IF-THEN-ELSE OVER THE ALGEBRA OF CONDITIONAL LOGIC

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To

Sadhguru

Certificate

This is to certify that the thesis entitled *If-then-else over the algebra of conditional logic* submitted by *Ms. Gayatri Panicker* to the Indian Institute of Technology Guwahati, for the award of the Degree of Doctor of Philosophy, is a record of the original bona fide research work carried out by her under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

This thesis aims at giving an axiomatization for the operation of `if-then-else` over algebras of non-halting programs and non-halting tests, and further, makes use of this axiomatization to study structural properties of the algebra of conditional logic.

To this aim the thesis introduces the notion of C-sets by considering the tests from a C-algebra. When the C-algebra is an ada, the axiomatization is shown to be complete through a subdirect representation. Further, this thesis gives an axiomatization for the equality test along with `if-then-else` through the notion of agreeable C-sets, which is complete for the class of agreeable C-sets where the C-algebra is an ada. The thesis also introduces the notion of C-monoids which consider the composition of programs as well as composition of programs with tests along with `if-then-else`. A Cayley-type theorem is obtained in that every C-monoid where the C-algebra is an ada is embeddable in a functional C-monoid.

The thesis also uses the `if-then-else` action to study the structure of C-algebras through the notions of annihilators and idempotence, through which a classification of elements of the C-algebra of transformations $\mathfrak{3}^X$ is achieved. The thesis also proposes the notions of atoms and atomicity in C-algebras and obtains a characterisation of atoms in $\mathfrak{3}^X$. Further, the thesis presents necessary or sufficient conditions for the atomicity of C-algebras and shows that the class of finite atomic C-algebras is precisely that of finite adas.

Contents

List of Symbols

Q	A Boolean algebra
$\overline{a}^{\theta}$	The equivalence class of a with respect to equivalence relation θ
Δ_A	The diagonal congruence on A
∇_A	The congruence $A \times A$
Y^X	The set of all functions from X to Y
$\mathcal{T}(X)$	The set of all functions from X to X
$X_\perp$	A pointed set $X \cup \{\perp\}$ with base point $\perp$
$\mathcal{T}_o(X_\perp)$	The set of all functions on $X_\perp$ which fix $\perp$
ζ_a	The constant function taking value a
$\wp(X)$	The power set of X
A^c	The complement of the set A
M	A C-algebra with T, F, U
$\mathfrak{Z}$	The three-element C-algebra
$\mathfrak{Z}^X$	The C-algebra of functions from X to $\mathfrak{Z}$

$M_{\#}$ The set $\{\alpha \in M : \alpha \vee \neg\alpha = T\}$ which is a Boolean algebra

$\overline{M^c_{\#}}$ The C-algebra $M^c_{\#} \cup \{T, F\}$

$\hat{M}$ The enveloping ada of C-algebra M

T, F, U `true`, `false` and `undefined` respectively

$\mathbf{T}, \mathbf{F}, \mathbf{U}$ The constant functions in $\mathfrak{Z}^X$ taking value T, F and U respectively

$(S_\perp, M)$ A C-set

$\_[\_,\_]$ The `if-then-else` action of the C-set $(S_\perp, M)$ or B-set (S, Q)

$\_[\![\_,\_]\!]$ The `if-then-else` action of the C-set (M, M) or B-set (Q, Q)

$\_ * \_$ The equality test on the agreeable C-set $(S_\perp, M)$ or B-set (S, Q)

$\_ \cdot \_$ The composition of elements over $S_\perp$

$\_ \circ \_$ The composition of elements of $S_\perp$ with M

s, t, u, v, q Elements of $S_\perp$

$\alpha, \beta, \gamma, \delta$ Elements of a C-algebra or Boolean algebra

$(S_\perp, \mathfrak{Z})$ A basic C-set

$\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$ A functional C-set

Ann The annihilator of an element or a set

$\mathfrak{I}$ The set of closed sets with respect to Ann^2

E_α The set of idempotent elements with respect to α

O The set of idempotent operations

O_α The set of idempotent operations with respect to α

$\oplus$	The finite join $\vee$ of elements which commute amongst themselves
$\varphi_{T,A}$	The element in $\mathfrak{Z}^X$ taking value T for $x \in A \subseteq X$ and F otherwise (φ may be replaced by any other Greek letter)
$\varphi_{U,A}$	The element in $\mathfrak{Z}^X$ taking value U for $x \in A \subseteq X$ and F otherwise (φ may be replaced by any other Greek letter)

Introduction

This thesis aims at giving an axiomatization for the operation of `if-then-else` over algebras of non-halting programs and non-halting tests, and further, makes use of this axiomatization to study structural properties of the algebra of conditional logic. The work presented in this thesis lies in the area of general algebra, with applications in the study of equivalence of programs in computer science.

We introduce the notion of C-sets to axiomatize the systems of `if-then-else` in which the tests are drawn from an abstract C-algebra – the algebra of conditional logic. Further, in order to axiomatize `if-then-else` systems with the equality test, we extend the concept of C-sets to agreeable C-sets. We then introduce the notion of C-monoids which include the composition of programs as well as composition of programs with tests. We also use the `if-then-else` action to study the structure of C-algebras.

The conditional expression `if-then-else` has received considerable importance in programming languages, playing a vital role in the study of program semantics. As noted in Bloom and Tindell [1983], `if-then-else` is present in almost all high level programming languages and is useful in the study of various aspects of computation (e.g., Courcelle and Nivat [1976], Manna and Vuillemin [1972]).

The most widely used interpretation of `if` p `then` f `else` g where p is a proposition usually evaluating to `true` or `false`, and f and g are program components evaluating to some finite output, gives output evaluating to that of f if p is `true`, and to that of g if p is `false`.

The juxtaposition of this construct in the context of real-world programming issues leads to the necessity of including a third truth value, viz., `undefined`. This is due to the fact that not all tests evaluate to only `true` or `false` because of programming errors or non-termination. For this reason, it is also imperative to consider program components that themselves might not halt. The concept of `if-then-else` is also closely related to that of the equality test. Indeed, various studies, including that of Sethi [1978], define the equality conditional in terms of the expression `if` $E = F$ `then` G `else` H where E, F, G, H might also themselves be expressions of the same kind. It is also of interest to require that the program components be combined via composition. Thus it would be natural to consider a monoid structure over the collection of elements and to achieve an axiomatization for `if-then-else` in this new setup.

Several authors have studied a suitable algebraic formalism to this construct which vary according to the treatment of the operation, the nature (halting/non-halting) of the tests and programs, and on the inclusion of an algebraic structure to the collection of tests.

One of the seminal works in the axiomatization of this conditional expression was by McCarthy [1963], where he gave an axiom schema for the determination of the semantic equivalence between any two conditional expressions. Since then several authors have studied the axiomatization of `if-then-else` in different contexts.

Following McCarthy's approach, Igarashi [1971] studied a formal system comprising ALGOL-like statements including various programming features along with `if-then-else` with predicates. The two systems were shown to be equivalent in de Bakker [1969], i.e., axioms of one could be derived from the other. As mentioned earlier, Sethi [1978] gave a different framework to determine the semantic equivalence of statements of the form `if` $E = F$ `then` G `else` H. Kennison [1981] defined comparison algebras as those equipped with a quaternary operation $C(s, t, u, v)$ satisfying certain identities modelling the equality test. He also showed that such algebras are simple if and only if C is the direct comparison operation C_0 given by $C_0(s, t, u, v)$ taking value u if $s = t$ and v otherwise. Pigozzi [1991] gave an axiomatization of the theory of equality test algebras appended with `if-then-else`, where the test is purely T (`true`) or F (`false`). He gave a finite axiom scheme for the quasi-equational theory of equality test algebras and another finite axiom scheme for the equational theory of `if-then-else` algebras. Bergman [1991] studied the sheaf-theoretic representation of sets equipped with an action of a Boolean algebra. This Boolean action was in fact the `if-then-else` function. This approach was adopted by Stokes [1998] who obtained a representation theorem for the Boolean algebra case of `if-then-else` algebras of Manes [1993]. Further Stokes [2010] extended the work of Kennison to semigroups and monoids. He showed that every comparison semigroup (monoid) is embeddable in the comparison semigroup (monoid) $\mathcal{T}(X)$ of all total functions from X to X, for some set X. He also obtained a similar result in terms of partial functions from X to X.

Jackson and Stokes [2009] gave a complete axiomatization of `if-then-else` over halting programs and tests. They also modelled composition of functions and of functions with predicates and called this object a B-monoid. They further showed that the more natural setting of only considering composition of functions would not admit a finite axiomatization. They proved that every B-monoid is embeddable in a functional B-monoid comprising total functions and halting tests and thus achieved a Cayley-type theorem for the class of B-monoids.

The work listed above mainly focus on halting tests (by assuming them to be of Boolean type) and halting programs. In the context of non-halting tests and programs much work has been done, besides the work by McCarthy [1963] and

Igarashi [1971]. Numerous studies have been made on the theory of partial functions equipped with composition. Along these lines, Jackson and Stokes [2003] introduced the notion of *agreeable semigroups*, which have an additional binary operation that models the equality test in the context of partial functions.

Bloom and Tindell [1983] studied four versions of `if-then-else` along with the equality test. In two cases they considered the halting scenario whilst in the other two they modelled possibly non-halting programs and tests. They provided an equationally complete proof system for each such framework while noting that none of the classes formed an equational class. In order to obtain similar results in the context of functional programming languages that have user-definable data types, Guessarian and Meseguer [1987] extended the proof system of Bloom and Tindell [1983] to heterogeneous algebras that have extra operations, predicates and equations. Another extension of Bloom and Tindell [1983] was by Mekler and Nelson [1987]. In this work the authors expanded the algebras in some equational class K by adding the `if-then-else` operation and found axioms for the equational class K^* generated by these algebras. They also showed that the equational theory for K^* is decidable if the word problem for K is decidable. On a slightly different track, Manes [1990] gave a transformational characterisation of `if-then-else` where the tests are Boolean but the functions on which they act could be non-halting. Further, Manes [1993] considered `if-then-else` algebras over Boolean algebras, C-algebras and adas (Algebra of Disjoint Alternatives). Here C-algebras and adas are algebras of non-halting tests, generalizing Boolean algebras to three-valued logics.

While there are several studies (e.g., Belnap [1970], Bergstra et al. [1995], Bochvar [1981], Heyting [1934], Kleene [1938], Lukasiewicz [1920]) on extending two-valued Boolean logic to three-valued logic, McCarthy's logic (cf. McCarthy [1963]) models the short-circuit evaluation exhibited by programming languages that evaluate expressions in sequential order, from left to right. Guzmán and Squier [1990] gave a complete axiomatization of McCarthy's three-valued logic and called the corresponding algebra a C-algebra, or the algebra of conditional logic. While studying `if-then-else` algebras, Manes [1993] defined an ada which is essentially a C-algebra equipped with an oracle for the halting problem.

Recently, Jackson and Stokes [2015] studied the algebraic theory of computable functions, which can be viewed as possibly non-halting programs, equipped with composition, `if-then-else` and `while-do`. In this work they assumed that the tests form a Boolean algebra. Further, they demonstrated how an algebra of non-halting tests could be constructed from Boolean tests in their setting. Jackson and Stokes proposed an alternative approach by considering an abstract collection of non-halting tests as in Manes [1993] and posed the following problem:

Characterize the algebras of computable functions associated with an abstract C-algebra of non-halting tests.

Aiming to solve this problem, in this thesis we axiomatize `if-then-else` over non-halting programs and non-halting tests by drawing the tests from an abstract C-algebra.

We also study the algebraic structure of C-algebras, aided by the presence of the `if-then-else` action. Various authors have studied structural properties of C-algebras. Vali et al. [2010] introduced the notion of ideals in C-algebras and studied various properties of ideals and principal ideals. Mandelker [1970] defined the notion of annihilators in lattices which was extended to various classes of lattices (cf. Cornish [1973]). The notion of annihilator ideals in C-algebras was introduced by Rao [2013] where he showed that the class of annihilator ideals forms a complete Boolean algebra. He further gave a list of equivalent conditions by which in the given C-algebra, every ideal is an annihilator ideal. For further reading on annihilators in C-algebras refer to Vali et al. [2015]. We introduce a notion of annihilators and idempotents in C-algebras through the `if-then-else` action and study their properties. Subsequently, we adopt the notion of atoms in Boolean algebras to

C-algebras and achieve a characterisation for the atomicity of finite C-algebras.

The work in this thesis has been divided into 2 parts comprising a total of 7 chapters in all. The first part is concerned with the axiomatization of `if-then-else` under different contexts. The second part is engaged with the study of C-algebras, first using the `if-then-else` action to obtain various structural properties, and second in terms of a notion of atoms. The organisation of this thesis is as follows:

Chapter 1: Preliminaries

Part I: Axiomatization of `if-then-else`

Chapter 2: C-sets

Chapter 3: Representation of C-sets

Chapter 4: Agreeable C-sets

Chapter 5: C-monoids

Part II: Structure of C-algebras

Chapter 6: Applications of `if-then-else`

Chapter 7: Atomicity

Chapter 1. In this chapter we present relevant background material including fundamentals of algebras, necessary material on the axiomatization of `if-then-else` over halting programs and tests from Jackson and Stokes [2009], and on the algebra of conditional logic and adas from Guzmán and Squier [1990] and Manes [1993] respectively.

Chapter 2. In this chapter we introduce the notion of a C-set to study an axiomatization of `if-then-else` that includes models of possibly non-halting programs and tests, where the tests are drawn from a C-algebra. We first define the axioms for C-sets in Section 2.1 and present the intuition behind the notion of a C-set and its axioms with respect to program constructs. Further, we present certain natural examples of C-sets. Subsequently, in Section 2.2 we investigate some properties of C-sets.

Chapter 3. Subdirect representations play an immense role in understanding structural properties of algebras. In Chapter 3, we obtain a subdirect representation for a class of C-sets in terms of simple or basic C-sets in which the tests arise from McCarthy's three-valued logic, $\mathbb{3}$ (cf. Theorem 3.2.4). In order to achieve such a representation, in this chapter we define a collection of congruences on the C-set such that each quotient algebra is simple (cf. Section 3.1). Further, in Section 3.2 the intersection of this family of congruences is shown to be trivial, from which we obtain the main theorem of this chapter, Theorem 3.2.4. Moreover, this result helps in establishing equivalence of programs under the current setup (cf. Corollary 3.2.5).

Chapter 4. In order to achieve an algebraic formalism for the equality test in this setup, in this chapter we extend the notion of C-sets to agreeable C-sets (cf. Section 4.1). We give natural interpretations for each of the agreeable C-set axioms along with some examples of agreeable C-sets. Further, in Section 4.2 we obtain a subdirect representation for the class of agreeable C-sets where the C-algebra is an ada (cf. Theorem 4.2.1). Using this, we obtain an alternative proof for Theorem 2.7 in Jackson and Stokes [2009] (cf. Theorem 4.2.4).

We await formal acceptance for publication for the work presented in chapters 2, 3 and 4 from the International Journal of Algebra and Computation.

Chapter 5. A natural line of thought would be to achieve an axiomatization for `if-then-else` over non-halting programs and tests which include the composition of programs. In Chapter 5 we consider a monoid structure which serves as an abstraction of programs with composition and give an axiomatization for `if-then-else` in this context. In Section 5.1 we introduce the notion of a C-monoid which is an extension of that of a C-set, and includes the composition of programs and that of programs with tests. Further, we obtain a Cayley-type theorem in that every C-monoid is embeddable in a functional C-monoid (cf. Theorem 5.2.1). In order to achieve this result we list some properties of maximal congruences of adas in Subsection 5.2.1. Following this, in Subsection 5.2.2 we construct a collection of homomorphisms from the C-monoid to functional C-monoids, which separate every distinct pair of elements. Using this, in Subsection 5.2.3 we construct a functional C-monoid and achieve an embedding from the C-monoid to this functional C-monoid (cf. Theorem 5.2.1). We conclude in Section 5.3 with some comments on C-sets, agreeable C-sets and C-monoids along with some avenues for further study.

The work presented in chapter 5 has been submitted to a journal for publication.

Chapter 6. In this chapter we invoke the inherent `if-then-else` action of the C-algebra M in order to study various structural properties of M. We treat the `if-then-else` action of M as a binary operation, viz., $\alpha[\![\,\_\,,\_\,]\!]$, for each $\alpha \in M$.

In Section 6.1 we introduce a notion of annihilators in C-algebras with T, F, U through the `if-then-else` action. The notion of Galois connection yields a closure operator in terms of annihilator, which in turn, yields closed sets. Further, in Section 6.2 we characterise the closed sets in the C-algebra of transformations $\mathbb{3}^X$. Additionally, we show that the collection of closed sets in $\mathbb{3}^X$ forms a complete Boolean algebra (cf. Theorem 6.2.5). Moreover, we obtain a classification of the elements of $\mathbb{3}^X$ where the elements of the Boolean algebra $\mathbb{2}^X$ form a distinct class (cf. Theorem 6.2.6). In Section 6.3 we define a notion of idempotent elements and that of idempotent operations through the `if-then-else` action and study their properties. We conclude this chapter with Section 6.4 by listing various unanswered problems.

Chapter 7. The concept of atoms in Boolean algebras is useful for achieving a structural representation of Boolean algebras. Indeed, a finite Boolean algebra can be represented in terms of the power set of its atoms. In Chapter 7 we adopt the notion of atoms in Boolean algebras to C-algebras and study structural properties of C-algebras. In Section 7.1 a partial order is given on the C-algebra M, following which the notions of atoms and atomic C-algebras are introduced. We also state some properties related to atomicity in Section 7.2. On studying the C-algebra $\mathfrak{Z}^X$, in Section 7.3 we obtain a characterisation of all atoms in $\mathfrak{Z}^X$ (cf. Theorem 7.3.1), using which we establish that the C-algebra $\mathfrak{Z}^X$ for finite X is atomic (cf. Theorem 7.3.5).

When the C-algebra M is a subalgebra of $\mathfrak{Z}^X$ for X finite, in view of Theorem 7.3.5, a natural line of thought would be to involve our understanding of the atomicity of $\mathfrak{Z}^X$ to gain an insight into that of M. In Section 7.4 we observe that all atoms of $\mathfrak{Z}^X$ that are in M are, in fact, atoms of M. However, the converse need not hold in general. The atoms of M that are also atoms of $\mathfrak{Z}^X$ are in some sense global, so that if every atom of M is an atom of $\mathfrak{Z}^X$ then M is closed under the global nature of its atoms. In this context we introduce the notion of M being globally closed in $\mathfrak{Z}^X$, or g-closed in short, and observe that such finite C-algebras are precisely $\mathfrak{Z}^X$ in Theorem 7.4.7. Subsequently, we present some necessary or sufficient conditions for the atomicity of C-algebras in Section 7.5 (cf. Theorems 7.5.1, 7.5.2, 7.5.4). Finally in Section 7.6 we obtain a characterisation of all finite atomic C-algebras and establish that they are precisely adas (cf. Theorem 7.6.3). We conclude this chapter in Section 7.7 with some avenues for further study.

1 Preliminaries

In this chapter we present relevant definitions and theorems from the literature which shall be used in later chapters. We begin with fundamental concepts of universal algebra in Section 1.1 and delve into the notions of closure operators and Galois connections in Section 1.2. Section 1.3 deals with concepts related to [Jackson and Stokes, 2009] along with some of the main theorems proved in the aforesaid paper (cf. Theorem 1.3.6, Theorem 1.3.11, Theorem 1.3.14). In Section 1.4 we fix the ternary logic over which we seek to axiomatize the operation of `if-then-else`, and the algebra associated with this logic, viz., C-algebra, as defined by Guzmán and Squier [1990]. We then present material on adas, defined by Manes [1993], which is a special class of C-algebras equipped with an oracle for the halting problem.

1.1 Elements of algebras

Definition 1.1.1. A *type* of algebras is a set $\mathscr{F}$ of *function symbols* such that a non-negative integer n is assigned to each member f of $\mathscr{F}$. This n is the *arity* of f and f is said to be an *n-ary function symbol*.

Definition 1.1.2. If $\mathscr{F}$ is a type of algebras, then an *algebra* $\mathbf{A}$ of type $\mathscr{F}$ is an ordered pair $\langle A, F\rangle$, where A is a non-empty set and F is a set of operations on A indexed by $\mathscr{F}$ such that there is a one-one correspondence between each n-ary function symbol f in $\mathscr{F}$ and an n-ary operation $f^{\mathbf{A}}$ on A.

The set A is referred to as the *universe* of $\mathbf{A} = \langle A, F\rangle$. The operations $f^{\mathbf{A}}$ will be called *fundamental operations of* $\mathbf{A}$. If $\mathscr{F}$ is finite, say $\mathscr{F} = \{f_1, f_2, \ldots, f_k\}$, we write $\langle A, F\rangle$ as $\langle A, f_1, f_2, \ldots, f_k\rangle$, where

$$\text{arity } f_1 \geq \text{arity } f_2 \geq \cdots \geq \text{arity } f_k.$$

We also say that algebra $\mathbf{A}$ is of type $(\text{arity } f_1, \text{arity } f_2, \ldots \text{arity } f_k)$. We denote the set of n-ary function symbols by $\mathscr{F}_n$.

The fundamental operations of the algebra could also satisfy certain axioms which are essentially identities or equational laws. Some algebras whose fundamental operations satisfy identities are given below. We list those algebras which are useful to us.

Definition 1.1.3. A *lattice* is an algebra $\langle L, \vee, \wedge\rangle$ of type $(2, 2)$ where both operations are commutative, associative and idempotent, and additionally satisfy the following axiom of absorption for all $x, y \in L$:

$$x = x \vee (x \wedge y),\ x = x \wedge (x \vee y) \tag{1.1}$$

Remark 1.1.4. Note that a lattice can be equivalently defined in terms of a partial ordering on the set L if for every $a, b \in L$ both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in L. Note that $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$.

Definition 1.1.5. A lattice L is *complete* if $\sup A$ and $\inf A$ exist in L for all $A \subseteq L$. We denote $\sup A$ by $\bigvee A$ and $\inf A$ by $\bigwedge A$.

Definition 1.1.6. If L is a lattice and $\emptyset \neq L' \subseteq L$ such that $a \vee b$ and $a \wedge b$ are in L' for every pair of elements $a, b \in L'$, then L' is said to be a *sublattice* of L.

Definition 1.1.7. A sublattice L' of a complete lattice L is said to be a *complete sublattice* of L if $\bigvee A$ and $\bigwedge A$ as defined in L are actually in L', for all $A \subseteq L'$.

Definition 1.1.8. A *distributive lattice* is a lattice $\langle L, \vee, \wedge \rangle$ of type $(2, 2)$ which satisfies either (and hence both) of the following axioms for all $x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1.2}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{1.3}$$

Definition 1.1.9. A *bounded lattice* is a lattice $\langle L, \vee, \wedge, 0, 1 \rangle$ of type $(2, 2, 0, 0)$ which satisfies the following axioms for all $x \in L$:

$$x \wedge 0 = 0 \tag{1.4}$$

$$x \vee 1 = 1 \tag{1.5}$$

Definition 1.1.10. A *Boolean algebra* is an algebra $\langle Q, \vee, \wedge, \neg, 0, 1 \rangle$ of type $(2, 2, 1, 0, 0)$ such that $\langle Q, \vee, \wedge \rangle$ is a distributive lattice, $\langle Q, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice and which satisfies the following axioms for all $x \in Q$:

$$x \wedge \neg x = 0 \tag{1.6}$$

$$x \vee \neg x = 1 \tag{1.7}$$

Definition 1.1.11. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same type, say $\mathscr{F}$. $\mathbf{B}$ is said to be a *subalgebra* of $\mathbf{A}$, denoted by $\mathbf{B} \leq \mathbf{A}$, if $B \subseteq A$ and $f^{\mathbf{B}} = f^{\mathbf{A}}|_B$ for all function symbol $f \in \mathscr{F}$.

Definition 1.1.12. Let $\mathbf{A}$ be an algebra of type $\mathscr{F}$. The equivalence relation θ on A is said to be a *congruence* on $\mathbf{A}$ if it satisfies the following compatibility or substitution property for all $f \in \mathscr{F}_n$ and $a_i, b_i \in A$ for $1 \leq i \leq n$:

$$\text{if } (a_i, b_i) \in \theta \text{ for all } 1 \leq i \leq n \text{ then } (f^{\mathbf{A}}(a_1, a_2, \ldots, a_n), f^{\mathbf{A}}(b_1, b_2, \ldots, b_n)) \in \theta.$$

Notation 1.1.13. We denote the diagonal congruence and the congruence $A \times A$ on $\mathbf{A}$ by Δ and ∇ respectively, i.e.,

$$\Delta = \{(a, a) \mid a \in A\}$$
$$\nabla = \{(a, b) \mid a, b \in A\} = A \times A$$

The equivalence class of a with respect to equivalence relation θ will be denoted by $\overline{a}^{\theta}$, i.e.,

$$\overline{a}^{\theta} = \{b \in A \mid (a, b) \in \theta\}.$$

Within a given context, if there is no ambiguity, we may simply denote the equivalence class by $\overline{a}$. The set of all equivalence classes with respect to equivalence relation θ will be denoted by A/θ. The *set of all congruences on an algebra* $\mathbf{A}$ is denoted by Con $\mathbf{A}$.

