FINITE ELEMENT METHODS FOR ELLIPTIC AND PARABOLIC OPTIMAL CONTROL PROBLEMS WITH MEASURE DATA

by

Pratibha Shakya

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI-781039, INDIA

August, 2018

FINITE ELEMENT METHODS FOR ELLIPTIC AND PARABOLIC OPTIMAL CONTROL PROBLEMS WITH MEASURE DATA

A thesis submitted

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

by

Pratibha Shakya

(Roll No. 136123009)

Department of Mathematics

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

August, 2018

To the loving memories of my parents

DECLARATION

It is certify that the work contained in the thesis entitled as **"Finite Element Methods for Elliptic and Parabolic Optimal Control Problems with Measure Data"** has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, under the guidance of **Prof. Rajen Kumar Sinha** for the award of Doctor of Philosophy and that this work has not been submitted elsewhere for a degree.

August, 2018

Pratibha Shakya
Department of Mathematics
Indian Institute of Technology Guwahati

CERTIFICATE

It is certify that the work contained in this thesis entitled as **"Finite Element Methods for Elliptic and Parabolic Optimal Control Problems with Measure Data"** submitted by **Pratibha Shakya (136123009)**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy, has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

August, 2018

Prof. Rajen Kumar Sinha
Department of Mathematics
Indian Institute of Technology Guwahati

Acknowledgement

This thesis would not have been possible without the kind support, active guidance, assistance and the encouragement of several individuals. Therefore, I take the pleasant opportunity to express my sincere gratitude and heartfelt obligation towards all the personages who have help me in this endeavor.

My first and foremost sincere gratitude with a deep sense of reverence goes to my supervisor Prof. Rajen Kumar Sinha, Department of Mathematics, IIT Guwahati, for his excellent guidance, caring, patience and providing me good materials (journals, research papers and books etc.). I am indebted to him for introducing me in to the realm of this interesting work, guiding me with the techniques, giving me freedom to thinking and learning about the Optimal Control Problems. My supervisor has constantly encouraged and motivated me to remain focused on achieving my goal. His advices and insight was invaluable to me. He has not only guided me in research, also his invaluable suggestion helped me to become a stronger person in life. I will remain grateful to him for all that he did for me.

I would also like to express my sincere gratitude to all my doctoral committee members Prof. R. Alam, Prof. N. Srinvasan and Dr. B. Deka for their continuous support, encouragement and valuable suggestions during my seminars and progress reviews that has lead to the successful completion of my thesis.

I owe my profound gratitude to the present and previous heads of the Department of Mathematics, IIT Guwahati, Prof. N. Selvaraju, Prof. S. N. Bora and Prof. B. K. Sharma for providing me the necessary facilities during my research work. My sincere thanks are due to other faculty members and office staff of the department for their help, encouragement and assistance in all official matters. I also thankful to Ministry of Human and Resource Development, Govt. of India, for offering me financial assistance to carry out my research work. I remain thankful to my seniors, juniors and specially thankful to Tamal, Tanushree, Shantiram and Ram Manohar for their help and cooperation at IIT Guwahati.

Lastly I am thankful to my family members whose continuous encouragement, support and *blessing helped me in successful completion of my research work. I owe my achievement to my parents, elder brother Vinay Kumar Shakya, younger brother Sachin Kumar Shakya and sister-in-law Mrs. Seema Shakya for their motivation. My greatest appreciation goes to Prabhakar and Ram Manohar who were always a great support in all my struggles and frustrations during my stay in IIT Guwahati. They were always there for cheering me up and stood for me in the good and bad times. It is also pleasure to mention my father Mr. Ram Kishore Shakya and mother Mrs. Kamala Shakya for their immense love which gives me a moral support in achieving my goal. Thus, through this acknowledgement I express my heartiest regards to my all well-wishers.*

August, 2018

With regards,
Pratibha Shakya
Department of Mathematics
Indian Institute of Technology Guwahati

Abstract

The aim of this thesis is to study *a priori* and *a posteriori* error analysis of finite element methods for elliptic and parabolic optimal control problems with measure data in a bounded convex domain in $\mathbb{R}^d (d = 2 \text{ or } 3)$. The control problems governed by elliptic partial differential equation with measure data are used to model the potential of an electric field with an electric charge distribution, whereas parabolic optimal control problems with measure data in space are used in the design and management of waste water treatment systems, mainly the disposal of sea outfalls discharging polluting effluent from a sewerage systems. On the other hand, parabolic optimal control problems with measure data in time appear in the optimality conditions of some optimal control problems with pointwise state constraints in time. The solution of the state equation of such type of problems exhibits low regularity due to the presence of measure data which introduces some difficulties for both theory and numerics of the finite element method. An effort has been made in this thesis to investigate both *a priori* and *a posteriori* error analysis of finite element method for these control problems. The strategy *optimize-then-discretize* is employed for the approximations of these control problems.

We first analyze elliptic optimal control problem with measure data and prove the existence, uniqueness and regularity of the solution to the optimal control problem. To discretize the control problem we use piecewise linear and continuous finite elements for the approximation of the state and co-state variables, whereas piecewise constant functions are used for the approximation of the control variable. We derive *a priori* error estimates for the state, co-state and control variables in the L^2-norm with an order of convergence $\mathcal{O}(h^{2-\frac{d}{2}})$. Further, global *a posteriori* upper bounds for the state, co-state and control variables in the L^2-norm are established. Moreover, local lower bounds for the errors in the state and co-state variables, and global lower bound for the error in the control variable are demonstrated in case of two space dimension ($d = 2$).

We next consider parabolic optimal control problems with measure data. Two kinds of problems, namely measure data in space and measure data in time are considered and analyzed. The existence, uniqueness and regularity of the solutions of both type of control problems are proved. The continuous piecewise linear functions are used for the approximations of the state and co-state variables, and piecewise constant functions for the approximation of the control

variable. Both spatially discrete and fully discrete finite element approximations of the control problems with measure data in space and time are analyzed. *A priori* error estimates of order $\mathcal{O}(h^{2-\frac{d}{2}})$ is demonstrated for the spatially discrete control problem with measure data in space whereas error estimate of order $\mathcal{O}(h^{2-\frac{d}{2}} + k^{1/2})$ is established for the fully discrete backward Euler time discretization. For parabolic optimal control problem with measure data in time, we have obtained error estimates of order $\mathcal{O}(h)$ for the state, costate and control variables for the spatially discrete problem. A time discretization scheme based on implicit backward-Euler method is analyzed and error estimates of order $\mathcal{O}(h + k^{1/2})$ are derived for the state, co-state and control variables. Further, we study the *a posteriori* error analysis for the space-time finite element discretization for both type of control problems. We derive global upper bounds for the errors in the state, co-state and control variables in the $L^2(0,T;L^2(\Omega))$-norm.

Finally, numerical results for two dimensional test problems are presented to illustrate our theoretical findings.

Contents

NOMENCLATURE

EOCP	Elliptic Optimal Control Problem
POCP	Parabolic Optimal Control Problem
PIDE	Parabolic Integro-Differential Equation
$\mathcal{A}$	Second order linear elliptic partial differential operator of the form $-\nabla \cdot (A\nabla) + a_0 I$
$\mathcal{A}^*$	Adjoint of the operator $\mathcal{A}$
$A = \{a_{ij}(x)\}$	Coefficient matrix corresponding to the operator $\mathcal{A}$
$\mathbb{R}^d$	d-dimensional Euclidean space
Ω	Bounded convex domain
T	Final time
$\partial\Omega$	Boundary of Ω
Ω_T	$\Omega \times (0, T]$
Γ_T	$\partial\Omega \times [0, T]$
y	Exact state
q	Exact control
z	Exact co-state
$y_0(x)$	Initial state
μ	Measure data
$L^p(\Omega), 1 \le p \le \infty$	Standard Lebesgue space of order p over Ω
$\|\cdot\|_{L^p(\Omega)}$	Norm on $L^p(\Omega)$
$(\cdot, \cdot)_\Omega$	Standard L^2-inner product on Ω
$(\cdot, \cdot)_{\Omega_T}$	L^2-inner product on Ω_T
$W^{m,p}(\Omega)$	Standard Sobolev space of order (m, p) over Ω
$\|\cdot\|_{W^{m,p}(\Omega)}$	Norm on $W^{m,p}(\Omega)$
$H^m(\Omega)$	Hilbert space $W^{m,2}(\Omega)$

$H_0^1(\Omega)$	Space of functions in $H^1(\Omega)$ that vanish on the boundary of Ω in the sense of trace
$\mathcal{C}^m(\overline{\Omega})$	Space of functions with continuous derivatives up to and including order m in $\overline{\Omega}$
$\mathcal{C}_0^m(\Omega)$	Space of all $C^m(\Omega)$ functions with compact support in Ω
$\mathcal{C}_0^\infty(\Omega)$	Space of all infinitely differentiable functions with compact support in Ω
$\mathcal{M}(\Omega)$	Space of real and regular Borel measures on Ω
$\mathcal{M}[0,T]$	Space of real and regular Borel measures in $[0,T]$
Q_{ad}^E	Set of admissible controls for EOCP
Q_{ad}^P	Set of admissible controls for POCP
$\mathcal{C}^\infty(\Omega_T)$	Space of infinitely differentiable functions in Ω_T
$\mathcal{D}(\Omega_T)$	Set of $\mathcal{C}^\infty(\Omega_T)$ functions with compact support in Ω_T
$\mathcal{D}'(\Omega_T)$	Dual space of $\mathcal{D}(\Omega_T)$
$\text{supp}(\phi)$	Support of ϕ
$a(\cdot,\cdot)$	Bilinear form corresponding to the elliptic operator $\mathcal{A}$
β	Coercivity constant for $a(\cdot,\cdot)$
β_1	Continuity constant for $a(\cdot,\cdot)$
$\mathcal{T}_h$	Shape regular, conforming triangulations of Ω
$\mathcal{T}_h^n$	Shape regular, conforming triangulations of Ω at $t = t_n$
$\text{diam}(K)$	Longest side of the element K
h_K	$\text{diam}(K)$
h	$\max\limits_{K \in \mathcal{T}_h} h_K$
$\mathcal{E}_h$	Set of internal edges(faces) of $\mathcal{T}_h$
$\mathcal{E}_h^n$	Set of internal edges(faces) of $\mathcal{T}_h^n$
w_K	Patch of the element K
w_e	Patch of the edge(face) e

$\mathcal{E}_h^K$	Collection of edges(faces) e of elements $K \subset \omega_K$
$\left[\frac{\partial v}{\partial n_{\mathcal{A}}}\right]\Big\vert_e$	Jump across the edge(face) e
I_n	n-th subinterval of $[0,T]$
k	Constant time step
$\mathcal{R}_h$	Ritz-projection operator
$\mathcal{L}_h$	L^2-projection operator
$\mathcal{L}_h^0$	L^2-projection operator at time $t = t_0$
$\mathcal{L}_h^n$	L^2-projection operator at time $t = t_n$
$\mathbb{P}_m$	Space of polynomials of *degree* $\leq m$
V_h	Finite dimensional subspace of $\mathcal{C}(\overline{\Omega})$ consist of piecewise linear polynomials
π_h	Standard Lagrange interpolation operator onto V_h
V_h^0	Finite element space for the state and co-state variables
Q_h^E	Finite element space of Q_{ad}^E
Q_h^P	Finite element space of Q_{ad}^P
V_h^n	Finite element space for the state and co-state variables at time $t = t_n$
$Q_{h,n}^P$	Finite element space for the control variable at time $t = t_n$
π_h^n	Standard Lagrange interpolation operator onto V_h^n
$J(\cdot,\cdot)$	Cost functional for EOCP
$\tilde{J}(\cdot,\cdot)$	Cost functional for POCP
$j(\cdot)$	Reduced cost functional for EOCP
$\tilde{j}(\cdot)$	Reduced cost functional for POCP
α	A fixed constant for EOCP
$\tilde{\alpha}$	A fixed constant for POCP
q_a	Lower bound of the control variable for EOCP
q_b	Upper bound of the control variable for EOCP
q_c	Lower bound of the control variable for POCP

q_d	Upper bound of the control variable for POCP
$y_d(x)$	Desired state for EOCP
$y_{\tilde{d}}(x,t)$	Desired state for POCP
γ_E	Positive constant related to second-order optimality condition for EOCP
γ_P	Positive constant related to second-order optimality condition for POCP
η_1	Control error estimator for EOCP
η_2	Data approximation error for EOCP
η_3	Space error estimator of the state variable for EOCP
η_4	Space error estimator of the co-state variable for EOCP
$\eta^n_{1,ms}$	Control error estimator for POCP with measure data in space
$\eta^n_{2,ms}$	Space error estimator of the co-state variable for POCP with measure data in space
$\eta^n_{3,ms}$	Error estimator for the error in the intermediate state variable for measure data in space
$\eta^n_{4,ms}$	Data approximation error of the co-state variable for POCP with measure data in space
$\eta^n_{5,ms}$	Time error estimator of the co-state variable for POCP with measure data in space
$\xi^n_{1,ms}$	Space error estimator of the state variable for POCP with measure data in space
$\xi^n_{2,ms}$	Time error estimator of the state variable for POCP with measure data in space
$\xi^n_{3,ms}$, $\xi^n_{4,ms}$	Data approximation errors of the state variable for POCP with measure data in space
$\eta^n_{1,mt}$	Control error estimator for POCP with measure data in time
$\eta^n_{2,mt}$	Space error estimator of the co-state variable for POCP with measure data in time

$\eta^n_{3,mt}$	Error estimator for the error in the intermediate state variable for measure data in time
$\eta^n_{4,mt}$	Data approximation error of the co-state variable for POCP with measure data in time
$\eta^n_{5,mt}$	Time error estimator of the co-state variable for POCP with measure data in time
$\xi^n_{1,mt}$	Space error estimator of the state variable for POCP with measure data in time
$\xi^n_{2,mt}$	Time error estimator of the state variable for POCP with measure data in time
$\xi^n_{3,mt}$, $\xi^n_{4,mt}$	Data approximation errors of the state variable for POCP with measure data in time

List of Figures

List of Tables

1 Introduction

The main objective of this thesis is to study *a priori* and *a posteriori* error analysis of finite element methods for elliptic and parabolic optimal control problems with measure data. This chapter is introductory and contains the description of the problems, a brief survey of the relevant literature and motivation for the present study. It also contains some notations and preliminary materials to be used in the thesis. The organization of the thesis is given in the last section of this chapter.

1.1 Description of the problems

Elliptic optimal control problem with measure data: Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded convex domain with smooth boundary $\partial\Omega$. Consider the following model elliptic optimal control problem (EOCP) with measure data:

$$\min_{q \in Q_{ad}^E} J(q, y) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to the state equation

$$\begin{cases} \mathcal{A}\, y = \mu + q & \text{in } \Omega, \\ \quad y = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and the control constraints

$$q_a \leq q(x) \leq q_b \quad \text{a.e. in } \Omega, \tag{1.3}$$

where $y = y(x)$ denotes the state variable and $q = q(x)$ is the control variable. The operator $\mathcal{A}$ is assumed to be a second order linear elliptic partial differential operator of the form

$$\mathcal{A}y = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{i,j}(x)\frac{\partial y}{\partial x_i}\Big) + a_0(x)y \tag{1.4}$$

with $a_0(x) \in L^\infty(\Omega)$, $a_0(x) \geq 0$ forall $x \in \Omega$, $a_{i,j}(x)(1 \leq i, j \leq d)$ is Lipschitz continuous on Ω and satisfies the following uniform ellipticity condition: There exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \geq c|\xi|^2, \qquad \forall \xi \in \mathbb{R}^d, \ \ x \in \Omega.$$

Moreover, $y_d(x) \in L^2(\Omega)$ is a given desired state, $\mu = g\,\omega$, where $g \in \mathcal{C}(\overline{\Omega})$, $\omega \in \mathcal{M}(\Omega)$ are given functions and $\alpha > 0$ is a fixed parameter. Here, $\mathcal{M}(\Omega)$ is the space of real and Borel measures on Ω, which can be defined as the dual space of $\mathcal{C}(\overline{\Omega})$. The set of admissible controls is defined by

$$Q_{ad}^E := \{q \in L^2(\Omega) : \ q_a \leq q(x) \leq q_b \ \ a.e. \ \text{in } \Omega\} \tag{1.5}$$

with $q_a, \ q_b \in \mathbb{R}$ fulfill $q_a < q_b$.

Parabolic optimal control problem with measure data: Let Ω be a bounded convex domain in $\mathbb{R}^d$ ($d = 2$ or 3) with smooth boundary $\partial\Omega$. Set $\Omega_T = \Omega \times (0, T]$ and $\Gamma_T = \partial\Omega \times [0, T]$ with $T < \infty$. We shall consider the following parabolic optimal control problem (POCP) with measure data:

$$\min_{q \in Q_{ad}^P} \tilde{J}(q, y) = \frac{1}{2}\int_0^T \|y - y_{\tilde{d}}\|^2_{L^2(\Omega)}\, d\tau + \frac{\tilde{\alpha}}{2}\int_0^T \|q\|^2_{L^2(\Omega)}\, d\tau \tag{1.6}$$

subject to the state equation

$$\begin{cases} y_t + \mathcal{A}y = \mu + q & \text{in } \ \Omega_T, \\ y(\cdot, 0) = y_0(x) & \text{in } \ \Omega, \\ y = 0 & \text{on } \ \Gamma_T, \end{cases} \tag{1.7}$$

and the control constraints

$$q_c \leq q(x, t) \leq q_d \quad a.e. \ \text{in } \ \Omega_T. \tag{1.8}$$

Here, $y = y(x, t)$ denotes the state variable, $q = q(x, t)$ is the control variable and $y_t = \frac{\partial y}{\partial t}$. The operator $\mathcal{A}$ is defined in (1.4) and $\tilde{\alpha} > 0$ is a fixed constant. Furthermore, the initial state $y_0(x) \in L^2(\Omega)$ and $y_{\tilde{d}}(x, t) \in L^2(0, T; L^2(\Omega))$ is a given desired state. We shall consider two kinds of POCPs with measure data. At first, we consider problem (1.6)-(1.8) with measure data in space, i.e., $\mu = g\omega$ with $g \in L^2(0, T; \mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the space of real and regular Borel measures on Ω. Next, we consider problem (1.6)-(1.8) with measure data in time, i.e., $\mu = g\omega$ with

$g \in \mathcal{C}([0,T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$, where $\mathcal{M}[0,T]$ is the space of real and regular Borel measures in $[0,T]$, which can be defined as the dual space of $\mathcal{C}[0,T]$.

The set of admissible controls is defined by

$$Q_{ad}^P := \{q \in L^2(0,T; L^2(\Omega)) : \ q_c \le q(x,t) \le q_d \ \ a.e. \ \text{in } \Omega_T\} \tag{1.9}$$

with $q_c,\ q_d \in \mathbb{R}$ fulfill $q_c < q_d$.

A classical control problem consists of finding a control function q which minimizes the cost functional (performance measure) $J(q,y)$ (or $\tilde{J}(q,y)$), where the pair (q,y) satisfies the state equation. The term $\frac{\alpha}{2}\|q\|^2_{L^2(\Omega)}$ (or $\frac{\tilde{\alpha}}{2}\int_0^T \|q\|^2_{L^2(\Omega)}\, d\tau$) is proportional to the consumed energy. Thus, minimizing $J($ or $\tilde{J})$ is a compromise between the energy consumption and finding q so that the state y is close to the desired state y_d (or $y_{\tilde{d}}$).

The optimal control problems of the form (1.1)-(1.3) can be used to model the potential of an electric field with an electric charge distribution. For the literature related to the elliptic equation with measure data, one may refer to [14] and [79]. The problems of the form (1.6)-(1.8) are used in the design and management of waste water treatment systems, mainly the disposal of sea outfalls discharging polluting effluent from a sewerage systems [68]. On the other hand, parabolic equations of the form (1.7) with measure data in time appear in the optimality conditions of some optimal control problems with pointwise state constraints in time (see [16] and [70]).

1.2 Notations and preliminaries

In this section, we shall introduce some standard notations and preliminary materials to be used in this thesis. All functions considered here are real valued. Let Ω be a bounded convex domain in $\mathbb{R}^d$ (d-dimensional Euclidean space) and let $\partial\Omega$ denote the boundary of Ω. Let $x = (x_1, x_2, \ldots, x_d) \in \Omega$, and let $dx = dx_1, \ldots, dx_d$. Further, let $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_d)$ be a d-tuple with nonnegative integer components. Denote the order of Υ as $|\Upsilon| = \Upsilon_1 + \ldots + \Upsilon_d$. Then, by $D^\Upsilon\phi$, we shall mean the Υth derivative of ϕ defined by

$$D^\Upsilon \phi := \frac{\partial^{|\Upsilon|}\phi}{\partial x_1^{\Upsilon_1} \ldots \partial x_d^{\Upsilon_d}}.$$

We shall make frequent reference to the following well-known function spaces. For $1 \le p < \infty$, $L^p(\Omega)$ denotes the linear space of equivalence classes of measurable functions ϕ in Ω such that $\int_\Omega |\phi(x)|^p\, dx$ exists and is finite. The norm on $L^p(\Omega)$ is given by

$$\|\phi\|_{L^p(\Omega)} := \Big(\int_\Omega |\phi(x)|^p\, dx\Big)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ denotes the space of functions ϕ on Ω such that

$$\|\phi\|_{L^\infty(\Omega)} := ess \sup_{x\in\Omega} |\phi(x)| < \infty.$$

When $p = 2$, $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)\, dx.$$

By support of a function ϕ, denoted by $supp(\phi)$, we mean the closure of all points x with $\phi(x) \neq 0$, i.e.,

$$supp(\phi) := \overline{\{x : \ \phi(x) \neq 0\}}.$$

For any nonnegative integer m, $\mathcal{C}^m(\overline{\Omega})$ denotes the space of functions with continuous derivatives upto and including order m in $\overline{\Omega}$. $\mathcal{C}_0^m(\Omega)$ is the space of all $\mathcal{C}^m(\Omega)$ functions with compact support in Ω and $\mathcal{C}_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω.

We now introduce the notion of Sobolev spaces. Let m be a nonnegative integer and let p be such that $1 \leq p < \infty$. The Sobolev space of order (m, p) on Ω, denoted by $W^{m,p}(\Omega)$, is defined as a linear space of functions (or equivalence class of functions) in $L^p(\Omega)$ whose distributional derivatives upto order m are also in $L^p(\Omega)$, i.e.,

$$W^{m,p}(\Omega) := \{\phi : \ D^\Upsilon\phi \in L^p(\Omega) \ \text{ for } \ 0 \leq |\Upsilon| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\begin{aligned}\|\phi\|_{W^{m,p}(\Omega)} &:= \Big(\sum_{0\leq|\Upsilon|\leq m} \int_\Omega |D^\Upsilon\phi(x)|^p\, dx\Big)^{\frac{1}{p}} \\ &:= \Big(\sum_{0\leq|\Upsilon|\leq m} \|D^\Upsilon\phi(x)\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}}, \qquad 1 \leq p < \infty.\end{aligned}$$

When $p = \infty$, the norm on the space $W^{m,\infty}(\Omega)$ is defined by

$$\|\phi\|_{W^{m,\infty}(\Omega)} := \max_{0\leq|\Upsilon|\leq m} \|D^\Upsilon\phi(x)\|_{L^\infty(\Omega)}.$$

Also, the semi-norms on $W^{m,p}(\Omega)$ are defined as

$$|\phi|_{W^{m,p}(\Omega)} := \sum_{|\Upsilon|=m} \|D^\Upsilon\phi\|_{L^p(\Omega)}.$$

For $p = 2$, we denote the spaces $W^{m,p}(\Omega) = H^m(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$. The space $H^m(\Omega)$ is a Hilbert space with natural inner product defined by

$$(\phi, \psi)_{H^m(\Omega)} := \sum_{0\leq|\Upsilon|\leq m} \int_\Omega D^\Upsilon\phi D^\Upsilon\psi\, dx, \qquad \phi, \psi \in H^m(\Omega).$$

The Sobolev space $H^m(\Omega)$ (respectively, $H^m_0(\Omega)$) is also defined as the closure of $\mathcal{C}^\infty(\Omega)$ (respectively, $\mathcal{C}^\infty_0(\Omega)$) with respect to the norm $\|\phi\|_{H^m(\Omega)} = \|\phi\|_{H^{m,2}(\Omega)}$ and semi-norm $|\phi|_{H^m(\Omega)} = |\phi|_{H^{m,2}(\Omega)}$. We denote the dual space of $H^1_0(\Omega)$ by $H^{-1}(\Omega)$ with norm

$$\|\phi\|_{H^{-1}(\Omega)} := \sup_{\psi \in H^1_0(\Omega),\, \psi \neq 0} \frac{(\phi, \psi)}{\|\psi\|_{H^1(\Omega)}}.$$

We denote $\mathcal{M}(\Omega)$ is the space of real and regular Borel measures on Ω, which can be defined as the dual space of $\mathcal{C}(\overline{\Omega})$ with its natural norm

$$\|\mu\|_{\mathcal{M}(\Omega)} := \sup\Big\{ \int_\Omega \phi \, d\mu : \;\; \phi \in \mathcal{C}(\overline{\Omega}) \;\text{ and }\; \|\phi\|_{\mathcal{C}(\overline{\Omega})} \leq 1 \Big\}.$$

For $1 \leq p \leq \infty$, we also define the standard Bochner spaces $L^p(0,T;\mathbf{B})$, where $\mathbf{B}$ is a real Banach space with norm $\|\cdot\|_{\mathbf{B}}$, consisting of all measurable functions $\phi : [0,T] \to \mathbf{B}$ for which

$$\begin{aligned} \|\phi\|_{L^p(0,T;\mathbf{B})} &:= \Big(\int_0^T \|\phi(\tau)\|^p_{\mathbf{B}} \, d\tau \Big)^{\frac{1}{p}} < \infty \qquad \text{for } \; 1 \leq p < \infty, \\ \|\phi\|_{L^\infty(0,T;\mathbf{B})} &:= \mathit{ess} \sup_{t \in (0,T)} \|\phi(t)\|_{\mathbf{B}} < \infty \qquad \text{for } \; p = \infty. \end{aligned}$$

In the subsequent chapters, we shall also use the following spaces. For a given Banach space $\mathbf{B}$, we define $H^1(0,T;\mathbf{B})$ as the space consisting of all measurable functions $\phi : (0,T) \to \mathbf{B}$ for which

$$\|\phi\|_{H^1(0,T;\mathbf{B})} := \Big(\int_0^T \|\phi(\tau)\|^2_{\mathbf{B}} \, d\tau + \int_0^T \|\phi_t(\tau)\|^2_{\mathbf{B}} \, d\tau \Big)^{\frac{1}{2}} < \infty.$$

When no risk of confusion exists we shall write $\|\cdot\|_{L^p(\mathbf{B})}$ for $\|\cdot\|_{L^p(0,T;\mathbf{B})}$, $\|\cdot\|_{L^\infty(\mathbf{B})}$ for $\|\cdot\|_{L^\infty(0,T;\mathbf{B})}$ and $\|\cdot\|_{H^1(\mathbf{B})}$ for $\|\cdot\|_{H^1(0,T;\mathbf{B})}$. Furthermore, $\mathcal{C}([0,T];\mathbf{B})$ is defined as the space of continuous functions $\phi : [0,T] \to \mathbf{B}$ with norm $\|\phi\|_{\mathcal{C}([0,T];\mathbf{B})} := \max_{t \in [0,T]} \|\phi(t)\|_{\mathbf{B}} < \infty$ and $\mathcal{M}[0,T]$ is the space of real and regular Borel measures in $[0,T]$, which can be defined as the dual space of $\mathcal{C}[0,T]$ with its natural norm

$$\|\mu\|_{\mathcal{M}[0,T]} := \sup\Big\{ \int_0^T \phi \, d\mu : \;\; \phi \in \mathcal{C}[0,T] \;\text{ and }\; \|\phi\|_{\mathcal{C}[0,T]} \leq 1 \Big\}.$$

Let $\mathcal{C}^\infty(\Omega_T)$ be the space of all infinitely differentiable functions in Ω_T. We denote $\mathcal{D}(\Omega_T)$ the set of $\mathcal{C}^\infty(\Omega_T)$ functions with compact support in Ω_T and $\mathcal{D}'(\Omega_T)$ is the dual space of $\mathcal{D}(\Omega_T)$. For a complete discussion on Sobolev spaces, see Adams and Fournier [1] and Grisvard [42]. From time to time we shall also use the following inequalities (see

Hardy et al. [43]):

Young's inequality. For a, $b \geq 0$ and $\epsilon > 0$, the following inequality

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon\, b^2}{2}$$

holds.

Cauchy-Schwarz inequality. For all $a, b \geq 0$, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

In integral form, if ϕ and ψ are both real valued functions with $\phi \in L^p(\Omega)$ and $\psi \in L^{p'}(\Omega)$, then

$$\int_\Omega \phi\, \psi\, dx \leq \|\phi\|_{L^p(\Omega)} \|\psi\|_{L^{p'}(\Omega)}.$$

For $p = p' = 2$, the above inequality is known as *Schwarz's inequality.*

Discrete version of the Cauchy-Schwarz inequality. Let ϕ_j, ψ_j, $j = 1, \ldots, d$ be positive real numbers. Then

$$\sum_{j=1}^{d} \phi_j \psi_j \leq \Big(\sum_{j=1}^{d} \phi_j^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{d} \psi_j^2\Big)^{\frac{1}{2}}.$$

Now, we state the Poincaré inequality which will be used frequently in this thesis.

Lemma 1.2.1 (Poincaré inequality)**.** *Let Ω be a bounded convex domain in $\mathbb{R}^d$. Then there exists a positive constant $C = C(\Omega)$ such that*

$$\|\phi\|_{L^2(\Omega)} \leq C \|\nabla \phi\|_{L^2(\Omega)}, \qquad \textit{for every } \phi \in H_0^1(\Omega).$$

In view of the Poincaré inequality, $\|\nabla(\cdot)\|_{L^2(\Omega)}$ defines a norm on $H_0^1(\Omega)$.

For the following definitions, we refer to [86].

Definition 1.2.1 (Directional derivative)**.** *Let X and Y be two normed spaces. Let X_0 be a non-empty open subset of X and $f : X_0 \to Y$ be a given mapping. If for two elements $x \in X_0$ and $v \in X$ the limit*

$$f'(x)(v) := \lim_{\nu \to 0} \frac{f(x + \nu v) - f(x)}{\nu}$$

exists, then $f'(x)(v)$ is called the directional derivative of f at x in the direction v. Moreover, if the limit

$$f''(x)(v, v) := \lim_{\nu \to 0} \frac{f'(x + \nu v)(v) - f'(x)(v)}{\nu}$$

exists, then $f''(x)(v, v)$ is called the second order directional derivative of f in the direction of v.

Definition 1.2.2 (Fréchet derivative). *Let X and Y be two normed spaces, X_0 be a non-empty open subset of X and $f : X_0 \to Y$ be a given mapping. Furthermore, let an element $x \in X_0$ be given. If there is a continuous linear mapping $f'(x) : X \to Y$ with the property*

$$\lim_{\|v\|_X \to 0} \frac{\|f(x+v) - f(x) - f'(x)v\|_Y}{\|v\|_X} = 0,$$

then $f'(x)$ is called the Fréchet derivative of f at x and f is said to be Fréchet differentiable at x.

Definition 1.2.3 (Convex functional). *Let $S \subset \mathbb{R}^d$ ($d = 2$ or 3) be a non-empty convex set. Then a functional $f : S \to \mathbb{R}$ is said to be convex if*

$$f(\nu x_1 + (1-\nu)x_2) \leq \nu f(x_1) + (1-\nu) f(x_2), \qquad \forall \nu \in [0,1] \text{ and } x_1, x_2 \in S.$$

The functional f is said to be strictly convex if the above condition holds with strict inequality whenever $x_1 \neq x_2$ and $\nu \in (0,1)$.

Let the bilinear form associated with the operator $\mathcal{A}$ on Ω and Ω_T be given by

$$a(v,w) = \sum_{i,j=1}^{d} \int_{\Omega} \Big(a_{i,j} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + a_0 v w \Big)\, dx, \qquad \forall v, w \in H_0^1(\Omega)$$

and

$$a(v,w)_{\Omega_T} = \sum_{i,j=1}^{d} \int_{\Omega_T} \Big(a_{i,j} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + a_0 v w \Big)\, dx d\tau, \qquad \forall v, w \in L^2(0,T; H_0^1(\Omega)),$$

respectively. The adjoint of the operator $\mathcal{A}$ is defined by

$$\mathcal{A}^* y = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \Big(a_{j,i}(x) \frac{\partial y}{\partial x_i} \Big) + a_0(x) y$$

and L^2-inner products on $L^2(\Omega)$ and $L^2(0,T; L^2(\Omega))$ be denoted by

$$\begin{aligned} (v,w) &= \int_{\Omega} v w \, dx, \qquad \forall v, w \in L^2(\Omega), \\ (v,w)_{\Omega_T} &= \int_{\Omega_T} v w \, dx d\tau, \qquad \forall v, w \in L^2(0,T; L^2(\Omega)), \end{aligned}$$

respectively. We assume that the bilinear $a(\cdot,\cdot)$ is coercive and continuous on $H_0^1(\Omega)$, i.e.,

$$a(v,v) \geq \beta \|v\|_{H^1(\Omega)}^2 \quad \text{and} \quad |a(v,w)| \leq \beta_1 \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}, \qquad \forall v, w \in H_0^1(\Omega)$$

with $\beta, \beta_1 \in \mathbb{R}^+$.

We now define some auxiliary elliptic problems which will be used in Chapters 2-3. For $f \in L^2(\Omega)$, let ϕ and ψ be the solutions of following problems

$$\begin{cases} \mathcal{A}\phi = f \ \text{ in } \Omega, \\ \phi = 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{1.10}$$

and

$$\begin{cases} \mathcal{A}^*\psi = f \text{ in } \Omega, \\ \psi = 0 \quad \text{ on } \partial\Omega, \end{cases} \tag{1.11}$$

respectively. Then, the following regularity results [42] hold true.

Lemma 1.2.2. *Let $\phi, \psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solutions of the Dirichlet problems (1.10) and (1.11), respectively. Then for $f \in L^2(\Omega)$, there exist positive constants C_R and $C_{\tilde{R}}$ such that*

$$\|\phi\|_{H^2(\Omega)} \leq C_R\|f\|_{L^2(\Omega)} \quad \text{and} \quad \|\psi\|_{H^2(\Omega)} \leq C_{\tilde{R}}\|f\|_{L^2(\Omega)}. \tag{1.12}$$

Now, we introduce some auxiliary parabolic problems which will be used in Chapters 4-6. For simplicity, we set

$$\begin{aligned} W(0,T) &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ X(0,T) &:= L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)). \end{aligned}$$

We note that $W(0,T) \hookrightarrow \mathcal{C}([0,T];L^2(\Omega))$ and $X(0,T) \hookrightarrow \mathcal{C}([0,T];H_0^1(\Omega))$ (see [61]). For $f \in L^2(0,T;L^2(\Omega))$, let ϕ and ψ, respectively, be the solutions of the following forward and backward in time standard parabolic problems:

$$\begin{cases} \phi_t + \mathcal{A}\phi = f \text{ in } \Omega_T, \\ \phi(\cdot,0) = 0 \quad \text{ in } \Omega, \\ \phi = 0 \quad\quad \text{ on } \Gamma_T \end{cases} \tag{1.13}$$

and

$$\begin{cases} -\psi_t + \mathcal{A}^*\psi = f \text{ in } \Omega \times [0,T), \\ \psi(\cdot,T) = 0 \quad\quad \text{ in } \Omega, \\ \psi = 0 \quad\quad\quad \text{ on } \Gamma_T. \end{cases} \tag{1.14}$$

Then the following regularity results hold true, see [61].

Lemma 1.2.3. *Let $\phi, \psi \in X(0,T)$ be the solutions of problems (1.13) and (1.14), respectively. Then, $X(0,T) \hookrightarrow \mathcal{C}([0,T]; H_0^1(\Omega))$ and satisfies*

$$\begin{aligned}\|\phi\|_{L^2(H^2(\Omega))} + \|\phi_t\|_{L^2(L^2(\Omega))} &\leq C_{R_1}\|f\|_{L^2(L^2(\Omega))},\\ \|\psi\|_{L^2(H^2(\Omega))} + \|\psi_t\|_{L^2(L^2(\Omega))} &\leq C_{R_2}\|f\|_{L^2(L^2(\Omega))},\end{aligned}$$

and

$$\|\phi(\cdot,T)\|_{H^1(\Omega)} \leq C_{R_1}\|f\|_{L^2(L^2(\Omega))}, \qquad \|\psi(\cdot,0)\|_{H^1(\Omega)} \leq C_{R_2}\|f\|_{L^2(L^2(\Omega))}$$

for some positive generic constants C_{R_1} and C_{R_2}.

1.3 Background and motivation

In this section, we shall discuss a brief survey of the relevant literature pertaining to the elliptic and parabolic optimal control problems and elucidates the motivation for the present study.

In optimal control problems, the general idea is to vary an input quantity (called control) in such a way that the output quantity (called state) minimizes the objective functional. The input can be a function prescribed on the boundary (called boundary control) or distributed all over the domain (called distributed control), and the output is the solution of the partial differential equation. Due to physical and technical limitations, one needs to impose some restrictions on the control or state. The theory of optimal control problems governed by partial differential equations was introduced by Lions in [60]. Among various numerical methods in the literature design for optimal control problems, the discretization procedure, namely finite element method is widely used in practice. We first detail a brief account of the literature concerning finite element method for optimal control problems.

Finite element method. The design of efficient numerical techniques to approximate the solution of an optimal control problem is of paramount importance in many practical applications. Over the last three decades, finite element method has become an important tool for approximating the solutions of control problems. There have been extensive theoretical and numerical studies for finite element approximations of optimal control problems. We refer to [47, 48, 66, 86] for an overview and an up-to-date discussion on numerical approximation to optimal control problems. The error analysis of finite element method is grouped into two categories: *A priori error analysis* and *a posteriori error analysis*. *A priori* error analysis yields bounds of the form

$$\|u - U\|_X \leq C(u)h^m, \tag{1.15}$$

where u and U are the exact and numerical solutions of some problem, respectively. Here, $C(u)$ is a positive constant depends on the exact solution u, h denotes the mesh parameter and $\|\cdot\|_X$ denotes a specified norm. The estimate (1.15) is not realistic in general as it depends on the exact solution which is unknown for most of the problems. Moreover, estimates of the form (1.15) can give asymptotic rates of convergence as the mesh parameter goes to zero, but are not designed to give an actual error estimate for a given mesh. The question of quantifying the error brings attention to a new error estimation method which is able to characterize explicitly the accuracy of approximate solutions and is known as *a posteriori* error estimation technique. An *a posteriori* error estimate is a computable quantity in terms of the finite element solution and data of the given problem, i.e., *a posteriori* error estimates employ the finite element solution and the data of the problem to derive estimates on the actual errors. On the contrary to *a priori* error analysis, *a posteriori* error analysis predicts bounds of the form

$$\|u - U\|_X \leq \eta(U, data),$$

where the estimator $\eta(U, data)$ is a computable quantity which depends on the numerical solution U and the data of the problem. The use of adaptive techniques based on *a posteriori* error estimation is well accepted in the context of finite element discretization of partial differential equations [29, 30] and [87]. Adaptive finite element method (AFEM) has been found to be able to save substantial computational work and ensures a higher density of nodes in certain area of the given domain where the solution is more difficult to approximate. The decision of whether further refinement of the meshes is necessary is based on the *a posteriori* error estimator. If further refinement is performed then the error estimator used as a guide to show how the refinement might be accomplish most efficiently. The application of these techniques to optimal control problems governed by partial differential equations is an active area of research. This thesis will focus on analyzing both *a priori* and *a posteriori* error analysis of finite element method for optimal control problems of the form (1.1)-(1.3) and (1.6)-(1.8). In order to put the results of this thesis in a proper prospective, we now give a brief account of the relevant literature concerning *a priori* and *a posteriori* error analysis of EOCPs and POCPs.

Finite element method for EOCPs. There are wide range of articles available in the literature regarding *a priori* error estimates for EOCPs. The two early papers on the numerical approximation for a class of linear EOCPs are due to [31] and [34]. In these papers, the authors have proved L^2-error estimates which reflect the H^1-regularity of the optimal control and optimal regularity of the state variable. Falk has considered distributed control in [31], while Geveci has concentrated on Neumann boundary control in [34]. An optimal control problem governed by elliptic equation with control constraints has been studied by Arnautu and Neittaanmäki in [8]. The authors have established the optimality condition and introduced the Ritz-Galerkin discretization for EOCP, and obtained error estimates for the control and state variables. Casas in [15] has studied convergence properties for a linear EOCPs with control in the coefficient of elliptic equation. Moreover, we refer to Arada and Raymond [6] for error estimates of relaxed optimal control problems governed by semilinear elliptic equations. In [5], Arada *et al.* have derived error estimates for the control in the L^∞ and L^2-norms for distributed control problems. For an analogous problem with finitely many state constraints, Casas in [17] has derived error estimates for the control in the L^∞-norm. Further, error estimates for Lagrange multipliers associated with the state constraints, state and co-state variables are also obtained in [17]. For a semilinear elliptic boundary control problem, Casas *et al.* in [20] have proved uniform convergence of discretized control to the optimal control under natural assumptions by taking piecewise constant controls. Subsequently, Casas and Raymond [21] have studied semilinear elliptic boundary control problem with pointwise constraints on the control. The authors have used continuous piecewise linear finite elements for the approximation of the state as well as control variables and the related error estimate for the control is derived. Moreover, for the approximation of EOCPs with control variable from measure spaces can be found in [18, 25, 76].

The *a posteriori* error analysis for optimal control problems governed by elliptic equations have been investigated by numerous researchers. AFEMs have been successfully applied to optimal control problems governed by partial differential equations. The pioneer work has been made by Liu and Yan [62] for residual based *a posteriori* error estimates and Becker *et al.* [10] for dual-weighted goal oriented adaptivity. We refer to [45, 46, 58, 63] and [65] for distributed and boundary control problems governed by elliptic equations. An essential ingredient of AFEM is *a posteriori* error estimators which provide information about the local quality of the approximate solution. AFEM aiming to distribute more mesh nodes around the area where singularity of the solution happens to save the computational cost has been first proposed by Babuška and Rheinboldt [9]. In [33], Gaevskaya *et al.* have studied the convergence of an AFEM for distributed optimal control problems with control constraints. Becker and Mao [11] have provided a convergence proof for the adaptive algorithm by viewing the control problems as a nonlinear elliptic system of the state and co-state variables. An adaptive algorithm presented in [11] involves marking of data oscillation. In [50], the authors have proved that the sequence of adaptively generated discrete solutions converged to the true solutions of optimal control problems. However, they have not shown convergence rate and optimality in [50]. Recently, Gong and Yan [40] have presented a rigorous proof for convergence and quasi-optimality of AFEM for an optimal control problem with pointwise control constraints by means of variational discretization technique. In [39], the authors have showed that the convergence of AFEM based on energy norm is suboptimal for the control variable and numerical experiments confirmed this sub-optimality. Demlow and Stevenson [27] have studied an AFEM for controlling L^2-errors for elliptic problems, where the convergence and quasi-optimality of the method are proved by keeping the meshes sufficiently mildly graded. Leng and Chen have extended the work of [27] to the optimal control problems in [56]. Moreover, for L^2-norm based AFEM, the authors of [41] have proved the contraction property and quasi-optimal complexity for the L^2-norm errors of the state, co-state and control variables. Their results improve the known result of [40] for energy norm based AFEM. The recovery type *a posteriori* error estimates for EOCPs are described in [59]. Apart from the residual and recovery type estimates new results on a class of functional type *a posteriori* error estimates are described in [52] and [89]. Very recently, Wolfmayr [89] has derived functional type *a posteriori* error estimates for EOCPs with control constraints.

Despite being vast literature on numerical approximations to EOCPs, the *a priori* and *a posteriori* error analysis of finite element methods for EOCPs with measure data remains unexplored. Our first attempt in this thesis is to study *a priori* error analysis for EOCP with measure data. The main source of difficulty is that the solution of the state equation has low regularity which makes the finite element error analysis somewhat difficult. In the case of standard elliptic problems with measure data, Casas [14] has derived error estimates of order $O(h^{2-\frac{d}{2}})$ in the L^2-norm, where h is the mesh parameter and d is the dimension of the domain Ω. Further literature on *a priori* error bounds for elliptic problem with measure data are contained in [79]. This thesis generalizes the *a priori* error analysis of finite element method for the standard elliptic problem with measure data to EOCP with measure data. To solve optimal control problem with measure data we use *optimize-then-discretize* approach. We prove the existence, uniqueness and regularity of the solution of EOCP with measure data. We use piecewise linear and continuous finite elements for the approximations of the state and co-state variables whereas piecewise constant functions are used for the approximation of the control variable. We derive convergence properties for the state, co-state and control variables in the L^2-norm with an order of convergence $\mathcal{O}(h^{2-\frac{d}{2}})$. Our theoretical results are supported by numerical experiments.

The next problem in this thesis is to study *a posteriori* error analysis of EOCP

(1.1)-(1.3) with measure data. The previous work on *a posteriori* error analysis with Dirac source term for the standard elliptic problems have been discussed in [7]. They have obtained global upper and local lower bounds in L^p norm and $W^{1,p}$ seminorm for $p < 2$. This thesis extends the work of [7] from elliptic problems to EOCP with measure data. In the context of EOCP with measure data (1.1)-(1.3), we have derived global upper bounds for the state, co-state and control variables in the L^2-norm. In case of two space dimension ($d = 2$), we derive local lower bounds for errors in the state and co-state variables, and global lower bound for error in the control variable. The key technical tools used in our analysis are the interpolation approximation properties, inverse estimates, element and edge bubble functions and their properties. Numerical assessment of the estimators are presented.

Finite element method for POCPs. Optimal control problems with time-dependent control play an important role in many applications, and the numerical treatment of these problems has been an active research topic in the recent years. The pioneering work of late 1970s and 1980s in the area of finite dimensional approximation for infinite dimensional POCPs can be found in [4, 54, 55] and [69], where the Ritz-Galerkin approximations and semigroup theory have been utilized. For an overview concerning *a priori* error analysis for finite element approximations for POCPs, we refer to [49, 74, 78, 88] and *a priori* error analysis for problems with state constraints can be found in [26, 36] and [70]. In [71, 72], the authors have developed *a priori* error analysis for space-time finite element discretization for POCPs, where they have applied discontinuous Galerkin schemes for temporal discretization. Recently, the authors of [73] have used Petrov-Galerkin Crank-Nicolson scheme for POCPs. However, all these papers have focused on distributed control problems. In many applications, control can only act locally at finitely many points of the domain, which is called pointwise control. Chryssoverghi [23] has studied convergence properties for the state and control variables of an optimal pointwise control problem for systems governed by a parabolic equation. Later, Droniou and Raymond have analyzed the optimal pointwise control of semilinear parabolic equations in [28]. We refer to [37] and [57] for *a priori* error estimates for finite element approximations of POCPs with pointwise control. Recently, *a priori* error estimates to sparse POCPs which utilize a formulation with control variable in measure spaces can be found in [19].

The research work on *a posteriori* error analysis for POCPs have been extensively analyzed in [53, 64, 82, 83, 84] and [90]. For residual type *a posteriori* error estimates of finite element methods for POCPs, we refer to [64] and [90]. Further, a recovery type *a posteriori* error estimate of fully discrete finite element approximation for POCP has been studied by Tang and Chen in [83]. Subsequently, Sun *et al.* [82] have derived both lower and upper bounds for the errors for POCPs. In [84], Tang and Hua have established upper bounds in the $L^\infty(0,T;L^2(\Omega))$-norm for the spatially discrete finite element approximations of POCP using elliptic reconstruction. Recently, functional type *a posteriori* error estimates for POCPs have been discussed in [53]. As opposed to the well established results on both *a priori* and *a posteriori* error estimates for POCPs, the development and analysis of both *a priori* and *a posteriori* error estimates for POCPs with measure data remains unexplored.

Our next attention in this thesis is to study *a priori* error analysis for POCP (1.6)-(1.8) with measure data. The mathematical difficulty of such type of problem is low regularity of the solution of the state equation which introduces some difficulties in both theoretical and numerical analysis. In this work, we mainly discuss two types of problem, namely measure data in space and measure data in time. For measure data in space, we take $\mu = g\omega$, where $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$ whereas for measure data in time $\mu = g\omega$ with $g \in \mathcal{C}([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. We employ *optimize-then-discretize* strategy to approximate the control problems. For the purpose of finite element formulation we rewrite the state equation in weak form using transposition techniques [61] and investigate the existence, uniqueness and regularity of the solutions for both type of problems. We have used continuous piecewise linear functions for the approximations of the state and co-state variables, and piecewise constant functions for the approximation of the control variable. We have analyzed both spatially discrete and fully discrete finite element approximations of POCPs with measure data in space and time. For measure data in space, *a priori* error estimates of order $\mathcal{O}(h^{2-\frac{d}{2}})$ are derived for the spatially discrete control problem whereas error estimates of order $\mathcal{O}(h^{2-\frac{d}{2}}+k^{1/2})$ are established for the fully discrete backward Euler time discretization scheme. Next, for POCPs with measure data in time, we have obtained error estimates of order $\mathcal{O}(h)$ for the state, costate and control variables for the spatially discrete problem. A time discretization scheme based on the backward-Euler method is analyzed and error estimates of order $\mathcal{O}(h+k^{1/2})$ are proved for the state, co-state and control variables. The key technical tools used in our analysis include nonstandard weak formulation of the state equation, interpolation approximation properties, embedding theorems, inverse estimates, optimality conditions and duality technique. The previous work on the nonlinear parabolic equations involving measure data, we refer to [12, 13]. Later, the author of [16] has studied semilinear parabolic problems with measure data. Very recently, Gong [35] has obtained *a priori* error estimates for a linear parabolic equation involving measure data and for *a posteriori* error analysis, we refer to [38].

Finally, we now turn our attention to study *a posteriori* error estimates for space-time finite element discretization of POCP (1.6)-(1.8) with measure data. Two kinds of control problems, namely measure data in space and measure data in time, are considered and analyzed. We use continuous piecewise linear functions for the approximations of the state and co-state variables, and piecewise constant functions for the approximation of the control variable whereas the time discretization is based on the backward-Euler implicit scheme. We use duality argument, first order optimality condition and interpolation approximation properties to derive global upper bounds for errors in the state, co-state and control variables in the $L^2(0,T;L^2(\Omega))$-norm. Numerical experiments are performed to study the performance of the derived error estimators.

1.4 Organization of the thesis

This thesis consists of eight chapters, and is organized as follows. Chapter 1 introduces the problems and it contains some basic notations and preliminary materials to be used throughout this thesis. A brief survey of the relevant literature and motivation for the present work are presented.

Chapter 2 considers optimal control problem governed by elliptic equation (1.1)-(1.3) with measure data. The existence, uniqueness and regularity results for the state and control variables are proved and *a priori* error estimates for the state, co-state and control variables are derived.

Chapter 3 is devoted to the *a posteriori* error analysis for EOCP (1.1)-(1.3) with measure data. We have derived the global upper bounds for the errors in the L^2-norm. Further, in the case of two space dimension ($d = 2$), local lower bounds are derived for errors in the state and co-state variables, and global lower bound for error in the control variable is established.

In Chapter 4, we study finite element approximations of POCP with measure data in space in a bounded convex domain. We prove the existence, uniqueness and regularity of the solution for the control problem. *A priori* error estimates for the state, co-state and control variables are derived for both spatially discrete and fully discrete approximations of the optimal control problem. Moreover, $L^2(0,T;L^2(\Omega))$ convergence properties for the state, co-state and control variables are established.

In Chapter 5, we consider optimal control problem governed by parabolic equations with measure data in time. We prove the existence, uniqueness and regularity of the solutions for this problem. The *a priori* error analysis for the finite element method is carried out and error estimates for the state, co-state and control variables for both spatially discrete and fully discrete schemes are derived.

Chapter 6 is concerned with residual type *a posteriori* error estimates of fully discrete finite element approximation for parabolic optimal control problem with measure data in a bounded convex domain. Two kinds of control problems, namely measure data in space and measure data in time, are considered and analyzed. We derive global upper bounds for the state, co-state and control variables in the $L^2(0,T;L^2(\Omega))$-norm.

Chapter 7 is devoted to the numerical assessments of both *a priori* and *a posteriori* error estimates for EOCPs and POCPs with measure data.

Finally, Chapter 8 discusses the critical evaluation of the results highlighting the contributions made by the thesis and scope of future investigations.

For clarity of presentation we have repeatedly given equations (1.1)-(1.3) or (1.6)-(1.8) at the beginning of the subsequent chapters.

2

EOCP with Measure Data: A Priori Error Analysis

In this chapter, we study the *a priori* error analysis of the control problem (1.1)-(1.3) with measure data in a bounded convex domain in $\mathbb{R}^d$ ($d = 2$ or 3). The state equation exhibits low regularity due to the presence of measure data which introduces some difficulties for both theoretical and numerical analysis. We prove the existence, uniqueness and regularity of the solution of the control problem and derive *a priori* error estimates for the state, co-state and control variables. To discretize the control problem we use piecewise linear and continuous finite elements for the approximations of the state and co-state variables whereas the control variable is approximated by piecewise constant functions.

