Some classical problems in harmonic analysis

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Declaration

I do hereby declare that this thesis entitled **"Some classical problems in harmonic analysis"** is a presentation of my original research work done under the supervision of **Dr. Jitendriya Swain**, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of doctor of philosophy. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

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Certificate

This is certified that the work contained in the thesis entitled **“Some classical problems in harmonic analysis”** by **Mr. Shyam Swarup Mondal** (Roll No. 166123103) has been carried out under my supervision. In my opinion, the thesis has reached the standard fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

Guwahati
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Dedicated

To

My Family


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Shyam Swarup Mondal

Abstract

This thesis focuses on certain classical problems in harmonic analysis in connection with mathematical physics. We begin with the Fourier analysis on the Euclidean space, discuss some well known results, basic definitions, and review of recent developments that motivates us to consider the problems discussed in the thesis.

We prove a restriction theorem for the Fourier-Hermite transform and obtain a Strichartz estimate for systems of orthonormal functions associated with the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ for the range $1 \leq q < \frac{n+1}{n-1}$ as an application. Besides, we show an optimal behavior of the constant in the Strichartz estimate as limit of a large number of functions.

Further, we prove a restriction theorem for the special Hermite transform and establish a Strichartz inequality as a by-product for the range $1 \leq q \leq 1 + \frac{1}{n}$, for systems of orthonormal functions associated with the special Hermite operator $\mathcal{L}$ on $\mathbb{C}^n$.

Next, we consider the Schrödinger operator $\mathcal{H} = -\Delta_{\mathbb{H}} + V$ on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}}$ is the full laplacian on $\mathbb{H}^n$ and V is a positive smooth potential grows like $|g|^\kappa, \kappa > 0$, for large value of $|g|$. We prove Szegö type limit theorem for $\mathcal{H}$ with respect to the multiplication operator $M_{\mathbf{b}}$, where $\mathbf{b}$ is a bounded real valued integrable function on $\mathbb{H}^n$. More preciously, we prove that, for any $f \in C(\mathbb{R})$,

$$\lim_{r \to \infty} \frac{\operatorname{Tr} f(\mathcal{P}_r M_{\mathbf{b}} \mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)} = \int_{\mathbb{H}^n} f(\mathbf{b}(g))\, dg,$$

where $\mathcal{P}_r$ denote the orthogonal projection of $L^2(\mathbb{H}^n)$ onto the space of eigenfunctions

of $\mathcal{H}$ with eigenvalue less than or equal to r. Further, we generalize the above result by taking a 0-th order self-adjoint pseudo-differential operator A on $L^2(\mathbb{H}^n)$ with symbol $a(g,\lambda)$ relative to the operator $1+|\lambda|H+V(g)$, where H is the Hermite operator on $L^2(\mathbb{R}^n)$ and $(g,\lambda)\in\mathbb{H}^n\times\mathbb{R}^*$, in place of the multiplication operator $M_{\mathbf{b}}$, and obtain the following Szegö type limit theorem:

$$\lim_{r\to\infty}\frac{\operatorname{Tr} f(\mathcal{P}_r A\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}=\lim_{r\to\infty}\frac{\int_{G^r} f(a_{g,\lambda}(\xi,x))\,d\xi\,dx\,dg\,d\mu(\lambda)}{\int_{G^r}\,d\xi\,dx\,dg\,d\mu(\lambda)}, \tag{0.0.1}$$

where $G^r=\{(g,\lambda,\xi,x)\in\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n : |\lambda|(1+|\xi|^2+|x|^2)+V(g)\leq r\}$, $a(g,\lambda)=Op^W(a_{g,\lambda})$, and $\mu(\lambda)$ is the Plancherel measure on the Heisenberg group, assuming one limit exists. We show that the above Szegö type limit theorem also holds under a perturbation of the Schrödinger operator $\mathcal{H}$ by a bounded self-adjoint operator on $L^2(\mathbb{H}^n)$. Further, we show that the right hand limit of (0.0.1) remains unaltered under a compact perturbation of the pseudo-differential operator A.

For a given compact (Hausdorff) group G and a closed subgroup H of G, we present symbolic criteria for pseudo-differential operators on compact homogeneous space G/H characterizing the Schatten-von Neumann classes $S_r(L^2(G/H))$, for all $0<r\leq\infty$. We provide a symbolic characterization for r-nuclear pseudo-differential operators with $0<r\leq 1$, on $L^p(G/H), 1\leq p<\infty$, along with applications to adjoint, product and trace formulae. Finally, as an application of the aforementioned results, we derive a trace formula and provide a criteria for the heat kernel to be r-nuclear on $L^p(G/H), 1\leq p<\infty$.

Abbreviation and Notation

$\mathbb{N}$	The set of all natural numbers
$\mathbb{Z}$	The set of all integer numbers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{C}$	The set of all complex numbers
$\mathbb{T}$	Unit circle in $\mathbb{R}$
$\mathbb{R}^*$	$\mathbb{R}\backslash\{0\}$
$\mathbb{Z}^n$	$\{(k_1, k_2, \ldots, k_n) \mid k_i \in \mathbb{Z}, i = 1, 2, \cdots, n\}$, $n \geq 1$
$\mathbb{R}^n$	$\{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$, $n \geq 1$
$\mathbb{C}^n$	$\{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C}, i = 1, 2, \cdots, n\}$, $n \geq 1$
Re z	The real part of $z \in \mathbb{C}$
Im z	The imaginary part of $z \in \mathbb{C}$
$\widehat{G}$	Unitary dual group of G
$\mathbb{S}^{n-1}$	The unit sphere in $\mathbb{R}^n$
$\mathbb{H}^n$	The Heisenberg group
$L^p(S)$	$\{f : S \to \mathbb{C} \mid f$ is measurable and $\int\limits_S \lvert f\rvert^p ds < \infty\}$
$S_r(\mathfrak{X})$	The r-Schatten-von Neumann classes on $\mathfrak{X}$
$B(\mathfrak{H})$	The class of bounded linear operators on a Hilbert space $\mathfrak{H}$
$C(\mathfrak{X})$	The set of all complex valued continuous functions on $\mathfrak{X}$
$S_1(\mathfrak{H})$	The collection of trace class operators on a Hilbert space $\mathfrak{H}$

$S_2(\mathfrak{H})$	The class of Hilbert-Schmidt operators on a Hilbert space $\mathfrak{H}$
Δ	Laplacian on $\mathbb{R}^n$
$\Delta_{\mathbb{Z}}$	Laplacian on $\mathbb{Z}^n$
H	Hermite operator on $\mathbb{R}^n$
$\mathcal{L}$	Special-Hermite operator on $\mathbb{C}^n$
$\mathcal{L}_{\mathbb{H}}$	Sublaplacian on the Heisenberg group $\mathbb{H}^n$
$\Delta_{\mathbb{H}}$	Full laplacian on the Heisenberg group $\mathbb{H}^n$
$\mathcal{L}_G$	Laplace-Beltrami operator on a compact group G
$\mathfrak{h}_n$	Lie algebra for $\mathbb{H}^n$
$f * g$	Convolution of f and g
$f \times g$	Twisted convolution of f and g
$\mu(\lambda)$	Plancherel measure on the Heisenberg group $\mathbb{H}^n$
$Op(\sigma), T_\sigma$	Pseudo-differential operator with symbol σ
$\mathrm{Tr}(A)$	Trace of an (trass class) operator A defined on some Hilbert space

Contents

CHAPTER 1

Introduction

In this thesis we focus our attention on three types of classical problems in harmonic analysis: Strichartz inequality for system of orthonormal functions associated with Hermite and special Hermite operator, Szegö type limit theorems on the Heisenberg group, and nuclearity of pseudo-differential operators on homogeneous space of compact groups. In this chapter, we provide basic definitions, notations, and some well known results (see [20, 28, 47, 84–86, 95, 100, 108]) that will be used throughout this thesis. To motivate the work presented in this thesis, we only outline the historical developments and results related to these topics.

1.1 Basic definitions

Let X and Y be two measurable spaces with positive measures μ and ν, respectively. The space $L^p(X)$ $(1 \leq p \leq \infty)$ is defined as follows:

$$\begin{aligned} L^p(X) &:= \left\{[f] : \int |f|^p d\mu(x) < \infty\right\}, \\ L^\infty(X) &:= \left\{[f] : \text{ess sup}|f|(x) < \infty\right\}, \end{aligned}$$

where $[\cdot]$ denotes the equivalence class of functions differing on a set of μ-measure zero. The mixed L^p-spaces is given by

$$L_{p,q}(X\times Y)=\left\{f: f \text{ is measurable on } X\times Y, \|f\|_{L^p_xL^q_y(X\times Y)}<\infty\right\},$$

where

$$\|f\|_{L^p_xL^q_y(X\times Y)}=\left(\int_X\left(\int_Y|f(x,y)|^q d\nu(y)\right)^{\frac{p}{q}}d\mu(x)\right)^{\frac{1}{p}}$$

is the norm in $L_{p,q}(X\times Y)$ for $1\leq p,q<\infty$. For $f\in L^1(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)dx,\quad \xi\in\mathbb{R}^n.$$

For $f\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, one has the Plancherel formula $||f||_2=||\hat{f}||_2$. Since $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the Fourier transform can be extended to functions in $L^2(\mathbb{R}^n)$. The inversion formula reads as

$$f(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi,\quad \text{for } a.e.\ x\in\mathbb{R}^n.$$

Definition 1.1.1. *Let $S(\mathbb{R}^n)$ denote the class of all infinitely differentiable functions on $\mathbb{R}^n$ such that*

$$\sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta\varphi(x)\right|<\infty,\quad \forall\ \alpha,\beta\in\mathbb{N}_0^n=\mathbb{N}^n\cup\{0\},$$

where $\alpha=(\alpha_1,\alpha_2,\cdots\alpha_n)$, $\beta=(\beta_1,\beta_2\cdots\beta_n)$, $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and $D^\beta=\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}\frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}}\cdots\frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$, for all $x=(x_1,x_2,\cdots x_n)$. The space $S(\mathbb{R}^n)$ is called Schwartz class of rapidly decreasing functions.

Let $C_0(\mathbb{R}^n)$ denote the class of continuous functions vanishing at infinity. Then $S(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, $1\leq p<\infty$. The Fourier transform $f\mapsto\hat{f}$ is a homeomorphism of $S(\mathbb{R}^n)$ onto itself. The collection $S'(\mathbb{R}^n)$ of all continuous linear functionals on $S(\mathbb{R}^n)$ is called the space of tempered distributions.

Let f be a function on $\mathbb{Z}^n$, and $e_j\in\mathbb{N}^n$ be such that e_j has 1 in the j-th entry and zeros elsewhere. The difference operator Δ_j is defined by

$$\Delta_j f(k)=f(k+e_j)-f(k),\quad k\in\mathbb{Z}^n,$$

and set $\Delta^\alpha=\Delta_1^{\alpha_1}\Delta_2^{\alpha_2}\ldots\Delta_n^{\alpha_n}$, for all $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}_0^n$.

Definition 1.1.2. *The Schwartz space $S(\mathbb{Z}^n)$, on the lattice $\mathbb{Z}^n$ is the space of all functions $\varphi : \mathbb{Z}^n \to \mathbb{C}$ such that*

$$\sup_{k\in\mathbb{Z}^n} \left|k^\alpha \left(\Delta^\beta \varphi\right)(k)\right| < \infty, \quad \forall \alpha, \beta \in \mathbb{N}_0^n,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n)$, $\beta = (\beta_1, \beta_2 \cdots \beta_n)$, $k^\alpha = k_1^{\alpha_1} k_2^{\alpha_2} \cdots k_n^{\alpha_n}$, and $\Delta^\beta = \Delta_1^{\beta_1} \Delta_2^{\beta_2} \ldots \Delta_n^{\beta_n}$, for all $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$.

Definition 1.1.3. *A topological group G is a group endowed with a topology such that the multiplication map $(g, h) \mapsto gh$ from $G \times G$ to G, and the inverse map $g \mapsto g^{-1}$ from G to G, are both continuous.*

Let G be a topological group and $\mathcal{H}$ be a Hilbert space. Denote $U(\mathcal{H})$ by the group of unitary operators on $\mathcal{H}$.

Definition 1.1.4. *A map π from G into the group $U(\mathcal{H})$ is called a homomorphism if $\pi(gh) = \pi(g)\pi(h)$, for all $g, h \in G$.*

Definition 1.1.5. *A homomorphism π from G into $U(\mathcal{H})$ is called strongly continuous if, for every x in $\mathcal{H}$, the map $g \mapsto \pi(g)x$ is continuous from G into $\mathcal{H}$.*

Definition 1.1.6. *A unitary representation of G is a strongly continuous homomorphism π of G into $U(\mathcal{H})$. In this case, the Hilbert space $\mathcal{H}$ is called the representation space of π and is denoted by $\mathcal{H}_\pi$. The dimension of $\mathcal{H}_\pi$ is called the dimension of the representation π.*

Definition 1.1.7. *Two unitary representations π and ρ of G are called equivalent if there exists an isometry T of $\mathcal{H}_\pi$ onto $\mathcal{H}_\rho$ such that $T\pi(g) = \rho(g)T$, for all g in G.*

Definition 1.1.8. *A subspace M of $\mathcal{H}_\pi$ is said to be invariant under the unitary representation π if e $\pi(g)M \subset M$, for all g in G.*

Definition 1.1.9. *A unitary representation π is said to be irreducible if the only π-invariant closed subspaces of $\mathcal{H}_\pi$ are $\{0\}$ and $\mathcal{H}_\pi$. The collection of equivalence classes of irreducible representations of G is denoted by $\widehat{G}$.*

Definition 1.1.10. *Let π be a representation of G. For every ξ, η in $\mathcal{H}_\pi$, the function $\pi_{\xi,\eta}(g) = \langle \pi(g)\xi, \eta\rangle$ is called representative function associated to π.*

Let $\mathcal{A}$ denote the set of all representative functions associated to all irreducible representations of G.

Theorem 1.1.11. [Peter-Weyl theorem] *Let G be a compact group. Then the following assertions holds.*

1. *Every irreducible unitary representation of G is finite dimensional.*

2. *Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of G, $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis of $\mathcal{H}_\pi$ and $\phi_{ij}(g) =< \pi(g)e_j, e_i >$. Let $\psi^i_j(g) = \sqrt{n}\ \phi_{ij}(g)$ and $E_i = \operatorname{span}\{\psi^i_1, \psi^i_2, \ldots, \psi^i_n\}$. Then $\oplus_{i=1}^{d_\pi} E_i = E_\pi \equiv \operatorname{span}\{\pi_{y,x} : x, y \in \mathcal{H}_\pi\}$ in $L^2(G)$ with $\dim(E_i) = \dim \pi = d_\pi$ and $\dim(E_\pi) = d_\pi^2$. Further, $L^2(G)$ decomposes into an orthogonal direct sum of all the irreducible representations of G, i.e.,*
$$L^2(G) = \oplus_{\lambda \in \hat{G}} E_\lambda \quad \textit{with } \dim(E_\lambda) = d_\pi^2.$$

3. *$\mathcal{A}$ is dense in $C(G)$, the space of continuous functions on G.*

4. *$\mathcal{A}$ is dense in $L^2(G)$.*

For more details regarding representation theory, we refer to [85, 95].

1.2 Orthonormal Strichartz inequality

A long-standing but persistent classical topic in harmonic analysis is the so-called restriction problem. Originally emerged by the works of Stein in the late 1960s, the restriction problem is a key problem for understanding the general oscillatory integral operators. The restriction problem and its applications are crucial from the point of view of their credible implementation in many areas of mathematical analysis, geometric measure theory, combinatorics, harmonic analysis, number theory, including the Bochner-Riesz conjecture, Kakeya conjecture, the estimation of solutions to the wave, Schrödinger, and the local smoothing conjecture for PDE's [98]. Given a surface S embedded in $\mathbb{R}^n$ with $n \geq 2$, the classical restriction problem is the following:

Problem 1: For which exponents $1 \leq p \leq 2, 1 \leq q \leq \infty$, the Fourier transform of a function $f \in L^p(\mathbb{R}^n)$ belongs to $L^q(S)$, where S is endowed with its $(n-1)$-dimensional Lebesgue measure $d\sigma$?

More precisely, if we define the restriction operator $\mathcal{R}_S$ as $\mathcal{R}_S f = \widehat{f}\Big|_S$, for all f in the Schwartz class of $\mathbb{R}^n$, then this question is equivalent to when $\mathcal{R}_S$ can be extended as a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(S)$. If $\mathcal{E}_S$ (Fourier extension operator) be the operator dual to $\mathcal{R}_S$ defined as

$$\mathcal{E}_S f(x) = (2\pi)^{-\frac{n}{2}} \int_S f(\xi) e^{i\xi\cdot x} d\sigma(\xi), \quad x \in \mathbb{R}^n,$$

for all $f \in L^1(S)$, then the restriction problem is thus equivalent to knowing when $\mathcal{E}_S$ is bounded from $L^{q'}(S)$ to $L^{p'}\left(\mathbb{R}^N\right)$, where p' and q' are the conjugate exponents of p and q, respectively. A model case of the restriction problem which is often considered in the literature is the case $q = 2$ (see [92, 94, 103]). Thus, Problem 1 can be also reframed as follows:

Problem 2: For which exponents $1 \leq p \leq 2$, the operator $\mathcal{E}_S f$ is bounded from $L^2(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$?

Since $\mathcal{E}_S$ is bounded from $L^2(S)$ to $L^{p'}(\mathbb{R}^n)$ if and only if $T_S := \mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, thus Problem 2 also can be re-written as follows:

Problem 3: For which exponents $1 \leq p \leq 2$, the operator $T_S := \mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$?

For smooth compact surfaces and quadratic surfaces, the restriction problem has been completely settled. In this context, the celebrated Stein-Tomas theorem for smooth compact surfaces with non-zero Gauss curvature states that the restriction problem has a positive answer if and only if $1 \leq p \leq \frac{2(n+1)}{(n+3)}$ (see [92, 103]). However, for quadratic surfaces, Strichartz in [94] gave a complete characterization depending on the type of the surfaces, such as paraboloid, cone, or spherical type. For a detailed study on the history of the restriction problem, we refer to the excellent survey of Tao [98]. There exists a vast literature on the restriction problem that is difficult to mention here. However, we refer to [3, 15, 64] for few recent developments and important works in this direction.

Generalization involving the orthonormal system is strongly motivated by the theory of many body quantum mechanics. In quantum mechanics, a system of N independent fermions in the Euclidean space $\mathbb{R}^n$ is described by a collection of N orthonormal functions $u_1, \ldots, u_N$ in $L^2(\mathbb{R}^n)$. It is then essential to obtain functional inequalities on these systems whose behavior is optimal in the finite number N of such orthonormal functions. For this particular reason, functional inequalities involving a large number of orthonormal functions are very useful in mathematical analysis of large quantum systems.

Therefore, it is natural to investigate generalization of Problem 2 in the framework of orthonormal systems. The question we want to address is a generalization of Problem 2 in the framework of orthonormal systems, whenever $\mathcal{E}_S f$ be the solution of Schrödinger equation associated with Hermite and special Hermite operators with initial data f. More precisely, let $(f_j)_{j\in J}$ be a (possibly infinite) system of orthonormal functions in $L^2(S)$, and let $(n_j)_{j\in J} \subset \mathbb{C}$ be a sequence of coefficients, then one can ask, for which exponents $1 \leq p \leq 2$, we have

$$\left\| \sum_{j\in J} n_j \left|\mathcal{E}_S f_j\right|^2 \right\|_{L^{\frac{p'}{2}}(\mathbb{R}^n)} \leq C \left(\sum_{j\in J} |n_j|^\alpha \right)^{\frac{1}{\alpha}}, \tag{1.2.1}$$

for some $\alpha > 1$ and for some positive constant C (independent of (f_j) and (n_j)). Using triangle inequality, Problem 2 leads to the estimate

$$\left\| \sum_{j\in J} n_j \left|\mathcal{E}_S f_j\right|^2 \right\|_{L^{\frac{p'}{2}}(\mathbb{R}^n)} \leq \sum_{j\in J} |n_j| \, \|\mathcal{E}_S f_j\|^2_{L^{p'}(\mathbb{R}^n)} \leq C \sum_{j\in J} |n_j|,$$

which is weaker than (1.2.1) (since $\alpha > 1$ in (1.2.1)). The estimate of the form (1.2.1) is important due to its applications to the Hartree equation modeling for infinitely many particles in a large quantum system [36, 66, 67].

The idea of extending functional inequalities involving a single function to a orthonormal systems of input functions is hardly a new topic. The first initiative work of such generalization goes back to the famous work established by Lieb and Thirring, known as Lieb-Thirring inequality [71, 72] and it states that for any $u_1, \cdots u_N$ orthonormal in $L^2(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^N |\nabla u_j(x)|^2 \right) dx \geq C \int_{\mathbb{R}^n} \left(\sum_{j=1}^N |u_j(x)|^2 \right)^{1+\frac{2}{n}} dx,$$

where $C(> 0)$ is independent of N, which generalizes the known Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \geq C' \int_{\mathbb{R}^n} |u(x)|^{2+\frac{4}{n}} dx$$

for an L^2-normalized function u. Importantly, Lieb-Thirring inequality (the sharp orthonormal inequality) is one of the fundamental tool to prove the stability of matter, see, for example [71] or the extensive survey by Lieb [70] for further details.

One more example of such type of generalization was proved by Lieb in [69], which states that for any N orthonormal functions $u_1, \ldots, u_N$ in $L^2(\mathbb{R}^n)$ and for any non-negative coefficients $n_1, n_2, \cdots, n_N$, we have

$$\left\| \sum_{j=1}^{N} n_j \left| (-\Delta)^{-\frac{s}{2}} u_j \right|^2 \right\|_{L^{\frac{n}{n-2s}}(\mathbb{R}^n)} \leq C \left(\sup_j n_j \right)^{\frac{2s}{n}} \left(\sum_{j=1}^{N} n_j \right)^{\frac{n-2s}{n}},$$

which generalizes the homogeneous Sobolev inequality

$$\left\| (-\Delta)^{-\frac{s}{2}} u \right\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \leq C \|u\|_{L^2(\mathbb{R}^n)},$$

for an L^2-function u and $0 < s < \frac{n}{2}$.

In 1977, Strichartz [94] proved the following remarkable estimate for the solution to inhomogeneous Schrödinger equation associated with Laplacian on $\mathbb{R}^n$ in connection with Fourier restriction theory:

Theorem 1.2.1. [94] *Let $f \in L^2(\mathbb{R}^n), g \in L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^n \times \mathbb{R})$ and u be the solution of inhomogeneous equation*

$$\begin{aligned} i\partial_t u(x,t) &= -\Delta u(x,t) + g(x,t), && x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(x,0) &= f(x), && x \in \mathbb{R}^n. \end{aligned} \tag{1.2.2}$$

Then $u \in L^{\frac{2(n+2)}{n}}(\mathbb{R}^n \times \mathbb{R})$ and satisfies the inequality

$$\|u\|_{L^{\frac{2(n+2)}{n}}(\mathbb{R}\times\mathbb{R}^n)} \leq C \left(\|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^n\times\mathbb{R})} \right).$$

The above inequality is popularly known as classical Strichartz inequality for the Schrödinger propagator $e^{it\Delta}$. In particular, when $g = 0$, $u = e^{it\Delta} f$ is the unique solution to the homogeneous initial value problem (1.2.2). In case of homogeneous Schrödinger equation, Theorem 1.2.1 can be extended to mixed norm setting (see [35]) as follows:

Theorem 1.2.2. *Let $f \in L^2(\mathbb{R}^n)$. If $p, q \geq 1$ satisfying $(p,q,n) \neq (1,\infty,2)$ and $\frac{2}{p} + \frac{n}{q} = n$, then $e^{it\Delta} f \in L_t^{2p} L_x^{2q}(\mathbb{R} \times \mathbb{R}^n)$ and satisfies the inequality*

$$\|e^{it\Delta} f\|_{L_t^{2p} L_x^{2q}(\mathbb{R}\times\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}.$$

The above inequality has been substantially generalized for a system of orthonormal functions in the works of Frank-Lewin-Lieb-Seiringer [35] and Frank-Sabin [36]. The result can be stated as follows:

Theorem 1.2.3. [35,36] *Assume that* $p, q, n \geq 1$ *such that*

$$1 \leq q < \frac{n+1}{n-1} \quad \text{and} \quad \frac{2}{p} + \frac{n}{q} = n.$$

For any (possibly infinite) system u_j *of orthonormal functions in* $L^2(\mathbb{R}^n)$ *and any coefficients* $(n_j) \subset \mathbb{C}$, *we have*

$$\left\| \sum_j n_j \left|e^{it\Delta} u_j\right|^2 \right\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^n)} \leq C_{n,q}^p \left(\sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}}, \tag{1.2.3}$$

where $C_{n,q}$ *is a universal constant which only depends on* n *and* q. *The exponent* $\frac{2q}{q+1}$, *in the right hand side of (1.2.3) is optimal.*

These generalized orthonormal Strichartz estimates (1.2.3) extensively used in the study of nonlinear evolution of quantum systems for many body particles [66,67]. It is important to note that Nakamura in [77] established the sharp orthonormal Strichartz inequality on $\mathbb{T}^n$, which generalizes Strichartz inequality on torus [15,16]. We also refer to [10] for the recent work in the framework of orthonormal families of initial data.

Further, Theorem 1.2.2 has been extended to the Schrödinger equation for the quantum harmonic oscillator associated with the Hermite operator $H = -\Delta + |x|^2$:

$$\begin{aligned} i\partial_t u(x,t) &= Hu(x,t), \quad && x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(x,0) &= f(x), && x \in \mathbb{R}^n. \end{aligned} \tag{1.2.4}$$

Assuming $f \in L^2(\mathbb{R}^n)$, the solution of the initial value problem (1.2.4) is given by $u(x,t) = e^{-itH} f(x)$. The classical Strichartz inequality in this case has been proved by Koch-Tataru [56] or Nandakumaran-Ratnakumar [76] resulting the following.

Theorem 1.2.4. [76] *Let* $f \in L^2(\mathbb{R}^n)$ *and* $u(x,t) = e^{-itH} f(x)$ *be the solution of the initial value problem (1.2.4). Then* u *is periodic in* t *and for*

$$1 < p < \infty \quad \text{and} \quad 2 \leq q < \Lambda = \begin{cases} \infty, & \text{if } n = 1, \\ \frac{2n}{n-2}, & \text{if } n \geq 2, \end{cases}$$

u satisfies the inequality

$$\|u\|_{L_t^p L_x^q([-\pi,\pi]\times\mathbb{R}^n)} \leq C_n \|f\|_{L^2(\mathbb{R}^n)}.$$

Recently, the above estimate has been substantially generalised to the context of orthonormal systems in the works of Bez-Hong-Lee-Nakamura-Sawano [9] as follows:

Theorem 1.2.5. [9] *Let $p, q, n \geq 1$ be such that*

$$1 \leq q < \frac{n+1}{n-1} \quad \text{and} \quad \frac{2}{p} + \frac{n}{q} = n.$$

For any (possibly infinite) system (u_j) of orthonormal functions in $L^2(\mathbb{R}^n)$ and any coefficients $(n_j) \subset \mathbb{C}$, we have

$$\left\| \sum_j n_j \left| e^{-itH} u_j \right|^2 \right\|_{L_t^p L_x^q((-\pi,\pi)\times\mathbb{R}^n)} \leq C_{n,q} \left(\sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}}, \tag{1.2.5}$$

where $C_{n,q}$ is a universal constant only depends on n and q.

Further, Theorem 1.2.4 has been extended for the Schrödinger equation associated with the special Hermite operator $\mathcal{L}$ defined on $L^2(\mathbb{C}^n)$. In this case, the Strichartz estimate has been considered by Ratnakumar [81] in the following initial value problem:

$$\begin{aligned} i\partial_t u(z,t) &= \mathcal{L}u(z,t), \quad && z \in \mathbb{C}^n, t \in \mathbb{R}, \\ u(z,0) &= f(z), && z \in \mathbb{C}^n. \end{aligned} \tag{1.2.6}$$

For $f \in L^2(\mathbb{C}^n)$, the solution of the initial value problem (1.2.6) is given by $u(z,t) = e^{-it\mathcal{L}} f(z)$ and satisfies the following Strichartz estimate.

Theorem 1.2.6. [81] *Let $f \in L^2(\mathbb{C}^n)$. If $1 < p < \infty$, $\frac{1}{p} \geq n\left(1 - \frac{1}{q}\right)$, or $\frac{1}{2} \leq p \leq 1$, $1 \leq q < \frac{n}{n-1}$, then*

$$\|e^{-it\mathcal{L}} f\|_{L_t^{2p} L_z^{2q}(\mathbb{T}\times\mathbb{C}^n)} \leq C\|f\|_{L^2(\mathbb{C}^n)}.$$

In this thesis, we aim to generalize Theorem 1.2.4 and Theorem 1.2.6 for a system of orthonormal functions associated with the Hermite and special Hermite operator, respectively. Note that, the Strichartz inequality for the system of orthonormal functions associated with Hermite operator has been proved in [9] using the classical Strichartz estimates for the free Schrödinger propagator for orthonormal systems [35,36] and the relation between the Schrödinger kernel and the Mehler kernel associated with the Hermite semigroup [90]. However, we obtain this result independently as a direct application of the Fourier-Hermite restriction theorem.

1.3 Pseudo-differential operators

The theory of pseudo-differential operators is one of the essential tools in recent interdisciplinary activities concerning mathematics analysis with important applications in applied mathematics and physics. Pseudo-differential operators are widely used in harmonic analysis, PDE, geometry, mathematical physics, time-frequency analysis, imaging, computations, and index theory [2,47]. Kohn and Nirenberg [57] introduced the theory of pseudo-differential operators and later used by Hörmander [47] for solving the problems in partial differential equations.

Let σ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Then the (global) pseudo-differential operator T_σ associated with σ is defined by

$$\left(T_\sigma f\right)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi, \quad x \in \mathbb{R}^n, \tag{1.3.1}$$

for all f in the Schwartz space $S\left(\mathbb{R}^n\right)$, provided the integral exists. The function $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ in (1.3.1) is called the symbol of the pseudo-differential operator T_σ. If the symbol σ does not depend on the variable x, then the function $\sigma = \sigma(\xi)$ is called the multiplier and T_σ is called the Fourier multiplier operator. In order to get a useful and tractable class of operators, it is necessary to impose certain conditions on the functions σ. The most fundamental question that arises in the field of pseudo-differential operators is to define a suitable class of symbols. In this regard, for $m \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$, Hörmander [47] introduced symbol class $S^m_{\rho,\delta}(\mathbb{R}^n)$, famously known as (ρ,δ)-Hörmander class, consisting of those functions $\sigma(\cdot,\cdot) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfy the following estimate:

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}\left(1+|\xi|^2\right)^{\frac{m-\delta|\alpha|+\rho|\beta|}{2}},$$

for all multi-indices $\alpha,\beta \in \mathbb{N}_0^n$. Here, m denotes the order of the symbol σ. The corresponding set of pseudo-differential operators with symbols in (ρ,δ)-classes are denoted as $\Psi^m_{\rho,\delta}(\mathbb{R}^n)$. For $\rho = 1$ and $\delta = 0$, the class $S^m_{1,0}(\mathbb{R}^n)$ is introduced by Kohn and Nirenberg [57]. The class $S^m_{1,0}(\mathbb{R}^n)$ is the most simplest and useful class of symbols to work. Eventually, such classes of pseudo-differential operators play a key role in the local solvability problem for differential operators (see [5]). Pseudo-differential operators on $\mathbb{R}^n$ satisfy the following important properties:

- Let $\sigma \in S^m_{1,0}(\mathbb{R}^n), m \in \mathbb{R}$ and $f \in S(\mathbb{R}^n)$, then $T_\sigma f \in S(\mathbb{R}^n)$, i.e., T_σ maps the Schwartz space S into itself.
- Let $\sigma \in S^0_{1,0}(\mathbb{R}^n)$. Then $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator.
- Let $\sigma \in S^{m_1}_{1,0}(\mathbb{R}^n)$ and $\tau \in S^{m_2}_{1,0}(\mathbb{R}^n)$. Then there exists a symbol $\lambda \in S^{m_1+m_2}_{1,0}(\mathbb{R}^n)$ such that $T_\lambda = T_\sigma T_\tau$.
- Let $\sigma \in S^m_{1,0}(\mathbb{R}^n), m \in \mathbb{R}$. Then there exists a symbol $\sigma^* \in S^m_{1,0}(\mathbb{R}^n)$ such that $T^*_\sigma = T_{\sigma^*}$, where T^*_σ is the formal L^2-adjoint of T_σ.

We refer to [85, 108] for several properties and symbolic calculus of pseudo-differential operators on $\mathbb{R}^n$.

We note that, the formation of a pseudo-differential operator is mainly based on the Fourier inversion formula given by

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

for all f in $S(\mathbb{R}^n)$. To define the pseudo-differential operators on other non-commutative groups, we first observe that $\mathbb{R}^n$ is a locally compact abelian group and its dual group is also $\mathbb{R}^n$. A pseudo-differential operator can also be defined using the inverse Fourier transform on $\mathbb{R}^n$. These observations allow us to extend the definition of pseudo-differential operators to other non-commutative groups, provided we have a Fourier inversion formula for the Fourier transform on the groups. Using this idea, pseudo-differential operators on various classes of groups, such as $\mathbb{S}^1, \mathbb{Z}$, finite abelian groups, locally compact abelian groups, affine groups, compact groups, compact Lie groups, homogeneous spaces of compact groups, Heisenberg group, and on general locally compact type I groups, have been defined and studied broadly by several researchers. We refer to [18–20, 22, 22, 24, 33, 34, 60, 61, 85, 86, 108] and references therein.

Ruzhansky and Turunen [85, 86] studied (global) pseudo-differential operators with matrix-valued symbols on compact (Lie) groups. They introduced symbol classes and studied symbolic calculus for matrix-valued symbols on compact Lie groups, and presented plentiful applications of this global theory. After that, the theory of pseudo-differential operators with matrix-valued symbols on compact (Hausdorff) groups, compact homogeneous spaces, compact manifolds is broadly studied by several authors [26, 28, 30, 40, 60, 74,75,108] in many different contexts. Further, pseudo-differential operators with matrix-valued symbols have been extended to non-compact non-abelian groups. In this direction, Ruzhansky and Fischer developed the global theory of pseudo-differential operators on the Heisenberg group, more generally on graded Lie groups [33, 34]. We refer to [73] for global quantization of pseudo-differential operators on nilpotent Lie groups.

1.3.1 Szegö limit theorem

The observable quantities in the classical system are described by real valued functions on the phase space, whereas in quantum systems they are given by self-adjoint operators on a Hilbert space. Therefore it is important to study the correspondence between classical and quantum statistical mechanics. Pseudo-differential operator theory provides a natural platform to relate classical and quantum mechanics. For instance in [109], Zelditch considered the Schrödinger operator on $\mathbb{R}^n$ of the form $\widetilde{H} = -\frac{1}{2}\Delta + V$, where V is a smooth positive function that grows like $V_0|x|^\kappa$, $\kappa > 0$, at infinity. He took a 0-th order self-adjoint pseudo-differential operator A associated with a symbol σ relative to Beals-Fefferman weights $\varphi_1(x,\xi) = 1, \varphi_2(x,\xi) = (1 + |\xi|^2 + V(x))^{1/2}$, and proved the following Szegö type theorem: For any continuous function f on $\mathbb{R}$,

$$\lim_{\lambda\to\infty} \frac{\mathrm{Tr} f(P_\lambda A P_\lambda)}{\mathrm{rank}(P_\lambda)} = \lim_{\lambda\to\infty} \frac{\int_{\widetilde{H}(x,\xi)\leq\lambda} f(\sigma(x,\xi))\, dx d\xi}{\mathrm{Vol}(\widetilde{H}(x,\xi) \leq \lambda)},$$

where $\widetilde{H}(x,\xi) = \frac{1}{2}|\xi|^2 + V(x)$ and P_λ is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the space of the eigenfunctions of $\widetilde{H}$ with eigenvalue less equal to λ, assuming one limit exists.

The classical Szegö limit theorem describes the asymptotic distribution of eigenvalues of the operator $P_n T_f P_n$, where T_f is the multiplication operator on $L^2((0, 2\pi))$ associated with a positive function $f \in C^{1+\alpha}[0, 2\pi], \alpha > 0$, and the orthogonal projections $\{P_n\}$ of $L^2[0, 2\pi]$ onto a linear subspace spanned by the functions $\{e^{im\theta} : 0 \leq m \leq n; 0 \leq \theta < 2\pi\}$. For such a triple $(f, T_f, \{P_n\})$, Szegö proved that

$$\lim_{n\to\infty} \frac{1}{n+1} \log\det P_n T_f P_n = \frac{1}{2\pi}\int_0^{2\pi} \log f(\theta) d\theta. \tag{1.3.2}$$

Equation (1.3.2) is well known as Szegö limit theorem. We refer to [41, 97] for details and related results. More specifically, for a bounded real-valued integrable function f, Szegö limit theorem can be generalized to any continuous function F (instead of the logarithm in

(1.3.2)) defined on $[\inf f, \sup f]$, containing the eigenvalues $\{\lambda_i^n\}_{i=1}^n$ of $P_n T_f P_n$ (see Section 5.3 of [41]). For such F, the following limit holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i^n) = \frac{1}{2\pi} \int_0^{2\pi} F(f(\theta))d\theta. \tag{1.3.3}$$

Notice that the left hand side can be seen as the limit of

$$\operatorname{Tr}(F(P_n T_f P_n)) / \operatorname{Tr}(P_n),$$

where $\operatorname{Tr}(X)$ denotes the trace of the operator X, with the asymptotic of the functional

$$\rho_\lambda(F) = \operatorname{Tr}(\pi_\lambda F(\pi_\lambda T_f \pi_\lambda)\pi_\lambda) = \sum_k F(\mu_k(\lambda))$$

being precisely the sum of Dirac measures located at the eigenvalues $\mu_k(\lambda)$ of the operator $\pi_\lambda T_f \pi_\lambda$. The above expression (1.3.3) roughly says that, as $n \to \infty$, the eigenvalues of $F(P_n T_f P_n)$ distribute like the values of $F(f(\theta))$ sampled at regularly spaced points in the interval $[0, 2\pi]$.

In [96], the authors consider the operator of the form $L = \Delta_{\mathbb{Z}} + V$ on the lattice, where the self adjoint discrete Laplacian operator $\Delta_{\mathbb{Z}}$ on $\ell^2(\mathbb{Z}^n)$ is defined as $(\Delta_{\mathbb{Z}} u)(k) = \sum_{|k-j|=1}(u(j) - u(k))$, and the operator V is the multiplication by a positive sequence $\{V(k), k \in \mathbb{Z}^n\}$ with $V(k) \to \infty$ as $|k| \to \infty$. They also considered 0-th order self-adjoint pseudo-differential operator B associated with symbol $b \in S_{1,0,\infty}(\mathbb{T}^n \times \mathbb{Z}^n)$, and proved the following Szegö type theorem on $\mathbb{Z}^n$: For any continuous function f on $\mathbb{R}$,

$$\lim_{\lambda\to\infty} \frac{\operatorname{Tr} f(\pi_\lambda B \pi_\lambda)}{\operatorname{rank}(\pi_\lambda)} = \lim_{\lambda\to\infty} \frac{1}{(2\pi)^n} \frac{\sum_{V(k)\le\lambda} \int_{\mathbb{T}^n} f(b(x,k))dx}{\sum_{V(k)\le\lambda} 1},$$

where π_λ is the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ onto the space of the eigenfunctions of L with eigenvalues less equal to λ, assuming one limit exists. Such asymptotic spectral formulae expressing the relation between functions of pseudo-differential operators and their symbols is an important and interesting problem in mathematical analysis. We refer to [46,48,49,89,96,107] for similar results available in the literature. There is an extensive work on the Szegö's theorem associated with orthogonal polynomials in $L^2(\mathbb{T}, d\mu)$ with some probability measure μ on $\mathbb{T}$, we refer to the monumental work of Barry Simon [89] for the details.

The main ingredient to establish Szegö type theorem is to consider the ratios of distribution functions associated to different measures and their asymptotic behavior. The asymptotic limit of such ratios is computed using a suitable theorem (Tauberian theorem), where some transform of these measures is considered and the limit is taken for such transforms. For example, Zelditch [109] used the Laplace transform via Karamata's Tauberian theorem (see [106]), whereas Robert [82] suggested the use of Stieltjes transform via Keldysh Tauberian theorem (see [50]). However, the authors in [96] considered Tauberian theorem of Grishin-Poedintseva (see [42]) and a theorem of Laptev-Safarov (see [63] and [62]) to compute the asymptotic limit of such ratios for estimating the errors.

Fischer and Ruzhansky in [34] (see also [33]) introduced and studied symbolic calculus for pseudo-differential operators on the Heisenberg group (more generally on nilpotent Lie groups). In this thesis we prove Szegö limit theorem on the Heisenberg group $\mathbb{H}^n$. We use the recent version of Tauberian theorem of Keldysh by Grishin-Poedintseva [42] and a theorem of Laptev-Safarov [62, 63] to estimate the error term.

1.3.2 Schatten class and nuclear pseudo-differential operators

The trace of an (trace class) operator on Hilbert spaces is the sum of its eigenvalues is equal to integration of its integral kernel over the diagonal. However, this property fails in Banach spaces. The importance of r-nuclear operators lies in the seminal work of Grothendieck [44, 45], who proved that, for 2/3-nuclear operators, the trace in L^p-spaces agrees with the sum of all the eigenvalues with multiplicities counted. Therefore, the notion of r-nuclear operators becomes useful. One of the interesting question is to find a good criteria for ensuring the r-nuclearity of operators on L^p-spaces. But this needs to be formulated differently than those on Hilbert spaces and has to take into account the impossibility of certain kernel formulations in view of Carleman's example [21] (also see [26]). In view of this, one should establish conditions imposed on symbols instead of kernels ensuring the r-nuclearity of the corresponding operators.

The initiative of finding necessary and sufficient conditions for pseudo-differential operators to be r-nuclear is due to Delgado and Wong [31]. The main tool used for such characterization was established by Delgado [25]. A multilinear version of this result was recently proved by Kumar and Cardona to study the nuclearity of multilinear pseudo-differential operators on the lattice and torus [19, 20]. Delgado and Ruzhansky [26] studied the L^p-nuclearity and traces of pseudo-differential operators on compact Lie groups using the global symbolic calculus developed by Ruzhansky and Turunen [85]. Later, Ruzhansky et. al extended these results to more general spaces such as compact homogeneous spaces and compact manifolds [25, 26, 28, 30]. On the other hand, Wong et. al extended the results of [31] in the settings of abstract compact groups without differential structure [39, 40]. In particular, characterizations of nuclear operators in terms of decomposition of symbol via Fourier transform were investigated by Ghaemi, Jamalpour Birgani, and Wong [39] for $\mathbb{S}^1$ and arbitrary compact groups [40].

It is well known that in the setting of Hilbert spaces, the class of r-nuclear operators agrees with the r-Schatten-von Neumann class of operators [79]. Over the years, considerable attention has been devoted by several researchers for finding good criteria for operators belonging to r-Schatten-von Neumann class and to the class of r-nuclear operators in terms of their symbols with lower regularity [17, 26, 29, 101, 102]. Ruzhansky and Delgado [26, 28] successfully drop the regularity condition in their setting using matrix-valued symbols. Ruzhansky and Delgado investigated this in detail in many different settings; for example, using the matrix-valued symbols on compact Lie groups in [26–29] they successfully characterized these classes of operators on compact Lie groups. Later, they with their collaborators extended these results to compact manifolds and to more general on Hilbert spaces [29, 30] using the non-harmonic analysis, developed by Ruzhansky and Tokmagambetov [83].

The homogeneous spaces of abstract compact groups play an important role in mathematical physics, geometric analysis, constructive approximation, and coherent state transform, see [51–55, 57] and the references therein. Let G is a compact (Hausdorff) group and H be a closed subgroup of G. Pseudo-differential operators on homogeneous spaces of compact groups G/H (without differential structure) was studied in [60] (see also [85]). Using the operator-valued Fourier transform on homogeneous spaces of compact groups developed by Ghani Farashahi [38], in this thesis, we define global pseudo-differential operators on homogeneous spaces of compact groups and study the r-Schatten-von Neumann class of operators on $L^2(G/H)$ and r-nuclear operators on L^p-spaces on compact homogeneous spaces.

1.4 Outline of the Thesis

This thesis consists of five chapters with the present chapter dealing with the basic definitions, review of recent developments, and our motivation to consider the problems discussed in the thesis.

Chapter 2 is mainly devoted to study orthonormal Strichartz inequality associated with Hermite operator H on $\mathbb{R}^n$. We generalize Theorem 1.2.4 and obtain the Strichartz estimate for $1 \leq q < \frac{n+1}{n-1}$, for the system of orthonormal functions associated with the Hermite operator as the restriction of the Hermite-Fourier transform to the discrete surface $S = \{(\mu, \nu) \in \mathbb{N}_0^n \times \mathbb{Z} : \nu = 2|\mu| + n\}$. As a key step to prove this, we obtain the duality principle in terms of Schatten bounds of the operator $We^{-itH}(e^{-itH})^*\overline{W}$ and give an affirmative answer to Problem 2, when $p = \frac{2\lambda_0}{1+\lambda_0}$, for some $\lambda_0 > 1$. We also prove the optimality of Schatten exponent.