Theorem 1.1.14. *The set of all equivalence relations on set A, denoted by $Eq(A)$ is a complete lattice with respect to the partial ordering $\subseteq$.*

Definition 1.1.15. Let $\theta \in$ Con $\mathbf{A}$. Then the *quotient algebra of* $\mathbf{A}$ *by* θ, denoted by $\mathbf{A}/\theta$, is the algebra with universe A/θ, and whose fundamental operations satisfy

$$f^{(\mathbf{A}/\theta)}(\overline{a_1}^{\theta}, \overline{a_2}^{\theta}, \ldots, \overline{a_n}^{\theta}) = \overline{(f^{\mathbf{A}}(a_1, a_2, \ldots, a_n))}^{\theta}$$

where $f \in \mathscr{F}_n$ and $a_1, a_2, \ldots, a_n \in A$.

Theorem 1.1.16. *The set Con* $\mathbf{A}$ *with respect to set inclusion* $\subseteq$ *is a complete sublattice of* $Eq(\mathbf{A})$.

Definition 1.1.17. An algebra $\mathbf{A}$ is *simple* if Con $\mathbf{A} = \{\Delta, \nabla\}$. A congruence θ on $\mathbf{A}$ is *maximal* if the interval $[\theta, \nabla]$ of Con $\mathbf{A}$ treated as a lattice has exactly two elements.

The following theorem is useful to us.

Theorem 1.1.18 (Correspondence Theorem, Birkhoff [1944])**.** *Let* $\mathbf{A}$ *be an algebra and* $\theta \in$ *Con* $\mathbf{A}$. *Then the lattice Con* $(\mathbf{A}/\theta)$ *is isomorphic to* $[\theta, \nabla]$.

Definition 1.1.19. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of type $\mathscr{F}$. A function $\phi : A \to B$ is a *homomorphism* if for all $f \in \mathscr{F}_n$ and $a_i \in A, 1 \leq i \leq n$ we have

$$\phi(f^{\mathbf{A}}(a_1, a_2, \ldots, a_n)) = f^{\mathbf{B}}(\phi(a_1), \phi(a_2), \ldots, \phi(a_n)).$$

Definition 1.1.20. Let $(\mathbf{A}_i)_{i\in I}$ be a family of algebras of type $\mathscr{F}$. The *direct product* of $(\mathbf{A}_i)_{i\in I}$, denoted by $\prod\limits_{i\in I} \mathbf{A}_i$, is an algebra with universe $\prod\limits_{i\in I} A_i$ such that given $f \in \mathscr{F}_n$ and elements $a_j \in \prod\limits_{i\in I} A_i$ for $1 \leq j \leq n$ and $i \in I$ we have the following:

$$f^{\prod_{i\in I}\mathbf{A}_i}(a_1, a_2, \ldots, a_n)(i) = f^{\mathbf{A}_i}(a_1(i), a_2(i), \ldots, a_n(i)).$$

Definition 1.1.21. Let $\mathbf{A} = \prod_{i\in I} \mathbf{A}_i$ be a direct product of algebras of type $\mathscr{F}$. For $j \in I$ we have the *projection map* $\pi_j : \prod_{i\in I} \mathbf{A}_i \longrightarrow \mathbf{A}_j$ defined by

$$\pi_j(a) = a(j).$$

The projection map is a surjective homomorphism.

Definition 1.1.22. An algebra $\mathbf{A}$ is said to be a *subdirect product* of a family of algebras $(\mathbf{A}_i)_{i\in I}$ if $\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$ and $\pi_i(\mathbf{A}) = \mathbf{A}_i$ for all $i \in I$.

Definition 1.1.23. An algebra $\mathbf{A}$ is said to be *subdirectly irreducible* if given any family of congruences $(\theta_i)_{i\in I}$ on $\mathbf{A}$ the following holds:

$$\bigwedge_{i\in I} \theta_i = \Delta \Rightarrow \exists i_\circ \in I \text{ such that } \theta_{i_\circ} = \Delta.$$

Theorem 1.1.24 (Birkhoff [1944]). *Given an algebra $\mathbf{A}$ and an indexed family $(\theta_i)_{i\in I}$ of congruences on $\mathbf{A}$ such that $\bigwedge_{i\in I} \theta_i = \Delta$, $\mathbf{A}$ is isomorphic to a subdirect product of $(\mathbf{A}/\theta_i)_{i\in I}$.*

The following concerns certain special classes of algebras and gives a characterisation in terms of equations or identities.

Definition 1.1.25. A non-empty class of algebras is called a *variety* if it is closed under subalgebras, homomorphic images and direct products.

Definition 1.1.26. Given a type of algebras $\mathscr{F}$ and a set X one can construct the set of *terms of type $\mathscr{F}$ over X* freely as syntactic objects. An *identity* of type $\mathscr{F}$ over X is an expression of the form $p = q$ where both p and q are terms over X. A *quasi-identity* of type $\mathscr{F}$ over X is an expression of the form $(p_1 = q_1) \wedge (p_2 = q_2) \wedge \cdots \wedge (p_N = q_N) \to p = q$ where p_i, q_i, p and q are terms over X. An algebra $\mathbf{A}$ of type $\mathscr{F}$ is said to *satisfy* identity $p = q$ if on substitution of elements of A in place of variables of p and q, both expressions evaluate to the same value. A class of algebras $\mathcal{K}$ satisfies the identity $p = q$ if every member of $\mathcal{K}$ satisfies $p = q$. Given a set of identities Σ, the class $\mathcal{K}$ satisfies Σ if it satisfies every identity $p = q$ in Σ.

Definition 1.1.27. Let Σ be a set of identities of type $\mathscr{F}$. Then $M(\Sigma)$ is defined to be the class of all algebras that satisfy Σ. If a class $\mathcal{K}$ of algebras is such that there is a set of identities Σ for which $\mathcal{K} = M(\Sigma)$ then $\mathcal{K}$ is said to be an *equational class*, or alternatively, that $\mathcal{K}$ is *defined* or *axiomatized* by Σ.

Theorem 1.1.28 (Birkhoff [1935]). *A non-empty class of algebras $\mathcal{K}$ is an equational class if and only if $\mathcal{K}$ is a variety.*

Let $I(\mathcal{K}), S(\mathcal{K}), H(\mathcal{K}), P(\mathcal{K})$ denote the class of algebras isomorphic to $\mathcal{K}$, the subalgebras of members of $\mathcal{K}$, the homomorphic images of members of $\mathcal{K}$, and the direct products of members of $\mathcal{K}$ ($\neq \emptyset$) respectively. Birkhoff [1935] proved the following interesting result.

Theorem 1.1.29 (Birkhoff [1935]). *All of the classes $\mathcal{K}, I(\mathcal{K}), S(\mathcal{K}), H(\mathcal{K})$ and $P(\mathcal{K})$, where $\mathcal{K} \neq \emptyset$, satisfy the same identities over any set of variables X.*

Remark 1.1.30. Suppose that each member of the class $\mathcal{K}$ can be written as a subdirect product of members of the class $\mathcal{K}'$ where $\mathcal{K}'$ consists of quotients of members of $\mathcal{K}$. Consequently, the class of identities (quasi-identities) satisfied by the class $\mathcal{K}$ is the same as the class of identities (quasi-identities) satisfied by the class $\mathcal{K}'$.

Birkhoff proved the following fundamental relation between tautologies in an equational class and theorems derived from the axiom set through term rewriting.

Theorem 1.1.31 (The Completeness Theorem for Equational Logic, Birkhoff [1935]). *Given a set of identities Σ and an identity $p = q$ over a set of variables X, Σ yields $p = q$ if and only if Σ proves $p = q$.*

We end this section with some definitions and statements regarding multi-sorted algebras from Birkhoff and Lipson [1970]. The notion of multi-sorted algebras is essentially a generalisation of that of algebras where one allows a collection of universes, called *sorts* along with a family of operations defined over subcollections of the sorts. In precise terms we have the following.

Definition 1.1.32. A *multi-sorted algebra* or *heterogeneous algebra* is a system $\mathbf{A} = (\mathcal{S}, F)$ in which:

1. $\mathcal{S} = \{S_i\}$ is a family of non-void sets S_i of different types of elements, each called a *sort* of the algebra $\mathbf{A}$. The sorts S_i are indexed by some set I; i.e., $S_i \in \mathcal{S}$ for $i \in I$.

2. $F = \{f_\alpha\}$ is a set of finitary operations, where each f_α is a mapping

$$f_\alpha : S_{i(1,\alpha)} \times S_{i(2,\alpha)} \times \cdots \times S_{i(n(\alpha),\alpha)} \rightarrow S_{r(\alpha)}$$

for some non-negative integer $n(\alpha)$, function $i_\alpha : k \rightarrow i(k, \alpha)$ from $n(\alpha) = \{1, 2, \ldots, n(\alpha)\}$ to I, and $r(\alpha) \in I$. The operations f_α are indexed by some set Ω; i.e., $f_\alpha \in F$ for $\alpha \in \Omega$.

Various notions in universal algebra have natural counterparts in this context, including those of congruences and homomorphisms. A (multi-sorted) congruence is one which has the compatibility or substitution property with respect to each of the operations of the algebra. More precisely we have the following.

Definition 1.1.33. A *congruence* on a heterogeneous algebra $\mathbf{A} = (\{S_i\}, F)$ is a family $\xi = \{E_i\}$ of equivalence relations, with E_i defined on S_i for each $i \in I$, which for each $f_\alpha \in F$ and $x_j, y_j \in S_{i(j,\alpha)}$ has the following compatibility or substitution property:

$$(x_j, y_j) \in E_{i(j,\alpha)} \ (j = 1, 2, \ldots, n(\alpha)) \text{ implies } (f_\alpha(x_1, \ldots x_{n(\alpha)}), f_\alpha(y_1, \ldots, y_{n(\alpha)}) \in E_{r(\alpha)}.$$

Along similar lines, a homomorphism over similar multi-sorted algebras is one which respects each of the operations of the algebras. For similar algebras $\mathbf{A} = (\{S_i\}, F)$ and $\mathbf{B} = (\{T_i\}, F)$, we have the following definition.

Definition 1.1.34. A *homomorphism* Φ from $\mathbf{A}$ to $\mathbf{B}$ is a set of functions

$$\phi_i : S_i \rightarrow T_i,$$

one for each $i \in I$ such that for any $f_\alpha \in F$, $n = n(\alpha)$,

$$f_\alpha \circ (\phi_{i(1,\alpha)} \times \cdots \times \phi_{i(n,\alpha)}) = \phi_{r(\alpha)} \circ f_\alpha.$$

We end this section with the following theorem.

Theorem 1.1.35 (Birkhoff and Lipson [1970])**.** *The epimorphic images of any algebra* $\mathbf{A}$ *are determined up to isomorphism by the quotient algebras* $\mathbf{A}/\xi$ *defined by the congruences* ξ *on* $\mathbf{A}$.

For further material related to multi-sorted algebras refer to Higgins [1963], Grätzer [1969], Goguen and Meseguer [1985] and Climent Vidal and Soliveres Tur [2015].

1.2 Closure operators

We also include some definitions related to partially ordered sets which are useful to us.

Definition 1.2.1. Let A and B be two partially ordered sets and $F : A \to B$ be a function. The map F is said to be *isotone* or *monotone* if $a_1 \leq a_2 \Rightarrow F(a_1) \leq F(a_2)$ for all $a_1, a_2 \in A$. The map F is said to be *antitone* if $a_1 \leq a_2 \Rightarrow F(a_1) \geq F(a_2)$ for all $a_1, a_2 \in A$.

Definition 1.2.2. Given a set X, a function $C : \wp(X) \to \wp(X)$ is termed a *closure operator* on X if for all $A, B \subseteq X$ it satisfies the following:

$$A \subseteq C(A) \qquad \text{(extensive)}$$

$$C^2(A) = C(A) \qquad \text{(idempotent)}$$

$$A \subseteq B \Rightarrow C(A) \subseteq C(B) \qquad \text{(isotone)}$$

A subset $A \subseteq X$ is called a *closed subset* if $C(A) = A$. The set of all closed sets of X ordered by set inclusion $\subseteq$ is a partially ordered set and is denoted by L_C.

Theorem 1.2.3. *If C is a closure operator on X then L_C forms a complete lattice.*

Definition 1.2.4. An *algebraic closure operator* on X is a closure operator C such that for every $A \subseteq X$ we have $C(A) = \bigcup\{C(B) : B \subseteq A \text{ and } B \text{ is finite}\}$.

Definition 1.2.5. An element a of lattice L is *compact* if whenever $a \leq \vee A$ for some subset A of L for which $\bigvee A$ exists, then there exists a finite subset $B \subseteq A$ such that $a \leq \bigvee B$. A lattice is *compactly generated* if every element is the sup of compact elements. An *algebraic lattice* is one that is both complete and compactly generated.

Theorem 1.2.6. *If C is an algebraic closure operator on X then L_C is an algebraic lattice, and the compact elements of L_C are precisely the closed sets $C(A)$ where A is a finite subset of X.*

Definition 1.2.7 (Mac Lane [1971])**.** Let A and B be posets and $F : A \to B$ and $G : B \to A$ be two antitone functions. The pair (F, G) is said to be an *antitone Galois connection* if for all $a \in A, b \in B$,

$$b \leq F(a) \Leftrightarrow a \leq G(b).$$

Theorem 1.2.8. *Given an antitone Galois connection (F, G) of posets A and B, the composite functions $FG : B \to B$ and $GF : A \to A$ form closure operators and are called the associated closure operators. Further, $FGF = F$ and $GFG = G$.*

1.3 B-sets and B-monoids

In this section, we list definitions and results based on which we have proposed the notions of C-set and C-monoid. Jackson and Stokes [2009] considered the notion of a B-set, which was introduced by Bergman [1991], in order to study the theory of halting programs equipped with the operation of `if-then-else`.

Definition 1.3.1. Let $\langle Q, \vee, \wedge, \neg, F, T\rangle$ be a Boolean algebra and S be a set. A *B-set* is a pair (S, Q), equipped with a function $\eta : Q \times S \times S \to S$, called *$B$-action*, where $\eta(\alpha, a, b)$ is denoted by $\alpha[a, b]$, read "`if` α `then` a `else` b", that satisfies the following axioms for all $\alpha, \beta \in Q$ and $a, b, c \in S$:

$$\alpha[a, a] = a \tag{1.8}$$

$$\alpha[\alpha[a, b], c] = \alpha[a, c] \tag{1.9}$$

$$\alpha[a, \alpha[b, c]] = \alpha[a, c] \tag{1.10}$$

$$F[a, b] = b \tag{1.11}$$

$$\neg\alpha[a, b] = \alpha[b, a] \tag{1.12}$$

$$(\alpha \wedge \beta)[a, b] = \alpha[\beta[a, b], b] \tag{1.13}$$

We recall the following examples from [Jackson and Stokes, 2009].

Example 1.3.2. For any Boolean algebra Q, the pair (Q, Q) is a B-set with the following action for all $\alpha, \beta, \gamma \in Q$:

$$\alpha[\beta, \gamma] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma).$$

We denote the action of the B-set (Q, Q) by double brackets $\_ [\![\_ , \_]\!]$.

Example 1.3.3. Consider the two-element Boolean algebra $\mathbb{2}$ with the universe $\{T, F\}$. For any set S, the pair $(S, \mathbb{2})$ is a B-set with the following action for all

$a, b \in S$:

$$T[a,b] = a,$$
$$F[a,b] = b.$$

These B-sets are called *basic* B-sets.

Notation 1.3.4. Let X and Y be two sets. The set of all functions from X to Y will be denoted by Y^X. The set of all functions from X to X will be denoted by $\mathcal{T}(X)$.

Example 1.3.5. For any set X, the pair $(\mathcal{T}(X), 2^X)$ is a B-set with the following action for all $\alpha \in 2^X$ and $g, h \in \mathcal{T}(X)$:

$$\alpha[g,h](x) = \begin{cases} g(x), & \text{if } \alpha(x) = T; \\ h(x), & \text{if } \alpha(x) = F. \end{cases}$$

Jackson and Stokes [2009] showed that every B-set can be represented in terms of basic B-sets.

Theorem 1.3.6 (Jackson and Stokes [2009]). *Every B-set is a subdirect product of basic B-sets.*

This result in Jackson and Stokes [2009] was achieved by constructing a family of multi-sorted congruences over B-sets whose respective intersections were trivial. Further, the Boolean algebra quotient was shown to be isomorphic to the two-element Boolean algebra 2, from which the result followed.

Remark 1.3.7. Theorems 1.3.6 and 1.1.35 imply that studying the identities or quasi-identities satisfied by the subclass of basic B-sets suffices to understand those satisfied by the entire class of B-sets. Checking the validity of any identity (quasi-identity) in a basic B-set involves merely checking the respective values for `true` and `false` and is thus far simpler than checking the same in an arbitrary B-set.

Further, Jackson and Stokes [2009] modelled the equality test based on the assumption that the tests arise from a Boolean algebra and that the functions are halting.

Definition 1.3.8. A B-set (S, Q) is said to be *agreeable* if it is equipped with an operation $* : S \times S \to Q$ satisfying the following axioms for all $s, t, u, v \in S$ and $\alpha \in Q$:

$$s * s = T \tag{1.14}$$

$$(s * t)[s, t] = t \tag{1.15}$$

$$\alpha[s, t] * \alpha[u, v] = \alpha[\![s * u, t * v]\!] \tag{1.16}$$

The following are examples of agreeable B-sets.

Example 1.3.9. The pair $(\mathcal{T}(X), 2^X)$ is an agreeable B-set with the operation $*$ defined as follows for all $f, g \in \mathcal{T}(X)$:

$$(f * g)(x) = \begin{cases} T, & \text{if } f(x) = g(x); \\ F, & \text{otherwise.} \end{cases}$$

Example 1.3.10. Let S be any set. The pair $(S, \mathbb{2})$ is an agreeable B-set under the operation $*$ defined in the following manner for all $s, t \in S$:

$$s * t = \begin{cases} T, & \text{if } s = t; \\ F, & \text{otherwise.} \end{cases}$$

These B-sets are called *basic agreeable B-sets* .

Jackson and Stokes [2009] proved the following result.

Theorem 1.3.11 (Jackson and Stokes [2009]). *Every agreeable B-set is a subdirect product of basic agreeable B-sets.*

Jackson and Stokes [2009] also considered the case of modelling `if-then-else` over a collection of programs with composition, where they included an operation abstracting the composition of programs with tests. The abstraction of this notion was their concept of a B-monoid, which is the monoid version of a B-set.

Definition 1.3.12. A *B-monoid* is a B-set (S, Q) for which $(S, \cdot)$ is a monoid with identity 1 and there is an operator $\circ : S \times Q \to Q$ satisfying the following axioms for all $a, b, c \in S$ and $\alpha, \beta \in Q$:

$$a \circ T = T \tag{1.17}$$

$$(a \circ \alpha) \wedge (a \circ \beta) = a \circ (\alpha \wedge \beta) \tag{1.18}$$

$$a \circ (\neg\alpha) = \neg(a \circ \alpha) \tag{1.19}$$

$$a \circ (b \circ \alpha) = (a \cdot b) \circ \alpha \tag{1.20}$$

$$(\alpha[a, b]) \cdot c = \alpha[a \cdot c, b \cdot c] \tag{1.21}$$

$$a \cdot (\alpha[b, c]) = (a \circ \alpha)[a \cdot b, a \cdot c] \tag{1.22}$$

$$\beta[a, b] \circ \alpha = \beta[\![a \circ \alpha, b \circ \alpha]\!] \tag{1.23}$$

$$1 \circ \alpha = \alpha \tag{1.24}$$

Example 1.3.13. Note that $\mathcal{T}(X)$ is a monoid under the usual operation of composition. The B-set $(\mathcal{T}(X), 2^X)$ is a B-monoid under the operation $\circ$ defined as follows for all $f \in \mathcal{T}(X)$ and $\alpha \in 2^X$:

$$(f \circ \alpha) = \{x \in X : f(x) \in \alpha\}.$$

Jackson and Stokes [2009] showed the following theorem.

Theorem 1.3.14 (Jackson and Stokes [2009]). *The B-monoid (S, Q) is embeddable as a two-sorted algebra into the B-monoid $(\mathcal{T}(X), 2^X)$ for some set X, with X finite* *if S and Q are finite.*

1.4 C-algebras and adas

Kleene [1952] discussed various three-valued logics that are extensions of Boolean logic. McCarthy [1963] first studied the three-valued non-commutative logic in the context of programming languages. This is the non-commutative regular extension of Boolean logic to three truth values. Here the third truth value U denotes the `undefined` state which is attained when a test diverges. In this new context, the evaluation of expressions is carried out sequentially from left to right, mimicking that of a majority of programming languages. A complete axiomatization for the class of algebras associated with this logic was given by Guzmán and Squier [1990]. They called the algebra associated with this logic a *C-algebra*.

Definition 1.4.1. A *C-algebra* is an algebra $\langle M, \vee, \wedge, \neg \rangle$ of type $(2,2,1)$, which satisfies the following axioms for all $\alpha, \beta, \gamma \in M$:

$$\neg\neg\alpha = \alpha \tag{1.25}$$

$$\neg(\alpha \wedge \beta) = \neg\alpha \vee \neg\beta \tag{1.26}$$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \tag{1.27}$$

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \tag{1.28}$$

$$(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee (\neg\alpha \wedge \beta \wedge \gamma) \tag{1.29}$$

$$\alpha \vee (\alpha \wedge \beta) = \alpha \tag{1.30}$$

$$(\alpha \wedge \beta) \vee (\beta \wedge \alpha) = (\beta \wedge \alpha) \vee (\alpha \wedge \beta) \tag{1.31}$$

Example 1.4.2. Every Boolean algebra is a C-algebra. In particular, $\mathbb{2}$ is a C-algebra.

Example 1.4.3. Let $\mathbb{3}$ denote the C-algebra with the universe $\{T, F, U\}$ and the following operations. This is, in fact, McCarthy's three-valued logic.

$\neg$	
T	F
F	T
U	U

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

Remark 1.4.4. In view of the fact that the class of C-algebras is a variety using Theorem 1.1.28, for any set X, $\mathbb{3}^X$ is a C-algebra with the operations defined pointwise. In fact, Guzmán and Squier [1990] showed that elements of $\mathbb{3}^X$ along with the C-algebra operations may be viewed in terms of *pairs of sets*. This is a pair (A, B) where $A, B \subseteq X$ and $A \cap B = \emptyset$. Akin to the well-known correlation between $\mathbb{2}^X$ and the power set $\wp(X)$ of X, for any element $\alpha \in \mathbb{3}^X$, associate the pair of sets (A, B) where $A = \{x \in X : \alpha(x) = T\}$ and $B = \{x \in X : \alpha(x) = F\}$. Conversely, for any pair of sets (A, B) where $A, B \subseteq X$ and $A \cap B = \emptyset$ associate the function α where $\alpha(x) = T$ if $x \in A$, $\alpha(x) = F$ if $x \in B$ and $\alpha(x) = U$ otherwise. With this correlation, the operations can be expressed as follows:

$$\neg(A_1, A_2) = (A_2, A_1)$$
$$(A_1, A_2) \wedge (B_1, B_2) = (A_1 \cap B_1, A_2 \cup (A_1 \cap B_2))$$
$$(A_1, A_2) \vee (B_1, B_2) = ((A_1 \cup (A_2 \cap B_1), A_2 \cap B_2)$$

Further, Guzmán and Squier showed that every C-algebra is a subalgebra of $\mathbb{3}^X$ for some X as stated below.

Theorem 1.4.5 (Guzmán and Squier [1990]). $\mathbb{3}$ *and* $\mathbb{2}$ *are the only subdirectly* *irreducible C-algebras. Hence, every C-algebra is a subalgebra of a product of copies of $\mathbb{3}$.*

Remark 1.4.6. Considering a C-algebra M as a subalgebra of $\mathbb{3}^X$, one may observe that $M_\# = \{\alpha \in M : \alpha \vee \neg\alpha = T\}$ forms a Boolean algebra under the induced operations.

Notation 1.4.7. A C-algebra with T, F, U is a C-algebra with nullary operations T, F, U, where T is the (unique) left-identity (and right-identity) for $\wedge$, F is the (unique) left-identity (and right-identity) for $\vee$ and U is the (unique) fixed point for $\neg$. Note that U is also a left-zero for both $\wedge$ and $\vee$ while F is a left-zero for $\wedge$.

Notation 1.4.8. The constants T, F, U of the C-algebra $\mathbb{3}^X$ will be denoted by $\mathbf{T}, \mathbf{F}, \mathbf{U}$ respectively, and they can be identified by the pairs of sets $(X, \emptyset), (\emptyset, X), (\emptyset, \emptyset)$ respectively.

