Localizing Spectra and Pseudospectra of Matrices and Matrix Polynomials

Ph.D. Thesis

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DECLARATION

I do hereby declare that this thesis entitled **Localizing Spectra and Pseudospectra of Matrices and Matrix Polynomials** is a presentation of my original research work carried out under the supervision of **Dr. Shreemayee Bora**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is to certify that the work contained in this thesis entitled **Localizing Spectra and Pseudospectra of Matrices and Matrix Polynomials** has been carried out by **Nandita Roy**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under my supervision for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

September, 2021

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Dedicated to my grandparents

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September, 2021 **Nandita Roy**

Abstract

This thesis considers different aspects of the study of sets that localize spectra and pseudospectra of matrices and matrix polynomials. Given a block upper triangular matrix, the literature has several results for outer approximations of the pseudospectra of the matrix by the pseudospectra of the diagonal blocks. The thesis provides inner approximations of the pseudopectra of a block upper triangular matrix in terms of pseudospectra of the diagonal blocks. Next the definitions of block Geršgorin sets, block Brualdi sets, block minimal Geršgorin sets and permuted pointwise minimal Geršgorin sets are extended to matrix polynomials in homogeneous form. The use of the homogeneous form is justified by its unified treatment of both the finite and infinite eigenvalues. Many properties of these sets are derived and efficient numerical methods for plotting the sets that can compare favourably with some existing methods under certain conditions, are proposed. In particular, the proposed methods may be used to plot all eigenvalue localization sets for matrix polynomials without laying a grid on any part of the complex plane.

Eigenvalue problems associated with the quadratic matrix polynomials have a wide range of applications. Localizations of eigenvalues of such polynomials are proposed via block Geršgorin sets that arise from some special linearizations. Several properties of these sets are derived and used to obtain some easily computable bounds on the eigenvalues of the matrix polynomial. They are also used to obtain sufficient conditions on the coefficient matrices of the polynomial for its eigenvalues to lie in particular regions of the complex plane that are important from the point of view of applications. The results also include various structured matrix polynomials. As an outcome of the analysis, several upper bounds on solutions of important distance problems associated with structured and unstructured quadratic matrix polynomials are derived for various choices of norms. In all cases, numerical experiments are performed to illustrate the results.

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1 Introduction

A polynomial of the form $P(z) = A_m z^m + \cdots + A_1 z + A_0,\ m \geq 1$, where A_i are $m \times n$ real or complex matrices is called a matrix polynomial or λ-matrix. It is of degree m if $A_m \neq 0$ and is said to be regular if it is square and $\det(P(z)) \neq 0$ for some $z \in \mathbb{C}$. Otherwise it is said to be singular or non-regular.

Given a square regular matrix polynomial $P(z)$, the polynomial eigenvalue problem consists of finding scalars λ and corresponding non-zero vectors x such that $P(\lambda)x = 0$. In such a case, λ is called an eigenvalue of $P(z)$, and x is called an eigenvector of $P(z)$ corresponding to λ. If the degree $m = 1$, then it corresponds to the generalized eigenvalue problem of finding scalars λ and non-zero vectors x that satisfy $(\lambda A_1 + A_0)x = 0$. Further, in such a case if $A_1 = -I$, we have the standard eigenvalue problem of finding a non-zero vector x and corresponding scalar λ such that $A_0 x = \lambda x$.

The polynomial $P(z)$ can also have eigenvalues at ∞. This happens whenever 0 is an eigenvalue of the reversal of $P(z)$ defined and denoted by $\mathrm{rev}P(z) := z^m P(1/z) = \sum_{i=0}^{m} A_{m-i} z^i$. The spectrum of a regular matrix polynomial is the set consisting of all finite and infinite eigenvalues. Only regular matrix polynomials are considered in the thesis.

Polynomial eigenvalue problems arise naturally in applications like vibration analysis of machines, buildings and vehicles, in control theory and linear systems theory [41, 44, 68]. This thesis considers three problems associated with matrix polynomials. The first problem is concerned with finding inner approximations of pseudospectra of matrices in block triangular forms. The second problem is concerned with finding inclusion or localization sets for eigenvalues of matrix polynomials in homogeneous form. The third problem merges the concept of linearizations of matrix polynomials with block Geršgorin-type localization sets of the eigenvalues of the polynomials to come up with results for locating eigenvalues of quadratic matrix polynomials that are important in applications.

For an arbitrary $\varepsilon > 0$, the ε-pseudospectrum of a square matrix A, denoted by $\Lambda_\varepsilon(A)$, is the set of all eigenvalues of real or complex matrices in a given ε-neighbourhood of A with respect to a choice of norm on the vector space of all $n \times n$ real or complex matrices.

The ε-pseudospectrum has long been recognized as an important tool for investigating the behaviour under perturbations of matrix-based models in dynamical system theory and algorithms in numerical linear algebra (see [69], and references therein).

Given a matrix A in block Schur form

$$A = U \begin{bmatrix} L & C \\ 0 & M \end{bmatrix} U^*$$

where L, M and C are complex matrices of sizes $\ell \times \ell$, $m \times m$ and $\ell \times m$ respectively, and U is a unitary matrix of size $(\ell + m) \times (\ell + m)$ such that the set of eigenvalues of L and M are disjoint, there are several results in literature [2, 15, 22, 71], which gives a function $g(\varepsilon)$ so that

$$\Lambda_\varepsilon(A) \subseteq \Lambda_{g(\varepsilon)}(L) \cup \Lambda_{g(\varepsilon)}(M),$$

forms an outer approximation of $\Lambda_\varepsilon(A)$ via the pseudospectra of the diagonal blocks L and M.

For the inner approximation, all we know is $\Lambda_\varepsilon(L) \cup \Lambda_\varepsilon(M) \subseteq \Lambda_\varepsilon(A)$. In order to have a tight approximation of $\Lambda_\varepsilon(A)$, via the pseudospectra of L and M, we would like to have a function $f(\varepsilon)$ preferably larger than ε, such that

$$\Lambda_{f(\varepsilon)}(L) \cup \Lambda_{f(\varepsilon)}(M) \subseteq \Lambda_\varepsilon(A).$$

The second chapter of the thesis addresses this problem with respect to the spectral norm $\|\cdot\|_2$. In particular, we find an optimal $f(\varepsilon)$ for the case that A is a 2×2 matrix. This yields an optimal $f(\varepsilon)$ for any general $n \times n$ matrix A which is unitary similar to matrices with 2×2 blocks on the diagonal. For the case of general $n \times n$ matrices of order $n (n \geq 3)$, we obtain such inner approximations for the ε-pseudospectrum of A under the assumption that the off-diagonal block C is of full rank.

The literature on localization of eigenvalues of matrix polynomials has expanded very rapidly in recent times [6, 7, 13, 36, 37, 39, 50–52, 54, 55, 63, 64]. In the year 1931, Geršgorin suggested a method which gives a bounded region on the complex plane containing all the eigenvalues of a matrix that can be computed easily by looking at the entries of the matrix A [20]. Because of its wide range of applications, this motivated further research to obtain various other related inclusion sets which are tighter than the Geršgorin sets such as Brualdi sets [75,76], minimal Geršgorin sets [72,75,76] and permuted minimal Geršgorin sets [43, 76] containing the eigenvalues of A. Such Geršgorin-type localizations for matrix pencils can be found in [36,39,54,63,64]. Inclusion sets such as Geršgorin sets, Ostrowski sets, Brauer sets, Dashnic-Zusmanovich sets and Householder sets for matrix polynomials are obtained in [13, 52]. Further, Geršgorin sets, Ostrowski sets and minimal Geršgorin sets are considered for nonlinear eigenvalue problems in [6, 37].

The block generalizations of Geršgorin's Theorem to the partitioned matrices were considered simultaneously and independently by Ostrowski [56], Feingold and Varga [17], and Fiedler and Pták [18]. The block generalizations of Brauer and Brualdi sets for partitioned matrices can be found in [76].

In the third chapter of the thesis, we generalize four localization sets, namely, the block Geršgorin sets, block minimal Geršgorin sets, block Brualdi sets and permuted pointwise minimal Geršgorin sets to matrix polynomials in homogeneous form. Several properties concerning these sets are obtained for these matrix polynomials. We provide numerical procedures for computing the sets wherein appropriate measures are taken for efficient computation. In particular, our procedure may be used to plot any eigenvalue localization sets for matrix polynomials without laying a grid on any part of the complex plane. We also demonstrate the advantages of incorporating the blocked versions of the localizations when computing these sets and provide blocking strategies.

In the final chapter of the thesis, we exploit the block Geršgorin sets in several contexts for the quadratic matrix polynomials. To start with, we derive the block Geršgorin sets arising from strong linearizations $L(z)$ of $P(z)$ that belong to any one of the three vector spaces of potential linearizations of $P(z)$, introduced by Mackey et. al. in [44, 46]. The resulting localization is a union of two sets with interesting properties. We obtain new upper and lower bounds on the eigenvalues of quadratic matrix polynomials arising from these localizations. These block Geršgorin sets lead to conditions on the coefficient matrices of quadratic matrix polynomials that can be easily computed and are sufficient for its eigenvalues to be located in particular regions of the complex plane. For a general quadratic matrix polynomial, we obtain sufficient conditions such that all its eigenvalues belong to the open left half of the complex plane. This provides upper bounds on the classical distance to stability for such polynomials.

Moreover, we also consider various structured quadratic matrix polynomials that arise in many applications like $*$-alternating, $*$-palindromic or Hermitian.

The coefficient matrices of $*$-alternating matrix polynomials alternate between being Hermitian and skew-Hermitian and are further divided into $*$-even and $*$-odd matrix polynomials. Their eigenvalues occur in pairs $\left(\lambda, -\bar{\lambda}\right)$ except on the imaginary axis where the pairing breaks down. Solutions of the associated quadratic eigenvalue problems are well studied for their important applications and interesting structure [41, 44, 46]. Here the presence of purely imaginary eigenvalues leads to numerical difficulties in applications [48, 58–60]. We find conditions on the coefficient matrices of quadratic $*$-alternating matrix polynomials that are sufficient for them to have no eigenvalues on the imaginary axis. This leads to an upper bound on the distance from a $*$-alternating quadratic matrix polynomial with purely imaginary eigenvalues to a nearest quadratic matrix polynomial of the same structure that does not have any such eigenvalues.

The eigenvalues of $*$-palindromic and $*$-antipalindromic matrix polynomials occur in pairs $\left(\lambda, 1/\bar{\lambda}\right)$ except on the unit circle where the pairing breaks down. The quadratic eigenvalue problem associated with $*$-palindromic and $*$-antipalindromic matrix polynomials has attracted interest due to its interesting structure and applications [44, 46]. Numerical difficulties arise in applications when such matrix polynomials have eigenvalues on the unit circle [48]. We find conditions on the coefficient matrices of quadratic $*$-palindromic and $*$-antipalindromic matrix polynomials that are sufficient for them to have no eigenvalues on the unit circle. This leads to an upper bound on the distance from a $*$-palindromic or $*$-antipalindromic quadratic matrix polynomial with eigenvalues on the unit circle to a nearest quadratic matrix polynomial of the same structure that does not have any such eigenvalues.

Matrix polynomials are said to be Hermitian when all their coefficient matrices are Hermitian. The theory and computation of eigenvalues of Hermitian matrix polynomials has been well studied (see for instance [26, 41, 42, 68, 79] and references therein). Their eigenvalues occur in pairs $\left(\lambda, \bar{\lambda}\right)$ except on the real axis where the symmetry breaks down. Hermitian matrix polynomials occur in structural mechanics as (M, D, K) problems where $A_2 = M$ corresponds to the mass matrix, $A_1 = D$ corresponds to damping effects and $A_0 = K$ denotes the stiffness in the structure. We find conditions on the coefficient matrices of Hermitian matrix polynomials that are sufficient for them to have no real eigenvalues. This leads to an upper bound on the distance from a Hermitian quadratic matrix polynomial with real eigenvalues to a nearest one of the same structure that does not have any such eigenvalues.

1.1 Notations

The following notations have been used throughout the thesis.

- $\mathbb{R}$ and $\mathbb{C}$ denotes the fields of real and complex numbers respectively. $\mathbb{N}$ denotes the set of all natural numbers.
- $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the vector spaces of real and complex column vectors of length n respectively.
- $\mathbb{C}^\infty := \mathbb{C} \cup \{\infty\}$ denote the one point compactification of $\mathbb{C}$.
- $\mathbb{R}^{m,n}$ and $\mathbb{C}^{m,n}$ respectively denotes the set of $m \times n$ real and complex matrices.
- $\det(A)$ denotes the determinant of the matrix A.
- A^{-1}, A^t and A^* denotes the inverse, transpose and conjugate transpose of a matrix A, respectively.
- $\mathbb{R}^n_+$ denotes the subset of $\mathbb{R}^n$ with positive entries.
- I and 0 respectively denotes the identity matrix and the zero matrix of appropriate size determined by the context.
- e_i denotes the i^{th} column of the identity matrix I where the length of e_i is clear from the context.
- For real matrices $A = [a_{ij}] \in \mathbb{R}^{m,n}$ and $B = [b_{ij}] \in \mathbb{R}^{m,n}$, $A \geq B$ if and only if $a_{ij} \geq b_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.
- $\Lambda(A)$ is the set of all eigenvalues of a matrix A and $\Lambda(P)$ denotes the eigenvalues of a matrix polynomial $P(z)$.
- $\lambda_{\min}(A)$ is the smallest eigenvalue of a matrix A.
- $\rho(A) = \max\{|\lambda| : \lambda \in \Lambda(A)\}$ denote the spectral radius of a matrix $A \in \mathbb{C}^{n,n}$.
- $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ is the smallest and the largest singular value of a matrix A, respectively.
- $\Lambda_\varepsilon(A)$ denotes the ε-pseudospectrum of a matrix A.
- $D(z_0, r)$ denotes the open disc with center $z_0 \in \mathbb{C}$ and radius $r > 0$ and D is the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$.
- $\mathbb{S} := \{(c, s) \in \mathbb{R} \times \mathbb{C} : c^2 + |s|^2 = 1\}$.
- δT denotes the boundary of a set T.
- $\|\cdot\|_p$ denotes the induced matrix norm for $1 \leq p \leq \infty$.

- $P(z)$ denotes the matrix polynomial $\sum_{i=0}^{m} A_i z^i$ where A_i are real or complex matrices of size $n \times n$.
- $\mathrm{rev}P(z) := \sum_{i=0}^{m} A_{m-i} z^i$ denotes the reversal of the matrix polynomial $P(z)$.
- $\mathcal{P}_n$ is the set of all $n \times n$ regular matrix polynomials of degree two.
- For a complex number z, sign(z) is defined by

$$\mathrm{sign}(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

- If i and j are integers, then the Kronecker delta $\delta_{i,j}$ is defined to be

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

- Given any $A \in \mathbb{C}^{n,n}$, the i^{th} deleted absolute row sum of A is defined and denoted by $r_i(A) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$.

1.2 Preliminaries

Throughout the thesis we consider a regular matrix polynomial of the form

$$P(z) = \sum_{i=0}^{m} A_i z^i, \; A_i \in \mathbb{C}^{n,n} \text{ and } A_m \neq 0. \tag{1.2.1}$$

The spectrum of $P(z)$ contains all points $z \in \mathbb{C}$ such that $\det P(z) = 0$. Furthermore, infinity is said to be an eigenvalue of $P(z)$ if zero is an eigenvalue of the reverse matrix polynomial $\mathrm{rev}P(z) = \sum_{i=0}^{m} A_{m-i} z^i$. Thus the spectrum of $P(z)$ is

$$\begin{cases} \{z \in \mathbb{C} : \det(P(z)) = 0\} & \text{if } A_m \text{ is non-singular}, \\ \{z \in \mathbb{C} : \det(P(z)) = 0\} \cup \{\infty\} & \text{if } A_m \text{ is singular}. \end{cases}$$

When the degree is one and $A_1 = -I$, then this reduces to the spectrum of the matrix A_0 consisting of all points $z \in \mathbb{C}$ so that $\det(A_0 - zI) = 0$.

In order to deal with both finite and infinite eigenvalues in a single framework, we consider the homogeneous form of the matrix polynomial $P(z)$ given by

$$P(c, s) = \sum_{i=0}^{m} A_i s^i c^{m-i}, \quad A_i \in \mathbb{C}^{n,n} \text{ and } A_m \neq 0.$$

Then the spectrum $\Lambda(P)$ is the collection of all pairs $(c, s) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $\det(P(c, s)) = 0$. An infinite eigenvalue is represented by $(0, s)$, where $s \neq 0$. Let $(c, s) := \{\tau(c, s) : \tau \in \mathbb{C}\}$ for $(c, s) \neq (0, 0)$, and consider the metric on $\mathbb{C}^2 \setminus \{(0,0)\}$ given by

$$\chi((c_1, s_1), (c_2, s_2)) = \frac{|c_1 s_2 - s_1 c_2|}{\sqrt{|c_1|^2 + |s_1|^2}\sqrt{|c_2|^2 + |s_2|^2}}.$$

Normalizing $(c, s) \in \Lambda(P)$ as $|c|^2 + |s|^2 = 1$, we identify $\Lambda(P)$ as a subset of the unit sphere $\mathbb{S}^1 = \{(c, s) \in \mathbb{C}^2 : |c|^2 + |s|^2 = 1\}$. Further, restricting c to be real, s to be complex with the normalization $c^2 + |s|^2 = 1$, the spectrum $\Lambda(P)$ is given by

$$\Lambda(P) := \{(c, s) \in \mathbb{S} : \det(P(c, s)) = 0\}.$$

Here we recall that $\mathbb{S} = \{(c, s) \in \mathbb{R} \times \mathbb{C} : c^2 + |s|^2 = 1\}$.

We begin with some important definitions for matrices and matrix polynomials.

Singular Value Decomposition of a matrix. Let A be a $n \times m$ complex matrix with rank r and let $q = \min\{m, n\}$. Then there exists unitary matrices $U \in \mathbb{C}^{n,n}$, $V \in \mathbb{C}^{m,m}$, and a diagonal matrix $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_q) \in \mathbb{R}^{n,m}$ where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q \geq 0$ such that

$$A = U\Sigma V^*. \tag{1.2.2}$$

If $r = q$, then $\sigma_q > 0$, else $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_q = 0$. The decomposition (1.2.2) is called the Singular Value Decomposition (SVD) of A. The non-negative quantities σ_i denote the i^{th} largest singular value of the matrix A, and u_i, v_i are corresponding left and right singular vectors associated with singular value σ_i of A, respectively.

ε-pseudopectra of matrices and matrix polynomials. Let $P(z)$ be a $n \times n$ matrix polyomial as given in 1.2.1. Then $P(z)$ belongs to the complex vector space of all $n \times n$ matrix polynomials of degree atmost m with respect to usual addition and scalar multiplication. Let $\|\cdot\|$ be a norm on this vector space. Suppose $\varepsilon > 0$ be arbitrary. Then the ε-pseudospectra of $P(z)$, defined and denoted by $\Lambda_\varepsilon(P)$, is the collection of all eigenvalues of perturbed polynomials $P(z) + \triangle P(z)$ where the perturbations $\triangle P(z)$ satisfy $\|\triangle P\| \leq \varepsilon$. The backward error of any $z \in \mathbb{C}^\infty$ as an approximate eigenvalue of $P(z)$ is defined as

$$\eta(z, P) := \inf\{\|\triangle P\| : z \in \Lambda(P + \triangle P)\}.$$

Note that we have $\eta(\infty, P) = \eta(0, \operatorname{rev}P)$. Then the set $\Lambda_\varepsilon(P)$ can be rewritten as

$$\Lambda_\varepsilon(P) = \{z \in \mathbb{C}^\infty : \eta(z, P) \leq \varepsilon\}.$$

For a matrix $A \in \mathbb{C}^{n,n}$, if $\|\cdot\|$ denotes any induced matrix norm, then with the convention that $\|(A-zI)^{-1}\| = \infty$ for $z \in \Lambda(A)$, we have $\eta(z,P) = \|(A-zI)^{-1}\|^{-1}$ and

$$\Lambda_\varepsilon(A) = \left\{ z \in \mathbb{C} : \left\|(A-zI)^{-1}\right\| \geq \frac{1}{\varepsilon} \right\}.$$

Expressions for the backward error $\eta(z,P)$ where $P(\lambda) = \sum_{i=0}^{m} A_i \lambda^i$ have been obtained in [1,67] for different choices of norms on the vector space of all $n \times n$ matrix polynomials of degree atmost m. For instance, with respect to the norm $\|P\|_2 = \left(\sum_{i=0}^{m} \|A_i\|_2^2 \right)^{\frac{1}{2}}$, we have

$$\eta(z,P) = \frac{\sigma_{\min}(P(z))}{\sqrt{\sum_{i=0}^{m} |z|^{2i}}}$$

so that

$$\Lambda_\varepsilon(P) = \left\{ z \in \mathbb{C}^\infty : \frac{\sigma_{\min}(P(z))}{\sqrt{\sum_{i=0}^{m} |z|^{2i}}} \leq \varepsilon \right\}.$$

The homogeneous form of the matrix polynomial $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ has been considered when obtaining expressions for the backward errors in [1]. Consider the norm

$$\|P\|_{p,\|\cdot\|} := \left\| \left[\|A_0\| \;\; \|A_1\| \;\; \cdots \;\; \|A_m\| \right] \right\|_p \quad \text{for } 1 \leq p \leq \infty \tag{1.2.3}$$

where $\|\cdot\|$ denote any norm on the linear space of all $n \times n$ matrices.

As a particular case, we have $\|P\|_2$ (mentioned earlier) for $\|\cdot\|_p = \|\cdot\| = \|\cdot\|_2$. If $\|\cdot\|$ denotes any induced matrix norm on $\mathbb{C}^{n,n}$ then for $(c,s) \in \mathbb{S}$, we have

$$\eta_{\|\cdot\|_{p,\|\cdot\|}}(c,s,P) = \frac{\|(P(c,s))^{-1}\|^{-1}}{\left\| [c^m \;\; c^{m-1}s \;\; \cdots \;\; s^m] \right\|_q} \tag{1.2.4}$$

with respect to the norm $\|\cdot\|_{p,\|\cdot\|}$, where $1 \leq q \leq \infty$ is such that $\frac{1}{p} + \frac{1}{q} = 1$ [1]. Therefore, $\Lambda_\varepsilon(P)$ in homogeneous form with respect to norm $\|\cdot\|_{p,\|\cdot\|}$ is given by

$$\Lambda_\varepsilon^{\|\cdot\|_{p,\|\cdot\|}}(P) = \left\{ (c,s) \in \mathbb{S} : \frac{\|(P(c,s))^{-1}\|^{-1}}{\left\| [c^m \;\; c^{m-1}s \;\; \cdots \;\; s^m] \right\|_q} \leq \varepsilon \right\}.$$

Next we define the point-wise classes of non-singular matrices that are used in the thesis.

Strictly diagonally dominant matrix. Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ with $n \geq 2$ be such that

$$|a_{ii}| > r_i(A) \text{ for each i} \in \{1, 2, \ldots, n\}$$

where $r_i(A)$ is defined to be the i^{th} deleted absolute row sum of A. Then A is called a strictly diagonally dominant (SDD) matrix.

$\mathbb{M}$-matrix. Given a real $n \times n$ matrix A having the form $A = sI - B$, where $B \geq 0$, $s > 0$, let $\rho(B) = \max\{|\lambda| : \lambda \in \Lambda(B)\}$ denote the spectral radius of B. The matrix A is called an $\mathbb{M}$-matrix if $\rho(B) \leq s$, and A is called a non-singular $\mathbb{M}$-matrix if $\rho(B) < s$.

$\mathbb{H}$-matrix. Given a matrix $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, the associated matrix $\langle A \rangle := [m_{i,j}]$ in $\mathbb{R}^{n,n}$, defined by

$$m_{ij} = \begin{cases} +|a_{i,j}| & \text{if } i = j, \\ -|a_{i,j}| & \text{if } i \neq j \end{cases}$$

is called the comparison matrix of A. A is called a $\mathbb{H}$-matrix if its comparison matrix $\langle A \rangle$ is an $\mathbb{M}$-matrix. Further, A is called a non-singular $\mathbb{H}$-matrix if $\langle A \rangle$ is a non-singular $\mathbb{M}$-matrix.

In particular, we use the following characterization of $\mathbb{H}$-matrices.

Theorem 1.2.1. *[4] A matrix $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ is a non-singular $\mathbb{H}$-matrix if and only if there exists a non-singular matrix $X = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ with positive diagonal entries such that AX is a SDD matrix.*

The following definitions from non-negative matrix theory plays an important role in the computations of the eigenvalue inclusion sets in the thesis.

Reducible matrix. For $n \geq 2$, a $n \times n$ complex matrix A is reducible if there exists an $n \times n$ permutation matrix P such that

$$PAP^t = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{12} & A_{22} \end{array}\right]$$

where A_{11} is a $r \times r$ submatrix and A_{22} is a $(n-r) \times (n-r)$ submatrix, $1 \leq r < n$. If no such permutation exists, then A is irreducible. If $A \in \mathbb{C}^{1,1}$, then A is irreducible if its single entry is non-zero, and reducible otherwise.

On recursively applying this algorithm for reducibility, we obtain a permutation matrix $P \in \mathbb{R}^{n,n}$ and a positive integer m, $2 \leq m \leq n$ such that

$$PAP^t = \begin{bmatrix} R_{1,1} & 0 & \ldots & 0 \\ R_{2,1} & R_{2,2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{m,1} & R_{m,2} & \ldots & R_{m,m} \end{bmatrix} \tag{1.2.5}$$

called the normal reduced form of A, where

$$\begin{cases} R_{j,j} \text{ is a } p_j \times p_j \text{ irreducible matrix with } p_j \geq 2, \\ \text{or,} \\ R_{j,j} \text{ is a } 1 \times 1 \text{ matrix with } R_{j,j} = [a_{k,k}] \text{ for some } k \in \{1, 2, \ldots, n\}. \end{cases}$$

Theorem 1.2.2. *[4][Perron-Frobenius Theorem for non-negative matrices]*
Given any $A = [a_{ij}] \in \mathbb{R}^{n,n}$ with $A \geq 0$, then

(i) A has a non-negative eigenvalue equal to its spectral radius $\rho(A)$;

(ii) to $\rho(A)$, there corresponds an eigenvector $x \geq 0$ with $x \neq 0$;

(iii) $\rho(A)$ may be a multiple eigenvalue of A;

(iv) the eigenvalue $\rho(A)$ of A satisfies

$$\rho(A) = \inf_{x>0} \max_{i \in \{1,\ldots,n\}} \frac{(Ax)_i}{x_i}.$$

Theorem 1.2.3. *[4][Perron-Frobenius Theorem for non-negative, irreducible matrices]*
Given any $A = [a_{ij}] \in \mathbb{R}^{n,n}$ with $A \geq 0$ and with A irreducible, then

(i) A has a positive eigenvalue equal to its spectral radius $\rho(A)$;

(ii) to $\rho(A)$, there corresponds an eigenvector $x > 0$;

(iii) $\rho(A)$ is a simple eigenvalue of A;

(iv) the eigenvalue $\rho(A)$ of A satisfies

$$\sup_{x>0} \min_{i \in \{1,\ldots,n\}} \frac{(Ax)_i}{x_i} = \rho(A) = \inf_{x>0} \max_{i \in \{1,\ldots,n\}} \frac{(Ax)_i}{x_i}.$$

Primitive matrix. [31] Let $A \in \mathbb{R}^{n,n}$ be such that A is non-negative and irreducible. If A has only a single eigenvalue of modulus $\rho(A)$, then A is called a primitive matrix.

Theorem 1.2.4. *[31]If $A \in \mathbb{R}^{n,n}$ is non-negative, then A is primitive if and only if $A^m > 0$ for some $m \geq 1$.*

Theorem 1.2.5. *[31]If $A \in \mathbb{R}^{n,n}$ is non-negative and irreducible, and if all its main diagonal entries are positive, then $A^{n-1} > 0$, so that A is primitive.*

Given a regular matrix polynomial $P(z)$, the associated polynomial eigenvalue problem consists of finding all its eigenvalues and corresponding eigenvectors. The most widely used approach for solving a polynomial eigenvalue problem is linearization.

Linearization. A $mn \times mn$ matrix pencil $L(z) = zX + Y$ is a linearization of an $n \times n$ matrix polynomial $P(z)$ of degree m if there exists two $mn \times mn$ unimodular matrix polynomials $E(z)$ and $F(z)$ such that

$$E(z)L(z)F(z) = \left[\begin{array}{c|c} P(z) & 0 \\ \hline 0 & I_{(m-1)n} \end{array}\right].$$

For example, given $P(z) = \sum_{i=0}^{m} A_i z^i$, the $nm \times nm$ matrix pencils

$$C_1(z) = z\begin{bmatrix} A_m & 0 & \dots & 0 \\ 0 & I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_n \end{bmatrix} + \begin{bmatrix} A_{m-1} & A_{m-2} & \dots & A_1 & A_0 \\ -I_n & 0 & \dots & 0 & 0 \\ 0 & -I_n & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -I_n & 0 \end{bmatrix} \tag{1.2.6}$$

$$\text{and} \quad C_2(z) = z\begin{bmatrix} A_m & 0 & \dots & 0 \\ 0 & I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_n \end{bmatrix} + \begin{bmatrix} A_{m-1} & -I_n & \dots & 0 & 0 \\ A_{m-2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_1 & 0 & \dots & 0 & -I_n \\ A_0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{1.2.7}$$

are called the first and second companion form linearizations of $P(z)$, respectively.

Strong linearization. A linearization $L(z) = zX + Y$ of a matrix polynomial $P(z)$ is called a strong linearization of $P(z)$ if $\mathrm{rev}L(z)$ is also a linearization of $\mathrm{rev}P(z)$.

$C_1(z)$ and $C_2(z)$ are strong linearizations of $P(z)$. The literature of linearizations is vast. In this thesis, we will focus on strong linearizations arising from three important vector spaces of potential strong linearizations for the matrix polynomial $P(z)$, the right and left ansatz vector spaces $\mathbb{L}_1(P)$, $\mathbb{L}_2(P)$, and the double ansatz space $\mathbb{DL}(P)$ were introduced in [44, 46]. These are defined below.

$$\mathbb{L}_1(P) = \quad \{L(z) = zX + Y : L(z)(\nabla \otimes I_n) = v \otimes P(z), v \in \mathbb{C}^m \setminus \{0\}\} \tag{1.2.8}$$

$$\mathbb{L}_2(P) = \quad \left\{L(z) = zX + Y : (\nabla^t \otimes I_n)L(z) = w^t \otimes P(z), w \in \mathbb{C}^m \setminus \{0\}\right\} \tag{1.2.9}$$

where $\nabla := [z^{m-1} \cdots z \ 1]^t$ and $\otimes$ denotes the Kronecker product. The defining identities in (1.2.8) and (1.2.9) are called the right and left ansatz equations respectively, and the corresponding vectors v in (1.2.8) and w in (1.2.9) are called the right and left ansatz vectors. The first companion form $C_1(z)$ of $P(z)$ belongs to $\mathbb{L}_1(P)$ with right ansatz vector e_1, while the second companion form $C_2(z)$ belongs to $\mathbb{L}_2(P)$ with left ansatz vector e_1.

The double ansatz space is the intersection $\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$. A pencil $L(z) \in \mathbb{DL}(P)$ if it satisfies both (1.2.8) and (1.2.9) with $v = w$.

Various classes of structured matrix polynomials are considered in this thesis. They are as follows.

$*$-even and $*$-odd matrix polynomials. A matrix polynomial $P(z)$ is said to be $*$-even if $(P(-z))^* = P(\bar{z})$ and $*$-odd if $(P(-z))^* = -P(\bar{z})$. The eigenvalues of $*$-even and $*$-odd matrix polynomials occur in pairs $\left(\lambda, -\bar{\lambda}\right)$. Moreover, if the polynomial is real, the eigenvalues occur in quadruples $\left(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}\right)$.

$*$-palindromic and $*$-antipalindromic matrix polynomials. A matrix polynomial $P(z)$ is said to be $*$-palindromic if $(P(z))^* = \mathrm{rev}P(\bar{z})$ and $*$-antipalindromic if $(P(z))^* = -\mathrm{rev}P(\bar{z})$. In either case, the eigenvalues of $P(z)$ occur in pairs $\left(\lambda, 1/\bar{\lambda}\right)$. Further, if $P(z)$ is real, then the eigenvalues occur in quadruples $\left(\lambda, 1/\lambda, \bar{\lambda}, 1/\bar{\lambda}\right)$.

Hermitian matrix polynomials. A matrix polynomial $P(z)$ is said to be Hermitian if $(P(z))^* = P(\bar{z})$. The eigenvalues of such a matrix polynomial $P(z)$ occur in pairs $\left(\lambda, \bar{\lambda}\right)$.

1.3 Literature survey

This thesis is about localizing spectra and pseudospectra of matrices and matrix polynomials. In the following subsections, we briefly survey some of the existing literature in these topics.

1.3.1 Localizing eigenvalues of matrices and matrix polynomials

For an arbitrary matrix $A \in \mathbb{C}^{n,n}$, the Geršgorin Theorem [20] for matrices gives a union of n closed discs in the complex plane which includes all the eigenvalues of A.

Theorem 1.3.1. *[20, 76] Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ with $n \geq 2$ and let λ be an eigenvalue of A. Then*

$$\Lambda(A) \subset \Gamma(A) := \bigcup_{1 \leq i \leq n} \Gamma_i(A)$$

where

$$\Gamma_i(A) := \left\{ z \in \mathbb{C} : |z - a_{i,i}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| \right\}, \quad i \in \{1, \ldots, n\}.$$

The set $\Gamma(A)$ is called the Geršgorin set for the matrix A, while $\Gamma_i(A)$ is called the i^{th} Geršgorin disc for A.

It has been proved that the Geršgorin inclusion result for the eigenvalues is equivalent to the non-singularity of the class of strictly diagonally dominant matrices. There are various results on eigenvalue localization sets published over the years, like Ostrowski sets, Brauer and Brualdi sets for matrices induced by the non-singularity of classes of more generalized versions of the class of strictly diagonally dominant matrices. Kostić has described a unified approach for these non-singular class of matrices in [34] where he introduced the concept of diagonally dominant type matrices and characterized the equivalent eigenvalue inclusion set.

Definition 1.3.2. *[34] Let $\mathbb{K} \subseteq \mathbb{C}^{n,n}$ be a non-empty class of matrices with the following properties.*

(i) For any $A \in \mathbb{K}$, the diagonal entries of A are non-zero.

(ii) For any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, $A \in \mathbb{K}$ if and only if $|A| \in \mathbb{K}$ where $|A| := [|a_{i,j}|]$.

(iii) For every $A \in \mathbb{K}$ and $B \in \mathbb{C}^{n,n}$, if $\langle B \rangle \geq \langle A \rangle$, then $B \in \mathbb{K}$.

Then $\mathbb{K}$ is called a diagonally dominant type (DD-type) class of matrices.

Theorem 1.3.3. *[34] Given a class of square matrices $K \in \mathbb{C}^{n,n}$, for any arbitrary $A \in \mathbb{C}^{n,n}$, define the set*

$$\theta^K(A) := \{z \in \mathbb{C} : zI - A \notin K\}.$$

Then the following two conditions are equivalent.

(i) All matrices from K are non-singular.

(ii) The set $\theta^K(A)$ contains all the eigenvalues of A i.e., $\Lambda(A) \subseteq \theta^K(A)$.

Thus the Geršgorin set $\Gamma(A)$ can be classified as the collection of those points $z \in \mathbb{C}$ such that $zI - A$ is not a strictly diagonally dominant matrix. The class of non-singular $\mathbb{H}$-matrices is also a DD-type class, and it is shown that this is the maximal non-singular class of the DD-type.

Theorem 1.3.4. *[34] If a diagonally dominant type class of matrices $\mathbb{K}$ is a class of non-singular matrices, then it is a subclass of the non-singular $\mathbb{H}$-matrices.*

As a consequence, the eigenvalue inclusion set arising from the class of non-singular $\mathbb{H}$-matrices is the smallest of all the inclusion sets corresponding to such non-singular DD-type matrices. The eigenvalue inclusion set corresponding to the class of non-singular $\mathbb{H}$-matrices is the minimal Geršgorin set.

Theorem 1.3.5. *[72, 76]. Let $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ and $x = [x_1\, x_2 \cdots x_n]^t \in \mathbb{R}^n_+$. The minimal Geršgorin set for A is defined as*

$$\mathcal{G}(A) := \bigcap_{x \in \mathbb{R}^n_+} \Gamma^x(A),$$

where

$$\Gamma^x(A) = \bigcup_{1 \le i \le n} \Gamma^x_i(A) = \bigcup_{1 \le i \le n} \left\{ z \in \mathbb{C} : |z - a_{i,i}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{i,j}| \frac{x_j}{x_i} \right\}$$

contains all the eigenvalues of A.

The Brauer set containing the eigenvalues of matrices is defined below.

Theorem 1.3.6. *[9, 76] For any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, $n \ge 2$ and any $\lambda \in \Lambda(A)$, there is a pair of distinct integers $i, j \in \{1, 2, \ldots, n\}$ such that*

$$\lambda \in \mathcal{K}_{i,j}(A) := \left\{ z \in \mathbb{C} : |z - a_{i,i}||z - a_{j,j}| \le \left(\sum_{\substack{k=1 \\ k \ne i}}^{n} |a_{i,k}| \right) \left(\sum_{\substack{k=1 \\ k \ne j}}^{n} |a_{j,k}| \right) \right\}.$$

Thus

$$\Lambda(A) \subseteq \mathcal{K}(A) := \bigcup_{\substack{1 \le i,j \le n \\ i \ne j}} \mathcal{K}_{i,j}(A).$$

The set $\mathcal{K}_{i,j}(A)$ is called the $(i, j)^{th}$ Brauer Cassini oval for the matrix A, while $\mathcal{K}(A)$ is called the Brauer set.

The following basic definitions from graph theory are required to introduce the Brualdi sets for matrices.

Definition 1.3.7. *[76] Given $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, the directed graph $\mathbb{G}(A)$ associated with A is defined as follows. The vertices of $\mathbb{G}(A)$ are numbered $1, 2, \ldots, n$. For each non-zero entry $a_{i,j}$ of A, there is an directed arc (i, j) form vertex i to vertex j. A path π from a vertex i to a vertex $j\, (i \ne j)$ is the sequence $i = i_0, i_1, \ldots, i_k = j$ of vertices where $(i_0, i_1), \ldots, (i_{k-1}, i_k)$ are arcs.*

Definition 1.3.8. *[76] A directed graph $\mathbb{G}(A)$ is strongly connected if for each ordered pair of vertices i and j, there is a directed path in $\mathbb{G}(A)$ with initial vertex i and terminal vertex j.*

The following theorem establishes connection between the concept of irreduciblity of matrices and strongly connected graphs.

Theorem 1.3.9. *[76] For any $A \in \mathbb{C}^{n,n}$, the matrix A is irreducible if and only if the directed graph $\mathbb{G}(A)$ is strongly connected.*

Definition 1.3.10. *[76] A strong cycle of $\mathbb{G}(A)$ is a sequence γ consisting of vertices $\{i_j\}_{j=1}^{p+1}$ of integers in $\{1,\ldots,n\}$ such that $p \geq 2$, the vertices $\{i_1, i_2, \ldots, i_p\}$ are all distinct with $i_{p+1} = i_1$ and $(i_1, i_2), \ldots, (i_{p-1}, i_p), (i_p, i_1)$ are arcs of $\mathbb{G}(A)$. We write $\gamma = (i_1, i_2, \ldots, i_p)$ and say $i_j \in \gamma$ for $j = 1, \ldots, p$. The positive integer p represents the length of cycle γ. If there is a vertex i of $\mathbb{G}(A)$ for which there is no strong cycle through i, then we define its associated weak cycle γ as $\gamma = (i)$, independent of whether or not $a_{i,i} = 0$. We denote the set of all cycles of $\mathbb{G}(A)$, by $\mathscr{C}(A)$.*

Definition 1.3.11. *[76] For each vertex i of A there is always a cycle (strong or weak) $\gamma \in \mathscr{C}(A)$ which passes through i. The deleted row sum at i with respect to γ is defined as $r_i^\gamma(A) := \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,j}|$, if γ is a strong cycle and $r_i^\gamma(A) = 0$, if γ is a weak cycle.*

Theorem 1.3.12. *[76] For any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ and any eigenvalue λ of A, there is a cycle γ in $\mathscr{C}(A)$ such that $\lambda \in \mathcal{B}_\gamma(A)$, where*

$$\mathcal{B}_\gamma(A) = \left\{ z \in \mathbb{C} : \prod_{i\in\gamma} |z - a_{i,i}| \leq \prod_{i\in\gamma} r_i^\gamma(A) \right\}.$$

Consequently, $\Lambda(A) \subseteq \bigcup_{\gamma\in\mathscr{C}(A)} \mathcal{B}_\gamma(A)$. The set $\mathcal{B}(A) = \bigcup_{\gamma\in\mathscr{C}(A)} \mathcal{B}_\gamma(A)$ is called the Brualdi set for the matrix A.

This eigenvalue inclusion set was first derived by Brualdi in [10] for weakly irreducible matrices, which are matrices having only strong cycles.

Definition 1.3.13. *[43, 76] For a matrix $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, let $x = [x_1\, x_2 \cdots x_n]^t$ in $\mathbb{R}_+^n$ so that $X := \operatorname{diag}(x_1, x_2, \ldots, x_n)$ is a non-singular matrix in $\mathbb{R}^{n,n}$. Let ϕ denote a permutation on the set $\{1, 2, \ldots, n\}$ and P_ϕ be the corresponding permutation matrix. Define a matrix $M \in \mathbb{C}^{n,n}$ by*

$$M := (X^{-1}AX - zI)P_\phi$$

Then for $z \in \Lambda(A)$, M is singular $\implies$ M is not a SDD matrix, that is, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$|M_{i,i}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |M_{i,j}|.$$

If $\phi(i) = i$*:* $\displaystyle |z - a_{i,i}| \leq \sum_{\substack{j=1\\j\neq i}}^{n} |a_{i,j}|\frac{x_j}{x_i}$,

if $\phi(i) \neq i$*:* $\displaystyle |z - a_{i,i}| \geq -\sum_{\substack{j=1\\j\neq i}}^{n} |a_{i,j}|\frac{x_j}{x_i} + 2|a_{i,\phi(i)}|\frac{x_{\phi(i)}}{x_i}$.

Thus if $\Gamma_\phi^x(A) := \{z \in \mathbb{C} : M$ *is not a SDD matrix*$\}$*. Then* $\Gamma_\phi^x(A) = \bigcup_{i=1}^{n} \Gamma_{i,\phi}^x(A)$*, where*

$$\Gamma_{i,\phi}^x(A) = \begin{cases} \left\{z \in \mathbb{C} : |z - a_{i,i}| \leq \displaystyle\sum_{\substack{j=1\\j\neq i}}^{n} |a_{i,j}|\frac{x_j}{x_i}\right\} & \text{if } \phi(i) = i, \\ \left\{z \in \mathbb{C} : |z - a_{i,i}| \geq -\displaystyle\sum_{\substack{j=1\\j\neq i}}^{n} |a_{i,j}|\frac{x_j}{x_i} + 2|a_{i,\phi(i)}|\frac{x_{\phi(i)}}{x_i}\right\} & \text{if } \phi(i) \neq i. \end{cases}$$

It follows that $\Lambda(A) \subseteq \Gamma_\phi^x(A)$*, for any* $x > 0$*. Hence,* $\Lambda(A) \subseteq \Gamma_\phi(A) := \bigcap_{x>0} \Gamma_\phi^x(A)$*, where* $\Gamma_\phi(A)$ *is called the minimal Geršgorin set relative to the permutation* ϕ*. If* ϕ *is the identity permutation on* $\{1, 2, \ldots, n\}$*, we get the pointwise minimal Geršgorin set.*

If Φ *denote the set of all* $n!$ *permutations on* $\{1, 2, \ldots, n\}$*, then on taking intersections over all permutations* $\phi \in \Phi$*,*

$$\Lambda(A) \subseteq \bigcap_{\phi\in\Phi} \Gamma_\phi(A) = \bigcap_{\phi\in\Phi} \bigcap_{x>0} \Gamma_\phi^x(A)$$

$\bigcap_{\phi\in\Phi} \Gamma_\phi(A)$ *is called permuted pointwise minimal Geršgorin set.*

Out of these $n!$ permutations, the trivial permutations $\phi \in \Phi$ for which the set $\Gamma_\phi(A)$ is the entire complex plane has no role to play in the intersection $\bigcap_{\phi\in\Phi} \Gamma_\phi(A)$. The next result characterizes the set of all non-trivial permutations for the matrix A.

Theorem 1.3.14. *[76] For any irreducible matrix* $A \in \mathbb{C}^{n,n}$*,* $n \geq 2$*, let* $\Phi(A)$ *denote the collection of permutations on the set* $\{1, 2, \ldots, n\}$ *derived from the cycle set* $\mathscr{C}(A)$ *of* A *by adjoining any singletons* $i \in \{1, 2, \ldots, n\} \setminus \gamma$ *to the cycle* $\gamma \in \mathscr{C}(A)$ *necessary to form a permutation of the set* $\{1, 2, \ldots, n\}$*. Then* $\Phi(A)$ *is the exact set of non-trivial permutations for* A*.*

For all the eigenvalue inclusion sets defined in this section, the following inclusion holds.

Theorem 1.3.15. *[76] For any matrix* $A = [a_{i,j}] \in \mathbb{C}^{n,n}$*, we have*

$$\bigcap_{\phi\in\Phi} \Gamma_\phi(A) \subseteq \mathcal{G}(A) \subseteq \mathcal{B}(A) \subseteq \mathcal{K}(A) \subseteq \Gamma(A).$$

The generalization of Geršgorin's result for the generalized eigenvalue problem $Ax = zBx$ was obtained by Stewart [63, 64] in terms of the chordal metric, by Nakatsukasa [54] in terms of the Euclidean metric under the assumption that each row of either A or B satisfies the SDD property, and by Kostić et. al in [36, 39]. The tighter one of the latter three is the one given by Kostić et. al in [36] where the generalized Geršgorin set for the matrix pencil $A - zB$ is defined as

$$\Gamma^K(A, B) := \{z \in \mathbb{C} : A - zB \text{ is not an } SDD \text{ matrix}\},$$

and the minimal Geršgorin set for pencil $A - zB$ is defined as

$$\mathcal{G}^K(A, B) := \{z \in \mathbb{C} : A - zB \text{ is not a non-singular } \mathbb{H}\text{-matrix}\}.$$

It has been proved [36] under certain conditions these sets will also contain the infinity eigenvalues of the matrix pencil $A - zB$.

Similar Geršgorin-type approximation sets for the eigenvalues of matrix polynomials can be found in Michailidou and Psarrakos [52]. Given a matrix polynomial $P(z) = \sum_{i=0}^{m} A_i z^i \in \mathbb{C}^{n,n}$ the Geršgorin sets in [52] are defined as follows

$$\Gamma^{MP}(P) = \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |(P(z))_{i,i}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |(P(z))_{i,j}| \right\}.$$

Similarly, the Brauer set for the polynomial $P(z)$ is defined by

$$\mathcal{K}^{MP}(P) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{i-1} \left\{ z \in \mathbb{C} : |(P(z))_{i,i}||(P(z))_{j,j}| \leq \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} |(P(z))_{i,k}| \right) \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} |(P(z))_{j,k}| \right) \right\}.$$

Other such inclusion sets for eigenvalues of matrices that are generalized to those for the matrix polynomials are the Ostrowski sets and Dashnic-Zusmanovich sets in [52] and Householder sets in [13].

Furthermore, generalized Geršgorin sets or the Ostrowski sets are defined for nonlinear eigenvalue problems in [6]. Additionally Kostić et. al. [37] have defined the Geršgorin sets, minimal Geršgorin sets and Ostrowski sets for nonlinear eigenvalue problems by using the approach of diagonal dominance.

Let Ω be a non-empty simply connected domain of $\mathbb{C}$ and $T : \Omega \to \mathbb{C}^{n,n}$ be a analytic and regular matrix valued function. Assume K be any of the non-singular DD-type class of matrices, namely, SDD matrices, non-singular $\mathbb{H}$-matrices or Ostrowski matrices, as defined in [37]. Then the permuted Geršgorin-type set [37] for T is defined as follows.

Let ϕ denote any permutation of the set $\{1, 2, \ldots, n\}$ and $P_\phi = \left[\delta_{i,\phi(j)}\right] \in \mathbb{R}^{n,n}$ be the associated permutation matrix, where $[\delta_{i,j}]$ is the usual Kronecker delta symbol. Then the permuted Geršgorin-type set of the matrix-valued function T is given by

$$\theta_\pi^K(A) := \{z \in \mathbb{C} : T(z)P_\phi \notin K\}.$$

Further, some additional definitions are introduced to take care of the infinity eigenvalue of T in [37].

The block generalizations of the strictly diagonally dominant matrices was considered independently and simultaneously in Ostrowski [56], Fiedlar and Pták [18], and Fiengold and Varga [17]. Given a matrix $A \in \mathbb{C}^{n,n}$, suppose $\pi = \{n_j\}_{j=0}^{\ell}$, where $0 = n_0 < n_1 < n_2 < \cdots < n_\ell = n$ is a partition of the set $\{1, 2, \ldots, n\}$. By this partition π, an $n \times n$ matrix A is partitioned into $\ell \times \ell$ blocks as given below

$$A = \left[\begin{array}{c|c|c|c} A_{1,1} & A_{1,2} & \ldots & A_{1,\ell} \\ \hline A_{2,1} & A_{2,2} & \ldots & A_{2,\ell} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{\ell,1} & A_{\ell,2} & \ldots & A_{\ell,\ell} \end{array}\right]. \tag{1.3.1}$$

We will use the notation $[A_{i,j}]_{\ell \times \ell}$ to denote the blocked matrix A with respect to the partition π. With the use of induced operator norm $\|\cdot\|_p (p \geq 1)$, construct a real $\ell \times \ell$ comparison matrix $\langle A \rangle_p^\pi = [\mu_{k,j}] \in \mathbb{R}^{\ell,\ell}$ as follows

$$\mu_{k,j} = \begin{cases} + \left\|A_{k,k}^{-1}\right\|_p^{-1} & \text{if } k = j, \\ - \|A_{k,j}\|_p & \text{if } k \neq j \end{cases} \tag{1.3.2}$$

where we use the convention that $\left\|A_{k,k}^{-1}\right\|_p^{-1} = 0$ if $\det(A_{k,k}) = 0$. This is used to define the notion of block strictly diagonally dominant matrices and block $\mathbb{H}$-matrices.

Definition 1.3.16. *[17] Given a partition π and a norm $\|.\|_p$, the block matrix $A = [A_{k,j}]_{\ell \times \ell}$ is a block strictly diagonally dominant matrix (B_p^π SDD) if $\langle A \rangle_p^\pi$ is a SDD matrix.*

Definition 1.3.17. *[35] Given a partition π and a norm $\|.\|_p$, the block matrix $A = [A_{k,j}]_{\ell \times \ell}$ is called a block $\mathbb{H}$-matrix (B_p^π $\mathbb{H}$-matrix) if $\langle A \rangle_p^\pi$ is a non-singular $\mathbb{H}$-matrix.*

It is proved that both the classes of matrices are non-singular.

Theorem 1.3.18. *[17] B_p^π SDD matrices are non-singular.*

Theorem 1.3.19. *[35] B_p^π $\mathbb{H}$-matrices are non-singular.*

The block analog of Geršgorin sets and minimal Geršgorin sets for matrices were defined in [17] and [35]. Here we adopt the terminology in [35].

Theorem 1.3.20. *[35] For any $p \geq 1$ and partition π, all eigenvalues of a given matrix A as in (1.3.1) belong to the block Geršgorin set $\Gamma_p^\pi(A)$, defined as*

$$\Gamma_p^\pi(A) = \left\{ z \in \mathbb{C} : \left[(zI - A)_{i,j} \right]_{\ell \times \ell} \text{ is not a } B_p^\pi \text{ SDD matrix} \right\}.$$

Theorem 1.3.21. *[35] All eigenvalues of a given matrix $A \in \mathbb{C}^{n,n}$, partitioned with respect to a partition π of $\{1, 2, \ldots, n\}$, as in (1.3.1), belong to the block minimal Geršgorin set $\mathcal{G}_p^\pi(A)$, defined as*

$$\mathcal{G}_p^\pi(A) = \left\{ z \in \mathbb{C} : \left[(zI - A)_{i,j} \right]_{\ell \times \ell} \text{ is not a } B_p^\pi \, \mathbb{H}\text{-matrix} \right\}.$$

Similarly, the block Brauer and block Brualdi localization sets for eigenvalues of matrices are defined as follows.

Theorem 1.3.22. *[76] Suppose $A \in \mathbb{C}^{n,n}$ is partitioned with respect to a partition π of $\{1, 2, \ldots, n\}$, as given in (1.3.1). Then for any eigenvalue λ of A there is a pair of distinct integers $i, j \in \{1, 2, \ldots, \ell\}$ such that*

$$\lambda \in \mathcal{K}_{p,i,j}^\pi(A) := \left\{ z \in \mathbb{C} : \left\| (zI - A_{i,i})^{-1} \right\|_p^{-1} \left\| (zI - A_{j,j})^{-1} \right\|_p^{-1} \leq \left(\sum_{\substack{k=1 \\ k \neq i}}^{\ell} \|A_{i,k}\|_p \right) \left(\sum_{\substack{k=1 \\ k \neq j}}^{\ell} \|A_{j,k}\|_p \right) \right\}$$

Consequently,

$$\Lambda(A) \subseteq \mathcal{K}_p^\pi(A) := \bigcup_{\substack{1 \leq i,j \leq n \\ i \neq j}} \mathcal{K}_{p,i,j}^\pi(A).$$

The set $\mathcal{K}_p^\pi(A)$ is called the partitioned Brauer set for A.