In Chapter 3, we investigate yet another Strichartz inequality for orthonormal functions, but for special Hermite operator $\mathcal{L}$ on $\mathbb{C}^n$. Adopting similar mathematical formulation as in Chapter 2, we generalize Theorem 1.2.6 and obtain the Strichartz estimate for $1 \leq q \leq 1 + \frac{1}{n}$, for systems of orthonormal functions associated with the special Hermite operator as the restriction of the special Hermite transform to the discrete surface $S = \{(\mu, \nu, \lambda) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{Z} : \lambda = 2|\nu| + n\}$.

In chapter 4, we prove Szegö type limit theorems on the Heisenberg group $\mathbb{H}^n$. We consider the Schrödinger operator $\mathcal{H} = -\Delta_{\mathbb{H}} + V$ on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}}$ is the full laplacian on $\mathbb{H}^n$ and V is a positive smooth potential, bounded below and grows like $|g|^\kappa, \kappa > 0$, for large $|g|$. First, we build up symbolic calculus for pseudo-differential operators relative to the operator $1 + |\lambda|H + V(g)$ on $L^2(\mathbb{H}^n)$, using the techniques developed in [33, 34]. Then we construct pseudo-differential approximations to the operator $(\mathcal{H} + u)^{-m}$ on $L^2(\mathbb{H}^n)$ and $(1 + |\lambda|(H + I) + V(g) + u)^{-m}$ on $L^2(\mathbb{R}^n)$ within the calculus of symbols defined related to $1 + |\lambda|H + V(g)$ and $1 + |\lambda|(1 + |\xi|^2 + |x|^2) + V(g)$, respectively. We first obtain Szegö type limit theorem for $\mathcal{H} = -\Delta_{\mathbb{H}} + V$ with respect to the multiplication operator $M_{\mathbf{b}}$, where $\mathbf{b}$ is a bounded real valued integrable function on $\mathbb{H}^n$. Further, we prove Szegö type limit theorem for $\mathcal{H} = -\Delta_{\mathbb{H}} + V$ by considering 0-th order self-adjoint pseudo-differential operator on $L^2(\mathbb{H}^n)$ relative to the operator $1 + |\lambda|H + V(g)$, where

$(g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^*$, in place of the multiplication operator $M_{\mathbf{b}}$. We show that the generalize Szegö limit theorem also holds under a perturbation of the Schrödinger operator $\mathcal{H}$ by a bounded self-adjoint operator on $L^2(\mathbb{H}^n)$. Further, we show that all the Szegö type limit theorems are also valid under a compact perturbation of the pseudo-differential operator A. Finally, we provide an alternative proof of the error estimate for $\kappa \in (0,1)$ without using pseudo-differential symbolic calculus, but the boundedness of the operators $[A, V]$ and $[A, \mathcal{L}]$ on $L^2(\mathbb{H}^n)$.

In Chapter 5, we consider homogeneous spaces of compact groups G/H, where G is a compact (Hausdorff) group and H be a closed subgroup of G. We present symbolic criteria for pseudo-differential operators on G/H characterizing the Schatten-von Neumann classes $S_r(L^2(G/H))$ for all $0 < r \leq \infty$. We provide a symbolic characterization for pseudo-differential operators on $L^p(G/H), 1 \leq p < \infty$, to be r-nuclear for $0 < r \leq 1$. We calculate the nuclear trace of related pseudo-differential operators. We also find symbols of the adjoint and product of r-nuclear pseudo-differential operators on G/H and provide a characterization for self-adjointness. In the end, we present an application of our results in the context of the heat kernel on G/H.

CHAPTER 2

Restriction theorem for the Fourier-Hermite transform and solution of the Hermite-Schrödinger equation

2.1 Introduction

In this chapter, we prove a restriction theorem for the Fourier-Hermite transform and obtain the full range Strichartz estimate for the system of orthonormal functions for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ as an application. We also show that the constant obtained in the Strichartz inequality is optimal in terms of the limit of a large number of functions.

The Strichartz inequality for the system of orthonormal functions for the Hermite operator has been proved in [9] using the classical Strichartz estimates for the free Schrödinger propagator for orthonormal systems [35, 36] and the link between the Schrödinger kernel and the Mehler kernel associated with the Hermite semigroup [90]. However, it is important to note that this result can also be obtained independently as a direct application of the Fourier-Hermite restriction theorem.

2.2 Preliminaries

In this section, we provide basic definitions and necessary background about the Hermite semigroup. We start with the definition of the Hermite operator.

2.2.1 Hermite operator and the spectral theory

Let $\mathbb{N}_0$ denote the set of all non-negative integers. For $z \in \mathbb{C}$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote real and imaginary parts of z, respectively. Let H_k denote the Hermite polynomial on $\mathbb{R}$, defined by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2}, \quad k \in \mathbb{N}_0,$$

and h_k denote the normalized Hermite functions on $\mathbb{R}$ defined by

$$h_k(x) = (2^k \sqrt{\pi} k!)^{-\frac{1}{2}} H_k(x) e^{-\frac{1}{2}x^2}, \quad k \in \mathbb{N}_0.$$

For $\alpha \in \mathbb{N}_0^n$, the higher dimensional Hermite functions Φ_α, are obtained by taking tensor product of one dimensional Hermite functions. Thus for any multi-index $\alpha \in \mathbb{N}_0^n$, and $x \in \mathbb{R}^n$, we define $\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j)$. The family $\{\Phi_\alpha\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. They are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$ corresponding to eigenvalues $(2|\alpha| + n)$, where $|\alpha| = \sum_{j=1}^n \alpha_j$. Given $f \in L^2(\mathbb{R}^n)$, we have the Hermite expansion

$$f = \sum_{\alpha \in \mathbb{N}_0^n} (f, \Phi_\alpha)\, \Phi_\alpha = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} (f, \Phi_\alpha)\, \Phi_\alpha = \sum_{k=0}^{\infty} P_k f,$$

where P_k denotes the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha : |\alpha| = k\}$. The operator H defines a semigroup called the Hermite semigroup $e^{-tH}, t > 0$, given by

$$e^{-tH} f = \sum_{k=0}^{\infty} e^{-(2k+n)t} P_k f$$

for $f \in L^2(\mathbb{R}^n)$. On a dense subspace, say the space of all Schwartz class functions, the above also can be written as

$$e^{-tH} f(x) = \int_{\mathbb{R}^n} f(y) K_t(x, y)\, dy,$$

where the kernel, $K_t(x, y)$ is given by the expansion

$$K_t(x, y) = \sum_{\alpha \in \mathbb{N}_0^n} e^{-(2|\alpha|+n)t} \Phi_\alpha(x) \Phi_\alpha(y), \quad x, y \in \mathbb{R}^n.$$

For $z' = r + it, r > 0, t \in \mathbb{R}$, the kernel of the operator $e^{-z'H}$ is given by

$$K_{z'}(x,y) = \sum_{k=0}^{\infty} e^{-z'(2k+n)} \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y), \quad x, y \in \mathbb{R}^n.$$

Using Mehler's formula, the kernel of the operator $e^{-z'H}$ can be obtained as (see [76])

$$K_{z'}(x,y) = \frac{1}{(2\pi \sinh 2z')^{\frac{n}{2}}} e^{\frac{1}{2}\left(-\coth 2z'(|x|^2+|y|^2) + \frac{2x\cdot y}{\sinh 2z'}\right)}. \tag{2.2.1}$$

For $t \in \mathbb{R} \setminus (\frac{\pi}{2})\mathbb{Z}$, letting $r \to 0$, the kernel of the operator e^{-itH} can be written as

$$K_{it}(x,y) = \frac{e^{-\frac{i\pi n}{4}}}{(2\pi \sin 2t)^{\frac{n}{2}}} e^{\frac{i}{2}\left(\cot 2t(|x|^2+|y|^2) - \frac{2x\cdot y}{\sin 2t}\right)}. \tag{2.2.2}$$

Also for $t \in \mathbb{R} \setminus (\frac{\pi}{2})\mathbb{Z}$, we have

$$K_{-it}(x,y) = \overline{K_{it}(x,y)} \quad \text{and} \quad K_{i(t+\frac{\pi}{2})}(x,y) = e^{-i\pi\frac{n}{2}} K_{it}(-x,y). \tag{2.2.3}$$

It is known that for real valued functions f, the $L^p(\mathbb{R}^n)$ norm of $e^{-itH}f$ is even and $\frac{\pi}{2}$-periodic as a function of t. We refer to [76,99] for a detailed study on the kernel associated with the operator e^{-itH}.

2.2.2 Schatten class

Let $\mathcal{H}$ be a complex and separable Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Let $T : \mathcal{H} \to \mathcal{H}$ be a compact operator and let T^* denotes the adjoint of T. For $1 \leq r < \infty$, the Schatten space, $S_r(\mathcal{H})$ is defined as the space of all compact operators T on $\mathcal{H}$ such that

$$\sum_{n=1}^{\infty} (s_n(T))^r < \infty,$$

where $s_n(T)$ denotes the singular values of T, i.e., the eigenvalues of $|T| = \sqrt{T^*T}$ counted according to multiplicity. For $T \in S_r(\mathcal{H})$, the Schatten r-norm is defined by

$$\|T\|_{S_r} = \left(\sum_{n=1}^{\infty} (s_n(T))^r \right)^{\frac{1}{r}}.$$

If $r = \infty$, we define $\|T\|_{S_\infty} = \|T\|$. An operator in $S_1(\mathcal{H})$ $(S_2(\mathcal{H}))$ is a Trace class operator (Hilbert-Schmidt operator). If $A : \mathcal{H} \to \mathcal{H}$ is bounded operator and $T : \mathcal{H} \to \mathcal{H}$ is a trace class operator then both AT and TA are trace class operators, and $|\operatorname{Tr}(AT)| \leq \|A\|\|T\|_{S_1}$. We refer to [87] for a detailed study on Schatten classes.

2.3 The Restriction theorem

In this section, we set a platform to prove the restriction theorem with respect to the Fourier-Hermite transform for a given discrete surface $S \subset \mathbb{N}_0^n \times \mathbb{Z}$. We start this section with the definition of Fourier-Hermite transform.

Let $f \in L^1(\mathbb{R}^n)$. The Hermite transform of f is defined by

$$\hat{f}(\mu) = \int_{\mathbb{R}^n} f(x)\Phi_\mu(x)\, dx, \quad \mu \in \mathbb{N}_0^n,$$

where Φ_μ's are the n-dimensional Hermite functions. If $f \in L^2(\mathbb{R}^n)$ then $\{\hat{f}(\mu)\} \in \ell^2(\mathbb{N}_0^n)$, and the Plancherel formula is of the form

$$\|f\|_2^2 = \sum_{\mu \in \mathbb{N}_0^n} |\hat{f}(\mu)|^2.$$

The inverse Hermite transform is given by

$$f(x) = \sum_{\mu \in \mathbb{N}_0^n} \hat{f}(\mu)\Phi_\mu(x), \quad x \in \mathbb{R}^n,$$

i.e., the orthonormal basis expansion of f with respect to $\{\Phi_\mu(x)\}$. Given a discrete surface S in $\mathbb{N}_0^n \times \mathbb{Z}$, we define the restriction operator $\mathcal{R}_S f := \{\hat{f}(\mu,\nu)\}_{(\mu,\nu)\in S}$ and the operator dual to $\mathcal{R}_S$ (called the extension operator) as

$$\mathcal{E}_S(\{\hat{f}(\mu,\nu)\}) := \sum_{(\mu,\nu)\in S} \hat{f}(\mu,\nu)\Phi_\mu(\cdot)e^{-i(\cdot)\nu}, \tag{2.3.1}$$

where the Fourier-Hermite transform of f is given by

$$\hat{f}(\mu,\nu) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^n}\int_{(-\pi,\pi)} f(t,x)\Phi_\mu(x)e^{it\nu}\, dxdt. \tag{2.3.2}$$

Then

$$\begin{aligned}
\mathcal{E}_S\mathcal{E}_S^* f(t,x) &= \sum_{(\mu,\nu)\in S} \hat{f}(\mu,\nu)\Phi_\mu(x)e^{-it\nu} \\
&= (2\pi)^{-\frac{1}{2}} \sum_{(\mu,\nu)\in S} \int_{\mathbb{R}^n}\int_{(-\pi,\pi)} f(s,y)\Phi_\mu(x)\Phi_\mu(y)e^{-i(t-s)\nu}\, dy\, ds \\
&= \int_{\mathbb{R}^n}\int_{(-\pi,\pi)} K(t-s,x,y)f(s,y)\, dy\, ds \\
&= \int_{\mathbb{R}^n} (K(\cdot,x,y) * f(\cdot,y))(t)\, dy,
\end{aligned} \tag{2.3.3}$$

where

$$K(t,x,y) = (2\pi)^{-\frac{1}{2}} \sum_{(\mu,\nu)\in S} \Phi_\mu(x)\Phi_\mu(y)e^{-i\nu t}, \tag{2.3.4}$$

and the convolution in (2.3.3) is on the circle group. Now we consider the following problem:

Problem 1: For which exponents $1 \leq p \leq 2$, the sequence of Fourier-Hermite transforms of a function $f \in L^p((-\pi,\pi)\times\mathbb{R}^n)$ belongs to $\ell^2(S)$?

This question can be reframed to the boundedness of the operator $\mathcal{E}_S$ from $\ell^2(S)$ to $L^{p'}((-\pi,\pi)\times\mathbb{R}^n)$, where p' is the conjugate exponent of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$. Since $\mathcal{E}_S$ is bounded from $\ell^2(S)$ to $L^{p'}((-\pi,\pi)\times\mathbb{R}^n)$ if and only if $T_S := \mathcal{E}_S\mathcal{E}_S^*$ is bounded from $L^p((-\pi,\pi)\times\mathbb{R}^n)$ to $L^{p'}((-\pi,\pi)\times\mathbb{R}^n)$, Problem 1 can be re-written as follows:

Problem 2: For which exponents $1 \leq p \leq 2$, the operator $T_S := \mathcal{E}_S\mathcal{E}_S^*$ is bounded from $L^p((-\pi,\pi)\times\mathbb{R}^n)$ to $L^{p'}((-\pi,\pi)\times\mathbb{R}^n)$?

Note that Hölder's inequality implies that the operator $T_S = \mathcal{E}_S\mathcal{E}_S^*$ is bounded from $L^p((-\pi,\pi)\times\mathbb{R}^n)$ to $L^{p'}((-\pi,\pi)\times\mathbb{R}^n)$ if and only if for any $W_1, W_2 \in L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)$, the operator $W_1T_SW_2$ (composition of the multiplication operator associated with W_1, T_S and the multiplication operator associated with W_2) is bounded on $L^2((-\pi,\pi)\times\mathbb{R}^n)$ with

$$\|W_1T_SW_2\|_{L^2((-\pi,\pi)\times\mathbb{R}^n)\to L^2((-\pi,\pi)\times\mathbb{R}^n)} \leq C\|W_1\|_{L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)}\|W_2\|_{L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)}, \tag{2.3.5}$$

for some constant $C > 0$.

Another motivation to consider Problem 1 is due to the connection to Frame theory: The Fourier-Hermite restriction and extension operators seem to be what is called analysis and synthesis operator in Gabor Analysis [12, 43]. More precisely, the extension operator as defined by (2.3.1) is not only a synthesis operator, it is already the frame operator for the union of modulated ONBs consisting of Hermite functions. The problem about the boundedness of the extension operator defined by (2.3.1) is therefore a reduced to the boundedness of the frame operator of a degenerated multi-window Gabor system. The degeneracy stems from the fact that no translations are used. The multi-window Gabor system is built from the eigenfunctions of a Daubechies localization operator [13] (see also [14]).

We introduce an analytic family of operators (T_z) defined on the strip, $a \le \text{Re } z \le b$ in the complex plane such that $T_S = T_c$ for some $c \in (a,b)$ and show that the operator $W_1T_SW_2$ belongs to a Schatten class with

$$\|W_1T_SW_2\|_{\mathcal{S}_\alpha(L^2((-\pi,\pi)\times\mathbb{R}^n))} \le C\|W_1\|_{L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)}\|W_2\|_{L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)},$$

for some $\alpha > 0$, which is a more general $L^p - L^{p'}$ boundedness result of T_S.

Recall that, a family of operators (T_z) on $\mathbb{R}^n$ defined on a strip $a \le \text{Re}(z) \le b$ with $a < b$, in the complex plane is analytic in the sense of Stein [91] if for all simple functions f, g on $\mathbb{R}^n$, the map $z \mapsto \langle g, T_z f\rangle$ is analytic in $a < \text{Re}(z) < b$, continuous in $a \le \text{Re}(z) \le b$, and if $\sup_{a\le x\le b} |\langle g, T_{x+is} f\rangle| \le C(s)$, for some $C(s)$ with at most a (double) exponential growth in s.

The following proposition assures an affirmative answer to Problem 2 under certain assumptions. We refer to Lemma 3 of [36] with appropriate modifications to obtain the following result. In order to obtain the Strichartz inequality for the system of orthonormal functions, we need the duality principle lemma in our context due to Frank-Sabin [36].

Lemma 2.3.1. [Duality principle] *Let $p, q \ge 1$ and $\alpha \ge 1$. Let A be a bounded linear operator from a separable Hilbert space $\mathcal{H}$ to $L_t^{q'}L_x^{p'}((-\pi,\pi)\times\mathbb{R}^n)$. Then the following statements are equivalent.*

1. *There is a constant $C > 0$ such that*

$$\left\|WAA^*\overline{W}\right\|_{\mathcal{S}_\alpha(L^2((-\pi,\pi)\times\mathbb{R}^n))} \le C\|W\|^2_{L_t^{\frac{2q}{2-q}}L_x^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)}, \tag{2.3.6}$$

for all $W \in L_t^{\frac{2q}{2-q}}L_x^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)$, where the function W is interpreted as an operator which acts by multiplication.

2. *For any orthonormal system $(f_j)_{j\in J}$ in $\mathcal{H}$ and any sequence $(n_j)_{j\in J} \subset \mathbb{C}$, there is a constant $C' > 0$ such that*

$$\left\|\sum_{j\in J} n_j\,|Af_j|^2\right\|_{L_t^{\frac{q'}{2}}L_x^{\frac{p'}{2}}((-\pi,\pi)\times\mathbb{R}^n)} \le C'\left(\sum_{j\in J}|n_j|^{\alpha'}\right)^{1/\alpha'}. \tag{2.3.7}$$

Proof. We note that (2.3.7) is equivalent to itself but with the additional condition on the scalers is that $n_j \ge 0, j \in J$, i.e.,

$$\left\|\sum_{j\in J} n_j\,|Af_j|^2\right\|_{L_t^{\frac{q'}{2}}L_x^{\frac{p'}{2}}((-\pi,\pi)\times\mathbb{R}^n)} \le C'\left(\sum_{j\in J}|n_j|^{\alpha'}\right)^{1/\alpha'}, \quad \forall n_j \ge 0, j \in J. \tag{2.3.8}$$

Thus to show equivalence of (2.3.6) and (2.3.7), it is enough to show that (2.3.6) is equivalent to (2.3.8). First we show that (2.3.6) implies (2.3.8). Let $(f_j)_{j\in J}$ be an orthonormal system in $\mathcal{H}$ and $(n_j)_{j\in J}\subset\mathbb{R}_+$.

We define an operator γ on $\mathcal{H}$ as

$$\gamma := \sum_j n_j \left|f_j\right\rangle \left\langle f_j\right|, \tag{2.3.9}$$

where the Dirac's notation $|u\rangle\langle v|$ stands for the rank-one operator $f\mapsto\langle v,f\rangle u$. Then (f_j) are the eigenfunctions of γ corresponding to the eigenvalues (n_j). Moreover, the estimate (2.3.6) is equivalent to

$$\left\|A^*|W|^2A\right\|_{S_\alpha(\mathcal{H})} \le C\|W\|^2_{L_t^{\frac{2q}{2-q}}L_x^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)}, \tag{2.3.10}$$

for all $W\in L_t^{\frac{2q}{2-q}}L_x^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)$. Using (2.3.10) and Hölder's inequality for Schatten spaces, we get

$$\begin{aligned}\operatorname{Tr}_{L^2((-\pi,\pi)\times\mathbb{R}^n)}\left(WA\gamma(WA)^*\right) &= \operatorname{Tr}_{\mathcal{H}}\left(\gamma A^*|W|^2A\right)\\ &\le \|\gamma\|_{S_{\alpha'}(\mathcal{H})}\left\|A^*|W|^2A\right\|_{S_\alpha(L^2(\mathbb{R}^n))}\\ &= C\left(\sum_{j\in J}|n_j|^{\alpha'}\right)^{1/\alpha'}\|W\|^2_{L_t^{\frac{2q}{2-q}}L_x^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)}.\end{aligned}$$

On the other hand, using the notation in (2.3.9), we have the following identity

$$\operatorname{Tr}_{L^2((-\pi,\pi)\times\mathbb{R}^n)}\left(WA\gamma(WA)^*\right) = \int_{(-\pi,\pi)}\int_{\mathbb{R}^n}\left(\sum_{j\in J}n_j\left|\left(Af_j\right)(x,t)\right|^2\right)|W(x,t)|^2\,dx\,dt.$$

So we can infer that, for all $V\in L_t^{\frac{q}{2-q}}L_x^{\frac{p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)$ with $V\ge 0$,

$$\begin{aligned}&\int_{(-\pi,\pi)}\int_{\mathbb{R}^n}\left(\sum_{j\in J}n_j\left|\left(Af_j\right)(x,t)\right|^2\right)|V(x,t)|dx\,dt\\ &\le C\left(\sum_{j\in J}|n_j|^{\alpha'}\right)^{1/\alpha'}\|V\|_{L_t^{\frac{q}{2-q}}L_x^{\frac{p}{2-p}}((-\pi,\pi)\times\mathbb{R}^n)}.\end{aligned}$$

Since $\left(\frac{p}{2-p}\right)'=\frac{p'}{2}$, the duality principle for L^p-spaces leads to (2.3.8). The proof of other part is similar. $\square$

Note that Lemma 2.3.1 is also valid in the domain $(-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n$.

Let S be the discrete surface $S = \{(\mu, \nu) \in \mathbb{N}_0^n \times \mathbb{Z} : R(\mu, \nu) = 0\}$, where $R(\mu, \nu)$ is a polynomial of degree one with respect to the counting measure.

For $\varepsilon > 0$, we define

$$T_\varepsilon f(t,x) = \int_{\mathbb{R}^n} \int_{(-\pi,\pi)} K_\varepsilon(t-s,x,y) g(s,y) \, dy \, ds,$$

where $K_\varepsilon(x,y,t) = 1_{\varepsilon<|t|<\pi} K(t,x,y)$ and $K(t,x,y)$ is defined in (2.3.4). We suppose that for $(t,x,y) \in (-\pi,\pi) \times \mathbb{R}^{2n}$, we have

$$|K_\varepsilon(t,x,y)| \le C|t|^{-\lambda}, \tag{2.3.11}$$

for some positive $\lambda \in \mathbb{R}$ and the constant C is independent of ε. We show that

$$\|W_1 T_\varepsilon W_2\|_{S_\alpha(L^2(-\pi,\pi)\times\mathbb{R}^n)} \le C \|W_1\|_{L_t^\beta L_x^\alpha((-\pi,\pi)\times\mathbb{R}^n)} \|W_2\|_{L_t^\beta L_x^\alpha((-\pi,\pi)\times\mathbb{R}^n)} \tag{2.3.12}$$

for some constant C independent of ε, where $\alpha, \beta \ge 1$ such that

$$\frac{2}{\beta} + \frac{2\lambda}{\alpha} = 1 \text{ and } \frac{1}{2(\lambda+1)} \le \frac{1}{\alpha} < \frac{1}{2\lambda+1}.$$

To prove (2.3.12), we apply Stein's complex interpolation, for $z \in \mathbb{C}$ with $\mathrm{Re}(z) \in [-1, \lambda]$, on the operator

$$T_{z,\varepsilon} f(t,x) = \int_{\mathbb{R}^n} \int_{(-\pi,\pi)} K_{z,\varepsilon}(t-s,x,y) g(s,y) \, dy \, ds,$$

where $K_{z,\varepsilon}(t,x,y) = |t|^z K_\varepsilon(t,x,y)$.

Then from (2.3.11), for any $(t,x,y) \in (-\pi,\pi) \times \mathbb{R}^{2n}$, we have

$$|K_{z,\varepsilon}(t,x,y)| \le C|t|^{\mathrm{Re}(z)-\lambda}. \tag{2.3.13}$$

Then, an application of the Hardy-Littlewood-Sobolev inequality (see page 39 in [6]) along with (2.3.13) yields

$$\begin{aligned}
&\|W_1 T_{z,\varepsilon} W_2\|^2_{S_2(L^2((\pi,\pi)\times\mathbb{R}^n))} \\
&= \int_{(\pi,\pi)^2} \int_{\mathbb{R}^{2n}} |W_1(t,x)|^2 \, |K_{z,\varepsilon}(t-t',x,x')|^2 \, |W_2\,(t',x')^2 \, dx dx' dt dt' \\
&\le C_1 \int_{(-\pi,\pi)^2} \int_{\mathbb{R}^{2n}} \frac{|W_1(t,x)|^2 |W_2\,(t',x')\,|^2}{|t-t'|^{2\lambda-2\,\mathrm{Re}(z)}} dx dx' dt dt' \\
&\le C_1 \int_{(-\pi,\pi)} \int_{(-\pi,\pi)} \frac{\|W_1(t)\|^2_{L_x^2(\mathbb{R}^n)} \, \|W_2\,(t')\|^2_{L_x^2(\mathbb{R}^n)}}{|t-t'|^{2\lambda-2\,\mathrm{Re}(z)}} dt dt'
\end{aligned}$$

$$\leq C_1 \left\| \|W_1\|^2_{L^2_x(\mathbb{R}^n)} \right\|_{L^{\tilde{u}}_t((-\pi,\pi))} \left\| \|W_2\|^2_{L^2_x(\mathbb{R}^n)} \right\|_{L^{\tilde{u}}_t((-\pi,\pi))},$$

provided we have $0 \leq 2\lambda - 2\operatorname{Re}(z) < 1$ and $\frac{1}{\tilde{u}} + (\lambda - \operatorname{Re}(z)) = 1$. Further, if we denote $2\tilde{u} = u$, then $\frac{1}{u} \in \left(\frac{1}{4}, \frac{1}{2}\right]$ and

$$\|W_1 T_{z,\varepsilon} W_2\|_{S_2(L^2((-\pi,\pi)\times\mathbb{R}^n))} \leq C\, \|W_1\|_{L^{\tilde{u}}_t L^2_x((-\pi,\pi)\times\mathbb{R}^n)} \|W_2\|_{L^{\tilde{u}}_t L^2_x((-\pi,\pi)\times\mathbb{R}^n)}, \tag{2.3.14}$$

provided $\frac{1}{u} = \frac{1}{2} + \frac{1}{2}(\operatorname{Re}(z) - \lambda)$ and $\operatorname{Re}(z) \in \left(\frac{2\lambda-1}{2}, \lambda\right]$.

Now we consider the case $\operatorname{Re}(z) = -1$. Suppose that $T_{z,\varepsilon} : L^2_{t,x}((-\pi,\pi)\times\mathbb{R}^n) \to L^2_{t,x}((-\pi,\pi)\times\mathbb{R}^n)$ is bounded with some constant that depends only on n and $\operatorname{Im}(z)$ exponentially, then, using the boundedness of $T_{z,\varepsilon}$ on $L^2_{t,x}((-\pi,\pi)\times\mathbb{R}^n)$ and the fact that S_∞-norm is the operator norm, we have

$$\|W_1 T_{z,\varepsilon} W_2\|_{S_\infty(L^2((-\pi,\pi)\times\mathbb{R}^n))} \leq C(\operatorname{Im}(z))\, \|W_1\|_{L^\infty_t L^\infty_x((-\pi,\pi)\times\mathbb{R}^n)} \|W_2\|_{L^\infty_t L^\infty_x((-\pi,\pi)\times\mathbb{R}^n)}, \tag{2.3.15}$$

for $\operatorname{Re}(z) = -1$. Finally, applying analytic interpolation (see [7,8]) between the estimates (2.3.14) and (2.3.15), we get the following Fourier-Hermite restriction theorem by letting $\varepsilon \to 0$ in (2.3.12).

Theorem 2.3.2. [Fourier-Hermite restriction theorem] *Let* $n \geq 1$ *and let* $S \subset \mathbb{N}^n_0 \times \mathbb{Z}$ *be a discrete surface. Suppose that*

(1) for $\operatorname{Re}(z) \in [-1, \lambda]$ *and for each* $(t,x,y) \in (-\pi,\pi)\times\mathbb{R}^{2n}$, $|K_\varepsilon(t,x,y)| \leq C|t|^{-\lambda}$, *for some positive real* λ, *and*

(2) for $\operatorname{Re}(z) = -1$, $T_{z,\varepsilon} : L^2_{t,x}((-\pi,\pi)\times\mathbb{R}^n) \to L^2_{t,x}((-\pi,\pi)\times\mathbb{R}^n)$ *is a bounded linear operator with the constant depends on* n *and* $\operatorname{Im}(z)$ *exponentially,*

then T_S *is bounded from* $L^p((-\pi,\pi)\times\mathbb{R}^n)$ *to* $L^{p'}((-\pi,\pi)\times\mathbb{R}^n)$ *for* $p = \frac{2(\lambda+1)}{\lambda+2}$.

Proof. In order to prove Theorem 2.3.2, by Lemma 2.3.1, it is enough to show that

$$\|W_1 T_S W_2\|_{S_\alpha(L^2(-\pi,\pi)\times\mathbb{R}^n)} \leq C\, \|W_1\|_{L^\beta_t L^\alpha_x((-\pi,\pi)\times\mathbb{R}^n)} \|W_2\|_{L^\beta_t L^\alpha_x((-\pi,\pi)\times\mathbb{R}^n)}$$

for $\alpha = \beta = 2(1+\lambda)$ at $z = 0$ and compare it with (2.3.5) to get $p = \frac{2(\lambda+1)}{\lambda+2}$. □

2.4 Strichartz inequality for system of orthonormal functions

Consider the Schrödinger equation associated with the Hermite operator $H = -\Delta + |x|^2$:

$$\begin{aligned} i\partial_t u(t,x) &= Hu(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(x,0) &= f(x), \qquad x \in \mathbb{R}^n. \end{aligned} \tag{2.4.1}$$

If $f \in L^2(\mathbb{R}^n)$, the solution of the initial value problem (2.4.1) is given by $u(t,x) = e^{-itH}f(x)$. The solution to the initial value problem (2.4.1) can be realized as the extension operator of some function f on $(-\pi,\pi)\times\mathbb{R}^n$. To estimate the solution to the initial value problem (2.4.1) is equivalent to obtain the Schatten bound (2.3.6) with $A = e^{-itH}$.

Let S be the discrete surface $S = \{(\mu,\nu) \in \mathbb{N}_0^n \times \mathbb{Z} : \nu = 2|\mu| + n\}$ with respect to the counting measure. Then for all f such that $\hat{f} \in \ell^1(S)$ and for all $(t,x) \in (-\pi,\pi)\times\mathbb{R}^n$, the extension operator can be written as

$$\mathcal{E}_S f(t,x) = \sum_{\mu,\nu\in S} \hat{f}(\mu,\nu)\Phi_\mu(x)e^{-it\nu}, \tag{2.4.2}$$

where $\hat{f}(\mu,\nu)$ is defined in (2.3.2). Choosing

$$\hat{f}(\mu,\nu) = \begin{cases} \hat{u}(\mu), & \text{if } \nu = 2|\mu| + n, \\ 0, & \text{otherwise,} \end{cases}$$

for some $u : \mathbb{R}^n \to \mathbb{C}$ in (2.4.2), we get

$$\begin{aligned} \mathcal{E}_S f(t,x) &= \sum_{\mu,\nu\in S} \hat{f}(\mu,\nu)\Phi_\mu(x)e^{-it\nu} \\ &= \int_{\mathbb{R}^n} \left(\sum_{\mu} \Phi_\mu(x)\Phi_\mu(y)e^{-it(2|\mu|+n)} \right) u(y)\, dy \\ &= e^{-itH}u(x). \end{aligned}$$

Now we prove our main result of this chapter.

Theorem 2.4.1. [Strichartz inequality for orthonormal functions for Hermite operator] *Let $p, q, n \geq 1$ such that*

$$1 \leq q < \frac{n+1}{n-1} \quad \textit{and} \quad \frac{2}{p} + \frac{n}{q} = n.$$

For any (possibly infinite) system (u_j) of orthonormal functions in $L^2(\mathbb{R}^n)$ and any coefficients $(n_j) \subset \mathbb{C}$, we have

$$\int_{(-\pi,\pi)} \left(\int_{\mathbb{R}^n} \left| \sum_j n_j \left| \left(e^{-itH} u_j\right)(x) \right|^2 \right|^q dx \right)^{\frac{p}{q}} dt \leq C_{n,q}^p \left(\sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{p(q+1)}{2q}}, \tag{2.4.3}$$

where $C_{n,q}$ is a universal constant which only depends on n and q.

Proof. To prove (2.4.3), by Lemma 2.3.1, is enough to show that

$$\|W_1 \mathcal{E}_S \mathcal{E}_S^* W_2\|_{S_\alpha(L^2((-\pi,\pi)\times\mathbb{R}^n))} \lesssim \|W_1\|_{L_t^\beta L_x^\alpha((-\pi,\pi)\times\mathbb{R}^n)} \|W_2\|_{L_t^\beta L_x^\alpha((-\pi,\pi)\times\mathbb{R}^n)}, \tag{2.4.4}$$

for all $\alpha, \beta \geq 1$ such that $\frac{2}{\beta} + \frac{n}{\alpha} = 1$ and $0 \leq \frac{1}{\alpha} < \frac{1}{n+1}$.

Using the fact that the operator e^{-itH} is unitary, the triangle inequality gives (2.4.3) for the pair $(p,q) = (\infty, 1)$. Equivalently, (2.4.4) is true for $(\beta, \alpha) = (2, \infty)$. Therefore it is enough to show (2.4.4) for the range $\frac{1}{n+2} \leq \frac{1}{\alpha} < \frac{1}{n+1}$. For $\varepsilon > 0$, we define

$$T_\varepsilon f(t,x) = \int_{\mathbb{R}^n} \int_{(-\frac{\pi}{4},\frac{\pi}{4})} K_\varepsilon(t-s,x,y) f(s,y)\, dy\, ds,$$

where $K_\varepsilon(t,x,y) = 1_{\varepsilon<|t|<\frac{\pi}{4}} K(t,x,y)$ and $K(t,x,y)$ is defined in (2.3.4). Once we can show that

$$\|W_1 T_\varepsilon W_2\|_{S_\alpha\left(L^2\left(\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\times\mathbb{R}^n\right)\right)} \leq C \|W_1\|_{L_t^\beta L_x^\alpha\left(\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\times\mathbb{R}^n\right)} \|W_2\|_{L_t^\beta L_x^\alpha\left(\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\times\mathbb{R}^n\right)}, \tag{2.4.5}$$

for some constant C independent of ε, then (2.4.4) follows by letting $\varepsilon \to 0$ and extending it to $(-\pi,\pi)$. In order to apply complex interpolation, for $z \in \mathbb{C}$ with $\operatorname{Re}(z) \in \left[-1, \frac{n}{2}\right]$, we further define

$$K_{z,\varepsilon}(t,x,y) = |t|^z K_\varepsilon(t,x,y)$$

and

$$T_{z,\varepsilon} f(t,x) = \int_{\mathbb{R}^n} \int_{(-\frac{\pi}{4},\frac{\pi}{4})} K_{z,\varepsilon}(t-s,x,y) f(s,y)\, dy\, ds.$$

But from (2.2.2), for every $x,y \in \mathbb{R}^n$ and $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$, we have

$$|K_{z,\varepsilon}(t,x,y)| \leq C|t|^{\operatorname{Re}(z)-\frac{n}{2}}. \tag{2.4.6}$$

Now, an application of the Hardy-Littlewood-Sobolev inequality (see page 39 in [6]) along with (2.4.6) yields

$$\|W_1 T_{z,\varepsilon} W_2\|^2_{S_2\left(L^2\left(\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\times\mathbb{R}^n\right)\right)}$$

$$
\begin{aligned}
&= \int_{(-\frac{\pi}{4},\frac{\pi}{4})^2}\int_{\mathbb{R}^{2n}} |W_1(t,x)|^2 \left|K_{z,\varepsilon}(t-t',x,x')\right|^2 \left|W_2\left(t',x'\right)^2 dx dx' dt dt'\\
&\leq C_1 \int_{(-\frac{\pi}{4},\frac{\pi}{4})^2}\int_{\mathbb{R}^{2n}} \frac{|W_1(t,x)|^2 |W_2\left(t',x'\right)|^2}{|t-t'|^{n-2\operatorname{Re}(z)}} dx dx' dt dt'\\
&\leq C_1 \int_{(-\frac{\pi}{4},\frac{\pi}{4})}\int_{(-\frac{\pi}{4},\frac{\pi}{4})} \frac{\|W_1(t)\|^2_{L^2_x(\mathbb{R}^n)} \left\|W_2\left(t'\right)\right\|^2_{L^2_x(\mathbb{R}^n)}}{|t-t'|^{n-2\operatorname{Re}(z)}} dt dt'\\
&\leq C_1 \left\| \|W_1\|^2_{L^2_x(\mathbb{R}^n)} \right\|_{L_t^{\tilde{u}}\left((-\frac{\pi}{4},\frac{\pi}{4})\right)} \left\| \|W_2\|^2_{L^2_x(\mathbb{R}^n)} \right\|_{L_t^{\tilde{u}}\left((-\frac{\pi}{4},\frac{\pi}{4})\right)},
\end{aligned}
$$

provided we have $0 \leq n - 2\operatorname{Re}(z) < 1$ and $\frac{2}{\tilde{u}} + (n - 2\operatorname{Re}(z)) = 2$. Further, if we denote $2\tilde{u} = u$, then $\frac{1}{u} \in \left(\frac{1}{4}, \frac{1}{2}\right]$ and

$$
\|W_1 T_{z,\varepsilon} W_2\|_{S_2(L^2((-\pi,\pi)\times\mathbb{R}^n))} \leq C \|W_1\|_{L_t^u L_x^2((-\pi,\pi)\times\mathbb{R}^n)} \|W_2\|_{L_t^u L_x^2((-\pi,\pi)\times\mathbb{R}^n)},
$$

provided $\frac{1}{u} = \frac{1}{2} + \frac{1}{2}\left(\operatorname{Re}(z) - \frac{n}{2}\right)$ and $\operatorname{Re}(z) \in \left(\frac{n-1}{2}, \frac{n}{2}\right]$.

Now we consider the case $\operatorname{Re}(z) = -1$. For $\operatorname{Re}(z) = -1$, we show that $T_{z,\varepsilon} : L^2_{t,x}\left((-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n\right) \to L^2_{t,x}\left((-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n\right)$ is bounded with some constant that only depends on the dimension n and $\operatorname{Im}(z)$ exponentially. Note that

$$
\begin{aligned}
T_{z,\varepsilon} f(t,x) &= \int_{\mathbb{R}^n}\int_{(-\frac{\pi}{4},\frac{\pi}{4})} K_{z,\varepsilon}(t-s,x,y) f(s,y)\, dy\, ds\\
&= (2\pi)^{-\frac{1}{2}} \sum_{\mu} \Phi_\mu(x) \int_{\varepsilon<|s|<\frac{\pi}{4}} s^z e^{-is(2|\mu|+n)} \left(\int_{\mathbb{R}^n} \Phi_\mu(y) f(t-s,y)\, dy\right) ds\\
&= (2\pi)^{-\frac{1}{2}} \sum_{\mu} \Phi_\mu(x) \int_{\varepsilon<|s|<\frac{\pi}{4}} s^z e^{-is(2|\mu|+n)} \hat{f}_2(t-s,\cdot)(\mu)\, ds,
\end{aligned}
$$

where $\hat{f}_2$ denotes the Hermite transform of f with respect to the second variable. Then using Plancherel's theorem, for each $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$, we have

$$
\begin{aligned}
\|T_{z,\varepsilon} f(\cdot,t)\|^2_{L^2_x(\mathbb{R}^n)} &= (2\pi)^{-1} \sum_{\mu} \left| \int_{\varepsilon<|s|<\frac{\pi}{4}} s^{-1+i\operatorname{Im}(z)} e^{-i(t-s)(2|\mu|+n)} \hat{f}_2(t-s,\cdot)(\mu)\, ds \right|^2\\
&= (2\pi)^{-1} \sum_{\mu} \left| \int_{\varepsilon<|s|<\frac{\pi}{4}} s^{-1+i\operatorname{Im}(z)} G_\mu\left(t-s\right)\, ds \right|^2 \qquad (2.4.7)
\end{aligned}
$$

where $G_\mu(t) = e^{-it(2|\mu|+n)} \hat{f}_2(t,\cdot)(\mu)$. If we define

$$
F_{z,\varepsilon} : G(t) \mapsto \int_{\varepsilon<|s|<\frac{\pi}{4}} s^{-1+i\operatorname{Im}(z)} G\left(t-s\right)\, ds,
$$

then (2.4.7) becomes

$$
\|T_{z,\varepsilon} f\|^2_{L^2_{t,x}((-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n)} = (2\pi)^{-1} \sum_{\mu} \|F_{z,\varepsilon} G_\mu\|^2_{L^2_t\left((-\frac{\pi}{4},\frac{\pi}{4})\right)}. \qquad (2.4.8)
$$

Since the operator $F_{z,\varepsilon}$ is the Hilbert transform up to $i\,\mathrm{Im}(z)$ (see [104]), the operator $F_{z,\varepsilon} : L^2 \to L^2$ is bounded with constant depends only on $\mathrm{Im}(z)$ exponentially. Thus, using the boundedness of $T_{z,\varepsilon} : L^2_{t,x}\left((-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n\right) \to L^2_{t,x}\left((-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n\right)$ and the fact that S_∞-norm is the operator norm, we have

$$\|W_1 T_{z,\varepsilon} W_2\|_{S_\infty\left(L^2\left((-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n\right)\right)} \leq C(\mathrm{Im}(z))\,\|W_1\|_{L^\infty_t L^\infty_x\left((-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n\right)}\,\|W_2\|_{L^\infty_t L^\infty_x\left((-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{R}^n\right)},$$

for $\mathrm{Re}(z) = -1$. Finally, applying Stein's analytic interpolation result [7, 8], the required inequality (2.4.4) holds for the range

$$\frac{1}{n+2} \leq \frac{1}{\alpha} < \frac{1}{n+1} \text{ with } \frac{2}{\beta} + \frac{n}{\alpha} = 1,$$

at $z = 0$. Using the kernel properties (2.2.3) of the semigroup e^{-itH}, the range of t can be extended to the interval $(-\pi, \pi)$.

□

2.5 Optimality of the Schatten exponent

In this section, we show that the power $\frac{p(q+1)}{2q}$ on the right hand side in (2.4.3) is optimal. The inequality (2.4.3) can also be written in terms of the operator

$$\gamma_0 := \sum_j n_j \,|u_j\rangle\,\langle u_j| \tag{2.5.1}$$

on $L^2(\mathbb{R}^n)$, where the Dirac's notation $|u\rangle\langle v|$ stands for the rank-one operator $f \mapsto \langle v, f\rangle u$. For such γ_0, let

$$\gamma(t) := e^{-itH}\gamma_0 e^{itH} = \sum_j n_j \left|e^{-itH}u_j\right\rangle\left\langle e^{-itH}u_j\right|.$$

Then the density of the operator $\gamma(t)$ is given by

$$\rho_{\gamma(t)} := \sum_j n_j \left|e^{-itH}u_j\right|^2. \tag{2.5.2}$$

With these notations (2.4.3) can be rewritten as

$$\left\|\rho_{\gamma(t)}\right\|_{L^p_t L^q_x((-\pi,\pi)\times\mathbb{R}^n)} \leq C_{n,q}\|\gamma_0\|_{S_{\frac{2q}{q+1}}}, \tag{2.5.3}$$

where $\|\gamma_0\|_{S_{\frac{2q}{q+1}}} = \left(\sum_j |n_j|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}}$.

Proposition 2.5.1. [Optimality of the Schatten exponent] *Assume that* $n, p, q \geq 1$ *satisfy* $\frac{2}{p}+\frac{n}{q}=n$. *Then we have*

$$\sup_{\gamma_0\in\mathcal{S}_r}\frac{\left\|\rho_{e^{-itH}\gamma_0 e^{itH}}\right\|_{L^p_t L^q_x((-\pi,\pi)\times\mathbb{R}^n)}}{\|\gamma_0\|_{\mathcal{S}_r}}=+\infty,$$

for all $r>\frac{2q}{q+1}$.