Let M be a C-algebra with T, F, U. When M is considered as a subalgebra of $\mathbb{3}^X$, the constants T, F, U of M will also be denoted by $\mathbf{T}, \mathbf{F}, \mathbf{U}$ respectively.

There is an important subclass of the variety of C-algebras. Manes [1993] introduced the notion of *ada* (algebra of disjoint alternatives) which is a C-algebra equipped with an oracle for the halting problem. He showed that the category of adas is equivalent to that of Boolean algebras. The C-algebra $\mathbb{3}$ is not functionally-complete. However, $\mathbb{3}$ is functionally-complete when treated as an ada. In fact, the variety of adas is generated by the ada $\mathbb{3}$.

Definition 1.4.9. An *ada* is a C-algebra M with T, F, U equipped with an additional unary operation $(\ )^{\downarrow}$ subject to the following equations for all $\alpha, \beta \in M$:

$$F^{\downarrow} = F \tag{1.32}$$

$$U^{\downarrow} = F \tag{1.33}$$

$$T^{\downarrow} = T \tag{1.34}$$

$$\alpha \wedge \beta^{\downarrow} = \alpha \wedge (\alpha \wedge \beta)^{\downarrow} \tag{1.35}$$

$$\alpha^{\downarrow} \vee \neg(\alpha^{\downarrow}) = T \tag{1.36}$$

$$\alpha = \alpha^{\downarrow} \vee \alpha \tag{1.37}$$

Example 1.4.10. The three-element C-algebra $\mathfrak{3}$ with the unary operation $(\ )^{\downarrow}$ defined as follows forms an ada.

$$T^{\downarrow} = T$$
$$U^{\downarrow} = F = F^{\downarrow}$$

We also use $\mathfrak{3}$ to denote this ada. One may easily resolve the notation overloading – whether $\mathfrak{3}$ is a C-algebra or an ada – depending on the context.

Manes [1993] showed that the three-element ada $\mathfrak{3}$ is the only subdirectly irreducible ada. In view of Theorem 1.1.28 for any set X, $\mathfrak{3}^X$ is an ada with operations defined pointwise. Note that the three element ada $\mathfrak{3}$ is also simple.

Remark 1.4.11. Since adas are C-algebras with an additional operation, every C-algebra M freely generates an ada $\hat{M}$. That is, there exists a C-algebra homomorphism $\phi : M \to \hat{M}$ with the universal property that for each ada A and C-algebra homomorphism $f : M \to A$ there exists a unique ada homomorphism $\psi : \hat{M} \to A$ with $\psi(\phi(x)) = f(x)$ for all $x \in M$. Manes [1993] called such an ada the *enveloping ada* of M.

Manes also showed the following result.

Proposition 1.4.12 (Manes [1993]). *Let A be an ada. Then $A^{\downarrow} = \{\alpha^{\downarrow} : \alpha \in A\}$ forms a Boolean algebra under the induced operations.*

Remark 1.4.13. In fact, $A^{\downarrow} = A_{\#}$. Also, $A^{\downarrow} = \{\alpha \in A : \alpha^{\downarrow} = \alpha\}$.

Further, as outlined in the following remark, Manes established that the category of adas and the category of Boolean algebras are equivalent.

Remark 1.4.14 (Manes [1993]). Let Q be a Boolean algebra. By Stone's representation of Boolean algebras, suppose Q is a subalgebra of $\mathtt{2}^X$ for some set X. Consider the subalgebra $Q^{\star}$ of the ada $\mathtt{3}^X$ with the universe $Q^{\star} = \{(E, F) : E \cap F = \emptyset\}$ given in terms of pairs of subsets of X. Note that the map $Q \mapsto (Q^{\star})_{\#}$ is a Boolean isomorphism. Similarly, for an ada A, the map $A \mapsto (A_{\#})^{\star}$ is an ada isomorphism. Hence, the functor based on the aforesaid assignment establishes that the category of adas and the category of Boolean algebras are equivalent.

Remark 1.4.15. In view of the fact that the only finite Boolean algebras are $\mathtt{2}^X$ for finite X and the equivalence of the categories of adas and Boolean algebras, we see that the only finite adas are $\mathtt{3}^X$ for finite X.

Notation 1.4.16. Let X be a set and $\perp \notin X$. The pointed set $X \cup \{\perp\}$ with base point $\perp$ is denoted by $X_{\perp}$. Note that the pointed set $X_{\perp}$ is also an algebra given by $\langle X \cup \{\perp\}, \perp\rangle$ of type (0). The set of all functions on $X_{\perp}$ which fix $\perp$ is denoted by $\mathcal{T}_o(X_{\perp})$, i.e. $\mathcal{T}_o(X_{\perp}) = \{f \in \mathcal{T}(X_{\perp}) \ : \ f(\perp) = \perp\}$.

Part I

Axiomatization of if-then-else

2

C-sets

In this chapter we introduce the notion of a C-set to study an axiomatization of `if-then-else` that includes models of possibly non-halting programs and tests, where the tests are drawn from a C-algebra. We first define the axioms for C-sets in Section 2.1 and present the intuition behind the notion of a C-set and its axioms with respect to program constructs. Further, we present natural examples of C-sets and detail the verification of the axioms in each. One of the examples presented is that of functional C-sets, which models non-halting programs and non-halting tests. Another example is that of basic C-sets where the C-algebra is McCarthy's three-valued logic $\mathbb{3}$. The notion of basic C-sets is useful for the results presented in this thesis. Finally, in Section 2.2 we investigate certain properties of C-sets.

2.1 Axiomatization and models

The concept of C-sets is an extension of that of B-sets, wherein the tests are drawn from a C-algebra instead of a Boolean algebra, and includes a non-halting or `error` state. For the definition of a C-algebra refer to Definition 1.4.1, while the term C-algebra with T, F, U is detailed in Notation 1.4.7.

Definition 2.1.1. Let $S_\perp$ be a pointed set with base point $\perp$ and M be a C-algebra with T, F, U. The pair $(S_\perp, M)$ equipped with an action

$$\_\,[\_\,,\_] : M \times S_\perp \times S_\perp \to S_\perp$$

is called a *C-set* if it satisfies the following axioms for all $\alpha, \beta \in M$ and $s, t, u, v \in S_\perp$:

$$U[s,t] = \perp \qquad (U\text{-axiom}) \qquad (2.1)$$

$$F[s,t] = t \qquad (F\text{-axiom}) \qquad (2.2)$$

$$(\neg\alpha)[s,t] = \alpha[t,s] \qquad (\neg\text{-axiom}) \qquad (2.3)$$

$$\alpha[\alpha[s,t],u] = \alpha[s,u] \qquad (\text{positive redundancy}) \qquad (2.4)$$

$$\alpha[s,\alpha[t,u]] = \alpha[s,u] \qquad (\text{negative redundancy}) \qquad (2.5)$$

$$(\alpha \wedge \beta)[s,t] = \alpha[\beta[s,t],t] \qquad (\wedge\text{-axiom}) \qquad (2.6)$$

$$\alpha[\beta[s,t],\beta[u,v]] = \beta[\alpha[s,u],\alpha[t,v]] \qquad (\text{premise interchange}) \qquad (2.7)$$

$$\alpha[s,t] = \alpha[t,t] \Rightarrow (\alpha \wedge \beta)[s,t] = (\alpha \wedge \beta)[t,t] \qquad (\wedge\text{-compatibility}) \qquad (2.8)$$

Remark 2.1.2. In view of equations (1.25) and (1.26) of C-algebras and (2.3) and (2.6) of C-sets, we have:

$$(\alpha \vee \beta)[s,t] = \neg(\neg(\alpha \vee \beta))[s,t] \qquad \text{from (1.25)}$$

$$= \neg(\neg\alpha \wedge \neg\beta)[s,t] \qquad \text{from (1.25) and (1.26)}$$

$$= (\neg\alpha \wedge \neg\beta)[t, s] \qquad \text{from (2.3)}$$

$$= (\neg\alpha)[(\neg\beta)[t, s], s] \qquad \text{from (2.6)}$$

$$= (\neg\alpha)[\beta[s, t], s] \qquad \text{from (2.3)}$$

$$= \alpha[s, \beta[s, t]] \qquad \text{from (2.3)}$$

Thus we have the following property in C-sets.

$$(\alpha \vee \beta)[s, t] = \alpha[s, \beta[s, t]] \text{ ($\vee$-axiom)}$$

We now present the intuition behind the notion of a C-set and its axioms with respect to program constructs. In order to include the possibility of non-halting tests, we assume that the tests form a C-algebra. A test diverges at a given input if it evaluates to U, `undefined`. When a test diverges or if the program throws up an `error` or does not halt, we say that the program evaluates to $\perp$. Thus a pointed set $S_\perp$ models the set of states and base point $\perp$ serves to denote the `error` state.

The U-axiom (2.1) essentially encapsulates the real-world requirement that if a test diverges, the output should be the `error` state. The F-axiom (2.2) is natural since when the test is `false`, the `else` part of the `if-then-else` construct is executed. The $\neg$-axiom (2.3) simply states that executing `if` *not* P `then` f `else` g is the same as executing `if` P `then` g `else` f. The axioms of positive redundancy and negative redundancy (2.4) and (2.5) encapsulate the *cascading* nature of `if-then-else`. The $\wedge$-axiom (2.6) states that evaluating the test P *AND* Q and then executing f else g works in exactly the same way as evaluating P first, which if `true`, executing `if` Q `then` f `else` g, and if `false` then simply executing g. The axiom of premise interchange (2.7) serves as a *switching* law. This states that the behaviour of the program where P is evaluated first and Q is a test situated within both the branches of the main `if-then-else`, is exactly the same as evaluating Q first with P in each branch, on suitably interchanging the programs situated at the leaves. The last axiom of $\wedge$-compatibility (2.8) loosely means that if f and g agree with regards to some domain, then they will agree on any subdomain.

Example 2.1.3. Let M be a C-algebra with T, F, U. By treating M as a pointed set with base point U, the pair (M, M) is a C-set under the following action for all $\alpha, \beta, \gamma \in M$:

$$\alpha[\beta, \gamma] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma).$$

Hereafter, the action of the C-set (M, M) will be denoted by double brackets $\_ [\![\_ , \_]\!]$. We now verify the axioms (2.1) – (2.8) in the following.

Axiom (2.1)**:** $U[\![\alpha, \beta]\!] = (U \wedge \alpha) \vee (\neg U \wedge \beta) = (U \wedge \alpha) \vee (U \wedge \beta) = U \vee U = U$.

Axiom (2.2)**:** $F[\![\alpha, \beta]\!] = (F \wedge \alpha) \vee (\neg F \wedge \beta) = (F \wedge \alpha) \vee (T \wedge \beta) = F \vee \beta = \beta$.

Axiom (2.3)**:** Note that $(\neg\alpha)[\![\beta, \gamma]\!] = (\neg\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ while $\alpha[\![\gamma, \beta]\!] = (\alpha \wedge \gamma) \vee (\neg\alpha \wedge \beta)$. Thus we have to check the validity of the identity $(\neg\alpha \wedge \beta) \vee (\alpha \wedge \gamma) = (\alpha \wedge \gamma) \vee (\neg\alpha \wedge \beta)$. In view of Theorem 1.4.5, this identity is valid in all C-algebras if it is valid in the three element C-algebra $\mathbb{3}$:

$\alpha = T$**:** In this case $(\neg\alpha \wedge \beta) \vee (\alpha \wedge \gamma) = (\neg T \wedge \beta) \vee (T \wedge \gamma) = (F \wedge \beta) \vee (T \wedge \gamma) = F \vee \gamma = \gamma = \gamma \vee F = (T \wedge \gamma) \vee (\neg T \wedge \beta) = (\alpha \wedge \gamma) \vee (\neg\alpha \wedge \beta)$.

$\alpha = F$**:** In this case $(\neg\alpha \wedge \beta) \vee (\alpha \wedge \gamma) = (\neg F \wedge \beta) \vee (F \wedge \gamma) = \beta \vee F = \beta = F \vee \beta = (F \wedge \gamma) \vee (\neg F \wedge \beta) = (\alpha \wedge \gamma) \vee (\neg\alpha \wedge \beta)$.

$\alpha = U$**:** This reduces to the verification of axiom (2.1), from which it follows that both expressions evaluate to U.

Axiom (2.4)**:** It is clear that $\alpha[\![\alpha[\![\beta, \gamma]\!], \delta]\!] = (\alpha \wedge ((\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma))) \vee (\neg\alpha \wedge \delta)$ while $\alpha[\![\beta, \delta]\!] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \delta)$. In view of Theorem 1.4.5 it suffices to check the validity of the identity $(\alpha \wedge ((\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma))) \vee (\neg\alpha \wedge \delta) = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \delta)$ in the C-algebra $\mathbb{3}$:

$\alpha = T$**:** In this case $(\alpha \wedge ((\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma))) \vee (\neg\alpha \wedge \delta) = (T \wedge ((T \wedge \beta) \vee (\neg T \wedge \gamma))) \vee (\neg T \wedge \delta) = (\beta \vee F) \vee F = \beta = \beta \vee F = (T \wedge \beta) \vee (\neg T \wedge \delta) = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \delta)$.

$\alpha = F$**:** We have $(\alpha \wedge ((\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma))) \vee (\neg\alpha \wedge \delta) = (F \wedge ((F \wedge \beta) \vee (\neg F \wedge \gamma))) \vee (\neg F \wedge \delta) = F \vee \delta = \delta = F \vee \delta = (F \wedge \beta) \vee (\neg F \wedge \delta) = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \delta)$.

$\alpha = U$**:** Since $\neg U = U$ and U is a left-identity for $\wedge$ and $\vee$ in this case, both expressions evaluate to U.

Axiom (2.5)**:** We have $\alpha[\![\beta, \alpha[\![\gamma, \delta]\!]]\!] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge ((\alpha \wedge \gamma) \vee (\neg\alpha \wedge \delta)))$ while $\alpha[\![\beta, \delta]\!] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \delta)$. As earlier, in view of Theorem 1.4.5 it suffices to consider the following three cases:

$\alpha = T$**:** Here $(\alpha \wedge \beta) \vee (\neg\alpha \wedge ((\alpha \wedge \gamma) \vee (\neg\alpha \wedge \delta))) = (T \wedge \beta) \vee (\neg T \wedge ((T \wedge \gamma) \vee (\neg T \wedge \delta))) = \beta \vee F = \beta = \beta \vee F = (T \wedge \beta) \vee (\neg T \wedge \delta) = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \delta)$.

$\alpha = F$**:** In this case $(\alpha \wedge \beta) \vee (\neg\alpha \wedge ((\alpha \wedge \gamma) \vee (\neg\alpha \wedge \delta))) = (F \wedge \beta) \vee (\neg F \wedge ((F \wedge \gamma) \vee (\neg F \wedge \delta))) = F \vee (F \vee \delta) = \delta = F \vee \delta = (F \wedge \beta) \vee (\neg F \wedge \delta) = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \delta)$.

$\alpha = U$**:** It is easy to see that both expressions in this case evaluate to U.

Axiom (2.6)**:** Here $(\alpha \wedge \beta)[\![\gamma, \delta]\!] = ((\alpha \wedge \beta) \wedge \gamma) \vee (\neg(\alpha \wedge \beta) \wedge \delta)$ while $\alpha[\![\beta[\![\gamma, \delta]\!], \delta]\!] = (\alpha \wedge ((\beta \wedge \gamma) \vee (\neg\beta \wedge \delta))) \vee (\neg\alpha \wedge \delta)$. It suffices to consider the following three cases:

$\alpha = T$**:** Then we have $((\alpha \wedge \beta) \wedge \gamma) \vee (\neg(\alpha \wedge \beta) \wedge \delta) = ((T \wedge \beta) \wedge \gamma) \vee (\neg(T \wedge \beta) \wedge \delta) = (\beta \wedge \gamma) \vee (\neg\beta \wedge \delta) = ((\beta \wedge \gamma) \vee (\neg\beta \wedge \delta)) \vee F = (T \wedge ((\beta \wedge \gamma) \vee (\neg\beta \wedge \delta))) \vee (\neg T \wedge \delta) = (\alpha \wedge ((\beta \wedge \gamma) \vee (\neg\beta \wedge \delta))) \vee (\neg\alpha \wedge \delta)$.

$\alpha = F$**:** Then we have $((\alpha \wedge \beta) \wedge \gamma) \vee (\neg(\alpha \wedge \beta) \wedge \delta) = ((F \wedge \beta) \wedge \gamma) \vee (\neg(F \wedge \beta) \wedge \delta) = F \vee \delta = \delta = F \vee \delta = (F \wedge ((\beta \wedge \gamma) \vee (\neg\beta \wedge \delta))) \vee (\neg F \wedge \delta) = (\alpha \wedge ((\beta \wedge \gamma) \vee (\neg\beta \wedge \delta))) \vee (\neg\alpha \wedge \delta)$.

$\alpha = U$**:** It is easy to see that both expressions in this case evaluate to U.

Axiom (2.7)**:** It is clear that $\alpha[\![\beta[\![\gamma,\delta]\!],\beta[\![\rho,\omega]\!]]\!] = (\alpha \wedge ((\beta\wedge\gamma)\vee(\neg\beta\wedge\delta)))\vee(\neg\alpha\wedge((\beta\wedge\rho)\vee(\neg\beta\wedge\omega)))$. On the other hand $\beta[\![\alpha[\![\gamma,\rho]\!],\alpha[\![\delta,\omega]\!]]\!] = (\beta\wedge((\alpha\wedge\gamma)\vee(\neg\alpha\wedge\rho)))\vee(\neg\beta\wedge((\alpha\wedge\delta)\vee(\neg\alpha\wedge\omega)))$. It suffices to consider the following three cases:

$\alpha = T$**:** In this case $(\alpha\wedge((\beta\wedge\gamma)\vee(\neg\beta\wedge\delta)))\vee(\neg\alpha\wedge((\beta\wedge\rho)\vee(\neg\beta\wedge\omega))) = (T\wedge((\beta\wedge\gamma)\vee(\neg\beta\wedge\delta)))\vee(\neg T\wedge((\beta\wedge\rho)\vee(\neg\beta\wedge\omega))) = ((\beta\wedge\gamma)\vee(\neg\beta\wedge\delta))\vee F = (\beta\wedge\gamma)\vee(\neg\beta\wedge\delta)$. Also $(\beta\wedge((\alpha\wedge\gamma)\vee(\neg\alpha\wedge\rho)))\vee(\neg\beta\wedge((\alpha\wedge\delta)\vee(\neg\alpha\wedge\omega))) = (\beta\wedge((T\wedge\gamma)\vee(\neg T\wedge\rho)))\vee(\neg\beta\wedge((T\wedge\delta)\vee(\neg T\wedge\omega))) = (\beta\wedge(\gamma\vee F))\vee(\neg\beta\wedge(\delta\vee F)) = (\beta\wedge\gamma)\vee(\neg\beta\wedge\delta)$.

$\alpha = F$**:** In this case $(\alpha\wedge((\beta\wedge\gamma)\vee(\neg\beta\wedge\delta)))\vee(\neg\alpha\wedge((\beta\wedge\rho)\vee(\neg\beta\wedge\omega))) = (F\wedge((\beta\wedge\gamma)\vee(\neg\beta\wedge\delta)))\vee(\neg F\wedge((\beta\wedge\rho)\vee(\neg\beta\wedge\omega))) = F\vee((\beta\wedge\rho)\vee(\neg\beta\wedge\omega)) = (\beta\wedge\rho)\vee(\neg\beta\wedge\omega)$. Similarly $(\beta\wedge((\alpha\wedge\gamma)\vee(\neg\alpha\wedge\rho)))\vee(\neg\beta\wedge((\alpha\wedge\delta)\vee(\neg\alpha\wedge\omega))) = (\beta\wedge((F\wedge\gamma)\vee(\neg F\wedge\rho)))\vee(\neg\beta\wedge((F\wedge\delta)\vee(\neg F\wedge\omega))) = (\beta\wedge(F\vee\rho))\vee(\neg\beta\wedge(F\vee\omega)) = (\beta\wedge\rho)\vee(\neg\beta\wedge\omega)$.

$\alpha = U$**:** In this case both expressions evaluate to U by checking casewise for $\beta\in\{T,F,U\}$.

Axiom (2.8)**:** Since $\alpha[\![\gamma,\delta]\!] = \alpha[\![\delta,\delta]\!]$ we have $(\alpha\wedge\gamma)\vee(\neg\alpha\wedge\delta) = (\alpha\wedge\delta)\vee(\neg\alpha\wedge\delta)$. By Theorem 1.4.5 it suffices to check the validity of this quasi-identity in $\mathbb{3}$:

$\alpha = T$**:** Then $(T\wedge\gamma)\vee(\neg T\wedge\delta) = (T\wedge\delta)\vee(\neg T\wedge\delta)$ so that $\gamma\vee F = \delta\vee F$ that is $\gamma=\delta$. Consequently $((T\wedge\beta)\wedge\gamma)\vee(\neg(T\wedge\beta)\wedge\delta) = (\beta\wedge\gamma)\vee(\neg\beta\wedge\delta) = (\beta\wedge\delta)\vee(\neg\beta\wedge\delta) = ((T\wedge\beta)\wedge\delta)\vee(\neg(T\wedge\beta)\wedge\delta)$.

$\alpha = F$**:** In this case $((F\wedge\beta)\wedge\gamma)\vee(\neg(F\wedge\beta)\wedge\delta) = (F\wedge\gamma)\vee(T\wedge\delta) = F\vee\delta = \delta = F\vee\delta = ((F\wedge\beta)\wedge\delta)\vee(\neg(F\wedge\beta)\wedge\delta)$.

$\alpha = U$**:** It is clear that $((U\wedge\beta)\wedge\gamma)\vee(\neg(U\wedge\beta)\wedge\delta) = U = ((U\wedge\beta)\wedge\delta)\vee(\neg(U\wedge\beta)\wedge\delta)$.

Hence, the pair (M,M) is a C-set.

We now present the motivating example of C-sets. Since the natural models of possibly non-halting programs are partial functions, we consider the model $\mathcal{T}_o(X_\perp)$ in view of the following one-to-one correspondence between $\mathcal{T}_o(X_\perp)$ and the set of partial functions on a set X. Each partial function f on X is represented by the total function $f' \in \mathcal{T}_o(X_\perp)$ where $f'(x) = f(x)$ when x is in the domain of f, and maps to $\perp$ otherwise. Conversely, each $g \in \mathcal{T}_o(X_\perp)$ is represented by the partial function g'' over X where $g''(x) = g(x)$ when $x \in X$ and $g(x) \neq \perp$, and is not defined elsewhere. The model $\mathcal{T}_o(X_\perp)$ can be seen to be a C-set under the action of the C-algebra $\mathfrak{Z}^X$ as shown in the following example.

Example 2.1.4. Consider $\mathcal{T}_o(X_\perp)$ as a pointed set with base point $\zeta_\perp$, the constant function taking the value $\perp$. The pair $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$ is a C-set with the following action for all $f, g \in \mathcal{T}_o(X_\perp)$ and $\alpha \in \mathfrak{Z}^X$:

$$\alpha[f,g](x) = \begin{cases} f(x), & \text{if } \alpha(x) = T; \\ g(x), & \text{if } \alpha(x) = F; \\ \perp, & \text{otherwise.} \end{cases} \tag{2.9}$$

Note that the execution of the first two cases, $\alpha(x) \in \{T, F\}$ demands that $x \in X$ as $\alpha \in \mathfrak{Z}^X$. These C-sets will be called *functional C-sets*.

In order to verify the axioms we rely on the pairs of sets representation of the C-algebra $\mathfrak{Z}^X$ as stated in Remark 1.4.4. Every $\alpha \in \mathfrak{Z}^X$ can be represented by the pair of sets $(A, B) = (\alpha^{-1}(T), \alpha^{-1}(F))$. In this representation $\mathbf{T} = (X, \emptyset), \mathbf{F} = (\emptyset, X)$ and $\mathbf{U} = (\emptyset, \emptyset)$. Thus we have the following:

$$\alpha[f,g](x) = (A,B)[f,g](x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B; \\ \perp, & \text{otherwise.} \end{cases}$$

We now verify the axioms (2.1) – (2.8) in the following. Assume that α and $\beta \in \mathfrak{Z}^X$

are represented by the pairs of sets (A, B) and (C, D) respectively.

Axiom (2.1)**:** In view of the above representation we have $\mathbf{U}[f, g](x) = (\emptyset, \emptyset)[f, g](x) = \bot$ for all $x \in X_\bot$. Thus $\mathbf{U}[f, g] = \zeta_\bot$.

Axiom (2.2)**:** We have $\mathbf{F}[f, g](x) = (\emptyset, X)[f, g](x) = g(x)$ for all $x \in X_\bot$. It follows that $\mathbf{F}[f, g] = g$.

