geometry of null-like surfaces in general relativity and its application to dynamics of gravitating matter. *Reports on Mathematical Physics* **46,** 399–418 (Dec. 2000).

126. Blagojevic, M. *Gravitation and gauge symmetries* (CRC Press, 2001).

127. Blagojević, M. & Hehl, F. *Gauge Theories of Gravitation: A Reader with Commentaries* ISBN: 9781848167261. https://books.google.co.in/books?id=C4DlTgEACAAJ (Imperial College Press, 2013).

128. Blagojevic, M. Three lectures on Poincare gauge theory. *SFIN A* **1** (eds Dragovich, B. & Sazdovic, B.) 147–172. arXiv: gr-qc/0302040 (2003).

129. Obukhov, Y. N. Poincare gauge gravity: Selected topics. *Int. J. Geom. Meth. Mod. Phys.* **03,** 95–138. arXiv: gr-qc/0601090 (2006).

130. Carter, B. Extended tensorial curvature analysis for embeddings and foliations. *Contemporary Mathematics* **203,** 207–220 (1997).

131. Misner, C. W., Thorne, K. S. & Wheeler, J. A. *Gravitation* ISBN: 978-0-7167-0344-0, 978-0-691-17779-3 (W. H. Freeman, San Francisco, 1973).

132. Brady, P. R., Droz, S., Israel, W. & Morsink, S. M. Covariant double null dynamics: (2+2) splitting of the Einstein equations. *Class. Quant. Grav.* **13,** 2211–2230. arXiv: gr-qc/9510040 (1996).

133. d'Inverno, R. A., Lambert, P. & Vickers, J. A. Hamiltonian analysis of the double null 2+2 decomposition of general relativity expressed in terms of self-dual bivectors. *Class. Quant. Grav.* **23,** 4511–4522. arXiv: gr-qc/0604084 (2006).

134. Jafari, G. Stress Tensor on Null Boundaries. *Phys. Rev. D* **99,** 104035. arXiv: 1901.04054 [hep-th] (2019).

135. Booth, I. & Fairhurst, S. Isolated, slowly evolving, and dynamical trapping horizons: Geometry and mechanics from surface deformations. *Phys. Rev. D* **75,** 084019. arXiv: gr-qc/0610032 (2007).

136. Speziale, S. Raychaudhuri and optical equations for null geodesic congruences with torsion. *Phys. Rev. D* **98,** 084029. arXiv: 1808.00952 [gr-qc] (2018).

137. Luz, P. & Vitagliano, V. Raychaudhuri equation in spacetimes with torsion. *Phys. Rev. D* **96,** 024021. https://link.aps.org/doi/10.1103/PhysRevD.96.024021 (2 2017).

138. Sachs, R. K. Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times. *Proc. Roy. Soc. Lond. A* **270,** 103–126 (1962).

139. Penrose, R. & Rindler, W. *SPINORS AND SPACE-TIME. VOL. 2: SPINOR AND TWISTOR METHODS IN SPACE-TIME GEOMETRY* ISBN: 978-0-521-34786-0, 978-0-511-86842-9 (Cambridge University Press, Apr. 1988).

140. Chandrasekhar, S. & Chandrasekhar, S. *The Mathematical Theory of Black Holes* ISBN: 9780198503705. https://books.google.co.in/books?id=LBOVcrzFfhsC (Clarendon Press, 1998).

141. Puetzfeld, D. & Obukhov, Y. N. Deviation equation in Riemann-Cartan spacetime. *Phys. Rev. D* **97,** 104069. arXiv: 1804.11106 [gr-qc] (2018).

142. Akhshabi, S. Light propagation and optical scalars in torsion theories of gravity. *Mod. Phys. Lett. A* **34,** 1950029. arXiv: 1805.07534 [gr-qc] (2019).

143. Kar, S. & SenGupta, S. The Raychaudhuri equations: A Brief review. *Pramana* **69,** 49. arXiv: gr-qc/0611123 (2007).

144. Boersma, S. & Dray, T. Parametric manifolds. 1. Extrinsic approach. *J. Math. Phys.* **36,** 1378–1393. arXiv: gr-qc/9407011 (1995).

145. Chakraborty, S., Bhattacharya, S. & Padmanabhan, T. Entropy of a generic null surface from its associated Virasoro algebra. *Phys. Lett. B* **763,** 347–351. arXiv: 1605.06988 [gr-qc] (2016).