Proof. Depending on the positive parameters β, L and μ, we construct the family of operators

$$\gamma_0=\frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\frac{x^2}{L^2}-\frac{\xi^2}{\mu}}\left|F_{x,\xi}\right\rangle\left\langle F_{x,\xi}\right| dxd\xi,$$

where $F_{x,\xi}(z)=(2\pi\beta)^{-\frac{n}{4}}e^{-\frac{(z-x)^2}{4\beta}}e^{i\xi\cdot z}$. The functions $F_{x,\xi}$ are normalized and satisfy

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{dxd\xi}{(2\pi)^n}\left|F_{x,\xi}\right\rangle\left\langle F_{x,\xi}\right|=1.$$

By Mehler's formula (2.2.1), we get

$$e^{itH}F_{x,\xi}(z)=(-2\pi i\sin 2t)^{-\frac{n}{2}}(2\pi\beta)^{-\frac{n}{4}}\int_{\mathbb{R}^n}e^{-\frac{i}{2}\cot 2t\left(z^2+y^2\right)+\frac{i}{\sin(2t)}z\cdot y}e^{-\frac{(y-x)^2}{4\beta}}e^{i\xi\cdot y}dy.$$

Therefore

$$\left|e^{itH}F_{x,\xi}(z)\right|=\left(\frac{2\beta}{\pi(4\beta^2\cos^2 2t+\sin^2 2t)}\right)^{\frac{n}{4}}e^{-\frac{\beta(z-x\cos 2t+\xi\sin 2t)^2}{4\beta^2\cos^2 2t+\sin^2 2t}},$$

and

$$\begin{aligned}\rho_{\gamma(t)}(z) &:= \rho_{e^{itH}\gamma_0 e^{-itH}}(z)\\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{dxd\xi}{(2\pi)^d}e^{-\frac{x^2}{L^2}-\frac{\xi^2}{\mu}}\left|e^{itH}F_{x,\xi}(z)\right|^2\\ &= \left(\frac{2\pi\beta\mu L^2}{(4\beta^2+2\beta L^2)\cos^2 2t+(1+2\mu\beta)\sin^2 2t}\right)^{\frac{n}{2}}e^{\frac{-2\beta z^2}{(4\beta^2+2\beta L^2)\cos^2 2t+(1+2\mu\beta)\sin^2 2t}}.\end{aligned}$$

So

$$\|\rho_{\gamma(t)}\|^q_{L^q_x(\mathbb{R}^n)} = \left(\frac{\pi}{q}\right)^{\frac{n}{2}}\left(\mu L^2\right)^{\frac{nq}{2}}\left(\frac{\beta}{(4\beta^2+2\beta L^2)\cos^2 2t+(1+2\mu\beta)\sin^2 2t}\right)^{\frac{n(q-1)}{2}}.$$

Using the fact that $n(q-1)p=2q$, we have

$$\begin{aligned}&\|\rho_{\gamma(t)}\|^p_{L^p_t L^q_x((-\pi,\pi)\times\mathbb{R}^n)}\\ &=\left(\frac{\pi}{q}\right)^{\frac{np}{2q}}\left(\mu L^2\right)^{\frac{np}{2}}\int_{[-\pi,\pi]}\frac{\beta}{(4\beta^2+2\beta L^2)\cos^2 2t+(1+2\mu\beta)\sin^2 2t}dt\end{aligned}$$

$$= \sqrt{2}\pi \left(\frac{\pi}{q}\right)^{\frac{np}{2q}} \left(\mu L^2\right)^{\frac{np}{2}} \frac{\beta}{\sqrt{2\beta^2 + \beta L^2\sqrt{1+2\mu\beta}}}.$$

Therefore

$$\begin{aligned}
&\|\rho_{\gamma(t)}\|_{L^p_tL^q_x((-\pi,\pi)\times\mathbb{R}^n)} \\
&= A_{n,p}\left(\mu L^2\right)^{\frac{n}{2}} (L^2)^{-\frac{1}{2p}}\mu^{-\frac{1}{2p}} \frac{1}{\left(\frac{2\beta}{L^2}+1\right)^{\frac{1}{2p}} \left(\frac{1}{\mu\beta}+2\right)^{\frac{1}{2p}}} \\
&= A_{n,p}\left(\mu L^2\right)^{\frac{n}{2}-\frac{1}{2p}} \frac{1}{\left(\frac{2\beta}{L^2}+1\right)^{\frac{1}{2p}} \left(\frac{1}{\mu\beta}+2\right)^{\frac{1}{2p}}}.
\end{aligned}$$

Using the fact that $\frac{n}{4}\left(1+\frac{1}{q}\right) = \frac{n}{2} - \frac{1}{2p}$ and choosing $1/\mu < \beta < L^2$, we obtain

$$\|\rho_{\gamma(t)}\|_{L^p_tL^q_x((-\pi,\pi)\times\mathbb{R}^n)} \geq A_{n,p}2^{-\frac{1}{2p}}N^{\frac{1+q}{2q}},$$

where

$$\begin{aligned}
N &= \int_{\mathbb{R}^n} \gamma_0(z,z)dz \\
&= \iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} \frac{dxd\xi}{(2\pi)^n} e^{-\frac{x^2}{2}-\frac{\xi^2}{\mu}} \left|F_{x\xi}(z)\right|^2 dz \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{dxd\xi}{(2\pi)^n} e^{-\frac{x^2}{2}-\frac{\xi^2}{\mu}} \\
&= A_nL^n\mu^{\frac{n}{2}}.
\end{aligned}$$

An application of the Berezin-Lieb inequality [68] gives that

$$\operatorname{Tr}\gamma_0^r \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{dxd\xi}{(2\pi)^d} e^{-\frac{rx^2}{2}-\frac{r\xi^2}{\mu}} = r^{-n}N,$$

where $r \geq 1$ and $N = \frac{(\mu L^2)^{\frac{n}{2}}}{2^n}$. Therefore

$$\frac{\left\|\rho_{e^{-itH}\gamma_0e^{itH}}\right\|_{L^p_tL^q_x((-\pi,\pi)\times\mathbb{R}^n)}}{\|\gamma_0\|_{S_r}} \geq \frac{A_{n,p}2^{-\frac{1}{2p}}}{r^{-\frac{n}{r}}} N^{\left(\frac{1+q}{2q}-\frac{1}{r}\right)}.$$

□


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CHAPTER 3

Strichartz inequality for orthonormal functions associated with special Hermite operator

3.1 Introduction

The objective of this chapter is to prove the restriction theorem for the special Hermite transform and obtain Strichartz inequality for the system of orthonormal functions associated with special Hermite operator $\mathcal{L}$ defined on $L^2(\mathbb{C}^n)$ as an application of the restriction theorem.

3.2 Preliminaries

In this section, we mainly discuss some basic definitions and provide necessary background information about the special Hermite semigroup. We start with the definition of the special Hermite functions.

3.2.1 Special Hermite operator and spectral theory

For each multi-index $\mu, \nu \in \mathbb{N}_0^n$ and $\zeta \in \mathbb{C}$, we define the special Hermite functions $\Phi_{\mu\nu}$ by

$$\Phi_{\mu\nu}(\zeta) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\Phi_\mu\left(\xi + \frac{y}{2}\right)\Phi_\nu\left(\xi - \frac{y}{2}\right) d\xi, \quad \zeta = x + iy \in \mathbb{C}^n,$$

where Φ_μ's are the higher dimensional Hermite functions defined in Subsection 2.2.1. The family of functions $\{\Phi_{\mu\nu}\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. These special Hermite functions are the eigenfunctions of the special Hermite operator $\mathcal{L}$ (or the twisted Laplacian) defined by

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{n}\left(Z_j\bar{Z}_j + \bar{Z}_j Z_j\right),$$

where $Z_j = \frac{\partial}{\partial\zeta_j} + \frac{1}{2}\bar{\zeta}_j, \bar{Z}_j = -\frac{\partial}{\partial\bar{\zeta}_j} + \frac{1}{2}\zeta_j, j = 1, 2, \ldots n$ with eigenvalues $(2|\nu| + n)$. The special Hermite operator $\mathcal{L}$ is self-adjoint and admits a spectral decomposition in terms of special Hermite functions. Given $f \in L^2(\mathbb{C}^n)$, the expansion

$$f = \sum_{\mu,\nu\in\mathbb{N}_0^n} \langle f, \Phi_{\mu\nu}\rangle\Phi_{\mu\nu} \tag{3.2.1}$$

converges to f in $L^2(\mathbb{C}^n)$. The above expansion also can be written as $f = \sum_{k=0}^{\infty} P_k f$, where

$$P_k = \sum_{\{\mu,\nu\in\mathbb{N}_0^n:|\nu|=k\}} \langle \cdot, \Phi_{\mu\nu}\rangle\Phi_{\mu\nu}$$

is the orthogonal projection of $L^2(\mathbb{C}^n)$ onto the eigenspace spanned by $\{\Phi_{\mu\nu} : |\nu| = k\}$. For each $k \in \mathbb{N}$, the spectral decomposition of $\mathcal{L}$ can be written as

$$\mathcal{L}f = \sum_{k=0}^{\infty}(2k + n)P_k f.$$

The twisted convolution of two functions f and g on $\mathbb{C}^n$ is defined by

$$f \times g(\zeta) = \int_{\mathbb{C}^n} f(\zeta - w)g(w)e^{\frac{i}{2}\operatorname{Im}(\zeta\cdot\bar{w})}dw, \quad \zeta \in \mathbb{C}^n.$$

The family $\{\Phi_{\mu\nu}\}$ satisfies the following orthogonality properties

$$\Phi_{\mu\nu} \times \Phi_{\alpha\beta} = \begin{cases} (2\pi)^{n/2}\Phi_{\mu\beta}, & \text{if } \nu = \alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.2}$$

Let L_k^α denote the Laguerre polynomial of degree k and of order $\alpha > -1$, defined by the generating function identity (see [81])

$$\sum_{k=0}^{\infty} L_k^\alpha(x)\omega^k = (1-\omega)^{-\alpha-1} e^{-\frac{\omega}{1-\omega}x}, \quad |\omega| < 1,$$

and let $\phi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2}$ be the Laguerre function of order $n-1$. Then the special Hermite functions $\Phi_{\nu\nu}$ are related to the Laguerre functions ϕ_k by the following relation

$$(2\pi)^{n/2} \sum_{|\nu|=k} \Phi_{\nu\nu} = \phi_k. \tag{3.2.3}$$

Now taking twisted convolution on both sides of (3.2.1) with $\Phi_{\alpha\alpha}$ and using the orthogonality property (3.2.2), we have

$$f \times \Phi_{\alpha\alpha} = (2\pi)^{n/2} \sum_{\mu \in \mathbb{N}_0^n} \langle f, \Phi_{\mu\alpha}\rangle \, \Phi_{\mu\alpha}, \quad \alpha \in \mathbb{N}_0^n. \tag{3.2.4}$$

Summing both sides of (3.2.4) with respect to all α such that $|\alpha| = k$ and using (3.2.3), the spectral projection P_k has the simpler representation

$$P_k f(\zeta) = (2\pi)^{-\frac{n}{2}} \sum_{|\alpha|=k} f \times \Phi_{\alpha\alpha}(\zeta) = (2\pi)^{-n} f \times \varphi_k(\zeta), \quad \zeta \in \mathbb{C}^n.$$

Then the special Hermite expansion takes the compact form

$$f(\zeta) = (2\pi)^{-n} \sum_k f \times \phi_k(\zeta), \quad \zeta \in \mathbb{C}^n.$$

The operator $\mathcal{L}$ defines a semigroup, called the special Hermite semigroup and denoted by $e^{-t\mathcal{L}}, t > 0$, by the expansion

$$e^{-t\mathcal{L}} f = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} f \times \phi_k,$$

for $f \in L^2(\mathbb{C}^n)$. For the auxiliary complex semigroup $\{e^{-\eta\mathcal{L}}\}$, $\eta = r + it, \ r > 0$, we write

$$e^{-\eta\mathcal{L}} f(\zeta) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-\eta(2k+n)} f \times \phi_k(\zeta), \quad \zeta \in \mathbb{C}^n.$$

Thus, $e^{-\eta\mathcal{L}}$ is a twisted convolution operator

$$e^{-\eta\mathcal{L}} f(\zeta) = \int_{\mathbb{C}^n} f(\zeta - w) K_\eta(w) e^{\frac{i}{2}\operatorname{Im}(\zeta\cdot\bar{w})} dw$$

with kernel (see [81])

$$K_\eta(\zeta)=(2\pi)^{-n}\sum_{k=0}^{\infty}e^{-\eta(2k+n)}\phi_k(\zeta)=(2\pi)^{-n}e^{-n\eta}(1-\omega)^{-n}e^{-\frac{1+\omega}{1-\omega}\frac{|\zeta|^2}{4}},$$

where $\omega=e^{-2\eta}$ with $\mathrm{Re}(\eta)>0$. Thus $K_{r+it}(\zeta)=K_{r+i(t+2\pi)}(\zeta)$ and for $\mathrm{Re}(\eta)>0$, K_η satisfies the uniform estimate

$$|K_\eta(\zeta)|\leq\frac{2}{|\sin t|^n},\quad \eta=r+it,\quad \zeta\in\mathbb{C}^n,\quad \text{for } t\in\mathbb{R}\setminus\pi\mathbb{Z}. \tag{3.2.5}$$

Also for $t\in\mathbb{R}\setminus\pi\mathbb{Z}$, we have

$$K_{-it}(\zeta)=\overline{K_{it}(\zeta)}\quad\text{and}\quad K_{i(t+\pi)}(\zeta)=e^{-in\pi}K_{it}(\zeta). \tag{3.2.6}$$

We refer to [81, 99] for a detailed study on special Hermite semigroup.

3.3 The Restriction theorem

In this section, we set a platform to prove the restriction theorem with respect to the special Hermite transform for a given discrete surface $S\subset\mathbb{N}_0^{2n}\times\mathbb{Z}$. We start this section with the definition of special Hermite transform.

Let $f\in L^1(\mathbb{C}^n)$. Define the special Hermite transform of f by

$$\hat{f}(\mu,\nu)=\int_{\mathbb{C}^n}f(z)\Phi_{\mu\nu}(z)\,dz,\quad \mu,\nu\in\mathbb{N}_0^n,$$

where $\Phi_{\mu\nu}$'s are the special Hermite functions on $\mathbb{C}^n$. If $f\in L^2(\mathbb{C}^n)$, then $\{\hat{f}(\mu,\nu)\}\in\ell^2(\mathbb{N}_0^{2n})$, and satisfies the following Plancherel formula

$$\|f\|_2^2=\sum_{(\mu,\nu)\in\mathbb{N}_0^{2n}}|\hat{f}(\mu,\nu)|^2.$$

The inverse special Hermite transform is given by

$$f(z)=\sum_{(\mu,\nu)\in\mathbb{N}_0^{2n}}\hat{f}(\mu,\nu)\Phi_{\mu\nu}(z),\quad z\in\mathbb{C}^n.$$

Given a discrete surface S in $\mathbb{N}_0^{2n}\times\mathbb{Z}$, we define the restriction operator $(\mathcal{R}_S f):=\{\hat{f}(\mu,\nu,\lambda)\}_{(\mu,\nu,\lambda)\in S}$ and the operator dual to $\mathcal{R}_S$ (called the extension operator) as

$$\mathcal{E}_S(\{\hat{f}(\mu,\nu,\lambda)\}):=\sum_{(\mu,\nu,\lambda)\in S}\hat{f}(\mu,\nu,\lambda)\Phi_{\mu\nu}(\cdot)e^{-i(\cdot)\lambda},$$

where the Fourier-special Hermite transform of f is given by

$$\hat{f}(\mu,\nu,\lambda) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{C}^n} \int_{(-\pi,\pi)} f(t,w)\Phi_{\mu\nu}(w)e^{i\lambda t}\,dtdw. \tag{3.3.1}$$

Using the similar notations in Chapter 2, we consider the following problem:

Problem 1: For which exponents $1 \leq p \leq 2$, the sequence of Fourier-special Hermite transform of a function $f \in L^p((-\pi,\pi)\times\mathbb{C}^n)$ belongs to $\ell^2(S)$?

This question can be reframed to the boundedness of the operator $\mathcal{E}_S$ from $\ell^2(S)$ to $L^{p'}((-\pi,\pi)\times\mathbb{C}^n)$, where p' is the conjugate exponent of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$. Since $\mathcal{E}_S$ is bounded from $\ell^2(S)$ to $L^{p'}((-\pi,\pi)\times\mathbb{C}^n)$ if and only if $T_S := \mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p((-\pi,\pi)\times\mathbb{C}^n)$ to $L^{p'}((-\pi,\pi)\times\mathbb{C}^n)$, Problem 1 can be re-written as follows:

Problem 2: For which exponents $1 \leq p \leq 2$, the operator $T_S := \mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p((-\pi,\pi)\times\mathbb{C}^n)$ to $L^{p'}((-\pi,\pi)\times\mathbb{C}^n)$?

Note that Hölder's inequality implies that the operator $T_S = \mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p((-\pi,\pi)\times\mathbb{C}^n)$ to $L^{p'}((-\pi,\pi)\times\mathbb{C}^n)$ if and only if for any $W_1, W_2 \in L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{C}^n)$, the operator $W_1T_SW_2$ is bounded on $L^2((-\pi,\pi)\times\mathbb{C}^n)$ with

$$\|W_1T_SW_2\|_{L^2((-\pi,\pi)\times\mathbb{C}^n)\to L^2((-\pi,\pi)\times\mathbb{C}^n)} \leq C\|W_1\|_{L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{C}^n)}\|W_2\|_{L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{C}^n)},$$

for some constant $C>0$. In this chapter, we consider the discrete surface S of the form $S = \{(\mu,\nu,\lambda) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{Z} : \lambda = 2|\nu|+n\}$ with respect to counting measure.

We introduce an analytic family of operators (T_z) defined on the strip $a \leq \operatorname{Re} z \leq b$ in the complex plane such that $T_S = T_c$ for some $c \in (a,b)$ and show that the operator $W_1T_SW_2$ belongs to a Schatten class with

$$\|W_1T_SW_2\|_{\mathcal{S}_\alpha(L^2((-\pi,\pi)\times\mathbb{C}^n))} \leq C\|W_1\|_{L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{C}^n)}\|W_2\|_{L^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{C}^n)},$$

for some $C>0$ and some $\alpha>0$, which is a more general $L^p - L^{p'}$ boundedness result of T_S.

The following proposition assures an affirmative answer to Problem 2 under certain assumptions. In order to obtain the Strichartz inequality for the system of orthonormal functions, we need the following duality principle lemma in our context. The proof follows along the similar lines of Lemma 2.3.1 of Chapter 2.

Lemma 3.3.1. [Duality principle] *Let* $p, q \geq 1$ *and* $\alpha \geq 1$. *Let* $Af(t,w) = e^{-it\mathcal{L}} f(w)$ *and if* A *be a bounded linear operator from* $L^2(\mathbb{C}^n)$ *to* $L_t^{q'} L_w^{p'}((-\pi,\pi)\times\mathbb{C}^n)$, *then the following statements are equivalent.*

1. *There is a constant* $C > 0$ *such that*

$$\left\| W A A^* \overline{W} \right\|_{\mathcal{S}_\alpha(L^2((-\pi,\pi)\times\mathbb{C}^n))} \leq C \|W\|^2_{L_t^{\frac{2q}{2-q}} L_w^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{C}^n)}, \tag{3.3.2}$$

for all $W \in L_t^{\frac{2q}{2-q}} L_w^{\frac{2p}{2-p}}((-\pi,\pi)\times\mathbb{C}^n)$, *where the function* W *is interpreted as an operator which acts by multiplication.*

2. *For any orthonormal system* $(f_j)_{j\in J}$ *in* $L^2(\mathbb{C}^n)$ *and any sequence* $(n_j)_{j\in J} \subset \mathbb{C}$, *there is a constant* $C' > 0$ *such that*

$$\left\| \sum_{j\in J} n_j \left| A f_j \right|^2 \right\|_{L_t^{\frac{q'}{2}} L_w^{\frac{p'}{2}}((-\pi,\pi)\times\mathbb{C}^n)} \leq C' \left(\sum_{j\in J} |n_j|^{\alpha'} \right)^{1/\alpha'}. \tag{3.3.3}$$

3.4 Strichartz inequality for system of orthonormal functions

Consider the Schrödinger equation associated with the special Hermite operator $\mathcal{L}$:

$$\begin{aligned} i\partial_t u(t,w) &= \mathcal{L} u(t,w), \quad && w \in \mathbb{C}^n, t \in \mathbb{R}, \\ u(0,w) &= f(w), && w \in \mathbb{C}^n. \end{aligned} \tag{3.4.1}$$

If $f \in L^2(\mathbb{C}^n)$, the solution of the initial value problem (3.4.1) is given by $u(t,w) = e^{-it\mathcal{L}} f(w)$. The solution to the initial value problem (3.4.1) can be realized as the extension operator of some function f on $(-\pi,\pi)\times\mathbb{C}^n$. Estimating the solution to the initial value problem (3.4.1) is equivalent to obtain the Schatten bound (3.3.2) with $A = e^{-it\mathcal{L}}$.

Let S be the discrete surface $S = \{(\mu,\nu,\lambda) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{Z} : \lambda = 2|\nu| + n\}$ with respect to counting measure. Then for all f such that $\hat{f} \in \ell^1(S)$ and for all $(t,\zeta) \in (-\pi,\pi)\times\mathbb{C}^n$, the extension operator can be written as

$$\mathcal{E}_S f(t,\zeta) = \sum_{(\mu,\nu,\lambda)\in S} \hat{f}(\mu,\nu,\lambda) \Phi_{\mu\nu}(\zeta) e^{-it\lambda}, \tag{3.4.2}$$

where $\hat{f}(\mu,\nu,\lambda)$ is defined in (3.3.1). Using the fact that

$$f \times \Phi_{\mu\mu} = (2\pi)^{\frac{n}{2}} \sum_{\nu} \langle f, \Phi_{\mu\nu}\rangle \Phi_{\mu\nu},$$

and choosing

$$\hat{f}(\mu,\nu,\lambda) = \begin{cases} (2\pi)^n \hat{u}(\mu,\nu), & \text{if } \lambda = 2|\nu| + n, \\ 0, & \text{otherwise,} \end{cases}$$

for some $u : \mathbb{C}^n \to \mathbb{C}$ in (3.4.2), we get

$$\begin{aligned} \mathcal{E}_S f(t,\zeta) &= (2\pi)^n \sum_{\mu,\nu} \hat{u}(\mu,\nu)\Phi_{\mu\nu}(\zeta) e^{-it(2|\nu|+n)} \\ &= (2\pi)^n \sum_{\nu} \left(\sum_{\mu} \langle u, \Phi_{\mu\nu}\rangle \Phi_{\mu\nu}(\zeta) \right) e^{-it(2|\nu|+n)} \\ &= (2\pi)^{\frac{n}{2}} \sum_{\nu} e^{-it(2|\nu|+n)} u \times \Phi_{\nu\nu}(\zeta) \\ &= (2\pi)^{\frac{n}{2}} \sum_{k=0}^{\infty} e^{-it(2k+n)} \left(u \times \sum_{|\nu|=k} \Phi_{\nu\nu}(\zeta) \right) \\ &= \sum_{k=0}^{\infty} e^{-it(2k+n)} u \times \phi_k(z) = e^{-it\mathcal{L}} u(\zeta). \end{aligned}$$

Again

$$\begin{aligned} \mathcal{E}_S \mathcal{E}_S^* f(t,\zeta) &= \sum_{(\mu,\nu,\lambda)\in S} \hat{f}(\mu,\nu,\lambda)\Phi_{\mu\nu}(\zeta) e^{-it\lambda} \\ &= (2\pi)^{-\frac{1}{2}} \sum_{(\mu,\nu,\lambda)\in S} \int_{(-\pi,\pi)} \langle f(s,\cdot), \Phi_{\mu\nu}\rangle \Phi_{\mu\nu}(\zeta) e^{-i(t-s)\lambda}\, dw\, ds \\ &= (2\pi)^{-\frac{1}{2}} \int_{(-\pi,\pi)} \sum_{\mu,\nu} \langle f(s,\cdot), \Phi_{\mu\nu}\rangle \Phi_{\mu\nu}(\zeta) e^{-i(t-s)(2|\nu|+n)}\, dw\, ds. \end{aligned} \tag{3.4.3}$$

Now from (3.2.4) and (3.2.3), we have

$$\begin{aligned} &\sum_{\mu,\nu} \langle f(s,\cdot), \Phi_{\mu\nu}\rangle \Phi_{\mu\nu}(\zeta) e^{-i(t-s)(2|\nu|+n)} \\ &= (2\pi)^{-\frac{n}{2}} \sum_{\nu} f(s,\cdot) \times \Phi_{\nu\nu}(\zeta) e^{-i(t-s)(2|\nu|+n)} \\ &= (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-i(t-s)(2k+n)} f(s,\cdot) \times \phi_k(\zeta) \\ &= (2\pi)^{-n} f(s,\cdot) \times \sum_{k=0}^{\infty} e^{-i(t-s)(2k+n)} \phi_k(\zeta). \end{aligned}$$

Then from (3.4.3), we get

$$\begin{aligned}\mathcal{E}_S\mathcal{E}_S^*f(t,\zeta) &= (2\pi)^{-(n+\frac{1}{2})}\int_{(-\pi,\pi)} f(s,\cdot,\cdot)\times\sum_{k=0}^{\infty}e^{-i(t-s)(2k+n)}\phi_k(\zeta)ds\\ &= (2\pi)^{-(n+\frac{1}{2})}\int_{(-\pi,\pi)}\int_{\mathbb{C}^n} H(t-s,\zeta-w)f(s,w)e^{-\frac{i}{2}\operatorname{Im}(\zeta\cdot\bar{w})}dsdw\\ &= (2\pi)^{-(n+\frac{1}{2})}\int_{\mathbb{C}^n}(H(\cdot,\zeta-w)*f(\cdot,w))(t)e^{-\frac{i}{2}\operatorname{Im}(\zeta\cdot\bar{w})}dw, \qquad (3.4.4)\end{aligned}$$

where

$$H(t,\zeta)=\sum_{k=0}^{\infty}e^{-it(2k+n)}\phi_k(\zeta). \qquad (3.4.5)$$

Next we prove our main result of this chapter.

Theorem 3.4.1. [Strichartz inequality for orthonormal functions for special Hermite operator] *Let $q, n \geq 1$ and $p > 1$ such that*

$$1\leq q<\frac{2n+1}{2n-1} \quad \textit{and} \quad \frac{1}{p}+\frac{n}{q}=n.$$

For any (possibly infinite) system (u_j) of orthonormal functions in $L^2(\mathbb{C}^n)$ and any coefficients $(n_j)\subset\mathbb{C}$, there exists a constant $C>0$ such that

$$\left\|\sum_j n_j\left|e^{-it\mathcal{L}}u_j\right|^2\right\|_{L_t^pL_w^q((-\pi,\pi)\times\mathbb{C}^n)}\leq C\left(\sum_j|n_j|^{\frac{2q}{q+1}}\right)^{\frac{(q+1)}{2q}}. \qquad (3.4.6)$$

Proof. To prove (3.4.6), by Lemma 3.3.1, is enough to show that

$$\|W_1\mathcal{E}_S\mathcal{E}_S^*W_2\|_{\mathcal{S}_\alpha(L^2((-\pi,\pi)\times\mathbb{C}^n))}\lesssim\|W_1\|_{L_t^\beta L_w^\alpha((-\pi,\pi)\times\mathbb{C}^n)}\|W_2\|_{L_t^\beta L_w^\alpha((-\pi,\pi)\times\mathbb{C}^n)}, \qquad (3.4.7)$$

for all $\alpha,\beta\geq 1$ such that $\frac{2}{\beta}+\frac{2n}{\alpha}=1$ and $0\leq\frac{1}{\alpha}<\frac{1}{2n+1}$.

Using the fact that the operator $e^{-it\mathcal{L}}$ is unitary, the triangle inequality gives (3.4.6) for the pair $(p,q)=(\infty,1)$. Equivalently, (3.4.7) is true for $(\beta,\alpha)=(2,\infty)$. Therefore it is enough to show (3.4.7) for the range $\frac{1}{2(n+1)}\leq\frac{1}{\alpha}<\frac{1}{2n+1}$.

For $\varepsilon>0$, we define

$$T_\varepsilon f(t,\zeta)=\int_{\mathbb{C}^n}\int_{(-\frac{\pi}{2},\frac{\pi}{2})}H_\varepsilon(t-s,\zeta-w)e^{-\frac{i}{2}\operatorname{Im}(\zeta\cdot\bar{w})}f(s,w)\,dw\,ds,$$

where $H_\varepsilon(t,\zeta)=1_{\varepsilon<|t|<\frac{\pi}{2}}H(t,\zeta)$ and $H(t,\zeta)$ is defined in (3.4.5). Once we can show that

$$\|W_1T_\varepsilon W_2\|_{\mathcal{S}_\alpha\left(L^2\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)\right)}\leq C\|W_1\|_{L_t^\beta L_w^\alpha\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)}\|W_2\|_{L_t^\beta L_w^\alpha\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)}, \qquad (3.4.8)$$

for some constant C independent of ε, then (3.4.7) follows by letting $\varepsilon \to 0$ and extending it to $(-\pi, \pi)$. In order to apply complex interpolation, for $z \in \mathbb{C}$ with $\mathrm{Re}(z) \in [-1, n]$, we further define

$$H_{z,\varepsilon}(t,\zeta) = |t|^z H_\varepsilon(t,\zeta)$$

and

$$T_{z,\varepsilon}f(t,\zeta) = \int_{(-\frac{\pi}{2},\frac{\pi}{2})}\int_{\mathbb{C}^n} H_{z,\varepsilon}(t-s,\zeta-w)e^{-\frac{i}{2}\mathrm{Im}(\zeta\cdot\bar{w})}f(s,w)\,dsdw.$$

But from (3.2.5), for every $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\zeta \in \mathbb{C}^n$, we have

$$|H_{z,\varepsilon}(t,\zeta)| \le C|t|^{\mathrm{Re}(z)-n}. \tag{3.4.9}$$

Now, an application of the Hardy-Littlewood-Sobolev inequality (see page 39 in [6]) along with (3.4.9) yields

$$\begin{aligned}
&\|W_1T_{z,\varepsilon}W_2\|^2_{S_2\left(L^2\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)\right)}\\
&= \int_{(-\frac{\pi}{2},\frac{\pi}{2})^2}\int_{\mathbb{C}^{2n}} |W_1(t,\zeta)|^2\,|H_{z,\varepsilon}(t-t',\zeta-\zeta')|^2\,|W_2(t',\zeta')|^2 d\zeta d\zeta' dtdt'\\
&\le C_1\int_{(-\frac{\pi}{2},\frac{\pi}{2})^2}\int_{\mathbb{C}^{2n}} \frac{|W_1(t,\zeta)|^2|W_2(t',\zeta')|^2}{|t-t'|^{2n-2\,\mathrm{Re}(z)}}d\zeta d\zeta' dtdt'\\
&\le C_1\int_{(-\frac{\pi}{2},\frac{\pi}{2})}\int_{(-\frac{\pi}{2},\frac{\pi}{2})} \frac{\|W_1(t)\|^2_{L^2_w(\mathbb{C}^n)}\,\|W_2(t')\|^2_{L^2_w(\mathbb{C}^n)}}{|t-t'|^{2n-2\,\mathrm{Re}(z)}}dtdt'\\
&\le C_1\left\|\|W_1\|^2_{L^2_w(\mathbb{C}^n)}\right\|_{L^{\tilde{u}}_t\left((-\frac{\pi}{2},\frac{\pi}{2})\right)}\left\|\|W_2\|^2_{L^2_w(\mathbb{C}^n)}\right\|_{L^{\tilde{u}}_t\left((-\frac{\pi}{2},\frac{\pi}{2})\right)},
\end{aligned}$$

provided we have $0 \le 2n - 2\,\mathrm{Re}(z) < 1$ and $\frac{1}{\tilde{u}} + (n - \mathrm{Re}(z)) = 1$. Further, if we denote $2\tilde{u} = u$, then $\frac{1}{u} \in \left(\frac{1}{4}, \frac{1}{2}\right]$ and

$$\|W_1T_{z,\varepsilon}W_2\|_{S_2\left(L^2\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)\right)} \le C\,\|W_1\|_{L^u_tL^2_w\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)}\|W_2\|_{L^u_tL^2_w\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)},$$

provided $\frac{1}{u} = \frac{1}{2} + \frac{1}{2}(\mathrm{Re}(z) - n)$ and $\mathrm{Re}(z) \in \left(\frac{2n-1}{2}, n\right]$.

Now we consider the case $\mathrm{Re}(z) = -1$. For $\mathrm{Re}(z) = -1$, we show that $T_{z,\varepsilon} : L^2_{t,w}\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right) \to L^2_{t,w}\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)$ is bounded with some constant that only depends on the dimension n and $\mathrm{Im}(z)$ exponentially. Note that

$$\begin{aligned}
T_{z,\varepsilon}f(t,\zeta) &= \int_{\mathbb{C}^n}\int_{(-\frac{\pi}{2},\frac{\pi}{2})} H_{z,\varepsilon}(t-s,\zeta-w)e^{-\frac{i}{2}\mathrm{Im}(\zeta\cdot\bar{w})}f(s,w)\,dw\,ds\\
&= \int_{\varepsilon<|s|<\frac{\pi}{2}} s^z\sum_{k=0}^{\infty}e^{-is(2k+n)}\left(\int_{\mathbb{C}^n}\phi_k(\zeta-w)e^{-\frac{i}{2}\mathrm{Im}(\zeta\cdot\bar{w})}f(t-s,w)\,dw\right)\,ds
\end{aligned}$$

$$= \int_{\varepsilon<|s|<\frac{\pi}{2}} s^z f(t-s,\cdot) \times \sum_{k=0}^{\infty} e^{-is(2k+n)}\phi_k(\zeta)\ ds$$
$$= (2\pi)^n \sum_{\mu,\nu} \Phi_{\mu\nu}(\zeta) \int_{\varepsilon<|s|<\frac{\pi}{2}} s^z e^{-is(2|\nu|+n)} \hat{f}_2(t-s,\cdot)(\mu,\nu)ds,$$

where $\hat{f}_2$ denotes the special Hermite transform of f with respect to the second variable. Then using Plancherel's theorem, for each $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$\|T_{z,\varepsilon}f(t,\cdot)\|^2_{L^2_w(\mathbb{C}^n)} = (2\pi)^{2n} \sum_{\mu,\nu} \left| \int_{\varepsilon<|s|<\frac{\pi}{2}} s^{-1+i\,\mathrm{Im}(z)} e^{-i(t-s)(2|\nu|+n)} \hat{f}_2(t-s,\cdot)(\mu,\nu)\ ds \right|^2$$
$$= (2\pi)^{2n} \sum_{\mu,\nu} \left| \int_{\varepsilon<|s|<\frac{\pi}{2}} s^{-1+i\,\mathrm{Im}(z)} G_{\mu\nu}\left(t-s\right)\ ds \right|^2 \tag{3.4.10}$$

where $G_{\mu\nu}(t) = e^{-it(2|\nu|+n)}\hat{f}_2(t,\cdot)(\mu,\nu)$. If we define

$$F_{z,\varepsilon} : G(t) \mapsto \int_{\varepsilon<|s|<\frac{\pi}{2}} s^{-1+i\,\mathrm{Im}(z)} G\left(t-s\right)\ ds,$$

then (3.4.10) becomes

$$\|T_{z,\varepsilon}f\|^2_{L^2_{t,w}((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n)} = (2\pi)^{2n} \sum_{\mu,\nu} \|F_{z,\varepsilon}G_{\mu\nu}\|^2_{L^2_t((-\frac{\pi}{2},\frac{\pi}{2}))}. \tag{3.4.11}$$

Since the operator $F_{z,\varepsilon}$ is just a Hilbert transform up to $i\,\mathrm{Im}(z)$, from [104], the operator $F_{z,\varepsilon} : L^2 \to L^2$ is bounded with constant depends only on $\mathrm{Im}(z)$ exponentially. Thus, using the boundedness of $T_{z,\varepsilon} : L^2_{t,w}\left((-\frac{\pi}{2},\frac{\pi}{2}) \times \mathbb{C}^n\right) \to L^2_{t,w}\left((-\frac{\pi}{2},\frac{\pi}{2}) \times \mathbb{C}^n\right)$ and the fact that S_∞-norm is the operator norm, we have

$$\|W_1T_{z,\varepsilon}W_2\|_{S_\infty\left(L^2\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)\right)} \leq C(\mathrm{Im}(z))\,\|W_1\|_{L^\infty_t L^\infty_w\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)}\,\|W_2\|_{L^\infty_t L^\infty_w\left((-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{C}^n\right)},$$

for $\mathrm{Re}(z) = -1$. Finally, applying Stein's analytic interpolation result [7, 8], the required inequality (3.4.7) holds for the range

$$\frac{1}{2(n+1)} \leq \frac{1}{\alpha} < \frac{1}{2n+1} \text{ with } \frac{2}{\beta} + \frac{2n}{\alpha} = 1,$$

at $z = 0$. Finally, using the kernel properties (3.2.6) of the semigroup $e^{-it\mathcal{L}}$, the range of t can be extended to the interval $(-\pi, \pi)$.

□

CHAPTER 4

Szegö type limit theorems on the Heisenberg group

4.1 Introduction

The observable quantities in the classical system are described by real valued functions on the phase space whereas in quantum system they are given by self-adjoint operators on a Hilbert space. Therefore it is important to study the correspondence between the classical and quantum statistical mechanics. Pseudo-differential operator theory provides a natural platform to relate the classical and quantum mechanics. In [109] Zelditch proved a Szegö limit theorem for the Schrödinger operator on $L^2(\mathbb{R}^n)$ with certain potentials comparing the quantum and classical mean values in the high energy limit. In this chapter, we prove a similar result on the Heisenberg group $\mathbb{H}^n$ for the Schrödinger operator $\mathcal{H}$ (defined in the subsection 4.2.2) with respect to the multiplication operator $M_{\mathbf{b}}$ in Theorem 4.5.1, where $\mathbf{b}$ is a bounded real valued integrable function on $\mathbb{H}^n$. Further, we generalize the Szegö type limit theorem for the Schrödinger operator $\mathcal{H}$ with respect to a 0-th order self-adjoint pseudo-differential operator on $L^2(\mathbb{H}^n)$ (relative to the operator $1 + |\lambda|H + V(g), g \in \mathbb{H}^n, \lambda \in \mathbb{R}^*$) with symbol $a(g, \lambda)$ in Theorem 4.5.4. We show that the conclusion of Theorem 4.5.4 remains unaltered under a perturbation of the Schrödinger operator $\mathcal{H}$ by bounded self-adjoint operators on $L^2(\mathbb{H}^n)$ in Theorem 4.6.1. We also show that our main results in Theorems 4.5.1, 4.5.4, and 4.6.1 remains unaltered under a compact perturbation of the pseudo-differential operator A.

4.2 Preliminaries

In this section we recall some basic definitions and important properties of pseudo-differential operators, Weyl operators on $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}^n$.

4.2.1 Pseudo-differential and Weyl quantized operator on $\mathbb{R}^n$

Given a reasonable function a on $\mathbb{R}^n \times \mathbb{R}^n$, the corresponding operator T_a associated with the function a given by

$$T_a f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi)\, d\xi, \quad \forall x \in \mathbb{R}^n,$$

for all Schwartz class functions f on $\mathbb{R}^n$, where the Fourier transform of f is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx, \quad \forall \xi \in \mathbb{R}^n.$$

The operator T_a is called pseudo-differential operator corresponding to the symbol a. Let $m \in \mathbb{R}, 0 \leq \delta < 1$ and $0 \leq \rho \leq 1$. Then the symbol class $S^m_{\rho,\delta}(\mathbb{R}^n)$ consists of those functions $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \left(1 + |\xi|^2\right)^{\frac{m-\delta|\alpha|+\rho|\beta|}{2}} \tag{4.2.1}$$

for all multi-indices α, β. Such m is called the order of the symbol a. We take $\rho = 1$ and $\delta = 0$ through out the paper and denote the symbol class $S^m_{1,0}(\mathbb{R}^n)$ by $S^m(\mathbb{R}^n)$. We refer to [85] for a detailed study on symbolic calculus and pseudo-differential operators on $\mathbb{R}^n$.

The Weyl quantization Op^W for a "reasonable" symbol a in $\mathbb{R}^n \times \mathbb{R}^n$ is given by

$$Op^W(a)f(u) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v)\cdot\xi} a\left(\frac{u+v}{2}, \xi\right) f(v)\, dv\, d\xi, \quad \forall u \in \mathbb{R}^n,$$

for all Schwartz class functions f on $\mathbb{R}^n$. The composition of two Weyl quantized operators $Op^W(a)$ and $Op^W(b)$ is given by $Op^W(a)Op^W(b) = Op^W(a\#b)$, where (see [65])

$$a\#b(\zeta,u) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-2i\{(\xi-\zeta)(y-u)-(\eta-\zeta)(x-u)\}} a(\xi,x) b(\eta,y) d\xi d\eta dx dy$$

and asymptotically

$$a\#b(x,\xi) \sim \sum_{j=0}^{N} \frac{1}{j!}\left(\frac{i}{2}\right)^{j} a(x,\xi)\left(\frac{\overleftarrow{\partial}}{\partial \xi}\frac{\overrightarrow{\partial}}{\partial u} - \frac{\overrightarrow{\partial}}{\partial \xi}\frac{\overleftarrow{\partial}}{\partial u}\right)^{j} b(x,\xi) + S_N(x,\xi) \tag{4.2.2}$$

(arrows point towards the factor to be differentiated) with $S_N \in S^{m_1+m_2-N}(\mathbb{R}^n)$.

Further, if $Op^W(a)$ is a trace class operator whose symbol $a \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$, then trace of the operator $Op^W(a)$ is given by

$$\mathrm{Tr}(Op^W(a)) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi)\, dx d\xi.$$

Moreover, the correspondence $a \to Op^W(a)$ is an isometry of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ onto the set of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. This yields

$$\mathrm{Tr}(AB^*) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (a\#\bar{b})(x,\xi)\, dx\, d\xi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi)\overline{b(x,\xi)}\, dx\, d\xi, \tag{4.2.3}$$

where $A = Op^W(a)$ and $B = Op^W(b)$. For a detailed study on pseudo-differential operators and Weyl operators on $\mathbb{R}^n$, we refer to [65, 85, 109].

4.2.2 The Heisenberg group

One of the simple and natural example of non-abelian, non-compact group is the famous Heisenberg group $\mathbb{H}^n$, which plays an important role in several branches of mathematics. The Heisenberg group $\mathbb{H}^n$ is a nilpotent Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ and the group operation is defined by

$$(x,y,t)(x',y',t') = (x+x', y+y', t+t' + \frac{1}{2}(xy' - x'y)),$$

where (x,y,t) and (x',y',t') are in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Moreover, $\mathbb{H}^n$ is a unimodular Lie group on which the Haar measure is the usual Lebesgue measure $dx\, dy\, dt$. The canonical basis for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}^n$ is given by the left-invariant vector fields:

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \quad j = 1, 2, \ldots n, \text{ and } T = \partial_t, \tag{4.2.4}$$

satisfying the commutator relation $[X_j, Y_j] = T, \quad j = 1, 2, \ldots n.$

The sublaplacian and the full laplacian on the Heisenberg group are defined as

$$\mathcal{L}_{\mathbb{H}} = \sum_{j=1}^{n}(X_j^2 + Y_j^2) = \sum_{j=1}^{n}\left(\left(\partial_{x_j} - \frac{y_j}{2}\partial_t\right)^2 + \left(\partial_{y_j} + \frac{x_j}{2}\partial_t\right)^2\right)$$

and

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{n} (X_j^2 + Y_j^2 + T_j^2),$$

respectively. Let $\mathcal{H} = -\Delta_{\mathbb{H}} + V$ be the Schrödinger operator on the Heisenberg group $\mathbb{H}^n$, where V is a positive smooth potential, bounded below and grows like $|g|^\kappa, \kappa > 0$ for large $|g|$ with

$$|g| := (|x|^4 + |t|^2)^{\frac{1}{4}}, \quad g = (x,t) \in \mathbb{H}^n, \tag{4.2.5}$$

defining the homogenous norm on $\mathbb{H}^n$.

Theorem 4.2.1. [Theorem 2, [88]] *Let μ be a measure on a locally compact space, X with $L^2(X, d\mu)$ separable. Let L_0 be a selfadjoint operator on $L^2(X, d\mu)$ so that its semigroup is ultracontractive, i.e., for some $s > 0, e^{-sL_0}$ maps L^2 to $L^\infty(X, d\mu)$. Suppose V is a nonnegative multiplication operator so that*

$$\mu(\{x \mid 0 \le V(x) < M\}) < \infty,$$

for all M. Then $L = L_0 + V$ has purely discrete spectrum.

From (2.2.1) of [58], the kernel of $e^{t\Delta_{\mathbb{H}}}$ can be computed as

$$k_t(x,u,\xi) = c_n \int_{\mathbb{R}} e^{-i\lambda\xi} e^{-t\lambda^2} \left(\frac{\lambda}{\sinh \lambda t} \right)^n e^{-\frac{1}{4}\lambda(\coth \lambda t)(x\cdot x + u\cdot u)} d\lambda$$

with $c_n = (4\pi)^{-n}$. Thus $e^{t\Delta_{\mathbb{H}}}$ is $L^2 - L^\infty$ bounded. Therefore, from the above result, we conclude that $\mathcal{H}$ has purely discrete spectrum whose eigenfunctions form a complete orthonormal basis for $L^2(\mathbb{H}^n)$.

By Stone-von Neumann theorem, the only infinite dimensional unitary irreducible representations (up to unitary equivalence) are given by π_λ, $\lambda \in \mathbb{R}^*$, where π_λ is defined by

$$\pi_\lambda(x,y,t) f(u) = e^{i\lambda(t + \frac{1}{2}xy)} e^{i\sqrt{\lambda} yu} f(u + \sqrt{|\lambda|} x), \quad (x,y,t) \in \mathbb{H}^n, u \in \mathbb{R}^n,$$

for all $f \in L^2(\mathbb{R}^n)$. We use the convention

$$\sqrt{\lambda} := \operatorname{sgn}(\lambda)\sqrt{|\lambda|} = \begin{cases} \sqrt{\lambda}, & \text{if } \lambda > 0, \\ -\sqrt{|\lambda|}, & \text{if } \lambda < 0. \end{cases}$$

For each $\lambda \in \mathbb{R}^*$, the group Fourier transform of $f \in L^1(\mathbb{H}^n)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$ defined by

$$\hat{f}(\lambda) \equiv \pi_\lambda(f) = \int_{\mathbb{H}^n} f(x,y,t)\pi_\lambda^*(x,y,t)\, dx\, dy\, dt.$$

We denote $B(L^2(\mathbb{R}^n))$ to be the set of all bounded operators on $L^2(\mathbb{R}^n)$. If $f \in L^2(\mathbb{H}^n)$, then $\hat{f}(\lambda)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$ and satisfies the Plancherel formula

$$\int_{\mathbb{R}^*} \|\hat{f}(\lambda)\|_{S_2}^2\, d\mu(\lambda) = \|f\|_{L^2(\mathbb{H}^n)},$$

where $\|.\|_{S_2}$ stands for the norm in the Hilbert space S_2 of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$ and $d\mu(\lambda) = c_n|\lambda|^n\, d\lambda$, c_n being a constant.