2.1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded convex domain with boundary $\partial\Omega$. We now recall the following EOCP:

$$\min_{q \in Q_{ad}^E} J(q, y) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)} \tag{2.1}$$

subject to the state equation

$$\begin{cases} \mathcal{A}\, y = \mu + q & \text{in } \Omega, \\ \quad y = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

and the control constraints

$$q_a \le q(x) \le q_b \quad \text{a.e. in } \Omega, \tag{2.3}$$

where $y = y(x)$ denotes the state variable and $q = q(x)$ is the control variable. The operator $\mathcal{A}$ is defined by (1.4) and $\alpha > 0$ is a fixed parameter. Moreover, $y_d(x) \in L^2(\Omega)$

is a given desired state, the function $\mu = g\,\omega$, where $g \in \mathcal{C}(\overline{\Omega})$ and $\omega \in \mathcal{M}(\Omega)$. The set of admissible controls is defined by

$$Q_{ad}^{E} := \{q \in L^2(\Omega) :\ q_a \leq q(x) \leq q_b \ \ a.e. \ \text{in}\ \Omega\} \tag{2.4}$$

with $q_a,\ q_b \in \mathbb{R}$ fulfill $q_a < q_b$.

The optimal control problems governed by elliptic equation with measure data play an important role in many applications such as modeling of the potential of an electric field with an electric charge distribution [60, 86]. An important feature of such problem is that the solution of the state equation exhibits low regularity which introduces some difficulties in the error analysis of finite element method. The solution of the state equation of the control problem does not even belongs to $H^1(\Omega)$. However, it belongs to the Sobolev space $W^{1,p}(\Omega)\,[1 \leq p < \frac{d}{d-1})$ of real valued functions together with all their partial distributional derivatives of first order belongs to $L^p(\Omega)$ (cf. [79]). The previous work on elliptic problem with measure data are contained in [14] and [79]. In [14], the author has derived convergence of order $\mathcal{O}(h^{2-\frac{d}{2}})$ for the finite element method of standard elliptic problem, where h is the mesh size of the space triangulation and d is the dimension of the domain Ω. This chapter extends the *a priori* error analysis of finite element method for standard elliptic problem with measure data to EOCP with measure data. The key technical tools used in our error analysis include duality technique, inverse estimate, optimality conditions and the properties of the interpolation operator.

The outline of this chapter is as follows. We prove the existence, uniqueness and regularity results for the state and control variables in Section 2.2. The finite element approximation of optimal control problem is discussed in Section 2.3. Finally, in Section 2.4, we derive *a priori* error estimates for the state, co-state and control variables.

Throughout this chapter, C denotes a positive generic constant independent of the mesh parameter h.

2.2 Existence, uniqueness and regularity results

In this section, we prove the existence, uniqueness and regularity results for the solution of optimal control problem (2.1)-(2.3). Let $\mathcal{C}(\overline{\Omega})$ be the space of real and continuous functions on $\overline{\Omega}$, endowed with the supremum norm $\|\cdot\|_{L^\infty(\Omega)}$, and let $\mathcal{C}_0(\Omega)$ be the subspace of $\mathcal{C}(\overline{\Omega})$ consisting of elements vanishing on the boundary. Note that the space $H^2(\Omega) \cap H_0^1(\Omega)$ is continuously embedded in $\mathcal{C}_0(\Omega)$, i.e., $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow \mathcal{C}_0(\overline{\Omega})$ (cf. [1]).

We now discuss the existence of solution of state equation (2.2). Let $y \in L^2(\Omega)$ be a solution of problem (2.2) in the sense that

$$\int_\Omega y\mathcal{A}^* v\,dx = \prec g\omega, v \succ + \int_\Omega qv\,dx, \qquad \forall v \in H^2(\Omega)\cap H_0^1(\Omega), \tag{2.5}$$

where $\prec g\omega, v \succ := \int_\Omega gv\,d\omega$. Note that the expression $\prec g\omega, v\succ$ is well defined because $v \in H^2(\Omega)\cap H_0^1(\Omega) \hookrightarrow \mathcal{C}_0(\overline{\Omega})$.

Theorem 2.2.1. *Let $q \in L^2(\Omega)$, $g \in \mathcal{C}(\overline{\Omega})$ and $\omega \in \mathcal{M}(\Omega)$. Then, the problem (2.2) admits a unique solution $y \in L^2(\Omega)$ in the sense that*

$$\int_\Omega y\mathcal{A}^* v\,dx = \prec g\omega, v \succ + \int_\Omega qv\,dx, \quad \forall v \in H^2(\Omega)\cap H_0^1(\Omega). \tag{2.6}$$

Further, there exists a positive constant C such that

$$\|y\|_{L^2(\Omega)} \le C\Big(\|g\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega)} + \|q\|_{L^2(\Omega)}\Big). \tag{2.7}$$

Moreover, $y \in W_0^{1,p}(\Omega)$ for $p \in [1, \frac{d}{d-1})$ and

$$\|y\|_{W_0^{1,p}(\Omega)} \le C\Big(\|g\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega)} + \|q\|_{L^2(\Omega)}\Big). \tag{2.8}$$

Proof. We borrow the proof technique of [14]. Let $T : L^2(\Omega) \to \mathcal{C}_0(\Omega)$ be a linear map such that $Tf = \phi_f$. Then, in view of (1.12) of Lemma 1.2.2, T is continuous.

Let $T^* : \mathcal{M}(\Omega) \to L^2(\Omega)$ be the adjoint operator of T. Take $y = y_1 + y_2$, where $y_1 = T^*\mu$ and $y_2 \in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of

$$\begin{cases} \mathcal{A}y_2 = q & \text{in } \Omega, \\ y_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, for $v \in H^2(\Omega)\cap H_0^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega y\mathcal{A}^* v\,dx &= \int_\Omega y_1\mathcal{A}^* v\,dx + \int_\Omega y_2\mathcal{A}^* v\,dx \\ &= \int_\Omega T^*\mu\mathcal{A}^* v\,dx + \int_\Omega \mathcal{A}y_2 v\,dx \\ &= \int_\Omega T(\mathcal{A}^* v)\,d\mu + \int_\Omega qv\,dx \\ &= \int_\Omega v\,d\mu + \int_\Omega qv\,dx. \end{aligned}$$

Since $\mu = g\omega$, we have

$$\int_\Omega y\mathcal{A}^* v\,dx = \prec g\omega, v \succ + \int_\Omega qv\,dx, \tag{2.9}$$

and hence (2.6) holds. The uniqueness follows as the problem is linear. To prove (2.7), we use the duality argument. For $f \in L^2(\Omega)$, let ψ be the solution of (1.11). Then, using (2.6) and Lemma 1.2.2 we obtain

$$\begin{aligned}\int_\Omega yf\,dx &= \int_\Omega y\,\mathcal{A}^*\psi\,dx = \prec g\omega, \psi \succ + \int_\Omega q\psi\,dx \\ &\leq C\Big(\|g\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega)}\|\psi\|_{L^\infty(\Omega)} + \|q\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}\Big) \\ &\leq C\Big(\|g\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega)} + \|q\|_{L^2(\Omega)}\Big)\|f\|_{L^2(\Omega)}. \end{aligned} \tag{2.10}$$

The definition of L^2-norm together with (2.10) gives the desired estimate. Next to prove (2.8), we take $\tilde{f} \in \mathcal{C}_0(\Omega)$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ to be the solution of problem:

$$\begin{cases} \mathcal{A}^* v = \tilde{f} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

From (2.9), we obtain

$$\begin{aligned}\Big|\int_\Omega y\tilde{f}\,dx\Big| &= \Big|\int_\Omega y\mathcal{A}^* v\,dx\Big| = \Big| \prec g\omega, v \succ + \int_\Omega qv\,dx\Big|, \\ &\leq \|g\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega)}\|v\|_{L^\infty(\Omega)} + \|q\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}. \end{aligned} \tag{2.11}$$

It follows from [75, Theorem 1.4] that there exists $c > 0$ such that

$$\|v\|_{L^\infty(\Omega)} \leq c\|\tilde{f}\|_{W^{-1,p'}(\Omega)}, \tag{2.12}$$

where $p' > d$ is arbitrary and $W^{-1,p'}(\Omega)$ denotes the dual of $W_0^{1,p}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. From (2.11) and (2.12), we obtain

$$\Big|\int_\Omega y\tilde{f}\,dx\Big| \leq c\|g\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega)}\|\tilde{f}\|_{W^{-1,p'}(\Omega)} + \|q\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.$$

Since $\mathcal{C}(\overline{\Omega})$ is dense in $W^{-1,p'}(\Omega)$ and $L^2(\Omega)$, it follows that $y \in W_0^{1,p}(\Omega)$ and satisfies (2.8). This completes the proof. □

Now, we write the optimal control problem (2.1)-(2.3) in the weak form: Find $(q, y) \in Q_{ad}^E \times L^2(\Omega)$ such that

$$\min_{q \in Q_{ad}^E} J(q, y) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)} \tag{2.13}$$

subject to

$$\int_\Omega y\mathcal{A}^* v\,dx = \prec g\omega, v \succ + (q, v), \qquad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.14}$$

In the next step, we introduce the reduced cost functional $j : L^2(\Omega) \to \mathbb{R}$ by

$$j(q) := J(q, y(q)),$$

where J is the cost function given by (2.13) and $y(q)$ is the weak solution of (2.2) as defined in (2.14) . The optimal control problem (2.13)-(2.14) can be equivalently reformulated as

$$\min_{q \in Q_{ad}^E} j(q), \tag{2.15}$$

where the set of admissible controls is given by (2.4).

The following theorem ensures the existence and uniqueness of solution of the minimization problem (2.15).

Theorem 2.2.2. *The optimal control problem (2.15) admits a unique solution.*

Proof. Let q_n be the minimizing sequences, i.e., $j(q_n) \to \inf_{q \in Q_{ad}^E} j(q)$. By virtue of the term $\alpha\|q\|_{L^2(\Omega)}$ in $j(q)$, q_n is bounded in Q_{ad}^E and let $y(q_n) = y_n$. From the *a priori* bound (2.7), we have

$$\|y_n\|_{L^2(\Omega)} \leq C,$$

where $C = C(\|g\|_{L^\infty(\Omega)}, \|\omega\|_{\mathcal{M}(\Omega)}, \|q\|_{L^2(\Omega)})$.

Hence, y_n and q_n has a subsequence denoted by y_{n_k} and q_{n_k} such that $y_{n_k} \to y$ in $L^2(\Omega)$ weakly. For $v \in \mathcal{D}(\Omega)$, we have

$$(q_{n_k}, v) = - < g\omega, v > + \int_\Omega (y_{n_k} - y)\mathcal{A}^* v\, dx + \int_\Omega y\mathcal{A}^* v\, dx,$$

as $n \to \infty$, $y_{n_k} \to y$ strongly and $(q_{n_k}, v) \to (q, v)$. Hence $q_{n_k} \to q$ in $\mathcal{D}'(\Omega)$ and y satisfies (2.14). Then, we have $y = y(q)$ and

$$j(q) \leq \liminf j(q_n) = \inf_{q \in Q_{ad}^E} j(q),$$

and therefore, q is the optimal control. □

In the following, we state the first order optimality condition for the optimal control problem (2.13)-(2.14) (cf. [60]).

Theorem 2.2.3. *Assume that $(q, y) \in Q_{ad}^E \times L^2(\Omega)$ is the solution of the problem (2.13)-(2.14). Then there exists a unique co-state $z \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying*

$$\begin{cases} \mathcal{A}^* z = y - y_d & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

Furthermore,

$$j'(q)(\hat{q} - q) = (\alpha q + z, \hat{q} - q) \geq 0, \quad \hat{q} \in Q_{ad}^E. \tag{2.17}$$

The optimality condition (2.17) can equivalently be written in the form [60, 86]:

$$q = P_{[q_a,q_b]}\Big(-\frac{1}{\alpha}z\Big), \tag{2.18}$$

where $P_{[q_a,q_b]}$ denotes the projection of $\mathbb{R}$ on to $[q_a, q_b]$ defined as

$$P_{[q_a,q_b]}(q) := \max(q_a, \min(q_b, q)).$$

The second derivative of $j''(q)(\cdot,\cdot)$ is independent of q and positive definite, i.e.,

$$j''(q)(r,r) \geq \gamma_E \|r\|^2_{L^2(\Omega)}, \quad \forall r \in L^2(\Omega), \tag{2.19}$$

where $\gamma_E > 0$ is a positive constant.

Using (2.18), we deduce the regularity results summarized below.

Theorem 2.2.4. *Let $(q, y) \in Q_{ad} \times L^2(\Omega)$ be the unique solution of (2.13)-(2.14), and z be the unique solution of (2.16). Then, we have*

$$y \in W_0^{1,p}(\Omega) \cap L^2(\Omega), \quad p \in [1, \frac{d}{d-1}),$$
$$z \in H^2(\Omega) \cap H_0^1(\Omega) \ \ \textit{and} \ \ q \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega).$$

Proof. It follows from Theorem 2.2.1 that $y \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$. By the standard regularity results of elliptic equation $z \in H^2(\Omega) \cap H_0^1(\Omega)$. Now the embedding theorem gives $z \in \mathcal{C}_0(\overline{\Omega})$, thus the control constraints (2.3) and the property (2.18) imply the stated regularity of the optimal control q. □

2.3 Discrete optimal control problem

This section considers finite element approximations of the control problem (2.13)-(2.14) to obtain discrete optimal control problem.

Let $\mathcal{T}_h$ be a family of triangulation of $\overline{\Omega}$ such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h}\overline{K}$. We assume that $\overline{\Omega}$ is the union of the elements of $\mathcal{T}_h$ so that element edges(faces) lying on the boundary may be curved. For each element $K \in \mathcal{T}_h$, we associate two parameters h_K and σ_K, where h_K denotes the diameter of the element K and σ_K is the supremum of the diameters of all circles contained in K. Define the mesh size $h = \max_K h_K$. Further, we assume that there exists positive constants C_1 and C_2 such that

$$\frac{h_K}{\sigma_K} \leq C_1, \qquad \frac{h}{h_K} \leq C_2$$

holds $\forall K \in \mathcal{T}_h$ and $\forall h_K > 0$ (cf. [24]). Associated with $\mathcal{T}_h$, let V_h be a finite dimensional subspace of $\mathcal{C}(\overline{\Omega})$, consisting of piecewise linear polynomials. We denote $V_h^0 = V_h \cap H_0^1(\Omega)$.

Since $\mathcal{T}_h$ is quasi-uniform, the following inverse estimates hold for all $v_h \in V_h$ [24]:

$$\|v_h\|_{H^{s_2}(\Omega)} \leq Ch^{s_1-s_2}\|v_h\|_{H^{s_1}(\Omega)}, \quad 0 \leq s_1 \leq s_2 \leq 1, \tag{2.20}$$

$$\|v_h\|_{L^\infty(\Omega)} \leq Ch^{-d/2}\|v_h\|_{L^2(\Omega)}. \tag{2.21}$$

We need the following approximation properties [24, 77].

Lemma 2.3.1. *Let $\pi_h : \mathcal{C}(\overline{\Omega}) \to V_h$ denote the standard Lagrange interpolation operator defined by*

$$\pi_h v := \sum_i v(a_i)\varphi_i,$$

where a_i are the nodes on $\overline{\Omega}$ and φ_i are the corresponding shape functions. Then, for $v \in H^2(\Omega)$, we have

$$\|v - \pi_h v\|_{L^2(\Omega)} + h\|v - \pi_h v\|_{H^1(\Omega)} \leq Ch^2\|v\|_{H^2(\Omega)}, \tag{2.22}$$

and

$$\|v - \pi_h v\|_{L^\infty(\Omega)} \leq Ch^{2-\frac{d}{2}}\|v\|_{H^2(\Omega)}. \tag{2.23}$$

Lemma 2.3.2. *Let $\mathcal{L}_h : L^2(\Omega) \to V_h$ be the L^2-projection operator defined by*

$$(\mathcal{L}_h v - v, v_h) = 0, \qquad \forall v_h \in V_h. \tag{2.24}$$

Then, we have

$$\|v - \mathcal{L}_h v\|_{H^{-1}(\Omega)} + h\|v - \mathcal{L}_h v\|_{L^2(\Omega)} \leq Ch^2\|v\|_{H^1(\Omega)}.$$

Let V_h^0 be the finite dimensional space for the state and co-state variables and we set

$$Q_h^E = \{\hat{q}_h \in Q_{ad}^E : \ \forall K \in \mathcal{T}_h, \ \hat{q}_h|_K = constant\}.$$

The finite element approximation of optimal control problem (2.13)-(2.14) is defined as follows: Find $(q_h, y_h) \in Q_h^E \times V_h^0$ such that

$$\min_{q_h \in Q_h^E} J(q_h, y_h) = \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q_h\|^2_{L^2(\Omega)} \tag{2.25}$$

subject to

$$a(y_h, v_h) = \prec g\omega, v_h \succ + (q_h, v_h), \qquad \forall v_h \in V_h^0, \tag{2.26}$$

where $\prec g\omega, v_h \succ := \int_\Omega g v_h \, d\omega$.

The following stability result holds true.

Lemma 2.3.3. *Let g and ω be functions such that $g \in \mathcal{C}(\overline{\Omega})$, $\omega \in \mathcal{M}(\Omega)$ with $\mu = g\omega$ and let $q \in L^2(\Omega)$. If $y_h(q) \in V_h^0$ is a unique solution of*

$$a(y_h(q), v_h) = \prec g\omega, v_h \succ + (q, v_h), \qquad \forall v_h \in V_h^0.$$

Then

$$\|y_h(q)\|_{L^2(\Omega)} \leq C\Big(\|g\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega)} + \|q\|_{L^2(\Omega)}\Big).$$

Analogous to the continuous case, we define the discrete reduced cost functional $j_h : L^2(\Omega) \to \mathbb{R}$ as

$$j_h(q) := J(q, y_h(q)).$$

The discrete optimal control problem (2.25)-(2.26) is read as

$$\min_{q_h \in Q_h^E} j_h(q_h). \tag{2.27}$$

With the standard arguments we prove the existence of a unique solution $q_h \in Q_h^E \subset Q_{ad}^E$ of (2.27). The first order optimality condition is written as

$$j_h'(q_h)(\hat{q}_h - q_h) \geq 0, \qquad \forall \hat{q}_h \in Q_h^E. \tag{2.28}$$

Similar to the continuous case, the directional derivative $j_h'(q_h)(\hat{q}_h - q_h)$ for given $q_h,\ \hat{q}_h \in Q_h^E$ can be expressed as

$$j_h'(q_h)(\hat{q}_h - q_h) = (\alpha q_h + z_h, \hat{q}_h - q_h), \tag{2.29}$$

where z_h be the solution of the discrete co-state equation

$$a(z_h, v_h) = (y_h - y_d, v_h), \qquad \forall v_h \in V_h^0. \tag{2.30}$$

The variational inequality (2.28) is equivalent to the following pointwise projection formula [60, 86]:

$$q_h = P_{[q_a,q_b]}\Big(-\frac{1}{\alpha}z_h\Big).$$

2.4 *A priori* error estimates

In this section, we derive *a priori* error estimates for the state, co-state and control variables of optimal control problem (2.1)-(2.3). For the subsequent error analysis it is convenient to introduce the following auxiliary problems: For $q \in Q_{ad}^E$, find $(y_h(q), z_h(q)) \in V_h^0 \times V_h^0$ such that

$$a(y_h(q), v_h) = \prec g\omega, v_h \succ + (q, v_h), \qquad \forall v_h \in V_h^0, \tag{2.31}$$
$$a(z_h(q), v_h) = (y_h(q) - y_d, v_h), \qquad \forall v_h \in V_h^0. \tag{2.32}$$

We now recall the following lemma associated with error estimates for the adjoint problem (1.11).

Lemma 2.4.1. *Assume that $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$. Let $\psi_h \in V_h^0$ be the solution of*

$$a(\psi_h, v_h) = a(\psi, v_h), \qquad \forall v_h \in V_h. \tag{2.33}$$

Then, we have

$$\|\psi - \psi_h\|_{L^2(\Omega)} \leq Ch^2 \|\psi\|_{H^2(\Omega)}, \tag{2.34}$$

$$\|\psi - \psi_h\|_{L^\infty(\Omega)} \leq Ch^{2-\frac{d}{2}} \|\psi\|_{H^2(\Omega)}. \tag{2.35}$$

Proof. The proof follows the idea of [24]. To prove (2.35), we split the error as follows

$$\|\psi - \psi_h\|_{L^\infty(\Omega)} \leq \|\psi - \pi_h\psi\|_{L^\infty(\Omega)} + \|\pi_h\psi - \psi_h\|_{L^\infty(\Omega)},$$

where π_h is the Lagrange interpolation operator defined in Lemma 2.3.1. Use of inverse estimate (2.21) together with Lemma 2.3.1 gives

$$\begin{aligned}
\|\psi - \psi_h\|_{L^\infty(\Omega)} &\leq Ch^{2-\frac{d}{2}} \|\psi\|_{H^2(\Omega)} + h^{-\frac{d}{2}} \|\pi_h\psi - \psi_h\|_{L^2(\Omega)} \\
&\leq Ch^{2-\frac{d}{2}} \|\psi\|_{H^2(\Omega)} + h^{-\frac{d}{2}} \|\psi - \psi_h\|_{L^2(\Omega)} \\
&\leq Ch^{2-\frac{d}{2}} \|\psi\|_{H^2(\Omega)}.
\end{aligned}$$

This completes the proof. □

In the following lemma we prove an intermediate error estimate for the state variable.

Lemma 2.4.2. *Assume that $q \in L^2(\Omega)$ and $\mu = g\omega$ with $g \in \mathcal{C}(\overline{\Omega})$, $\omega \in \mathcal{M}(\Omega)$. Let y and $y_h(q)$ be the solutions of (2.14) and (2.31), respectively. Then, there exists a constant $C > 0$ independent of h such that the following error estimate holds:*

$$\|y - y_h(q)\|_{L^2(\Omega)} \leq C\Big(h^{2-\frac{d}{2}} \|g\|_{L^\infty(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)} + h^2 \|q\|_{L^2(\Omega)}\Big).$$

Proof. For $f \in L^2(\Omega)$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (1.11). Then, use of Green's formula and (2.14) yields

$$\begin{aligned}
\int_\Omega (y - y_h(q)) f \, dx &= \int_\Omega (y - y_h(q)) \mathcal{A}^* \psi \, dx \\
&= \int_\Omega y \mathcal{A}^* \psi \, dx - \int_\Omega y_h(q) \mathcal{A}^* \psi \, dx \\
&= \prec g\omega, \psi \succ + (q, \psi) - a(y_h(q), \psi).
\end{aligned}$$

With an add of (2.33) and (2.31) we obtain

$$\begin{aligned}\int_\Omega (y - y_h(q)) f\,dx &= \prec g\omega, \psi \succ + (q, \psi) - a(y_h(q), \psi_h) \\ &= \prec g\omega, \psi - \psi_h \succ + (q, \psi - \psi_h).\end{aligned}$$

An application of the Cauchy-Schwarz inequality gives

$$\Big| \int_\Omega (y - y_h(q)) f\,dx \Big| \le \|g\|_{L^\infty(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)} \|\psi - \psi_h\|_{L^\infty(\Omega)} + \|q\|_{L^2(\Omega)} \|\psi - \psi_h\|_{L^2(\Omega)}.$$

Using the estimates of Lemmas 2.4.1 and 1.2.2, we obtain

$$\Big| \int_\Omega (y - y_h(q)) f\,dx \Big| \le C\Big(h^{2-\frac{d}{2}} \|g\|_{L^\infty(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)} + h^2 \|q\|_{L^2(\Omega)} \Big) \|f\|_{L^2(\Omega)}.$$

Now, from the definition of L^2-norm, we have

$$\begin{aligned}\|y - y_h(q)\|_{L^2(\Omega)} &= \sup_{f \in L^2(\Omega), f \neq 0} \frac{(f, y - y_h(q))}{\|f\|_{L^2(\Omega)}} \\ &\le C\Big(h^{2-\frac{d}{2}} \|g\|_{L^\infty(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)} + h^2 \|q\|_{L^2(\Omega)} \Big),\end{aligned}$$

and this completes the proof. □

Now, we proceed to estimate $\|z - z_h(q)\|_{L^2(\Omega)}$.

Lemma 2.4.3. *Let z and $z_h(q)$ be the solutions of (2.16) and (2.32), respectively. Then there exists a constant $C > 0$ such that*

$$\|z - z_h(q)\|_{L^2(\Omega)} \le Ch^2 \Big(\|y\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} \Big) + \|y - y_h(q)\|_{L^2(\Omega)}.$$

Proof. We shall use the standard duality argument to prove this. Let $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (1.10) with $f \in L^2(\Omega)$. We multiply (2.16) by ϕ and form L^2-inner product over Ω. Then, use of Green's formula, (2.32) and (2.33) yields

$$\begin{aligned}\Big| \int_\Omega (z - z_h(q)) f\,dx \Big| &= \Big| \int_\Omega (z - z_h(q)) \mathcal{A}\phi\,dx \Big| \\ &= \Big| a(z, \phi) - a(z_h(q), \phi) \Big| \\ &= \Big| a(z, \phi) - a(z_h(q), \phi_h) \Big| \\ &= \Big| (y - y_d, \phi) - (y_h(q) - y_d, \phi_h) \Big|.\end{aligned}$$

Add and subtract the term $(y - y_d, \phi_h)$ and an application of the Cauchy-Schwarz inequality together with Lemma 2.4.1 gives

$$\begin{aligned}\Big| \int_\Omega (z - z_h(q)) f\,dx \Big| &= \Big| (y - y_d, \phi - \phi_h) + (y - y_h(q), \phi_h) \Big| \\ &\le \|y - y_d\|_{L^2(\Omega)} \|\phi - \phi_h\|_{L^2(\Omega)} + \|y - y_h(q)\|_{L^2(\Omega)} \|\phi_h\|_{L^2(\Omega)} \\ &\le \Big(Ch^2 \|y - y_d\|_{L^2(\Omega)} + \|y - y_h(q)\|_{L^2(\Omega)} \Big) \|f\|_{L^2(\Omega)}.\end{aligned}$$

In the above, we have used the stability estimate $\|\phi_h\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ and Lemma 1.2.2. The definition of L^2-norm gives the desired estimate. This completes the proof of the lemma. □

The following lemma yields the error between the derivative of continuous and discrete reduced cost functionals.

Lemma 2.4.4. *Let $j'(q)(r)$ and $j_h'(q)(r)$ be given by (2.17) and (2.29) with $q_h = q$, respectively. Then*

$$\left|j'(q)(r) - j_h'(q)(r)\right| \leq \hat{C}h^{2-\frac{d}{2}}\|r\|_{L^2(\Omega)}, \quad \forall r \in L^2(\Omega),$$

where

$$\hat{C} = C\Big(\|g\|_{L^\infty(\Omega)}, \|\omega\|_{\mathcal{M}(\Omega)}, \|y_d\|_{L^2(\Omega)}, \|q\|_{L^2(\Omega)}\Big). \tag{2.36}$$

Proof. Use of (2.17) and (2.29) together with the Cauchy-Schwarz inequality gives

$$\begin{aligned} |j'(q)(r) - j_h'(q)(r)| &= |(z - z_h(q), r)| \\ &\leq \|z - z_h(q)\|_{L^2(\Omega)}\|r\|_{L^2(\Omega)}. \end{aligned}$$

An application of Lemma 2.4.3 completes the rest of the proof. □

We are now in a position to derive one of the main result of this chapter, namely the error between the continuous control q and discrete control q_h.

Theorem 2.4.1. *Let q and q_h be the optimal controls of (2.15) and (2.27), respectively. Assume that $j''(q)(\cdot,\cdot)$ satisfies (2.19). Then, we have*

$$\|q - q_h\|_{L^2(\Omega)} \leq \tilde{C}\frac{h}{\sqrt{\gamma_E}} + \hat{C}\frac{h^{2-\frac{d}{2}}}{\gamma_E},$$

where $\hat{C}$ is given by (2.36) and

$$\tilde{C} = C\Big(\|g\|_{L^\infty(\Omega)}, \|\omega\|_{\mathcal{M}(\Omega)}, \|y_d\|_{L^2(\Omega)}, \|q\|_{L^2(\Omega)}, \alpha\Big). \tag{2.37}$$

Proof. With $r \in L^2(\Omega)$, we have

$$j''(q)(r,r) \geq \gamma_E\|r\|^2_{L^2(\Omega)}, \quad \forall q \in Q^E_{ad}, \tag{2.38}$$

and

$$j_h''(q_h)(r,r) \geq \gamma_E\|r\|^2_{L^2(\Omega)}, \quad \forall q_h \in Q^E_h. \tag{2.39}$$

We now formulate the following auxiliary problem:

$$\min_{q_h \in Q^E_h} j(q_h), \tag{2.40}$$

where we only discretize the control variable. Let $\tilde{q}_h$ be the solution of problem (2.40). We decompose the error as follows

$$q - q_h = (q - \tilde{q}_h) + (\tilde{q}_h - q_h), \tag{2.41}$$

and proceed to estimate each term separately.

In view of (2.38), we have for $\lambda \in [0,1]$ with $\xi = \lambda q + (1-\lambda)\tilde{q}_h$ and h sufficiently small,

$$\begin{aligned}
\gamma_E \|q - \tilde{q}_h\|^2_{L^2(\Omega)} &\leq j''(\xi)(q - \tilde{q}_h, q - \tilde{q}_h) \\
&= j'(q)(q - \tilde{q}_h) - j'(\tilde{q}_h)(q - \tilde{q}_h) \\
&= j'(q)(q - \tilde{q}_h) - j'(\tilde{q}_h)(q - \mathcal{L}_h q) - j'(\tilde{q}_h)(\mathcal{L}_h q - \tilde{q}_h).
\end{aligned}$$

The necessary optimality condition imply for h sufficiently small

$$j'(q)(q - \tilde{q}_h) \leq 0 \quad \text{and} \quad -j'(\tilde{q}_h)(\mathcal{L}_h q - \tilde{q}_h) \leq 0,$$

which together with the properties of $\mathcal{L}_h$ and the Young's inequality yields

$$\begin{aligned}
\gamma_E \|q - \tilde{q}_h\|^2_{L^2(\Omega)} &\leq -j'(\tilde{q}_h)(q - \mathcal{L}_h q) \\
&= -\Big(\alpha \tilde{q}_h + z(\tilde{q}_h), q - \mathcal{L}_h q\Big) \\
&= -(z(\tilde{q}_h) - \mathcal{L}_h z(\tilde{q}_h), q - \mathcal{L}_h q) \\
&\leq \Big(\frac{1}{2}\|z(\tilde{q}_h) - \mathcal{L}_h z(\tilde{q}_h)\|^2_{L^2(\Omega)} + \frac{1}{2}\|q - \mathcal{L}_h q\|^2_{L^2(\Omega)}\Big).
\end{aligned}$$

Therefore, we obtain

$$\|q - \tilde{q}_h\|_{L^2(\Omega)} \leq \Big(\frac{C}{\sqrt{\gamma_E}}\|z(\tilde{q}_h) - \mathcal{L}_h z(\tilde{q}_h)\|_{L^2(\Omega)} + \frac{C}{\sqrt{\gamma_E}}\|q - \mathcal{L}_h q\|_{L^2(\Omega)}\Big).$$

Using Lemma 2.3.2, we immediately have

$$\begin{aligned}
\|q - \tilde{q}_h\|_{L^2(\Omega)} &\leq \Big(\frac{C}{\sqrt{\gamma_E}} h \|z(\tilde{q}_h)\|_{H^1(\Omega)} + \frac{C}{\sqrt{\gamma_E}} h \|q\|_{H^1(\Omega)}\Big), \\
&\leq \frac{\tilde{C}}{\sqrt{\gamma_E}} h,
\end{aligned}$$

where $\tilde{C}$ is given by (2.37). To estimate the second term in (2.41), we use the necessary optimality condition which leads to the following relation

$$j_h'(q_h)(q_h - r_h) \leq 0 \leq j'(\tilde{q}_h)(r_h - \tilde{q}_h), \quad \forall r_h \in Q_h^E.$$

With $\xi = \lambda q_h + (1-\lambda)\tilde{q}_h$, $\lambda \in [0,1]$ and h sufficiently small, we have from (2.39)

$$\begin{aligned}\gamma_E \|q_h - \tilde{q}_h\|^2_{L^2(\Omega)} &\leq j''_h(\xi)(q_h - \tilde{q}_h, q_h - \tilde{q}_h) \\ &= j'_h(q_h)(q_h - \tilde{q}_h) - j'_h(\tilde{q}_h)(q_h - \tilde{q}_h) \\ &\leq j'(\tilde{q}_h)(q_h - \tilde{q}_h) - j'_h(\tilde{q}_h)(q_h - \tilde{q}_h) \\ &\leq \hat{C} h^{2-\frac{d}{2}} \|q_h - \tilde{q}_h\|_{L^2(\Omega)},\end{aligned}$$

where the last step follows from Lemma 2.4.4 and $\hat{C}$ is given by (2.36). This completes the proof. □

The following theorem yields the error between the continuous and approximate state solutions.

Theorem 2.4.2. *Let $y \in L^2(\Omega)$ and $y_h \in V_h^0$ be the solutions of (2.14) and (2.26), respectively. Assume that $\mu = g\omega$, g and ω are given functions such that $g \in \mathcal{C}(\overline{\Omega})$ and $\omega \in \mathcal{M}(\Omega)$. Then, we have*

$$\|y - y_h\|_{L^2(\Omega)} \leq C\Big(h^{2-\frac{d}{2}} \|g\|_{L^\infty(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)} + h^2 \|q\|_{L^2(\Omega)}\Big) + \|q - q_h\|_{L^2(\Omega)}.$$

Proof. Let ψ be the solution of (1.11) with $f \in L^2(\Omega)$. Then, using (2.14), (2.26) and Lemma 2.4.1, we have

$$\begin{aligned}\int_\Omega (y - y_h) f \, dx &= \prec g\omega, \psi - \psi_h \succ + (q, \psi - \psi_h) + (q - q_h, \psi_h) \\ &\leq \|g\|_{L^\infty(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)} \|\psi - \psi_h\|_{L^\infty(\Omega)} + \|q\|_{L^2(\Omega)} \|\psi - \psi_h\|_{L^2(\Omega)} \\ &\quad + \|q - q_h\|_{L^2(\Omega)} \|\psi_h\|_{L^2(\Omega)} \\ &\leq C\Big(h^{2-\frac{d}{2}} \|g\|_{L^\infty(\Omega)} \|\omega\|_{\mathcal{M}(\Omega)} + h^2 \|q\|_{L^2(\Omega)} + \|q - q_h\|_{L^2(\Omega)}\Big) \|f\|_{L^2(\Omega)},\end{aligned}$$

where we have used the fact $\|\psi_h\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ and Lemma 1.2.2. The definition of L^2-norm gives the desired estimate. This completes the proof. □

Now, we write the error for the co-state variable.

Theorem 2.4.3. *Let z and z_h be the solutions of (2.16) and (2.30), respectively. Then there exists a constant $C > 0$ such that*

$$\|z - z_h\|_{L^2(\Omega)} \leq Ch^2\Big(\|y\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\Big) + \|y - y_h\|_{L^2(\Omega)}.$$

Proof. The proof is similar to Lemma 2.4.3. Hence, we omit the details. □

Concluding remarks. In this chapter, we consider the optimal control problem governed by elliptic equation with measure data and investigate the existence, uniqueness and regularity results for the control problem. To discretize the control problem we use piecewise linear and continuous finite elements for the approximations of the state and co-state variables, whereas piecewise constant functions are used for the approximation of the control variable. We have derived *a priori* error estimates for the state, co-state and control variables (see Theorems 2.4.1-2.4.3). Numerical experiment is performed to validate our theoretical results in Chapter 7 (see Example 7.1).

3

EOCP with Measure Data: A Posteriori Error Analysis

This chapter is concerned with *a posteriori* error analysis of EOCP (1.1)-(1.3) with measure data in a bounded convex domain in $\mathbb{R}^d$ ($d = 2$ or 3). We derive global upper bounds for errors in the state, co-state and control variables. Moreover, local lower bounds are established for errors in the state and co-state variables, and global lower bound for error in the control variable is demonstrated in the case of two space dimension ($d = 2$). The present work extends the results of standard elliptic problem [7] to EOCP (1.1)-(1.3) with measure data.

3.1 Introduction

We now recall the following EOCP with measure data:

$$\min_{q\in Q_{ad}^E} J(q,y) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)} \tag{3.1}$$

subject to the state equation

$$\begin{cases} \mathcal{A}y = \mu + q & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

and control constraints

$$q_a \le q(x) \le q_b \ \text{ a.e. in } \Omega, \tag{3.3}$$

where Ω is a bounded convex domain in $\mathbb{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega$. The operator $\mathcal{A}$ is defined by (1.4) and $\alpha > 0$ is a fixed parameter. Further, $y_d \in L^2(\Omega)$ is a given desired state and $\mu = g\omega$ with $g \in \mathcal{C}(\overline{\Omega})$, $\omega \in \mathcal{M}(\Omega)$. The set of admissible controls is defined by

$$Q_{ad}^E := \{q \in L^2(\Omega) : \ q_a \le q(x) \le q_b \ \textit{a.e.} \text{ in } \Omega\} \tag{3.4}$$

with $q_a, q_b \in \mathbb{R}$ fulfill $q_a < q_b$.

For the purpose of finite element approximation we write the weak formulation of EOCP (3.1)-(3.3) as follows: Find a pair $(q, y) \in Q_{ad}^E \times L^2(\Omega)$ such that

$$\min_{q \in Q_{ad}^E} J(q, y) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)} \tag{3.5}$$

subject to

$$\int_\Omega y\mathcal{A}^* v\, dx = \prec g\omega, v \succ + (q, v), \qquad \forall v \in H^2(\Omega) \cap H_0^1(\Omega), \tag{3.6}$$

where $\prec g\omega, v \succ := \int_\Omega gv\, d\omega$.

In the next step, we introduce the reduced cost functional $j : L^2(\Omega) \to \mathbb{R}$ by

$$j(q) := J(q, y(q)),$$

where J is the cost function given by (3.5) and $y(q)$ is the weak solution of (3.2) as defined in (3.6). The optimal control problem can then be equivalently reformulated as

$$\min_{q \in Q_{ad}^E} j(q), \tag{3.7}$$

where the set of admissible controls is given by (3.4).

In the following theorem we state the necessary optimality condition for the optimal control problem (3.5)-(3.6) (cf. [60]).

Theorem 3.1.1. *Assume that $(q, y) \in Q_{ad}^E \times L^2(\Omega)$ is the solution of problem (3.5)-(3.6). Then there exists a unique co-state $z \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying*

$$\begin{cases} \mathcal{A}^* z = y - y_d & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Furthermore,

$$j'(q)(\hat{q} - q) = (\alpha q + z, \hat{q} - q) \geq 0, \qquad \hat{q} \in Q_{ad}^E. \tag{3.9}$$

Since this is a linear control problem, the reduced cost functional j is convex. Furthermore, $j(\cdot)$ is strictly convex in the sense that there is a positive constant c_1 independent of h such that

$$(j'(q) - j'(\hat{q}), q - \hat{q}) \geq c_1\|q - \hat{q}\|^2_{L^2(\Omega)}, \qquad \forall \hat{q},\ q \in Q_{ad}^E. \tag{3.10}$$

We now consider the finite element discretization of the control problem (3.5)-(3.6).

Let $\mathcal{T}_h$ be a family of triangulation of $\overline{\Omega}$ such that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} \overline{K}$. We assume that $\overline{\Omega}$ is the union of the elements of $\mathcal{T}_h$ so that element edges(faces) lying on the boundary may be curved. For each element $K \in \mathcal{T}_h$, we associate two parameters h_K and σ_K, where h_K denotes the diameter of the element K and σ_K is the supremum of the diameters of all circles contained in K. Further, we assume that there exists a positive constant C_1 $\frac{h_K}{\sigma_K} \leq C_1$ holds $\forall K \in \mathcal{T}_h$ and $\forall h_K > 0$ (cf. [24]). Let V_h be a finite dimensional subspace of $\mathcal{C}(\overline{\Omega})$ consisting of piecewise linear polynomials. Set $V_h^0 = V_h \cap H_0^1(\Omega)$. Let $\mathcal{E}_h$ be the set of interelement edges(faces) in the interior of the mesh. The quantity

$$\left[\frac{\partial v}{\partial n_{\mathcal{A}}}\right]\Big|_e = (A\nabla v)_K \cdot n_K + (A\nabla v)_{K'} \cdot n_{K'}$$

defined on the edge(face) $e \in \mathcal{E}_h$, $e = \overline{K} \cap \overline{K'}$, measures the jump of ∇v across the element edge(face) e. Here n_K denotes the unit outward normal vector to ∂K and $A = (a_{i,j}(x))_{d\times d}$. Let h_e denotes the length of the edge(face) e.

Set Q_h^E as follows:

$$Q_h^E = \{\hat{q}_h \in Q_{ad}^E : \ \forall K \in \mathcal{T}_h, \ \hat{q}_h|_K = constant\}.$$

The finite element approximation of the optimal control problem (3.5)-(3.6) is defined as follows: Find a pair $(q_h, y_h) \in Q_h^E \times V_h^0$ such that

$$\min_{q_h \in Q_h^E} J(q_h, y_h) = \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q_h\|^2_{L^2(\Omega)} \tag{3.11}$$

subject to

$$a(y_h, v_h) = \prec g\omega, v_h \succ + (q_h, v_h), \qquad \forall v_h \in V_h^0. \tag{3.12}$$

Analogous to the continuous case, we define the discrete reduced cost functional $j_h : L^2(\Omega) \to \mathbb{R}$ as

$$j_h(q) := J(q, y_h(q)).$$

The discrete optimal control problem is then read as

$$\min_{q_h \in Q_h^E} j_h(q_h). \tag{3.13}$$

With the standard arguments we prove the existence of a unique solution $q_h \in Q_h^E \subset Q_{ad}^E$ of (3.13), see [60]. The first optimality condition is then written as

$$j_h'(q_h)(\hat{q}_h - q_h) \geq 0, \qquad \forall \hat{q}_h \in Q_h^E. \tag{3.14}$$

Similar to the continuous case, the directional derivative $j_h'(q_h)(\hat{q}_h - q_h)$ for given $q_h, \ \hat{q}_h \in Q_h^E$ can be expressed as

$$j_h'(q_h)(\hat{q}_h - q_h) = (\alpha q_h + z_h, \hat{q}_h - q_h), \tag{3.15}$$

where z_h be the solution of the discrete co-state equation

$$a(z_h, v_h) = (y_h - y_d, v_h), \qquad \forall v_h \in V_h^0. \tag{3.16}$$

The main focus of this work is to study *a posteriori* error analysis of the finite element method for EOCP with measure data. It is known that *a posteriori* estimates are key ingredients for the adaptive algorithm. The low regularity of the solution of the state equation due to presence of measure data introduces some difficulties for both theory and numerics of finite element method. Therefore, an efficient numerical technique is used to solve these control problems and the development of AFEM suits to this kind of problem for better accuracy. The aim of this chapter is to construct *a posteriori* error bounds for EOCP with measure data. In the context of EOCP with measure data (3.1)-(3.3), we have derived global upper bounds for errors in the state, co-state and control variables in the L^2-norm. In case of two space dimension ($d = 2$), we have obtained local lower bounds for errors in the state and co-state variables, and global lower bound for error in the control variable. The key technical tools used in our analysis are the interpolation approximation properties, inverse estimates, optimality conditions, element and edge bubble functions and their properties. The previous work on *a posteriori* error analysis with Dirac source term for the standard elliptic problem has been discussed in [7]. They have obtained global upper and local lower bounds in L^p-norm and $W^{1,p}$ seminorm for $p < 2$.

The rest of the chapter is organized as follows. In Section 3.2, we derive upper bounds for errors in the state, co-state and control variables. Section 3.3 is devoted to local lower bounds for errors in the state and co-state variables, and global lower bound for the control variable.

3.2 Upper error bounds

This section considers *a posteriori* error analysis for the control problem (3.5)-(3.6) and derive upper bounds for errors in the state, co-state and control variables.

In the following, we define some auxiliary problems: For $q_h \in Q_h^E$, let $(y(q_h), z(q_h))$ be the solutions of the following equations:

$$\int_\Omega y(q_h)\mathcal{A}^* v\,dx = \prec g\omega, v \succ + (q_h, v), \qquad \forall v \in H^2(\Omega) \cap H_0^1(\Omega), \tag{3.17}$$

$$a(z(q_h), v) = (y(q_h) - y_d, v), \qquad \forall v \in H_0^1(\Omega). \tag{3.18}$$

From (3.9) and (3.15), we have

$$(j'(q), r) = (\alpha q + z, r), \qquad \forall r \in L^2(\Omega), \tag{3.19}$$

$$(j'(q_h), r) = (\alpha q_h + z(q_h), r), \qquad \forall r \in L^2(\Omega), \tag{3.20}$$

and

$$(j_h'(q_h), r) = (\alpha q_h + z_h, r), \qquad \forall r \in L^2(\Omega). \tag{3.21}$$

We need the following lemmas in deriving residual-type *a posteriori* error estimates. For a proof, see [24, 51].

Lemma 3.2.1. *Let $\pi_h : \mathcal{C}_0(\overline{\Omega}) \to V_h$ be the standard Lagrange interpolation operator defined in Lemma 2.3.1. Then, for $m = 0, 1$, we have*

$$\|v - \pi_h v\|_{H^m(K)} \leq C_{I,m} h_K^{2-m} |v|_{H^2(K)}, \qquad \forall v \in H^2(K),$$

and

$$\|v - \pi_h v\|_{L^\infty(K)} \leq C_{I,\infty} h_K^{2-d/2} |v|_{H^2(K)}, \qquad \forall v \in H^2(K).$$

Lemma 3.2.2. *For all $v \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, we have*

$$\|v\|_{L^p(e)} \leq C_{I,e}\Big(h_K^{-1/p}\|v\|_{L^p(K)} + h_K^{1-1/p}|v|_{W^{1,p}(K)}\Big).$$

In the following lemma, we derive upper bound for error in the control variable.

Lemma 3.2.3. *Let q and q_h be the solutions of (3.7) and (3.13), respectively. Assume that $(\alpha q_h + z_h)|_K \in H^1(K)$, $\forall K \in \mathcal{T}_h$, and there exists a positive constant $C_3 > 0$, and $r_h \in Q_h^E$ such that*

$$|(\alpha q_h + z_h, r_h - q)| \leq C_3 \sum_{K \in \mathcal{T}_h} h_K |\alpha q_h + z_h|_{H^1(K)} \|q - q_h\|_{L^2(K)}. \tag{3.22}$$

Then

$$\|q - q_h\|^2_{L^2(\Omega)} \leq \left(\frac{3C_3^2}{2(2c_1 - 1)}\eta_1^2 + \frac{3}{2(2c_1 - 1)}\|z_h - z(q_h)\|^2_{L^2(\Omega)}\right), \tag{3.23}$$

where $c_1 > \frac{1}{2}$ and

$$\eta_1 := \Big(\sum_{K \in \mathcal{T}_h} h_K^2 |\alpha q_h + z_h|^2_{H^1(K)}\Big)^{1/2}.$$

Proof. Using (3.9), (3.10) and (3.22), we have

$$\begin{aligned}
c_1\|q - q_h\|^2_{L^2(\Omega)} &\leq (j'(q), q - q_h) - (j'(q_h), q - q_h) \\
&\leq -(j'(q_h), q - q_h) = (j_h'(q_h), q_h - q) + (j_h'(q_h) - j'(q_h), q - q_h) \\
&\leq C_3 \sum_{K \in \mathcal{T}_h} h_K |\alpha q_h + z_h|_{H^1(K)} \|q - q_h\|_{L^2(K)} + |(j_h'(q_h) - j'(q_h), q - q_h)| \\
&\leq \frac{3C_3^2}{4} \sum_{K \in \mathcal{T}_h} h_K^2 |\alpha q_h + z_h|^2_{H^1(K)} + \frac{1}{4}\|q - q_h\|^2_{L^2(\Omega)} \\
&\quad + |(j_h'(q_h) - j'(q_h), q - q_h)|.
\end{aligned} \tag{3.24}$$

From (3.20) and (3.21), it is easy to show that

$$\begin{aligned} |(j_h'(q_h) - j'(q_h), q - q_h)| &= |(z_h - z(q_h), q - q_h)| \\ &\leq \frac{3}{4}\|z_h - z(q_h)\|^2_{L^2(\Omega)} + \frac{1}{4}\|q - q_h\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.25}$$

Combine (3.24) and (3.25) to complete the rest of the proof. □

In the following lemma, we derive intermediate error estimates for the state and co-state variables.

Lemma 3.2.4. *Let (y_h, z_h) and $(y(q_h), z(q_h))$ be the solutions of (3.12), (3.16) and (3.17)-(3.18), respectively. Then,*

$$\|y(q_h) - y_h\|_{L^2(\Omega)} \leq C_4(\eta_2 + \eta_3), \tag{3.26}$$

and

$$\|z(q_h) - z_h\|_{L^2(\Omega)} \leq C_4(\eta_2 + \eta_3) + C_5\,\eta_4, \tag{3.27}$$

where

$$\begin{aligned} \eta_2^2 &:= \sum_{K\in\mathcal{T}_h} h_K^{4-d}\|g\|^2_{L^\infty(K)}\|\omega\|^2_{\mathcal{M}(K)}, \\ \eta_3^2 &:= \sum_{K\in\mathcal{T}_h} h_K^4\|q_h - \mathcal{A}y_h\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_h} h_e^3\Big\|\Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|^2_{L^2(e)}, \\ \eta_4^2 &:= \sum_{K\in\mathcal{T}_h} h_K^4\|y_h - y_d - \mathcal{A}^* z_h\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_h} h_e^3\Big\|\Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big]\Big\|^2_{L^2(e)}, \end{aligned}$$

$C_4 = C_1 C_{\tilde{R}}\max\{C_{I,\infty}, C_{I,0}, C_{I,e}\}$ *and* $C_5 = \sqrt{2C_R^2 C_1^2\max\{C_{I,0}^2, C_{I,e}^2\}}$.