146. Bhattacharya, K. & Majhi, B. R. Noncommutative Heisenberg algebra in the neighbourhood of a generic null surface. *Nucl. Phys. B* **934,** 557–577. arXiv: 1802.02862 [gr-qc] (2018).

147. Maitra, M., Maity, D. & Majhi, B. R. Symmetries near a generic charged null surface and associated algebra: an off-shell analysis. *Phys. Rev. D* **97,** 124065. arXiv: 1802.07108 [gr-qc] (2018).

148. Baccetti, V. & Visser, M. Clausius entropy for arbitrary bifurcate null surfaces. *Class. Quant. Grav.* **31,** 035009. arXiv: 1303.3185 [gr-qc] (2014).

149. Jacobson, T. Entanglement Equilibrium and the Einstein Equation. *Phys. Rev. Lett.* **116,** 201101. arXiv: 1505.04753 [gr-qc] (2016).

150. Smarr, L. Mass formula for Kerr black holes. *Phys. Rev. Lett.* **30.** [Erratum: Phys.Rev.Lett. 30, 521–521 (1973)], 71–73 (1973).

151. Padmanabhan, T. Equipartition of energy in the horizon degrees of freedom and the emergence of gravity. *Mod. Phys. Lett. A* **25,** 1129–1136. arXiv: 0912.3165 [gr-qc] (2010).

152. Banerjee, R. & Majhi, B. R. Statistical Origin of Gravity. *Phys. Rev. D* **81,** 124006. arXiv: 1003.2312 [gr-qc] (2010).

153. Banerjee, R., Majhi, B. R., Modak, S. K. & Samanta, S. Killing Symmetries and Smarr Formula for Black Holes in Arbitrary Dimensions. *Phys. Rev. D* **82,** 124002. arXiv: 1007.5204 [gr-qc] (2010).

154. Jacobson, T., Kang, G. & Myers, R. C. Increase of black hole entropy in higher curvature gravity. *Phys. Rev. D* **52,** 3518–3528. arXiv: gr-qc/9503020 (1995).

155. Chatterjee, A. & Sarkar, S. Physical process first law and increase of horizon entropy for black holes in Einstein-Gauss-Bonnet gravity. *Phys. Rev. Lett.* **108,** 091301. arXiv: 1111.3021 [gr-qc] (2012).

156. Majhi, B. R. Thermodynamics of Sultana-Dyer Black Hole. *JCAP* **05,** 014. arXiv: 1403.4058 [gr-qc] (2014).

157. Hayward, S. A. Quasilocal gravitational energy. *Phys. Rev. D* **49,** 831–839. arXiv: gr-qc/9303030 (1994).

158. Prain, A., Vitagliano, V., Faraoni, V. & Lapierre-Léonard, M. Hawking–Hayward quasi-local energy under conformal transformations. *Class. Quant. Grav.* **33,** 145008. arXiv: 1501.02977 [gr-qc] (2016).

159. Clifton, T., Ferreira, P. G., Padilla, A. & Skordis, C. Modified Gravity and Cosmology. *Phys. Rept.* **513,** 1–189. arXiv: 1106.2476 [astro-ph.CO] (2012).

160. Nojiri, S., Odintsov, S. D. & Oikonomou, V. K. Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution. *Phys. Rept.* **692,** 1–104. arXiv: 1705.11098 [gr-qc] (2017).

161. Callan Jr., C. G., Martinec, E. J., Perry, M. J. & Friedan, D. Strings in Background Fields. *Nucl. Phys. B* **262,** 593–609 (1985).

162. Esposito-Farese, G. *Scalar tensor theories and cosmology and tests of a quintessence Gauss-Bonnet coupling* in *38th Rencontres de Moriond on Gravitational Waves and Experimental Gravity* (June 2003). arXiv: gr-qc/0306018.

163. Elizalde, E., Nojiri, S. & Odintsov, S. D. Late-time cosmology in (phantom) scalar-tensor theory: Dark energy and the cosmic speed-up. *Phys. Rev. D* **70,** 043539. arXiv: hep-th/0405034 (2004).

164. Saridakis, E. N. & Tsoukalas, M. Cosmology in new gravitational scalar-tensor theories. *Phys. Rev. D* **93,** 124032. arXiv: 1601.06734 [gr-qc] (2016).