Theorem 4.2.2. *For all Schwartz class functions on $\mathbb{H}^n$, the following inversion formula holds:*

$$f(g) = \int_{\mathbb{R}^*} \mathrm{Tr}(\pi_\lambda(g)\hat{f}(\lambda))\, d\mu(\lambda), \quad \forall g \in \mathbb{H}^n.$$

Definition 4.2.3. *Let $\sigma : \mathbb{H}^n \times \mathbb{R}^* \to B(L^2(\mathbb{R}^n))$ be a operator valued function. Then the pseudo-differential operator T_σ corresponding to σ is defined by*

$$T_\sigma f(g) = \int_{\mathbb{R}^*} \mathrm{Tr}(\pi_\lambda(g)\sigma(g,\lambda)\hat{f}(\lambda))\, d\mu(\lambda), \quad g \in \mathbb{H}^n, \tag{4.2.6}$$

for all $f \in S(\mathbb{H}^n)$. The operator valued function σ is called the symbol of the pseudo-differential operator T_σ. We also often denote the pseudo-differential operator T_σ as $Op(\sigma)$.

For several important properties of pseudo-differential operators on the Heisenberg group, we refer to [33, 34]. We also refer to Thangavelu [100] for a complete account of representation theory on the Heisenberg group $\mathbb{H}^n$.

4.3 Symbolic calculus relative to $1+|\lambda|H+V(g)$ on $\mathbb{H}^n$

In this section we develop the symbolic calculus relative to $1+|\lambda|H+V(g)$ on the Heisenberg group $\mathbb{H}^n$, that will be useful for our study. We start this section by recalling the definition of Weyl-Hörmander pseudo-differential calculus and obtain the $(\lambda, V(g))$-Shubin classes $\Sigma^m_{\rho,\lambda,V}(\mathbb{R}^n)$. This classes depends in both the parameter λ and $V(g), g \in \mathbb{H}^n$ which will be of particular importance to us.

4.3.1 Weyl-Hörmander calculus

We present the main elements of the Weyl-Hörmander calculus relevant to the pseudo-differential analysis for $(\lambda, V(g))$-Shubin classes. We refer to [65] for the details on the underlying theory. Identifying the cotangent bundle of $\mathbb{R}^n$ with $\mathbb{R}^{2n}$, the canonical symplectic form on $\mathbb{R}^{2n}$ is ω, defined as

$$\omega(T, T') = x \cdot \xi' - x' \cdot \xi, \quad T = (\xi, x), T' = (\xi', x') \in \mathbb{R}^{2n}.$$

If q is a positive quadratic form on $\mathbb{R}^{2n}$, then its conjugate q^ω is defined by

$$q^\omega(T) := \sup_{T' \in \mathbb{R}^{2n} \setminus \{0\}} \frac{|\omega(T, T')|^2}{q(T')}, \quad \forall T \in \mathbb{R}^{2n}.$$

Also, the gain factor of q is defined as

$$\Lambda_q := \inf_{T \in \mathbb{R}^{2n} \setminus \{0\}} \frac{q^\omega(T)}{q(T)}.$$

Definition 4.3.1. *A metric is a family of positive quadratic forms* $\tilde{g} = \{g_X, X \in \mathbb{R}^{2n}\}$ *that depends smoothly on* $X \in \mathbb{R}^{2n}$. *Then*

- *the metric* $\tilde{g}$ *is uncertain if* $\Lambda_{\tilde{g}_X} \geq 1$ *for every* $X \in \mathbb{R}^{2n}$.
- *The metric* $\tilde{g}$ *is said to be slowly varying if there exists a positive constant* $\bar{C}$ *such that for any* $X, X' \in \mathbb{R}^{2n}$, $g_X(X - X') \leq \bar{C}^{-1}$ *implies that*

  $$\sup_{T \in \mathbb{R}^{2n} \setminus \{0\}} \left(\frac{g_X(T)}{g_{X'}(T)} + \frac{g_{X'}(T)}{g_X(T)} \right) \leq \bar{C}.$$

- *The metric* $\tilde{g}$ *is called temperate if there are constants* $\bar{C} > 0$ *and* $\bar{N} > 0$ *such that for any* $X, X' \in \mathbb{R}^{2n}$ *and* $T \in \mathbb{R}^{2n} \setminus \{0\}$, *we have*

  $$\frac{g_X(T)}{g_{X'}(T)} \leq \bar{C}\left(1 + g^\omega_X(X - X')\right)^{\bar{N}}.$$

A metric $\tilde{g}$ *is of Hörmander type if it is uncertain, slowly varying and temperate. For a detailed study on Weyl-Hörmander calculus, we refer to Section 6.4 of [34].*

Now we define the Shubin metric $\tilde{g}^{(\rho,\lambda,V(g))}_{\xi,u}$ depending on both the parameter $\lambda\in\mathbb{R}^*$ and $V(g), g\in\mathbb{H}^n$ on $\mathbb{R}^{2n}$ as

$$\tilde{g}^{(\rho,\lambda,V(g))}_{\xi,u}(d\xi,du):=\left(\frac{|\lambda|}{1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)}\right)^{\rho}\left(d\xi^2+du^2\right).$$

The associated positive function $M^{(\lambda,V(g))}$ on $\mathbb{R}^{2n}$ is

$$M^{(\lambda,V(g))}(\xi,u):=\left(1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)\right)^{\frac{1}{2}}.$$

We consider these $(\lambda,V(g))$-families of metrics for the case $\rho=1$ as introduced in [4].

Proposition 4.3.2. *For each* $\lambda\in\mathbb{R}^*$ *and* $g\in\mathbb{H}^n$*, the metric* $\tilde{g}^{(\rho,\lambda,V(g))}$ *is of Hörmander type, i.e.,* $\tilde{g}$ *is uncertain, slowly varying and temperate, where the conjugate of* $\tilde{g}^{(\rho,\lambda,V(g))}_{\xi,u}$ *is* $\left(\tilde{g}^{(\rho,\lambda,V(g))}_{\xi,u}\right)^{\omega}$ *given by*

$$\left(\tilde{g}^{(\rho,\lambda,V(g))}_{\xi,u}\right)^{\omega}(d\xi,du)=\left(\frac{1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)}{|\lambda|}\right)^{\rho}\left(d\xi^2+du^2\right).$$

Moreover, the gain is given by

$$\Lambda_{\tilde{g}^{(\rho,\lambda,V(g))}_{\xi,u}}=\left(\frac{1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)}{|\lambda|}\right)^{2\rho}.$$

Proof. The proof of the proposition follows exactly as in Proposition 1.20 of [4] for $\rho=1$. □

Definition 4.3.3. *Let* $\tilde{g}$ *be a metric of Hörmander type. A positive function* M *defined on* $\mathbb{R}^{2n}$ *is said to be a* $\tilde{g}$*-weight when there are positive constants* $\bar{C}'$ *and* $\bar{N}'$ *satisfying: for any* $X,Y\in\mathbb{R}^{2n}$*,* $\tilde{g}_X(X-Y)\leq\bar{C}'^{-1}$ *implies that*

$$\frac{M(X)}{M(Y)}+\frac{M(Y)}{M(X)}\leq\bar{C}',$$

and

$$\frac{M(X)}{M(Y)}\leq\bar{C}\left(1+\tilde{g}^{\omega}_X(X-Y)\right)^{\bar{N}'}.$$

Definition 4.3.4. [Hörmander symbol class $S(M,\tilde{g})$]. *Let* $\tilde{g}$ *be a metric of Hörmander type and* M *be a* $\tilde{g}$*-weight on* $\mathbb{R}^{2n}$*. The symbol class* $S(M,\tilde{g})$ *is the set of functions* $a\in C^\infty(\mathbb{R}^{2n})$ *such that for each integer* $l\in\mathbb{N}_0$*, the quantity*

$$\|a\|_{S(M,\tilde{g}),l}:=\sup_{\substack{l'\leq l,X\in\mathbb{R}^{2n}\\ g_X(T_{l'})\leq 1}}\frac{|\partial_{T_1}\cdots,\partial_{T_{l'}}a(X)|}{M(X)}$$

is finite.

We refer to Chapters 1 and 2 of [65] for a detailed study of the Hörmander symbol class $S(M,\tilde{g})$. For each parameters $\lambda \in \mathbb{R}^*$ and $V(g), g \in \mathbb{H}^n$, we define the $(\lambda, V(g))$-Shubin symbol class (with $m \in \mathbb{R}$ and the fixed parameter $\rho \in (0,1]$) on $\mathbb{R}^n$ as:

$$\Sigma^m_{\rho,\lambda,V(g)}\left(\mathbb{R}^n\right) := S\left(\left(1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)\right)^{\frac{m}{2}}, \frac{|\lambda|^\rho\left(d\xi^2+du^2\right)}{\left(1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)\right)^\rho}\right).$$

That means a symbol $a_{g,\lambda} \in C^\infty(\mathbb{R}^{2n})$ is in $\Sigma^m_{\rho,\lambda,V(g)}\left(\mathbb{R}^n\right)$ if and only if for each $N \in \mathbb{N}_0$, the quantity $\|a_{g,\lambda}\|_{\Sigma^m_{\rho,\lambda,V(g)},N} < \infty$, where

$$\begin{aligned}
&\|a_{g,\lambda}\|_{\Sigma^m_{\rho,\lambda,V(g)},N} \\
&:= \sup_{\substack{(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n \\ |\alpha|,|\beta|\le N}} |\lambda|^{-\rho\frac{|\alpha|+|\beta|}{2}}\left(1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)\right)^{-\frac{m-\rho(|\alpha|+|\beta|)}{2}}\left|\partial_\xi^\alpha\partial_u^\beta a_{g,\lambda}(\xi,u)\right|.
\end{aligned}$$

In other words, a symbol $a_{g,\lambda} = \{a_{g,\lambda}(\xi,u)\}$ is in $\Sigma^m_{\rho,\lambda,V(g)}\left(\mathbb{R}^n\right)$ if and only if for all $\alpha, \beta \in \mathbb{N}_0^n$ and for all $(\xi,u) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists a constant $C = C_{\alpha,\beta} > 0$ such that

$$\left|\partial_\xi^\alpha\partial_u^\beta a_{g,\lambda}(\xi,u)\right| \le C|\lambda|^{\rho\frac{|\alpha|+|\beta|}{2}}\left(1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)\right)^{\frac{m-\rho(|\alpha|+|\beta|)}{2}}.$$

4.3.2 Difference operators

In this subsection we discuss the difference operators and some of its properties to describe the symbolic calculus on the Heisenberg group. We only state the main results of difference operators that can be found in Section 5.2 of [34].

Definition 4.3.5. *Let $\mathbb{H}^n$ be the Heisenberg group with Lie algebra $\mathfrak{h}_n$, i.e., the Lie algebra equipped with the Lie bracket given by the commutator relations of its canonical basis $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, T\}$:*

$$[X_j, Y_j] = T \quad \textit{for} \ \ j = 1, \cdots, n,$$

and all the other Lie brackets (apart from those obtained by anti-symmetry) are trivial.

1. *A function P on $\mathbb{H}^n$ is a polynomial if the composition $P \circ exp^{\mathbb{H}^n}$ is a polynomial on $\mathfrak{h}_n$, where $exp^{\mathbb{H}^n}$ defines the exponential map from $\mathfrak{h}_n$ onto $\mathbb{H}^n$.*

2. *We denote by P, the set of all polynomials on $\mathbb{H}^n$. For any $M \geq 0$ we denote $P_{\leq M}$ the set of polynomials on $\mathbb{H}^n$ whose homogeneous degree $DP \leq M$, where the homogeneous degree of the polynomial $P = \sum_{\alpha\in\mathbb{N}_0^{2n+1}} c_\alpha g^\alpha$ (with all but finitely many of the*

coefficients $c_\alpha \in \mathbb{C}$ *vanish) is defined as* $DP := \{\max[\alpha] : \alpha \in \mathbb{N}_0^{2n+1} \ \text{ with } \ c_\alpha \neq 0\}$, *where* $[\alpha] := \alpha_{11}+\alpha_{12}+\cdots+\alpha_{1n}+\alpha_{21}+\cdots+\alpha_{2n}+2\alpha_3$ *defining the homogeneous degree of the multi-index* $\alpha = (\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1n}, \alpha_{21}, \cdots, \alpha_{2n}, \alpha_3) \in \mathbb{N}_0^{2n+1}$. *Similarly we define* $P_{=M}$. *For* $g = (x_1, \cdots, x_n, y_1, \cdots, y_n, t) \in \mathbb{H}^n$, *we have used the notation* $g^\alpha = x_1^{\alpha_{11}}, x_2^{\alpha_{12}}, \cdots, x_n^{\alpha_{1n}}, y_1^{\alpha_{21}}, y_2^{\alpha_{22}} \cdots, y_n^{\alpha_{2n}}, t^{\alpha_3}$.

Example 4.3.1. *On three dimensional Heisenberg group* $\mathbb{H}^1$, *any element can be described as* $q = (x, y, t) \in \mathbb{R}^3$ *with the degree 1 polynomials are* $q^{(1,0,0)} = x$, $q^{(0,1,0)} = y$, *and* $q^{(0,0,1)} = t$, *where as degree 2 polynomials are* $q^{(2,0,0)} = x^2$, $q^{(0,2,0)} = y^2$, *and* $q^{(1,1,0)} = xy$.

Let $\mathcal{W}$ be the set of all possible homogeneous degrees $[\alpha], \alpha \in \mathbb{N}_0^{2n+1}$, i.e.,

$$\mathcal{W} := \{|\alpha_1| + |\alpha_2| + 2\alpha_3 : \alpha \in \mathbb{N}_0^{2n+1}\},$$

where $\alpha_1 = (\alpha_{11}, \cdots, \alpha_{1n}) \in \mathbb{N}_0^n$, $\alpha_2 = (\alpha_{21}, \cdots, \alpha_{2n}) \in \mathbb{N}_0^n$ and $\alpha_3 \in \mathbb{N}_0$. We define the difference operators associated with the polynomials appearing with the Taylor expansions:

Proposition 4.3.6. [34]

1. *For each* $\alpha = (\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1n}, \alpha_{21}, \cdots, \alpha_{2n}, \alpha_3) \in \mathbb{N}_0^{2n+1}$, *there exists a unique homogeneous polynomial* q_α *of degree* $[\alpha]$ *satisfying*

$$\forall \beta \in \mathbb{N}_0^{2n+1} \quad X^\beta q_\alpha(0) = \begin{cases} 1, & if \quad \beta = \alpha, \\ 0, & otherwise, \end{cases}$$

where $X^\alpha = X^{\alpha_1} Y^{\alpha_2} T^{\alpha_3}$, $X^{\alpha_1} = X_1^{\alpha_{11}} X_2^{\alpha_{12}} \ldots X_n^{\alpha_{1n}}$ and $Y^{\alpha_1} = Y_1^{\alpha_{11}} Y_2^{\alpha_{12}} \cdots Y_n^{\alpha_{1n}}$.

2. *The polynomials* $q_\alpha, \alpha \in \mathbb{N}_0^{2n+1}$, *form a basis for* P. *Furthermore, for each* $M \in \mathcal{W}$, *the polynomials* q_α, $[\alpha] = M$, *form a basis of* $P_{[\alpha]=M}$.

3. *The Taylor polynomial of a suitable function* f *at a point* $g \in \mathbb{H}^n$ *of homogeneous degree* $M \in \mathcal{W}$ *is*

$$P^f_{g,M}(h) = \sum_{[\alpha] \leq M} q_\alpha(h) X^\alpha f(g), \quad h \in \mathbb{H}^n. \tag{4.3.1}$$

Definition 4.3.7. *For $\alpha \in \mathbb{N}_0^{2n+1}$, the difference operators are defined as*

$$\Delta^\alpha := \Delta_{\tilde{q}_\alpha},$$

where $\tilde{q}_\alpha(g) = q_\alpha(g^{-1}),\ g \in \mathbb{H}^n$ and $q_\alpha \in P_{=[\alpha]}$ defined in Proposition 4.3.6.

Let us collect some properties of difference operators in the following proposition.

Proposition 4.3.8. [34]

1. *For any $\alpha \in \mathbb{N}_0^{2n+1}$, the operator Δ^α is linear and*

$$\Delta^\alpha \mathcal{F}(\mathcal{S}(\mathbb{H}^n)) \subset \mathcal{F}(\mathcal{S}(\mathbb{H}^n)),$$

where $\mathcal{F}$ and $\mathcal{S}(\mathbb{H}^n)$ denote the Fourier transform and the collection of all Schwartz class functions on $\mathbb{H}^n$, respectively.

2. *For any $\alpha_1, \alpha_2 \in \mathbb{N}_0^{2n+1}$, there exists constants $c_{\alpha_1,\alpha_2,\alpha} \in \mathbb{R}$ with $[\alpha] = [\alpha_1] + [\alpha_2]$ such that for any $\phi \in \mathcal{S}(\mathbb{H}^n)$, we have*

$$\Delta^{\alpha_1}(\Delta^{\alpha_2}\hat{\phi}) = \Delta^{\alpha_2}(\Delta^{\alpha_1}\hat{\phi}) = \sum_{[\alpha]=[\alpha_1]+[\alpha_2]} c_{\alpha_1,\alpha_2,\alpha}\Delta^\alpha\hat{\phi},$$

where $\hat{\phi}$ is the Fourier transform of ϕ and the sum is taken over all $\alpha \in \mathbb{N}_0^{2n+1}$ satisfying $[\alpha] = [\alpha_1] + [\alpha_2]$.

3. *For any $\alpha \in \mathbb{N}_0^{2n+1}$, there exists constants $c_{\alpha,\alpha_1,\alpha_2} \in \mathbb{R}$ with $[\alpha] = [\alpha_1] + [\alpha_2]$ such that for any $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{H}^n)$, we have*

$$\Delta^\alpha(\hat{\phi}_1\hat{\phi}_2) = \sum_{[\alpha]=[\alpha_1]+[\alpha_2]} c_{\alpha,\alpha_1,\alpha_2}\Delta^{\alpha_1}\hat{\phi}_1\Delta^{\alpha_2}\hat{\phi}_2,$$

where the sum is taken over all $\alpha_1, \alpha_2 \in \mathbb{N}_0^{2n+1}$ satisfying $[\alpha_1] + [\alpha_2] = [\alpha]$.

4.3.3 Computations of difference operators

For $j = 1, 2, \cdots, n$, the difference operators $\Delta_{x_j}, \Delta_{y_j}$ and Δ_t are defined via

$$\Delta_{x_j}\hat{\kappa}(\pi_\lambda) := \pi_\lambda(x_j\kappa), \quad \Delta_{y_j}\hat{\kappa}(\pi_\lambda) := \pi_\lambda(y_j\kappa), \quad \Delta_t\hat{\kappa}(\pi_\lambda) := \pi_\lambda(t\kappa)$$

for suitable distributions κ defined on $\mathbb{H}^n$. The following properties of difference operators in the setting of Heisenberg group are well known and can be found in Section 6.3 of [34]. For a Schwartz class function h on $\mathbb{R}^n$, we have

$$\begin{aligned}
(\Delta_{x_j}\hat{\kappa}(\pi_\lambda))h(u) &= \frac{1}{\sqrt{|\lambda|}}(u_j(\hat{\kappa}(\pi_\lambda)h)(u) - (\hat{\kappa}(\pi_\lambda)(u_j h))(u)),\\
(\Delta_{y_j}\hat{\kappa}(\pi_\lambda))h(u) &= \frac{1}{i\sqrt{\lambda}}(-\partial_{u_j}(\hat{\kappa}(\pi_\lambda)h)(u) + (\hat{\kappa}(\pi_\lambda)(\partial_{u_j} h))(u)),\\
(\Delta_t\hat{\kappa}(\pi_\lambda))h(u) &= i\partial_\lambda \pi_\lambda(\kappa)h(u) + \frac{1}{2}\sum_{j=1}^{n}\Delta_{x_j}\Delta_{y_j}\pi_\lambda(\kappa)h(u)\\
&\quad + \frac{i}{2\lambda}\sum_{j=1}^{n}\{\pi_\lambda(Y_j)\Delta_{y_j}\pi_\lambda(\kappa) + \Delta_{x_j}\pi_\lambda(\kappa)\pi_\lambda(X_j)\}h(u).
\end{aligned}$$

If $\widehat{\kappa}(\pi_\lambda) \equiv \pi_\lambda(\kappa) = \mathrm{Op}^W(a_\lambda)$ and $a_\lambda = \{a_\lambda(\xi,u)\}$, then the difference operators $\Delta_{x_j}, \Delta_{y_j}$, and Δ_t satisfies

$$\begin{aligned}
\Delta_{x_j}\widehat{\kappa}(\pi_\lambda) &:= \pi_\lambda(x_j\kappa) = \mathrm{Op}^W\left(\frac{i}{\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda\right),\\
\Delta_{y_j}\widehat{\kappa}(\pi_\lambda) &:= \pi_\lambda(y_j\kappa) = \mathrm{Op}^W\left(\frac{i}{\sqrt{\lambda}}\partial_{u_j}a_\lambda\right),\\
\Delta_t\widehat{\kappa}(\pi_\lambda) &:= \pi_\lambda(t\kappa) \quad = i\mathrm{Op}^W\left(\tilde{\partial}_{\lambda,\xi,u}a_\lambda\right),
\end{aligned}$$

where

$$\tilde{\partial}_{\lambda,\xi,u} := \partial_\lambda - \frac{1}{2\lambda}\sum_{j=1}^{n}\left(u_j\partial_{u_j} + \xi_j\partial_{\xi_j}\right). \tag{4.3.2}$$

Further, an easy calculation gives

$$\begin{aligned}
&\Delta_{x_j}\pi_\lambda(Y_k) = \Delta_{x_j}\pi_\lambda(T) = \Delta_{y_j}\pi_\lambda(X_k) = \Delta_{y_j}\pi_\lambda(T) = 0,\\
&\Delta_{x_j}\pi_\lambda(X_k) = \Delta_{y_j}\pi_\lambda(Y_k) = -\delta_{j,k}I, \qquad \Delta_t\pi_\lambda(T) = -I,\\
&\Delta_{x_j}\pi_\lambda(\mathcal{L}) = -2\pi_\lambda(X_j), \qquad \Delta_{y_j}\pi_\lambda(\mathcal{L}) = -2\pi_\lambda(Y_j), \qquad \Delta_t\pi_\lambda(\mathcal{L}) = 0.
\end{aligned}$$

Lemma 4.3.9. *Let* $a_\lambda = \{a_\lambda(\xi,u)\}$ *be a family of Weyl symbols depending smoothly on* $\lambda \neq 0$. *If* $\tilde{a}_\lambda$ *is the renormalization obtained via* $a_\lambda(\xi,u) := \tilde{a}_\lambda(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u)$, *then*

$$\tilde{\partial}_{\lambda,\xi,u}a_\lambda(\xi,u) = \{\partial_\lambda\tilde{a}_\lambda(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u)\},$$

$$\frac{1}{i\sqrt{|\lambda|}}\partial_{\xi_j}a_\lambda(\xi,u) = \partial_{\xi_j}\tilde{a}_\lambda(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u),$$

and

$$\frac{1}{i\sqrt{\lambda}}\partial_{u_j} a_\lambda(\xi,u) = \partial_{u_j}\tilde{a}_\lambda(\sqrt{|\lambda|}\xi, \sqrt{\lambda}u).$$

Consequently,

$$\Delta_{x_j}\pi_\lambda(\kappa) = iOp^W(\partial_{\xi_j}\tilde{a}_\lambda), \quad \Delta_{y_j}\pi_\lambda(\kappa) = iOp^W(\partial_{u_j}\tilde{a}_\lambda), \textit{and} \quad \Delta_t\pi_\lambda(\kappa) = iOp^W(\partial_\lambda\tilde{a}_\lambda).$$

4.3.4 The symbol class $S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$

In this subsection we define the symbol class $S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$ relative to the operator $1+|\lambda|H+V(g)$. First we recall some important formulae that will be used to define the symbol class $S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$. We mainly adopt the notation and terminology given in [34]. The Schrödinger infinitesimal representation of π_λ acts on the canonical basis of $\mathfrak{h}_n$ via

$$\begin{aligned} \pi_\lambda\left(X_j\right) = \sqrt{|\lambda|}\partial_{u_j} &= \mathrm{Op}^W\left(i\sqrt{|\lambda|}\xi_j\right), \\ \pi_\lambda\left(Y_j\right) = i\sqrt{\lambda}u_j \quad &= \mathrm{Op}^W\left(i\sqrt{\lambda}, u_j\right) \\ \pi_\lambda(T) = i\lambda \mathrm{I} \qquad &= \mathrm{Op}^W(i\lambda), \end{aligned}$$

for $j=1,\ldots,n$. Thus $\pi_\lambda(\mathcal{L}) = |\lambda|\sum_{j=1}^n\left(\partial^2_{u_j} - u_j^2\right) = \mathrm{Op}^W\left(|\lambda|\sum_{j=1}^n\left(-\xi_j^2 - u_j^2\right)\right)$. Now we are in a position to define the symbol class $S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$ relative to the operator $1+|\lambda|H+V(g)$ (see Section 6.5.2 of [34]) by the following family of seminorms which are finite:

$$\|\sigma\|_{S^m_{\rho,\delta,\mathcal{H}},a,b,c} := \sup_{g\in\mathbb{H}^n,\lambda\in\mathbb{R}^*}\|\sigma(g,\lambda)\|_{S^m_{\rho,\delta,\mathcal{H},\lambda,V}a,b,c}, \quad a,b,c\in\mathbb{N}_0,$$

where

$$\begin{aligned} &\|\sigma(g,\lambda)\|_{S^m_{\rho,\delta,\mathcal{H},\lambda,V}a,b,c} \qquad (4.3.3) \\ &:= \sup_{[\alpha]\le a,[\beta]\le b,|\gamma|\le c}\|(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{\frac{\rho[\alpha]-m-\delta[\beta]+\gamma}{2}}X_g^\beta\Delta'^\alpha\sigma(g,\lambda)(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{-\frac{\gamma}{2}}\|_{op} \end{aligned}$$

with $\alpha = (\alpha_1,\alpha_2,\alpha_3) = (\alpha_{11},\alpha_{12}\ldots\alpha_{1n},\alpha_{21},\alpha_{22}\ldots\alpha_{2n},\alpha_3)\in\mathbb{N}_0^n\times\mathbb{N}_0^n\times\mathbb{N}_0$, $\|\cdot\|_{op}$ denote the operator norm on $B(L^2(\mathbb{R}^n))$ and the difference operators are

$$\Delta'^\alpha := \Delta_x^{\alpha_1}\Delta_y^{\alpha_2}\Delta_t^{\alpha_3}, \quad \text{where } \Delta_x^{\alpha_1} = \Delta_{x_1}^{\alpha_{11}}\Delta_{x_2}^{\alpha_{12}}\cdots\Delta_{x_n}^{\alpha_{1n}},\ \Delta_y^{\alpha_2} = \Delta_{y_1}^{\alpha_{21}}\Delta_{y_2}^{\alpha_{22}}\ldots\Delta_{y_n}^{\alpha_{2n}},$$

and

$$X^\alpha = X^{\alpha_1}Y^{\alpha_2}T^{\alpha_3}, \text{ where } X^{\alpha_1} = X_1^{\alpha_{11}}X_2^{\alpha_{12}}\ldots X_n^{\alpha_{1n}} \text{ and } Y^{\alpha_1} = Y_1^{\alpha_{11}}Y_2^{\alpha_{12}}\cdots Y_n^{\alpha_{1n}}.$$

4.3.5 Characterisation of $S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$

In this subsection we describe the symbol in $S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$ in terms of scalar-valued $(\lambda, V(g))$-symbols. More precisely, the symbols $\sigma=\sigma(g,\lambda)$ in $S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$ are all of the form

$$\sigma(g,\lambda)=Op^W(a_{g,\lambda}(\xi,u)),$$

with the $(\lambda, V(g))$-symbols $a_{g,\lambda}$ satisfying some properties described below in terms of the family of $(\lambda, V(g))$-Shubin classes.

Theorem 4.3.10. *Let $m,\rho,\delta\in\mathbb{R}$ such that $1\geq\rho\geq\delta\geq 0$ and $(\rho,\delta)\neq(0,0)$. If $\sigma=\sigma(g,\lambda)$ is in $S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$, then there exist a smooth function $a=a(g,\lambda,\xi,u)=a_{g,\lambda}(\xi,u)$ on $\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n$ such that*

$$\sigma(g,\lambda)=Op^W(a_{g,\lambda})$$

with $\tilde{\partial}^{\alpha_3}_{\lambda,\xi,u}X^\beta_g a_{g,\lambda}\in\Sigma^{m-2\rho\alpha_3+\delta|\beta|}_{\rho,\lambda,V(g)}(\mathbb{R}^n)$ for each $(g,\lambda)\in\mathbb{H}^n\times\mathbb{R}^*$ satisfying

$$\sup_{(g,\lambda)\in\mathbb{H}^n\times\mathbb{R}^*}\left\|\tilde{\partial}^{\alpha_3}_{\lambda,\xi,u}X^\beta_g a_{g,\lambda}\right\|_{\Sigma^{m-2\rho\alpha_3+\delta|\beta|}_{\rho,\lambda,V(g)}(\mathbb{R}^n),N}<\infty, \tag{4.3.4}$$

for every $N\in\mathbb{N}_0$. More precisely, for every $N\in\mathbb{N}_0$ there exist $C>0$ and a,b,c such that

$$\sup_{(g,\lambda)\in\mathbb{H}^n\times\mathbb{R}^*}\left\|\tilde{\partial}^{\alpha_3}_{\lambda,\xi,u}X^\beta_g a_{g,\lambda}\right\|_{\Sigma^{m-2\rho\alpha_3+\delta|\beta|}_{\rho,\lambda,V(g)}(\mathbb{R}^n),N}\leq C\|\sigma\|_{S^m_{\rho,\delta,\mathcal{H}},a,b,c},$$

where the operator $\tilde{\partial}^{\alpha_3}_{\lambda,\xi,u}$ is defined in (4.3.2).

Conversely, if $a=\{a_{(g,\lambda,\xi,u)}=a_{g,\lambda}(\xi,u)\}$ is a smooth function on $\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n$ satisfying (4.3.4) for every $N\in\mathbb{N}_0$, then there exist a unique symbol $\sigma\in S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$ such that $\sigma(g,\lambda)=Op^W(a_{g,\lambda})$. Furthermore, for every a,b,c there exists $C>0$ and $N\in\mathbb{N}_0$ such that

$$\|\sigma\|_{S^m_{\rho,\delta,\mathcal{H}},a,b,c}\leq C\sup_{(g,\lambda)\in\mathbb{H}^n\times\mathbb{R}^*}\left\|\tilde{\partial}^{\alpha_3}_{\lambda,\xi,u}X^\beta_g a_{g,\lambda}\right\|_{\Sigma^{m-2\rho\alpha_3+\delta|\beta|}_{\rho,\lambda,V(g)}(\mathbb{R}^n),N}. \tag{4.3.5}$$

Proof. The proof is similar to the proof of Theorem 6.5.1 of [34]. ☐

In other words, Theorem 4.3.10 yields that $\sigma\in S^m_{\rho,\delta,\mathcal{H}}(\mathbb{H}^n)$ is equivalent to $\sigma(g,\lambda)=Op^W(a_{g,\lambda})$ for each $(g,\lambda)\in\mathbb{H}^n\times\mathbb{R}^*$ with $a_{g,\lambda}\in C^\infty(\mathbb{R}^{2n})$ satisfying: for any $\alpha_1\in\mathbb{N}^{2n+1}_0$, there exists a constant $C > 0$ such that for every $(g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^*$ and for every $(\xi, u) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$|\partial_\xi^\alpha \partial_u^\beta \tilde{\partial}_{\lambda,\xi,u}^{\bar{\alpha}} X_g^{\bar{\beta}} a_{g,\lambda}(\xi, u)| \leq C_{\alpha,\beta,\bar{\alpha},\bar{\beta}} |\lambda|^{\rho \frac{|\alpha|+|\beta|}{2}} (1 + |\lambda|(1 + |\xi|^2 + |u|^2) + V(g))^{\frac{m-\rho|\alpha_1|+\delta|\bar{\beta}|}{2}}.$$

We take $\rho = 1$ and $\delta = 0$ throughout this chapter and denote the symbol classes $S^m_{1,0,\mathcal{H}}(\mathbb{H}^n)$ by $S^m_{\mathcal{H}}(\mathbb{H}^n)$.

Example 4.3.2. *For any $\beta \in \mathbb{R}$, $\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}), V(g)^\beta, (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}) + V(g))^\beta$ and $(1 + \lambda^2)^\beta$ are symbols with order $2, 2\beta, 2\beta$ and 4β, respectively.*

Remark 4.3.11. *Let $\sigma \in S^m_{\mathcal{H}}(\mathbb{H}^n)$. Then we have the following properties.*

1. *If $\beta_o \in \mathbb{N}_0^n$ then the symbol $\left\{X_x^{\beta_o}\sigma(g, \lambda), (g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^*\right\}$ is in $S^m_{\mathcal{H}}(\mathbb{H}^n)$ and*

$$\left\|X_g^{\beta_o}\sigma(g, \lambda)\right\|_{S^m_{\mathcal{H}},a,b,c} \leq C_{b,\beta_o} \|\sigma(g, \lambda)\|_{S^m_{\mathcal{H}},a,b+[\beta_o],c}.$$

2. *If $\alpha_o \in \mathbb{N}_0^n$ then the symbol $\{\Delta^{\alpha_o}\sigma(g, \lambda), (g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^*\}$ is in $S^{m-[\alpha_o]}_{\mathcal{H}}(\mathbb{H}^n)$ and*

$$\|\Delta^{\alpha_o}\sigma(g, \lambda)\|_{S^{m-[\alpha_o]}_{\mathcal{H}},a,b,c} \leq C_{a,\alpha_o} \|\sigma(g, \lambda)\|_{S^m_{\mathcal{H}},a+[\alpha_o],b,c}.$$

3. *If $\sigma_1 \in S^\mu_{\mathcal{H}}(\mathbb{H}^n)$ and $\sigma_2 \in S^\nu_{\mathcal{H}}(\mathbb{H}^n)$ then $\sigma(g, \lambda) = \sigma_1(g, \lambda)\sigma_2(g, \lambda) \in S^{\mu+\nu}_{\mathcal{H}}(\mathbb{H}^n)$ and*

$$\|\sigma(g, \lambda)\|_{S^{\mu+\nu}_{\mathcal{H}},a,b,c} \leq \|\sigma_1(g, \lambda)\|_{S^\mu_{\mathcal{H}},a,b,c+a+|\nu|} \|\sigma_2(g, \lambda)\|_{S^\nu_{\mathcal{H}},a,b,c}.$$

4. *If $\sigma_1 \in S^\mu_{\mathcal{H}}(\mathbb{H}^n)$ and $\sigma_2 \in S^\nu_{\mathcal{H}}(\mathbb{H}^n)$ then $\Delta^\alpha \sigma_1 X_x^\beta \sigma_2 \in S^{\mu+\nu-[\alpha]}_{\mathcal{H}}(\mathbb{H}^n)$ for $\alpha, \beta \in \mathbb{N}_0^n$.*

Lemma 4.3.12. *If A is a trace class pseudo-differential operator on $L^2(\mathbb{H}^n)$ with symbol $\sigma(\cdot, \cdot) \in L^1(\mathbb{H}^n \times \mathbb{R}^*, S_1, d\mu(\lambda))$, then*

$$\mathrm{Tr}(A) = \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} \mathrm{Tr}(\sigma(g, \lambda))\, dg\, d\mu(\lambda). \tag{4.3.6}$$

Proof. For all $f \in L^2(\mathbb{H}^n)$, we have

$$\begin{aligned}
(Af)(g) &= \int_{\mathbb{R}^*} \mathrm{Tr}(\pi_\lambda(g)\sigma(g, \lambda)\hat{f}(\lambda))\, d\mu(\lambda) \\
&= \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} \mathrm{Tr}(\pi_\lambda(g)\sigma(g, \lambda)\pi_\lambda(g_1)^*)\, d\mu(\lambda) f(g_1)\, dg_1 \\
&= \int_{\mathbb{H}^n} K(g, g_1) f(g_1)\, dg_1,
\end{aligned}$$

with

$$K(g,g_1)=\int_{\mathbb{R}^*}\operatorname{Tr}(\pi_\lambda(g)\sigma(g,\lambda)\pi_\lambda(g_1)^*)\,d\mu(\lambda).$$

Therefore

$$\operatorname{Tr}(A)=\int_{\mathbb{H}^n}K(g,g)\,dg=\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\operatorname{Tr}(\sigma(g,\lambda))\,dg\,d\mu(\lambda),$$

where the last step of the above computation is justified by Corollary 3.2 of [11]. □

For an operator valued symbol σ on $\mathbb{H}^n$, the correspondence $\sigma\to Op(\sigma)$ is an isometry from $L^2\left(\mathbb{H}^n\times\mathbb{R}^*,S_2,d\mu(\lambda)\right)$ onto the set of Hilbert-Schmidt operators on $L^2(\mathbb{H}^n)$ via square integrable kernels [80]. This allows us to write

$$\begin{aligned}\operatorname{Tr}(Op(\sigma)\circ Op(\tau)^*)&=\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\operatorname{Tr}(\sigma\#_{\mathbb{H}^n}\tau^{(*)})(g,\lambda)\,dg\,d\mu(\lambda)\\&=\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\operatorname{Tr}(\sigma(g,\lambda)\tau^{(*)}(g,\lambda))\,dg\,d\mu(\lambda),\end{aligned}\tag{4.3.7}$$

where $\sigma\#_{\mathbb{H}^n}\tau$ is the symbol of the composition $Op(\sigma)\circ Op(\tau)$ (defined in Theorem 4.3.18) and $\tau^{(*)}$ is the symbol of $Op(\tau)^*$, the adjoint of $Op(\tau)$ (see page 365 of [34]).

Now as in the proof of Calderón-Vaillancourt theorem (Theorem 5.7.1 of [34]), we get the following Calderón-Vaillancourt theorem for the symbol class $S^0_{\mathcal{H}}(\mathbb{H}^n)$.

Theorem 4.3.13 (The Calderón-Vaillancourt theorem). *Let $\sigma\in S^0_{\mathcal{H}}(\mathbb{H}^n)$. Then $Op(\sigma)$ extends a bounded operator on $L^2(\mathbb{H}^n)$. Moreover, there exist a constant $C>0$ and a seminorm $\|\cdot\|_{S^0_{\mathcal{H}},a,b,c}$ with computable integers $a,b,c\in\mathbb{N}_0$ independent of $Op(\sigma)$ such that*

$$\|Op(\sigma)\phi\|_{L^2(\mathbb{H}^n)}\le C\|\sigma\|_{S^0_{\mathcal{H}},a,b,c}\|\phi\|_{L^2(\mathbb{H}^n)},\quad \phi\in S(\mathbb{H}^n).$$

Definition 4.3.14 (Kernel associated with a symbol). *If σ is a symbol, then the tempered distribution*

$$\kappa_g:=\mathcal{F}^{-1}\{\sigma(g,\cdot)\}\in\mathcal{S}'(\mathbb{H}^n)$$

is called its associated kernel or the right convolution kernel. We also call the smooth map $\mathbb{H}^n\ni g\to\kappa_g\in\mathcal{S}'(\mathbb{H}^n)$ or the map $(g,h)\to\kappa_g(h)=\kappa(g,h)$ the kernel associated with the symbol σ.

Remark 4.3.15. 1. *If $\sigma=\{\sigma(g,\lambda)\}$ is a symbol of the operator $Op(\sigma)$ acting on the Heisenberg group with kernel κ_g then for any $\beta\in\mathbb{N}_0^{2n+1}$, $X^\beta\sigma:=\{X_g^\beta\sigma(g,\lambda)\}$, $\tilde{X}^\beta\sigma:=$*

$\{\tilde{X}_g^\beta \sigma(g,\lambda)\}$ and $\partial_g^\beta \sigma := \{\partial_g^\beta \sigma(g,\lambda)\}$ are symbols with respective kernels $X_g^\beta \kappa_g, \tilde{X}_g^\beta \kappa_g$ and $\partial_g^\beta \kappa_g$, where $\tilde{X}_g$ is the right invariant vector fields related with the left invariant vector fields X_g on $\mathbb{H}^n$ by $\tilde{X}_g f(g) = -(X\tilde{f})(g^{-1})$ with $\tilde{f}(g) = f(g^{-1})$.

2. *The quantization defined in (4.2.6) makes sense for any symbol $\sigma = \{\sigma(g,\lambda)\}$. More precisely, for any $\phi \in \mathcal{S}(\mathbb{H}^n)$ and $g \in \mathbb{H}^n$, we have*

$$Op(\sigma)\phi(g) = \int_{\mathbb{R}^*} Tr(\pi_\lambda(g)\sigma(g,\lambda)\hat{\phi}(\lambda))d\mu(\lambda) = \phi * \kappa_g(g), \tag{4.3.8}$$

where κ_g denotes the kernel of σ.

4.3.6 Composition of symbols

Let $a \in S_{\mathcal{H}}^{m_1}(\mathbb{H}^n)$ and $b \in S_{\mathcal{H}}^{m_2}(\mathbb{H}^n)$. Then the composition of pseudo-differential operators corresponding to the symbols a and b defines a pseudo-differential operator and the symbol σ of the composition is given by the following asymptotic expansion (4.3.9). We add constraints on V (see [59] and [109]) which guarantees the asymptotic expansion (4.3.9).

Definition 4.3.16. *The potential V is said to be temperate potential if there exists a constant $C > 0$ such that*

$$\|(\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}) + V(g))^{-1}(\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}) + V(gg_1))\|_{op} \leq C|g_1|^k,$$

for all $g, g_1 \in \mathbb{H}^n$ and for some constant $k > 0$.

Definition 4.3.17. *The potential V is said to be a regular potential if $V \in C^\infty(\mathbb{H}^n)$ and*

$$|X_g^\alpha V(g)| \leq C_\alpha(V(g) + 1)$$

for all $g \in \mathbb{H}^n$, for some $C_\alpha > 0$.

Theorem 4.3.18 (Composition formula). *Let $a \in S_{\mathcal{H}}^{m_1}(\mathbb{H}^n)$ and $b \in S_{\mathcal{H}}^{m_2}(\mathbb{H}^n)$. Then the composition $Op(a) \circ Op(b)$ is a pseudo-differential operator with symbol $\sigma \in S_{\mathcal{H}}^{m_1+m_2}(\mathbb{H}^n)$ having asymptotic expansion*

$$\sigma(g,\lambda) \sim \sum_\alpha \Delta^\alpha a(g,\lambda) X_g^\alpha b(g,\lambda), \tag{4.3.9}$$

where the asymptotic expansion means that for every $M \in \mathbb{N}$, *we have*

$$\sigma - \sum_{[\alpha]\leq M} \Delta^\alpha a X_g^\alpha b \in S_{\mathcal{H}}^{m_1+m_2-M}(\mathbb{H}^n),$$

where $[\alpha] = |\alpha_1| + |\alpha_2| + 2\alpha_3$ *defining the homogeneous degree of the multi-index* α *(see the notations used in (4.3.3)).*

Now we are in a position to estimate the reminder term in composition formula in the following lemma.