Proof. To derive *a posteriori* error estimate for $\|y(q_h) - y_h\|_{L^2(\Omega)}$, we use the duality argument. Let ψ be the solution of the adjoint problem (1.11) with $f = y(q_h) - y_h$. Then, from (3.17), we have

$$\begin{aligned} \|y(q_h) - y_h\|^2_{L^2(\Omega)} &= \int_\Omega (y(q_h) - y_h)\mathcal{A}^*\psi\,dx \\ &= \int_\Omega y(q_h)\mathcal{A}^*\psi\,dx - a(y_h, \psi) \\ &= \prec g\omega, \psi \succ + (q_h, \psi) - a(y_h, \psi). \end{aligned}$$

Using interpolation operator π_h, (3.12) and Green's formula, we obtain

$$\begin{aligned}\|y(q_h)-y_h\|^2_{L^2(\Omega)} &= \prec g\omega, \psi-\pi_h\psi \succ + (q_h, \psi-\pi_h\psi) - a(y_h, \psi-\pi_h\psi) \\ &= \Big\{ \prec g\omega, \psi-\pi_h\psi \succ \Big\} + \Big\{ \sum_{K\in\mathcal{T}_h} \int_K (q_h - \mathcal{A}y_h)(\psi-\pi_h\psi)\,dx \\ &\quad - \sum_{e\in\mathcal{E}_h} \int_e \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big](\psi-\pi_h\psi)\,de \Big\} \\ &=: E_1 + E_2. \end{aligned} \tag{3.28}$$

Now, we need to estimate E_i, $i=1,2$. From Lemmas 3.2.1 and 3.2.2, we have

$$\|\psi-\pi_h\psi\|_{L^2(K)} \le C_{I,0} h_K^2 |\psi|_{H^2(K)}, \tag{3.29}$$

$$\begin{aligned}\|\psi-\pi_h\psi\|_{L^2(e)} &\le C_{I,e}\Big(h_K^{-1/2}\|\psi-\pi_h\psi\|_{L^2(K)} + h_K^{1/2}|\psi-\pi_h\psi|_{H^1(K)}\Big) \\ &\le C_{I,e} h_K^{3/2}|\psi|_{H^2(K)},\end{aligned} \tag{3.30}$$

$$\|\psi-\pi_h\psi\|_{L^\infty(K)} \le C_{I,\infty} h_K^{2-\frac{d}{2}} |\psi|_{H^2(K)}, \tag{3.31}$$

where $e\subset\overline{K}$. For E_1, use of property (3.31) yields

$$\begin{aligned}|E_1| &\le \sum_{K\in\mathcal{T}_h} \|g\|_{L^\infty(K)}\|\omega\|_{\mathcal{M}(K)}\|\psi-\pi_h\psi\|_{L^\infty(K)} \\ &\le C_{I,\infty}\sum_{K\in\mathcal{T}_h} h_K^{2-\frac{d}{2}}\|g\|_{L^\infty(K)}\|\omega\|_{\mathcal{M}(K)}\|\psi\|_{H^2(K)},\end{aligned} \tag{3.32}$$

and for E_2, use of (3.29), (3.30) together with the Cauchy-Schwarz inequality and shape regularity of $\mathcal{T}_h$ gives

$$\begin{aligned}|E_2| &= \Big| \sum_{K\in\mathcal{T}_h} \int_K (q_h-\mathcal{A}y_h)(\psi-\pi_h\psi)\,dx - \sum_{e\in\mathcal{E}_h}\int_e \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big](\psi-\pi_h\psi)\,de \Big| \\ &\le C_1 \max\{C_{I,0}, C_{I,e}\}\Big\{ \sum_{K\in\mathcal{T}_h} h_K^4 \|q_h-\mathcal{A}y_h\|^2_{L^2(K)} \\ &\quad + \sum_{e\in\mathcal{E}_h} h_e^3 \Big\|\Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|^2_{L^2(e)}\Big\}^{\frac{1}{2}} \|\psi\|_{H^2(\Omega)},\end{aligned} \tag{3.33}$$

Using the definition of L^2-norm, (3.32), (3.33) and Lemma 1.2.2 in (3.28), we obtain

$$\begin{aligned}\|y(q_h)-y_h\|_{L^2(\Omega)} &\le C_1 C_{\tilde{R}} C_{I,\infty} \sum_{K\in\mathcal{T}_h} h_K^{2-\frac{d}{2}}\|g\|_{L^\infty(K)}\|\omega\|_{\mathcal{M}(K)} + C_1 C_{\tilde{R}} \max\{C_{I,0}, C_{I,e}\} \\ &\quad \times\Big(\sum_{K\in\mathcal{T}_h} h_K^4\|q_h-\mathcal{A}y_h\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_h} h_e^3 \Big\|\Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|^2_{L^2(e)}\Big)^{\frac{1}{2}},\end{aligned}$$

which proves (3.26). It remains to estimate $\|z(q_h) - z_h\|_{L^2(\Omega)}$. Let ϕ be the solution of (1.10) with $f = z(q_h) - z_h$. Then, use of (3.16) and (3.18) yields

$$\begin{aligned}
\|z(q_h) - z_h\|^2_{L^2(\Omega)} &= (z(q_h) - z_h, \mathcal{A}\phi) \\
&= a(z(q_h), \phi) - a(z_h, \phi) \\
&= (y(q_h) - y_d, \phi) - a(z_h, \phi) \\
&= (y(q_h) - y_d, \phi) - a(z_h, \phi - \pi_h\phi) - (y_h - y_d, \pi_h\phi) \\
&= \sum_{K\in\mathcal{T}_h} \int_K (y_h - y_d - \mathcal{A}^* z_h)(\phi - \pi_h\phi)\, dx \\
&\quad - \sum_{e\in\mathcal{E}_h} \int_e \Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big](\phi - \pi_h\phi)\, de + (y(q_h) - y_h, \phi).
\end{aligned}$$

Using (3.29)-(3.30), Lemma 1.2.2 and the Young's inequality, we obtain

$$\begin{aligned}
\|z(q_h) - z_h\|^2_{L^2(\Omega)} &\le C_1^2 C_R^2 \max\{C_{I,0}^2, C_{I,e}^2\}\Big\{ \sum_{K\in\mathcal{T}_h} h_K^4 \int_K (y_h - y_d - \mathcal{A}^* z_h)^2\, dx \\
&\quad + \sum_{e\in\mathcal{E}_h} h_e^3 \int_e \Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big]^2 de\Big\} + \frac{1}{2}\|y(q_h) - y_h\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2}\|z(q_h) - z_h\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore, we have

$$\|z(q_h) - z_h\|^2_{L^2(\Omega)} \le 2\, C_1^2 C_R^2 \max\{C_{I,0}^2, C_{I,e}^2\}\, \eta_4^2 + \|y(q_h) - y_h\|^2_{L^2(\Omega)}.$$

This completes the proof of the lemma. □

Now, we are in a position to derive the estimates for $\|y_h - y\|_{L^2(\Omega)}$ and $\|z_h - z\|_{L^2(\Omega)}$. We now write

$$\|y_h - y\|_{L^2(\Omega)} \le \|y(q_h) - y_h\|_{L^2(\Omega)} + \|y(q_h) - y\|_{L^2(\Omega)},$$

and

$$\|z_h - z\|_{L^2(\Omega)} \le \|z(q_h) - z_h\|_{L^2(\Omega)} + \|z(q_h) - z\|_{L^2(\Omega)}.$$

Using Lemma 1.2.2, we have

$$\|y_h - y\|_{L^2(\Omega)} \le \|y(q_h) - y_h\|_{L^2(\Omega)} + \|q_h - q\|_{L^2(\Omega)},$$

and

$$\begin{aligned}
\|z_h - z\|_{L^2(\Omega)} &\le \|z(q_h) - z_h\|_{L^2(\Omega)} + \|y(q_h) - y\|_{L^2(\Omega)} \\
&\le \|z(q_h) - z_h\|_{L^2(\Omega)} + \|q_h - q\|_{L^2(\Omega)}.
\end{aligned}$$

The following theorem provides upper bounds for errors in the state, co-state and control variables.

Theorem 3.2.1. *Let (y, q) and (y_h, q_h) be the solutions of (3.5)-(3.6) and (3.11)-(3.12), respectively. Let z and z_h be the solutions of the co-state equations (3.8) and (3.16), respectively. Then,*

$$\|y_h - y\|^2_{L^2(\Omega)} + \|z_h - z\|^2_{L^2(\Omega)} + \|q_h - q\|^2_{L^2(\Omega)} \leq \frac{3}{2(2c_1-1)}\Big\{C_3^2\eta_1^2 + C_4^2(\eta_2^2+\eta_3^2) + C_5^2\,\eta_4^2\Big\},$$

where η_1 is defined in Lemma 3.2.3 and $\eta_i|_{i=2,\dots,4}$ are defined in Lemma 3.2.4.

Remark 3.2.1. *Note that Theorem 3.2.1 is valid when $\omega = \delta_{x_c}$, where δ_{x_c} denotes the Dirac delta function concentrated at a point x_c. If $x_c \in K$ then the bound for the state reduces to*

$$\|y_h - y\|^2_{L^2(\Omega)} \leq \frac{3}{2(2c_1-1)}\Big\{C_3^2\eta_1^2 + C_4^2(\tilde{\eta}_2^2+\eta_3^2) + C_5^2\,\eta_4^2\Big\},$$

where $\tilde{\eta}_2 := h_K\|g\|_{L^\infty(K)}$, η_1 is defined in Lemma 3.2.3 and $\eta_i|_{i=3,4}$ are defined in Lemma 3.2.4. If x_c is the vertex of the element K then the bound for the state reduces to

$$\|y_h - y\|^2_{L^2(\Omega)} \leq \frac{3}{2(2c_1-1)}\Big(C_3^2\eta_1^2 + \hat{C}_4^2\eta_3^2 + C_5^2\,\eta_4^2\Big),$$

where $\hat{C}_4 = C_{\tilde{R}}C_1\max\{C_{I,0}, C_{I,e}\}$.

3.3 Lower error bounds

This section is devoted to the estimation of lower bounds for the errors in the case of two space dimension $(d = 2)$. We shall use the element and edge bubble functions and their properties (cf. [2, 7, 87]) to derive the main results. We take $\omega = \delta_{x_c}$ and define the bubble functions for the state and co-state separately.

Bubble functions for the state: Given $e \in \mathcal{E}_h$, let $b_e(x)$ be the edge bubble function having support

$$w_e := \cup\{K : \partial K \supset e\}$$

is defined by

$$b_e(x) := \begin{cases} \left(\lambda^{K_1}_{P_2}\lambda^{K_1}_{P_3}\lambda^{K_2}_{P_2}\lambda^{K_2}_{P_3}\right)^2 \frac{|x-x_c|^2}{h_e^2}, & \text{if } x_c \in w_e^\circ, \\ \left(\lambda^{K_1}_{P_2}\lambda^{K_1}_{P_3}\lambda^{K_2}_{P_2}\lambda^{K_2}_{P_3}\right)^2, & \text{otherwise,} \end{cases}$$

where w_e° is the interior of w_e and $\lambda_{P_i}^{K_j}$ is the barycentric coordinate of x associated with the triangle K_j and the point P_i, extended to the whole w_e.

The following lemma provides some properties of the edge bubble function $b_e(x)$ which will be used in deriving lower bound for the state variable.

Lemma 3.3.1. *For $e \in \mathcal{E}_h$, let $b_e(x)$ and w_e be defined as above. Then,*

$$\begin{aligned}
\frac{\partial b_e(x)}{\partial n_e} &= 0 \quad on \ w_e, \\
C_6\, h_e &\leq \int_e b_e(x)\, de \leq C_7\, h_e, \\
|b_e(x)|_{H^m(w_e)} &\leq C_8\, h_e^{1-m}, \qquad m = 1, 2, \\
\|b_e(x)\|_{L^2(w_e)} &\leq C_9,
\end{aligned}$$

where $C_i|_{i=6,\dots,9}$ are positive constants depend on the polynomial degree and shape parameter of $\mathcal{T}_h$, and n_e is the unit outward normal to the edge e.

We now define the element bubble function as follows: Let K be a triangle of $\mathcal{T}_h$ containing x_c (if x_c lies on an inner edge, any of the two triangles sharing the edge can be chosen as K). Let

$$w_K := \cup\{K' \in \mathcal{T}_h : \ K' \cap K \neq \emptyset\}, \tag{3.34}$$

and $L := dist(x_c, \partial w_K)$, where ∂w_K denotes the boundary of w_K. Notice that, because of the shape regularity of the mesh, $h_K \leq CL$. Let $b_{x_c}(x)$ be a smooth bubble function defined on Ω with support in w_K and satisfying

$$\begin{cases}
0 \leq b_{x_c}(x) \leq 1, & \forall x \in \Omega, \\
b_{x_c}(x) = 1, \quad |x - x_c| \leq L/4, & \forall x \in \Omega, \\
b_{x_c}(x) = 0, \quad |x - x_c| \geq 3L/4, & \forall x \in \Omega.
\end{cases} \tag{3.35}$$

Lemma 3.3.2. *Let $b_{x_c}(x)$ and w_K be defined as above and $x_c \in K$. Then, we have*

$$\begin{aligned}
\|b_{x_c}(x)\|_{L^\infty(w_K)} &\leq C_{10}, \\
|b_{x_c}(x)|_{H^m(w_K)} &\leq C_{11}\, h_K^{1-m}, \qquad m = 1, 2, \\
\|b_{x_c}(x)\|_{L^2(w_K)} &\leq C_{12},
\end{aligned}$$

where the positive constants $C_i|_{i=10,11,12}$ depend on the polynomial degree and shape regularity of the triangulation $\mathcal{T}_h$.

Bubble functions for the co-state: Let $b_K(x)$ be the standard third order polynomial on K scaled such that $b_K = \lambda_1\lambda_2\lambda_3$, where $\{\lambda_1, \lambda_2, \lambda_3\}$ denote the barycentric coordinates on K. Then, $b_K(x)$ satisfies the following properties:

$$supp\, b_K(x) \subset K, \;\; 0 \le b_K(x) \le 1, \;\; \max_{x\in K} b_K(x) = 1,$$

Let $\beta_K = \varphi b_K$ for all polynomials φ of degree 1, then $\beta_K \in H^2(K)$ and

$$C_{13}\|\varphi\|_{L^2(K)} \le \|b_K^{\frac{1}{2}}\varphi\|_{L^2(K)} \tag{3.36}$$

$$\|\beta_K\|_{H^2(K)} \le C_{14}h_K^{-2}\|\beta_K\|_{L^2(K)}, \quad \forall K \in \mathcal{T}_h, \tag{3.37}$$

where C_{13} and C_{14} are positive constants depend on the polynomial degree and shape regularity of the triangulation $\mathcal{T}_h$.

We need to introduce the edge bubble function for the co-state. Let $\tilde{b}_e(x)$ be the fourth order polynomial for the edge e, where $e = \partial K_1 \cap \partial K_2$ and K_1 be the triangle with vertices $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2)$, K_2 be the triangle with vertices $(x_0, y_0),\ (x_2, y_2),\ (x_3, y_3)$ and e be the common edge joining (x_0, y_0), (x_2, y_2). So, we define the edge bubble function $\tilde{b}_e(x)$ as follows:

$$\tilde{b}_e(x) := \begin{cases} \hat{\lambda}_0\hat{\lambda}_2\hat{\lambda}'_0\hat{\lambda}'_2, & K_1 \cup K_2, \\ 0, & \Omega \setminus K_1 \cup K_2, \end{cases}$$

where $\hat{\lambda}_0, \hat{\lambda}_2$ corresponds to the triangle K_1 and $\hat{\lambda}'_0, \hat{\lambda}'_2$ corresponds to the triangle K_2.

Denote $\Delta_1 := \begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$, $\Delta_2 := \begin{vmatrix} x_0 & y_0 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, $\Delta_{01} := \begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ \bar{x} & \bar{y} & 1 \end{vmatrix}$, $\Delta_{12} := \begin{vmatrix} \bar{x} & \bar{y} & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$, $\Delta_{23} := \begin{vmatrix} \bar{x} & \bar{y} & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ and $\Delta_{03} := \begin{vmatrix} x_0 & y_0 & 1 \\ \bar{x} & \bar{y} & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Then, set

$$\hat{\lambda}_0 := \frac{\Delta_{12}}{\Delta_1 + \Delta_2}, \quad \hat{\lambda}_2 := \frac{\Delta_{01}}{\Delta_1 + \Delta_2}, \quad \hat{\lambda}'_0 := \frac{\Delta_{23}}{\Delta_1 + \Delta_2} \text{ and } \hat{\lambda}'_2 := \frac{\Delta_{03}}{\Delta_1 + \Delta_2}.$$

Let $\tilde{K} = K_1 \cup K_2$. Then for $\beta_e = \tilde{\varphi}\tilde{b}_e(x)$ satisfies $\frac{\partial \beta_e}{\partial n_e} = 0$ on $\partial\tilde{K}$, where n_e is the unit normal to the edge e, for all polynomials $\tilde{\varphi}$ of degree 1. So, we have $\beta_e \in H^2_0(\tilde{K})$.

Then, from the standard scaling arguments, it can be shown that (cf. [3, 87])

$$\|\beta_e\|_{L^2(\tilde{K})} \leq C_{15} h_e^{1/2} \|\beta_e\|_{L^2(e)}, \tag{3.38}$$

$$C_{16}\|\tilde{\varphi}\|_{L^2(e)} \leq \|\tilde{b}_e^{\frac{1}{2}}\tilde{\varphi}\|_{L^2(e)}, \tag{3.39}$$

$$\|\beta_e\|_{H^2(\tilde{K})} \leq C_{17} h_e^{-2} \|\beta_e\|_{L^2(\tilde{K})}, \tag{3.40}$$

where C_{15}, C_{16} and C_{17} are positive constants depend on the polynomial degree and shape regularity of the triangulation $\mathcal{T}_h$.

Now, we are in a position to derive local lower bounds for error in the state variable.

Theorem 3.3.1. *Let $x_c \in K$ and w_K be defined in (3.34). Let $\mathcal{E}_h^K$ be the set of edges e of triangles $K \subset w_K$ such that $e \nsubseteq \partial w_K$. Then*

$$\begin{aligned} h_K g_K(x_c) \leq\ & h_K \Big\{ C_{10}\|g_K - g\|_{L^\infty(w_K)} + C_{12}\Big(\|q - q_h\|_{L^2(w_K)} + \|q_h - \mathcal{A}y_h\|_{L^2(w_K)}\Big)\Big\} \\ & + C_{11}\|y - y_h\|_{L^2(w_K)} + \sum_{e \in \mathcal{E}_h^K} h_e^2 \Big\| \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)}, \end{aligned}$$

and

$$h_e^2 \Big\| \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)} \leq \frac{1}{C_6}\Big\{ C_8\|y - y_h\|_{L^2(w_e)} + C_9\, h_e\Big(\|q_h - \mathcal{A}y_h\|_{L^2(w_e)} + \|q - q_h\|_{L^2(w_e)}\Big)\Big\},$$

where g_K denotes the mean value of g on the element K, i.e., $g_K(x) := \frac{1}{|K|}\int_K g(x)\,dx$.

Proof. Let b_{x_c} be the bubble function defined by (3.35). Using (3.6) and integration by parts, we obtain

$$\begin{aligned} g_K(x_c) &= \prec g_K \delta_{x_c}, b_{x_c} \succ \\ &= \prec (g_K - g)\delta_{x_c}, b_{x_c} \succ + \int_{w_K} y \mathcal{A}^* b_{x_c}\,dx - \int_{w_K} q\, b_{x_c}\,dx \\ &= \prec (g_K - g)\delta_{x_c}, b_{x_c} \succ + \int_{w_K} (y - y_h)\mathcal{A}^* b_{x_c}\,dx + \int_{w_K} \mathcal{A}y_h\, b_{x_c}\,dx \\ &\quad + \sum_{e \in \mathcal{E}_h^K} \int_e \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big] b_{x_c}\,de - \sum_{e \in \mathcal{E}_h^K} \int_e \Big[\frac{\partial b_{x_c}}{\partial n_{\mathcal{A}^*}}\Big] y_h\,de - \int_{w_K} q\, b_{x_c}\,dx. \end{aligned}$$

Since $\Big[\frac{\partial b_{x_c}}{\partial n_{\mathcal{A}^*}}\Big]\Big|_e = 0$, it now follows that

$$\begin{aligned} g_K(x_c) =\ & \prec (g_K - g)\delta_{x_c}, b_{x_c} \succ + \int_{w_K} (y - y_h)\mathcal{A}^* b_{x_c}\,dx + \int_{w_K} (\mathcal{A}y_h - q_h)\, b_{x_c}\,dx \\ & + \sum_{e \in \mathcal{E}_h^K} \int_e \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big] b_{x_c}\,de + \int_{w_K} (q_h - q)\, b_{x_c}\,dx. \end{aligned}$$

An application of the Cauchy Schwarz inequality and the shape regularity of the mesh yields

$$\begin{aligned} g_K(x_c) \;\leq\; & \|g_K - g\|_{L^\infty(w_K)}\|b_{x_c}\|_{L^\infty(w_K)} + \|y - y_h\|_{L^2(w_K)}\|\mathcal{A}^* b_{x_c}\|_{L^2(w_K)} \\ & + \|q_h - \mathcal{A} y_h\|_{L^2(w_K)}\|b_{x_c}\|_{L^2(w_K)} + \sum_{e\in\mathcal{E}_h^K} h_e \Big\| \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)} \\ & + \|q - q_h\|_{L^2(w_K)}\|b_{x_c}\|_{L^2(w_K)}. \end{aligned}$$

An use of Lemma 3.3.2 implies

$$\begin{aligned} g_K(x_c) \;\leq\; & C_{10}\|g_K - g\|_{L^\infty(w_K)} + C_{11}\, h_K^{-1}\|y - y_h\|_{L^2(w_K)} + C_{12}\,\|q_h - \mathcal{A} y_h\|_{L^2(w_K)} \\ & + \sum_{e\in\mathcal{E}_h^K} h_e \Big\| \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)} + C_{12}\,\|q - q_h\|_{L^2(w_K)}, \end{aligned}$$

which completes the proof of the first inequality. To prove the second inequality, we use (3.6) and integration by parts to obtain

$$\begin{aligned} \int_{w_e} (y - y_h)\mathcal{A}^* b_e\, dx + \int_{w_e} (q_h - q)\, b_e\, dx \;=\; & \prec g\delta_{x_c}, b_e \succ - \int_{w_e} y_h \mathcal{A}^* b_e\, dx + \int_{w_e} q_h\, b_e\, dx \\ =\; & \prec g\delta_{x_c}, b_e \succ + \int_{w_e} (q_h - \mathcal{A} y_h)\, b_e\, dx \\ & - \int_e \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big] b_e\, de + \int_e \Big[\frac{\partial b_e}{\partial n_{\mathcal{A}^*}}\Big] y_h\, de. \end{aligned}$$

Using the fact $b_e(x_c) = 0$ and $\Big[\frac{\partial b_e}{\partial n_{\mathcal{A}^*}}\Big]\Big|_e = 0$, we have

$$\int_{w_e} (y - y_h)\, \mathcal{A}^* b_e\, dx + \int_{w_e} (q_h - q)\, b_e\, dx = \int_{w_e} (q_h - \mathcal{A} y_h)\, b_e\, dx - \int_e \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big] b_e\, de. \quad (3.41)$$

Since

$$\Big|\int_e \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big] b_e\, de\Big| \;\geq\; C_6 h_e \Big\| \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)},$$

the equation (3.41) becomes

$$\begin{aligned} C_6\, h_e \Big\| \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)} \;\leq\; & \Big|\int_{w_e} (q_h - \mathcal{A} y_h)\, b_e\, dx\Big| + \Big|\int_{w_e} (y - y_h)\, \mathcal{A}^* b_e\, dx\Big| + \Big|\int_{w_e} (q - q_h)\, b_e\, dx\Big| \\ \leq\; & \|q_h - \mathcal{A} y_h\|_{L^2(w_e)}\|b_e\|_{L^2(w_e)} + \|y - y_h\|_{L^2(w_e)}\|\mathcal{A}^*\, b_e\|_{L^2(w_e)} \\ & + \|q - q_h\|_{L^2(w_e)}\|b_e\|_{L^2(w_e)}. \end{aligned}$$

Finally, using Lemma 3.3.1 we obtain

$$C_6\, h_e \Big\| \Big[\frac{\partial y_h}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)} \leq C_8\, h_e^{-1}\|y - y_h\|_{L^2(w_e)} + C_9 \Big\{\|q_h - \mathcal{A} y_h\|_{L^2(w_e)} + \|q - q_h\|_{L^2(w_e)}\Big\},$$

and this completes the proof of the theorem. □

In the following theorem, we derive local lower bounds for the co-state variable.

Theorem 3.3.2. *Let (y, q) and (y_h, q_h) be the solutions of (3.5)-(3.6) and (3.11)-(3.12), respectively. Let z and z_h be the solutions of the co-state equations (3.8) and (3.16), respectively. Then,*

$$h_K^2 \|(\bar{y}_h - \bar{y}_d) - \mathcal{A}^* z_h\|_{L^2(K)} \leq C_{13}^{-2} \Big\{ C_{14} \|z - z_h\|_{L^2(K)} + h_K^2 \Big(\|y_h - y_d - (\bar{y}_h - \bar{y}_d)\|_{L^2(K)} + \|y - y_h\|_{L^2(K)} \Big) \Big\},$$

and

$$h_e^{\frac{3}{2}} \Big\| \Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big] \Big\|_{L^2(e)} \leq C_{16}^{-2} C_{15} \Big\{ h_e^{\frac{1}{2}} \Big(\|(\bar{y}_h - \bar{y}_d) - \mathcal{A}^* z_h\|_{L^2(\tilde{K})} + \|(y_h - y_d) - (\bar{y}_h - \bar{y}_d)\|_{L^2(\tilde{K})} + \|y - y_h\|_{L^2(\tilde{K})} \Big) + C_{17} \|z - z_h\|_{L^2(\tilde{K})} \Big\},$$

where $\bar{v}$ is the mean value of v on the element K such that $\bar{v}|_K = \frac{1}{|K|} \int_K v(x)\, dx$.

Proof. Multiplying (3.8) by $v \in H_0^1(\Omega)$. We form L^2-inner product over Ω. Then, an application of the Green's theorem leads to

$$\begin{aligned} a(z - z_h, v) &= (y - y_d, v) - a(z_h, v) \\ &= (y - y_d, v) - \sum_{K \in \mathcal{T}_h} \int_K \mathcal{A}^* z_h \, v\, dx - \sum_{e \in \mathcal{E}_h} \int_e \Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big] v\, de \\ &= (y - y_h, v) + \sum_{K \in \mathcal{T}_h} \int_K ((\bar{y}_h - \bar{y}_d) - \mathcal{A}^* z_h)\, v\, dx - \sum_{e \in \mathcal{E}_h} \int_e \Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big] v\, de \\ &\quad + \sum_{K \in \mathcal{T}_h} \int_K (y_h - y_d - (\bar{y}_h - \bar{y}_d))\, v\, dx. \end{aligned} \tag{3.42}$$

Note that

$$a(z - z_h, v) = \sum_{K \in \mathcal{T}_h} \int_K \mathcal{A} v\, (z - z_h)\, dx + \sum_{e \in \mathcal{E}_h} \int_e \Big[\frac{\partial v}{\partial n_{\mathcal{A}}}\Big] (z - z_h)\, de. \tag{3.43}$$

Set $\beta_K = b_K((\bar{y}_h - \bar{y}_d) - \mathcal{A}^* z_h)$ and choose $v = \beta_K$ in (3.42) and (3.43), and compare both the equations to obtain

$$\begin{aligned} \int_K \mathcal{A} \beta_K (z - z_h)\, dx &= \int_K ((\bar{y}_h - \bar{y}_d) - \mathcal{A}^* z_h)^2\, b_K\, dx + \int_K (y_h - y_d - (\bar{y}_h - \bar{y}_d))\, \beta_K\, dx \\ &\quad + \int_K (y - y_h)\, \beta_K\, dx, \end{aligned}$$

where we have used $supp\,\beta_K \subset K$. An application of the triangle inequality yields

$$\begin{aligned}\Big|\int_K ((\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h)^2\, b_K\, dx\Big| &= \Big|\int_K \mathcal{A}\beta_K(z-z_h)\,dx - \int_K (y-y_h)\,\beta_K\,dx \\ &\quad -\int_K (y_h-y_d-(\bar{y}_h-\bar{y}_d))\,\beta_K\,dx\Big| \\ &\le \|\beta_K\|_{H^2(K)}\|z-z_h\|_{L^2(K)} + \|y-y_h\|_{L^2(K)}\|\beta_K\|_{L^2(K)} \\ &\quad +\|y_h-y_d-(\bar{y}_h-\bar{y}_d)\|_{L^2(K)}\|\beta_K\|_{L^2(K)}.\end{aligned}$$

The property of β_K, $\|\beta_K\|_{H^2(K)} \le C_{14} h_K^{-2}\|\beta_K\|_{L^2(K)}$ gives

$$\begin{aligned}\Big|\int_K ((\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h)^2\, b_K\, dx\Big| &\le C_{14}h_K^{-2}\|\beta_K\|_{L^2(K)}\|z-z_h\|_{L^2(K)} \\ &\quad +\|y-y_h\|_{L^2(K)}\|\beta_K\|_{L^2(K)} \\ &\quad +\|y_h-y_d-(\bar{y}_h-\bar{y}_d)\|_{L^2(K)}\|\beta_K\|_{L^2(K)}, \qquad (3.44)\end{aligned}$$

and using (3.36), we find that

$$\int_K ((\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h)^2\, b_K\, dx \ge C_{13}^2\|(\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h\|^2_{L^2(K)}. \qquad (3.45)$$

We now combine (3.44) and (3.45) to have

$$\begin{aligned}C_{13}^2\|(\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h\|^2_{L^2(K)} &\le \Big(C_{14}h_K^{-2}\|z-z_h\|_{L^2(K)} + \|y-y_h\|_{L^2(K)} \\ &\quad +\|y_h-y_d-(\bar{y}_h-\bar{y}_d)\|_{L^2(K)}\Big)\|(\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h\|_{L^2(K)},\end{aligned}$$

where we have used the fact $\|\beta_K\|_{L^2(K)} \le \|(\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h\|_{L^2(K)}$. This proves the first inequality. Next, to prove the second inequality, let $\beta_e = \left[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\right]\tilde{b}_e$ and choose $v=\beta_e$ in (3.42) and (3.43). We compare both the equations to have

$$\begin{aligned}\int_{\tilde{K}} \mathcal{A}\beta_e(z-z_h)\,dx &= \int_{\tilde{K}} (y-y_h)\,\beta_e\,dx + \int_{\tilde{K}} ((\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h)\beta_e\,dx \\ &\quad -\int_e \Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big]^2\tilde{b}_e\,de + \int_{\tilde{K}} (y_h-y_d-(\bar{y}_h-\bar{y}_d))\beta_e\,dx \\ &\quad -\int_e \Big[\frac{\partial \beta_e}{\partial n_{\mathcal{A}}}\Big] z_h\,de.\end{aligned}$$

Since $\left[\frac{\partial \beta_e}{\partial n_{\mathcal{A}}}\right]\Big|_e = 0$, it now leads to

$$\begin{aligned}\Big|\int_e \Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big]^2\tilde{b}_e\,de\Big| &= \Big|\int_{\tilde{K}} ((\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h)\beta_e\,dx + \int_{\tilde{K}} (y_h-y_d-(\bar{y}_h-\bar{y}_d))\beta_e\,dx \\ &\quad +\int_{\tilde{K}} (y-y_h)\beta_e\,dx - \int_{\tilde{K}} \mathcal{A}\beta_e(z-z_h)\,dx\Big| \\ &\le \Big(\|(\bar{y}_h-\bar{y}_d)-\mathcal{A}^* z_h\|_{L^2(\tilde{K})} + \|y_h-y_d-(\bar{y}_h-\bar{y}_d)\|_{L^2(\tilde{K})} \\ &\quad +\|y-y_h\|_{L^2(\tilde{K})}\Big)\|\beta_e\|_{L^2(\tilde{K})} + \|\beta_e\|_{H^2(\tilde{K})}\|z-z_h\|_{L^2(\tilde{K})}.\end{aligned}$$

An application of (3.38) and (3.40) yields

$$\begin{aligned}\Big|\int_e\Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big]^2\tilde{b}_e\,de\Big| &\leq \Big(\|(\bar{y}_h-\bar{y}_d)-\mathcal{A}^*z_h\|_{L^2(\tilde{K})}+\|y_h-y_d-(\bar{y}_h-\bar{y}_d)\|_{L^2(\tilde{K})}\\ &\quad+\|y-y_h\|_{L^2(\tilde{K})}+C_{17}h_e^{-2}\|z-z_h\|_{L^2(\tilde{K})}\Big)\|\beta_e\|_{L^2(\tilde{K})}\\ &\leq C_{15}h_e^{\frac{1}{2}}\Big(\|(\bar{y}_h-\bar{y}_d)-\mathcal{A}^*z_h\|_{L^2(\tilde{K})}+\|y_h-y_d-(\bar{y}_h-\bar{y}_d)\|_{L^2(\tilde{K})}\\ &\quad+\|y-y_h\|_{L^2(\tilde{K})}+C_{17}h_e^{-2}\|z-z_h\|_{L^2(\tilde{K})}\Big)\|\beta_e\|_{L^2(e)}\\ &\leq C_{15}h_e^{\frac{1}{2}}\Big(\|(\bar{y}_h-\bar{y}_d)-\mathcal{A}^*z_h\|_{L^2(\tilde{K})}+\|y_h-y_d-(\bar{y}_h-\bar{y}_d)\|_{L^2(\tilde{K})}\\ &\quad+\|y-y_h\|_{L^2(\tilde{K})}+C_{17}h_e^{-2}\|z-z_h\|_{L^2(\tilde{K})}\Big)\Big\|\Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big]\Big\|_{L^2(e)}. \quad (3.46)\end{aligned}$$

In view of (3.39), we have

$$\int_e\Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big]^2\tilde{b}_e\,de \geq C_{16}^2\Big\|\Big[\frac{\partial z_h}{\partial n_{\mathcal{A}^*}}\Big]\Big\|^2_{L^2(e)},$$

which combine with (3.46) completes the rest of the proof. □

Now, we derive global lower bound for error in the control variable.

Theorem 3.3.3. *Let (y,q) and (y_h,q_h) be the solutions of (3.5)-(3.6) and (3.11)-(3.12), respectively. Let z and z_h be the solutions of the co-state equations (3.8) and (3.16), respectively. Then,*

$$\eta_1^2 \leq C^*\Big(\alpha^2\|q-q_h\|^2_{L^2(\Omega)}+\|z-z_h\|^2_{L^2(\Omega)}\Big),$$

where C^* is a positive constant and η_1 is defined in Lemma 3.2.3.

Proof. From the optimality condition (3.9), we deduce that $\alpha q + z = 0$ on Q_{ad}^E. An application of inverse property [24] yields

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}h_K^2|\alpha q_h+z_h|^2_{H^1(K)} &\leq \sum_{K\in\mathcal{T}_h}h_K^2\|\alpha q_h+z_h\|^2_{H^1(K)}\\ &\leq C^*\sum_{K\in\mathcal{T}_h}\|\alpha q_h+z_h\|^2_{L^2(K)}\\ &= C^*\|\alpha q_h+z_h\|^2_{L^2(\Omega)}\\ &= C^*\|\alpha q_h+z_h-\alpha q-z\|^2_{L^2(\Omega)}\\ &\leq C^*\Big(\alpha^2\|q-q_h\|^2_{L^2(\Omega)}+\|z-z_h\|^2_{L^2(\Omega)}\Big),\end{aligned}$$

and this completes the proof. □

Concluding remarks. In this chapter, *a posteriori* error estimators for the state, co-state and control variables are obtained for EOCP with measure data. The error estimators obtained in Theorem 3.2.1 are contributed from the approximation error of the state, co-state and control variables. Among them η_1 mainly indicates the approximation error for the control, η_2 and η_3 are contributed by the state equation whereas η_4 is contributed by the co-state equation. The local lower bound for the state and co-state variables are derived in Theorems 3.3.1-3.3.2. Further, a global lower bound for the control variable is derived in Theorem 3.3.3. The performance of our error estimators is reported in Chapter 7. These error estimators work satisfactorily in guiding the mesh refinement and save substantial computational work (see Table 7.2, Chapter 7).

4

POCP with Measure Data in Space: A Priori Error Analysis

In this chapter, we analyze finite element approximation for POCP (1.6)-(1.8) with measure data in space in a bounded convex domain in $\mathbb{R}^d$ $(d = 2$ or $3)$. The main mathematical difficulty of this problem is that the solution of the state equation exhibits low regularity due to the presence of measure data in the source term, which makes the the convergence analysis somewhat cumbersome. We prove the existence, uniqueness and regularity of the solution of the control problem. *A priori* error estimates for the state, co-state and control variables are derived for both spatially discrete and fully discrete approximations of optimal control problems. Moreover, $L^2(0,T;L^2(\Omega))$ convergence properties for the state, co-state and control variables are established. We use piecewise linear and continuous finite elements for approximations of the state and co-state variables whereas the control variable is approximated by piecewise constant functions.

4.1 Introduction

Let $\Omega_T = \Omega \times (0,T]$ and $\Gamma_T = \partial\Omega \times [0,T]$, where Ω is a bounded convex domain in $\mathbb{R}^d$ $(d = 2$ or $3)$ with boundary $\partial\Omega$ and $T < \infty$. We now recall the following POCP:

$$\min_{q \in Q_{ad}^P} \tilde{J}(q,y) = \frac{1}{2}\int_0^T \left\{ \|y - y_{\tilde{d}}\|^2_{L^2(\Omega)} + \tilde{\alpha}\|q\|^2_{L^2(\Omega)} \right\} d\tau \tag{4.1}$$

subject to the state equation

$$\begin{cases} y_t + \mathcal{A}y = \mu + q \ \text{ in } \ \Omega_T, \\ y(\cdot,0) = y_0(x) \ \text{ in } \ \Omega, \\ y = 0 \ \text{ on } \ \Gamma_T \end{cases} \tag{4.2}$$

and the control constraints

$$q_c \le q(x,t) \le q_d \ \text{ a.e. in } \Omega_T, \tag{4.3}$$

where $q_c,\ q_d \in \mathbb{R}$ fulfill $q_c < q_d$ and $y_t = \frac{\partial y}{\partial t}$. The operator $\mathcal{A}$ is defined in (1.4) and $\tilde{\alpha} > 0$ is a fixed constant. Further, $y_0(x) \in L^2(\Omega)$ and $y_{\tilde{d}}(x,t) \in L^2(0,T;L^2(\Omega))$ are given functions. The function $\mu = g\omega$ with $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$.

The set of adimissible controls is defined by

$$Q_{ad}^P := \{q \in L^2(0,T;L^2(\Omega)) : \ q_c \le q(x,t) \le q_d \ \ a.e. \ \text{in } \Omega_T\}. \tag{4.4}$$

Optimal control problems with time-dependent control play an important role in many applications, and the numerical treatment of these problems has been an active research topic in the recent years. POCPs with measure data in space are used in the modeling of acoustic monopoles and transport equations for effluent discharge in aquatic media and the design and management of waste water treatment systems, mainly the disposal of sea outfalls discharging polluting effluent from a sewerage system [68]. For the purpose of finite element formulation we rewrite the state equation in weak form using transposition techniques [61] and investigate the existence, uniqueness and regularity of the solution of the state variable for measure data in space. We have used continuous piecewise linear functions for the approximations of the state and co-state variables, and piecewise constant functions for the control variable. We have analyzed both spatially discrete and fully discrete finite element approximations of POCP with measure data in space. *A priori* error estimates of order $\mathcal{O}(h^{2-\frac{d}{2}})$ is derived for the spatially discrete control problem whereas error estimate of order $\mathcal{O}(h^{2-\frac{d}{2}} + k^{1/2})$ is established for the fully discrete backward Euler time discretization. The key technical tools used in our analysis include nonstandard weak formulation of the state equation, interpolation approximation properties, inverse estimates, optimality conditions, embedding theorems and duality technique. The earlier work on standard nonlinear parabolic problems involving measure data, we refer to [12, 13]. Later, the author of [16] has studied semilinear parabolic problems with measure data. Recently, Gong has derived error estimates for linear parabolic equation involving measure data in [35]. The purpose of this chapter is to extent the results of standard linear parabolic problem [35] to POCP (1.6)-(1.8) with measure data in space.

The plan of this chapter is as follows. In section 4.2, we discuss the existence, uniqueness and regularity results of the optimal control problems. Section 4.3 deals with the spatially discrete finite element approximations of the control problem, and derives error estimates for the state, co-state and control variables. Section 4.4 is devoted to the fully discrete error analysis for the control problem and the related convergence results are proved.

Throughout this chapter C denotes a positive generic constant independent of the mesh parameters h and k.

4.2 Existence, uniqueness and regularity results

This section concerns the existence, uniqueness and regularity results of POCP (4.1)-(4.3). In the following theorem we prove the existence, uniqueness and regularity results of the solution to problem (4.2). We recall the following notations

$$\begin{aligned} W(0,T) &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ X(0,T) &:= L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)). \end{aligned}$$

Theorem 4.2.1. *Assume that $y_0 \in L^2(\Omega)$, $q \in L^2(0,T;L^2(\Omega))$, $\mu = g\omega$, g and ω are given functions such that $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$. Further, the problem (4.2) admits a unique solution $y \in L^2(0,T;L^2(\Omega))$ in the sense that*

$$-(y,v_t)_{\Omega_T} + (y,\mathcal{A}^*v)_{\Omega_T} = <\mu, v>_{\Omega_T} + (y_0, v(\cdot,0)) + (q,v)_{\Omega_T}, \quad \forall v \in X(0,T) \tag{4.5}$$

with $v(\cdot,T) = 0$, where

$$<\mu, v>_{\Omega_T} = \int_\Omega \Big(\int_0^T g(x,\tau)v(x,\tau)\,d\tau \Big)\,d\omega(x), \quad \forall v \in L^2(0,T;\mathcal{C}(\overline{\Omega})).$$

Then there exists a positive constant C such that

$$\|y\|_{L^2(L^2(\Omega))} \le C\Big(\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)} + \|y_0\|_{L^2(\Omega)}\Big) + \|q\|_{L^2(L^2(\Omega))}. \tag{4.6}$$

Moreover, we have $y \in L^1(0,T;W^{1,p}(\Omega)) \cap \mathcal{C}([0,T];(W^{1,p'}(\Omega))')$ and
$y_t \in L^1(0,T;(W^{1,p'}(\Omega))')$ with

$$\|y\|_{L^1(W^{1,p}(\Omega))} \le C\Big(\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)} + \|y_0\|_{L^2(\Omega)}\Big) + \|q\|_{L^2(L^2(\Omega))}, \tag{4.7}$$

where $p \in [1, \frac{d}{d-1})$ and p' is the conjugate number of p such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. The case $\mu \equiv 0$ and $q \equiv 0$ is contained in [61]. We set $y_0 \equiv 0$ and let $\{\omega_n\}_n \subset \mathcal{C}(\overline{\Omega})$ be the sequence converging weak* to ω in $\mathcal{M}(\Omega)$ and satisfies

$$\|\omega_n\|_{L^1(\Omega)} \leq \|\omega\|_{\mathcal{M}(\Omega)}.$$

Let y_n be the solutions of

$$\begin{cases} y_{n,t} + \mathcal{A}y_n = g\omega_n + q \quad \text{in } \Omega_T, \\ y_n(\cdot,0) = 0 \quad \text{in } \Omega, \\ y_n = 0 \quad \text{on} \quad \Gamma_T. \end{cases} \tag{4.8}$$

Then $y_n \in X(0,T)$. For $f \in \mathcal{D}(\Omega_T)$, let ψ be the solution of problem (1.14). From the regularity of the parabolic problem $\psi \in X(0,T)$. Using (4.8), we find that

$$\begin{aligned} \int_{\Omega_T} f y_n \, dx d\tau &= \int_{\Omega_T} (-\psi_t + \mathcal{A}^*\psi) y_n \, dx d\tau \\ &= \int_{\Omega_T} g\omega_n \psi \, dx d\tau + \int_{\Omega_T} q\psi \, dx d\tau \\ &\leq C\Big(\|g\|_{L^2(L^\infty(\Omega))} \|\omega_n\|_{L^1(\Omega)} \|\psi\|_{L^2(\mathcal{C}(\overline{\Omega}))} + \|q\|_{L^2(L^2(\Omega))} \|\psi\|_{L^2(L^2(\Omega))} \Big). \end{aligned}$$

Embedding theorem gives $L^2(0,T;H^2(\Omega)) \hookrightarrow L^2(0,T;\mathcal{C}(\overline{\Omega}))$. By standard estimates, we have

$$\|\psi\|_{L^2(H^2(\Omega))} \leq C\|f\|_{L^2(L^2(\Omega))}. \tag{4.9}$$

In view of (4.9) we conclude that the solution sequence $\{y_n\}_n$ is bounded in the space $L^2(0,T;L^2(\Omega))$. Thus we can extract a subsequence y_{n_k} such that $y_{n_k} \to y$ weakly in $L^2(0,T;L^2(\Omega))$ and hence (4.6) is satisfied.

Next to prove (4.5), let $\psi \in X(0,T)$ and $\psi(\cdot,T) = 0$. Multiplying (4.8) by ψ and integrating by parts over Ω_T, we obtain

$$\begin{aligned} \int_0^T \Big(\int_\Omega g\psi \, \omega_n(x) \, dx \Big) d\tau + \int_{\Omega_T} q\psi \, dx d\tau = -\int_{\Omega_T} y_n \psi_t \, dx d\tau \\ + \int_{\Omega_T} \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial y_n}{\partial x_i} \frac{\partial \psi}{\partial x_j} + a_0 y_n \psi \Big) dx d\tau. \end{aligned} \tag{4.10}$$

Passing to the limit in (4.10) we get (4.5). Finally, the uniqueness follows from the fact that the only solution for the zero data of (4.5) is $y = 0$.

Furthermore, the inclusion $W^{1,p}(\Omega) \subset \mathcal{M}(\overline{\Omega}) \subset (W^{1,p'}(\Omega))'$ and [16, Theorem 6.3] imply the existence of a unique solution $y \in L^1(0,T;W^{1,p}(\Omega))$, $\forall \, p \in [1, \frac{d}{d-1})$ and $y_t \in L^1(0,T;(W^{1,p'}(\Omega))')$ in the sense of (4.5), such that (4.7) is satisfied. Thus, $y \in \mathcal{C}([0,T];(W^{1,p'}(\Omega))')$ after a modification on a set of zero measure. $\square$

The weak form of POCP with measure data (4.1)-(4.3) is defined as follows:

$$\min_{q\in Q_{ad}^P} \tilde{J}(q,y) = \frac{1}{2}\int_0^T \Big(\|y-y_{\tilde{d}}\|^2_{L^2(\Omega)} + \tilde{\alpha}\|q\|^2_{L^2(\Omega)}\Big)\, d\tau \tag{4.11}$$

subject to

$$-(y,v_t)_{\Omega_T} + (y,\mathcal{A}^*v)_{\Omega_T} =< \mu, v >_{\Omega_T} + (y_0, v(\cdot,0)) + (q,v)_{\Omega_T}, \quad \forall v\in X(0,T). \tag{4.12}$$

In the next step, we introduce the reduced cost functional $\tilde{j} : L^2(0,T;L^2(\Omega)) \to \mathbb{R}$ by

$$\tilde{j}(q) := \tilde{J}(q, y(q)),$$

where $\tilde{J}$ is the cost function given by (4.11) and $y(q)$ is the weak solution of (4.12). Then the optimal control problem (4.11)-(4.12) can be equivalently reformulated as

$$\min_{q\in Q_{ad}^P} \tilde{j}(q). \tag{4.13}$$

The following theorem ensures the existence and uniqueness of the solution of the minimization problem (4.13).

Theorem 4.2.2. *The optimal control problem (4.13) admits a unique solution.*

Proof. Let q_n be the minimizing sequences, i.e., $\tilde{j}(q_n) \to \inf_{q\in Q_{ad}^P} \tilde{j}(q)$. By virtue of the term $\tilde{\alpha}\|q\|_{L^2(L^2(\Omega))}$ in $\tilde{j}(q)$, q_n is bounded in Q_{ad}^P and let $y(q_n) = y_n$. From the *a priori* bound (4.6), we have

$$\|y_n\|_{L^2(L^2(\Omega))} \le C,$$

where

$$C = C\Big(\|g\|_{L^2(L^\infty(\Omega))},\ \|\omega\|_{\mathcal{M}(\Omega)},\ \|q\|_{L^2(L^2(\Omega)},\ \|y_0\|_{L^2(\Omega)}\Big).$$

Since $L^2(0,T;L^2(\Omega))$ is a Hilbert spaces and the dual space of $L^2(0,T;L^2(\Omega))$ is itself, it follows that y_n has a subsequence, denoted by y_{n_k}, such that $y_{n_k} \to y$ strongly in $L^2(0,T;L^2(\Omega))$. For $v \in \mathcal{D}(\Omega_T)$, we have

$$\begin{aligned}
(q_{n_k}, v)_{\Omega_T} &= -(y_{n_k}, v_t)_{\Omega_T} + (y_{n_k}, \mathcal{A}^*v)_{\Omega_T} - < \mu, v >_{\Omega_T} - (y_{n_k}(\cdot,0), v(\cdot,0)) \\
&= -(y_{n_k} - y, v_t)_{\Omega_T} + (y_{n_k} - y, \mathcal{A}^*v)_{\Omega_T} - (y_{n_k}(\cdot,0) - y(\cdot,0), v(\cdot,0)) \\
&\quad -(y, v_t)_{\Omega_T} + (y,\mathcal{A}^*v)_{\Omega_T} - < \mu, v >_{\Omega_T} - (y(\cdot,0), v(\cdot,0)).
\end{aligned}$$

As $n\to\infty$, $y_{n_k} \to y$ strongly and $(q_{n_k}, v)_{\Omega_T} \to (q,v)_{\Omega_T}$. Then, $q_{n_k} \to q$ in $\mathcal{D}'(\Omega_T)$ and consequently y satisfies (4.12). Hence $y = y(q)$ and $\liminf \tilde{j}(q_{n_k}) \ge \tilde{j}(q)$ and therefore, q is the optimal control. □

In the following theorem, we state the necessary optimality condition for the control problem (4.11)-(4.12).

Theorem 4.2.3. *Let* $(q, y) \in Q_{ad}^P \times L^2(0,T;L^2(\Omega))$ *be a unique solution of (4.11)-(4.12). Then there exists a unique co-state* $z \in X(0,T)$ *satisfying*

$$\begin{cases} -z_t + \mathcal{A}^* z = y - y_{\tilde{d}} \quad in \ \ \Omega \times [0,T), \\ z(\cdot, T) = 0 \quad in \ \ \Omega, \\ z = 0 \quad on \ \ \Gamma_T. \end{cases} \tag{4.14}$$

Moreover,

$$\tilde{j}'(q)(\hat{q} - q) = \int_0^T (\tilde{\alpha} q + z, \hat{q} - q)\, d\tau \geq 0, \qquad \forall \hat{q} \in Q_{ad}^P. \tag{4.15}$$

The optimality condition (4.15) can be equivalently written in the form [60, 86]

$$q = P_{[q_c, q_d]}\Big(-\frac{1}{\tilde{\alpha}} z(q) \Big), \tag{4.16}$$

where $P_{[q_c,q_d]}$ denotes the projection of $\mathbb{R}$ onto $[q_c, q_d]$ defined as

$$P_{[q_c,q_d]}(q) := max(q_c, min(q_d, q)).$$

The second derivative $\tilde{j}''(q)(\cdot,\cdot)$ is independent of q and positive definite, i.e.,

$$\tilde{j}''(q)(r,r) \geq \gamma_P \|r\|^2_{L^2(L^2(\Omega))}, \quad \forall r \in L^2(0,T;L^2(\Omega)), \tag{4.17}$$

where $\gamma_P > 0$ is a positive constant.

Using (4.16), we deduce the regularity results summarized below.

Theorem 4.2.4. *Let* $(q, y) \in Q_{ad}^P \times L^2(0,T;L^2(\Omega))$ *be the unique solution of (4.11)-(4.12), and* z *be the unique solution of (4.14). Then, we have*

$$\begin{aligned} & y \in L^1(0,T;W_0^{1,p}(\Omega)) \cap L^2(0,T;L^2(\Omega)) \cap \mathcal{C}([0,T];(W^{1,p'}(\Omega))'), \\ & z \in X(0,T) \quad and \quad q \in X(0,T) \cap L^\infty(0,T;L^\infty(\Omega)). \end{aligned}$$

Proof. It follows from Theorem 4.2.1 that

$$y \in L^1(0,T;W_0^{1,p}(\Omega)) \cap L^2(0,T;L^2(\Omega)) \cap \mathcal{C}([0,T];(W^{1,p'}(\Omega))').$$

By standard regularity results for the parabolic equation, we have $z \in X(0,T)$. Now the embedding theorem gives $z \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$, thus the control constraints (4.3) and the property (4.16) imply the stated regularity for the optimal control q. □

4.3 Spatially discrete approximations of POCP

In this section, we consider the continuous time finite element approximation of (4.11)-(4.12) and derive error estimates for the state, co-state and control variables for spatially discrete optimization problem.

Let $\mathcal{T}_h$ be the triangulation of Ω as described in Section 2.3 of Chapter 2. Associated with $\mathcal{T}_h$, let V_h be a finite dimensional subspace of $\mathcal{C}(\overline{\Omega})$, consisting of piecewise linear polynomials. We denote $V_h^0 = V_h \cap H_0^1(\Omega)$, and let

$$Q_h^P := \{\hat{q}_h(t) \in Q_{ad}^P : \ \hat{q}_h(t)|_K = constant, \ \forall \ K \in \mathcal{T}_h, \ t \in (0,T]\}.$$

Since $\mathcal{T}_h$ is quasi-uniform, the following inverse estimates hold for all $v_h \in V_h$ [24]:

$$\|v_h\|_{H^{s_2}(\Omega)} \leq Ch^{s_1-s_2}\|v_h\|_{H^{s_1}(\Omega)}, \quad 0 \leq s_1 \leq s_2 \leq 1, \tag{4.18}$$

$$\|v_h\|_{L^\infty(\Omega)} \leq Ch^{-\frac{d}{2}}\|v_h\|_{L^2(\Omega)}, \tag{4.19}$$

$$\|v_h\|_{L^\infty(\Omega)} \leq C\rho(d,h)\|v_h\|_{H^1(\Omega)}, \tag{4.20}$$

where

$$\rho(d,h) = \begin{cases} \sqrt{|log\, h|}, & d = 2, \\ h^{-\frac{1}{2}}, & d = 3. \end{cases}$$

In the following lemma, we recall the approximation results from [24] and [77]. Let $\mathcal{L}_h : L^2(\Omega) \to V_h$ be the L^2-projection operator defined by

$$(\mathcal{L}_h v - v, v_h) = 0, \qquad \forall v_h \in V_h, \tag{4.21}$$

and let $\mathcal{R}_h : H_0^1(\Omega) \to V_h^0$ denote the Ritz projection operator defined as

$$a(\mathcal{R}_h v - v, v_h) = 0, \qquad \forall v_h \in V_h^0. \tag{4.22}$$

Lemma 4.3.1. *Let $\mathcal{L}_h$ and $\mathcal{R}_h$ be defined in (4.21) and (4.22), respectively. Then, we have*

$$\|v - \mathcal{L}_h v\|_{H^{-1}(\Omega)} + h\|v - \mathcal{L}_h v\|_{L^2(\Omega)} \leq Ch^2\|v\|_{H^1(\Omega)},$$

$$\|v - \mathcal{R}_h v\|_{L^2(\Omega)} + h\|v - \mathcal{R}_h v\|_{H^1(\Omega)} \leq Ch^2\|v\|_{H^2(\Omega)}.$$

Moreover,

$$\|v - \mathcal{R}_h v\|_{L^\infty(\Omega)} \leq Ch^{2-\frac{d}{2}}\|v\|_{H^2(\Omega)}.$$

Note that the solution of the problem (4.12) exhibits low regularity. To estimate error between the solution of the continuous problem and the spatially discrete problem in the $L^2(0,T;L^2(\Omega))$-norm, we introduce the spatially discrete finite element approximation of the forward and backward parabolic problems (1.13) and (1.14):

$$\begin{cases}(\phi_{h,t}, v_h)_{\Omega_T} + a(\phi_h, v_h)_{\Omega_T} = (f, v_h)_{\Omega_T}, & \forall v_h \in V_h^0,\\ \phi_h(\cdot,0) = 0, \quad \forall v_h \in V_h^0, \end{cases} \tag{4.23}$$

and

$$\begin{cases}-(\psi_{h,t}, v_h)_{\Omega_T} + a(\psi_h, v_h)_{\Omega_T} = (f, v_h)_{\Omega_T}, & \forall v_h \in V_h^0,\\ \psi_h(\cdot,T) = 0, \quad \forall v_h \in V_h^0, \end{cases} \tag{4.24}$$

where $\phi_h(t)$, $\psi_h(t) \in H^1(0,T;V_h^0)$.