165. Crisostomi, M., Koyama, K. & Tasinato, G. Extended Scalar-Tensor Theories of Gravity. *JCAP* **04,** 044. arXiv: 1602.03119 [hep-th] (2016).

166. Langlois, D., Saito, R., Yamauchi, D. & Noui, K. Scalar-tensor theories and modified gravity in the wake of GW170817. *Phys. Rev. D* **97,** 061501. arXiv: 1711.07403 [gr-qc] (2018).

167. Faraoni, V. & Gunzig, E. Einstein frame or Jordan frame? *Int. J. Theor. Phys.* **38,** 217–225. arXiv: astro-ph/9910176 (1999).

168. Faraoni, V., Gunzig, E. & Nardone, P. Conformal transformations in classical gravitational theories and in cosmology. *Fund. Cosmic Phys.* **20,** 121. arXiv: gr-qc/9811047 (1999).

169. Faraoni, V. Black hole entropy in scalar-tensor and f(R) gravity: An Overview. *Entropy* **12,** 1246. arXiv: 1005.2327 [gr-qc] (2010).

170. Faraoni, V. & Nadeau, S. The (pseudo)issue of the conformal frame revisited. *Phys. Rev. D* **75,** 023501. arXiv: gr-qc/0612075 (2007).

171. Saltas, I. D. & Hindmarsh, M. The dynamical equivalence of modified gravity revisited. *Class. Quant. Grav.* **28,** 035002. arXiv: 1002.1710 [gr-qc] (2011).

172. Capozziello, S., Martin-Moruno, P. & Rubano, C. Physical non-equivalence of the Jordan and Einstein frames. *Phys. Lett. B* **689,** 117–121. arXiv: 1003.5394 [gr-qc] (2010).

173. Padilla, A. & Sivanesan, V. Boundary Terms and Junction Conditions for Generalized Scalar-Tensor Theories. *JHEP* **08,** 122. arXiv: 1206.1258 [gr-qc] (2012).

174. Koga, J.-i. & Maeda, K.-i. Equivalence of black hole thermodynamics between a generalized theory of gravity and the Einstein theory. *Phys. Rev. D* **58,** 064020. arXiv: gr-qc/9803086 (1998).

175. Jacobson, T. & Kang, G. Conformal invariance of black hole temperature. *Class. Quant. Grav.* **10,** L201–L206. arXiv: gr-qc/9307002 (1993).

176. Kang, G. On black hole area in Brans-Dicke theory. *Phys. Rev. D* **54,** 7483–7489. arXiv: gr-qc/9606020 (1996).

177. Deser, S. & Tekin, B. Conformal Properties of Charges in Scalar-Tensor Gravities. *Class. Quant. Grav.* **23,** 7479–7482. arXiv: gr-qc/0609111 (2006).

178. Dehghani, M. H., Pakravan, J. & Hendi, S. H. Thermodynamics of charged rotating black branes in Brans-Dicke theory with quadratic scalar field potential. *Phys. Rev. D* **74,** 104014. arXiv: hep-th/0608197 (2006).

179. Sheykhi, A. & Yazdanpanah, M. M. Thermodynamics of charged Brans-Dicke AdS black holes. *Phys. Lett. B* **679,** 311–316. arXiv: 0904.1777 [hep-th] (2009).

180. Steinwachs, C. F. & Kamenshchik, A. Y. One-loop divergences for gravity non-minimally coupled to a multiplet of scalar fields: calculation in the Jordan frame. I. The main results. *Phys. Rev. D* **84,** 024026. arXiv: 1101.5047 [gr-qc] (2011).

181. Kamenshchik, A. Y. & Steinwachs, C. F. Question of quantum equivalence between Jordan frame and Einstein frame. *Phys. Rev. D* **91,** 084033. arXiv: 1408.5769 [gr-qc] (2015).

182. Banerjee, N. & Majumder, B. A question mark on the equivalence of Einstein and Jordan frames. *Phys. Lett. B* **754,** 129–134. arXiv: 1601.06152 [gr-qc] (2016).

183. Pandey, S. & Banerjee, N. Equivalence of Jordan and Einstein frames at the quantum level. *Eur. Phys. J. Plus* **132,** 107. arXiv: 1610.00584 [gr-qc] (2017).