Axiom (2.3)**:** As $\alpha \in \mathfrak{Z}^X$ is represented by (A, B) then $\neg\alpha$ is represented by (B, A). Thus $(\neg\alpha)[f, g](x) = (B, A)[f, g](x)$. Thus

$$(B, A)[f, g](x) = \begin{cases} f(x), & \text{if } x \in B; \\ g(x), & \text{if } x \in A; \\ \bot, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} g(x), & \text{if } x \in A; \\ f(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$
$$= (A, B)[g, f](x).$$

Thus $(\neg\alpha)[f, g] = \alpha[g, f]$.

Axiom (2.4)**:** Since $(A, B)[f, g](x) = f(x)$ when $x \in A$ we have the following:

$$(A, B)[(A, B)[f, g], h](x) = \begin{cases} (A, B)[f, g](x), & \text{if } x \in A; \\ h(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(x), & \text{if } x \in A; \\ h(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

$$= (A, B)[f, h](x).$$

Thus $\alpha[\alpha[f, g], h] = \alpha[f, h]$.

Axiom (2.5)**:** Since $(A, B)[g, h](x) = h(x)$ when $x \in B$ we have the following:

$$(A, B)[f, (A, B)[g, h]](x) = \begin{cases} f(x), & \text{if } x \in A; \\ (A, B)[g, h](x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(x), & \text{if } x \in A; \\ h(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

$$= (A, B)[f, h](x).$$

Thus $\alpha[f, \alpha[g, h]] = \alpha[f, h]$.

Axiom (2.6)**:** The pair of sets representing $\alpha \wedge \beta \in \mathfrak{Z}^X$ is given by $(A, B) \wedge (C, D) = (A \cap C, B \cup (A \cap D))$. Thus we have the following:

$$(\alpha \wedge \beta)[f, g](x) = (A \cap C, B \cup (A \cap D))[f, g](x) = \begin{cases} f(x), & \text{if } x \in A \cap C; \\ g(x), & \text{if } x \in B \cup (A \cap D); \\ \bot, & \text{otherwise.} \end{cases}$$

Similarly we have

$$(A,B)[(C,D)[f,g],g](x) = \begin{cases} (C,D)[f,g](x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(x), & \text{if } x \in A \cap C; \\ g(x), & \text{if } x \in A \cap D; \\ \bot, & \text{if } x \in A \cap (X \setminus (C \cup D)); \\ g(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(x), & \text{if } x \in A \cap C; \\ g(x), & \text{if } x \in B \cup (A \cap D); \\ \bot, & \text{otherwise.} \end{cases}$$

Thus $(A \cap C, B \cup (A \cap D))[f,g](x) = (A,B)[(C,D)[f,g],g](x)$ for all $x \in X_\bot$ and so $(\alpha \wedge \beta)[f,g] = \alpha[\beta[f,g],g]$.

Verification continues on the next page . . . . . .

Axiom (2.7)**:** We have the following:

$$(A,B)[(C,D)[f,g],(C,D)[h,k]](x) = \begin{cases} (C,D)[f,g](x), & \text{if } x \in A; \\ (C,D)[h,k](x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(x), & \text{if } x \in A \cap C; \\ g(x), & \text{if } x \in A \cap D; \\ \bot, & \text{if } x \in A \cap (X \setminus (C \cup D)); \\ h(x), & \text{if } x \in B \cap C; \\ k(x), & \text{if } x \in B \cap D; \\ \bot, & \text{if } x \in B \cap (X \setminus (C \cup D)); \\ \bot, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(x), & \text{if } x \in A \cap C; \\ g(x), & \text{if } x \in A \cap D; \\ h(x), & \text{if } x \in B \cap C; \\ k(x), & \text{if } x \in B \cap D; \\ \bot, & \text{otherwise.} \end{cases}$$

On the other hand we have the following:

$$(C,D)[(A,B)[f,h],(A,B)[g,k]](x) = \begin{cases} (A,B)[f,h](x), & \text{if } x \in C; \\ (A,B)[g,k](x), & \text{if } x \in D; \\ \bot, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(x), & \text{if } x \in C \cap A; \\ h(x), & \text{if } x \in C \cap B; \\ \perp, & \text{if } x \in C \cap (X \setminus (A \cup B)); \\ g(x), & \text{if } x \in D \cap A; \\ k(x), & \text{if } x \in D \cap B; \\ \perp, & \text{if } x \in D \cap (X \setminus (A \cup B)); \\ \perp, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} f(x), & \text{if } x \in C \cap A; \\ h(x), & \text{if } x \in C \cap B; \\ g(x), & \text{if } x \in D \cap A; \\ k(x), & \text{if } x \in D \cap B; \\ \perp, & \text{otherwise.} \end{cases}$$

Thus $(A,B)[(C,D)[f,g],(C,D)[h,k]](x) = (C,D)[(A,B)[f,h],(A,B)[g,k]](x)$ for all $x \in X_\perp$ and so $\alpha[\beta[f,g],\beta[h,k]] = \beta[\alpha[f,h],\alpha[g,k]]$.

Axiom (2.8)**:** Note that $\alpha \wedge \beta$ is represented by $(A \cap C, B \cup (A \cap D))$. For $f, g \in \mathcal{T}_o(X_\perp)$, $\alpha[f,g] = \alpha[g,g]$ implies that $\alpha[f,g](x) = \alpha[g,g](x)$ for all $x \in X_\perp$. Consider the following:

$$(A,B)[f,g](x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B; \\ \perp, & \text{otherwise.} \end{cases}$$

Similarly we have the following:

$$(A,B)[g,g](x) = \begin{cases} g(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

Given that $(A,B)[f,g](x) = (A,B)[g,g](x)$ for all $x \in X_\bot$ as a consequence we have $f(x) = g(x)$ for all $x \in A$. Moreover $f(x) = g(x)$ for all $x \in A \cap C \subseteq A$. Thus

$$\begin{aligned}(A\cap C, B \cup (A \cap D))[f,g](x) &= \begin{cases} f(x), & \text{if } x \in A\cap C; \\ g(x), & \text{if } x \in B \cup (A\cap D); \\ \bot, & \text{otherwise.} \end{cases} \\ &= \begin{cases} g(x), & \text{if } x \in A\cap C; \\ g(x), & \text{if } x \in B \cup (A\cap D); \\ \bot, & \text{otherwise.} \end{cases}\end{aligned}$$

Thus $(\alpha \wedge \beta)[f,g] = (\alpha\wedge\beta)[g,g]$ and so the quasi-identity (2.8) holds. Consequently the pair $\left(\mathcal{T}_o(X_\bot), \mathfrak{Z}^X\right)$ is a C-set.

Example 2.1.5. Consider $S_\bot^X$, the set of all functions from X to $S_\bot$, as a pointed set with base point $\zeta_\bot$. The pair $\left(S_\bot^X, \mathfrak{Z}^X\right)$ is a C-set under the action given in (2.9), where $f, g \in S_\bot^X$ and $\alpha \in \mathfrak{Z}^X$. The axioms (2.1) – (2.8) can be verified along the same lines as in Example 2.1.4.

Example 2.1.6. Consider $\mathcal{T}(X_\bot)$, the set of all total functions on $X_\bot$, as a pointed set with base point $\zeta_\bot$. The pair $\left(\mathcal{T}(X_\bot), \mathfrak{Z}^X\right)$ is a C-set under the action given in (2.9), where $f, g \in \mathcal{T}(X_\bot)$ and $\alpha \in \mathfrak{Z}^X$. The axioms (2.1) – (2.8) can be verified along the same lines as in Example 2.1.4.

We believe that the C-set given in Example 2.1.6 does not occur naturally in the context of programs as this would include elements that terminate even when the input diverges, i.e. the input is $\perp$.

We now present a fundamental example of a C-set, where we only consider the basic tests, `true, false, undefined`.

Example 2.1.7. Let $S_\perp$ be a pointed set with base point $\perp$. The pair $(S_\perp, \mathfrak{Z})$ is a C-set with respect to the following action for all $a, b \in S_\perp$ and $\alpha \in \mathfrak{Z}$:

$$\alpha[a,b] = \begin{cases} a, & \text{if } \alpha = T; \\ b, & \text{if } \alpha = F; \\ \perp, & \text{if } \alpha = U. \end{cases}$$

These C-sets are called *basic C-sets*. In the following we verify the axioms (2.1) – (2.8) by considering α to be T, F and U casewise.

Axiom (2.1)**:** It is clear that $U[s,t] = \perp$.

Axiom (2.2)**:** It is easy to see that $F[s,t] = t$.

Axiom (2.3)**:** It suffices to consider the following three cases:

$\alpha = T$**:** In this case $(\neg T)[s,t] = F[s,t] = t = T[t,s]$.

$\alpha = F$**:** In this case $(\neg F)[s,t] = T[s,t] = s = F[t,s]$.

$\alpha = U$**:** In this case $(\neg U)[s,t] = U[s,t] = \perp = U[t,s]$.

Axiom (2.4)**:** Consider the following three cases:

$\alpha = T$**:** In this case $T[T[s,t],u] = T[s,t] = s = T[s,u]$.

$\alpha = F$**:** In this case $F[F[s,t],u] = u = F[s,u]$.

$\alpha = U$**:** In this case $U[U[s,t],u] = \perp = U[s,u]$.

Axiom (2.5)**:** Consider the following three cases:

$\alpha = T$**:** In this case $T[s, T[t, u]] = s = T[s, u]$.

$\alpha = F$**:** In this case $F[s, F[t, u]] = F[t, u] = u = F[s, u]$.

$\alpha = U$**:** In this case $U[s, U[t, u]] = \bot = U[s, u]$.

Axiom (2.6)**:** It suffices to consider the following three cases:

$\alpha = T$**:** In this case $(T \wedge \beta)[s, t] = \beta[s, t] = T[\beta[s, t], t]$.

$\alpha = F$**:** In this case $(F \wedge \beta)[s, t] = F[s, t] = t = F[\beta[s, t], t]$.

$\alpha = U$**:** In this case $(U \wedge \beta)[s, t] = U[s, t] = \bot = U[\beta[s, t], t]$.

Axiom (2.7)**:** It suffices to consider the following three cases:

$\alpha = T$**:** In this case $T[\beta[s, t], \beta[u, v]] = \beta[s, t] = \beta[T[s, u], T[t, v]]$.

$\alpha = F$**:** In this case $F[\beta[s, t], \beta[u, v]] = \beta[u, v] = \beta[F[s, u], F[t, v]]$.

$\alpha = U$**:** In this case $U[\beta[s, t], \beta[u, v]] = \bot$. Consider $\beta \in \{T, F, U\}$. It is easy to see that in each case we have $\beta[\bot, \bot] = \bot$. In other words $U[\beta[s, t], \beta[u, v]] = \bot = \beta[\bot, \bot] = \beta[U[s, u], U[t, v]]$.

Axiom (2.8)**:** For verification of this quasi-identity we again consider the following three cases:

$\alpha = T$**:** The hypothesis $\alpha[s, t] = \alpha[t, t]$ gives $T[s, t] = T[t, t]$ that is $s = t$. It is easy to see that for each $\beta \in \{T, F, U\}$ we have $\beta[s, t] = \beta[t, t]$ that is $(T \wedge \beta)[s, t] = (T \wedge \beta)[t, t]$.

$\alpha = F$**:** In this case $(F \wedge \beta)[s, t] = F[s, t] = t = F[t, t] = (F \wedge \beta)[t, t]$.

$\alpha = U$**:** In this case $(U \wedge \beta)[s, t] = U[s, t] = \bot = U[t, t] = (U \wedge \beta)[t, t]$.

2.2 Properties of C-sets

Henceforth, unless explicitly mentioned otherwise, an arbitrary C-algebra with T, F, U is always denoted by M and an arbitrary C-set by $(S_\perp, M)$. In this section, we prove certain properties of C-sets.

Proposition 2.2.1. *The following statements hold for all $\alpha, \beta \in M$ and $s, t, r \in S_\perp$:*

(i) $\alpha[\perp, \perp] = \perp$.

(ii) *If $\alpha[s, u] = \alpha[t, q]$ for some $u, q \in S_\perp$ then $\alpha[s, v] = \alpha[t, v]$ for all $v \in S_\perp$.*

(iii) *If $\alpha[s, u] = \alpha[r, r]$ for some $u \in S_\perp$ then $\alpha[s, r] = \alpha[r, r]$.*

(iv) *If $\alpha[s, u] = \alpha[t, u]$ for some $u \in S_\perp$ then $\alpha[s, v] = \alpha[t, v]$ for all $v \in S_\perp$.*

(v) *If $\alpha[s, t] = \alpha[t, t]$ then $(\beta \wedge \alpha)[s, t] = (\beta \wedge \alpha)[t, t]$.*

Proof.

(i) Using (2.1) and (2.7), $\alpha[\perp, \perp] = \alpha[U[\perp, \perp], U[\perp, \perp]] = U[\alpha[\perp, \perp], \alpha[\perp, \perp]] = \perp$.

(ii) Using (2.4), $\alpha[s, v] = \alpha[\alpha[s, u], v] = \alpha[\alpha[t, q], v] = \alpha[t, v]$.

(iii) Using Proposition 2.2.1(ii), putting $t = q = v = r$, $\alpha[s, r] = \alpha[r, r]$.

(iv) Using Proposition 2.2.1(ii), putting $q = u$, $\alpha[s, v] = \alpha[t, v]$.

(v) Using (2.6), $(\beta \wedge \alpha)[s, t] = \beta[\alpha[s, t], t] = \beta[\alpha[t, t], t] = (\beta \wedge \alpha)[t, t]$.

□

Remark 2.2.2.

(i) The C-set axioms from (2.2) to (2.6) are the same as the ones in the definition of B-set. In view of (2.1), the only B-set axiom that does not carry over in the context of C-sets is (1.8). To illustrate this, consider the basic C-set $(S_\perp, \mathbb{3})$ where $S_\perp = \{a, \perp\}$. Then $U[a, a] = \perp$ $(\neq a)$.

(ii) Bergman [1991] showed that the axiom of premise interchange (2.7) holds in B-sets.

(iii) Following the proof given in Proposition 2.2.1(v) and using the commutativity of $\wedge$ in the context of B-sets, it can be observed that the axiom of $\wedge$-compatibility (2.8) holds in B-sets.

Proposition 2.2.3. *For each $\alpha \in M_{\#}$ and $s \in S_{\perp}$, we have $\alpha[s, s] = s$.*

Proof. Let $\alpha \in M_{\#}$ and $s \in S_{\perp}$.

$$
\begin{aligned}
s &= T[s, s] && \text{from } (2.3), (2.2) \\
&= (\alpha \vee (\neg\alpha))[s, s] && \text{since } \alpha \in M_{\#} \\
&= \alpha[s, (\neg\alpha)[s, s]] && \text{from } (2.1.2) \\
&= \alpha[s, \alpha[s, s]] && \text{from } (2.3) \\
&= \alpha[s, s] && \text{from } (2.5)
\end{aligned}
$$

□

In view of Proposition 2.2.3, the axiom (1.8) of B-sets holds for the elements of Boolean algebra $M_{\#}$. Hence, we have the following corollary.

Corollary 2.2.4. *The pair $(S_{\perp}, M_{\#})$ is a B-set.*

Remark 2.2.5. The proof of Proposition 2.2.3 also shows us that axiom (1.8) is redundant in the definition of a B-set.

We make the following observations on C-sets.

Remark 2.2.6.

(i) Given C-set $(S_{\perp}, M)$ where $\alpha[s, t] = \beta[s, t]$ for all $s, t \in S_{\perp}$, α need not be equal to β.

Let $X = \{1,2\}$. Consider $(S_\perp, M) \leq (\mathcal{T}_o(X_\perp), \mathfrak{Z}^X)$ given by the following. Take $M = \{(T,T),(F,F),(U,U),(F,U),(T,U)\}$ and $S_\perp = \{\zeta_\perp, f_1, f_2\}$ where $f_1(1) = 1$ and $f_1(x) = \perp$ for $x \neq 1$ and $f_2(1) = 2$ and $f_1(x) = \perp$ for $x \neq 1$. One can see that this pair is closed under the usual action. Identify each element of M with the pair of sets given below.

$$(T,T) = (\{1,2\}, \emptyset),$$
$$(F,F) = (\emptyset, \{1,2\}),$$
$$(U,U) = (\emptyset, \emptyset),$$
$$(F,U) = (\emptyset, \{1\}),$$
$$(T,U) = (\{1\}, \emptyset).$$

Thus $(T,T)[f,g] = (\{1,2\},\emptyset)[f,g] = f$, $(F,F)[f,g] = (\emptyset,\{1,2\})[f,g] = g$ and $(U,U)[f,g] = (\emptyset,\emptyset)[f,g] = \zeta_\perp$. We also have the following:

$$(F,U)[f,g](x) = (\emptyset,\{1\})[f,g](x) = \begin{cases} g(1), & \text{if } x = 1; \\ \perp, & \text{otherwise.} \end{cases}$$

For the elements of $S_\perp$ we have $(F,U)[f,g] = g$. Similarly $(T,U)[f,g] = f$ and so it follows that this is a subalgebra of the C-set $(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X)$. Thus in this C-set $(S_\perp, M)$ we have $(T,U)[f,f] = f = (F,U)[f,f]$ for all $f \in S_\perp$, however $(T,U) \notin M_\#$ and $(F,U) \notin M_\#$. Along similar lines take $\alpha = (T,T)$ and $\beta = (T,U)$. Then for all $f,g \in S_\perp$ we have $\alpha[f,g] = f = \beta[f,g]$ but $\alpha \neq \beta$.

(ii) Along similar lines the statement $\alpha[s,s] = s \Leftrightarrow \alpha \in M_\#$ need not hold in an arbitrary C-set $(S_\perp, M)$.

Consider the C-set $(S_\perp, M)$ as defined in Remark 2.2.6(i) with $\alpha = (F,U)$. It is straightforward to see that $(F,U)[f,f] = f$ for all $f \in S_\perp$, however $\alpha \notin M_\#$.

We know that if M is a C-algebra with T, F, U then $M_\#$ is a Boolean algebra under the induced operations. A natural question that arises is what structure $M_\#^c$ has.

Proposition 2.2.7. *Let M be a C-algebra with T, F, U. Then $M_\#^c$ is a C-algebra under the induced operations of M.*

Proof. Let $\alpha \in M_\#^c$. Then $\neg\alpha \vee \neg\neg\alpha = \neg\alpha \vee \alpha = \alpha \vee \neg\alpha \neq T$ and so $\neg\alpha \in M_\#^c$. Let $\alpha, \beta \in M_\#^c$. Then considering $M \leq \mathbf{3}^X$ for some set X it follows that there exists $x \in X$ such that $\alpha(x) = U$. Therefore $(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x) = U \wedge \beta(x) = U$ and so $\alpha \wedge \beta \in M_\#^c$. Thus $M_\#^c$ is closed under $\neg$ and $\wedge$, and therefore under $\vee$. The result follows. □

Remark 2.2.8. Note that although $M_\#^c$ is a C-algebra under the induced operations of M, it is not closed under the constants T and F, and is therefore not a subalgebra of M (with T, F, U). It is therefore natural to consider $\overline{M_\#^c} = M_\#^c \cup \{T, F\}$, which is clearly closed with respect to T, F, U.

Thus every C-algebra with T, F, U can be written in terms of a Boolean algebra and another C-algebra with T, F, U. More precisely,

$$M = M_\# \cup \overline{M_\#^c}$$

where $M_\# \cap \overline{M_\#^c} = \{T, F\}$.

Remark 2.2.9. Hence every C-set $(S_\perp, M)$ can be written in terms of the B-set $(S_\perp, M_\#)$ and the C-set $(S_\perp, \overline{M_\#^c})$.

In view of this, we question what the image of the `if-then-else` action under elements from $M_\#$ (and $\overline{M_\#^c}$) is. Let $(S_\perp, M)$ be a C-set. Consider the following subsets of $S_\perp$.

$$J = \{r \in S_\perp : \alpha[s,t] = r \text{ for some } s,t \in S_\perp \text{ and some } \alpha \in M_\#\},$$
$$K = \{r \in S_\perp : \alpha[s,t] = r \text{ for some } s,t \in S_\perp \text{ and some } \alpha \in \overline{M^c_\#}\}.$$

Remark 2.2.10.

(i) $J = S_\perp$ since for $r \in S_\perp$ and any $\alpha \in M_\#(\neq \emptyset)$ using Proposition 2.2.3 we have $\alpha[r,r] = r$.

(ii) $K = S_\perp$ since for $r \in S_\perp$ we have $T[r,r] = r$ for $T \in \overline{M^c_\#}$.

Since both J and K are simply $S_\perp$ consider the subset of $S_\perp$ defined by

$$L = \{r \in S_\perp : \alpha[s,t] = r \text{ for some } s,t \in S_\perp \text{ and some } \alpha \in M^c_\#\}.$$

We observe that $\perp \in L$ since $U \in M^c_\#$ so that $U[s,t] = \perp$. We make the following observations on the set L for various examples of C-sets.

Remark 2.2.11. In the basic C-set $(S_\perp, \mathfrak{Z})$ we have $L = \{\perp\}$ since $M^c_\# = \{U\}$.

Remark 2.2.12. In the functional C-set $(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X)$ we have $L = \{h \in \mathcal{T}_o(X_\perp) : h(x) = \perp \text{ for some } x \in X\}$. This is because $M^c_\# = \{\alpha \in \mathfrak{Z}^X : \alpha(x) = U \text{ for some } x \in X\}$.

Remark 2.2.13. Consider the C-set (M, M). We show that $L = M^c_\#$. Let $\delta \in L$. Then $\delta = \alpha[\![\beta,\gamma]\!]$ for some $\alpha \in M^c_\#$ and $\beta,\gamma \in M$ where $\alpha[\![\beta,\gamma]\!] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma)$. Consider $M \leq \mathfrak{Z}^X$ for some set X. Then $\alpha \in M^c_\#$ implies that $\alpha(x) = U$ for some $x \in X$. Thus $(\alpha[\![\beta,\gamma]\!])(x) = U$ so that $\delta \in M^c_\#$.

For the reverse inclusion consider $\delta \in M^c_\#$. Then $\delta[\![\delta,\delta]\!] \in L$ and $\delta[\![\delta,\delta]\!] = (\delta \wedge \delta) \vee (\neg\delta \wedge \delta) = \delta \vee (\neg\delta \wedge \delta)$. The identity $\delta \vee (\neg\delta \wedge \delta) = \delta$ can be easily seen to hold in $\mathfrak{Z}$ so that it holds in all C-algebras. Thus $\delta[\![\delta,\delta]\!] = \delta$ so that $\delta \in L$. Consequently $L = M^c_\#$.

3

Representation of C-sets

With the aim of studying structural properties of C-sets, in this chapter we obtain a subdirect representation in terms of basic C-sets for the class of C-sets in which the C-algebras are adas (cf. Theorem 3.2.4). Note that except in Example 2.1.3, the C-algebras in all other examples of C-sets given in Section 2.1 are adas (cf. Example 1.4.10). In Section 3.1 we produce an equivalence relation E_θ on $S_\perp$ such that (E_θ, θ) is a congruence on $(S_\perp, M)$ for each maximal congruence θ on M. Further, in Section 3.2 we show that the intersection of the collection of congruences (E_θ, θ) is trivial, using which we prove Theorem 3.2.4.

Let $(S_\perp, M)$ be a C-set, where M is an ada. In the following results we consistently use α, β, γ for the elements of M, and r, s, t, u, v for the elements of $S_\perp$.

3.1 A family of congruences

Considering the C-sets as two-sorted algebras, we now define congruences on them.

Definition 3.1.1. A *congruence* of a C-set is a pair (σ, τ), where σ is an equivalence relation on $S_\perp$ and τ is a congruence on the ada M such that

$$(s,t),(u,v) \in \sigma \;\text{ and }\; (\alpha,\beta) \in \tau \Rightarrow (\alpha[s,u], \beta[t,v]) \in \sigma.$$

In order to give a subdirect representation of the C-set $(S_\perp, M)$, we consider the collection of all maximal congruences on the ada M so that for each such congruence θ, we have $M/\theta \cong \mathfrak{3}$ using Theorem 1.1.18. We produce an equivalence relation E_θ on $S_\perp$ such that (E_θ, θ) is a congruence on $(S_\perp, M)$ for each θ, and the intersection of the collection of congruences (E_θ, θ) is trivial. Consequently we show that $(S_\perp, M)$ is a subdirect product of basic C-sets $(S_\perp/E_\theta, M/\theta)$.

Definition 3.1.2. For each maximal congruence θ on M, we define a relation on $S_\perp$ by

$$E_\theta = \{(s,t) \in S_\perp \times S_\perp : \; \beta[s,t] = \beta[t,t] \;\text{ for some } \beta \in \overline{T}^\theta\}.$$

Lemma 3.1.3. *The relation E_θ is an equivalence on $S_\perp$.*

Proof. Since $T[s,s] = T[s,s]$ and $T \in \overline{T}^\theta$, we have $(s,s) \in E_\theta$ so that the binary relation E_θ on $S_\perp$ is reflexive.