Theorem 1.3.23. *[76] Given a partition $\pi = \{n_j\}_{j=0}^{\ell}$ of $\mathbb{C}^n$, let $A \in \mathbb{C}^{n,n}$ be partitioned as in (1.3.1). Suppose $\mathscr{C}(\langle A \rangle_p^\pi)$ denote the cycle set associated with the comparison matrix $\langle A \rangle_p^\pi$ of A. If $\lambda \in \Lambda(A)$, there is a cycle $\gamma \in \mathscr{C}(\langle A \rangle_p^\pi)$ such that*

$$\lambda \in \mathcal{B}_{p,\gamma}^\pi(A) := \left\{ z \in \mathbb{C} : \prod_{i \in \gamma} \left\| (zI - A_{i,i})^{-1} \right\|_p^{-1} \leq \prod_{i \in \gamma} r_i^\gamma(\langle A \rangle_p^\pi) \right\}.$$

Thus $\Lambda(A) \subseteq \mathcal{B}_p^\pi(A) = \bigcup_{\gamma \in \mathscr{C}(\langle A \rangle_p^\pi)} \mathcal{B}_{p,\gamma}^\pi(A)$. The set $\mathcal{B}_p^\pi(A)$ is called the partitioned Brualdi set for A.

The next result summarizes the relation between these three block sets for matrices.

Theorem 1.3.24. *[76] Given any partition $\pi = \{n_j\}_{j=0}^{\ell}$ of $\mathbb{C}^n$, let $A \in \mathbb{C}^{n,n}$ be partitioned as in (1.3.1). Then we have*

$$\mathcal{G}_p^{\pi}(A) \subseteq \mathcal{B}_p^{\pi}(A) \subseteq \mathcal{K}_p^{\pi}(A) \subseteq \Gamma_p^{\pi}(A).$$

1.3.2 Outer approximations of pseudospectra of block triangular matrices

Given a matrix $A \in \mathbb{C}^{n,n}$ in block upper triangular form,

$$A = \left[\begin{array}{c|c} L & C \\ \hline 0 & M \end{array}\right], \quad L \in \mathbb{C}^{\ell,\ell},\ M \in \mathbb{C}^{m,m} \tag{1.3.3}$$

where $\Lambda(L) \cap \Lambda(M) = \emptyset$, the results in literature [2,15,22,61], give a function $g(\epsilon)$ so that $\Lambda_{g(\varepsilon)}(L) \cup \Lambda_{g(\varepsilon)}(M)$ forms an inclusion region for $\Lambda_\epsilon(A)$ via the pseudospectra of L and M.

Theorem 1.3.25. *[22] Suppose that the matrices L, M, C and A are as given by (1.3.3). Then*

$$\Lambda_\varepsilon(A) \subseteq \Lambda_{g(\varepsilon)}(L) \cup \Lambda_{g(\varepsilon)}(M),$$

where $g(\varepsilon) = \varepsilon\sqrt{1 + \frac{\|C\|_2}{\varepsilon}}$.

In order to state the next two theorems we introduce some further notations. Since $\Lambda(L) \cap \Lambda(M) = \emptyset$, by elementary theory of Sylvester equations(see [14,15,62]) there is a unique matrix $R \in \mathbb{C}^{\ell,m}$ such that

$$RM - LR = C. \tag{1.3.4}$$

If $V_1 = [I \quad 0]^t$ and $V_2 = [R^t \quad I]^t$, then $AV_1 = V_1 L$ and $AV_2 = V_2 M$. Thus, the columns of V_1 and V_2 span complementary invariant subspaces of A. The associated projection onto V_1 along V_2 is given by the matrix $P = \begin{bmatrix} I & -R \\ 0 & 0 \end{bmatrix}$. Defining the terms

$$p = \sqrt{1 + \|R\|_2^2} \text{ and } \kappa = p + \|R\|_2, \tag{1.3.5}$$

it can be shown that κ is the condition number of the matrix $T = \begin{bmatrix} I & R/p \\ 0 & I/p \end{bmatrix}$, where $TAT^{-1} = \begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}$.

Theorem 1.3.26. *[15] Let A, L and M be as defined in* (1.3.3). *With the use of notations in (1.3.5), we have*

$$\Lambda_\varepsilon(A) \subseteq \Lambda_{\kappa\varepsilon}(L) \cup \Lambda_{\kappa\varepsilon}(M).$$

Theorem 1.3.27. *[2] Let A, L and M satisfy* (1.3.3), *and R be the the solution of the Sylvester equation (1.3.4). Then for a operator norm $\|\cdot\|$ satisfying the maximum property, we have*

$$\Lambda_\varepsilon(A) \subseteq \Lambda_{g_i(\varepsilon)}(L) \cup \Lambda_{g_i(\varepsilon)}(M),\ i = 1, 2$$

where $g_1(\varepsilon) = (2\|R\| + 1)\varepsilon$ *and* $g_2(\varepsilon) = \frac{\varepsilon + \sqrt{\varepsilon^2 + 4\|C\|\varepsilon}}{2}$.

Varah [71] defined the separation of the matrices $L \in \mathbb{C}^{\ell,\ell}$ and $M \in \mathbb{C}^{m,m}$ by

$$\mathrm{sep}_\lambda(L, M) = \min\{\, \varepsilon_1 + \varepsilon_2 |\ \ \varepsilon_1, \varepsilon_2 \geq 0,\ \Lambda_{\varepsilon_1}(L) \cap \Lambda_{\varepsilon_2}(M) \neq \emptyset\,\}.$$

Obviously, $\mathrm{sep}_\lambda(L, M) > 0$ if and only if the matrices L and M have disjoint spectra. Moreover, if $\varepsilon < \frac{\mathrm{sep}_\lambda(L,M)}{2}$, then the pseudospectra $\Lambda_\varepsilon(L)$ and $\Lambda_\varepsilon(M)$ are disjoint. The next result provides improvement of the bounds in Theorem 1.3.26 for values of ε less than $\mathrm{sep}_\lambda(L, M)/2\kappa$.

Theorem 1.3.28. *[61] Suppose A, L and M satisfy* (1.3.3), *with* $\Lambda(L) \cap \Lambda(M) = \emptyset$. *Let s be such that $0 < s \leq \mathrm{sep}_\lambda(L, M)$ and $\varepsilon \leq s/(2\kappa)$. Then, we have*

$$\Lambda_\varepsilon(A) \subseteq \Lambda_{g(\varepsilon)\,\varepsilon}(L) \cup \Lambda_{g(\varepsilon)\,\varepsilon}(M),\ \text{where } g(\varepsilon) = \frac{p - \varepsilon/s}{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\varepsilon}{s}\left(p - \frac{\varepsilon}{s}\right)}}.$$

2 Inner approximations for pseudospectra of block triangular matrices

Throughout this chapter, we assume that the underlying matrix norm is the spectral norm $\|\cdot\|_2$. Given an $\varepsilon > 0$, the ε-pseudospectra of a matrix $A \in \mathbb{C}^{n,n}$ with respect to the 2-norm satisfies

$$\begin{aligned}\Lambda_\varepsilon(A) &= \left\{ z \in \mathbb{C} : \left\|(A - zI)^{-1}\right\|_2 \geq \frac{1}{\varepsilon} \right\} \\ &= \{ z \in \mathbb{C} \,:\, \sigma_{\min}(A - zI) \leq \varepsilon \},\end{aligned}$$

where we use the convention that $\|(A - zI)^{-1}\|_2 = \infty$ for any eigenvalue z of A. We consider two matrices A and $\widehat{A}$ given by

$$A = \left[\begin{array}{c|c} L & C \\ \hline 0 & M \end{array}\right] \in \mathbb{C}^{n,n}, \text{ and } \widehat{A} = \left[\begin{array}{c|c} L & 0 \\ \hline 0 & M \end{array}\right] \in \mathbb{C}^{n,n} \tag{2.0.1}$$

where $L \in \mathbb{C}^{\ell,\ell}$, $M \in \mathbb{C}^{m,m}$ and $C \in \mathbb{C}^{\ell,m}$ such that $\Lambda(L) \cap \Lambda(M) = \emptyset$. Our goal is to provide inner approximations for the pseudospectra of A in terms of the pseudospectra of $\widehat{A}$. Using the singular value inequality [30, Corollary 3.1.3], we obtain $\sigma_{\min}(zI - A) \leq \sigma_{\min}(zI - L)$ and $\sigma_{\min}(zI - A) \leq \sigma_{\min}(zI - L)$. Thus a trivial lower bound for the ε-pseudospectra of a matrix A is given by $\Lambda_\varepsilon(L) \cup \Lambda_\varepsilon(M) \subseteq \Lambda_\varepsilon(A)$. Therefore, our aim is to find $f(\varepsilon) > \varepsilon$ so that

$$\Lambda_{f(\varepsilon)}(L) \cup \Lambda_{f(\varepsilon)}(M) \subseteq \Lambda_\varepsilon(A).$$

2.1 The 2×2 case

In this section, we consider the matrix $A \in \mathbb{C}^{2,2}$ given by

$$A = \begin{bmatrix} \lambda & c \\ 0 & \mu \end{bmatrix}. \tag{2.1.1}$$

We seek an optimal radius $f(\varepsilon)\varepsilon$ such that

$$D(\lambda, f(\varepsilon)\varepsilon) \cup D(\mu, f(\varepsilon)\varepsilon) \subseteq \Lambda_\epsilon(A). \tag{2.1.2}$$

The following lemma provides an elementary result about singular values of 2×2 upper triangular matrices, a proof of which is included just for the sake of completeness.

Lemma 2.1.1. *If $\sigma_{\min}(\lambda, \mu, c)$ and $\sigma_{\max}(\lambda, \mu, c)$ respectively denotes the maximum and minimum singular values associated with the matrix A in (2.1.1), then*

$$\sigma_{\min}(\lambda, \mu, c) = \frac{1}{2}\left(\sqrt{(|\lambda| + |\mu|)^2 + |c|^2} - \sqrt{(|\lambda| - |\mu|)^2 + |c|^2}\right), \tag{2.1.3}$$

$$\sigma_{\max}(\lambda, \mu, c) = \frac{1}{2}\left(\sqrt{(|\lambda| + |\mu|)^2 + |c|^2} + \sqrt{(|\lambda| - |\mu|)^2 + |c|^2}\right). \tag{2.1.4}$$

Both the numbers equal the non-negative zeros of the polynomials

$$p(\sigma) = \sigma^4 - (|\lambda|^2 + |\mu|^2 + |c|^2)\sigma^2 + |\lambda\mu|^2, \tag{2.1.5}$$

$$q(\sigma) = \sigma^2 - \left(\sqrt{(|\lambda| + |\mu|)^2 + |c|^2}\right)\sigma + |\lambda\mu|. \tag{2.1.6}$$

Proof. The squares of the singular values of A are the zeros of the characteristic polynomial of A^*A given by $\chi(z) = z^2 - (|\lambda|^2 + |\mu|^2 + |c|^2)\, z + |\lambda\mu|^2$. Thus, $\sigma \geq 0$ is a singular value of A if and only if $0 = \chi(\sigma^2) = p(\sigma)$. The polynomial $p(\sigma)$ can further be decomposed into factors $p(\sigma) = q(\sigma)\hat{q}(\sigma)$, where

$$\begin{aligned} q(\sigma) &= \sigma^2 - \left(\sqrt{(|\lambda| + |\mu|)^2 + |c|^2}\right)\sigma + |\lambda\mu| \\ \text{and} \quad \hat{q}(\sigma) &= \sigma^2 + \sqrt{(|\lambda| + |\mu|)^2 + |c|^2}\,\sigma + |\lambda\mu|. \end{aligned}$$

Since the roots of $\hat{q}(\sigma)$ are both non-positive, it follows that the singular values are the non-negative zeros of q. This yields (2.1.3) and (2.1.4). □

The singular values of Lemma 2.1.1 have the following monotonicity properties.

Lemma 2.1.2. *Let $\sigma_{\min}(\lambda, \mu, c)$ and $\sigma_{\max}(\lambda, \mu, c)$ be as described in Lemma 2.1.1. For $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}$, $k = 1, 2$, the following hold.*
(i) If $|\lambda_1| \leq |\lambda_2|, |\mu_1| \leq |\mu_2|$ and $|c_1| \leq |c_2|$ then $\sigma_{\max}(\lambda_1, \mu_1, c_1) \leq \sigma_{\max}(\lambda_2, \mu_2, c_2)$.
(ii) If $|\lambda_1| \leq |\lambda_2|, |\mu_1| \leq |\mu_2|$ and $|c_2| \leq |c_1|$ then $\sigma_{\min}(\lambda_1, \mu_1, c_1) \leq \sigma_{\min}(\lambda_2, \mu_2, c_2)$.

Proof. We show (ii) as the proof of (i) will follow in a similar manner. Since the singular values depend only on absolute values we may assume that λ_k, μ_k, c_k are non-negative. We show that for fixed $\mu, c \geq 0$, the function $\lambda \mapsto f(\lambda) := \sigma_{\min}(\lambda, \mu, c)$

is increasing. This is obvious if $c = 0$, since then $f(\lambda) = \min\{|\lambda|, |\mu|\}$. Let $c > 0$. We show that the derivative

$$f'(\lambda) = \frac{1}{2}\left(\frac{\lambda+\mu}{\sqrt{(\lambda+\mu)^2+c^2}} - \frac{\lambda-\mu}{\sqrt{(\lambda-\mu)^2+c^2}}\right)$$

is non-negative. This is immediate if $\lambda \leq \mu$. If $\lambda \geq \mu$ the non-negativity of f' follows from the inequality $(\lambda-\mu)^2((\lambda+\mu)^2+c^2) \leq (\lambda+\mu)^2((\lambda-\mu)^2+c^2)$ by taking square roots and reordering terms. Thus, f is increasing. Analogously, $\mu \mapsto \sigma_{\min}(\lambda,\mu,c)$ is increasing. It remains to show that $\sigma_{\min}(\lambda,\mu,c)$ is a decreasing function of $c > 0$. This is seen from

$$\begin{aligned}\sigma_{\min}(\lambda,\mu,c) &= \frac{\left(\sqrt{(\lambda+\mu)^2+c^2} - \sqrt{(\lambda-\mu)^2+c^2}\right)\left(\sqrt{(\lambda+\mu)^2+c^2} + \sqrt{(\lambda-\mu)^2+c^2}\right)}{2\left(\sqrt{(\lambda+\mu)^2+c^2} + \sqrt{(\lambda-\mu)^2+c^2}\right)} \\ &= \frac{2\,\lambda\,\mu}{\sqrt{(\lambda+\mu)^2+c^2} + \sqrt{(\lambda-\mu)^2+c^2}}.\end{aligned}$$

□

Therefore, the pseudospectrum of the 2×2 matrix A in (2.1.1) is given by

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} \,:\, \sigma_{\min}(A - zI) \leq \varepsilon\}$$

where by Lemma 2.1.1,

$$\sigma_{\min}(A - zI) = \frac{1}{2}\left(\sqrt{(|z-\lambda|+|z-\mu|)^2+|c|^2} - \sqrt{(|z-\lambda|-|z-\mu|)^2+|c|^2}\right).$$

Notice that the disc $D(\lambda,\varepsilon)$ equals the pseudospectrum $\Lambda_\varepsilon([\lambda])$. If $\lambda = \mu$ then both the discs in (2.1.2) coincide and equals the pseudospectrum of A. This is immediate from the following result.

Proposition 2.1.3. *If $\lambda = \mu$ then $\Lambda_\epsilon(A) = D(\lambda, g(\varepsilon)\varepsilon)$, where $g(\varepsilon) = \sqrt{1+|c|/\varepsilon}$.*

Proof. Under the assumption $\lambda = \mu$, the inequality $\sigma_{\min}(A - zI) \leq \varepsilon$ is equivalent to $\frac{1}{2}(\sqrt{4|z-\lambda|^2+|c|^2} - |c|) \leq \varepsilon$ which in turn implies $|z-\lambda| \leq \sqrt{\varepsilon^2+\varepsilon\,|c|} = g(\varepsilon)\,\varepsilon$. □

Next, we consider the case $\lambda \neq \mu$. In accordance with the notations in (1.3.5) of Chapter 1, for the matrix A in (2.1.1) we set

$$s = |\lambda-\mu|, \qquad R = c/(\mu-\lambda), \qquad p = \sqrt{1+|R|^2}, \qquad \kappa = p + |R|. \tag{2.1.7}$$

Then $v_\lambda = [1 \;\; 0]^\top$ and $v_\mu = [R \;\; 1]^\top$ are (right) eigenvectors of A associated with λ and μ, respectively, s equals the separation of the eigenvalues, p equals the norm of the projector onto v_λ along v_μ, and $\kappa = \|T\|_2\,\|T^{-1}\|_2$ is the condition number of the matrix $T = \begin{bmatrix} 1 & R/p \\ 0 & 1/p \end{bmatrix}$ which satisfies $T^{-1}AT = \operatorname{diag}(\lambda,\mu)$.

Theorem 2.1.4. *Suppose the eigenvalues λ and μ of A as given by (2.1.1) are distinct. Let*

$$f(\epsilon) := \sup\{r \geq 0 \mid D(\lambda, r\varepsilon) \cup D(\mu, r\varepsilon) \subseteq \Lambda_\varepsilon(A)\},$$

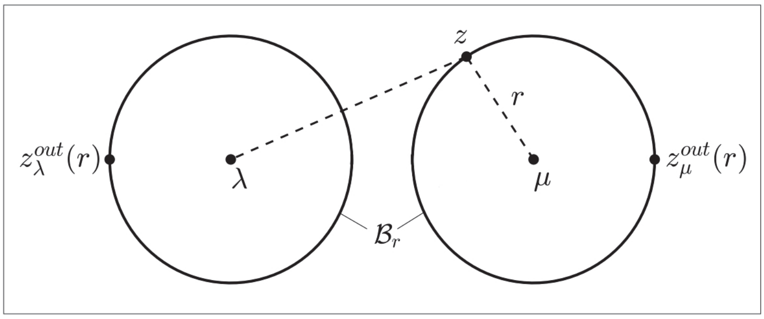
Then with the notations in (2.1.7),

$$f(\varepsilon) = \frac{p + \varepsilon/s}{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\varepsilon}{s}\left(p + \frac{\varepsilon}{s}\right)}}.$$

For proving Theorem 2.1.4, we first determine the maximum value of the function $\sigma_{\min}(A - zI)$ for $z \in \mathcal{B}_r$, the boundary of $D(\lambda, r) \cup D(\mu, r)$. We distinguish two cases which are displayed in Figure 2.1.1. If $r \leq s/2$ then $\mathcal{B}_r$ is the union of the circles of radius r with centers λ and μ. If $r > s/2$, then $\mathcal{B}_r$ is the union of the circular arcs of radius r about λ and μ. In both cases, $\mathcal{B}_r$ contain the points

$$z_\lambda^{out}(r) = \lambda - r\,e \quad \text{and} \quad z_\mu^{out}(r) = \mu + r\,e$$

where $e = (\mu - \lambda)/|\mu - \lambda|$.



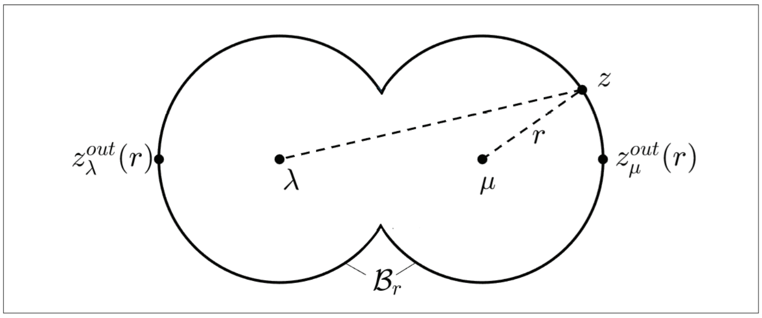


Figure 2.1.1: Illustration to the proof of Lemma 2.1.5. We are looking for z on $\mathcal{B}_r$ such that $|z - \lambda|$ or $|z - \mu|$ becomes extremal. The extremal are respectively attained at $z_\mu^{out}(r)$ and $z_\lambda^{out}(r)$.

Lemma 2.1.5. *For $r \geq 0$, let $\hat{h}(r) = \max_{z \in \mathcal{B}_r} \sigma_{\min}(A - zI)$. Then*

$$\hat{h}(r) = \sigma_{\min}(A - z_\lambda^{out}(r)I) = \sigma_{\min}(A - z_\mu^{out}(r)I) = \frac{1}{2}\left(\sqrt{(s+2r)^2 + |c|^2} - \sqrt{s^2 + |c|^2}\right).$$

Proof. Let $z \in \mathcal{B}_r$. Then $|z - \lambda| = r$ or $|z - \mu| = r$. Suppose the latter. Then with the $\sigma_{\min}$ notation introduced in Lemma 2.1.1,

$$\sigma_{\min}(A - zI) = \sigma_{\min}(|z - \lambda|, r, |c|).$$

From the monotonicity properties of $\sigma_{\min}$ (see Lemma 2.1.2) it follows that

$$\sigma_{\min}(A - z_\mu^{out}(r)I) \geq \sigma_{\min}(A - zI) \quad \forall\ z \in \mathcal{B}_r.$$

Thus using (2.1.3), we get

$$\hat{h}(r) = \sigma_{\min}(A - z_\mu^{out}(r)I) = \frac{1}{2}\left(\sqrt{(s+2r)^2 + |c|^2} - \sqrt{s^2 + |c|^2}\right).$$

Similarly, if $z \in \mathcal{B}_r$ is such that $|z - \lambda| = r$, then $\hat{h}(r) = \sigma_{\min}(A - z_\lambda^{out}(r)I)$. This completes the proof. $\square$

Observe that $\hat{h}(r)$ is continuous and a strictly increasing function of r. Now, the proof of Theorem 2.1.4 proceeds as follows.
Proof of Theorem 2.1.4 For any $r > 0$,

$$\begin{aligned}
D(\lambda, r) \cup D(\mu, r) \subseteq \Lambda_\varepsilon(A) \quad &\Leftrightarrow \quad \sigma_{\min}(A - zI) \leq \varepsilon \text{ for all } z \in \mathcal{B}_r \\
&\Leftrightarrow \quad \hat{h}(r) \leq \varepsilon \\
&\Leftrightarrow \quad \frac{1}{2}\left(\sqrt{(s+2r)^2 + |c|^2} - \sqrt{s^2 + |c|^2}\right) \leq \varepsilon \\
&\Leftrightarrow \quad r^2 + sr \leq (\varepsilon^2 + \varepsilon\sqrt{s^2 + |c|^2}) \\
&\Leftrightarrow \quad r \leq \frac{-s + \sqrt{s^2 + 4(\varepsilon^2 + \varepsilon\sqrt{s^2 + |c|^2})}}{2} = f(\varepsilon)\varepsilon. \qquad \square
\end{aligned}$$

The following examples illustrates the bounds in Theorem 2.1.4.

Example 2.1.6. *For the matrices*

$$\widehat{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 9+5i \\ 0 & 1 \end{bmatrix}$$

with $\varepsilon = 0.0950$, Theorem 2.1.4 gives $\tilde{\varepsilon} = f(\varepsilon)\varepsilon = 0.4161$ as shown in Figure 2.1.2(left).

Example 2.1.7. *Consider the matrices*

$$\widehat{A} = \begin{bmatrix} -2 & 0 \\ 0 & 1+i \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 9+8i \\ 0 & 1+i \end{bmatrix}$$

with $\varepsilon = 1.7$*, the value of* $\tilde{\varepsilon} = f(\varepsilon)\varepsilon$ *from Theorem 2.1.4 is calculated to be* 3.5720*. This is shown in Figure 2.1.2(right).*

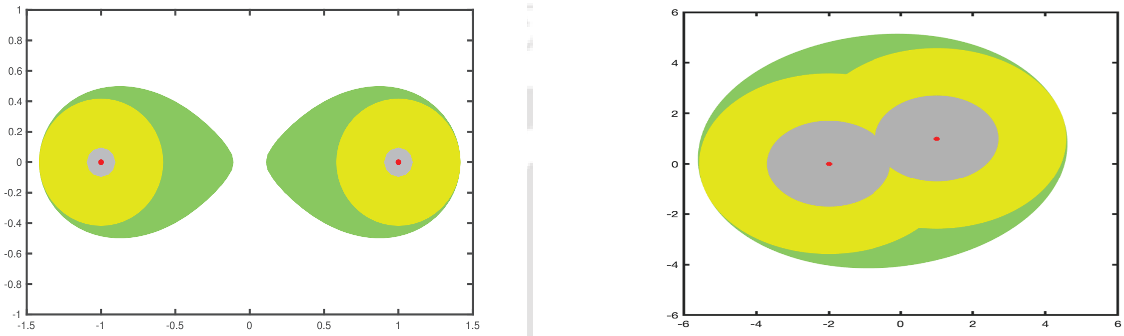


Figure 2.1.2: $\blacksquare\,\Lambda_\varepsilon(\widehat{A})$, $\blacksquare\,\Lambda_{\tilde{\varepsilon}}(\widehat{A})$ and $\blacksquare\,\Lambda_\varepsilon(A)$ along with the spectrum $\bullet$ for Example 2.1.6(left) and Example 2.1.7(right).

2.2 The general case

In this section, we consider the case when A and $\widehat{A}$ are $n \times n$ matrices, $n \geq 3$. Theorem 2.1.4 provides the best possible inner approximation of the pseudospectra $\Lambda_\varepsilon(A)$ by the pseudospectra $\Lambda_{\tilde{\varepsilon}}(\widehat{A})$ for all 2×2 matrices. Thus for general $n \times n$ matrices which are unitarily similar to matrices with 2×2 blocks on the diagonal, the aforesaid result helps us in finding an inner approximation of the $n \times n$ upper triangular block matrix via the pseudospectra of the diagonal blocks. For example, consider the 6×6 matrix

$$A = \left[\begin{array}{ccc|ccc} a_{11} & & & c_{11} & & \\ & a_{22} & & & c_{22} & \\ & & a_{33} & & & c_{33} \\ \hline & & & b_{11} & & \\ & & & & b_{22} & \\ & & & & & b_{33} \end{array}\right].$$

Then there exists a permutation matrix P such that $P^t M P = \widehat{A}$, where

$$\widehat{A} = \left[\begin{array}{cc|cc|cc} a_{11} & c_{11} & & & & \\ 0 & b_{11} & & & & \\ \hline & & a_{22} & c_{22} & & \\ & & 0 & b_{22} & & \\ \hline & & & & a_{33} & c_{33} \\ & & & & 0 & b_{33} \end{array}\right] \text{ and } P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consequently,

$$\sigma_{\min}(A - zI) = \sigma_{\min}(\widehat{A} - zI) = \min_{1\leq j\leq 3} \sigma_{\min}(T_j - zI),$$

where $T_j = \begin{bmatrix} a_{jj} & c_{jj} \\ 0 & b_{jj} \end{bmatrix}$; $j \in \{1, 2, 3\}$ so that $\Lambda_\varepsilon(A) = \Lambda_\varepsilon(\widehat{A}) = \bigcup_{j=1}^{3} \Lambda_\varepsilon(T_j)$. Let $\varepsilon > 0$ be arbitrary. For each $j = 1, 2, 3$, let $\varepsilon_j > \varepsilon$ be the optimal value from Theorem 2.1.4 such that $D_{\varepsilon_j}(a_{jj}) \cup D_{\varepsilon_j}(b_{jj}) \subseteq \Lambda_\varepsilon(T_j)$. If $\varepsilon_0 := \min\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_3\}$, then $\bigcup_{j=1}^{3} (D_{\varepsilon_0}(a_{jj}) \cup D_{\varepsilon_0}(b_{jj})) \subseteq \Lambda_\varepsilon(A)$ is an optimal inner approximation of $\Lambda_\varepsilon(A)$ via pseudospectra levels corresponding to its diagonal entries.

However, there are cases where it can be proved that $\Lambda_\varepsilon(\widehat{A})$ is the optimal inner approximation of $\Lambda_\varepsilon(A)$. One such situation is identified in the next theorem.

Theorem 2.2.1. *Suppose $L \in \mathbb{C}^{m,m}, M \in \mathbb{C}^{m,m}$ such that $\widehat{A} = \left[\begin{array}{c|c} L & 0 \\ \hline 0 & M \end{array}\right]$ is a normal matrix. If one of the eigenvectors of L belongs to the left null space of C or one of the eigenvectors of M belongs to the right null space of C, then for $A = \left[\begin{array}{c|c} L & C \\ \hline 0 & M \end{array}\right]$ and any given $\varepsilon > 0$, $\Lambda_\varepsilon(\widehat{A}) \subseteq \Lambda_\varepsilon(A)$ is such that $\Lambda_\varepsilon(\widehat{A})$ has a common boundary point with $\Lambda_\varepsilon(A)$.*

Proof. Suppose that L has an eigenvector v corresponding to an eigenvalue λ such that $v^*C = 0$. As $\widehat{A}$ is a normal matrix, so are L and M. Therefore, there exist unitary matrices V_1 and V_2 and diagonal matrices D_L and D_M such that $L = V_1 D_L V_1^*$ and $M = V_2 D_M V_2^*$. Also we may assume without loss of generality that v is the first column of V_1. Therefore, setting $S = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$, and $\widehat{C} = V_1^* C V_2$, it is easy to see that the first row and first column of S^*AS are given by $\begin{bmatrix} \lambda & 0 & \cdots & 0 \end{bmatrix}$ and $\begin{bmatrix} \lambda & 0 & \cdots & 0 \end{bmatrix}^T$ respectively. This implies that the disc with centre at λ and radius ε is common to both $\Lambda_\varepsilon(A)$ and $\Lambda_\varepsilon(\widehat{A})$. The proof for the case that an eigenvector of M, belongs to the right nullspace of C follows with similar arguments. Hence, the proof. $\square$

If A satisfies the hypothesis of Theorem 2.2.1, then $\Lambda_\varepsilon(\widehat{A})$ is a good inner approximation of $\Lambda_\varepsilon(A)$ as it consists of discs of radius ε centered at the eigenvalues of A that reach upto the boundary of $\Lambda_\varepsilon(A)$. Thus without loss of generality, we consider an A that does not satisfy the properties of Theorem 2.2.1. Theorem 2.2.2, Lemma 2.2.3 and Corollary 2.2.4 provide a means of finding the desired inner approximations for the case when the diagonal blocks L, M are square matrices of the same size. Throughout we assume C to be non-singular.

Theorem 2.2.2. *Suppose $\widehat{A}$ and A are as given in (2.0.1) with $\ell = m$ and $C \in \mathbb{C}^{m,m}$ is non-singular. Then*

$$\left\|(A - zI)^{-1}\right\|_2 \geq \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + \|C^{-1}(L - zI)\|_2^{-2}}, \tag{2.2.1}$$

$$\text{and} \quad \left\|(A - zI)^{-1}\right\|_2 \geq \left\|(L - zI)^{-1}\right\|_2 \sqrt{1 + \|C^{-1}(M - zI)\|_2^{-2}}. \tag{2.2.2}$$

Proof. For $z \in \mathbb{C} \setminus (\Lambda(L) \cup \Lambda(M))$, let $v_{\min}(z)$, $u_{\min}(z) \in \mathbb{C}^m$ denote right and left singular vectors of $M - zI$ corresponding to $\sigma_{\min}(M - zI)$ respectively. Then

$$\begin{aligned}
\left\|(A - zI)^{-1}\right\|_2 &= \left\| \left[\begin{array}{c|c} (L - zI)^{-1} & -(L - zI)^{-1}C(M - zI)^{-1} \\ \hline 0 & (M - zI)^{-1} \end{array}\right] \right\|_2 \\
&\geq \left\| \left[\begin{array}{c|c} (L - zI)^{-1} & -(L - zI)^{-1}C(M - zI)^{-1} \\ \hline 0 & (M - zI)^{-1} \end{array}\right] \left[\begin{array}{c} 0 \\ \hline u_{\min}(z) \end{array}\right] \right\|_2 \\
&= \left\| \left[\begin{array}{c} -\|(M - zI)^{-1}\|_2 \, (L - zI)^{-1}Cv_{\min}(z) \\ \hline \|(M - zI)^{-1}\|_2 \, v_{\min}(z) \end{array}\right] \right\|_2 \\
&= \left\|(M - zI)^{-1}\right\|_2 \sqrt{\|v_{\min}(z)\|_2^2 + \|(L - zI)^{-1}Cv_{\min}(z)\|_2^2} \\
&= \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + \|(L - zI)^{-1}Cv_{\min}(z)\|_2^2} \\
&\geq \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + \sigma_{\min}^2((L - zI)^{-1}C)} \\
&\geq \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + \|C^{-1}(L - zI)\|_2^{-2}}.
\end{aligned}$$

In a similar way, one can derive the second inequality. □

For $z \in \Lambda_\varepsilon(A)$, $\|C^{-1}(L - zI)\|_2 \leq \|C^{-1}\|_2 (\|A\|_2 + \|L\|_2 + \varepsilon)$. Since our aim is to approximate the ε-pseudospectra of A by $\tilde{\varepsilon}(> \varepsilon)$ level pseudospectra of both L and M. So for the matrix M, if $\gamma(z) := \sqrt{1 + \|C^{-1}(L - zI)\|_2^{-2}}$ then the aim is to obtain a lower bound $\gamma > 1$ of $\gamma(z)$ such that $\|(M - zI)^{-1}\|_2 \geq \frac{1}{\gamma\varepsilon}$ implies that $\|(A - zI)^{-1}\|_2 \geq \frac{1}{\varepsilon}$. With this aim in view, we have the following result.

Lemma 2.2.3. *Suppose L, M and C are as described in Theorem 2.2.2. Let $\sigma_{\min}(C) > 0$ and $z_0 \in \mathbb{C}$. For $\alpha > 1$, if*

$$r(\alpha) := \sigma_{\min}(C) \left(\frac{1}{\sqrt{\alpha^2 - 1}} - \left\|C^{-1}(L - z_0 I)\right\|_2 \right), \tag{2.2.3}$$

then for any $\varepsilon > 0$,

$$D(z_0, r(\alpha)) \cap \Lambda_{\alpha\varepsilon}(M) \subseteq \Lambda_\varepsilon\left(\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}\right).$$

Proof. Let $z \in D(z_0, r(\alpha))$. Then,

$$\begin{aligned}\left\|C^{-1}(L-zI)\right\|_2 \leq |z-z_0|\left\|C^{-1}\right\|_2 + \left\|C^{-1}(L-z_0I)\right\|_2 &\leq \frac{r(\alpha)}{\sigma_{\min}(C)} + \left\|C^{-1}(L-z_0I)\right\|_2 \\ &= \frac{1}{\sqrt{\alpha^2-1}} := R.\end{aligned}$$

Therefore, $\alpha = \sqrt{1+\frac{1}{R^2}}$ and using Theorem 2.2.2, we obtain for $z \in D(z_0, r(\alpha))$,

$$\begin{aligned}\left\|\begin{bmatrix} L-zI & C \\ 0 & M-zI \end{bmatrix}^{-1}\right\|_2 &\geq \left\|(M-zI)^{-1}\right\|_2 \sqrt{1+\frac{1}{\|C^{-1}(L-zI)\|_2^2}} \\ &\geq \left\|(M-zI)^{-1}\right\|_2 \sqrt{1+\frac{1}{R^2}} \\ &= \alpha\left\|(M-zI)^{-1}\right\|_2.\end{aligned}$$

Therefore, $z \in D(z_0, r(\alpha)) \cap \Lambda_{\alpha\varepsilon}(M) \implies \left\|\begin{bmatrix} L-zI & C \\ 0 & M-zI \end{bmatrix}^{-1}\right\|_2 \geq \frac{1}{\varepsilon}$. □

Corollary 2.2.4. *Suppose* $\widehat{A}$ *and* A *are as given in Theorem 2.2.2. Let* $\varepsilon > 0$ *and* z_0, $\hat{z}_0 \in \mathbb{C}$. *Choose* $R_L(\alpha) > 0$ *and* $R_M(\alpha) > 0$ *such that*

$$\Lambda_{\alpha\varepsilon}(L) \subseteq D(z_0, R_L(\alpha)) \text{ and } \Lambda_{\alpha\varepsilon}(M) \subseteq D(\hat{z}_0, R_M(\alpha)).$$

Choose α_1 *such that* $R_L(\alpha_1) = \sigma_{\min}(C)\left(\frac{1}{\sqrt{\alpha_1^2-1}} - \|C^{-1}(M-z_0I)\|_2\right)$ *and* α_2 *such that* $R_M(\alpha_2) = \sigma_{\min}(C)\left(\frac{1}{\sqrt{\alpha_2^2-1}} - \|C^{-1}(L-\hat{z}_0I)\|_2\right)$. *Then* $\alpha_0 := \min\{\alpha_1, \alpha_2\} > 1$ *and* $\Lambda_{\alpha_0\varepsilon}(\widehat{A}) \subseteq \Lambda_\varepsilon(A)$.

Proof. As $R_M(\alpha) > 0$ is such that $\Lambda_{\alpha\varepsilon}(M) \subseteq D(\hat{z}_0, R_M(\alpha))$, by Lemma 2.2.3 for this particular choice of α_2 we have $\Lambda_{\alpha_2\varepsilon}(M) \subseteq \Lambda_\varepsilon(A)$. Similarly, by interchanging the roles of L and M, we get α_1 such that $\Lambda_{\alpha_1\varepsilon}(L) \subseteq \Lambda_\varepsilon(A)$. From simple calculations, it is clear that

$$\alpha_1 = \sqrt{1+\left(\frac{\sigma_{\min}(C)}{R_L(\alpha_1) + \sigma_{\min}(C)\left\|C^{-1}(M-z_0I)\right\|_2}\right)^2}$$

and

$$\alpha_2 = \sqrt{1+\left(\frac{\sigma_{\min}(C)}{R_M(\alpha_2) + \sigma_{\min}(C)\left\|C^{-1}(L-\hat{z}_0I)\right\|_2}\right)^2}.$$

Therefore, $\Lambda_{\alpha_0\varepsilon}(\widehat{A}) \subseteq \Lambda_\varepsilon(A)$ for $\alpha_0 = \min\{\alpha_1, \alpha_2\} > 1$. □

We seek a suitable centre z_0 and radii $R_L(\alpha_1)$, $R_M(\alpha)$ for the discs containing the $\alpha\varepsilon$-pseudospectra of the diagonal blocks. From the definition of $r(\alpha)$, it can be seen that α is inversely proportional to $r(\alpha)$. So in order to increase α, we choose the point $\hat{z}_0$ and $R_M(\alpha)$ suitably so that $D(\hat{z}_0, R_M(\alpha))$ is a tight outer approximation for $\Lambda_{\alpha\varepsilon}(M)$.

The $\alpha\varepsilon$-pseudospectrum of M is contained in the disc centered at $\hat{z}_0$ of radius $\|M - \hat{z}_0 I\|_2 + \alpha\varepsilon$. This follows since for $z \in \Lambda_{\alpha\varepsilon}(M)$ we have $z \in \Lambda(M + E)$ for some $E \in \mathbb{C}^{n,n}$ with $\|E\|_2 \leq \varepsilon$. This gives $z - \hat{z}_0 \in \Lambda(M + E - \hat{z}_0 I)$ which implies $|z - \hat{z}_0| \leq \|M - \hat{z}_0 I\|_2 + \alpha\varepsilon$. Thus,

$$\Lambda_{\alpha\varepsilon}(M) = \Lambda_{\alpha\varepsilon}(M) \cap D(\hat{z}_0, \|M - \hat{z}_0 I\|_2 + \alpha\varepsilon) \subseteq \Lambda_\varepsilon\left(\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}\right), \tag{2.2.4}$$

and also replacing $\alpha\varepsilon$ by a larger quantity $\left(\sqrt{\alpha^2 - 1} + 1\right)\varepsilon$,

$$\Lambda_{\alpha\varepsilon}(M) = \Lambda_{\alpha\varepsilon}(M) \cap D(\hat{z}_0, \|M - \hat{z}_0 I\|_2 + \varepsilon(\sqrt{\alpha^2 - 1} + 1)) \subseteq \Lambda_\varepsilon\left(\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}\right). \tag{2.2.5}$$

For finding a suitable α, we first take $R_M(\alpha) = \|M - \hat{z}_0 I\|_2 + \alpha\varepsilon$ and equate it with $r(\alpha)$. Then any solution of $R_M(\alpha) = r(\alpha)$ is the largest positive solution of the quartic equation

$$x^4 + 2p_3 x^3 + p_2 x^2 + 2p_1 x + p_0 = 0 \tag{2.2.6}$$

where

$$\begin{aligned} p_3 &= \frac{\sigma_{\min}(C)\,\|C^{-1}(L - \hat{z}_0 I)\|_2 + \|M - \hat{z}_0 I\|_2}{\varepsilon}, \\ p_2 &= \frac{(\sigma_{\min}(C)\,\|C^{-1}(L - \hat{z}_0 I)\|_2 + \|M - \hat{z}_0 I\|_2)^2 - \varepsilon^2}{\varepsilon^2}, \\ p_1 &= -\frac{\sigma_{\min}(C)\,\|C^{-1}(L - \hat{z}_0 I)\|_2 + \|M - \hat{z}_0 I\|_2}{\varepsilon}, \\ \text{and} \quad p_0 &= -\frac{(\sigma_{\min}(C)\,\|C^{-1}(L - \hat{z}_0 I)\|_2 + \|M - \hat{z}_0 I\|_2)^2 + \sigma_{\min}^2(C)}{\varepsilon^2}. \end{aligned}$$

Since $p_3 > 0$ and $p_0 < 0$, there exists $\alpha_0 > 0$ such that $R_M(\alpha_0) = r(\alpha_0)$. Similarly, taking $R_M(\alpha) = \|M - \hat{z}_0 I\|_2 + \varepsilon(\sqrt{\alpha^2 - 1} + 1)$, we have $R_M(\alpha_0) = r(\alpha_0)$ for

$$\alpha_0 = \sqrt{\beta^2 + 1}, \tag{2.2.7}$$

where $\beta = -\frac{F}{2} + \frac{1}{2}\sqrt{F^2 + 4\frac{\sigma_{\min}(C)}{\varepsilon}}$ for $F := \frac{\sigma_{\min}(C)\left\|C^{-1}(L - \hat{z}_0 I)\right\|_2 + \|M - \hat{z}_0 I\|_2 + \varepsilon}{\varepsilon}$.

A suitable $\alpha > 1$ such that $\Lambda_{\alpha\varepsilon}(L) \subseteq \Lambda_\varepsilon(A)$ is computed by using an identical strategy. One can also obtain α by replacing the discs $D(\hat{z}_0, \|M - \hat{z}_0 I\|_2 + \alpha\varepsilon)$ and $D(\hat{z}_0, \|M - \hat{z}_0 I\|_2 + \varepsilon(\sqrt{\alpha^2 - 1} + 1))$ in (2.2.4) and (2.2.5) by discs arising from

Bauer-Fike and Gersgorin inclusion regions. However, the quality of the inner approximations using the inclusions from (2.2.4) and (2.2.5) appear to be better than the ones from these strategies. Clearly, the value of α also depends on the choice of the point $\hat{z}_0$. One strategy is to choose $\hat{z}_0$ as the center of the smallest disc containing the eigenvalues of the matrix M. We obtain this by using the CVX toolbox [23] in Matlab. Notice that in (2.2.7), α is a monotonically increasing function of $\sigma_{\min}(C)\,\|C^{-1}(L-\hat{z}_0 I)\|_2 + \|M-\hat{z}_0 I\|_2$. So another strategy is to choose $\hat{z}_0$ to be the point which minimizes the convex function $\sigma_{\min}(C)\,\|C^{-1}(L-zI)\|_2 + \|M-zI\|_2$. The following examples illustrate the outcome of both strategies.

Example 2.2.5. *Let A and $\widehat{A}$ be as in* (2.0.1) *where*

$$L = \begin{bmatrix} -77.29063 & 0.6551 & 0.9597 & 0.7513 \\ 0.7547 & -77.8374 & 0.3404 & 0.2551 \\ 0.2760 & 0.1190 & -77.4147 & 0.5060 \\ 0.6797 & 0.4984 & 0.2238 & -77.3009 \end{bmatrix}, \quad M = \begin{bmatrix} -55.1091 & 0.1493 & 0.8143 & 0.1966 \\ 0.9593 & -55.7425 & 0.2435 & 0.2511 \\ 0.5472 & 0.8407 & -55.0707 & 0.6160 \\ 0.1386 & 0.2543 & 0.3500 & -55.5267 \end{bmatrix},$$

$$\text{and } C = \begin{bmatrix} -88.6483 & 0.9172 & 0.3804 & 0.5308 \\ 0.8308 & -88.7142 & 0.5678 & 0.7792 \\ 0.5853 & 0.7572 & -88.9241 & 0.9340 \\ 0.5497 & 0.7537 & 0.0540 & -88.8701 \end{bmatrix}$$

and $\varepsilon = 1.5$. We choose z_0 and $\hat{z}_0$ to be the center of the smallest disc containing the eigenvalues of L and M respectively. The value of $\tilde{\varepsilon} = \alpha_0\varepsilon$ where α_0 is a root of (2.2.6) is 4.6936. If $\tilde{\varepsilon} = \alpha_0\varepsilon$ where α_0 is given by (2.2.7) then $\tilde{\varepsilon} = 4.5421$. The corresponding pseudospectra levels are shown in Figure 2.2.1 and Figure 2.2.2.

Next, we select z_0 to be the point minimizing the function $\sigma_{\min}(C)\,\|C^{-1}(M-zI)\|_2 + \|L-zI\|_2$ and $\hat{z}_0$ to be the one that minimizes $\sigma_{\min}(C)\,\|C^{-1}(L-zI)\|_2 + \|M-zI\|_2$. Then $\tilde{\varepsilon} = \alpha_0\varepsilon = 4.7131$ when α_0 is a root of equation (2.2.6), while $\tilde{\varepsilon} = \alpha_0\varepsilon = 4.5603$ when α_0 satisfies (2.2.7). This is illustrated in Figure 2.2.3 and Figure 2.2.4, respectively.

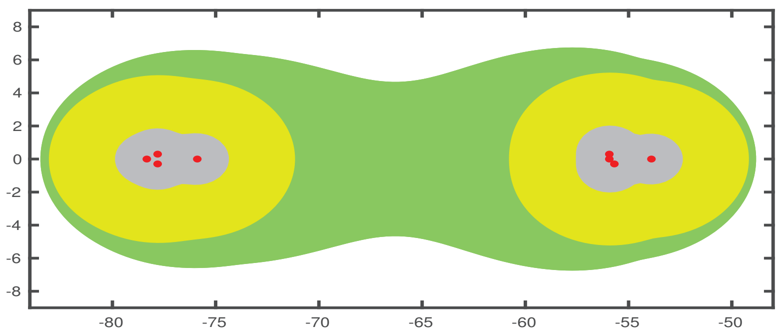


Figure 2.2.1: $\Lambda_\varepsilon(\widehat{A})$, $\Lambda_\varepsilon(A)$ and $\Lambda_{\alpha_0\varepsilon}(\widehat{A})$, for $\varepsilon = 1.5$ and $\alpha_0\varepsilon = 4.6936$. The eigenvalues of A are marked by $\bullet$.

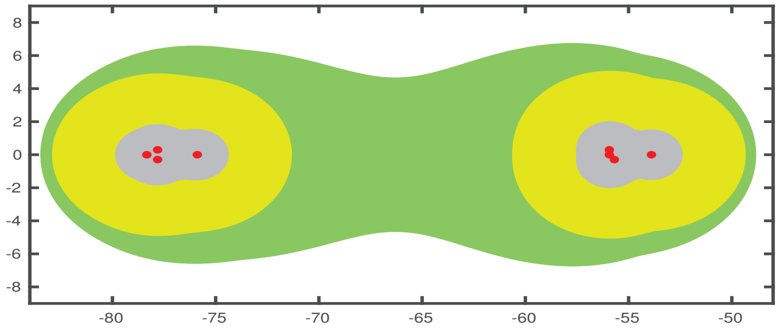


Figure 2.2.2: $\Lambda_\varepsilon(\widehat{A})$, $\Lambda_\varepsilon(A)$ and $\Lambda_{\alpha_0\varepsilon}(\widehat{A})$, where $\varepsilon = 1.5$ and $\alpha_0\varepsilon = 4.5421$. The eigenvalues of A are marked by $\bullet$.

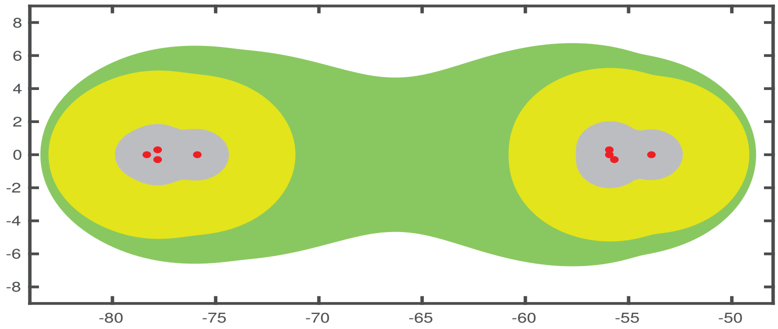


Figure 2.2.3: $\Lambda_\varepsilon(\widehat{A})$, $\Lambda_\varepsilon(A)$ and $\Lambda_{\alpha_0\varepsilon}(\widehat{A})$, where $\varepsilon = 1.5$ and $\alpha_0\varepsilon = 4.7131$ for Example 2.2.5. The eigenvalues of A are marked by $\bullet$.

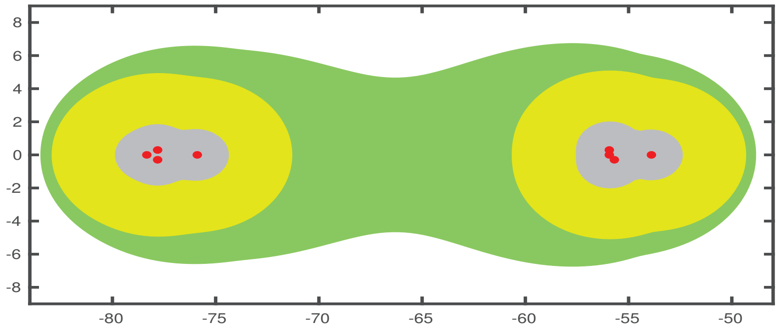


Figure 2.2.4: $\Lambda_\varepsilon(\widehat{A})$, $\Lambda_\varepsilon(A)$ and $\Lambda_{\alpha_0\varepsilon}(\widehat{A})$, where $\varepsilon = 1.5$ and $\alpha_0\varepsilon = 4.5603$ for Example 2.2.5. The eigenvalues of A are marked by $\bullet$.

If M is a normal matrix and $\hat{z}_0$ is chosen to be the center of the smallest disc containing the eigenvalues of M, then $D(\hat{z}_0, \|M - \hat{z}_0 I\|_2 + \alpha\varepsilon)$ is also the smallest disc containing $\Lambda_{\alpha\varepsilon}(M)$. This ensures that the radius $r(\alpha)$ cannot be made smaller than $\|M - \hat{z}_0 I\|_2 + \alpha\varepsilon$ for normal matrices.

Proposition 2.2.6. *Let $M \in \mathbb{C}^{m,m}$ be a normal matrix and suppose $z_0 \in \mathbb{C}$ is the center of the smallest disc containing the eigenvalues of M. Then $D(z_0, \|M - z_0 I\|_2 + \alpha\varepsilon)$ is the smallest disc containing $\Lambda_{\alpha\varepsilon}(M)$.*

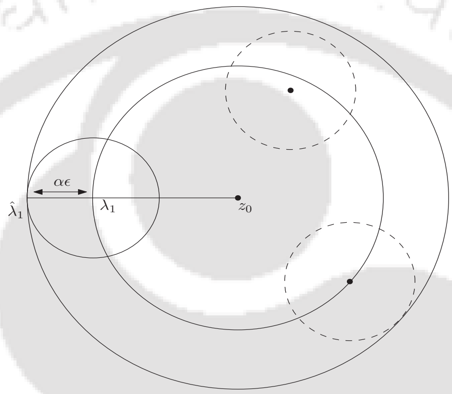


Figure 2.2.5: Spectrum and pseudospectrum for a normal matrix.

Proof. Since $\Lambda_{\alpha\varepsilon}(M) \subseteq D(z_0, \|M - z_0 I\|_2 + \alpha\varepsilon)$, it suffices to show there are points on the boundary of the disc $D(z_0, \|M - z_0 I\|_2 + \alpha\varepsilon)$ belonging to $\Lambda_{\alpha\varepsilon}(M)$. Let λ_1 be any eigenvalue of M on the boundary of the smallest disc centered at z_0 that contains all the eigenvalues of M as shown in Figure 2.2.5. The existence of at least one such eigenvalue of M is ensured by the choice of z_0. Since $\lambda_1 - z_0 \in \Lambda(M - z_0 I)$ and $M - z_0 I$ is normal, so $|\lambda_1 - z_0| = \|M - z_0 I\|_2$. Consider the point $\hat{\lambda}_1$ on the boundary of the disc $D(z_0, \|M - z_0 I\|_2 + \alpha\varepsilon)$ obtained by extending the straight line joining λ_1 and z_0. Clearly, λ_1 is the closest eigenvalue of M to $\hat{\lambda}_1$ and $\left|\hat{\lambda}_1 - \lambda_1\right| = \alpha_0\varepsilon$. Let $E := -\sigma_{\min}(M - \hat{\lambda}_1 I)uv^*$, where u and v are the left and right singular vectors corresponding to the minimum singular value of $M - \hat{\lambda}_1 I$, then $\hat{\lambda}_1 \in \Lambda(M + E)$ with

$$\|E\|_2 = \sigma_{\min}(M - \hat{\lambda}_1 I) = \min_{\lambda \in \Lambda(M)} |\lambda - \hat{\lambda}_1| = |\lambda_1 - \hat{\lambda}_1| = \alpha\varepsilon.$$

So $\hat{\lambda}_1 \in \Lambda_{\alpha\varepsilon}(M)$ and hence the proof. □

We have the following example as an illustration of Proposition 2.2.6.