184. Ruf, M. S. & Steinwachs, C. F. Quantum equivalence of $f(R)$ gravity and scalar-tensor theories. *Phys. Rev. D* **97,** 044050. arXiv: 1711.07486 [gr-qc] (2018).

185. Karam, A., Pappas, T. & Tamvakis, K. Frame-dependence of higher-order inflationary observables in scalar-tensor theories. *Phys. Rev. D* **96,** 064036. arXiv: 1707.00984 [gr-qc] (2017).

186. Bahamonde, S., Odintsov, S. D., Oikonomou, V. K. & Tretyakov, P. V. Deceleration versus acceleration universe in different frames of $F(R)$ gravity. *Phys. Lett. B* **766,** 225–230. arXiv: 1701.02381 [gr-qc] (2017).

187. Karam, A., Lykkas, A. & Tamvakis, K. Frame-invariant approach to higher-dimensional scalar-tensor gravity. *Phys. Rev. D* **97,** 124036. arXiv: 1803.04960 [gr-qc] (2018).

188. Bhattacharya, K. & Majhi, B. R. Fresh look at the scalar-tensor theory of gravity in Jordan and Einstein frames from undiscussed standpoints. *Phys. Rev. D* **95,** 064026. arXiv: 1702.07166 [gr-qc] (2017).

189. Bhattacharya, K., Das, A. & Majhi, B. R. Noether and Abbott-Deser-Tekin conserved quantities in scalar-tensor theory of gravity both in Jordan and Einstein frames. *Phys. Rev. D* **97,** 124013. arXiv: 1803.03771 [gr-qc] (2018).

190. Bhattacharya, K., Majhi, B. R. & Singleton, D. Fluid-gravity correspondence in the scalar-tensor theory of gravity: (in)equivalence of Einstein and Jordan frames. *JHEP* **07,** 018. arXiv: 2002.04743 [hep-th] (2020).

191. Bhattacharya, K. *Thermodynamic aspects and phase transition of black holes* PhD thesis (Gauhati U., 2020).

192. Sakharov, A. D. Vacuum quantum fluctuations in curved space and the theory of gravitation. *Doklady Akademii Nauk* **177,** 70–71 (1967).

193. Ghosh, R. & Sarkar, S. Black Hole Zeroth Law in Higher Curvature Gravity. *Phys. Rev. D* **102,** 101503. arXiv: 2009.01543 [gr-qc] (2020).

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194. Racz, I. & Wald, R. M. Global extensions of space-times describing asymptotic final states of black holes. *Class. Quant. Grav.* **13,** 539–553. arXiv: gr-qc/9507055 (1996).

195. Velásquez, J. & Castañeda, L. Equivalence between Scalar-Tensor theories and $f(R)$-gravity: from the action to cosmological perturbations. *J. Phys. Comm.* **4,** 055007. arXiv: 1808.05615 [gr-qc] (2020).

196. Padmanabhan, T. Surface Density of Spacetime Degrees of Freedom from Equipartition Law in theories of Gravity. *Phys. Rev. D* **81,** 124040. arXiv: 1003.5665 [gr-qc] (2010).

197. Chakraborty, S. & Padmanabhan, T. Evolution of Spacetime arises due to the departure from Holographic Equipartition in all Lanczos-Lovelock Theories of Gravity. *Phys. Rev. D* **90,** 124017. arXiv: 1408.4679 [gr-qc] (2014).

198. Chakraborty, S. & Padmanabhan, T. Geometrical variables with direct thermodynamic significance in Lanczos-Lovelock gravity. *Phys. Rev. D* **90,** 084021. arXiv: 1408.4791 [gr-qc] (2014).

199. Cartan, E. Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie). fr. *Annales scientifiques de l'École Normale Supérieure* **3e série, 40,** 325–412. http://www.numdam.org/articles/10.24033/asens.751/ (1923).

200. Kibble, T. W. Lorentz invariance and the gravitational field. *Journal of mathematical physics* **2,** 212–221 (1961).