For symmetry, let $(s,t) \in E_\theta$. Then there exists $\beta \in \overline{T}^\theta$ such that $\beta[s,t] = \beta[t,t]$. Using (2.4), we have $\beta[t,s] = \beta[\beta[t,t],s] = \beta[\beta[s,t],s] = \beta[s,s]$ so that $(t,s) \in E_\theta$.

Let $(s,t),(t,r) \in E_\theta$. Then there exist $\alpha, \beta \in \overline{T}^\theta$, such that $\alpha[s,t] = \alpha[t,t]$ and $\beta[t,r] = \beta[r,r]$. As θ is a congruence on M, $(\alpha, T), (\beta, T) \in \theta$ implies $(\alpha \wedge \beta, T) \in \theta$

so that $(\alpha \wedge \beta) \in \overline{T}^\theta$. Note that

$$\begin{aligned}
(\alpha \wedge \beta)[s,r] &= (\alpha \wedge \beta)[(\alpha \wedge \beta)[s,t],r] && \text{from (2.4)} \\
&= (\alpha \wedge \beta)[(\alpha \wedge \beta)[t,t],r] && \text{from } \alpha[s,t] = \alpha[t,t] \text{ and (2.8)} \\
&= (\alpha \wedge \beta)[t,r] && \text{from (2.4)} \\
&= (\alpha \wedge \beta)[r,r] && \text{from } \beta[t,r] = \beta[r,r] \text{ and Proposition 2.2.1(v)}.
\end{aligned}$$

Hence $(s,r) \in E_\theta$ so that E_θ is transitive. □

Remark 3.1.4. Note that as θ is a maximal congruence on the ada M, using Theorem 1.1.18, M/θ must be simple, i.e., $M/\theta \cong \mathfrak{3}$. Further, the quotient set $S_\perp/E_\theta$ can be treated as a pointed set with base point $\overline{\perp}$. Thus $(S_\perp/E_\theta, M/\theta)$ is a basic C-set under the action

$$\overline{\alpha}^\theta[\overline{s}^{E_\theta}, \overline{t}^{E_\theta}] = \begin{cases} \overline{s}^{E_\theta}, & \text{if } \alpha \in \overline{T}^\theta; \\ \overline{t}^{E_\theta}, & \text{if } \alpha \in \overline{F}^\theta; \\ \overline{\perp}^{E_\theta}, & \text{if } \alpha \in \overline{U}^\theta. \end{cases}$$

Proposition 3.1.5. *For any* $\alpha \in M$, $\beta = \neg(\alpha^\downarrow \vee (\neg\alpha)^\downarrow) \vee U$ *satisfies* $\beta \wedge \alpha = U$. *Moreover, if* $(\alpha, U) \in \theta$ *then* $(\beta, T) \in \theta$.

Proof. Since $\mathfrak{3}$ is the only subdirectly irreducible ada, it is sufficient to check the validity of the identity $\beta \wedge \alpha = U$ in $\mathfrak{3}$.

$\alpha = T$**:** Then $\beta = \neg(T^\downarrow \vee F^\downarrow) \vee U = \neg(T \vee F) \vee U = F \vee U = U$. Thus $\beta \wedge \alpha = U \wedge T = U$.

$\alpha = F$**:** Then $\beta = \neg(F^\downarrow \vee T^\downarrow) \vee U = \neg(F \vee T) \vee U = F \vee U = U$. Thus $\beta \wedge \alpha = U \wedge F = U$.

$\alpha = U$**:** Then $\beta = \neg(U^\downarrow \vee U^\downarrow) \vee U = \neg(F \vee F) \vee U = T \vee U = T$. Thus $\beta \wedge \alpha = T \wedge U = U$.

Hence in all these three cases we see that $\beta \wedge \alpha = U$.

Suppose $(\alpha, U) \in \theta$. Since θ is a congruence, we have $(\alpha^{\downarrow}, U^{\downarrow}) = (\alpha^{\downarrow}, F) \in \theta$. Also, we have $(\neg\alpha, \neg U) = (\neg\alpha, U) \in \theta$ so that $((\neg\alpha)^{\downarrow}, F) \in \theta$. Now, by substitution with respect to $\vee$, we have $(\alpha^{\downarrow} \vee (\neg\alpha)^{\downarrow}, F) \in \theta$.

This further implies $(\neg(\alpha^{\downarrow} \vee (\neg\alpha)^{\downarrow}), \neg F) = (\neg(\alpha^{\downarrow} \vee (\neg\alpha)^{\downarrow}), T) \in \theta$. However since $(U, U) \in \theta$ we have $(\neg(\alpha^{\downarrow} \vee (\neg\alpha)^{\downarrow}) \vee U, T \vee U) = (\neg(\alpha^{\downarrow} \vee (\neg\alpha)^{\downarrow}) \vee U, T) \in \theta$. Hence $(\beta, T) \in \theta$. □

Proposition 3.1.6. *For each $\alpha \in M$ and each $s, t \in S_{\perp}$, we have the following:*

(i) $(\alpha, T) \in \theta \Rightarrow (\alpha[s,t], s) \in E_{\theta}$.

(ii) $(\alpha, F) \in \theta \Rightarrow (\alpha[s,t], t) \in E_{\theta}$.

(iii) $(\alpha, U) \in \theta \Rightarrow (\alpha[s,t], \perp) \in E_{\theta}$.

Proof.

(i) From (2.4), we have $\alpha[\alpha[s,t], s] = \alpha[s,s]$. Hence $(\alpha[s,t], s) \in E_{\theta}$ as $\alpha \in \overline{T}^{\theta}$.

(ii) Note that $(\alpha, F) \in \theta$ implies $(\neg\alpha, T) \in \theta$. Using (2.5) and (2.3), $(\neg\alpha)[\alpha[s,t], t] = \alpha[t, \alpha[s,t]] = \alpha[t,t] = (\neg\alpha)[t,t]$. Thus $(\alpha[s,t], t) \in E_{\theta}$.

(iii) If $(\alpha, U) \in \theta$, then by Proposition 3.1.5, $\beta = \neg(\alpha^{\downarrow} \vee (\neg\alpha)^{\downarrow}) \vee U \in \overline{T}^{\theta}$, and $\beta \wedge \alpha = U$. Note that

$$\begin{aligned}
\beta[\alpha[s,t], t] &= (\beta \wedge \alpha)[s,t] && \text{from (2.6)} \\
&= U[s,t] && \text{from Proposition 3.1.5} \\
&= \perp && \text{from (2.1)} \\
&= \beta[\perp, \perp] && \text{from Proposition 2.2.1(i).}
\end{aligned}$$

Consequently, by Proposition 2.2.1(iii), we have $\beta[\alpha[s,t],\bot] = \beta[\bot,\bot]$. Hence $(\alpha[s,t],\bot) \in E_\theta$.

□

Lemma 3.1.7. *The pair (E_θ, θ) is a C-set congruence.*

Proof. In view of Remark 3.1.4, $(S_\bot/E_\theta, M/\theta)$ is a basic C-set. Consider the canonical maps $\nu_1 : S_\bot \to S_\bot/E_\theta$, given by $\nu_1(s) = \overline{s}^{E_\theta}$, and $\nu_2 : M \to M/\theta \cong \mathfrak{Z}$, given by $\nu_2(\alpha) = \overline{\alpha}^\theta$. We show that the pair (ν_1, ν_2) is a C-set homomorphism so that $\ker(\nu_1,\nu_2) = (E_\theta, \theta)$ is a C-set congruence.

It is straightforward to see that $\nu_1(\bot) = \overline{\bot}^{E_\theta}$ and thus ν_1 is a homomorphism of pointed sets. It is also clear that ν_2 is a homomorphism of adas. Additionally, we require that $\nu_1(\alpha[s,t]) = (\nu_2(\alpha))[\nu_1(s), \nu_1(t)]$. In order to prove this, it suffices to consider the following three cases in view of the maximality of congruence θ.

Case I: If $\alpha \in \overline{T}^\theta$, then we effectively need to show that $\overline{\alpha[s,t]}^{E_\theta} = \overline{\alpha}^\theta[\overline{s}^{E_\theta}, \overline{t}^{E_\theta}]$. From Remark 3.1.4 and the fact that $\alpha \in \overline{T}^\theta$, we have $\overline{\alpha}^\theta[\overline{s}^{E_\theta}, \overline{t}^{E_\theta}] = \overline{s}^{E_\theta}$. This reduces to showing that $(\alpha[s,t], s) \in E_\theta$, which follows from Proposition 3.1.6(i).

Case II: In a similar vein, if $\alpha \in \overline{F}^\theta$, we need to show that $(\alpha[s,t], t) \in E_\theta$, which follows from Proposition 3.1.6(ii).

Case III: Similarly, if $\alpha \in \overline{U}^\theta$, we require that $(\alpha[s,t],\bot) \in E_\theta$, which is precisely Proposition 3.1.6(iii).

This completes the proof. □

Lemma 3.1.8. *For the C-set (M,M) the equivalence E_θ on M, denoted by E_{θ_M}, is a subset of θ.*

Proof. Let $(\alpha,\beta) \in E_{\theta_M}$. Then there exists $\gamma \in \overline{T}^\theta$ such that $\gamma[\![\alpha,\beta]\!] = \gamma[\![\beta,\beta]\!]$. In other words,

$$(\gamma \wedge \alpha) \vee (\neg\gamma \wedge \beta) = (\gamma \wedge \beta) \vee (\neg\gamma \wedge \beta) \tag{3.1}$$

Since $\gamma \in \overline{T}^\theta$, we have $(\gamma, T) \in \theta$. Moreover, $(\alpha,\alpha) \in \theta$ as θ is reflexive. It follows that $(\gamma \wedge \alpha, T \wedge \alpha) = (\gamma \wedge \alpha, \alpha) \in \theta$. Similarly, $(\gamma, T) \in \theta$ implies that $(\neg\gamma, F) \in \theta$, and as $(\beta, \beta) \in \theta$, we have $(\neg\gamma \wedge \beta, F \wedge \beta) = (\neg\gamma \wedge \beta, F) \in \theta$. Consequently $((\gamma \wedge \alpha) \vee (\neg\gamma \wedge \beta), \alpha \vee F) = ((\gamma \wedge \alpha) \vee (\neg\gamma \wedge \beta), \alpha) \in \theta$. Following a similar procedure, using $(\gamma, T), (\neg\gamma, F), (\beta, \beta) \in \theta$, we obtain $((\gamma \wedge \beta) \vee (\neg\gamma \wedge \beta), \beta) \in \theta$. Now using (3.1), the symmetry and transitivity of θ, we have $(\alpha, \beta) \in \theta$ so that $E_{\theta_M} \subseteq \theta$. □

3.2 A subdirect representation

In this section we show that the intersection of this family of congruences is trivial using which we achieve a subdirect representation for this class of C-sets in terms of basic C-sets. In order to obtain the main theorem we have the following result.

Proposition 3.2.1. *If $\alpha[s,t] = \alpha[t,t]$, then $\alpha^{\downarrow}[s,t] = \alpha^{\downarrow}[t,t]$.*

Proof. Using (1.37) and (2.1.2), $\alpha[s,t] = (\alpha^{\downarrow} \vee \alpha)[s,t] = \alpha^{\downarrow}[s, \alpha[s,t]] = \alpha^{\downarrow}[s, \alpha[t,t]]$. On the other hand, observe that $\alpha[s,t] = \alpha[t,t] = (\alpha^{\downarrow} \vee \alpha)[t,t] = \alpha^{\downarrow}[t, \alpha[t,t]]$ so that $\alpha^{\downarrow}[s, \alpha[t,t]] = \alpha^{\downarrow}[t, \alpha[t,t]]$. Consequently, we have $\alpha^{\downarrow}[s,t] = \alpha^{\downarrow}[t,t]$ by Proposition 2.2.1(iv). □

We now show that the intersection of all equivalence relations E_θ is trivial. We proceed by first considering the pair $(S_\perp, M_\#)$ which is a B-set. Consequently, Theorem 1.3.6 allows us to consider the pair $(S_\perp, M_\#)$ as a subalgebra of a product of basic B-sets and using which we may treat elements of $S_\perp$ as functions. We then construct a family of maximal congruences on M which, along with the functional representation helps prove that for (s,t) in the intersection of all E_θ we must have $s = t$. Consequently, the intersection of all equivalence relations E_θ is trivial.

Lemma 3.2.2. $\bigcap_{\theta} E_\theta = \Delta_{S_\perp}$*, where θ ranges over all maximal congruences on M.*

Proof. By Corollary 2.2.4, $(S_\perp, M_\#)$ is a B-set so that $(S_\perp, M_\#)$ is a subdirect product of basic B-sets (cf. Theorem 1.3.6). Hence, $(S_\perp, M_\#)$ is a subalgebra of a product of basic B-sets $(S_x, 2)$, where x ranges over some set X. That is,

$$\begin{aligned}(S_\perp, M_\#) &\leq \prod_{x\in X}(S_x, 2)\\ &= \left(\prod_{x\in X} S_x, 2^X\right)\\ &\leq \left(\left(\bigcup_{x\in X} S_x\right)^X, 2^X\right)\end{aligned}$$

Note that the action in both $\left(\prod_{x\in X} S_x, 2^X\right)$ and $\left((\bigcup_{x\in X} S_x)^X, 2^X\right)$ is

$$(\alpha[s,t])(x) = \begin{cases} s(x), & \text{if } \alpha(x) = T;\\ t(x), & \text{if } \alpha(x) = F.\end{cases}$$

Also note that the action in $\left(\prod_{x\in X} S_x, 2^X\right)$ is simply a restriction of that on $\left((\bigcup_{x\in X} S_x)^X, 2^X\right)$. Since $(S_\perp, M_\#)$ is a subalgebra of $\left((\bigcup_{x\in X} S_x)^X, 2^X\right)$, we can see that $M_\#$ is a subalgebra of 2^X. Using the construction mentioned in Remark 1.4.14, $M \cong (M_\#)^\star \leq 3^X$.

Now for any $x_o \in X$, treating M as a subalgebra of 3^X we define maximal congruences on M as follows.

$$(\alpha, \beta) \in \theta_{x_o} \Leftrightarrow \alpha(x_o) = \beta(x_o).$$

Such θ_{x_o} is indeed a maximal congruence on M. It is clearly an equivalence relation on M. Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \theta_{x_o}$. Then $\alpha_1(x_o) = \beta_1(x_o)$ and $\alpha_2(x_o) = \beta_2(x_o)$. Thus $\alpha_1(x_o) \wedge \alpha_2(x_o) = \beta_1(x_o) \wedge \beta_2(x_o)$. Thus $(\alpha_1 \wedge \alpha_2, \beta_1 \wedge \beta_2) \in \theta_{x_o}$. Similarly θ_{x_o} is compatible with the other operations on M, viz., $\vee, \neg$ and $^\downarrow$. Note that M has only the following three equivalence classes with respect to θ_{x_o}.

$$\overline{\mathbf{T}}^{\theta} = \{\alpha \in M : \alpha(x_o) = T\}$$
$$\overline{\mathbf{F}}^{\theta} = \{\alpha \in M : \alpha(x_o) = F\}$$
$$\overline{\mathbf{U}}^{\theta} = \{\alpha \in M : \alpha(x_o) = U\}$$

Thus M/θ_{x_o} is simple and so θ_{x_o} is maximal.

We now show that $\bigcap E_\theta = \Delta_{S_\perp}$. Let $(s,t) \in \bigcap E_\theta$. Then for every maximal congruence θ on M, there exists a $\beta_\theta \in \overline{T}^{\theta}$ such that $\beta_\theta[s,t] = \beta_\theta[t,t]$. On using Proposition 3.2.1 we have $\beta_\theta^{\downarrow}[s,t] = \beta_\theta^{\downarrow}[t,t]$. Note that if β_θ is in $\overline{T}^{\theta}$, so is $\beta_\theta^{\downarrow}$. Moreover, $\beta_\theta^{\downarrow} \in M_\#$.

As $S_\perp \leq (\bigcup_{x\in X} S_x)^X$, we may treat s,t as functions $s',t' \in (\bigcup_{x\in X} S_x)^X$. Note that the `if-then-else` action in $(S_\perp, M_\#)$ can be treated as a restriction of that on $((\bigcup_{x\in X} S_x)^X, 2^X)$. Considering the maximal congruences defined above, for each $x_o \in X$ there exists $\beta_{\theta_{x_o}}^{\downarrow} \in \overline{T}^{\theta_{x_o}}$, that is, $\beta_{\theta_{x_o}}^{\downarrow}(x_o) = T$ and $\beta_{\theta_{x_o}}^{\downarrow}[s',t'] = \beta_{\theta_{x_o}}^{\downarrow}[t',t']$. In other words, for each $x \in X$, $(\beta_{\theta_{x_o}}^{\downarrow}[s',t'])(x) = (\beta_{\theta_{x_o}}^{\downarrow}[t',t'])(x)$. In particular for $x = x_o$, $(\beta_{\theta_{x_o}}^{\downarrow}[s',t'])(x_o) = (\beta_{\theta_{x_o}}^{\downarrow}[t',t'])(x_o)$.

However $(\beta_{\theta_{x_o}}^{\downarrow}[s',t'])(x_o) = s'(x_o)$ as $\beta_{\theta_{x_o}}^{\downarrow}(x_o) = T$. Similarly, $(\beta_{\theta_{x_o}}^{\downarrow}[t',t'])(x_o) = t'(x_o)$.

This tells us that for each $x_o \in X$, $s'(x_o) = t'(x_o)$, that is, $s' \equiv t'$ which means that $s = t$ in $S_\perp$. This completes the proof. □

We now ascertain that the intersection of all maximal congruences on M must be trivial in the following.

Remark 3.2.3. Let $\alpha, \beta \in M$ with $\alpha \neq \beta$. Treating M as a subalgebra of $\mathfrak{Z}^X$ for some X, there exists $x_o \in X$ such that $\alpha(x_o) \neq \beta(x_o)$. Then θ_{x_o}, as in the previous proof, is a maximal congruence which clearly separates α and β. Since the intersection of all such congruences is Δ_M, the intersection of all maximal congruences on M

$$\bigcap_{\theta \text{ maximal}} \theta = \Delta_M$$

We now prove the main theorem of this chapter.

Theorem 3.2.4. *Every C-set $(S_\perp, M)$ where M is an ada is a subdirect product of basic C-sets.*

Proof. Let $(S_\perp, M)$ be a C-set where M is an ada and $\{\theta\}$ be the collection of all maximal congruences on M. By Lemma 3.1.7, for each θ, the pair (E_θ, θ) is a C-set congruence on $(S_\perp, M)$ and by Remark 3.1.4 $(S_\perp/E_\theta, M/\theta)$ is a basic C-set. Further, by Lemma 3.2.2 and Remark 3.2.3, the intersection of all congruences (E_θ, θ) is trivial. Hence, $(S_\perp, M)$ is a subdirect product of $(S_\perp/E_\theta, \mathbb{3})$, where θ varies over maximal congruences on M. □

The following consequence of Theorem 3.2.4 is useful to establish the equivalence between programs which admit the current setup.

Corollary 3.2.5. *An identity (quasi-identity) is satisfied in every C-set $(S_\perp, M)$ where M is an ada if and only if it is satisfied in all basic C-sets.*

4

Agreeable C-sets

In this chapter we describe an algebraic formalism for the equality test over possibly non-halting programs, where the tests are drawn from a C-algebra. In Section 4.1 we extend the notion of C-sets to agreeable C-sets and give natural interpretations for each of the agreeable C-set axioms in terms of the functional model, along with some examples of agreeable C-sets. In Section 4.2 we give a representation theorem for the class of agreeable C-sets where the C-algebra is an ada (cf. Theorem 4.2.1). We achieve this result by showing that each C-set congruence (E_θ, θ) as defined in Section 3.1 is also an agreeable C-set congruence. Using this we achieve an alternative proof for Theorem 2.7 in Jackson and Stokes [2009].

4.1 Axiomatization of equality test

The equality test over the functions $f, g \in \mathcal{T}_o(X_\perp)$ can be naturally described by the following:

$$(f * g)(x) = \begin{cases} T, & \text{if } f(x) = g(x) \text{ and } f(x) \neq \perp \neq g(x); \\ F, & \text{if } f(x) \neq g(x) \text{ and } f(x) \neq \perp \neq g(x); \\ U, & \text{otherwise.} \end{cases} \tag{4.1}$$

For simplicity of notation, we denote the condition $f(x) = g(x)$ and $f(x) \neq \perp \neq g(x)$ by $f(x) = g(x)$ $(\neq \perp)$ and the condition $f(x) \neq g(x)$ and $f(x) \neq \perp \neq g(x)$ by $f(x) \neq g(x)$ $(\neq \perp)$. Consequently, $f * g$ can be identified with the pair of sets (A, B) on X, where $A = \{x \in X : f(x) = g(x)\ (\neq \perp)\}$ and $B = \{x \in X : f(x) \neq g(x)\ (\neq \perp)\}$.

Keeping this model in mind, we extend the notion of agreeable B-sets, given in Jackson and Stokes [2009], and define agreeable C-sets as follows.

Definition 4.1.1. A C-set $(S_\perp, M)$ equipped with a function

$$* : S_\perp \times S_\perp \to M$$

is said to be *agreeable* if it satisfies the following axioms for all $s, t, u, v \in S_\perp$ and $\alpha \in M$:

$$(s * s)[s, \perp] = s \qquad \text{(domain axiom)} \tag{4.2}$$

$$\perp * s = U = s * \perp \qquad (\perp\text{-comparison}) \tag{4.3}$$

$$(s * t)[s, t] = (s * t)[t, t] \qquad \text{(equality on conclusions)} \tag{4.4}$$

$$\alpha[s,t] * \alpha[u,v] = \alpha[\![s * u, t * v]\!] \qquad \text{(operation interchange)} \qquad (4.5)$$

$$((s * s = T) \wedge (s * t = U)) \Rightarrow t = \bot \qquad \text{(totality condition)} \qquad (4.6)$$

While the operation interchange axiom (4.5) is indeed an axiom in the context of agreeable B-sets (cf. axiom (1.16)), one can verify that axiom (4.4) holds in agreeable B-sets. However, the other axioms are specific to the current scenario of the non-halting case. These axioms can be justified along the following lines by considering equality of functions over the functional model $(\mathcal{T}_o(X_\bot), \mathfrak{Z}^X)$ of C-sets.

In $\mathcal{T}_o(X_\bot)$, the domain of a function is considered in the spirit of a partial function, i.e., all those points whose image is not $\bot$. In the model $(\mathcal{T}_o(X_\bot), \mathfrak{Z}^X)$, the partial predicate $s * s$ represents the domain of s. The domain axiom (4.2) captures the behaviour of `if-then-else` with respect to the domain of s. For instance, we expect $s * s$ takes truth value T in the domain of s so that $(s * s)[s, \bot] = s$. Also, in the complement of the domain of s, $s * s$ should take value U so that $(s * s)[s, \bot] = U[s, \bot] = \bot = s$.

Note that we check the equality of two functions over their domains. Thus the $\bot$-comparison axiom (4.3) states that comparing the `error` state $\bot$ with any element s results in the `undefined` predicate U.

The axiom of equality on conclusions (4.4) exhibits the behaviour of equality test $*$ on conclusions of the `if-then-else` action of the C-set. Indeed, when the partial predicate $s * t = T$, $(s * t)[s, t] = s = t = (s * t)[t, t]$ and similarly if $s * t = F$, then $(s * t)[s, t] = t = (s * t)[t, t]$. Further, if $s * t = U$, then $(s * t)[s, t] = \bot = (s * t)[t, t]$.

The axiom of operation interchange (4.5) describes how $*$ and the `if-then-else` action relate to the action on the C-set (M, M). The totality condition (4.6) is a quasi-identity, in which if s is a total function but $s * t$ is undefined, then it must follow that t is the empty function, i.e., $t = \zeta_\bot$.

We have the following example of agreeable C-sets.

Example 4.1.2. The pair $\left(\mathcal{T}_o(X_\bot), \mathfrak{Z}^X\right)$ is an agreeable C-set under the operation

$*$ defined in (4.1). Such agreeable C-sets are called *agreeable functional C-sets*. We now verify the axioms (4.2) – (4.6) in the following.

Axiom (4.2)**:** We show that $(f*f)[f,\zeta_\perp] = f$. For any $f \in \mathcal{T}_o(X_\perp)$, $f*f = (A,B)$ where $A = \{x \in X : f(x) \neq \perp\}$ and $B = \emptyset$. Thus

$$(A,B)[f,\zeta_\perp](x) = \begin{cases} f(x), & \text{if } x \in A; \\ \perp, & \text{if } x \notin A. \end{cases}$$
$$= \begin{cases} f(x), & \text{if } f(x) \neq \perp; \\ \perp, & \text{if } f(x) = \perp. \end{cases}$$
$$= f(x)$$

Axiom (4.3)**:** For any $f \in \mathcal{T}_o(X_\perp)$, $\zeta_\perp * f = (A,B)$ where $A = \{x \in X : \zeta_\perp(x) = f(x)\ (\neq \perp)\}$ and $B = \{x \in X : \zeta_\perp(x) \neq f(x)\ (\neq \perp)\}$. Thus $\zeta_\perp * f = (A,B) = (\emptyset,\emptyset) = U = f * \zeta_\perp$.