Lemma 4.3.19. *Let* $m_1, m_2 \in \mathbb{R}$, $\beta_0 \in \mathbb{N}_0^n$, *and* $M, M_1 \in \mathbb{N}_0$. *Suppose that*

$$\begin{cases} m_2 \leq 2M_1 < M - m_1 + v_1, \\ m_2 \leq 2M_1 < -m_1 - M. \end{cases} \tag{4.3.10}$$

If $M \geq 2M_1$, *then only the second condition may be assumed. Then there exist a constant* $C > 0$ *and two pseudo-norms* $\|\cdot\|_{S_{\mathcal{H}}^{m_1,R},a_1,b_1}$, $\|\cdot\|_{S_{\mathcal{H}}^{m_2},0,b_2,0}$ *such that for any two symbol* a, b *and for any* $(g,\lambda) \in \mathbb{H}^n \times \mathbb{R}^*$, *we have*

$$\left\| X_g^{\beta_0} \left(a \circ b(g,\lambda) - \sum_{[\alpha]\leq M} \Delta^\alpha a(g,\lambda) X_g^\alpha b(g,\lambda) \right) \right\| \leq C \|a\|_{S_{\mathcal{H}}^{m_1,R},a_1,b_1} \|b\|_{S_{\mathcal{H}}^{m_2},0,b_2,0}.$$

Proof. Let k_1 and k_2 are the kernels of $Op(a)$ and $Op(b)$ respectively. Then for $\phi \in \mathcal{S}(\mathbb{H}^n)$

$$\begin{aligned} Op(a) \circ Op(b)\phi(g) &= \int_{\mathbb{H}^n} Op(b)\phi(h) \kappa_1(g, h^{-1}g) dh \\ &= \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \phi(g_1) \kappa_2(h, g_1^{-1}h) \kappa_1(g, h^{-1}g) dh dg_1 \\ &= \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \phi(g_1) \kappa_2(g g_2^{-1}, g_1^{-1} g g_2^{-1}) \kappa_1(g, g_2) dg_1 dg_2 \\ &= \int_{\mathbb{H}^n} \phi(g_1) \kappa(g, g_1^{-1} g) dg_1 = \phi * \kappa(g,g), \end{aligned}$$

where $\kappa(g, g_1^{-1}g) = \kappa_2(h, g_1^{-1}h)\kappa_1(g, h^{-1}g)$ with a change of variable $h^{-1}g = g_2$. If $Op(a) \circ Op(b) = Op(\sigma)$, then

$$\begin{aligned} \sigma(g,\lambda) = \hat{\kappa}(g,\lambda) &= \int_{\mathbb{H}^n} \kappa(g, h_1) \pi_\lambda(h_1)^* dh_1 \\ &= \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \kappa_2(gh^{-1}, h_1 h^{-1}) \kappa_1(g,h) \pi_\lambda(h)^* \pi_\lambda(h_1 h^{-1})^* dh_1 dh \\ &= \int_{\mathbb{H}^n} k_1(g,h) \pi(h)^* b\left(gh^{-1}, \lambda\right) dh. \end{aligned} \tag{4.3.11}$$

By the Taylor series expansion (see (4.3.1)) of b in the first variable we have

$$b(gh^{-1},\lambda) \approx \sum_{\alpha} q_\alpha(h^{-1}) X_g^\alpha b(g,\lambda).$$

Implementing this in (4.3.11), we get

$$\begin{aligned} \sigma(g,\lambda) &\approx \int_{\mathbb{H}^n} k_1(g,h)\pi(h)^* \sum_{\alpha} q_\alpha(h^{-1}) X_g^\alpha b(g,\lambda) dh \qquad (4.3.12)\\ &= \sum_{\alpha} q_\alpha(h^{-1}) k_1(g,h)\pi(h)^* X_g^\alpha b(g,\lambda) dh \\ &= \sum_{\alpha} \Delta^\alpha a(g,\lambda) X_g^\alpha b(g,\lambda). \end{aligned}$$

First we consider the case when $\beta_0 = 0$. Thus by (4.3.11) and (4.3.12), we have

$$\begin{aligned} &\sigma(g,\lambda) - \sum_{[\alpha]\le M} \Delta^\alpha a(g,\lambda) X_g^\alpha b(g,\lambda) \\ &= \int_{\mathbb{H}^n} k_1(g,h)\pi_\lambda(h)^* (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g))^{M_1} (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g))^{-M_1} \\ &\qquad \times \left(b\left(gh^{-1},\lambda\right) - \sum_{[\alpha]\le M} q_\alpha\left(h^{-1}\right) X_g^\alpha b(g,\lambda) \right) dh \\ &= \sum_{j=0}^{M_1} \int_{\mathbb{H}^n} k_1(g,h)\pi_\lambda(h)^* (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}))^j V(g)^{M_1-j} (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g))^{-M_1} \\ &\qquad \times \left(b\left(gh^{-1},\lambda\right) - \sum_{[\alpha]\le M} q_\alpha\left(h^{-1}\right) X_g^\alpha b(g,\lambda) \right) dh \\ &= \sum_{j=0}^{M_1} \overline{\sum}_{[\beta_{11}]+[\beta_{22}]\le 2j} \int_{\mathbb{H}^n} \tilde{X}_h^{\beta_{11}} k_1(g,h) V(g)^{M_1-j} \pi_\lambda(h)^* \\ &\qquad \times \tilde{X}_h^{\beta_{22}} (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g))^{-M_1} R_{g,M}^{b(\cdot,\lambda)}\left(h^{-1}\right) dh, \end{aligned}$$

where $R_{g,M}^{b(\cdot,\lambda)}(h) = b\left(gh,\lambda\right) - \sum_{[\alpha]\le M} q_\alpha\left(h\right) X_g^\alpha b(g,\lambda)$ and $\overline{\sum} a_j := c_j a_j$ denotes a finite linear combination with some constants c_j. Taking the operator norm on $B(L^2(\mathbb{R}^n))$, we have

$$\begin{aligned} &\|\sigma(g,\lambda) - \sum_{[\alpha]\le M} \Delta^\alpha a(g,\lambda) X_g^\alpha b(g,\lambda)\|_{op} \\ &\le \sum_{j=0}^{M_1} \sum_{[\beta_{11}]+[\beta_{22}]\le 2j} \int_{\mathbb{H}^n} \left| \tilde{X}_h^{\beta_{11}} k_1(g,h) V(g)^{M_1-j} \right| \end{aligned}$$

$$\times \left\|\tilde{X}_h^{\beta_{22}}(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{-M_1}R_{g,M}^{b(\cdot,\lambda)}\left(h^{-1}\right)\right\|_{op}dh.$$

Using Taylor's estimate for vector-valued functions given in Proposition 3.1.40, Remark 3.1.52(3), Proposition 3.1.46 and Corollary 3.1.53 of [34], there is a constant $C>0$ (depending on M) such that

$$\begin{aligned}
&\left\|\tilde{X}_h^{\beta_{22}}(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{-M_1}R_{g,M}^{b(\cdot,\lambda)}\left(h^{-1}\right)\right\|_{op}\\
&=\left\|(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{-M_1}R_{g,M}^{X_g^{\beta_{22}}b(\cdot,\lambda)}\left(h^{-1}\right)\right\|_{op}\\
&\leq\left\|(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(gh^{-1}))^{-M_1}X_g^{\beta_{22}}b\left(gh^{-1},\lambda\right)-(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{-M_1}\sum_{[\alpha]\leq M}q_\alpha\left(h^{-1}\right)X_g^{\alpha}X_g^{\beta_{22}}b(g,\lambda)\right\|_{op}\\
&+\left\|(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{-M_1}X_g^{\beta_{22}}b\left(gh^{-1},\lambda\right)-(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(gh^{-1}))^{-M_1}X_g^{\beta_{22}}b\left(gh^{-1},\lambda\right)\right\|_{op}\\
&\leq C_M\sum_{\substack{|\gamma|\leq(M-[\beta_{22}])+1\\|\gamma|>(M-[\beta_{22}])}}|h|^{[\gamma]}\sup_{|g_1|\leq c_1|h|}\left\|X^{\gamma}(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(gg_1))^{-M_1}X^{\beta_{22}}b\left(gg_1,\lambda\right)\right\|_{op}\\
&+\|(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{-M_1}(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(gh^{-1}))^{M_1}-I\|_{op}\|(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(gh^{-1}))^{-M_1}X_g^{\beta_{22}}b\left(gh^{-1},\lambda\right)\|_{op}.
\end{aligned}$$

Using the fact that V is a regular temperate potential (see definitions 4.3.16 and 4.3.17), we have

$$\begin{aligned}
&\left\|\tilde{X}_h^{\beta_{22}}(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g))^{-M_1}R_{g,M}^{b(\cdot,\lambda)}\left(h^{-1}\right)\right\|_{op}\\
&\leq C_M\sum_{\substack{|\gamma|\leq(M-[\beta_{22}])+1\\|\gamma|>(M-[\beta_{22}])}}|h|^{[\gamma]}\sum_{[\gamma_1]\leq[\gamma]}\sup_{|g_1|\leq c_1|h|}\left\|(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(gg_1))^{-M_1}X^{\gamma_1}X^{\beta_{22}}b\left(gg_1,\lambda\right)\right\|_{op}\\
&+(|h|^{kM_1}+1)\|(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(gh^{-1}))^{-M_1}X_g^{\beta_{22}}b\left(gh^{-1},\lambda\right)\|_{op}.
\end{aligned}$$

Let $a_1(g,\lambda)=V(g)^{M_1-j}a(g,\lambda)$. Then $a_1\in S_{\mathcal{H}}^{m_1+2(M_1-j)}$ with kernel $\tilde{k}_g=V(g)K_1(g,\cdot)$. Thus $a_1=\pi(\tilde{k}_g)$. Choosing M,M_1 such that it satisfies (4.3.10) and the conditions of Lemma 5.5.6 in [34]. Therefore

$$\begin{aligned}
&\|\sigma(g,\lambda)-\sum_{[\alpha]\leq M}\Delta^{\alpha}a(g,\lambda)X_g^{\alpha}b(g,\lambda)\|_{op}\\
&\leq\sum_{j=0}^{M_1}\sum_{[\beta_{11}]+[\beta_{22}]\leq 2j}C_M\sum_{\substack{|\gamma|\leq(M-[\beta_{22}])+1\\|\gamma|>(M-[\beta_{22}])}}\int_{\mathbb{H}^n}(|h|^{[\gamma]}+|h|^{kM_1}+1)\left|\tilde{X}_h^{\beta_{11}}k_1(g,h)V(g)^{M_1-j}\right|dh\\
&\qquad\times\|b\|_{S_{\mathcal{H}}^{m_2},0,b_2,0}\\
&\leq C_2\|a_1\|_{S_{\mathcal{H}}^{m_1+2M_1,R_2},0,b_2,0}\times\|b\|_{S_{\mathcal{H}}^{m_2},0,b_2,0}\\
&\leq C\|a\|_{S_{\mathcal{H}}^{m_1,R},a_1,b_1}\times\|b\|_{S_{\mathcal{H}}^{m_2},0,b_2,0},
\end{aligned}$$

where $C = C_1 \|V\|_{S^{2M_2,R}_{\mathcal{H}},a_1,b_1}$. The general case $\beta_0 \neq 0$ follows by adopting the proof of Lemma 5.5.5 in [34]. $\square$

Proof of theorem 4.3.18. Let $T = Op(a) \circ Op(b)$. Then

$$Tf(g) = \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} f(h) k_2(g_1, h^{-1} g_1) k_1(g,h) dg_1 dh,$$

where k_1 and k_2 are the kernels of $Op(a)$ and $Op(b)$, respectively. Furthermore, we have $Op(a) \circ Op(b) = Op(\sigma)$, where

$$\sigma(g,\lambda) = \int_{\mathbb{H}^n} k_1(g,h) \pi_\lambda(h)^* b\left(gh^{-1}, \lambda\right) dh.$$

By the Taylor series expansion (see (4.3.1)) of b in the first variable, we get

$$\sigma(g,\lambda) \sim \sum_\alpha \Delta^\alpha a(g,\lambda) X_g^\alpha b(g,\lambda).$$

The reminder term is estimated similar to Theorem 5.5.3 of [34] with few modifications. We will only indicate the main steps with modifications in our setting. Let $m = m_1 + m_2$, $\beta_0 \in \mathbb{N}_0$ and $M_0 \in \mathbb{N}$. By Theorem 4.3.13, we have

$$\begin{aligned} &\left\| X_g^{\beta_0} \tau_M(g,\lambda) (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}) + V(g))^{-\frac{m-M_0}{2}} \right\|_{op} \\ &= \left\| X_g^{\beta_0} \tau_M(g,\lambda) (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}))^{-\frac{m-M_0}{2}} \left[(\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}))^{\frac{m-M_0}{2}} (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}) + V(g))^{-\frac{m-M_0}{2}} \right] \right\|_{op} \\ &\leq \left\| X_g^{\beta_0} \tau_M(g,\lambda) (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}))^{-\frac{m-M_0}{2}} \right\| \left\| (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}))^{\frac{m-M_0}{2}} (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}) + V(g))^{-\frac{m-M_0}{2}} \right\|_{op} \\ &\leq C \left\| X_g^{\beta_0} \tau_M(g,\lambda) (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}))^{-\frac{m-M_0}{2}} \right\|_{op}, \end{aligned} \tag{4.3.13}$$

where $\tau_M = a \circ b - \sum_{[\alpha] \leq M} \Delta^\alpha a X_g^\alpha b$. We fix $m_2' := -m_1 + M_0$. Then we can find $M \geq \max(M_0, v_1)$ such that $-m_1 + M - m_2' \geq 2$. This shows that we can find M_1 satisfying the second condition of (4.3.10) for m_1, m_2' and therefore also the first. Hence we can apply Lemma 4.3.19 to M, M_1 and the symbols a and $b(\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}))^{-\frac{m-M_0}{2}}$, with orders m_1 and m_2'. Thus by (4.3.13) and Theorem 4.3.13, we obtain

$$\begin{aligned} &\left\| X_g^{\beta_0} \tau_M(g,\lambda) (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}) + V(g))^{-\frac{m-M_0}{2}} \right\|_{op} \\ &\leq \|a\|_{S^{m_1,R}_{\mathcal{H}},a_1,b_1} \left\| b\, (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}))^{-\frac{m-M_0}{2}} \right\|_{S^{m_2'}_{\mathcal{H}},0,b_2,0} \\ &= \|a\|_{S^{m_1,R}_{\mathcal{H}},a_1,b_1} \left\| (\pi_\lambda(I - \mathcal{L}_{\mathbb{H}} + V(g)))^{-\frac{m_2'}{2}} b(\pi_\lambda(I - \mathcal{L}_{\mathbb{H}}))^{-\frac{m-M_0}{2}} \right\|_{op} \end{aligned}$$

$$\begin{aligned}
&= \|a\|_{S^{m_1,R}_{\mathcal{H}},a_1,b_1} \Big\| (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}+V(g)))^{-\frac{m_2'}{2}} b\ (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}+V(g)))^{-\frac{m-M_0}{2}} \\
&\quad \times (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}+V(g)))^{-\frac{m-M_0}{2}} (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}))^{-\frac{m-M_0}{2}} \Big\|_{op} \\
&\leq \|a\|_{S^{m_1,R}_{\mathcal{H}},a_1,b_1} \Big\| (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}+V(g)))^{-\frac{m_2'}{2}} b\ (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}+V(g)))^{-\frac{m-M_0}{2}} \Big\|_{op} \\
&\quad \times \Big\| (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}+V(g)))^{-\frac{m-M_0}{2}} (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}))^{-\frac{m-M_0}{2}} \Big\|_{op} \\
&\leq C \|a\|_{S^{m_1,R}_{\mathcal{H}},a_1,b_1} \Big\| b\ (\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}+V(g)))^{-\frac{m-M_0}{2}} \Big\|_{S^{m_2'}_{\mathcal{H}},0,b_2,0} \\
&\leq C \|a\|_{S^{m_1,R}_{\mathcal{H}},a_1,b_1} \|b\|_{S^{m_2}_{\mathcal{H}},0,b_2,c_2} .
\end{aligned}$$

The rest part of proof follows along the similar lines of Theorem 5.5.3 in [34]. □

4.4 Symbolic calculus relative to $(1+|\lambda|H+V(g)+|w|)$ on $\mathbb{H}^n$

Let $\Gamma \subset \mathbb{C}$ be a curve enclosing $\mathbb{R}^+$ and w vary over Γ. In particular, let us consider the curve Γ be made up of two half-lines hinged at -1 and makes an angles of $\pm\frac{\pi}{4}$ with respect to the real axis. In order to construct the pseudo-differential approximation to the operator $(\mathcal{H}+u)^{-m}$, we need to define the following symbol class.

4.4.1 The symbol class $S^m_{\rho,\delta,\mathcal{H},w}(\mathbb{H}^n)$

We define the symbol class $S^m_{\rho,\delta,\mathcal{H},w}(\mathbb{H}^n)$ relative to the operator $1+|\lambda|H+V(g)+|w|$ as in Subsection 4.3.4 and $(\lambda, V(g))$-Shubin class defined as in Section 4.3 relative to the weight $1+|\lambda|(|\xi|^2+|x|^2+1)+V(g)+|w|$. Also we get the similar result for the symbol class $S^m_{\rho,\delta,\mathcal{H},w}(\mathbb{H}^n)$ as in Theorem 4.3.10. When $\rho=1$ and $\delta=0$, we denote the symbol classes $S^m_{1,0,\mathcal{H},w}(\mathbb{H}^n)$ by $S^m_{\mathcal{H},w}(\mathbb{H}^n)$.

Proposition 4.4.1. *Let* $a_{g,\lambda,w}(\xi,u) = \left(|\lambda|(1+|\xi|^2+|u|^2)+V(g)-w\right)^s, s \in \mathbb{R}$, *and* $\sigma(g,\lambda,w) = Op^W(a_{g,\lambda,w})$. *Then* $\sigma \in S^{2s}_{\mathcal{H},w}(\mathbb{H}^n)$.

Proof. By Theorem 4.3.10, $Op^W(|\lambda|(1+|\xi|^2+|u|^2)+V(g)-w) \in S^2_{\mathcal{H},w}(\mathbb{H}^n)$. Now

$$\partial_\xi^\alpha \partial_u^\beta \tilde{\partial}^{\tilde{\alpha}}_{\lambda,\xi,u} X_g^{\tilde{\beta}} a_{g,\lambda,w}(\xi,u)$$

$$= \sum_{\substack{1\leq\theta\leq|\alpha|+|\beta|+|\tilde{\alpha}|+|\tilde{\beta}| \\ |\mu_1|+\cdots+|\mu_\theta|=|\alpha| \\ |\nu_1|+\cdots+|\nu_\theta|=|\beta| \\ |\tilde{\mu_1}|+\cdots+|\tilde{\mu_\theta}|=|\tilde{\alpha}| \\ |\tilde{\nu_1}|+\cdots+|\tilde{\nu_\theta}|=|\tilde{\beta}|}} \left(|\lambda|(1+|\xi|^2+|u|^2)+V(g)-w\right)^{s-\theta}$$
$$\times \prod_{j=1}^{\theta} \partial_\xi^{\mu_j}\partial_u^{\nu_j}\tilde{\partial}_{\lambda,\xi,u}^{\tilde{\mu_j}}X_g^{\tilde{\nu_j}}\left(|\lambda|(1+|\xi|^2+|u|^2)+V(g)-w\right).$$

Since each term is bounded by a constant times

$$\begin{aligned}&\left(|\lambda|(1+|\xi|^2+|u|^2)+V(g)-w\right)^{s-\theta}\prod_{j=1}^{\theta}|\lambda|^{\frac{|\mu_j|+|\nu_j|}{2}} \\ &\quad\times\left(1+|\lambda|(1+|\xi|^2+|u|^2)+V(g)+|w|\right)^{\frac{2-2|\tilde{\mu}_j|-(|\mu_j|+|\nu_j|)}{2}} \\ &\leq |\lambda|^{\frac{|\alpha|+|\beta|}{2}}\left(1+|\lambda|(1+|\xi|^2+|u|^2)+V(g)+|w|\right)^{\frac{-2-2|\tilde{\alpha}|-(|\alpha|+|\beta|)}{2}},\end{aligned}$$

thus for any $w\in\Gamma$, we have

$$\left|\partial_\xi^\alpha\partial_u^\beta\tilde{\partial}_{\lambda,\xi,u}^{\tilde{\alpha}}X_g^{\tilde{\beta}}a_{g,\lambda,w}(\xi,u)\right|\leq C|\lambda|^{\frac{|\alpha|+|\beta|}{2}}\left(1+|\lambda|(1+|\xi|^2+|u|^2)+V(g)+|w|\right)^{\frac{-2-2|\tilde{\alpha}|-(|\alpha|+|\beta|)}{2}}.$$

Now

$$\begin{aligned}&\left\|\tilde{\partial}_{\lambda,\xi,u}^{\tilde{\alpha}}X_g^{\tilde{\beta}}a_{g,\lambda,w}\right\|_{\Sigma_{\rho,\lambda,V(g)}^{-2-2\tilde{\alpha}}(\mathbb{R}^n),N} \\ &= \sup_{\substack{(\xi,u)\in\mathbb{R}^n\times\mathbb{R}^n \\ |\alpha|,|\beta|\leq N}} |\lambda|^{-\frac{|\alpha|+|\beta|}{2}}\left(1+|\lambda|\left(1+|\xi|^2+|u|^2\right)+V(g)+w\right)^{-\frac{-2-2|\tilde{\alpha}|-(|\alpha|+|\beta|)}{2}} \\ &\qquad\times\left|\partial_\xi^\alpha\partial_u^\beta\tilde{\partial}_{\lambda,\xi,u}^{\tilde{\alpha}}X_g^{\tilde{\beta}}a_{g,\lambda,w}(\xi,u)\right|\leq C_{\tilde{\alpha},\tilde{\beta},N}.\end{aligned}$$

Thus $\sigma=\sigma(g,\lambda,w)=Op^W(a_{g,\lambda,w})\in S_{\mathcal{H},w}^{2s}(\mathbb{H}^n)$ by (4.3.5). □

Construct a symbol $R_N(g,\lambda,w)$ such that $(\mathcal{H}-w)\circ Op(R_N(g,\lambda,w))=I_{L^2(\mathbb{H}^n)}+Op(S_N(g,\lambda,w))$, where $S_N\in S_{\mathcal{H},w}^{-N}(\mathbb{H}^n)$ or equivalently $(|\lambda|(H+I)+V(g)-w)\#_{\mathbb{H}^n}R_N(g,\lambda,w)=I_{L^2(\mathbb{R}^n)}+S_N(g,\lambda,w)$. By substituting the expansion $R_N=R_{-2}+R_{-3}+\cdots+R_{-N}$ with the property that $R_{-2-\ell}\in S_{\mathcal{H},w}^{-2-\ell}(\mathbb{H}^n)$ into the asymptotic expansion (4.3.9), we get

$$\begin{aligned}&(|\lambda|(H+I)+V(g)-w)\#_{\mathbb{H}^n}R_N(g,\lambda,w) \\ &=\sum_{[\alpha]\leq N}\Delta^\alpha(|\lambda|(H+I)+V(g)-w)\;X_g^\alpha R_N(g,\lambda,w)+S_N(g,\lambda,w) \\ &=I_{L^2(\mathbb{R}^n)}+S_N(g,\lambda,w).\end{aligned}$$

Now solving for $R_{-2-\ell}$ recursively by comparing the order by order of the symbols so that the sum equals to 1, we get

$$R_{-2}(g,\lambda,w)=(|\lambda|(H+I)+V(g)-w)^{-1} \tag{4.4.1}$$

and

$$\begin{aligned}R_{-2-\ell}(g,\lambda,w)&=(|\lambda|(H+I)+V(g)-w)^{-1}\\ &\quad\times\sum_{\substack{|k|+|j|=\ell\\|k|<\ell}}\Delta^j(|\lambda|(H+I)+V(g)-w)\ X_g^jR_{-2-|k|}(g,\lambda,w)\end{aligned} \tag{4.4.2}$$

for $w\in\Gamma$. To understand the dependence on w, we express the symbol $R_{-2-\ell}$ differently in the following proposition.

Proposition 4.4.2. *Let $w\in\Gamma$. Then*

$$\begin{aligned}R_{-2-\ell}(g,\lambda,w)&=(|\lambda|(H+I)+V(g)-w)^{-1}\\ &\quad\times\sum_{[\frac{\ell}{2}]\le M\le\ell}R_{\ell,M}(g,\lambda)(|\lambda|(H+I)+V(g)-w)^{-M},\end{aligned} \tag{4.4.3}$$

where $[\frac{\ell}{2}]$ denotes the least integer grater than $\frac{\ell}{2}$ and $R_{\ell,M}(g,\lambda)\in S^{2M-\ell}_{\mathcal{H},w}(\mathbb{H}^n)$ is a polynomial in $\pi(X)$ and $X_g^\alpha V, |\alpha|\le\ell$.

Proof. We prove the proposition by induction on ℓ. When $\ell=0$, the expression is trivial from (4.4.1). Assume that the expression (4.4.3) holds for $k\le\ell-1$. From (4.4.2), the difference operator Δ contributes only some possible factors of $\pi(X)$ but no w. However, the differential operator X_g either acts on $(|\lambda|(H+I)+V(g)-w)^{-M}$ or $R_{\ell,M}$ (after substituting (4.4.3) for $k\le\ell-1$ in (4.4.2)) resulting the expressions as in (4.4.3). It is easy to check that each term in the sum for $R_{-2-\ell}$ lies in $S^{-2-\ell}_{\mathcal{H},w}(\mathbb{H}^n)$ after expanding by Leibniz rule. Since $(|\lambda|(H+I)+V(g)-w)^{-1}R_{\ell,M}(g,\lambda)(|\lambda|(H+I)+V(g)-w)^{-M}\in S^{-2-\ell}_{\mathcal{H},w}(\mathbb{H}^n)$, this implies that $R_{\ell,M}\in S^{2M-\ell}_{\mathcal{H},w}(\mathbb{H}^n)$. So $R_{\ell,M}$ is a polynomial in $\pi(X)$ and $X_g^\alpha V$. The highest power of $(|\lambda|(H+I)+V(g)-w)^{-1}$ in the right comes out when we throw all derivatives on factors of $(|\lambda|(H+I)+V(g)-w)^{-1}$ and count this number which is essentially ℓ.

□

4.4.2 Approximation of symbols

Let f be a holomorphic function. Then by the holomorphic functional calculus for unbounded operators, all pseudo-differential approximations can be written in the following way:

$$f(\mathcal{H}) = \frac{1}{2\pi i}\int_\Gamma f(w)(\mathcal{H}-w)^{-1}\,dw$$

and define the pseudo-differential operator $f_N(\mathcal{H}) = \frac{1}{2\pi i}\int_\Gamma f(w)Op(R_N(g,\lambda,w))\,dw$ with symbol

$$f_N(g,\lambda) = \sum_{\ell=0}^{N}\sum_{[\frac{\ell}{2}]\le M\le \ell}\left(\frac{1}{2\pi i}\int_\Gamma f(w)(|\lambda|(H+I)+V(g)-w)^{-1}R_{\ell,M}(g,\lambda)\right.$$
$$\left.\times\,(|\lambda|(H+I)+V(g)-w)^{-M}\,dw\right)$$

by formally computing the residue. However, the error term in the approximation of $f(\mathcal{H})$ is given by

$$\frac{1}{2\pi i}\int_\Gamma f(w)(\mathcal{H}-w)^{-1}Op(S_N(g,\lambda,w))\,dw. \tag{4.4.4}$$

In particular, letting $f(w) = (w+u)^s$ for some fixed $u>0$, the pseudo-differential approximation to $(\mathcal{H}+u)^s$ is $Op(\sigma_{s,N}(g,\lambda))$, where

$$\sigma_{s,N}(g,\lambda) = \sum_{\ell=0}^{N}\sum_{[\frac{\ell}{2}]\le M\le \ell}\left(\frac{1}{2\pi i}\int_\Gamma (w+u)^s(|\lambda|(H+I)+V(g)-w)^{-1}R_{\ell,M}(g,\lambda)\right.$$
$$\left.\times\,(|\lambda|(H+I)+V(g)-w)^{-M}\,dw\right).$$

Let A be a linear operator on $L^2(\mathbb{H}^n)$. Observe that $|\operatorname{Tr}(A)| \le \|(I+\mathcal{H})^s A\||\operatorname{Tr}(I+\mathcal{H})^{-s}|$. If $\|(I+\mathcal{H})^s A\|$ is finite and $(I+\mathcal{H})^{-s}$ is also a trace class operator then A is trace class on $L^2(\mathbb{H}^n)$. From the above discussion, we can write $(I+\mathcal{H})^{-\frac{s}{2}} = Op((1+|\lambda|H+V(g))^{-\frac{s}{2}})+Op(F_{\frac{s}{2}}(g,\lambda))$. So $(I+\mathcal{H})^{-s}$ is a trace class operator when $(I+\mathcal{H})^{-\frac{s}{2}}$ is Hilbert-Schmidt operator. That means, if $Op((1+|\lambda|H+V(g))^{-\frac{s}{2}})$ and $Op(F_{\frac{s}{2}}(g,\lambda))$ are Hilbert-Schmidt operators or equivalently $(1+|\lambda|H+V(g))^{-\frac{s}{2}}, F_{\frac{s}{2}}(g,\lambda) \in L^2\left(\mathbb{H}^n\times\mathbb{R}^*, S_2, d\mu(\lambda)\right)$, $(I+\mathcal{H})^{-s}$ is a trace class operator. By generalized Minkowski's inequality, we have

$$\|(1+|\lambda|H+V(g))^{-\frac{s}{2}}\|_{L^2(\mathbb{H}^n\times\mathbb{R}^*,S_2,d\mu(\lambda))}$$

$$
\begin{aligned}
&= \left(\int_{\mathbb{H}^n}\int_{\mathbb{R}^*} |\operatorname{Tr}(1+|\lambda|H+V(g))^{-\frac{s}{2}}|^2\, dg\, d\mu(\lambda)\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{H}^n}\int_{\mathbb{R}^*} \left|\sum_{\alpha} \frac{1}{(1+|\lambda|(2|\alpha|+n)+V(g))^{\frac{s}{2}}}\right|^2 dg\, d\mu(\lambda)\right)^{\frac{1}{2}} \\
&\leq C\sum_{\alpha}\left(\int_{\mathbb{H}^n}\int_{\mathbb{R}^*} \frac{\lambda^n}{(1+|\lambda|(2|\alpha|+n)+V(g))^{s}}\, dg\, d\lambda\right)^{\frac{1}{2}} \\
&= C\sum_{\alpha} \frac{1}{(2|\alpha|+n)^{\frac{n+1}{2}}}\left(\int_{\mathbb{H}^n}\int_{1+V(g)}^{\infty} \frac{(u-1-V(g))^n}{u^s}\, du\, dg\right)^{\frac{1}{2}} \\
&= C\sum_{\alpha} \frac{1}{(2|\alpha|+n)^{\frac{n+1}{2}}}\left(\int_{\mathbb{H}^n} \frac{1}{(1+V(g))^{s-n-1}}\, dg\right)^{\frac{1}{2}}. \qquad (4.4.5)
\end{aligned}
$$

Under the assumption $V(g) \sim V_0|g|^k$ as $|g| \to \infty$, the function $\frac{1}{(1+V(g))^{s-n-1}}$ is integrable if we choose $(s-n-1)k > 1$. A similar argument gives $F_{\frac{s}{2}}$ is also a Hilbert-Schmidt operator for large N. Indeed $(\mathcal{H}-w)^{-1} = Op(R_N)+(\mathcal{H}-w)^{-1}Op(S_N(g,\lambda,w))$ is compact and hence has discrete spectrum.

Proposition 4.4.3. *Let $u > 0$ and $m \in \mathbb{N}$ be sufficiently large such that $(\mathcal{H}+u)^{-m}$ is a trace class operator on $L^2(\mathbb{H}^n)$. Then for such m, we have*

$$(\mathcal{H}+u)^{-m} = Op\left((|\lambda|(H+I)+V(g)+u)^{-m}\right) + Op(E(g,\lambda,u))$$

such that

$$
\begin{aligned}
&\left|\operatorname{Tr}(\mathcal{H}+u)^{-m} - \operatorname{Tr}\left(Op\left((|\lambda|(H+I)+V(g)+u)^{-m}\right)\right)\right| = |\operatorname{Tr}(Op(E(g,\lambda,u)))| \\
&\leq \psi_1(u)\left|\operatorname{Tr}\left(Op\left((|\lambda|(H+I)+V(g)+u)^{-m}\right)\right)\right|
\end{aligned}
$$

with $\psi_1(u) \to 0$ as $u \to \infty$.

Proof. From the discussions in the previous subsections, we write

$$(\mathcal{H}+u)^{-m} = Op\left((|\lambda|(H+I)+V(g)+u)^{-m}\right) + Op(E(g,\lambda,u)), \qquad (4.4.6)$$

where

$$
\begin{aligned}
&E(g,\lambda,u) \\
&= \sum_{\ell=1}^{N}\sum_{[\frac{\ell}{2}]\leq M\leq \ell} \frac{1}{2\pi i}\int_{\Gamma}(w+u)^{-m}(|\lambda|(H+I)+V(g)-w)^{-1}R_{\ell,M}(g,\lambda)
\end{aligned}
$$

$$\times\,(|\lambda|(H+I)+V(g)-w)^{-M}\,dw+\frac{1}{2\pi i}\int_{\Gamma}(w+u)^{-m}(\mathcal{H}-w)^{-1}Op(S_N(g,\lambda,w))\,dw.$$

For large N, choose $0<s<N$ such that $(I+\mathcal{H})^{-\frac{s}{2}}$ is a trace class operator. Then $Op(S_N(g,\lambda,w))$ is a trace class operator with

$$\begin{aligned}|\mathrm{Tr}(Op(S_N(g,\lambda,w)))| &\leq \left|\mathrm{Tr}\left((I+\mathcal{H})^{-\frac{s}{2}}(I+\mathcal{H})^{\frac{s}{2}}Op(S_N(g,\lambda,w))\right)\right|\\ &\leq \left|\mathrm{Tr}\left((I+\mathcal{H})^{-\frac{s}{2}}\right)\right|\left\|(I+\mathcal{H})^{\frac{s}{2}}Op(S_N(g,\lambda,w))\right\|. \end{aligned}\tag{4.4.7}$$

But from Theorem 4.3.13, we have

$$\begin{aligned}&\left\|(I+\mathcal{H})^{\frac{s}{2}}Op(S_N(g,\lambda,w))\right\|\\ &=\left\|(I+\mathcal{H})^{\frac{s}{2}}(\mathcal{H}+|w|)^{-\frac{N}{2}}(\mathcal{H}+|w|)^{\frac{N}{2}}Op(S_N(g,\lambda,w))\right\|\\ &=\left\|(I+\mathcal{H})^{\frac{s}{2}}(\mathcal{H}+|w|)^{-\frac{N}{2}}\right\|\left\|(\mathcal{H}+|w|)^{\frac{N}{2}}Op(S_N(g,\lambda,w))\right\|\\ &\leq C\left\|(I+\mathcal{H})^{\frac{s}{2}}(\mathcal{H}+|w|)^{-\frac{s}{2}}(\mathcal{H}+|w|)^{\frac{s}{2}}(\mathcal{H}+|w|)^{-\frac{N}{2}}\right\|\\ &\leq C\left\|(I+\mathcal{H})^{\frac{s}{2}}(\mathcal{H}+|w|)^{-\frac{s}{2}}\right\|\left\|(\mathcal{H}+|w|)^{\frac{(s-N)}{2}}\right\|\\ &=O\left(|w|^{\frac{(s-N)}{2}}\right).\end{aligned}\tag{4.4.8}$$

Therefore, from (4.4.7) and (4.4.8), we obtain

$$\left|\mathrm{Tr}\left(\int_{\Gamma}(w+u)^{-m}(\mathcal{H}-w)^{-1}Op(S_N(g,\lambda,w))\,dw\right)\right|\leq Cu^{1-m}\to 0 \text{ as } u\to\infty.$$

Consequently this part of the error is negligible and the pseudo-differential part of $E(g,\lambda,u)$ is a trace class operator because it has smooth rapidly decaying symbol. By Lemma 4.3.12, each term of $\mathrm{Tr}(Op(E(g,\lambda,u)))$ is of the form

$$\begin{aligned}&\mathrm{Tr}\left[Op\left(\frac{1}{2\pi i}\int_{\Gamma}(w+u)^{-m}(|\lambda|(H+I)+V(g)-w)^{-1}R_{\ell,M}(g,\lambda)\right.\right.\\ &\qquad\left.\left.\times\,(|\lambda|(H+I)+V(g)-w)^{-M}dw\right)\right]\\ &=\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\mathrm{Tr}\left[\left(\frac{1}{2\pi i}\int_{\Gamma}(w+u)^{-m}(|\lambda|(H+I)+V(g)-w)^{-1-M}dw\right)R_{\ell,M}(g,\lambda)\right]dg\,d\mu(\lambda)\\ &=C_{m,M}\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\mathrm{Tr}\left((|\lambda|(H+I)+V(g)+u)^{-m-M}R_{\ell,M}(g,\lambda)\right)dg\,d\mu(\lambda)\\ &=C_{m,M}u^{-m-M}\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\mathrm{Tr}\left((u^{-1}|\lambda|(H+I)+u^{-1}V(g)+1)^{-m-M}R_{\ell,M}(g,\lambda)\right)dg\,d\mu(\lambda)\\ &\sim C_{m,M}u^{-m-M+n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}}\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\mathrm{Tr}\left((|\lambda|(H+I)+|g|^{\kappa}+1)^{-m-M}R_{\ell,M}(\tilde{g},u\lambda)\right)|\lambda|^n\,dg\,d\lambda,\end{aligned}$$

where $\widetilde{g} = (u^{\frac{1}{\kappa}}x_1, u^{\frac{1}{\kappa}}x_2, \cdots, u^{\frac{1}{\kappa}}x_{2n}, u^{\frac{2}{\kappa}}t)$. Since $R_{\ell,M} \in S^{2M-\ell}_{\mathcal{H},w}(\mathbb{H}^n)$,

$$\left\| R_{\ell,M}(\tilde{g}, u\lambda)(u|\lambda|(H+I)+u|\tilde{g}|^{k}+1)^{\frac{-2M+\ell}{2}} \right\|_{op}$$

is uniformly bounded and so

$$\begin{aligned}
&\left|\mathrm{Tr}\left(Op\left((|\lambda|(H+I)+V(g)+u)^{-m-M}R_{\ell,M}(g,\lambda)\right)\right)\right| \\
&\sim C\frac{u^{n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}}}{u^{m+M}}\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\left|\mathrm{Tr}\Big((u|\lambda|(H+I)+u|g|^{\kappa}+1)^{\frac{2M-\ell}{2}}\right. \\
&\qquad \left.\times(|\lambda|(H+I)+|g|^{\kappa}+1)^{-m-M}\Big)\right||\lambda|^n dg\,d\lambda \qquad (4.4.9) \\
&\leq Cu^{-m-M+n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}}\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\sum_k\frac{(u|\lambda|(2|k|+n+1)+u|g|^{\kappa}+1)^{\frac{2M-\ell}{2}}}{(|\lambda|(2|k|+n+1)+|g|^{\kappa}+1)^{m+M}}|\lambda|^n\ dg\,d\lambda \\
&\leq Cu^{-m-M+n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}+\frac{2M-\ell}{2}}\sum_k\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\frac{(|\lambda|(2|k|+n+1)+|g|^{\kappa}+1)^{\frac{2M-\ell}{2}}}{(|\lambda|(2|k|+n+1)+|g|^{\kappa}+1)^{m+M}}|\lambda|^n\ dg\,d\lambda \\
&\leq Cu^{-m+n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}-\frac{\ell}{2}}\sum_k\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\frac{|\lambda|^n}{(|\lambda|(2|k|+n+1)+|g|^{\kappa}+1)^{m+\frac{\ell}{2}}}\ dg\,d\lambda \\
&\leq Cu^{-m+n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}-\frac{\ell}{2}}\sum_k\int_{\mathbb{R}^*}\frac{|\lambda|^n\,d\lambda}{(|\lambda|(2|k|+n+1)+1)^{\frac{m}{2}+\frac{\ell}{4}}}\int_{\mathbb{H}^n}\frac{dg}{(|g|^{\kappa}+1)^{\frac{m}{2}+\frac{\ell}{4}}} \\
&\leq Cu^{-m+n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}-\frac{\ell}{2}}\sum_k\frac{1}{(2|k|+n+1)^{n+1}} \\
&\approx u^{-m+n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}-\frac{\ell}{2}}. \qquad (4.4.10)
\end{aligned}$$

Similarly, we have

$$\left|\mathrm{Tr}\left(Op\left((|\lambda|(H+I)+V(g)+u)^{-m}\right)\right)\right| \approx u^{-m+n+1+\frac{2n}{\kappa}+\frac{2}{\kappa}}. \qquad (4.4.11)$$

Thus applying trace in (4.4.6) and using (4.4.9), (4.4.11), we get

$$\begin{aligned}
\left|\frac{\mathrm{Tr}(Op(E(g,\lambda,u)))}{\mathrm{Tr}\left(Op\left((|\lambda|(H+I)+V(g)+u)^{-m}\right)\right)}\right| &= \left|\frac{\mathrm{Tr}\left((\mathcal{H}+u)^{-m}\right)}{\mathrm{Tr}\left(Op\left((|\lambda|(H+I)+V(g)+u)^{-m}\right)\right)}-1\right| \\
&\leq C\psi_1(u)\to 0 \text{ as } u\to\infty, \qquad (4.4.12)
\end{aligned}$$

where $\psi_1(u) = \frac{1}{u^{m-1}} + \sum_{\ell=1}^{N} u^{-\frac{\ell}{2}}$. Note that when $\ell = 0, M = 0$ and $R_{\ell,M} = 1$, (4.4.9), (4.4.11) has same decay. If $\ell \geq 1$ then (4.4.12) also holds. ☐

Let w be the complex number varying over the curve Γ (defined in Section 4.4). For fixed $(g,\lambda) \in \mathbb{H}^n \times \mathbb{R}^*$, the class $S^m_{w,g,\lambda}(\mathbb{R}^n)$ defined as

$S^m_{w,g,\lambda}(\mathbb{R}^n)$

$$= \left\{ a_{g,\lambda} \in C^\infty(\mathbb{R}^{2n} \times \Gamma) : |\partial_\xi^\alpha \partial_x^\beta a_{g,\lambda}(x,\xi)| \leq C_{\alpha,\beta}(1 + |\lambda|(1+|\xi|^2+|x|^2) + V(g) + |w|)^{\frac{m-|\alpha|}{2}} \right\}. \tag{4.4.13}$$

We obtain the following result as in Proposition 4.4.3.

Proposition 4.4.4. *Let $m > 0$ be a sufficiently large such that $\big(|\lambda|(H+I) + V(g)+u\big)^{-m}$ is in trace class. Then for a fixed $(g,\lambda) \in \mathbb{H}^n \times \mathbb{R}^*$, we have*

$$\big(|\lambda|(H+I) + V(g)+u\big)^{-m} = Op^W\left(\big(|\lambda|(1+|\xi|^2+|x|^2) + V(g) + u\big)^{-m}\right) + Op^W(E_{g,\lambda}(u)),$$

where

$$\begin{aligned}
&\left|\operatorname{Tr}\big(|\lambda|(H+I) + V(g) + u\big)^{-m} - \operatorname{Tr}\left(Op^W\left(\big(|\lambda|(1+|\xi|^2+|x|^2) + V(g) + u\big)^{-m}\right)\right)\right| \\
&= |\operatorname{Tr}(Op^W(E_{g,\lambda}(u)))| \\
&\leq \psi_2(u) \left|\operatorname{Tr}\left(Op^W\left(\big(|\lambda|(1+|\xi|^2+|x|^2) + V(g) + u\big)^{-m}\right)\right)\right|
\end{aligned}$$

with $\psi_2(u) \to 0$ as $u \to \infty$.