From (1.13), (4.23), (1.14) and (4.24), we have the following error equations

$$(\phi_t - \phi_{h,t}, v_h)_{\Omega_T} + a(\phi - \phi_h, v_h)_{\Omega_T} = 0, \quad \forall v_h \in V_h^0, \tag{4.25}$$

and

$$-(\psi_t - \psi_{h,t}, v_h)_{\Omega_T} + a(\psi - \psi_h, v_h)_{\Omega_T} = 0, \quad \forall v_h \in V_h^0. \tag{4.26}$$

We now prove some error estimates associated with the problems (1.13), (4.23), (1.14) and (4.24), which will play a crucial role in the derivation of our main results.

Lemma 4.3.2. *Let $\psi \in X(0,T) \hookrightarrow \mathcal{C}([0,T];H_0^1(\Omega))$ and $\psi_h(t) \in H^1(0,T;V_h^0)$ be the solutions of (1.14) and (4.24), respectively. Then, we have the following error estimates:*

$$\|\psi(t) - \psi_h(t)\|_{L^\infty(L^2(\Omega))} \leq Ch\Big(\|\psi\|_{L^2(H^2(\Omega))} + \|\psi_t\|_{L^2(L^2(\Omega))}\Big), \tag{4.27}$$

$$\|\psi(t) - \psi_h(t)\|_{L^2(L^2(\Omega))} \leq Ch^2\Big(\|\psi\|_{L^2(H^2(\Omega))} + \|\psi_t\|_{L^2(L^2(\Omega))}\Big), \tag{4.28}$$

and

$$\|\psi(t) - \psi_h(t)\|_{L^2(L^\infty(\Omega))} \leq Ch^{2-\frac{d}{2}}\Big(\|\psi\|_{L^2(H^2(\Omega))} + \|\psi_t\|_{L^2(L^2(\Omega))}\Big). \tag{4.29}$$

Similarly, $\phi \in X(0,T)) \hookrightarrow \mathcal{C}([0,T];H_0^1(\Omega))$ and $\phi_h(t) \in H^1(0,T;V_h^0)$ be the solutions of (1.13) and (4.23), respectively. Then,

$$\|\phi(t) - \phi_h(t)\|_{L^2(L^2(\Omega))} \leq Ch^2\Big(\|\phi\|_{L^2(H^2(\Omega))} + \|\phi_t\|_{L^2(L^2(\Omega))}\Big). \tag{4.30}$$

Proof. Following the arguments as in [22] it is not difficult to prove the following *a priori* error estimates under minimal regularity assumptions for the backward parabolic equations

$$\|\psi(t)-\psi_h(t)\|_{L^2(\Omega)}+\|\psi-\psi_h\|_{L^2(H^1(\Omega))}\leq Ch\Big(\|\psi\|_{L^2(H^2(\Omega))}+\|\psi_t\|_{L^2(L^2(\Omega))}\Big)$$

and

$$\|\psi-\psi_h\|_{L^2(L^2(\Omega))}\leq Ch^2\Big(\|\psi\|_{L^2(H^2(\Omega))}+\|\psi_t\|_{L^2(L^2(\Omega))}\Big).$$

In [22], the L^2-projection play a key role instead of the Ritz-projection used in the other literatures like [67, 85]. To prove (4.29), let $\pi_h\psi$ be the piecewise linear interpolant of ψ defined in Lemma 2.3.1. Then, using (4.19) we obtain

$$\begin{aligned}\|\psi-\psi_h\|_{L^2(L^\infty(\Omega))} &\leq \|\psi-\pi_h\psi\|_{L^2(L^\infty(\Omega))}+\|\pi_h\psi-\psi_h\|_{L^2(L^\infty(\Omega))}\\ &\leq Ch^{2-\frac{d}{2}}\|\psi\|_{L^2(H^2(\Omega))}+Ch^{-\frac{d}{2}}\|\pi_h\psi-\psi_h\|_{L^2(L^2(\Omega))}\\ &\leq Ch^{2-\frac{d}{2}}\|\psi\|_{L^2(H^2(\Omega))}+Ch^{-\frac{d}{2}}\|\psi-\psi_h\|_{L^2(L^2(\Omega))},\end{aligned}\tag{4.31}$$

where we have used Lemma 2.3.1. Now, (4.31) together with (4.28) implies (4.29). Similarly, we have the results for the forward parabolic problem (1.13) and (4.23). ☐

We now define the spatially discrete finite element approximation of (4.11)-(4.12) as follows: Find a pair $(q_h, y_h) : [0,T] \to Q_h^P \times V_h^0$ such that

$$\min_{q_h\in Q_h^P}\tilde{J}(q_h,y_h)=\frac{1}{2}\int_0^T\Big\{\|y_h-y_{\tilde{d}}\|^2_{L^2(\Omega)}+\tilde{\alpha}\|q_h\|^2_{L^2(\Omega)}\Big\}\,d\tau\tag{4.32}$$

subject to

$$-(y_h,v_{h,t})_{\Omega_T}+a(y_h,v_h)_{\Omega_T}=<\mu,v_h>_{\Omega_T}+(q_h,v_h)_{\Omega_T}+(y_{h,0},v_h(\cdot,0)),\tag{4.33}$$

$\forall v_h\in H^1(0,T;V_h^0)$ with $y_{h,0}=\mathcal{L}_h y_0$ and $v_h(\cdot,T)=0$. Here

$$<\mu,v_h>_{\Omega_T}=\int_{\Omega_T}v_h\,d\mu=\int_0^T\Big(\int_\Omega g(x,\tau)v_h(x)\,d\omega(x)\Big)\,d\tau,\qquad\forall v_h\in H^1(0,T;V_h^0).$$

Analogous to Theorem 4.2.1, we have the following stability result.

Lemma 4.3.3. *Assume that $\mu = g\omega$, g and ω are given functions such that $g\in L^2(0,T;\mathcal{C}(\overline{\Omega}))$, $\omega\in\mathcal{M}(\Omega)$, $y_0\in L^2(\Omega)$ and $q\in L^2(0,T;L^2(\Omega))$. Let $y_h(q)\in L^2(0,T;V_h^0)$ be the unique solution of*

$$-(y_h(q),v_{h,t})_{\Omega_T}+a(y_h(q),v_h)_{\Omega_T}=<\mu,v_h>_{\Omega_T}+(q,v_h)_{\Omega_T}+(y_{h,0}(q),v_h(\cdot,0)),$$

$\forall v_h\in H^1(0,T;V_h^0)$ with $v_h(\cdot,T)=0$ and $y_{h,0}(q)=\mathcal{L}_h y_0$. Then

$$\|y_h(q)\|_{L^2(L^2(\Omega))}\leq C\Big(\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}+\|q\|_{L^2(L^2(\Omega))}+\|y_{h,0}(q)\|_{L^2(\Omega)}\Big).$$

Similar to the continuous case, the problem (4.32)-(4.33) admits a unique solution (q_h, y_h) if and only if there exists a co-state variable z_h such that the triplet (y_h, q_h, z_h) satisfies the following optimality conditions for all $v_h \in H^1(0,T;V_h^0)$:

$$\begin{aligned} -(y_h, v_{h,t})_{\Omega_T} + a(y_h, v_h)_{\Omega_T} &= <\mu, v_h>_{\Omega_T} + (q_h, v_h)_{\Omega_T} \\ &\quad + (y_{h,0}, v_h(\cdot, 0)), &(4.34)\\ -(z_{h,t}, v_h)_{\Omega_T} + a(z_h, v_h)_{\Omega_T} &= (y_h - y_{\tilde{d}}, v_h)_{\Omega_T}, &(4.35)\\ z_h(\cdot, T) &= 0, &(4.36)\\ (\tilde{\alpha} q_h + z_h, \hat{q}_h - q_h) &\geq 0, \ \ \forall \hat{q}_h \in Q_h^P. &(4.37)\end{aligned}$$

We introduce the discrete reduced cost functional $\tilde{j}_h : L^2(0,T;L^2(\Omega)) \to \mathbb{R}$ by

$$\tilde{j}_h(q) := \tilde{J}(q, y_h(q)). \tag{4.38}$$

Then, the discrete optimal control problem (4.32)-(4.33) can be rewritten as

$$\min_{q_h \in Q_h^P} \tilde{j}_h(q_h). \tag{4.39}$$

The derivative of the discrete reduced cost functional is given by

$$\tilde{j}_h'(q_h)(\hat{q}_h - q_h) = \int_0^T (\tilde{\alpha} q_h + z_h, \hat{q}_h - q_h)\, d\tau \geq 0, \qquad \forall \hat{q}_h \in Q_h^P. \tag{4.40}$$

For the purpose of error analysis it is convenient to introduce the following two auxiliary problems: For $q \in Q_{ad}^P$, find a pair $(y_h(q), z_h(q)) \in L^2(0,T;V_h^0) \times H^1(0,T;V_h^0)$ satisfying

$$\begin{aligned} -(y_h(q), v_{h,t})_{\Omega_T} + a(y_h(q), v_h)_{\Omega_T} &= <\mu, v_h>_{\Omega_T} + (q, v_h)_{\Omega_T} \\ &\quad + (y_{h,0}(q), v_h(\cdot, 0)), &(4.41)\\ -(z_{h,t}(q), v_h)_{\Omega_T} + a(z_h(q), v_h)_{\Omega_T} &= (y_h(q) - y_{\tilde{d}}, v_h)_{\Omega_T}, &(4.42)\\ z_h(q)(\cdot, T) &= 0, &(4.43)\end{aligned}$$

$\forall v_h \in H^1(0,T;V_h^0)$.

The following lemma provide some auxiliary error estimate for the state variable.

Lemma 4.3.4. *Assume that $\mu = g\omega$ with $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$, $\omega \in \mathcal{M}(\Omega)$ and $q \in L^2(0,T;L^2(\Omega))$. Let y and $y_h(q)$ be the solutions of (4.12) and (4.41), respectively. Then, we have*

$$\|y - y_h(q)\|_{L^2(L^2(\Omega))} \leq Ch^{2-\frac{d}{2}} \Big(\|g\|_{L^2(L^\infty(\Omega))} \|\omega\|_{\mathcal{M}(\Omega)} + \|y_0\|_{L^2(\Omega)} + \|q\|_{L^2(L^2(\Omega))} \Big).$$

Proof. Let ψ be the solution of (1.14) with $f \in L^2(0,T;L^2(\Omega))$. Then, from (4.12) together with (4.26), (4.41) and (4.21), we have

$$\begin{aligned}
\int_{\Omega_T}(y-y_h(q))f\,dxd\tau &= \int_0^T\int_\Omega (y-y_h(q))(-\psi_t+\mathcal{A}^*\psi)\,dxd\tau \\
&= -(y,\psi_t)_{\Omega_T}+(y,\mathcal{A}^*\psi)_{\Omega_T}+(y_h(q),\psi_t)_{\Omega_T}-a(y_h(q),\psi)_{\Omega_T} \\
&= <\mu,\psi>_{\Omega_T}+(y_0,\psi(\cdot,0))+(q,\psi)_{\Omega_T}+(y_h(q),\psi_{h,t})_{\Omega_T} \\
&\quad -a(y_h(q),\psi_h)_{\Omega_T} \\
&= <\mu,\psi>_{\Omega_T}+(y_0,\psi(\cdot,0))+(q,\psi)_{\Omega_T}-<\mu,\psi_h>_{\Omega_T} \\
&\quad -(y_{h,0}(q),\psi_h(\cdot,0))-(q,\psi_h)_{\Omega_T} \\
&= <\mu,\psi-\psi_h>_{\Omega_T}+(y_0,\psi(\cdot,0)-\psi_h(\cdot,0))+(q,\psi-\psi_h)_{\Omega_T} \\
&= \int_0^T\Big(\int_\Omega g(x,\tau)(\psi-\psi_h)\,d\omega(x)\Big)\,d\tau+(y_0,\psi(\cdot,0)-\psi_h(\cdot,0)) \\
&\quad +(q,\psi-\psi_h)_{\Omega_T} \\
&\le \Big(\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}\|\psi-\psi_h\|_{L^2(L^\infty(\Omega))} \\
&\quad +\|y_0\|_{L^2(\Omega)}\|\psi-\psi_h\|_{\mathcal{C}([0,T];L^2(\Omega))}+\|q\|_{L^2(L^2(\Omega))}\|\psi-\psi_h\|_{L^2(L^2(\Omega))}\Big).
\end{aligned}$$

Using Lemmas 1.2.3 and 4.3.2, we obtain

$$\begin{aligned}
\int_{\Omega_T}(y-y_h(q))f\,dxd\tau &\le \Big(Ch^{2-\frac{d}{2}}\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}+Ch\|y_0\|_{L^2(\Omega)} \\
&\quad +Ch^2\|q\|_{L^2(L^2(\Omega))}\Big)\Big(\|\psi\|_{L^2(H^2(\Omega))}+\|\psi_t\|_{L^2(L^2(\Omega))}\Big) \\
&\le Ch^{2-\frac{d}{2}}\Big(\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}+\|y_0\|_{L^2(\Omega)} \\
&\quad +\|q\|_{L^2(L^2(\Omega))}\Big)\|f\|_{L^2(L^2(\Omega))}.
\end{aligned}$$

Finally, the definition of $L^2(0,T;L^2(\Omega))$-norm gives the desired estimate. □

In the following lemma we derive an auxiliary error estimate for the co-state variable.

Lemma 4.3.5. *Let z and $z_h(q)$ be the solutions of (4.14) and (4.42)-(4.43), respectively. Then, we have*

$$\|z-z_h(q)\|_{L^2(L^2(\Omega))}\le Ch^2\Big(\|y\|_{L^2(L^2(\Omega))}+\|y_{\tilde d}\|_{L^2(L^2(\Omega))}\Big)+\|y-y_h(q)\|_{L^2(L^2(\Omega))}.$$

Proof. Let ϕ be the solution of (1.13) with $f \in L^2(0,T;L^2(\Omega))$. Then, multiply (4.14) by ϕ and form an L^2-inner product over Ω_T. Then, using Green's formula together with

(4.25), (4.42) and (4.43), we obtain

$$\begin{aligned}\int_{\Omega_T}(z - z_h(q))f\,dxd\tau &= \int_0^T\int_\Omega (z-z_h(q))(\phi_t+\mathcal{A}\phi)\,dxd\tau\\ &= (z,\phi_t)_{\Omega_T}+a(z,\phi)_{\Omega_T}-(z_h(q),\phi_t)_{\Omega_T}-a(z_h(q),\phi)_{\Omega_T}\\ &= -(z_t,\phi)_{\Omega_T}+a(z,\phi)_{\Omega_T}-(z_h(q),\phi_{h,t})_{\Omega_T}-a(z_h(q),\phi_h)_{\Omega_T}\\ &= (y-y_{\tilde d},\phi)_{\Omega_T}+(z_{h,t}(q),\phi_h)_{\Omega_T}-a(z_h(q),\phi_h)_{\Omega_T}\\ &= (y-y_{\tilde d},\phi)_{\Omega_T}-(y_h(q)-y_{\tilde d},\phi_h)_{\Omega_T}\\ &= (y-y_{\tilde d},\phi-\phi_h)_{\Omega_T}+(y-y_h(q),\phi_h)_{\Omega_T}\\ &\le \|y-y_{\tilde d}\|_{L^2(L^2(\Omega))}\|\phi-\phi_h\|_{L^2(L^2(\Omega))}+\|y-y_h(q)\|_{L^2(L^2(\Omega))}\|\phi_h\|_{L^2(L^2(\Omega))}.\end{aligned}$$

An application of Lemmas 1.2.3 and 4.3.2 yields

$$\begin{aligned}\int_{\Omega_T}(z-z_h(q))f\,dxd\tau \;&\le\; Ch^2\Big(\|y\|_{L^2(L^2(\Omega))}+\|y_{\tilde d}\|_{L^2(L^2(\Omega))}\Big)\|f\|_{L^2(L^2(\Omega))}\\ &\quad+\|\phi_h\|_{L^2(L^2(\Omega))}\|y-y_h(q)\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Using the stability result $\|\phi_h\|_{L^2(L^2(\Omega))}\le C\|f\|_{L^2(L^2(\Omega))}$ and the definition of $L^2(0,T;L^2(\Omega))$-norm yields the desired estimate. □

In the next lemma, we present the difference between the derivative of continuous reduced cost functional and discrete reduced cost functional.

Lemma 4.3.6. *Let $\tilde{j}'(q)(r)$ and $\tilde{j}'_h(q)(r)$ be given by (4.15) and (4.40) with $q_h = q$ respectively. Then*

$$|\tilde{j}'(q)(r)-\tilde{j}'_h(q)(r)|\le \hat{C}_1 h^{2-\frac{d}{2}}\|r\|_{L^2(L^2(\Omega))},\qquad \forall r\in L^2(0,T;L^2(\Omega)),$$

where

$$\hat{C}_1 = C\Big(\|g\|_{L^2(L^\infty(\Omega))},\ \|\omega\|_{\mathcal{M}(\Omega)},\ \|y_0\|_{L^2(\Omega)},\ \|q\|_{L^2(L^2(\Omega))},\|y_{\tilde d}\|_{L^2(L^2(\Omega))}\Big). \tag{4.44}$$

Proof. Using (4.15) and (4.40) with $q_h = q$, we have

$$\begin{aligned}|\tilde{j}'(q)(r)-\tilde{j}'_h(q)(r)| \;&\le\; \Big|\int_0^T (z-z_h(q),r)\,d\tau\Big|\\ &\le\; \|z-z_h(q)\|_{L^2(L^2(\Omega))}\|r\|_{L^2(L^2(\Omega))}.\end{aligned}$$

An application of Lemma 4.3.5 gives the desired estimate. This completes the proof. □

In the following theorem, we write the error between the continuous control q and the discrete control q_h.

Theorem 4.3.1. *Let q and q_h be the optimal controls of (4.13) and (4.39), respectively. Assume that the second order optimality condition (4.17) is valid. Then the following error estimate holds:*

$$\|q - q_h\|_{L^2(L^2(\Omega))} \leq \frac{\tilde{C}_1}{\sqrt{\gamma_P}} h + \frac{\hat{C}_1}{\gamma_P} h^{2-\frac{d}{2}},$$

where $\hat{C}_1$ is given by (4.44) and

$$\tilde{C}_1 = C\Big(\|g\|_{L^2(L^\infty(\Omega))},\ \|\omega\|_{\mathcal{M}(\Omega)},\ \|y_0\|_{L^2(\Omega)},\ \|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}, \tilde{\alpha}\Big). \tag{4.45}$$

Proof. With $r \in L^2(0,T;L^2(\Omega))$, we have

$$\tilde{j}''(q)(r,r) \geq \gamma_P \|r\|^2_{L^2(L^2(\Omega))} \tag{4.46}$$

and

$$\tilde{j}''_h(q_h)(r,r) \geq \gamma_P \|r\|^2_{L^2(L^2(\Omega))}. \tag{4.47}$$

We now formulate the following auxiliary problem:

$$\min_{q_h \in Q_h^P} \tilde{j}(q_h), \tag{4.48}$$

where we only discretize the control variable. Suppose $\tilde{q}_h$ be the solution of problem (4.48). We now decompose the error as

$$q(t) - q_h(t) = (q(t) - \tilde{q}_h(t)) + (\tilde{q}_h(t) - q_h(t)), \tag{4.49}$$

and proceed to estimate each term separately. In view of (4.46), we have for $\tilde{\lambda} \in [0,1]$ with $\xi = \tilde{\lambda} q + (1-\tilde{\lambda})\tilde{q}_h$ and h sufficiently small,

$$\begin{aligned} \gamma_P \|q - \tilde{q}_h\|^2_{L^2(L^2(\Omega))} &\leq \tilde{j}''(\xi)(q - \tilde{q}_h, q - \tilde{q}_h) \\ &= \tilde{j}'(q)(q - \tilde{q}_h) - \tilde{j}'(\tilde{q}_h)(q - \tilde{q}_h) \\ &= \tilde{j}'(q)(q - \tilde{q}_h) - \tilde{j}'(\tilde{q}_h)(q - \mathcal{L}_h q) - \tilde{j}'(\tilde{q}_h)(\mathcal{L}_h q - \tilde{q}_h). \end{aligned}$$

The necessary optimality condition imply, for h sufficiently small,

$$\tilde{j}'(q)(q - \tilde{q}_h) \leq 0 \quad \text{and} \quad -\tilde{j}'(\tilde{q}_h)(\mathcal{L}_h q - \tilde{q}_h) \leq 0,$$

which together with the properties of $\mathcal{L}_h$ and the Young's inequality yields

$$\begin{aligned} \gamma_P \|q - \tilde{q}_h\|^2_{L^2(L^2(\Omega))} &\leq -\tilde{j}'(\tilde{q}_h)(q - \mathcal{L}_h q) \\ &= -\int_0^T \Big(\tilde{\alpha}\tilde{q}_h + z(\tilde{q}_h), q - \mathcal{L}_h q\Big)\, d\tau \\ &= -\int_0^T (z(\tilde{q}_h) - \mathcal{L}_h z(\tilde{q}_h), q - \mathcal{L}_h q)\, d\tau \\ &\leq \int_0^T \Big(\frac{1}{2}\|z(\tilde{q}_h) - \mathcal{L}_h z(\tilde{q}_h)\|^2_{L^2(\Omega)} + \frac{1}{2}\|q - \mathcal{L}_h q\|^2_{L^2(\Omega)}\Big)\, d\tau. \end{aligned}$$

Therefore, we have

$$\|q-\tilde{q}_h\|_{L^2(L^2(\Omega))} \leq \int_0^T \Big(\frac{C}{\sqrt{\gamma_P}}\|z(\tilde{q}_h)-\mathcal{L}_h z(\tilde{q}_h)\|_{L^2(\Omega)} + \frac{C}{\sqrt{\gamma_P}}\|q-\mathcal{L}_h q\|_{L^2(\Omega)}\Big)\, d\tau.$$

An application of Lemma 4.3.1 yields

$$\|q-\tilde{q}_h\|_{L^2(L^2(\Omega))} \leq \int_0^T \Big(\frac{C}{\sqrt{\gamma_P}}h\|z(\tilde{q}_h)\|_{H^1(\Omega)} + \frac{C}{\sqrt{\gamma_P}}h\|q\|_{H^1(\Omega)}\Big)\, d\tau \leq \frac{\tilde{C}_1}{\sqrt{\gamma_P}}h,$$

where $\tilde{C}_1$ is given by (4.45). To estimate the second term in (4.49), we use the necessary optimality condition (4.40) which leads to the following relation:

$$\tilde{j}'_h(q_h)(q_h-r_h) \leq 0 \leq j'(\tilde{q}_h)(r_h-\tilde{q}_h), \qquad \forall r_h \in Q_h^P.$$

With $\xi = \tilde{\lambda}q_h + (1-\tilde{\lambda})\tilde{q}_h$, $\tilde{\lambda} \in [0,1]$ and h sufficiently small, from (4.47) we have

$$\begin{aligned}
\gamma_P\|q_h-\tilde{q}_h\|^2_{L^2(L^2(\Omega))} &\leq \tilde{j}''_h(\xi)(q_h-\tilde{q}_h, q_h-\tilde{q}_h) \\
&= \tilde{j}'_h(q_h)(q_h-\tilde{q}_h) - \tilde{j}'_h(\tilde{q}_h)(q_h-\tilde{q}_h) \\
&\leq \tilde{j}'(\tilde{q}_h)(q_h-\tilde{q}_h) - \tilde{j}'_h(\tilde{q}_h)(q_h-\tilde{q}_h) \\
&\leq \hat{C}_1 h^{2-\frac{d}{2}}\|q_h-\tilde{q}_h\|_{L^2(L^2(\Omega))}.
\end{aligned}$$

The last step follows from Lemma 4.3.6 and $\hat{C}_1$ is given by (4.44). This completes the proof of the theorem. □

Now, we write the error between the continuous and the spatially discrete state variables in the $L^2(0,T;L^2(\Omega))$-norm.

Theorem 4.3.2. *Assume that $\mu = g\omega$, g and ω are given functions such that $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$. Let y and y_h be the solutions of (4.12) and (4.33), respectively. Then, we have*

$$\begin{aligned}
\|y-y_h\|_{L^2(L^2(\Omega))} \leq\ & Ch^{2-\frac{d}{2}}\Big(\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)} + \|y_0\|_{L^2(\Omega)} + \|q\|_{L^2(L^2(\Omega))}\Big) \\
& + \|q-q_h\|_{L^2(L^2(\Omega))}.
\end{aligned}$$

Proof. let ψ be the solution of the problem (1.14) with $f \in L^2(0,T;L^2(\Omega))$. Then from (4.12), (4.21), (4.26) and (4.33), we obtain

$$\begin{aligned}
\int_{\Omega_T}(y-y_h)f\,dxd\tau &= \int_0^T\Big(\int_\Omega g(x,\tau)(\psi-\psi_h)\,d\omega(x)\Big)\,d\tau + (y_0,\psi(\cdot,0)-\psi_h(\cdot,0)) \\
&\quad +(q,\psi)_{\Omega_T} - (q_h,\psi_h)_{\Omega_T} \\
&\leq C\Big(\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}\|\psi-\psi_h\|_{L^2(L^\infty(\Omega))} \\
&\quad +\|y_0\|_{L^2(\Omega)}\|\psi-\psi_h\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|q\|_{L^2(L^2(\Omega))}\|\psi-\psi_h\|_{L^2(L^2(\Omega))} \\
&\quad +\|q-q_h\|_{L^2(L^2(\Omega))}\|\psi_h\|_{L^2(L^2(\Omega))}\Big).
\end{aligned}$$

Applications of Lemmas 1.2.3 and 4.3.2 yield

$$\begin{aligned}\int_{\Omega_T}(y-y_h)f\,dxd\tau &\leq \Big(Ch^{2-\frac{d}{2}}\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}+Ch\|y_0\|_{L^2(\Omega)}+Ch^2\|q\|_{L^2(L^2(\Omega))}\Big)\\ &\quad\times\Big(\|\psi\|_{L^2(H^2(\Omega))}+\|\psi_t\|_{L^2(L^2(\Omega))}\Big)+\|q-q_h\|_{L^2(L^2(\Omega))}\|\psi_h\|_{L^2(L^2(\Omega))}\\ &\leq Ch^{2-\frac{d}{2}}\Big\{\|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}+\|y_0\|_{L^2(\Omega)}+\|q\|_{L^2(L^2(\Omega))}\Big\}\|f\|_{L^2(L^2(\Omega))}\\ &\quad+\|q-q_h\|_{L^2(L^2(\Omega))}\|f\|_{L^2(L^2(\Omega))}.\end{aligned}$$

The definition of $L^2(0,T;L^2(\Omega))$-norm and the stability result $\|\phi_h\|_{L^2(L^2(\Omega))}\leq C\|f\|_{L^2(L^2(\Omega))}$ gives the desired estimate. This completes the proof. □

The following theorem presents the error estimate between the continuous and discrete co-state variables.

Theorem 4.3.3. *Let z and z_h be the solutions of (4.14) and (4.35)-(4.36), respectively. Then*

$$\|z-z_h\|_{L^2(L^2(\Omega))}\leq Ch^2\Big(\|y\|_{L^2(L^2(\Omega))}+\|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}\Big)+\|y-y_h\|_{L^2(L^2(\Omega))}.$$

Proof. Following the lines of arguments of Lemma 4.3.5, it is easy to derive the result and hence, the details are thus omitted. □

4.4 Fully discrete approximations of POCP

In this section, we consider the fully discrete approximation of spatially discrete parabolic optimal control problem (4.32)-(4.33) using the backward Euler scheme in time.

Let $0=t_0<t_1<...<t_{N-1}<t_N=T$ be a partition of $[0,T]$ with time step-size $k:=t_n-t_{n-1}$ and $I_n:=(t_{n-1},t_n]$ for $n=1,2,\dots,N$. For any function $\chi\in C([0,T];L^2(\Omega))$, define $\chi^n:=\chi(x,t_n)$ and $\bar{\partial}\chi^n:=\frac{1}{k}(\chi^n-\chi^{n-1})$. For $n=1,2,\dots,N$, construct the finite element spaces $V_h^n\subset H_0^1(\Omega)$ and $Q_{h,n}^P$ with the mesh $\mathcal{T}_h^n$. For our error analysis, we set $k=\mathcal{O}(h^d)$.

The fully discrete approximation of (4.32)-(4.33) is to find $(q_h^n,y_h^n)\in Q_{h,n}^p\times V_h^n$, $n=1,2,\dots,N$, such that

$$\min_{q_h^n\in Q_{h,n}^P}\tilde{J}(q_h^n,y_h^n)=\frac{1}{2}\sum_{n=1}^N\int_{I_n}\Big\{\|y_h^n-y_{\tilde{d}}^n\|^2_{L^2(\Omega)}+\tilde{\alpha}\|q_h^n\|^2_{L^2(\Omega)}\Big\}d\tau \tag{4.50}$$

subject to

$$\begin{cases}(\bar{\partial}y_h^n,v_h)+a(y_h^n,v_h)=<\mu,v_h>_{I_n}+(q_h^n,v_h), & \forall v_h\in V_h^n,\\ y_h^0(x)=y_{h,0}(x), & x\in\Omega.\end{cases} \tag{4.51}$$

Here,

$$< \mu, v_h >_{I_n} = \frac{1}{k}\int_{\Omega\times(t_{n-1},t_n]} v_h \, d\mu = \frac{1}{k}\int_{t_{n-1}}^{t_n}\left(\int_\Omega g(x,\tau)v_h(x)\, d\omega(x)\right) d\tau, \ \ \forall v_h \in V_h^n.$$

The optimal control problem (4.50)-(4.51) has a unique solution (q_h^n, y_h^n), $n = 1, 2, \ldots, N$, such that $(y_h^n, q_h^n, z_h^{n-1})$ satisfies the following optimality conditions:

$$(\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) = < \mu, v_h >_{I_n} + (q_h^n, v_h), \quad \forall v_h \in V_h^n, \ \ n \geq 1, \tag{4.52}$$

$$y_h^0(x) = y_{h,0}(x), \quad x \in \Omega, \tag{4.53}$$

$$-(\bar{\partial} z_h^n, v_h) + a(z_h^{n-1}, v_h) = (y_h^n - y_{\tilde{d}}^n, v_h), \quad \forall v_h \in V_h^n, \tag{4.54}$$

$$z_h^N(x) = 0, \quad x \in \Omega, \tag{4.55}$$

$$(\tilde{\alpha} q_h^n + z_h^{n-1}, \hat{q}_h^n - q_h^n) \geq 0, \quad \forall \hat{q}_h^n \in Q_{h,n}^P. \tag{4.56}$$

Analogous to the spatially discrete case, we reformulate the fully discrete optimal control problem (4.50)-(4.51) as:

$$\min_{q_h^n \in Q_{h,n}^P} \tilde{j}_h^n(q_h^n) \tag{4.57}$$

for $n = 1, \ldots, N$.

We now proceed to derive error estimates for the fully discrete optimization problem. Let $\mathcal{Y}_h$ and $\mathcal{Z}_h$ be the fully discrete finite element approximations of y and z, respectively. Further, $\mathcal{Y}_h$ and $\mathcal{Z}_h$ are piecewise constant in time and piecewise linear in space on each time interval. To begin with, we first derive the stability estimate for (4.51).

Lemma 4.4.1. *Assume that g and ω are given functions such that $g \in L^2(0, T; \mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$, $y_0 \in L^2(\Omega)$. Let $y_h^n \in V_h^n$, $n = 1, 2, \ldots, N$ be the solutions of fully discrete scheme (4.51) and $y_h^0 = \mathcal{L}_h y_0$. Then there exists a constant C independent of h, k and the data (g, ω, y_0) such that*

$$\sum_{n=1}^N \|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + k\|y_h^N\|^2_{H^1(\Omega)} \leq C\Big(kh^{-2}\|y_0\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(L^\infty(\Omega))}\|\omega\|^2_{\mathcal{M}(\Omega)}\Big) + C\sum_{n=1}^N \|kq_h^n\|^2_{L^2(\Omega)}, \tag{4.58}$$

and

$$\|y_h^N\|^2_{L^2(\Omega)} + \sum_{n=1}^N k\|y_h^n\|^2_{H^1(\Omega)} \leq C\Big(\|y_0\|^2_{L^2(\Omega)} + \rho^2(d,h)\|g\|^2_{L^2(L^\infty(\Omega))}\|\omega\|^2_{\mathcal{M}(\Omega)}\Big) + C\sum_{n=1}^N \|kq_h^n\|^2_{L^2(\Omega)}. \tag{4.59}$$

Proof. Setting $v_h = k(y_h^n - y_h^{n-1})$ in (4.51), we get

$$(y_h^n - y_h^{n-1}, y_h^n - y_h^{n-1}) + ka(y_h^n, y_h^n - y_h^{n-1}) = k < \mu, y_h^n - y_h^{n-1} >_{I_n} + k(q_h^n, y_h^n - y_h^{n-1}).$$

Use of coercivity, continuity of $a(\cdot,\cdot)$ and the Cauchy-Schwarz inequality yields

$$\begin{aligned}\|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + k\beta\|y_h^n\|^2_{H^1(\Omega)} &\leq k\,a(y_h^n, y_h^{n-1}) + \int_{t_{n-1}}^{t_n}\Big(\int_\Omega g(x,\tau)(y_h^n - y_h^{n-1})d\omega(x)\Big)\,d\tau \\ &\quad + k(q_h^n, y_h^n - y_h^{n-1}) \\ &\leq \beta_1\frac{k}{2}\|y_h^n\|^2_{H^1(\Omega)} + \beta_1\frac{k}{2}\|y_h^{n-1}\|^2_{H^1(\Omega)} \\ &\quad + \int_{t_{n-1}}^{t_n}\|y_h^n - y_h^{n-1}\|_{L^\infty(\Omega)}\|g(\tau)\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega)}\,d\tau \\ &\quad + \|kq_h^n\|_{L^2(\Omega)}\|y_h^n - y_h^{n-1}\|_{L^2(\Omega)}.\end{aligned}$$

Apply the inequality $ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$ with $a,\ b \geq 0$ and $\epsilon > 0$, to obtain

$$\begin{aligned}\|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + k\|y_h^n\|^2_{H^1(\Omega)} &\leq Ck\|y_h^{n-1}\|^2_{H^1(\Omega)} + C\|g\|^2_{L^2(t_{n-1},t_n;L^\infty(\Omega))}\|\omega\|^2_{\mathcal{M}(\Omega)} \\ &\quad + C\|kq_h^n\|^2_{L^2(\Omega)},\end{aligned}$$

where we have used the inverse estimate (4.19) to obtain

$$\sqrt{k}\|y_h^n - y_h^{n-1}\|_{L^\infty(\Omega)} \leq C\sqrt{k}h^{-\frac{d}{2}}\|y_h^n - y_h^{n-1}\|_{L^2(\Omega)} \leq C\|y_h^n - y_h^{n-1}\|_{L^2(\Omega)}.$$

Summing over n from $n = 1$ to N and using the inverse estimate (4.18), we get

$$\begin{aligned}\sum_{n=1}^N \|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} &+ k\|y_h^N\|^2_{H^1(\Omega)} \leq Ck\|y_h^0\|^2_{H^1(\Omega)} + C\sum_{n=1}^N \|g\|^2_{L^2(t_{n-1},t_n;L^\infty(\Omega))}\|\omega\|^2_{\mathcal{M}(\Omega)} \\ &+ C\sum_{n=1}^N \|kq_h^n\|^2_{L^2(\Omega)} \\ &\leq Ck\|\mathcal{L}_h y_0\|^2_{H^1(\Omega)} + C\|g\|^2_{L^2(L^\infty(\Omega))}\|\omega\|^2_{\mathcal{M}(\Omega)} + C\sum_{n=1}^N \|kq_h^n\|^2_{L^2(\Omega)} \\ &\leq Ckh^{-2}\|y_0\|^2_{L^2(\Omega)} + C\|g\|^2_{L^2(L^\infty(\Omega))}\|\omega\|^2_{\mathcal{M}(\Omega)} + C\sum_{n=1}^N \|kq_h^n\|^2_{L^2(\Omega)},\end{aligned}$$

which proves (4.58). Similarly, setting $v_h = ky_h^n$ in (4.51), we have

$$(y_h^n - y_h^{n-1}, y_h^n) + ka(y_h^n, y_h^n) = k < \mu, y_h^n >_{I_n} + k(q_h^n, y_h^n),$$

and hence,

$$\begin{aligned}(y_h^n, y_h^n) + ka(y_h^n, y_h^n) &= (y_h^{n-1}, y_h^n) + \int_{t_{n-1}}^{t_n}\Big(\int_\Omega g(x,\tau)y_h^n\,d\omega(x)\Big)\,d\tau + k(q_h^n, y_h^n) \\ &\leq (y_h^{n-1}, y_h^n) + \int_{t_{n-1}}^{t_n}\|y_h^n\|_{L^\infty(\Omega)}\|g\|_{L^\infty(\Omega)}\|\omega\|_{\mathcal{M}(\Omega))}\,d\tau + k(q_h^n, y_h^n).\end{aligned}$$

Using (4.20), coercive property of the bilinear form and the inequality $ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$ with $a,\ b \geq 0$ and $\epsilon > 0$, we obtain

$$\begin{aligned}\|y_h^n\|^2_{L^2(\Omega)} + k\beta\|y_h^n\|^2_{H^1(\Omega)} &\leq \|y_h^{n-1}\|^2_{L^2(\Omega)} + C\rho^2(d,h)\|g\|^2_{L^2(t_{n-1},t_n;L^\infty(\Omega))}\|\omega\|^2_{\mathcal{M}(\Omega))} \\ &\quad +\|kq_h^n\|^2_{L^2(\Omega)}.\end{aligned}$$

Summation over n from $n=1$ to N and use of the fact $y_h^0 = \mathcal{L}_h y_0$ proves (4.59). □

Now, we introduce some intermediate variables $(y_h^n(q), z_h^{n-1}(q))$ which satisfy the following equations for $n = 1, 2, \ldots, N$.

$$\begin{aligned}(\bar{\partial} y_h^n(q), v_h) + a(y_h^n(q), v_h) &= \ < \mu, v_h >_{I_n} + (q, v_h),\ \ \forall v_h \in V_h^n, &(4.60)\\ y_h^0(q)(x) &= y_{h,0}(x),\ \ x \in \Omega, &(4.61)\\ -(\bar{\partial} z_h^n(q), v_h) + a(z_h^{n-1}(q), v_h) &= (y_h^n(q) - y_{\tilde{d}}^n, v_h),\ \ \forall v_h \in V_h^n, &(4.62)\\ z_h^N(q)(x) &= 0,\ \ x \in \Omega. &(4.63)\end{aligned}$$

The following two lemmas provide some intermediate error estimates of the state and co-state variables which will be useful in the derivation of the main results.

Lemma 4.4.2. *Assume that g and ω are given functions such that $g \in L^2(0,T;\mathcal{C}(\overline{\Omega})) \cap H^{\frac{1}{2}}(0,T;L^\infty(\Omega))$, $\omega \in \mathcal{M}(\Omega)$, $y_0 \in L^2(\Omega)$ and $q \in L^2(0,T;L^2(\Omega))$. Let $y \in L^2(0,T;L^2(\Omega))$ be the solution of (4.12), and let $\mathcal{Y}_h(q)$ be the solution of (4.60)-(4.61). Then, we have*

$$\begin{aligned}\|y - \mathcal{Y}_h(q)\|_{L^2(L^2(\Omega))} &\leq C(h^{2-\frac{d}{2}} + k^{\frac{1}{2}})\Big(\|g\|_{H^{\frac{1}{2}}(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)} \\ &\quad +\|y_0\|_{L^2(\Omega)} + \|q\|_{L^2(L^2(\Omega))}\Big).\end{aligned}$$

Proof. To prove this lemma, we shall use the duality argument. Let ψ be the solution of problem (1.14) with $f \in L^2(0,T;L^2(\Omega))$. Then, from (4.12) and $\psi(\cdot,T)=0$, we have

$$\begin{aligned}
\int_{\Omega_T}(y \;-\; \mathcal{Y}_h(q))f\,dxd\tau &= \int_0^T\int_\Omega (y-\mathcal{Y}_h(q))(-\psi_t+\mathcal{A}^*\psi)\,dxd\tau \\
&= -(y,\psi_t)_{\Omega_T}+(y,\mathcal{A}^*\psi)_{\Omega_T}+\sum_{n=1}^N\int_{I_n}\{(y_h^n(q),\psi_t)-a(y_h^n(q),\psi)\}\,d\tau \\
&= <\mu,\psi>_{\Omega_T}+(y_0,\psi(\cdot,0))+(q,\psi)_{\Omega_T} \\
&\quad +\sum_{n=1}^N\int_{I_n}\{k^{-1}(y_h^n(q),\psi^n-\psi^{n-1})-a(y_h^n(q),\psi)\}\,d\tau \\
&= <\mu,\psi>_{\Omega_T}+(y_0,\psi(\cdot,0))-\sum_{n=1}^N\int_{I_n}\{k^{-1}(y_h^n(q)-y_h^{n-1}(q),\psi^{n-1}) \\
&\quad +a(y_h^n(q),\psi)\}\,d\tau+(y_h^N(q),\psi^N)-(y_h^0(q),\psi(\cdot,0))+(q,\psi)_{\Omega_T} \\
&= -\sum_{n=1}^N\int_{I_n}\{k^{-1}(y_h^n(q)-y_h^{n-1}(q),\psi^{n-1})+a(y_h^n(q),\psi)\}\,d\tau \\
&\quad +<\mu,\psi>_{\Omega_T}+(y_0-y_h^0(q),\psi(\cdot,0))+(q,\psi)_{\Omega_T}.
\end{aligned} \tag{4.64}$$

From (4.60), we note that

$$\sum_{n=1}^N\{(\bar\partial y_h^n(q),\bar{\mathcal{R}}_h\psi)+a(y_h^n(q),\bar{\mathcal{R}}_h\psi)\}=\sum_{n=1}^N<\mu,\bar{\mathcal{R}}_h\psi>_{I_n}+\sum_{n=1}^N(q,\bar{\mathcal{R}}_h\psi), \tag{4.65}$$

where $\bar{\mathcal{R}}_h\psi\in V_h^n$ is defined on I_n as

$$\bar{\mathcal{R}}_h\psi=\bar{\mathcal{R}}_h\psi^n=\frac{1}{k}\int_{I_n}\mathcal{R}_h\psi(\cdot,\tau)\,d\tau,\qquad n>0, \tag{4.66}$$

and $\bar{\mathcal{R}}_h\psi^N=\bar{\mathcal{R}}_h\psi(\cdot,T)$. Let $\bar\psi$ denote the average of ψ in I_n as defined in (4.66) for all $\psi\in L^1(I_n)$. Then, it is easy to see that

$$\int_{I_n}(\psi-\bar\psi)\,d\tau=0. \tag{4.67}$$

Integrate (4.65) in time and add the resulting equation to (4.64). An application of (4.67) yields

$$\begin{aligned}
\int_{\Omega_T}(y \;-\; \mathcal{Y}_h(q))f\,dxd\tau &= -\sum_{n=1}^N\int_{I_n}\{k^{-1}(y_h^n(q)-y_h^{n-1}(q),\psi^{n-1}-\bar{\mathcal{R}}_h\psi) \\
&\quad +a(y_h^n(q),\bar\psi-\bar{\mathcal{R}}_h\psi)\}\,d\tau+\{<\mu,\psi>_{\Omega_T}-\sum_{n=1}^N\int_{I_n}<\mu,\bar{\mathcal{R}}_h\psi>_{I_n}\,d\tau\} \\
&\quad +\{(y_0-y_h^0(q),\psi(\cdot,0))\}+\{(q,\psi)_{\Omega_T}-\sum_{n=1}^N\int_{I_n}(q,\bar{\mathcal{R}}_h\psi)\,d\tau\}, \\
&=: \hat E_1+\hat E_2+\hat E_3+\hat E_4.
\end{aligned} \tag{4.68}$$

Now, we estimate $\hat{E}_i|_{i=1,\dots,4}$. For $\hat{E}_1$, we note that $y_h^n(q) \in V_h^n$ and from the definition of Ritz-projection we have

$$\int_{I_n} a(y_h^n(q), \bar{\psi} - \bar{\mathcal{R}}_h\psi)\, d\tau = 0. \tag{4.69}$$

Then, an application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} |\hat{E}_1| &= \Big| -\sum_{n=1}^{N} \int_{I_n} \{k^{-1}(y_h^n(q) - y_h^{n-1}(q), \psi^{n-1} - \bar{\mathcal{R}}_h\psi)\} d\tau \Big| \\ &\leq \hat{F}_1 \cdot \hat{F}_2, \end{aligned}$$

where

$$\hat{F}_1 = \left(\sum_{n=1}^{N} \|y_h^n(q) - y_h^{n-1}(q)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}, \qquad \hat{F}_2 = \left(\sum_{n=1}^{N} \|\psi^{n-1} - \bar{\mathcal{R}}_h\psi\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

From (4.58) of Lemma 4.4.1 and the fact $k = \mathcal{O}(h^d)$, we have

$$\begin{aligned} \hat{F}_1 &\leq C \left(k^{\frac{1}{2}} h^{-1} \|y_0\|_{L^2(\Omega)} + \|g\|_{L^2(L^\infty(\Omega))} \|\omega\|_{\mathcal{M}(\Omega)} + k\|q\|_{L^2(L^2(\Omega))} \right) \\ &\leq C \left(h^{\frac{d-2}{2}} \|y_0\|_{L^2(\Omega)} + \|g\|_{L^2(L^\infty(\Omega))} \|\omega\|_{\mathcal{M}(\Omega)} + k\|q\|_{L^2(L^2(\Omega))} \right). \end{aligned}$$

Use of Lemma 4.3.2 and some standard error estimate yields

$$\begin{aligned} \|\psi^{n-1} - \bar{\mathcal{R}}_h\psi\|_{L^2(\Omega)} &\leq \|\psi^{n-1} - \bar{\psi}\|_{L^2(\Omega)} + \|\bar{\psi} - \bar{\mathcal{R}}_h\psi\|_{L^2(\Omega)} \\ &\leq \|\psi^{n-1} - \bar{\psi}\|_{L^2(\Omega)} + Ch^2 \|\bar{\psi}\|_{H^2(\Omega)}, \end{aligned}$$

and

$$\|\psi^{n-1} - \bar{\psi}\|_{L^2(\Omega)} \leq k^{\frac{1}{2}} \|\psi_t\|_{L^2(t_{n-1},t_n;L^2(\Omega))}.$$

Since

$$\|\bar{\psi}\|_{H^2(\Omega)} \leq k^{-\frac{1}{2}} \|\psi\|_{L^2(t_{n-1},t_n;H^2(\Omega))},$$

we have

$$\begin{aligned} \hat{F}_2 &\leq C \Big(\sum_{n=1}^{N} \{h^4 \|\bar{\psi}\|^2_{H^2(\Omega)} + k\|\psi_t\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}\} \Big)^{\frac{1}{2}} \\ &\leq C \Big(\sum_{n=1}^{N} \{h^4 k^{-1} \|\psi\|^2_{L^2(t_{n-1},t_n;H^2(\Omega))} + k\|\psi_t\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}\} \Big)^{\frac{1}{2}} \\ &\leq C \Big(\sum_{n=1}^{N} \{h^{4-d} \|\psi\|^2_{L^2(t_{n-1},t_n;H^2(\Omega))} + k\|\psi_t\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}\} \Big)^{\frac{1}{2}} \\ &\leq C \Big(h^{2-\frac{d}{2}} \|\psi\|_{L^2(H^2(\Omega))} + k^{\frac{1}{2}} \|\psi_t\|_{L^2(L^2(\Omega))} \Big). \end{aligned} \tag{4.70}$$

Thus

$$
\begin{aligned}
|\hat{E}_1| &\leq C(h^{2-\frac{d}{2}} + k^{\frac{1}{2}})\Big(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)} \\
&\quad + \|q\|_{L^2(L^2(\Omega))}\Big)\|f\|_{L^2(L^2(\Omega))}. \qquad (4.71)
\end{aligned}
$$

To estimate $\hat{E}_2$, we have

$$
\begin{aligned}
|\hat{E}_2| &= \Big|\sum_{n=1}^{N}\int_\Omega \Big(\int_{t_{n-1}}^{t_n} g(x,\tau)(\psi - \bar{\mathcal{R}}_h\psi)(x,\tau)\,d\tau\Big)\,d\omega(x)\Big| \\
&= \Big|\sum_{n=1}^{N}\int_\Omega \Big(\int_{t_{n-1}}^{t_n} g(x,\tau)\psi(x,\tau) - \bar{g}(x,\tau)\mathcal{R}_h\psi(x,\tau)\,d\tau\Big)\,d\omega(x)\Big| \\
&\leq \Big|\sum_{n=1}^{N}\int_\Omega \Big(\int_{t_{n-1}}^{t_n} \psi(x,\tau)(g(x,\tau) - \bar{g}(x,\tau))\,d\tau\Big)\,d\omega(x)\Big| \\
&\quad + \Big|\sum_{n=1}^{N}\int_\Omega \Big(\int_{t_{n-1}}^{t_n} \bar{g}(x,\tau)(\psi - \mathcal{R}_h\psi)(x,\tau)\,d\tau\Big)\,d\omega(x)\Big| \\
&\leq C\|g - \bar{g}\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}\|\psi\|_{L^2(L^\infty(\Omega))} \\
&\quad + C\|\bar{g}\|_{L^2(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}\|\psi - \mathcal{R}_h\psi\|_{L^2(L^\infty(\Omega))}. \qquad (4.72)
\end{aligned}
$$

Standard error estimate yields

$$
\|g - \bar{g}\|_{L^2(L^\infty(\Omega))} \leq Ck^{\frac{1}{2}}\|g\|_{H^{\frac{1}{2}}(L^\infty(\Omega))},
$$

and

$$
\|\psi - \mathcal{R}_h\psi\|_{L^2(L^\infty(\Omega))} \leq Ch^{2-\frac{d}{2}}\|\psi\|_{L^2(H^2(\Omega))}.
$$

Hence

$$
|\hat{E}_2| \leq C(k^{\frac{1}{2}} + h^{2-\frac{d}{2}})\|g\|_{H^{\frac{1}{2}}(L^\infty(\Omega))}\|f\|_{L^2(L^2(\Omega))}. \qquad (4.73)
$$

For $\hat{E}_3$, we have

$$
\begin{aligned}
|\hat{E}_3| &= |(y_0 - y_h^0(q), \psi(\cdot,0))| \leq \|y_0 - y_h^0(q)\|_{H^{-1}(\Omega)}\|\psi(\cdot,0)\|_{H^1(\Omega)} \\
&\leq Ch\|y_0\|_{L^2(\Omega)}\|f\|_{L^2(L^2(\Omega))}. \qquad (4.74)
\end{aligned}
$$

Finally, for $\hat{E}_4$, we find that

$$
\begin{aligned}
|\hat{E}_4| &= \Big|\sum_{n=1}^{N}\int_{I_n}(q, \psi - \bar{\mathcal{R}}_h\psi)\,d\tau\Big| \\
&\leq C\|q\|_{L^2(L^2(\Omega))}\|\psi - \bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))}.
\end{aligned}
$$

Note that

$$\begin{aligned}\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^\infty(L^2(\Omega))} &\le \|\psi-\bar{\psi}\|_{L^\infty(L^2(\Omega))}+\|\bar{\psi}-\bar{\mathcal{R}}_h\psi\|_{L^\infty(L^2(\Omega))}\\ &\le Ck^{\frac{1}{2}}\|\psi\|_{H^1(0,T;L^2(\Omega))}+Ch\|\bar{\psi}\|_{L^\infty(H^1(\Omega))}\\ &\le C(k^{\frac{1}{2}}+h)\|\psi\|_{L^2(H^2(\Omega))}. \qquad (4.75)\end{aligned}$$

Using the fact $\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))}\le\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^\infty(L^2(\Omega))}$ and using Lemma 1.2.3 together with (4.75) yields

$$|\hat{E}_4|\le C(h+k^{\frac{1}{2}})\|q\|_{L^2(L^2(\Omega))}\|f\|_{L^2(L^2(\Omega))}. \qquad (4.76)$$

Combining the estimates of $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$,$\hat{E}_4$ and the definition of $L^2(0,T;L^2(\Omega))$-norm, we complete the rest of the proof. ☐

Lemma 4.4.3. *Let z and $\mathcal{Z}_h(q)$ be the solutions of (4.14) and (4.62)-(4.63), respectively. Then, for $n\ge 1$, we have*

$$\|z-\mathcal{Z}_h(q)\|_{L^2(L^2(\Omega))}\le\|y-\mathcal{Y}_h(q)\|_{L^2(L^2(\Omega))}+C(h+k^{\frac{1}{2}})\Big(\|z\|_{L^2(H^2(\Omega))}+\|z_t\|_{L^2(L^2(\Omega))}\Big).$$

Proof. In order to estimate the error $z-\mathcal{Z}_h(q)$. We split the error on I_n as follows:

$$\begin{aligned}z(t_{n-1})-z_h^{n-1}(q) &= (z(t_{n-1})-\bar{\mathcal{R}}_h z(t_{n-1}))-(z_h^{n-1}(q)-\bar{\mathcal{R}}_h z(t_{n-1}))\\ &=: \eta^{n-1}-\zeta^{n-1}(q).\end{aligned}$$

By the standard arguments we have

$$\sum_{n=1}^{N}k\|\eta^n\|^2_{L^2(\Omega)}\le C\Big(h^2\|z\|_{L^2(H^2(\Omega))}+k\|z_t\|_{L^2(L^2(\Omega))}\Big). \qquad (4.77)$$

To bound $\zeta^{n-1}(q)$, we follow the idea of [32]. Define the fully-discrete approximation of (1.13) at time $t=t_n$ with $f^n=\zeta^{n-1}(q)$ as

$$k^{-1}(\phi^n-\phi^{n-1},v_h)+a(\phi^n,v_h)=(\zeta^{n-1}(q),v_h)\qquad\forall v_h\in V_h^n. \qquad (4.78)$$

Choose $v_h=\zeta^{n-1}(q)$ in (4.78) and integrate in time to have

$$\begin{aligned}\sum_{n=1}^{N}k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{N}\int_{I_n}\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)}\,d\tau\\ &= \sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n-\phi^{n-1},\zeta^{n-1}(q))+a(\phi^n,\zeta^{n-1}(q))\}\,d\tau\\ &= \sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n-\phi^{n-1},z_h^{n-1}(q))+a(\phi^n,z_h^{n-1}(q))\}\,d\tau\\ &\quad -\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n-\phi^{n-1},\bar{\mathcal{R}}_h z(t_{n-1}))+a(\phi^n,\bar{\mathcal{R}}_h z(t_{n-1}))\}\,d\tau.\end{aligned}$$

Using summation by parts to obtain

$$\begin{aligned}\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)} &= -\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n, z_h^n(q)-z_h^{n-1}(q))-a(\phi^n, z_h^{n-1}(q))\}\,d\tau\\ &+\sum_{n=1}^{N}\int_{I_n} k^{-1}[(\phi^n, z_h^n(q))-(\phi^{n-1}, z_h^{n-1}(q))]\,d\tau\\ &+\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n, \bar{\mathcal{R}}_h z(t_n)-\bar{\mathcal{R}}_h z(t_{n-1}))-a(\phi^n, \bar{\mathcal{R}}_h z(t_{n-1}))\}\,d\tau\\ &-\sum_{n=1}^{N}\int_{I_n} k^{-1}[(\phi^n, \bar{\mathcal{R}}_h z(t_n))-(\phi^{n-1}, \bar{\mathcal{R}}_h z(t_{n-1}))]\,d\tau.\end{aligned}$$

Using (4.62)-(4.63) and $\phi(\cdot,0)=0$, we obtain

$$\begin{aligned}\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{N}\int_{I_n}(\phi^n, y_h^n(q)-y_{\tilde{d}}^n)\,d\tau+\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n, \bar{\mathcal{R}}_h z(t_n)-\bar{\mathcal{R}}_h z(t_{n-1}))\\ &\quad -a(\phi^n, \bar{\mathcal{R}}_h z(t_{n-1}))\}\,d\tau\\ &= \sum_{n=1}^{N}\int_{I_n}(\phi^n, y_h^n(q)-y(t_n))\,d\tau-\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n, \eta^n-\eta^{n-1})\\ &\quad -a(\phi^n, \eta^{n-1})\}d\tau-\sum_{n=1}^{N}\int_{I_n} a(\phi^n, z(t_{n-1})-z(t_n))\,d\tau,\end{aligned}$$

where in the last step we subtracted the equation

$$\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(v, z(t_n)-z(t_{n-1}))-a(v, z(t_n))\}\,d\tau=-\sum_{n=1}^{N}\int_{I_n}(v, y(t_n)-y_{\tilde{d}}^n)\,d\tau$$

for $v=\phi^n$. Summation by parts implies

$$\begin{aligned}\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{N}\int_{I_n}(\phi^n, y_h^n(q)-y(t_n))\,d\tau+\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n-\phi^{n-1}, \eta^n)\\ &\quad +a(\phi^n, \eta^{n-1})\}d\tau-\sum_{n=1}^{N}\int_{I_n} a(\phi^n, z(t_{n-1})-z(t_n))\,d\tau.\end{aligned}$$

Using the fact $\int_{I_n} a(\phi^n, \eta^{n-1})\,d\tau=0$ in the above equation, we obtain

$$\begin{aligned}\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{N}\int_{I_n}(\phi^n, y_h^n(q)-y(t_n))\,d\tau+\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(\phi^n-\phi^{n-1}, \eta^n)d\tau\\ &\quad -\sum_{n=1}^{N}\int_{I_n} a(\phi^n, z(t_{n-1})-z(t_n))\,d\tau.\end{aligned}$$

An application of the Cauchy-Schwarz inequality and continuity of the bilinear form gives

$$\begin{aligned}\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)} \leq & \Big(\sum_{n=1}^{N} k\|\phi^n\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{N} k\|y_h^n(q)-y(t_n)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\ & +\Big(\sum_{n=1}^{N} k^{-1}\|\phi^n-\phi^{n-1}\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{N} k\|\eta^n\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\ & +C\Big(\sum_{n=1}^{N} k\|\phi^n\|^2_{H^1(\Omega)}\Big)^{\frac{1}{2}} \\ & \times\Big(\sum_{n=1}^{N} k\{\|z(t_n)\|^2_{H^1(\Omega)}+\|z(t_{n-1})\|^2_{H^1(\Omega)}\}\Big)^{\frac{1}{2}}. \qquad (4.79)\end{aligned}$$

By setting $v_h = \phi^n - \phi^{n-1}$ in (4.78), using the Cauchy-Schwarz inequality and Young's inequality together with the summation by parts gives

$$\begin{aligned}\sum_{n=1}^{N} k^{-1}\|\phi^n-\phi^{n-1}\|^2_{L^2(\Omega)}+\|\phi^N\|^2_{H^1(\Omega)} &\leq \|\phi^0\|^2_{H^1(\Omega)}+C\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)} \\ &\leq C\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)}, \qquad (4.80)\end{aligned}$$

where we have used $\phi^0 = 0$. Similarly set $v_h = k\phi^n$ in (4.78) and applying the Cauchy-Schwarz inequality, the Young's inequality together with the summation by parts and $\phi^0 = 0$ yields

$$\|\phi^N\|^2_{L^2(\Omega)}+\sum_{n=1}^{N} k\|\phi^n\|_{H^1(\Omega)} \leq C\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)}, \qquad (4.81)$$

Use (4.80) and (4.81) in (4.79) leads to

$$\begin{aligned}\Big(\sum_{n=1}^{N} k\|\zeta^{n-1}(q)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \leq & C\Big[\Big(\sum_{n=1}^{N} k\|y_h^n(q)-y(t_n)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}+\Big(\sum_{n=1}^{N} k\|\eta^n\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\ & +\Big(\sum_{n=1}^{N} k\{\|z(t_n)\|^2_{H^1(\Omega)}+\|z(t_{n-1})\|^2_{H^1(\Omega)}\}\Big)^{\frac{1}{2}}\Big].\end{aligned}$$

This completes the proof. □

In the following lemma, we prove the error between the derivative of the continuous and fully discrete reduced cost functions.