Axiom (4.4)**:** Let $f, g \in \mathcal{T}_o(X_\perp)$ and $f*g = (A,B)$. Then

$$(f*g)[f,g](x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B; \\ \perp, & \text{otherwise.} \end{cases}$$

Thus for $x \in A$ we have $f(x) = g(x)\ (\neq \perp)$ and so $(f*g)[f,g](x) = f(x) = g(x) = (f*g)[g,g](x)$. Similarly when $x \in B$ we have $(f*g)[f,g](x) = g(x) = (f*g)[g,g](x)$. If $x \in (A \cup B)^{\mathsf{c}}$ then $(f*g)[f,g](x) = \perp = (f*g)[g,g](x)$.

Axiom (4.5)**:** We show that $\alpha[f,g] * \alpha[h,k] = \alpha[\![f*h, g*k]\!]$. Let $\alpha = (A,B)$ and

$\mathscr{F}_1 = \alpha[f,g]$, where

$$\mathscr{F}_1(x) = \alpha[f,g](x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

Similarly let $\mathscr{F}_2 = \alpha[h,k]$ where

$$\mathscr{F}_2(x) = \alpha[h,k](x) = \begin{cases} h(x), & \text{if } x \in A; \\ k(x), & \text{if } x \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

Let $\mathscr{F}_1 * \mathscr{F}_2 = (C,D)$ where $C = \{x \in X : \mathscr{F}_1(x) = \mathscr{F}_2(x)(\neq \bot)\}$ and $D = \{x \in X : \mathscr{F}_1(x) \neq \mathscr{F}_2(x)(\neq \bot)\}$. Let $f * h = (E,F)$ where $E = \{x \in X : f(x) = h(x)(\neq \bot)\}$ and $F = \{x \in X : f(x) \neq h(x)(\neq \bot)\}$. Let $g * k = (G,H)$ where $G = \{x \in X : g(x) = k(x)(\neq \bot)\}$ and $H = \{x \in X : g(x) \neq k(x)(\neq \bot)\}$. Then we have

$$\begin{aligned}(A,B)[\![(E,F),(G,H)]\!] &= \big((A,B) \wedge (E,F)\big) \vee \big(\neg((A,B)) \wedge (G,H)\big) \\ &= \big((A \cap E, B \cup (A \cap F)\big) \vee \big(B \cap G, A \cup (B \cap H)\big) \\ &= \Big((A \cap E) \cup \big((B \cup (A \cap F)) \cap (B \cap G)\big), \\ &\quad \big(B \cup (A \cap F)\big) \cap \big(A \cup (B \cap H)\big)\Big)\end{aligned}$$

In effect we need to show that

$$(C,D) = \Big((A \cap E) \cup \big((B \cup (A \cap F)) \cap (B \cap G)\big), \big(B \cup (A \cap F)\big) \cap \big(A \cup (B \cap H)\big)\Big).$$

Claim: $(A \cap E) \cup \big((B \cup (A \cap F)) \cap (B \cap G)\big) \subseteq C$.

Let $x \in (A \cap E) \cup \big((B \cup (A \cap F)) \cap (B \cap G)\big)$. Then $x \in (A \cap E)$ or $x \in \big((B \cup (A \cap F)) \cap (B \cap G)\big)$. Suppose $x \in A \cap E$. It follows that $x \in A$ and so $\mathscr{F}_1(x) = f(x)$, $\mathscr{F}_2(x) = h(x)$. It is also true that $x \in E$ and so $f(x) = h(x)(\neq \perp)$. Thus $\mathscr{F}_1(x) = \mathscr{F}_2(x)(\neq \perp)$ from which it follows that $x \in C$.

Suppose $x \in \big((B \cup (A \cap F)) \cap (B \cap G)\big)$. Then $x \in B \cap G$ and hence $x \in B$, from which it follows that $\mathscr{F}_1(x) = g(x)$ and $\mathscr{F}_2(x) = k(x)$. Moreover as $x \in G$, $g(x) = k(x)(\neq \perp)$. Thus $\mathscr{F}_1(x) = \mathscr{F}_2(x)(\neq \perp)$ and so $x \in C$.

Claim: $C \subseteq (A \cap E) \cup \big((B \cup (A \cap F)) \cap (B \cap G)\big)$.
Let $x \in C$. Then $\mathscr{F}_1(x) = \mathscr{F}_2(x)(\neq \perp)$.

Case I: $x \in A$*:* From the fact that $\mathscr{F}_1(x) = \mathscr{F}_2(x)(\neq \perp)$, on assuming that $x \in A$, we have $\mathscr{F}_1(x) = f(x)$ and $\mathscr{F}_2(x) = h(x)$, that is, $f(x) = h(x)(\neq \perp)$, that is, $x \in E$. Hence $x \in (A \cap E)$ and so $x \in (A \cap E) \cup \big((B \cup (A \cap F)) \cap (B \cap G)\big)$.

Case II: $x \in B$*:* If $x \in B$ then $\mathscr{F}_1(x) = g(x)$, $\mathscr{F}_2(x) = k(x)$ and as $\mathscr{F}_1(x) = \mathscr{F}_2(x)(\neq \perp)$, it follows that $g(x) = k(x)(\neq \perp)$. Hence $x \in B \cap G \subseteq B \subseteq (B \cup (A \cap F))$ from which the result follows.

Case III: $x \in (A \cup B)^{\mathsf{c}}$*:* Then $\mathscr{F}_1(x) = \perp = \mathscr{F}_2(x)$. However, our assumption states that $\mathscr{F}_1(x) = \mathscr{F}_2(x)(\neq \perp)$, a contradiction. It follows that this case cannot occur under the given hypothesis.

Claim: $\big(B \cup (A \cap F)\big) \cap \big(A \cup (B \cap H)\big) \subseteq D$.
Let $x \in \big(B \cup (A \cap F)\big) \cap \big(A \cup (B \cap H)\big)$. Then $x \in \big(B \cup (A \cap F)\big)$ and $x \in \big(A \cup (B \cap H)\big)$. From the fact that $x \in \big(B \cup (A \cap F)\big)$ we have $x \in B$ or $x \in (A \cap F)$.

If $x \in B$ then from the fact that $A \cap B = \emptyset$, we have $x \notin A$. However, as it is given that $x \in \big(A \cup (B \cap H)\big)$, it must be the case that $x \in B \cap H$. Moreover,

$x \in B$ implies that $\mathscr{F}_1(x) = g(x)$ and $\mathscr{F}_2(x) = k(x)$, and $x \in H$ implies that $g(x) \neq k(x)(\neq \perp)$. Consequently $\mathscr{F}_1(x) \neq \mathscr{F}_2(x)(\neq \perp)$, that is, $x \in D$.

On the other hand, if $x \in (A \cap F)$, then $x \in A$ and so $\mathscr{F}_1(x) = f(x)$ and $\mathscr{F}_2(x) = h(x)$ and $x \in F$ means that $f(x) \neq h(x)(\neq \perp)$. Thus $\mathscr{F}_1(x) \neq \mathscr{F}_2(x)(\neq \perp)$, which means that $x \in D$.

Claim: $D \subseteq \big(B \cup (A \cap F)\big) \cap \big(A \cup (B \cap H)\big)$. Let $x \in D$. Then $\mathscr{F}_1(x) \neq \mathscr{F}_2(x)(\neq \perp)$.

Case I: $x \in A$*:* Then $\mathscr{F}_1(x) = f(x)$ and $\mathscr{F}_2(x) = h(x)$. From the hypothesis, we have $f(x) \neq h(x)(\neq \perp)$ which implies that $x \in F$. Hence $x \in (A \cap F) \subseteq A$, which gives that $x \in \big(B \cup (A \cap F)\big) \cap \big(A \cup (B \cap H)\big)$.

Case II: $x \in B$*:* In this case $\mathscr{F}_1(x) = g(x)$ and $\mathscr{F}_2(x) = k(x)$. From the hypothesis, it follows that $g(x) \neq k(x)(\neq \perp)$ which implies that $x \in H$. Thus $x \in B \cap H \subseteq B$ which implies that $x \in \big(B \cup (A \cap F)\big) \cap \big(A \cup (B \cap H)\big)$.

Case III: $x \in (A \cup B)^{\mathsf{c}}$*:* Then $\mathscr{F}_1(x) = \perp = \mathscr{F}_2(x)$, a contradiction to our assumption. It follows that this case cannot occur.

Thus $(C, D) = \Big((A \cap E) \cup \big((B \cup (A \cap F)) \cap (B \cap G)\big), \big(B \cup (A \cap F)\big) \cap \big(A \cup (B \cap H)\big)\Big)$.

Axiom (4.6)**:** Let $f * f = (X, \emptyset)$ and $f * g = (\emptyset, \emptyset)$ for $f, g \in \mathcal{T}_o(X_\perp)$. It follows from $f * f = (X, \emptyset)$ that $\{x \in X : f(x) \neq \perp\} = X$ that is, for all $x \in X$, $f(x) \neq \perp$. Consider $f * g = (A, B) = (\emptyset, \emptyset)$ where $A = \{x \in X : f(x) = g(x)(\neq \perp)\}$ and $B = \{x \in X : f(x) \neq g(x)(\neq \perp)\}$. Suppose that for some $x \in X$, we have $g(x) \neq \perp$. Using the fact that $f(x) \neq \perp$ and that $A = \emptyset$, we can deduce that $f(x) \neq g(x)$, and so by definition, $x \in B$. However, it is given that $B = \emptyset$, a contradiction. Hence, $g(x) = \perp$ for each $x \in X$, that is $g = \zeta_\perp$. Therefore the last quasi-identity holds so that the pair $\big(\mathcal{T}_o(X_\perp), 3^X\big)$ is an agreeable C-set.

Example 4.1.3. Every basic C-set is agreeable under the operation given by

$$s * t = \begin{cases} T, & \text{if } s = t \ (\neq \bot); \\ F, & \text{if } s \neq t \ (\neq \bot); \\ U, & \text{if } s = \bot \text{ or } t = \bot. \end{cases} \tag{4.7}$$

Such agreeable C-sets will be called *agreeable basic C-sets*. The verification of axioms (4.2) – (4.6) is given below.

Axiom (4.2)**:** For $s \in S_\bot \setminus \{\bot\}$, $s * s = T$. Thus $(s * s)[s, \bot] = T[s, \bot] = s$. If $s = \bot$ then $s * s = \bot * \bot = U$. Thus $(s * s)[s, \bot] = U[\bot, \bot] = \bot = s$.

Axiom (4.3)**:** $\bot * s = U = s * \bot$ by definition.

Axiom (4.4)**:** For $s, t \in S_\bot$, we have the following:

$s * t = T$**:** This occurs if and only if $s = t$ and $s \neq \bot \neq t$. Thus $(s * t)[s, t] = T[s, t] = s = t = T[t, t] = (s * t)[t, t]$.

$s * t = F$**:** In this case, we have $(s * t)[s, t] = F[s, t] = t = F[t, t] = (s * t)[t, t]$.

$s * t = U$**:** Here we have $(s * t)[s, t] = U[s, t] = \bot = U[t, t] = (s * t)[t, t]$.

Axiom (4.5)**:** It suffices to consider the following three cases:

$\alpha = T$**:** Then $\alpha[s, t] * \alpha[u, v] = T[s, t] * T[u, v] = s * u$. And $\alpha[\![s * u, t * v]\!] = (T \wedge (s * u)) \vee (F \wedge (t * v)) = s * u$.

$\alpha = F$**:** Then $\alpha[s, t] * \alpha[u, v] = F[s, t] * F[u, v] = t * v$. And $\alpha[\![s * u, t * v]\!] = (F \wedge (s * u)) \vee (T \wedge (t * v)) = t * v$.

$\alpha = U$**:** Then $\alpha[s, t] * \alpha[u, v] = U[s, t] * U[u, v] = \bot * \bot = U$. And $\alpha[\![s * u, t * v]\!] = (U \wedge (s * u)) \vee (U \wedge (t * v)) = U \vee U = U$.

Axiom (4.6)**:** Let $s, t \in S_\bot$ such that $s * s = T$ and $s * t = U$. Clearly $s \neq t$. As $s * s = T$, this means that $s \neq \bot$. If $t \neq \bot$ then in view of the fact that $s \neq t$

we have $s * t = F$ by definition, which is a contradiction to our assumption that $s*t = U$. Thus $t = \bot$. Therefore every basic C-set is an agreeable C-set.

Proposition 4.1.4. *The operation defined in* (4.7) *is the only possible operation under which a basic C-set can be made agreeable.*

Proof. We show that for a basic C-set, axioms (4.2) to (4.6) restrict the operation $*$ to precisely (4.7). Let $(S_\bot, \mathfrak{Z})$ be a basic C-set which is agreeable, that is, it is equipped with an operation $* : S_\bot \times S_\bot \to \mathfrak{Z}$ which satisfies (4.2) - (4.6). Consider the following cases:

(i) *Case I*: $s = \bot$ or $t = \bot$: Then from (4.3), we have $s * t = U$.

(ii) *Case II*: $s = t$ $(\neq \bot)$: We show that neither $s*t = F$ nor $s*t = U$ is possible. Consequently, it must be the case that $s * t = T$.

Assume that $s * t = F$. This, in conjunction with the hypothesis $s = t$ and (4.2), gives that $\bot = F[s, \bot] = (s * t)[s, \bot] = (s * s)[s, \bot] = s$, a contradiction to our assumption that $s \neq \bot$. If $s * t = U$, along similar lines, we obtain $\bot = U[s, \bot] = (s * s)[s, \bot] = s$, a contradiction.

(iii) *Case III*: $s \neq t$ $(\neq \bot)$: Along similar lines, we show that $s*t \notin \{T, U\}$, which would imply that $s * t = F$.

Assume that $s * t = T$. It follows from (4.4) that $s = T[s, t] = (s * t)[s, t] = (s*t)[t, t] = T[t, t] = t$, a contradiction to the hypothesis $s \neq t$. Note that if $S_\bot$ has exactly two distinct elements then this case would be redundant. Suppose that $s*t = U$. *Case II* proved above, in conjunction with the hypothesis that $s \neq \bot$, gives that $s * s = T$. From the statements $s * s = T$, $s * t = U$ and quasi-identity (4.6), we have $t = \bot$, a contradiction.

Thus the operation $*$ must be as defined in (4.7). □

Example 4.1.5. The C-set (M, M) is agreeable under the operation

$$\alpha * \beta = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta).$$

The operation can be equivalently expressed in terms of the `if-then-else` action by

$$\alpha * \beta = \alpha[\![\beta, \neg\beta]\!].$$

The verification of the axioms (4.2) – (4.6) is given below.

Axiom (4.2)**:** For $\alpha \in M$, $\alpha*\alpha = (\alpha\wedge\alpha)\vee(\neg\alpha\wedge\neg\alpha) = \alpha\vee\neg\alpha$. Hence $(\alpha*\alpha)[\![\alpha, U]\!] = (\alpha \vee \neg\alpha)[\![\alpha, U]\!] = ((\alpha \vee \neg\alpha) \wedge \alpha) \vee (\neg\alpha \wedge \alpha \wedge U)$. This reduces to checking the validity of the identity $((\alpha \vee \neg\alpha) \wedge \alpha) \vee (\neg\alpha \wedge \alpha \wedge U) = \alpha$ in the C-algebra M. In view of Theorem 1.4.5, it suffices to check over three elements T, F, U.

$\alpha = T$**:** $((\alpha\vee\neg\alpha)\wedge\alpha)\vee(\neg\alpha\wedge\alpha\wedge U) = ((T\vee F)\wedge T)\vee(F\wedge T\wedge U) = T\vee F = T = \alpha$.

$\alpha = F$**:** $((\alpha\vee\neg\alpha)\wedge\alpha)\vee(\neg\alpha\wedge\alpha\wedge U) = ((F\vee T)\wedge F)\vee(T\wedge F\wedge U) = F\vee F = F = \alpha$.

$\alpha = U$**:** $((\alpha\vee\neg\alpha)\wedge\alpha)\vee(\neg\alpha\wedge\alpha\wedge U) = ((U\vee U)\wedge U)\vee(U\wedge U\wedge U) = U\vee U = U = \alpha$.

Axiom (4.3)**:** For $\alpha \in M$, $U * \alpha = (U \wedge \alpha) \vee (\neg U \wedge \neg\alpha) = U \vee U = U$. On the other hand $\alpha * U = (\alpha \wedge U) \vee (\neg\alpha \wedge \neg U)$. It suffices to consider the following three cases:

$\alpha = T$**:** $\alpha * U = T * U = (T \wedge U) \vee (F \wedge \neg U) = U \vee F = U$.

$\alpha = F$**:** $\alpha * U = F * U = (F \wedge U) \vee (T \wedge \neg U) = F \vee U = U$.

$\alpha = U$**:** $\alpha * U = U * U = (U \wedge U) \vee (U \wedge \neg U) = U \vee U = U$.

Axiom (4.4)**:** For $\alpha, \beta \in M$ we have $(\alpha * \beta)[\![\alpha, \beta]\!] = ((\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta))[\![\alpha, \beta]\!] = (((\alpha\wedge\beta)\vee(\neg\alpha\wedge\neg\beta))\wedge\alpha)\vee(\neg((\alpha\wedge\beta)\vee(\neg\alpha\wedge\neg\beta))\wedge\beta)$. On the other hand

$(\alpha * \beta)\llbracket \beta, \beta \rrbracket = (((\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)) \wedge \beta) \vee (\neg((\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)) \wedge \beta)$. It suffices to check the validity of this identity in the following three cases:

$\alpha = T$**:** Here $(\alpha * \beta)\llbracket \alpha, \beta \rrbracket = (((T \wedge \beta) \vee (F \wedge \neg\beta)) \wedge T) \vee (\neg((T \wedge \beta) \vee (F \wedge \neg\beta)) \wedge \beta) = ((\beta \vee F) \wedge T) \vee (\neg(\beta \vee F) \wedge \beta) = \beta \vee (\neg\beta \wedge \beta) = \beta$. Also $(\alpha * \beta)\llbracket \beta, \beta \rrbracket = (((T \wedge \beta) \vee (F \wedge \neg\beta)) \wedge \beta) \vee (\neg((T \wedge \beta) \vee (F \wedge \neg\beta)) \wedge \beta) = ((\beta \vee F) \wedge \beta) \vee (\neg(\beta \vee F) \wedge \beta) = \beta \vee (\neg\beta \wedge \beta) = \beta$.

$\alpha = F$**:** In this case $(\alpha * \beta)\llbracket \alpha, \beta \rrbracket = (((F \wedge \beta) \vee (T \wedge \neg\beta)) \wedge F) \vee (\neg((F \wedge \beta) \vee (T \wedge \neg\beta)) \wedge \beta) = ((F \vee \neg\beta) \wedge F) \vee (\neg(F \vee \neg\beta) \wedge \beta) = (\neg\beta \wedge F) \vee (\beta \wedge \beta) = (\neg\beta \wedge \beta) \vee \beta$ using the fact that $x \wedge F = x \wedge \neg x$ holds in all C-algebras. This expression reduces to $(\neg\beta \wedge \beta) \vee \beta = \beta$. Also $(\alpha * \beta)\llbracket \beta, \beta \rrbracket = (((F \wedge \beta) \vee (T \wedge \neg\beta)) \wedge \beta) \vee (\neg((F \wedge \beta) \vee (T \wedge \neg\beta)) \wedge \beta) = ((F \vee \neg\beta) \wedge \beta) \vee (\neg(F \vee \neg\beta) \wedge \beta) = (\neg\beta \wedge \beta) \vee (\beta \wedge \beta) = (\neg\beta \wedge \beta) \vee \beta = \beta$.

$\alpha = U$**:** In this case $(\alpha * \beta)\llbracket \alpha, \beta \rrbracket = (((U \wedge \beta) \vee (U \wedge \neg\beta)) \wedge U) \vee (\neg((U \wedge \beta) \vee (U \wedge \neg\beta)) \wedge \beta) = ((U \vee U) \wedge U) \vee (\neg(U \vee U) \wedge \beta) = U$. Also $(\alpha * \beta)\llbracket \beta, \beta \rrbracket = (((U \wedge \beta) \vee (U \wedge \neg\beta)) \wedge \beta) \vee (\neg((U \wedge \beta) \vee (U \wedge \neg\beta)) \wedge \beta) = ((U \vee U) \wedge \beta) \vee (\neg(U \vee U) \wedge \beta) = U$.

Axiom (4.5)**:** We need to check the validity of the identity $\alpha\llbracket \beta, \gamma \rrbracket * \alpha\llbracket \delta, \rho \rrbracket = \alpha\llbracket \beta * \delta, \gamma * \rho \rrbracket$. It suffices to consider the following three cases:

$\alpha = T$**:** In this case $\alpha\llbracket \beta, \gamma \rrbracket * \alpha\llbracket \delta, \rho \rrbracket = T\llbracket \beta, \gamma \rrbracket * T\llbracket \delta, \rho \rrbracket = \beta * \delta$. On the other hand $\alpha\llbracket \beta * \delta, \gamma * \rho \rrbracket = T\llbracket \beta * \delta, \gamma * \rho \rrbracket = \beta * \delta$.

$\alpha = F$**:** Here $\alpha\llbracket \beta, \gamma \rrbracket * \alpha\llbracket \delta, \rho \rrbracket = F\llbracket \beta, \gamma \rrbracket * F\llbracket \delta, \rho \rrbracket = \gamma * \rho$. Also $\alpha\llbracket \beta * \delta, \gamma * \rho \rrbracket = F\llbracket \beta * \delta, \gamma * \rho \rrbracket = \gamma * \rho$.

$\alpha = U$**:** We have $\alpha\llbracket \beta, \gamma \rrbracket * \alpha\llbracket \delta, \rho \rrbracket = U\llbracket \beta, \gamma \rrbracket * U\llbracket \delta, \rho \rrbracket = U$. Also $\alpha\llbracket \beta * \delta, \gamma * \rho \rrbracket = U\llbracket \beta * \delta, \gamma * \rho \rrbracket = U$.

Axiom (4.6)**:** In order to verify quasi-identity (4.6), we recall from Theorem 1.4.5 that M is a subalgebra of $\mathfrak{3}^X$ for some set X. Suppose that $\alpha * \alpha = \mathbf{T}$ and

$\alpha * \beta = \mathbf{U}$ for some $\alpha, \beta \in M$, that is, $\alpha \vee \neg\alpha = \mathbf{T}$ and $(\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta) = \mathbf{U}$. Treating α, β as elements of $\mathfrak{Z}^X$, we have $\alpha(x) \vee \neg(\alpha(x)) = T$, and $(\alpha(x) \wedge \beta(x)) \vee (\neg\alpha(x) \wedge \neg\beta(x)) = U$ for each $x \in X$. Since $\alpha(x) \vee \neg\alpha(x) = T$, where $\alpha(x) \in \{T, F, U\}$, there are only two possible cases:

$\alpha(x) = T$**:** Since $(\alpha * \beta)(x) = U$ we have $(\alpha(x) \wedge \beta(x)) \vee (\neg\alpha(x) \wedge \neg\beta(x)) = (T \wedge \beta(x)) \vee (F \wedge \neg\beta(x)) = \beta(x) = U$.

$\alpha(x) = F$**:** Since $(\alpha * \beta)(x) = U$ we have $(\alpha(x) \wedge \beta(x)) \vee (\neg\alpha(x) \wedge \neg\beta(x)) = (F \wedge \beta(x)) \vee (T \wedge \neg\beta(x)) = \neg\beta(x) = U$, from which it follows that $\beta(x) = U$.

Thus for each $x \in X$, $\beta(x) = U$ that is, $\beta = \mathbf{U}$. Therefore, the quasi-identity (4.6) holds. Equivalently, we could have simply verified the validity of the quasi-identity in $\mathfrak{Z}$ which follows along similar lines as the reasoning followed above. Thus the pair (M, M) is an agreeable C-set.

Remark 4.1.6. If the C-algebra M is $\mathfrak{Z}^X$, the equality test on the agreeable C-set (M, M) coincides with that of the functional case, as shown below:

$$(\alpha * \beta)(x) = \begin{cases} T, & \text{if } \alpha(x) = \beta(x) \ (\neq U); \\ F, & \text{if } \alpha(x) \neq \beta(x) \ (\neq U); \\ U, & \text{otherwise.} \end{cases}$$

4.2 A subdirect representation

We now prove a representation theorem of agreeable C-sets along the lines of Theorem 3.2.4 for the class of C-sets where the C-algebra is an ada. We proceed by showing that the pair (E_θ, θ) as defined in Definition 3.1.2 for a maximal congruence θ on ada M is also an agreeable C-set congruence, from which Theorem 4.2.1 follows.