201. Sciama, D. W. Recent Developments in General Relativity. *Pergamon Press and PWN, Warsaw,* 415 (1962).

202. Sciama, D. W. The physical structure of general relativity. *Reviews of Modern Physics* **36,** 463 (1964).

203. Sciama, D. Erratum: The physical structure of general relativity. *Reviews of Modern Physics* **36,** 1103 (1964).

204. Hehl, F. W. How does one measure torsion of space-time? *Physics Letters A* **36,** 225–226 (1971).

205. De Sabbata, V. & Gasperini, M. *Introduction to gravitation* (World Scientific Publishing Company, 1986).

206. De Sabbata, V. & Sivaram, C. *Spin and torsion in gravitation* (world scientific, 1994).

207. Kleinert, H. *Gauge Fields in Condensed Matter Gauge Fields in Condensed Matter* **v. 2.** ISBN: 9789971502119. https://books.google.co.in/books?id=PgJRygEACAAJ (World Scientific, 1989).

208. Shapiro, I. L. Physical aspects of the space-time torsion. *Phys. Rept.* **357,** 113. arXiv: hep-th/0103093 (2002).

209. Hensh, S. & Liberati, S. Raychaudhuri equations and gravitational collapse in Einstein-Cartan theory. *Phys. Rev. D* **104,** 084073. https://link.aps.org/doi/10.1103/PhysRevD.104.084073 (8 2021).

210. Hochberg, D. & Visser, M. Dynamic wormholes, antitrapped surfaces, and energy conditions. *Phys. Rev. D* **58,** 044021. https://link.aps.org/doi/10.1103/PhysRevD.58.044021 (4 1998).

211. Padmanabhan, T. Classical and quantum thermodynamics of horizons in spherically symmetric spacetimes. *Classical and Quantum Gravity* **19,** 5387 (2002).

212. Pilling, T. G. *Gauge torsion gravity, string theory, and antisymmetric tensor interactions* Other thesis (2002).

213. Chakraborty, S. & Dey, R. Noether Current, Black Hole Entropy and Spacetime Torsion. *Phys. Lett. B* **786,** 432–441. arXiv: 1806.05840 [gr-qc] (2018).

214. Fabbri, L. On a completely antisymmetric Cartan torsion tensor. *Annales Fond. Broglie* **32,** 215–228. arXiv: gr-qc/0608090 (2007).

215. SenGupta, S. & Sur, S. Spherically symmetric solutions of gravitational field equations in Kalb-Ramond background. *Phys. Lett. B* **521,** 350–356. arXiv: gr-qc/0102095 (2001).

216. Price, R. H. & Thorne, K. S. Membrane viewpoint on black holes: Properties and evolution of the stretched horizon. *Phys. Rev. D* **33,** 915–941. https://link.aps.org/doi/10.1103/PhysRevD.33.915 (4 1986).

217. Eling, C., Fouxon, I. & Oz, Y. The Incompressible Navier-Stokes Equations From Membrane Dynamics. *Phys. Lett. B* **680,** 496–499. arXiv: 0905.3638 [hep-th] (2009).

218. Gourgoulhon, E. & Jaramillo, J. L. New theoretical approaches to black holes. *New Astron. Rev.* **51,** 791–798. arXiv: 0803.2944 [astro-ph] (2008).

219. Bhattacharyya, S., Minwalla, S. & Wadia, S. R. The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity. *JHEP* **08,** 059. arXiv: 0810.1545 [hep-th] (2009).

220. Bhattacharyya, S., Hubeny, V. E., Minwalla, S. & Rangamani, M. Nonlinear Fluid Dynamics from Gravity. *JHEP* **02,** 045. arXiv: 0712.2456 [hep-th] (2008).

221. Bhattacharyya, S. *et al.* Forced Fluid Dynamics from Gravity. *JHEP* **02,** 018. arXiv: 0806.0006 [hep-th] (2009).

222. Policastro, G., Son, D. T. & Starinets, A. O. From AdS / CFT correspondence to hydrodynamics. *JHEP* **09,** 043. arXiv: hep-th/0205052 (2002).

223. Bredberg, I., Keeler, C., Lysov, V. & Strominger, A. From Navier-Stokes To Einstein. *JHEP* **07,** 146. arXiv: 1101.2451 [hep-th] (2012).

224. Compere, G., McFadden, P., Skenderis, K. & Taylor, M. The Holographic fluid dual to vacuum Einstein gravity. *JHEP* **07,** 050. arXiv: 1103.3022 [hep-th] (2011).