Lemma 4.4.4. *Let $\tilde{j}'(q)(r)$ and $(\tilde{j}_h^n)'(q)(r)$ be given by*

$$\tilde{j}'(q)(r) = \int_0^T (\tilde{\alpha}q(\tau) + z(\tau), r)\, d\tau, \qquad \forall r \in L^2(0,T;L^2(\Omega)),$$

and

$$(\tilde{j}_h^n)'(q)(r) = \sum_{n=1}^{N} \int_{I_n} (\tilde{\alpha}q(\tau) + z_h^{n-1}(q), r)\, d\tau, \qquad \forall r \in L^2(0,T;L^2(\Omega)),$$

respectively. Then

$$|\tilde{j}'(q)(r) - (\tilde{j}_h^n)'(q)(r)| \le \bar{C}_1(h^{2-\frac{d}{2}} + k^{\frac{1}{2}})\|r\|_{L^2(L^2(\Omega))},$$

where

$$\bar{C}_1 = C\Big(\|y_0\|_{L^2(\Omega)},\ \|g\|_{H^{\frac{1}{2}}(L^\infty(\Omega))},\ \|\omega\|_{\mathcal{M}(\Omega)}, \|q\|_{L^2(L^2(\Omega))},\ \|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}\Big). \tag{4.82}$$

Proof. The proof follows by straightforward calculation. □

The following theorem estimates the error in the control variable.

Theorem 4.4.1. *Let q and q_h^n be the optimal controls of (4.13) and (4.57), respectively. Let the second order optimality condition (4.17) holds true. Then we have*

$$\|q - q_h^n\|_{L^2(L^2(\Omega))} \le \tilde{C}_1 \frac{h}{\sqrt{\gamma_P}} + \frac{\bar{C}_1}{\gamma_P}(h^{2-\frac{d}{2}} + k^{\frac{1}{2}}), \tag{4.83}$$

where $\tilde{C}_1$ and $\bar{C}_1$ are given by (4.45) and (4.82).

Proof. For $r \in L^2(0,T;L^2(\Omega))$, we have

$$\tilde{j}''(q)(r,r) \ge \gamma_P \|r\|^2_{L^2(L^2(\Omega))} \tag{4.84}$$

and

$$(\tilde{j}_h^n)''(q_h^n)(r,r) \ge \gamma_P \|r\|^2_{L^2(L^2(\Omega))}. \tag{4.85}$$

We now formulate the following auxiliary problem:

$$\min_{q_h^n \in Q_{h,n}^P} \tilde{j}(q_h^n), \tag{4.86}$$

where we only discretize the control variable. Let $\tilde{q}_h^n$ be the solution of problem (4.86). By triangle inequality, we have

$$\|q - q_h^n\|_{L^2(L^2(\Omega))} \le \|q - \tilde{q}_h^n\|_{L^2(L^2(\Omega))} + \|\tilde{q}_h^n - q_h^n\|_{L^2(L^2(\Omega))}. \tag{4.87}$$

With $\xi = \tilde{\lambda} q + (1-\tilde{\lambda})\tilde{q}_h^n$, $\tilde{\lambda} \in [0,1]$ and h sufficiently small, we have by (4.84)

$$\begin{aligned}
\gamma_P \|q - \tilde{q}_h^n\|^2_{L^2(L^2(\Omega))} &\leq \tilde{j}''(\xi)(q - \tilde{q}_h^n, q - \tilde{q}_h^n), \\
&= \tilde{j}'(q)(q - \tilde{q}_h^n) - \tilde{j}'(\tilde{q}_h^n)(q - \tilde{q}_h^n) \\
&= \tilde{j}'(q)(q - \tilde{q}_h^n) - \tilde{j}'(\tilde{q}_h^n)(q - \mathcal{L}_h q) - j'(\tilde{q}_h^n)(\mathcal{L}_h q - \tilde{q}_h^n).
\end{aligned}$$

The necessary optimality condition, for sufficiently small h, imply

$$\tilde{j}'(q)(q - \tilde{q}_h^n) \leq 0 \quad \text{and} \quad -\tilde{j}'(\tilde{q}_h^n)(\mathcal{L}_h q - \tilde{q}_h^n) \leq 0,$$

and hence, using the properties of $\mathcal{L}_h$ and the Young's inequality, we have

$$\begin{aligned}
\gamma_P \|q - \tilde{q}_h^n\|^2_{L^2(L^2(\Omega))} &\leq -\tilde{j}'(\tilde{q}_h^n)(q - \mathcal{L}_h q) \\
&= -\int_0^T \Big(\tilde{\alpha}\tilde{q}_h^n + z(\tilde{q}_h^n), q - \mathcal{L}_h q\Big)\, d\tau \\
&= -\int_0^T \Big(z(\tilde{q}_h^n) - \mathcal{L}_h z(\tilde{q}_h^n), q - \mathcal{L}_h q\Big)\, d\tau \\
&\leq \int_0^T \Big(\frac{1}{2}\|z(\tilde{q}_h^n) - \mathcal{L}_h z(\tilde{q}_h^n)\|^2_{L^2(\Omega)} + \frac{1}{2}\|q - \mathcal{L}_h q\|^2_{L^2(\Omega)}\Big)\, d\tau.
\end{aligned}$$

An application of Lemma 4.3.1 leads to

$$\|q - \tilde{q}_h^n\|_{L^2(L^2(\Omega))} \leq \int_0^T \Big(\frac{C}{\sqrt{\gamma_P}} h \|\nabla z(\tilde{q}_h^n)\|_{L^2(\Omega)} + \frac{C}{\sqrt{\gamma_P}} h \|\nabla q\|_{L^2(\Omega)}\Big)\, d\tau \leq \frac{\tilde{C}_1}{\sqrt{\gamma_P}} h,$$

where $\tilde{C}_1$ is given by (4.45). To estimate the second term in (4.87), we use the necessary optimality condition leading to the following relation

$$(\tilde{j}_h^n)'(q_h^n)(q_h^n - r_h) \leq 0 \leq \tilde{j}'(\tilde{q}_h^n)(r_h - \tilde{q}_h^n), \quad \forall r_h \in Q_{h,n}^P.$$

Again, with $\xi = \tilde{\lambda} q_h^n + (1-\tilde{\lambda})\tilde{q}_h^n$, $\tilde{\lambda} \in [0,1]$ and sufficiently small h, we obtain using (4.85)

$$\begin{aligned}
\gamma_P \|q_h^n - \tilde{q}_h^n\|^2_{L^2(L^2(\Omega))} &\leq (\tilde{j}_h^n)''(\xi)(q_h^n - \tilde{q}_h^n, q_h^n - \tilde{q}_h^n) \\
&= (\tilde{j}_h^n)'(q_h^n)(q_h^n - \tilde{q}_h^n) - (\tilde{j}_h^n)'(\tilde{q}_h^n)(q_h^n - \tilde{q}_h^n) \\
&\leq \tilde{j}'(\tilde{q}_h^n)(q_h^n - \tilde{q}_h^n) - (\tilde{j}_h^n)'(\tilde{q}_h^n)(q_h^n - \tilde{q}_h^n) \\
&\leq \bar{C}_1 (h^{2-\frac{d}{2}} + k^{\frac{1}{2}}) \|q_h^n - \tilde{q}_h^n\|_{L^2(L^2(\Omega))},
\end{aligned}$$

where $\bar{C}_1$ as in (4.82). The last step in the above follows from Lemma 4.4.4 and this completes the proof. □

With the above preparations we are now ready to estimate the error between the solution y of the continuous problem (4.12) and the solution $\mathcal{Y}_h$ of the fully discrete problem (4.51).

Theorem 4.4.2. *Assume that g and ω are given functions such that*

$$g \in L^2(0,T;\mathcal{C}(\overline{\Omega})) \cap H^{\frac{1}{2}}(0,T;L^\infty(\Omega)) \text{ and } \omega \in \mathcal{M}(\Omega),\ y_0 \in L^2(\Omega).$$

Let $y \in L^2(0,T;L^2(\Omega))$ and $\mathcal{Y}_h$ be the solutions of the problems (4.12) and (4.51), respectively. Then, we have

$$\begin{aligned}\|y-\mathcal{Y}_h\|_{L^2(L^2(\Omega))} &\leq C(h^{2-\frac{d}{2}}+k^{\frac{1}{2}})\Big(\|g\|_{H^{\frac{1}{2}}(L^\infty(\Omega))}\|\omega\|_{\mathcal{M}(\Omega)}+\|y_0\|_{L^2(\Omega)}+\|q\|_{L^2(L^2(\Omega))}\Big)\\ &\quad +\|q-q_h^n\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Proof. We use the duality argument to prove this result. Let ψ be the solution of the problem (1.14) with $f \in L^2(0,T;L^2(\Omega))$. Then, from (4.12), we obtain

$$\begin{aligned}\int_{\Omega_T}(y-\mathcal{Y}_h)f\,dxd\tau &= \int_0^T\int_\Omega (y-\mathcal{Y}_h)(-\psi_t+\mathcal{A}^*\psi)\,dxd\tau\\ &= -\sum_{n=1}^N\int_{I_n}\{k^{-1}(y_h^n-y_h^{n-1},\psi^{n-1})+a(y_h^n,\psi)\}\,d\tau+<\mu,\psi>_{\Omega_T}\\ &\quad +(y_0-y_h^0,\psi(\cdot,0))+(q,\psi)_{\Omega_T}.\end{aligned}\tag{4.88}$$

From (4.51), we have

$$\sum_{n=1}^N\{(\bar{\partial}y_h^n,\bar{\mathcal{R}}_h\psi)+a(y_h^n,\bar{\mathcal{R}}_h\psi)\}=\sum_{n=1}^N<\mu,\bar{\mathcal{R}}_h\psi>_{I_n}+\sum_{n=1}^N(q_h^n,\bar{\mathcal{R}}_h\psi),\tag{4.89}$$

where $\bar{\mathcal{R}}_h\psi \in V_h^n$ is defined in (4.66).

Integrate (4.89) in time and add the resulting equation to (4.88) gives

$$\begin{aligned}\int_{\Omega_T}(y-\mathcal{Y}_h)f\,dxd\tau &= -\sum_{n=1}^N\int_{I_n}\{k^{-1}(y_h^n-y_h^{n-1},\psi^{n-1}-\bar{\mathcal{R}}_h\psi)+a(y_h^n,\bar{\psi}-\bar{\mathcal{R}}_h\psi)\}\,d\tau\\ &\quad +\{<\mu,\psi>_{\Omega_T}-\sum_{n=1}^N\int_{I_n}<\mu,\bar{\mathcal{R}}_h\psi>_{I_n}\,d\tau\}+\{(y_0-y_h^0,\psi(\cdot,0))\}\\ &\quad +\{(q,\psi)_{\Omega_T}-\sum_{n=1}^N\int_{I_n}(q_h^n,\bar{\mathcal{R}}_h\psi)\,d\tau\},\\ &=: \hat{E}_1+\hat{E}_2+\hat{E}_3+\check{E}_4.\end{aligned}$$

Note that $\hat{E}_1,\ \hat{E}_2,\ \hat{E}_3$ are estimated in Lemma 4.4.2. To estimate $\check{E}_4$, we have

$$\begin{aligned}|\check{E}_4| &= \Big|\sum_{n=1}^{N}\int_{I_n}(q,\psi)\,d\tau - \sum_{n=1}^{N}\int_{I_n}(q_h^n,\bar{\mathcal{R}}_h\psi)\,d\tau\Big| \\ &= \Big|\sum_{n=1}^{N}\int_{I_n}(q,\psi-\bar{\mathcal{R}}_h\psi)\,d\tau + \sum_{n=1}^{N}\int_{I_n}(q-q_h^n,\bar{\mathcal{R}}_h\psi)\,d\tau\Big|.\end{aligned}$$

Then, use of the Cauchy-Schwarz inequality and the stability result $\|\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))}\leq C\|\psi\|_{L^2(L^2(\Omega))}$ together with Lemma 1.2.3 gives

$$\begin{aligned}|\check{E}_4| &\leq C\|q\|_{L^2(L^2(\Omega))}\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))} + \|q-q_h^n\|_{L^2(L^2(\Omega))}\|\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))} \\ &\leq C\Big\{(h+k^{\frac{1}{2}})\|q\|_{L^2(L^2(\Omega))} + \|q-q_h^n\|_{L^2(L^2(\Omega))}\Big\}\|f\|_{L^2(L^2(\Omega))},\end{aligned}$$

where we have used (4.75) and the fact $\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))}\leq\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^\infty(L^2(\Omega))}$. The estimates of $\hat{E}_1,\ \hat{E}_2,\ \hat{E}_3,\ \check{E}_4$ and the definition of $L^2(0,T;L^2(\Omega))$-norm completes the rest of the proof. □

Now, we write the error between the continuous and fully-discrete co-state variables.

Theorem 4.4.3. *Let z and $\mathcal{Z}_h$ be the solutions of (4.14) and (4.54)-(4.55), respectively. Then, we have*

$$\begin{aligned}\|z-\mathcal{Z}_h\|_{L^2(L^2(\Omega))} &\leq C\|y-\mathcal{Y}_h\|_{L^2(L^2(\Omega))} + C(h+k^{\frac{1}{2}})\Big(\|z\|_{L^2(H^2(\Omega))} \\ &\quad +\|z_t\|_{L^2(L^2(\Omega))}\Big).\end{aligned}$$

Proof. Following the lines of arguments of Lemma 4.4.3, it is easy to derive the desired estimate. □

Concluding remarks. In this chapter we study the *a priori* error analysis for the finite element approximations of POCP with measure data in space. The spatial discretization of the state and co-state variables are based on piecewise linear and continuous finite elements whereas the control variable is approximated by piecewise constant functions. *A priori* error bounds of order $\mathcal{O}(h^{2-\frac{d}{2}})$ are shown for the state, co-state and control variables for the spatially discrete approximations of POCP with measure data in space (see Theorems 4.3.1-4.3.3). The time discritization is based on the backward-Euler implicit scheme and *a priori* error estimates of order $\mathcal{O}(h^{2-\frac{d}{2}}+k^{\frac{1}{2}})$ for the state, co-state and control variables are established (see Theorems 4.4.1-4.4.3). Numerical results are provided in Chapter 7 (see Example 7.2) to support our theoretical findings.

5

POCP with Measure Data in Time: A Priori Error Analysis

This chapter considers finite element approximations for POCP (1.6)-(1.8) with measure data in time, i.e., $\mu = g\omega$ with $g \in \mathcal{C}([0,T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. We prove the existence, uniqueness and regularity results for the control problem. *A priori* error estimates for both spatially discrete and fully discrete approximations for the state, co-state and control variables are derived. The problem is spatially discretized by using finite element approximation scheme with continuous piecewise linear functions for the state, co-state variables and piecewise constant functions for the control variable. The time discretization scheme is based on the backward Euler implicit method.

5.1 Introduction

Let $\Omega_T = \Omega \times (0,T]$ and $\Gamma_T = \partial\Omega \times [0,T]$, where Ω is a bounded convex domain in $\mathbb{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega$ and $T < \infty$. We now recall the following POCP:

$$\min_{q \in Q_{ad}^P} \tilde{J}(q,y) = \frac{1}{2}\int_0^T \left\{ \|y - y_{\tilde{d}}\|^2_{L^2(\Omega)} + \tilde{\alpha}\|q\|^2_{L^2(\Omega)} \right\} d\mathcal{T} \tag{5.1}$$

subject to the state equation

$$\begin{cases} y_t + \mathcal{A}y = \mu + q \ \text{ in } \ \Omega_T, \\ y(\cdot,0) = y_0(x) \ \text{ in } \ \Omega, \\ y = 0 \ \text{ on } \ \Gamma_T \end{cases} \tag{5.2}$$

and the control constraints

$$q_c \le q(x,t) \le q_d \ \text{ a.e. in } \Omega_T, \tag{5.3}$$

where $q_c,\ q_d \in \mathbb{R}$ fulfill $q_c < q_d$ and $y_t = \frac{\partial y}{\partial t}$. The operator $\mathcal{A}$ is defined in (1.4) and $\tilde{\alpha} > 0$ is a fixed constant. Further, $y_0(x) \in L^2(\Omega)$ and $y_{\tilde{d}}(x,t) \in L^2(0,T;L^2(\Omega))$ are given functions. The function $\mu = g\omega$, where $g \in \mathcal{C}([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. The set of admissible controls is defined by

$$Q_{ad}^P := \{q \in L^2(0,T;L^2(\Omega)) :\ q_c \leq q(x,t) \leq q_d \ \ a.e.\ \text{in}\ \Omega_T\}. \tag{5.4}$$

The POCP (5.1)-(5.3) with measure data in time occurs in optimal control problems with state constraints pointwise in time [16]. In this work, we prove the existence, uniqueness and regularity of the solution of the control problem, and investigate the convergence properties of finite element approximations for the state, co-state and control variables. Both spatially discrete and fully discrete optimization problems are analyzed and error estimates for the state, co-state and control variables are derived in the $L^2(0,T;L^2(\Omega))$-norm. We obtain error estimates of order $\mathcal{O}(h)$ for the state, co-state and control variables for the spatially discrete control problem. A time discretization scheme based on the backward-Euler method is analyzed and error estimates of order $\mathcal{O}(h + k^{1/2})$ are proved for the state, co-state and control variables.

A brief outline of this chapter is as follows. In Section 5.2, we discuss the existence, uniqueness and regularity results of the control problem. Section 5.3 deals with the spatially discrete finite element approximations of problem and derive error estimates for the state, co-state and control variables. Section 5.4 is devoted to the fully discrete error analysis for the control problem and the related convergence results are derived.

Throughout this chapter C denotes a positive generic constant independent of h and k.

5.2 Existence, uniqueness and regularity results

This section is concerned with the existence, uniqueness and regularity results of the control problem (5.1)-(5.3).

We recall the following notations:

$$\begin{aligned} W(0,T) &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ X(0,T) &:= L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \end{aligned}$$

It is straightforward that $W(0,T) \hookrightarrow \mathcal{C}([0,T];L^2(\Omega))$ and $X(0,T) \hookrightarrow \mathcal{C}([0,T];H_0^1(\Omega))$ (see [61]). The following theorem provides the results concerning the existence, uniqueness and the regularity of the solution to problem (5.2).

Theorem 5.2.1. *Assume that* $q \in L^2(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $y_0 \in L^2(\Omega)$, $\mu = g\omega$, g *and* ω *are given functions such that* $g \in \mathcal{C}([0,T];L^2(\Omega))$, $\omega \in \mathcal{M}[0,T]$. *Then, the problem (5.2) admits a unique solution* $y \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ *such that*

$$-(y,v_t)_{\Omega_T} + a(y,v)_{\Omega_T} = << \mu, v >>_{\Omega_T} + (q,v)_{\Omega_T} + (y_0, v(\cdot,0)), \quad \forall v \in W(0,T) \tag{5.5}$$

with $v(\cdot,T) = 0$ *and*

$$\|y\|_{L^2(H_0^1(\Omega))} + \|y\|_{L^\infty(L^2(\Omega))} \le C\Big(\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|q\|_{L^2(L^2(\Omega))} + \|y_0\|_{L^2(\Omega)}\Big), \tag{5.6}$$

where

$$<< \mu, v >>_{\Omega_T} = \int_{\Omega_T} v\,d\mu = \int_0^T \Big(\int_\Omega g(x,\tau)v(x,\tau)\,dx\Big)\,d\omega(\tau), \quad \forall v \in \mathcal{C}([0,T];L^2(\Omega)).$$

Proof. We borrow the proof techniques from [16]. Since the problem is linear, it is enough to consider either $y_0 \equiv 0$ or $\mu \equiv 0$ and $q \equiv 0$. If $\mu \equiv 0$, $q \equiv 0$ and $y_0 \in L^2(\Omega)$, it is straightforward that the problem (5.2) admits a unique solution $y \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ satisfying the *a priori* estimates [61]

$$\|y\|_{L^2(H_0^1(\Omega))} + \|y\|_{L^\infty(L^2(\Omega))} \le C\|y_0\|_{L^2(\Omega)}.$$

Now we assume $y_0 \equiv 0$, let $\{\omega_n\}_n \subset \mathcal{C}[0,T]$ be a sequence converging weak* to ω in $\mathcal{M}[0,T]$ and satisfy

$$\|\omega_n\|_{L^1[0,T]} \le \|\omega\|_{\mathcal{M}[0,T]}.$$

Let y_n be the solutions of

$$\begin{cases} y_{n,t} + \mathcal{A}y_n = g\omega_n + q \quad \text{in } \Omega_T, \\ y_n(\cdot,0) = 0 \quad \text{in } \Omega, \\ y_n = 0 \text{ on } \quad \Gamma_T, \end{cases} \tag{5.7}$$

Then $y_n \in X(0,T)$. For $f \in \mathcal{D}(\Omega_T)$, let ψ be the solution of problem (1.14). Then $\psi \in X(0,T) \hookrightarrow \mathcal{C}([0,T];H_0^1(\Omega))$. Now, from (5.7) we have

$$\begin{aligned}\int_{\Omega_T} f y_n\,dxd\tau &= \int_{\Omega_T} (-\psi_t + \mathcal{A}^*\psi)y_n\,dxd\tau \\ &= \int_{\Omega_T} g\omega_n\psi\,dxd\tau + \int_{\Omega_T} q\psi\,dxd\tau \\ &\le C\Big(\|g\|_{L^\infty(L^2(\Omega))}\|\omega_n\|_{L^1[0,T]}\|\psi\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|q\|_{L^2(L^2(\Omega))}\|\psi\|_{L^2(L^2(\Omega))}\Big).\end{aligned}$$

It is quite standard to prove the following bounds [61]:

$$\|\psi\|_{\mathcal{C}([0,T];L^2(\Omega))} \leq C\|f\|_{L^1(L^2(\Omega))}, \tag{5.8}$$

$$\|\psi\|_{\mathcal{C}([0,T];L^2(\Omega))} \leq C\|f\|_{L^2(H^{-1}(\Omega))}, \tag{5.9}$$

and

$$\|\psi\|_{L^2(L^2(\Omega))} \leq C\|f\|_{L^2(L^2(\Omega))}. \tag{5.10}$$

In view of (5.8) we conclude that the solution sequence $\{y_n\}_n$ is bounded in the space $L^\infty(0,T;L^2(\Omega))$. By (5.9) we note that the sequence $\{y_n\}_n$ is also bounded in the space $L^2(0,T;H_0^1(\Omega))$. Thus we can extract a subsequence y_{n_k} such that $y_{n_k} \to y$ weakly in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and hence (5.6) holds.

Next to prove (5.5), let $\psi \in W(0,T)$ and $\psi(\cdot,T) = 0$. Multiplying (5.7) by ψ and integrating by parts over Ω_T we obtain

$$\int_0^T \Big(\int_\Omega g\psi\,dx\Big)\omega_n(\tau)d\tau + \int_{\Omega_T} q\psi\,dxd\tau = -\int_{\Omega_T} y_n\psi_t\,dxd\tau$$
$$+\int_{\Omega_T}\Big(\sum_{i,j=1}^d a_{ij}\frac{\partial y_n}{\partial x_i}\frac{\partial \psi}{\partial x_j} + a_0 y_n\psi\Big)dxd\tau. \tag{5.11}$$

Passing to the limit in (5.11) we get (5.5). Finally, the uniqueness follows from the fact that the only solution for the zero data of (5.5) is $y = 0$. □

The weak form of the model problem (5.1)-(5.3) is as follows:

$$\min_{q\in Q_{ad}^P} \tilde{J}(q,y) = \frac{1}{2}\int_0^T \Big\{\|y - y_{\tilde{d}}\|^2_{L^2(\Omega)} + \tilde{\alpha}\|q\|^2_{L^2(\Omega)}\Big\}\,d\tau \tag{5.12}$$

subject to

$$-(y,v_t)_{\Omega_T} + a(y,v)_{\Omega_T} = \ll \mu, v \gg_{\Omega_T} + (q,v)_{\Omega_T} + (y_0, v(\cdot,0)), \quad \forall v \in W(0,T). \tag{5.13}$$

In the next step, we introduce the reduced cost functional $\tilde{j} : L^2(0,T;L^2(\Omega)) \to \mathbb{R}$ by

$$\tilde{j}(q) := \tilde{J}(q, y(q)),$$

where $\tilde{J}$ is the cost function given by (5.12) and $y(q)$ is the weak solution of (5.2) as defined in (5.13). The optimal control problem (5.12)-(5.13) can be reformulated as

$$\min_{q\in Q_{ad}^P} \tilde{j}(q), \tag{5.14}$$

where the set of admissible controls is given by (5.4).

The following theorem ensures the existence and uniqueness of solution of minimization problem (5.14).

Theorem 5.2.2. *The optimal control problem (5.14) admits a unique solution.*

Proof. Let q_n be the minimizing sequences, i.e., $\tilde{j}(q_n) \to \inf_{q\in Q^P_{ad}} \tilde{j}(q)$. By virtue of the term $\tilde{\alpha}\|q\|_{L^2(L^2(\Omega))}$ in $\tilde{j}(q)$, q_n is bounded in Q^P_{ad} and let $y(q_n) = y_n$. From the regularity estimate (5.6), we have

$$\|y_n\|_{L^2(H^1_0(\Omega))} \leq C,$$

where

$$C = C(\|g\|_{L^\infty(L^2(\Omega))}, \|\omega\|_{\mathcal{M}[0,T]}, \|q\|_{L^2(L^2(\Omega))}, \|y_0\|_{L^2(\Omega)}).$$

Since the injection map $H^1_0(\Omega) \to L^2(\Omega)$ is compact, it follows that y_n has a subsequence, denoted by y_{n_k} such that

$$\begin{aligned} y_{n_k} \to y \quad &\text{weakly in } L^2(0,T;H^1_0(\Omega)),\\ \text{and } y_{n_k} \to y \ &\text{strongly in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

For $v \in \mathcal{D}(\Omega_T)$, we have

$$\begin{aligned} (q_{n_k}, v)_{\Omega_T} &= -(y_{n_k}, v_t)_{\Omega_T} + a(y_{n_k}, v)_{\Omega_T} - << \mu, v >>_{\Omega_T} - (y_{n_k}(\cdot,0), v(\cdot,0))\\ &= -(y_{n_k} - y, v_t)_{\Omega_T} + a(y_{n_k} - y, v)_{\Omega_T} - (y_{n_k}(\cdot,0) - y(\cdot,0), v(\cdot,0))\\ &\quad -(y, v_t)_{\Omega_T} + a(y,v)_{\Omega_T} - << \mu, v >>_{\Omega_T} - (y(\cdot,0), v(\cdot,0)). \end{aligned}$$

As $n \to \infty$, $y_{n_k} \to y$ strongly and $(q_{n_k}, v)_{\Omega_T} \to (q,v)_{\Omega_T}$. Then, $q_{n_k} \to q$ in $\mathcal{D}'(\Omega_T)$ and consequently y satisfies (5.13). Hence $y = y(q)$ and $\liminf \tilde{j}(q_{n_k}) \geq \tilde{j}(q)$ and therefore, q is the optimal control. □

The first order optimality condition is characterized by

$$\tilde{j}'(q)(\hat{q} - q) \geq 0, \quad \forall \hat{q} \in Q^P_{ad}. \tag{5.15}$$

The derivative $\tilde{j}'(q)(\hat{q} - q)$ for given $q,\ \hat{q} \in Q^P_{ad}$ can be expressed as

$$\tilde{j}'(q)(\hat{q} - q) = \int_0^T (\tilde{\alpha} q + z, \hat{q} - q)\, d\tau, \tag{5.16}$$

where $z = z(q)$ is the solution of the co-state problem

$$\begin{cases} -(z_t, v)_{\Omega_T} + a(z,v)_{\Omega_T} = (y - y_{\tilde{d}}, v)_{\Omega_T}, \quad \forall v \in L^2(0,T;H^1_0(\Omega)),\\ z(\cdot,T) = 0 \qquad \text{in } \Omega, \end{cases} \tag{5.17}$$

and $y = y(q)$ on the right hand side of (5.17) is the solution of the state equation (5.13). Let the co-state variable corresponds to the optimal control q is denoted by $z(q)$. Then the optimality condition (5.15) reads

$$q = P_{[q_c,q_d]}\Big(-\frac{1}{\tilde{\alpha}} z(q)\Big), \tag{5.18}$$

where $P_{[q_c,q_d]}$ denotes the projection of $\mathbb{R}$ on to $[q_c, q_d]$ defined as

$$P_{[q_c,q_d]}(q) := \max(q_c, \min(q_d, q)). \tag{5.19}$$

The second derivative of $\tilde{j}''(q)(\cdot,\cdot)$ is independent of q and positive definite, i.e.,

$$\tilde{j}''(q)(r,r) \geq \gamma_P \|r\|^2_{L^2(L^2(\Omega))}, \quad \forall r \in L^2(0,T; L^2(\Omega)). \tag{5.20}$$

Using property (5.19), we provide an important regularity result.

Theorem 5.2.3. *Assume that (q, y) be the solution of the optimization problem (5.12)-(5.13) and z is the co-state variable. Then, we have*

$$\begin{aligned} y &\in L^2(0,T; H^1_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega)), \\ z &\in X(0,T), \\ q &\in X(0,T) \cap L^\infty(0,T; L^\infty(\Omega)). \end{aligned}$$

Proof. The regularity of y follows directly from Theorem 5.2.1 and regularity of z follows by Lemma 1.2.3. The property (5.18) imply the desired result for q. □

5.3 Spatially discrete approximations of POCP

In this section, we consider the continuous time finite element approximation of (5.12)-(5.13) and derive error estimates of the continuous and the spatially discrete state, co-state and control variables under suitable norm.

Associated with $\mathcal{T}_h$, let V_h be a finite dimensional subspace of $\mathcal{C}(\overline{\Omega})$, consisting of piecewise linear polynomials as described in Section 2.3 of Chapter 2 . We denote $V_h^0 = V_h \cap H^1_0(\Omega)$ and let

$$Q_h^P := \{\hat{q}_h(t) \in Q_{ad}^P : \ \hat{q}_h(t)|_K = constant, \ \forall K \in \mathcal{T}_h, \, t \in (0,T]\}.$$

Now we define the spatially discrete finite element approximation of (5.12)-(5.13) as follows: Find a pair $(q_h, y_h) : [0,T] \to Q_h^P \times V_h^0$ such that

$$\min_{q_h \in Q_h^P} \tilde{J}(q_h, y_h) = \frac{1}{2}\int_0^T \Big\{ \|y_h - y_{\tilde{d}}\|^2_{L^2(\Omega)} + \tilde{\alpha}\|q_h\|^2_{L^2(\Omega)} \Big\}\, d\tau \tag{5.21}$$

subject to

$$-(y_h, v_{h,t})_{\Omega_T} + a(y_h, v_h)_{\Omega_T} =<< \mu, v_h >>_{\Omega_T} + (q_h, v_h)_{\Omega_T} + (y_{h,0}, v_h(\cdot, 0)), \qquad (5.22)$$

$\forall v_h \in H^1(0,T;V_h^0)$ with $y_{h,0} = \mathcal{L}_h y_0$ and $v_h(\cdot,T) = 0$, where $\mathcal{L}_h$ is the L^2-projection defined by (4.21). Here,

$$<< \mu, v_h >>_{\Omega_T} = \int_{\Omega_T} v_h \, d\mu = \int_0^T \Big(\int_\Omega g(x,\tau) v_h(x) \, dx \Big) \, d\omega(\tau), \qquad \forall v_h \in H^1(0,T;V_h^0).$$

Analogous to Theorem 5.2.1, we have the following stability result.

Lemma 5.3.1. *Let $q \in L^2(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\mu = g\omega$, g and ω are given functions such that $g \in \mathcal{C}([0,T];L^2(\Omega))$, $\omega \in \mathcal{M}[0,T]$. Let $y_h(q) \in L^2(0,T;V_h^0)$ be the unique solution of*

$$-(y_h(q), v_{h,t})_{\Omega_T} + a(y_h(q), v_h)_{\Omega_T} =<< \mu, v_h >>_{\Omega_T} + (q, v_h)_{\Omega_T} + (y_{h,0}(q), v_h(\cdot,0)), \quad (5.23)$$

$\forall v_h \in H^1(0,T;V_h^0)$ with $v_h(\cdot,T) = 0$ and $y_{h,0}(q) = \mathcal{L}_h y_0$. Then

$$\begin{aligned} \|y_h(q)\|_{L^2(H_0^1(\Omega))} + \|y_h(q)\|_{L^\infty(L^2(\Omega))} \;\; &\le \;\; C\Big(\|g\|_{L^\infty(L^2(\Omega))} \|\omega\|_{\mathcal{M}[0,T]} + \|q\|_{L^2(L^2(\Omega))} \\ &\quad + \|y_{h,0}(q)\|_{L^2(\Omega)} \Big). \end{aligned} \qquad (5.24)$$

Analogous to the continuous case, we introduce the discrete reduced cost functional $\tilde{j}_h : L^2(0,T;L^2(\Omega)) \to \mathbb{R}$ by

$$\tilde{j}_h(q) := \tilde{J}(q, y_h(q)).$$

The discretized optimization problem (5.21)-(5.22) is stated as follows:

$$\min_{q_h \in Q_h^P} \tilde{j}_h(q_h). \qquad (5.25)$$

The existence of a unique solution $q_h \in Q_h^P$ of (5.25) follows from the standard arguments of [60]. Then the first order optimality condition may be read as

$$\tilde{j}_h'(q_h)(\hat{q}_h - q_h) \ge 0, \; \forall \, \hat{q}_h \in Q_h^P, \qquad (5.26)$$

where the directional derivative $j_h'(q_h)(\hat{q}_h - q_h)$, for given $q_h, \; \hat{q}_h \in Q_h^P$, can be expressed as

$$\tilde{j}_h'(q_h)(\hat{q}_h - q_h) = \int_0^T (\tilde{\alpha} q_h + z_h)(\hat{q}_h - q_h) \, d\tau \qquad (5.27)$$

with $z_h = z_h(q_h)$ is the solution of the discrete co-state problem:

$$\begin{cases} -(z_{h,t}, v_h)_{\Omega_T} + a(z_h, v_h)_{\Omega_T} = (y_h - y_{\tilde{d}}, v_h)_{\Omega_T}, \\ z_h(\cdot, T) = 0, \end{cases} \qquad (5.28)$$

$\forall v_h \in H^1(0,T;V_h^0)$. The variational inequality (5.26) is equivalent to the following pointwise projection formula:

$$q_h = P_{[q_c,q_d]}\Big(-\frac{1}{\tilde{\alpha}} z_h\Big), \tag{5.29}$$

where the projection $P_{[q_c,q_d]}$ is defined by (5.19).

For the purpose of error estimates, it is convenient to introduce the following two auxiliary problems: For $q \in Q_{ad}^P$, let a pair $(y_h(q), z_h(q)) \in L^2(0,T;V_h^0) \times H^1(0,T;V_h^0)$ satisfy

$$\begin{aligned}
-(y_h(q), v_{h,t})_{\Omega_T} + a(y_h(q), v_h)_{\Omega_T} &= << \mu, v_h >>_{\Omega_T} + (q, v_h)_{\Omega_T} \\
&\quad + (y_{h,0}(q), v_h(\cdot,0)), && (5.30)\\
-(z_{h,t}(q), v_h)_{\Omega_T} + a(z_h(q), v_h)_{\Omega_T} &= (y_h(q) - y_{\tilde{d}}, v_h)_{\Omega_T}, && (5.31)\\
z_h(q)(\cdot,T) &= 0, && (5.32)
\end{aligned}$$

$\forall v_h \in H^1(0,T;V_h^0)$. Since $q \in Q_{ad}^P$, we conclude from Lemma 5.3.1 that problem (5.30) admits a unique solution $y_h(q) \in L^2(0,T;V_h^0)$. Similarly, (5.31)-(5.32) admits a unique solution $z_h(q) \in H^1(0,T;V_h^0)$.

In the following two lemmas, we prove some intermediate error estimates for the state and co-state variables.

Lemma 5.3.2. *Assume that $q \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^2(\Omega))$ and $\mu = g\omega$ with $g \in \mathcal{C}([0,T];L^2(\Omega))$, $\omega \in \mathcal{M}[0,T]$. Let y and $y_h(q)$ be the solutions of (5.13) and (5.30), respectively. Then, we have*

$$\|y - y_h(q)\|_{L^2(L^2(\Omega))} \le Ch\Big(\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|y_0\|_{L^2(\Omega)} + \|q\|_{L^2(L^2(\Omega))}\Big).$$

Proof. The proof will proceed by the duality argument. Let ψ be the solution of the problem (1.14) with $f \in L^2(0,T;L^2(\Omega))$. Then, using (5.13), (4.26) and (5.30), we obtain

$$\begin{aligned}
\int_{\Omega_T} (y - y_h(q)) f \, dx d\tau &= \int_0^T \int_\Omega (y - y_h(q))(-\psi_t + \mathcal{A}^*\psi) \, dx d\tau \\
&= -(y, \psi_t)_{\Omega_T} + a(y,\psi)_{\Omega_T} + (y_h(q), \psi_t)_{\Omega_T} - a(y_h(q), \psi)_{\Omega_T} \\
&= << \mu, \psi >>_{\Omega_T} + (y_0, \psi(\cdot,0)) + (q,\psi)_{\Omega_T} \\
&\quad + (y_h(q), \psi_{h,t})_{\Omega_T} - a(y_h(q), \psi_h)_{\Omega_T} \\
&= << \mu, \psi >>_{\Omega_T} + (y_0, \psi(\cdot,0)) + (q,\psi)_{\Omega_T} - << \mu, \psi_h >>_{\Omega_T} \\
&\quad - (y_{h,0}(q), \psi_h(\cdot,0)) - (q, \psi_h)_{\Omega_T}.
\end{aligned}$$

Use of (4.21) gives

$$\begin{aligned}\int_{\Omega_T}(y-y_h(q))f\,dxd\tau &= << \mu,\psi-\psi_h >>_{\Omega_T} +(y_0,\psi(\cdot,0)-\psi_h(\cdot,0))+(q,\psi-\psi_h)_{\Omega_T}\\ &= \int_0^T\Big(\int_\Omega g(x,\tau)(\psi-\psi_h)\,dx\Big)\,d\omega(\tau)+(y_0,\psi(\cdot,0)-\psi_h(\cdot,0))\\ &\quad +(q,\psi-\psi_h)_{\Omega_T}\\ &\le C\Big(\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+\|y_0\|_{L^2(\Omega)}\Big)\|\psi-\psi_h\|_{L^\infty(L^2(\Omega))}\\ &\quad +\|q\|_{L^2(L^2(\Omega))}\|\psi-\psi_h\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Apply Lemma 4.3.2 and regularity results from Lemma 1.2.3 to obtain

$$\begin{aligned}\int_{\Omega_T}(y-y_h(q))f\,dxd\tau &\le \Big(Ch\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+Ch\|y_0\|_{L^2(\Omega)}\\ &\quad +Ch^2\|q\|_{L^2(L^2(\Omega))}\Big)\Big(\|\psi\|_{L^2(H^2(\Omega))}+\|\psi_t\|_{L^2(L^2(\Omega))}\Big)\\ &\le Ch\Big(\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+\|y_0\|_{L^2(\Omega)}+\|q\|_{L^2(L^2(\Omega))}\Big)\|f\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Then, from the definition of $L^2(0,T;L^2(\Omega))$-norm, we have

$$\begin{aligned}\|y-y_h(q)\|_{L^2(L^2(\Omega))} &= \sup_{f\in L^2(0,T;L^2(\Omega)),f\neq 0}\frac{(f,y-y_h(q))_{\Omega_T}}{\|f\|_{L^2(L^2(\Omega))}}\\ &\le Ch\Big(\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+\|y_0\|_{L^2(\Omega)}+\|q\|_{L^2(L^2(\Omega))}\Big).\end{aligned}$$

This completes the proof of the lemma. □

Remark 5.3.1. *In Lemma 5.3.2, we have obtained intermediate error estimate for the state variable of order $\mathcal{O}(h)$ for the spatially discrete approximation of POCP with measure data in time. Compared to the estimate of Lemma 4.3.4 this estimate does not depend on the dimension of the domain Ω.*

Lemma 5.3.3. *Let z and $z_h(q)$ be the solutions of (5.17) and (5.31)-(5.32), respectively. Then, we have*

$$\|z-z_h(q)\|_{L^2(L^2(\Omega))}\le Ch^2\Big(\|y\|_{L^2(L^2(\Omega))}+\|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}\Big)+\|y-y_h(q)\|_{L^2(L^2(\Omega))}.$$

Proof. Let ϕ be the solution of (1.13) with $f\in L^2(0,T;L^2(\Omega))$. Then, from (5.17), (4.25), (5.31) and (5.32), we obtain

$$\begin{aligned}\int_{\Omega_T}(z - z_h(q))f\,dxd\tau &= \int_0^T\int_\Omega (z-z_h(q))(\phi_t+\mathcal{A}\phi)\,dxd\tau \\ &= (z,\phi_t)_{\Omega_T} + a(z,\phi)_{\Omega_T} - (z_h(q),\phi_t)_{\Omega_T} - a(z_h(q),\phi)_{\Omega_T} \\ &= -(z_t,\phi)_{\Omega_T} + a(z,\phi)_{\Omega_T} - (z_h(q),\phi_{h,t})_{\Omega_T} - a(z_h(q),\phi_h)_{\Omega_T} \\ &= (y-y_{\tilde{d}},\phi)_{\Omega_T} + (z_{h,t}(q),\phi_h)_{\Omega_T} - a(z_h(q),\phi_h)_{\Omega_T} \\ &= (y-y_{\tilde{d}},\phi)_{\Omega_T} - (y_h(q)-y_{\tilde{d}},\phi_h)_{\Omega_T} \\ &= (y-y_{\tilde{d}},\phi-\phi_h)_{\Omega_T} + (y-y_h(q),\phi_h)_{\Omega_T} \\ &\leq \|y-y_{\tilde{d}}\|_{L^2(L^2(\Omega))}\|\phi-\phi_h\|_{L^2(L^2(\Omega))} + \|y-y_h(q)\|_{L^2(L^2(\Omega))}\|\phi_h\|_{L^2(L^2(\Omega))}.\end{aligned}$$

An application of Lemmas 1.2.3 and 4.3.2 yields

$$\begin{aligned}\int_{\Omega_T}(z-z_h(q))f\,dxd\tau \;\leq\; & Ch^2\Big(\|y\|_{L^2(L^2(\Omega))} + \|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}\Big)\|f\|_{L^2(L^2(\Omega))} \\ & +\|\phi_h\|_{L^2(L^2(\Omega))}\|y-y_h(q)\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Using the stability estimate $\|\phi_h\|_{L^2(L^2(\Omega))} \leq C\|f\|_{L^2(L^2(\Omega))}$ and the definition of $L^2(0,T;L^2(\Omega))$-norm yields the desired estimate. □

In the next lemma, we derive the error between the derivatives of continuous and discrete reduced cost functionals.

Lemma 5.3.4. *Let $r \in L^2(0,T;L^2(\Omega))$, and let the derivatives of $\tilde{j}(q)$ and $\tilde{j}_h(q)$ be given by (5.16) and (5.27) with $q_h = q$, respectively. Then, we have*

$$|\tilde{j}'(q)(r) - \tilde{j}_h'(q)(r)| \leq \hat{C}_2 h\|r\|_{L^2(L^2(\Omega))},$$

where

$$\hat{C}_2 = C\Big(\|g\|_{L^\infty(L^2(\Omega))},\ \|\omega\|_{\mathcal{M}[0,T]},\ \|y_0\|_{L^2(\Omega)},\ \|q\|_{L^2(L^2(\Omega))}, \|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}\Big). \tag{5.33}$$

Proof. For $r \in L^2(0,T;L^2(\Omega))$, we have

$$\tilde{j}'(q)(r) = \int_0^T(\tilde{\alpha}q(\tau)+z(\tau),r)\,d\tau,$$

and

$$\tilde{j}_h'(q)(r) = \int_0^T(\tilde{\alpha}q(\tau)+z_h(q)(\tau),r)\,d\tau.$$

Then, we have

$$\tilde{j}'(q)(r) - \tilde{j}_h'(q)(r) = \int_0^T(z(\tau)-z_h(q)(\tau),r)\,d\tau,$$

and hence, we obtain

$$\begin{aligned}|\tilde{j}'(q)(r) - \tilde{j}_h'(q)(r)| &\leq \left| \int_0^T (z(\tau) - z_h(q)(\tau), r)\, d\tau \right| \\ &\leq \|z - z_h(q)\|_{L^2(L^2(\Omega))} \|r\|_{L^2(L^2(\Omega))} \\ &\leq \hat{C}_2 h \|r\|_{L^2(L^2(\Omega))},\end{aligned}$$

where we have used Lemma 5.3.3. This completes the proof. □

We now present one of the the main result of this section namely, the error between the continuous control q of (5.14) and discrete control q_h of (5.25).