Example 2.2.7. *Let A and $\widehat{A}$ be as in (2.0.1) where L and M are normal matrices given by*

$$L = \begin{bmatrix} 4.5976 & 0.7910 & 0.0366 & -0.2508 & 0.0065 \\ 0.7910 & 4.8848 & -0.4335 & -0.1206 & -0.5339 \\ 0.0366 & -0.4335 & 4.6619 & -0.0631 & 0.2296 \\ -0.2508 & -0.1206 & -0.0631 & 4.3691 & 0.8024 \\ 0.0065 & -0.5339 & 0.2296 & 0.8024 & 4.7638 \end{bmatrix},$$

$$M = \begin{bmatrix} -6.0069 & -0.3399 & 0.1542 & -0.2773 & 0.1958 \\ -0.3399 & -7.3751 & -0.7132 & -0.6721 & -0.8094 \\ 0.1542 & -0.7132 & -6.2306 & -1.0102 & -0.5780 \\ -0.2773 & -0.6721 & -1.0102 & -7.2276 & -0.5225 \\ 0.1958 & -0.8094 & -0.5780 & -0.5225 & -5.7072 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 120.9716 & 0.0396 & 0.9554 & 0.0050 & 0.7673 \\ 0.3609 & 120.4694 & 0.7242 & 0.7825 & 0.9971 \\ 0.6442 & 0.1501 & 120.5809 & 0.9269 & 0.2277 \\ 0.0679 & 0.9913 & 0.5403 & 120.0083 & 0.9195 \\ 0.20791 & 0.4271 & 0.7054 & 0.8246 & 120.6420 \end{bmatrix}.$$

Let $\varepsilon = 0.1$, and z_0 and $\hat{z}_0$ be the centers of optimal discs containing eigenvalues of L and M respectively. Then using (2.2.6) and (2.2.7) gives the values of $\tilde{\varepsilon}$ as 0.7638 and 0.7595 respectively such that $\Lambda_{\tilde{\varepsilon}}(\widehat{A}) \subseteq \Lambda_\varepsilon(A)$.

If z_0 is a minimizer of the convex function $\sigma_{\min}(C)\,\|C^{-1}(M - zI)\|_2 + \|L - zI\|_2$ while $\hat{z}_0$ minimizes $\sigma_{\min}(C)\,\|C^{-1}(L - zI)\|_2 + \|M - zI\|_2$, using equations (2.2.6) and (2.2.7) also give $\tilde{\varepsilon}$ to be 0.7638 and 0.7595 respectively such that $\Lambda_{\tilde{\varepsilon}}(\widehat{A}) \subseteq \Lambda_\varepsilon(A)$. The corresponding pseudospectra levels for $\widehat{A}$ and A are shown in Figures 2.2.6 and 2.2.7.

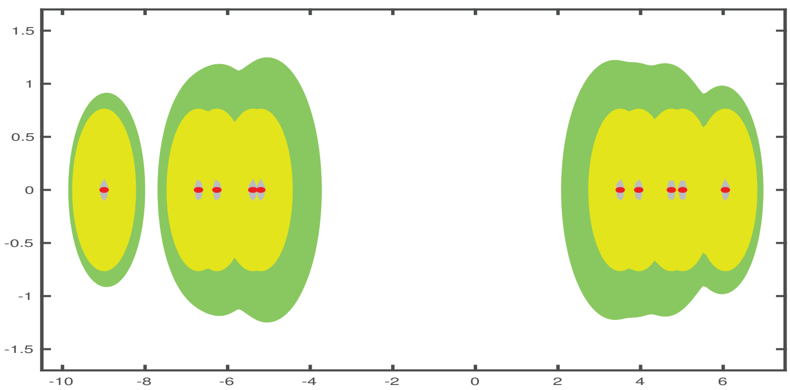


Figure 2.2.6: $\Lambda_\varepsilon(\widehat{A})$, $\Lambda_\varepsilon(A)$ and $\Lambda_{\alpha_0\varepsilon}(\widehat{A})$, for $\varepsilon = 0.1$ and $\alpha_0\varepsilon = 0.7638$. The eigenvalues of A are marked by $\bullet$.

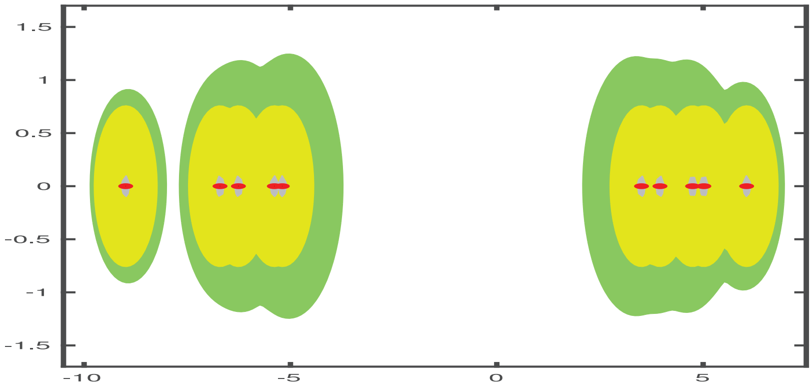


Figure 2.2.7: $\blacksquare\,\Lambda_\varepsilon(\widehat{A})$, $\blacksquare\,\Lambda_\varepsilon(A)$ and $\blacksquare\,\Lambda_{\alpha_0\varepsilon}(\widehat{A})$, where $\varepsilon = 0.1$ and $\alpha_0\varepsilon = 0.7595$. The eigenvalues of A are marked by $\bullet$.

Now we come to the case when the diagonal blocks L and M of A as in (2.0.1) differ in size and C is of full rank i.e., $\operatorname{rank} C = \min\{\ell, m\}$. The following theorem allows us to obtain an inner approximation of $\Lambda_\varepsilon(A)$ via some higher level ε-pseudospectra of the smaller of the two diagonal blocks.

Theorem 2.2.8. *Suppose $\widehat{A}$ and A are as given in (2.0.1) with $\ell < m$ and $C \in \mathbb{C}^{\ell,m}$ such that $\operatorname{rank} C = \ell$. Then*

$$\left\|(A - zI)^{-1}\right\|_2 \geq \left\|(L - zI)^{-1}\right\|_2 \sqrt{1 + \left(\sigma_{\min}(C)/\left\|M - zI\right\|_2\right)^2}. \tag{2.2.8}$$

Proof. For $z \in \mathbb{C} \setminus (\Lambda(L) \cup \Lambda(M))$, and right, left singular vectors $v_{\min}(z), u_{\min}(z)$ in $\mathbb{C}^m$ of $L - zI$ corresponding to $\sigma_{\min}(L - zI)$, we have

$$\begin{aligned}
\left\|(A - zI)^{-1}\right\|_2 &= \left\|\left[\begin{array}{c|c} (L - zI)^{-1} & -(L - zI)^{-1}C(M - zI)^{-1} \\ \hline 0 & (M - zI)^{-1} \end{array}\right]\right\|_2 \\
&\geq \left\|\left[\begin{array}{cc} v_{\min}^*(z) & 0 \end{array}\right]\left[\begin{array}{c|c} (L - zI)^{-1} & -(L - zI)^{-1}C(M - zI)^{-1} \\ \hline 0 & (M - zI)^{-1} \end{array}\right]\right\|_2 \\
&= \left\|\left[\begin{array}{cc} \|(L - zI)^{-1}\|_2\, u_{\min}^*(z) & -\|(L - zI)^{-1}\|_2\, u_{\min}^*(z)C(M - zI)^{-1} \end{array}\right]\right\|_2 \\
&= \left\|(L - zI)^{-1}\right\|_2 \sqrt{1 + \|u_{\min}^*(z)C(M - zI)^{-1}\|_2^2} \\
&\geq \left\|(L - zI)^{-1}\right\|_2 \sqrt{1 + \left(\sigma_{\min}(C)/\left\|M - zI\right\|_2\right)^2}.
\end{aligned}$$

$\square$

Lemma 2.2.9. *Let $\widehat{A}$ and A be as described in the statement of Theorem 2.2.8 and suppose $\sigma_{\min}(C) > 0$. Choose any $z_0 \in \mathbb{C}$. For $\alpha > 1$, let*

$$r(\alpha) = \left(\sigma_{\min}(C)/\sqrt{\alpha^2 - 1}\right) - \left\|M - z_0 I\right\|_2. \tag{2.2.9}$$

Then for any $\varepsilon > 0$, we have $D(z_0, r(\alpha)) \cap \Lambda_{\alpha\varepsilon}(L) \subseteq \Lambda_\varepsilon(A)$.

Proof. The proof is identical to that of Lemma 2.2.3. □

Corollary 2.2.10. *Let $\widehat{A}$ and A be as in* (2.0.1) *with $\ell < m$ and $C \in \mathbb{C}^{\ell,m}$ such that* $\operatorname{rank} C = \ell$. *Also let $r(\alpha)$ be as in Lemma 2.2.9. Suppose $\varepsilon > 0$, $z_0 \in \mathbb{C}$ and $R_L(\alpha) > 0$ is chosen so that $\Lambda_{\alpha\varepsilon}(L) \subseteq D(z_0, R_L(\alpha))$ for all $\alpha > 1$. Let α_0 be the solution of $r(\alpha) = R_L(\alpha)$. Then $\alpha_0 > 1$ and we have $\Lambda_{\alpha_0\varepsilon}(L) \subseteq \Lambda_\varepsilon(A)$.*

Proof. The proof is similar to that of Corollary 2.2.4. □

To be able to use the idea of the proof of Theorem 2.2.8, to find $\tilde{\varepsilon} > \varepsilon$ such that $\Lambda_{\tilde{\varepsilon}}(M) \subseteq \Lambda_\varepsilon(A)$ when L, M and C are as assumed in its statement, it is necessary to ensure that $Cv_{\min}(z) \neq 0$ for any right singular vector $v_{\min}(z)$ of $M - zI$ corresponding to $\sigma_{\min}(M - zI)$ for $z \notin \Lambda(M)$. However, this need not hold in general. To see this, let U and V be unitary matrices such that $U^*CV = \begin{bmatrix} \Sigma_{\ell\times\ell} & 0_{\ell\times(m-\ell)} \end{bmatrix}$ is an SVD of C. Since $\Lambda_\varepsilon(A)$ with respect to the norm $\|\cdot\|_2$ is invariant with respect to multiplication of A by unitary matrices, we assume without loss of generality that

$$A = \left[\begin{array}{c|c} L & \begin{bmatrix} \Sigma_{\ell\times\ell} & 0_{\ell\times(m-\ell)} \end{bmatrix} \\ \hline 0 & M \end{array}\right].$$

Suppose $M = \begin{bmatrix} M_{1,1} & \cdots & \cdots & M_{1,m-1} & 0 \\ 0 & \ddots & & \vdots & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & M_{m-1,m-1} & \vdots \\ 0 & \cdots & & 0 & M_{m,m} \end{bmatrix}$. If the point z is chosen close to $M_{m,m}$, the right singular vector corresponding to $\sigma_{\min}(M - zI)$ is $v_{\min}(z) = e_m$, therefore $Cv_{\min}(z) = 0$.

However, it appears that only special values of z satisfy $Cv_{\min}(z) = 0$. To see this, assume that the unitary matrix V in the Singular Value Decomposition of $C \in \mathbb{C}^{\ell,m}$ is partitioned as $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ with $V_1 \in \mathbb{C}^{m,\ell}$ and $V_2 \in \mathbb{C}^{m,m-\ell}$ where $\operatorname{rank} C = \ell$. If $v_{\min}(z), u_{\min}(z) \in \mathbb{C}^m$ are right, left singular vectors of $M - zI$ corresponding to $\sigma_{\min}(M - zI)$ respectively, then $Cv_{\min}(z) = 0$ if and only if $v_{\min}(z)^*V_1 = 0$. In cases when none of the $z \in \Lambda_\varepsilon(A)$ satisfy this condition, it may be possible to have an inner approximation of $\Lambda_\varepsilon(A)$ via $\Lambda_{\alpha\varepsilon}(M)$ for some $\alpha > 1$.

Theorem 2.2.11. *Let L, M and C be as described in Theorem 2.2.8, and $\alpha_0 > 1$ be as obtained from Corollary 2.2.10. Suppose D_M is a disc centered at some $\hat{z} \in \mathbb{C}$ that is large enough, so that $\Lambda_\varepsilon(M) \subseteq \Lambda_{\alpha_0\varepsilon}(M) \subset D_M$. For $z \in D_M$, let $v_{\min}(z)$ be any right singular vector of $M - zI$ corresponding to $\sigma_{\min}(M - zI)$ and define*

$\zeta = \inf_{z \in D_M} \|Cv_{\min}(z)\|_2$. *If* $\zeta > 0$, *then for* $z \in D_M$, *we have*

$$\left\|(A - zI)^{-1}\right\|_2 \geq \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + (\zeta / \|L - zI\|_2)^2}.$$

Proof. Let $u_{\min}(z) \in \mathbb{C}^m$ denote the left singular vector of $M - zI$ corresponding to $\sigma_{\min}(M - zI)$. Then for $z \in \mathbb{C} \setminus (\Lambda(L) \cup \Lambda(M))$, we have

$$\begin{aligned}
\left\|(A - zI)^{-1}\right\|_2 &= \left\| \left[\begin{array}{c|c} (L - zI)^{-1} & -(L - zI)^{-1}C(M - zI)^{-1} \\ \hline 0 & (M - zI)^{-1} \end{array}\right] \right\|_2 \\
&\geq \left\| \left[\begin{array}{c|c} (L - zI)^{-1} & -(L - zI)^{-1}C(M - zI)^{-1} \\ \hline 0 & (M - zI)^{-1} \end{array}\right] \left[\begin{array}{c} 0 \\ u_{\min}(z) \end{array}\right] \right\|_2 \\
&= \left\| \left[\begin{array}{c} -\|(M - zI)^{-1}\|_2 (L - zI)^{-1}Cv_{\min}(z) \\ \hline \|(M - zI)^{-1}\|_2 v_{\min}(z) \end{array}\right] \right\|_2 \\
&= \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + \|(L - zI)^{-1}Cv_{\min}(z)\|_2^2} \\
&\geq \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + (\|Cv_{\min}(z)\|_2 / \|L - zI\|_2)^2}.
\end{aligned}$$

which remains valid for all $z \in D_M$. Since $\|Cv_{\min}(z)\|_2 \geq \zeta$ for all $z \in D_M$, the desired inequality follows. $\square$

Lemma 2.2.12. *Let* L, M *and* C *be as in the statement of Theorem 2.2.8 and suppose* $\sigma_{\min}(C) > 0$. *For* $\beta > 1$ *and* $z_1 \in \mathbb{C}$, *let*

$$\hat{r}(\beta) = \frac{\zeta}{\sqrt{\beta^2 - 1}} - \|L - z_1 I\|_2.$$

Also let D_M *be as in Theorem 2.2.11. Then for any* $\varepsilon > 0$, *we have*

$$D(z_1, \hat{r}(\beta)) \cap (\Lambda_{\beta\varepsilon}(M) \cap D_M) \subseteq \Lambda_\varepsilon(A).$$

Proof. Let $z \in D(z_1, \hat{r}(\beta))$. Then, $\|L - zI\|_2 \leq \hat{r}(\beta) + \|L - z_1 I\|_2 = \frac{\zeta}{\sqrt{\beta^2 - 1}}$. Defining $R = \frac{\zeta}{\sqrt{\beta^2 - 1}}$, we have $\beta = \sqrt{1 + \left(\frac{\zeta}{R}\right)^2}$. Therefore, using Theorem 2.2.11, we obtain for $z \in D(z_1, \hat{r}(\beta)) \cap D_M$,

$$\begin{aligned}
\left\| \begin{bmatrix} L - zI & C \\ 0 & M - zI \end{bmatrix}^{-1} \right\|_2 &\geq \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + (\zeta / \|L - zI\|_2)^2}. \\
&\geq \left\|(M - zI)^{-1}\right\|_2 \sqrt{1 + \left(\frac{\zeta}{R}\right)^2} \\
&= \beta \left\|(M - zI)^{-1}\right\|_2.
\end{aligned}$$

Thus $z \in (D(z_1, \hat{r}(\beta)) \cap D_M) \cap \Lambda_{\beta\varepsilon}(M) \implies \left\| \begin{bmatrix} L - zI & C \\ 0 & M - zI \end{bmatrix}^{-1} \right\|_2 \geq \frac{1}{\varepsilon}$. $\square$

Now we use the preceding analysis to formulate a strategy for finding a $\beta > 1$ such that $\Lambda_{\beta\varepsilon}(M) \subseteq \Lambda_{\varepsilon}(A)$. Let α_0 be as in Corollary 2.2.10 so that $\Lambda_{\alpha_0\varepsilon}(L) \subseteq \Lambda_{\varepsilon}(A)$. Then choosing $D_M = D(\hat{z}, \|M - \hat{z}I\|_2 + \alpha_0\varepsilon)$, we have $\Lambda_{\alpha_0\varepsilon}(M) \subseteq D_M$. If we select $R_M(\beta) = \|M - z_1 I\|_2 + \beta\varepsilon$, then $\Lambda_{\beta\varepsilon}(M) \cap D_M \subseteq D(z_1, R_M(\beta))$. Let β_1 be the largest positive solution of the equation $R_M(\beta) = \hat{r}(\beta)$ where $\hat{r}(\beta)$ is as in Lemma 2.2.12. In view of this lemma, $\Lambda_{\beta_1\varepsilon}(M) \cap D_M \subseteq \Lambda_{\varepsilon}(A)$. Alternatively,

$$D_M = D(\hat{z}, \|M - \hat{z}I\|_2 + \alpha_0\varepsilon) \subseteq D(z_1, \|M - \hat{z}I\|_2 + |\hat{z} - z_1| + \alpha_0\varepsilon).$$

Let $\beta_2 > 1$ be such that $\hat{r}(\beta_2) = \|M - \hat{z}I\|_2 + |\hat{z} - z_1| + \alpha_0\varepsilon$, so that Lemma 2.2.12 implies $\Lambda_{\beta_2\varepsilon}(M) \cap D_M \subseteq \Lambda_{\varepsilon}(A)$. Let $\beta_0 := \max\{\beta_1, \beta_2\}$, since $\Lambda_{\alpha_0\varepsilon}(M) \subseteq D_M$, we have $\Lambda_{\alpha_0\varepsilon}(M) \subseteq \Lambda_{\varepsilon}(A)$ if $\alpha_0 \leq \beta_0$ and $\Lambda_{\beta_0\varepsilon}(M) \subseteq \Lambda_{\varepsilon}(A)$ otherwise.

In numerical experiments, computing the values of α_0 and β_1 is similar to the ones obtained for the case when $\ell = m$. For finding the value of α_0 for L, if we take $R_L(\alpha) = \|L - z_0 I\|_2 + \alpha\varepsilon$ and equate the $r(\alpha)$ in (2.2.9) with $R_L(\alpha)$, so that α is the largest positive solution of the equation

$$x^4 + 2p_3x^3 + p_2x^2 + 2p_1x + p_0 = 0 \tag{2.2.10}$$

where

$$\begin{aligned}
p_3 &= \left(\|L - z_0I\|_2 + \|M - z_0I\|_2\right)/\varepsilon,\\
p_2 &= \left[\left(\|L - z_0I\|_2 + \|M - z_0I\|_2\right)^2 - \varepsilon^2\right]/\varepsilon^2,\\
p_1 &= -\left(\|L - z_0I\|_2 + \|M - z_0I\|_2\right)/\varepsilon,\\
\text{and}\quad p_0 &= -\left[\left(\|L - z_0I\|_2 + \|M - z_0I\|_2\right)^2 + \sigma_{\min}^2(C)\right]/\varepsilon^2,
\end{aligned}$$

with z_0 being chosen as the point minimizing the convex function $\|L - zI\|_2 + \|M - zI\|_2$. Similarly, β_1 is the largest positive solution of the quartic equation

$$x^4 + 2p_3x^3 + p_2x^2 + 2p_1x + \widehat{p_0} = 0 \tag{2.2.11}$$

where

$$\widehat{p_0} = -\left[\left(\|L - z_0I\|_2 + \|M - z_0I\|_2\right)^2 + \zeta^2\right]/\varepsilon^2.$$

Moreover, it is easy to see that

$$\beta_2 = \sqrt{\left(\frac{\zeta}{\|L - z_1I\|_2 + \|M - \hat{z}I\|_2 + |z_1 - \hat{z}| + \alpha_0\varepsilon}\right)^2 + 1} \tag{2.2.12}$$

is the solution of $\hat{r}(\beta) = \|M - \hat{z}I\|_2 + |\hat{z} - z_1| + \alpha_0\varepsilon$, where z_1 can be chosen to be the point minimizing the function $\|L - z_1I\|_2 + |z_1 - \hat{z}|$. This is illustrated in the following example.

Example 2.2.13. *Let A and $\widehat{A}$ be as in Theorem* (2.2.8)*, where*

$$L = \begin{bmatrix} 24.9500 & 0.1411 & 0.7321 & 0.5208 \\ 0.1582 & 24.5121 & 0.7498 & 0.2191 \\ 0.2864 & 0.7213 & 24.4073 & 0.8424 \\ 0.6871 & 0.9288 & 0.2395 & 24.6629 \end{bmatrix}, \; C = \begin{bmatrix} 23.8312 & 0.5386 & 0.0763 & 0.2681 & 0.7039 & 0.3808 \\ 0.9223 & 23.4633 & 0.7087 & 0.8325 & 0.9323 & 0.6346 \\ 0.32702 & 0.8208 & 23.2349 & 0.9954 & 0.6876 & 0.3632 \\ 0.8041 & 0.9519 & 0.3989 & 23.6497 & 0.56835 & 0.4076 \end{bmatrix},$$

$$M = \begin{bmatrix} 20.8162 & 0.7699 & 0.8651 & 0.10704 & 0.5767 & 0.8288 \\ 0.7938 & 20.8295 & 0.06802 & 0.7242 & 0.94402 & 0.3225 \\ 0.4691 & 0.70608 & 20.9685 & 0.61369 & 0.8714 & 0.97615 \\ 0.3095 & 0.5953 & 0.0987 & 20.7829 & 0.5076 & 0.2782 \\ 0.6875 & 0.7528 & 0.5469 & 0.5666 & 20.7888 & 0.07283 \\ 0.9868 & 0.4967 & 0.4029 & 0.8113 & 0.4730 & 20.7512 \end{bmatrix},$$

and $\varepsilon = 0.9$. *Let* z_0 *be the point minimizing the function* $\|L - zI\|_2 + \|M - zI\|_2$. *Solving equation (2.2.10), we obtain* $\alpha_0\,\varepsilon = 2.4110$ *such that* $\Lambda_{\alpha_0\varepsilon}(L) \subseteq \Lambda_\varepsilon(A)$. *For the inner approximation via a suitable* ε*-pseudospectra level of* M*, first we choose* $D_M = D(21.5, \|M - 21.5I\|_2 + 2.411)$ *and* z_1 *as a minimizer of the convex function* $\|L - zI\|_2 + \|M - zI\|_2$. *Then solving the equation (2.2.11), gives* $\beta_1\,\varepsilon = 2.1512$ *so that* $\Lambda_{\beta_1\varepsilon}(M) \subseteq \Lambda_\varepsilon(A)$. *Next choosing* z_1 *to be the point minimizing the function* $\|L - z_1 I\|_2 + |z_1 - \hat{z}|$, *equation (2.2.12) gives* $\beta_2\varepsilon = 2.0196$ *so that* $\Lambda_{\beta_2\varepsilon}(M) \subseteq \Lambda_\varepsilon(A)$. *Thus if* $\beta_0\varepsilon = \max\{\beta_1\varepsilon, \beta_2\varepsilon\} = 2.1512$*, and* $\tilde{\varepsilon} = \min\{\alpha_0\varepsilon, \beta_0\varepsilon\} = \beta_0\varepsilon = 2.1512$*, then* $\Lambda_{\tilde{\varepsilon}}(\widehat{A}) \subseteq \Lambda_\varepsilon(A)$. *The corresponding pseudospectra levels are shown in Figure 2.2.8.*

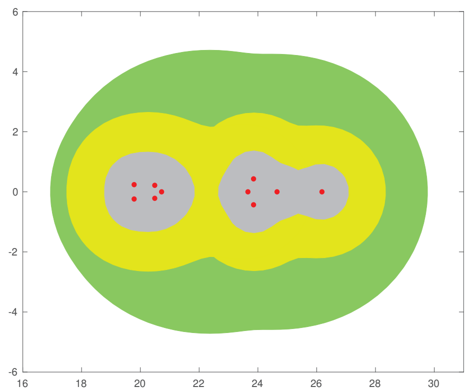


Figure 2.2.8: ■ $\Lambda_\varepsilon(\widehat{A})$, ■ $\Lambda_\varepsilon(A)$ and ■ $\Lambda_{\tilde{\varepsilon}}(\widehat{A})$, where $\varepsilon = 0.9$ and $\tilde{\varepsilon} = 2.1512$ for Example 2.2.13. • denotes the eigenvalues.

3

Localization sets for eigenvalues of homogeneous matrix polynomials

In this chapter, we define the block Geršgorin sets, block minimal Geršgorin sets, block Brualdi sets and permuted pointwise minimal Geršgorin sets for matrix polynomials in homogeneous form. This allows for finite and infinite eigenvalues to be localized in a single framework. Several properties of these sets are proved and strategies for numerically computing the sets are provided. The advantages of considering the blocked versions of the localization sets in computations are also demonstrated.

3.1 Block Geršgorin-type localizations for eigenvalues of homogeneous matrix polynomials

In this section, we extend the notions of Geršgorin sets, Brualdi sets and minimal Geršgorin sets to their blocked versions for matrix polynomials in the homogeneous framework with the aim of finding a localization that includes any potential eigenvalue at infinity.

Consider a matrix polynomial

$$P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i},\ A_i \in \mathbb{C}^{n,n}. \tag{3.1.1}$$

Let $\pi = \{n_j\}_{j=0}^{\ell}$ be a partition on the set $\{1,\ldots,n\}$. Suppose $P(c,s) \in \mathbb{C}^{n,n}$ is partitioned with respect to π as

$$P(c,s) = \left[\begin{array}{c|c|c|c} P_{1,1}(c,s) & P_{1,2}(c,s) & \ldots & P_{1,\ell}(c,s) \\ \hline P_{2,1}(c,s) & P_{2,2}(c,s) & \ldots & P_{2,\ell}(c,s) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline P_{\ell,1}(c,s) & P_{\ell,2}(c,s) & \ldots & P_{\ell,\ell}(c,s) \end{array}\right] \tag{3.1.2}$$

for each $(c,s) \in \mathbb{S}$. We will refer to the partitioned matrix polynomial $[P_{k,j}(c,s)]_{\ell\times\ell}$ as a block matrix polynomial. Consider the comparison matrix $\langle P(c,s)\rangle_p^\pi = [\mu_{k,j}(c,s)]$ in $\mathbb{R}^{\ell,\ell}$ defined by

$$\mu_{k,j}(c,s) = \begin{cases} +\,\|(P_{k,k}(c,s))^{-1}\|_p^{-1} & \text{if } k=j, \\ -\,\|P_{k,j}(c,s)\|_p & \text{if } k\neq j \end{cases}$$

where we use the convention that $\|(P_{k,k}(c,s))^{-1}\|_p^{-1} = 0$ if $\det(P_{k,k}(c,s)) = 0$. For each $(c,s) \in \mathbb{S}$, define the non-negative sums $r_k(\langle P(c,s)\rangle_p^\pi) := \sum_{\substack{j=1\\ j\neq k}}^{\ell} \|P_{k,j}(c,s)\|_p$, for $k \in \{1,2,\ldots,\ell\}$.

Definition 3.1.1. *The block Geršgorin set for the homogeneous matrix polynomial $P(c,s)$ (as in (3.1.1)) with respect to the partition $\pi = \{n_j\}_{j=0}^{\ell}$, is defined as*

$$\begin{aligned} H\Gamma_p^\pi(P) &= \{(c,s)\in\mathbb{S} : P(c,s) \textit{ is not a } B_p^\pi \textit{ SDD matrix}\} \\ &= \{(c,s)\in\mathbb{S} : \langle P(c,s)\rangle_p^\pi \textit{ is not a SDD matrix}\}. \end{aligned}$$

Now $\langle P(c,s)\rangle_p^\pi$ is not $SDD \implies \exists\, k\in\{1,2,\ldots,\ell\}$ such that

$$\left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} \leq r_k(\langle P(c,s)\rangle_p^\pi).$$

Therefore,

$$H\Gamma_p^\pi(P) = \bigcup_{k=1}^{\ell} H\Gamma_{p,k}^\pi(P), \tag{3.1.3}$$

where

$$H\Gamma_{p,k}^\pi(P) := \left\{(c,s)\in\mathbb{S} : \left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} \leq r_k(\langle P(c,s)\rangle_p^\pi)\right\}. \tag{3.1.4}$$

It is evident that $(c,s)\in\Lambda(P)$ if and only if $P(c,s)$ is singular. Therefore, $\langle P(c,s)\rangle_p^\pi$ is not SDD, so that $(c,s)\in H\Gamma_p^\pi(P)$. Therefore, $\Lambda(P)\subset H\Gamma_p^\pi(P)$.

We now establish the following properties of the block Geršgorin set for the matrix polynomial $P(c,s)$. Some of them are adaptation of the properties for Geršgorin sets for matrix pencils and matrix polynomials in [36, 52].

Theorem 3.1.2. *Given a $n\times n$ homogeneous matrix polynomial $P(c,s)$ partitioned as in (3.1.2) with respect to a partition π of $\{1,\ldots,n\}$, and a choice of matrix p-norm, the following statements hold for the block Geršgorin set $H\Gamma_p^\pi(P)$.*

(i) $(0,1)\in H\Gamma_{p,k}^\pi(P) \iff \left\|(A_m)_{k,k}^{-1}\right\|_p^{-1} \leq \sum_{j\neq k} \|(A_m)_{k,j}\|_p$ *for some* $k\in\{1,2,\ldots,\ell\}$.

(ii) $(0,1) \notin H\Gamma^{\pi}_{p}(P) \iff A_m$ *is* B^{π}_{p} *SDD.*

(iii) $(1,0) \in H\Gamma^{\pi}_{p,k}(P) \iff \left\|(A_0)^{-1}_{k,k}\right\|_p^{-1} \leq \sum_{j\neq k} \|(A_0)_{k,j}\|_p$ *for some* $k \in \{1,2,\ldots,\ell\}$.

(iv) $(1,0) \notin H\Gamma^{\pi}_{p}(P) \iff A_0$ *is* B^{π}_{p} *SDD.*

(v) If for some k, $(A_m)_{k,k} = 0$ *and there exists* $j \neq k$ *such that* $(A_m)_{k,j} = 0$ *and*

$$\left\|\left(\sum_{i=0}^{m-1} A_i c^{m-1-i} s^i\right)^{-1}_{k,k}\right\|_p^{-1} \leq \sum_{\substack{j\neq k \\ (A_m)_{k,j}=0}} \left\|\left(\sum_{i=0}^{m-1} A_i c^{m-1-i} s^j\right)\right\|_p,$$

then $H\Gamma^{\pi}_{p,k}(P) = \mathbb{S}$.

(vi) If for some $k \in \{1,\ldots,\ell\}$, $P_{k,k}(c,s)$ *is singular then* $H\Gamma^{\pi}_{p,k}(P) = \mathbb{S}$.

(vii) If all the coefficient matrices have their k^{th} *block row real, then* $(c,s) \in H\Gamma^{\pi}_{p,k}(P)$ *if and only if* $(c,\bar{s}) \in H\Gamma^{\pi}_{p,k}(P)$ *i.e., the set* $H\Gamma^{\pi}_{p,k}(P)$ *on the sphere is symmetric with respect to the xz-plane.*

Proof. (i) $(0,1) \in H\Gamma^{\pi}_{p,k}(P) \iff \|(P_{k,k}(0,1))^{-1}\|_p^{-1} \leq \sum_{j\neq k} \|(P_{k,j}(0,1))\|_p$ for some $k \in \{1,2,\ldots,\ell\}$ i.e., $\left\|(A_m)^{-1}_{k,k}\right\|_p^{-1} \leq \sum_{j\neq k} \|(A_m)_{k,j}\|_p$ for some $k \in \{1,2,\ldots,\ell\}$.

(ii) $(0,1) \notin H\Gamma^{\pi}_{p}(P)$ if and only if $(0,1) \notin H\Gamma^{\pi}_{p,k}(P)$ for all $k \in \{1,2,\ldots,\ell\}$. Therefore, from (i), it follows that $\left\|(A_m)^{-1}_{k,k}\right\|_p^{-1} > \sum_{j\neq k} \|(A_m)_{k,j}\|_p \ \forall\ k \in \{1,2,\ldots,\ell\}$. Hence, the result.

The proofs of (iii) and (iv) follow by arguing as in the proofs of (i) and (ii) respectively.

(v) Since $(A_m)_{k,k} = 0$, we have

$$\begin{aligned} H\Gamma^{\pi}_{p,k}(P) &= \left\{(c,s)\in\mathbb{S} : \left\|\left(\sum_{i=0}^{m-1} A_i c^{m-i} s^i\right)^{-1}_{k,k}\right\|_p^{-1} \leq \sum_{j\neq k} \left\|\left(\sum_{i=0}^{m} A_i c^{m-i} s^i\right)_{k,j}\right\|_p\right\} \\ &= \{(0,s)\} \cup \left\{(c,s)\in\mathbb{S}\setminus\{(0,s)\} : \left\|\left(\sum_{i=0}^{m-1} A_i c^{m-1-i} s^i\right)^{-1}_{k,k}\right\|_p^{-1} - \right. \\ &\qquad \left. \sum_{\substack{j\neq k \\ (A_m)_{k,j}=0}} \left\|\left(\sum_{i=0}^{m-1} A_i c^{m-1-i} s^i\right)_{k,j}\right\|_p \leq \sum_{\substack{j\neq k \\ (A_m)_{k,j}\neq 0}} \frac{1}{|c|} \left\|\left(\sum_{i=0}^{m} A_i c^{m-i} s^i\right)_{k,j}\right\|_p\right\} \\ &= \mathbb{S}. \end{aligned}$$

(vi) The proof follows from the definition of $H\Gamma^{\pi}_{p,k}(P)$.

(vii)

$$
\begin{aligned}
(c,s) \in H\Gamma^{\pi}_{p,k}(P) \quad &\iff \quad \left\| \left(\sum_{j=0}^{m} A_j c^{m-j} s^j \right)^{-1}_{k,k} \right\|_p^{-1} \leq \sum_{j \neq k} \left\| \left(\sum_{j=0}^{m} A_j c^{m-j} s^j \right)_{k,j} \right\|_p \\
&\iff \quad \left\| \overline{\left(\sum_{j=0}^{m} A_j c^{m-j} s^j \right)^{-1}_{k,k}} \right\|_p^{-1} \leq \sum_{j \neq k} \left\| \overline{\left(\sum_{j=0}^{m} A_j c^{m-j} s^j \right)_{k,j}} \right\|_p \\
&\iff \quad \left\| \left(\sum_{j=0}^{m} A_j c^{m-j} \bar{s}^j \right)^{-1}_{k,k} \right\|_p^{-1} \leq \sum_{j \neq k} \left\| \left(\sum_{j=0}^{m} A_j c^{m-j} \bar{s}^j \right)_{k,j} \right\|_p \\
&\iff \quad (c,\bar{s}) \in H\Gamma^{\pi}_{p,k}(P)
\end{aligned}
$$

Therefore, the set $H\Gamma^{\pi}_{p,k}(P)$ on the sphere is symmetric with respect to the xz-plane. Since each point $(c,s) \in \mathbb{S} \setminus \{(0,s)\}$ is mapped to $z = s/c \in \mathbb{C}$. The projection of such a set on the complex plane is symmetric with respect to real axis.

$\square$

In the next definition, we extend the notion of minimal Geršgorin sets to block matrix polynomials.

Definition 3.1.3. *Let $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ be an $n \times n$ matrix polynomial partitioned with respect to a partition π of the set $\{1,\ldots,n\}$ as in (3.1.2) and $\|\cdot\|_p$ be any matrix p-norm. The block minimal Geršgorin set for $P(c,s)$ with respect to partition π and norm $\|\cdot\|_p$, denoted by $H\mathcal{G}^{\pi}_p(P)$ is defined as*

$$
\begin{aligned}
H\mathcal{G}^{\pi}_p(P) &= \{(c,s) \in \mathbb{S} : [P_{k,j}(c,s)]_{\ell \times \ell} \text{ is not a } B^{\pi}_p\mathbb{H}\text{-matrix}\} \\
&= \{(c,s) \in \mathbb{S} : \langle P(c,s) \rangle^{\pi}_p \text{ is not a non-singular } \mathbb{M}\text{-matrix}\}.
\end{aligned}
$$

Theorem 3.1.4. *Given a $n \times n$ homogeneous matrix polynomial $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$, let $\pi = \{n_j\}_{j=0}^{\ell}$ denote a partition of $\{1,\ldots,n\}$ and $\|\cdot\|_p$ be any induced matrix p-norm. Define $\mathcal{S} = \{\operatorname{diag}(x_1 I_1, \ldots, x_\ell I_\ell) : x_k > 0, k = \{1,2,\ldots,\ell\}\}$, where the size of the identity matrix I_k, $k = 1,\ldots,\ell$, is the same as that of the k^{th} diagonal block of $P(c,s)$ induced by the partition π. Then*

$$
H\mathcal{G}^{\pi}_p(P) = \bigcap_{X \in \mathcal{S}} H\Gamma^{\pi}_p(X^{-1}PX)
$$

where

$$
H\Gamma^{\pi}_p(X^{-1}PX) := \left\{ (c,s) \in \mathbb{S} : \left\langle X^{-1}P(c,s)X \right\rangle^{\pi}_p \text{ is not a SDD matrix} \right\}.
$$

Proof. Let $(c,s) \in H\mathcal{G}_p^{\pi}(P)$. Then by definition, this is equivalent to $\langle P(c,s)\rangle_p^{\pi}$ being a singular $\mathbb{M}$-matrix. By Theorem 1.2.1, this in turn is equivalent to $\langle P(c,s)\rangle_p^{\pi} X$ not being a SDD matrix for every $X \in \mathcal{S}$ i.e., for every $[x_1 \cdots x_\ell]^t \in \mathbb{R}_+^\ell$, there exists $k \in \{1,\dots,\ell\}$ such that

$$\left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} \leq \frac{x_j}{x_k}\sum_{j\neq k}\|(P_{k,j}(c,s))\|_p .$$

Therefore, $\langle X^{-1}P(c,s)X\rangle_p^{\pi}$ is not SDD for every $X \in \mathcal{S}$. Hence,

$$\begin{aligned}(c,s) \in H\mathcal{G}_p^{\pi}(P) &\iff (c,s) \in H\Gamma_p^{\pi}(X^{-1}PX)\ \forall\ X \in \mathcal{S}\\ &\iff (c,s) \in \bigcap_{X\in\mathcal{S}} H\Gamma_p^{\pi}(X^{-1}PX).\end{aligned}$$

□

From Theorem 3.1.4, it is clear that $\Lambda(P) \subseteq H\mathcal{G}_p^{\pi}(P)$. Also as

$$H\Gamma_p^{\pi}(X^{-1}PX) := \bigcup_{k=1}^{\ell}\left\{(c,s) \in \mathbb{S} : \left\|((P_{k,k}(c,s))^{-1}\right\|_p^{-1} \leq \sum_{j\neq k}\frac{x_j}{x_k}\|(P_{k,j}(c,s))\|_p\right\},$$

where $x = [x_1 \cdots x_\ell]^t \in \mathbb{R}_+^\ell$, therefore setting

$$H\Gamma_{p,k}^{\pi,x}(P) = \left\{(c,s) \in \mathbb{S} : \left\|((P_{k,k}(c,s))^{-1}\right\|_p^{-1} \leq \sum_{j\neq k}\frac{x_j}{x_k}\|(P_{k,j}(c,s))\|_p\right\}, \tag{3.1.5}$$

we have

$$H\mathcal{G}_p^{\pi}(P) = \bigcap_{x\in\mathbb{R}_+^\ell}\bigcup_{k=1}^{\ell} H\Gamma_{p,k}^{\pi,x}(P). \tag{3.1.6}$$

As in Theorem 3.1.2, it is easy to verify the following properties for the block minimal Geršgorin set.

Theorem 3.1.5. *Given a $n\times n$ homogeneous matrix polynomial $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ partitioned as in (3.1.2) with respect to a partition π of the set $\{1,\dots,n\}$ and a choice of matrix p-norm, the following statements hold for the block minimal Geršgorin set $H\mathcal{G}_p^{\pi}(P)$.*

(i) $(0,1) \notin H\mathcal{G}_p^{\pi}(P) \iff A_m$ *is a* $B_p^{\pi}\,\mathbb{H}$*- matrix.*

(ii) $(1,0) \notin H\mathcal{G}_p^{\pi}(P) \iff A_0$ *is a* $B_p^{\pi}\,\mathbb{H}$*-matrix.*

(iii) If $\exists\ k \in \{1,\dots,\ell\}$ *such that* $(A_m)_{k,k} = 0$*, then* $\left(\sum_{i=0}^{m-1} A_i c^{m-1-i}s^i\right)_{k,k}$ *is singular if and only if* $H\Gamma_{p,k}^{\pi,x}(P) = \mathbb{S}$ *for any* $x \in \mathbb{R}_+^\ell$*, where* $H\Gamma_{p,k}^{\pi,x}(P)$ *is as in (3.1.5).*

(iv) *The set* $H\mathcal{G}_p^{\pi}(P) = \mathbb{S}$ *if there exists some* $k \in \{1, 2, \ldots, \ell\}$ *such that* $P_{k,k}(c, s)$ *is a singular matrix polynomial.*

(v) *If all the coefficient matrices of* $P(c, s)$ *have real entries in the* k^{th} *block row, then* $(c, s) \in H\Gamma_{p,k}^{\pi,x}(P)$ *if and only if* $(c, \bar{s}) \in H\Gamma_{p,k}^{\pi,x}(P)$ *i.e., the set* $H\Gamma_{p,k}^{\pi}(P)$ *on the sphere is symmetric with respect to the xz-plane.*

Proof. (i) We have,

$$\begin{aligned}
(0,1) \notin H\mathcal{G}_p^{\pi}(P) &\iff (0,1) \notin H\Gamma_p^{\pi}(X^{-1}PX) \text{ for some } X \in \mathcal{S}.\\
&\iff X^{-1}A_mX \text{ is } B_p^{\pi}SDD \text{ for some } X \in \mathcal{S}.\\
&\iff \langle X^{-1}A_mX\rangle_p^{\pi} \text{ is } SDD \text{ for some } X \in \mathcal{S}.\\
&\iff \text{for some } [x_1 \cdots x_\ell]^t \in \mathbb{R}_+^{\ell},\\
&\qquad \left\|(A_m)_{k,k}^{-1}\right\|_p^{-1} > \sum_{j\neq k} \frac{x_j}{x_k} \left\|(A_m)_{k,j}\right\|_p, \forall\ k \in \{1, \ldots, \ell\}.\\
&\iff \langle A_m\rangle_p^{\pi}X \text{ is } SDD \text{ for some } X = \operatorname{diag}(x_1, \ldots, x_\ell), \text{ where } x_i \in \mathbb{R}_+.\\
&\iff \langle A_m\rangle_p^{\pi} \text{ is a non-singular } \mathbb{M}\text{-matrix}.\\
&\iff A_m \text{ is a } B_p^{\pi}\mathbb{H}\text{-matrix}.
\end{aligned}$$

The proof of (ii) follows by arguing as in the proof of (i).

(iii) Suppose $(A_m)_{k,k} = 0$ and $(A_0c^{m-1} + A_1c^{m-2}s + \cdots + A_{m-1}s^{m-1})_{k,k}$ is singular. Then $P_{k,k}(c, s)$ is also singular as it equals $c\left(\sum_{i=0}^{m-1} A_ic^{m-1-i}s^i\right)_{k,k}$ which is a singular matrix polynomial. Hence, $H\Gamma_{p,k}^{\pi,x}(P)$ as defined in (3.1.5) is equal to $\mathbb{S}$.

Conversely, assume that $H\Gamma_{p,k}^{\pi,x}(P) = \mathbb{S}$ for any $x \in \mathbb{R}_+^{\ell}$ and $(A_m)_{k,k} = 0$. Suppose $(A_0c^{m-1} + A_1c^{m-2}s + \ldots + A_{m-1}s^{m-1})_{k,k}$ is regular. Then there exists an element $(c_0, s_0) \in \mathbb{S}$, $c_0 \neq 0$ such that

$$r := |c_0| \left\|(A_0c_0^{m-1} + A_1c_0^{m-2}s_0 + \ldots + A_{m-1}s_0^{m-1})_{k,k}^{-1}\right\|_p^{-1} > 0.$$

Let $t = \sum_{j\neq k} \left\|(A_0c_0^m + A_1c_0^{m-1}s_0 + \ldots + A_{m-1}s_0^{m-1}c_0 + A_ms_0^m)_{k,j}\right\|_p$. Then $t \geq 0$.
Now there exists $n \in \mathbb{N}$ such that $nr > t \implies \frac{t}{n} < r$.
Thus, choosing $\hat{x} = [\hat{x}_1 \cdots \hat{x}_\ell]^t \in \mathbb{R}_+^{\ell}$ such that $\hat{x}_k = n$, $\hat{x}_j = 1$ for $j \neq k$, we get $(c_0, s_0) \notin H\Gamma_{p,k}^{\pi,\hat{x}}$, a contradiction.

(iv) Clearly, if there exists $k \in \{1, 2, \cdots, \ell\}$ such that $P_{k,k}(c, s)$ is singular, then by definition of $H\Gamma_{p,k}^{\pi,x}(P)$ in 3.1.5, we have $H\Gamma_{p,k}^{\pi,x}(P) = \mathbb{S}$ for every $x \in \mathbb{R}_+^{\ell}$ so that $H\mathcal{G}_p^{\pi}(P) = \mathbb{S}$.

(v) The proof follows by arguing as in the proof of Theorem 3.1.2(vii). □

However, the converse of the above Theorem 3.1.5(iv) is not true i.e., $H\mathcal{G}_p^\pi(P) = \mathbb{S}$ does not necessarily imply that one of $P_{k,k}(c,s)$, has to be singular. For example, consider the matrix polynomial $P(c,s) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} p(c,s)$ where $p(c,s) = c^2 + 2cs + 3s^2$ for $(c,s) \in \mathbb{S}$. Then the block minimal Geršgorin set with respect to the partition $\pi = \{0, 1, 2\}$ is given by $H\mathcal{G}_p^\pi(P) = \{(c,s) \in \mathbb{S} : |p(c,s)|^2 \leq 2|p(c,s)|^2\} = \mathbb{S}$, but each $p_{i,i}(c,s)$ is regular.

Next we define the block Brualdi sets for homogeneous matrix polynomials.

Definition 3.1.6. *Let* $P(c,s) := \sum_{i=0}^{m} A_i s^i c^{m-i}$ *of size* $n \times n$ *be partitioned with respect to the partition* $\pi = \{n_j\}_{j=0}^{\ell}$ *of* $\{1, \ldots, n\}$.

Given a norm $\|\cdot\|_p$, *let* C_π *denote the cycle set* $\mathscr{C}(\langle P(c,s)\rangle_p^\pi)$ *of the comparison matrix* $\langle P(c,s)\rangle_p^\pi$ *as in Definition 1.3.10. For a cycle* $\gamma \in C_\pi$, *define*

$$H\mathcal{B}_{\gamma,p}^\pi(P) = \left\{ (c,s) \in \mathbb{S} : \prod_{j\in\gamma} \left\| (P_{j,j}(c,s))^{-1} \right\|_p^{-1} \leq \prod_{j\in\gamma} r_j^\gamma(\langle P(c,s)\rangle_p^\pi) \right\}, \tag{3.1.7}$$

where $r_j^\gamma(\langle P(c,s)\rangle_p^\pi) := \sum_{\substack{k=1 \\ k\neq j}}^{\ell} \|P_{j,k}(c,s)\|_p$ *if* γ *is a strong cycle for vertex* j *in* $\langle P(c,s)\rangle_p^\pi$, *and by* $r_j^\gamma(\langle P(c,s)\rangle_p^\pi) = 0$ *if* γ *is a weak cycle. Then, the block Brualdi set is defined as*

$$H\mathcal{B}_p^\pi(P) = \bigcup_{\gamma\in C_\pi} H\mathcal{B}_{\gamma,p}^\pi(P). \tag{3.1.8}$$

Next we have the following theorem which is an adaptation of the proof of Theorem 1.3.12 in [76].

Theorem 3.1.7. *Suppose a homogeneous matrix polynomial* $P(c,s)$ *as in (3.1.1) is partitioned with respect to a partition* π *as in (3.1.2) and* $\|\cdot\|_p$ *be any induced matrix p-norm. Then for any eigenvalue* $(c,s) \in \Lambda(P)$, *there is a cycle* $\gamma \in C_\pi$ *such that* $(c,s) \in H\mathcal{B}_{\gamma,p}^\pi(P)$. *Consequently,* $\Lambda(P) \subseteq H\mathcal{B}_p^\pi(P)$.

Proof. If $\pi = \{0, n\}$, then $\langle P(c,s)\rangle_p^\pi = \|(P(c,s))^{-1}\|_p^{-1}$. So C_π consists of the weak cycle $\gamma = (1)$ and $H\mathcal{B}_p^\pi(P) = \left\{(c,s) \in \mathbb{S} : \|(P(c,s))^{-1}\|_p^{-1} \leq 0\right\} = \Lambda(P)$. Assume that $\pi = \{n_j\}_{j=0}^{\ell}$, $\ell \geq 2$ and $(c,s) \in \Lambda(P)$ such that $(c,s) \in \Lambda(P_{k,k})$ for some $k \in \{1, 2, \ldots, \ell\}$. Now there always exists a strong or weak cycle in C_π through vertex k. If γ is a weak cycle through k, then $H\mathcal{B}_{\gamma,p}^\pi(P) = \Lambda(P_{k,k})$. If γ is a strong cycle through vertex k, then $\|(P_{k,k}(c,s))^{-1}\|_p^{-1} = 0 \implies (c,s) \in H\mathcal{B}_{\gamma,p}^\pi(P)$.
Next we assume $(c,s) \in \Lambda(P)$ such that $(c,s) \notin \Lambda(P_{k,k})$, for any $k \in \{1, 2, \ldots, \ell\}$. So

there exists $x \in \mathbb{C}^n$, $x \neq 0$ with

$$P(c,s)\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \vdots\\ \hat{x}_\ell\end{bmatrix}=0$$

where $\hat{x}_k$ are vectors of length equal to the sizes of the diagonal block $P_{k,k}(c,s)$ for each $k=1,2,\ldots,\ell$, that constitute a partition of x conformal to the blocks of $[P_{k,j}(c,s)]_{\ell\times\ell}$. Assume $\hat{x}_j \neq 0$, then

$$\begin{aligned} P_{j,j}(c,s)\hat{x}_j = -\sum_{\substack{k=1\\k\neq j}}^{\ell} P_{j,k}(c,s)\hat{x}_k &\implies \hat{x}_j = -P_{j,j}(c,s)^{-1}\sum_{\substack{k=1\\k\neq j}}^{\ell} P_{j,k}(c,s)\hat{x}_k\\ &\implies \left\|(P_{j,j}(c,s))^{-1}\right\|_p^{-1}\|\hat{x}_j\|_p \leq \sum_{\substack{k=1\\k\neq j}}^{\ell}\|P_{j,k}(c,s)\hat{x}_k\|_p. \end{aligned}$$

Since $\|(P_{j,j}(c,s))^{-1}\|_p^{-1}\|\hat{x}_j\|_p \neq 0$ so there exists $r\neq j$ such that $P_{j,r}(c,s)\hat{x}_r \neq 0$. Let

$$\|\hat{x}_r\|_p = \max\left\{\|\hat{x}_k\|_p : k\in\{1,2,\ldots,\ell\}\setminus\{j\} \text{ and } P_{j,k}(c,s)\hat{x}_k\neq 0\right\}.$$

Thus $\|\hat{x}_r\|_p > 0$ and

$$\left\|(P_{j,j}(c,s))^{-1}\right\|_p^{-1}\|\hat{x}_j\|_p \leq \sum_{\substack{k=1\\k\neq j}}^{\ell}\|P_{j,k}(c,s)\|_p\|\hat{x}_k\|_p \leq r_j(\langle P(c,s)\rangle_p^{\pi})\|\hat{x}_r\|_p.$$

Setting $i_1=j$, $i_2=r$, and repeating the above process for i_2, we get

$$\left\|(P_{i_2,i_2}(c,s))^{-1}\right\|_p^{-1}\|\hat{x}_{i_2}\|_p \leq r_{i_2}(\langle P(c,s)\rangle_p^{\pi})\|\hat{x}_{i_3}\|_p$$

where $\|\hat{x}_{i_3}\|_p = \max\{\|\hat{x}_k\|_p : k\in\{1,\ldots,\ell\}\setminus\{i_2\}$ and $P_{i_2,k}(c,s)\hat{x}_k\neq 0\}$.
If $i_3=i_1$, we have a cycle $\gamma=(i_1\ i_2)\in C_\pi$. If $i_3\neq i_1$, this process will continue, but eventually terminate as $\{1,\ldots,\ell\}$ is finite when we find an i_{p+1} which equals some i_q, $q\in\{1,\ldots,\ell\}$ obtained earlier. Therefore, we get a sequence $\{i_j\}_{j=q}^{p}$ of distinct integers in $\{1,2,\ldots,\ell\}$ such that $i_{p+1}=i_q$ and the entries $P_{i_q,i_{q+1}}(c,s), P_{i_{q+1},i_{q+2}}(c,s)$, $\ldots, P_{i_p,i_{p+1}}(c,s)$ are all non-zero. Then $\gamma=(i_1,i_2,\ldots,i_p)\in C_\pi$ is a strong cycle such that

$$\left\|(P_{i_j,i_j}(c,s))^{-1}\right\|_p^{-1}\left\|\hat{x}_{i_j}\right\|_p \leq r_{i_j}^{\gamma}(\langle P(c,s)\rangle_p^{\pi})\left\|\hat{x}_{i_{j+1}}\right\|_p, \text{ for } j=q,\ldots,p.$$

Taking products of the above, since $\prod_{j=q}^{p}\left\|\hat{x}_{i_j}\right\|_p = \prod_{j=q}^{p}\left\|\hat{x}_{i_{j+1}}\right\|_p > 0$,

$$\prod_{j\in\gamma}\left\|(P_{j,j}(c,s))^{-1}\right\|_p^{-1} \leq \prod_{j\in\gamma} r_j^{\gamma}(\langle P(c,s)\rangle_p^{\pi}) \implies (c,s)\in H\mathcal{B}_{\gamma,p}^{\pi}(P).$$

□

Below, we derive a few properties for the block Brualdi sets.