Proof. The proof is based on the similar idea as in Proposition 4.4.3. For fixed $(g,\lambda) \in \mathbb{H}^n \times \mathbb{R}^*$, there exists $m \in \mathbb{N}$ such that $\big(|\lambda|(H+I) + V(g)+u\big)^{-m}$ is a trace class operator on $L^2(\mathbb{R}^n)$. We refer to page 72-75 of [109] for similar pseudo-differential approximation to $(|\lambda|(H+I) + V(g) + u)^{-m}$ on $L^2(\mathbb{R}^n)$. However, we will only indicate some intermediate steps. Now

$$\big(|\lambda|(H+I) + V(g) + u\big)^{-m} = Op^W\left(\big(|\lambda|(1+|\xi|^2+|x|^2) + V(g) + u\big)^{-m}\right) + Op^W(E_{g,\lambda}(u)), \tag{4.4.14}$$

where

$$\begin{aligned}
E_{g,\lambda}(u) = &\sum_{\ell=1}^{N} \sum_{[\frac{\ell}{2}] \leq M \leq \ell} \frac{\Gamma(s+1)}{\Gamma(s-M)\Gamma(M+1)} R_{\ell,M}^{(g,\lambda)}(\xi,x)\big((|\lambda|(1+|\xi|^2+|x|^2) + V(g) + u\big)^{-m-M} \\
&+ \frac{1}{2\pi i}\int_\Gamma (w+u)^{-m} S_N^{(g,\lambda)}(w)(|\lambda|(H+I) + V(g) - w)^{-1}\, dw
\end{aligned}$$

with $S_N^{(g,\lambda)} \in S_{w,g,\lambda}^{-N}(\mathbb{R}^n)$ and $R_{\ell,M}^{(g,\lambda)} \in S_{w,g,\lambda}^{2\ell-M}(\mathbb{R}^n)$. Let $0 < s < N$ and $(I + |\lambda|(H+I) + V(g))^{-\frac{s}{2}}$ is a trace class operator on $L^2(\mathbb{R}^n)$. Then imitating the similar calculations in page 75 of [109], we have

$$\left|\operatorname{Tr}\left(\int_\Gamma (w+u)^{-m} Op^W(S_N^{(g,\lambda)}(w))(|\lambda|(H+I) + V(g) - w)^{-1}\, dw\right)\right| \to 0 \text{ as } u \to \infty,$$

and

$$\begin{aligned}
&\left|\mathrm{Tr}\left(Op^W\left(R^{(g,\lambda)}_{\ell,M}(\xi,x)\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\right)^{-m-M}\right)\right)\right| \\
&= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} R^{(g,\lambda)}_{\ell,M}(\xi,x)\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\right)^{-m-M}\, dx\, d\xi\right| \\
&\leq u^{-m-M}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|R^{(g,\lambda)}_{\ell,M}(\xi,x)\right|\left(u^{-1}|\lambda|(1+|\xi|^2+|x|^2)+u^{-1}V(g)+1\right)^{-m-M}\, dx\, d\xi \\
&\leq u^{-m-M}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|R^{(g,\lambda)}_{\ell,M}(\xi,x)\right|\left(u^{-1}|\lambda|(|\xi|^2+|x|^2)+1\right)^{-m-M}\, dx\, d\xi \\
&= u^{-m-M+n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|R^{(g,\lambda)}_{\ell,M}(u^{\frac{1}{2}}\xi,u^{\frac{1}{2}}x)\right|\left(|\lambda|(|\xi|^2+|x|^2)+1\right)^{-m-M}\, dx\, d\xi \\
&\leq u^{-m-M+n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|(1+|\lambda|(1+u|\xi|^2+u|x|^2)+V(g)+|w|)^{\frac{2M-\ell}{2}}\right| \\
&\qquad \times\left(|\lambda|(|\xi|^2+|x|^2)+1\right)^{-m-M}\, dx\, d\xi \\
&\leq Cu^{-m+n-\frac{\ell}{2}}.
\end{aligned}$$

Note that if $\ell = 1$ and $M = 1$, $R^{(g,\lambda)}_{1,1}(x,\xi) = 1$. So for $\ell \geq 2$, we have

$$\left|\mathrm{Tr}\left(Op^W\left(R^{(g,\lambda)}_{\ell,M}(\xi,x)\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\right)^{-m-M}\right)\right)\right| \approx u^{-m+n-1}.$$

Similarly, we have

$$\left|\mathrm{Tr}\left(Op^W\left(\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\right)^{-m-M}\right)\right)\right| \approx u^{-m+n}.$$

Thus

$$\begin{aligned}
&\left|\frac{\mathrm{Tr}(Op^W(E_{g,\lambda}(u)))}{\mathrm{Tr}\left(Op^W\left(\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\right)^{-m}\right)\right)}\right| \\
&= \left|\frac{\mathrm{Tr}\left(Op\left((|\lambda|(H+I)+V(g)+u)^{-m}\right)\right)}{\mathrm{Tr}\left(Op^W\left(\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\right)^{-m}\right)\right)}-1\right| \leq \psi_2(u) \to 0 \text{ as } u \to \infty,
\end{aligned}$$

where $\psi_2(u) = \frac{1}{u^{m-1}} + \sum_{\ell=1}^{N} u^{-1}$. ☐

Remark 4.4.5. *Note that for sufficiently large $m \in \mathbb{N}$, the operator*

$$Op\left(Op^W\left(R^{(g,\lambda)}_{\ell,M}(\xi,x)\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\right)^{-m-M}\right)\right)$$

is a trace class operator on $L^2(\mathbb{H}^n)$, since from Proposition 4.4.4, we have

$$\left|\mathrm{Tr}\left(Op\left(Op^W\left(R^{(g,\lambda)}_{\ell,M}(\xi,x)\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\right)^{-m-M}\right)\right)\right)\right|$$

$$
\begin{aligned}
&\leq \int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\left|\operatorname{Tr}\left(Op^W\left(R^{(g,\lambda)}_{\ell,M}(\xi,x)\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+1\right)^{-m-M}\right)\right)\right| dg\, d\mu(\lambda)\\
&\leq \int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\int_{\mathbb{R}^{2n}}\left|(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+1)^{\frac{2M-\ell}{2}}\right|\\
&\qquad \times\left(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+1\right)^{-m-M}\, dx\, d\xi\, dg\, d\mu(\lambda)<\infty.\\
&\leq \int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\int_{\mathbb{R}^{2n}}(1+|\lambda|(1+|\xi|^2+|x|^2)+V(g))^{\frac{-m-\ell}{2}}\, dx\, d\xi\, dg\, d\mu(\lambda)\\
&\leq C\int_{\mathbb{R}^{2n}}\int_{\mathbb{H}^n}\int_{\mathbb{R}^*}\frac{\lambda^n}{(1+|\lambda|(1+|\xi|^2+|x|^2)+V(g)+1)^{\frac{m+\ell}{2}}}\, dx\, d\xi\, dg\, d\lambda\\
&= C\int_{\mathbb{R}^{2n}}\frac{1}{(1+|\xi|^2+|x|^2)^{n+1}}\, dx\, d\xi\int_{\mathbb{H}^n}\int_{1+V(g)}^{\infty}\frac{(u-1-V(g))^n}{u^{\frac{m+\ell}{2}}}\, du\, dg\\
&= C\int_{\mathbb{R}^{2n}}\frac{1}{(1+|\xi|^2+|x|^2)^{n+1}}\, dx\, d\xi\int_{\mathbb{H}^n}\frac{1}{(1+V(g))^{\frac{m+\ell}{2}-n-1}}\, dg<\infty.
\end{aligned}
$$

Similarly it can be shown that $Op\left(Op^W\left((|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u)^{-m}\right)\right)$ *is a trace class operator on* $L^2(\mathbb{H}^n)$ *for sufficiently large* $m\in\mathbb{N}$.

We take the positive integer m such that m-th power of the operators discussed earlier is a trace class operator.

4.5 Szegö type limit theorems for $\mathcal{H}$

In this section we aim to prove Szegö type limit theorems for $\mathcal{H}$ (defined on Subsection 4.2.2). Let $A=Op(Op^W(a_{g,\lambda}))$ be a 0-th order self-adjoint pseudo-differential operator on $L^2(\mathbb{H}^n)$ relative to the operator $1+|\lambda|H+V(g)$ (defined in Subsection 4.3.4) and $\mathcal{P}_r$ be the orthogonal projection of $L^2(\mathbb{H}^n)$ onto the space of eigenfunctions of $\mathcal{H}$ with eigenvalue $\leq r$. For each $r>0$, $\mathcal{P}_rA\mathcal{P}_r$ is a finite rank symmetric operator with spectral measure defined as the sum of Dirac delta functions at its eigen values. We show (in Theorem 4.5.4) that the sequence of measures $\frac{\operatorname{Tr} f(\mathcal{P}_rA\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}$ converges to the weak limit $\frac{\int_{G^r} f(a_{g,\lambda}(\xi,x))\, d\xi\, dx\, dg\, d\mu(\lambda)}{\int_{G^r} d\xi\, dx\, dg\, d\mu(\lambda)}$, where $G^r=\{(g,\lambda,\xi,x)\in\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n:|\lambda|(1+|\xi|^2+|x|^2)+V(g)\leq r\}$ and $\mu(\lambda)$ is the Plancherel measure on the Heisenberg group. In particular, if $\mathbf{b}$ is a bounded real valued integrable function on $\mathbb{H}^n$ then we obtain the following result with respect to the operator of multiplication $M_{\mathbf{b}}$:

Theorem 4.5.1. *Consider the Schrödinger operator of the form* $\mathcal{H}=-\Delta_{\mathbb{H}}+V$ *on the Heisenberg group* $\mathbb{H}^n$. *Let* $\mathcal{P}_r$ *be the orthogonal projection of* $L^2(\mathbb{H}^n)$ *onto the space of eigenfunctions of* $\mathcal{H}$ *with eigenvalue* $\leq r$. *Let* $\boldsymbol{b}$ *be a bounded real valued integrable function* *on $\mathbb{H}^n$ and $M_{\boldsymbol{b}}$ be the operator of multiplication by $\boldsymbol{b}$ on $L^2(\mathbb{H}^n)$. Then for any $f \in C(\mathbb{R})$, the space of all continuous function defined on $\mathbb{R}$, we have*

$$\lim_{r\to\infty} \frac{\mathrm{Tr} f(\mathcal{P}_r M_{\boldsymbol{b}} \mathcal{P}_r)}{\mathrm{Tr}(\mathcal{P}_r)} = \int_{\mathbb{H}^n} f(\boldsymbol{b}(g))\, dg.$$

In order to prove Theorem 4.5.1, we need the following lemmas.

Lemma 4.5.2. *Let $M_{\boldsymbol{b}}$ be the multiplication operator defined in Theorem 4.5.1, then* $\mathrm{Tr} f(\mathcal{P}_r M_{\boldsymbol{b}} \mathcal{P}_r) = \mathrm{Tr}(\mathcal{P}_r f(M_{\boldsymbol{b}})\mathcal{P}_r)$ *for any $f \in C(\mathbb{R})$.*

Proof. Notice that $\|(I-\mathcal{P}_r)M_{\mathbf{b}}\mathcal{P}_r\|_{S_2}^2 = \mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}}\mathcal{P}_r) = \mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}}^2 \mathcal{P}_r) - \mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}} \mathcal{P}_r)^2$. Also $\mathcal{P}_r M_{\mathbf{b}}^2 \mathcal{P}_r$ is an operators on $L^2(\mathbb{H}^n)$ with kernel $K_1(g, g_1) = \sum\limits_{k_1,k_2 \le r} \langle \mathbf{b}^2 e_{k_1}, e_{k_2}\rangle e_{k_2}(g)\overline{e_{k_1}(g_1)}$, for any orthonormal basis $\{e_k\}$ of $L^2(\mathbb{H}^n)$. Therefore $\mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}}^2 \mathcal{P}_r) = \int_{\mathbb{H}^n} K_1(g,g)dg = \sum\limits_{k\le r} \langle \mathbf{b}^2 e_k, e_k\rangle$. Further, $\mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}}\mathcal{P}_r)^2 = \int_{\mathbb{H}^n} K_2(g,g)dg = \sum\limits_{k\le r}\langle \mathbf{b}^2 e_k, e_k\rangle$, where the operator $\mathcal{P}_r M_{\mathbf{b}} \mathcal{P}_r M_{\mathbf{b}} \mathcal{P}_r$ is an integral operator with kernel

$$K_2(g, g_1) = \sum_{k_1,k_2,k_3 \le r} \langle \mathbf{b} e_{k_1}, e_{k_2}\rangle \langle \mathbf{b} e_{k_2}, e_{k_3}\rangle e_{k_3}(g)\overline{e_{k_1}(g_1)}.$$

So $\mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}}^2 \mathcal{P}_r) = \mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}}\mathcal{P}_r)^2$. Therefore $\|(I-\mathcal{P}_r)M_{\mathbf{b}}\mathcal{P}_r\|_{S_2}^2 = 0$. Observe that for each $n \in \mathbb{N}$, $\mathcal{P}_r M_{\mathbf{b}}^n \mathcal{P}_r = \mathcal{P}_r M_{\mathbf{b}}(\mathcal{P}_r + (I-\mathcal{P}_r))M_{\mathbf{b}} \cdots M_{\mathbf{b}}\mathcal{P}_r = (\mathcal{P}_r M_{\mathbf{b}} \mathcal{P}_r)^n +$ terms with a factor of $(I-\mathcal{P}_r)M_{\mathbf{b}}\mathcal{P}_r$. By Cauchy-Schwarz inequality, Tr(terms with a factor of $(I-\mathcal{P}_r)M_{\mathbf{b}}$) is dominated by some constant (depending on $\mathbf{b}$) times $\|(I-\mathcal{P}_r)M_{\mathbf{b}}\mathcal{P}_r\|_{S_2}$. Therefore $|\mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}}^n \mathcal{P}_r) - \mathrm{Tr}(\mathcal{P}_r M_{\mathbf{b}}\mathcal{P}_r)^n| = 0$. Thus $\mathrm{Tr} f(\mathcal{P}_r M_{\mathbf{b}}\mathcal{P}_r) = \mathrm{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r)$ for $f(x) = x^n$, $\forall n \in \mathbb{N}$, and this result can be extended to continuous functions as an application of the Weierstrass approximation theorem and spectral theorem. □

Lemma 4.5.3. *For $r > 0$ define $I_r : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ by*

$$I_r(\phi)(g) = \int_{-r}^{r} \mathrm{Tr}(\pi_\lambda(g)\hat{\phi}(\lambda))d\mu(\lambda), \quad g \in \mathbb{H}^n.$$

Then

$$\lim_{r\to\infty} \frac{\mathrm{Tr}(\mathcal{P}_r f(M_{\boldsymbol{b}})\mathcal{P}_r)}{\mathrm{Tr}(\mathcal{P}_r)} = \lim_{r\to\infty} \frac{\mathrm{Tr}(\mathcal{P}_r I_r f(M_{\boldsymbol{b}})\mathcal{P}_r I_r)}{\mathrm{Tr}(\mathcal{P}_r I_r)}. \tag{4.5.1}$$

Proof. We know that if X is a positive trace class operator and Y is a bounded operator on $L^2(\mathbb{H}^n)$ then $|\,\mathrm{Tr}(XYX)| \le \|Y\|\,\mathrm{Tr}(X^2)$. Using this inequality, we get

$$\big|\,\mathrm{Tr}(\mathcal{P}_r) - \mathrm{Tr}(\mathcal{P}_r I_r)\big| = \big|\,\mathrm{Tr}(\mathcal{P}_r(I - I_r))\big| \le \|I - I_r\|\,\mathrm{Tr}(\mathcal{P}_r)|.$$

But for $\psi \in L^2(\mathbb{H}^n)$, an application of Plancherel formula gives

$$\|(I-I_r)\psi\|_2^2 = \int_{|\lambda|>r} \|\hat{\psi}(\lambda)\|_{S_2}^2 \, d\mu(\lambda) \to 0 \text{ as } r \to \infty.$$

Therefore,

$$\left| \frac{\operatorname{Tr}(\mathcal{P}_r I_r)}{\operatorname{Tr}(\mathcal{P}_r)} - 1 \right| \le \|I - I_r\| \to 0 \text{ as } r \to \infty. \tag{4.5.2}$$

We add a suitable constant to make the operator $M_{\mathbf{b}}$ positive and any $f \in C(\mathbb{R})$ can be written as the difference of two positive functions namely the positive and the negative part of f. So without loss of generality we take $f(M_{\mathbf{b}})$ as a positive operator. Further,

$$\begin{aligned}
&\left|\operatorname{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r) - \operatorname{Tr}(\mathcal{P}_r I_r f(M_{\mathbf{b}})\mathcal{P}_r I_r)\right| \\
&= \left|\operatorname{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r (I - I_r)) - \operatorname{Tr}(\mathcal{P}_r (I - I_r) f(M_{\mathbf{b}})\mathcal{P}_r I_r)\right| \\
&\le \left|\operatorname{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r)\right| \|I - I_r\| + \left|\operatorname{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r I_r)\right| \|I - I_r\|.
\end{aligned}$$

Therefore,

$$\left| \frac{\operatorname{Tr}(\mathcal{P}_r I_r f(M_{\mathbf{b}})\mathcal{P}_r I_r)}{\operatorname{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r)} - 1 \right| \le \left(1 + \left| \frac{\operatorname{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r I_r)}{\operatorname{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r)} \right| \right) \|I - I_r\| \to 0, \tag{4.5.3}$$

as $r \to \infty$. Combining (4.5.2) and (4.5.3), we get (4.5.1). □

Now we are in a position to prove Szegö limit theorem for the multiplication operator $M_{\mathbf{b}}$.

Proof of theorem 4.5.1. The operator $\mathcal{P}_r I_r f(M_{\mathbf{b}})\mathcal{P}_r I_r$ is an integral operator with kernel

$$K_r(g, g_1) = \int_{-r}^{r} \operatorname{Tr}(\pi_\lambda(g) I_{r\times r} f(\mathbf{b}(g_1))\pi_\lambda(g_1)) \, d\mu(\lambda).$$

Therefore

$$\operatorname{Tr}(\mathcal{P}_r I_r f(M_{\mathbf{b}})\mathcal{P}_r I_r) = \int_{\mathbb{H}^n} K_r(g, g) \, dg = r \int_{-r}^{r} d\mu(\lambda) \int_{\mathbb{H}^n} f(\mathbf{b}(g)) \, dg,$$

and

$$\operatorname{Tr}(\mathcal{P}_r I_r) = \sum_{i \le r} \langle \widehat{I_r \phi_i}, \hat{\phi}_i \rangle = \sum_{i \le r} \langle \hat{\phi}_i \mathcal{X}_{[-r,r]}, \hat{\phi}_i \rangle = \sum_{i \le r} \int_{-r}^{r} \operatorname{Tr}(\hat{\phi}_i^{\,*}(\lambda)\hat{\phi}_i(\lambda)) \, d\mu(\lambda) = r \int_{-r}^{r} d\mu(\lambda),$$

for any orthonormal basis $\{\phi_i\}_{i=1}^{\infty}$ of $L^2(\mathbb{H}^n)$. Thus

$$\lim_{r\to\infty} \frac{\operatorname{Tr}(\mathcal{P}_r f(M_{\mathbf{b}})\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)} = \lim_{r\to\infty} \frac{\operatorname{Tr}(\mathcal{P}_r I_r f(M_{\mathbf{b}})\mathcal{P}_r I_r)}{\operatorname{Tr}(\mathcal{P}_r I_r)} = \int_{\mathbb{H}^n} f(\mathbf{b}(g)) \, dg.$$

□

We can generalize Theorem 4.5.1 by taking a 0-th order self-adjoint pseudo-differential operator on $L^2(\mathbb{H}^n)$ relative to the operator $1+|\lambda|H+V(g)$, where H is the Hermite operator on $L^2(\mathbb{R}^n)$ and $(g,\lambda)\in\mathbb{H}^n\times\mathbb{R}^*$, in place of the multiplication operator $M_{\mathbf{b}}$ and obtain the following Szegö type limit theorem:

Theorem 4.5.4. *Consider the Schrödinger operator of the form* $\mathcal{H}=-\Delta_{\mathbb{H}}+V$ *on the Heisenberg group* $\mathbb{H}^n$*. Let* $\mathcal{P}_r$ *be the orthogonal projection of* $L^2(\mathbb{H}^n)$ *onto the space of eigenfunctions of* $\mathcal{H}$ *with eigenvalue* $\leq r$*; let* A *be a 0-th order self-adjoint pseudo-differential operator relative to the operator* $1+|\lambda|H+V(g)$ *on* $L^2(\mathbb{H}^n)$ *with symbol* $a(g,\lambda)$*, where* $g\in\mathbb{H}^n,\lambda\in\mathbb{R}^*$*. Then for any* $f\in C(\mathbb{R})$*, we have*

$$\lim_{r\to\infty}\frac{\operatorname{Tr} f(\mathcal{P}_rA\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}=\lim_{r\to\infty}\frac{\int_{G^r} f(a_{g,\lambda}(\xi,x))\,d\xi\,dx\,dg\,d\mu(\lambda)}{\int_{G^r}\,d\xi\,dx\,dg\,d\mu(\lambda)},\tag{4.5.4}$$

(Assuming one limit exists)

where $G^r=\{(g,\lambda,\xi,x)\in\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n:|\lambda|(1+|\xi|^2+|x|^2)+V(g)\leq r\}$*,* $a(g,\lambda)=Op^W(a_{g,\lambda})$*, and* $\mu(\lambda)$ *is the Plancherel measure on the Heisenberg group.*

We prove Theorem 4.5.4 under certain assumptions on the symbol $a(g,\lambda)$ to ensure the existence the RHS limit in Theorem 4.5.4 (see [109]). We assume

$$\lim_{E\to\infty}\bar{a}(E)=a,\tag{4.5.5}$$

where $\bar{a}(E)=\dfrac{1}{S(E)}\displaystyle\int_{G_E}a_{g,\lambda}(\xi,x)\,d\xi\,dx\,dg\,d\mu(\lambda)$ and $S(E)=\displaystyle\int_{G_E}d\xi\,dx\,dg\,d\mu(\lambda)$, with $G_E=\{(g,\lambda,\xi,x)\in\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n:|\lambda|(1+|\xi|^2+|x|^2)+V(g)=E\}$ and

$$V(g)\sim V_0|g|^\kappa\quad\text{as }|g|\to\infty\tag{4.5.6}$$

for real $\kappa>0$ in the sense that $V(g)=V_0|g|^\kappa+W(g)$, where $W(g)=o(|g|^\kappa)$.

Proposition 4.5.5. *Let* $G^E=\{(g,\lambda,\xi,x)\in\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n:|\lambda|(1+|\xi|^2+|x|^2)+V(g)\leq E\}$*. Then volume of* $G^E=v(E)\approx E^{n+1+\frac{2(n+1)}{\kappa}}$ *as* $E\to\infty$*.*

Proof. Using the homogeneous norm on $\mathbb{H}^n$, we have

$$\begin{aligned}v(E)&=\int_{G^E}d\xi\,dx\,dg\,d\mu(\lambda)\\&=C_n\int_{\mathbb{H}^n}\iint_{\mathbb{R}^{2n}}\left(\int_{|\lambda|\leq\frac{(E-V(g))_+}{1+|x|^2+|\xi|^2}}|\lambda|^n\,d\lambda\right)d\xi\,dx\,dg\end{aligned}$$

$$\begin{aligned}
&= 2C_n \int_{\mathbb{H}^n} (E - V(g))_+^{n+1}\, dg \iint_{\mathbb{R}^{2n}} \left(\frac{1}{1+|x|^2+|\xi|^2} \right)^{n+1} d\xi\, dx \\
&= C_n' E^{n+1} \int_{\mathbb{H}^n} \left(1 - E^{-1}V(g)\right)_+^{n+1} dg \\
&\sim C_n' E^{n+1} \int_{\mathbb{H}^n} \left(1 - E^{-1}(V_0|g|^\kappa + W(g))\right)_+^{n+1} dg \\
&= C_n' E^{n+1+\frac{2(n+1)}{\kappa}} \int_{\mathbb{H}^n} \left(1 - V_0|g|^k - E^{-1}W(\widetilde{g})\right)_+^{n+1} dg, \qquad (4.5.7)
\end{aligned}$$

where $\widetilde{g} = (E^{\frac{1}{\kappa}}x_1, E^{\frac{1}{\kappa}}x_2, \cdots, E^{\frac{1}{\kappa}}x_{2n}, E^{\frac{2}{\kappa}}t)$ for $g = (x_1, x_2 \cdots, x_{2n}, t) \in \mathbb{H}^n$. Since $\lim_{E\to\infty} E^{-1}W(\widetilde{g}) = 0$, the right hand side of the above integral converges to $\int_{\mathbb{H}^n} (1 - V_0|g|^\kappa)_+^{n+1}\, dg$ by dominated convergence theorem. ☐

Lemma 4.5.6. *Let $\phi(r) = \mathrm{Tr}(\mathcal{P}_r)$ and $\psi(r) = \mathrm{Tr}(\mathcal{P}_r A \mathcal{P}_r)$. Then under the assumption (4.5.5) and (4.5.6), we have*

$$\Phi(u) = \int_0^\infty \frac{\phi(r)}{(r+u)^{m+1}}\, dr = \int_0^\infty \frac{\int_{G^E} d\xi\, dx\, dg\, d\mu(\lambda)}{(E+u)^{m+1}}\, dE + E_1(u)$$

and

$$\Psi(u) = \int_0^\infty \frac{\psi(r)}{(r+u)^{m+1}}\, dr = \int_0^\infty \frac{\int_{G^E} a_{g,\lambda}(\xi,x)\, d\xi\, dx\, dg\, d\mu(\lambda)}{(E+u)^{m+1}}\, dE + E_2(u),$$

with $|E_i(u)| \to 0$ as $u \to \infty$, $i = 1, 2$.

Proof. The operator $\mathcal{H}$ has discrete spectrum of eigenvalues $0 \le c_1 \le c_2 \cdots \infty$. Let $\{\psi_j\}_{j=1}^\infty$ be the complete set of eigenfunctions on corresponding to the eigenvalues $\{c_j\}$ on $L^2(\mathbb{H}^n)$. Then $\psi(r) = \mathrm{Tr}(\mathcal{P}_r A \mathcal{P}_r) = \sum_{c_j \le r} \langle A\psi_j, \psi_j\rangle$ and $\psi'(r) = \sum_{j=1}^\infty \langle A\psi_j, \psi_j\rangle \delta(r - c_j)$. Now

$$\begin{aligned}
\Psi(u) = \int_0^\infty \frac{\psi(r)}{(r+u)^{m+1}}\, dr &= m \sum_{j=1}^\infty \langle A\psi_j, \psi_j\rangle \int_0^\infty \frac{\delta(r-c_j)}{(r+u)^m}\, dr \\
&= m \sum_{j=1}^\infty \langle A\psi_j, \psi_j\rangle \frac{1}{(c_j+u)^m} = m\ \mathrm{Tr}\left(A(\mathcal{H}+u)^{-m}\right).
\end{aligned}$$

By Proposition 4.4.3, we obtain

$$\begin{aligned}
\Psi(u) &= m\ \mathrm{Tr}\left(A(\mathcal{H}+u)^{-m}\right) \\
&= m\ \mathrm{Tr}\left(A\ Op(|\lambda|(H+I)+V(g)+u)^{-m}\right) + m\ \mathrm{Tr}\left(A\ Op(E(g,\lambda,u))\right)
\end{aligned}$$

with $|\mathrm{Tr}\,(A\ Op(E(g,\lambda,u)))\,| \leq \|A\|\,|\mathrm{Tr}\,(Op(E(g,\lambda,u)))| \to 0\ u \to \infty$. Thus for large u, using (4.3.7) and (4.2.3), we have

$$\begin{aligned}
\Psi(u) &= m\int_{\mathbb{H}^n\times\mathbb{R}^*} \mathrm{Tr}\left(a(g,\lambda)\big(|\lambda|(H+I)+V(g)+u\big)^{-m}\right) dg\, d\mu(\lambda)\\
&= m\int_{\mathbb{H}^n\times\mathbb{R}^*} \mathrm{Tr}\Big(a(g,\lambda)Op^W\left(\big(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\big)^{-m}\right)\\
&\qquad + a(g,\lambda)Op^W(E_{g,\lambda}(u))\Big)\, dg\, d\mu(\lambda)\\
&= m\int_{\mathbb{H}^n\times\mathbb{R}^*}\int_{\mathbb{R}^n\times\mathbb{R}^n} \left(a_{g\lambda}(\xi,x)\big(|\lambda|(1+|\xi|^2+|x|^2)+V(g)+u\big)^{-m}\right) d\xi\, dx\, dg\, d\mu(\lambda) + E_1(u)\\
&= \int_0^\infty \frac{\displaystyle\int_{G_E} a_{g\lambda}(\xi,x)\, d\xi\, dx\, dg\, d\mu(\lambda)}{(E+u)^{m+1}}\, dE + E_1(u),
\end{aligned}$$

where $E_1(u) = m\displaystyle\int_{\mathbb{H}^n\times\mathbb{R}^*} \mathrm{Tr}\left(a(g,\lambda)Op^W(E_{g,\lambda}(u))\right) dg\ \ d\mu(\lambda)$. From Remark 4.4.5, we conclude that $|E_1(u)| \to 0$ as $u \to \infty$ by dominated convergence theorem. Similarly taking $A = I$, we get $\phi(r) = \mathrm{Tr}(\mathcal{P}_r)$, and in this case, for large u, we have

$$\Phi(u) = \int_0^\infty \frac{\displaystyle\int_{G_E} d\xi\, dx\, dg\, d\mu(\lambda)}{(E+u)^{m+1}}\, dE + E_2(u)$$

with $|E_2(u)| \to 0$ as $u \to \infty$. □

In order to prove the Szegö limit theorem for the Schrödinger operator $\mathcal{H}$, we need to estimate the asymptotic growth of the measures $\mathrm{Tr}(\mathcal{P}_r A\mathcal{P}_r)$ and $\mathrm{Tr}(\mathcal{P}_r)$. We apply Keldysh Tauberian Theorem (see Theorem 5.4 in Appendix) to compare the measures. Before that, first we recall the definition of multiplicatively continuous of a function.

Definition 4.5.7. *A function φ is said to be multiplicatively continuous at infinity if it satisfies* $\displaystyle\lim_{\substack{r\to\infty\\ \tau\to 1}} \frac{\varphi(\tau r)}{\varphi(r)} = 1$.

Corollary 4.5.8. *Consider the self-adjoint operator $\mathcal{P}_r$ and $v(r)$ as given in Theorem 4.5.4 and Proposition 4.5.5, respectively. Let $\phi(r) = \mathrm{Tr}(\mathcal{P}_r)$ and $\psi(r) = \mathrm{Tr}(\mathcal{P}_r A\mathcal{P}_r)$, then we have the following asymptotic:*

1. *$v(r) \approx r^{n+1+\frac{2(n+1)}{\kappa}}$ as $r \to \infty$.*

2. *v is multiplicatively continuous.*

3. $\mathrm{Tr}(\mathcal{P}_r) \approx r^{n+1+\frac{2(n+1)}{\kappa}}$ *as* $r \to \infty$.

4. $\sup\limits_{\mu \leq r}[\mathrm{Tr}(\mathcal{P}_{\mu+r_1}) - \mathrm{Tr}(\mathcal{P}_\mu)] \leq \mathrm{Tr}(\mathcal{P}_r)\left[\left(n+1+\frac{2(n+1)}{\kappa}\right)\frac{r_1}{r} + \mathcal{O}\left(\frac{1}{r}\right)^2\right]$, *as* $r \to \infty$.

5. ψ *is multiplicatively continuous.*

Proof. Clearly (1) directly follows from Proposition 4.5.5. Now

$$\lim_{\substack{r\to\infty\\ \tau\to 1}} \frac{v(\tau r)}{v(r)} = \lim_{\substack{r\to\infty\\ \tau\to 1}} \frac{(\tau r)^{n+1+\frac{2(n+1)}{\kappa}}}{r^{n+1+\frac{2(n+1)}{\kappa}}} = \lim_{\tau\to 1} \tau^{n+1+\frac{2(n+1)}{\kappa}} = 1.$$

Therefore v is multiplicatively continuous function. We choose sufficiently large m such that the operator $(\mathcal{H}+uI)^{-m}$ is a trace class operator. Therefore by Lemma 4.5.6 and Theorem 8 of Grishin-Poedintseva [42], we get $\phi(r)/v(r) \to 1$ as $r \to \infty$. This proves (3). Using the asymptotic in (3), it is easy to check that

$$\sup_{\mu \leq r}[\mathrm{Tr}(\mathcal{P}_{\mu+r_1}) - \mathrm{Tr}(\mathcal{P}_\mu)] \leq \mathrm{Tr}(\mathcal{P}_r)\left[\left(n+1+\frac{2(n+1)}{\kappa}\right)\frac{r_1}{r} + \mathcal{O}\left(\frac{1}{r}\right)^2\right].$$

To prove (5), notice that if φ and χ are two distribution functions satisfying $\lim\limits_{r\to\infty}\dfrac{\varphi(r)}{\chi(r)} = 1$, then φ is multiplicatively continuous whenever χ is. Therefore ψ is also a multiplicatively continuous function. ☐

Theorem 4.5.9. *Under the assumption (4.5.5) and (4.5.6), we have*

$$\lim_{r\to\infty} \frac{\mathrm{Tr}(\mathcal{P}_r A \mathcal{P}_r)}{\mathrm{Tr}(\mathcal{P}_r)} = \lim_{r\to\infty} \frac{\int_{G^r} a_{g,\lambda}(\xi,x)\, d\xi\, dx\, dg\, d\mu(\lambda)}{\int_{G^r} d\xi\, dx\, dg\, d\mu(\lambda)}.$$

Proof. The proof follows directly by Lemma 4.5.6, as all the requirements (by our assumption (4.5.5) on the symbol $a(g,\lambda)$) of Theorem 8 of Grishin-Poedintseva [42] are satisfied. ☐

Corollary 4.5.10. *Let P be a polynomial in $\mathbb{R}$. Then*

$$\lim_{r\to\infty} \frac{\mathrm{Tr}(\mathcal{P}_r P(A) \mathcal{P}_r)}{\mathrm{Tr}(\mathcal{P}_r)} = \lim_{r\to\infty} \frac{\int_{G^r} P(a_{g,\lambda}(\xi,x))\, d\xi\, dx\, dg\, d\mu(\lambda)}{\int_{G^r} d\xi\, dx\, dg\, d\mu(\lambda)}$$

Proof. From the asymptotic expression of Theorem 4.3.18 along with Remark 4.3.11, we see that the operator $P(A)$ has symbol $P(a(g,\lambda))+E(g,\lambda)+E_{-1}(g,\lambda)$, where $E(g,\lambda), E_{-1}(g,\lambda) \in S^{-1}_{\mathcal{H}}(\mathbb{H}^n)$ (the term associated with $[\alpha]=1$ is $E(g,\lambda)$ and $E_{-1}(g,\lambda)$ is the remaining terms with $[\alpha] > 1$ in the asymptotic expansion). The proof will be complete if we can show that

$$\lim_{r\to\infty} \frac{\int_{G^r} \tilde{E}_{g,\lambda}(\xi,x)\, d\xi\, dx\, dg\, d\mu(\lambda)}{\int_{G^r} d\xi\, dx\, dg\, d\mu(\lambda)} = 0, \tag{4.5.8}$$

where $E(g,\lambda) + E_{-1}(g,\lambda) = Op^W(\tilde{E}_{g,\lambda}(\xi,x))$. Now proceeding as in Proposition 4.5.5, we get

$$\begin{aligned}
&\int_{G^r} |\tilde{E}_{g,\lambda}(\xi,x)|\, d\xi\, dx\, dg\, d\mu(\lambda)\\
&\leq C \int_{\mathbb{H}^n} \iint_{\mathbb{R}^{2n}} \left(\int_{|\lambda| \leq \frac{(r-V(g))_+}{1+|x|^2+|\xi|^2}} (1+|\lambda|(1+|\xi|^2+|x|^2)+V(g))^{-\frac{1}{2}} |\lambda|^n\, d\lambda \right) d\xi\, dx\, dg\\
&\leq C \int_{\mathbb{H}^n} \iint_{\mathbb{R}^{2n}} \left(\int_{|\lambda| \leq \frac{(r-V(g))_+}{1+|x|^2+|\xi|^2}} |\lambda|^{n-\frac{1}{2}}\, d\lambda \right) d\xi\, dx\, dg\\
&= C \int_{\mathbb{H}^n} (r-V(g))_+^{n+\frac{1}{2}}\, dg \iint_{\mathbb{R}^{2n}} \left(\frac{1}{1+|x|^2+|\xi|^2} \right)^{n+\frac{1}{2}} d\xi\, dx\\
&= C r^{n+\frac{1}{2}} \int_{\mathbb{H}^n} (1-r^{-1}V(g))_+^{n+\frac{1}{2}}\, dg\\
&\approx r^{n+\frac{1}{2}+\frac{2(n+1)}{\kappa}} \qquad (\text{see } (4.5.7)).
\end{aligned}$$

On the other hand, from Corollary 4.5.8, we have $\mathrm{Tr}(\mathcal{P}_r) \approx r^{n+1+\frac{2(n+1)}{\kappa}}$. Thus we get (4.5.8). □

Lemma 4.5.11. *Let $\mathcal{H}, A$ be the operators defined in Theorem 4.5.4. Then*

(a) $\sqrt{\mathcal{H}} = \mathcal{H}_{\frac{1}{2}} + C$, where $\mathcal{H}_{\frac{1}{2}} = Op(H_{\frac{1}{2}}(g,\lambda))$ and C is a bounded operator on $L^2(\mathbb{H}^n)$,

(b) the operator $[\sqrt{\mathcal{H}}, A]$ is bounded on $L^2(\mathbb{H}^n)$,

(c) under the assumptions of Theorem 4.5.4, we have

$$\left| \frac{\mathrm{Tr} f(\mathcal{P}_r A \mathcal{P}_r)}{\mathrm{Tr}(\mathcal{P}_r)} - \frac{\mathrm{Tr}(\mathcal{P}_r f(A) \mathcal{P}_r)}{\mathrm{Tr}(\mathcal{P}_r)} \right| \to 0 \text{ as } r \to \infty.$$

Proof. (a) Let $f(w) = w^{\frac{1}{2}}$. Proceeding as in Subsection 4.4.2, we get $\sqrt{\mathcal{H}} = \mathcal{H}_{\frac{1}{2}} + F_{\frac{1}{2}}$, where $\mathcal{H}_{\frac{1}{2}} = Op(H_{\frac{1}{2}}(g,\lambda))$ with $H_{\frac{1}{2}}(g,\lambda) \in S^1_{\mathcal{H}}(\mathbb{H}^n)$ and $F_{\frac{1}{2}}$ is defined in (4.4.4) with $S_N \in S^{-N}_{\mathcal{H},w}(\mathbb{H}^n)$. We choose $N > 0$ such that the integral (4.4.4) converges in the norm on $B(L^2(\mathbb{R}^n))$. Denoting $C = F_{\frac{1}{2}}$, we have $\sqrt{\mathcal{H}} = \mathcal{H}_{\frac{1}{2}} + C$ as desired.

(b) From part (a), we have $\sqrt{\mathcal{H}} = \mathcal{H}_{\frac{1}{2}} + C$. Since C is bounded, $[\sqrt{\mathcal{H}}, A]$ is bounded if $[\mathcal{H}_{\frac{1}{2}}, A]$ is bounded on $L^2(\mathbb{H}^n)$. Now using the composition formula (4.3.9) of Theorem

4.3.18, there exist two symbols $R_1(g,\lambda), R_2(g,\lambda) \in S^0_{\mathcal{H}}(\mathbb{H}^n)$ such that

$$\begin{aligned}
[\mathcal{H}_{\frac{1}{2}}, A] &= \mathcal{H}_{\frac{1}{2}} A - A\mathcal{H}_{\frac{1}{2}} \\
&= Op\left(H_{\frac{1}{2}}(g,\lambda) \#_{\mathbb{H}^n} a(g,\lambda)\right) - Op\left(a(g,\lambda) \#_{\mathbb{H}^n} H_{\frac{1}{2}}(g,\lambda)\right) \\
&= Op\left(\ H_{\frac{1}{2}}(g,\lambda) a(g,\lambda) + \Delta^\alpha H_{\frac{1}{2}}(g,\lambda)\ X_g^\alpha a(g,\lambda) + R_1(g,\lambda)\right) \\
&\qquad - Op\left(a(g,\lambda) H_{\frac{1}{2}}(g,\lambda) + \Delta^\alpha a(g,\lambda) X_g^\alpha H_{\frac{1}{2}}(g,\lambda) + R_2(g,\lambda)\right) \\
&= Op\left(Op^W(a_{g,\lambda}(\xi,u) \# H_{\frac{1}{2},g,\lambda}(\xi,u) - H_{\frac{1}{2},g,\lambda}(\xi,u) \# a_{g,\lambda}(\xi,u)\right) \\
&\qquad + Op\left(\Delta^\alpha H_{\frac{1}{2}}(g,\lambda)\ X_g^\alpha a(g,\lambda) - \Delta^\alpha a(g,\lambda) X_g^\alpha H_{\frac{1}{2}}(g,\lambda)\right) + Op\left(R_1(g,\lambda) - R_2(g,\lambda)\right) \\
&= Op\left(Op^W(F^1_{g,\lambda}(\xi,u) + F^2_{g,\lambda}(\xi,u))\right) + Op\left(R_1(g,\lambda) - R_2(g,\lambda)\right) \\
&\qquad + Op\left(\Delta^\alpha H_{\frac{1}{2}}(g,\lambda)\ X_g^\alpha a(g,\lambda) - \Delta a(g,\lambda) X_g H_{\frac{1}{2}}(g,\lambda)\right), \qquad (4.5.9)
\end{aligned}$$

where $[\alpha] = 1$ and $F^1_{g,\lambda}(\xi,u), F^2_{g,\lambda}(\xi,u) \in S^0(\mathbb{R}^n)$ (the term associated with $j = 1$ is $F^1_{g,\lambda}$ and $F^2_{g,\lambda}$ is the remaining terms with $j > 1$ in the asymptotic expansion (4.2.2)). Therefore $Op^W(F^1_{g,\lambda}(\xi,u) + F^2_{g,\lambda}(\xi,u)) \in S^0_{\mathcal{H}}(\mathbb{H}^n)$. Since each symbol in the last equality of the expression (4.5.9) belongs to the $S^0_{\mathcal{H}}(\mathbb{H}^n)$ class, by Theorem 4.3.13, the operator $[\mathcal{H}_{\frac{1}{2}}, A]$ is bounded on $L^2(\mathbb{H}^n)$.

(c) Since A is bounded self-adjoint, the spectrum of A, $\sigma(A)$, is a compact subset of $\mathbb{R}$. Since any continuous function can be approximated in the supremum norm by smooth functions, it is enough to assume that $f \in C^2(\sigma(A))$. By Theorem 1.6 of Laptev-Safarov [63], by setting $A = \sqrt{\mathcal{H}}, B = A, \chi = 0, \psi = f$, and $P_\lambda = \mathcal{P}_{r^2}$, we get

$$\begin{aligned}
&|\mathrm{Tr}(\mathcal{P}_{r^2} f(A) \mathcal{P}_{r^2} - \mathcal{P}_{r^2} f(\mathcal{P}_{r^2} A \mathcal{P}_{r^2}) \mathcal{P}_{r^2})| \\
&\leq \frac{1}{2} \|f''\|_\infty N_{r_1}(r^2) \Big(\|\mathcal{P}_{r^2 - r_1, r^2} A\|^2 + \frac{\pi^2}{6 r_1^2} \|\mathcal{P}_{r^2 - r_1} [A, \sqrt{\mathcal{H}}]\|^2 \Big).
\end{aligned}$$

Dividing both sides by $\mathrm{Tr}(\mathcal{P}_{r^2})$ and setting $r_1 = r^{2\alpha}, \alpha \in (0,1)$, we get

$$\frac{|\mathrm{Tr}(\mathcal{P}_{r^2} f(A) \mathcal{P}_{r^2} - \mathcal{P}_{r^2} f(\mathcal{P}_{r^2} A \mathcal{P}_{r^2}) \mathcal{P}_{r^2})|}{\mathrm{Tr}(\mathcal{P}_{r^2})} \leq C \frac{N_{r^{2\alpha}}(r)}{\mathrm{Tr}(\mathcal{P}_{r^2})} \approx r^{2\alpha - 2}.$$

Thus

$$\frac{|\mathrm{Tr}(\mathcal{P}_r f(A) \mathcal{P}_r - \mathcal{P}_r f(\mathcal{P}_r A \mathcal{P}_r) \mathcal{P}_r)|}{\mathrm{Tr}(\mathcal{P}_r)} \leq C \frac{N_{r^\alpha}(r)}{\mathrm{Tr}(\mathcal{P}_r)} \approx r^{\alpha - 1} \to 0 \text{ as } r \to \infty,$$

by part (3) and part (4) of Corollary 4.5.8, where $N_{r_1}(r) = \sup_{\mu \leq r} \left(\mathrm{Tr}(\mathcal{P}_{\mu + r_1} - \mathcal{P}_\mu)\right)$. ☐

Proof of theorem 4.5.4. The proof of Theorem 4.5.4 follows from Corollary 4.5.10 and part (c) of Lemma 4.5.11. ☐

We show that the right hand limit in (4.5.4) remains unaltered under a perturbation of the Schrödinger operator by a bounded self-adjoint operator $\tilde{B}$ on $L^2(\mathbb{H}^n)$. So we establish Szegö type limit theorem for the operator $\mathcal{H}_1 = \tilde{B} + \mathcal{H}$ on $L^2(\mathbb{H}^n)$.

4.6 Szegö type limit theorem for $\mathcal{H}_1$

Note that the operator $e^{-t(\tilde{B}+\mathcal{H})} = e^{-t\tilde{B}}e^{-t\mathcal{H}}$ is a compact operator as $e^{-t\tilde{B}}$ is a bounded operator for any $t > 0$ (see Theorem 2 of [88]). So $\mathcal{H}_1$ has discrete spectrum. Since the operators $e^{-t\mathcal{H}_1}$ and $e^{-t\mathcal{H}}$ are compact for $t > 0$, we choose a suitable $m \in \mathbb{N}$ such that $(\mathcal{H}_1 + rI)^{-m}$ and $(\mathcal{H} + rI)^{-m}$ are trace class operators on $L^2(\mathbb{H}^n)$ for $r > 0$. Then we have the following Szegö type limit theorem for $\mathcal{H}_1$.

Theorem 4.6.1. *Consider the operator $\mathcal{H}_1 = \tilde{B} + \mathcal{H}$ on the Heisenberg group $\mathbb{H}^n$, where $\tilde{B}$ is a bounded self-adjoint operator on $\mathbb{H}^n$ such that $\mathcal{H}_1$ has purely discrete spectrum and the eigenfunctions of $\mathcal{H}_1$ form a complete orthogonal basis for $L^2(\mathbb{H}^n)$. Let $\mathcal{P}_r'$ be the orthogonal projection of $L^2(\mathbb{H}^n)$ onto the space of eigenfunctions of $\mathcal{H}_1$ with eigenvalue $\leq r$; let A be a 0-th order self-adjoint pseudo-differential operator relative to the operator $1 + |\lambda|H + V(g)$ on $L^2(\mathbb{H}^n)$ with symbol $a(g, \lambda)$, where $g \in \mathbb{H}^n, \lambda \in \mathbb{R}^*$. Then for any $f \in C(\mathbb{R})$, we have*

$$\lim_{r\to\infty} \frac{\operatorname{Tr} f(\mathcal{P}_r' A \mathcal{P}_r')}{\operatorname{Tr}(\mathcal{P}_r')} = \lim_{r\to\infty} \frac{\int_{G^r} f(a_{g,\lambda}(\xi, x))\, d\xi\, dx\, dg\, d\mu(\lambda)}{\int_{G^r} d\xi\, dx\, dg\, d\mu(\lambda)},$$

(Assuming one limit exists)

where $G^r = \{(g, \lambda, \xi, x) \in \mathbb{H}^n \times \mathbb{R}^* \times \mathbb{R}^n \times \mathbb{R}^n : |\lambda|(1 + |\xi|^2 + |x|^2) + V(g) \leq r\}$, $a(g, \lambda) = Op^W(a_{g,\lambda})$, and $\mu(\lambda)$ is the Plancherel measure on the Heisenberg group.

We observe the following facts before proving Theorem 4.6.1.