Theorem 5.3.1. *Let q and q_h be the optimal controls of (5.14) and (5.25), respectively. Assume that the second order optimality condition (5.20) is valid. Then the following error estimate holds:*

$$\|q - q_h\|_{L^2(L^2(\Omega))} \leq \frac{\tilde{C}_2}{\sqrt{\gamma_P}} h + \frac{\hat{C}_2}{\gamma_P} h,$$

where $\hat{C}_2$ is given by (5.33) and

$$\tilde{C}_2 = C\Big(\|g\|_{L^\infty(L^2(\Omega))},\ \|\omega\|_{\mathcal{M}[0,T]},\ \|y_0\|_{L^2(\Omega)},\ \|q\|_{L^2(L^2(\Omega))},\ \|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}, \tilde{\alpha} \Big). \tag{5.34}$$

Proof. The proof will follow in a manner similar to Theorem 4.3.1. But, for the sake of completeness we present the proof. With $r \in L^2(0,T;L^2(\Omega))$, we have

$$\tilde{j}''(q)(r,r) \geq \gamma_P \|r\|^2_{L^2(L^2(\Omega))}, \tag{5.35}$$

and

$$\tilde{j}_h''(q_h)(r,r) \geq \gamma_P \|r\|^2_{L^2(L^2(\Omega))}. \tag{5.36}$$

We now formulate the following auxiliary problem:

$$\min_{q_h \in Q_h^P} \tilde{j}(q_h), \tag{5.37}$$

where we only discretize the control variable. Suppose $\tilde{q}_h$ be the solution of problem (5.37). We decompose the error as follows

$$q - q_h = (q - \tilde{q}_h) + (\tilde{q}_h - q_h), \tag{5.38}$$

and proceed to estimate each term separately. In view of (5.35), we have for $\tilde{\lambda} \in [0,1]$ with $\xi = \tilde{\lambda} q + (1-\tilde{\lambda})\tilde{q}_h$ and h sufficiently small,

$$\begin{aligned}\gamma_P \|q - \tilde{q}_h\|^2_{L^2(L^2(\Omega))} &\leq \tilde{j}''(\xi)(q - \tilde{q}_h, q - \tilde{q}_h) \\ &= \tilde{j}'(q)(q - \tilde{q}_h) - \tilde{j}'(\tilde{q}_h)(q - \tilde{q}_h) \\ &= \tilde{j}'(q)(q - \tilde{q}_h) - \tilde{j}'(\tilde{q}_h)(q - \mathcal{L}_h q) - j'(\tilde{q}_h)(\mathcal{L}_h q - \tilde{q}_h).\end{aligned}$$

The necessary optimality condition imply, for h sufficiently small

$$\tilde{j}'(q)(q-\tilde{q}_h) \leq 0 \quad \text{and} \quad -\tilde{j}'(\tilde{q}_h)(\mathcal{L}_h q-\tilde{q}_h) \leq 0,$$

which together with the properties of $\mathcal{L}_h$ and the Young's inequality yields

$$\begin{aligned}
\gamma_P\|q-\tilde{q}_h\|^2_{L^2(L^2(\Omega))} &\leq -\tilde{j}'(\tilde{q}_h)(q-\mathcal{L}_h q)\\
&= -\int_0^T \Big(\tilde{\alpha}\tilde{q}_h + z(\tilde{q}_h), q-\mathcal{L}_h q\Big)\,d\tau\\
&= -\int_0^T (z(\tilde{q}_h)-\mathcal{L}_h z(\tilde{q}_h), q-\mathcal{L}_h q)\,d\tau\\
&\leq \int_0^T \Big(\frac{1}{2}\|z(\tilde{q}_h)-\mathcal{L}_h z(\tilde{q}_h)\|^2_{L^2(\Omega)} + \frac{1}{2}\|q-\mathcal{L}_h q\|^2_{L^2(\Omega)}\Big)\,d\tau.
\end{aligned}$$

Therefore, we have

$$\|q-\tilde{q}_h\|_{L^2(L^2(\Omega))} \leq \int_0^T \Big(\frac{C}{\sqrt{\gamma_P}}\|z(\tilde{q}_h)-\mathcal{L}_h z(\tilde{q}_h)\|_{L^2(\Omega)} + \frac{C}{\sqrt{\gamma_P}}\|q-\mathcal{L}_h q\|_{L^2(\Omega)}\Big)\,d\tau.$$

An application of Lemma 4.3.1 yields

$$\|q-\tilde{q}_h\|_{L^2(L^2(\Omega))} \leq \int_0^T \Big(\frac{C}{\sqrt{\gamma_P}}h\|z(\tilde{q}_h)\|_{H^1(\Omega)} + \frac{C}{\sqrt{\gamma_P}}h\|q\|_{H^1(\Omega)}\Big)\,d\tau \leq \frac{\tilde{C}_2}{\sqrt{\gamma_P}}h,$$

where $\tilde{C}_2$ is given by (5.34). To estimate the second term in (5.38), we use the necessary optimality condition (5.26), which leads to the following relation:

$$\tilde{j}_h'(q_h)(q_h-r_h) \leq 0 \leq \tilde{j}'(\tilde{q}_h)(r_h-\tilde{q}_h), \qquad \forall r_h \in Q_h^P.$$

With $\xi = \tilde{\lambda}q_h + (1-\tilde{\lambda})\tilde{q}_h$, $\tilde{\lambda} \in [0,1]$ and h sufficiently small, we have from (5.36)

$$\begin{aligned}
\gamma_P\|q_h-\tilde{q}_h\|^2_{L^2(L^2(\Omega))} &\leq j_h''(\xi)(q_h-\tilde{q}_h, q_h-\tilde{q}_h)\\
&= \tilde{j}_h'(q_h)(q_h-\tilde{q}_h) - \tilde{j}_h'(\tilde{q}_h)(q_h-\tilde{q}_h)\\
&\leq \tilde{j}'(\tilde{q}_h)(q_h-\tilde{q}_h) - \tilde{j}_h'(\tilde{q}_h)(q_h-\tilde{q}_h)\\
&\leq \hat{C}_2 h\|q_h-\tilde{q}_h\|_{L^2(L^2(\Omega))}.
\end{aligned}$$

The last step follows from Lemma 5.3.4 and $\hat{C}_2$ is given by (5.33). This completes the proof of the theorem. □

The final result of this section gives the error between the continuous and spatially discrete state variables in the $L^2(0,T;L^2(\Omega))$-norm.

Theorem 5.3.2. *Let $y \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $y_h \in L^2(0,T;V_h^0)$ be the solutions of (5.13) and (5.22), respectively. Assume that $\mu = g\omega$, g and ω are given functions such that $g \in \mathcal{C}([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. Then, we have*

$$\begin{aligned}\|y-y_h\|_{L^2(L^2(\Omega))} &\leq Ch\Big(\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|y_0\|_{L^2(\Omega)} + \|q\|_{L^2(L^2(\Omega))}\Big)\\ &\quad +\|q-q_h\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Proof. Let ψ be the solution of the problem (1.14) with $f \in L^2(0,T;L^2(\Omega))$. Then from (5.13), (4.21), (4.26) and (5.22), we obtain

$$\begin{aligned}\int_{\Omega_T}(y-y_h)f\,dxd\tau &= \int_0^T\Big(\int_\Omega g(x,\tau)(\psi-\psi_h)\,dx\Big)\,d\omega(\tau) + (y_0,\psi(\cdot,0)-\psi_h(\cdot,0))\\ &\quad +(q,\psi-\psi_h)_{\Omega_T} + (q-q_h,\psi_h)_{\Omega_T}\\ &\leq C\Big(\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|y_0\|_{L^2(\Omega)}\Big)\|\psi-\psi_h\|_{L^\infty(L^2(\Omega))}\\ &\quad +\|q\|_{L^2(L^2(\Omega))}\|\psi-\psi_h\|_{L^2(L^2(\Omega))} + \|q-q_h\|_{L^2(L^2(\Omega))}\|\psi_h\|_{L^2(L^2(\Omega))}.\end{aligned}$$

From Lemma 4.3.2, we have

$$\begin{aligned}\int_{\Omega_T}(y-y_h)f\,dxd\tau &\leq \Big(Ch\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + Ch\|y_0\|_{L^2(\Omega)}\\ &\quad +Ch^2\|q\|_{L^2(L^2(\Omega))}\Big)\Big(\|\psi\|_{L^2(H^2(\Omega))} + \|\psi_t\|_{L^2(L^2(\Omega))}\Big)\\ &\quad +\|q-q_h\|_{L^2(L^2(\Omega))}\|\psi_h\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Using the stability estimate $\|\psi_h\|_{L^2(L^2(\Omega))} \leq C\|f\|_{L^2(L^2(\Omega))}$ and Lemma 1.2.3, we obtain

$$\begin{aligned}\int_{\Omega_T}(y-y_h)f\,dxd\tau &\leq Ch\Big\{\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|y_0\|_{L^2(\Omega)} + \|q\|_{L^2(L^2(\Omega))}\Big\}\|f\|_{L^2(L^2(\Omega))}\\ &\quad +\|q-q_h\|_{L^2(L^2(\Omega))}\|f\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Finally, the definition of $L^2(0,T;L^2(\Omega))$-norm gives the desired estimate. This completes the proof of the theorem. □

The following theorem presents the error between the continuous and discrete co-state variables.

Theorem 5.3.3. *Let z and z_h be the solutions of (5.17) and (5.28), respectively. Then*

$$\|z-z_h\|_{L^2(L^2(\Omega))} \leq Ch^2\Big(\|y\|_{L^2(L^2(\Omega))} + \|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}\Big) + \|y-y_h\|_{L^2(L^2(\Omega))}.$$

Proof. The proof follows by same lines of Lemma 5.3.3. So, we omit the details. □

5.4 Fully discrete approximations of POCP

In this section, we shall consider the fully discrete approximations of (5.21)-(5.22) based on the backward-Euler scheme for the time discretization and the continuous piecewise linear finite elements for the space discretization. The control variable is approximated by the piecewise constant functions.

Let $0 = t_0 < t_1 < ... < t_{N-1} < t_N = T$ be a partition of $[0, T]$ with time step-size $k := t_n - t_{n-1}$ and $I_n := (t_{n-1}, t_n]$ for $n = 1, 2, \dots, N$. For any function $\chi \in C([0, T]; L^2(\Omega))$, define $\chi^n := \chi(x, t_n)$ and $\bar{\partial}\chi^n := \frac{\chi^n - \chi^{n-1}}{k}$. For $n = 1, 2, \dots, N$, construct the finite element spaces $V_h^n \subset H_0^1(\Omega)$ and $Q_{h,n}^P$ with the mesh $\mathcal{T}_h^n$. For our error analysis, we set $k = \mathcal{O}(h^2)$ throughout this section.

The fully discrete approximation of (5.21)-(5.22) is to find $(q_h^n, y_h^n) \in Q_{h,n}^P \times V_h^n,\ n = 1, 2, \dots, N$, such that

$$\min_{q_h^n \in Q_{h,n}^P} \tilde{J}(q_h^n, y_h^n) = \frac{1}{2} \sum_{n=1}^{N} \int_{I_n} \left\{ \|y_h^n - y_{\tilde{d}}^n\|_{L^2(\Omega)}^2 + \tilde{\alpha} \|q_h^n\|_{L^2(\Omega)}^2 \right\} d\tau \tag{5.39}$$

subject to

$$\begin{cases} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) = << \mu, v_h >>_{I_n} + (q_h^n, v_h), \quad \forall v_h \in V_h^n, \\ y_h^0(x) = y_{h,0}(x), \qquad x \in \Omega. \end{cases} \tag{5.40}$$

Here

$$<< \mu, v_h >>_{I_n} = \frac{1}{k} \int_{\Omega \times (t_{n-1}, t_n]} v_h \, d\mu = \frac{1}{k} \int_{t_{n-1}}^{t_n} \left(\int_{\Omega} g(x, \tau) v_h(x) \, dx \right) d\omega(\tau), \quad \forall v_h \in V_h^n.$$

The optimal control problem (5.39)-(5.40) has a unique solution (q_h^n, y_h^n), $n = 1, 2, \dots, N$, such that $(y_h^n, q_h^n, z_h^{n-1})$ satisfies the following optimality conditions:

$$(\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) = << \mu, v_h >>_{I_n} + (q_h^n, v_h), \quad \forall v_h \in V_h^n, \quad n \geq 1, \tag{5.41}$$

$$y_h^0(x) = y_{h,0}(x), \qquad x \in \Omega, \tag{5.42}$$

$$-(\bar{\partial} z_h^n, v_h) + a(z_h^{n-1}, v_h) = (y_h^n - y_{\tilde{d}}^n, v_h), \quad \forall v_h \in V_h^n, \tag{5.43}$$

$$z_h^N(x) = 0, \qquad x \in \Omega, \tag{5.44}$$

$$(\tilde{\alpha} q_h^n + z_h^{n-1}, \hat{q}_h^n - q_h^n) \geq 0, \qquad \forall \hat{q}_h^n \in Q_{h,n}^P. \tag{5.45}$$

Analogous to the spatially discrete case, we reformulate the fully discrete optimal control problem (5.39)-(5.40) as:

$$\min_{q_h^n \in Q_{h,n}^P} \tilde{j}_h^n(q_h^n) \tag{5.46}$$

for $n = 1, \ldots, N$.

The derivative of the discretized reduced cost functional is given by

$$(\tilde{j}_h^n)'(q_h^n)(r) = \sum_{n=1}^{N} \int_{I_n} (\tilde{\alpha} q_h^n + z_h^{n-1}, r)\, d\tau. \tag{5.47}$$

As in the case of spatially discrete error analysis, we shall first obtain some intermediate error estimates which will be then used to derive error estimates for the fully discrete scheme. Let $\mathcal{Y}_h$ and $\mathcal{Z}_h$ be the fully discrete finite element approximations of y and z, respectively, which is piecewise constant in time and piecewise linear in space on each time interval I_n. For this, we first derive some stability estimates for (5.40).

Lemma 5.4.1. *Assume that g and ω are given functions such that $g \in \mathcal{C}([0,T]; L^2(\Omega))$, $\omega \in \mathcal{M}([0,T])$ and $y_0 \in L^2(\Omega)$. Let $y_h^n \in V_h^n$, $n = 1, 2 \ldots, N$, be the solutions of fully discrete problem (5.40) with $y_{h,0} = \mathcal{L}_h y_0$. Then there exists a constant C independent of h, k and the data (g, ω, y_0) such that*

$$\begin{aligned} \sum_{n=1}^{N} \|y_h^n - y_h^{n-1}\|_{L^2(\Omega)}^2 + k\|y_h^N\|_{H^1(\Omega)}^2 \;&\leq\; C\Big(\|y_0\|_{L^2(\Omega)}^2 + \|g\|_{L^\infty(L^2(\Omega))}^2 \|\omega\|_{\mathcal{M}[0,T]}^2\Big) \\ &\quad + C\sum_{n=1}^{N} \|kq_h^n\|_{L^2(\Omega)}^2 \end{aligned} \tag{5.48}$$

and

$$\begin{aligned} \|y_h^N\|_{L^2(\Omega)}^2 + \sum_{n=1}^{N} k\|y_h^n\|_{H^1(\Omega)}^2 \;&\leq\; C\Big(\|y_0\|_{L^2(\Omega)}^2 + \|g\|_{L^\infty(L^2(\Omega))}^2 \|\omega\|_{\mathcal{M}[0,T]}^2\Big) \\ &\quad + C\sum_{n=1}^{N} \|kq_h^n\|_{L^2(\Omega)}^2. \end{aligned} \tag{5.49}$$

Proof. Choose $v_h = k(y_h^n - y_h^{n-1})$ in (5.40) to obtain

$$(y_h^n - y_h^{n-1}, y_h^n - y_h^{n-1}) + ka(y_h^n, y_h^n - y_h^{n-1}) = k << \mu, y_h^n - y_h^{n-1} >>_{I_n} + k(q_h^n, y_h^n - y_h^{n-1}).$$

Using coercivity, continuity of $a(\cdot,\cdot)$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|y_h^n - y_h^{n-1}\|_{L^2(\Omega)}^2 + k\beta\|y_h^n\|_{H^1(\Omega)}^2 \;&\leq\; ka(y_h^n, y_h^{n-1}) + \int_{t_{n-1}}^{t_n} (g(\tau), y_h^n - y_h^{n-1})\, d\omega(\tau) \\ &\quad + k(q_h^n, y_h^n - y_h^{n-1}) \\ &\leq\; \beta_1 \frac{k}{2}\|y_h^n\|_{H^1(\Omega)}^2 + \beta_1 \frac{k}{2}\|y_h^{n-1}\|_{H^1(\Omega)}^2 \\ &\quad + \|y_h^n - y_h^{n-1}\|_{L^2(\Omega)} \int_{t_{n-1}}^{t_n} \|g(\tau)\|_{L^2(\Omega)}\, d\omega(\tau) \\ &\quad + k\|q_h^n\|_{L^2(\Omega)}\|y_h^n - y_h^{n-1}\|_{L^2(\Omega)}. \end{aligned}$$

Applying the inequality $ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$, $a, b \geq 0$, $\epsilon > 0$, we obtain

$$\|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + k\|y_h^n\|^2_{H^1(\Omega)} \leq Ck\|y_h^{n-1}\|^2_{H^1(\Omega)} + C\Big(\int_{t_{n-1}}^{t_n} \|g(\tau)\|_{L^2(\Omega)}\, d\omega(\tau)\Big)^2 + C\|kq_h^n\|^2_{L^2(\Omega)}.$$

Summation over n from $n = 1$ to N leads to

$$\sum_{n=1}^{N} \|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + k\|y_h^N\|^2_{H^1(\Omega)} \leq Ck\|\mathcal{L}_h y_0\|^2_{H^1(\Omega)} + C\sum_{n=1}^{N}\left(\int_{t_{n-1}}^{t_n} \|g(\tau)\|_{L^2(\Omega)}\, d\omega(\tau)\right)^2 + C\sum_{n=1}^{N} \|kq_h^n\|^2_{L^2(\Omega)}.$$

Now, using the inverse estimate (4.18) in $k\|\mathcal{L}_h y_0\|^2_{H^1(\Omega)} \leq kh^{-2}\|\mathcal{L}_h y_0\|^2_{L^2(\Omega)} \leq C\|y_0\|^2_{L^2(\Omega)}$, we have

$$\sum_{n=1}^{N} \|y_h^n - y_h^{n-1}\|^2_{L^2(\Omega)} + k\|y_h^N\|^2_{H^1(\Omega)} \leq C\Big(\|y_0\|^2_{L^2(\Omega)} + \|g\|^2_{L^\infty(L^2(\Omega))}\|\omega\|^2_{\mathcal{M}[0,T]}\Big) + C\sum_{n=1}^{N} \|kq_h^n\|^2_{L^2(\Omega)},$$

which proves (5.48). To prove the second part of the lemma, set $v_h = ky_h^n$ in (5.40) and use of coercive property of the bilinear form together with the Cauchy-Schwarz inequality and Young's inequality we have

$$\begin{aligned}\|y_h^n\|^2_{L^2(\Omega)} + \beta k\|y_h^n\|^2_{H^1(\Omega)} &\leq (y_h^{n-1}, y_h^n) + \int_{t_{n-1}}^{t_n} (g(\tau), y_h^n)\, d\omega(\tau) + (kq_h^n, y_h^n) \\ &\leq \frac{1}{2}\|y_h^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|y_h^{n-1}\|^2_{L^2(\Omega)} + \|y_h^n\|_{L^2(\Omega)} \int_{t_{n-1}}^{t_n} \|g(\tau)\|_{L^2(\Omega)}\, d\omega(\tau) \\ &\quad + C\|kq_h^n\|_{L^2(\Omega)}\|y_h^n\|_{L^2(\Omega)}.\end{aligned}$$

Using the inequality $ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$, $a, b \geq 0$ and $\epsilon > 0$, and the standard kickback argument we obtain

$$\|y_h^n\|^2_{L^2(\Omega)} + k\|y_h^n\|^2_{H^1(\Omega)} \leq C\|y_h^{n-1}\|^2_{L^2(\Omega)} + C\left(\int_{t_{n-1}}^{t_n} \|g(\tau)\|_{L^2(\Omega)}\, d\omega(\tau)\right)^2 + C\|kq_h^n\|^2_{L^2(\Omega)}.$$

Summing the above over n from $n = 1$ to N, it now leads to

$$\|y_h^N\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} k\|y_h^n\|^2_{H^1(\Omega)} \leq C\|\mathcal{L}_h y_0\|^2_{L^2(\Omega)} + C\sum_{n=1}^{N}\left(\int_{t_{n-1}}^{t_n} \|g(\tau)\|_{L^2(\Omega)}\, d\omega(\tau)\right)^2 + C\sum_{n=1}^{N} \|kq_h^n\|^2_{L^2(\Omega)},$$

and this completes the proof of (5.49). □

In the following lemma, we shall investigate the error between the continuous solution y and intermediate solution $y_h^n(q)$ of

$$\begin{cases} (\bar{\partial} y_h^n(q), v_h) + a(y_h^n(q), v_h) = << \mu, v_h >>_{I_n} + (q, v_h), \quad \forall v_h \in V_h^n,\ n \geq 1, \\ y_h^0(q)(x) = y_{h,0}(x), \quad x \in \Omega. \end{cases} \tag{5.50}$$

Lemma 5.4.2. *Assume that $q \in L^2(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\mu = g\omega$, g and ω are given functions such that $g \in \mathcal{C}([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. Let $y \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be the solution of (5.13), and let $\mathcal{Y}_h(q)$ be the solution of (5.50). Then, we have*

$$\|y - \mathcal{Y}_h(q)\|_{L^2(L^2(\Omega))} \leq C(h + k^{\frac{1}{2}})\Big(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|q\|_{L^2(L^2(\Omega))}\Big),$$

where the constant C is independent of h, k and the given data $(g,\ \omega,\ y_0)$.

Proof. Let ψ be the solution of problem (1.14) with $f \in L^2(0,T;L^2(\Omega))$. It follows from (5.13) and $\psi^N = 0$ that

$$\begin{aligned}
\int_{\Omega_T}(y - \mathcal{Y}_h(q)) f\, dx d\tau &= \int_0^T \int_\Omega (y - \mathcal{Y}_h(q))(-\psi_t + \mathcal{A}^*\psi)\, dx d\tau \\
&= -(y, \psi_t)_{\Omega_T} + a(y,\psi)_{\Omega_T} + \sum_{n=1}^N \int_{I_n} \{(y_h^n(q), \psi_t) - a(y_h^n(q), \psi)\}\, d\tau \\
&= << \mu, \psi >>_{\Omega_T} + (y_0, \psi(\cdot,0)) + (q,\psi)_{\Omega_T} \\
&\quad + \sum_{n=1}^N \int_{I_n} \{k^{-1}(y_h^n(q), \psi^n - \psi^{n-1}) - a(y_h^n(q), \psi)\}\, d\tau \\
&= << \mu, \psi >>_{\Omega_T} + (y_0, \psi(\cdot,0)) - \sum_{n=1}^N \int_{I_n} \{k^{-1}(y_h^n(q) - y_h^{n-1}(q), \psi^{n-1}) \\
&\quad + a(y_h^n(q), \psi)\}\, d\tau + (y_h^N(q), \psi^N) - (y_h^0(q), \psi(\cdot,0)) + (q,\psi)_{\Omega_T} \\
&= -\sum_{n=1}^N \int_{I_n} \{k^{-1}(y_h^n(q) - y_h^{n-1}(q), \psi^{n-1}) + a(y_h^n(q), \psi)\}\, d\tau \\
&\quad + << \mu, \psi >>_{\Omega_T} + (y_0 - y_h^0(q), \psi(\cdot,0)) + (q,\psi)_{\Omega_T}.
\end{aligned} \tag{5.51}$$

From (5.50), we have

$$\sum_{n=1}^N \{(\bar{\partial} y_h^n(q), \bar{\mathcal{R}}_h\psi) + a(y_h^n(q), \bar{\mathcal{R}}_h\psi)\} = \sum_{n=1}^N << \mu, \bar{\mathcal{R}}_h\psi >>_{I_n} + \sum_{n=1}^N (q, \bar{\mathcal{R}}_h\psi), \tag{5.52}$$

where $\bar{\mathcal{R}}_h\psi \in V_h^n$ in (4.66). Integrate (5.52) in time and add the resulting equation to (5.51) together with (4.67), we obtain

$$\begin{aligned}\int_{\Omega_T}(y \quad - \quad & \mathcal{Y}_h(q))f\,dxd\tau = -\sum_{n=1}^{N}\int_{I_n}\{k^{-1}(y_h^n(q)-y_h^{n-1}(q),\psi^{n-1}-\bar{\mathcal{R}}_h\psi)\\ & +a(y_h^n(q),\bar{\psi}-\bar{\mathcal{R}}_h\psi)\}\,d\tau+\{<<\mu,\psi>>_{\Omega_T}-\sum_{n=1}^{N}\int_{I_n}<<\mu,\bar{\mathcal{R}}_h\psi>>_{I_n}\ d\tau\}\\ & +\{(y_0-y_h^0(q),\psi(\cdot,0))\}+\{(q,\psi)_{\Omega_T}-\sum_{n=1}^{N}\int_{I_n}(q,\bar{\mathcal{R}}_h\psi)\,d\tau\}\\ =: \quad & \tilde{E}_1+\tilde{E}_2+\tilde{E}_3+\tilde{E}_4. \end{aligned} \tag{5.53}$$

Now, we need to estimate $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$ and $\tilde{E}_4$. To estimate $\tilde{E}_1$, (4.69) and the Cauchy-Schwarz inequality yields

$$|\tilde{E}_1| \leq \tilde{F}_1 \cdot \tilde{F}_2,$$

where

$$\tilde{F}_1=\left(\sum_{n=1}^{N}\|y_h^n(q)-y_h^{n-1}(q)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}},\ \tilde{F}_2=\left(\sum_{n=1}^{N}\|\psi^{n-1}-\bar{\mathcal{R}}_h\psi\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Using (5.48) of Lemma 5.4.1 gives

$$\tilde{F}_1 \leq C\Big(\|y_0\|_{L^2(\Omega)}+\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+\|q\|_{L^2(L^2(\Omega))}\Big).$$

Similar to (4.70), we conclude that

$$\begin{aligned}|\tilde{F}_2| \quad \leq \quad & C\Big(\sum_{n=1}^{N}\{h^4\|\bar{\psi}\|^2_{H^2(\Omega)}+k\|\psi_t\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}\}\Big)^{\frac{1}{2}}\\ \leq \quad & C\Big(\sum_{n=1}^{N}\{h^4k^{-1}\|\psi\|^2_{L^2(t_{n-1},t_n;H^2(\Omega))}+k\|\psi_t\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}\}\Big)^{\frac{1}{2}}\\ \leq \quad & C\Big(h\|\psi\|_{L^2(H^2(\Omega))}+k^{\frac{1}{2}}\|\psi_t\|_{L^2(L^2(\Omega))}\Big),\end{aligned}$$

and use of Lemma 1.2.3 yields

$$|\tilde{E}_1| \leq C(h+k^{\frac{1}{2}})\|f\|_{L^2(L^2(\Omega))}\Big(\|y_0\|_{L^2(\Omega)}+\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+\|q\|_{L^2(L^2(\Omega))}\Big). \tag{5.54}$$

For $\tilde{E}_2$, we note that

$$\begin{aligned}|\tilde{E}_2| \quad = \quad & \Big|\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\Big(\int_{\Omega}g(x,\tau)(\psi-\bar{\mathcal{R}}_h\psi)(x)\,dx\Big)d\omega(\tau)\Big|\\ \leq \quad & C\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^\infty(L^2(\Omega))}.\end{aligned} \tag{5.55}$$

Use (4.75) and Lemma 1.2.3 in (5.55) to obtain

$$\begin{aligned}|\tilde{E}_2| &\leq C(k^{\frac{1}{2}}+h)\|\psi\|_{L^2(H^2(\Omega))}\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} \\ &\leq C(k^{\frac{1}{2}}+h)\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}\|f\|_{L^2(L^2(\Omega))}. \end{aligned} \tag{5.56}$$

For $\tilde{E}_3$, we find that

$$\begin{aligned}|\tilde{E}_3| &= |(y_0-y_h^0(q),\psi(\cdot,0))| \leq \|y_0-\mathcal{L}_h y_0\|_{H^{-1}(\Omega)}\|\psi(\cdot,0)\|_{H^1(\Omega)} \\ &\leq Ch\|y_0\|_{L^2(\Omega)}\|f\|_{L^2(L^2(\Omega))}. \end{aligned} \tag{5.57}$$

Finally, for $\tilde{E}_4$, an application of the Cauchy-Schwarz inequality, (4.75) and Lemma 1.2.3 gives

$$\begin{aligned}|\tilde{E}_4| &= \Big|(q,\psi)_{\Omega_T}-\sum_{n=1}^{N}\int_{I_n}(q,\bar{\mathcal{R}}_h\psi)\,d\tau\Big| \\ &= \Big|\sum_{n=1}^{N}\int_{I_n}(q,\psi-\bar{\mathcal{R}}_h\psi)\,d\tau\Big| \\ &\leq \|q\|_{L^2(L^2(\Omega))}\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))} \\ &\leq C(h+k^{\frac{1}{2}})\|q\|_{L^2(L^2(\Omega))}\|f\|_{L^2(L^2(\Omega))}, \end{aligned} \tag{5.58}$$

where we have used the fact $\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))} \leq \|\psi-\bar{\mathcal{R}}_h\psi\|_{L^\infty(L^2(\Omega))}$ and (4.75). It follows from (5.54), (5.56)-(5.58) and (5.53) that

$$\begin{aligned}\|y-\mathcal{Y}_h(q)\|_{L^2(L^2(\Omega))} &= \sup_{f\in L^2(0,T;L^2(\Omega)),f\neq 0}\frac{(f,y-\mathcal{Y}_h)_{\Omega_T}}{\|f\|_{L^2(L^2(\Omega))}} \\ &\leq C(h+k^{\frac{1}{2}})\Big(\|y_0\|_{L^2(\Omega)}+\|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+\|q\|_{L^2(L^2(\Omega))}\Big).\end{aligned}$$

This completes the proof of the lemma. □

In order to estimate $\|z-\mathcal{Z}_h(q)\|_{L^2(L^2(\Omega))}$, we define some auxiliary problem as follows: Let $z_h^{n-1}(q)\in V_h^n$ satisfy

$$\begin{cases}-(\bar{\partial}z_h^n(q),v_h)+a(z_h^{n-1}(q),v_h)=(y_h^n(q)-y_{\tilde{d}}^n,v_h), & \forall v_h\in V_h^n,\\ z_h^N(q)(x)=0, & x\in\Omega.\end{cases} \tag{5.59}$$

Lemma 5.4.3. *Let z and $\mathcal{Z}_h(q)$ be the solutions of (5.17) and (5.59), respectively. Then, for $n\geq 1$, we have*

$$\begin{aligned}\|z-\mathcal{Z}_h(q)\|_{L^2(L^2(\Omega))} &\leq C(h+k^{\frac{1}{2}})\Big(\|z\|_{L^2(H^2(\Omega))}+\|z_t\|_{L^2(L^2(\Omega))}\Big) \\ &\quad +\|y-\mathcal{Y}_h(q)\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Proof. Following the lines of arguments of Lemma 4.4.3, the estimate can be easily derived. Therefore, we omit the details. □

In the following lemma, we derive the error between the derivative of the continuous and fully discrete reduced cost functionals.

Lemma 5.4.4. *Let the derivative of the continuous and fully discrete cost functionals be given by (5.16) and (5.47), respectively. Then*

$$|\tilde{j}'(q)(r) - (\tilde{j}_h^n)'(q)(r)| \leq \bar{C}_2(h + k^{\frac{1}{2}})\|r\|_{L^2(L^2(\Omega))},$$

where

$$\bar{C}_2 = C\Big(\|y_0\|_{L^2(\Omega)},\ \|g\|_{L^\infty(L^2(\Omega))},\ \|\omega\|_{\mathcal{M}[0,T]}, \|q\|_{L^2(L^2(\Omega))}, \|y_{\tilde{d}}\|_{L^2(L^2(\Omega))}\Big). \tag{5.60}$$

Proof. From (5.16), for $r \in L^2(0,T;L^2(\Omega))$, we have

$$\tilde{j}'(q)(r) = \int_0^T (\tilde{\alpha}q(\tau) + z(\tau), r)\, d\tau,$$

and the derivative of the discretized reduced cost functional

$$(\tilde{j}_h^n)'(q)(r) = \sum_{n=1}^{N} \int_{I_n} (\tilde{\alpha}q(\tau) + z_h^{n-1}(q), r)\, d\tau.$$

Then

$$\begin{aligned}
\Big|\tilde{j}'(q)(r) - (\tilde{j}_h^n)'(q)(r)\Big| &= \Big|\sum_{n=1}^{N} \int_{I_n} (\tilde{\alpha}q(\tau) + z(\tau), r)\, d\tau - \sum_{n=1}^{N} \int_{I_n} (\tilde{\alpha}q(\tau) + z_h^{n-1}(q), r)\, d\tau\Big| \\
&\leq \Big|\sum_{n=1}^{N} \int_{I_n} (z(\tau) - z_h^{n-1}(q), r)\, d\tau\Big| \\
&\leq \|z - \mathcal{Z}_h(q)\|_{L^2(L^2(\Omega))}\|r\|_{L^2(L^2(\Omega))} \\
&\leq \bar{C}_2(h + k^{\frac{1}{2}})\|r\|_{L^2(L^2(\Omega))},
\end{aligned}$$

where $\bar{C}_2$ is given by (5.60), and the last step follows by Lemma 5.4.3. This completes the proof. □

The following theorem estimates the error in the control variable.

Theorem 5.4.1. *Let q and q_h^n be the optimal controls of (5.14) and (5.46), respectively. Assume that the second order optimality condition (5.20) is valid. Then we have*

$$\|q - q_h^n\|_{L^2(L^2(\Omega))} \leq \tilde{C}_2 \frac{h}{\sqrt{\gamma_P}} + \bar{C}_2 \frac{(h + k^{\frac{1}{2}})}{\gamma_P},$$

where $\tilde{C}_2$ and $\bar{C}_2$ is given by (5.34) and (5.60).

Proof. The proof follows by the same lines of arguments of Theorem 4.4.1. So we refrain from giving the details. □

With the above preparations we are ready to estimate the error between the solution y of the continuous problem (5.13) and the solution $\mathcal{Y}_h$ of the fully discrete problem (5.40).

Theorem 5.4.2. *Assume that $\mu = g\omega$, g and ω are given functions such that $g \in \mathcal{C}([0,T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. Let $y \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ be the solution of (5.13), and let $\mathcal{Y}_h$ be the solution of (5.40). Then, we have*

$$\begin{aligned}\|y - \mathcal{Y}_h\|_{L^2(L^2(\Omega))} \;\le\;& C(h + k^{\frac{1}{2}})\Big(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|q\|_{L^2(L^2(\Omega))}\Big)\\ &+\|q - q_h^n\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Proof. Let ψ be the solution of the problem (1.14) with $f \in L^2(0,T; L^2(\Omega))$. It follows from (5.13) that

$$\begin{aligned}\int_{\Omega_T}(y - \mathcal{Y}_h) f \, dx d\tau &= \int_0^T \int_\Omega (y - \mathcal{Y}_h)(-\psi_t + \mathcal{A}^*\psi)\, dx d\tau\\ &= -(y, \psi_t)_{\Omega_T} + a(y,\psi)_{\Omega_T} + \sum_{n=1}^N \int_{I_n} \{(y_h^n, \psi_t) - a(y_h^n, \psi)\}\, d\tau\\ &= -\sum_{n=1}^N \int_{I_n} \{k^{-1}(y_h^n - y_h^{n-1}, \psi^{n-1}) + a(y_h^n, \psi)\}\, d\tau + << \mu, \psi >>_{\Omega_T}\\ &\quad +(y_0 - y_h^0, \psi(\cdot, 0)) + (q,\psi)_{\Omega_T}. \qquad (5.61)\end{aligned}$$

From (5.40), we note that

$$\sum_{n=1}^N \{(\bar\partial y_h^n, \bar{\mathcal{R}}_h\psi) + a(y_h^n, \bar{\mathcal{R}}_h\psi)\} = \sum_{n=1}^N << \mu, \bar{\mathcal{R}}_h\psi >>_{I_n} + \sum_{n=1}^N (q_h^n, \bar{\mathcal{R}}_h\psi), \qquad (5.62)$$

where $\bar{\mathcal{R}}_h\psi \in V_h^n$ is defined in (4.66). Integrate (5.62) with respect to time and combine the resulting equation to (5.61) we obtain

$$\begin{aligned}\int_{\Omega_T}(y - \mathcal{Y}_h) f \, dx d\tau &= -\sum_{n=1}^N \int_{I_n} \{k^{-1}(y_h^n - y_h^{n-1}, \psi^{n-1} - \bar{\mathcal{R}}_h\psi) + a(y_h^n, \bar\psi - \bar{\mathcal{R}}_h\psi)\}\, d\tau\\ &\quad +\{<< \mu, \psi >>_{\Omega_T} - \sum_{n=1}^N \int_{I_n} << \mu, \bar{\mathcal{R}}_h\psi >>_{I_n} \, d\tau\} + \{(y_0 - y_h^0, \psi(\cdot, 0))\}\\ &\quad +\{(q,\psi)_{\Omega_T} - \sum_{n=1}^N \int_{I_n} (q_h^n, \bar{\mathcal{R}}_h\psi)\, d\tau\}\\ &=: \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \hat{\tilde{E}}_4. \qquad (5.63)\end{aligned}$$

The terms $\tilde{E}_1,\ \tilde{E}_2,\ \tilde{E}_3$ are estimated as in Lemma 5.4.2. For the last term $\hat{\tilde{E}}_4$, we have

$$
\begin{aligned}
|\hat{\tilde{E}}_4| &= \Big|\sum_{n=1}^{N}\int_{I_n}(q,\psi)d\tau - \sum_{n=1}^{N}\int_{I_n}(q_h^n,\bar{\mathcal{R}}_h\psi)\,d\tau\Big| \\
&= \Big|\sum_{n=1}^{N}\int_{I_n}(q,\psi-\bar{\mathcal{R}}_h\psi)d\tau + \sum_{n=1}^{N}\int_{I_n}(q-q_h^n,\bar{\mathcal{R}}_h\psi)\,d\tau\Big|.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality, (4.75) and the estimate $\|\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))} \leq C\|\psi\|_{L^2(L^2(\Omega))}$, it now follows that

$$
\begin{aligned}
|\tilde{E}_4| &\leq \|q\|_{L^2(L^2(\Omega))}\|\psi-\bar{\mathcal{R}}_h\psi\|_{L^2(L^2(\Omega))} + C\|q-q_h^n\|_{L^2(L^2(\Omega))}\|\psi\|_{L^2(L^2(\Omega))} \\
&\leq C\Big\{(h+k^{\frac{1}{2}})\|q\|_{L^2(L^2(\Omega))} + \|q-q_h^n\|_{L^2(L^2(\Omega))}\Big\}\|f\|_{L^2(L^2(\Omega))}.
\end{aligned}
$$

Putting the estimates of $\tilde{E}_1,\ \tilde{E}_2,\ \tilde{E}_3$ and $\hat{\tilde{E}}_4$ in (5.63), we arrive at

$$
\begin{aligned}
\|y-\mathcal{Y}_h\|_{L^2(L^2(\Omega))} &= \sup_{f\in L^2(0,T;L^2(\Omega)),f\neq 0}\frac{(f,y-\mathcal{Y}_h)_{\Omega_T}}{\|f\|_{L^2(L^2(\Omega))}} \\
&\leq C(h+k^{\frac{1}{2}})\Big(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^\infty(L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|q\|_{L^2(L^2(\Omega))}\Big) \\
&\quad +\|q-q_h^n\|_{L^2(L^2(\Omega))}.
\end{aligned}
$$

This completes the proof of the theorem. □

The following theorem presents the error between the continuous and fully discrete co-state variables.

Theorem 5.4.3. *Let z and $\mathcal{Z}_h$ be the solutions of (5.17) and (5.43)-(5.44), respectively. Then, for $n \geq 1$, we have*

$$
\|z-\mathcal{Z}_h\|_{L^2(L^2(\Omega))} \leq C(h+k^{\frac{1}{2}})\Big(\|z\|_{L^2(H^2(\Omega))} + \|z_t\|_{L^2(L^2(\Omega))}\Big) + \|y-\mathcal{Y}_h\|_{L^2(L^2(\Omega))}.
$$

Proof. The proof is similar to that of Lemma 4.4.3. The details are thus omitted. □

Concluding remarks. This chapter is devoted to the development of finite element approximations for POCP with measure data in time. We derive *a priori* error bounds for both spatially discrete and fully discrete finite element methods. The order of convergence for the state and co-state and control variables are shown to be of $\mathcal{O}(h)$ (see Theorems 5.3.1-5.3.3) for the spatially discrete approximations of POCP with measure data in time. The error estimates for the fully discrete approximations of optimal control problem are presented in Theorems 5.4.1-5.4.3. The order of convergence with respect to the time discretization is $\mathcal{O}(k^{\frac{1}{2}})$ for the state, co-state and control variables. Numerical experiment is performed in Chapter 7 (see Example 7.3) to illustrate our theoretical results.

6

POCP with Measure Data in Space and Time: A Posteriori Error Analysis

In this chapter, we derive *a posteriori* error estimates for the space-time finite element discretization of POCP (1.6)-(1.8) with measure data. Two kinds of control problems, namely measure data in space and measure data in time, are considered and analyzed. We use continuous piecewise linear functions for the approximations of the state and co-state variables and piecewise constant functions for the control variable, whereas the time discretization is based on the backward Euler implicit scheme. We derive global upper bounds for the errors in the state, co-state and control variables in the $L^2(0,T;L^2(\Omega))$-norm.

6.1 Introduction

Let $\Omega_T = \Omega \times (0,T]$ and $\Gamma_T = \partial\Omega \times [0,T]$, where Ω is a bounded convex domain in $\mathbb{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega$ and $T < \infty$ be a real number. We recall the following model problem:

$$\min_{q \in Q_{ad}^P} \tilde{J}(q,y) = \frac{1}{2}\int_0^T \Big(\|y - y_{\tilde{d}}\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)} \Big)\, d\tau \tag{6.1}$$

subject to the state equation

$$\begin{cases} y_t + \mathcal{A}y = \mu + q \quad \text{in } \Omega_T, \\ y(\cdot,0) = y_0(x) \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \Gamma_T, \end{cases} \tag{6.2}$$

and the control constraints

$$q_c \le q(x,t) \le q_d \ \text{ a.e. in } \Omega_T, \tag{6.3}$$

where $q_c, q_d \in \mathbb{R}$ fulfill $q_c < q_d$ and $y_t = \frac{\partial y}{\partial t}$. The operator $\mathcal{A}$ is defined in (1.4). Moreover, the initial state $y_0(x) \in L^2(\Omega)$ and $y_{\tilde{d}}(x,t) \in L^2(0,T;L^2(\Omega))$ is a given desired state. For ease of exposition we set $\tilde{\alpha} = 1$. The set of admissible controls is defined by

$$Q_{ad}^P := \{q \in L^2(0,T;L^2(\Omega)) : \; q_c \leq q(x,t) \leq q_d \; \text{ a.e. in } \Omega_T\}. \tag{6.4}$$

In this chapter, we consider the control problem (6.1)-(6.3) with measure data in space and time separately. These problems are discretized using finite element method and *a posteriori* error estimates are derived assessing the error with respect to the cost functional. The main intent of this work is to extend the *a posteriori* error analysis of parabolic problem [38] to POCP with measure data. We consider the control problems with measure data in space and time separately and derive *a posteriori* error estimates for the state, co-state and control variables. We use finite element discretization for the approximations of the spatial variable and the backward-Euler implicit scheme for the temporal discretization. While continuous piecewise linear functions are used to approximate the state and co-state variables, piecewise constant functions are employed for the approximation of the control variable. We derive *a posteriori* error estimates for the state, co-state and control variables in the $L^2(0,T;L^2(\Omega))$-norm for both control problems. The key ingredient in our analysis includes the interpolation approximation properties, first order optimality condition and the duality trick.

The layout of this chapter is as follows. In Section 6.2, we derive *a posteriori* error estimates for the control problem with measure data in space. Section 6.3 is concerned with *a posteriori* error estimates for the control problem with measure data in time.

6.2 POCP with measure data in space

In this section, we consider the POCP (6.1)-(6.3) with measure data in space. In this case, we take $\mu = g\omega$ with $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$. The existence, uniqueness and regularity of the solution to problem (6.2) is already discussed in Chapter 4. We recall the following notations:

$$\begin{aligned} W(0,T) &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)),\\ X(0,T) &:= L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)). \end{aligned}$$

The weak formulation of optimal control problem (6.1)-(6.3) can be defined as follows: Find a pair $(q,y) \in Q_{ad}^P \times L^2(0,T;L^2(\Omega))$ such that

$$\min_{q \in Q_{ad}^P} \tilde{J}(q,y) = \frac{1}{2}\int_0^T \left(\|y - y_{\tilde{d}}\|_{L^2(\Omega)}^2 + \|q\|_{L^2(\Omega)}^2 \right) d\tau \tag{6.5}$$

subject to

$$-(y, v_t)_{\Omega_T} + (y, \mathcal{A}^* v)_{\Omega_T} = \langle \mu, v \rangle_{\Omega_T} + (y_0, v(\cdot, 0)) + (q, v)_{\Omega_T}, \quad \forall v \in X(0,T), \tag{6.6}$$

where

$$\langle \mu, v \rangle_{\Omega_T} = \int_\Omega \Big(\int_0^T g(x,\tau) v(x,\tau)\, d\tau \Big)\, d\omega(x), \qquad \forall v \in L^2(0,T; \mathcal{C}(\overline{\Omega})).$$

The optimal control problem (6.5)-(6.6) admits a unique solution if the following optimality conditions are satisfied.

Theorem 6.2.1. *A pair $(q,y) \in Q^P_{ad} \times L^2(0,T;L^2(\Omega))$ is the solution of the optimal control problem (6.5)-(6.6) if and only if there exists a co-state $z \in X(0,T)$ such that the triplet (y,q,z) satisfies the following optimality conditions:*

$$-(y, v_t)_{\Omega_T} + (y, \mathcal{A}^* v)_{\Omega_T} = \langle \mu, v \rangle_{\Omega_T} + (q, v)_{\Omega_T} + (y_0, v(\cdot,0)), \quad \forall v \in X(0,T), \tag{6.7}$$

$$-(z_t, v)_{\Omega_T} + a(v,z)_{\Omega_T} = (y - y_{\tilde{d}}, v)_{\Omega_T}, \qquad \forall v \in L^2(0,T; H^1_0(\Omega)), \tag{6.8}$$

$$z(\cdot, T) = 0, \tag{6.9}$$

$$(q + z, \hat{q} - q) \geq 0, \qquad \forall \hat{q} \in Q^P_{ad}. \tag{6.10}$$

In the following, we introduce the reduced cost functional as:

$$\begin{aligned} \tilde{j} : L^2(0,T;L^2(\Omega)) &\to \mathbb{R} \\ q &\mapsto \tilde{j}(q) := \tilde{J}(q, y(q)), \end{aligned}$$

where $y(q)$ is the solution of the state equation (6.6). Hence the parabolic optimal control problem (6.5)-(6.6) can be equivalently reformulated as

$$\min_{q \in Q^P_{ad}} \tilde{j}(q). \tag{6.11}$$

The following lemma presents *a priori* bounds for the co-state variable. For a proof, we refer to [61].

Lemma 6.2.1. *With $q \in L^2(0,T;L^2(\Omega))$, let $y = y(q)$ and $z = z(q)$ be the solutions of (6.6) and the co-state equation (6.8)-(6.9), respectively. Then $z \in X(0,T)$ and satisfies*

$$\|z\|_{L^2(H^2(\Omega))} + \|z_t\|_{L^2(L^2(\Omega))} \leq C\Big(\|y\|_{L^2(L^2(\Omega))} + \|y_{\tilde{d}}\|_{L^2(L^2(\Omega))} \Big).$$

6.2.1 Finite element approximations

Let V_h be a finite dimensional subspace of $\mathcal{C}(\overline{\Omega})$ consisting of piecewise linear polynomials as discussed in Chapter 3. Set $V_h^0 = V_h \cap H_0^1(\Omega)$. Let $\mathcal{E}_h$ be the set of interelement edges(faces) in the interior of the mesh. The quantity

$$\Big[\frac{\partial v}{\partial n_{\mathcal{A}}}\Big] = (A\nabla v)_K \cdot n_K + (A\nabla v)_{K'} \cdot n_{K'},$$

defined on the edge(face) $e \in \mathcal{E}_h$, $e = \overline{K} \cap \overline{K'}$, measures the jump of ∇v across the element edge(face) e. Let Q_h^P be the finite element subspace of Q_{ad}^P defined by

$$Q_h^P := \{\hat{q}_h \in Q_{ad}^P : \ \hat{q}_h|_K = constant, \quad \forall K \in \mathcal{T}_h\}.$$

Next, let $0 = t_0 < \ldots < t_{N-1} < t_N = T$ be a partition of $[0,T]$. For $n \in [1:N]$, let $k_n := t_n - t_{n-1}$ be the time step and $I_n := (t_{n-1}, t_n]$. We set $\psi^n = \psi^n(x) = \psi(x, t_n)$ and $\bar{\partial}\psi^n = \frac{\psi^n - \psi^{n-1}}{k_n}$. Let V_h^n be the finite element space associated with the mesh $\mathcal{T}_h^n$. Similar to $\mathcal{E}_h$ we also denote $\mathcal{E}_h^n$ as the union of interelement edges(faces) of $\mathcal{T}_h^n$. On each I_n, we define the projection operator $\mathcal{P}_n$ onto the piecewise constant space $\mathbb{P}_0(I_n)$ as $\mathcal{P}_n v := \frac{1}{k_n}\int_{I_n} v\, d\tau$. Let $\mathcal{L}_h^n : L^2(\Omega) \to V_h^n$ be the L^2-projection operator defined by

$$(\mathcal{L}_h^n y - y, v_h) = 0, \quad \forall v_h \in V_h^n. \tag{6.12}$$

Similarly, we construct the finite element space $Q_{h,n}^P$.

Then the fully discrete finite element approximation of problem (6.5)-(6.6) is to find $(q_h^n, y_h^n) \in Q_{h,n}^P \times V_h^n$, $n = 1, 2, \ldots, N$, such that

$$\min_{q_h^n \in Q_{h,n}^P} \tilde{J}(q_h^n, y_h^n) = \frac{1}{2}\sum_{n=1}^{N}\int_{I_n}\Big(\|y_h^n - y_d^n\|^2 + \|q_h^n\|^2\Big)\, d\tau \tag{6.13}$$

subject to

$$\begin{cases} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) = <\mu, v_h>_{I_n} + (q_h^n, v_h), & \forall v_h \in V_h^n, \\ y_h^0(x) = y_{h,0}(x), & x \in \Omega, \end{cases} \tag{6.14}$$

where $y_{h,0} := \mathcal{L}_h^0 y_0$ is an approximation of y_0 and

$$<\mu, v_h>_{I_n} = \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\Big(\int_\Omega g(x,\tau)\, v_h(x)\, d\omega(x)\Big)\, d\tau, \quad \forall v_h \in V_h^n.$$

The optimal control problem (6.13)-(6.14) has a unique solution (q_h^n, y_h^n), $n = 1, 2, \ldots, N$, if and only if there is a co-state z_h^{n-1} $(n = 1, 2, \ldots, N)$ such that the triplet $(y_h^n, q_h^n, z_h^{n-1})$ satisfies the following optimality conditions:

$$
\begin{aligned}
(\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) &= <\mu, v_h>_{I_n} + (q_h^n, v_h), \quad \forall v_h \in V_h^n, \quad n \geq 1, && (6.15)\\
y_h^0(x) &= y_{h,0}(x), \quad x \in \Omega, && (6.16)\\
-(\bar{\partial} z_h^n, v_h) + a(z_h^{n-1}, v_h) &= (y_h^n - y_{\tilde{d}}^n, v_h), \quad \forall v_h \in V_h^n, && (6.17)\\
z_h^N(x) &= 0, \quad x \in \Omega, && (6.18)\\
(q_h^n + z_h^{n-1}, \hat{q}_h^n - q_h^n) &\geq 0, \quad \forall \hat{q}_h^n \in Q_{h,n}^P. && (6.19)
\end{aligned}
$$

For $t \in I_n$ $(n = 1, 2, \ldots, N)$, let

$$
\begin{aligned}
Y_h(t) &:= y_h^n,\\
Z_h(t) &:= ((t_n - t) z_h^{n-1} + (t - t_{n-1}) z_h^n)/k_n,\\
\tilde{\mathcal{Q}}_h(t) &:= q_h^n.
\end{aligned}
$$

For any function $\chi \in \mathcal{C}([0, T]; L^2(\Omega))$, let

$$
\chi^n|_{t \in I_n} = \chi(x, t_n), \qquad \chi^{n-1}|_{t \in I_n} = \chi(x, t_{n-1}).
$$

Then, for $n = 1, 2, \ldots, N$, the optimality conditions (6.15) to (6.19) can be restated as

$$
\begin{aligned}
(\bar{\partial} Y_h^n, v_h) + a(Y_h^n, v_h) &= <\mu, v_h>_{I_n} + (\tilde{\mathcal{Q}}_h, v_h), \quad \forall v_h \in V_h^n, \quad n \geq 1, && (6.20)\\
Y_h^0(x) &= y_{h,0}(x), \quad x \in \Omega, && (6.21)\\
-(\bar{\partial} Z_h^n, v_h) + a(Z_h^{n-1}, v_h) &= (Y_h^n - y_{\tilde{d}}^n, v_h), \quad \forall v_h \in V_h^n, && (6.22)\\
Z_h^N(x) &= 0, \quad x \in \Omega, && (6.23)\\
(\tilde{\mathcal{Q}}_h + Z_h^{n-1}, \hat{q}_h^n - \tilde{\mathcal{Q}}_h) &\geq 0, \quad \forall \hat{q}_h^n \in Q_{h,n}^P. && (6.24)
\end{aligned}
$$

Analogous to the continuous case, we reformulate the fully discrete optimal control problem (6.13)-(6.14) as:

$$
\min_{q_h^n \in Q_{h,n}^P} \tilde{j}_h^n(q_h^n) \tag{6.25}
$$

for $n = 1, \ldots, N$, where $\tilde{j}_h^n := \tilde{J}(q_h^n, y_h^n)$.