Theorem 3.1.8. *Let $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ be a $n \times n$ matrix polynomial partitioned with respect to the partition π of $\{1,\ldots,n\}$ as in (3.1.2). Suppose $\|\cdot\|_p$ be any induced matrix p-norm. Then the following results hold for the Brualdi sets in (3.1.8).*

(i) $(0,1) \in H\mathcal{B}_p^\pi(P)$ *if and only if there exists a cycle* $\gamma \in \mathscr{C}(\langle A_m\rangle_p^\pi)$ *such that* $\prod_{j\in\gamma} \left\|(A_m)_{j,j}^{-1}\right\|_p^{-1} \leq \prod_{j\in\gamma} r_j^\gamma(\langle A_m\rangle_p^\pi)$.

(ii) $(1,0) \in H\mathcal{B}_p^\pi(P)$ *if and only if there exists a cycle* $\gamma \in \mathscr{C}(\langle A_0\rangle_p^\pi)$ *such that* $\prod_{j\in\gamma} \left\|(A_0)_{j,j}^{-1}\right\|_p^{-1} \leq \prod_{j\in\gamma} r_j^\gamma(\langle A_0\rangle_p^\pi)$.

(iii) If for some $k \in \{1,\ldots,\ell\}$, $(A_m)_{k,k} = 0$ *and* $\left(\sum_{i=0}^{m-1} A_i c^{m-1-i} s^i\right)_{k,k}$ *is singular then* $H\mathcal{B}_p^\pi(P) = \mathbb{S}$.

(iv) If for some $k \in \{1,\ldots,\ell\}$, $P_{k,k}(c,s)$ *is singular then* $H\mathcal{B}_p^\pi(P) = \mathbb{S}$.

(v) If all the coefficient matrices have their entries real in the block rows j corresponding to $j \in \gamma$ in C_π, then $(c,s) \in H\mathcal{B}_{\gamma,p}^\pi(P)$ if and only if $(c,\bar{s}) \in H\mathcal{B}_{\gamma,p}^\pi(P)$ i.e., the set $H\mathcal{B}_{\gamma,p}^\pi(P)$ on the sphere is symmetric with respect to the xz-plane.

Proof. (i) We have,

$$\begin{aligned}
(0,1) \in H\mathcal{B}_p^\pi(P) \iff & \text{ there exists } \gamma \in \mathscr{C}(\langle P(0,1)\rangle_p^\pi) \text{ such that} \\
& \prod_{j\in\gamma} \left\|(P_{j,j}(0,1))^{-1}\right\|_p^{-1} \leq \prod_{j\in\gamma} r_j^\gamma(\langle P(0,1)\rangle_p^\pi) \\
\iff & \text{ there exists } \gamma \in \mathscr{C}(\langle A_m\rangle_p^\pi) \text{ such that} \\
& \prod_{j\in\gamma} \left\|(A_m)_{j,j}^{-1}\right\|_p^{-1} \leq \prod_{j\in\gamma} r_j^\gamma(\langle A_m\rangle_p^\pi).
\end{aligned}$$

(ii) The proof follows from (i) by using the fact that $(1,0) \in H\mathcal{B}_p^\pi(P)$ if and only if $(0,1) \in H\mathcal{B}_p^\pi(\mathrm{rev}P)$.

(iii) If γ is any cycle(strong or weak) corresponding to vertex k, then the given condition implies $\left\|(A_0c^{m-1} + A_1c^{m-2}s + \cdots + A_{m-1}s^{m-1})_{k,k}^{-1}\right\|_p^{-1} = 0$, so that $H\mathcal{B}_{\gamma,p}^\pi(P) = \mathbb{S}$.

(iv) Suppose $P_{k,k}(c,s)$ is a singular matrix polynomial. Then $\|(P_{k,k}(c,s))^{-1}\|_p^{-1} = 0$ for all $(c,s) \in \mathbb{S}$. Now for each $k \in \{1,2,\ldots,\ell\}$ there is a cycle γ in C_π which passes through k. Consequently $H\mathcal{B}_{\gamma,p}^\pi(P) = \mathbb{S} \implies H\mathcal{B}_p^\pi(P) = \mathbb{S}$.

(v) The proof follows by arguing in a similar way as in the proof of Theorem 3.1.2(vii).

□

Observe that if the partition $\pi = \{1, 2, \ldots, n\}$, then we get the pointwise Brualdi sets for the homogeneous matrix polynomials. The block minimal Brualdi sets for matrix polynomials may also be defined. In fact they may be seen as an alternative definition of the block minimal Geršgorin sets as the two sets can be established to be equal.

To see this consider a $n \times n$ matrix polynomial $P(c, s)$ as in (3.1.1), partitioned with respect to a partition π of the set $\{1, \ldots, n\}$ as in (3.1.2). Suppose $\|\cdot\|_p$ is any induced matrix norm and define the set $\mathcal{S}$ as in Theorem 3.1.4. Then the block minimal Brualdi set for the homogeneous matrix polynomial $P(c, s)$, denoted by $H\beta_p^\pi(P)$, is defined as

$$\begin{aligned} H\beta_p^\pi(P) &= \bigcap_{X \in \mathcal{S}} H\mathcal{B}_p^\pi(X^{-1}PX) \\ &= \bigcap_{X \in \mathcal{S}} \bigcup_{\gamma \in C_\pi} H\mathcal{B}_{\gamma,p}^{\pi,x}(P) \end{aligned}$$

where C_π is the cycle set $\mathscr{C}(\langle P(c, s)\rangle_p^\pi)$ and

$$H\mathcal{B}_{\gamma,p}^{\pi,x}(P) = \left\{(c, s) \in \mathbb{S} : \prod_{j \in \gamma} \left\|(P_{j,j}(c, s))^{-1}\right\|_p^{-1} \leq \prod_{j \in \gamma} \hat{r}_j^\gamma(\langle P(c, s)\rangle_p^\pi)\right\}$$

$$\text{where } \hat{r}_j^\gamma(\langle P(c, s)\rangle_p^\pi) := \begin{cases} \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{\ell} \frac{x_k}{x_j} \|P_{j,k}(c, s)\|_p & \text{if } \gamma \text{ is a strong cycle,} \\ 0 & \text{if } \gamma \text{ is a weak cycle.} \end{cases}$$

The equality of the block minimal Brualdi and block minimal Geršgorin sets of an $n \times n$ partitioned matrix $[A_{i,j}]_{\ell \times \ell}$ is well known and is a consequence of the fact that $\langle A\rangle_p^\pi = [\mu_{k,j}] \in \mathbb{R}^{\ell,\ell}$(as defined in 1.3.2) is a non-singular $\mathbb{M}$-matrix if and only if there exists a diagonal matrix X with positive diagonal entries such that $X^{-1}\langle A\rangle_p^\pi X$ satisfies

$$\prod_{i \in \gamma} \mu_{i,i} > \prod_{i \in \gamma} \hat{r}_i^\gamma(\langle A\rangle_p^\pi) \quad \forall\, \gamma \in \mathscr{C}(\langle A\rangle_p^\pi),$$

$$\text{where } \hat{r}_i^\gamma(\langle A\rangle_p^\pi) := \begin{cases} \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_j}{x_i} |\mu_{i,j}| & \text{if } \gamma \text{ is a strong cycle,} \\ 0 & \text{if } \gamma \text{ is a weak cycle.} \end{cases}$$

A proof of this follows from [10, Theorem 3.3].

Likewise the equality $H\beta_p^\pi(P) = H\mathcal{G}_p^\pi(P)$ is a simple consequence of applying this result to every partitioned matrix $[P_{j,k}(c,s)]_{\ell\times\ell}$, $(c,s)\in\mathbb{S}$. In fact as the block Brualdi and block minimal Brualdi sets are identical when the partition $\pi=\{0,n_1,n\}$ with $n_1<n$, in these cases, the block minimal Geršgorin sets and the block Brualdi sets are equal.

The block Brauer sets for the matrix polynomial $P(c,s)$ partitioned as in (3.1.2) with respect to the any induced matrix p-norm is defined as follows.
For each $(c,s)\in\mathbb{S}$, consider the set of all two length cycles $(i\ j)$, $i,j\in\{1,2,\ldots,\ell\}$ and $i\neq j$ of $\langle P(c,s)\rangle_p^\pi$, regardless of whether the $(i,j)^{th}$ entry in the comparison matrix $\langle P(c,s)\rangle_p^\pi$ is zero or not. Then the set

$$H\mathcal{K}_p^\pi(P) = \bigcup_{\substack{i,j=1\\ i\neq j}}^{\ell} H\mathcal{K}_{p,i,j}^\pi(P) \tag{3.1.9}$$

where

$$H\mathcal{K}_{p,i,j}^\pi(P) = \left\{(c,s)\in\mathbb{S} : \prod_{\substack{k=i,j\\ i\neq j}} \left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} \leq \prod_{\substack{k=i,j\\ i\neq j}} r_k^\gamma\left(\langle P(c,s)\rangle_p^\pi\right)\right\} \tag{3.1.10}$$

is called the block Brauer set for the matrix poynomial $P(c,s)$. Clearly, the block Brauer set contains all the eigenvalues of the polynomial $P(c,s)$ and properties analogous to those of the block Brualdi sets as in Theorem 3.1.8 also hold for the block Brauer sets.

Now the relations

$$H\mathcal{G}_p^\pi(P) \subseteq H\mathcal{B}_p^\pi(P) \subseteq H\mathcal{K}_p^\pi(P) \subseteq H\Gamma_p^\pi(P)$$

are a simple consequence of applying Theorem 1.3.24 to the partitioned matrix polynomial $[P_{j,k}(c,s)]_{\ell\times\ell}$ for each $(c,s)\in\mathbb{S}$.

Morever as in the case of matrices, [76, pp.55], it can be shown that if all the off-diagonal entries of the comparison matrix $\langle P(c,s)\rangle_p^\pi$ are non-zero for every $(c,s)\in\mathbb{S}$, then the block Brualdi set is identical to the block Brauer set. This is stated below along with a proof as it has important implications for plotting block Brualdi sets.

Lemma 3.1.9. *Let $P(c,s)$ be an $n\times n$ homogeneous matrix polynomial, and $\|\cdot\|_p$ be any induced matrix norm. Let π be a partition of the set $\{1,2,\ldots,n\}$ such that all the off-diagonal entries of the comparison matrix $\langle P(c,s)\rangle_p^\pi$ are non-zero. For such a partition π, the block Brualdi set $H\mathcal{B}_p^\pi(P)$ equals the set $H\mathcal{K}_p^\pi(P)$.*

Proof. Since each set of the union in $H\mathcal{K}_p^\pi(P)$ corresponds to a 2-cycle, it is contained in $H\mathcal{B}_p^\pi(P)$ as none of the off-diagonal entries is zero. For the other inclusion, let $(c,s) \in H\mathcal{B}_p^\pi(P)$. Then there exists $\gamma \in \mathscr{C}(\langle P(c,s)\rangle_p^\pi)$ such that

$$\prod_{j\in\gamma} \left\|(P_{j,j}(c,s))^{-1}\right\|_p^{-1} \leq \prod_{j\in\gamma} r_j^\gamma(\langle P(c,s)\rangle_p^\pi). \tag{3.1.11}$$

If γ is a weak cycle, then $r_j^\gamma(\langle P(c,s)\rangle_p^\pi) = 0$ for some $j \in \{1,2,\ldots,\ell\}$ and thus $(c,s) \in H\mathcal{K}_{p,i,j}^\pi(P)$. Therefore, $(c,s) \in H\mathcal{K}_p^\pi(P)$. Next, assume that γ is a strong cycle, say $\gamma = (i_1\ i_2 \ldots i_q)$ with $i_{q+1} = i_1$. Then from (3.1.11),

$$\begin{aligned}\left\|(P_{i_1,i_1}(c,s))^{-1}\right\|_p^{-1} \left\|(P_{i_2,i_2}(c,s))^{-1}\right\|_p^{-1} \ldots \left\|(P_{i_q,i_q}(c,s))^{-1}\right\|_p^{-1} &\leq \\ r_{i_1}(\langle P(c,s)\rangle_p^\pi) r_{i_2}(\langle P(c,s)\rangle_p^\pi) \ldots r_{i_q}(\langle P(c,s)\rangle_p^\pi).\end{aligned}$$

Squaring both sides and rearranging the terms, we get

$$\begin{aligned}&\frac{\|(P_{i_1,i_1}(c,s))^{-1}\|_p^{-1} \|(P_{i_2,i_2}(c,s))^{-1}\|_p^{-1}}{r_{i_1}(\langle P(c,s)\rangle_p^\pi) r_{i_2}(\langle P(c,s)\rangle_p^\pi)} \frac{\|(P_{i_2,i_2}(c,s))^{-1}\|_p^{-1} \|(P_{i_3,i_3}(c,s))^{-1}\|_p^{-1}}{r_{i_2}(\langle P(c,s)\rangle_p^\pi) r_{i_3}(\langle P(c,s)\rangle_p^\pi)} \cdots \\ &\qquad \frac{\left\|(P_{i_q,i_q}(c,s))^{-1}\right\|_p^{-1} \|(P_{i_1,i_1}(c,s))^{-1}\|_p^{-1}}{r_{i_q}(\langle P(c,s)\rangle_p^\pi) r_{i_1}(\langle P(c,s)\rangle_p^\pi)} \leq 1\end{aligned}$$

Therefore, there exists an $t \in \{1,2,\ldots,q\}$ such that

$$\frac{\|(P_{i_t,i_t}(c,s))^{-1}\|_p^{-1} \left\|(P_{i_{t+1},i_{t+1}}(c,s))^{-1}\right\|_p^{-1}}{r_{i_t}(\langle P(c,s)\rangle_p^\pi) r_{i_{t+1}}(\langle P(c,s)\rangle_p^\pi)} \leq 1.$$

Hence, the result follows. □

Remark 3.1.10. *From Lemma 3.1.9 it follows that if the partition $\pi = \{0, n_1, n\}$ with $0 < n_1 < n$ be such that all the off-diagonal entries of $\langle P(c,s)\rangle_p^\pi$ are non-zero for every $(c,s) \in \mathbb{S}$, then the block minimal Geršgorin and the block Brauer sets are equal.*

The next result provides information about the number of eigenvalues in each disjoint part of the localization sets.

Theorem 3.1.11. *Let $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ where $A_i \in \mathbb{C}^{n,n}$ be partitioned with respect to the partition π as in(3.1.2). Let $S(P)$ be either a block Geršgorin set, block Brauer set, block Brualdi set or a block minimal Geršgorin set with respect to π and matrix p-norm. If there exists closed disjoint sets $U, V \subset \mathbb{C}^\infty$ such that*

$$S(P) = U \cup V$$

then the number of eigenvalues of the matrix polynomial $P(c, s)$ in U is equal to the sum of the number of eigenvalues of the polynomials $P_{k,k}(c, s) := \sum_{i=0}^{m}(A_i)_{k,k}s^i c^{m-i}$ on the diagonal of the partitioned $P(c, s)$ in U for each $k \in \{1, 2, \ldots, \ell\}$, counting multiplicities.

Proof. We prove the theorem for block Geršgorin sets as the proof for the other sets follow in an identical manner. Assume each A_i is partitioned conformally into $\ell \times \ell$ blocks with respect to π as in (1.3.1). For each $i \in \{0, 1, \ldots, m\}$, let

$$D^{A_i} = \operatorname{diag}((A_i)_{1,1}, \ldots, (A_i)_{\ell,\ell}) \text{ and } F^{A_i} = A_i - D^{A_i}.$$

Let $A_i(t) = D^{A_i} + tF^{A_i}$, so that $P(t)(c, s) = \sum_{i=0}^{m} D^{A_i}s^i c^{m-i} + t\left(\sum_{i=0}^{m} F^{A_i}s^i c^{m-i}\right)$ for $t \in [0, 1]$. First we show that $H\Gamma_p^\pi(P(t)) \subset H\Gamma_p^\pi(P)$, $\forall\ 0 \leq t \leq 1$.
Let, $t \in [0, 1)$ and $(c, s) \in H\Gamma_p^\pi(P(t))$. Then $\langle P(t)(c, s)\rangle_p^\pi$ is not a SDD matrix.
If possible, assume $(c, s) \notin H\Gamma_p^\pi(P)$. Then for all $k \in \{1, 2, \ldots, \ell\}$,

$$\left\|(P_{k,k}(t)(c, s))^{-1}\right\|_p^{-1} = \left\|(P_{k,k}(c, s))^{-1}\right\|_p^{-1} > \sum_{j\neq k} \|(P_{k,j}(c, s))\|_p \geq t\sum_{j\neq k} \|(P_{k,j}(c, s))\|_p .$$

But this clearly contradicts the fact that $\langle P(t)(c, s)\rangle_p^\pi$ is not SDD. Thus we have $(c, s) \in H\Gamma_p^\pi(P)$ and therefore, $H\Gamma_p^\pi(P(t)) \subset H\Gamma_p^\pi(P)$ for all $t \in [0, 1]$.
Since, $A_i(0) = D_i$, so

$$H\Gamma_p^\pi(P(0)) = \bigcup_{k=1}^{\ell}\{(c, s) \in \mathbb{S} : \left\|(P_{k,k}(c, s))^{-1}\right\|_p^{-1} = 0\} = \bigcup_{k=1}^{\ell} \Lambda((P_{k,k})).$$

Therefore,

$$H\Gamma_p^\pi(P(0)) = \bigcup_{k=1}^{\ell}\{(c, s) \in \mathbb{S} : (c, s) \in \Lambda((P_{k,k}))\} \text{ where } P_{k,k}(c, s) = \sum_{i=0}^{m}(A_i)_{k,k}s^i c^{m-i}.$$

Let $\lambda(0)$ be an eigenvalue of $P(0)(c, s)$ belonging to U. Then it is an eigenvalue of some $P_{k,k}(c, s)$, $1 \leq k \leq \ell$. Now the eigenvalue curves $\{\lambda(t) : t \in [0, 1]\}$ are continuous on the Riemann sphere which is the geometrical interpretation of $\mathbb{C}^\infty$. Therefore, as $\lambda(0) \in U$, and U and V are disjoint, we have $\lambda(1) \in U$. Consequently the number of eigenvalues of $P(c, s)$ in U equals the sum of the number of eigenvalues of $P_{k,k}(c, s)$ in U, counting multiplicities.

□

3.1.1 Approximating the block Geršgorin and block minimal Geršgorin sets via pseudospectra

Given $A \in \mathbb{C}^{n,n}$, from Theorem 1.3.20 it is clear that the block Geršgorin set $\Gamma_p^\pi(A)$ with respect to a partition π on $\{1, \ldots, n\}$ and the induced matrix norm $\|\cdot\|_p$ is the union of ε-pseudospectra levels of the diagonal blocks $A_{k,k}$ for $k = 1, \ldots, \ell$ of the partitioned matrix A. In fact, we have $\Gamma_p^\pi(A) = \bigcup_{k=1}^{\ell} \Lambda_{\varepsilon_k}(A_{k,k})$, where $\varepsilon_k = \sum_{\substack{j=1 \\ j\neq k}}^{\ell} \|A_{k,j}\|_p$ for $k = 1, \ldots, \ell$ and matrix p-norm. For a $n \times n$ matrix polynomial $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ partitioned as in (3.1.2) with respect to a partition π of $\{1, 2, \ldots, n\}$, we approximate the block minimal Geršgorin set $H\mathcal{G}_p^\pi(P)$ and block Geršgorin set $H\Gamma_p^\pi(P)$ via the pseudospectra levels of the diagonal blocks $P_{k,k}(c,s)$ in (3.1.2) with respect to the norm $\|\!|\cdot|\!\|_{p,\|\cdot\|_p}$ defined in (1.2.3).

If $(c,s) \in H\mathcal{G}_p^\pi(P)$, then for all $x = [x_1 \cdots x_\ell]^t \in \mathbb{R}_+^\ell$ there exists $k \in \{1, \ldots, \ell\}$ such that

$$\begin{aligned}
\left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} &\leq \sum_{\substack{j=1 \\ j\neq k}}^{\ell} \frac{x_j}{x_k} \|P_{k,j}(c,s)\|_p \\
&= \sum_{\substack{j=1 \\ j\neq k}}^{\ell} \frac{x_j}{x_k} \left\| \sum_{i=0}^{m} (A_i)_{k,j}\, s^i c^{m-i} \right\|_p \\
&\leq \sum_{\substack{j=1 \\ j\neq k}}^{\ell} \frac{x_j}{x_k} \left[\sum_{i=0}^{m} \left\|(A_i)_{k,j}\right\|_p |s|^i |c|^{m-i} \right] \\
&\leq \sum_{\substack{j=1 \\ j\neq k}}^{\ell} \frac{x_j}{x_k} \left\| \left[\|(A_0)_{k,j}\|_p \ \cdots \ \|(A_m)_{k,j}\|_p \right] \right\|_p \left\| \left[c^m \ c^{m-1}s \ \cdots \ s^m \right] \right\|_q \\
&= \sum_{\substack{j=1 \\ j\neq k}}^{\ell} \frac{x_j}{x_k} \|\!|P_{k,j}|\!\|_p \left\| \left[c^m \ c^{m-1}s \ \cdots \ s^m \right] \right\|_q .
\end{aligned}$$

Let $\eta_p(c,s,P_{k,k})$ denote the backward error of the point (c,s) as an approximate eigenvalue of $P_{k,k}(c,s)$ in homogeneous form with respect to the $\|\!|\cdot|\!\|_{p,\|\cdot\|_p}$. Then by (1.2.4),

$$\begin{aligned}
\eta_p(c,s,P_{k,k}) = \frac{\|(P_{k,k}(c,s))^{-1}\|_p^{-1}}{\|[c^m \ c^{m-1}s \ \cdots \ s^m]\|_q} &\implies \eta_p(c,s,P_{k,k}) \leq \sum_{\substack{j=1 \\ j\neq k}}^{\ell} \frac{x_j}{x_k} \|\!|P_{k,j}|\!\|_p \\
&\implies (c,s) \in \Lambda_{\varepsilon_k(x)}^{\|\!|\cdot|\!\|_{p,\|\cdot\|_p}}(P_{k,k}),
\end{aligned}$$

where

$$\varepsilon_k(x) := \sum_{\substack{j=1 \\ j\neq k}}^{\ell} \frac{x_j}{x_k} \|\!|P_{k,j}|\!\|_p. \tag{3.1.12}$$

Hence, $H\mathcal{G}_p^\pi(P) \subseteq \bigcap_{x\in\mathbb{R}_+^\ell} \bigcup_{k=1}^{\ell} \Lambda_{\varepsilon_k(x)}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k})$. Let,

$$\widehat{\varepsilon}_k(x) := \inf_{(c,s)\in\mathbb{S}} \sum_{\substack{j=1\\ j\neq k}}^{\ell} \frac{(x_j/x_k)\,\|P_{k,j}(c,s)\|_p}{\|[c^m\ c^{m-1}s\ \cdots\ s^m]\|_q}. \tag{3.1.13}$$

If $(c,s) \in \Lambda_{\widehat{\varepsilon}_k(x)}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k})$, then

$$\frac{\left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1}}{\|[c^m\ c^{m-1}s\ \cdots\ s^m]\|_q} \leq \widehat{\varepsilon}_k(x) \leq \sum_{\substack{j=1\\ j\neq k}}^{\ell} \frac{(x_j/x_k)\,\|P_{k,j}(c,s)\|_p}{\|[c^m\ c^{m-1}s\ \cdots\ s^m]\|_q}$$

$$\implies \left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} \leq \sum_{\substack{j=1\\ j\neq k}}^{\ell} \frac{x_j}{x_k}\,\|P_{k,j}(c,s)\|_p$$

$$\implies (c,s) \in H\Gamma_{p,k}^{\pi,x}(P).$$

Therefore, $\Lambda_{\widehat{\varepsilon}_k(x)}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k}) \subseteq H\Gamma_{p,k}^{\pi,x}(P)$ for each $k \in \{1,\ldots,\ell\}$, so that

$$\bigcup_{k=1}^{\ell} \Lambda_{\widehat{\varepsilon}_k(x)}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k}) \subseteq \bigcup_{k=1}^{\ell} H\Gamma_{p,k}^{\pi,x}(P) \ \text{ for all } x \in \mathbb{R}_+^\ell$$

$$\implies \bigcap_{x\in\mathbb{R}_+^\ell} \bigcup_{k=1}^{\ell} \Lambda_{\widehat{\varepsilon}_k(x)}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k}) \subseteq H\mathcal{G}_p^\pi(P).$$

Thus we have the following theorem.

Theorem 3.1.12. *Let $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ be an $n \times n$ matrix polynomial partitioned with respect to π as in (3.1.2). For any matrix p-norm $\|\cdot\|_p$, the associated block minimal Geršgorin set $H\mathcal{G}_p^\pi(P)$ satisfies*

$$\bigcap_{x\in\mathbb{R}_+^\ell} \bigcup_{k=1}^{\ell} \Lambda_{\widehat{\varepsilon}_k(x)}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k}) \subseteq H\mathcal{G}_p^\pi(P) \subseteq \bigcap_{x\in\mathbb{R}_+^\ell} \bigcup_{k=1}^{\ell} \Lambda_{\varepsilon_k(x)}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k}),$$

where $\varepsilon_k(x)$ and $\widehat{\varepsilon}_k(x)$ are given by (3.1.12) and (3.1.13) respectively. In particular, for $\varepsilon_k = \varepsilon_k(x_0)$ and $\widehat{\varepsilon}_k = \widehat{\varepsilon}_k(x_0)$ where $x_0 = [1 \cdots 1]^t$, the block Geršgorin set $H\Gamma_p^\pi(P)$ satisfies

$$\bigcup_{k=1}^{\ell} \Lambda_{\widehat{\varepsilon}_k}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k}) \subseteq H\Gamma_p^\pi(P) \subseteq \bigcup_{k=1}^{\ell} \Lambda_{\varepsilon_k}^{\|\cdot\|_{p},\|\cdot\|_p}(P_{k,k}).$$

3.1.2 Equivalent characterization of block minimal Geršgorin sets for matrix polynomials

The minimal Geršgorin set $H\mathcal{G}_p^\pi(P)$ is not in general easy to determine numerically. In this section we adopt the approach of Kostić et al. [38, 77] to characterize the block minimal Geršgorin sets for matrix polynomials. These characterizations form the basis for computing them numerically in Section 3.3. For given $(c, s) \in \mathbb{S}$, define

$$Q(c,s) = -\langle P(c,s)\rangle_p^{\pi}. \tag{3.1.14}$$

Let, $\gamma(c,s) = \max\limits_{1\leq k\leq \ell} \left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1}$ and

$$R(c,s) = Q(c,s) + \gamma(c,s)I_\ell. \tag{3.1.15}$$

Then $R(c,s) \geq 0$. By Perron-Frobenius Theorem [Theorem 1.2.2], $R(c,s)$ possesses a real, non-negative eigenvalue $\rho(R(c,s))$ and a corresponding eigenvector $\hat{x} \geq 0$ such that $R(c,s)\hat{x} = \rho(R(c,s))\hat{x}$.
Set,

$$\mu(c,s) = \rho(R(c,s)) - \gamma(c,s). \tag{3.1.16}$$

Then $Q(c,s)\hat{x} = \mu(c,s)\hat{x}$ i.e., $\mu(c,s)$ is an eigenvalue of $Q(c,s)$ with corresponding eigenvector $\hat{x}$. Further, $\mu(c,s)$ is also the eigenvalue with largest real part among all eigenvalues of $Q(c,s)$. Suppose $x = [x_1 \cdots x_\ell]^t \in \mathbb{R}_+^\ell$. Then by Theorem 1.2.2,

$$\begin{aligned}
\rho(R(c,s)) &= \inf_{x>0} \max_{k\in\{1,\ldots,\ell\}} \frac{(R(c,s)x)_k}{x_k} \\
\Longrightarrow \mu(c,s) &= \inf_{x>0} \max_{k\in\{1,\ldots,\ell\}} \sum_{j\neq k} \frac{x_j}{x_k} \left\|(P_{k,j}(c,s))\right\|_p - \left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} \\
\Longrightarrow \mu(c,s) &= \inf_{x>0} \max_{k\in\{1,\ldots,\ell\}} \frac{(Q(c,s)x)_k}{x_k}
\end{aligned}$$

where $\mu(c,s)$ being a continuous function of the entries of $Q(c,s)$ is in turn a continuous function of (c,s).

Theorem 3.1.13. *Let $P(c,s) = \sum\limits_{i=0}^{m} A_i s^i c^{m-i}$ be an $n\times n$ matrix polynomial partitioned with respect to a partition π of $\{1,\ldots,n\}$ and $\|\cdot\|_p$ be any matrix p-norm. For $(c,s)\in\mathbb{S}$, let $Q(c,s)$ be as defined in (3.1.14) and $\mu(c,s)$ be as in (3.1.16). Then*

$$(c,s) \in H\mathcal{G}_p^{\pi}(P) \iff \mu(c,s) \geq 0$$

Proof. We have $(c,s) \in H\mathcal{G}_p^{\pi}(P) \iff (c,s) \in H\Gamma_{p,k}^{\pi,x}(P)$ as defined in (3.1.5) for some $k = 1, 2, \ldots, \ell$ where k depends on x. Thus,

$$\begin{aligned}
&\sum_{j\neq k} \frac{x_j}{x_k} \left\|(P_{k,j}(c,s))\right\|_p - \left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} \geq 0 \\
\Longrightarrow\ &\inf_{x>0} \max_{k\in\{1,\ldots,\ell\}} \sum_{j\neq k} \frac{x_j}{x_k} \left\|(P_{k,j}(c,s))\right\|_p - \left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1} \geq 0 \\
\Longrightarrow\ &\mu(c,s) \geq 0.
\end{aligned}$$

Conversely, assume that $\mu(c,s) \geq 0$. Then for each $x \in \mathbb{R}^{\ell}_{+}$, $\exists\ k \in \{1,\dots,\ell\}$ (dependent on x) such that

$$\sum_{j\neq k} \frac{x_j}{x_k} \left\|(P(c,s))_{k,j}\right\|_p - \left\|(P(c,s))^{-1}_{k,k}\right\|_p^{-1} \geq 0.$$

But this implies that for each $x \in \mathbb{R}^{\ell}_{+}$, there exists a $k \in \{1,\dots,\ell\}$ such that $(c,s) \in H\Gamma^{\pi,x}_{p,k}(P) \subseteq \bigcup_{k=1}^{\ell} H\Gamma^{\pi,x}_{p,k}(P)$. Hence, $(c,s) \in \bigcap_{x\in\mathbb{R}^{\ell}_{+}} \bigcup_{k=1}^{\ell} H\Gamma^{\pi,x}_{p,k}(P) = H\mathcal{G}^{\pi}_{p}(P)$. $\square$

Theorem 3.1.14. *Let* $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ *be an* $n\times n$ *matrix polynomial partitioned with respect to a partition* π *of the set* $\{1,\dots,n\}$ *as in (3.1.2) and* $\|\cdot\|_p$ *be any matrix p-norm. Suppose* $(c,s)\in\mathbb{S}$ *such that* $P(c,s)$ *is irreducible, then its associated comparison matrix* $\langle P(c,s)\rangle^{\pi}_{p}$ *is also irreducible. Let* $Q(c,s)$ *and* $\mu(c,s)$ *be as defined in (3.1.14) and (3.1.16) respectively. Then for each* $x \in \mathbb{R}^{\ell}_{+}$, *either*

$$Q(c,s)x = \mu(c,s)x,$$

or,

$$\min_{k\in\{1,\dots,\ell\}} \frac{(Q(c,s)x)_k}{x_k} \leq \mu(c,s) \leq \max_{k\in\{1,\dots,\ell\}} \frac{(Q(c,s)x)_k}{x_k}.$$

Proof. If possible, assume $\pi = \{n_j\}_{j=0}^{\ell}$ is a partition of the set $\{1,2,\dots,n\}$ such that $\langle P(c,s)\rangle^{\pi}_{p} \in \mathbb{R}^{\ell,\ell}$ is reducible. Then $\exists$ a $\ell\times\ell$ permutation matrix $T = [t_{ij}]$ such that

$$[T\langle P(c,s)\rangle^{\pi}_{p} T^t]_{(1,\ell)} = 0.$$

The presence of any zero entry in the comparison matrix $\langle P(c,s)\rangle^{\pi}_{p}$ is due to the zero block matrices in $P(c,s)$, of size specified by the partition π.

Consider the permutation matrix $\widehat{T} = \left[T_{(i,j)}\right]_{\ell\times\ell}$ where each $\widehat{T}_{(i,j)}$ is a block matrix defined by

$$\widehat{T}_{(i,j)} = \begin{cases} I_{\ell_j\times\ell_j} & \text{if } t_{ij} = 1,\\ 0 & \text{elsewhere,}\end{cases}$$

where $\ell_j \times \ell_j$ is the size of the (j,j) diagonal block in $P(c,s)$ and the sizes of the zero blocks are such that $\widehat{T}$ has the same size as $P(c,s)$. As an illustration, suppose $P(c,s) \in \mathbb{C}^{4,4}$ and $\pi = \{0,1,3,4\}$ is a partition of the set $\{0,1,2,3,4\}$. Let the matrix polynomial $P(c,s)$ is given by

$$P(c,s) = \left[\begin{array}{c|cc|c} p_{1,1}(c,s) & p_{1,2}(c,s) & p_{1,3}(c,s) & p_{1,4}(c,s) \\ \hline p_{2,1}(c,s) & p_{2,2}(c,s) & p_{2,3}(c,s) & p_{2,4}(c,s) \\ p_{3,1}(c,s) & p_{3,2}(c,s) & p_{3,3}(c,s) & p_{3,4}(c,s) \\ \hline p_{4,1}(c,s) & p_{4,2}(c,s) & p_{4,3}(c,s) & p_{4,4}(c,s) \end{array}\right] \in \mathbb{C}^{4,4},$$

$$\text{and } \langle P(c,s)\rangle_p^\pi = \left[\begin{array}{c|c|c} c_{1,1} & 0 & c_{1,3} \\ \hline c_{2,1} & c_{2,2} & c_{2,3} \\ \hline c_{3,1} & 0 & c_{3,3} \end{array}\right] \in \mathbb{R}^{3,3}$$

is the comparison matrix, then $p_{1,2}(c,s) = p_{1,3}(c,s) = p_{4,2}(c,s) = p_{4,3}(c,s) = 0$. Suppose $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is a permutation matrix such that

$$T\langle P(c,s)\rangle_p^\pi T^t = \left[\begin{array}{c|c|c} c_{1,1} & c_{1,3} & 0 \\ \hline c_{3,1} & c_{3,3} & 0 \\ \hline c_{2,1} & c_{2,3} & c_{2,2} \end{array}\right].$$

Define

$$\widehat{T} = \left[\begin{array}{c|c|c} I_{1\times 1} & 0 & 0 \\ \hline 0 & 0 & I_{1\times 1} \\ \hline 0 & I_{2\times 2} & 0 \end{array}\right] \in \mathbb{R}^{4,4}.$$

If the pre-multiplication of T interchanges rows i and j of the comparison matrix $\langle P(c,s)\rangle_p^\pi$, then the same block rows i and j of the blocked matrix $[P_{k,j}(c,s)]_{\ell\times\ell}$ are interchanged in $\widehat{T}P(c,s)$. Similarly, if post multiplication by T^t interchanges columns i and j in $T\langle P(c,s)\rangle_p^\pi$, then the same block columns i and j will be interchanged via post multiplication of $\widehat{T}P(c,s)$ by $\widehat{T}^t$. Therefore, we have $[\widehat{T}P(c,s)\widehat{T}^t]_{(1,\ell)} = 0$ and $P(c,s)$ is reducible.

Since $Q(c,s) = -\langle P(c,s)\rangle_p^\pi$ is irreducible, $R(c,s) = Q(c,s) + \gamma(c,s)I_\ell$, where $\gamma(c,s) = \max\limits_{1\le k\le \ell} \left\|(P_{k,k}(c,s))^{-1}\right\|_p^{-1}$ is non-negative and irreducible.

Therefore, by Perron-Frobenius Theorem [Theorem 1.2.3], there exists $x = [x_1 \cdots x_\ell]^t$ in $\mathbb{R}_+^\ell$ such that

$$R(c,s)x = \rho(R(c,s))x$$

and

$$\sup_{x>0} \min_{k\in\{1,\ldots,\ell\}} \frac{(R(c,s)x)_k}{x_k} = \rho(R(c,s)) = \inf_{x>0} \max_{k\in\{1,\ldots,\ell\}} \frac{(R(c,s)x)_k}{x_k}.$$

Therefore, for $\mu(c,s) = \rho(R(c,s)) - \gamma(c,s)$, we get for each $x = [x_1 \cdots x_\ell]^t \in \mathbb{R}_+^\ell$,

$$\begin{aligned} &\text{either, } Q(c,s)x = \mu(c,s)x \\ &\text{or, } \sup_{x>0} \min_{k\in\{1,\ldots,\ell\}} \frac{(Q(c,s)x)_k}{x_k} = \mu(c,s) = \inf_{x>0} \max_{k\in\{1,\ldots,\ell\}} \frac{(Q(c,s)x)_k}{x_k} \\ &\implies \min_{k\in\{1,\ldots,\ell\}} \frac{(Q(c,s)x)_k}{x_k} \le \mu(c,s) \le \max_{k\in\{1,\ldots,\ell\}} \frac{(Q(c,s)x)_k}{x_k}. \end{aligned}$$

□

3.2 Permuted pointwise minimal Geršgorin set for homogeneous matrix polynomials

Permuted minimal Geršgorin sets are introduced for matrices in [43]. In this section, we extend them to the matrix polynomials.

Let $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ be a $n \times n$ matrix polynomial of degree m and $p_{i,j}(c,s)$ denote the $(i,j)^{th}$ entry of $P(c,s)$. Assume $(c,s) \in \mathbb{S}$ and Φ denote the set of all $n!$ permutations on $\{1, 2, \cdots, n\}$. Suppose T_ϕ is the permutation matrix corresponding to a permutation $\phi \in \Phi$ and consider the matrix $X = \mathrm{diag}(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n,n}$ where $x = [x_1\, x_2\, \cdots\, x_n]^t \in \mathbb{R}^n_+$. If $M(c,s) := X^{-1}P(c,s)XT_\phi$ is singular then $M(c,s)$ is not SDD i.e., $\exists\ i \in \{1, 2, \ldots, n\}$ such that

$$|M_{i,i}(c,s)| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |M_{i,j}(c,s)|.$$

If $\phi(i) = i$: $\quad |p_{i,i}(c,s)| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_j}{x_i} |p_{i,j}(c,s)|,$

if $\phi(i) \neq i$: $\quad |p_{i,\phi(i)}(c,s)| \frac{x_{\phi(i)}}{x_i} \leq \sum_{\substack{j=1 \\ j \neq i,\phi(i)}}^{n} \frac{x_j}{x_i} |p_{i,j}(c,s)| + |p_{i,i}(c,s)|.$

Thus if $H\Gamma^x_\phi(P) := \{(c,s) \in \mathbb{S} : M(c,s) \text{ is not a } SDD \text{ matrix}\}$. Then

$$H\Gamma^x_\phi(P) = \bigcup_{i=1}^{n} H\Gamma^x_{i,\phi}(P),$$

where

$$H\Gamma^x_{i,\phi}(P) = \begin{cases} \left\{ (c,s) \in \mathbb{S} : |p_{i,i}(c,s)| \leq \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_j}{x_i} |p_{i,j}(c,s)| \right\} & \text{if } \phi(i) = i, \\ \left\{ (c,s) \in \mathbb{S} : |p_{i,i}(c,s)| \geq -\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_j}{x_i} |p_{i,j}(c,s)| + 2|p_{i,\phi(i)}(c,s)| \frac{x_{\phi(i)}}{x_i} \right\} & \text{if } \phi(i) \neq i. \end{cases}$$

Let $H\Gamma_\phi(P) := \bigcap_{x \in \mathbb{R}^n_+} H\Gamma^x_\phi(P)$. Then $H\Gamma_\phi(P)$ is called the minimal Geršgorin set relative to the permutation ϕ and $\bigcap_{\phi \in \Phi} H\Gamma_\phi(P)$ is called the permuted pointwise minimal Geršgorin set. Since $M(c,s)$ is a singular matrix if and only if $(c,s) \in \Lambda(P)$, clearly $\Lambda(P) \subseteq \bigcap_{\phi \in \Phi} H\Gamma_\phi(P)$.

Equivalent characterization of the set $\mathbf{H\Gamma_\phi(P)}$ Here we adapt the procedure in [43, 76] for deriving an equivalent criteria for a $(c,s) \in \mathbb{S}$ to be inside the set $H\Gamma_\phi(P)$.

For a given $(c,s) \in \mathbb{S}$ and permutation ϕ on the set $\{1, \ldots, n\}$, let T_ϕ denote the permutation matrix associated with ϕ. Define $Q_\phi(c,s) = \langle P(c,s)T_\phi \rangle$ and let $q_{i,j}(c,s)$ denote the $(i,j)^{th}$ entry of $Q_\phi(c,s)$. Then

$$q_{i,j}(c,s) = (-1)^{\delta_{i,j}} |p_{i,\phi(j)}(c,s)|. \tag{3.2.1}$$

Let,

$$\gamma_\phi(c,s) = \max_{1\le i\le n} |q_{i,i}(c,s)| \quad \text{and} \quad R_\phi(c,s) = Q_\phi(c,s) + \gamma_\phi(c,s)I. \tag{3.2.2}$$

Then $R_\phi(c,s) \ge 0$. By Perron-Frobenius Theorem [Theorem 1.2.2], it has a non-negative eigenvalue $\rho(R_\phi(c,s))$ and a corresponding eigenvector $\hat{x} \in \mathbb{R}^n$ with $\hat{x} \ge 0$. Set,

$$\mu_\phi(c,s) = \rho(R_\phi(c,s)) - \gamma_\phi(c,s). \tag{3.2.3}$$

Then $Q_\phi(c,s)x = \mu_\phi(c,s)x$, and $\mu_\phi(c,s)$ has the largest real part among all eigenvalues of $Q_\phi(c,s)$. Also, by Theorem 1.2.2,

$$\mu_\phi(c,s) = \inf_{\substack{x\in\mathbb{R}^n\\ x>0}} \max_{k\in\{1,\dots,n\}} \frac{(Q_\phi(c,s)x)_k}{x_k} \tag{3.2.4}$$

where $\mu_\phi(c,s)$ being a continuous function of entries of $Q_\phi(c,s)$ is in turn a continuous function of (c,s).
For each $(c,s) \in \mathbb{S}$ and $x = [x_1\, x_2 \cdots x_n]^t \in \mathbb{R}^n_+$, define the term $s^x_{i,\phi}(c,s)$ by

$$s^x_{i,\phi}(c,s) = \begin{cases} -|p_{i,i}(c,s)| + \sum\limits_{j\ne i} \dfrac{x_j}{x_i}|p_{i,j}(c,s)| & \text{if } \phi(i)=i, \\ |p_{i,i}(c,s)| + \sum\limits_{j\ne i} \dfrac{x_j}{x_i}|p_{i,j}(c,s)| - 2|p_{i,\phi(i)}(c,s)|\dfrac{x_{\phi(i)}}{x_i} & \text{if } \phi(i)\ne i. \end{cases} \tag{3.2.5}$$

Then the set $H\Gamma^x_{i,\phi}(P)$ can be rewritten as $H\Gamma^x_{i,\phi}(P) = \{(c,s)\in\mathbb{S} : s^x_{i,\phi}(c,s) \ge 0\}$.

Lemma 3.2.1. *Let $P(c,s) = \sum\limits_{i=0}^{m} A_i s^i c^{m-i}$ be a $n\times n$ matrix polynomial and ϕ denote a permutation on the set $\{1,2,\cdots,n\}$. Let $Q_\phi(c,s)$ and $s^x_{i,\phi}(c,s)$ be as given in (3.2.1) and (3.2.5) respectively. For any $x\in\mathbb{R}^n_+$, define $w\in\mathbb{R}^n_+$ by $w_i = x_{\phi(i)}$. Then*

$$s^x_{i,\phi}(c,s) = \frac{x_{\phi(i)}}{x_i}\frac{(Q_\phi(c,s)w)_i}{w_i} = \frac{(Q_\phi(c,s)w)_i}{x_i}.$$

Proof. By definition of $Q_\phi(c,s)$ and w, if $\phi(i)=i$, then

$$\frac{(Q_\phi(c,s)w)_i}{x_i} = \frac{-|p_{i,i}(c,s)|x_i + \sum\limits_{j\ne i}|p_{i,j}(c,s)|x_j}{x_i} = s^x_{i,\phi}(c,s)$$

and if $\phi(i)\ne i$, then

$$\begin{aligned}\frac{(Q_\phi(c,s)w)_i}{x_i} &= \frac{-|p_{i,\phi(i)}(c,s)|x_{\phi(i)} + \sum\limits_{j\ne\phi(i)}|p_{i,j}(c,s)|x_j}{x_i} \\ &= |p_{i,i}(c,s)| + \sum_{j\ne\phi(i)}\frac{x_j}{x_i}|p_{i,j}(c,s)| - |p_{i,\phi(i)}(c,s)|\frac{x_{\phi(i)}}{x_i} = s^x_{i,\phi}(c,s)\end{aligned}$$

for all $(c,s)\in\mathbb{S}$. ∎

Theorem 3.2.2. *Given a homogeneous matrix polynomial* $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ *where* $A_i \in \mathbb{C}^{n,n}$, *let* ϕ *denote a permutations on the set* $\{1, 2, \cdots, n\}$. *Then we have*

$$(c, s) \in H\Gamma_{\phi}(P) \iff \mu_{\phi}(c, s) \geq 0.$$

Proof. The proof follows from (3.2.4) by arguing as in the proof of Theorem 3.1.13. □

Trivial permutations for $\mathbf{P(c,s)}$ Given a homogeneous matrix polynomial $P(c,s)$, let ϕ be a permutation on the set $\{1,\ldots,n\}$ such that for each $x \in \mathbb{R}^n_+$, there is an $i \in N$ such that $H\Gamma^x_{i,\phi}(P) = \mathbb{S}$. Then $H\Gamma^x_{\phi}(P) = \mathbb{S}$, implying $H\Gamma_{\phi}(P) = \mathbb{S}$. Such permutations have no role in determining $\bigcap_{\phi\in\Phi} H\Gamma_{\phi}(P)$ and are called trivial permutations for the matrix polynomial $P(c,s)$.

Clearly, if $\phi \in \Phi$ is such that $p_{i,\phi(i)}(c,s) = 0$ for all $(c,s) \in \mathbb{S}$, then ϕ is a trivial permutation. Unlike the case of matrices, as stated in [76, pp.120], it is not true that if $\phi \in \Phi$ is a trivial permutation then there exists $i \in \{1, 2, \ldots, n\}$ such that $\phi(i) \neq i$ and $|p_{i,\phi(i)}(c,s)| = 0$ for all $(c,s) \in \mathbb{S}$. For example, consider the matrix polynomial $P(c,s) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} p(c,s)$ where $p(c,s)$ denote the scalar homogeneous polynomial $p(c,s) = c^3 + c^2 s + cs^2 + s^3$ for $(c,s) \in \mathbb{S}$. Then $\phi = (1\,2)$ is a trivial permutation for $P(c,s)$. This follows since for $x = [x_1\, x_2]^t \in \mathbb{R}^2_+$, the set $H\Gamma^x_{\phi}(P)$ is the union of sets

$$H\Gamma^x_{1,\phi}(P) = \left\{(c,s) \in \mathbb{S} : |p(c,s)| \leq 2|p(c,s)|\frac{x_1}{x_2}\right\}$$
$$\text{and} \quad H\Gamma^x_{2,\phi}(P) = \left\{(c,s) \in \mathbb{S} : |p(c,s)| \leq 3|p(c,s)|\frac{x_2}{x_1}\right\}.$$

Therefore, $H\Gamma^x_{1,\phi}(P) = \mathbb{S}$, if $x_1 \geq x_2$ and $H\Gamma^x_{2,\phi}(P) = \mathbb{S}$, otherwise.

Below we derive a few properties of the permuted pointwise minimal Geršgorin sets.

Theorem 3.2.3. *Let* $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ *be a* $n \times n$ *homogeneous matrix polynomial and* Φ *be the set of all permutations on the set* $\{1, 2, \ldots, n\}$. *Also let* $\mu_{\phi}(c,s)$ *be as defined in (3.2.3) for* $(c,s) \in \mathbb{S}$ *and* T_{ϕ} *be the corresponding permutation* $\phi \in \Phi$. *Then the following hold.*

(i) $(0,1) \in \bigcap_{\phi\in\Phi} H\Gamma_{\phi}(P)$ *if and only if the rightmost eigenvalue of* $-\langle A_m T_{\phi}\rangle$ *is non-negative for every non-trivial permutations* $\phi \in \Phi$.

(ii) $(1,0) \in \bigcap_{\phi\in\Phi} H\Gamma_\phi(P)$ *if and only if the rightmost eigenvalue of* $-\langle A_0 T_\phi\rangle$ *is non-negative for every non-trivial permutations* $\phi \in \Phi$.

Proof. The proof follows from the Theorem 3.2.2 and the fact that for each $(c,s) \in \mathbb{S}$, μ_ϕ is the rightmost eigenvalue of $Q_\phi(c,s)$ $(= \langle P(c,s)T_\phi\rangle)$. □

3.3 Plotting eigenvalue localization sets for matrix polynomials

The approach in the literature for plotting eigenvalue localization sets for matrix polynomials is to lay a grid on the complex plane and determine whether the grid points belong to the set. This assumes some prior knowledge of the location of the eigenvalues and eigenvalue bounds are typically used for this. But these bounds exist only when all the eigenvalues are finite or equivalently only when the leading coefficient matrix of the polynomial is non-singular. Also the plots of the localization sets on this grid provide information about the sets only on a particular section of the complex plane and is unsatisfactory in cases when the localization set is unbounded despite all eigenvalues being finite.

To address these issues we take a different approach of laying a grid on the Riemann sphere. This does not require any prior knowledge of the eigenvalue locations. We then determine whether the grid points belong to the localization set and this results in a plot on the Riemann sphere. This plot is then projected back onto the complex plane by using the fact that any point $(x,y,z) \neq (0,0,1)$ on the Riemann sphere corresponds to the point $w := \frac{x+iy}{1-z}$ on the complex plane, while the point $(0,0,1)$ corresponds to the point at ∞. This process may in fact be used to efficiently plot all eigenvalue localization sets for matrix polynomials.

Computing the block Geršgorin sets and block Brauer sets for the $n \times n$ matrix polynomial $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$ is much simpler as it only involves checking whether the homogeneous representation (c_i, s_j) of a grid point satisfies the corresponding relations.

Therefore, our focus is on the procedure for plotting the block minimal Geršgorin set $H\mathcal{G}_p^\pi(P)$ of $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$. But before that a few points about the method for plotting the pointwise minimal Geršgorin sets in the literature needs to be mentioned. This is because they provide the tightest possible Geršgorin-type localization of the eigenvalues of the matrix polynomial and also because as they are a special case of the block minimal Geršgorin sets, consequently there is a common approach for plotting both the sets. This is based on Theorem 3.1.13. Let $(c_i, s_j) \in \mathbb{S}$ be the homogeneous representation of a grid point on the Riemann sphere. If $\pi = \{0, 1, \ldots, n\}$, by Theorem 3.1.13, we have

$$(c_i, s_j) \in H\mathcal{G}_p^{\pi}(P) \iff \mu(c_i, s_j) \geq 0 \iff \rho(R(c_i, s_j)) \geq \gamma(c_i, s_j),$$

where $\gamma(c_i, s_j) = \max\limits_{1 \leq k \leq n} |p_{k,k}(c_i, s_j)|$ and $p_{k,k}(c, s)$ denotes the k^{th} diagonal entry of $P(c, s)$ for $k = 1, 2, \ldots, n$. But it is important to note that the grid is placed on $\mathbb{C}$ and the non-homogeneous form of the matrix polynomial is used in the literature to execute this check [37]. Another strategy is to use the concept of star shaped subsets of pointwise minimal Geršgorin sets to trace its boundary via a curve tracing algorithm [36, 38, 77]. This requires the very significant assumption that the leading coefficient A_m is a non-singular $\mathbb{H}$-matrix. In either case, the main work involves estimating $\rho(R(c_i, s_j))$ which is done by assuming that $P(c_i, s_j)$ is irreducible. The curve tracing algorithm [38] further requires finding eigenvalues at each step of the iteration and can be expensive for large values of n. Recently, an implicit algorithm for computing pointwise minimal Geršgorin sets for matrices has been proposed in [53] to replace the eigenvalue computations with LU decompositions in the iterations but this can still cost $O(n^3)$ flops per iteration.