225. Brattan, D., Camps, J., Loganayagam, R. & Rangamani, M. CFT dual of the AdS Dirichlet problem : Fluid/Gravity on cut-off surfaces. *JHEP* **12,** 090. arXiv: 1106.2577 [hep-th] (2011).

226. Pinzani-Fokeeva, N. & Taylor, M. Towards a general fluid/gravity correspondence. *Phys. Rev. D* **91,** 044001. arXiv: 1401.5975 [hep-th] (2015).

227. Huang, T.-Z., Ling, Y., Pan, W.-J., Tian, Y. & Wu, X.-N. From Petrov-Einstein to Navier-Stokes in Spatially Curved Spacetime. *JHEP* **10,** 079. arXiv: 1107.1464 [gr-qc] (2011).

228. De, S. & Majhi, B. R. Fluid description of gravity on a timelike cut-off surface: beyond Navier-Stokes equation. *JHEP* **01,** 044. arXiv: 1810.07017 [hep-th] (2019).

229. Zhang, C.-Y., Ling, Y., Niu, C., Tian, Y. & Wu, X.-N. Magnetohydrodynamics from gravity. *Phys. Rev. D* **86,** 084043. arXiv: 1204.0959 [hep-th] (2012).

230. Chirco, G., Eling, C. & Liberati, S. Higher Curvature Gravity and the Holographic fluid dual to flat spacetime. *JHEP* **08,** 009. arXiv: 1105.4482 [hep-th] (2011).

231. Bai, X., Hu, Y.-P., Lee, B.-H. & Zhang, Y.-L. Holographic Charged Fluid with Anomalous Current at Finite Cutoff Surface in Einstein-Maxwell Gravity. *JHEP* **11,** 054. arXiv: 1207.5309 [hep-th] (2012).

232. Cai, R.-G., Li, T.-J., Qi, Y.-H. & Zhang, Y.-L. Incompressible Navier-Stokes Equations from Einstein Gravity with Chern-Simons Term. *Phys. Rev. D* **86,** 086008. arXiv: 1208.0658 [hep-th] (2012).

233. Zou, D.-C., Zhang, S.-J. & Wang, B. Holographic charged fluid dual to third order Lovelock gravity. *Phys. Rev. D* **87,** 084032. arXiv: 1302.0904 [hep-th] (2013).

234. Hu, Y.-P., Tian, Y. & Wu, X.-N. Bulk Viscosity of dual Fluid at Finite Cutoff Surface via Gravity/Fluid correspondence in Einstein-Maxwell Gravity. *Phys. Lett. B* **732,** 298–304. arXiv: 1311.3891 [hep-th] (2014).

235. Cai, R.-G., Li, L. & Zhang, Y.-L. Non-Relativistic Fluid Dual to Asymptotically AdS Gravity at Finite Cutoff Surface. *JHEP* **07,** 027. arXiv: 1104.3281 [hep-th] (2011).

236. Huang, T.-Z., Ling, Y., Pan, W.-J., Tian, Y. & Wu, X.-N. Fluid/gravity duality with Petrov-like boundary condition in a spacetime with a cosmological constant. *Phys. Rev. D* **85,** 123531. arXiv: 1111.1576 [hep-th] (2012).

237. Anninos, D., Anous, T., Bredberg, I. & Ng, G. S. Incompressible Fluids of the de Sitter Horizon and Beyond. *JHEP* **05,** 107. arXiv: 1110.3792 [hep-th] (2012).

238. Ling, Y., Niu, C., Tian, Y., Wu, X.-N. & Zhang, W. Note on the Petrov-like boundary condition at finite cutoff surface in gravity/fluid duality. *Phys. Rev. D* **90,** 043525. arXiv: 1306.5633 [gr-qc] (2014).

239. Eling, C., Meyer, A. & Oz, Y. The Relativistic Rindler Hydrodynamics. *JHEP* **05,** 116. arXiv: 1201.2705 [hep-th] (2012).

240. Berkeley, J. & Berman, D. S. The Navier-Stokes equation and solution generating symmetries from holography. *JHEP* **04,** 092. arXiv: 1211.1983 [hep-th] (2013).

241. Lysov, V. Dual Fluid for the Kerr Black Hole. *JHEP* **06,** 080. arXiv: 1712.08079 [hep-th] (2018).

242. Wu, X., Ling, Y., Tian, Y. & Zhang, C. Fluid/Gravity Correspondence For General Non-rotating Black Holes. *Class. Quant. Grav.* **30,** 145012. arXiv: 1303.3736 [hep-th] (2013).