6.2.2 *A posteriori* error estimates

In this section, we derive *a posteriori* error estimates for the control problem (6.5)-(6.6). We first introduce the following auxiliary problems.

For $\tilde{\mathcal{Q}}_h \in Q^P_{h,n}$, let $(y(\tilde{\mathcal{Q}}_h), z(\tilde{\mathcal{Q}}_h)) \in L^2(0,T;L^2(\Omega)) \times X(0,T)$ be the solutions of following equations:

$$
\begin{aligned}
-(y(\tilde{\mathcal{Q}}_h), v_t)_{\Omega_T} + (y(\tilde{\mathcal{Q}}_h), \mathcal{A}^* v)_{\Omega_T} &= <\mu, v>_{\Omega_T} + (y_0, v(\cdot,0)) \\
&\quad + (\tilde{\mathcal{Q}}_h, v)_{\Omega_T}, \quad \forall v \in X(0,T), \quad (6.26)\\
-(z_t(\tilde{\mathcal{Q}}_h), v)_{\Omega_T} + a(z(\tilde{\mathcal{Q}}_h), v)_{\Omega_T} &= (y(\tilde{\mathcal{Q}}_h) - y_{\tilde{d}}, v)_{\Omega_T}, \ v \in L^2(0,T;H^1_0(\Omega)), \quad (6.27)\\
z(\tilde{\mathcal{Q}}_h)(\cdot,T) &= 0, \quad x \in \Omega. \quad (6.28)
\end{aligned}
$$

First, we derive the bounds for the error in the control variable.

Lemma 6.2.2. *Let (y,q,z) and $(Y_h, \tilde{\mathcal{Q}}_h, Z_h)$ be the solutions of (6.7)-(6.10) and (6.20)-(6.24), respectively. Assume that $(\tilde{\mathcal{Q}}_h + Z_h^{n-1})|_K \in H^1(K)$ and there exists a positive constant C_{18}, and $w_h \in Q^P_{h,n}$ such that*

$$
\Big| \int_0^T (\tilde{\mathcal{Q}}_h + Z_h^{n-1}, w_h - q)\, d\tau \Big| \le C_{18} \int_0^T \sum_{K \in \mathcal{T}_h} h_K |\tilde{\mathcal{Q}}_h + Z_h^{n-1}|_{H^1(K)} \|q - \tilde{\mathcal{Q}}_h\|_{L^2(K)}\, d\tau. \quad (6.29)
$$

Then, we have

$$
\|q - \tilde{\mathcal{Q}}_h\|^2_{L^2(L^2(\Omega))} \le C_{19}^2 \Big(\eta^n_{1,ms} + \|Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))} \Big), \quad (6.30)
$$

where $C_{19} = \sqrt{\frac{3}{2}} \max\{1, C_{18}\}$, $\eta^n_{1,ms} := \Big(\int_0^T \sum_{K \in \mathcal{T}_h} h_K^2 |\tilde{\mathcal{Q}}_h + Z_h^{n-1}|^2_{H^1(K)}\, d\tau \Big)$, and $z(\tilde{\mathcal{Q}}_h)$ is the solution of (6.27)-(6.28).

Proof. From (6.10), we have

$$
(q, q - \tilde{\mathcal{Q}}_h) \le -(z, q - \tilde{\mathcal{Q}}_h). \quad (6.31)
$$

Then use of (6.31) gives

$$
\begin{aligned}
\|q - \tilde{\mathcal{Q}}_h\|^2_{L^2(L^2(\Omega))} &= \int_0^T (q - \tilde{\mathcal{Q}}_h, q - \tilde{\mathcal{Q}}_h)\, d\tau \\
&\le \int_0^T \{-(z, q - \tilde{\mathcal{Q}}_h) - (\tilde{\mathcal{Q}}_h, q - \tilde{\mathcal{Q}}_h)\}\, d\tau \\
&= - \int_0^T (Z_h^{n-1} + \tilde{\mathcal{Q}}_h, q - w_h)\, d\tau - \int_0^T (\tilde{\mathcal{Q}}_h + Z_h^{n-1}, w_h - \tilde{\mathcal{Q}}_h)\, d\tau \\
&\quad + \int_0^T (Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h), q - \tilde{\mathcal{Q}}_h)\, d\tau + \int_0^T (z(\tilde{\mathcal{Q}}_h) - z, q - \tilde{\mathcal{Q}}_h)\, d\tau.
\end{aligned}
$$

An application of (6.24) yields

$$
\begin{aligned}
\|q - \tilde{\mathcal{Q}}_h\|^2_{L^2(L^2(\Omega))} &\le \int_0^T (\tilde{\mathcal{Q}}_h + Z_h^{n-1}, w_h - q)\, d\tau + \int_0^T (Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h), q - \tilde{\mathcal{Q}}_h)\, d\tau \\
&\quad + \int_0^T (z(\tilde{\mathcal{Q}}_h) - z, q - \tilde{\mathcal{Q}}_h)\, d\tau \\
&=: I_1 + I_2 + I_3. \quad (6.32)
\end{aligned}
$$

From the assumption (6.29), it is easy to see that

$$\begin{aligned} |I_1| &= \Big|\int_0^T (\tilde{\mathcal{Q}}_h + Z_h^{n-1}, w_h - q)\, d\tau\Big| \\ &\le \int_0^T \sum_{K\in\mathcal{T}_h} C_{18}\, h_K |\tilde{Q}_h + Z_h^{n-1}|_{H^1(K)} \|q - \tilde{Q}_h\|_{L^2(K)}\, d\tau \\ &\le \frac{3\,C_{18}^2}{4}\eta_{1,ms}^n + \frac{1}{4}\|q - \tilde{\mathcal{Q}}_h\|^2_{L^2(L^2(\Omega))}. \end{aligned} \tag{6.33}$$

Moreover, it is clear that

$$\begin{aligned} |I_2| &= \Big|\int_0^T (Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h), q - \tilde{\mathcal{Q}}_h)\, d\tau\Big| \\ &\le \frac{3}{4}\|Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))} + \frac{1}{4}\|q - \tilde{\mathcal{Q}}_h\|^2_{L^2(L^2(\Omega))}. \end{aligned} \tag{6.34}$$

Now, it remains to estimate I_3. For this, we note that

$$y(x,0) - y(\tilde{\mathcal{Q}}_h)(x,0) = 0 \quad \text{and} \quad z(x,T) - z(\tilde{\mathcal{Q}}_h)(x,T) = 0,$$

using (6.7) and (6.26), we write I_3 as

$$\begin{aligned} I_3 &= \int_0^T (q - \tilde{\mathcal{Q}}_h, z(\tilde{\mathcal{Q}}_h) - z)\, d\tau \\ &= \int_0^T \{-(y - y(\tilde{\mathcal{Q}}_h), z_t(\tilde{\mathcal{Q}}_h) - z_t) + (y - y(\tilde{\mathcal{Q}}_h), \mathcal{A}^*(z(\tilde{\mathcal{Q}}_h) - z))\}\, d\tau, \end{aligned}$$

which combine with (6.8) and (6.27) yields

$$I_3 = -\int_0^T \|y - y(\tilde{\mathcal{Q}}_h)\|^2\, d\tau \le 0. \tag{6.35}$$

Altogether (6.32)-(6.35) completes the rest of the proof. □

The following two lemmas provide intermediate error estimates which will be used to derive the main theorem of this section.

Lemma 6.2.3. *Let (Y_h, Z_h) and $(y(\tilde{\mathcal{Q}}_h), z(\tilde{\mathcal{Q}}_h))$ be the solutions of (6.20)-(6.23) and (6.26)-(6.28), respectively. Then, we have*

$$\|Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))} \le C_{20}^2 \sum_{i=2}^{5} \eta_{i,ms}^n,$$

where

$$\begin{aligned}
\eta^n_{2,ms} &:= \Big\{ \int_0^T \sum_{K\in\mathcal{T}_h^n} h_K^4 \| - Z_{h,t} + \mathcal{A}^* Z_h^{n-1} - Y_h^n + y_{\tilde{d}}^n \|^2_{L^2(K)}\, d\tau \\
&\quad + \int_0^T \sum_{e\in\mathcal{E}_h^n} h_e^3 \Big\| \Big[\frac{\partial Z_h^{n-1}}{\partial n_{\mathcal{A}^*}} \Big] \Big\|^2_{L^2(e)} d\tau \Big\}, \\
\eta^n_{3,ms} &:= \|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))}, \\
\eta^n_{4,ms} &:= \|y_{\tilde{d}} - y_{\tilde{d}}^n\|^2_{L^2(L^2(\Omega))}, \\
\eta^n_{5,ms} &:= \|Z_h - Z_h^{n-1}\|^2_{L^2(H^1(\Omega))},
\end{aligned}$$

and $C_{20} = C_{R_1} \max\{1, C_1 \max\{C_{I,0}, C_{I,e}\}\}$.

Proof. Let ϕ be the solution of (1.13) with $f = Z_h - z(\tilde{\mathcal{Q}}_h)$, and let $\phi_I = \pi_h\phi$ be the Lagrange interpolation of ϕ defined in Lemma 3.2.1. Then, we have

$$\begin{aligned}
\|Z_h - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))} &= \int_0^T (Z_h - z(\tilde{\mathcal{Q}}_h), \phi_t + \mathcal{A}\phi)\, d\tau \\
&= \int_0^T \Big\{ - (Z_{h,t} - z_t(\tilde{\mathcal{Q}}_h), \phi) + a(Z_h - z(\tilde{\mathcal{Q}}_h), \phi) \Big\}\, d\tau.
\end{aligned}$$

Add and subtract the terms to have

$$\begin{aligned}
\|Z_h - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))} &= \int_0^T \Big\{ - (Z_{h,t} - z_t(\tilde{\mathcal{Q}}_h), \phi - \phi_I) + a(Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h), \phi - \phi_I) \Big\}\, d\tau \\
&\quad + \int_0^T \Big\{ - (Z_{h,t} - z_t(\tilde{\mathcal{Q}}_h), \phi_I) + a(Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h), \phi_I) \Big\}\, d\tau \\
&\quad + \int_0^T a(Z_h - Z_h^{n-1}, \phi)\, d\tau.
\end{aligned}$$

After rearranging the terms, we have

$$\begin{aligned}
\|Z_h - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))} &= \int_0^T \Big(- Z_{h,t} - (y(\tilde{\mathcal{Q}}_h) - y_{\tilde{d}}), \phi - \phi_I \Big)\, d\tau + \int_0^T a(Z_h^{n-1}, \phi - \phi_I)\, d\tau \\
&\quad + \int_0^T (Y_h^n - y(\tilde{\mathcal{Q}}_h), \phi_I)\, d\tau + \int_0^T (y_{\tilde{d}} - y_{\tilde{d}}^n, \phi_I)\, d\tau \\
&\quad + \int_0^T a(Z_h - Z_h^{n-1}, \phi)\, d\tau,
\end{aligned}$$

and hence,

$$\begin{aligned}
\|Z_h - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))} &= \Big\{ \int_0^T (-Z_{h,t} + \mathcal{A}^* Z_h^{n-1} - (Y_h^n - y_{\tilde{d}}^n), \phi - \phi_I)\, d\tau \\
&\quad + \int_0^T \sum_{e\in\mathcal{E}_h^n} \int_e \Big[\frac{\partial Z_h^{n-1}}{\partial n_{\mathcal{A}^*}}\Big](\phi - \phi_I)\, de\, d\tau \Big\} + \int_0^T (Y_h^n - y(\tilde{\mathcal{Q}}_h), \phi)\, d\tau \\
&\quad + \int_0^T (y_{\tilde{d}} - y_{\tilde{d}}^n, \phi)\, d\tau + \int_0^T a(Z_h - Z_h^{n-1}, \phi)\, d\tau \\
&=: F_1^n + F_2^n + F_3^n + F_4^n. \qquad (6.36)
\end{aligned}$$

Now, we estimate $F_i^n|_{i=1,\dots,4}$. An application of Lemmas 3.2.1-3.2.2 gives

$$\begin{aligned}
|F_1^n| &= \Big| \int_0^T (-Z_{h,t} + \mathcal{A}^* Z_h^{n-1} - Y_h^n + y_{\tilde{d}}^n, \phi - \phi_I)\, d\tau + \int_0^T \sum_{e\in\mathcal{E}_h} \int_e \Big[\frac{\partial Z_h^{n-1}}{\partial n_{\mathcal{A}}}\Big](\phi - \phi_I)\, ded\tau \Big| \\
&\le \int_0^T \Big\{ \sum_{K\in\mathcal{T}_h^n} C_{I,0}\, h_K^2 \| - Z_{h,t} + \mathcal{A}^* Z_h^{n-1} - Y_h^n + y_{\tilde{d}}^n\|_{L^2(K)} \|\phi\|_{H^2(K)} \Big\} d\tau \\
&\quad + \int_0^T \sum_{e\in\mathcal{E}_h^n} C_{I,e}\, h_e^{3/2} \Big\| \Big[\frac{\partial Z_h^{n-1}}{\partial n_{\mathcal{A}^*}}\Big]\Big\|_{L^2(e)} \|\phi\|_{H^2(K)}\, d\tau
\end{aligned}$$

Using shape regularity of $\mathcal{T}_h^n$ to obtain

$$\begin{aligned}
|F_1^n| &\le C_1 \max\{C_{I,0}, C_{I,e}\} \Big\{ \int_0^T \Big(\sum_{K\in\mathcal{T}_h^n} h_K^4 \| - Z_{h,t} + \mathcal{A}^* Z_h^{n-1} - Y_h^n + y_{\tilde{d}}^n\|^2_{L^2(K)} \Big) d\tau \\
&\quad + \int_0^T \sum_{e\in\mathcal{E}_h^n} h_e^3 \Big\| \Big[\frac{\partial Z_h^{n-1}}{\partial n_{\mathcal{A}^*}}\Big]\Big\|^2_{L^2(e)} d\tau \Big\}^{\frac{1}{2}} \|\phi\|_{L^2(H^2(\Omega))}. \qquad (6.37)
\end{aligned}$$

Use of Lemma 1.2.3 gives

$$\begin{aligned}
|F_1^n| &\le C_1 C_{R_1} \max\{C_{I,0}, C_{I,e}\} \Big\{ \int_0^T \Big(\sum_{K\in\mathcal{T}_h^n} h_K^4 \| - Z_{h,t} + \mathcal{A}^* Z_h^{n-1} - Y_h^n + y_{\tilde{d}}^n\|^2_{L^2(K)} \Big) d\tau \\
&\quad + \int_0^T \sum_{e\in\mathcal{E}_h^n} h_e^3 \Big\| \Big[\frac{\partial Z_h^{n-1}}{\partial n_{\mathcal{A}}}\Big]\Big\|^2_{L^2(e)} d\tau \Big\}^{\frac{1}{2}} \|Z_h - z(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))}. \qquad (6.38)
\end{aligned}$$

To estimate F_2^n and F_3^n, an application of the Cauchy-Schwarz inequality and Lemma 1.2.3 gives

$$\begin{aligned}
|F_2^n| &\le \|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))} \|\phi\|_{L^2(L^2(\Omega))} \\
&\le C_{R_1} \|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))} \|Z_h - z(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))}, \qquad (6.39)
\end{aligned}$$

and

$$\begin{aligned}|F_3^n| &\leq \|y_{\tilde{d}} - y_{\tilde{d}}^n\|_{L^2(L^2(\Omega))}\|\phi\|_{L^2(L^2(\Omega))}\\ &\leq C_{R_1}\|y_{\tilde{d}} - y_{\tilde{d}}^n\|_{L^2(L^2(\Omega))}\|Z_h - z(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))}. \quad (6.40)\end{aligned}$$

Finally, to estimate F_4^n, we have

$$\begin{aligned}|F_4^n| &\leq \|Z_h - Z_h^{n-1}\|_{L^2(H^1(\Omega))}\|\phi\|_{L^2(H^1(\Omega))}\\ &\leq C_{R_1}\|Z_h - Z_h^{n-1}\|_{L^2(H^1(\Omega))}\|Z_h - z(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))}. \quad (6.41)\end{aligned}$$

Combining (6.36)-(6.41) we complete the rest of the proof. □

Lemma 6.2.4. *Assume that $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$, $\omega \in \mathcal{M}(\Omega)$ and $y_0 \in L^2(\Omega)$. Let $y(\tilde{\mathcal{Q}}_h) \in L^2(0,T;L^2(\Omega))$ be the solution of (6.26), and let $Y_h^n \in V_h^n$ $(n = 1,2,\ldots,N)$ be the solution of (6.20)-(6.21). Then we have*

$$\sum_{n=1}^{N}\int_{I_n}\|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|_{L^2(\Omega)}^2\,d\tau \leq C_{21}^2\sum_{n=1}^{N}\{k_n(\xi_{1,ms}^n + \xi_{2,ms}^n + \xi_{4,ms}^n) + \xi_{3,ms}^n\},$$

where

$$\begin{aligned}\xi_{1,ms}^n &:= \sum_{K\in\mathcal{T}_h^n} h_K^4\|k_n^{-1}(Y_h^n - Y_h^{n-1}) + \mathcal{A}Y_h^n - \tilde{\mathcal{Q}}_h\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h^n} h_e^3\Big\|\Big[\frac{\partial Y_h^n}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)}^2,\\ \xi_{2,ms}^n &:= \|Y_h^n - Y_h^{n-1}\|_{L^2(\Omega)}^2,\\ \xi_{3,ms}^n &:= \|y_0 - Y_h^0\|_{L^2(\Omega)}^2,\\ \xi_{4,ms}^n &:= \frac{1}{k_n}\sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{T}_h^n}\Big(\|\omega\|_{\mathcal{M}(K)}^2\{\|g - \mathcal{P}_n g\|_{L^\infty(K)}^2 + h_K^{4-d}\|g\|_{L^\infty(K)}^2\}\Big)\Big\}\,d\tau,\end{aligned}$$

and $C_{21} = C_{R_2}\max\{1, C_I, C_1\max\{C_{I,2}, C_{I,\infty}, C_{I,0}, C_{I,e}\}\}$.

Proof. To derive *a posteriori* error estimates we use the duality argument. Let ψ be the solution of the problem (1.14) with $f \in L^2(0,T;L^2(\Omega))$. Note that $\psi = 0$ on $\partial\Omega$, $\psi^N = \psi(\cdot,T) = 0$, it follows from (6.26) and integrating by parts that

$$\begin{aligned}\int_{\Omega_T}(Y_h^n - y(\tilde{\mathcal{Q}}_h))f\,dxd\tau &= \int_0^T\int_\Omega(Y_h^n - y(\tilde{\mathcal{Q}}_h))(-\psi_t + \mathcal{A}^*\psi)\,dxd\tau\\ &= (y(\tilde{\mathcal{Q}}_h),\psi_t)_{\Omega_T} - (y(\tilde{\mathcal{Q}}_h),\mathcal{A}^*\psi)_{\Omega_T} - \sum_{n=1}^{N}\int_{I_n}((Y_h^n,\psi_t) - a(Y_h^n,\psi))\,d\tau\\ &= -<\mu,\psi>_{\Omega_T} - (y_0 - Y_h^0,\psi(\cdot,0)) - (\tilde{\mathcal{Q}}_h,\psi)_{\Omega_T}\\ &\quad + \sum_{n=1}^{N}\int_{I_n}\Big\{k_n^{-1}(Y_h^n - Y_h^{n-1},\psi^{n-1}) + a(Y_h^n,\psi)\Big\}\,d\tau. \quad (6.42)\end{aligned}$$

We define ψ_I such that $\psi_I|_{I_n} := \pi_h^n(\mathcal{P}_n\psi) \in V_h^n$ on each time interval I_n with π_h^n the standard Lagrange interpolation operator onto V_h^n. Note that $\psi \in L^2(0,T;H^2(\Omega)) \hookrightarrow L^2(0,T;\mathcal{C}(\overline{\Omega}))$, so that Lagrange interpolation is well defined. From equation (6.20), we obtain

$$\sum_{n=1}^{N}\Big\{k_n^{-1}(Y_h^n - Y_h^{n-1},\psi_I) + a(Y_h^n,\psi_I)\Big\} = \sum_{n=1}^{N} < \mu,\psi_I >_{I_n} + \sum_{n=1}^{N}(\tilde{\mathcal{Q}}_h,\psi_I). \tag{6.43}$$

Then we have

$$\begin{aligned}\int_{\Omega_T}(Y_h^n - y(\tilde{\mathcal{Q}}_h))f\,dxd\tau \;=\;& \sum_{n=1}^{N}\int_{I_n} k_n^{-1}(Y_h^n - Y_h^{n-1},\psi^{n-1}-\psi_I)\,d\tau + \sum_{n=1}^{N}\int_{I_n}\{a(Y_h^n,\psi-\psi_I)\\ & -(\tilde{\mathcal{Q}}_h,\psi-\psi_I)\}\,d\tau - (y_0 - Y_h^0,\psi(\cdot,0))\\ & -\Big\{ < \mu,\psi >_{\Omega_T} - \sum_{n=1}^{N}\int_{I_n} < \mu,\psi_I >_{I_n}\, d\tau\Big\}\\ =:\;& E_1^n + E_2^n + E_3^n + E_4^n.\end{aligned} \tag{6.44}$$

Now, we estimate $E_i^n|_{i=1,\dots,4}$. Integrating by parts and using Lemma 3.2.1, we obtain

$$\begin{aligned}|E_1^n| \;=\;& \Big|\sum_{n=1}^{N}\int_{I_n} k_n^{-1}(Y_h^n - Y_h^{n-1},\psi^{n-1} - \mathcal{P}_n\psi + \mathcal{P}_n(\psi - \pi_h^n\psi))\,d\tau\Big|\\ =\;& \Big|\sum_{n=1}^{N}\int_{I_n}\Big\{\int_{\Omega} k_n^{-1}(Y_h^n - Y_h^{n-1})(\psi^{n-1}-\mathcal{P}_n\psi)\,dx\\ & + \int_{I_n} k_n^{-1}(Y_h^n - Y_h^{n-1})\mathcal{P}_n(\psi - \pi_h^n\psi)\,dx\Big\}\,d\tau\Big|\\ \le\;& \max\{C_I, C_1C_{I,0}\}\Big[k_n\Big(\|Y_h^n - Y_h^{n-1}\|^2_{L^2(\Omega)} + \sum_{K\in\mathcal{T}_n^h} h_K^4\|k_n^{-1}(Y_h^n - Y_h^{n-1})\|^2_{L^2(K)}\Big)\Big]^{\frac{1}{2}}\\ & \times\Big(\|\psi_t\|_{L^2(L^2(\Omega))} + \|\psi\|_{L^2(H^2(\Omega))}\Big),\end{aligned} \tag{6.45}$$

where we have used the following properties of $\mathcal{P}_n$:

$$\|\psi^{n-1} - \mathcal{P}_n\psi\|_{L^2(L^2(\Omega))} \le C_I k_n \|\psi_t\|_{L^2(L^2(\Omega))},$$

and

$$\|\mathcal{P}_n(\psi - \pi_h^n\psi)\|_{L^2(K)} \le \|\psi - \pi_h^n\psi\|_{L^2(K)} \le C_{I,0}\, h_K^2\|\psi\|_{L^2(H^2(K))}.$$

We now proceed to estimate E_2^n. From the definition of $\mathcal{P}_n$, we have

$$\begin{aligned}
|E_2^n| &= \Big|\sum_{n=1}^{N}\int_{I_n}\{a(Y_h^n,\psi-\psi_I)-(\tilde{\mathcal{Q}}_h,\psi-\psi_I)\}\,d\tau\Big| \\
&= \Big|\sum_{n=1}^{N}\int_{I_n}\{a(Y_h^n,\psi-\pi_h^n\psi)-(\tilde{\mathcal{Q}}_h,\psi-\pi_h^n\psi)\}\,d\tau\Big| \\
&\leq \sum_{n=1}^{N}\int_{I_n}\Big(\sum_{K\in\mathcal{T}_h^n}\|\mathcal{A}Y_h^n-\tilde{\mathcal{Q}}_h\|_{L^2(K)}\|\psi-\pi_h^n\psi\|_{L^2(K)} \\
&\quad+\sum_{e\in\mathcal{E}_h^n}\Big\|\Big[\frac{\partial Y_h^n}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)}\|\psi-\pi_h^n\psi\|_{L^2(e)}\Big)\,d\tau. \qquad (6.46)
\end{aligned}$$

The shape regularity of $\mathcal{T}_h^n$ and Lemmas 3.2.1-3.2.2 gives

$$\begin{aligned}
|E_2^n| &\leq C_1\max\{C_{I,0},C_{I,e}\}\Big[\sum_{n=1}^{N}k_n\Big(\sum_{K\in\mathcal{T}_n^h}h_K^4\|\mathcal{A}Y_h^n-\tilde{\mathcal{Q}}_h\|^2_{L^2(K)} \\
&\quad+\sum_{e\in\mathcal{E}_h^n}h_e^3\Big\|\Big[\frac{\partial Y_h^n}{\partial n_{\mathcal{A}}}\Big]\Big\|^2_{L^2(e)}\Big)\Big]^{\frac{1}{2}}\|\psi\|_{L^2(H^2(\Omega))}. \qquad (6.47)
\end{aligned}$$

To estimate E_3^n, we note that

$$|E_3^n| = |(y_0-Y_h^0,\psi(\cdot,0))| \leq \|y_0-Y_h^0\|_{L^2(\Omega)}\|\psi(\cdot,0)\|_{L^2(\Omega)}.$$

From the structure of E_4^n and Lemma 3.2.1, we have

$$\begin{aligned}
|E_4^n| &= \Big|\sum_{n=1}^{N}\int_{I_n}\Big(\int_{\Omega}g(\psi-\psi_I)\,d\omega(x)\Big)\,d\tau\Big| \\
&= \sum_{n=1}^{N}\Big\{\int_{I_n}\Big(\int_{\Omega}g(x,\tau)(\psi-\mathcal{P}_n\psi)\,d\omega(x)\Big)\,d\tau \\
&\quad+\int_{I_n}\Big(\int_{\Omega}g(x,\tau)\,\mathcal{P}_n(\psi-\pi_h^n\psi)\,d\omega(x)\Big)\,d\tau\Big\} \\
&\leq \sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{T}_h^n}\Big(\|g-\mathcal{P}_n\,g\|_{L^\infty(K)}\|\psi-\mathcal{P}_n\psi\|_{L^\infty(K)}\|\omega\|_{\mathcal{M}(K)} \\
&\quad+\|g\|_{L^\infty(K)}\|\mathcal{P}_n(\psi-\pi_h^n\psi)\|_{L^\infty(K)}\|\omega\|_{\mathcal{M}(K)}\Big)\Big\}\,d\tau.
\end{aligned}$$

Using the fact $\|\psi-\mathcal{P}_n\psi\|_{L^\infty(K)} \leq C_{I,2}\|\psi\|_{H^2(K)}$ and

$$\|\mathcal{P}_n(\psi-\pi_h^n\psi)\|_{L^\infty(K)} \leq C_{I,\infty}h^{2-\frac{d}{2}}\|\psi\|_{H^2(K)},$$

$$|E_4^n| \leq C_1 \max\{C_{I,2}, C_{I,\infty}\}\Big[\sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{T}_h^n}\Big(\|g-\mathcal{P}_n g\|_{L^\infty(K)}^2\|\omega\|_{\mathcal{M}(K)}^2 + h_K^{4-d}\|g\|_{L^\infty(K)}^2\|\omega\|_{\mathcal{M}(K)}^2\Big)\Big\}\,d\tau\Big]^{\frac{1}{2}}\|\psi\|_{L^2(H^2(\Omega))}.$$

Combining the estimates of $E_i^n|_{i=1,\dots,4}$ and Lemma 1.2.3, we obtain

$$\|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))} \leq C_{21}\Big[\Big\{\sum_{n=1}^{N} k_n\Big(\sum_{K\in\mathcal{T}_h^n} h_K^4\|k_n^{-1}(Y_h^n - Y_h^{n-1}) + \mathcal{A}Y_h^n - \tilde{\mathcal{Q}}_h\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h^n} h_e^3\Big\|\Big[\frac{\partial Y_h^n}{\partial n_{\mathcal{A}}}\Big]\Big\|_{L^2(e)}^2\Big)\Big\}^{\frac{1}{2}} + \Big\{\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\Big(\sum_{K\in\mathcal{T}_h^n}\Big\{\|\omega\|_{\mathcal{M}(K)}^2(\|g-\mathcal{P}_n g\|_{L^\infty(K)}^2 + h_K^{4-d}\|g\|_{L^\infty(K)}^2)\Big\}\Big)\,d\tau\Big\}^{\frac{1}{2}} + \|y_0 - Y_h^0\|_{L^2(\Omega)} + \Big(\sum_{n=1}^{N} k_n\|Y_h^n - Y_h^{n-1}\|_{L^2(\Omega)}^2\Big)^{\frac{1}{2}}\Big].$$

This completes the proof. □

Using Lemmas 6.2.2-6.2.4, we can obtain the following *a posteriori* error bounds for measure data in space.

Theorem 6.2.2. *Let (y, q, z) and $(Y_h, \tilde{\mathcal{Q}}_h, Z_h)$ be the solutions of (6.7)-(6.10) and (6.20)-(6.24), respectively. Assume that all the conditions in Lemmas 6.2.2-6.2.4 are valid. Then, we have*

$$\sum_{n=1}^{N}\int_{I_n}\|Y_h^n - y\|_{L^2(\Omega)}^2\,d\tau + \sum_{n=1}^{N}\int_{I_n}\|Z_h^{n-1} - z\|_{L^2(\Omega)}^2\,d\tau + \sum_{n=1}^{N}\int_{I_n}\|\tilde{Q}_h - q\|_{L^2(\Omega)}^2\,d\tau$$
$$\leq C_{19}^2\,\eta_{1,ms}^n + C_{20}^2\sum_{n=1}^{N}\{\eta_{2,ms}^n + \eta_{4,ms}^n + \eta_{5,ms}^n\} + C_{21}^2\sum_{n=1}^{N}\{k_n(\xi_{1,ms}^n + \xi_{2,ms}^n + \xi_{4,ms}^n) + \xi_{3,ms}^n\},$$

where $\eta_{1,ms}^n$ is defined in Lemma 6.2.2, $\eta_{i,ms}^n$, $i = 2, 4, 5$ are defined in Lemma 6.2.3 and $\xi_{i,ms}^n$, $i = 1, \dots, 4$ are defined in Lemma 6.2.4.

Proof. Note that

$$\|Y_h^n - y\|_{L^2(L^2(\Omega))} \leq \|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))} + \|y(\tilde{\mathcal{Q}}_h) - y\|_{L^2(L^2(\Omega))},$$
$$\|Z_h^{n-1} - z\|_{L^2(L^2(\Omega))} \leq \|Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))} + \|z(\tilde{\mathcal{Q}}_h) - z\|_{L^2(L^2(\Omega))}.$$

Applications of Lemma 1.2.3 and Lemmas 6.2.2-6.2.4 complete the rest of the proof. □

Remark 6.2.1. *The estimators presented in Theorem 6.2.2 are contributed from the approximation errors of the state, co-state and control variables. Among them, $\eta_{1,ms}^n$ mainly* *indicates the approximation error for the control, and the other estimators mainly reflect the approximation errors for the state and co-state. The estimators* $\eta^n_{2,ms}$, $\eta^n_{4,ms}$, $\eta^n_{5,ms}$ *are contributed from the co-state equation and the estimators* $\xi^n_{1,ms}, \dots, \xi^n_{4,ms}$ *are contributed by the state equation. These estimators consist of three parts: the estimators* $\xi^n_{2,ms}$ *and* $\eta^n_{5,ms}$ *are due to the time discretization, the estimators* $\xi^n_{1,ms}$ *and* $\eta^n_{2,ms}$ *are due to the space discretization, the estimators* $\eta^n_{4,ms}$, $\xi^n_{3,ms}$ *and* $\xi^n_{4,ms}$ *corresponds to the data approximation error. These estimators are very useful to guide adaptive procedure.*

Remark 6.2.2. *In particular, one may choose* $\omega = \delta_{\gamma(t)}$, *where* $\gamma(t)$ *is a Lipschitz-continuous* m*-dimensional manifold in* Ω *with* $0 \leq m \leq d-1$ *for all* $t \in [0,T]$ *and* $\delta_{\gamma(t)}$ *is the Dirac delta function concentrated on* $\gamma(t)$. *Note that for* $m = 0$, $\gamma(t)$ *represents a single point or a finite number of singular points, for* $m = 1$, $\gamma(t)$ *is a* $\mathcal{C}^2$*-curve and for* $m = 2$, $\gamma(t)$ *is a* $\mathcal{C}^2$*-surface. With this choice of* ω, *the term* $\xi^n_{4,ms}$ *in Theorem 6.2.2 takes the form*

$$\xi^n_{4,ms} := \begin{cases} \frac{1}{k_n}\int_{I_n}\sum_{K\in\mathcal{T}_h^n}\left(\|g-\mathcal{P}_n g(\gamma(t_n),\tau)\|^2_{L^\infty(K)} + h_K^{4-d}\|g\|^2_{L^\infty(K)}\right)d\tau, & \text{if } m=0,\\ \frac{1}{k_n}\int_{I_n}\left(\|g-\mathcal{P}_n g(\gamma(t_n),\tau)\|^2_{L^2(\gamma(t))} + \sum_{K\in\mathcal{T}_h^n} h_K^{4-d}\|_{L^2(\gamma(t)\cap K)}\right)d\tau, & \text{if } m>0. \end{cases}$$

6.3 POCP with measure data in time

Now we consider the POCP (6.1)-(6.3) with measure data in time. Here $\mu = g\omega$ with $g \in \mathcal{C}([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. The weak formulation of POCP (6.1)-(6.3) with measure data in time is defined as follows:

$$\min_{q\in Q^P_{ad}} \tilde{J}(q,y) = \frac{1}{2}\int_0^T \left(\|y-y_{\tilde{d}}\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)}\right)d\tau \tag{6.48}$$

subject to

$$\begin{aligned} -(y,v_t)_{\Omega_T} + a(y,v)_{\Omega_T} &= << \mu, v >>_{\Omega_T} + (y_0, v(\cdot,0)) \\ &\quad + (q,v)_{\Omega_T}, \qquad \forall v \in W(0,T), \end{aligned} \tag{6.49}$$

where

$$<< \mu, v >>_{\Omega_T} = \int_{\Omega_T} v\, d\mu = \int_0^T \left(\int_\Omega g(x,\tau)v(x,\tau)\,dx\right)d\omega(\tau), \qquad \forall v \in \mathcal{C}([0,T];L^2(\Omega)).$$

Note that the problem (6.49) admits a unique solution (cf. Theorem 5.2.1). Similar to (6.5)-(6.6), the parabolic optimal control problem (6.48)-(6.49) admits a unique solution $(q,y) \in Q^P_{ad} \times L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ if and only if there exists a co-state

$z \in X(0,T)$ such that the triplet (y,q,z) satisfies the following optimality conditions:

$$\begin{aligned} -(y,v_t)_{\Omega_T} + a(y,v)_{\Omega_T} &= << \mu, v >>_{\Omega_T} + (y_0, v(\cdot,0)) \\ &\quad + (q,v)_{\Omega_T}, \ \forall v \in W(0,T) \quad (6.50) \\ -(z_t,v)_{\Omega_T} + a(z,v)_{\Omega_T} &= (y - y_{\tilde{d}}, v)_{\Omega_T}, \ \forall v \in L^2(0,T;H_0^1(\Omega)), \quad (6.51) \\ z(\cdot,T) &= 0, \quad (6.52) \\ (q+z, \hat{q}-q) &\geq 0, \quad \forall \hat{q} \in Q_{ad}^P, \quad (6.53) \end{aligned}$$

where $z = z(q)$ is the solution of (6.51) and $y = y(q)$ is the solution of the state equation (6.50). We reformulate the parabolic optimal control problem (6.48)-(6.49) as

$$\min_{q \in Q_{ad}^P} \tilde{j}(q) := \tilde{J}(q, y(q)), \quad (6.54)$$

where $y(q)$ is the solution of problem (6.50).

6.3.1 Finite element approximations

This section considers the fully discrete approximations to POCP with measure data in time. The time discretization is based on the backward Euler scheme. The fully discrete approximation of (6.48)-(6.49) is to find $(q_h^n, y_h^n) \in Q_{h,n}^P \times V_h^n$, $n \geq 1$, such that

$$\min_{q_h^n \in Q_{h,n}^P} \tilde{J}(q_h^n, y_h^n) = \frac{1}{2} \sum_{n=1}^{N} \int_{I_n} \left(\|y_h^n - y_{\tilde{d}}^n\|^2 + \|q_h^n\|^2 \right) d\tau \quad (6.55)$$

subject to

$$\begin{cases} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) = << \mu, v_h >>_{I_n} + (q_h^n, v_h), \ \forall v_h \in V_h^n, \\ y_h^0(x) = y_{h,0}(x), \quad x \in \Omega, \end{cases} \quad (6.56)$$

where $y_{h,0}$ is the L^2-projection of y_0 in V_h^0 and

$$<< \mu, v_h >>_{I_n} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \left(\int_\Omega g(x,\tau)\, v_h(x)\, dx \right) d\omega(\tau), \quad \forall v_h \in V_h^n.$$

The optimal control problem (6.55)-(6.56) has a unique solution (q_h^n, y_h^n), $n = 1, 2, \ldots, N$, such that the triplet $(y_h^n, q_h^n, z_h^{n-1})$ satisfies the following optimality conditions:

$$\begin{aligned} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) &= << \mu, v_h >>_{I_n} + (q_h^n, v_h), \ \forall v_h \in V_h^n, \ n \geq 1, \quad (6.57) \\ y_h^0(x) &= y_{h,0}(x), \quad x \in \Omega, \quad (6.58) \\ -(\bar{\partial} z_h^n, v_h) + a(z_h^{n-1}, v_h) &= (y_h^n - y_{\tilde{d}}^n, v_h), \ \forall v_h \in V_h^n, \quad (6.59) \\ z_h^N(x) &= 0, \quad x \in \Omega, \quad (6.60) \\ (q_h^n + z_h^{n-1}, \hat{q}_h^n - q_h^n) &\geq 0, \quad \forall \hat{q}_h^n \in Q_{h,n}^P. \quad (6.61) \end{aligned}$$

As before, we reformulate the fully discrete optimal control problem (6.55)-(6.56) as:

$$\min_{q_h^n \in Q_{h,n}^P} \tilde{j}_h^n(q_h^n), \tag{6.62}$$

for $n = 1, \ldots, N$. Similar to measure data in space we rewrite the optimality conditions (6.57)-(6.61) as follows:

$$(\bar{\partial} Y_h^n, v_h) + a(Y_h^n, v_h) = << \mu, v_h >>_{I_n} + (\tilde{\mathcal{Q}}_h, v_h), \ \forall v_h \in V_h^n, \tag{6.63}$$

$$Y_h^0(x) = y_{h,0}(x), \quad x \in \Omega, \tag{6.64}$$

$$-(\bar{\partial} Z_h^n, v_h) + a(Z_h^{n-1}, v_h) = (Y_h^n - y_{\tilde{d}}^n, v_h), \ \forall v_h \in V_h^n, \tag{6.65}$$

$$Z_h^N(x) = 0, \quad x \in \Omega, \tag{6.66}$$

$$(\tilde{\mathcal{Q}}_h + Z_h^{n-1}, \hat{q}_h^n - \tilde{\mathcal{Q}}_h) \geq 0, \quad \forall \hat{q}_h^n \in Q_{h,n}^P. \tag{6.67}$$

6.3.2 *A posteriori* error estimates

This section is devoted to *a posteriori* error estimate of parabolic optimal control problem with measure data in time.

In order to derive the error between the continuous solution y of (6.49) and the fully discrete solution Y_h^n of (6.63)-(6.64), we introduce some intermediate variables. For $\tilde{\mathcal{Q}}_h \in Q_{h,n}^P$, let $(y(\tilde{\mathcal{Q}}_h), z(\tilde{\mathcal{Q}}_h))$ satisfy the following equations:

$$-(y(\tilde{\mathcal{Q}}_h), v_t)_{\Omega_T} + a(y(\tilde{\mathcal{Q}}_h), v)_{\Omega_T} = << \mu, v >>_{\Omega_T} + (y_0, v(\cdot, 0)) + (\tilde{\mathcal{Q}}_h, v)_{\Omega_T}, \ \forall v \in W(0,T), \tag{6.68}$$

$$-(z_t(\tilde{\mathcal{Q}}_h), v)_{\Omega_T} + a(z(\tilde{\mathcal{Q}}_h), v)_{\Omega_T} = (y(\tilde{\mathcal{Q}}_h) - y_{\tilde{d}}, v)_{\Omega_T}, \ \forall v \in L^2(0,T; H_0^1(\Omega)), \tag{6.69}$$

$$z(\tilde{\mathcal{Q}}_h)(\cdot, T) = 0, \quad x \in \Omega. \tag{6.70}$$

Now, we split the error as follows:

$$\|Y_h^n - y\|_{L^2(L^2(\Omega))} \leq \|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))} + \|y(\tilde{\mathcal{Q}}_h) - y\|_{L^2(L^2(\Omega))},$$

$$\|Z_h^{n-1} - z\|_{L^2(L^2(\Omega))} \leq \|Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))} + \|z(\tilde{\mathcal{Q}}_h) - z\|_{L^2(L^2(\Omega))}.$$

In the following lemma, we derive an intermediate error estimate for the state variable.

Lemma 6.3.1. *Assume that $g \in \mathcal{C}([0,T]; L^2(\Omega))$, $\omega \in \mathcal{M}[0,T]$ and $y_0 \in L^2(\Omega)$. Let $y(\tilde{\mathcal{Q}}_h) \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ be the solution of (6.68), and let $Y_h^n \in V_h^n$ $(n = 1, 2, \ldots, N)$ be the solution of (6.63)-(6.64). Then, we have*

$$\sum_{n=1}^{N} \int_{I_n} \|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|_{L^2(\Omega)}^2 \, d\tau \leq C_{22}^2 \Big\{ \sum_{n=1}^{N} k_n(\xi_{1,mt}^n + \xi_{2,mt}^n) + \xi_{3,mt}^n + \xi_{4,mt}^n \Big\},$$

where

$$
\begin{aligned}
\xi_{1,mt}^n &:= \sum_{K\in\mathcal{T}_h^n} h_K^4 \|k_n^{-1}(Y_h^n - Y_h^{n-1}) + \mathcal{A}Y_h^n - \tilde{\mathcal{Q}}_h\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_h^n} h_e^3 \Big\| \Big[\frac{\partial Y_h^n}{\partial n_{\mathcal{A}}}\Big] \Big\|_{L^2(e)}^2,\\
\xi_{2,mt}^n &:= \|Y_h^n - Y_h^{n-1}\|_{L^2(\Omega)}^2,\\
\xi_{3,mt}^n &:= \|y_0 - Y_h^0\|_{L^2(\Omega)}^2,\\
\xi_{4,mt}^n &:= \sum_{n=1}^N \Big\{ \sum_{K\in\mathcal{T}_h^n} \|\omega\|_{\mathcal{M}(I_n)}^2 \Big(k_n \|g - \mathcal{P}_n\, g\|_{L^\infty(I_n;L^2(K))}^2 + h_K^2 \|g\|_{L^\infty(I_n;L^2(K))}^2 \Big) \Big\}.
\end{aligned}
$$

and $C_{22} = C_{R_2} \max\{1, C_I, C_1 \max\{C_{I,4}, C_{I,3}, C_{I,0}, C_{I,e}\}\}$.

Proof. Let ψ be the solution of problem (1.14) with $f \in L^2(0,T;L^2(\Omega))$, and let $\psi_I|_{I_n} := \pi_h^n \mathcal{P}_n \psi$ be the corresponding space-time interpolation. Proceeding as in the proof of Lemma 6.2.4, we have

$$
\begin{aligned}
\int_{\Omega_T} (Y_h^n - y(\tilde{\mathcal{Q}}_h)) f \, dx d\tau &= \sum_{n=1}^N \int_{I_n} (k_n^{-1}(Y_h^n - Y_h^{n-1}, \psi^{n-1} - \psi_I))\, d\tau\\
&+ \sum_{n=1}^N \int_{I_n} \{a(Y_h^n, \psi - \psi_I)) - (\tilde{\mathcal{Q}}_h, \psi - \psi_I)\}\, d\tau - (y_0 - Y_h^0, \psi(\cdot,0))\\
&- \Big\{ << \mu, \psi >>_{\Omega_T} - \sum_{n=1}^N \int_{I_n} << \mu, \psi_I >>_{I_n} \, d\tau \Big\}\\
&=: \check{E}_1^n + \check{E}_2^n + \check{E}_3^n + \check{E}_4^n. \qquad (6.71)
\end{aligned}
$$

The estimates of $\check{E}_1^n$, $\check{E}_2^n$ and $\check{E}_3^n$ are similar to E_1^n, E_2^n, E_3^n as in the proof of Lemma 6.2.4. For $\check{E}_4^n$, we have

$$
\begin{aligned}
|\check{E}_4^n| &= \Big| \sum_{n=1}^N \int_\Omega \Big(\int_{I_n} g(\psi - \psi_I)\, d\omega(\tau) \Big) dx \Big|\\
&= \Big| \sum_{n=1}^N \Big\{ \int_\Omega \Big(\int_{I_n} g(x,\tau)(\psi - \mathcal{P}_n\psi)\, d\omega(\tau) \Big) dx\\
&\quad + \int_\Omega \Big(\int_{I_n} g(x,\tau)\, \mathcal{P}_n(\psi - \pi_h^n\psi)\, d\omega(\tau) \Big) dx \Big\} \Big|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\check{E}_4^n| \le \sum_{n=1}^N \Big\{ \sum_{K\in\mathcal{T}_h^n} \Big(&\|g - \mathcal{P}_n\, g\|_{L^\infty(I_n;L^2(K))} \|\psi - \mathcal{P}_n\psi\|_{L^\infty(I_n;L^2(K))} \|\omega\|_{\mathcal{M}(I_n)}\\
&+ \|g\|_{L^\infty(I_n;L^2(K))} \|\mathcal{P}_n(\psi - \pi_h^n\psi)\|_{L^\infty(I_n;L^2(K))} \|\omega\|_{\mathcal{M}(I_n)} \Big) \Big\}.
\end{aligned}
$$

Which can be estimated as

$$|\check{E}_4^n| \leq C_1 \max\{C_{I,3}, C_{I,4}\} \sum_{n=1}^{N} \Big\{ \sum_{K\in\mathcal{T}_h^n} \Big(k_n \|g - \mathcal{P}_n\, g\|^2_{L^\infty(I_n;L^2(K))} \|\omega\|^2_{\mathcal{M}(I_n)}$$

$$+ h_K^2 \|g\|^2_{L^\infty(I_n;L^2(K))} \|\omega\|^2_{\mathcal{M}(I_n)} \Big) \Big\}^{\frac{1}{2}} \times \Big(\|\psi\|_{L^\infty(H^1(\Omega))} + \|\psi\|_{H^1(L^2(\Omega))} \Big), \qquad (6.72)$$

where we have used the following approximation properties:

$$\begin{aligned} \|\psi - \mathcal{P}_n\psi\|_{L^\infty(I_n;L^2(\Omega))} &\leq C_{I,3}\, k_n^{\frac{1}{2}} \|\psi\|_{H^1(L^2(\Omega))}, \\ \|\psi - \pi_h^n\psi\|_{L^\infty(I_n;L^2(K))} &\leq C_{I,4}\, h_K \|\psi\|_{L^\infty(I_n;H^1(K))}. \end{aligned}$$

Combining (6.45)-(6.47), (6.71) and (6.72), we obtain

$$\|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|_{L^2(L^2(\Omega))} \leq C_{22} \Bigg[\Big\{ \sum_{n=1}^{N} k_n \Big(\sum_{K\in\mathcal{T}_h^n} h_K^4 \|k_n^{-1}(Y_h^n - Y_h^{n-1}) + \mathcal{A}Y_h^n - \tilde{\mathcal{Q}}_h\|^2_{L^2(K)}$$

$$+ \sum_{e\in\mathcal{E}_h^n} h_e^3 \Big\| [\frac{\partial Y_h^n}{\partial n_{\mathcal{A}}}] \Big\|^2_{L^2(e)} \Big) \Big\}^{\frac{1}{2}} + \|y_0 - Y_h^0\|_{L^2(\Omega)} + \Big(\sum_{n=1}^{N} k_n \|Y_h^n - Y_h^{n-1}\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}$$

$$+ \sum_{n=1}^{N} \Big\{ \sum_{K\in\mathcal{T}_h^n} \Big(k_n \|g - \mathcal{P}_n\, g\|^2_{L^\infty(I_n;L^2(K))} \|\omega\|^2_{\mathcal{M}(I_n)} + h_K^2 \|g\|^2_{L^\infty(I_n;L^2(K))} \|\omega\|^2_{\mathcal{M}(I_n)} \Big) \Big\}^{\frac{1}{2}} \Bigg].$$

This completes the proof. □

In the following lemma, we obtain a bound for the error in the control variable. Following the proof of Lemma 6.2.2 together with (6.50)-(6.53), (6.67) and (6.68)-(6.70), it is easy to derive the result.

Lemma 6.3.2. *Let (y, q, z) and $(Y_h, \tilde{\mathcal{Q}}_h, Z_h)$ be the solutions of (6.50)-(6.53) and (6.63)-(6.67), respectively. Assume that $\forall n \in [1:N]$, $Q^P_{h,n} \subset Q^P_{ad}$, $(\tilde{\mathcal{Q}}_h + Z_h^{n-1})|_K \in H^1(K)$ and there is a constant $C_{18} > 0$, and $w_h \in Q^P_{h,n}$ such that*

$$|(\tilde{\mathcal{Q}}_h + Z_h^{n-1}, w_h - q)| \leq C_{18} \sum_{K\in\mathcal{T}_h} h_K |\tilde{\mathcal{Q}}_h + Z_h^{n-1}|_{H^1(K)} \|q - \tilde{\mathcal{Q}}_h\|_{L^2(K)}.$$

Then, we have

$$\|q - \tilde{\mathcal{Q}}_h\|^2_{L^2(L^2(\Omega))} \leq C_{19}^2 \eta^n_{1,mt} + \|Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))},$$

where $C_{19} = \sqrt{\frac{3}{2}} \max\{1, C_{18}\}$, $\eta^n_{1,mt} := \Big(\int_0^T \sum_{K\in\mathcal{T}_h} h_K^2 |\tilde{\mathcal{Q}}_h + Z_h^{n-1}|^2_{H^1(K)}\, d\tau \Big)$, and $z(\tilde{\mathcal{Q}}_h)$ satisfies the problem (6.69)-(6.70).

Analogous to Lemma 6.2.3, the following lemma gives an intermediate error estimate for the co-state variable.

Lemma 6.3.3. *Let (Y_h, Z_h) and $(y(\tilde{\mathcal{Q}}_h), z(\tilde{\mathcal{Q}}_h))$ be the solutions of (6.63)-(6.66) and (6.68)-(6.70), respectively. Then, we have*

$$\|Z_h^{n-1} - z(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))} \leq C_{20}^2 \sum_{i=2}^{5} \eta^n_{i,mt}, \tag{6.73}$$

where

$$\begin{aligned}
\eta^n_{2,mt} &:= \Big\{ \int_0^T \sum_{K\in\mathcal{T}_h^n} h_K^4 \| - Z_{h,t} + \mathcal{A}^* Z_h^{n-1} - Y_h^n + y_{\tilde{d}}^n\|^2_{L^2(K)}\, d\tau \\
&\quad + \int_0^T \sum_{e\in\mathcal{E}_h^n} h_e^3 \Big\| \Big[\frac{\partial Z_h^{n-1}}{\partial n_{\mathcal{A}^*}} \Big] \Big\|^2_{L^2(e)} d\tau \Big\}, \\
\eta^n_{3,mt} &:= \|Y_h^n - y(\tilde{\mathcal{Q}}_h)\|^2_{L^2(L^2(\Omega))}, \\
\eta^n_{4,mt} &:= \|y_{\tilde{d}} - y^n_{\tilde{d}}\|^2_{L^2(L^2(\Omega))}, \\
\eta^n_{5,mt} &:= \|Z_h - Z_h^{n-1}\|^2_{L^2(H^1(\Omega))},
\end{aligned}$$

and the constant C_{20} is defined in Lemma 6.2.3.

We are now in a position to present *a posteriori* error bound in the $L^2(0,T;L^2(\Omega))$-norm for measure data in time.