Theorem 4.2.1. *Every agreeable C-set $(S_\perp, M)$ where M is an ada is a subdirect product of agreeable basic C-sets.*

Proof. Let $(S_\perp, M)$ be an agreeable C-set where M is an ada. For every maximal congruence θ on M, consider the pair (E_θ, θ) as in Definition 3.1.2. By Lemma 3.1.7, we have already ascertained that for each θ, the pair (E_θ, θ) is a C-set congruence on $(S_\perp, M)$ and by Remark 3.1.4 that $(S_\perp/E_\theta, M/\theta)$ is a basic C-set. In order to ascertain that this pair is indeed a congruence in the context of agreeable C-sets, it is sufficient to show that

$$(a_1, a_2), (b_1, b_2) \in E_\theta \Rightarrow (a_1 * b_1, a_2 * b_2) \in \theta.$$

Let $(a_1, a_2), (b_1, b_2) \in E_\theta$. Then there exist α and $\beta \in \overline{T}^\theta$ such that

$$\alpha[a_1, a_2] = \alpha[a_2, a_2] \tag{4.8}$$

$$\beta[b_1, b_2] = \beta[b_2, b_2] \tag{4.9}$$

Note that $(\alpha, T), (\beta, T) \in \theta$ implies that $(\alpha \wedge \beta, T \wedge T) = (\alpha \wedge \beta, T) \in \theta$. Applying (2.8) on (4.8) and Proposition 2.2.1(v) on (4.9), we have, for $(\alpha \wedge \beta) \in \overline{T}^\theta$

$$(\alpha \wedge \beta)[a_1, a_2] = (\alpha \wedge \beta)[a_2, a_2] \tag{4.10}$$

$$(\alpha \wedge \beta)[b_1, b_2] = (\alpha \wedge \beta)[b_2, b_2] \tag{4.11}$$

These imply that

$$(\alpha \wedge \beta)[a_1, a_2] * (\alpha \wedge \beta)[b_1, b_2] = (\alpha \wedge \beta)[a_2, a_2] * (\alpha \wedge \beta)[b_2, b_2]. \tag{4.12}$$

From (4.5) it follows that

$$(\alpha \wedge \beta)[\![a_1 * b_1, a_2 * b_2]\!] = (\alpha \wedge \beta)[\![a_2 * b_2, a_2 * b_2]\!], \tag{4.13}$$

so that $(a_1 * b_1, a_2 * b_2) \in E_{\theta_M} \subseteq \theta$, by Lemma 3.1.8. Further, by Lemma 3.2.2 and Remark 3.2.3, the intersection of all congruences (E_θ, θ), where θ ranges over all maximal congruences of M, is trivial. This completes the proof. □

Corollary 4.2.2. *An identity (quasi-identity) is satisfied in every agreeable C-set $(S_\perp, M)$ where M is an ada if and only if it is satisfied in all agreeable basic C-sets.*

In view of Corollary 4.2.2 and (4.7), we have the following result.

Corollary 4.2.3. *In every agreeable C-set $(S_\perp, M)$ where M is an ada we have $s * t = t * s$.*

Note that the only axiom of agreeable C-sets that plays a role in the proof of Theorem 4.2.1 is (4.5). The remaining axioms have been included in order that the operation on agreeable basic C-sets be uniquely defined. The proof of Theorem 4.2.1 suggests an alternative proof for Theorem 1.3.11, without using the commutativity of $*$, which we now present.

Theorem 4.2.4 (Jackson and Stokes [2009]). *Every agreeable B-set (S, B) is a subdirect product of basic agreeable B-sets.*

Proof. Let F be an ultrafilter of B. Consider the relation $E_F = \{(s,t) \in S \times S : \gamma[s,t] = t$ for some $\gamma \in F\}$ as defined in Jackson and Stokes [2009]. The pair (E_F, F) is a B-set congruence. In order to show that the pair (E_F, F) is a congruence on agreeable B-sets, we show that

$$(a_1, a_2), (b_1, b_2) \in E_F \Rightarrow (a_1 * b_1, a_2 * b_2) \in \theta_F,$$

where θ_F is the congruence on B induced by the ultrafilter F.

Since $(a_1, a_2), (b_1, b_2) \in E_F$, there exist $\alpha, \beta \in F$ such that

$$\alpha[a_1, a_2] = a_2 \tag{4.14}$$

$$\beta[b_1, b_2] = b_2 \tag{4.15}$$

In view of the commutativity of $\wedge$, (1.13), (4.14) and (1.8), we obtain $(\alpha\wedge\beta)[a_1, a_2] = (\beta \wedge \alpha)[a_1, a_2] = \beta[\alpha[a_1, a_2], a_2] = \beta[a_2, a_2] = a_2$. Similarly we can obtain $(\alpha \wedge \beta)[b_1, b_2] = b_2$. This implies that

$$(\alpha \wedge \beta)[a_1, a_2] * (\alpha \wedge \beta)[b_1, b_2] = a_2 * b_2$$

From axiom (1.16), we can deduce that

$$(\alpha \wedge \beta)[\![a_1 * b_1, a_2 * b_2]\!] = a_2 * b_2.$$

Since F is an ultrafilter of B, it suffices to ascertain that

$$a_1 * b_1 \in F \Leftrightarrow a_2 * b_2 \in F.$$

Assume that $a_1 * b_1 \in F$. Since $\alpha\wedge\beta \in F$, we have $(\alpha\wedge\beta)\wedge(a_1*b_1) \in F$. Further, as F is a filter and $(\alpha \wedge \beta) \wedge (a_1 * b_1) \leq ((\alpha \wedge \beta) \wedge (a_1 * b_1)) \vee (\neg(\alpha \wedge \beta) \wedge (a_2 * b_2)) = (\alpha \wedge \beta)[\![a_1 * b_1, a_2 * b_2]\!]$ we have $(\alpha \wedge \beta)[\![a_1 * b_1, a_2 * b_2]\!] = a_2 * b_2 \in F$.

Conversely, assume that $a_2 * b_2 \in F$. The symmetry of equivalence relation E_F implies that $(a_2, a_1), (b_2, b_1) \in E_F$. Along similar lines as above, we obtain $a_1*b_1 \in F$. This completes the proof. □

5

C-monoids

In this chapter we consider a monoid structure which serves as an abstraction of programs with composition, over which we give an axiomatization for `if-then-else`. In Section 5.1 we introduce the notion of a C-monoid, which includes the composition of programs and that of programs with tests. Further, we obtain a Cayley-type theorem in that every C-monoid is embeddable in a functional C-monoid (cf. Theorem 5.2.1). In order to achieve this result we list some properties of maximal congruences of adas in Subsection 5.2.1. In Subsection 5.2.2 we construct a collection of homomorphisms from the C-monoid to functional C-monoids, which separate every distinct pair of elements. Using this, in Subsection 5.2.3 we construct a functional C-monoid and give an embedding from the C-monoid to this functional C-monoid.

5.1 Axiomatization over monoids

We consider the case where the composition of two elements of the base set and of an element with a predicate is allowed. Our motivating example is again $\big(\mathcal{T}_o(X_\perp), 3^X\big)$, where $\mathcal{T}_o(X_\perp)$ is considered to be a monoid with zero by equipping it with composition of functions. The composition will be written from left to right, i.e., $(f \cdot g)(x) = g(f(x))$. The monoid identity in $\mathcal{T}_o(X_\perp)$ is the identity function $id_{X_\perp}$ and the zero element is the constant function $\zeta_\perp$ satisfying $\zeta_\perp \cdot f = f \cdot \zeta_\perp = \zeta_\perp$. We also include composition of functions with predicates via the natural interpretation given by the following for all $f \in \mathcal{T}_o(X_\perp)$ and $\alpha \in 3^X$:

$$(f \circ \alpha)(x) = \begin{cases} T, & \text{if } \alpha(f(x)) = T; \\ F, & \text{if } \alpha(f(x)) = F; \\ U, & \text{otherwise.} \end{cases} \tag{5.1}$$

Note that if the composition takes value T or F at some point $x \in X_\perp$ then as $\alpha \in 3^X$ this implies that $f(x) \neq \perp$.

With this example in mind we define a C-monoid as follows.

Definition 5.1.1. Let $(S_\perp, \cdot)$ be a monoid with identity element 1 and zero element $\perp$ where $\perp \cdot s = \perp = s \cdot \perp$ for all $s \in S_\perp$. Let M be a C-algebra with T, F, U and $(S_\perp, M)$ be a C-set with $\perp$ as the base point of the pointed set $S_\perp$. The pair $(S_\perp, M)$ equipped with a function

$$\circ : S_\perp \times M \to M$$

is said to be a *C-monoid* if it satisfies the following axioms for all $s, t, r, u \in S_\perp$ and

$\alpha, \beta \in M$:

$$\bot \circ \alpha = U \qquad (\bot\text{-}\circ\text{-axiom}) \qquad (5.2)$$

$$t \circ U = U \qquad (U\text{-}\circ\text{-axiom}) \qquad (5.3)$$

$$1 \circ \alpha = \alpha \qquad (1\text{-}\circ\text{-axiom}) \qquad (5.4)$$

$$s \circ (\neg\alpha) = \neg(s \circ \alpha) \qquad (\neg\text{-}\circ\text{-axiom}) \qquad (5.5)$$

$$s \circ (\alpha \wedge \beta) = (s \circ \alpha) \wedge (s \circ \beta) \qquad (\wedge\text{-}\circ\text{-axiom}) \qquad (5.6)$$

$$(s \cdot t) \circ \alpha = s \circ (t \circ \alpha) \qquad (\text{semigroup action}) \qquad (5.7)$$

$$\alpha[s, t] \cdot u = \alpha[s \cdot u, t \cdot u] \qquad (\text{right composition}) \qquad (5.8)$$

$$r \cdot \alpha[s, t] = (r \circ \alpha)[r \cdot s, r \cdot t] \qquad (\text{left composition}) \qquad (5.9)$$

$$\alpha[s, t] \circ \beta = \alpha[\![s \circ \beta, t \circ \beta]\!] \qquad (\circ\text{-interchange}) \qquad (5.10)$$

Note that (1.17) does not hold in C-monoids in view of (5.2), since $\bot \circ T = U$ $(\neq T)$. The following are examples of C-monoids.

Example 5.1.2. The C-set $\big(\mathcal{T}_o(X_\bot), \mathfrak{Z}^X\big)$ equipped with the operation $\circ$ given in (5.1) and with $\mathcal{T}_o(X_\bot)$ treated as a monoid with zero is a C-monoid. Such C-monoids will be called *functional C-monoids.*

We use the pairs of sets representation as stated in Remark 1.4.4 and identify $\alpha \in \mathfrak{Z}^X$ with a *pair of sets* (A, B) of X where $A = \alpha^{-1}(T)$ and $B = \alpha^{-1}(F)$. In this representation $\mathbf{T} = (X, \emptyset), \mathbf{F} = (\emptyset, X)$ and $\mathbf{U} = (\emptyset, \emptyset)$. Thus the operation $\circ$ is given as follows:

$$(f \circ \alpha)(x) = \begin{cases} T, & \text{if } f(x) \in A; \\ F, & \text{if } f(x) \in B; \\ U, & \text{otherwise.} \end{cases}$$

In other words $f \circ \alpha$ can be identified with the pair of sets (C, D) where $C = \{x \in X : f(x) \in A\}$ and $D = \{x \in X : f(x) \in B\}$. We now verify the axioms (5.2) –

(5.10) in the following.

Axiom (5.2)**:** Let α be identified with the pair of sets (A, B). Then $\zeta_\perp \circ \alpha = (\emptyset, \emptyset) = \mathbf{U}$ as $\zeta_\perp(x) = \perp \notin (A \cup B)$.

Axiom (5.3)**:** Consider $\mathbf{U} = (\emptyset, \emptyset)$. Then $f \circ \mathbf{U} = (\emptyset, \emptyset) = \mathbf{U}$.

Axiom (5.4)**:** We have

$$(1 \circ \alpha)(x) = \begin{cases} T, & \text{if } id_{X_\perp}(x) \in A; \\ F, & \text{if } id_{X_\perp}(x) \in B; \\ U, & \text{otherwise} \end{cases}$$
$$= \alpha(x).$$

Thus $1 \circ \alpha = \alpha$.

Axiom (5.5)**:** Let α be identified with the pair of sets (A, B). Then $f \circ \alpha = (C, D)$ where $C = \{x \in X : f(x) \in A\}$ and $D = \{x \in X : f(x) \in B\}$. Thus $\neg(f \circ \alpha) = (D, C)$. Also $f \circ (\neg\alpha) = f \circ (B, A) = (E, F)$ where $E = \{x \in X : f(x) \in B\}$ and $F = \{x \in X : f(x) \in A\}$. It follows that $(E, F) = (D, C)$.

Axiom (5.6)**:** Let α, β be represented by the pairs of sets (A_1, A_2) and (B_1, B_2) respectively. Then $\alpha \wedge \beta = (A_1 \cap B_1, A_2 \cup (A_1 \cap B_2))$. Also let $f \circ \alpha = (C_1, C_2)$ where $C_1 = \{x \in X : f(x) \in A_1\}$ and $C_2 = \{x \in X : f(x) \in A_2\}$, and $f \circ \beta = (D_1, D_2)$ where $D_1 = \{x \in X : f(x) \in B_1\}$ and $D_2 = \{x \in X : f(x) \in B_2\}$. Then $(C_1, C_2) \wedge (D_1, D_2) = (C_1 \cap D_1, C_2 \cup (C_1 \cap D_2))$. Thus $C_1 \cap D_1 = \{x \in X : f(x) \in A_1 \cap B_1\}$ and $C_2 \cup (C_1 \cap D_2) = \{x \in X : f(x) \in A_2 \cup (A_1 \cap B_2)\}$. Hence $f \circ (\alpha \wedge \beta) = (f \circ \alpha) \wedge (f \circ \beta)$.

Axiom (5.7)**:** Consider $f, g \in \mathcal{T}_o(X_\perp)$ and $\alpha \in \mathfrak{Z}^X$ represented by the pair of sets

(A, B).

$$((f \cdot g) \circ \alpha)(x) = \begin{cases} T, & \text{if } g(f(x)) \in A; \\ F, & \text{if } g(f(x)) \in B; \\ U, & \text{otherwise.} \end{cases}$$

Let $g \circ \alpha = (C, D)$ where $C = \{x \in X : g(x) \in A\}$ and $D = \{x \in X : g(x) \in B\}$.

$$(f \circ (g \circ \alpha))(x) = \begin{cases} T, & \text{if } f(x) \in C; \\ F, & \text{if } f(x) \in D; \\ U, & \text{otherwise.} \end{cases}$$

We may consider the following three cases.

Case I: *$x \in X$ such that $g(f(x)) \in A$:* Then $((f \cdot g) \circ \alpha)(x) = T$. Also $f(x) \in C$ as $g(f(x)) \in A$. Thus $(f \circ (g \circ \alpha))(x) = T$.

Case II: *$x \in X$ such that $g(f(x)) \in B$:* Then $((f \cdot g) \circ \alpha)(x) = F$. Similarly $g(f(x)) \in B$ means that $f(x) \in D$. Thus $(f \circ (g \circ \alpha))(x) = F$.

Case III: *$x \in X$ such that $g(f(x)) \notin (A \cup B)$:* Then $((f \cdot g) \circ \alpha)(x) = U$. Since $f(x)$ is in neither C nor D it follows that $(f \circ (g \circ \alpha))(x) = U$.

Axiom (5.8)**:** Consider $\alpha \in \mathfrak{Z}^X$ represented by the pair of sets (A, B).

$$(\alpha[f, g] \cdot h)(x) = h(\alpha[f, g](x)) = \begin{cases} h(f(x)), & \text{if } x \in A; \\ h(g(x)), & \text{if } x \in B; \\ \perp, & \text{otherwise.} \end{cases}$$

Hence $\alpha[f, g] \cdot h = \alpha[f \cdot h, g \cdot h]$.

Axiom (5.9)**:** Let $\alpha \in \mathfrak{Z}^X$ be represented by the pair of sets (A, B).

$$(h \cdot \alpha[f,g])(x) = \alpha[f,g](h(x)) = \begin{cases} f(h(x)), & \text{if } h(x) \in A; \\ g(h(x)), & \text{if } h(x) \in B; \\ \bot, & \text{otherwise.} \end{cases}$$

Let $h \circ \alpha$ be represented by the pair of sets (C, D) where $C = \{x \in X : h(x) \in A\}$ and $D = \{x \in X : h(x) \in B\}$.

$$\begin{aligned}(h \circ \alpha)[h \cdot f, h \cdot g](x) &= \begin{cases} (h \cdot f)(x), & \text{if } x \in C; \\ (h \cdot g)(x), & \text{if } x \in D; \\ \bot, & \text{otherwise} \end{cases} \\ &= \begin{cases} f(h(x)), & \text{if } h(x) \in A; \\ g(h(x)), & \text{if } h(x) \in B; \\ \bot, & \text{otherwise.} \end{cases}\end{aligned}$$

Thus $h \cdot \alpha[f,g] = (h \circ \alpha)[h \cdot f, h \cdot g]$.

Axiom (5.10)**:** Let $\alpha, \beta \in \mathfrak{Z}^X$ be represented by the pairs of sets (A_1, A_2) and (B_1, B_2) respectively. For $f, g \in \mathcal{T}_o(X_\bot)$ we have the following:

$$h(x) = \alpha[f,g](x) = \begin{cases} f(x), & \text{if } x \in A_1; \\ g(x), & \text{if } x \in A_2; \\ \bot, & \text{otherwise.} \end{cases}$$

Also $h \circ \beta = (C_1, C_2)$ where $C_1 = \{x \in X : h(x) \in B_1\}$ and $C_2 = \{x \in X : h(x) \in B_2\}$. Similarly $f \circ \beta = (D_1, D_2)$ where $D_1 = \{x \in X : f(x) \in B_1\}$ and $D_2 = \{x \in X : f(x) \in B_2\}$. Let $g \circ \beta = (E_1, E_2)$ where $E_1 = \{x \in$

$X : g(x) \in B_1\}$ and $E_2 = \{x \in X : g(x) \in B_2\}$. Thus $\alpha[\![f \circ \beta, g \circ \beta]\!] = \big((A_1, A_2) \wedge (D_1, D_2)\big) \vee \big(\neg(A_1, A_2) \wedge (E_1, E_2)\big)$.

This evaluates to

$$\begin{aligned}\alpha[\![f \circ \beta, g \circ \beta]\!] &= \big(A_1 \cap D_1, A_2 \cup (A_1 \cap D_2)\big) \vee \big(A_2 \cap E_1, A_1 \cup (A_2 \cap E_2)\big) \\ &= \Big((A_1 \cap D_1) \cup \big((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)\big), \\ &(A_2 \cup (A_1 \cap D_2)) \cap (A_1 \cup (A_2 \cap E_2))\Big) \\ &= (S_1, S_2) \text{ (say)}\end{aligned}$$

We show that $(C_1, C_2) = (S_1, S_2)$ by standard set theoretic arguments.

First we prove that $C_1 \subseteq S_1$. Let $x \in C_1$. Then $h(x) \in B_1$. Consider the following cases:

Case I: $x \in A_1$: Then $h(x) = f(x) \in B_1$ hence $x \in D_1$. Therefore $x \in A_1 \cap D_1$ and so $x \in S_1$.

Case II: $x \in A_2$: Then $h(x) = g(x) \in B_1$ hence $x \in E_1$. Hence $x \in A_2 \cap E_1 \subseteq A_2$ we have $x \in S_1$.

Case III: $x \notin (A_1 \cup A_2)$: Then $h(x) = \perp \notin B_1$ a contradiction to our assumption that $h(x) \in B_1$. It follows that this case cannot occur.

We show that $S_1 \subseteq C_1$. Let $x \in S_1$. Thus $x \in A_1 \cap D_1$ or $x \in \big((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)\big)$. If $x \in A_1 \cap D_1$ then $h(x) = f(x)$ as $x \in A_1$ and $f(x) \in B_1$ as $x \in D_1$. Thus $h(x) \in B_1$ and so $x \in C_1$. If $x \in \big((A_2 \cup (A_1 \cap D_2)) \cap (A_2 \cap E_1)\big)$, then $x \in (A_2 \cap E_1)$. Thus $h(x) = g(x)$ as $x \in A_2$ and $g(x) \in B_1$ as $x \in E_1$. Hence $h(x) \in B_1$, thus $x \in C_1$.

We show that $C_2 \subseteq S_2$. Let $x \in C_2$ hence $h(x) \in B_2$. Consider the following cases:

Case I: $x \in A_1$: Then $h(x) = f(x) \in B_2$, therefore $x \in D_2$. Hence $x \in A_1 \cap D_2 \subseteq A_1$ and so $x \in S_2$.

Case II: $x \in A_2$: Then $h(x) = g(x) \in B_2$ therefore $x \in E_2$. Thus $x \in A_2 \cap E_2 \subseteq A_2$ and so $x \in S_2$.

Case III: $x \notin (A_1 \cup A_2)$: Then $h(x) = \bot \notin B_2$ which is a contradiction. It follows that this case cannot occur.

Finally we show that $S_2 \subseteq C_2$. Since $A_1 \cap A_2 = \emptyset$ it follows that $x \in A_1 \cap D_2$ or $x \in A_2 \cap E_2$. If $x \in A_1 \cap D_2$ then $h(x) = f(x) \in B_2$ and hence $x \in C_2$. If $x \in A_2 \cap E_2$ then $h(x) = g(x) \in B_2$ hence $x \in C_2$.

Thus $\alpha[f, g] \circ \beta = \alpha[\![f \circ \beta, g \circ \beta]\!]$ and so the pair $\left(\mathcal{T}_o(X_\bot), \mathfrak{Z}^X\right)$ is a C-monoid.

Example 5.1.3. Let $S_\bot$ be a non-trivial monoid with identity 1 and zero $\bot$ and no non-zero zero-divisors, i.e., $s \cdot t = \bot \Rightarrow s = \bot$ or $t = \bot$. Then $S_\bot^X$ is also a monoid with zero for any set X with operations defined pointwise. For $f, g \in S_\bot^X$ define $(f \cdot g)(x) = f(x) \cdot g(x)$. The identity of $S_\bot^X$ is the constant function ζ_1 taking the value 1. The zero and base point of $S_\bot^X$ is the constant function $\zeta_\bot$ taking the value $\bot$. The C-set $\left(S_\bot^X, \mathfrak{Z}^X\right)$ is a C-monoid with $\circ$ defined as follows for all $f \in S_\bot^X$ and $\alpha \in \mathfrak{Z}^X$:

$$(f \circ \alpha)(x) = \begin{cases} \alpha(x), & \text{if } f(x) \neq \bot; \\ U, & \text{otherwise.} \end{cases}$$

We now verify the axioms (5.2) – (5.10) in the following. Let $f, g, h \in S_\bot^X$ and $\alpha, \beta \in \mathfrak{Z}^X$.

Axiom (5.2)**:** It is easy to see that $(\zeta_\bot \circ \alpha)(x) = U$ for all $x \in X$.

Axiom (5.3)**:** It is clear that $(f \circ \mathbf{U})(x) = U$.

Axiom (5.4)**:** Since $S_\bot$ is non-trivial we must have $1 \neq \bot$. If not then for $a \in S_\bot \setminus \{\bot\}$ we have $a = a \cdot 1 = a \cdot \bot = \bot$ a contradiction. It follows that $\zeta_1 \neq \zeta_\bot$.

Hence $(\zeta_1 \circ \alpha)(x) = \alpha(x)$ as $\zeta_1(x) = 1 \neq \bot$.

Axiom (5.5)**:** We have

$$(f \circ (\neg\alpha))(x) = \begin{cases} (\neg\alpha)(x), & \text{if } f(x) \neq \bot; \\ U, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \neg(\alpha(x)), & \text{if } f(x) \neq \bot; \\ U, & \text{otherwise} \end{cases}$$
$$= \neg(f \circ \alpha)(x).$$

Thus $f \circ (\neg\alpha) = \neg(f \circ \alpha)$.

Axiom (5.6)**:** We have

$$(f \circ (\alpha \wedge \beta))(x) = \begin{cases} (\alpha \wedge \beta)(x), & \text{if } f(x) \neq \bot; \\ U, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \alpha(x) \wedge \beta(x), & \text{if } f(x) \neq \bot; \\ U \wedge U, & \text{otherwise} \end{cases}$$
$$= (f \circ \alpha)(x) \wedge (f \circ \beta)(x).$$

Thus $f \circ (\alpha \wedge \beta) = (f \circ \alpha) \wedge (f \circ \beta)$.