Lemma 4.6.2. *Consider the self-adjoint operators $\mathcal{H}$ and $\mathcal{H}_1$ as defined in Theorem 4.6.1. Then*

(a)

$$\left| \frac{\operatorname{Tr}((\mathcal{H}_1 + rI)^{-m})}{\operatorname{Tr}((\mathcal{H} + rI)^{-m})} - 1 \right| \to 0 \text{ as } r \to \infty.$$

(b) If B is any bounded operator on $L^2(\mathbb{H}^n)$, then

$$\left| \frac{\operatorname{Tr}(B(\mathcal{H}_1 + rI)^{-m})}{\operatorname{Tr}(B(\mathcal{H} + rI)^{-m})} - 1 \right| \to 0 \text{ as } r \to \infty.$$

Proof. Without loss of generality we prove the result for the positive operator B by adding a suitable constant $c > 0$ which makes the operator $B + cI$ positive.

(a) Since B and $(\mathcal{H} + rI)^{-1}$ are bounded and positive operators, we have

$$(\mathcal{H}_1 + rI) = (\mathcal{H} + rI)^{\frac{1}{2}}((\mathcal{H} + rI)^{-\frac{1}{2}}(B)(\mathcal{H} + rI)^{-\frac{1}{2}} + 1)(\mathcal{H} + rI)^{\frac{1}{2}}. \tag{4.6.1}$$

Therefore

$$(\mathcal{H}_1 + rI)^{-m} = (\mathcal{H} + rI)^{-m} + (\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1)(\mathcal{H} + rI)^{-\frac{m}{2}}, \tag{4.6.2}$$

where $K_r = (\mathcal{H}+rI)^{-\frac{1}{2}}B(\mathcal{H}+rI)^{-\frac{1}{2}}$. Here K_r is a positive operator and $\|(I+K_r)^{-1}\| \leq 1$, for any $r > 0$. Thus

$$\begin{aligned}
\left|\operatorname{Tr}\left((\mathcal{H}_1 + rI)^{-m}\right) - \operatorname{Tr}((\mathcal{H} + rI)^{-m})\right| &= \left|\operatorname{Tr}\left((\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1)(\mathcal{H} + rI)^{-\frac{m}{2}}\right)\right| \\
&\leq \operatorname{Tr}\left((\mathcal{H} + rI)^{-m}\right)\left\|((1 + K_r)^{-m} - 1)\right\| \\
&\leq m\left\|K_r\right\| \operatorname{Tr}\left((\mathcal{H} + rI)^{-m}\right) \\
&\leq m\left\|B\right\|\left\|(\mathcal{H} + rI)^{-1}\right\| \operatorname{Tr}\left((\mathcal{H} + rI)^{-m}\right).
\end{aligned}$$

Therefore,

$$\left|\frac{\operatorname{Tr}((\mathcal{H}_1 + rI)^{-m})}{\operatorname{Tr}((\mathcal{H} + rI)^{-m})} - 1\right| \leq m\left\|B\right\|\left\|(\mathcal{H} + rI)^{-1}\right\| \to 0 \text{ as } r \to \infty.$$

(b) Using the relation (4.6.2), we have

$$\begin{aligned}
&\left|\operatorname{Tr}\left(B(\mathcal{H}_1 + rI)^{-m}\right) - \operatorname{Tr}(B(\mathcal{H} + rI)^{-m})\right| \\
&= \left|\operatorname{Tr}\left(B(\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1)(\mathcal{H} + rI)^{-\frac{m}{2}}\right)\right| \\
&= \left|\operatorname{Tr}\left((\mathcal{H} + rI)^{-\frac{m}{2}}B(\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1)\right)\right| \\
&= \left|\operatorname{Tr}\left(W_r((1 + K_r)^{-m} - 1)\right)\right| \\
&= \left|\operatorname{Tr}\left(W_r^{\frac{1}{2}}((1 + K_r)^{-m} - 1)W_r^{\frac{1}{2}}\right)\right| \\
&\leq m\left\|B\right\|\left\|(\mathcal{H} + rI)^{-1}\right\| \operatorname{Tr}\left(B(\mathcal{H} + rI)^{-m}\right),
\end{aligned}$$

where $W_r = (\mathcal{H} + rI)^{-\frac{m}{2}}B(\mathcal{H} + rI)^{-\frac{m}{2}}$ is a positive and trace class operator on $L^2(\mathbb{H}^n)$.

Therefore,

$$\left|\frac{\operatorname{Tr}(B(\mathcal{H}_1 + rI)^{-m})}{\operatorname{Tr}(B(\mathcal{H} + rI)^{-m})} - 1\right| \leq m\left\|B\right\|\left\|(\mathcal{H} + rI)^{-1}\right\| \to 0 \text{ as } r \to \infty.$$

□

Lemma 4.6.3. *Let $\mathcal{H}$ and $\mathcal{H}_1$ defined as in Theorem 4.6.1 and B is a bounded operator on $L^2(\mathbb{H}^n)$, then for $r>0$, we have*

$$\lim_{r\to\infty}\frac{\mathrm{Tr}(B(\mathcal{H}_1+rI)^{-m})}{\mathrm{Tr}((\mathcal{H}_1+rI)^{-m})}=\lim_{r\to\infty}\frac{\mathrm{Tr}(B(\mathcal{H}+rI)^{-m})}{\mathrm{Tr}((\mathcal{H}+rI)^{-m})}.$$

The above equality valid in the sense that if one of limits exist then the other also does and the limits are the same.

Proof. For each $r>0$, we have

$$\frac{\left(\frac{\mathrm{Tr}(B(\mathcal{H}_1+rI)^{-m})}{\mathrm{Tr}(B(\mathcal{H}+rI)^{-m})}\right)}{\left(\frac{\mathrm{Tr}((\mathcal{H}_1+rI)^{-m})}{\mathrm{Tr}((\mathcal{H}+rI)^{-m})}\right)}=\frac{\left(\frac{\mathrm{Tr}(B(\mathcal{H}_1+rI)^{-m})}{\mathrm{Tr}((\mathcal{H}_1+rI)^{-m})}\right)}{\left(\frac{\mathrm{Tr}((B(\mathcal{H}+rI)^{-m})}{\mathrm{Tr}((\mathcal{H}+rI)^{-m})}\right)}. \tag{4.6.3}$$

Since the left hand side of (4.6.3) has limit 1 (by part (b) of Lemma 4.6.2), the right hand side limit in (4.6.3) exists and equal to 1. Therefore, if the numerator or the denominator in the fraction in the right hand side has a limit in (4.6.3), then the other also has a limit and they both agree. Therefore, $\lim_{r\to\infty}\frac{\mathrm{Tr}(B(\mathcal{H}_1+rI)^{-m})}{\mathrm{Tr}((\mathcal{H}_1+rI)^{-m})}=\lim_{r\to\infty}\frac{\mathrm{Tr}(B(\mathcal{H}+rI)^{-m})}{\mathrm{Tr}((\mathcal{H}+rI)^{-m})}$. □

Proof of theorem 4.6.1. Without loss of generality add a suitable constant to make the function f positive. Then $f(A)$ is a positive operator. Setting $\phi_{\mathcal{H}}(r)=\mathrm{Tr}(\mathcal{P}_r)$, $\phi_{\mathcal{H}_1}(r)=\mathrm{Tr}(\mathcal{P}_r^{'})$, $\phi_{\mathcal{H},f}(r)=\mathrm{Tr}(\mathcal{P}_rf(A)\mathcal{P}_r)$, and $\phi_{\mathcal{H}_1,f}(r)=\mathrm{Tr}(\mathcal{P}_r^{'}f(A)\mathcal{P}_r^{'})$, we have

$$\begin{aligned}\lim_{r\to\infty}\frac{\mathrm{Tr}(\mathcal{P}_r^{'}f(A)\mathcal{P}_r^{'})}{\mathrm{Tr}(\mathcal{P}_r^{'})} &= \lim_{r\to\infty}\frac{\int_0^\infty\frac{\phi_{\mathcal{H}_1,f}(u)}{(1+\frac{u}{r})^{m+1}}du}{\int_0^\infty\frac{\phi_{\mathcal{H}_1}(u)}{(1+\frac{u}{r})^{m+1}}du}\\ &= \lim_{r\to\infty}\frac{\int_0^\infty\frac{\phi_{\mathcal{H},f}(u)}{(1+\frac{u}{r})^{m+1}}du}{\int_0^\infty\frac{\phi_{\mathcal{H}}(u)}{(1+\frac{u}{r})^{m+1}}du}\\ &= \lim_{r\to\infty}\frac{\mathrm{Tr}(\mathcal{P}_rf(A)\mathcal{P}_r)}{\mathrm{Tr}(\mathcal{P}_r)}\\ &= \lim_{r\to\infty}\frac{\int_{G^r}f(a_{g,\lambda}(\xi,x))\,d\xi\,dx\,dg\,d\mu(\lambda)}{\int_{G^r}\,d\xi\,dx\,dg\,d\mu(\lambda)},\end{aligned}$$

(Assuming one limit exists)

where $G^r=\{(g,\lambda,\xi,x)\in\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n:|\lambda|(1+|\xi|^2+|x|^2)+V(g)\leq r\}$ and $a(g,\lambda)=Op^W(a_{g,\lambda})$. We use Lemma 4.6.3 for the middle equality and Theorem 4.7.3 (see Appendix) for the extreme left equalities. The extreme right equality follows from Lemma 4.5.11. □

In the next corollary, we show that all the Szegö type limit theorems are valid under a compact perturbation of the pseudo-differential operator A.

Corollary 4.6.4. *Theorems 4.5.1, 4.5.4 and 4.6.1 also hold under the compact perturbation of the pseudo-differential operator A.*

Proof. To prove the above result, it is enough to show that

$$\lim_{r\to\infty}\frac{\operatorname{Tr}(\mathcal{P}_r A^n\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}=\lim_{r\to\infty}\frac{\operatorname{Tr}(\mathcal{P}_r (A+K)^n\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}$$

for any compact operator K on $L^2(\mathbb{H}^n)$. Notice that $(A+K)^n = A^n +$ terms with factor A^pK^{n-p} or K^pA^{n-p}, where $p\in\{1,2,\cdots,n\}$. Since the class of compact operators form a two sided ideal of the class of bounded operators, we can write $(A+K)^n = A^n +$ a compact operator. We are done if we can prove that for a compact operator T, $\lim\limits_{r\to\infty}\dfrac{\operatorname{Tr}(\mathcal{P}_rT\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}=0$. Since T is a compact operator, for given $\epsilon>0$ there exist a finite rank operator T_k such that $\|T_k-T\|\to 0$ as $k\to\infty$. Thus

$$\left|\frac{\operatorname{Tr}(\mathcal{P}_rT\mathcal{P}_r)-\operatorname{Tr}(\mathcal{P}_rT_k\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}\right|\leq\|T-T_k\|\ \to 0 \text{ as } k\to\infty.$$

Therefore for given $\epsilon>0$, there exist $N_0\in\mathbb{N}$ such that $\left|\dfrac{\operatorname{Tr}(\mathcal{P}_rT\mathcal{P}_r)-\operatorname{Tr}(\mathcal{P}_rT_k\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}\right|<\dfrac{\epsilon}{2}$ for $k\geq N_0$. Further, $\left|\frac{\operatorname{Tr}(\mathcal{P}_rT_{N_0}\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}\right|\to 0$ as $r\to\infty$, i.e., for given $\epsilon>0$, there exists $N_1\in\mathbb{N}$ such that $\left|\frac{\operatorname{Tr}(\mathcal{P}_rT_{N_0}\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}\right|<\dfrac{\epsilon}{2}$, for all $r>N_1$. Thus

$$\left|\frac{\operatorname{Tr}(\mathcal{P}_rT\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}\right|\leq\left|\frac{\operatorname{Tr}(\mathcal{P}_rT_{N_0}\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}\right|+\|T-T_{N_0}\|<\epsilon$$

for all $r\geq N_1$. This completes the proof of the theorem. □

In the next result, we provide an alternative proof of the error estimate for $\kappa\in(0,1)$ without using pseudo-differential symbolic calculus, but by proving the boundedness of the operators $[A,V]$ and $[A,\mathcal{L}_{\mathbb{H}}]$ on $L^2(\mathbb{H}^n)$.

Remark 4.6.5. *The proof of part (c) of Lemma 4.5.11 can also be achieved for $\kappa\in(0,1)$ by proving the boundedness of the operators $[A,V]$ and $[A,\mathcal{L}_{\mathbb{H}}]$ on $L^2(\mathbb{H}^n)$. Now for any $h\in L^2(\mathbb{H}^n)$, we have*

$$[A,V]h(g)=(AV-VA)h(g)=\int_{\mathbb{H}^n}K_3(g,g_1)h(g_1)\,dg_1,$$

where

$$K_3(g,g_1)=\big(V(g_1)-V(g)\big)\int_{\mathbb{R}^*}\operatorname{Tr}\big(\pi_\lambda(g)\ a(g,\lambda)\pi_\lambda^*(g_1)\big)\,d\mu(\lambda). \tag{4.6.4}$$

We note that (see Corollary 5.2.25 of [34])

$$\begin{aligned}
a(g,\lambda)&=\widehat{k_g}(\lambda)\\
&=\int_{\mathbb{H}^n}k_g(g_1)\pi_\lambda^*(g_1)dg_1\\
&=\int_{\mathbb{H}^n}k_g(g_1)(I-T^2)^N(1+\lambda^2I)^{-N}\pi_\lambda^*(g_1)dg_1\\
&=(-1)^N(1+\lambda^2I)^{-N}\int_{\mathbb{H}^n}(I-T^2)^Nk_g(g_1)\pi_\lambda^*(g_1)dg_1\\
&=(-1)^N(1+\lambda^2I)^{-N}((I-\widehat{T^2)^{4N}k_g})(\lambda)\\
&=(-1)^N(1+\lambda^2I)^{-N}\sum_{\beta\leq 2N}\pi(T)^\beta a(g,\lambda).
\end{aligned}\tag{4.6.5}$$

Then, using the identity (4.6.5), we have

$$\begin{aligned}
&\big|\operatorname{Tr}\big(\pi_\lambda(g)\ a(g,\lambda)\pi_\lambda^*(g_1)\big)\big|=\big|\operatorname{Tr}\left(\pi_\lambda(g_1^{-1}g)\ a(g,\lambda)\right)\big|\\
&=\bigg|\operatorname{Tr}\bigg(\pi_\lambda(g_1^{-1}g)\left(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g_1^{-1}g)\right)^{-4N}\left(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g_1^{-1}g)\right)^{4N}\\
&\qquad\times(1+\lambda^2I)^{-4N}\sum_{\beta\leq 4N}\pi(T)^\beta a(g,\lambda)\bigg)\bigg|\\
&\leq\left|\operatorname{Tr}\big(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g_1^{-1}g)\big)^{-4N}\right|\left\|\big(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g_1^{-1}g)\big)^{4N}(1+\lambda^2)^{-4N}\right\|_{op}\\
&\qquad\times\left\|\sum_{\beta\leq 4N}\pi(T)^\beta a(g,\lambda)\right\|_{op}\\
&\leq C\left|\operatorname{Tr}\big(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}})+V(g_1^{-1}g)\big)^{-4N}\right|,
\end{aligned}\tag{4.6.6}$$

where the second and the third terms are bounded by Theorem 4.3.13. Now

$$\begin{aligned}
\operatorname{Tr}\ (1+|\lambda|H+V(g_1^{-1}g))^{-4N}&=\sum_\alpha\langle(1+|\lambda|H+V(g_1^{-1}g))^{-4N}\Phi_\alpha,\Phi_\alpha\rangle\\
&=\sum_\alpha\frac{1}{(1+|\lambda|(2|\alpha|+n)+V(g_1^{-1}g))^{4N}},
\end{aligned}$$

and consequently

$$\int_{\mathbb{R}^*}\operatorname{Tr}\big|\big(\pi_\lambda(g)\ a(g,\lambda)\pi_\lambda^*(g_1)\big)\big|\,d\mu(\lambda)$$

$$
\begin{aligned}
&\le C\sum_{\alpha}\int_0^{\infty}\frac{\lambda^n}{(1+\lambda(2|\alpha|+n)+V(g_1^{-1}g))^{4N}}\,d\lambda\\
&= C\sum_{\alpha}\frac{1}{(2|\alpha|+n)^{n+1}}\int_{1+V(g_1g^{-1})}^{\infty}\frac{(u-1-V(g_1^{-1}g))^n}{u^{4N}}\,du\\
&= C_2\frac{1}{(1+V(g_1^{-1}g))^{4N-n-1}}. \qquad (4.6.7)
\end{aligned}
$$

Thus from (4.6.4), (4.6.6), and (4.6.7), we get

$$|K_3(g,g_1)| \le \frac{c|(V(g_1)-V(g)|}{(1+V(g_1^{-1}g))^{4N-n-1}}.$$

For large $|g|$ and $|g_1|$, using the triangle inequality for the homogeneous norm (see page 113 of [34]) and the fact that $\kappa\in(0,1)$, we have

$$
\begin{aligned}
\|[A,V]h\|_2^2 &\le \int_{\mathbb{H}^n}\left|\int_{\mathbb{H}^n}K_3(g,g_1)h(g_1)\,dg_1\right|^2 dg\\
&\le c_N\int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n}\left|\frac{||g_1|^k-|g|^k|h(g_1)}{(1+\left|g_1^{-1}g\right|^k)^{4N-n-1}}\right|dg_1\right)^2 dg\\
&\le c_N\int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n}\left|\frac{||g_1|-|g||h(g_1)}{(1+\left|g_1^{-1}g\right|^k)^{4N-n-1}}\right|dg_1\right)^2 dg\\
&\le c_N\int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n}\left|\frac{\left|g_1^{-1}g\right|h(g_1)}{(1+\left|g_1^{-1}g\right|^k)^{4N-n-1}}\right|dg_1\right)^2 dg\\
&\le c_N\int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n}\left|\frac{h(g_1)}{(1+\left|g_1^{-1}g\right|^k)^{4N-n-2}}\right|dg_1\right)^2 dg = \||h|*K\|_2^2,
\end{aligned}
$$

where $K(g)=\dfrac{1}{(1+|g|^k)^{4N-n-2}}$. Since for a sufficiently large $N\in\mathbb{N}$, $K\in L^1(\mathbb{H}^n)$, an application of Minkowski's inequality gives $\|[A,V]h\|_2\le C\|K\|_1\|h\|_2$.

If $|g|$ and $|g_1|$ are lying in some compact set $\mathcal{K}\subset\mathbb{R}$, then $\int_{\mathcal{K}}\left|\int_{\mathcal{K}}K_3(g,g_1)h(g_1)\,dg_1\right|^2 dg\le C_{\mathcal{K}}\|h\|_2$. If $|g|$ (or $|g_1|$) lies in K and $|g_1|$ (or $|g|$) is large, an application of Cauchy-Schwarz inequality gives $\|[A,V]h\|_2\le\|h\|_2\int_{\mathcal{K}}\int_{|g|}|K_3(g,g_1)|^2\,dg_1\,dg\le C_{\mathcal{K}}\|h\|_2$.

Now for any $\kappa\in(0,1)$, clearly the operator $[V,A]$ is bounded. The boundedness of the operator $[\mathcal{L}_{\mathbb{H}},A]$ will imply boundedness of the operator $[\mathcal{H},A]$ on $L^2(\mathbb{H}^n)$ as $[T^2,A]=0$. Using the identity (4.6.5), we get

$$A\mathcal{L}_{\mathbb{H}}h(g)=\int_{\mathbb{R}^*}\operatorname{Tr}\left(\pi_\lambda^*(g)a(g,\lambda)\widehat{\mathcal{L}_{\mathbb{H}}h}(\lambda)\right)\,d\mu(\lambda)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^*} \operatorname{Tr}\left(\pi_\lambda^*(g)(1+\lambda^2 I)^{-2N} \sum_{\beta \le 4N} \pi(T)^\beta a(g,\lambda)\ \hat{h}(\lambda)|\lambda| H \right) d\mu(\lambda) \\
&= \int_{\mathbb{R}^*} \operatorname{Tr}\Big(\left(\pi_\lambda^*(g)\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g)\right)^{-4N+1} \left(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g)\right)^{4N} (1+\lambda^2 I)^{-2N} \\
&\qquad \times \sum_{\beta \le 4N} \pi(T)^\beta a(g,\lambda)\ \hat{h}(\lambda)|\lambda| H \left(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g)\right)^{-1} \Big) d\mu(\lambda).
\end{aligned}
$$

Arguing as in (4.6.6) and (4.6.7), we obtain

$$
\begin{aligned}
|A\mathcal{L}_{\mathbb{H}} h(g)| &\le \int_{\mathbb{R}^*} \operatorname{Tr}\left(\left(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g)\right)^{-4N+1} \right) \|\hat{h}(\lambda)\|_{op}\, d\mu(\lambda) \\
&\le \int_{\mathbb{R}^*} \operatorname{Tr}\left(\left(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g)\right)^{-4N+1} \right) \|\hat{h}(\lambda)\|_{S_2}\, d\mu(\lambda) \\
&\le \left(\int_{\mathbb{R}^*} \left| \operatorname{Tr}\left(\left(\pi_\lambda(I-\mathcal{L}_{\mathbb{H}}) + V(g)\right)^{-4N+1} \right) \right|^2 d\mu(\lambda) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^*} \|\hat{h}(\lambda)\|_{S_2}^2\, d\mu(\lambda) \right)^{\frac{1}{2}} \\
&\le C\|h\|_2 (1+V(g))^{\frac{-8N+n+3}{2}}.
\end{aligned}
$$

For sufficiently large N*, an application of Cauchy-Schwarz inequality gives* $\|A\mathcal{L}_{\mathbb{H}} h\|_2 \le M_1 \|h\|_2$*. Further,*

$$
\begin{aligned}
&\mathcal{L}_{\mathbb{H}} A h(g) \\
&= \int_{\mathbb{R}^*} \operatorname{Tr}\left(\pi_\lambda^*(g) \widehat{\mathcal{L}_{\mathbb{H}} A h}(\lambda) \right) d\mu(\lambda) \\
&= \int_{\mathbb{R}^*} \operatorname{Tr}\left(\pi_\lambda^*(g) \widehat{Ah}(\lambda)|\lambda| H \right) d\mu(\lambda) \\
&= \int_{\mathbb{R}^*} \int_{\mathbb{H}^n} Ah(g_1)\ \operatorname{Tr}\Big(\pi_\lambda^*(g)\pi_\lambda(g_1)|\lambda| H \Big) d\mu(\lambda)\, dg_1 \\
&= \int_{\mathbb{R}^*} \int_{\mathbb{H}^n} \left(\int_{\mathbb{R}^*} \operatorname{Tr}\left(\pi_{\lambda_1}^*(g_1) a(g_1,\lambda_1) \widehat{h}(\lambda_1) \right) d\mu(\lambda_1) \right) \operatorname{Tr}\Big(\pi_\lambda^*(g)\pi_\lambda(g_1)|\lambda| H \Big) d\mu(\lambda)\, dg_1.
\end{aligned}
$$

Proceeding as in (4.6.7), we get

$$
\int_{\mathbb{R}^*} \left| \operatorname{Tr}\left(\pi_{\lambda_1}^*(g_1) a(g_1,\lambda_1) \widehat{h}(\lambda_1) \right) \right| d\mu(\lambda_1) \le C\|h\|_2 (1+V(g_1))^{\frac{-8N+n+1}{2}}.
$$

Moreover, a similar way as in (4.6.7), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^*} \int_{\mathbb{H}^n} \left| \operatorname{Tr}\left(\pi_\lambda^*(g)\pi_\lambda(g_1)|\lambda| H \right) \right| (1+V(g_1))^{\frac{-8N+n+1}{2}}\, d\mu(\lambda)\, dg_1 \\
&\le \int_{\mathbb{H}^n} (1+V(g_1))^{\frac{-8N+n+1}{2}} dg_1 \int_{\mathbb{R}^*} \left| \operatorname{Tr}\left(|\lambda| H\ \pi_\lambda^*(g) \right) \right| d\mu(\lambda) \\
&\le \int_{\mathbb{R}^*} \operatorname{Tr}\left((1+\lambda^2)^{-4N} \sum_{\beta \le 8N} \pi(T)^\beta |\lambda| H \right) d\mu(\lambda)
\end{aligned}
$$

$$\leq \quad C(1+V(g))^{-8N+n}.$$

Therefore $\|[A,\mathcal{L}_{\mathbb{H}}]h\|_2 \leq M\|h\|_2$ *and so the operator* $[A,\mathcal{H}]$ *is bounded on* $L^2(\mathbb{H}^n)$.

Now setting $A=\mathcal{H}, B=A, \chi=0, \psi=f,$ *and* $P_\lambda=\pi_r$ *in Theorem 1.6 of Laptev-Safarov [63], we get*

$$\begin{aligned}&|\mathrm{Tr}(\mathcal{P}_r f(A)\mathcal{P}_r-\mathcal{P}_r f(\mathcal{P}_r A\mathcal{P}_r)\mathcal{P}_r)|\\ &\leq \frac{1}{2}\|f''\|_\infty N_r(r)\Big(\|\pi_r A\|^2+\frac{\pi^2}{6r^2}\|\pi_{r-r}[A,\mathcal{H}]\|^2\Big).\end{aligned}$$

Dividing both sides by $\mathrm{Tr}(\mathcal{P}_r)$ *and setting* $r=r^\alpha, \alpha\in(0,1)$, *the boundness of* $A,[A,\mathcal{H}]$ *together with (4) of Corollary 4.5.8 implies that*

$$\frac{|\mathrm{Tr}(\mathcal{P}_r f(A)\mathcal{P}_r-\mathcal{P}_r f(\mathcal{P}_r A\mathcal{P}_r)\mathcal{P}_r)|}{\mathrm{Tr}(\mathcal{P}_r)}\leq C\frac{N_{r^\alpha}(r)}{\mathrm{Tr}(\mathcal{P}_r)}\to 0 \text{ as } r\to\infty,$$

where $N_r(r)=\sup_{\mu\leq r}(\mathrm{Tr}(\pi_\mu-\pi_{\mu-r}))$.

4.7 Appendix

We collect few definitions and theorems of Grishin-Poedintseva [42], that we use in this chapter for the reader's convenience.

Definition 4.7.1. *Let* ϕ *be a positive function on the half line* $[0,\infty)$. *Let*

$$S=\{\alpha:\exists M,R \text{ with } \phi(\mathrm{tr})\leq Mt^\alpha\phi(r), \text{ for all } t\geq 1, r\geq R\}$$

and

$$G=\{\beta:\exists M,R \text{ with } \phi(tr)\geq Mt^\beta\phi(r), \text{ for all } t\geq 1, r\geq R\}.$$

Then the numbers $\alpha(\phi):=\inf S$ *and* $\beta(\phi):=\sup G$ *are called the* **upper** *and* **lower Matushevskaya index** *of* ϕ, *respectively.*

Theorem 4.7.2 ([42], Theorem 2). *Let* $m>-1$. *Assume that* φ *is positive measurable function on* $[0,\infty)$ *that does not vanish identically in any neighborhood of infinity. Let* $\Phi(r)=\int_0^\infty\frac{\varphi(rt)}{(1+t)^{m+1}}dt$ *be finite. Then the functions* φ *and* Φ *have same growth at infinity if and only if* $\beta(\varphi)>-1$ *and* $\alpha(\varphi)<m$.

Theorem 4.7.3 ([42], Theorem 8). *Let* φ *and* ψ *be positive functions on* $[0,\infty)$ *satisfying the following conditions:*

1. *the functions* φ *and* ψ *do not vanish identically in any neighborhood of infinity;*

2. *the function* φ *is multiplicatively continuous at infinity and* $\beta(\varphi) > -1$*;*

3. *the function* ψ *is increasing;*

4. *at least one of the inequalities* $\alpha(\varphi) < m$ *and* $\alpha(\psi) < m$ *holds, where* $m > -1$*;*

5. *the functions*

$$\Phi(r) = \int_0^\infty \frac{\varphi(ru)}{(1+u)^{m+1}}du \quad \textit{and} \quad \Psi(r) = \int_0^\infty \frac{\psi(ru)}{(1+u)^{m+1}}du$$

are finite, and if $\lim\limits_{r\to\infty} \dfrac{\Psi(r)}{\Phi(r)} = 1$*, then* $\lim\limits_{r\to\infty} \dfrac{\psi(r)}{\varphi(r)} = 1$*.*

CHAPTER 5

Schatten class and nuclear pseudo-differential operators on homogeneous spaces of compact groups

5.1 Introduction

Let G be a compact (Hausdorff) group and H be a closed subgroup of G. In this chapter, we present symbolic criteria for pseudo-differential operators on the compact homogeneous space G/H characterizing the Schatten-von Neumann classes $S_r(L^2(G/H))$ for all $0 < r \leq \infty$. We provide a symbolic characterization for r-nuclear, $0 < r \leq 1$, pseudo-differential operators on $L^p(G/H)$ with applications to adjoint, product, and trace formulae. The criteria here are given in terms of matrix-valued symbols defined on noncommutative analogue of phase space $G/H \times \widehat{G/H}$. Finally, we present an application of aforementioned results in the context of heat kernel.

5.2 Fourier analysis and the global quantization on homogeneous spaces of compact groups

We begin this section by recalling some basic and important concepts of harmonic analysis on homogeneous spaces of compact (Hausdorff) groups from [38], which is almost similar to the theory given in [1] and [105] (see also [23,29,78] for homogeneous spaces of compact Lie groups).

Throughout the chapter, we assume that G is a compact (Hausdorff) group with the normalized Haar measure dx and H is a closed subgroup of G with the probability Haar measure dh. The left coset space G/H can be seen as a homogeneous space with respect to the action of G on G/H given by left multiplication. Let $C(\Omega)$ denote the space of all continuous functions on a compact Hausdorff space Ω. Define $T_H : C(G) \to C(G/H)$ by

$$T_H(f)(xH) = \int_H f(xh)\, dh, \quad xH \in G/H.$$

Then T_H is onto. The homogeneous space G/H has a unique normalized G-invariant positive Radon measure μ such that the Weil formula

$$\int_{G/H} T_H(f)(xH)\, d\mu(xH) = \int_G f(x)\, dx$$

holds. The map T_H can be extended to $L^2(G/H, \mu)$ and is a partial isometry on $L^2(G/H)$ with $\langle T_H(f), T_H(g)\rangle_{L^2(G/H,\mu)} = \langle f, g\rangle_{L^2(G)}$, for all $f, g \in L^2(G)$.

Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of a compact group G on a Hilbert space $\mathcal{H}_\pi$. It is well-known that any irreducible representation $(\pi, \mathcal{H}_\pi)$ of G is finite dimensional with the dimension d_π (say). Consider an operator-valued integral

$$T_H^\pi := \int_H \pi(h)\, dh$$

defined in the weak sense, i.e., $\langle T_H^\pi u, v\rangle = \int_H \langle \pi(h)u, v\rangle\, dh$, for all $u, v \in \mathcal{H}_\pi$. Note that T_H^π is a bounded linear operator on $\mathcal{H}_\pi$ with norm bounded by one. Further, T_H^π is a partial isometric orthogonal projection and is an identity operator if and only if $\pi(h) = I$ for all $h \in H$ (see [38]).

Definition 5.2.1. *Let H be a closed subgroup of a compact group G. Then the dual $\widehat{G/H}$ of G/H is a subset of $\widehat{G}$ given by*

$$\widehat{G/H} := \left\{\pi \in \widehat{G} : T_H^\pi \neq 0\right\} = \left\{\pi \in \widehat{G} : \int_H \pi(h)\, dh \neq 0\right\}.$$

We note that the set $\widehat{G/H}$ is the set of all type 1 representations of G with respect to H which was denoted by $\widehat{G_0}$ in [78, 105].

Let $\pi \in \widehat{G/H}$. Then the functions $\pi^H_{\zeta,\xi} : G/H \to \mathbb{C}$ defined by

$$\pi^H_{\zeta,\xi}(xH) := \langle \pi(x)T^\pi_H \zeta, \xi \rangle, \qquad xH \in G/H,$$

for $\xi, \zeta \in \mathcal{H}_\pi$, are called H-matrix elements of $(\pi, \mathcal{H}_\pi)$. If $\{e_1, e_2, \dots, e_{d_\pi}\}$ is an orthonormal basis for $\mathcal{H}_\pi$ then we denote $\langle \pi(x)T^\pi_H e_i, e_j\rangle$ by $\pi^H_{ij}(xH)$. Using the orthogonality relation of matrix coefficients of G and the fact that $T_H(\pi_{\zeta,\xi}) = \pi^H_{\zeta,\xi}$, we have

$$\left\langle \pi^H_{i,j}, \xi^H_{k,l} \right\rangle_{L^2(G/H,\mu)} = \frac{1}{d_\pi}\delta_{\pi\xi}\delta_{ik}\delta_{jl}.$$

Let $\varphi \in L^1(G/H, \mu)$ and $\pi \in \widehat{G/H}$. Then the group Fourier transform $\mathcal{F}_{G/H}(\varphi)$ of φ at π defined by

$$\mathcal{F}_{G/H}(\varphi)(\pi) = \hat{\varphi}(\pi) := \int_{G/H} \varphi(xH)\Gamma_\pi(xH)^*\, d\mu(xH) \tag{5.2.1}$$

is a bounded linear operator on the Hilbert space $\mathcal{H}_\pi$, where for $xH \in G/H$ the notation $\Gamma_\pi(xH)$ stands for a bounded linear operator on $\mathcal{H}_\pi$ satisfying

$$\langle \zeta, \Gamma_\pi(xH)\xi\rangle = \langle \zeta, \pi(x)T^\pi_H \xi\rangle$$

for all $\zeta, \xi \in \mathcal{H}_\pi$. Note that, from the notation of $\Gamma_\pi(xH)$, the H-matrix coefficients $\pi^H_{i,j}(xH)$ are same as $\Gamma_\pi(xH)_{ij}$. Moreover, if $\varphi \in L^2(G/H)$, $\hat{\varphi}(\pi)$ is a Hilbert-Schmidt operator on $\mathcal{H}_\pi$ and satisfies the following Plancherel formula as stated in next theorem.

Theorem 5.2.2. *For $\varphi \in L^2(G/H, \mu)$, we have*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \|\hat{\varphi}(\pi)\|^2_{S_2} = \|\varphi\|^2_{L^2(G/H,\mu)},$$

where $\|.\|_{S_2}$ stands for the Hilbert-Schmidt norm on the space of all Hilbert-Schmidt operators on $\mathcal{H}_\pi$.

Theorem 5.2.3. *For $\varphi \in L^2(G/H, \mu)$, the following Fourier inversion formula holds*

$$\varphi(xH) = \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{Tr}[\hat{\varphi}(\pi)\pi(x)T^\pi_H], \qquad \text{for } \mu-a.e.\ xH \in G/H. \tag{5.2.2}$$

We would like to record the following lemma whose proof is similar to [26, Lemma 2.5] by using the fact that the operator T^π_H is norm bounded by one.

Lemma 5.2.4. *Let G/H be a compact homogeneous space with the normalized measure μ and let $\pi \in \widehat{G/H}$. Then, for all $1 \leq i, j \leq d_\pi$, we have*

$$\|\Gamma_\pi(\cdot)_{ij}\|_{L^q(G/H)} \leq \begin{cases} d_\pi^{-\frac{1}{q}}, & \text{if} \quad 2 \leq q \leq \infty, \\ d_\pi^{-\frac{1}{2}}, & \text{if} \quad 1 \leq q \leq 2, \end{cases}$$

with the convention that for $q = \infty$ we have $d_\pi^{-\frac{1}{q}} = 1$.

Given a continuous linear operator $T : C(G/H) \to C(G/H)$, its *matrix-valued global symbol* $\sigma_T(xH, \pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ is defined by

$$T_H^\pi \sigma_T(xH, \pi) = \pi(x)^*(T\Gamma_\pi)(xH), \qquad (xH, \pi) \in G/H \times \widehat{G/H}. \tag{5.2.3}$$

where $T\Gamma_\pi$ stands for the action of T on the matrix components of $\Gamma_\pi(xH)$. Setting $(T\Gamma_\pi(xH))_{mn} = (T(\Gamma_{\pi_{mn}}))(xH)$, we have

$$(T_H^\pi \sigma_T(xH, \pi))_{mn} := \sum_{k=1}^{d_\pi} \overline{\pi_{km}(x)} (T\Gamma_\pi(xH))_{kn},$$

where $1 \leq m, n \leq d_\pi$.

Assume that σ_T is a matrix-valued global symbol for the continuous linear operator $T : C(G/H) \to C(G/H)$ as above. Then we can recover the operator T by using the Fourier inversion formula as follows:

$$\begin{aligned} Tf(xH) &= T\left(\sum_{[\pi] \in \widehat{G/H}} d_\pi \operatorname{Tr}(\pi(x) T_H^\pi \hat{f}(\pi)) \right) \\ &= \sum_{[\pi] \in \widehat{G/H}} d_\pi \operatorname{Tr}(T\Gamma_\pi(xH)\, \hat{f}(\pi)). \end{aligned}$$

Using (5.2.3) and the relation $\pi(x) T_H^\pi = \Gamma_\pi(xH)$, we get

$$Tf(xH) = \sum_{[\pi] \in \widehat{G/H}} d_\pi \operatorname{Tr}(\Gamma_\pi(xH) \sigma_T(xH, \pi) \hat{f}(\pi)) \tag{5.2.4}$$

for all $f \in C(G/H)$, μ-a.e. $xH \in G/H$, and the sum is independent of the representation from each equivalence class $[\pi] \in \widehat{G/H}$. We will also write $T = \operatorname{Op}(\sigma_T)$ for the operator T given by the formula (5.2.4). The operator $T = \operatorname{Op}(\sigma_T)$ will be called *pseudo-differential operator* corresponding to matrix-valued symbol σ_T. For more details and consistent development of this quantization on compact Lie group and the corresponding symbolic calculus, we refer to [85] and [86].

Remark 5.2.5. *Let H be a closed normal subgroup of the compact group G and μ be the normalized G-invariant measure over the left quotient group G/H associated to the Weil formula. Then μ is a Haar measure on the compact (quotient) group G/H and $\widehat{G/H} = H^{\perp} := \{\pi \in \widehat{G} : \pi(h) = I \text{ for all } h \in H\}$. Moreover, the Fourier transform (5.2.1), inverse Fourier transform (5.2.2), and pseudo-differential operator defined by (5.2.4) coincide with the classical Fourier transform, inverse Fourier transform, and pseudo-differential operator on the compact group G/H, respectively.*

5.3 r-Schatten-von Neumann class of pseudo-differential operators on $L^2(G/H)$

This section is devoted to the study of r-Schatten-von Neumann class of pseudo-differential operators on the Hilbert space $L^2(G/H)$. We begin this section with the definition of $L^2(G/H \times \widehat{G/H})$ space.

Let $L^2(G/H \times \widehat{G/H})$ denotes the space of all matrix-valued functions σ_A on $G/H \times \widehat{G/H}$ such that

$$\|\sigma_A\|_{L^2(G/H\times\widehat{G/H})} = \left(\int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \|\sigma_A(xH,\xi)T_H^\xi\|_{S_2}^2 \, d\mu(xH)\right)^{\frac{1}{2}} < \infty.$$

The following theorem gives a characterization of Hilbert-Schmidt pseudo-differential operators on G/H. We remark here that the following theorem is already proved by Kumar in [60] using a different method.

Theorem 5.3.1. *Let $T : L^2(G/H) \to L^2(G/H)$ be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then T is a Hilbert-Schmidt operator if and only if $\sigma_T \in L^2(G/H \times \widehat{G/H})$. Moreover, we have*

$$\|T\|_{S_2} = \|\sigma_T\|_{L^2(G/H\times\widehat{G/H})}.$$

Proof. For all $f \in L^2(G/H)$, we have

$$Tf(xH) = \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\hat{f}(\xi))$$

$$
\begin{aligned}
&= \int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(wH)^*) f(wH)\, d\mu(wH) \\
&= \int_{G/H} K(xH,wH) f(wH)\, d\mu(wH),
\end{aligned}
$$

where the kernel $K(xH,wH)$ is given by

$$K(xH,wH) = \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(wH)^*), \quad xH, wH \in G/H.$$

Then

$$
\begin{aligned}
\|T\|_{S_2}^2 &= \int_{G/H}\int_{G/H} |K(xH,yH)|^2\, d\mu(xH)\, d\mu(yH) \\
&= \int_{G/H}\int_{G/H} |K(xH,xz^{-1}H)|^2\, d\mu(xH)\, d\mu(zH).
\end{aligned}
$$

Note that

$$
\begin{aligned}
K(xH,xz^{-1}H) &= \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}\left(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(xz^{-1}H)^*\right) \\
&= \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}\left(\Gamma_\xi(zH)\sigma_T(xH,\xi)T_H^\xi\right) \\
&= \left(\mathcal{F}^{-1}\tau(xH,\cdot)\right)(zH),
\end{aligned}
$$

where $\tau(xH,\xi) = \sigma_T(xH,\xi)T_H^\xi$. Therefore, using Plancherel's formula, we have

$$
\begin{aligned}
\|T\|_{S_2}^2 &= \int_{G/H}\int_{G/H} |K(xH,xz^{-1}H)|^2 d\mu(xH)\, d\mu(zH) \\
&= \int_{G/H}\int_{G/H} |\mathcal{F}^{-1}\tau(xH,\cdot)(zH)|^2 d\mu(xH)\, d\mu(zH) \\
&= \int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \|\tau(xH,\xi)\|_{S_2}^2\, d\mu(xH) \\
&= \int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \|\sigma_T(xH,\xi)T_H^\xi\|_{S_2}^2\, d\mu(xH) \\
&= \|\sigma_T\|_{L^2(G/H\times\widehat{G/H})}.
\end{aligned}
$$

□

The following lemma is a consequence of the definition of r-Schatten-von Neumann class (see [28]).

Lemma 5.3.2. *Let $A : \mathcal{H} \to \mathcal{H}$ be a compact linear operator. Let $0 < r, t < \infty$. Then $A \in S_r(\mathcal{H})$ if and only if $|A|^{\frac{r}{t}} \in S_t(\mathcal{H})$. Moreover, $\|A\|_{S_r}^r = \||A|^{\frac{r}{t}}\|_{S_t}^t$.*

The corollary below is the main result of this section which present a characterization of a pseudo-differential operator on $L^2(G/H)$ to be in r-Schatten-von Neumann class of operators. The proof follows from Lemma 5.3.2 with $t = 2$ and Theorem 5.3.1.

Corollary 5.3.3. *Let $T : L^2(G/H) \to L^2(G/H)$ be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then $T \in S_r\left(L^2(G/H)\right)$ if and only if*

$$\int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \|\sigma_{|T|^{\frac{r}{2}}}(xH, \xi) T_H^\xi\|_{S_2}^2 \, d\mu(xH) < \infty.$$

5.4 Characterizations and traces of r-nuclear, $0 < r \leq 1$, pseudo-differential operators on $L^p(G/H)$

This section is devoted to the study of r-nuclear operators on Banach spaces $L^p(G/H)$. Here we present a symbolic characterization of r-nuclear operators and give a formula for the nuclear trace of such operators. We begin this section by recalling the basic notions of nuclear operators on Banach spaces.

Let $0 < r \leq 1$ and T be a bounded linear operator from a complex Banach space X into another complex Banach space Y such that there exist sequences $\{x_n'\}_{n=1}^\infty$ in the dual space X' of X and $\{y_n\}_{n=1}^\infty$ in Y such that $\sum_{n=1}^\infty \|x_n'\|_{X'}^r \|y_n\|_Y^r < \infty$ and

$$Tx = \sum_{n=1}^\infty x_n'(x) y_n, \quad x \in X.$$

Then we call $T : X \to Y$ a r-nuclear operator and if $X = Y$, then its nuclear trace $\mathrm{Tr}(T)$ is given by

$$\mathrm{Tr}(T) = \sum_{n=1}^\infty x_n'(y_n).$$

The definition of r-nuclear operators is independent of the choices of the sequences $\{x_n'\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$. An 1-nuclear operators will be simply called a nuclear operator. The following theorem is a characterization of r-nuclear operators on σ-finite measure spaces [25].

Theorem 5.4.1. *Let $0 < r \leq 1$. Let (X_1, μ_1) and (X_2, μ_2) be two σ-finite measure spaces. Then a bounded linear operator $T : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2), 1 \leq p_1, p_2 < \infty$, is r-nuclear if and only if there exist sequences $\{g_n\}_{n=1}^{\infty}$ in $L^{p_1'}(X_1, \mu_1)$ and $\{h_n\}_{n=1}^{\infty}$ in $L^{p_2}(X_2, \mu_2)$ such that for all $f \in L^{p_1}(X_1, \mu_1)$,*

$$(Tf)(x) = \int_{X_1} K(x, y) f(y) d\mu_1(y), \quad x \in X_2,$$

where

$$K(x, y) = \sum_{n=1}^{\infty} h_n(x) g_n(y), \quad x \in X_2, y \in X_1,$$

and

$$\sum_{n=1}^{\infty} \|g_n\|^r_{L^{p_1'}(X_1,\mu_1)} \|h_n\|^r_{L^{p_2}(X_2,\mu_2)} < \infty.$$

Let $0 < r \leq 1$ and (X, μ) be a σ-finite measure space. Let $T : L^p(X, \mu) \to L^p(X, \mu)$, $1 \leq p < \infty$, be a r-nuclear operator. Then by Theorem 5.4.1, we can find sequences $\{g_n\}_{n=1}^{\infty}$ in $L^{p'}(X, \mu)$ and $\{h_n\}_{n=1}^{\infty}$ in $L^p(X, \mu)$ such that

$$\sum_{n=1}^{\infty} \|g_n\|^r_{L^{p'}(X,\mu)} \|h_n\|^r_{L^p(X,\mu)} < \infty,$$

and for all $f \in L^p(X, \mu)$, we have

$$(Tf)(x) = \int_X K(x, y) f(y) \, d\mu(y), \quad x \in X,$$

where

$$K(x, y) = \sum_{n=1}^{\infty} h_n(x) g_n(y), \quad x, y \in X,$$

and it satisfies

$$\int_X |K(x, y)| \, d\mu(y) \leq \sum_{n=1}^{\infty} \|g_n\|^r_{L^{p'}(X,\mu)} \|h_n\|^r_{L^p(X,\mu)} .$$

The nuclear trace $\mathrm{Tr}(T)$ of $T : L^p(X, \mu) \to L^p(X, \mu)$ is given by

$$\mathrm{Tr}(T) = \int_X K(x, x) \, d\mu(x). \tag{5.4.1}$$

Now, we present a characterization of r-nuclear pseudo-differential operators from $L^{p_1}(G/H)$ into $L^{p_2}(G/H)$.