We do not plot the block minimal Geršgorin set by tracing its boundary as we make no assumption about A_m. Instead we use a bisection type algorithm which is an adaptation of the algorithms proposed in [77] to approximate $\rho(R(c_i, s_j))$ where $R(c, s) \in \mathbb{R}^{\ell,\ell}$ corresponds to the partitioned matrix polynomial $P(c, s)$ as in (3.1.2), and determine whether the corresponding $\mu(c_i, s_j) \geq 0$. This algorithm works under the additional assumption that $\langle P(c_i, s_j) \rangle_p^{\pi}$ is irreducible, so that $R(c_i, s_j)$ is non-negative and irreducible. It is based on the following theorem.

Theorem 3.3.1. *[74] Let $A \geq 0$ be an $n \times n$ irreducible matrix, and let $x^{(0)} > 0$ be an arbitrary column vector in $\mathbb{R}^n$.*
Defining $x^{(t)} = Ax^{(t-1)} = A^t x^{(0)}$, $t \geq 1$. Let,

$$\underline{\lambda_t} := \min_{i \in \{1, \ldots, n\}} \frac{x_i^{(t+1)}}{x_i^{(t)}} \quad \text{and} \quad \overline{\lambda_t} := \max_{i \in \{1, \ldots, n\}} \frac{x_i^{(t+1)}}{x_i^{(t)}}$$

Then

$$\underline{\lambda_0} \leq \underline{\lambda_1} \leq \underline{\lambda_2} \leq \cdots \leq \rho(A) \leq \cdots \leq \overline{\lambda_2} \leq \overline{\lambda_1} \leq \overline{\lambda_0}.$$

Further, if A is primitive, then

$$\lim_{t \to \infty} \underline{\lambda_t} = \rho(A) = \lim_{t \to \infty} \overline{\lambda_t}.$$

Observe that if $\langle P(c_i, s_j)\rangle_p^\pi$ is irreducible, then $R(c_i, s_j)$ fails to be primitive only if all its diagonal entries are zero and it may be made primitive by adding a very small positive number to all of its diagonal entries [74]. So assuming without loss of generality that $R(c_i, s_j)$ is primitive, we have the following algorithm for determining whether $(c_i, s_j) \in H\mathcal{G}_p^\pi(P)$.

Step 1 Choose arbitary $x^{(0)}$ in $\mathbb{R}_+^\ell$.

Step 2 Compute $\gamma(c_i, s_j) = \max\limits_{1\le k\le \ell} \left\| (P_{k,k}(c_i, s_j))^{-1}\right\|_p^{-1}$.

Step 3 For $t = 1, 2, \ldots$ do

$$x^{(t)} = R(c_i, s_j)x^{(0)}$$

$$\underline{\lambda_{t-1}} = \min_{i\in\{1,\ldots,\ell\}} \frac{x_i^{(t)}}{x_i^{(t-1)}}$$

$$\overline{\lambda_{t-1}} = \max_{i\in\{1,\ldots,\ell\}} \frac{x_i^{(t)}}{x_i^{(t-1)}}$$

$$x^{(0)} = x^{(t)}$$

if $\underline{\lambda_t} > \gamma(c_i, s_j)$

set $(c_i, s_j) \in H\mathcal{G}_p^\pi(P)$ and exit.

else if $\overline{\lambda_t} < \gamma(c_i, s_j)$

set $(c_i, s_j) \in \mathbb{S} \setminus H\mathcal{G}_p^\pi(P)$ and exit.

else if $\overline{\lambda_t} - \underline{\lambda_t} < \varepsilon$ for given tolerance $\varepsilon > 0$

set $\rho(R(c_i, s_j)) = \frac{\overline{\lambda_t}+\underline{\lambda_t}}{2}$;

if $\gamma(c_i, s_j) \le \rho(R((c_i, s_j))$,

set $(c_i, s_j) \in H\mathcal{G}_p^\pi(P)$.

else

$(c_i, s_j) \in \mathbb{S} \setminus H\mathcal{G}_p^\pi(P)$.

end

end

Since the algorithm works if $\langle P(c_i, s_j)\rangle_p^\pi$ is irreducible at each grid point, we need a strategy for determining this and dealing with situations where the $\langle P(c_i, s_j)\rangle_p^\pi$ is reducible in an efficient manner. It is worth mentioning here that while plotting minimal Geršgorin sets $\mathcal{G}(A)$ for matrices $A \in \mathbb{C}^{n,n}$, the irreducibility of comparison matrix $\langle A - zI\rangle$ needs to be determined at each grid point $z \in \mathbb{C}$. Since moving from one grid point to another affects only the diagonal entries of A, this is equivalent to the irreducibilty of A. This may be assumed without loss of generality as if A is reducible, then localizing the eigenvalues of A is equivalent to localizing the eigenvalues of the irreducible matrices that form the diagonal blocks of a normal reduced form of A [38].

But changing the grid points affects off-diagonal entries of $\langle P(c_i, s_j)\rangle_p^\pi$. Therefore, we proceed as follows. Let R be the set of all points $(c, s) \in \mathbb{S}$ that are roots of the non-zero scalar polynomial entries of $P(c, s)$ that are not on the diagonal. Then R is a finite set with atmost $mn(n-1)$ elements. Consider $(c_0, s_0) \in \mathbb{S} \setminus R$. Since R is finite, in practice almost any arbitrary point in $\mathbb{S}$ will satisfy this and it is not necessary to compute R to determine (c_0, s_0). In fact, we may choose (c_0, s_0) to be one of the homogeneous representations (c_i, s_j) of the grid points on the Riemann sphere. Since the zero entries (if any) of $P(c_0, s_0)$ and $P(c, s)$ are exactly the same for all $(c, s) \in \mathbb{S} \setminus R$, this is also true of the zero entries of $\langle P(c_0, s_0)\rangle_p^\pi$ and $\langle P(c, s)\rangle_p^\pi$ for all $(c, s) \in \mathbb{S} \setminus R$. Therefore, we have the following two cases.

Case I $\langle P(c_0, s_0)\rangle_p^\pi$ is irreducible. In this case we form the comparison matrices $\langle P(c_i, s_j)\rangle_p^\pi$ corresponding to each grid point and check if any of them have zero entries at positions different from those of $\langle P(c_0, s_0)\rangle_p^\pi$. If this is the case, then $(c_i, s_j) \in R$. Note that very few grid points are likely to satisfy this and if this happens then we include them in the plot. At all other grid points, $\langle P(c_i, s_j)\rangle_p^\pi$ is irreducible and we use the algorithm stated earlier to plot $H\mathcal{G}_p^\pi(P)$ on the Riemann sphere and its projection on the complex plane.

Case II $\langle P(c_0, s_0)\rangle_p^\pi$ is reducible. In this case we find a permutation $T \in \mathbb{R}^{\ell,\ell}$ such that $T\langle P(c_0, s_0)\rangle_p^\pi T^t$ is in normal reduced form and construct the permutation $\widehat{T} \in \mathbb{R}^{n,n}$ as in the proof of Theorem 3.1.14. Then $\widehat{T}P(c_0, s_0)\widehat{T}^t$ has the same structure as a normal reduced form of $P(c_0, s_0)$ except that some of the blocks on the diagonal may be reducible matrices. For each such reducible block which is expected to be much smaller in size than n, a suitable permutation may be found that transforms it to its normal reduced form. This leads to a permutation $\widetilde{T} \in \mathbb{R}^{n,n}$ such that $\widetilde{T}P(c, s)\widetilde{T}^t$ is in normal reduced form for each $(c, s) \in \mathbb{S}$. The eigenvalue localization problem for $P(c, s)$ now breaks up into the equivalent problem of localizing the eigenvalues of the matrix polynomials on the diagonals of $\widetilde{T}P(c, s)\widetilde{T}^t$. These are much smaller in size than n and in fact if their size is 2×2, then we may use a minimal Geršgorin type localization without blocking to localize their eigenvalues. For all other matrix polynomials say $\widetilde{R}_{k,k}(c, s)$ that form the larger blocks on the diagonal of $\widetilde{T}P(c, s)\widetilde{T}^t$, $\widetilde{R}_{k,k}(c, s)$ is irreducible at a homogeneous representation (c_i, s_j) of a grid point if $(c_i, s_j) \in \mathbb{S} \setminus R$. By Theorem 3.1.14, the corresponding comparison matrix is also irreducible for any choice of partition π for each of these matrix polynomials. Therefore, we plot their block minimal Geršgorin sets by following the procedure in Case I.

Observe that in the above strategy we plot the block minimal Geršgorin set $H\mathcal{G}_p^\pi(P)$ together with possibly a few more grid points that belong to R. In the process we need to check the irreducibility of the comparison matrix $\langle P(c,s)\rangle_p^\pi$ and compute the normal reduced form of $P(c,s)$ if $\langle P(c,s)\rangle_p^\pi$ is reducible (via the process described in Case II) only at one point on the grid. This is done by applying Tarjan's algorithm [66] to $\langle P(c,s)\rangle_p^\pi \in \mathbb{R}^{\ell,\ell}$ which costs $O(\ell+e)$ flops, e being the number of non-zero off-diagonal entries of $\langle P(c,s)\rangle_p^\pi$. The cost of forming the matrix $P(c,s)$ at each grid point is atmost $O(mn^2)$. Computing the diagonal entries $\|(P_{k,k}(c_i,s_j))^{-1}\|_p^{-1}$ of the matrices $\langle P(c_i,s_j)\rangle_p^\pi$ may be costly. To reduce this cost, we initially check if $P(c_i,s_j)$ is an SDD matrix at each grid point. In such a case, (c_i,s_j) cannot be an eigenvalue of $P(c,s)$ and so we do not construct $\langle P(c_i,s_j)\rangle_p^\pi$ at such points. For all other points (c_i,s_j), we use the 1-norm to reduce the cost of computation and estimate $\|(P_{k,k}(c_i,s_j))^{-1}\|_1^{-1}$ by using the algorithms in [25,27]. If $|P(c_i,s_j)|$ is already computed and the sizes of the diagonal blocks $P_{k,k}(c_i,s_j)$ are $m_k\times m_k$, $k=1,\ldots,\ell$, then computing $\langle P(c_i,s_j)\rangle_p^\pi$ costs an additional $\sum_{k=1}^{\ell} O(m_k^3)+n^2$. Having formed $\langle P(c_i,s_j)\rangle_p^\pi$, the cost of implementing the algorithm for deciding if $\mu(c_i,s_j)\geq 0$ is another additional $O(\ell^2)$ flops per iteration. However, these costs will be much lesser if we are in Case II and/or $P(c,s)$ is a sparse matrix polynomial.

When plotting the pointwise minimal Geršgorin sets, Tarjan's algorithm for deciding whether the $n\times n$ matrix $P(c_i,s_j)$ is irreducible will cost more than the corresponding check for the block minimal Geršgorin set which involves the much smaller comparison matrix $\langle P(c_i,s_j)\rangle_p^\pi \in \mathbb{R}^{\ell,\ell}$. The algorithm for determining whether $\mu(c_i,s_j)\geq 0$ at each grid point can cost $2n^2$ flops at each iteration if $\langle P(c_i,s_j)\rangle$ is irreducible and dense. In contrast, the corresponding flop count per iteration for the block minimal Geršgorin sets will be much lower due to the smaller size of the comparison matrix. However, from the preceding discussion it is clear that the cost of computing $\langle P(c_i,s_j)\rangle_p^\pi$ exceeds that of computing $\langle P(c_i,s_j)\rangle$ by $\sum_{k=1}^{\ell} O(m_k^3)+n^2$ flops. But if $\langle P(c_i,s_j)\rangle$ is irreducible and the partition π is chosen in such a way that $\sum_{k=1}^{\ell} m_k^3 < n^2$, then the flop count for determining whether (c_i,s_j) belongs to $H\mathcal{G}_p^\pi(P)$ can be significantly lower than the flop count for determining whether it belongs to the pointwise minimal Geršgorin set. In particular, if $\pi=\{0,n_1,n\}$, $0<n_1<n$ so that $\langle P(c,s)\rangle_p^\pi \in \mathbb{R}^{2,2}$ with non-zero off-diagonal entries at each $(c,s)\in\mathbb{S}\setminus R$, then by Remark 3.1.10 and Lemma 3.1.9, plotting $H\mathcal{G}_p^\pi(P)$ involves simply checking whether the homogeneous representation (c_i,s_j) of each grid point belongs to the corresponding block Brauer set $H\mathcal{K}_p^\pi(P)$. In such a case, if the structures of the two diagonal blocks in the partitioned matrix polynomial are such that the diagonal entries of $\langle P(c_i,s_j)\rangle_p^\pi$ can be cheaply estimated via the algorithm in [25,27], (say, for

instance, they are upper or lower triangular) but the comparison matrix $\langle P(c_i, s_j)\rangle$ is irreducible, then the cost of determining whether $(c_i, s_j) \in H\mathcal{G}_p^{\pi}(P)$ may once again be cheaper than the cost of determining whether it belongs to the pointwise minimal Geršgorin set.

Due to Lemma 3.1.9, the block Brualdi and block Brauer sets coincide if the off-diagonal entries of $\langle P(c,s)\rangle_p^{\pi}$ are all non-zero. We use this fact to compute the block Brualdi sets if this condition is satisfied at all $(c,s) \in \mathbb{S} \setminus R$. If this does not hold for a partition π, then we have to compute the cycle set of $\langle P(c,s)\rangle_p^{\pi}$ for a $(c,s) \in \mathbb{S} \setminus R$. This may be done via Johnson's algorithm [32]. If $\langle P(c,s)\rangle_p^{\pi} \in \mathbb{R}^{\ell,\ell}$, e is the number of non-zero off-diagonal entries and c is the number of cycles in $\langle P(c,s)\rangle_p^{\pi}$, then this costs $O(\ell + e)(c+1)$ flops.

Next we come to the computation of the permuted pointwise minimal Geršgorin sets $\bigcap_{\phi \in \Phi} H\Gamma_{\phi}(P)$ of $P(c,s) = \sum_{i=0}^{m} A_i s^i c^{m-i}$. Let $\widehat{R}$ be the collection of all points $(c,s) \in \mathbb{S}$ that are not roots of any non-zero scalar polynomial entries of $P(c,s)$ and $(c_0, s_0) \in \mathbb{S} \setminus \widehat{R}$. To construct a class of permutations on $\{1, 2, \ldots, n\}$ that includes all non-trivial permutations of $P(c,s)$ we proceed as follows. We consider all the strong cycles of $P(c_0, s_0)$ and extend them if necessary to form permutations on $\{1, 2, \ldots, n\}$. We add the identity permutation to the resulting collection of permutations and remove from it any permutation ϕ such that $p_{i,\phi(i)}(c_0, s_0) = 0$. If Ψ be this set of permutations, then Ψ contains the class of non-trivial permutations on $P(c,s)$.

The permuted pointwise Geršgorin set may be constructed by determining whether the homogeneous representation (c_i, s_j) of each grid point belongs to $H\Gamma_{\phi}(P)$ for each $\phi \in \Psi$. By Theorem 3.2.2 this happens if $\mu_{\phi}(c_i, s_j) \geq 0$ for each $\phi \in \Psi$. In view of the relations (3.2.1)-(3.2.4), this can be achieved by using the same strategy and algorithm used to compute the block minimal Geršgorin sets. Note that computing the comparison matrix $\langle P(c_i, s_j)\rangle_p^{\pi}$ is much less flop count intensive in comparision to that for the block Geršgorin-type localizations. However, the process has to be repeated for each $\phi \in \Psi$ and will be efficient if there are only a few strong cycles in $P(c_0, s_0)$. The strong cycles of $P(c_0, s_0)$ are computed via Johnson's algorithm which costs $O(n + e)(c + 1)$ where e is the number of non-zero off-diagonal entries and c is the number of strong cycles of $P(c_0, s_0)$. If all the off-diagonal entries of $P(c_0, s_0)$ are non-zero then $c = \sum_{k=2}^{n} {}^{n}C_k (k-1)!$ which grows faster than 2^n for $n \geq 4$. Therefore, in practice, the permuted pointwise minimal Geršgorin sets may be computed efficiently for sparse matrix polynomials that have only a few strong cycles when evaluated at $(c,s) \in \mathbb{S} \setminus \widehat{R}$.

3.3.1 Numerical Experiments

We illustrate some of the localization sets obtained in this chapter via numerical experiments. All the programming for generating the figures is performed by using Matlab R2018b. The first three examples are chosen to be ones already considered in the literature in order to compare the localization sets with those from the literature.

Example 3.3.2. *[37] Consider the quadratic eigenvalue problem Bilby [5] arising in a quasi-birth-death process model of the population given by* $T(z) = z^2A + zB + C$*, where*

$$A = \begin{bmatrix} 0 & 0.05 & 0.055 & 0.08 & 0.1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.22 & 0 & 0 \\ 0 & 0 & 0 & 0.32 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0.01 & 0.02 & 0.01 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0.04 & -1 & 0 & 0 \\ 0 & 0 & 0.08 & -1 & 0 \\ 0 & 0 & 0 & 0.04 & -1 \end{bmatrix},$$

$$\text{and} \quad C = \begin{bmatrix} 0.1 & 0.4 & 0.0250 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}.$$

Since the leading coefficient matrix A is singular, infinity is an eigenvalue of $T(z)$. We use the partition $\pi = \{0, 1, 5\}$ and $p = 1$ norm to plot the block Geršgorin set on the Riemann sphere which can be seen in Figure 3.3.1. As may be expected from Theorem 3.1.2(vii), the block Geršgorin set is symmetric with respect to the xz-plane.

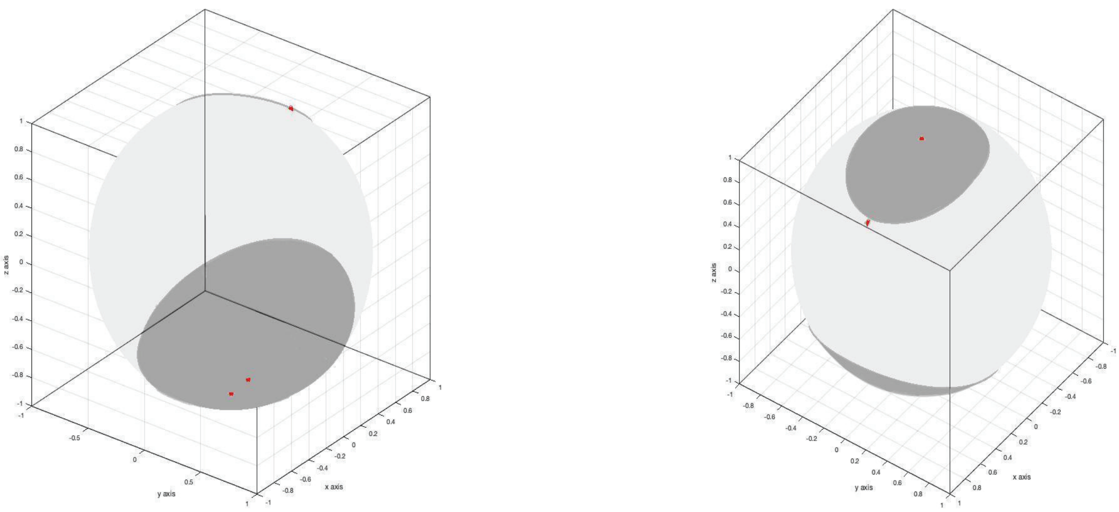


Figure 3.3.1: The shaded regions on the Riemann sphere showing the full block Geršgorin set $H\Gamma_1^{\pi}(T)$ corresponding to the partition $\pi = \{0, 1, 5\}$ for $T(z)$ in Example 3.3.2. The eigenvalues are represented by $\color{red}{\bullet}$.

For this choice of partition π, the block Brualdi set equals the block minimal Geršgorin set. Figure 3.3.2 represents the block minimal Geršgorin set on the Riemann sphere which is in fact exactly the set of eigenvalues of $T(z)$. Figure 3.3.3 (left) and Figure 3.3.3 (right) constitute the projections of the block Geršgorin sets and the block minimal Geršgorin sets on the complex plane respectively.

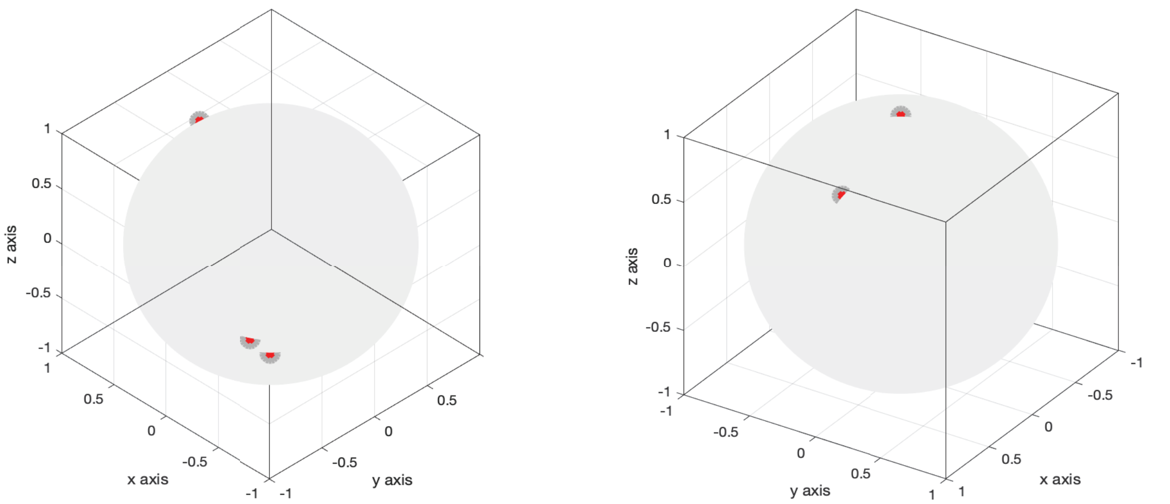


Figure 3.3.2: The block minimal Geršgorin set $H\mathcal{G}_1^\pi(T)$ for $T(z)$ in Example 3.3.2 on the Riemann sphere with respect to $\pi = \{0, 1, 5\}$. The $\bullet$ represent eigenvalues.

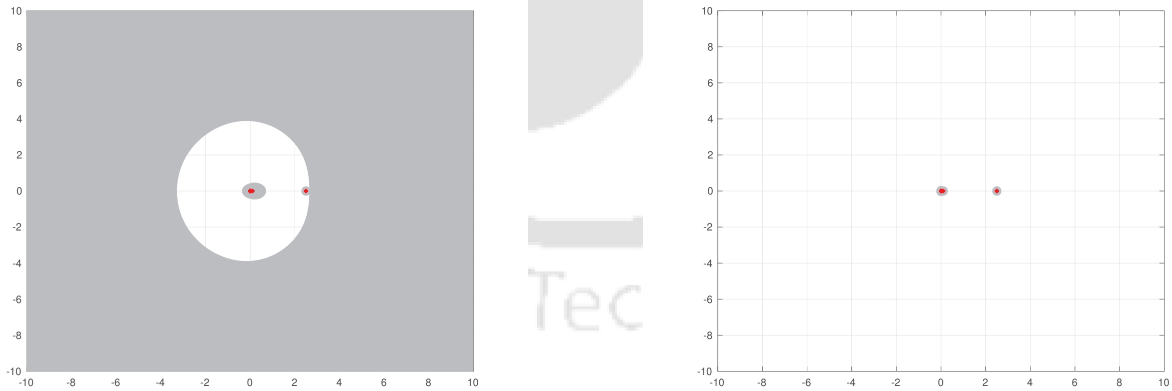


Figure 3.3.3: Projection of the block Geršgorin set $H\Gamma_1^\pi(T)$ (left) and pointwise minimal Geršgorin set $\mathcal{G}(T)$/block minimal Geršgorin set $H\mathcal{G}_1^\pi(T)$ (right) for $T(z)$ in Example 3.3.2 on the complex plane with $\pi = \{0, 1, 5\}$. The $\bullet$ denote the eigenvalues.

Since for every permutation ϕ on the set $\{1, 2, \ldots, 5\}$ with $\phi(i) \neq i$, the entry $t_{i,\phi(i)}(z)$ or $t_{i,\phi(i)}(z)$ is identically zero, every permutation on the set $\{1, 2, \ldots, 5\}$ other than the identity permutation is a trivial permutation. Therefore, the permuted *pointwise minimal Geršgorin set equals the minimal Geršgorin set. All the vertices of $T(z)$ being disconnected, the minimal Geršgorin set (or minimal Brualdi set) is equal to the collection of roots of the polynomials $-z+0.1$, z and $0.4z^2-z$, which in this example is identical to the block minimal Geršgorin set. The pointwise Geršgorin set is shown in Figure 3.3.4.*

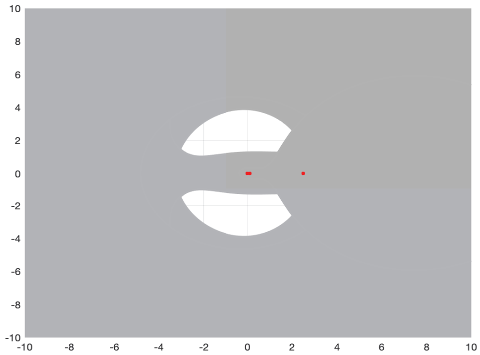


Figure 3.3.4: Pointwise Geršgorin set [37] for $T(z)$ in Example 3.3.2 on the complex plane. $\bullet$ denote the eigenvalues.

Example 3.3.3. *[52] Consider the matrix polynomial*

$$P(z) = \begin{bmatrix} 8iz^2-2iz+2 & 2iz^2+iz+(1+2i) & (-1+i)z^2+z+2 \\ -3iz^2+5iz+(1+i) & -6iz^2+3iz+4i & (2-2i)z^2-4z-5i \\ (0.8-i)z^2+iz+(1-i) & 6iz^2-iz & (6-2i)z^2+2i \end{bmatrix}.$$

With the use of partition $\pi=\{0,1,3\}$ and $p=2$ norm, the block Geršgorin set on the Riemann sphere can be seen in Figure 3.3.5. The projection of this set on the complex plane is shown in Figure 3.3.6 (left). The projections of the block minimal Geršgorin sets (or, the block Brualdi sets) on the complex plane are shown in Figure 3.3.7. Six permutations have to be considered for plotting the permuted pointwise minimal Geršgorin set. The resulting plot is in Figure 3.3.8 (left). The pointwise Geršgorin set [52] and pointwise minimal Geršgorin set are shown in Figure 3.3.6 (right) and Figure 3.3.8 (right) respectively. The block minimal Geršgorin set provides the best localization for the eigenvalues of $P(z)$.

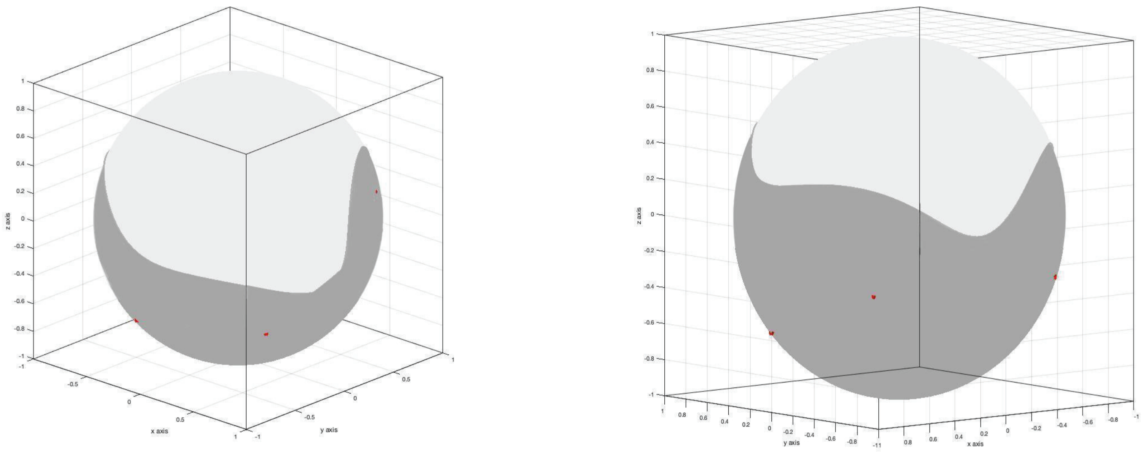


Figure 3.3.5: Shaded regions on the Riemann sphere representing the complete block Geršgorin set corresponding to the partition $\pi = \{0, 1, 3\}$ for $P(z)$ in Example 3.3.3. The eigenvalues are denoted by $\bullet$.

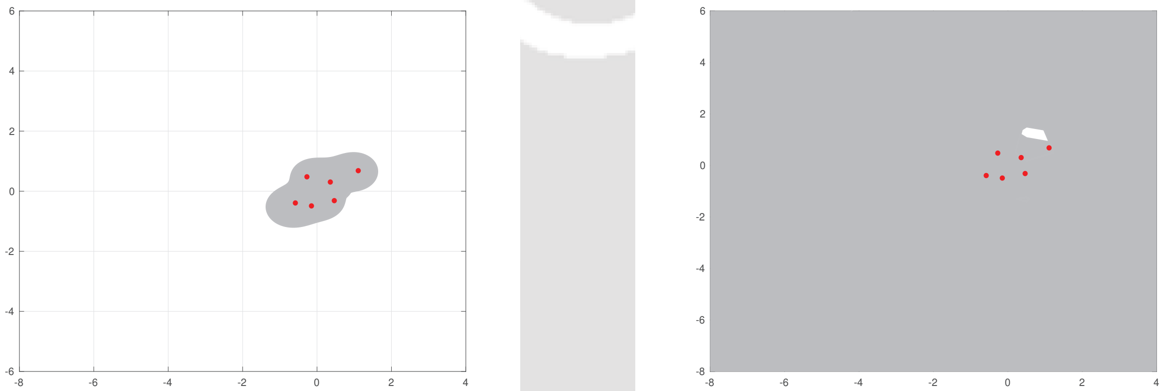


Figure 3.3.6: Projection of the block Geršgorin set $H\Gamma_2^{\pi}(P)$ (left) and pointwise Geršgorin set (right) [52] for $P(z)$ in Example 3.3.3 on the complex plane. The $\bullet$ denote eigenvalues.

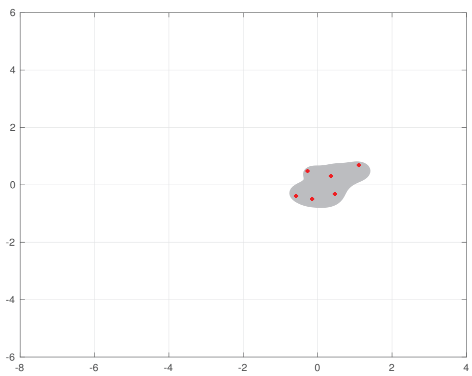


Figure 3.3.7: Block minimal Geršgorin set $H\mathcal{G}_2^{\pi}(P)$ for $P(z)$ in Example 3.3.3 on the complex plane with $\pi = \{0, 1, 3\}$. The eigenvalues are denoted by $\bullet$.

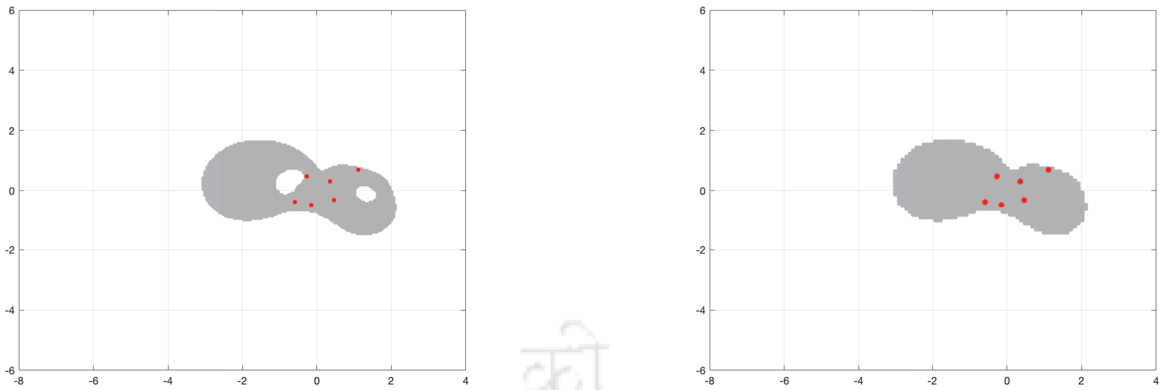
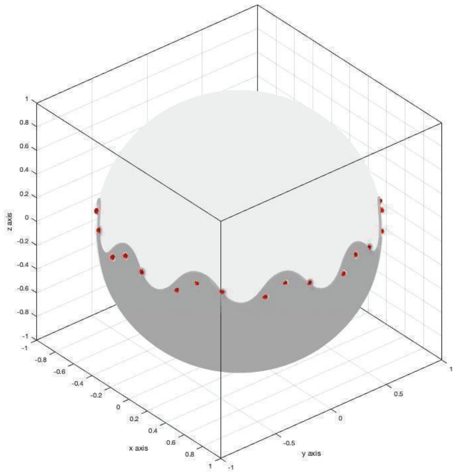


Figure 3.3.8: Permuted pointwise minimal Geršgorin set (left) and pointwise minimal Geršgorin set (right) for $P(z)$ in Example 3.3.3 on the complex plane. • denotes the eigenvalues.

Example 3.3.4. *[52] Consider the complex matrix polynomial $Q(z)$ given by*

$$Q(z) \quad = \quad Iz^{10} - A = \begin{bmatrix} z^{10} + 3.5i & 3i & 0 \\ 2 & z^{10} - 2i & 0 \\ 0.5 + 2i & -7i & z^{10} + 5 - 3i \end{bmatrix}.$$

The matrix $Q(z)$ is in normal reduced form for every $z \in \mathbb{C}$. We select the partition $\pi = \{0, 2, 3\}$ and $p = 2$ norm to plot the block localization sets for the eigenvalues. In keeping with our computation strategy for block minimal Geršgorin sets, plotting the block minimal Geršgorin set is equivalent to plotting the pointwise minimal Geršgorin set of the leading 2×2 block on the diagonal of $P(z)$ and the roots of $z^{10} + 5 - 3i$ which is illustrated in Figure 3.3.10. Thus the pointwise minimal Geršgorin set is same as the block minimal Geršgorin set. The block Geršgorin set plotted on the Riemann sphere is shown in Figure 3.3.9 (left) and its corresponding projection on the complex plane can be seen in Figure 3.3.9 (right). The only cycle of $\langle Q(z) \rangle$ being $(1\ 2)$, it can be shown that the Brualdi set is identical to the pointwise minimal Geršgorin set for $Q(z)$. The set Ψ necessary for plotting the permuted minimal Geršgorin set consists of permutations $\psi_1 = (1\ 2)\,(3)$ and $\psi_2 = (1)$. Accordingly we plot the permuted pointwise minimal Geršgorin set which can be seen in Figure 3.3.11. The pointwise Geršgorin set is shown in Figure 3.3.12.



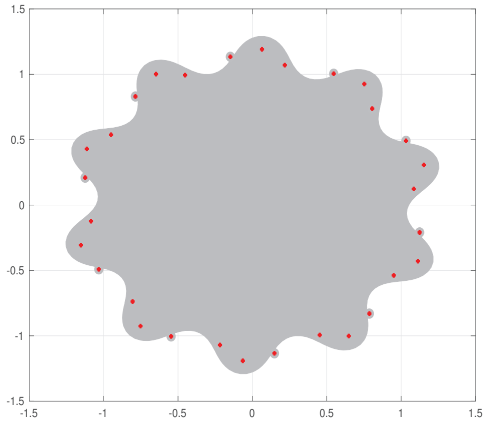


Figure 3.3.9: The shaded regions represent the block Geršgorin set corresponding to the partition $\pi = \{0, 2, 3\}$ on the Riemann sphere (left) and its corresponding projection on the complex plane (right) for $Q(z)$ given in Example 3.3.4. ● denote the eigenvalues.

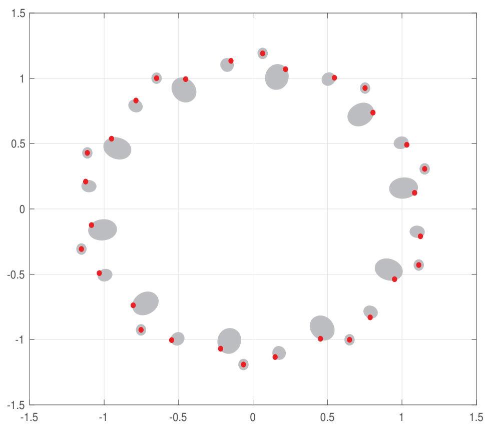


Figure 3.3.10: The pointwise minimal Geršgorin set/block minimal Geršgorin set for $Q(z)$ in Example 3.3.4 on the complex plane. The eigenvalues are denoted by ●.

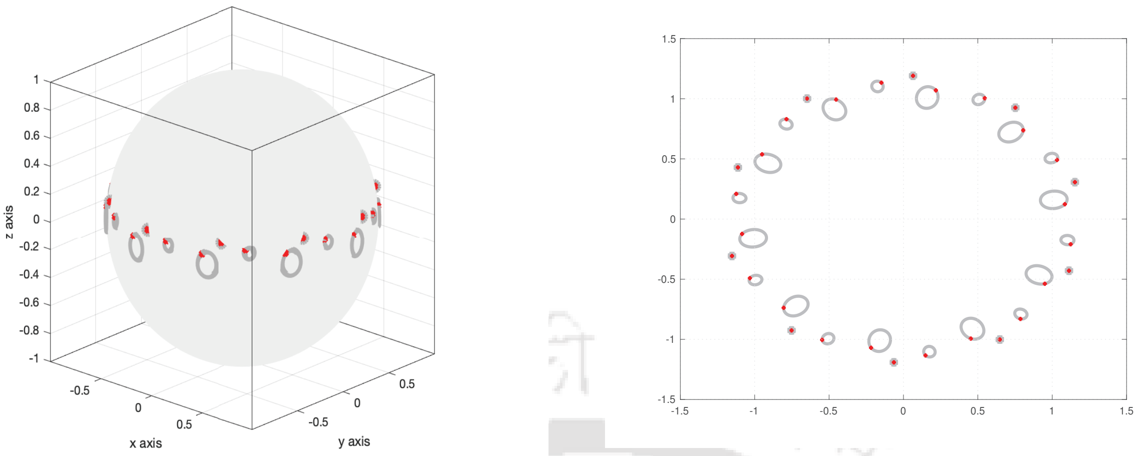


Figure 3.3.11: Permuted pointwise minimal Geršgorin set for $Q(z)$ on the Riemann sphere (left) and its projection on the complex plane (right). $\color{red}{\bullet}$ denote the eigenvalues.

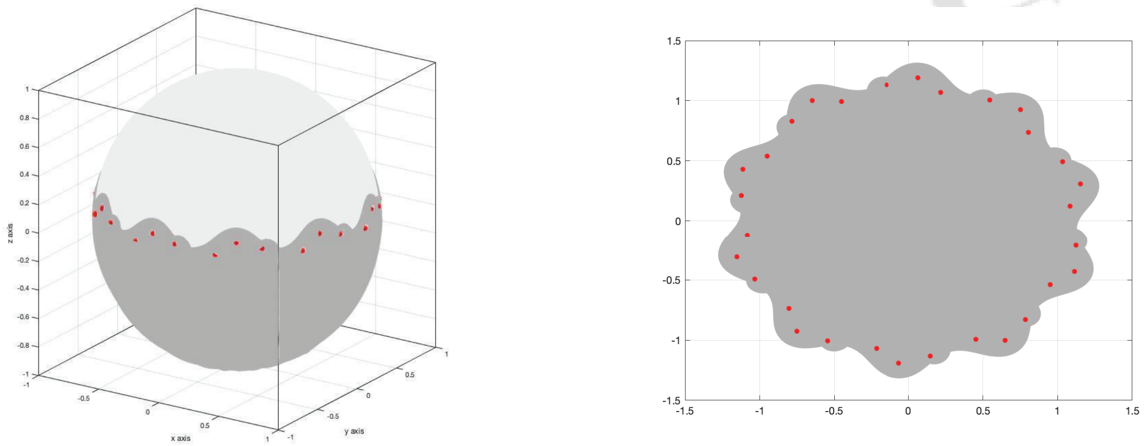


Figure 3.3.12: The shaded regions represent the pointwise Geršgorin set on the Riemann sphere (left) and its corresponding projection on the complex plane (right) for $Q(z)$ given in Example 3.3.4. The eigenvalues are represented by $\color{red}{\bullet}$.

Example 3.3.5. *Consider the matrix polynomial* $P(z) = Az^2 + Bz + C$ *where*

$$A = 0.5I,$$

$$B = \begin{bmatrix} 0 & -0.7467 & 0 & 0.3922 & 0 & 0 & 0 & 0 & 0 & 0.7431 \\ 0 & 0 & -1.3333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9231 & 0.6787 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6555 & 0 & 0 & 0 & -2.3758 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.2222 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8711 & 0.7577 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9.0462 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.8718 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11.8588 \\ 0.2828 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.8667 \end{bmatrix},$$

$$C = \begin{bmatrix} 2.5167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10.0670 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 22.6507 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 40.2680 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 62.9187 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 90.6030 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 123.3207 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 161.0719 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 203.8567 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 251.6749 \end{bmatrix}.$$

We use the partition $\pi = \{0, 3, 6, 10\}$ *and* $p = 1$ *norm to plot the block Geršgorin set* $H\Gamma_1^\pi(P)$ *on the Riemann sphere which can be seen in Figure 3.3.13. The projection of this set on the complex plane is shown in Figure 3.3.14.*

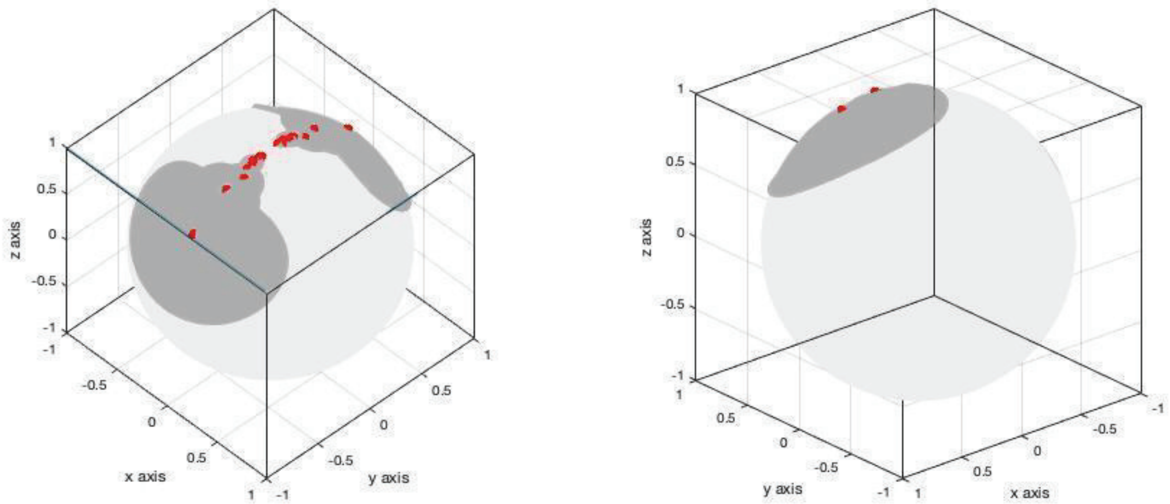


Figure 3.3.13: Shaded regions on the Riemann sphere representing the block Geršgorin set $H\Gamma_1^\pi(P)$ corresponding to the partition $\pi = \{0, 3, 6, 10\}$ for $P(z)$ in Example 3.3.5. The eigenvalues are denoted by $\color{red}{\bullet}$.

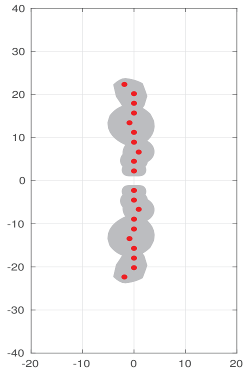


Figure 3.3.14: Projection of the block Geršgorin set $H\Gamma_1^{\pi}(P)$ on the complex plane. $\bullet$ denote the eigenvalues.

For this choice of partition, the projection of the block Brualdi sets and block minimal Geršgorin set are shown in Figure 3.3.15 (left) and Figure 3.3.15 (right) respectively. We plot the projection of the pointwise Geršgorin set and pointwise minimal Geršgorin set on the complex plane in Figure 3.3.16 (left) and Figure 3.3.16 (right) respectively. The set Ψ *necessary for plotting the permuted pointwise Geršgorin set consists of the six permutations* $\psi_1 = (1\ 4)\,(2)$, $\psi_2 = (1\ 10)\,(2)$, $\psi_3 = (1\ 2\ 3\ 4)\,(5)$, $\psi_4 = (1\ 4\ 5\ 6\ 7\ 8\ 9\ 10)\,(2)$, $\psi_5 = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)$ *and the identity permutation. The permuted pointwise minimal Geršgorin set is identical to the block minimal Geršgorin set.*

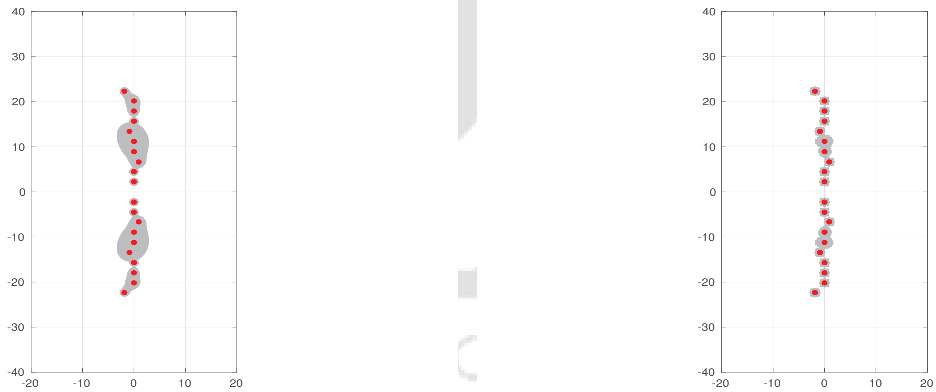


Figure 3.3.15: Projection of the block Brauldi sets $H\mathcal{B}_1^{\pi}(P)$ (left) and block minimal Geršgorin set $H\mathcal{G}_1^{\pi}(P)$/permuted pointwise minimal Geršgorin set (right) for $P(z)$ in Example 3.3.5 on the complex plane. $\bullet$ denote the eigenvalues.

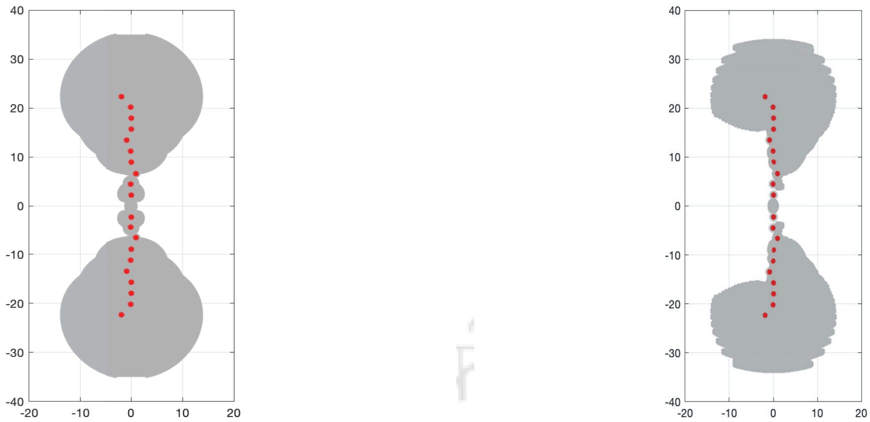


Figure 3.3.16: Projection of the pointwise Geršgorin set (left) and pointwise minimal Geršgorin set (right) for $P(z)$ in Example 3.3.5 on the complex plane. The eigenvalues are represented by $\bullet$.

4 Applications of block Geršgorin sets arising from linearizations of quadratic matrix polynomials

In this chapter we consider eigenvalue localization sets for quadratic matrix polynomials due to their extensive use in applications. The focus is on localizations obtained via block Geršgorin sets constructed from certain linearizations of the matrix polynomials. These linearizations are chosen from well-known vector spaces of potential linearizations of the matrix polynomial introduced in [44, 46]. We derive properties of these sets and use them to provide simple upper and lower bounds on the eigenvalues of the matrix polynomials. These new localization sets constructed via the linearizations of the quadratic matrix polynomials also lead to simple conditions on the coefficient matrices of the polynomials so that their eigenvalues are located in particular regions of the complex plane. These regions are chosen to be ones that are important in applications. The analysis results in upper bounds on the solutions of a number of distance problems associated with these polynomials.

In Chapter 3, we used the homogeneous form of the matrix polynomial to treat both finite and infinite eigenvalues in a single framework as our primary concern was localization of eigenvalues and visualization of the eigenvalue localization sets. In this chapter, our emphasis shifts to finding bounds for the finite eigenvalues of the matrix polynomials and deriving conditions on the coefficient matrices of the polynomial so that its eigenvalues lie in certain parts of the complex plane. We use the non-homogeneous form of the matrix polynomial for this purpose as it is more convenient for the necessary analysis.

As described in Chapter 3, let a $n \times n$ matrix polynomial $P(z) = \sum_{i=0}^{m} A_i z^i$ be partitioned with respect to the partition $\pi = \{n_j\}_{j=0}^{\ell}$ of the set $\{1, \ldots, n\}$ as in (3.1.2). Using the convention that

$$\infty \in \Gamma^{\pi}_{p,k}(P) \text{ if and only if } \left\| ((\mathrm{rev}P)_{k,k}(0))^{-1} \right\|_p^{-1} \leq r_k(\langle \mathrm{rev}P(0) \rangle_p^{\pi}),$$

let

$$\Gamma^{\pi}_{p,k}(P) := \left\{ z \in \mathbb{C}^{\infty} : \left\| (P_{k,k}(z))^{-1} \right\|_p^{-1} \leq r_k(\langle P(z) \rangle^{\pi}_p) \right\}.$$

Then the block Geršgorin set for the matrix polynomial $P(z)$ with respect to the p-norm and partition π is defined as

$$\Gamma^{\pi}_p(P) = \bigcup_{k=1}^{\ell} \Gamma^{\pi}_{p,k}(P).$$

The following result which is a restatement of Theorem 3.1.11 in the non-homogeneous setting for the particular case of $\Gamma^{\pi}_p(P)$, will be important in our analysis.

Theorem 4.0.1. *All the eigenvalues of* $P(z) = \sum_{i=0}^{m} A_i z^i$, $A_i \in \mathbb{C}^{n,n}$ *lie inside the set* $\Gamma^{\pi}_p(P)$. *Moreover, if there exist closed disjoint sets* $U, V \subset \mathbb{C}^{\infty}$ *such that*

$$\Gamma^{\pi}_p(P) = U \cup V$$

then one of them, say U*, is compact in* $\mathbb{C}$*. Also the number of finite eigenvalues of the matrix polynomial* $P(z)$ *in* U *is equal to the sum of the number of eigenvalues of the polynomials* $P_{k,k}(z) := \sum_{i=0}^{m} (A_i)_{k,k} z^i$ *in* U *for each* $k \in \{1, 2, \ldots, \ell\}$*, counting multiplicities.*

Throughout this chapter, we use the notation $\mathcal{P}_n$ to denote the set of all $n \times n$ matrix polynomials of degree two, and assume $P(z)$ to be a matrix polynomial in $\mathcal{P}_n$ given by

$$P(z) = A_2 z^2 + A_1 z + A_0, \;\; A_i \in \mathbb{C}^{n,n}, \;\; A_2 \neq 0. \tag{4.0.1}$$

4.1 Linearizations of matrix polynomials arising from vector spaces

The following theorems restate results from [44] and [46] that are essential for the work in the subsequent sections for the particular case of $P(z) \in \mathcal{P}_n$.

Theorem 4.1.1. *Let* $P(z) \in \mathcal{P}_n$ *be as in (4.0.1). The general form of a strong linearization* $L(z)$ *in* $\mathbb{L}_1(P)$ *with right ansatz vector* $v = \alpha e_1$, $\alpha \in \mathbb{C} \setminus \{0\}$ *is given by*

$$L(z) = z \left[\begin{array}{c|c} \alpha A_2 & -W_1 \\ \hline 0 & -W_2 \end{array} \right] + \left[\begin{array}{c|c} W_1 + \alpha A_1 & \alpha A_0 \\ \hline W_2 & 0 \end{array} \right]. \tag{4.1.1}$$

Similarly, if $v = \alpha e_2$, $\alpha \in \mathbb{C} \setminus \{0\}$, *then* $L(z)$ *in* $\mathbb{L}_1(P)$ *with right ansatz vector* v *is given by*

$$L(z) = z \left[\begin{array}{c|c} 0 & -W_2 \\ \hline \alpha A_2 & -W_1 \end{array} \right] + \left[\begin{array}{c|c} W_2 & 0 \\ \hline W_1 + \alpha A_1 & \alpha A_0 \end{array} \right]. \tag{4.1.2}$$

In either case $W_2 \in \mathbb{C}^{n,n}$ is non-singular and $W_1 \in \mathbb{C}^{n,n}$ is arbitrary.