243. Chou, C.-J., Wu, X., Yang, Y. & Yuan, P.-H. Rotating Black Holes and Coriolis Effect. *Phys. Lett. B* **761,** 131–135. arXiv: 1511.08691 [hep-th] (2016).

244. Hubeny, V. E. The Fluid/Gravity Correspondence: a new perspective on the Membrane Paradigm. *Class. Quant. Grav.* **28** (eds Marolf, D. & Sudarsky, D.) 114007. arXiv: 1011.4948 [gr-qc] (2011).

245. Rangamani, M. Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence. *Class. Quant. Grav.* **26** (ed Uranga, A. M.) 224003. arXiv: 0905.4352 [hep-th] (2009).

246. Vardoulakis, I. *Cosserat continuum mechanics* (Springer, Cham, Switzerland, 2019).

247. Gogala, B. Torsion and related concepts: An introductory overview. *International Journal of Theoretical Physics* **19,** 573–586 (1980).

248. Von Der Heyde, P. The equivalence principle in the U4 theory of gravitation. *Lettere al Nuovo Cimento (1971-1985)* **14,** 250–252. ISSN: 1827-613X. https://doi.org/10.1007/BF02745635 (1975).

249. Bhattacharya, S. & Shankaranarayanan, S. Fluctuations in horizon-fluid lead to negative bulk viscosity. *Phys. Rev. D* **93,** 064030. arXiv: 1511.01377 [gr-qc] (2016).

250. Bhattacharya, S. & Shankaranarayanan, S. Negative specific heat of black-holes from Fluid-Gravity Correspondence. *Class. Quant. Grav.* **34,** 075005. arXiv: 1702.03682 [gr-qc] (2017).

251. Cropp, B., Bhattacharya, S. & Shankaranarayanan, S. Hints of quantum gravity from the horizon fluid. *Phys. Rev. D* **95,** 024006. arXiv: 1607.06222 [gr-qc] (2017).

252. Aldrovandi, R. & Pereira, J. *Teleparallel Gravity: An Introduction* ISBN: 9789400751439. https://books.google.co.in/books?id=2dZ5zuKe_fUC (Springer Netherlands, 2012).

253. Scholz, E. E. Cartan's attempt at bridge-building between Einstein and the Cosserats - or how translational curvature became to be known as torsion. *European Physical Journal H* **44.** arXiv: 1810.03872 [math.HO] (Feb. 2019).

254. Ashtekar, A. & Krishnan, B. Dynamical horizons and their properties. *Phys. Rev. D* **68,** 104030. arXiv: gr-qc/0308033 (2003).

255. Booth, I. & Fairhurst, S. The First law for slowly evolving horizons. *Phys. Rev. Lett.* **92,** 011102. arXiv: gr-qc/0307087 (2004).

256. Altas, E. & Tekin, B. Basics of Apparent horizons in black hole physics. *J. Phys. Conf. Ser.* **2191,** 012002. arXiv: 2108.05119 [gr-qc] (2022).

257. Beltrán Jiménez, J., Heisenberg, L. & Koivisto, T. Coincident General Relativity. *Phys. Rev. D* **98,** 044048. arXiv: 1710.03116 [gr-qc] (2018).

258. Strominger, A. Black hole entropy from near horizon microstates. *JHEP* **02,** 009. arXiv: hep-th/9712251 (1998).

259. Bondi, H., van der Burg, M. G. J. & Metzner, A. W. K. Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems. *Proc. Roy. Soc. Lond. A* **269,** 21–52 (1962).

260. Sachs, R. Asymptotic Symmetries in Gravitational Theory. *Phys. Rev.* **128,** 2851–2864. https://link.aps.org/doi/10.1103/PhysRev.128.2851 (6 1962).

261. Averin, A., Dvali, G., Gomez, C. & Lust, D. Goldstone origin of black hole hair from supertranslations and criticality. *Mod. Phys. Lett. A* **31,** 1630045. arXiv: 1606.06260 [hep-th] (2016).

262. Eling, C. & Oz, Y. On the Membrane Paradigm and Spontaneous Breaking of Horizon BMS Symmetries. *JHEP* **07,** 065. arXiv: 1605.00183 [hep-th] (2016).