Theorem 5.4.2. *Let $0 < r \leq 1$ and let $T : L^{p_1}(G/H) \to L^{p_2}(G/H), 1 \leq p_1, p_2 < \infty$, be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Suppose that σ_T satisfies*

$$\sum_{[\xi]\in\widehat{G/H}} d_\xi^{2+\frac{r}{\tilde{p_1}}} \| \|\sigma_T(\cdot,\xi)^t\|_{op(\ell^\infty,\ell^\infty)} \|^r_{L^{p_2}(G/H)} < \infty,$$

where $\tilde{p_1} = min\{2, p_1\}$ and $\sigma_T(\cdot,\xi)^t$ denotes the transpose of the operator $\sigma_T(\cdot,\xi)^t$. Then the operator T is r-nuclear.

Proof. Since the operator T can be written as

$$Tf(xH) = \int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(wH)^*) f(wH)\, d\mu(wH),$$

the kernel of T is given by

$$K(xH, wH) = \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(wH)^*).$$

Now we write

$$\operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(wH)^*) = \sum_{i,j=1}^{d_\xi} (\Gamma_\xi(xH)\sigma_T(xH,\xi))_{ij}\overline{\Gamma_\xi(wH)}_{ij},$$

and set that $h_{\xi,ij}(xH) = d_\xi(\Gamma_\xi(xH)\sigma_T(xH,\xi))_{ij}$, and $g_{\xi,ij}(wH) = (\Gamma_\xi(wH)^*)_{ji} = \overline{\Gamma_\xi(wH)}_{ij}$. We observe that

$$(\Gamma_\xi(xH)\sigma_T(xH,\xi))_{ij} = \sum_{k=1}^{d_\xi} \Gamma_\xi(xH)_{ik}\, \sigma_T(xH,\xi)_{kj} = \sum_{k=1}^{d_\xi} (\sigma_T(xH,\xi))^t_{jk}\, \Gamma_\xi(xH)_{ik}.$$

By taking into account that $|\Gamma_\xi(xH)_{ik}| \leq 1$, we get

$$\begin{aligned} |(\Gamma_\xi(xH)\sigma_T(xH,\xi))_{ij}| &= \left| \sum_{k=1}^{d_\xi} (\sigma_T(xH,\xi))^t_{jk}\, \Gamma_\xi(xH)_{ik} \right| \\ &\leq \|\sigma_T(xH,\xi)^t\|_{op(\ell^\infty,\ell^\infty)} \|(\Gamma_\xi(xH)_{i1}, \Gamma_\xi(xH)_{i2}, \dots, \Gamma_\xi(xH)_{id_\xi})\|_{\ell^\infty} \\ &\leq \|\sigma_T(xH,\xi)^t\|_{op(\ell^\infty,\ell^\infty)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|h_{\xi,ij}(\cdot)\|^r_{L^{p_2}(G/H)} &= \|d_\xi\, (\Gamma_\xi(\cdot)\sigma_T(\cdot,\xi))_{ij}\|^r_{L^{p_2}(G/H)} \\ &\leq d_\xi^r \| \|\sigma_T(\cdot,\xi)^t\|_{op(\ell^\infty,\ell^\infty)} \|_{L^{p_2}(G/H)}. \end{aligned}$$

If p_1' denotes the Lebesgue conjugate of p_1, then we have $\frac{1}{\tilde{p}_1}+\frac{1}{\tilde{q}_1}=1$, where $\tilde{q}_1=\max\{2,p_1'\}$. By Lemma 5.2.4, we have $\|\Gamma_\xi(\cdot)\|^r_{L^{p_1'}(G/H)}\le d_\xi^{-\frac{r}{\tilde{p}_1'}}$. Thus

$$\begin{aligned}\sum_{[\xi],i,j}\|g_{\xi,ij}(\cdot)\|^r_{L^{p_1'}(G/H)}\|h_{\xi,ij}(\cdot)\|^r_{L^{p_2}(G/H)} &\le \sum_{[\xi]} d_\xi^{-\frac{r}{\tilde{p}_1'}} d_\xi^r d_\xi^2 \| \|\sigma_T(\cdot,\xi)^t\|_{op(\ell^\infty,\ell^\infty)}\|_{L^{p_2}(G/H)}\\ &\le \sum_{[\xi]} d_\xi^{2+\frac{r}{\tilde{p}_1}} \| \|\sigma_T(\cdot,\xi)^t\|_{op(\ell^\infty,\ell^\infty)}\|_{L^{p_2}(G/H)}<\infty.\end{aligned}$$

Hence, by invoking Theorem 5.4.1, it follows that T is r-nuclear. ☐

Next theorem gives a necessary and sufficient condition for an operator to be r-nuclear in terms of its symbolic decomposition.

Theorem 5.4.3. *Let $0<r\le 1$ and let $T:L^{p_1}(G/H)\to L^{p_2}(G/H), 1\le p_1,p_2<\infty$, be a continuous linear operator with matrix-valued symbol σ_T on $G/H\times\widehat{G/H}$. Then T is r-nuclear if and only if there exist sequences $\{g_k\}_{k=1}^\infty\in L^{p_1'}(G/H)$ and $\{h_k\}_{k=1}^\infty\in L^{p_2}(G/H)$ such that*

$$\sum_{k=1}^\infty \|g_k\|^r_{L^{p_1'}(G/H)}\,\|h_k\|^r_{L^{p_2}(G/H)}<\infty$$

and

$$T_H^\xi\sigma_T(xH,\xi)=\xi(x)^*\sum_{k=1}^\infty h_k(xH)\widehat{\overline{g_k}}(\xi)^*,\quad (xH,\xi)\in G/H\times\widehat{G/H}.$$

Proof. Suppose that $T:L^{p_1}(G/H)\to L^{p_2}(G/H)$ is r-nuclear for $1\le p_1,p_2<\infty$. Then by Theorem 5.4.1, there exist sequences $\{g_k\}_{k=1}^\infty$ in $L^{p_1'}(G/H)$ and $\{h_k\}_{k=1}^\infty$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^\infty \|g_k\|^r_{L^{p_1'}(G/H)}\,\|h_k\|^r_{L^{p_2}(G/H)}<\infty$$

and for all $f\in L^{p_1}(G/H)$, we have

$$\begin{aligned}(Tf)(xH) &= \sum_{[\pi]\in\widehat{G/H}} d_\pi\,\mathrm{Tr}(\Gamma_\pi(xH)\sigma_T(xH,\pi)\hat{f}(\pi))\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \sum_{i,j=1}^{d_\pi}(\Gamma_\pi(xH)\sigma_T(xH,\pi))_{ij}\hat{f}(\pi)_{ji}\\ &= \int_{G/H}\sum_{[\pi]\in\widehat{G/H}} d_\pi\sum_{i,j=1}^{d_\pi}(\Gamma_\pi(xH)\sigma_T(xH,\pi))_{ij}\overline{\Gamma_\pi(wH)_{ij}}f(wH)\,d\mu(wH)\\ &= \int_{G/H}\left(\sum_{k=1}^\infty h_k(xH)g_k(wH)\right)f(wH)\,d\mu(wH) \qquad (5.4.2)\end{aligned}$$

for all $xH \in G/H$. Let ξ be a fixed but arbitrary element in $\widehat{G/H}$. Then for $1 \leq m, n \leq d_\xi$, we define the function f on G/H by

$$f(wH) = \Gamma_\xi(wH)_{nm}, \quad wH \in G/H.$$

Since

$$\int_{G/H} \Gamma_\xi(wH)_{nm} \overline{\Gamma_\pi(wH)_{ij}} \, d\mu(wH) = \frac{1}{d_\xi}$$

if and only if $\pi = \xi$, $i = n$ and $j = m$, and is zero otherwise, it follows from (5.4.2) that

$$\begin{aligned}(\Gamma_\xi(xH)\sigma_T(xH,\xi))_{nm} &= \sum_{k=1}^{\infty} h_k(xH) \left(\int_{G/H} g_k(wH)\, \Gamma_\xi(wH)_{nm} \, d\mu(wH) \right) \\ &= \sum_{k=1}^{\infty} h_k(xH) \overline{\left(\widehat{g_k}(\xi)\right)_{mn}}.\end{aligned}$$

Therefore,

$$T_H^\xi \sigma_T(xH,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(xH) \widehat{\overline{g_k}}(\xi)^*, \quad (xH,\xi) \in G/H \times \widehat{G/H}.$$

Conversely, suppose that there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(G/H)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|^r_{L^{p_1'}(G/H)} \|h_k\|^r_{L^{p_2}(G/H)} < \infty$$

and

$$T_H^\xi \sigma_T(xH,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(xH) \widehat{\overline{g_k}}(\xi)^*, \quad (xH,\xi) \in G/H \times \widehat{G/H}.$$

Then, for all $f \in L^{p_1}(G/H)$, we have

$$\begin{aligned}(Tf)(xH) &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\hat{f}(\xi)) \\ &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH)\sigma_T(xH,\xi))_{nm} \hat{f}(\xi)_{mn} \\ &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} \left(\sum_{k=1}^{\infty} h_k(xH) \widehat{\overline{g_k}}(\xi)^*{}_{nm} \right) \hat{f}(\xi)_{mn} \\ &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} \left(\sum_{k=1}^{\infty} h_k(xH) \overline{\widehat{g_k}(\xi)_{mn}} \right) \hat{f}(\xi)_{mn}\end{aligned}$$

$$= \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} \sum_{k=1}^{\infty} h_k(xH) \left(\int_{G/H} g_k(wH)\Gamma_\xi(wH)_{nm}\, d\mu(wH) \right) \hat{f}(\xi)_{mn}$$

$$= \int_{G/H} \left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} \Gamma_\xi(wH)_{nm} \hat{f}(\xi)_{mn} \right) \sum_{k=1}^{\infty} h_k(xH) g_k(wH)\, d\mu(wH)$$

$$= \int_{G/H} \left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(wH)\hat{f}(\xi)) \right) \sum_{k=1}^{\infty} h_k(xH) g_k(wH)\, d\mu(wH)$$

$$= \int_{G/H} \left(\sum_{k=1}^{\infty} h_k(xH) g_k(wH) \right) f(wH)\, d\mu(wH)$$

for all $xH \in G/H$. Therefore by Theorem 5.4.1, it follows that T is r-nuclear. □

In the next theorem, we will give another characterization of r-nuclear operators from $L^{p_1}(G/H)$ into $L^{p_2}(G/H)$ in order to find the trace of r-nuclear operators from $L^p(G/H)$ into $L^p(G/H)$.

Theorem 5.4.4. *Let $0 < r \leq 1$ and let $T : L^{p_1}(G/H) \to L^{p_2}(G/H), 1 \leq p_1, p_2 < \infty$, be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then the operator T is r-nuclear if and only if there exist sequences $\{g_k\}_{k=1}^\infty$ in $L^{p_1'}(G/H)$ and $\{h_k\}_{k=1}^\infty$ in $L^{p_2}(G/H)$ such that*

$$\sum_{k=1}^{\infty} \|g_k\|^r_{L^{p_1'}(G/H)} \|h_k\|^r_{L^{p_2}(G/H)} < \infty$$

and

$$\sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(yH)^*) = \sum_{k=1}^{\infty} h_k(xH) g_k(yH).$$

Proof. Suppose that $T : L^{p_1}(G/H) \to L^{p_2}(G/H)$ is a r-nuclear operator for $1 \leq p_1, p_2 < \infty$. Then by Theorem 5.4.3, there exist sequences $\{g_k\}_{k=1}^\infty$ in $L^{p_1'}(G/H)$ and $\{h_k\}_{k=1}^\infty$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|^r_{L^{p_1'}(G/H)} \|h_k\|^r_{L^{p_2}(G/H)} < \infty$$

and

$$(\Gamma_\xi(xH)\sigma_T(xH,\xi))_{nm} = \sum_{k=1}^{\infty} h_k(xH) \overline{\left(\widehat{g_k}(\xi)\right)}_{mn}, \quad (xH,\xi) \in G/H \times \widehat{G/H},$$

for all n, m with $1 \leq n, m \leq d_\xi$. Let $yH \in G/H$. Then

$$\begin{aligned}(\Gamma_\xi(xH)\sigma_T(xH,\xi))_{nm}\overline{\Gamma_\xi(yH)}_{nm} &= \sum_{k=1}^{\infty} h_k(xH)\overline{\left(\widehat{g_k}(\xi)\right)}_{mn}\overline{\Gamma_\xi(yH)}_{nm} \\ &= \int_{G/H} \Gamma_\xi(zH)_{nm}\overline{\Gamma_\xi(yH)}_{nm} \sum_{k=1}^{\infty} h_k(xH)g_k(zH)d\mu(zH).\end{aligned}$$

Consequently,

$$\begin{aligned}&\sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH)\sigma_T(xH,\xi))_{nm}\overline{\Gamma_\xi(yH)}_{nm} \\ &= \int_{G/H} \left(\sum_{m,n=1}^{d_\xi} \Gamma_\xi(zH)_{nm}\overline{\Gamma_\xi(yH)}_{nm} \right) \sum_{k=1}^{\infty} h_k(xH)g_k(zH)d\mu(zH).\end{aligned}$$

Therefore, for all xH, $yH \in G/H$, we get

$$\begin{aligned}&\sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(yH)^*) \\ &= \int_{G/H} \left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}\left(\Gamma_\xi(zH)\Gamma_\xi(yH)^*\right) \right) \sum_{k=1}^{\infty} h_k(xH)g_k(zH)d\mu(zH) \\ &= \int_{G/H} \left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \overline{\operatorname{Tr}\left(\Gamma_\xi(yH)\Gamma_\xi(zH)^*\right)} \right) \sum_{k=1}^{\infty} h_k(xH)g_k(zH)d\mu(zH) \\ &= \sum_{k=1}^{\infty} h_k(xH) \int_{G/H} \left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \overline{\operatorname{Tr}\left(\Gamma_\xi(yH)\Gamma_\xi(zH)^*\right)} g_k(zH) \right) \\ &= \sum_{k=1}^{\infty} h_k(xH) \int_{G/H} \left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \overline{\operatorname{Tr}\left(\Gamma_\xi(yH)\Gamma_\xi(zH)^*\overline{g_k(zH)}\right)} \right) \\ &= \sum_{k=1}^{\infty} h_k(xH) \overline{\left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}\left(\Gamma_\xi(yH)\widehat{\overline{g}}(\xi)\right) \right)} = \sum_{k=1}^{\infty} h_k(xH)\overline{\left(\overline{g(yH)}\right)} \\ &= \sum_{k=1}^{\infty} h_k(xH)g_k(yH).\end{aligned}$$

Conversely, let $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ be sequences in $L^{p_1'}(G/H)$ and $L^{p_2}(G/H)$, respectively such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p_1'}(G/H)}^r \|h_k\|_{L^{p_2}(G/H)}^r < \infty$$

and for all xH and yH in G/H, we have

$$\sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma(xH,\xi)\Gamma_\xi(yH)^*) = \sum_{k=1}^{\infty} h_k(xH)g_k(yH).$$

Then, for all $f \in L^{p_1}(G/H)$, we get

$$\begin{aligned}
(T_\sigma f)(xH) &= \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma(xH,\xi)\hat{f}(\xi)) \\
&= \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH)\sigma(xH,\xi))_{mn}\hat{f}(\xi)_{nm} \\
&= \int_{G/H} \left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH)\sigma(xH,\xi))_{mn}\overline{\Gamma_\xi(yH)_{mn}} \right) f(yH)\, d\mu(yH) \\
&= \int_{G/H} \left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma(xH,\xi)\Gamma_\xi(yH)^*) \right) f(yH)\, d\mu(yH) \\
&= \int_{G/H} \left(\sum_{k=1}^{\infty} h_k(xH)g_k(yH) \right) f(yH)\, d\mu(yH)
\end{aligned}$$

for all $xH \in G/H$. This completes the proof of the theorem. □

An immediate consequence of Theorem 5.4.4 gives the trace of a r-nuclear pseudo-differential operator on $L^p(G/H)$ for $1 \le p < \infty$. Indeed, we have the following result.

Corollary 5.4.5. *Let $0 < r \le 1$ and let $T : L^p(G/H) \to L^p(G/H), 1 \le p < \infty$, be a r-nuclear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then the nuclear trace of T is given by*

$$\operatorname{Tr}(T) = \int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(T_H^\xi \sigma_T(xH,\xi))d\mu(xH).$$

Proof. Using trace formula (5.4.1) and Theorem 5.4.4, we have

$$\begin{aligned}
\operatorname{Tr}(T) &= \int_{G/H} \sum_{k=1}^{\infty} h_k(xH)g_k(xH)d\mu(xH) \\
&= \int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH,\xi)\Gamma_\xi(xH)^*)d\mu(xH).
\end{aligned}$$

Since T_H^ξ is an orthogonal projection, we obtain

$$\mathrm{Tr}\,(T) = \int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \,\mathrm{Tr}(T_H^\xi \sigma_T(xH,\xi))d\mu(xH).$$

$\square$

5.5 Adjoint and product of r-nuclear pseudo-differential operators

Let $T : L^{p_1}(G/H) \to L^{p_2}(G/H), 1 \leq p_1, p_2 < \infty$, be a r-nuclear continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then the adjoint operator $T^* : L^{p_2'}(G/H) \to L^{p_1'}(G/H)$ of T is also nuclear (in general on Banach spaces, if T is nuclear then its adjoint T^* is also nuclear). In this section, we give a formula for symbols of the adjoints of r-nuclear pseudo-differential operators from $L^{p_1}(G/H)$ into $L^{p_2}(G/H)$ for $1 \leq p_1, p_2 < \infty$, where G is a compact Hausdorff group and H be a closed subgroup of G.

Theorem 5.5.1. *Let $0 < r \leq 1$ and let $T : L^{p_1}(G/H) \to L^{p_2}(G/H), 1 \leq p_1, p_2 < \infty$, be a r-nuclear continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then the symbol τ of T^*, the adjoint operator of T is given by*

$$T_H^\xi \tau(xH,\xi) = \xi(x)^* \sum_{k=1}^{\infty} \overline{g_k}(xH)\widehat{h}_k(\xi)^*, \quad (xH,\xi) \in G/H \times \widehat{G/H},$$

where $\{g_k\}_{k=1}^\infty$ and $\{h_k\}_{k=1}^\infty$ are two sequences in $L^{p_1'}(G/H)$ and $L^{p_2}(G/H)$, respectively, such that

$$\sum_{k=1}^{\infty} \|g_k\|^r_{L^{p_1'}(G/H)} \|h_k\|^r_{L^{p_2}(G/H)} < \infty.$$

Proof. For $f \in L^{p_1}(G/H)$ and $g \in L^{p_2'}(G/H)$, from the definition of the adjoint of an operator, we have

$$\int_{G/H} (Tf)\,(xH)\overline{g(xH)}\; d\mu(xH) = \int_{G/H} f(xH)\overline{(T^*g)\,(xH)}\; d\mu(xH).$$

Therefore,

$$\int_{G/H}\Bigg(\int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH)\sigma_T(xH,\xi))_{mn}$$

$$
\begin{aligned}
&\times \overline{\Gamma_\xi(yH)_{mn}} f(yH)\; d\mu(yH)\Bigg)\overline{g(xH)}\; d\mu(xH) \qquad (5.5.1)\\
&= \int_{G/H} f(xH) \overline{\Bigg(\int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH)\tau(xH,\xi))_{mn}}\\
&\qquad \overline{\times \overline{\Gamma_\xi(yH)_{mn}} g(yH)\; d\mu(yH)\Bigg)}\; d\mu(xH).
\end{aligned}
$$

Let γ and η be elements in $\widehat{G/H}$. Then for $1 \le i, j \le d_\gamma$ and $1 \le p, q \le d_\eta$, we define the functions f and g on G/H by

$$f(xH) = \Gamma_\gamma(xH)_{ij}, \quad xH \in G/H,$$

and

$$g(xH) = \Gamma_\eta(xH)_{pq}, \quad xH \in G/H.$$

Thus, from the relation (5.5.1), it follows that

$$
\begin{aligned}
&\int_{G/H} (\Gamma_\gamma(xH)\sigma_T(xH,\gamma))_{ij}\overline{\Gamma_\eta(xH)}_{pq}\; d\mu(xH)\\
&= \int_{G/H} \Gamma_\gamma(xH)_{ij}\overline{(\Gamma_\eta(xH)\tau(xH,\eta))}_{pq}\; d\mu(xH),
\end{aligned}
$$

and so

$$
\begin{aligned}
&\overline{\int_{G/H} (\Gamma_\gamma(xH)\sigma_T(xH,\gamma))_{ij}\overline{\Gamma_\eta(xH)_{pq}}\; d\mu(xH)}\\
&= \int_{G/H} (\Gamma_\eta(xH)\tau(xH,\eta))_{pq}\overline{\Gamma_\gamma(xH)_{ij}}\; d\mu(xH).
\end{aligned}
$$

This implies that

$$\overline{(((\Gamma_\gamma(\cdot)\sigma_T(\cdot,\gamma))_{ij})^\wedge(\eta))_{qp}} = (((\Gamma_\eta(\cdot)\tau(\cdot,\eta))_{pq})^\wedge(\gamma))_{ji}, \qquad (5.5.2)$$

for $1 \le i, j \le d_\gamma$, $1 \le p, q \le d_\eta$, and $\gamma, \eta \in \widehat{G/H}$. Since $T : L^{p_1}(G/H) \to L^{p_2}(G/H)$ is r-nuclear, from Theorem 5.4.3, there exist sequences $\{g_k\}_{k=1}^\infty$ in $L^{p_1'}(G/H)$ and $\{h_k\}_{k=1}^\infty$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^\infty \|g_k\|^r_{L^{p_1'}(G/H)} \|h_k\|^r_{L^{p_2}(G/H)} < \infty$$

and for all $(yH,\gamma) \in G/H \times \widehat{G/H}$, we have

$$(\Gamma_\gamma(yH)\sigma_T(yH,\gamma))_{ij} = \sum_{k=1}^{\infty} h_k(yH)\overline{\left(\widehat{\overline{g_k}}(\gamma)\right)}_{ji}.$$

Then

$$\begin{aligned}(\Gamma_\eta(xH)\tau(xH,\eta))_{pq} &= \sum_{[\gamma]\in\widehat{G/H}} d_\gamma \operatorname{Tr}[\Gamma_\gamma(xH)((\Gamma_\eta(\cdot)\tau(\cdot,\eta))_{pq})^\wedge(\gamma)] \\ &= \sum_{[\gamma]\in\widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma(\Gamma_\gamma(xH))_{ij}(((\Gamma_\eta(\cdot)\tau(\cdot,\eta))_{pq})^\wedge(\gamma))_{ji}\end{aligned}$$

for all $(xH,\eta) \in G/H \times \widehat{G/H}$. Thus by (5.5.2), for all $(xH,\eta) \in G/H \times \widehat{G/H}$, we get

$$\begin{aligned}&(\Gamma_\eta(xH)\tau(xH,\eta))_{pq} \\ &= \sum_{[\gamma]\in\widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma(\Gamma_\gamma(xH))_{ij}\overline{(((\Gamma_\gamma(\cdot)\sigma_T(\cdot,\gamma))_{ij})^\wedge(\eta))_{qp}} \\ &= \sum_{[\gamma]\in\widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma(\Gamma_\gamma(xH))_{ij} \int_{G/H} \overline{(\Gamma_\gamma(yH)\sigma_T(yH,\gamma))_{ij}}\Gamma_\eta(yH)_{pq}\, d\mu(yH) \\ &= \sum_{[\gamma]\in\widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma(\Gamma_\gamma(xH))_{ij} \int_{G/H} \sum_{k=1}^{\infty} \overline{h_k(yH)}(\widehat{\overline{g_k}}(\gamma))_{ji}\Gamma_\eta(yH)_{pq}\, d\mu(yH) \\ &= \sum_{k=1}^{\infty}\left(\int_{G/H} \overline{h_k(yH)}\Gamma_\eta(yH)_{pq}\, d\mu(yH)\right) \sum_{[\gamma]\in\widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma(\Gamma_\gamma(xH))_{ij}(\widehat{\overline{g_k}}(\gamma))_{ji} \\ &= \sum_{k=1}^{\infty} \overline{\widehat{h_k}(\eta)_{qp}} \sum_{[\gamma]\in\widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma(\Gamma_\gamma(xH))_{ij}(\widehat{\overline{g_k}}(\gamma))_{ji} \\ &= \sum_{k=1}^{\infty} \overline{\widehat{h_k}(\eta)_{qp}} \sum_{[\gamma]\in\widehat{G/H}} d_\gamma \operatorname{Tr}(\Gamma_\gamma(xH)\widehat{\overline{g_k}}(\gamma)) \\ &= \sum_{k=1}^{\infty} \overline{\widehat{h_k}(\eta)_{qp}}\overline{g_k}(xH) = \sum_{k=1}^{\infty} \widehat{h_k}(\eta)^*_{pq}\overline{g_k}(xH)\end{aligned}$$

for all $1 \leq p,q \leq d_\eta$. Thus, for all $(xH,\eta) \in G/H \times \widehat{G/H}$, we have

$$\Gamma_\eta(xH)\tau(xH,\eta) = \sum_{k=1}^{\infty} \widehat{h_k}(\eta)^*\ \overline{g_k}(xH)$$

and hence

$$T_H^\eta\tau(xH,\eta) = \eta(x)^* \sum_{k=1}^{\infty} \widehat{h_k}(\eta)^*\ \overline{g_k}(xH).$$

□

As an application of Theorem 5.4.3 and Theorem 5.5.1, in the next corollary, we give a criterion for the self-adjointness of r-nuclear pseudo-differential operators.

Corollary 5.5.2. *Let $0 < r \leq 1$ and let $T : L^2(G/H) \to L^2(G/H)$ be a r-nuclear continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then T is self-adjoint if and only if there exist sequences $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ in $L^2(G/H)$ such that*

$$\sum_{k=1}^{\infty} \|h_k\|^r_{L^2(G/H)} \|g_k\|^r_{L^2(G/H)} < \infty,$$

$$\sum_{k=1}^{\infty} h_k(xH)\widehat{g_k}(\xi)^* = \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \; \overline{g_k}(xH), \quad (xH,\xi) \in G/H \times \widehat{G/H},$$

and

$$T^{\xi}_{H}\sigma_T(xH,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(xH)\widehat{g_k}(\xi)^*, \quad (xH,\xi) \in G/H \times \widehat{G/H}.$$

We can give another characterization of symbols for the adjoints of r-nuclear operators. Indeed, we have the following theorem.

Theorem 5.5.3. *Let $0 < r \leq 1$. Let σ_T be a matrix-valued function on $G/H \times \widehat{G/H}$ such that the corresponding pseudo-differential operator $T : L^{p_1}(G/H) \to L^{p_2}(G/H)$ is r-nuclear for $1 \leq p_1, p_2 < \infty$. Then the symbol τ of the adjoint $T^* : L^{p_2'}(G/H) \to L^{p_1'}(G/H)$ is given by*

$$T^{\xi}_{H}\tau(xH,\xi) = \xi(x)^* \sum_{[\eta]\in\widehat{G/H}} d_\eta \int_{G/H} \mathrm{Tr}[(\Gamma_\eta(yH)\sigma_T(yH,\eta))^*\Gamma_\eta(xH)]\Gamma_\xi(yH)\,d\mu(yH),$$

which is eventually same as

$$T^{\xi}_{H}\tau(xH,\xi) = \xi(x)^* \sum_{[\eta]\in\widehat{G/H}} d_\eta \left(\overline{\mathrm{Tr}(\sigma_T(\cdot,\eta)^*\Gamma_\eta(\cdot)^*\Gamma_\eta(xH))}^{\wedge}(\xi)\right)^*$$

for all $(xH,\xi) \in G/H \times \widehat{G/H}$.

Proof. Suppose that $T : L^{p_1}(G/H) \to L^{p_2}(G/H)$ is r-nuclear operator for $1 \leq p_1, p_2 < \infty$. Then by Theorem 5.4.3, there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p_1'}(G/H)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|^r_{L^{p_1'}(G/H)} \|h_k\|^r_{L^{p_2}(G/H)} < \infty$$

and for all $(yH, \eta) \in G/H \times \widehat{G/H}$, we have

$$\Gamma_\eta(yH)\sigma_T(yH,\eta) = \sum_{k=1}^{\infty} h_k(yH)\widehat{g_k}(\eta)^*,$$

or,

$$(\Gamma_\eta(yH)\sigma_T(yH,\eta))^* = \sum_{k=1}^{\infty} \overline{h_k(yH)}\widehat{g_k}(\eta).$$

Let $(xH, \xi) \in G/H \times \widehat{G/H}$. Then

$$\begin{aligned}
&\int_{G/H} \mathrm{Tr}\left[(\Gamma_\eta(yH)\sigma_T(yH,\eta))^*\Gamma_\eta(xH)\right]\Gamma_\xi(yH)\, d\mu(yH) \\
&= \int_{G/H} \mathrm{Tr}\left[\sum_{k=1}^{\infty} \overline{h_k(yH)}\widehat{g_k}(\eta)\Gamma_\eta(xH)\right]\Gamma_\xi(yH)\, d\mu(yH) \\
&= \sum_{k=1}^{\infty} \mathrm{Tr}(\widehat{g_k}(\eta)\Gamma_\eta(xH)) \int_{G/H} \overline{h_k(yH)}\Gamma_\xi(yH)\, d\mu(yH) \\
&= \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^*\, \mathrm{Tr}[\widehat{g_k}(\eta)\Gamma_\eta(xH)].
\end{aligned}$$

Thus by Theorem 5.5.1, we obtain

$$\begin{aligned}
&\sum_{[\eta]\in\widehat{G/H}} d_\eta \int_{G/H} \mathrm{Tr}[(\Gamma_\eta(yH)\sigma_T(yH,\eta))^*\Gamma_\eta(xH)]\Gamma_\xi(yH)\, d\mu(yH) \\
&= \sum_{[\eta]\in\widehat{G/H}} d_\eta \left(\sum_{k=1}^{\infty} \widehat{h_k}(\xi)^*\, \mathrm{Tr}[\widehat{g_k}(\eta)\Gamma_\eta(xH)]\right) \\
&= \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \sum_{[\eta]\in\widehat{G/H}} d_\eta\, \mathrm{Tr}[\widehat{g_k}(\eta)\Gamma_\eta(xH)] \\
&= \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^*\overline{g_k}(xH) = \Gamma_\xi(xH)\tau(xH,\xi)
\end{aligned}$$

for all $(xH, \xi) \in G/H \times \widehat{G/H}$. □

Another criterion for the self-adjointness of r-nuclear pseudo-differential operators on homogeneous space of compact groups is as follows.

Corollary 5.5.4. *Let $0 < r \leq 1$. Let σ_T be a matrix-valued function on $G/H \times \widehat{G/H}$ such that $T : L^2(G/H) \to L^2(G/H)$ is r-nuclear. Then $T : L^2(G/H) \to L^2(G/H)$ is self-adjoint if and only if*

$$T_H^{\xi}\sigma_T(xH,\xi) = \xi(x)^* \sum_{[\eta]\in\widehat{G/H}} d_\eta \left(\overline{\mathrm{Tr}(\sigma_T(\cdot,\eta)^*\Gamma_\eta(\cdot)^*\Gamma_\eta(xH))}^{\wedge}(\xi)\right)^*$$

for all $(xH,\xi) \in G/H \times \widehat{G/H}$.

Next, we show that the product of a nuclear pseudo-differential operator on $L^p(G/H)$ with a bounded operator again a nuclear pseudo-differential operator on $L^p(G/H)$ for $1 \leq p < \infty$, where G is a compact (Hausdorff) group and H is a closed subgroup of G. We present a formula for the symbol of the product operator.

Theorem 5.5.5. *Let $T : L^p(G/H) \to L^p(G/H)$, $1 \leq p < \infty$, be a nuclear operator with matrix valued symbol σ_T and let $S : L^p(G/H) \to L^p(G/H)$ be a bounded linear operator with symbol σ_S. Then the symbol λ of the nuclear operator $ST : L^p(G/H) \to L^p(G/H)$ is given by*

$$T_H^{\xi}\lambda(xH,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k'(xH)\widehat{\overline{g_k}}(\xi)^*$$

for all $(xH,\xi) \in G/H \times \widehat{G/H}$, where $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are two sequences in $L^{p'}(G/H)$ and $L^p(G/H)$, respectively, such that $\sum_{k=1}^{\infty} \|g_k\|_{L^{p'}(G/H)} \|h_k\|_{L^p(G/H)} < \infty$ with

$$h_k'(xH) = \sum_{[\eta]\in\widehat{G/H}} d_\eta \,\mathrm{Tr}\left[\Gamma_\eta(xH)\sigma_S(xH,\eta)\widehat{h_k}(\eta)\right], \quad xH \in G/H.$$

Proof. Since $T : L^p(G/H) \to L^p(G/H)$ is a nuclear pseudo-differential operator for $1 \leq p < \infty$, by Theorem 5.4.3, there exist sequences $\{g_k\}_{k=1}^{\infty} \in L^{p'}(G/H)$ and $\{h_k\}_{k=1}^{\infty} \in L^p(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p'}(G/H)} \|h_k\|_{L^p(G/H)} < \infty$$

and

$$T_H^{\xi}\sigma_T(xH,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(xH)\widehat{\overline{g_k}}(\xi)^*, \quad (xH,\xi) \in G/H \times \widehat{G/H}.$$

Let $f \in L^p(G/H)$. Then

$$(STf)(xH) = \sum_{[\eta]\in\widehat{G/H}} d_\eta \,\mathrm{Tr}(\Gamma_\eta(xH)\sigma_S(xH,\eta)\widehat{Tf}(\eta))$$

$$\begin{aligned}
&= \sum_{[\eta]\in\widehat{G/H}} d_\eta \operatorname{Tr}\left[\Gamma_\eta(xH)\sigma_S(xH,\eta)\left(\int_{G/H} Tf(yH)\Gamma_\eta(yH)^* d\mu(yH)\right)\right] \\
&= \sum_{[\eta]\in\widehat{G/H}} d_\eta \operatorname{Tr}\Bigg[\Gamma_\eta(xH)\sigma_S(xH,\eta) \\
&\qquad \times \int_{G/H}\left(\sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(yH)\sigma_T(yH,\xi)\widehat{f}(\xi))\right)\Gamma_\eta(yH)^* d\mu(yH)\Bigg]
\end{aligned}$$

for all $xH\in G/H$. Using the nuclearity of T, we have

$$\begin{aligned}
(STf)(xH) &= \sum_{[\eta]\in\widehat{G/H}} d_\eta \operatorname{Tr}\Bigg[\Gamma_\eta(xH)\sigma_S(xH,\eta)\int_{G/H}\sum_{[\xi]\in\widehat{G/H}} d_\xi \\
&\qquad \times \operatorname{Tr}\left(\sum_{k=1}^{\infty} h_k(yH)\widehat{g_k}(\xi)^*\widehat{f}(\xi)\right)\Gamma_\eta(yH)^* d\mu(yH)\Bigg] \\
&= \sum_{[\eta]\in\widehat{G/H}} d_\eta \operatorname{Tr}\Bigg[\Gamma_\eta(xH)\sigma_S(xH,\eta) \sum_{[\xi]\in\widehat{G/H}}\sum_{k=1}^{\infty} d_\xi \operatorname{Tr}\left(\widehat{g_k}(\xi)^*\widehat{f}(\xi)\right) \\
&\qquad \times \int_{G/H} h_k(yH)\Gamma_\eta(yH)^* d\mu(yH)\Bigg] \\
&= \sum_{[\eta]\in\widehat{G/H}} d_\eta \operatorname{Tr}\left[\Gamma_\eta(xH)\sigma_S(xH,\eta) \sum_{[\xi]\in\widehat{G/H}}\sum_{k=1}^{\infty} d_\xi \widehat{h_k}(\eta) \operatorname{Tr}\left(\widehat{g_k}(\xi)^*\widehat{f}(\xi)\right)\right] \\
&= \sum_{[\eta]\in\widehat{G/H}}\sum_{k=1}^{\infty}\sum_{[\xi]\in\widehat{G/H}} d_\xi d_\eta \operatorname{Tr}\left[\Gamma_\eta(xH)\sigma_S(xH,\eta)\widehat{h_k}(\eta)\right]\operatorname{Tr}\left(\widehat{g_k}(\xi)^*\widehat{f}(\xi)\right) \\
&= \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}\left(\Gamma_\xi(xH)\lambda(xH,\xi)\widehat{f}(\xi)\right),
\end{aligned}$$

where

$$\begin{aligned}
T^{\xi}_{H}\lambda(xH,\xi) &= \xi(x)^*\sum_{k=1}^{\infty}\sum_{[\eta]\in\widehat{G/H}} d_\eta \operatorname{Tr}\left[\Gamma_\eta(xH)\sigma_S(xH,\eta)\widehat{h_k}(\eta)\right]\widehat{g_k}(\xi)^* \\
&= \xi(x)^*\sum_{k=1}^{\infty} h_k'(xH)\widehat{g_k}(\xi)^*
\end{aligned}$$

for all $(xH,\xi)\in G/H\times\widehat{G/H}$, and

$$h_k'(xH) = \sum_{[\eta]\in\widehat{G/H}} d_\eta \operatorname{Tr}\left[\Gamma_\eta(xH)\sigma_S(xH,\eta)\widehat{h_k}(\eta)\right], \quad xH\in G/H.$$

□

5.6 Application to the heat kernel on G/H

In this section, we assume that G is a compact Lie group and H is a closed subgroup of G. Let $\mathcal{L}_G$ be the Laplace-Beltrami operator (or the Casimir element of the universal enveloping algebra) on G. For every $[\xi] \in \widehat{G}$, the matrix elements of ξ are the eigenfunctions of $\mathcal{L}_G$ with same eigenvalue denoted by $-\lambda^2_{[\xi]}$. Therefore,

$$-\mathcal{L}_G \xi_{ij} = \lambda^2_{[\xi]} \xi_{ij} \quad \text{for all } 1 \leq i,\, j \leq d_\xi.$$

Let $-\mathcal{L}_{G/H} : C^\infty(G/H) \to C^\infty(G/H)$ be the differential operator on G/H obtained by $-\mathcal{L}_G$ acting on functions that are constant on cosets of G, i.e., such that $-\widetilde{\mathcal{L}_{G/H} f} = -\mathcal{L}_G \widetilde{f}$ for $f \in C^\infty(G/H)$, where for $f \in C^\infty(G/H)$, $\tilde{f} \in C^\infty(G)$ is the lifting of f given by $\tilde{f}(x) = f(xH)$. The operator $-\mathcal{L}_{G/H}$ has the eigenfunctions $\Gamma_{\xi_{ij}}(xH)$ for $1 \leq i,\, j \leq d_\xi$ corresponding to the common eigenvalue $\lambda^2_{[\xi]}$. For more details on $-\mathcal{L}_{G/H}$, see [78]. We make use the symbol of the heat kernel $e^{-t\mathcal{L}_{G/H}}$. Indeed, by taking into account $\sigma_{e^{-t\mathcal{L}_{G/H}}}(xH, \xi) = e^{-t|\xi|^2} T^\xi_H$, where $|\xi| = \lambda^2_{[\xi]}$, we have

$$\begin{aligned} e^{-t\mathcal{L}_{G/H}} f(xH) &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH) \sigma_{e^{-t\mathcal{L}_{G/H}}}(xH, \xi) \widehat{f}(\xi)) \\ &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH) e^{-t\lambda^2_{[\xi]}} T^\xi_H \widehat{f}(\xi)) \\ &= \sum_{[\xi] \in \widehat{G/H}} d_\xi e^{-t\lambda^2_{[\xi]}} \operatorname{Tr}(\Gamma_\xi(xH) \widehat{f}(\xi)). \end{aligned}$$

Now, we show the nuclearity of the heat kernel on L^p-spaces.

Theorem 5.6.1. *Let G be a compact Lie group and let H be closed subgroup of G. Then the heat kernel $e^{-t\mathcal{L}_{G/H}} : L^{p_1}(G/H) \to L^{p_2}(G/H)$ is nuclear for every $t > 0$ and all $1 \leq p_1, p_2 < \infty$. Moreover, if $0 < r \leq 1$, then $e^{-t\mathcal{L}_{G/H}} : L^p(G/H) \to L^p(G/H)$ is r-nuclear operator for every $t > 0$ and $1 \leq p < \infty$. In particular, on each $L^p(G/H)$, we have the following nuclear trace formula*

$$\operatorname{Tr}(e^{-t\mathcal{L}_{G/H}}) = \sum_{[\xi] \in \widehat{G/H}} d_\xi e^{-t\lambda^2_{[\xi]}} \operatorname{Tr}(T^\xi_H).$$

Proof. The kernel of $e^{-t\mathcal{L}_{G/H}}$ is given by

$$K_t(x, y) = \sum_{[\xi] \in \widehat{G/H}} d_\xi e^{-t\lambda^2_{[\xi]}} \operatorname{Tr}(\Gamma_\xi(xH) \Gamma(yH)^*)$$

$$= \sum_{[\xi]\in\widehat{G/H}} d_\xi e^{-t\lambda^2_{[\xi]}} \operatorname{Tr}(\Gamma_\xi(xH)\xi(y)^*)$$

with

$$\operatorname{Tr}(\Gamma_\xi(xH)\xi(y)^*) = \sum_{i,j}^{d_\xi} \Gamma_\xi(xH)_{ij}\,\overline{\xi(y)}_{ij}.$$

We set

$$h_{\xi,ij} = d_\xi e^{-t\lambda^2_{[\xi]}}\Gamma_\xi(xH)_{ij}, \qquad g_{\xi,ij} = \overline{\xi(y)}_{ij}.$$

Let p_1' denotes the Lebesgue conjugate of p_1 and $\tilde{q}_1 = \max\{2, p_1'\}$. Then by Lemma 5.2.4, we get

$$\|g_{\xi,ij}\|_{L^{p_1'}(G/H)} = \|\overline{\xi}_{ij}\|_{L^{p_1'}(G/H)} \leq \|\Gamma_\xi(\cdot)_{ij}\|_{L^{p_1'}(G/H)} \leq d_\xi^{-\frac{1}{\tilde{q}_1}}.$$

Also, we have

$$\begin{aligned}\|h_{\xi,ij}\|_{L^{p_2}(G/H)} &= \|d_\xi e^{-t\lambda^2_{[\xi]}}\Gamma_\xi(xH)_{ij}\|_{L^{p_2}(G/H)}\\ &\leq \|d_\xi e^{-t\lambda^2_{[\xi]}}\|\Gamma_\xi(\cdot)\|_{op}\|_{L^{p_2}(G/H)} \leq d_\xi e^{-t\lambda^2_{[\xi]}}.\end{aligned}$$

Therefore,

$$\sum_{[\xi],i,j} \|g_{\xi,ij}(\cdot)\|_{L^{p_1'}(G/H)}\|h_{\xi,ij}(\cdot)\|_{L^{p_2}(G/H)} \leq \sum_{[\xi]\in\widehat{G/H}} d_\xi^2 d_\xi^{-\frac{1}{\tilde{q}_1}} e^{-t\lambda^2_{[\xi]}} < \infty,$$

where the last convergence follows from any of the Weyl formula, see, for example [23]. Therefore, $e^{-t\mathcal{L}_{G/H}}$ is a nuclear operator. Similarly, one can prove r-nuclearity of $e^{-t\mathcal{L}_{G/H}}$. By Corollary 5.4.5 and using the fact that measure μ on G/H is normalized, the nuclear trace formula of $e^{-t\mathcal{L}_{G/H}}$ given by

$$\operatorname{Tr}(e^{-t\mathcal{L}_{G/H}}) = \int_{G/H} \sum_{[\xi]\in\widehat{G/H}} d_\xi \operatorname{Tr}(e^{-t\lambda^2_{[\xi]}}T_H^\xi)d\mu(xH) = \sum_{[\xi]\in\widehat{G/H}} d_\xi e^{-t\lambda^2_{[\xi]}} \operatorname{Tr}(T_H^\xi).$$

□

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Publications

Based on the work in this thesis, the following research articles are published or communicated.

1. V. Kumar and S. S. Mondal, Schatten class and nuclear pseudo-differential operators on homogeneous spaces of compact groups, *Monatsh. Math.* 197(1), 149-176 (2021).
2. S. S. Mondal and J. Swain, Restriction theorem for the Fourier-Hermite transform and solution of the Hermite-Schrödinger equation, *Adv. Oper. Theory* 7, Article number: 44 (2022).
3. S. S. Mondal and J. Swain, Strichartz inequality for orthonormal functions associated with special Hermite operator, preprint arXiv:2103.12586 (2021).
4. S. S. Mondal and J. Swain, Szegö type limit theorems on the Heisenberg group, preprint arXiv:2111.10224 (2021).