Further, if $v(\neq \alpha e_j) \in \mathbb{C}^2$, $j = 1, 2$, and $M \in \mathbb{C}^{2,2}$ is a non-singular matrix such that $Mv = \alpha e_j$, then any strong linearization $L_v(z) \in \mathbb{L}_1(P)$ with right ansatz vector v satisfies $L_v(z) = (M \otimes I_n)^{-1} L(z)$ where $L(z)$ is given by (4.1.1) if $j = 1$ and (4.1.2) if $j = 2$.

Remark 4.1.2. *Observe that the pencil in (4.1.1) with $\alpha = 1$, $W_1 = 0$ and $W_2 = -I_n$ corresponds to the first companion form $C_1(z)$ in (1.2.6).*

Theorem 4.1.3. *Let $P(z)$ be as in (4.0.1). The general form of a strong linearization $L(z)$ in $\mathbb{L}_2(P)$ with left ansatz vector $v = \alpha e_1$, $\alpha \in \mathbb{C} \setminus \{0\}$ is given by*

$$L(z) = z \left[\begin{array}{c|c} \alpha A_2 & 0 \\ \hline -W_1 & -W_2 \end{array}\right] + \left[\begin{array}{c|c} W_1 + \alpha A_1 & W_2 \\ \hline \alpha A_0 & 0 \end{array}\right]. \tag{4.1.3}$$

Similarly, the pencil $L(z)$ in $\mathbb{L}_2(P)$ with left ansatz vector $v = \alpha e_2$, $\alpha \in \mathbb{C} \setminus \{0\}$, is given by

$$L(z) = z \left[\begin{array}{c|c} 0 & \alpha A_2 \\ \hline -W_2 & -W_1 \end{array}\right] + \left[\begin{array}{c|c} W_2 & W_1 + \alpha A_1 \\ \hline 0 & \alpha A_0 \end{array}\right]. \tag{4.1.4}$$

In both the cases $W_2 \in \mathbb{C}^{n,n}$ is non-singular and $W_1 \in \mathbb{C}^{n,n}$ is arbitrary.

Further, if $v(\neq \alpha e_j) \in \mathbb{C}^2$, $j = 1, 2$, and $M \in \mathbb{C}^{2,2}$ is a non-singular matrix such that $Mv = \alpha e_j$, then any strong linearization $L_v(z) \in \mathbb{L}_2(P)$ with left ansatz vector v satisfies $L_v(z) = L(z)(M^t \otimes I_n)^{-1}$ where $L(z)$ is given by (4.1.3) if $j = 1$ and by (4.1.4) if $j = 2$.

Remark 4.1.4. *With $\alpha = 1$, $W_1 = 0$ and $W_2 = -I_n$, the pencil in (4.1.3) corresponds to the second companion form $C_2(z)$ in (1.2.7).*

Theorem 4.1.5. *Let $P(z) \in \mathcal{P}_n$ be as given in (4.0.1) and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}^2$. Any pencil in $\mathbb{DL}(P)$ with ansatz vector v is of the form*

$$L(z) = z \left[\begin{array}{c|c} v_1 A_2 & v_2 A_2 \\ \hline v_2 A_2 & v_2 A_1 - v_1 A_0 \end{array}\right] + \left[\begin{array}{c|c} v_1 A_1 - v_2 A_2 & v_1 A_0 \\ \hline v_1 A_0 & v_2 A_0 \end{array}\right].$$

It is a strong linearization of $P(z)$ if and only if no root of $p(x) = v_1 x + v_2$ is an eigenvalue of $P(z)$, the convention being that ∞ is a root of $p(x)$ if $v_1 = 0$.

4.2 Block Geršgorin sets arising from linearizations of quadratic matrix polynomials

In this section we examine the block Geršgorin sets arising from the matrix pencils $L(z)$ belonging to the vector spaces $\mathbb{L}_1(P)$, $\mathbb{L}_2(P)$ and $\mathbb{DL}(P)$ that are strong linearizations of $P(z) \in \mathcal{P}_n$ as given in (4.0.1). The partition π is taken to be the canonical partitioning of blocks as stated in Theorems 4.1.1, 4.1.3 and 4.1.5. With this partition π and induced operator norm $\|.\|_p$, for the sake of simplicity, throughout this chapter we denote the block Geršgorin set $\Gamma_p^\pi(L)$ obtained via the linearization $L(z)$ of $P(z)$ by the notation $\Gamma(P)$.

Theorems 4.2.1 and 4.2.3 provide the block Geršgorin sets $\Gamma(P)$ corresponding to strong linearizations in $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ with ansatz vectors αe_1 and αe_2 respectively.

Theorem 4.2.1. *Let $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$ with $A_0 \neq 0$. Then the block Geršgorin set $\Gamma(P)$ containing the spectrum of $P(z)$ is given by*

$$\Gamma(P) = R_1 \cup R_2$$

where the sets R_1 and R_2 take different forms depending on the choice of linearization $L(z)$. The details are as follows.

(i) If $L(z)$ is the first companion form linearization, then

$$R_1 = \{z \in \mathbb{C}^\infty : \left\|(zA_2 + A_1)^{-1}\right\|_p^{-1} \leq \|A_0\|_p\}, \tag{4.2.1}$$
$$R_2 = \{z \in \mathbb{C} : |z| \leq 1\}. \tag{4.2.2}$$

(ii) If $L(z)$ is the second companion form linearization, then

$$R_1 = \{z \in \mathbb{C}^\infty : \left\|(zA_2 + A_1)^{-1}\right\|_p^{-1} \leq 1\},$$
$$R_2 = \{z \in \mathbb{C} : |z| \leq \|A_0\|_p\}.$$

(iii) If $L(z) \in \mathbb{L}_1(P)$ with right ansatz vector $v = \alpha e_1$, $\alpha \in \mathbb{C} \setminus \{0\}$ is as in (4.1.1), then

$$R_1 = \left\{z \in \mathbb{C}^\infty : \left\|(\alpha(zA_2 + A_1) + W_1)^{-1}\right\|_p^{-1} \leq \|\alpha A_0 - zW_1\|_p\right\},$$
$$R_2 = \{z \in \mathbb{C} : |z| \leq \kappa\}$$

where $\kappa = \|W_2\|_p \left\|W_2^{-1}\right\|_p$ is the condition number of the non-singular matrix $W_2 \in \mathbb{C}^{n,n}$ and $W_1 \in \mathbb{C}^{n,n}$ is arbitrary.

(iv) If $L(z) \in \mathbb{L}_2(P)$ with left ansatz vector $v = \alpha e_1$, $\alpha \in \mathbb{C} \setminus \{0\}$ as in (4.1.3) then

$$R_1 = \left\{z \in \mathbb{C}^\infty : \left\|(\alpha(zA_2 + A_1) + W_1)^{-1}\right\|_p^{-1} \leq \|W_2\|_p\right\},$$
$$R_2 = \left\{z \in \mathbb{C} : |z| \leq \left\|W_2^{-1}\right\|_p \|\alpha A_0 - zW_1\|_p\right\}$$

where $W_2 \in \mathbb{C}^{n,n}$ is non-singular and $W_1 \in \mathbb{C}^{n,n}$ is arbitrary.

Proof. We first prove (iii). From (4.1.1), the linearization $L(z) \in \mathbb{L}_1(P)$ with ansatz vector αe_1 takes the form

$$L(z) = \left[\begin{array}{c|c} \alpha(zA_2 + A_1) + W_1 & \alpha A_0 - zW_1 \\ \hline W_2 & -zW_2 \end{array}\right].$$

Applying the definition of block Geršgorin sets to $L(z)$, we have $\Gamma(P) = R_1 \cup R_2$ where

$$\begin{aligned} R_1 &= \left\{ z \in \mathbb{C}^{\infty} : \left\| (\alpha(zA_2 + A_1) + W_1)^{-1} \right\|_p^{-1} \leq \|\alpha A_0 - zW_1\|_p \right\}, \text{ and} \\ R_2 &= \left\{ z \in \mathbb{C}^{\infty} : |z| \left\| W_2^{-1} \right\|_p^{-1} \leq \|W_2\|_p \right\} = \left\{ z \in \mathbb{C} : |z| \leq \|W_2\|_p \left\| W_2^{-1} \right\|_p \right\}. \end{aligned}$$

Using Theorem 4.0.1, $\Lambda(L) \subseteq \Gamma(L)$. Consequently, $\Lambda(P) = \Lambda(L) \subseteq \Gamma(L)$.

In order to determine whether the eigenvalue ∞ belongs to the set R_1 or R_2, we look at the block Geršgorin set arising from $\mathrm{rev}L(z)$ given by

$$z \left[\begin{array}{c|c} W_1 + \alpha A_1 & \alpha A_0 \\ \hline W_2 & 0 \end{array}\right] + \left[\begin{array}{c|c} \alpha A_2 & -W_1 \\ \hline 0 & -W_2 \end{array}\right],$$

where $W_2 \in \mathbb{C}^{n,n}$ is non-singular and $W_1 \in \mathbb{C}^{n,n}$ is arbitrary. Clearly, 0 belongs to the set $R_1^{\mathrm{rev}L} := \{z \in \mathbb{C} : \left\| (\alpha(zA_1 + A_2) + zW_1)^{-1} \right\|_p^{-1} \leq \|\alpha z A_0 - W_1\|_p\}$ if A_2 is singular. Also conversely if $0 \in R_1^{\mathrm{rev}L}$ for every $W_1 \in \mathbb{C}^{n,n}$, then A_2 is a singular matrix. Therefore we associate ∞ with R_1.

(iv) can be derived in a similar manner using the form of $L(z)$ from (4.1.3). Also since for the eigenvalue at ∞, we have

$$\mathrm{rev}L(z) = z \left[\begin{array}{c|c} W_1 + \alpha A_1 & W_2 \\ \hline \alpha A_0 & 0 \end{array}\right] + \left[\begin{array}{c|c} \alpha A_2 & 0 \\ \hline -W_1 & -W_2 \end{array}\right]$$

and infinity is associated with R_1 as

$$0 \in R_1^{\mathrm{rev}L} := \left\{ z \in \mathbb{C} : \left\| (\alpha(zA_1 + A_2) + zW_1)^{-1} \right\|_p^{-1} \leq |z| \, \|W_2\|_p \right\}$$

for all admissible choices of W_1 and W_2 if and only if A_2 is singular.

With $\alpha = 1$, $W_1 = 0$ and $W_2 = -I_n$ in (iii) and (iv) respectively, we get (i) and (ii). $\square$

Remark 4.2.2. *Observe that in case A_2 is singular, the set R_1 corresponding to the first and second companion form is always unbounded.*

Theorem 4.2.3. *For a quadratic matrix polynomial $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$, the block Geršgorin sets corresponding to the linearizations in $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ with* *ansatz vector $v = \alpha e_2$ are given by the union of sets R_1 and R_2 where*

$$R_1 = \left\{ z \in \mathbb{C}^\infty : |z| \geq \frac{1}{\|W_2\|_p \left\|W_2^{-1}\right\|_p} \right\},$$
$$R_2 = \left\{ z \in \mathbb{C} : \left\|(\alpha A_0 - zW_1)^{-1}\right\|_p^{-1} \leq \|\alpha(zA_2 + A_1) + W_1\|_p \right\},$$

if $L(z) \in \mathbb{L}_1(P)$ is of the form (4.1.2), and

$$R_1 = \left\{ z \in \mathbb{C}^\infty : \left\|W_2^{-1}\right\|_p^{-1} \leq \|\alpha(zA_2 + A_1) + W_1\|_p \right\}, \tag{4.2.3}$$
$$R_2 = \left\{ z \in \mathbb{C} : \left\|(\alpha A_0 - zW_1)^{-1}\right\|_p^{-1} \leq |z| \, \|W_2\|_p \right\}, \tag{4.2.4}$$

if $L(z) \in \mathbb{L}_2(P)$ is of the form as given in (4.1.4). In either case $W_2 \in \mathbb{C}^{n,n}$ is non-singular and $W_1 \in \mathbb{C}^{n,n}$ is arbitrary.

Proof. The proof follows by arguing as in the proof of Theorem 4.2.1. Note that here the sets R_1 and R_2 may be unbounded in each case. For both the linearizations, 0 belongs to the set R_1 corresponding to rev$L(z)$ for all choices of W_1 and W_2. But this is not the case for R_2. Hence ∞ is associated with R_1 in both instances. □

It is evident from Theorem 4.2.1 that if the pencil $zA_2 + A_1$ is singular, then the block Geršgorin sets corresponding to the first and second companion forms are the entire complex plane. We conjecture that the converse also holds.

The block Geršgorin sets in Theorems 4.2.1 and 4.2.3 are obtained via linearizations with respect to ansatz vectors αe_1 and αe_2, where $\alpha \in \mathbb{C} \setminus \{0\}$. Now we consider the general case of an ansatz vector v that is not a multiple of e_1 or e_2.

Theorem 4.2.4. *Let $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$. Suppose $L(z)$ is a linearization of $P(z)$ in $\mathbb{L}_1(P)$ with corresponding ansatz vector $v \in \mathbb{C}^2$ which is not a multiple of e_1 or e_2. Let $M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathbb{C}^{2,2}$ be non-singular.*

(i) If $Mv = \alpha e_1$, $\alpha \in \mathbb{C} \setminus \{0\}$ then the block Geršgorin sets $\Gamma(P) = R_1 \cup R_2$ arising from $L(z)$ is such that

$$R_1 = \left\{ z \in \mathbb{C}^\infty : \left\|(zA_2 + A_1 + \mathscr{W}_1)^{-1}\right\|_p^{-1} \leq \|A_0 - z\mathscr{W}_1\|_p \right\}, \tag{4.2.5}$$
$$R_2 = \left\{ z \in \mathbb{C} : \left\|(A_0 - z\mathscr{W}_2)^{-1}\right\|_p^{-1} \leq \|zA_2 + A_1 + \mathscr{W}_2\|_p \right\}. \tag{4.2.6}$$

Here $\mathscr{W}_1 = W_1 - \frac{m_2}{m_4} W_2$ and $\mathscr{W}_2 = W_1 - \frac{m_1}{m_3} W_2$ where $W_1, W_2 \in \mathbb{C}^{n,n}$ with W_2 non-singular such that

$$L(z) = (\alpha M^{-1} \otimes I_n) \left(z \left[\begin{array}{c|c} A_2 & -W_1 \\ \hline 0 & -W_2 \end{array} \right] + \left[\begin{array}{c|c} W_1 + A_1 & A_0 \\ \hline W_2 & 0 \end{array} \right] \right).$$

(ii) If $Mv = \alpha e_2$, $\alpha \in \mathbb{C}\setminus\{0\}$ then $\Gamma(P) = R_1 \cup R_2$ is the block Geršgorin set associated with $L(z)$ where R_1 and R_2 are given by (4.2.5) and (4.2.6) respectively, with the only difference being that $\mathscr{W}_1 = W_1 - \frac{m_4}{m_2}W_2$ and $\mathscr{W}_2 = W_1 - \frac{m_3}{m_1}W_2$ where W_1, $W_2 \in \mathbb{C}^{n,n}$ with W_2 non-singular such that

$$L(z) = (\alpha M^{-1} \otimes I_n)\left(z\left[\begin{array}{c|c} 0 & -W_2 \\ \hline A_2 & -W_1 \end{array}\right] + \left[\begin{array}{c|c} W_2 & 0 \\ \hline W_1 + A_1 & A_0 \end{array}\right]\right).$$

Proof. Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}^2$. Then by the hypothesis v_1 and v_2 are both non-zero. First we prove (i). By Theorem 4.1.1, $\left(\frac{M}{\alpha} \otimes I_n\right) L(z) \in \mathbb{L}_1(P)$ with ansatz vector e_1. Therefore, there exists W_1, $W_2 \in \mathbb{C}^{n,n}$ with W_2 non-singular such that

$$\begin{aligned} L(z) &= (\alpha M^{-1} \otimes I_n)\left(z\left[\begin{array}{c|c} A_2 & -W_1 \\ \hline 0 & -W_2 \end{array}\right] + \left[\begin{array}{c|c} W_1 + A_1 & A_0 \\ \hline W_2 & 0 \end{array}\right]\right) \\ &= \frac{\alpha}{\det(M)}\left[\begin{array}{c|c} m_4 I_n & -m_2 I_n \\ \hline -m_3 I_n & m_1 I_n \end{array}\right]\left(z\left[\begin{array}{c|c} A_2 & -W_1 \\ \hline 0 & -W_2 \end{array}\right] + \left[\begin{array}{c|c} W_1 + A_1 & A_0 \\ \hline W_2 & 0 \end{array}\right]\right) \\ &= \frac{\alpha}{\det(M)}\left[\begin{array}{c|c} m_4(zA_2 + A_1 + W_1) - m_2 W_2 & m_4(A_0 - zW_1) + m_2 z W_2 \\ \hline -m_3(zA_2 + A_1 + W_1) + m_1 W_2 & -m_3(A_0 - zW_1) - m_1 z W_2 \end{array}\right]. \end{aligned}$$

Therefore, the block Geršgorin region $\Gamma(P) = R_1 \cup R_2$ arising from $L(z)$ is such that

$$R_1 = \left\{ z \in \mathbb{C}^\infty : \left\| (m_4(zA_2 + A_1 + W_1) - m_2 W_2)^{-1} \right\|_p^{-1} \leq \|m_4(A_0 - zW_1) + m_2 z W_2\|_p \right\},$$
$$R_2 = \left\{ z \in \mathbb{C} : \left\| (m_3(A_0 - zW_1) + m_1 z W_2)^{-1} \right\|_p^{-1} \leq \|m_3(zA_2 + A_1 + W_1) - m_1 W_2\|_p \right\}$$

as 0 belongs to the set R_1 associated with $\mathrm{rev}L(z)$ for all choices of $\mathscr{W}_1$ if and only if A_2 is singular. Suppose $m_4 = 0$. Then $Mv = e_1 \implies m_3 v_1 = 0 \implies m_3 = 0$ or $v_1 = 0$. Both cases are impossible as M is non-singular and $v_1 \neq 0$. Hence, $m_4 \neq 0$. Similarly, if $m_3 = 0$ then $Mv = e_1 \implies m_4 v_4 = 0 \implies m_3 = 0$ or $v_2 = 0$, which are both impossible as M is non-singular and $v_2 \neq 0$. Therefore,

$$R_1 = \left\{ z \in \mathbb{C}^\infty : \left\| \left(zA_2 + A_1 + W_1 - \frac{m_2}{m_4} W_2 \right)^{-1} \right\|_p^{-1} \leq \left\| A_0 - zW_1 + z\frac{m_2}{m_4} W_2 \right\|_p \right\},$$
$$R_2 = \left\{ z \in \mathbb{C} : \left\| \left(A_0 - zW_1 + \frac{m_1}{m_3} W_2 \right)^{-1} \right\|_p^{-1} \leq \left\| zA_2 + A_1 + W_1 - \frac{m_1}{m_3} W_2 \right\|_p \right\}.$$

Now (4.2.5) and (4.2.6) follow by setting $\mathscr{W}_1 = W_1 - \frac{m_2}{m_4}W_2$ and $\mathscr{W}_2 = W_1 - \frac{m_1}{m_3}W_2$. The proof of (ii) follows by identical arguments as in this case by (4.1.2), there exists W_1, $W_2 \in \mathbb{C}^{n,n}$ with W_2 non-singular such that

$$\left(\frac{M}{\alpha} \otimes I_n\right) L(z) = z\left[\begin{array}{c|c} 0 & -W_2 \\ \hline A_2 & -W_1 \end{array}\right] + \left[\begin{array}{c|c} W_2 & 0 \\ \hline W_1 + A_1 & A_0 \end{array}\right].$$

□

Theorem 4.2.5. *Let $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$. Suppose $L(z)$ is a linearization of $P(z)$ in $\mathbb{L}_2(P)$ with corresponding ansatz vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}^2$ which is not a multiple of e_1 or e_2. Let $M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \in \mathbb{C}^{2,2}$ be non-singular.*

(i) If $Mv = \alpha e_1$, $\alpha \in \mathbb{C}\setminus\{0\}$ then the block Geršgorin set $\Gamma(P) = R_1 \cup R_2$ arising from $L(z)$ is such that

$$R_1 = \left\{ z \in \mathbb{C}^\infty : \left\| (zA_2 + A_1 + \mathscr{W}_1)^{-1} \right\|_p^{-1} \leq \left| \frac{v_2}{v_1} \right| \|zA_2 + A_1 + \mathscr{W}_2\|_p \right\}, \tag{4.2.7}$$

$$R_2 = \left\{ z \in \mathbb{C} : \left\| (A_0 - z\mathscr{W}_2)^{-1} \right\|_p^{-1} \leq \left| \frac{v_1}{v_2} \right| \|A_0 - z\mathscr{W}_1\|_p \right\}. \tag{4.2.8}$$

Here $\mathscr{W}_1 = W_1 - \frac{m_2}{m_4}W_2$ and $\mathscr{W}_2 = W_1 - \frac{m_1}{m_3}W_2$ where $W_1, W_2 \in \mathbb{C}^{n,n}$ with W_2 non-singular such that

$$L(z) = \left(z \left[\begin{array}{c|c} A_2 & 0 \\ \hline -W_1 & -W_2 \end{array} \right] + \left[\begin{array}{c|c} W_1 + A_1 & W_2 \\ \hline A_0 & 0 \end{array} \right] \right) (\alpha M^{-t} \otimes I_n).$$

(ii) If $Mv = \alpha e_2$, $\alpha \in \mathbb{C}\setminus\{0\}$ then $\Gamma(P) = R_1 \cup R_2$ is the block Geršgorin set associated with $L(z)$ where R_1 and R_2 are as given in part (i) except for the fact that $\mathscr{W}_1 = W_1 - \frac{m_4}{m_2}W_2$ and $\mathscr{W}_2 = W_1 - \frac{m_3}{m_1}W_2$ where $W_1, W_2 \in \mathbb{C}^{n,n}$ with W_2 non-singular such that

$$L(z) = \left(z \left[\begin{array}{c|c} 0 & A_2 \\ \hline -W_2 & -W_1 \end{array} \right] + \left[\begin{array}{c|c} W_2 & W_1 + A_1 \\ \hline 0 & A_0 \end{array} \right] \right) (\alpha M^{-t} \otimes I_n).$$

Proof. Note that, $Mv = \alpha e_1 \Rightarrow m_3v_1 + m_4v_2 = 0 \Rightarrow \left|\frac{m_3}{m_4}\right| = \left|\frac{v_2}{v_1}\right|$. Also we have, $Mv = \alpha e_2 \Rightarrow m_1v_1 + m_2v_2 = 0 \Rightarrow \left|\frac{m_1}{m_2}\right| = \left|\frac{v_2}{v_1}\right|$. The rest of the proof now follows from that of Theorem 4.2.4 due to the fact that $\mathbb{L}_2(P) = [(\mathbb{L}_1(P^t))]^t$. Also the point $z = 0$ belongs to the set R_1 associated with $\mathrm{rev}L(z)$ for all admissible choices of $\mathscr{W}_1, \mathscr{W}_2 \in \mathbb{C}^{n,n}$ and ansatz vectors $v \in \mathbb{C}^2$ if and only if A_2 is singular. Hence ∞ can belong to R_1. $\square$

Remark 4.2.6. *Observe that in Theorems 4.2.4 and 4.2.5, either*

$$\begin{bmatrix} \mathscr{W}_1 \\ \mathscr{W}_2 \end{bmatrix} = \left(\begin{bmatrix} 1 & -\frac{m_2}{m_4} \\ 1 & -\frac{m_1}{m_3} \end{bmatrix} \otimes I_n \right) \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \tag{4.2.9}$$

$$\text{or} \quad \begin{bmatrix} \mathscr{W}_1 \\ \mathscr{W}_2 \end{bmatrix} = \left(\begin{bmatrix} 1 & -\frac{m_4}{m_2} \\ 1 & -\frac{m_3}{m_1} \end{bmatrix} \otimes I_n \right) \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \tag{4.2.10}$$

where $\begin{bmatrix} 1 & -\frac{m_2}{m_4} \\ 1 & -\frac{m_1}{m_3} \end{bmatrix}$ *and* $\begin{bmatrix} 1 & -\frac{m_4}{m_2} \\ 1 & -\frac{m_3}{m_1} \end{bmatrix}$ *are non-singular as* M *is non-singular. Also, as* W_2 *is non-singular, therefore in each case* $\mathscr{W}_2 - \mathscr{W}_1$ *is non-singular. This has important implications for forming the sets* R_1 *and* R_2 *in Theorem 4.2.4 and Theorem 4.2.5, because it means that if* $\mathscr{W}_1, \mathscr{W}_2 \in \mathbb{C}^{n,n}$ *are arbitrarily chosen such that* $\mathscr{W}_2 - \mathscr{W}_1$ *is non-singular, then infinitely many corresponding pairs of matrices* $W_1, W_2 \in \mathbb{C}^{n,n}$ *with* W_2 *non-singular can be formed via the relations (4.2.9) and (4.2.10) by making different choices of* M *and right ansatz vectors* v*. Therefore, once* R_1 *and* R_2 *as in Theorem 4.2.4 and Theorem 4.2.5 are constructed with such choices of* $\mathscr{W}_1$ *and* $\mathscr{W}_2$*, their union form block Geršgorin sets* $\Gamma(P)$ *that may be associated with infinitely many linearizations in* $\mathbb{L}_1(P)$ *and* $\mathbb{L}_2(P)$ *respectively.*

The next theorem gives the eigenvalue localization sets for the spectra of a quadratic matrix polynomial $P(z)$ arising from strong linearizations in $\mathbb{DL}(P)$ with respect to any ansatz vector v in $\mathbb{C}^2$.

Theorem 4.2.7. *Consider a quadratic matrix polynomial* $P(z) \in \mathcal{P}_n$ *as in (4.0.1). Let* $L(z) \in \mathbb{DL}(P)$ *with corresponding ansatz vector* $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}^2$ *be a strong linearization of* $P(z)$*. The block Geršgorin set corresponding to* $L(z)$ *is given by* $\Gamma(P) = R_1 \cup R_2$ *where*

$$\begin{aligned} R_1 &= \left\{ z \in \mathbb{C}^\infty : \left\| (v_1(zA_2 + A_1) - v_2 A_2)^{-1} \right\|_p^{-1} \le \| z v_2 A_2 + v_1 A_0 \|_p \right\}, \\ R_2 &= \left\{ z \in \mathbb{C} : \left\| (v_2(zA_1 + A_0) - z v_1 A_0)^{-1} \right\|_p^{-1} \le \| z v_2 A_2 + v_1 A_0 \|_p \right\}. \end{aligned}$$

Proof. The proof follows immediately by using the definition of $\Gamma(P)$ and Theorem 4.1.5. Note that 0 belongs to the set R_1 corresponding to $\mathrm{rev}P(z)$ for all choices of ansatz vectors v if and only if A_2 is singular. Hence ∞ is associated with R_1. □

Remark 4.2.8. *Since* $\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$*, it is natural to expect that the block Geršgorin set* $R_1 \cup R_2$ *given in Theorem 4.2.7 is a particular case of the sets* R_1 *and* R_2 *obtained earlier with respect to strong linearizations in* $\mathbb{L}_1(P)$ *and* $\mathbb{L}_2(P)$*. Indeed it is easy to see that if* $v_1 = 0$*, then* R_1 *and* R_2 *in Theorem 4.2.7 are obtained by choosing* $W_1 = -v_2 A_1$ *and* $W_2 = A_2$ *in the sets* R_1 *and* R_2 *in Theorem 4.2.3. The choice of* W_2 *is justified by the fact that if* $L(z) \in \mathbb{DL}(P)$ *with ansatz vector* $v = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$ *is a strong linearization of* $P(z)$*, then* A_2 *is non-singular.*

Similarly, if $v_2 = 0$*, then* R_1 *and* R_2 *as in Theorem 4.2.7 are obtained by choosing* $W_1 = 0$ *and* $W_2 = v_1 A_0$ *in the descriptions of* R_1 *and* R_2 *as given in points (iii) and (iv) of Theorem 4.2.1.*

If $v_1v_2 \neq 0$, then the set $R_1 \cup R_2$ from Theorem 4.2.7 is the same set as the one obtained in Theorem 4.2.4 by choosing $\mathscr{W}_1 = -\frac{v_2}{v_1}A_2$ in (4.2.5) and $\mathscr{W}_2 = \frac{v_1}{v_2}A_0 - A_1$ in (4.2.6). It is also the same set as the one obtained from Theorem 4.2.5 by making the same choices of $\mathscr{W}_1$ and $\mathscr{W}_2$ in (4.2.7) and (4.2.8). Observe that in both instances, the choices of $\mathscr{W}_1$ and $\mathscr{W}_2$ are justified by the fact that if a pencil in $\mathbb{DL}(P)$ with ansatz vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ where $v_1v_2 \neq 0$ is a strong linearization of $P(z)$, then the difference $\mathscr{W}_1 - \mathscr{W}_2 = -\frac{v_1}{v_2}P\left(-\frac{v_2}{v_1}\right)$ has to be non-singular.

The remaining results in this section depict important properties of block Geršgorin sets for $P(z) \in \mathcal{P}_n$ arising from strong linearizations in $\mathbb{L}_1(P)$ or $\mathbb{DL}(P)$. In the next lemma, we state a well known inequality which will be frequently used in the remaining sections of the chapter, a proof of which is included for the sake of completeness.

Lemma 4.2.9. *For any A, $B \in \mathbb{C}^{n,n}$,*

$$\left\|A^{-1}\right\|_p^{-1} - \|B\|_p \leq \left\|(A+B)^{-1}\right\|_p^{-1} \leq \left\|A^{-1}\right\|_p^{-1} + \|B\|_p.$$

Proof. For any X, $Y \in \mathbb{C}^{n,n}$, $Y^{-1} - X^{-1} = Y^{-1}(X-Y)X^{-1}$. Therefore, taking norm $\|\cdot\|_p$ on both sides,

$$\left|\left\|Y^{-1}\right\|_p - \left\|X^{-1}\right\|_p\right| \leq \left\|Y^{-1} - X^{-1}\right\|_p \leq \left\|X^{-1}\right\|_p \left\|Y^{-1}\right\|_p \|X-Y\|_p.$$

Dividing both sides by $\|X^{-1}\|_p \|Y^{-1}\|_p$, we obtain

$$\left|\left\|X^{-1}\right\|_p^{-1} - \left\|Y^{-1}\right\|_p^{-1}\right| \leq \|X-Y\|_p.$$

The desired inequality now follows by replacing X by A and Y by $A+B$. □

Theorem 4.2.10. *Let $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$. Then for any strong linearization $L(z)$ in $\mathbb{L}_1(P)$ or $\mathbb{DL}(P)$,*

$$\{z \in \Lambda(P) : |z| \geq 1\} \subseteq R_1 \quad \textit{and} \quad \{z \in \Lambda(P) : |z| \leq 1\} \subseteq R_2.$$

Proof. In view of Remark 4.2.8, the block Geršgorin sets arising from linearizations in $\mathbb{DL}(P)$ are particular cases of those arising from linearizations in $\mathbb{L}_1(P)$. Thus it is enough to prove the theorem for strong linearization $L(z) \in \mathbb{L}_1(P)$. If $\lambda = \infty$, then by (4.2.5), $\lambda \in R_1$. So we assume without loss of generality that $\lambda \in \mathbb{C}$. Then, we have

$$\begin{aligned}
\left\|(\lambda A_2 + A_1 + \mathscr{W}_1)^{-1}\right\|_p^{-1} &= \left\|((P(\lambda) - A_0)/\lambda + \mathscr{W}_1)^{-1}\right\|_p^{-1} \\
&\leq |\lambda|^{-1}\left(\left\|(P(\lambda))^{-1}\right\|_p^{-1} + \|A_0 - \lambda\mathscr{W}_1\|_p\right) \\
&= |\lambda|^{-1}\|A_0 - \lambda\mathscr{W}_1\|_p \leq \|A_0 - \lambda\mathscr{W}_1\|_p \ \text{(since } |\lambda| \geq 1)
\end{aligned}$$

Thus $\lambda \in R_1$ where R_1 is as in (4.2.5). Now suppose $\lambda \in \Lambda(P)$ with $|\lambda| \leq 1$. Then,

$$\begin{aligned}\left\|(A_0 - \lambda \mathscr{W}_2)^{-1}\right\|_p^{-1} &= \left\|(P(\lambda) - \lambda(A_2\lambda + A_1 + \mathscr{W}_2))^{-1}\right\|_p^{-1} \\ &\leq \left\|(P(\lambda))^{-1}\right\|_p^{-1} + |\lambda| \, \|A_2\lambda + A_1 + \mathscr{W}_2\|_p \\ &\leq \|A_2\lambda + A_1 + \mathscr{W}_2\|_p \qquad (\text{as } |\lambda| \leq 1)\end{aligned}$$

Thus $\lambda \in R_2$ where R_2 is as in (4.2.6). Since $\|S\|_p \|S^{-1}\|_p \geq 1$, for any non-singular matrix S, the proofs for the case of the other linearizations in $\mathbb{L}_1(P)$ with ansatz vectors αe_1 or αe_2 follow in a similar way. □

Theorem 4.2.10 has the important consequence of making the inclusion regions containing the eigenvalues of $P(z) \in \mathcal{P}_n$ even more tight than the original block Geršgorin sets $\Gamma(P)$.

Corollary 4.2.11. *Let $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ and D be the closed unit disc. Let $\Gamma(P) = R_1 \cup R_2$ be a block Geršgorin region where R_1 and R_2 are associated with linearizations in $\mathbb{L}_1(P)$ or $\mathbb{DL}(P)$. Then*

$$\Lambda(P) \subseteq (R_1 \cap D^c) \cup (R_2 \cap D). \tag{4.2.11}$$

Moreover, if R_1 is given by (4.2.5), and the corresponding R_2 is given by (4.2.6), then

$$\Lambda(P) = \bigcap \{(R_1 \cap D^c) \cup (R_2 \cap D) : (\mathscr{W}_1, \mathscr{W}_2) \in \mathcal{T}\} \tag{4.2.12}$$

where $\mathcal{T} = \{(X, Y) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n} : X - Y \text{ is non-singular}\}$.

Proof. Since (4.2.11) is an immediate consequence of Theorem 4.2.10, we prove (4.2.12). So let,

$$\begin{aligned} R_1 &= \left\{ z \in \mathbb{C}^\infty : \left\|(zA_2 + A_1 + \mathscr{W}_1)^{-1}\right\|_p^{-1} \leq \|A_0 - z\mathscr{W}_1\|_p \right\}, \\ R_2 &= \left\{ z \in \mathbb{C} : \left\|(A_0 - z\mathscr{W}_2)^{-1}\right\|_p^{-1} \leq \|zA_2 + A_1 + \mathscr{W}_2\|_p \right\} \end{aligned}$$

where $(\mathscr{W}_1, \mathscr{W}_2) \in \mathcal{T}$, in view of Remark 4.2.6. Since $\Lambda(P) \subseteq (R_1 \cap D^c) \cup (R_2 \cap D)$ for all $(\mathscr{W}_1, \mathscr{W}_2) \in \mathcal{T}$, to complete the proof, we show that

$$\bigcap_{(\mathscr{W}_1, \mathscr{W}_2) \in \mathcal{T}} (R_1 \cap D^c) \cup (R_2 \cap D) \subseteq \Lambda(P).$$

Let $z \in (R_1 \cap D^c) \cup (R_2 \cap D)$ for all $(\mathscr{W}_1, \mathscr{W}_2) \in \mathcal{T}$. If $|z| > 1$, then $z \in R_1 \cap D^c$. Let $\mathscr{W}_1 = z^{-1}A_0$ and $\mathscr{W}_2 = \mathscr{W}_1 + M$ for some non-singular $M \in \mathbb{C}^{n,n}$, so that $(\mathscr{W}_1, \mathscr{W}_2) \in \mathcal{T}$. Since $z \in R_1$ for any $(\mathscr{W}_1, \mathscr{W}_2) \in \mathcal{T}$, we have

$$\left\|(P(z))^{-1}\right\|_p^{-1} = |z| \left\|(zA_2 + A_1 + A_0/z)^{-1}\right\|_p^{-1} \leq |z| \, \|A_0 - z(A_0/z)\|_p = 0.$$

If $|z| \leq 1$, then $z \in R_2 \cap D$. Since $z \in R_2$ for any $(\mathscr{W}_1, \mathscr{W}_2) \in \mathfrak{T}$, choosing $(\mathscr{W}_2 + M, \mathscr{W}_2) \in \mathfrak{T}$ where $\mathscr{W}_2 = -(A_1 + zA_2)$ and $M \in \mathbb{C}^{n,n}$ is non-singular,

$$\left\|(P(z))^{-1}\right\|_p^{-1} = \left\|(A_0 - z\mathscr{W}_2)^{-1}\right\|_p^{-1} \leq \|zA_2 + A_1 + \mathscr{W}_2\|_p = 0.$$

Therefore, $z \in \Lambda(P)$. Hence, $\bigcap\limits_{(\mathscr{W}_1,\mathscr{W}_2)\in\mathfrak{T}} (R_1 \cap D^c) \cup (R_2 \cap D) \subseteq \Lambda(P)$ and the proof follows. $\square$

To conclude this section we illustrate some of the block Geršgorin regions formed above. In view of Corollary 4.2.11, when the underlying linearization is from $\mathbb{L}_1(P)$ or $\mathbb{DL}(P)$, we will be plotting the inclusion regions as given by (4.2.11).

Example 4.2.12. *Consider the quadratic matrix polynomial* $T(z) = A_2z^2 + A_1z + A_0$, *where*

$$A_2 = \begin{bmatrix} 1.6171 & 1.8699 & -1.1765 \\ -1.3891 & 1.9302 & 1.1206 \\ 1.6517 & 0.0896 & 2.1788 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1.8463 & 2.4104 & -2.0470 \\ -2.9724 & 3.3995 & 1.4926 \\ 2.2453 & -0.7627 & 2.8893 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 0.7922 & 0.0357 & 0.6787 \\ 0.9595 & 0.8491 & 0.7577 \\ 0.6557 & 0.9339 & 0.7431 \end{bmatrix}.$$

Here we plot the localization sets with respect to the norm $\|.\|_2$. *Since the leading coefficient matrix* A_2 *is non-singular, all the eigenvalues of* $T(z)$ *are finite. Figure 4.2.1 (left) plots block Geršgorin sets associated with linearizations in* $\mathbb{L}_2(P)$ *given by Theorem 4.2.5. In view of Remark 4.2.6, we choose* $\mathscr{W}_1 = 0$ *and* $\mathscr{W}_2 = -A_1$, *and* $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}^2$ *as* $v_1 = 2\|A_2\|_2$, $v_2 = \sigma_{\min}(A_2)$. *Figure 4.2.1 (right) plots block Geršgorin sets associated with the second companion linearization given by Theorem 4.2.1(ii).*

Figures 4.2.2 and 4.2.3 plot inclusion sets given by Corollary 4.2.11 for various forms of R_1 *and* R_2.

The choices of R_1 *and* R_2 *in Figure 4.2.2 (left) are given by (4.2.5) and (4.2.6) respectively with* $\mathscr{W}_1 = 0$ *and* $\mathscr{W}_2 = -A_1$ *(as* A_1 *is non-singular).*

The sets R_1 *and* R_2 *in Figure 4.2.2 (right) arise from linearizations in* $\mathbb{DL}(P)$ *as given by Theorem 4.2.7. Here since* $\frac{-2}{\sigma_{\min}(A_2)\|A_2\|_2}$ *is not an eigenvalue of* $T(z)$, *we choose* $v_1 = \|A_2\|_2$ *and* $v_2 = \frac{2}{\sigma_{\min}(A_2)}$.

Finally, the R_1 *and* R_2 *in Figure 4.2.3 are associated with the first companion linearization given by Theorem 4.2.1(i).*

For comparison we also plot the Geršgorin-type localization sets for $T(z)$ *as obtained in Chapter 3 without going to its linearization. For this purpose consider a* *partition* $\pi = \{0, 2, 3\}$ *of* $T(z)$ *and norm* $\|\cdot\|_2$. *Then the block Geršgorin set is illustrated in Figure 4.2.4 (left). The block minimal Geršgorin set, the block Brualdi set and the block Brauer set are identical and are as shown in Figure 4.2.4 (right).*

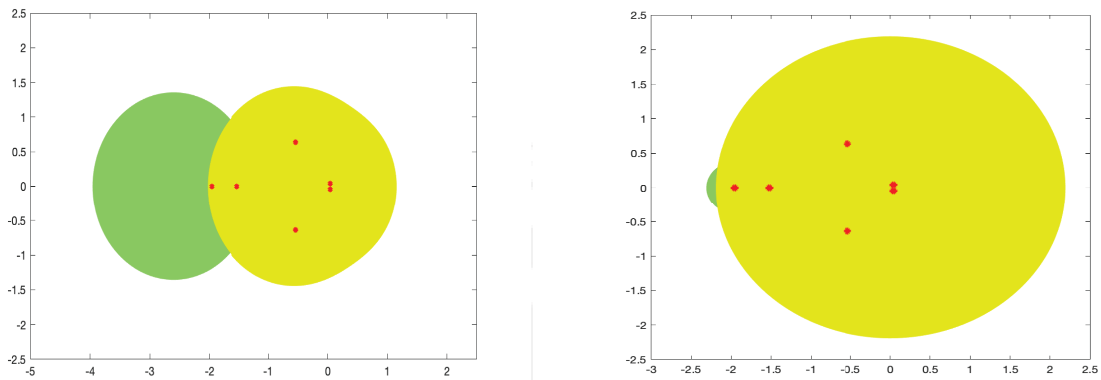


Figure 4.2.1: Block Geršgorin sets for $T(z)$ corresponding to linearizations in $\mathbb{L}_2(P)$ (left) and second companion form (right) for $T(z)$. The eigenvalues are represented by $\bullet$.

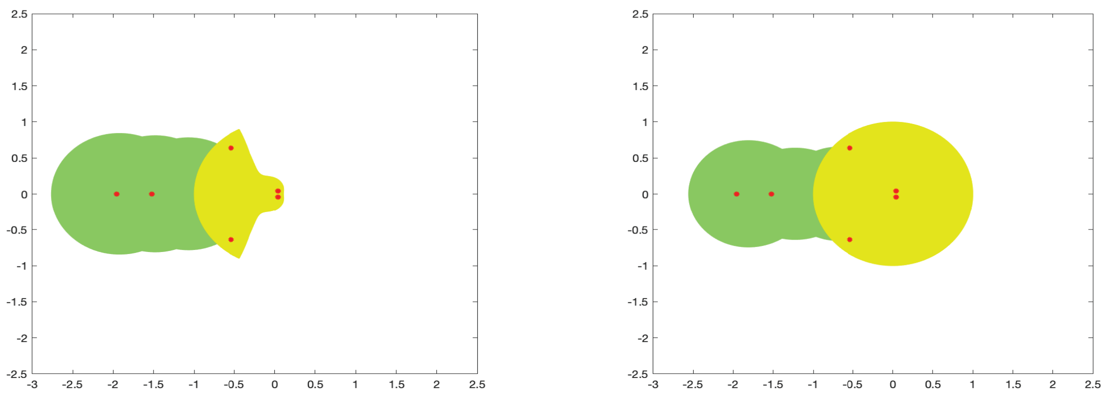


Figure 4.2.2: Inclusion sets given by Corollary 4.2.11 where R_1 and R_2 are associated with linearizations in $\mathbb{L}_1(P)$ (left) and with linearizations in $\mathbb{DL}(P)$ (right) for $T(z)$. The symbol $\bullet$ denote the eigenvalues.

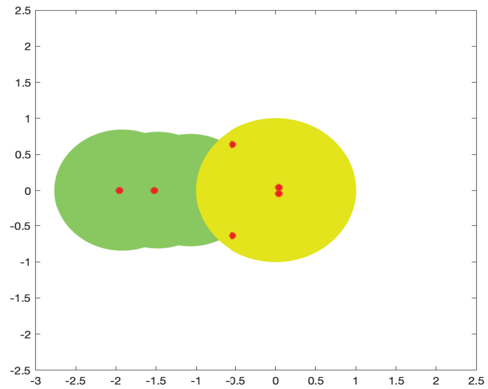


Figure 4.2.3: Inclusion set given by Corollary 4.2.11 where R_1 and R_2 are associated with the first companion linearization for $T(z)$. The eigenvalues are denoted by $\bullet$.

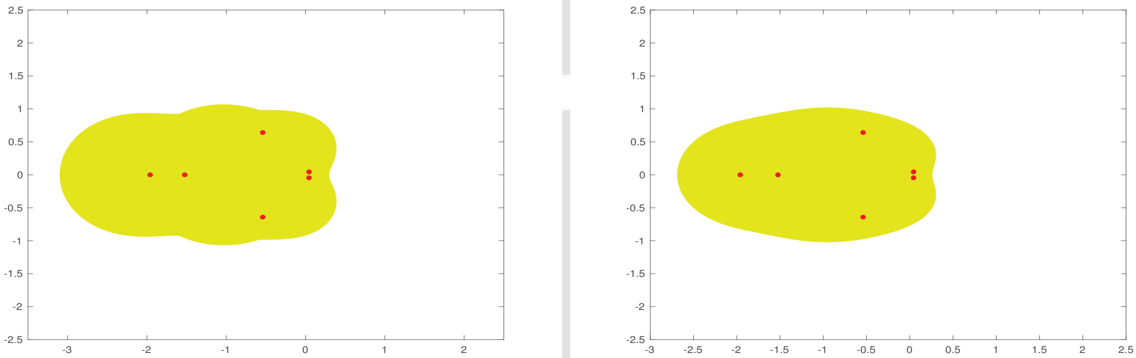


Figure 4.2.4: Block Geršgorin set (left) and block minimal Geršgorin set (right) corresponding to the partition $\pi = \{0, 2, 3\}$ for $T(z)$. The eigenvalues are represented by $\bullet$.

4.3 Bounds for eigenvalues of quadratic matrix polynomials

In this section, we derive bounds on the eigenvalues of quadratic matrix polynomials that arise from the block Geršgorin sets obtained in section 4.2. The bounds for eigenvalues have a wide range of applications. The idea of the location of the eigenvalues not only allows better initial guesses or choices of shifts in iterative algorithms but also guides the selection of an appropriate region for plotting the pseudospectra. The following lemma will be useful in deriving the bounds.

Lemma 4.3.1. *Let* $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$. *Let* $\mathscr{W}_1, \mathscr{W}_2 \in \mathbb{C}^{n,n}$ *such that* $\mathscr{W}_1 - \mathscr{W}_2$ *is non-singular and* $\Gamma(P) = R_1 \cup R_2$ *be the corresponding block Geršgorin* *set with* R_1 *and* R_2 *given by (4.2.5) and (4.2.6) respectively. Then,*

$$R_1 \subseteq \left\{ z \in \mathbb{C} : |z| \leq \frac{\|A_0\|_p + \|A_1 + \mathscr{W}_1\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|\mathscr{W}_1\|_p} \right\} \tag{4.3.1}$$

whenever $\left\|A_2^{-1}\right\|_p^{-1} > \|\mathscr{W}_1\|_p$. *Also,*

$$R_2 \subseteq \left\{ z \in \mathbb{C} : |z| \geq \frac{\left\|A_0^{-1}\right\|_p^{-1} - \|A_1 + \mathscr{W}_2\|_p}{\|A_2\|_p + \|\mathscr{W}_2\|_p} \right\}. \tag{4.3.2}$$

Proof. Due to Remark 4.2.6, the proof follows by applying the inequalities in Lemma 4.2.9 to the relations that define R_1 and R_2. □

Theorem 4.3.2. *Let* $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ *where* A_2 *is non-singular. Assume* λ *be any eigenvalue of* $P(z)$.

(i) Then

$$l_1 \leq |\lambda| \leq \min\{u_1, u_2\} \tag{4.3.3}$$

where

$$l_1 := \min\left\{1, \frac{1}{\left\|A_0^{-1}\right\|_p (\|A_2\|_p + \|A_1\|_p)}\right\},$$

$$u_1 := \max\left\{1, \left\|A_2^{-1}\right\|_p (\|A_1\|_p + \|A_0\|_p)\right\},$$

$$\text{and } u_2 := \left\|A_2^{-1}\right\|_p \max\left\{\|A_0\|_p, \|A_2\|_p + \|A_1\|_p\right\}.$$

(ii) If A_2 *and* A_1 *satisfy* $\left\|A_2^{-1}\right\|_p^{-1} > \|A_1\|_p$, *then*

$$|\lambda| \geq \frac{\min\left\{\left\|A_0^{-1}\right\|_p^{-1}, \left\|A_2^{-1}\right\|_p^{-1} - \|A_1\|_p\right\}}{\|A_2\|_p}. \tag{4.3.4}$$

Proof. First we prove the bounds in (4.3.3). By setting $\mathscr{W}_1 = 0$ and $\mathscr{W}_2$ to be any non-singular matrix in (4.3.1) and (4.3.2) respectively, and using Theorem 4.2.10, we have $|\lambda| \leq u_1$. The lower bound $l_1 \leq |\lambda|$ follows with the same choices of $\mathscr{W}_1$ and $\mathscr{W}_2$ applied to rev$P(z)$ while assuming without loss of generality that A_0 is non-singular. The eigenvalues of rev$P(z)$ belong to $R_1 \cup R_2$ where

$$\begin{aligned} R_1 &= \{\mu \in \mathbb{C}^\infty : \left\|W_2^{-1}\right\|_p^{-1} \leq \|\mu A_0 + A_1 + W_1\|_p\}, \\ \text{and } R_2 &= \{\mu \in \mathbb{C} : \left\|(A_2 - \mu W_1)^{-1}\right\|_p^{-1} \leq |\mu| \, \|W_2\|_p\}, \end{aligned}$$

obtained by setting $\alpha = 1$ in (4.2.3) and (4.2.4) for $\mathrm{rev}P(z)$. By Lemma 4.2.9,

$$\mu \in R_1 \implies |\mu| \geq \frac{\left\|W_2^{-1}\right\|_p^{-1} - \|A_1 + W_1\|_p}{\|A_0\|_p}$$
$$\mu \in R_2 \implies |\mu| \geq \frac{\left\|A_2^{-1}\right\|_p^{-1}}{\|W_1\|_p + \|W_2\|_p}.$$

Combining both the inequalities and setting $W_2 = A_2$ and $W_1 = -A_1$, any eigenvalue μ of $\mathrm{rev}P(z)$ satisfies

$$|\mu| \geq \min\left\{\frac{\left\|A_2^{-1}\right\|_p^{-1}}{\|A_0\|_p}, \frac{\left\|A_2^{-1}\right\|_p^{-1}}{\|A_1\|_p + \|A_2\|_p}\right\},$$

so that for eigenvalues $\lambda \in \Lambda(P)$, $|\lambda| \leq u_2$.