Theorem 6.3.1. *Let (y, q, z) and $(Y_h, \tilde{\mathcal{Q}}_h, Z_h)$ be the solutions of (6.50)-(6.53) and (6.63)-(6.67), respectively. Assume that all the conditions in Lemmas 6.3.1-6.3.3 are valid. Then, we have*

$$\begin{aligned}
&\sum_{n=1}^{N} \int_{I_n} \|Y_h^n - y\|^2_{L^2(\Omega)}\, d\tau + \sum_{n=1}^{N} \int_{I_n} \|Z_h^{n-1} - z\|^2_{L^2(\Omega)}\, d\tau + \sum_{n=1}^{N} \int_{I_n} \|\tilde{Q}_h - q\|^2_{L^2(\Omega)}\, d\tau \\
\leq\ & C_{19}^2 \eta^n_{1,mt} + C_{20}^2 \sum_{n=1}^{N} (\eta^n_{2,mt} + \eta^n_{4,mt} + \eta^n_{5,mt}) \\
&+ C_{22}^2 \Big\{ \sum_{n=1}^{N} k_n(\xi^n_{1,mt} + \xi^n_{2,mt}) + \xi^n_{3,mt} + \xi^n_{4,mt} \Big\},
\end{aligned}$$

where $\xi^n_{i,mt}|_{i=1,\ldots 4}$ are defined in Lemma 6.3.1, $\eta^n_{1,mt}$ is defined in Lemma 6.3.2, $\eta^n_{2,mt}$, $\eta^n_{4,mt}$ and $\eta^n_{5,mt}$ are defined in Lemma 6.3.3.

Remark 6.3.1. *If we have $\omega = \delta(t - t_*)$, where $\delta(t - t_*)$ denotes the Dirac measure in time concentrated at time $t = t_*$. Then, $\xi^n_{4,mt}$ in Theorem 6.3.1 reduces to*

$$\xi^n_{4,mt} := \sum_{K\in\mathcal{T}_{h,\mathcal{I}}} \Big(k_{\mathcal{I}} \|g(\cdot, t_*)\|^2_{L^2(K)} + h_K^2 \|g(\cdot, t_*)\|^2_{L^2(K)} \Big),$$

with $t_* \in (t_{n-1}, t_n]$ *for some* $n \in \mathbb{N}$ *and* $\mathcal{I}$ *be the set of indices for time partitions where measure data* $\delta(t - t_*)$ *is concentrated on.*

Concluding remarks. The main concern of this chapter is to study *a posteriori* error estimates for the space-time finite element discretizations of POCP (6.1)-(6.3) with measure data. Here, we have considered two kinds of problems. First, we consider the problem (6.1)-(6.3) with measure data in space, i.e., $\mu = g\omega$ where $g \in L^2(0, T; \mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$. Next, we consider the POCP (6.1)-(6.3) with measure data in time, i.e., $\mu = g\omega$ with $g \in \mathcal{C}([0, T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0, T]$. The finite element method is used to discretized these problems and *a posteriori* error estimators are derived. These estimators are used to obtain quantitative information on the discretization error as well as guiding an adaptive algorithm for local mesh refinement. The implementation of these estimators are carried out for the test problems in Chapter 7 with promising numerical results (see Examples 7.2 and 7.4).

7 Numerical Assessments

In this chapter, we shall present numerical results for two dimensional test problems to illustrate our theoretical findings. All computations are carried using the software FreeFem++ [44]. A projection gradient algorithm is used to solve the optimal control problems [65]. For each example, we compute the errors in the state, co-state and control variables.

Example 7.1. *In this example, we consider the following EOCP of the form*

$$\min_{q\in Q_{ad}^E} J(q,y) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)}$$

subject to the state equation

$$\begin{aligned} -\Delta y &= g\omega + q \quad in\ \Omega, \\ y &= 0 \qquad\quad on\ \partial\Omega, \end{aligned}$$

where $g \in \mathcal{C}(\overline{\Omega})$ and $\omega \in \mathcal{M}(\Omega)$. The co-state problem is given by

$$\begin{aligned} -\Delta z &= y - y_d \quad in\ \Omega, \\ z &= 0 \qquad\quad on\ \partial\Omega, \end{aligned}$$

where y_d is a given desired state.

Here, $\Omega = B(0,1)$, where $B(0,1)$ is the unit circle centered at zero with radius 1 and $\alpha = 3$. The exact solutions are chosen as follows:

$$\begin{aligned} y(x) &= -\frac{1}{2\pi} log|x|, \\ z(x) &= -cos(\frac{\pi}{2}|x|^2), \\ q(x) &= max(q_a, min(q_b, -z(x)/\alpha), \\ q_a &= 0.1 \quad q_b = 0.8. \end{aligned}$$

Then after a simple calculation, we have

$$\begin{aligned} g\omega &= \delta_{x_0} - q(x), \\ y_d(x) &= 2\pi sin(\frac{\pi}{2}|x|^2) + \pi^2|x|^2 cos(\frac{\pi}{2}|x|^2) - \frac{1}{2\pi}log|x|, \end{aligned}$$

where δ_{x_0} *is the Dirac function centered at* $x_0 = (0, 0)$.

First we validate the results obtain in Chapter 2. In this numerical implementation, the errors $\|y - y_h\|_{L^2(\Omega)}$, $\|z - z_h\|_{L^2(\Omega)}$ and $\|q - q_h\|_{L^2(\Omega)}$ obtained on a sequence of uniformly refined triangular meshes are presented in Table 7.1. Further, the rate of convergence for the state variable is less compared to the co-state variable see Table 7.1. To find the errors we use different degree of freedoms (DOF). The errors in the state, co-state and control variables decreases with the increase of DOF. The profiles of the exact and approximate solutions are plotted in Figures 7.1-7.3. The rate is calculated by the following formula:

$$Rate \simeq \frac{log(E_i^*/E_{i+1}^*)}{log(h_i/h_{i+1})},$$

where i corresponds to the spatial partition, and E_i^* denotes the error norm.

Table 7.1: *The errors for the state, co-state and control variables in the* L^2*-norm.*

DOF	$\|y - y_h\|_{L^2(\Omega)}$	Rate	$\|z - z_h\|_{L^2(\Omega)}$	Rate	$\|q - q_h\|_{L^2(\Omega)}$	Rate
25	8.27569e-4	-	6.34391e-3	-	2.50618e-3	-
81	2.78924e-4	0.92	7.12468e-4	1.86	7.25218e-4	1.05
144	1.5646e-4	1.033	2.72908e-4	1.714	3.9496e-4	1.08
289	7.57631e-5	1.046	8.12089e-5	1.748	1.7089e-4	1.2

Next, we validate the results obtained in Chapter 3. More precisely, we implement the estimators derived in Theorem 3.2.1 for Example 7.1. All constants involved in the estimators are taken to be 1. For this, we compute the errors on a uniform mesh and an adaptive mesh, respectively. Here the solution of the state equation has a singularity at the origin. In Table 7.2, the mesh information is displayed with L^2-norm approximation errors for the state, co-state and control variables. It can be clearly seen that on adaptive mesh one may use fewer nodes and elements in comparison to the uniform mesh. Since the main computational loads in solving the control problem come from repeatedly solving the state and co-state equations, the adaptive mesh can therefore save much computation.

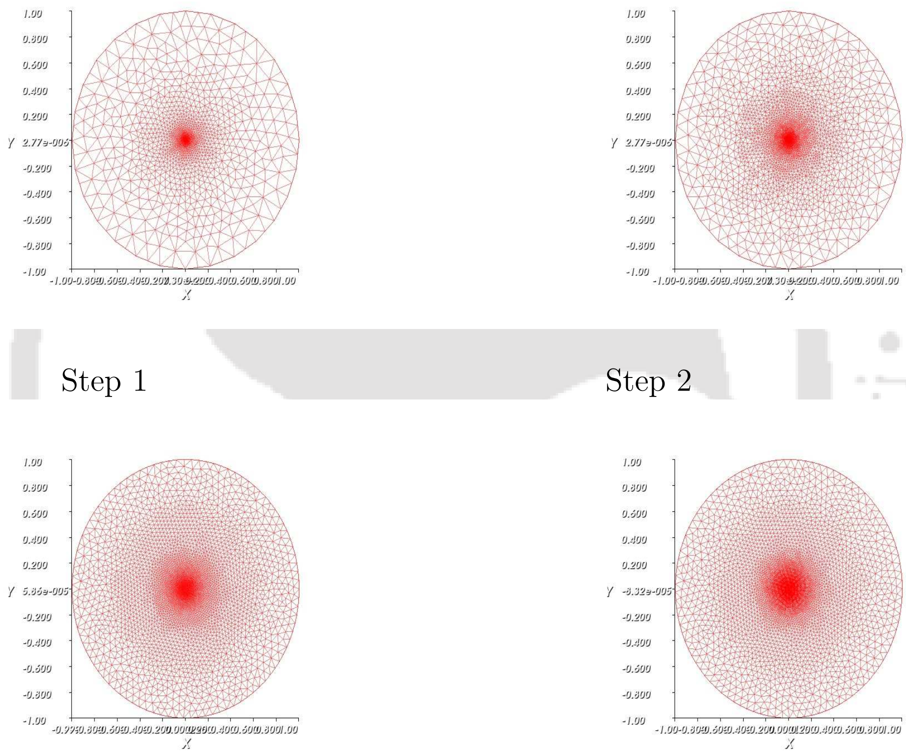
Figure 7.1: *The profile of the approximate state(left) and the exact state(right) with 1089 degree of freedoms.*

Figure 7.2: *The profile of the approximate co-state(left) and the exact co-state(right) with 1089 degree of freedoms.*

Figure 7.3: *The profile of the approximate control(left) and the exact control(right) with 1089 degree of freedoms.*

Table 7.2: *The performance of the estimators obtained in Theorem 3.2.1.*

	on uniform mesh			on adaptive mesh		
	y	z	q	y	z	q
Nodes	21733	21733	21733	5934	8995	8998
Elements	42964	42964	42964	11743	17630	14738
Error	1.37e-1	2.36331e-2	8.09986e-1	1.36946e-1	2.39682e-2	8.47418e-1



Step 1 Step 2

Step 3 Step 4

Figure 7.4: *The profile of the adaptive mesh for the state at different steps.*

The *a posteriori* error estimators obtained in Theorem 3.2.1 are used to detect the location of the singularity and distributed more mesh nodes around the area, where the singularity happen to save the computational cost. The mesh adapt very well at the neighborhood of the singular point, and a higher density of nodes are indeed distributed in the neighborhood, see Figure 7.4.

To validate the theoretical results obtained in Chapters 4-6, we consider the following POCP with measure data as follows:

$$\min_{q\in Q_{ad}^P} \tilde{J}(q,y) = \frac{1}{2}\int_0^T \{\|y - y_{\tilde{d}}\|^2_{L^2(\Omega)} + \tilde{\alpha}\|q\|_{L^2(\Omega)}\}\, d\tau \tag{7.1}$$

subject to the state equation

$$\begin{cases} y_t - \Delta y = g\omega + q & \text{in } \Omega_T, \\ y(\cdot,0) = y_0(x) & \text{in } \Omega, \\ y(x,t) = 0 & \text{on } \Gamma_T. \end{cases} \tag{7.2}$$

We introduce the co-state problem as:

$$\begin{cases} -z_t - \Delta z = y - y_{\tilde{d}} & \text{in } \Omega \times [0,T), \\ z(\cdot,T) = 0 & \text{in } \Omega, \\ z(x,t) = 0 & \text{on } \Gamma_T, \end{cases} \tag{7.3}$$

where $y_{\tilde{d}}$ is the given desired state.

To validate the theoretical results obtained in Chapter 4, we consider the following example.

Example 7.2. *Consider the problem (7.1)-(7.3) with $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$. We choose the domain $\Omega_T = B(0,1) \times [0,0.1]$, where $B(0,1)$ is the unit circle centered at zero with radius 1. The exact solutions are chosen as follows:*

$$\begin{aligned} y(x,t) &= -\frac{1}{2\pi} log|x| \cdot t(exp(t) - exp(T)), \\ z(x,t) &= -cos(\frac{\pi}{2}|x|^2) \cdot (exp(t) - exp(T)), \\ q(x,t) &= max(q_c, min(q_d, -z(x,t))). \\ q_c &= -0.08 \quad q_d = 0.1. \end{aligned}$$

Then, the data is given by

$$\begin{aligned} g\omega &= -\frac{1}{2\pi} log|x| \cdot (t \cdot exp(t) + exp(t) - exp(T)) \\ &\quad + t \cdot (exp(t) - exp(T))\delta_{x_0} - q(x,t), \\ y_{\tilde{d}}(x,t) &= -\frac{1}{2\pi} log|x| \cdot t(exp(t) - exp(T)) - cos(\frac{\pi}{2}|x|^2) \cdot exp(t) \\ &\quad + \Big(2\pi sin(\frac{\pi}{2}|x|^2) + \pi^2|x|^2 cos(\frac{\pi}{2}|x|^2)\Big) \cdot (exp(t) - exp(T)), \end{aligned}$$

where δ_{x_0} is the Dirac delta function concentrated at the point $x_0 = (0,0)$. We set $\tilde{\alpha} = 1$, $k = 0.01$, where k is the time step size and h is the mesh size of the spatial triangulation.

In Table 7.3, the errors $\|y - y_h\|_{L^2(L^2(\Omega))}$, $\|z - z_h\|_{L^2(L^2(\Omega))}$, $\|q - q_h\|_{L^2(L^2(\Omega))}$ are computed using piecewise linear and continuous finite elements for the state and co-state variables whereas the control variable is approximated by piecewise constant functions. The numerical results at final time $T = 0.1$ with different DOF are shown in Table 7.3 which validate the theoretical results. The profile of the exact and approximate solutions are shown in Figures 7.5-7.7.

Table 7.3: *The errors for the state, co-state and control variables in the $L^2(L^2(\Omega))$-norm for Example 7.2.*

DOF	$\|y - y_h\|_{L^2(L^2(\Omega))}$	Rate	$\|z - z_h\|_{L^2(L^2(\Omega))}$	Rate	$\|q - q_h\|_{L^2(L^2(\Omega))}$	Rate
25	7.57256e-2	-	1.55701e-1	-	2.01236e-3	-
81	2.3841e-2	0.98	4.5872e-2	1.03	5.9889e-4	1.029
144	1.2483e-2	1.15	2.32036e-2	1.21	2.95175e-4	1.026
289	6.02956e-3	1.05	1.12188e-2	1.05	1.4469e-4	1.029

Figure 7.5: *The profile of the approximate state(left) and the exact state(right) at $t = 0.08$ with 625 degrees of freedoms.*

Figure 7.6: *The profile of the approximate co-state(left) and the exact co-state(right) at* $t = 0.08$ *with 625 degrees of freedoms.*

Figure 7.7: *The profile of the approximate control(left) and the exact control(right) at* $t = 0.08$ *with 625 degrees of freedoms.*

The following example validates the results of Chapter 5.

Example 7.3. *Consider the problem (7.1)-(7.3) with* $g \in \mathcal{C}([0,T]; L^2(\Omega))$ *and* $\omega \in \mathcal{M}[0,T]$. *We choose the domain* $\Omega_T = B(0,1) \times [0,1]$, *where* $B(0,1)$ *is the unit circle centered at zero with radius* 1. *We set* $\tilde{\alpha} = 1$, $k = 0.01$, *where* k *is the time step size and* h *is the mesh size of spatial triangulation. The exact solution are chosen as follows:*

$$\begin{aligned} y(x,t) &= sin(\pi|x|^2) \cdot \begin{cases} t^2, & t < 0.5, \\ t^2 + 2\,t, & t \geq 0.5, \end{cases} \\ z(x,t) &= sin(\pi|x|^2) \cdot t, \\ q(x,t) &= max(q_c, min(q_d, -z(x,t))), \\ q_c &= -0.5, \qquad q_d = 0.1. \end{aligned}$$

A short calculation shows the data is given by

$$g\omega = sin(\pi|x|^2)\cdot\delta(t-0.5) - q(x,t) + sin(\pi|x|^2)\cdot\gamma(t)$$
$$+\Big(-4\pi cos(\pi|x|^2) + 4\,\pi^2|x|^2 sin(\pi|x|^2)\Big)\cdot\begin{cases} t^2, & t<0.5,\\ t^2+2\,t, & t\geq 0.5,\end{cases}$$

and

$$y_{\tilde{d}}(x,t) = sin(\pi|x|^2) + \Big(4\pi cos(\pi|x|^2) - 4\,\pi^2|x|^2 sin(\pi|x|^2)\Big)\cdot t$$
$$+sin(\pi|x|^2)\cdot\begin{cases} t^2, & t<0.5,\\ t^2+2\,t, & t\geq 0.5,\end{cases}$$

where $\delta(t-t^*)$ denotes the Dirac measure with respect to the time variable t concentrated at $t=t^*$, and

$$\gamma(t) = \begin{cases} 2\,t, & t<0.5,\\ 2\,t+2, & t\geq 0.5.\end{cases}$$

To validate the results obtained in Chapter 5, we compute the errors in sequence of uniformly refined triangular meshes. The errors $\|y-y_h\|_{L^2(L^2(\Omega))}$, $\|z-z_h\|_{L^2(L^2(\Omega))}$ $\|q-q_h\|_{L^2(L^2(\Omega))}$ are computed at final time $T=1$ with different degrees of freedom. The state and co-state are approximated by the continuous piecewise linear functions and the control is approximated by piecewise constant functions. To find the errors, we fix our time discretization with 100 time steps and use different degree of freedoms to validate our results. Table 7.4 shows the errors for the state, co-state and control variables in the $L^2(0,T;L^2(\Omega))$-norm. The profile of the exact and approximate controls are shown in Figure 7.8.

We now proceed to validate the results obtained in Chapter 6. For this, we have taken Example 7.2. To study the performance of the estimators obtained in Theorem 6.2.2, we perform the experiment on a uniform mesh and an adaptive mesh, respectively. We have considered the singularity of the state solution at the origin. The state and co-state variables are approximated by the continuous piecewise linear functions while piecewise constant functions are used to approximate the control. All constants involved in the estimators are taken to be 1. It can be clearly seen from Table 7.5 that on adaptive mesh we use fewer nodes, elements compared to the uniform mesh at final time $T=0.1$ with

Table 7.4: *The errors for the state, co-state and control variables in the $L^2(L^2(\Omega))$-norm for Example 7.3.*

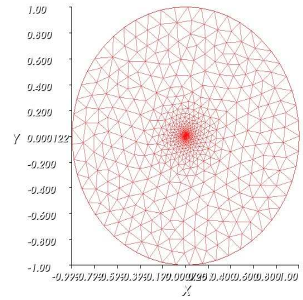
DOF	$\|y - y_h\|_{L^2(L^2(\Omega))}$	Rate	$\|z - z_h\|_{L^2(L^2(\Omega))}$	Rate	$\|q - q_h\|_{L^2(L^2(\Omega))}$	Rate
25	1.73004e-1	-	7.3917e-2	-	3.62604e-1	-
81	5.6486e-2	0.95	2.191e-2	1.03	7.87327e-2	1.29
169	2.7658e-2	0.97	8.7945e-3	1.24	3.701e-2	1.02
225	2.1146e-2	0.96	6.2789e-3	1.21	2.72908e-2	1.10

Figure 7.8: *The profile of the approximate control(left) and the exact control(right) at $T = 1$ with 625 degrees of freedoms.*

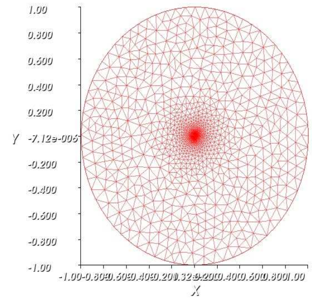
time step size $k = 0.01$. The main computational load in solving the control problem is due to repeated solving the state and the co-state equations and hence, the adaptivity saves much computation. The errors for the state, co-state and control variables are computed in the $L^2(0, T; L^2(\Omega))$-norm.

Table 7.5: *The performance of the estimators obtained in Theorem 6.2.2.*

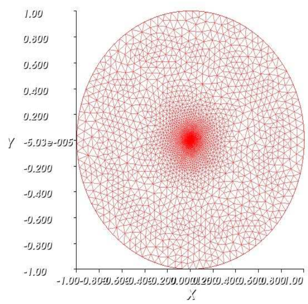
	on uniform mesh			on adaptive mesh		
	y	z	q	y	z	q
Nodes	14059	14059	14059	5920	6958	9000
Elements	27716	27716	27716	11776	13676	17266
Error	2.21556e-1	1.62374e-1	1.37254e-1	2.20165e-1	1.62368e-1	1.37419e-1



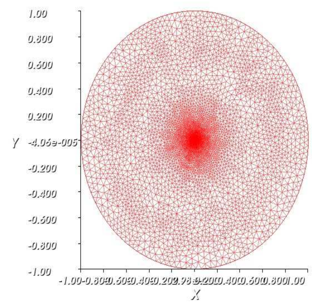
Adaptive mesh at step 1



Adaptive mesh at step 2



Adaptive mesh at step 3



Adaptive mesh at step 4

Figure 7.9: *The profile of the adaptive mesh for the state at different steps at time* $T = 0.1$.

Figure 7.9 shows the adaptive mesh for the state for different steps. In this figure, it is easy to see that the mesh for the state variable adapts very well in the neighborhood of the singular point, and a higher density of nodes are indeed distributed in the neighborhood.

In the following example, we verify the performance of the error estimators obtained in Theorem 6.3.1 for POCP with measure data in time.

Example 7.4. *Consider the problem (7.1)-(7.3) with the data as follows. Let* $\Omega_T = [-1,-1]^2 \times [0,1]$, $\gamma \in (0,1)$ *and* $\hat{\lambda} \in \mathbb{R}$. *Let*

$$\epsilon(t) = (e^{-\hat{\lambda}t} - e^{\frac{\hat{\lambda}}{2}}).$$

The exact solutions are chosen as follows:

$$y(x,t) \;=\; 0.1\,exp(-25|x-(t-0.5)|^2)(1.01-t)\frac{e^{\hat{\lambda}t}}{\hat{\lambda}(1-\gamma)}\cdot\begin{cases}\epsilon(0)^{1-\gamma}, & t \geq 0.5,\\ \epsilon(0)^{1-\gamma}-\epsilon(t)^{1-\gamma}, & t < 0.5.\end{cases}$$

$$\begin{aligned} z(x,t) &= sin(\pi|x|^2)\cdot t,\\ q(x,t) &= max(q_c, min(q_d, -z)),\\ q_c &= -0.05, \qquad q_d = 0.1. \end{aligned}$$

A short calculation shows the data

$$g\omega = 0.1\,exp(-25|x-(t-0.5)|^2)(1.01-t)\delta(t) - q,$$

and

$$\begin{aligned} y_{\tilde{d}}(x,t) &= sin(\pi|x|^2) + \Big(4\pi cos(\pi|x|^2) - 4\,\pi^2|x|^2 sin(\pi|x|^2)\Big)\cdot t\\ &+0.1\,exp(-25|x-(t-0.5)|^2)(1.01-t)\frac{e^{\hat{\lambda}t}}{\hat{\lambda}(1-\gamma)}\cdot\begin{cases}\epsilon(0)^{1-\gamma}, & t\geq 0.5,\\ \epsilon(0)^{1-\gamma}-\epsilon(t)^{1-\gamma}, & t<0.5,\end{cases} \end{aligned}$$

where

$$\delta(t) = \begin{cases} 0, & t\geq 0.5,\\ \epsilon(t)^{-\gamma}, & t<0.5. \end{cases}.$$

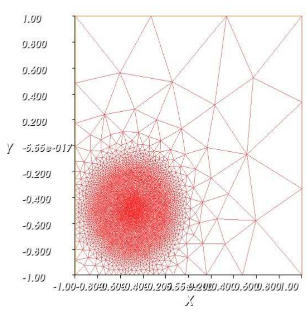
We set $\hat{\lambda} = 1$ *and* $\gamma = 0.6$.

We compute Example 7.4 on a uniform mesh and an adaptive mesh, respectively. In this example, the solution of the state equation y has singularity in time, i.e., as time vary singularity vary. The piecewise linear and continuous finite elements are used for the approximations of the state and co-state variables while the control variable is approximated by piecewise constant functions. All constants involved in the estimators are taken to be 1. In Table 7.6, the mesh information is displayed with $L^2(0,T;L^2(\Omega))$ approximation errors for the state, costate and control variables at time $T=1$ with time step size $k=0.01$. It can be clearly seen that on adaptive mesh one may use less number of nodes compare to the uniform mesh, thus the adaptive mesh can substantially save much computation.

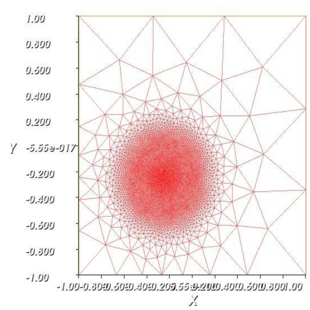
The *a posteriori* error estimators obtained in Theorem 6.3.1 are used to detect the location of the singularity and distributed more mesh nodes around the area, where the singularity happen to save the computational cost. The mesh adapt very well at the point, where the singularity occurs. Figure 7.10 shows that the singularity moves as time changes, while $t=0.00025$, $t=0.25$ and $t=0.95$.

Table 7.6: *The performance of the estimators obtained in Theorem 6.3.1.*

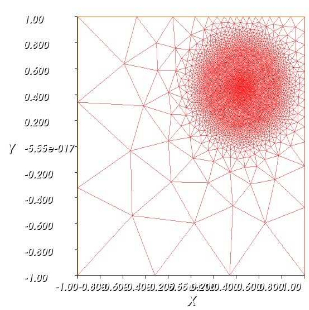
	on uniform mesh			on adaptive mesh		
	y	z	q	y	z	q
Nodes	11987	11987	11987	3177	5536	16146
Elements	23572	23572	23572	6160	10730	9000
Error	7.4367e-2	2.06573e-2	1.32832e-2	7.43054e-2	1.32726e-2	1.40755e-2



t=.00025



t=0.25



t=0.95

Figure 7.10: *The profile of adaptive mesh at different time levels.*

8

Conclusions and Extensions

This chapter is devoted to the critical assessment of the results highlighting the contributions made by this thesis and the techniques used in deriving these. It also provides information for the scope of possible extensions and future investigations.

8.1 Critical review of the results

In this thesis, we have considered finite element method for the approximations of optimal control problems governed by elliptic and parabolic equations with measure data. Our main focus is on the derivation of *a priori* and *a posteriori* error estimates in suitable norms for the unknown variables that appeared in the formulation. Moreover, the theoretical results are supported by numerical experiments. We now present critical review of the results obtained in each chapter.

In Chapter 2, we have studied finite element approximations of optimal control problems governed by elliptic equations with measure data. Unlike standard finite element methods, the weak solution of the problem (2.1)-(2.3) is defined via nonstandard weak formulation (2.5) of the state equation. Under the assumptions that $\mu = g\omega$, where $g \in \mathcal{C}(\overline{\Omega})$ and $\omega \in \mathcal{M}(\Omega)$, we prove the existence, uniqueness as well as regularity of the solutions to the problem (2.2)(cf. Theorem 2.2.1). We derive *a priori* error bounds for the control, state and co-state variables in the L^2-norm (see Theorems 2.4.1-2.4.3). The main tools used in our error analysis are duality technique, optimality conditions, error estimates associated with the adjoint problem (1.11) (Lemma 2.4.1) and second order optimality condition.

In Chapter 3, we have discussed *a posteriori* error analysis for EOCP (3.1)-(3.3) with measure data. The global upper bound estimators are derived for ($d = 2$ or 3) in the L^2-norm with $\mu = g\omega$, where $g \in \mathcal{C}(\overline{\Omega})$ and $\omega \in \mathcal{M}(\Omega)$. We have used residual type estimators to bound the errors. The upper bounds for the state, co-state and control variables in the L^2-norm are derived in Theorem 3.2.1. The local lower bounds for the errors in the state and co-state variables (Theorems 3.3.1 and 3.3.2) and global lower bound for the control variable (Theorem 3.3.3) are proved in the case of two space dimension ($d = 2$). The element and edge bubble functions play a key role in deriving lower bounds for the state and co-state variables, whereas inverse estimates play a key role for deriving lower bound for the control variable.

Chapter 4 is devoted to the development of finite element method for POCP (4.1)-(4.3) with measure data in space. With the assumption that $\mu = g\omega$, where $g \in L^2(0, T; \mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$, the weak solution of the problem (4.2) is defined by transposition techniques (see Lions and Magenes [61]). The existence, uniqueness and regularity of the solutions of problem (4.2) are proved in Theorem 4.2.1. The key ingredients used in the proof of Theorem 4.2.1 include embedding results and duality argument. We derive *a priori* error estimates for both semidiscrete and fully discrete finite element methods. The spatial discretization uses piecewise linear and continuous finite elements for the state and co-state variables whereas piecewise constant functions for the control variable. The backward Euler scheme is employed for the time discretization. The duality trick in conjunction with the approximation properties associated with the L^2-projection and Ritz projection (cf. Lemma 4.3.1), inverse estimates, interpolation properties are used to derive error bounds for the state, co-state and control variables in the $L^2(0, T; L^2(\Omega))$-norm. The first and second order optimality condition plays a crucial role in deriving the error in the control variable (cf. Theorems 4.3.1 and 4.4.1).

Chapter 5 deals with the finite element approximations of POCP with measure data in time (5.1)-(5.3), i.e., $\mu = g\omega$ with $g \in \mathcal{C}([0, T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0, T]$. The existence, uniqueness as well as regularity of the solution of the state equation (5.2) are studied and related *a priori* bounds for the solution are derived in Theorem 5.2.1. We derive *a priori* error estimates for both semidiscrete and fully discrete finite element methods. We use piecewise linear and continuous finite elements for the approximation of the state and co-state variables whereas the control variable is approximated by piecewise constant functions. The embedding results, duality technique and approximation properties (Lemma 4.3.1) play key technical role in deriving these estimates. For the estimation of error in the control variable we have used the first and second order optimality condition.

Chapter 6 concerns with *a posteriori* error estimates for the space-time finite element discretization of POCP (6.1)-(6.3) with measure data. Here, we consider two kinds of problems. At first, we consider problem (6.1)-(6.3) with measure data in space, i.e., $\mu = g\omega$, $g \in L^2(0, T; \mathcal{C}(\overline{\Omega}))$ and $\omega \in \mathcal{M}(\Omega)$. Next, we consider the POCP (6.1)-(6.3) with measure data in time, i.e., $\mu = g\omega$, $g \in \mathcal{C}([0,T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. The solution of state equation of such problem exhibits low regularity due to the presence of measure data and hence the development of AFEM fits to this kind of problems for the sake of accuracy enhancement. Since *a posteriori* error estimates play an important role to guide the adaptive procedure, *a posteriori* error estimators are thus constructed in this chapter. We derive *a posteriori* error bounds for the state, co-state and control variables in the $L^2(0,T;L^2(\Omega))$-norm for both type of problems (see Theorems 6.2.2 and 6.3.1). The estimators presented in Theorems 6.2.2 and 6.3.1 consist of the approximation errors of the state, co-state and control variables. The key technical tools include interpolation error estimates for the element and edge (Lemmas 3.2.1 and 3.2.2), duality technique and the first order optimality condition (see Lemma 6.2.2).

Chapter 7 is concerned with numerical assessments of our theoretical results derived in Chapters 2-6 with two dimensional test problems. All computations are carried out using the software FreeFem++ [44]. The optimization problems are solved by a projection gradient algorithm [65]. Numerical results are shown to be in agreement with our derived results.

8.2 Possible extensions

In this section, we make some informal observations pertaining to the possible extensions of our results to new research directions and propose some open problems for future investigation. Below, we shall briefly outline some interesting problems to be persuaded in future.

***A priori* and *a posteriori* error analysis of EOCP with measure data in nonconvex domains.** Let Ω be a non-convex polygonal domain in $\mathbb{R}^2$. Consider the following problem:

$$\min_{q \in Q_{ad}^E} J(q,y) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)}$$

subject to

$$\begin{cases} \mathcal{A}\,y = \mu + q \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and the control constraints

$$q_a \le q(x) \le q_b \quad a.e. \text{ in } \Omega,$$

where $y = y(x)$ denotes the state variable and $q = q(x)$ is the control variable. The operator $\mathcal{A}$ is defined by (1.4). We assume, for simplicity of presentation, that exactly one interior angle θ is reentrant, i.e., $\theta \in (\pi, 2\pi)$. We set $\tilde{\beta} = \frac{\pi}{\theta}$ and note that $\frac{1}{2} < \tilde{\beta} < 1$. In the special case of an L-shaped domain, $\theta = \frac{3\pi}{2}$ and $\tilde{\beta} = \frac{2}{3}$. The given function $\mu = g\omega$, $g \in \mathcal{C}(\overline{\Omega})$ and $\omega \in \mathcal{M}(\Omega)$. Furthermore, $y_d(x) \in L^2(\Omega)$ is the given desired state and $\alpha > 0$ is a fixed parameter. The bounds $q_a,\ q_b \in \mathbb{R}$ fulfill $q_a < q_b$.

It would be challenging to study the convergence analysis of the finite element method for EOCP with measure data in a nonconvex domain. We strongly believe that the techniques used in Chapters 2 and 3 will play a crucial role to tackle this issue.

***A priori* and *a posteriori* error analysis of POCP with measure data in nonconvex domains.** Let Ω be a nonconvex domain in $\mathbb{R}^2$. Consider the following optimal control problem governed by parabolic equation with control constraints

$$\min_{q \in Q_{ad}^P} \tilde{J}(q, y) = \frac{1}{2} \int_0^T \{\|y - y_{\tilde{d}}\|^2_{L^2(\Omega)} + \tilde{\alpha}\|q\|^2_{L^2(\Omega)}\}\, d\tau$$

subject to

$$\begin{cases} y_t + \mathcal{A}y = \mu + q \ \text{ in } \ \Omega_T, \\ y(\cdot, 0) = y_0(x) \ \text{ in } \ \Omega, \\ y = 0 \quad \text{on } \ \Gamma_T, \end{cases}$$

and the control constraints

$$q_c \leq q(x,t) \leq q_d \quad a.e. \ \text{ in } \ \Omega_T,$$

where $y = y(x,t)$ denotes the state variable, $q = q(x,t)$ denotes the control variable and $y_t = \frac{\partial y}{\partial t}$. The operator $\mathcal{A}$ is defined by (1.4). We assume, for simplicity of presentation, that exactly one interior angle θ is reentrant, i.e., $\theta \in (\pi, 2\pi)$. We set $\tilde{\beta} = \frac{\pi}{\theta}$ and note that $\frac{1}{2} < \tilde{\beta} < 1$. In the special case of an L-shaped domain, $\theta = \frac{3\pi}{2}$ and $\tilde{\beta} = \frac{2}{3}$. The constant $\tilde{\alpha} > 0$ is a fixed constant and the bounds $q_c, q_d \in \mathbb{R}$ fulfill $q_c < q_d$. Moreover, $y_0(x)$ and $y_{\tilde{d}}(x,t)$ are sufficiently smooth in their respective domains. Here, we shall consider two kinds of problems. First, measure data in space, i.e., $\mu = g\omega$, $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$, $\omega \in \mathcal{M}(\Omega)$. Next, we shall analyze measure data in time, i.e., $\mu = g\omega$, $g \in \mathcal{C}([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$.

We note that the error analysis in Chapters 4 and 5 rely on the inverse estimates, approximation properties of the L^2-projection and Ritz projection (Lemma 4.3.1). Further, Lemma 4.3.1 is valid under the assumption that Ω is convex, so the approximation properties in Lemma 4.3.1 do not hold when Ω is a nonconvex domain. Therefore, it would be interesting to see how the results of Chapters 4 and 5 can be extended to the case when the domain Ω is nonconvex.

Adaptive finite element methods for EOCPs and POCPs with measure data. Adaptive finite element method (AFEM) is one of the useful numerical method to approximate optimal control problems. It is known fact that the solution of the state equations of EOCP (1.1)-(1.3) and POCP (1.6)-(1.8) with measure data exhibit low regularity. Thus, the development of adaptive algorithms for finite element method fit to these kind of problems for the sake of accuracy enhancement. Moreover, *a posteriori* error estimators are key ingredients for the design of adaptive algorithms. We believe that *a posteriori* error estimators constructed in Chapters 3 and 6 would be useful to guide the adaptive procedure for these problems.

Optimal control problems governed by parabolic integro-differential equation (PIDE) with measure data. Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded convex domain with boundary $\partial\Omega$. Consider the following problem:

$$\min_{q \in Q_{ad}^P} \tilde{J}(q, y) = \frac{1}{2} \int_0^T \{\|y - y_{\tilde{d}}\|^2_{L^2(\Omega)} + \tilde{\alpha}\|q\|^2_{L^2(\Omega)}\}\, d\tau \tag{8.1}$$

subject to the linear PIDE of the form

$$\begin{cases} y_t + \mathcal{A}y = \int_0^t \mathcal{B}(t,\tau)y(x,\tau)\, d\tau + \mu + q \quad \text{in } \Omega_T, \\ y(\cdot, 0) = y_0(x) \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \Gamma_T, \end{cases} \tag{8.2}$$

and the control constraints

$$q_c \leq q(x,t) \leq q_d \ \ a.e. \text{ in } \Omega_T, \tag{8.3}$$

where q_c, $q_d \in \mathbb{R}$ fulfill $q_c < q_d$, $y_t = \frac{\partial y}{\partial t}$, $\tilde{\alpha} > 0$ is a fixed constant. The operator $\mathcal{A}$ is defined by (1.4) and the operator $\mathcal{B}$ is a second order partial differential operator of the form

$$\mathcal{B}(t,\tau) = -\nabla \cdot (B(t,s)\nabla y),$$

where ∇ denotes the spatial gradient. The coefficient matrix $B(t,s) = \{b_{ij}(x,t;s)\}$, $0 \leq s \leq t$, is a $d \times d$ real-valued matrix assumed to be in $L^\infty(\Omega)^{d\times d}$ in the space variable. Moreover, the elements of $B(t,s)$ are assumed to be smooth in both t and s. The given functions $y_0(x)$ and $y_{\tilde{d}}(x,t)$ are assumed to be smooth for our purpose. We shall consider two kinds of problem, i.e., measure data in space and measure data in time. The given function $\mu = g\omega$, where $g \in L^2(0,T;\mathcal{C}(\overline{\Omega}))$, $\omega \in \mathcal{M}(\Omega)$ for measure data in space and $g \in \mathcal{C}([0,T];L^2(\Omega))$, $\omega \in \mathcal{M}[0,T]$ for measure data in time.

Optimal control problems governed by PIDE occur in many applications, such as heat conduction control of materials with memory, population dynamics control, and control in elastic-plastic mechanics (see [80, 81]). The state equation for optimal control problems governed by PIDE (8.2) can be thought of as perturbation of the associated state equation (1.7) of POCP. We would like to see how the error analysis developed in Chapters 4-6 can be extended to optimal control problems governed by PIDE. We wish to investigate both *a priori* and *a posteriori* error analysis of the optimal control problems governed by (8.1)-(8.3).

Bibliography

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, Elsevier, Amsterdam, 2003.

[2] M. Ainsworth and J. T. Oden, *A unified approach to a posteriori error estimation using element residual methods*, Numer. Math., 65 (1993), pp. 23–50.

[3] M. Ainsworth and J. T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley-Interscience, New York, 2000.

[4] W. Alt and U. Mackenroth, *Convergence of finite element approximations to state constrained convex parabolic boundary control problems*, SIAM J. Control Optim., 27 (1989), pp. 718–736.

[5] N. Arada, E. Casas, and F. Tröltzsch, *Error estimates for the numerical approximation of a semilinear elliptic control problem*, Comput. Optim. Appl., 23 (2002), pp. 201–229.

[6] N. Arada and J. P. Raymond, *Approximation of Young measures for state-constrained optimal control problems*, Numer. Funct. Anal. Optim., 21 (2000), pp. 193–216.

[7] R. Araya, E. Behrens, and R. Rodríguez, *A posteriori error estimates for elliptic problems with Dirac delta source term*, Numer. Math., 105 (2006), pp. 193–216.

[8] V. Arnautu and P. Neittaanmäki, *Discretization estimates for an elliptic control problem*, Numer. Funct. Anal. Optim., 19 (1998), pp. 431–464.

[9] I. Babuška and W. C. Rheinboldt, *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal., 15 (1978), pp. 736–754.

[10] R. Becker, H.Kapp, and R. Rannacher, *Adaptive finite element methods for optimal control governed by partial differential equations: Basic concept*, SIAM J. Control Optim., 39 (2000), pp. 113–132.

[11] R. Becker and S. P. Mao, *Quasi-optimality of an adaptive finite element method for an optimal control problem*, Comput. Methods Appl. Math., 11 (2011), pp. 107–128.

[12] L. Boccardo, A. Dall'Aglio, and T. Gallouët, *Nonlinear parabolic equations with measure data*, J. Funct. Anal., 147 (1997), pp. 237–258.

[13] L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal., 87 (1989), pp. 149–169.

[14] E. Casas, *L^2 estimates for the finite element method for the Dirichlet problem with singular data*, Numer. Math., 47 (1985), pp. 627–632.

[15] ——, *Optimal control in coefficients of elliptic equations with state constraints*, Appl. Math. Optim., 26 (1992), pp. 21–37.

[16] ——, *Pontryagin's principle for state constrained boundary control problems of semilinear parabolic equations*, SIAM J. Control Optim., 35 (1997), pp. 1297–1327.

[17] ——, *Error estimates for the numerical approximation of semilinear elliptic control problems with finitely many state constraints*, ESAIM Control Optim. Calc. Var., 8 (2002), pp. 345–374.

[18] E. Casas, C. Clason, and K. Kunisch, *Approximation of elliptic control problems in measure spaces with sparse solutions*, SIAM J. Control Optim., 50 (2012), pp. 1735–1752.

[19] ——, *Parabolic control problems in measure spaces with sparse solutions*, SIAM J. Control Optim., 51 (2013), pp. 28–63.

[20] E. Casas, M. Mateos, and F. Tröltzsch, *Error estimates for the numerical approximation of boundary semilinear elliptic control problems*, Comput. Optim. Appl., 31 (2005), pp. 193–219.

[21] E. Casas and J. P. Raymond, *Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations*, SIAM J. Control Optim., 45 (2006), pp. 1586–1611.

[22] K. Chrysafinos and L. S. Hou, *Error estimates for semidiscrete finite element approximations of linear and semilinear parabolic equations under minimal regularity assumptions*, SIAM J. Numer. Anal., 40 (2002), pp. 282–306.

[23] I. Chryssoverghi, *Approximate methods for optimal pointwise control of parabolic systems*, Systems Control Lett., 1 (1982), pp. 216–229.

[24] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, SIAM, Philadelphia, 2002.

[25] C. Clason and K. Kunisch, *A duality-based approach to elliptic control problems in non reflexive Banach spaces*, ESAIM Control Optim. Calc. Var., 17 (2011), pp. 243–266.

[26] K. Deckelnick and M. Hinze, *Variational discretization of parabolic control problems in the presence of pointwise state constraints*, J. Comput. Math., 29 (2011), pp. 1–16.

[27] A. Demlow and R. Stevenson, *Convergence and quasi-optimality of an adaptive finite element method for controlling L^2-errors*, Numer. Math, 117 (2011), pp. 185–218.

[28] J. Droniou and J. P. Raymond, *Optimal pointwise control of semilinear parabolic equations*, Nonlinear Anal., 39 (2000), pp. 135–156.

[29] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, *Introduction to Adaptive Methods for Differential Equations*, Cambridge Univ. Press, 1995.

[30] K. Eriksson and C. Johnson, *An adaptive finite element method for linear elliptic problems*, Math. Comp., 50 (1988), pp. 361–383.

[31] R. Falk, *Approximation of a class of optimal control problems with order of convergence estimates*, J. Math. Anal. Appl., 44 (1973), pp. 28–47.

[32] D. A. French and J. T. King, *Analysis of a robust finite element approximation for a parabolic equation with rough boundary data*, Math. Comp., 60 (1993), pp. 79–104.

[33] A. Gaevskaya, R. H. W. Hoppe, Y. Iliash, and M. Kieweg, *Convergence analysis of an adaptive finite element method for distributed control problems with control constraints*, Internat. Ser. Numer. Math., 155 (2007), pp. 47–68.

[34] T. Geveci, *On the approximation of the solution of an optimal control problem governed by an elliptic equation*, RAIRO Anal. Numér., 13 (1979), pp. 313–328.

[35] W. Gong, *Error estimates for finite element approximations of parabolic equations with measure data*, Math. Comp., 82 (2013), pp. 69–98.

[36] W. Gong and M. Hinze, *Error estimates for parabolic optimal control problems with control and state constraints*, Comput. Optim. Appl., 56 (2013), pp. 131–151.

[37] W. Gong, M. Hinze, and Z. Zhou, *A priori error analysis for finite element approximation of parabolic optimal control problems with pointwise control*, SIAM J. Control Optim., 52 (2014), pp. 97–119.

[38] W. Gong, H. Liu, and N. Yan, *Adaptive finite element method for parabolic equations with Dirac measure*, Comput. Methods Appl. Mech. Engrg., 328 (2018), pp. 217–241.

[39] W. Gong, H. H. Xie, and N. N. Yan, *Adaptive multilevel correction method for finite element approximations of elliptic optimal control problems*, J. Sci. Comput., 72 (2017), pp. 820–841.

[40] W. Gong and N. Yan, *Adaptive finite element method for elliptic optimal control problems: convergence and optimality*, Numer. Math., 135 (2017), pp. 1121–1170.

[41] W. Gong, N. Yan, and Z. Zhou, *Convergence of L^2 norm based adaptive finite element method for elliptic optimal control problems*, arXiv:1608.08699, (2016).

[42] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman(Advanced Publishing Program), Boston, MA, 1985.

[43] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1988.

[44] F. Hecht, *New development in FreeFem++*, J. Numer. Math., 20 (2012), pp. 251–265.

[45] M. Hintermüller and R. H. W. Hoppe, *Goal-oriented adaptivity in control constrained optimal control of partial differential equations*, SIAM J. Control Optim., 47 (2008), pp. 1721–1743.

[46] M. Hintermüller, R. H. W. Hoppe, Y. Iliash, and M. Kieweg, *An a posteriori error analysis of adaptive finite element methods for distributed elliptic control problems with control constraints*, ESAIM: Control Optim. Calc. Var., 14 (2008), pp. 540–560.

[47] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich, *Optimization with PDE Constraints*, Math. Model. Theo. Appl., Springer, New York, 2009.

[48] M. Hinze and F. Tröltzsch, *Discrete concepts versus error analysis in PDE-constrained optimization*, GAMM-MITT, 33 (2010), pp. 148–162.

[49] G. Knowles, *Finite element approximation of parabolic time optimal control problems*, SIAM J. Control Optim., 20 (1982), pp. 414–427.

[50] K. Kohls, A. Rösch, and K. G. Siebert, *Convergence of adaptive finite elements for optimal control problems with control constraints*, Internat. Ser. Numer. math., 165 (2015), pp. 403–419.

[51] A. Kufner, O. John, and S. Fucik, *Function Spaces*, Nordhoff, Leyden, The Netherlands, 1977.

[52] U. Langer, S. Repin, and M. Wolfmayr, *Functional a posteriori error estimates for parabolic time-periodic boundary value problems*, Comput. Methods Appl. Math., 15 (2015), pp. 353–372.

[53] ——, *Functional a posteriori error estimates for parabolic time-periodic parabolic optimal control problems*, Numer. Funct. Anal. Optim., 37 (2016), pp. 1267–1294.

[54] I. Lasiecka, *RitzGalerkin approximation of the time optimal boundary control problem for parabolic systems with Dirichlet boundary conditions*, SIAM J. Control Optim., 22 (1984), pp. 477–500.

[55] I. Lasiecka and K. Malanowski, *On discrete-time Ritz-Galerkin approximation of control constrained optimal control problems for parabolic systems*, Control Cybernet., 7 (1978), pp. 21–36.

[56] H. Leng and Y. Chen, *Convergence and quasi-optimality of an adaptive finite element method for optimal control problems on L^2-errors*, J. Sci. Comput., 73 (2017), pp. 438–458.

[57] D. Leykekhman and B. Vexler, *Optimal a priori error estimates of parabolic optimal control problems with pointwise control*, SIAM J. Numer. Anal., 51 (2013), pp. 2797–2821.

[58] R. Li, W. Liu, H. P. Ma, and T. Tang, *Adaptive finite element approximation for distributed elliptic optimal control problems*, SIAM J. Control. Optim., 41 (2002), pp. 1321–1349.

[59] R. Li, W. Liu, and N. Yan, *A posteriori error estimates of recovery type for disributed convex optimal control problems*, J. Sci. Comput., 33 (2007), pp. 155–182.

[60] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971.

[61] J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin, 1972.

[62] W. Liu and N. Yan, *A posteriori error analysis for distributed convex optimal control problems*, Adv. Comp. math., 15 (2001), pp. 285–309.

[63] ——, *A posteriori error estimates for convex boundary control problems*, SIAM J. Numer. Anal., 39 (2001), pp. 73–99.

[64] ——, *A posteriori error estimates for optimal control problems governed by parabolic equations*, Numer. Math., 93 (2003), pp. 497–521.

[65] ——, *Adaptive Finite Element Methods for Optimal Control Governed by PDEs*, Science Press, Beijing, 2008.

[66] J. D. los Reyes, *Numerical PDE-Constrained Optimization*, Springer, Berlin, 2015.

[67] M. Luskin and R. Rannacher, *On the smoothing property of the Galerkin method for parabolic equations*, SIAM J. Numer. Anal., 19 (1982), pp. 93–113.

[68] A. Martínez, C. Rodríguez, and M. E. Vaźquez-Méndez, *Theoretical and numerical analysis of an optimal control problem related to waste water treatment*, SIAM J. Control Optim., 38 (2000), pp. 1534–1553.

[69] R. S. McKnight and W. E. Bosarge, *Jr., The Ritz-Galerkin procedure for parabolic control problems*, SIAM J. Control Optim., 11 (1973), pp. 510–524.

[70] D. Meidner, R. Rannacher, and B. Vexler, *A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time*, SIAM J. Control Optim., 49 (2011), pp. 1961–1997.

[71] D. Meidner and B. Vexler, *A priori error estimates for space-time finite element discretization of parabolic optimal control problems Part I: Problems without control constraints*, SIAM J. Control Optim., 47 (2008), pp. 1150–1177.

[72] ——, *A priori error estimates for space-time finite element discretization of parabolic optimal control problems Part II: Problems with control constraints*, SIAM J. Control Optim., 47 (2008), pp. 1301–1329.

[73] ——, *A priori error analysis of the Petro-Galerkin Crank-Nicolson scheme for parabolic optimal control problems*, SIAM J. Control Optim., 49 (2011), pp. 2183–2211.

[74] I. Neitzel and B. Vexler, *A priori error estimates for space-time finite element discretization of semilinear parabolic optimal control problems*, Numer. Math., 120 (2012), pp. 345–386.

[75] J. Nečas, S. Načasová, and C. G. Simader, *Direct Methods in the Theory of Elliptic Equations*, Springer-Verlag, Berlin, 2012.

[76] K. Pieper and B. Vexler, *A priori error analysis for discretization of sparse elliptic optimal control problems in measure space*, SIAM J. Control Optim., 51 (2013), pp. 2788–2808.

[77] R. Rannacher and R. Scott, *Some optimal error estimates for piecewise linear finite element approximations*, Math. Comp., 38 (1982), pp. 437–445.

[78] A. Rösch, *Error estimates for parabolic optimal control problems with control constraints*, Z. Anal. Anwendungen, 23 (2004), pp. 353–376.

[79] R. Scott, *Finite element convergence for singular data*, Numer. Math., 21 (1973), pp. 317–327.

[80] W. Shen, L. Ge, and D. Yang, *Sharp a posteriori error estimates for optimal control governed by parabolic integro-differential equations*, J. Sci. Comput., 65 (2015), pp. 1–33.

[81] W. Shen, L. Ge, D. Yang, and W. Liu, *A priori error estimates of finite element methods for linear parabolic integro-differential optimal control problems*, Adv. Appl. Math. Mech., 6 (2013), pp. 552–569.

[82] T. Sun, L. Ge, and W. Liu, *Equivalent a posteriori error estimates for a constrained optimal control problem governed by parabolic equations*, Inter. J. Numer. Anal. Model., 10 (2013), pp. 1–23.

[83] Y. Tang and Y. Chen, *Recovery type a posteriori error estimates of fully discrete finite element methods for general convex parabolic optimal control problems*, Numer. Math. Theor. Meth. Appl., 5 (2012), pp. 573–591.

[84] Y. Tang and Y. Hua, *Elliptic reconstruction and a posteriori error estimates for parabolic optimal control problems*, J. Applied Anal. Comput., 4 (2014), pp. 295–306.

[85] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, Berlin, 2006.

[86] F. Tröltzsch, *Optimal Control of Partial Differential Equations*, AMS, Providence, RI, 2010.

[87] R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, NewYork, 1996.

[88] R. Winther, *Error estimates for a Galerkin approximation of a parabolic control problem*, Ann. Mat. Pura Appl., 117 (1978), pp. 173–206.

[89] M. Wolfmayr, *A note on functional a posteriori estimates for elliptic optimal control problems*, Numer. Methods Partial Differential Equations, 33 (2016), pp. 403–424.

[90] C. Xiong and Y. Li, *A posteriori error estimates for optimal distributed control governed by the evolution equations*, Appl. Numer. Math., 61 (2011), pp. 181–200.