To prove (ii) we use the fact that $\Lambda(\mathrm{rev}P) \subset R_1 \cup R_2$ where

$$\begin{aligned} R_1 &= \{\mu \in \mathbb{C}^\infty : |\mu| \left\|A_0^{-1}\right\|_p^{-1} \leq \|A_2\|_p\}, \\ \text{and } R_2 &= \{\mu \in \mathbb{C} : |\mu| \leq \left\|A_2^{-1}\right\|_p \|A_2 + \mu A_1\|_p\} \end{aligned}$$

are given by setting $\alpha = 1$, $W_1 = -A_1$ and $W_2 = A_2$ in Theorem 4.2.1(iv). Therefore, if $\left\|A_2^{-1}\right\|_p^{-1} > \|A_1\|_p$, then by Lemma 4.2.9 any eigenvalue μ of $\mathrm{rev}P(z)$ satisfies

$$|\mu| \leq \max\left\{\frac{\|A_2\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|A_1\|_p}, \frac{\|A_2\|_p}{\left\|A_0^{-1}\right\|_p^{-1}}\right\},$$

where we assume without loss of generality that A_0 is non-singular. Therefore for any eigenvalue λ of $P(z)$,

$$|\lambda| \geq \frac{\min\left\{\left\|A_0^{-1}\right\|_p^{-1}, \left\|A_2^{-1}\right\|_p^{-1} - \|A_1\|_p\right\}}{\|A_2\|_p}.$$

$\square$

Remark 4.3.3. *Another lower bound* $\hat{l}_1 := \min\left\{1, \frac{\left\|A_0^{-1}\right\|_p^{-1} - \|A_1\|_p}{\|A_2\|_p}\right\}$ *on the eigenvalues of* $P(z)$ *as in Theorem 4.3.2 may be obtained by setting* $\mathscr{W}_2 = 0$ *and* $\mathscr{W}_1$ *to be any non-singular matrix in (4.3.1) and (4.3.2) respectively. However, this is not better than* l_1 *as it is easy to see that* $l_1 = \hat{l}_1 = 1$ *(which is the optimal value of both lower bounds) if* $\|A_1\|_p + \|A_2\|_p \leq \left\|A_0^{-1}\right\|_p^{-1}$ *whereas* $l_1 > \hat{l}_1$ *if* $\|A_1\|_p + \|A_2\|_p > \left\|A_0^{-1}\right\|_p^{-1}$*. Similarly, if* $\left\|A_2^{-1}\right\|_p^{-1} > \|A_1\|_p$*, then an upper bound* $\hat{u}_1 := \max\left\{1, \frac{\|A_0\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|A_1\|_p}\right\}$

may be obtained by choosing $\mathscr{W}_2 = 0$ *and* $\mathscr{W}_1$ *to be any non-singular matrix in (4.3.2) and (4.3.1) respectively applied to* $\mathrm{rev}P(z)$*. However, this is not better than* u_1 *as again it is easy to see that* $u_1 = \hat{u}_1 = 1$ *(which is the best value of both upper bounds) if* $\|A_0\|_p + \|A_1\|_p \leq \left\|A_2^{-1}\right\|_p^{-1}$*, whereas* $u_1 < \hat{u}_1$ *if* $\|A_0\|_p + \|A_1\|_p > \left\|A_2^{-1}\right\|_p^{-1}$*. In summary, the best possible upper and lower bounds that arise from choosing one of the matrices* $\mathscr{W}_1$ *and* $\mathscr{W}_2$ *to be zero in Lemma 4.3.1 are* l_1 *and* u_1 *respectively.*

Now suppose $\mathscr{W}_1, \mathscr{W}_2 \in \mathbb{C}^{n,n}$ *in Lemma 4.3.1 are such that* $\mathscr{W}_1 - \mathscr{W}_2$ *is non-singular,* $\mathscr{W}_1 \neq 0$ *and* $\|\mathscr{W}_1\|_p < \left\|A_2^{-1}\right\|_p^{-1}$*. Then relation (4.3.1) provides the upper bound* $\widetilde{u}_1 := \max\left\{1, \frac{\|A_0\|_p + \|A_1 + \mathscr{W}_1\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|\mathscr{W}_1\|_p}\right\}$*. Suppose* $\widetilde{u}_1 < u_1$*. Then* $u_1 > 1$ *which implies*

$$\left\|A_2^{-1}\right\|_p^{-1} < \|A_0\|_p + \|A_1\|_p \tag{4.3.5}$$

and at least one of the following holds.

$$\frac{\|A_0\|_p + \|A_1 + \mathscr{W}_1\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|\mathscr{W}_1\|_p} \leq 1, \tag{4.3.6}$$

$$\text{and} \quad \frac{\|A_0\|_p + \|A_1 + \mathscr{W}_1\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|\mathscr{W}_1\|_p} < \left\|A_2^{-1}\right\|_p (\|A_1\|_p + \|A_0\|_p). \tag{4.3.7}$$

If (4.3.6) holds, then $\left\|A_2^{-1}\right\|_p^{-1} > \|A_0\|_p + \|A_1 + \mathscr{W}_1\|_p + \|\mathscr{W}_1\|_p$*. Since (4.3.5) holds, this implies that* $\|A_1 + \mathscr{W}_1\|_p + \|\mathscr{W}_1\|_p < \|A_1\|_p$*. But this is impossible as* $\mathscr{W}_1 \neq 0$*. Similarly, if (4.3.7) holds then once again by applying (4.3.5), we obtain* $\|A_1 + \mathscr{W}_1\|_p + \|\mathscr{W}_1\|_p < \|A_1\|_p$ *which is also impossible as* $\mathscr{W}_1 \neq 0$*. Therefore, an optimal choice of* $\mathscr{W}_1$ *to obtain an upper bound from (4.3.1) is* $\mathscr{W}_1 = 0$*. Another upper bound* $\breve{u}_1 := \max\left\{1, \frac{\|A_0\|_p + \|\mathscr{W}_2\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|A_1 + \mathscr{W}_2\|_p}\right\}$ *arises by choosing* $\mathscr{W}_1, \mathscr{W}_2 \in \mathbb{C}^{n,n}$ *satisfying the conditions of Lemma 4.3.1 such that* $\mathscr{W}_2 \neq 0$ *and* $\|A_1 + \mathscr{W}_2\|_p < \left\|A_2^{-1}\right\|_p^{-1}$*. If* $\breve{u}_1 < u_1$*, then* $u_1 > 1$ *so that (4.3.5) holds and at least one of the following is also true.*

$$\frac{\|A_0\|_p + \|\mathscr{W}_2\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|A_1 + \mathscr{W}_2\|_p} \leq 1, \tag{4.3.8}$$

$$\text{and} \quad \frac{\|A_0\|_p + \|\mathscr{W}_2\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|A_1 + \mathscr{W}_2\|_p} < \left\|A_2^{-1}\right\|_p (\|A_1\|_p + \|A_0\|_p). \tag{4.3.9}$$

Since (4.3.5) holds, both (4.3.8) and (4.3.9) imply that $\|A_1 + \mathscr{W}_2\|_p + \|\mathscr{W}_2\|_p < \|A_1\|_p$*. But this is impossible for any* $\mathscr{W}_2 \neq 0$*. As already established earlier, choosing* $\mathscr{W}_2 = 0$*, results in* $\breve{u} = \hat{u}$ *which is also not better than* u_1*. Therefore,* u_1 *is the optimal choice of an upper bound from Lemma 4.3.1 for the eigenvalues of* $P(z)$*. As a consequence,*

$\max\left\{1, \left\|A_0^{-1}\right\|_p (\|A_1\|_p + \|A_2\|_p)\right\}$ *is an optimal upper bound on the eigenvalues of* $\mathrm{rev}P(z)$ *from Lemma 4.3.1 so that* l_1 *is the optimal lower bound from Lemma 4.3.1 on the eigenvalues of* $P(z)$.

Note that the relations in Lemma 4.3.1 arise from applying Lemma 4.2.9 to block Geršgorin localizations associated with strong linearizations of $P(z)$ *belonging to* $\mathbb{L}_1(P)$ *with ansatz vector* $v \in \mathbb{C}^2$ *that is not a multiple of* e_1 *or* e_2 *(Theorem 4.2.4) and the eigenvalue bounds are obtained by using Theorem 4.2.10. Therefore, with this approach, the linearizations of* $P(z)$ *from this class that provide the tightest eigenvalue bounds are obtained by choosing* W_1, $W_2 \in \mathbb{C}^{n,n}$ *and ansatz vector* $v \in \mathbb{C}^2$ *in such a way that one among* $\mathscr{W}_1$ *and* $\mathscr{W}_2$ *is zero and the other is non-singular. Observe that the upper bound* u_1 *and the lower bound* l_1 *are also obtained from the sets* R_1 *in (4.2.1) arising from the first companion linearizations associated with* $P(z)$ *and* $\mathrm{rev}P(z)$ *respectively.*

If the linearization in $\mathbb{L}_1(P)$ *corresponds to the ansatz vector* αe_1*, then the corresponding block Geršgorin set* $R_1 \cup R_2$ *is given by Theorem 4.2.1(iii) and due to Theorem 4.2.10, all upper and lower bounds on the eigenvalues of* $P(z)$ *arise from considering the set* R_1 *for* $P(z)$ *and* $\mathrm{rev}P(z)$*. Clearly,* R_1 *for* $P(z)$ *satisfies*

$$R_1 \subseteq \left\{ z \in \mathbb{C} : |z| \leq \frac{\|A_0\|_p + \|A_1 + \alpha^{-1}W_1\|_p}{\left\|A_2^{-1}\right\|_p^{-1} - \|\alpha^{-1}W_1\|_p} \right\}$$

for any $W_1 \in \mathbb{C}^{n,n}$ *such that* $\|W_1\|_p < |\alpha| \left\|A_2^{-1}\right\|_p^{-1}$*. By arguing as in the proof of Theorem 4.3.2 and in the preceding lines of this remark, the optimal upper and lower bounds on the eigenvalues of* $P(z)$ *obtained from this approach are those for which* $W_1 = 0$ *and are in fact* u_1 *and* l_1*.*

If Lemma 4.2.9 is used on the block Geršgorin set $R_1 \cup R_2$ *arising from linearizations in* $\mathbb{L}_1(P)$ *with ansatz vector* αe_2 *(as given in Theorem 4.2.3), then*

$$R_1 \subseteq \{z \in \mathbb{C} : |z| \geq 1\} \quad \textit{and} \quad R_2 \subseteq \left\{ z \in \mathbb{C} : |z| \geq \frac{\left\|A_0^{-1}\right\|_p^{-1} - \|A_1 + \alpha^{-1}W_1\|_p}{\|A_2\|_p + \|\alpha^{-1}W_1\|_p} \right\}.$$

Clearly, any useful eigenvalue bounds can only arise from using the inclusion involving R_2 *which is identical to (4.3.2) with* $\mathscr{W}_2$ *replaced by* $\alpha^{-1}W_1$*. But the preceding arguments show that such bounds are not better than* l_1 *and* u_1*.*

Therefore, if we take the approach of applying Theorem 4.2.10 and inequalities in Lemma 4.2.9, then l_1 *and* u_1 *are the optimal eigenvalue bounds for* $P(z)$ *arising from block Geršgorin sets associated with strong linearizations of* $P(z)$ *in* $\mathbb{L}_1(P)$ *with respect to the natural partition induced by the linearization.*

The bounds in Theorem 4.3.2 are easy to compute. The quality of such bounds is usually inversely proportional to the computational effort. Now we compare the bounds in Theorem 4.3.2 with other well-known bounds in the literature that also depend upon the norms of the coefficient matrices of $P(z)$. Most of the upper bounds are obtained by assuming A_2 to be non-singular and considering the matrix polynomial $P_U(z) := Iz^2 + U_1 z + U_0$ where

$$U_1 = A_2^{-1} A_1 \text{ and } U_0 = A_2^{-1} A_0, \tag{4.3.10}$$

while the lower bounds are obtained by assuming that A_0 is non-singular and considering the matrix polynomial $P_L(z) := Iz^2 + L_1 z + L_2$ where

$$L_1 = A_0^{-1} A_1 \text{ and } L_2 = A_0^{-1} A_2. \tag{4.3.11}$$

The following bounds were obtained in Lemmas 2.2 and 2.3 of [28] by considering the first companion linearizations $C_U(z)$ and $C_L(z)$ of $P_U(z)$ and $P_L(z)$, respectively.

$$\left(1 + \|L_1\|_p + \|L_2\|_p\right)^{-1} \leq |\lambda| \leq 1 + \|U_0\|_p + \|U_1\|_p, \quad 1 \leq p \leq \infty \tag{4.3.12}$$

$$\max\left(\|L_2\|_1, 1 + \|L_1\|_1\right)^{-1} \leq |\lambda| \leq \max\left(\|U_0\|_1, 1 + \|U_1\|_1\right) \tag{4.3.13}$$

$$\max\left(1, \|L\|_\infty\right)^{-1} \leq |\lambda| \leq \max\left(1, \|U\|_\infty\right) \tag{4.3.14}$$

$$\|I + LL^*\|_2^{-\frac{1}{2}} \leq |\lambda| \leq \|I + UU^*\|_2^{\frac{1}{2}} \tag{4.3.15}$$

where $U := [U_0 \; U_1]$ and $L := [L_2 \; L_1]$. We will also be considering the following bounds that were established for $p = 2$ in [7] and generalized for all subordinate matrix norms in [55].

$$\underbrace{\frac{\sqrt{\|L_1\|_p^2 + 4\|L_2\|_p} - \|L_1\|_p}{2\|L_2\|_p}}_{:=l_p} \leq |\lambda| \leq \underbrace{\frac{\|U_1\|_p + \sqrt{\|U_1\|_p^2 + 4\|U_0\|_p}}{2}}_{:=u_p} \tag{4.3.16}$$

where U_0, U_1 are as in (4.3.10) and L_1, L_2 are as in (4.3.11).

The bounds in (4.3.13), (4.3.14) and (4.3.15) are always tighter than the ones in (4.3.12). Therefore the bounds in (4.3.12) are not considered in the comparisons. Observe that the best possible values of the upper and lower bounds from both Theorem 4.3.2 and the bounds in (4.3.13), (4.3.14), (4.3.15) is 1. The upper (respectively, lower) bounds in (4.3.13), (4.3.14) and (4.3.15) are always strictly greater (respectively, smaller) than 1 unless at least one among A_1 and A_0 (respectively, A_1 and A_2) are zero. But this is not the case for the bounds in (4.3.3). However, the bounds in (4.3.14) are always better than the ones in Theorem 4.3.2 for $p = \infty$. Therefore, in numerical experiments we consider only the cases $p = 1$ and $p = 2$.

Example 4.3.4. *Consider* $P(z) = A_2z^2 + A_1z + A_0$*, where*

$$A_2 = \begin{bmatrix} 7.114 & 0.595 & 0.448 \\ -3.783 & 1.155 & -6.67 \\ -1.379 & -5.849 & -3.688 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2.546 & -0.089 & 0.578 \\ 2.37 & -1.813 & -2.015 \\ -0.022 & 1.809 & -0.968 \end{bmatrix},$$

$$\text{and } A_0 = \begin{bmatrix} 28.13 & 5.558 & -2.259 \\ -0.186 & -15.96 & -23.51 \\ -0.358 & -22.54 & 18.64 \end{bmatrix}.$$

Here both A_0 *and* A_2 *are non-singular. Tables 4.3.1 and 4.3.2 compares the various eigenvalues bounds of* $P(z)$ *for the* 1*-norm and* 2*-norm respectively.*

Source	Upper bounds	Lower bounds
(4.3.3)	10.6190	1
(4.3.13)	7.6905	0.8231
(4.3.16)	3.4005	1.2915

Table 4.3.1: Comparison of bounds for $P(z)$ in Example 4.3.4 for $p = 1$.

Source	Upper bounds	Lower bounds
(4.3.3)	6.3998	1
(4.3.15)	6.1668	0.9442
(4.3.16)	2.8804	1.5218

Table 4.3.2: Comparison of bounds for $P(z)$ in Example 4.3.4 for $p = 2$.

Example 4.3.5. *Consider* $P(z) = A_2z^2 + A_1z + A_0$*, where*

$$A_2 = \begin{bmatrix} 28.131 & 5.558 & -2.259 \\ -0.186 & -15.955 & -23.505 \\ -0.358 & -22.541 & 18.637 \end{bmatrix}, A_1 = \begin{bmatrix} 2.944 & -0.313 & 0.416 \\ 3.080 & -2.213 & -2.303 \\ -0.132 & 1.871 & -0.924 \end{bmatrix},$$

$$\text{and } A_0 = \begin{bmatrix} 9.053 & -0.827 & 0.521 \\ -5.081 & 1.305 & -7.382 \\ -0.282 & -8.692 & -5.333 \end{bmatrix}.$$

Here too both the coefficient matrices A_2 *and* A_0 *are non-singular and satisfy* $\left\|A_2^{-1}\right\|_p > \|A_1\|_p$*, for* $p = 1, 2$*. Tables 4.3.3 and 4.3.4 shows the various eigenvalue bounds for* $P(z)$ *with respect to* $\|\cdot\|_p$*,* $p = 1, 2$*, respectively.*

Source	Upper bounds	Lower bounds
(4.3.3)	1.1315	0.0844
(4.3.4)	–	0.0961
(4.3.13)	1.2645	0.1285
(4.3.16)	0.8928	0.2795

Table 4.3.3: Comparison of bounds for $P(z)$ in Example 4.3.5 for $p = 1$.

Source	Upper bounds	Lower bounds
(4.3.3)	1	0.1351
(4.3.4)	–	0.1562
(4.3.15)	1.0844	0.1636
(4.3.16)	0.7330	0.3360

Table 4.3.4: Comparison of bounds for $P(z)$ in Example 4.3.5 for $p = 2$.

In the above examples, the bounds from(4.3.16) are much tighter than those in Theorem 4.3.2. In general, the upper (respectively, lower) bounds in (4.3.16) have the advantage that they can by smaller (respectively, greater) than 1, unlike the bounds in Theorem 4.3.2 or even the ones in (4.3.13) and (4.3.15). The following examples show that if $\min\{u_1, u_2\}$ and l_1 in Theorem 4.3.2 are close to their optimal value (which is 1), then they can be quite close to u_p and l_p from (4.3.16).

Example 4.3.6. *Consider $P(z) = A_2 z^2 + A_1 z + A_0$, where*

$$A_2 = \begin{bmatrix} -0.6404 & -0.3577 & -0.6797 \\ 0.3121 & -0.9298 & 0.1952 \\ -0.7018 & -0.0872 & 0.7071 \end{bmatrix}, A_1 = \begin{bmatrix} 0.2515 & -0.2446 & 0.0653 \\ -0.2446 & -0.1394 & 0.5245 \\ 0.0653 & 0.5245 & -0.4158 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 0.1289 & 0.0150 & -0.0023 \\ 0.0150 & 0.1212 & 0.0142 \\ -0.0023 & 0.0142 & 0.1177 \end{bmatrix}.$$

The coefficient matrices A_0 and A_2 are non-singular. Tables 4.3.5 and 4.3.6 compares the various eigenvalues bounds of $P(z)$ with respect to the 1-norm and 2-norm respectively.

Source	Upper bounds	Lower bounds
(4.3.3)	1.9394	0.0356
(4.3.13)	2.1245	0.0619
(4.3.16)	1.2961	0.0944

Table 4.3.5: Comparison of bounds for $P(z)$ in Example 4.3.6 for $p = 1$.

Source	Upper bounds	Lower bounds
(4.3.3)	1	0.0538
(4.3.15)	1.3218	0.0789
(4.3.16)	1	0.1097

Table 4.3.6: Comparison of bounds for $P(z)$ in Example 4.3.6 for $p = 2$.

Example 4.3.7. *Consider $P(z) = A_2 z^2 + A_1 z + A_0$, where*

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -0.0532 & 0.1306 & -0.0780 \\ 0.0516 & -0.1687 & -0.2835 \\ -0.1381 & -0.1572 & 0.1385 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 0.0663 & -0.2776 & 0.0533 \\ -0.1538 & -0.0166 & 0.0523 \\ 0.2792 & -0.0392 & -0.1444 \end{bmatrix}.$$

The coefficient matrices A_2 and A_0 are both non-singular and satisfies $\left\|A_2{}^{-1}\right\|_p > \|A_1\|_p$, for $p = 1, 2$. Tables 4.3.7 and 4.3.8 compares the various eigenvalue bounds for $P(z)$ with respect to norms $\|\cdot\|_p$, $p = 1, 2$, respectively.

Source	Upper bounds	Lower bounds
(4.3.3)	1	0.0140
(4.3.4)	–	0.0210
(4.3.13)	1.5000	0.0210
(4.3.16)	0.9995	0.0660

Table 4.3.7: Comparison of bounds for $P(z)$ in Example 4.3.7 for $p = 1$.

Source	Upper bounds	Lower bounds
(4.3.3)	1	0.0219
(4.3.4)	–	0.0294
(4.3.15)	1.1053	0.0282
(4.3.16)	0.8107	0.0788

Table 4.3.8: Comparison of bounds for $P(z)$ in Example 4.3.7 for $p = 2$.

Observe that the set of bounds involving U_0, U_1, L_1 and L_2 are costlier to compute and in theory, the lower bounds involving L_1 and L_2 cannot be derived unless A_0 is non-singular. In contrast, since we assume that $\left\|A_0^{-1}\right\|_p^{-1} = 0$ if A_0 is a singular matrix, the lower bounds in Theorem 4.3.2 can be formed without assuming A_0 to be non-singular.

4.4 What do coefficient matrices of quadratic matrix polynomials reveal about their eigenvalues?

In this section we provide various sufficient conditions on the coefficient matrices of $P(z) \in \mathcal{P}_n$ for its eigenvalues to be located in particular regions of the complex plane. Although the statements hold for $\|\cdot\|_p$ for any p, $1 \leq p \leq \infty$, they are stated for the most commonly used norms $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_\infty$.

Given a scalar quadratic polynomial $ax^2 + bx + c$, if $|a| > |b| + |c|$, then both the roots lie in the interior of the closed unit disc. If $|c| > |b| + |a|$, then they lie in the exterior of the closed unit disc while if $|b| > |a| + |c|$, then one of them lies in the exterior of the closed unit disc while the other lies in the interior. The following theorem may be seen as a generalization of these statements for the eigenvalues of a quadratic matrix polynomial.

Theorem 4.4.1. *Let $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$. For $p = 1$, 2 or ∞*

(i) If $\left\|A_2^{-1}\right\|_p^{-1} > \|A_0\|_p + \|A_1\|_p$, then all eigenvalues of $P(z)$ belong to the interior of D.

(ii) If $\left\|A_1^{-1}\right\|_p^{-1} > \|A_0\|_p + \|A_2\|_p$, then $P(z)$ has n eigenvalues in D^c, counting multiplicities.

(iii) If $\left\|A_0^{-1}\right\|_p^{-1} > \|A_1\|_p + \|A_2\|_p$, then all eigenvalues of $P(z)$ belong to D^c.

(iv) If all eigenvalues of $P(z)$ belong either to the interior of D or to D^c, then $\left\|A_1^{-1}\right\|_p^{-1} \leq \|A_0\|_p + \|A_2\|_p$.

Proof. To prove (i), suppose $\left\|A_2^{-1}\right\|_p^{-1} > \|A_0\|_p + \|A_1\|_p$ and assume that there exists $\lambda \in \Lambda(P)$ such that $|\lambda| \geq 1$. From Theorem 4.2.1(i), $\Lambda(P) \subset R_1 \cup D$ where

$$R_1 = \left\{z \in \mathbb{C}^\infty : \left\|(zA_2 + A_1)^{-1}\right\|_p^{-1} \leq \|A_0\|_p\right\}.$$

Note that by Theorem 4.2.10, if $P(z)$ has any eigenvalues on the unit circle, then they belong to R_1. Therefore, $\Lambda(P) \subseteq R_1 \cup \{z \in \mathbb{C} : |z| < 1\}$, and $\lambda \in R_1$. Since $|\lambda| \geq 1$, applying Lemma 4.2.9 gives

$$-\|A_1\|_p + \left\|A_2^{-1}\right\|_p^{-1} \leq \|A_0\|_p \implies \left\|A_2^{-1}\right\|_p^{-1} \leq \|A_0\|_p + \|A_1\|_p ,$$

which contradicts the hypothesis. Hence, all the eigenvalues of $P(z)$ belong to the interior of D.

Next we prove (ii). Suppose $R_1 \cap D \neq \emptyset$. Then $\exists\, z_0 \in D$ such that

$$\begin{aligned}\left\|(z_0A_2 + A_1)^{-1}\right\|_p^{-1} \leq \|A_0\|_p \quad &\implies \quad -|z_0|\,\|A_2\|_p + \left\|A_1^{-1}\right\|_p^{-1} \leq \|A_0\|_p \\ &\implies \quad \left\|A_1^{-1}\right\|_p^{-1} \leq \|A_0\|_p + |z_0|\,\|A_2\|_p \\ &\implies \quad \left\|A_1^{-1}\right\|_p^{-1} \leq \|A_0\|_p + \|A_2\|_p .\end{aligned}$$

Therefore, if $\left\|A_1^{-1}\right\|_p^{-1} > \|A_0\|_p + \|A_2\|_p$, then $R_1 \cap D = \emptyset$. Since R_1 and R_2 arise from the first companion linearization, by Theorem 4.0.1 the number of eigenvalues of $P(z)$ in D is the same as the number of eigenvalues of the matrix pencil zI_n in D counting multiplicities. Clearly, these are n in number. Therefore, the remaining n eigenvalues of $P(z)$ counting multiplicities and including any possible eigenvalue at ∞ belong to R_1. This completes the proof of (ii) as $R_1 \subset D^c$.

The proof of (iii) follows by replacing $P(z)$ by $\text{rev}P(z)$ in the proof of (i). Finally (iv) is a simple corollary of (ii). □

Remark 4.4.2. *It is easy to see that all the eigenvalues of a square matrix pencil $A + zB$, lie in the interior of the unit disc if $\|A\|_p < \|B^{-1}\|_p^{-1}$ while they all lie in the exterior if $\|B\|_p < \|A^{-1}\|_p^{-1}$. Parts (i) and (iii) of Theorem 4.4.1 can also be seen as generalizations of these statements to the quadratic matrix polynomials.*

If a matrix polynomial $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ is $*$-palindromic then $A_2^* = A_0$ and $A_1^* = A_1$ while if $P(z)$ is $*$-antipalindromic, then $A_2^* = -A_0$ and $A_1^* = -A_1$. The following corollary of Theorem 4.4.1 gives a sufficient condition for quadratic $*$-palindromic and $*$-antipalindromic polynomials to have no eigenvalues on the unit circle.

Corollary 4.4.3. *Let $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ be a $*$-palindromic or $*$-antipalindromic matrix polynomial. If any one of the following conditions hold, then $P(z)$ does not have any eigenvalues on the unit circle.*

(i) $|\lambda_{\min}(A_1)| > 2\,\|A_2\|_2$.

(ii) $\left\|A_1^{-1}\right\|_1^{-1} > \|A_j\|_1 + \|A_j\|_\infty$, $j = 0$ *or* 2.

(iii) $\left\|A_1^{-1}\right\|_\infty^{-1} > \|A_j\|_1 + \|A_j\|_\infty$, $j = 0$ *or* 2.

Proof. Here $\sigma_{\min}(A_1) = |\lambda_{\min}(A_1)|$ and $\|A_0\|_2 + \|A_2\|_2 = 2\|A_2\|_2$ due to the structure of $P(z)$. Also $\|A_2\|_1 = \|A_0^*\|_1 = \|A_0\|_\infty$ and $\|A_2\|_\infty = \|A_0^*\|_\infty = \|A_0\|_1$. Therefore, if any of the statements (i), (ii) and (iii) hold, then by Theorem 4.4.1(ii) $P(z)$ has n eigenvalues in D^c. Due to the eigenvalue pairing, the remaining n eigenvalues of $P(z)$ belong to the interior of D. Hence, $P(z)$ has no eigenvalues on the unit circle. □

To prove the other results of this section, we will need the concept of Cayley transforms. Consider the Cayley transform $z \mapsto \frac{1+z}{1-z}$. Its inverse is the Cayley transform $\mu \mapsto \frac{\mu-1}{\mu+1}$. It is well known [46] that the polynomial,

$$Q(\mu) = (1+\mu)^2 P\left(\frac{\mu-1}{\mu+1}\right) := B_2\mu^2 + B_1\mu + B_0 \tag{4.4.1}$$

where $B_2 = A_2 + A_1 + A_0$, $B_1 = 2(A_0 - A_2)$, $B_0 = A_2 - A_1 + A_0$, is such that

(i) $\lambda \neq -1$ is an eigenvalue of $Q(\mu)$ if and only if $\frac{\lambda-1}{\lambda+1}$ is an eigenvalue of $P(z)$.

(ii) $Q(\mu)$ has -1 as an eigenvalue if and only if ∞ is an eigenvalue of $P(z)$.

(iii) The number of eigenvalues of $Q(\mu)$ in D^c, interior of D and on the unit circle excluding -1 are equal to the number of eigenvalues of $P(z)$ on the open right half of the complex plane, the open left half of the complex plane and on the imaginary axis, respectively.

(iv) $Q(\mu)$ has an eigenvalue at ∞ if and only if 1 is an eigenvalue of $P(z)$.

(v) $Q(\mu)$ is $*$-palindromic if $P(z)$ is $*$-even, i.e., $(P(-z))^* = P(\bar{z})$. Further, $Q(\mu)$ is $*$-antipalindromic if $P(z)$ is $*$-odd, i.e., $(P(-z))^* = -P(\bar{z})$.

For a detailed study of the effect of such transforms on matrix polynomials, see [47].

Corollary 4.4.4. *Let* $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$. *For* $p = 1, 2$ *or* ∞

(i) If $\|(A_0 + A_1 + A_2)^{-1}\|_p^{-1} > \|A_0 - A_1 + A_2\|_p + 2\|A_0 - A_2\|_p$, *then all the eigenvalues of* $P(z)$ *belong to the open left half of the complex plane.*

(ii) If $\|(A_0 - A_1 + A_2)^{-1}\|_p^{-1} > \|A_0 + A_1 + A_2\|_p + 2\|A_0 - A_2\|_p$, *then all eigenvalues of* $P(z)$ *belong to the open right half of the complex plane.*

(iii) If all eigenvalues of $P(z)$ *lie on the open left or open right half of the complex plane, then* $2\|(A_0 - A_2)^{-1}\|_p^{-1} \leq \|A_0 + A_1 + A_2\|_p + \|A_0 - A_1 + A_2\|_p$.

Proof. Consider $Q(\mu)$ as defined in (4.4.1). The proofs of (i), (ii) and (iii) follow by applying Theorem 4.4.1(i), (ii) and (iv) to $Q(\mu)$ respectively. $\square$

Corollary 4.4.4 also provides a sufficient condition for any quadratic matrix polynomial to have no eigenvalues on the imaginary axis which is important in many applications [21, 79].

If $P(z) = A_2z^2 + A_1z + A_0$ is $*$-even, then A_2 and A_0 are Hermitian matrices and A_1 is skew-Hermitian whereas if $P(z)$ is $*$-odd, then A_2 and A_0 are skew-Hermitian and A_1 is Hermitian. The $*$-even and $*$-odd matrix polynomials are together referred to as $*$-alternating matrix polynomials. The following corollary of Theorem 4.4.1 gives a sufficient condition for $*$-alternating polynomials in $\mathcal{P}_n$ to have no eigenvalues on the imaginary axis.

Corollary 4.4.5. *Let $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ be $*$-even or $*$-odd. Then $P(z)$ does not have any purely imaginary eigenvalues if any one of the following holds.*

(i) $|\lambda_{\min}(A_0 - A_2)| > \|A_0 + A_1 + A_2\|_2$.

(ii) $2\,\|(A_0 - A_2)^{-1}\|_p^{-1} > \|A_0 + A_1 + A_2\|_1 + \|A_0 + A_1 + A_2\|_\infty,\ p = 1$ *or* ∞.

Proof. Let $P(z)$ be $*$-even. Then $Q(\mu)$ as in (4.4.1) is $*$-palindromic. By Corollary 4.4.3, if any one of (i) or (ii) holds, then $Q(\mu)$ does not have any eigenvalues on the unit circle. Consequently, $P(z)$ does not have any purely imaginary eigenvalues. If $P(z)$ is $*$-odd, then the proof follows from the fact that $iP(z)$ is $*$-even. $\square$

The spectral symmetry associated with $*$-even or $*$-odd matrix polynomials is called Hamiltonian spectral symmetry as it is also associated with Hamiltonian matrices. Besides $*$-alternating matrix polynomials, Hamiltonian spectral symmetry is also displayed by matrix polynomials whose coefficient matrices alternate between being Hamiltonian and skew-Hamiltonian. In fact there are a host of other quadratic operator eigenvalue problems that display Hamiltonian spectral symmetry [57]. Corollary 4.4.5 is likely to be applicable in all such settings.

The following corollary of Theorem 4.4.1 gives a sufficient condition for a Hermitian matrix polynomial in $\mathcal{P}_n$ to have no real eigenvalues.

Corollary 4.4.6. *Let $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ be a Hermitian matrix polynomial. Then $P(z)$ has no real eigenvalues if any one of the following statements hold.*

(i) $|\lambda_{\min}(A_0 + A_2)| > \|A_0 + iA_1 - A_2\|_2$.

(ii) $2\,\|(A_0 + A_2)^{-1}\|_p^{-1} > \|A_0 + iA_1 - A_2\|_1 + \|A_0 + iA_1 - A_2\|_\infty;\ p = 1$ *or* ∞.

Proof. The proof follows from Corollary 4.4.5, due to the fact that any $\lambda \in \mathbb{R}$ is an eigenvalue of $P(z)$ if and only if $i\lambda$ is an eigenvalue of the $*$-even polynomial $-A_2z^2 - iA_1z + A_0$. $\square$

Remark 4.4.7. *Clearly, for $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$, $P(1) = A_0 + A_1 + A_2$, $P(-1) = A_0 - A_1 + A_2$, $P(i) = A_0 + iA_1 - A_2$, $P(i) + P(-i) = 2(A_0 - A_2)$ and $P(1) + P(-1) = 2(A_0 + A_2)$. Therefore, Corollaries 4.4.4-4.4.6 may be restated in terms of $P(1), P(-1), P(i)$ and $P(-i)$. In fact the quantities in the LHS of the inequalities in Corollary 4.4.4(i) and Corollary 4.4.4(ii) are proportional to the backward errors of 1 and -1 respectively considered as approximate eigenvalues of $P(z)$, the constant of proportionality being a function of the choice of the norm on $P(z)$ (see for example, [1]). Therefore, these statements essentially imply that if appropriate backward errors are large enough, then all the eigenvalues of $P(z)$ lie on one side of the imaginary axis or the other.*

The following examples illustrate the results in this section. The first example illustrates some of the statements in Theorem 4.4.1.

Example 4.4.8. *The polynomial $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ where*

$$A_0 = \begin{bmatrix} -0.311 & -0.693 & 0.364 \\ 0.645 & 0.325 & 0.88 \\ -0.808 & 0.712 & -0.578 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.415 & -0.782 & 0.307 \\ 0.341 & 0.189 & 0.702 \\ -0.523 & 0.208 & 0.641 \end{bmatrix},$$

$$\text{and } A_2 = \begin{bmatrix} 2.817 & 2.883 & -0.161 \\ 2.157 & -2.042 & 2.012 \\ 0.79 & -0.329 & -4.487 \end{bmatrix},$$

satisfies $\|A_1\|_2 + \|A_0\|_2 < \sigma_{\min}(A_2)$. All its eigenvalues lie in the interior of the unit disc as they are $0.6764, 0.4260, -0.0866 + 0.5486i, -0.0866 - 0.5486i, -0.3652 + 0.0854i, -0.3652 - 0.0854i$.

If the coefficients A_2 and A_1 are swapped, then $\|A_2\|_2 + \|A_0\|_2 < \sigma_{\min}(A_1)$ holds for the resulting polynomial. Three eigenvalues -4.467, 6.2457 and 3.6363 lie in the exterior of the unit disc while the other three -0.3386, 0.2734 and 0.1597 are in the interior. If the leading coefficient A_2 of $P(z)$ is replaced by

$$A_2 = \begin{bmatrix} 1.77 & 5.487 & 17.699 \\ 2.655 & -1.77 & 26.549 \\ -3.982 & 2.655 & 10.18 \end{bmatrix},$$

then the resulting matrix polynomial satisfies $\left\|A_2^{-1}\right\|_1^{-1} > \|A_0\|_1 + \|A_1\|_1$. All its eigenvalues, viz., $0.5084, -0.0631 + 0.4428i, -0.0631 - 0.4428i, -0.4125, -0.0085 + 0.1419i$ and $-0.0085 - 0.1419i$ lie in the interior of the unit disc.

The next two examples illustrate Corollary 4.4.3.

Example 4.4.9. *The $*$-palindromic polynomial $P(z) = A_2 z^2 + A_1 z + A_2^* \in \mathcal{P}_n$ where*

$$A_2 = \begin{bmatrix} -0.6543 & -0.6258 & -0.1447 \\ 0.6668 & -0.1989 & -0.1155 \\ 0.2859 & -1.0094 & 1.0301 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 3.8977 & -0.1465 & -0.1406 \\ -0.1465 & 4.4135 & -0.6894 \\ -0.1406 & -0.6894 & 4.1887 \end{bmatrix}$$

satisfies $|\lambda_{\min}(A_1)| > 2\|A_2\|_2$. As predicted in Corollary 4.4.3, none of the eigenvalues of $P(z)$ lie on the unit circle as they are $2.5497+3.6448i$, $2.5497-3.6448i$, $0.1289+0.1842i$, $0.1289-0.1842i$, -3.7798 *and* -0.2646.

Example 4.4.10. *The $*$-palindromic polynomial $P(z) = A_2 z^2 + A_1 z + A_2^* \in \mathcal{P}_n$ where*

$$A_2 = \begin{bmatrix} -0.1227 & -0.0293 & -0.1085 \\ 0.5001 & -0.0373 & -0.3465 \\ 0.0134 & -0.0118 & 0.1931 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 16.8404 & -0.1103 & -0.0499 \\ -0.8820 & 4.7986 & -0.2890 \\ -2.3204 & -1.6826 & 3.1313 \end{bmatrix}$$

satisfies $\left\|A_1^{-1}\right\|_\infty^{-1} > \|A_2\|_\infty + \|A_2\|_1$. As predicted in Corollary 4.4.3, three eigenvalues of $P(z)$, viz., $23.7278+31.9674i, 23.7278-31.9674i$ *and* -30.7621 *lie in the exterior of the unit disc while the remaining three eigenvalues, viz.,* $0.0063+0.0172i, 0.0063-0.0172i$ *and* -0.0609 *lie in the interior of the unit disc.*

The next example illustrates Corollary 4.4.4.

Example 4.4.11. *The polynomial $A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$ where*

$$A_0 = \begin{bmatrix} 4.83 & -9.792 & 6.132 \\ 7.41 & 2.809 & 14.04 \\ -7.20 & 2.750 & 3.778 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4.251 & -23.75 & 11.599 \\ 8.6 & 7.464 & 16.26 \\ -12.87 & 1.504 & 25.108 \end{bmatrix},$$

$$\text{and } A_2 = \begin{bmatrix} 6.68 & -6.137 & 4.332 \\ 5.172 & 0.49 & 10.061 \\ -4.39 & -0.513 & 5.636 \end{bmatrix}$$

satisfies $\|A_0 - A_1 + A_2\|_2 + 2\|A_0 - A_2\|_2 < \sigma_{\min}(A_0 + A_1 + A_2)$. Its eigenvalues $-3.4448+1.2125i, -3.4448-1.2125i, -0.7286+0.8613i, -0.7286-0.8613i, -0.5310$ *and* -0.3053 *all lie on the open left half of $\mathbb{C}$.*

The next example illustrates Corollary 4.4.5.

Example 4.4.12. *The $*$-even polynomial $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$ where*

$$A_0 = \begin{bmatrix} 6.473 & -0.264 & 0.002 \\ -0.264 & 10.039 & -4.536 \\ 0.002 & -4.536 & 15.622 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -6.463 & -2.153 \\ 6.463 & 0 & 4.470 \\ 2.153 & -4.47 & 0 \end{bmatrix},$$

$$\text{and } A_2 = \begin{bmatrix} -13.02 & 0.469 & 0.704 \\ 0.469 & -12.03 & -1.089 \\ 0.704 & -1.089 & -5.321 \end{bmatrix},$$

satisfies $|\lambda_{\min}(A_0 - A_2)| > \|A_0 + A_1 + A_2\|_2$. As predicted in Corollary 4.4.5, none of the eigenvalues of $P(z)$ lie on the imaginary axis as they are -1.7195, $-0.7257+0.2816i$, $-0.7257-0.2816i$, $0.7257+0.2816i$, $0.7257-0.2816i$ *and* 1.7195.

The final example illustrates Corollary 4.4.6.

Example 4.4.13. *The Hermitian polynomial* $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ *where*

$$A_0 = \begin{bmatrix} -1.848 & -9.299 & 0.394 \\ -9.299 & 4.409 & 3.893 \\ 0.394 & 3.893 & 8.998 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1.1 & -6.693 & 5.858 \\ -6.693 & 2.2 & 0.839 \\ 5.858 & 0.839 & 3.3 \end{bmatrix},$$

$$\text{and } A_2 = \begin{bmatrix} 1.257 & -9.063 & 2.615 \\ -9.063 & 1.158 & -4.070 \\ 2.615 & -4.07 & 14.776 \end{bmatrix},$$

satisfies $|\lambda_{\min}(A_0 + A_2)| > \|A_0 + iA_1 - A_2\|_2$. *As predicted in Corollary 4.4.6,* $P(z)$ *does not have any real eigenvalues as they are* $0.0443 + 1.4007i, 0.0443 - 1.4007i$, $-0.5436 + 0.9411i$, $-0.5436 - 0.9411i$, $-0.2422 + 0.5467i$ *and* $-0.2422 - 0.5467i$.

4.5 Upper bounds on some distances associated with quadratic matrix polynomials

In this section we use the results of the previous section to obtain upper bounds of several distances associated with polynomials in $\mathcal{P}_n$. The distances are measured with respect to the following norms

$$\begin{aligned} \|P\|_{2,2} &:= \|P\|_{2,\|\cdot\|_2} = \sqrt{\|A_2\|_2^2 + \|A_1\|_2^2 + \|A_0\|_2^2}, \\ \|P\|_{\infty,1} &:= \|P\|_{\infty,\|\cdot\|_1} = \max\left\{\|A_2\|_1, \|A_1\|_1, \|A_0\|_1\right\}, \\ \text{and} \quad \|P\|_{1,\infty} &:= \|P\|_{1,\|\cdot\|_\infty} = \|A_2\|_\infty + \|A_1\|_\infty + \|A_0\|_\infty \end{aligned}$$

for $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$. First we consider the classical distance to stability for $P(z) \in \mathcal{P}_n$.

Theorem 4.5.1. *Let* $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$. *If* $P(z)$ *has an eigenvalue in the closed right half of the complex plane, then*

$$\sigma_{\min}(A_0 + A_1 + A_2) \le \|A_0 - A_1 + A_2\|_2 + 2\|A_0 - A_2\|_2.$$

If $\sigma_{\min}(A_0 + A_1 + A_2) = \|A_0 - A_1 + A_2\|_2 + 2\|A_0 - A_2\|_2$, *then given* $\epsilon > 0$, *there exists* $\delta P(z) \in \mathcal{P}_n$ *such that* $(P + \delta P)(z)$ *has no eigenvalues in the closed right half of the complex plane.*

Further, if $\sigma_{\min}(A_0 + A_1 + A_2) < \|A_0 - A_1 + A_2\|_2 + 2\|A_0 - A_2\|_2$, *then the distance to a nearest polynomial in* $\mathcal{P}_n$ *with all eigenvalues on the open left half of the complex plane with respect to the norm* $\|\cdot\|_{2,2}$ *is bounded above by*

$$\frac{1}{2\sqrt{2}}\left(\|A_0 - A_1 + A_2\|_2 + 2\|A_0 - A_2\|_2 - \sigma_{\min}(A_0 + A_1 + A_2)\right)$$

if $\sigma_{\min}(A_0 + A_1 + A_2) \geq \|A_0 - A_1 + A_2\|_2$ *and by*

$$\frac{\sqrt{3}}{2\sqrt{2}} \left(\|A_0 - A_1 + A_2\|_2 + 2\|A_0 - A_2\|_2 - \sigma_{\min}(A_0 + A_1 + A_2)\right) \tag{4.5.1}$$

otherwise.

Proof. Let

$$F(M) = \sigma_{\min}(M_0 + M_1 + M_2) - \|M_0 - M_1 + M_2\|_2 - 2\|M_0 - M_2\|_2 \tag{4.5.2}$$

for any $M(z) = M_2 z^2 + M_1 z + M_0 \in \mathcal{P}_n$. Since $P(z)$ has an eigenvalue in the closed right half of the complex plane, by Corollary 4.4.4(i), $F(P) \leq 0$. Suppose $F(P) = 0$. Let $\varepsilon > 0$ be arbitrary and set $B_0 = A_0 - A_1 + A_2$, $B_1 = 2(A_0 - A_2)$ and $B_2 = A_0 + A_1 + A_2$. If B_0 and B_1 are zero matrices, then $P(z) = (z+1)^2 A_0$ and A_0 is a singular matrix so that $P(z)$ has eigenvalues $-1, 0$ and ∞. Now there exists a perturbation δA_0 such that $A_0 + \delta A_0$ is non-singular and $\|\delta A_0\|_2 < \varepsilon/\sqrt{6}$. Therefore, $\delta P(z) := (z+1)^2 \delta A_0$ is a perturbation with $|\!|\!|\delta P|\!|\!|_{2,2} < \epsilon$ such that $F(P + \delta P) > 0$.

Next suppose $B_1 \neq 0$ and let $\sigma_1 = \cdots = \sigma_m = \|B_1\|_2 > \sigma_{m+1} \geq \cdots \geq \sigma_n$ be singular values of B_1 with corresponding singular vectors u_i, v_i ; $i = 1, 2, \ldots, n$ such that $B_1 v_i = \sigma_i u_i$ and $\|u_i\|_2 = \|v_i\|_2 = 1$ for all $i = 1, 2, \ldots, n$. Choose $\widehat{\varepsilon} > 0$ such that $\widehat{\varepsilon} < \min\{\varepsilon, \|B_1\|_2 - \sigma\}$ where $\sigma = \sigma_{m+1}$ if $1 \leq m \leq n-1$ and $\sigma = 0$ if $m = n$. Let $U_m = [u_1\, u_2 \cdots u_m]$ and $V_m = [v_1\, v_2 \cdots v_m]$. Setting $\delta A_2 = -\delta A_0 = \frac{\widehat{\varepsilon}}{4} U_m V_m^*$, $\delta A_1 = 0$ and $\hat{A}_i = A_i + \delta A_i$, for $i = 0, 1, 2$,

$$\sigma_{\min}(\hat{A}_0 + \hat{A}_1 + \hat{A}_2) - \left\|\hat{A}_0 - \hat{A}_1 + \hat{A}_2\right\|_2 = \sigma_{\min}(B_2) - \|B_0\|_2 = \|B_1\|_2, \tag{4.5.3}$$

$$2\|(A_0 + \delta A_0) - (A_2 + \delta A_2)\|_2 = \|2(A_0 - A_2) - \widehat{\varepsilon} U_m V_m^*\|_2. \tag{4.5.4}$$

Since $\|2(A_0 - A_2) - \widehat{\varepsilon} U_m V_m^*\|_2 = \|B_1 - \widehat{\varepsilon} U_m V_m^*\|_2 = \|B_1\|_2 - \widehat{\varepsilon} < \|B_1\|_2$, from (4.5.3)-(4.5.4) $(P + \delta P)(z) := \hat{A}_2 z^2 + \hat{A}_1 z + \hat{A}_0 \in \mathcal{P}_n$ satisfies $F(P + \delta P) > 0$.

Hence, by Corollary 4.4.4(i), $(P + \delta P)(z)$ has no eigenvalues in the closed right half of the complex plane. Also $\delta P(z) = \frac{\widehat{\varepsilon}}{4} U_m V_m^* (z^2 - 1) \in \mathcal{P}_n$ clearly satisfies $|\!|\!|\delta P|\!|\!|_{2,2} = \frac{\widehat{\varepsilon}}{2\sqrt{2}} < \varepsilon$.

Now if $B_1 = 0$ but $B_0 \neq 0$, then given $\epsilon > 0$ we may construct an perturbation $\delta P(z)$ such that $|\!|\!|\delta P|\!|\!|_{2,2} < \epsilon$ and $F(P + \delta P) > 0$ by replacing the role of B_1 by B_0 in the above arguments. The only difference is that in this case, we choose the perturbation $\delta P(z) := -\frac{\widehat{\varepsilon}}{4} U_m V_m^* (z-1)^2$. Therefore, in all cases, by Corollary 4.4.4(i), $(P + \delta P)(z) \in \mathcal{P}_n$ has no eigenvalues in the closed right half of the complex plane and $|\!|\!|\delta P|\!|\!|_{2,2} < \epsilon$.

Next suppose $F(P) < 0$ and $\sigma_{\min}(B_2) \geq \|B_0\|_2$. Define $s = F(P)$. In view of the preceding arguments, to complete the proof it is enough to construct $\delta P(z)$ satisfying $(P + \delta P)(z) \in \mathcal{P}_n$, $F(P + \delta P) = 0$ and $|\!|\!|\delta P|\!|\!|_{2,2} = |s|/(2\sqrt{2})$.

Let $1 \leq j_0 \leq n$ be the largest value of j such that $\sigma_j > \sigma_{\min}(B_2) - \|B_0\|_2$. Let $s_j = \sigma_{\min}(B_2) - \|B_0\|_2 - \sigma_j$ for $j = m+1, \ldots, j_0$ and consider the matrix $\delta B_1 = sU_m V_m^* + \sum_{j=m+1}^{j_0} s_j u_j v_j^*$. Since $|s| > |s_{m+1}| \geq \cdots \geq |s_{j_0}|$, $\|\delta B_1\|_2 = |s|$ and $\|B_1 + \delta B_1\|_2 = \sigma_{\min}(B_2) - \|B_0\|_2$. So for $\delta P(z) = \frac{\delta B_1}{4}(z^2 - 1)$, we have $(P + \delta P)(z)$ belongs to $\mathcal{P}_n$ with $F(P + \delta P) = 0$ and $|||\delta P|||_{2,2} = \|\delta B_1\|_2 / (2\sqrt{2}) = |s|/(2\sqrt{2})$ which completes the proof for this case.

Finally suppose $F(P) < 0$ and $\sigma_{\min}(B_2) < \|B_0\|_2$. Assuming $B_2 = U\Sigma V^*$ to be a singular value decomposition of B_2, let us define $\delta A_0 = \delta A_2 = -(s/4)UV^*$ and $\delta A_1 = -(s/2)UV^*$. Then, $\delta A_0 + \delta A_1 + \delta A_2 = -sUV^*$ and,

$$\sigma_{\min}\left(\sum_{j=0}^{2}(A_j + \delta A_j)\right) = \|A_0 - A_1 + A_2\|_2 + 2\|A_0 - A_2\|_2,$$
$$\|A_0 - A_1 + A_2\|_2 = \|A_0 + \delta A_0 - (A_1 + \delta A_1) + A_2 + \delta A_2\|_2$$
$$\text{and} \quad 2\|A_0 - A_2\|_2 = 2\|A_0 + \delta A_0 - (A_2 + \delta A_2)\|_2.$$

Therefore, for $\delta P(z) = -sUV^*(z^2 + 2z + 1)/4$, $(P + \delta P)(z) \in \mathcal{P}_n$ and $F(P + \delta P) = 0$. The proof now follows from the fact that $|||\delta P|||_{2,2} = (\sqrt{3}|s|)/(2\sqrt{2})$. $\square$

Next we derive upper bounds on the distance to instability with respect to the norms $|||\cdot|||_{\infty,1}$ and $|||\cdot|||_{1,\infty}$.

Theorem 4.5.2. *Let $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$. If $P(z)$ has an eigenvalue in the closed right half of the complex plane, then*

$$\left\|(A_0 + A_1 + A_2)^{-1}\right\|_p^{-1} \leq \|A_0 - A_1 + A_2\|_p + 2\|A_0 - A_2\|_p, \quad p = 1, \infty.$$

If $\|(A_0 + A_1 + A_2)^{-1}\|_p^{-1} = \|A_0 - A_1 + A_2\|_p + 2\|A_0 - A_2\|_p$, then the distance from $P(z)$ to a nearest matrix polynomial in $\mathcal{P}_n$ having no eigenvalues in the closed right half of the complex plane is zero with respect to the norm $|||\cdot|||_{\infty,1}$ if $p = 1$, and the norm $|||\cdot|||_{1,\infty}$ if $p = \infty$.

Further, if $\|(A_0 + A_1 + A_2)^{-1}\|_p^{-1} < \|A_0 - A_1 + A_2\|_p + 2\|A_0 - A_2\|_p$, then for $p = 1$ the distance to a nearest polynomial in $\mathcal{P}_n$ with all eigenvalues on the open left half of the complex plane with respect to the norm $|||\cdot|||_{\infty,1}$ is bounded above by

$$\frac{1}{2}\|(\|A_0 - A_1 + A_2\|_1 + 2\|A_0 - A_2\|_1) I - (A_0 + A_1 + A_2)\|_1, \tag{4.5.5}$$

while for $p = \infty$, the same distance with respect to the norm $|||\cdot|||_{1,\infty}$ is bounded above by

$$\|(\|A_0 - A_1 + A_2\|_\infty + 2\|A_0 - A_2\|_\infty) I - (A_0 + A_1 + A_2)\|_\infty. \tag{4.5.6}$$

Proof. We prove the theorem with respect to the norm $\|\cdot\|_{\infty,1}$ as the proof for the norm $\|\cdot\|_{1,\infty}$ follows with identical arguments.

Let

$$F(M) = \left\|(M_0 + M_1 + M_2)^{-1}\right\|_1^{-1} - \|M_0 - M_1 + M_2\|_1 - 2\,\|M_0 - M_2\|_1$$

for any $M(z) = M_2z^2 + M_1z + M_0 \in \mathcal{P}_n$. Since $P(z)$ has an eigenvalue in the closed right half of the complex plane, by Corollary 4.4.4(i), $F(P) \leq 0$. Suppose $F(P) = 0$ and $\varepsilon > 0$ be arbitrary. The proof in this case is complete by showing the existence of a polynomial $\delta P(z)$ of degree atmost two such that $(P + \delta P)(z) \in \mathcal{P}_n$ has no eigenvalues in the closed right half of the complex plane and $\|\delta P\|_{\infty,1} < \varepsilon$.

If $A_0 - A_1 + A_2$ and $A_0 - A_2$ are zero matrices, then by arguing as in the proof of Theorem 4.5.1 we can construct a perturbation $\delta P(z)$ to $P(z)$ such that the perturbed polynomial $(P+\delta P)(z) \in \mathcal{P}_n$ has no eigenvalues in the closed right half of the complex plane and $\|\delta P\|_{\infty,1} < \varepsilon$.

Assume $B := A_0 - A_1 + A_2 \neq 0$. Choose $\widehat{\varepsilon} > 0$ such that $\widehat{\varepsilon} < 2\varepsilon/n$ and $\widehat{\varepsilon} < \min\{|b_{i,j}| : b_{i,j} \neq 0,\ 1 \leq i,j \leq n\}$, where $b_{i,j}$ denotes the (i,j)-th entry of B. Define $E = [e_{i,j}] \in \mathbb{C}^{n,n}$ by

$$e_{i,j} = \begin{cases} -\widehat{\varepsilon}\,\mathrm{sign}(b_{i,j}) & \text{if } b_{i,j} \neq 0, \\ 0 & \text{if } b_{i,j} = 0. \end{cases}$$

Then $\|E\|_1 \leq n\widehat{\varepsilon} < 2\varepsilon$ and $\|B + E\|_1 < \|B\|_1$. Setting $\delta A_2 = \delta A_0 = \frac{E}{4}$, $\delta A_1 = -\frac{E}{2}$, and $\hat{A}_i = A_i + \delta A_i$ for $i = 0, 1, 2$,

$$\begin{aligned} \left\|(\hat{A}_0 + \hat{A}_1 + \hat{A}_2)^{-1}\right\|_1^{-1} - 2\left\|\hat{A}_0 - \hat{A}_2\right\|_1 &= \left\|(A_0 + A_1 + A_2)^{-1}\right\|_1^{-1} - 2\,\|A_0 - A_2\|_1\,, \\ \left\|\hat{A}_0 - \hat{A}_1 + \hat{A}_2\right\|_1 = \|B + E\|_1 < \|B\|_1 &= \|A_0 - A_1 + A_2\|_1\,. \end{aligned}$$

Therefore, the matrix polynomial $(P + \delta P)(z) := \hat{A}_2z^2 + \hat{A}_1z + \hat{A}_0 \in \mathcal{P}_n$ satisfies $F(P + \delta P) > 0$ so that by Corollary 4.4.4(i), $(P + \delta P)(z)$ has no eigenvalues in the closed right half of the complex plane. Also $\|\delta P\|_{\infty,1} = \frac{\|E\|_1}{2} < \varepsilon$.

If $A_0 + A_2 - A_1 = 0$ but $A_0 \neq A_2$, then the desired perturbation $\delta P(z)$ is obtained by replacing $A_0 + A_2 - A_1$ by $2(A_0 - A_2)$ in the preceding arguments, and constructing the matrix E in a identical way. In this case the perturbation $\delta P(z) = \sum_{i=0}^{2} \delta A_i z^i$ is chosen as $\delta A_2 = \delta A_0 = -\frac{E}{2}$ and $\delta A_1 = 0$.

Next suppose $F(P) < 0$. In view of the preceding arguments, to complete the proof it is enough to construct a $\delta P(z)$ satisfying $(P + \delta P)(z) \in \mathcal{P}_n$, $F(P + \delta P) = 0$ and $\|\delta P\|_{\infty,1} = \frac{1}{2}\,\|(\|A_0 - A_1 + A_2\|_1 + 2\,\|A_0 - A_2\|_1)\,I - (A_0 + A_1 + A_2)\|_1\,.$

Define $s = \|A_0 - A_1 + A_2\|_1 + 2\|A_0 - A_2\|_1$ and let $E = sI - (A_0 + A_1 + A_2)$. Suppose $\delta P(z) = \delta A_2 z^2 + \delta A_1 z + \delta A_0$ where $\delta A_2 = \delta A_0 = \frac{E}{4}$ and $\delta A_1 = \frac{E}{2}$. Then $(P + \delta P)(z) = \hat{A}_2 z^2 + \hat{A}_1 z + \hat{A}_0 \in \mathcal{P}_n$, where $\hat{A}_i = A_i + \delta A_i$ satisfies

$$\begin{aligned}\left\|(\hat{A}_0 + \hat{A}_1 + \hat{A}_2)^{-1}\right\|_1^{-1} &= \left\|(A_0 + A_1 + A_2 + E)^{-1}\right\|_1^{-1} \\ &= \left\|\hat{A}_0 - \hat{A}_1 + \hat{A}_2\right\|_1 + 2\left\|\hat{A}_0 - \hat{A}_2\right\|_1 = s,\end{aligned}$$

so that $F(P + \delta P) = 0$. Also, $(P + \delta P)(z) \in \mathcal{P}_n$ and

$$\|\delta P\|_{\infty,1} = \frac{\|E\|_1}{2} = \frac{1}{2}\|sI - (A_0 + A_1 + A_2)\|_1 .$$

□

It is well known (see for instance [3, 49]) that if a Hamiltonian matrix has purely imaginary eigenvalues, then an arbitrary small Hamiltonian perturbation may not remove them from the imaginary axis. The following theorem throws light on this problem when $P(z)$ is a $*$-even or $*$-odd matrix polynomial.

Theorem 4.5.3. *Let $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$ be a $*$-even or $*$-odd matrix polynomial. If $P(z)$ has a purely imaginary eigenvalue, then*

$$|\lambda_{\min}(A_0 - A_2)| \leq \|A_0 + A_1 + A_2\|_2 .$$

If $|\lambda_{\min}(A_0 - A_2)| = \|A_0 + A_1 + A_2\|_2$, then given $\varepsilon > 0$, there exists a structure preserving perturbation $\delta P(z)$ such that $\|\delta P\|_{2,2} < \varepsilon$ and $(P + \delta P)(z)$ has no purely imaginary eigenvalues.

If $|\lambda_{\min}(A_0 - A_2)| < \|A_0 + A_1 + A_2\|_2$, then the distance to a nearest $*$-even or $*$-odd matrix polynomial with respect to the norm $\|\cdot\|_{2,2}$ having no purely imaginary eigenvalues is bounded above by

$$\sqrt{\frac{3}{2}}\left(\|A_0 + A_1 + A_2\|_2 - |\lambda_{\min}(A_0 - A_2)|\right). \tag{4.5.7}$$

Proof. Let $P(z)$ be $*$-even and define $F(T) = |\lambda_{\min}(T_0 - T_2)| - \|T_0 + T_1 + T_2\|_2$ for any $*$-even $T(z) = T_2 z^2 + T_1 z + T_0 \in \mathcal{P}_n$. Since $P(z)$ has a purely imaginary eigenvalue, by Corollary 4.4.5(i), $F(P) \leq 0$. Suppose $F(P) = 0$ and $\varepsilon > 0$ be arbitrarily chosen. If $A_0 + A_1 + A_2 = 0$, then $A_1 = 0$ and $A_2 = -A_0$, with both matrices being singular. Therefore, $P(z) = (1 - z^2)A_0$ with eigenvalues $\pm 1, 0$ and ∞. Let δA_0 be a Hermitian perturbation to A_0 such that $A_0 + \delta A_0$ is non-singular and $\|\delta A_0\|_2 < \varepsilon/\sqrt{2}$. Then $\delta P(z) := \delta A_0(1 - z^2)$ is a structure preserving perturbation of $P(z)$ such that $F(P + \delta P) > 0$ and $\|\delta P\|_{2,2} < \varepsilon$.

Suppose $A_0 + A_1 + A_2 \neq 0$ and let the singular values of $A_0 + A_1 + A_2$ be $\sigma_1 \,(= \|A_0 + A_1 + A_2\|_2) \geq \sigma_2 \geq \cdots \geq \sigma_n$ with corresponding left and right singular vectors $u_i,\ v_i;\ i = 1, 2, \ldots, n$ such that $\|u_i\|_2 = \|v_i\|_2 = 1$ and $(A_0+A_1+A_2)v_i = \sigma_i u_i$ for all $i = 1, 2, \ldots, n$. Suppose $\|A_0 + A_1 + A_2\|_2 = \sigma_1 = \cdots = \sigma_m (m \leq n)$ and $\widehat{\varepsilon} > 0$ be so chosen such that $\widehat{\varepsilon} < \sqrt{\frac{2}{3}}\varepsilon$ and $\widehat{\varepsilon} < \|A_0 + A_1 + A_2\|_2 - \sigma$ where $\sigma = 0$ if $m = n$ and $\sigma = \sigma_{m+1}$ otherwise. Further, suppose that $\delta P(z) = \delta A_2 z^2 + \delta A_1 z + \delta A_0$ where

$$\delta A_2 = \delta A_0 = -\widehat{\varepsilon}\left(U_m V_m^* + V_m U_m^*\right)/4 \quad \text{and} \quad \delta A_1 = -\widehat{\varepsilon}\left(U_m V_m^* - V_m U_m^*\right)/2$$

with $U_m = [u_1\, u_2\, \cdots\, u_m]$ and $V_m = [v_1\, v_2\, \cdots\, v_m]$. Then $\delta P(z)$ is clearly $*$-even and $\delta A_0 + \delta A_1 + \delta A_2 = -\widehat{\varepsilon} U_m V_m^*$. Also, the perturbed polynomial $(P + \delta P)(z)$ belongs to the set $\mathcal{P}_n$ and satisfies $F(P + \delta P) = F(P) + \widehat{\epsilon} > 0$. Therefore, by Corollary 4.4.5(i), $(P + \delta P)(z)$ is a $*$-even matrix polynomial that does not have any purely imaginary eigenvalues. Clearly, $\delta A_2 = \delta A_0 = -\frac{\widehat{\varepsilon}}{2}\,\mathrm{Herm}(U_m V_m^*)$ and $\delta A_1 = -\widehat{\varepsilon}\,\mathrm{SHerm}(U_m V_m^*)$, where $\mathrm{Herm}(B) := (B + B^*)/2$ and $\mathrm{SHerm}(B) := (B - B^*)/2$ denote the Hermitian and skew-Hermitian parts of a square matrix B respectively. Therefore,

$$\|\delta P\|_{2,2} = \widehat{\varepsilon}\left(\|\mathrm{Herm}(U_m V_m^*)\|_2^2/2 + \|\mathrm{SHerm}(U_m V_m^*)\|_2^2\right)^{1/2} \leq \widehat{\varepsilon}\sqrt{\frac{3}{2}}\,\|U_m V_m^*\|_2 < \varepsilon.$$

Next suppose $F(P) < 0$. Let $s = |\lambda_{\min}(A_0 - A_2)| - \|A_0 + A_1 + A_2\|_2$. Due to the preceding arguments, the proof in this case is complete if there exists a $*$-even $\delta P(z) \in \mathcal{P}_n$ satisfying $(P + \delta P)(z) \in \mathcal{P}_n$, $F(P + \delta P) = 0$ and $\|\delta P\|_{2,2} \leq \sqrt{3/2}|s|$.

Let $1 \leq j_0 \leq n$ be the largest value of j such that $\sigma_j > |\lambda_{\min}(A_0 - A_2)|$. Define $s_j := |\lambda_{\min}(A_0 - A_2)| - \sigma_j\,;\ j = m+1, \ldots, j_0$ and set $\delta A_2 = \delta A_0 = \mathrm{Herm}(B)/2$ and $\delta A_1 = \mathrm{SHerm}(B)$ where $B = sU_m V_m^* + \sum_{j=m+1}^{j_0} s_j u_j v_j^*$. Then $\delta P(z)$ is $*$-even. Also $\|A_0 + \delta A_0 + A_1 + \delta A_1 + A_2 + \delta A_2\|_2 = |\lambda_{\min}(A_0 - A_2)|$. Therefore, $(P + \delta P)(z) \in \mathcal{P}_n$ satisfies $F(P + \delta P) = 0$ as $|\lambda_{\min}(A_0 - A_2)| = |\lambda_{\min}(A_0 + \delta A_0 - A_2 - \delta A_2)|$. Clearly, $\|B\|_2 = |s|$ as $|s| > |s_{m+1}| \geq \cdots \geq |s_{j_0}|$. This completes the proof for the $*$-even case as

$$\|\delta P\|_{2,2} = \left(\|\mathrm{Herm}(B)\|_2^2/2 + \|\mathrm{SHerm}(B)\|_2^2\right)^{1/2} \leq \sqrt{3/2}\,\|B\|_2 = \sqrt{3/2}|s|.$$

The proof for $*$-odd matrix polynomials follows by identical arguments as in such a case $iP(z)$ is $*$-even. □

Corollary 4.5.4. *Let $P(z) = A_2 z^2 + A_1 z + A_0$ be a $*$-even or $*$-odd matrix polynomial having a purely imaginary eigenvalue. Suppose $\|A_0 + A_1 + A_2\|_2$ is a simple singular value of $A_0 + A_1 + A_2$ and $|\lambda_{\min}(A_0 - A_2)|$ is not less than any other singular value of $A_0 + A_1 + A_2$ except $\|A_0 + A_1 + A_2\|_2$. Then the distance to a nearest matrix polynomial of the same structure that does not have any purely imaginary eigenvalues with respect to the norm $\|\cdot\|_{2,2}$ is bounded above by $\|A_0 + A_1 + A_2\|_2 - |\lambda_{\min}(A_0 - A_2)|$.*

Proof. It follows from the proof of Theorem 4.5.3 that under the given conditions, a $*$-even matrix polynomial $\delta P(z) = \delta A_2 z^2 + \delta A_1 z + \delta A_0$ satisfying

$$\|A_0 + \delta A_0 + A_1 + \delta A_1 + A_2 + \delta A_2\|_2 = |\lambda_{\min}\left((A_0 + \delta A_0) - (A_2 + \delta A_2)\right)|$$

is obtained by choosing $\delta A_2 = \delta A_0 = \frac{s}{4}(u_1 v_1^* + v_1 u_1^*)$ and $\delta A_1 = \frac{s}{2}(u_1 v_1^* - v_1 u_1^*)$ where $s = |\lambda_{\min}(A_0 - A_2)| - \|A_0 + A_1 + A_2\|_2$. Now,

$$\begin{aligned}
|||\delta P|||_{2,2} &= |s|\sqrt{\|u_1 v_1^* + v_1 u_1^*\|_2^2/8 + \|u_1 v_1^* - v_1 u_1^*\|_2^2/4} \\
&< (|s|/2)\sqrt{\|u_1 v_1^* + v_1 u_1^*\|_F^2 + \|u_1 v_1^* - v_1 u_1^*\|_F^2} \\
&= (|s|/2)\sqrt{2\|u_1 v_1^*\|_F^2 + 2\|v_1 u_1^*\|_F^2} \\
&= s \text{ (since } \|u_1 v_1^*\|_F = \|v_1 u_1^*\|_F = 1) \\
&= \|A_0 + A_1 + A_2\|_2 - |\lambda_{\min}(A_0 - A_2)|
\end{aligned}$$

and this completes the proof. $\square$

The next two theorems are counterparts of Theorem 4.5.3 with respect to the norms $|||\cdot|||_{\infty,1}$ and $|||\cdot|||_{1,\infty}$.

Theorem 4.5.5. *Let $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$ be a $*$-even matrix polynomial. If $P(z)$ has a purely imaginary eigenvalue, then*

$$2\left\|(A_0 - A_2)^{-1}\right\|_p^{-1} \leq \|A_0 + A_1 + A_2\|_1 + \|A_0 + A_1 + A_2\|_\infty, \; p = 1, \infty.$$

If $2\|(A_0 - A_2)^{-1}\|_p^{-1} = \|A_0 + A_1 + A_2\|_1 + \|A_0 + A_1 + A_2\|_\infty$, then the distance from $P(z)$ to a nearest $*$-even matrix polynomial in $\mathcal{P}_n$ having no purely imaginary eigenvalues is zero with respect to the norm $|||\cdot|||_{\infty,1}$ if $p = 1$, and $|||\cdot|||_{1,\infty}$ if $p = \infty$.

If

$$2\left\|(A_0 - A_2)^{-1}\right\|_p^{-1} < \|A_0 + A_1 + A_2\|_1 + \|A_0 + A_1 + A_2\|_\infty, \tag{4.5.8}$$

then for $p = 1$ the distance to a nearest $*$-even matrix polynomial having no purely imaginary eigenvalues with respect to $|||\cdot|||_{\infty,1}$ is bounded above by

$$\max\left\{\left\|\left\|(A_0 - A_2)^{-1}\right\|_1^{-1} I - (A_0 + A_2)\right\|_1/2, \|A_1\|_1\right\}, \tag{4.5.9}$$

and for $p = \infty$, the same distance with respect to the norm $|||\cdot|||_{1,\infty}$ is bounded above by

$$\left\|\left\|(A_0 - A_2)^{-1}\right\|_\infty^{-1} I - (A_0 + A_2)\right\|_\infty + \|A_1\|_\infty. \tag{4.5.10}$$

Further, if $A_0 - A_2$ is non-singular and (4.5.8) holds for $p = 1$, the distance of $P(z)$ to a nearest $*$-even matrix polynomial having no purely imaginary eigenvalues *with respect to* $\|\!|\cdot|\!\|_{\infty,1}$ *is bounded above by*

$$\frac{1}{2}\left(\frac{\|A_0+A_1+A_2\|_1+\|A_0+A_1+A_2\|_\infty}{2}\left\|(A_0-A_2)^{-1}\right\|_1-1\right)\|A_0-A_2\|_1, \tag{4.5.11}$$

whereas if (4.5.8) holds for $p=\infty$, *then it is bounded above by*

$$\left(\frac{\|A_0+A_1+A_2\|_1+\|A_0+A_1+A_2\|_\infty}{2}\left\|(A_0-A_2)^{-1}\right\|_\infty-1\right)\|A_0-A_2\|_\infty \tag{4.5.12}$$

with respect to the norm $\|\!|\cdot|\!\|_{1,\infty}$.

Proof. We prove the bounds for the norm $\|\!|\cdot|\!\|_{\infty,1}$ as the proof for the case when distances are measured with respect to the norm $\|\!|\cdot|\!\|_{1,\infty}$ will follow in a similar manner.

Let $P(z)$ be $*$-even and define

$$F(T)=2\left\|(T_0-T_2)^{-1}\right\|_1^{-1}-\|T_0+T_1+T_2\|_1-\|T_0+T_1+T_2\|_\infty$$

for any $*$-even $T(z)=T_2z^2+T_1z+T_0\in\mathcal{P}_n$. Since $P(z)$ has a purely imaginary eigenvalue, by Corollary 4.4.5(ii), $F(P)\leq 0$. Suppose $F(P)=0$ and $\varepsilon>0$ be arbitrary. The proof is complete by showing the existence of a $*$-even polynomial $\delta P(z)$ of degree atmost two such that $(P+\delta P)(z)\in\mathcal{P}_n$ is $*$-even with no purely imaginary eigenvalues and $\|\!|\delta P|\!\|_{\infty,1}<\varepsilon$.

If $A_0+A_1+A_2=0$, we argue as in the proof of Theorem 4.5.3 to construct a $*$-even polynomial $(P+\delta P)(z)\in\mathcal{P}_n$ having no purely imaginary eigenvalues where $\|\!|\delta P|\!\|_{\infty,1}<\varepsilon$.

Let $B=A_0+A_1+A_2\neq 0$ and $\widehat{b}=\min\{|b_{i,j}|: b_{i,j}\neq 0,\ 1\leq i,j\leq n\}$ where $b_{i,j}$ denotes the (i,j)-th entry of B. Choose $\widehat{\varepsilon}>0$ such that $\widehat{\varepsilon}<\min\{\varepsilon/n,\widehat{b}\}$. Define a matrix $E=[e_{i,j}]\in\mathbb{C}^{n,n}$ by

$$e_{i,j}=\begin{cases}-\widehat{\varepsilon}\,\mathrm{sign}(b_{i,j}) & \text{if } b_{i,j}\neq 0,\\ 0 & \text{if } b_{i,j}=0.\end{cases}$$

Then $\|E\|_1\leq n\widehat{\varepsilon}<\varepsilon$ and $\|B+E\|_1+\|B+E\|_\infty<\|B\|_1+\|B\|_\infty$.

Define $\delta P(z)=\sum_{i=0}^{2}\delta A_i z^i$ where $\delta A_2=\delta A_0=\mathrm{Herm}(E)/2$ and $\delta A_1=\mathrm{SHerm}(E)$. Clearly, $\delta P(z)$ is $*$-even and $(P+\delta P)(z):=\hat{A}_2z^2+\hat{A}_1z+\hat{A}_0\in\mathcal{P}_n$, where $\hat{A}_i=A_i+\delta A_i$ satisfies

$$\begin{aligned}2\left\|(\hat{A}_0-\hat{A}_2)^{-1}\right\|_1^{-1}&=2\left\|(A_0-A_2)^{-1}\right\|_1^{-1},\\ \left\|\hat{A}_0+\hat{A}_1+\hat{A}_2\right\|_1+\left\|\hat{A}_0+\hat{A}_1+\hat{A}_2\right\|_\infty&=\|B+E\|_1+\|B+E\|_\infty\\ &<\|A_0+A_1+A_2\|_1+\|A_0+A_1+A_2\|_\infty.\end{aligned}$$

Hence, by Corollary 4.4.5(ii), $F(P+\delta P) > 0$ so that $(P+\delta P)(z)$ is a $*$-even matrix polynomial having no purely imaginary eigenvalue. Also

$$\|\delta P\|_{\infty,1} = \max\{\|\mathrm{Herm}(E)/2\|_1, \|\mathrm{SHerm}(E)\|_1\} \leq n\hat{\varepsilon} < \varepsilon.$$

Next assume $F(P) < 0$. Let $s = \|(A_0 - A_2)^{-1}\|_1^{-1}$. In view of the preceding arguments, to complete the proof it is enough to construct a $*$-even $\delta P(z)$ satisfying $(P+\delta P)(z) \in \mathcal{P}_n$, $F(P+\delta P) = 0$ and $\|\delta P\|_{\infty,1} = \max\left\{\frac{\|sI-(A_0+A_2)\|_1}{2}, \|A_1\|_1\right\}$.

Let $\delta A_2 = \delta A_0 = \frac{sI-(A_0+A_2)}{2}$ and $\delta A_1 = -A_1$. Then $\delta P(z) = \delta A_2 z^2 + \delta A_1 z + \delta A_0$ is $*$-even and $(P+\delta P)(z) = \hat{A}_2 z^2 + \hat{A}_1 z + \hat{A}_0 \in \mathcal{P}_n$, where $\hat{A}_i = A_i + \delta A_i$ satisfies

$$\begin{aligned} 2\left\|(\hat{A}_0 - \hat{A}_2)^{-1}\right\|_1^{-1} &= 2\left\|(A_0 - A_2)^{-1}\right\|_1^{-1} \\ &= \left\|\hat{A}_0 + \hat{A}_1 + \hat{A}_2\right\|_1 + \left\|\hat{A}_0 + \hat{A}_1 + \hat{A}_2\right\|_\infty \end{aligned}$$

so that $(P+\delta P)(z)$ is $*$-even and $F(P+\delta P) = 0$. Also

$$\|\delta P\|_{\infty,1} = \max\left\{\frac{\|sI - (A_0 + A_2)\|_1}{2}, \|A_1\|_1\right\}.$$

Alternatively, if $A_0 - A_2$ is non-singular such that $F(P) < 0$, define

$$\begin{aligned} s_1 &= \|A_0 + A_1 + A_2\|_1 + \|A_0 + A_1 + A_2\|_\infty, \\ \text{and } \triangle &= \left(\frac{s_1 \|(A_0 - A_2)^{-1}\|_1}{2} - 1\right)(A_0 - A_2). \end{aligned} \tag{4.5.13}$$

Let $\delta P(z) = \delta A_2 z^2 + \delta A_1 z + \delta A_0$ where $\delta A_0 = -\delta A_2 = \triangle/2$ and $\delta A_1 = 0$. Then $\delta P(z)$ is $*$-even and $\hat{A}_i = A_i + \delta A_i$ satisfies

$$\begin{aligned} 2\left\|(\hat{A}_0 - \hat{A}_2)^{-1}\right\|_1^{-1} &= 2\left\|(A_0 - A_2 + \triangle)^{-1}\right\|_1^{-1} \\ &= \left\|\hat{A}_0 + \hat{A}_1 + \hat{A}_2\right\|_1 + \left\|\hat{A}_0 + \hat{A}_1 + \hat{A}_2\right\|_\infty = s_1. \end{aligned}$$

Hence, $(P+\delta P)(z) \in \mathcal{P}_n$ is $*$-even with $F(P+\delta P) = 0$. This completes the proof as

$$\|\delta P\|_{\infty,1} = \|\triangle\|_1/2 = \frac{1}{2}\left(\frac{s_1}{2}\left\|(A_0 - A_2)^{-1}\right\|_1 - 1\right)\|A_0 - A_2\|_1.$$

□

Theorem 4.5.6. *Let $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$ be a $*$-odd matrix polynomial. If $P(z)$ has a purely imaginary eigenvalue, then*

$$2\left\|(A_0 - A_2)^{-1}\right\|_p^{-1} \leq \|A_0 + A_1 + A_2\|_1 + \|A_0 + A_1 + A_2\|_\infty, \ p = 1, \infty.$$

If $2\|(A_0 - A_2)^{-1}\|_p^{-1} = \|A_0 + A_1 + A_2\|_1 + \|A_0 + A_1 + A_2\|_\infty$, then the distance from $P(z)$ to a nearest $*$-odd matrix polynomial in $\mathcal{P}_n$ having no purely imaginary *eigenvalues is zero with respect to the norm* $|||\cdot|||_{\infty,1}$ *if* $p=1$*, and the norm* $|||\cdot|||_{1,\infty}$ *if* $p=\infty$*.*

If

$$2\left\|(A_0-A_2)^{-1}\right\|_p^{-1} < \|A_0+A_1+A_2\|_1 + \|A_0+A_1+A_2\|_\infty, \tag{4.5.14}$$

then for $p=1$ *the distance to a nearest* $*$*-odd matrix polynomial having no purely imaginary eigenvalues with respect to the norm* $|||\cdot|||_{\infty,1}$ *is bounded above by*

$$\max\left\{\left\|\left\|(A_0-A_2)^{-1}\right\|_1^{-1} I - i(A_0+A_2)\right\|_1/2, \|A_1\|_1\right\},$$

whereas if $p=\infty$*, then it is bounded above by*

$$\left\|\left\|(A_0-A_2)^{-1}\right\|_\infty^{-1} I - i(A_0+A_2)\right\|_\infty + \|A_1\|_\infty$$

if the distance is measured with respect to the norm $|||\cdot|||_{1,\infty}$*.*

Further, if A_0-A_2 *is non-singular and (4.5.14) holds with* $p=1$*, then the distance of* $P(z)$ *to a nearest* $*$*-odd matrix polynomial having no imaginary eigenvalues is bounded above by (4.5.11) with respect to the norm* $|||\cdot|||_{\infty,1}$*, whereas if (4.5.14) holds with* $p=\infty$*, then the distance is bounded above by (4.5.12) with respect to the norm* $|||\cdot|||_{1,\infty}$*.*

Proof. The proof for $*$-odd matrix polynomials $P(z)$ follows by identical arguments as in the proof of Theorem 4.5.5 since $iP(z)$ is $*$-even. $\square$

The corresponding results for Hermitian matrix polynomials with respect to real eigenvalues are as follows.

Theorem 4.5.7. *Let* $P(z)=A_2z^2+A_1z+A_0\in\mathcal{P}_n$ *be a Hermitian matrix polynomial having a real eigenvalue. Then* $|\lambda_{\min}(A_0+A_2)|\le\|A_0+iA_1-A_2\|_2$*.*

If $|\lambda_{\min}(A_0+A_2)|=\|A_0+iA_1-A_2\|_2$*, then given* $\varepsilon>0$*, there exists a Hermitian matrix polynomial* $\delta P(z)$ *satisfying* $|||\delta P|||_{2,2}<\varepsilon$ *such that* $(P+\delta P)(z)\in\mathcal{P}_n$ *has no real eigenvalues.*

If $|\lambda_{\min}(A_0+A_2)|<\|A_0+iA_1-A_2\|_2$*, then the distance to a nearest Hermitian matrix polynomial having no real eigenvalues with respect to the norm* $|||\cdot|||_{2,2}$ *is bounded above by*

$$\sqrt{\frac{3}{2}}\left(\|A_0+iA_1-A_2\|_2 - |\lambda_{\min}(A_0+A_2)|\right). \tag{4.5.15}$$

Proof. The proof follows by replacing $A_0+A_1+A_2$ by $A_0+iA_1-A_2$ and $\lambda_{\min}(A_0-A_2)$ by $\lambda_{\min}(A_0+A_2)$ in the proof of Theorem 4.5.3 and making identical arguments. The only difference is in the construction of $\delta P(z)$.

In the case that $|\lambda_{\min}(A_0 + A_2)| = \|A_0 + iA_1 - A_2\|_2$ and $A_0 + iA_1 - A_2 = 0$, we choose the perturbation $\delta P(z) := \delta A_0(z^2+1)$ where δA_0 is a Hermitian perturbation to A_0 such that $A_0 + \delta A_0$ is non-singular and $\|\delta A_0\|_2 < \varepsilon/\sqrt{2}$. Then $(P + \delta P)(z)$ is a Hermitian matrix polynomial without any real eigenvalues and $\|\!|\delta P|\!\|_{2,2} < \varepsilon$. The proof for the case $|\lambda_{\min}(A_0+A_2)| = \|A_0 + iA_1 - A_2\|_2$, with $A_0+iA_1-A_2 \neq 0$ follows by choosing $\delta P(z) = \delta A_2 z^2 + \delta A_1 z + \delta A_0$ where $\delta A_0 = \frac{-\widehat{\varepsilon}}{4}(U_m V_m^* + V_m U_m^*) = -\delta A_2$ and $\delta A_1 = \frac{-\widehat{\varepsilon}}{2i}(U_m V_m^* - V_m U_m^*)$, U_m and V_m being isometries as chosen in Theorem 4.5.3.

Similarly, the proof for the case that $|\lambda_{\min}(A_0+A_2)| < \|A_0 + iA_1 - A_2\|_2$, follows by choosing $\delta A_0 = -\delta A_2 = \mathrm{Herm}(B)/2$ and $\delta A_1 = -i\,\mathrm{SHerm}(B)$, where B is chosen as in Theorem 4.5.3 but with $A_0+A_1+A_2$ replaced by $A_0+iA_1-A_2$ and $\lambda_{\min}(A_0-A_2)$ by $\lambda_{\min}(A_0 + A_2)$. □

The following corollary of Theorem 4.5.7 is immediate.

Corollary 4.5.8. *Let $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ be a Hermitian matrix polynomial having a real eigenvalue. Suppose $\|A_0 + iA_1 - A_2\|_2$ is a simple singular value of $A_0 + iA_1 - A_2$ and $|\lambda_{\min}(A_0 + A_2)|$ is not less than any other singular value of $A_0 + iA_1 - A_2$ except $\|A_0 + iA_1 - A_2\|_2$. Then the distance to a nearest matrix polynomial having no real eigenvalues with respect to the norm $\|\!|\cdot|\!\|_{2,2}$ is bounded above by $\|A_0 + iA_1 - A_2\|_2 - |\lambda_{\min}(A_0 + A_2)|$.*

Theorem 4.5.9. *Let $P(z) = A_2z^2+A_1z+A_0 \in \mathcal{P}_n$ be a Hermitian matrix polynomial having a real eigenvalue. Then*

$$2\left\|(A_0 + A_2)^{-1}\right\|_p^{-1} \leq \|A_0 + iA_1 - A_2\|_1 + \|A_0 + iA_1 - A_2\|_\infty ;\ p = 1,\ \infty.$$

If $2\,\|(A_0 + A_2)^{-1}\|_p^{-1} = \|A_0 + iA_1 - A_2\|_1 + \|A_0 + iA_1 - A_2\|_\infty$, then the distance from $P(z)$ to a nearest Hermitian matrix polynomial in $\mathcal{P}_n$ having no real eigenvalues is zero with respect to the norm $\|\!|\cdot|\!\|_{\infty,1}$ if $p = 1$, and $\|\!|\cdot|\!\|_{1,\infty}$ if $p = \infty$.

If

$$2\left\|(A_0 + A_2)^{-1}\right\|_p^{-1} < \|A_0 + iA_1 - A_2\|_1 + \|A_0 + iA_1 - A_2\|_\infty, \tag{4.5.16}$$

then for $p = 1$, the distance to a nearest Hermitian matrix polynomial having no real eigenvalues with respect to the norm $\|\!|\cdot|\!\|_{\infty,1}$ is bounded above by

$$\max\left\{\frac{\left\|\|(A_0 + A_2)^{-1}\|_1^{-1} I - (A_0 - A_2)\right\|_1}{2}, \|A_1\|_1\right\}, \tag{4.5.17}$$

while for $p = \infty$, it is bounded above by

$$\left\|\left\|(A_0 + A_2)^{-1}\right\|_\infty^{-1} I - (A_0 - A_2)\right\|_\infty + \|A_1\|_\infty \tag{4.5.18}$$

if the norm is $|||\cdot|||_{1,\infty}$.

Further, if A_0+A_2 *is non-singular and (4.5.16) holds with* $p=1$*, then the distance of* $P(z)$ *to a nearest Hermitian matrix polynomial having no imaginary eigenvalues is bounded above by*

$$\frac{1}{2}\left(\frac{\|A_0+iA_1-A_2\|_1+\|A_0+iA_1-A_2\|_\infty}{2}\left\|(A_0+A_2)^{-1}\right\|_1-1\right)\|A_0+A_2\|_1 \tag{4.5.19}$$

if the norm is $|||\cdot|||_{\infty,1}$*, whereas if (4.5.16) holds with* $p=\infty$*, then it is bounded above by*

$$\left(\frac{\|A_0+iA_1-A_2\|_1+\|A_0+iA_1-A_2\|_\infty}{2}\left\|(A_0+A_2)^{-1}\right\|_\infty-1\right)\|A_0+A_2\|_\infty \tag{4.5.20}$$

with respect to the norm $|||\cdot|||_{1,\infty}$.

Proof. The proof follows by making identical arguments as in Theorem 4.5.5 by replacing $A_0+A_1+A_2$ by $A_0+iA_1-A_2$ and A_0-A_2 by A_0+A_2. The only difference is in the construction of $\delta P(z)$. Considering the norm $|||\cdot|||_{\infty,1}$, in the case that $2\,\|(A_0+A_2)^{-1}\|_1^{-1} = \|A_0+iA_1-A_2\|_1+\|A_0+iA_1-A_2\|_\infty$ and the matrix $A_0+iA_1-A_2=0$, we argue in a similar way as in Theorem 4.5.7 by choosing the perturbation $\delta P(z) := \delta A_0(z^2+1)$ where δA_0 is a Hermitian perturbation to A_0 such that $A_0+\delta A_0$ is non-singular and $\|\delta A_0\|_1<\varepsilon$. Then $(P+\delta P)(z)\in\mathcal{P}_n$ is a Hermitian matrix polynomial without any real eigenvalues and $|||\delta P|||_{\infty,1}<\varepsilon$. The proof for the case $2\,\|(A_0+A_2)^{-1}\|_1^{-1} = \|A_0+iA_1-A_2\|_1+\|A_0+iA_1-A_2\|_\infty$ with $A_0+iA_1-A_2\neq 0$ follows by choosing $\delta P(z)=\delta A_2 z^2+\delta A_1 z+\delta A_0$ with $\delta A_0=\frac{\mathrm{Herm}(E)}{2}=-\delta A_2$ and $\delta A_1=-i\mathrm{SHerm}(E)$, where the matrix E is constructed as in the proof of Theorem 4.5.5 with $A_0+A_1+A_2$ and A_0-A_2 being replaced by $A_0+iA_1-A_2$ and A_0+A_2 respectively. Similarly, the required Hermitian perturbation $\delta P(z)=\delta A_2z^2+\delta A_1z+\delta A_0$ for proving the upper bound in (4.5.17) is given by $\delta A_0=\frac{sI-(A_0-A_2)}{2}=-\delta A_2$ and $\delta A_1=-A_1$ where $s=\|(A_0+A_2)^{-1}\|_1^{-1}$.

If A_0+A_2 is non-singular the proof follows by forming $\delta P(z)=\frac{\triangle}{2}(z^2+1)$ where $\triangle$ is as in (4.5.13) with A_0-A_2 replaced by A_0+A_2 and $A_0+A_1+A_2$ replaced by $A_0+iA_1-A_2$. $\square$

The analogous result for $*$-palindromic or $*$-antipalindromic matrix polynomials vis-a-vis eigenvalues on the unit disc is as follows.

Theorem 4.5.10. *Let* $P(z)=A_2z^2+A_1z+A_0\in\mathcal{P}_n$ *be a* $*$*-palindromic or* $*$*-antipalindromic matrix polynomial having eigenvalues on the unit circle. Then we have* $|\lambda_{\min}(A_1)|\leq 2\,\|A_2\|_2$.

If $|\lambda_{\min}(A_1)| = 2\|A_2\|_2$ and $\varepsilon > 0$ is arbitrarily chosen, then there exists a structure preserving perturbation $\delta P(z) = \delta A_2 z^2 + \delta A_1 z + \delta A_0$ satisfying $|||\delta P|||_{2,2} < \varepsilon$ such that $(P + \delta P)(z) \in \mathcal{P}_n$ has no eigenvalues on the unit circle.

If $|\lambda_{\min}(A_1)| < 2\|A_2\|_2$, then the distance of $P(z)$ to a nearest $*$-palindromic or $*$-antipalindromic matrix polynomial that has no eigenvalues on the unit circle with respect to the norm $|||\cdot|||_{2,2}$ is bounded above by

$$\frac{1}{\sqrt{2}}\left(2\|A_2\|_2 - |\lambda_{\min}(A_1)|\right). \tag{4.5.21}$$

Proof. Let $P(z)$ be a $*$-palindromic polynomial having an eigenvalue on the unit circle. The proof for the $*$-palindromic case follows by replacing $A_0 + A_1 + A_2$ by A_2 and $\lambda_{\min}(A_0 - A_2)$ by $\lambda_{\min}(A_1)/2$ in the proof of Theorem 4.5.3. Note that in this case $A_2 \neq 0$. The difference is in the construction of $*$-palindromic polynomial $\delta P(z)$. This is chosen as $\delta P(z) = Bz^2 + B^*$ where $B = -\widehat{\varepsilon} U_m V_m^*$ when $|\lambda_{\min}(A_1)| = 2\|A_2\|_2$. When $|\lambda_{\min}(A_1)| < 2\|A_2\|_2$, this is chosen as $\delta P(z) = \widehat{B}z^2 + \widehat{B}^*$ where

$$\widehat{B} = \left(\frac{|\lambda_{\min}(A_1)|}{2} - \|A_2\|_2\right) U_m V_m^* + \sum_{j=m+1}^{j_0}\left(\frac{|\lambda_{\min}(A_1)|}{2} - \sigma_j\right) u_j v_j^*,$$

U_m and V_m being isometries as chosen in Theorem 4.5.3. For the $*$-antipalindromic case, the required $\delta P(z)$ is obtained by replacing B^* and $\widehat{B}^*$ by $-B^*$ and $-\widehat{B}^*$ respectively in the above construction of $\delta P(z)$. □

Theorem 4.5.11. *Let $P(z) = A_2 z^2 + A_1 z + A_0 \in \mathcal{P}_n$ be a $*$-palindromic (or $*$-antipalindromic) matrix polynomial having eigenvalues on the unit circle. Then*

$$\left\|A_1^{-1}\right\|_p^{-1} \leq \|A_0\|_1 + \|A_0\|_\infty, \; p = 1, \infty.$$

If $\left\|A_1^{-1}\right\|_p^{-1} = \|A_0\|_1 + \|A_0\|_\infty$, then the distance from $P(z)$ to a nearest $*$-palindromic (or $*$-antipalindromic) matrix polynomial in $\mathcal{P}_n$ having no eigenvalues on the unit circle is zero with respect to the norm $|||\cdot|||_{\infty,1}$ if $p = 1$, and $|||\cdot|||_{1,\infty}$ if $p = \infty$.

If

$$\left\|A_1^{-1}\right\|_p^{-1} < \|A_0\|_1 + \|A_0\|_\infty, \tag{4.5.22}$$

then for $p = 1$ the distance to a nearest $*$-palindromic (or $*$-antipalindromic) matrix polynomial that has no eigenvalues on the unit circle with respect to the norm $|||\cdot|||_{\infty,1}$ is bounded above by

$$\max\left\{\left\|\left(\left\|A_1^{-1}\right\|_1^{-1}/2\right) I - A_0\right\|_1, \left\|\left(\left\|A_1^{-1}\right\|_1^{-1}/2\right) I - A_0\right\|_\infty\right\}, \tag{4.5.23}$$

whereas for $p = \infty$, the upper bound is

$$\left\|\left(\left\|A_1^{-1}\right\|_\infty^{-1}/2\right) I - A_0\right\|_1 + \left\|\left(\left\|A_1^{-1}\right\|_\infty^{-1}/2\right) I - A_0\right\|_\infty \tag{4.5.24}$$

if the distance is measured with respect to the norm $|||\cdot|||_{1,\infty}$.

Further, if A_1 is non-singular and (4.5.22) holds for $p = 1$, then the distance of $P(z)$ to a nearest $*$-palindromic or $*$-antipalindromic matrix polynomial having no eigenvalues on the unit circle is bounded above by

$$\left[(\|A_0\|_1 + \|A_0\|_\infty)\left\|A_1^{-1}\right\|_1 - 1\right]\|A_1\|_1, \tag{4.5.25}$$

with respect to the norm $|||\cdot|||_{\infty,1}$, whereas if (4.5.22) holds for $p = \infty$, then it is bounded above by

$$\left[(\|A_0\|_1 + \|A_0\|_\infty)\left\|A_1^{-1}\right\|_\infty - 1\right]\|A_1\|_\infty \tag{4.5.26}$$

with respect to the norm $|||\cdot|||_{1,\infty}$.

Proof. Suppose $P(z) \in \mathcal{P}_n$ is $*$-palindromic and has an eigenvalue on the unit circle. The proof follows by making identical arguments as in Theorem 4.5.5 by replacing $A_0 + A_1 + A_2$ by A_0 and $2(A_0 - A_2)$ by A_1. Note that in this case $A_0 = A_2^* \neq 0$. The only difference is in the construction of $\delta P(z)$. With respect to the norm $|||\cdot|||_{\infty,1}$ the proof for the case $\left\|A_1^{-1}\right\|_1^{-1} = \|A_0\|_1 + \|A_0\|_\infty$ follows by choosing $\delta P(z) = E^* z^2 + E$, where the matrix E is constructed as in the proof of Theorem 4.5.5, with $A_0 + A_1 + A_2$ being replaced by A_0 and $2(A_0 - A_2)$ by A_1. To prove the upper bound (4.5.23), the structure preserving perturbation is chosen to be $\delta P(z) = E^* z^2 + E$ where $E = \left(\left\|A_1^{-1}\right\|_1^{-1}/2\right) I - A_0$. For the $*$-antipalindromic case, the required $\delta P(z)$ is obtained by replacing E^* by $-E^*$ in the above construction of $\delta P(z)$.

If A_1 is non-singular the bounds in (4.5.25) and (4.5.26) follow by forming $\delta P(z) = \delta A_2 z^2 + \delta A_1 z + \delta A_0$ with $\delta A_0 = \delta A_2 = 0$ and $\delta A_1 = \triangle$, where $\triangle$ is as in (4.5.13) with $2(A_0 - A_2)$ replaced by A_1 and $A_0 + A_1 + A_2$ by A_0. For the $*$-antipalindromic case, we choose $\delta P(z) = \triangle z$ in the above construction of $\delta P(z)$. □

An important observation from this section is that if the quadratic matrix polynomial $P(z)$ is appropriately multiplied by invertible matrices from the left and/or from the right, then the upper bounds on the distances obtained in the preceding results can be much smaller for the new polynomial than the ones for the original polynomial. This is primarily because of the fact that eigenvalues of $P(z)$ are invariant under such changes but the respective upper bounds are affected by them. For example, the polynomial in Example 4.3.4 has eigenvalues with positive real parts and the upper bound on the distance to a nearest matrix polynomial that has no such eigenvalues (from Theorem 4.5.1) is 41.77. However, if all the coefficient matrices are multiplied by A_0^{-1}, then the upper bound is 1.3. Indeed, in fact the upper bounds can be made smaller than any given $\epsilon > 0$ with appropriate scaling but this can make unfeasible changes to the coefficient matrices in most applications. This highlights the role of admissible scaling strategies when calculating these upper bounds. However, if the distances are measured in a relative sense with respect to the norm of the polynomial, then the variations in the values of the upper bounds with respect to such changes reduce. For instance in Example 4.3.4, the upper bound from Theorem 4.5.1 divided by $\|P\|_{2,2}$ is approximately 1.24 for the original polynomial and 1.23 when the polynomial is multiplied by A_0^{-1}. But there can be greater variations with respect to such changes even when the perturbations are measured with respect to the norm of the polynomial. Moreover, all the coefficient matrices of the polynomial cannot be equally perturbed in many applications. New techniques for deriving the bounds need to be explored in such cases.

Now we provide examples which illustrates the bounds obtained in this section. In each case the distances are assumed to be measured relative to the norm of the polynomial and the reported values of the bounds are the ones derived in the respective theorems divided by the corresponding norm of the polynomial.

Example 4.5.12. *The matrix polynomial* $P(z) = A_2z^2 + A_1z + A_0$, *where*

$$A_2 = \begin{bmatrix} 0.9828 & 0.1624 & 0.0877 \\ 0.1842 & -0.8310 & -0.5249 \\ 0.0124 & -0.5320 & 0.8466 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -0.8882 & 0.8064 & -0.3165 \\ -0.7704 & 0.6995 & -0.2745 \\ -0.8170 & 0.7418 & -0.2911 \end{bmatrix},$$

$$\text{and } A_1 = \begin{bmatrix} 0.9927 + 0.0598i & 2.1284 - 0.0094i & 0.2054 - 0.1917i \\ -1.5183 + 0.1767i & -1.0108 - 0.0277i & -0.4240 - 0.5666i \\ -0.9020 - 0.2316i & -0.2216 + 0.0364i & -0.2704 + 0.7427i \end{bmatrix}$$

satisfies $\sigma_{\min}(A_0 + A_1 + A_2) < \|A_0 - A_1 + A_2\|_2$. *The eigenvalues are* $0, 0, 2, -1-3$ *and* $-i$. *Table 4.5.1 demostrates the upper bounds of the distance of* $P(z)$ *to a nearest polynomial in* $\mathcal{P}_n$ *having no eigenvalues in the closed right half of the complex plane, with respect to three different choices of norms.*

Norm	Upper bound
$\|\cdot\|_{2,2}$	1.0779 (4.5.1)
$\|\cdot\|_{\infty,1}$	2.0351 (4.5.5)
$\|\cdot\|_{1,\infty}$	1.9581 (4.5.6)

Table 4.5.1: Upper bounds on distances for various choices of norms for $P(z)$ in Example 4.5.12

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Example 4.5.13. *The $*$-even matrix polynomial $P(z) = A_2z^2 + A_1z + A_0$, where*

$$A_2 = \begin{bmatrix} 16.8357 & 9.0992 & -2.9527 \\ 9.0992 & 7.0360 & -5.0081 \\ -2.9527 & -5.0081 & 15.1283 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1.0614i & -1.0261i & 1.9047i \\ -1.0261i & -0.7410i & 0.8121i \\ 1.9047i & 0.8121i & 0.8024i \end{bmatrix},$$

$$\text{and } A_0 = \begin{bmatrix} 12.9631 & 4.3771 & 1.4985 \\ 4.3771 & 6.7916 & -1.9497 \\ 1.4985 & -1.9497 & 15.2454 \end{bmatrix}$$

has eigenvalues $-2i$, $2i$, $1.0351i$, $-1.1890i - 0.7169i$ and $+0.8369i$, all purely imaginary. Table 4.5.2 demostrates the upper bounds of the distance of $P(z)$ to a nearest $*$-even polynomial in $\mathcal{P}_n$ having no purely imaginary eigenvalues with respect to three different choices of norms. Since $A_0 - A_2$ is invertible, row 3 reports the minimum of the upper bounds in (4.5.9), (4.5.11) and row 4 represents the minimum of bounds in (4.5.10) and (4.5.12).

Norm	Upper bound
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{2,2}$	1.2362 (4.5.7)
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{\infty,1}$	0.7387 (4.5.9)
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{1,\infty}$	0.9024 (4.5.10)

Table 4.5.2: Upper bounds on distances for various choices of norms for $P(z)$ in Example 4.5.13

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Example 4.5.14. *The Hermitian polynomial $P(z) = A_2z^2 + A_1z + A_0 \in \mathcal{P}_n$ where*

$$A_2 = \begin{bmatrix} 1.9353 & -0.5475 & -0.7698 \\ -0.5475 & 2.3288 & 0.1293 \\ -0.7698 & 0.1293 & 1.7359 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1.0371 & 1.0688 & -1.3876 \\ 1.0688 & 3.8293 & -2.2993 \\ -1.3876 & -2.2993 & 2.1335 \end{bmatrix},$$

$$\text{and } A_0 = \begin{bmatrix} -1.1068 & -0.6592 & -7.7584 \\ -0.6592 & 7.7823 & -4.6825 \\ -7.7584 & -4.6825 & 0.3245 \end{bmatrix}$$

has eigenvalues ± 3, $-1.5 + 1.8028i$, $-1.5 - 1.8028i$, $-0.1667 + 1.2802i$ and $-0.1667 - 1.2802i$, of which two eigenvalues are real. Table 4.5.3 demostrates the upper bounds of the distance of $P(z)$ to a nearest Hermitian quadratic matrix polynomial having no real eigenvalues with respect to three choices of norms. As $A_0 + A_2$ is non-singular, row 3 represents the minimum of the upper bounds in (4.5.17), (4.5.19) and row 4 represents the minimum of the bounds in (4.5.18) and (4.5.20).

Norm	Upper bound
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{2,2}$	0.2677 (4.5.15)
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{\infty,1}$	0.7414 (4.5.17)
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{1,\infty}$	0.9445 (4.5.20)

Table 4.5.3: Upper bounds on distances for various choices of norms for $P(z)$ in Example 4.5.14

Example 4.5.15. *The $*$-palindromic polynomial $P(z) = A_2z^2 + A_1z + A_2^* \in \mathcal{P}_n$ where*

$$A_2 = \begin{bmatrix} 0.8701 + 1.3637i & -0.9254 + 0.0810i & 0.7453 - 1.4916i \\ -0.9254 + 0.0810i & 3.4842 + 0.0048i & -0.6569 - 0.0886i \\ 0.7453 - 1.4916i & -0.6569 - 0.0886i & 0.6457 + 1.6315i \end{bmatrix},$$

$$\text{and } A_1 = \begin{bmatrix} 3.0347 & -0.4910 & -1.8234 \\ -0.4910 & 2.6636 & -0.5757 \\ -1.8234 & -0.5757 & 3.3017 \end{bmatrix}$$

has eigenvalues $-0.5000 - 0.8660i$, $-0.3750 - 0.9270i$, $0.5528 + 0.8333i$, $0.5528 + 0.8333i$, $-0.3750 + 0.9270i$, $-0.5528 + 0.8333i$ *and* $-0.5000 + 0.8660i$, *all being on the unit circle. Table 4.5.4 demostrates the upper bounds of the distance of $P(z)$ to a nearest $*$-palindromic polynomial in $\mathcal{P}_n$ having no eigenvalues on the unit circle with respect to three choices of norms. The coefficient matrix A_1 being non-singular, row* 3 *represents the minimum of the upper bounds in (4.5.23), (4.5.25) and row* 4 *represents the minimum of bounds in (4.5.24) and (4.5.26).*

Norm	Upper bound
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{2,2}$	0.6556 (4.5.21)
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{\infty,1}$	0.8095 (4.5.23)
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{1,\infty}$	0.5822 (4.5.24)

Table 4.5.4: Upper bounds on distances for various choices of norms for $P(z)$ in Example 4.5.15

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Conclusion

In this thesis we have studied the various aspects of the localization sets for the spectra and pseudospectra of matrices and matrix polynomials. Given a block upper triangular matrix $A \in \mathbb{C}^{n,n}$, we have derived inner approximations for the pseudospectra of A via the pseudospectra of its diagonal blocks, under the assumption that the non-zero off-diagonal block matrix is of full rank. The derived inner approximations are optimal for the case when A is a 2×2 matrix. Next we have generalized the definitions of four localization sets for eigenvalues, namely, block Geršgorin sets, block Brualdi sets, block minimal Geršgorin sets and permuted pointwise minimal Geršgorin sets to matrix polynomials in homogeneous form. The use of the homogeneous form has the advantage of treating both the finite and infinite eigenvalues in a single framework. Several properties of these sets have also been derived. Given a $n \times n$ homogeneous matrix polynomial $P(c, s)$ partitioned with respect to a partition $\pi = \{n_j\}_{j=0}^{\ell}$ of the set $\{1, 2, \ldots, n\}$, we have approximated the associated block Geršgorin set and block minimal Geršgorin set via appropriate pseudospectra levels of the polynomials $P_{k,k}(c, s)$, $k = 1, 2, \ldots, \ell$ on the diagonal of the partitioned polynomial. We have identified partitions for which computing the block minimal Geršgorin sets is likely to be less computationally intensive than the pointwise minimal Geršgorin sets for certain matrix polynomials. We have provided a detailed strategy for numerically plotting the localization sets. The sets are plotted by laying a grid on the Riemann sphere and then projecting the plot onto the complex plane, thus avoiding the need for any prior knowledge of the location of the spectrum of the polynomial. The plots are illustrated for different choices of matrix polynomials.

Finally we have studied the localization sets for the eigenvalues of quadratic matrix polynomials using the concept of block Geršgorin sets applied to linearizations of the polynomial. The linearizations are chosen from the well known vector spaces of potential strong linearizations $\mathbb{L}_1(P)$, $\mathbb{L}_2(P)$ and $\mathbb{DL}(P)$. The resulting localization is a union of two sets. The properties of these sets have been analyzed and used to obtain new and simple upper and lower bounds on the eigenvalues of the matrix polynomial. These bounds have been compared with another set of bounds from the literature obtained with comparable computational effort. The analysis of the block Geršgorin sets also lead to conditions on the coefficient matrices of the polynomial that are sufficient for the eigenvalues to lie in certain parts of the complex plane. The regions are described with respect to the imaginary axis, the unit circle and the real line and are important in applications. These conditions are illustrated via numerical experiments. As a consequence, we have also derived various upper bounds on several important distances associated with certain structured and unstructured quadratic matrix polynomials for various choices of norms. These bounds are also computed for a number of different matrix polynomials.

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