Axiom (5.7)**:** Since $S_\bot$ has no zero-divisors we have $f(x) \cdot g(x) = \bot \Leftrightarrow f(x) = \bot$ or $g(x) = \bot$. Consequently

$$((f \cdot g) \circ \alpha)(x) = \begin{cases} \alpha(x), & \text{if } (f \cdot g)(x) \neq \bot; \\ U, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \alpha(x), & \text{if } f(x) \cdot g(x) \neq \bot; \\ U, & \text{otherwise} \end{cases}$$
$$= \begin{cases} \alpha(x), & \text{if } f(x) \neq \bot \text{ and } g(x) \neq \bot; \\ U, & \text{otherwise} \end{cases}$$
$$= (f \circ (g \circ \alpha))(x).$$

Thus $(f \cdot g) \circ \alpha = f \circ (g \circ \alpha)$.

Axiom (5.8)**:** We have

$$(\alpha[f,g] \cdot h)(x) = \alpha[f,g](x) \cdot h(x) = \begin{cases} f(x) \cdot h(x), & \text{if } \alpha(x) = T; \\ g(x) \cdot h(x), & \text{if } \alpha(x) = F; \\ \bot, & \text{otherwise} \end{cases}$$
$$= \alpha[f \cdot h, g \cdot h](x).$$

Thus $\alpha[f,g] \cdot h = \alpha[f \cdot h, g \cdot h]$.

Axiom (5.9)**:** Consider

$$h \cdot \alpha[f,g](x) = h(x) \cdot \alpha[f,g](x) = \begin{cases} h(x) \cdot f(x), & \text{if } \alpha(x) = T; \\ h(x) \cdot g(x), & \text{if } \alpha(x) = F; \\ \bot, & \text{otherwise.} \end{cases}$$

On the other hand

$$(h\circ\alpha)[h\cdot f,h\cdot g](x)=\begin{cases}h(x)\cdot f(x), & \text{if } (h\circ\alpha)(x)=T;\\ h(x)\cdot g(x), & \text{if } (h\circ\alpha)(x)=F;\\ \bot, & \text{otherwise.}\end{cases}$$

Note that if $h(x)=\bot$ then $h\cdot\alpha[f,g](x)=\bot=(h\circ\alpha)[h\cdot f,h\cdot g](x)$. Suppose that $h(x)\neq\bot$ then $(h\circ\alpha)(x)=\alpha(x)$. It is clear that in this case as well $h\cdot\alpha[f,g](x)=(h\circ\alpha)[h\cdot f,h\cdot g](x)$ holds. Thus $h\cdot\alpha[f,g]=(h\circ\alpha)[h\cdot f,h\cdot g]$.

Axiom (5.10)**:** Consider

$$\begin{aligned}(\alpha[f,g]\circ\beta)(x)&=\begin{cases}\beta(x), & \text{if } \alpha[f,g](x)\neq\bot;\\ U, & \text{otherwise}\end{cases}\\ &=\begin{cases}\beta(x), & \text{if } (f(x)\neq\bot,\alpha(x)=T) \text{ or } (g(x)\neq\bot,\alpha(x)=F);\\ U, & \text{otherwise.}\end{cases}\end{aligned}$$

We have $(\alpha[\![f\circ\beta,g\circ\beta]\!])(x)=(\alpha(x)\wedge(f\circ\beta)(x))\vee(\neg\alpha(x)\wedge(g\circ\beta)(x))$.

If $f(x)\neq\bot$ and $\alpha(x)=T$ we have $(\alpha[\![f\circ\beta,g\circ\beta]\!])(x)=(T\wedge\beta(x))\vee(F\wedge(g\circ\beta)(x))=\beta(x)\vee F=\beta(x)=(\alpha[f,g]\circ\beta)(x)$.

If $g(x)\neq\bot$ and $\alpha(x)=F$ we have $(\alpha[\![f\circ\beta,g\circ\beta]\!])(x)=(F\wedge(f\circ\beta)(x))\vee(T\wedge\beta(x))=F\vee\beta(x)=\beta(x)=(\alpha[f,g]\circ\beta)(x)$.

In all other cases it can be easily ascertained that $(\alpha[\![f\circ\beta,g\circ\beta]\!])(x)=U=(\alpha[f,g]\circ\beta)(x)$. Thus $\alpha[f,g]\circ\beta=\alpha[\![f\circ\beta,g\circ\beta]\!]$. Thus the pair $\left(S_\bot^X,\mathfrak{Z}^X\right)$ is a C-monoid.

Example 5.1.4. Let $S_\bot$ be a non-trivial monoid with zero $\bot$ and no non-zero zero-divisors, i.e., $s\cdot t=\bot\Rightarrow s=\bot$ or $t=\bot$. The basic C-set $(S_\bot,\mathfrak{Z})$ equipped with

$\circ : S_\perp \times \mathfrak{Z} \to \mathfrak{Z}$ defined below for all $s \in S_\perp$ and $\alpha \in \mathfrak{Z}$ is a C-monoid.

$$s \circ \alpha = \begin{cases} \alpha, & \text{if } s \neq \perp; \\ U, & \text{if } s = \perp. \end{cases}$$

We now verify the axioms (5.2) – (5.10) in the following.

Axiom (5.2)**:** It is clear that $\perp \circ \alpha = U$.

Axiom (5.3)**:** It is obvious that $t \circ U = U$.

Axiom (5.4)**:** Since $S_\perp$ is non-trivial it follows that $1 \neq \perp$. Consequently $1 \circ \alpha = \alpha$.

Axiom (5.5)**:** If $s = \perp$ then $s \circ (\neg\alpha) = U = \neg(s \circ \alpha)$. If $s \neq \perp$ then $s \circ (\neg\alpha) = \neg\alpha = \neg(s \circ \alpha)$. Thus $s \circ (\neg\alpha) = \neg(s \circ \alpha)$.

Axiom (5.6)**:** If $s = \perp$ then $s \circ (\alpha \wedge \beta) = U$ and $(s \circ \alpha) \wedge (s \circ \beta) = U \wedge U = U$. If $s \neq \perp$ then $s \circ (\alpha \wedge \beta) = \alpha \wedge \beta = (s \circ \alpha) \wedge (s \circ \beta)$. Thus $s \circ (\alpha \wedge \beta) = (s \circ \alpha) \wedge (s \circ \beta)$.

Axiom (5.7)**:** Consider $s, t \in S_\perp$ such that $s \cdot t = \perp$. Then $(s \cdot t) \circ \alpha = \perp \circ \alpha = U$. Since $S_\perp$ has no non-zero zero-divisors we have $s = \perp$ or $t = \perp$ and so $s \circ (t \circ \alpha) = U$ in either case. If $s \cdot t \neq \perp$ then $(s \cdot t) \circ \alpha = \alpha$ and $s \circ (t \circ \alpha) = t \circ \alpha = \alpha$ as neither s nor t are $\perp$. Thus $(s \cdot t) \circ \alpha = s \circ (t \circ \alpha)$.

Axiom (5.8)**:** As $\alpha \in \{T, F, U\}$ we consider the following three cases:

$\alpha = T$**:** Then $\alpha[s,t] \cdot u = T[s,t] \cdot u = s \cdot u = T[s \cdot u, t \cdot u]$.

$\alpha = F$**:** Then $\alpha[s,t] \cdot u = F[s,t] \cdot u = t \cdot u = F[s \cdot u, t \cdot u]$.

$\alpha = U$**:** Then $\alpha[s,t] \cdot u = U[s,t] \cdot u = \perp \cdot u = \perp = U[s \cdot u, t \cdot u]$.

Thus $\alpha[s,t] \cdot u = \alpha[s \cdot u, t \cdot u]$.

Axiom (5.9)**:** Consider the following cases:

Case I: $r = \perp$: Then $r \cdot \alpha[s,t] = \perp \cdot \alpha[s,t] = \perp = U[r \cdot s, r \cdot t] = (\perp \circ \alpha)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Case II: $r \neq \perp$: We again consider the following three cases:

$\alpha = T$**:** Then $r \cdot \alpha[s,t] = r \cdot T[s,t] = r \cdot s = T[r \cdot s, r \cdot t] = (r \circ T)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

$\alpha = F$**:** Then $r \cdot \alpha[s,t] = r \cdot F[s,t] = r \cdot t = F[r \cdot s, r \cdot t] = (r \circ F)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

$\alpha = U$**:** Then $r \cdot \alpha[s,t] = r \cdot U[s,t] = r \cdot \perp = \perp = U[r \cdot s, r \cdot t] = (r \circ U)[r \cdot s, r \cdot t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Thus $r \cdot \alpha[s,t] = (r \circ \alpha)[r \cdot s, r \cdot t]$.

Axiom (5.10)**:** Consider the following three cases:

$\alpha = T$**:** Then $\alpha[s,t] \circ \beta = T[s,t] \circ \beta = s \circ \beta = T[\![s \circ \beta, t \circ \beta]\!]$.

$\alpha = F$**:** Then $\alpha[s,t] \circ \beta = F[s,t] \circ \beta = t \circ \beta = F[\![s \circ \beta, t \circ \beta]\!]$.

$\alpha = U$**:** Then $\alpha[s,t] \circ \beta = U[s,t] \circ \beta = \perp \circ \beta = U = U[\![s \circ \beta, t \circ \beta]\!]$.

Thus $\alpha[s,t] \circ \beta = \alpha[\![s \circ \beta, t \circ \beta]\!]$. Thus the pair $(S_\perp, \mathfrak{Z})$ is a C-monoid.

5.2 A Cayley-type theorem

In this section we obtain a Cayley-type theorem for a class of C-monoids as stated in the following main theorem.

Theorem 5.2.1. *Every C-monoid $(S_\perp, M)$ where M is an ada is embeddable in the C-monoid $\left(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\right)$ for some set X. Moreover, if both $S_\perp$ and M are finite then so is X.*

Sketch of the proof. For each maximal congruence θ of M, we consider the C-set congruence (E_θ, θ) of $(S_\perp, M)$. Corresponding to each such congruence, we construct a homomorphism of C-monoids from $(S_\perp, M)$ to the functional C-monoid over the set $S_\perp/E_\theta$. This collection of homomorphisms has the property that every distinct pair of elements from each component of the C-monoid will be separated by some homomorphism from this collection. We then set X to be the disjoint union of $S_\perp/E_\theta$'s excluding the equivalence class $\overline{\perp}^{E_\theta}$. We complete the proof by constructing a monomorphism – by pasting together each of the individual homomorphisms from the collection defined earlier – from the C-monoid $(S_\perp, M)$ to the functional C-monoid over the pointed set $X_\perp$ with a new base point $\perp$.

The proof of Theorem 5.2.1 will be developed through various subsections. First in Subsection 5.2.1, we study some properties of maximal congruences of adas. We then present a collection of homomorphisms which separate every distinct pair of elements from each component of $(S_\perp, M)$ in Subsection 5.2.2. In Subsection 5.2.3, we construct the required functional C-monoid and establish an embedding from $(S_\perp, M)$. Finally, we consolidate the proof in Subsection 5.2.4.

In what follows $(S_\perp, M)$ is a C-monoid with M as an ada. Let θ be a maximal congruence on M and E_θ be the equivalence on $S_\perp$ as defined in Definition 3.1.2 so that the pair (E_θ, θ) is a congruence on $(S_\perp, M)$ (cf. Lemma 3.1.7). We denote the quotient set $S_\perp/E_\theta$ by $S_{\theta_\perp}$ and use S_θ to denote the set $S_{\theta_\perp} \setminus \{\overline{\perp}^{E_\theta}\}$. Further, we use q, s, t, u, v to denote elements of $S_\perp$ and α, β, γ to denote elements of the ada M.

5.2.1 Properties of maximal congruences

The following properties are useful in proving the main theorem.

Proposition 5.2.2. *No two elements of $\{T, F, U\}$ are related under θ. That is, $(T, F) \notin \theta$, $(T, U) \notin \theta$ and $(F, U) \notin \theta$.*

Proof. If $(T, F) \in \theta$ then we show that $\theta = \nabla$ contradicting the maximality of θ. Suppose $(T, F) \in \theta$ and let $\alpha, \beta \in M$. Then $(T, F), (\alpha, \alpha) \in \theta \Rightarrow (T \wedge \alpha, F \wedge \alpha) \in \theta$ that is $(\alpha, F) \in \theta$. Similarly $(\beta, F) \in \theta$ and so using the symmetry and transitivity of θ we have $(\alpha, \beta) \in \theta$ and consequently $\theta = \nabla$. The proof of $(T, U) \notin \theta$ follows along similar lines. Finally since $(F, U) \in \theta \Leftrightarrow (T, U) \in \theta$, the result follows. □

Proposition 5.2.3. *For each $q \in S_\perp$, we have*

(i) $(q \circ T)[q, \perp] = q$.

(ii) $(q \circ T, F) \notin \theta$.

(iii) $(q \circ T, U) \in \theta \Leftrightarrow (q, \perp) \in E_\theta$.

(iv) $(q \circ T, T) \in \theta \Leftrightarrow (q \circ F, F) \in \theta \Leftrightarrow (q, \perp) \notin E_\theta$.

(v) $(s, t) \in E_\theta \Rightarrow (s \circ \alpha, t \circ \alpha) \in \theta$ *for all* $\alpha \in M$.

(vi) $(1, \perp) \notin E_\theta$.

Proof.

(i) Using (5.9) we have $q = q \cdot 1 = q \cdot T[1, \perp] = (q \circ T)[q \cdot 1, q \cdot \perp] = (q \circ T)[q, \perp]$.

(ii) We prove the result by contradiction. Suppose $(q \circ T, F) \in \theta$. Using the fact that θ is a congruence on M and (5.5) we have $(q \circ T, F) \in \theta \Rightarrow (\neg(q \circ T), \neg F) \in \theta \Rightarrow (q \circ (\neg T), \neg F) \in \theta \Rightarrow (q \circ F, T) \in \theta$. Similarly using the fact that θ is a congruence, (5.6) and (5.5) we have $((q \circ F) \vee (q \circ T), (T \vee F)) \in \theta \Rightarrow (q \circ (F \vee T), (T \vee F)) \in \theta \Rightarrow (q \circ T, T) \in \theta$. Thus we have $(q \circ T, F) \in \theta$ and $(q \circ T, T) \in \theta$. From the symmetry and transitivity of θ it follows that $(T, F) \in \theta$, a contradiction by Proposition 5.2.2. The result follows.

(iii) We first show that $(q \circ T, U) \in \theta \Rightarrow (q, \perp) \in E_\theta$. Let $(q \circ T, U) \in \theta$. Using Proposition 3.1.6(iii) we can say that for any choice of $s, t \in S_\perp$ we have

$((q \circ T)[s,t], \perp) \in E_\theta$. On choosing $s = q, t = \perp$ and using Proposition 5.2.3(i) we have $((q \circ T)[q, \perp], \perp) \in E_\theta$ that is $(q, \perp) \in E_\theta$ which is the required result. Conversely if $(q, \perp) \in E_\theta$ there exists $\beta \in \overline{T}^\theta$ such that $\beta[q, \perp] = \beta[\perp, \perp] = \perp$ using Proposition 2.2.1(i). Thus

$$\begin{aligned} \beta[q,\perp] \circ T &= \perp \circ T \\ \beta[\![q \circ T, \perp \circ T]\!] &= U && \text{from } (5.10), (5.2) \\ \beta[\![q \circ T, U]\!] &= \beta[\![U, U]\!] && \text{from Proposition 2.2.1(i) on } (M, M) \end{aligned}$$

Thus $(q \circ T, U) \in E_{\theta_M}$ and so from Lemma 3.1.8, $(q \circ T, U) \in \theta$. Thus $(q \circ T, U) \in \theta \Leftrightarrow (q, \perp) \in E_\theta$.

(iv) We first show that $(q \circ T, T) \in \theta \Leftrightarrow (q \circ F, F) \in \theta$ by making use of the substitution property of the congruence θ with respect to $\neg$, the fact that $\neg$ is an involution and (5.5). Thus $(q \circ T, T) \in \theta \Leftrightarrow (\neg(q \circ T), \neg T) \in \theta \Leftrightarrow (q \circ (\neg T), \neg T) \in \theta \Leftrightarrow (q \circ F, F) \in \theta$. Using Proposition 5.2.2, Proposition 5.2.3(ii) and Proposition 5.2.3(iii) we show the equivalence $(q \circ T, T) \in \theta \Leftrightarrow (q, \perp) \notin E_\theta$. We have $(q \circ T, T) \in \theta \Rightarrow (q \circ T, U) \notin \theta \Rightarrow (q, \perp) \notin E_\theta$. Conversely $(q, \perp) \notin E_\theta \Rightarrow (q \circ T, U) \notin \theta$. Using Proposition 5.2.3(ii) it follows that $(q \circ T, F) \notin \theta$. Since θ is a maximal congruence the only remaining possibility is that $(q \circ T, T) \in \theta$ which completes the proof.

(v) Consider $(s,t) \in E_\theta$ and $\alpha \in M$. Then there exists $\beta \in \overline{T}^\theta$ such that $\beta[s,t] = \beta[t,t]$. Thus $\beta[s,t] \circ \alpha = \beta[t,t] \circ \alpha$. Using (5.10) we have $\beta[\![s \circ \alpha, t \circ \alpha]\!] = \beta[\![t \circ \alpha, t \circ \alpha]\!]$ from which it follows that $(s \circ \alpha, t \circ \alpha) \in E_{\theta_M} \subseteq \theta$ by Lemma 3.1.8.

(vi) Suppose that $(1, \perp) \in E_\theta$. Using Proposition 5.2.3(v), (5.4), (5.2) we have $(1, \perp) \in E_\theta \Rightarrow (1 \circ T, \perp \circ T) \in \theta$ and so $(T, U) \in \theta$ a contradiction by Proposition 5.2.2.

□

5.2.2 A class of homomorphisms separating pairs of elements

For each maximal congruence θ on M, in this subsection, we present homomorphisms $\phi_\theta : S_\perp \to \mathcal{T}_o(S_{\theta_\perp})$ and $\rho_\theta : M \to \mathfrak{Z}^{S_\theta}$. Then we establish that $(\phi_\theta, \rho_\theta)$ is a homomorphism from $(S_\perp, M)$ to the functional C-monoid $\big(\mathcal{T}_o(S_{\theta_\perp}), \mathfrak{Z}^{S_\theta}\big)$. Further, we ascertain that every pair of elements in $S_\perp$ (or M) are separated by some ϕ_θ (or ρ_θ).

Proposition 5.2.4. *The function $\phi_\theta : S_\perp \to \mathcal{T}_o(S_{\theta_\perp})$ given by $\phi_\theta(s) = \psi_\theta^s$, where $\psi_\theta^s(\overline{t}^{E_\theta}) = \overline{t \cdot s}^{E_\theta}$, is a monoid homomorphism that maps the zero (and base point) of $S_\perp$ to that of $\mathcal{T}_o(S_{\theta_\perp})$, that is $\perp \mapsto \zeta_{\overline{\perp}}$.*

Proof. Claim: ϕ_θ is well-defined. It suffices to show that ψ_θ^s is well-defined and that $\psi_\theta^s \in \mathcal{T}_o(S_{\theta_\perp})$, that is $\psi_\theta^s(\overline{\perp}) = \overline{\perp}$. In order to show the well-definedness of ψ_θ^s we consider $\overline{u} = \overline{t}$ that is $(u,t) \in E_\theta$. Then there exists $\beta \in \overline{T}^\theta$ such that $\beta[u,t] = \beta[t,t]$. Consequently

$$\begin{aligned} \beta[u \cdot s, t \cdot s] &= \beta[u,t] \cdot s && \text{from (5.8)} \\ &= \beta[t,t] \cdot s \\ &= \beta[t \cdot s, t \cdot s] && \text{from (5.8)} \end{aligned}$$

Thus $(u \cdot s, t \cdot s) \in E_\theta$ and so $\psi_\theta^s(\overline{u}) = \psi_\theta^s(\overline{t})$. Also $\psi_\theta^s(\overline{\perp}) = \overline{\perp \cdot s} = \overline{\perp}$. Thus $\psi_\theta^s \in \mathcal{T}_o(S_{\theta_\perp})$.

Claim: $\phi_\theta(\perp) = \zeta_{\overline{\perp}}$. We have $\phi_\theta(\perp) = \psi_\theta^\perp$ where $\psi_\theta^\perp(\overline{t}) = \overline{t \cdot \perp} = \overline{\perp}$. Thus $\phi_\theta(\perp) = \zeta_{\overline{\perp}}$.

Claim: $\phi_\theta(1) = id_{S_{\theta_\perp}}$. We have $\phi_\theta(1) = \psi_\theta^1$ where $\psi_\theta^1(\overline{t}) = \overline{t \cdot 1} = \overline{t}$.

Claim: ϕ_θ is a semigroup homomorphism. Consider $\phi_\theta(s \cdot t) = \psi_\theta^{s \cdot t}$ where $\psi_\theta^{s\cdot t}(\overline{u}) = \overline{u \cdot (s \cdot t)} = \overline{(u \cdot s) \cdot t} = \psi_\theta^t(\overline{u \cdot s}) = \psi_\theta^t(\psi_\theta^s(\overline{u})) = (\psi_\theta^s \cdot \psi_\theta^t)(\overline{u})$. Thus $\phi_\theta(s \cdot t) = \phi_\theta(s) \cdot \phi_\theta(t)$. ∎

Proposition 5.2.5. *The function* $\rho_\theta : M \rightarrow \mathfrak{Z}^{S_\theta}$ *given by*

$$\rho_\theta(\alpha) = \begin{cases} (S_\theta, \emptyset), & \text{if } \alpha = T; \\ (\emptyset, S_\theta), & \text{if } \alpha = F; \\ (A_\theta^\alpha, B_\theta^\alpha), & \text{otherwise} \end{cases}$$

where $A_\theta^\alpha = \{\overline{t}^{E_\theta} : t \circ \alpha \in \overline{T}^\theta\}$ *and* $B_\theta^\alpha = \{\overline{t}^{E_\theta} : t \circ \alpha \in \overline{F}^\theta\}$*, is a homomorphism of* C*-algebras with* T, F, U*.*

Proof. Claim: ρ_θ *is well-defined.* If $\alpha \in \{T, F\}$ then the proof is obvious. If $\alpha \notin \{T, F\}$ then we show that $A_\theta^\alpha \cap B_\theta^\alpha = \emptyset$ and that $A_\theta^\alpha, B_\theta^\alpha \subseteq S_\theta$, that is, $\overline{\bot} \notin A_\theta^\alpha \cup B_\theta^\alpha$. Let $\overline{t} \in A_\theta^\alpha \cap B_\theta^\alpha$. Then $t \circ \alpha \in \overline{T}^\theta$ and $t \circ \alpha \in \overline{F}^\theta$ and so $(T, F) \in \theta$ which is a contradiction to Proposition 5.2.2. Using (5.2) we have $\bot \circ \alpha = U$ and so if $\overline{\bot} \in A_\theta^\alpha \cup B_\theta^\alpha$ we would have $\bot \circ \alpha = U \in \{\overline{T}^\theta, \overline{F}^\theta\}$, a contradiction to Proposition 5.2.2. Finally we show that the image under ρ_θ is independent of the representative of the equivalence class chosen. Using Proposition 5.2.3(v) we have $\overline{s} = \overline{t} \Rightarrow (s \circ \alpha, t \circ \alpha) \in \theta$. The result follows.

Claim: ρ_θ *preserves the constants* T, F, U*.* It is clear that $\rho_\theta(T) = (S_\theta, \emptyset)$, $\rho_\theta(F) = (\emptyset, S_\theta)$ and, using (5.3) and Proposition 5.2.2, that $\rho_\theta(U) = (A_\theta^U, B_\theta^U) = (\emptyset, \emptyset)$ from which the result follows.

Claim: ρ_θ *is a* C*-algebra homomorphism.* We show that $\rho_\theta(\neg\alpha) = \neg(\rho_\theta(\alpha))$. If $\alpha \in \{T, F\}$ the proof is obvious. Suppose that $\alpha \notin \{T, F\}$. Then we have the following.

$$\begin{aligned} \rho_\theta(\neg\alpha) &= (A_\theta^{\neg\alpha}, B_\theta^{\neg\alpha}) \\ &= (\{\overline{t} : t \circ (\neg\alpha) \in \overline{T}^\theta\}, \{\overline{t} : t \circ (\neg\alpha) \in \overline{F}^\theta\}) \\ &= (\{\overline{t} : \neg(t \circ \alpha) \in \overline{T}^\theta\}, \{\overline{t} : \neg(t \circ \alpha) \in \overline{F}^\theta\}) && \text{using (5.5)} \end{aligned}$$

$$= (\{\bar{t} : t \circ \alpha \in \overline{F}^{\theta}\}, \{\bar{t} : t \circ \alpha \in \overline{T}^{\theta}\})$$
$$= (B_{\theta}^{\alpha}, A_{\theta}^{\alpha})$$
$$= \neg(\rho_{\theta}(\alpha))$$

Finally we show that $\rho_{\theta}(\alpha \wedge \beta) = \rho_{\theta}(\alpha) \wedge \rho_{\theta}(\beta)$. Note that the proof of $\rho_{\theta}(\alpha \vee \beta) = \rho_{\theta}(\alpha) \vee \rho_{\theta}(\beta)$ follows using the double negation and De Morgan's laws, viz., (1.25) and (1.26) respectively in conjunction with the fact that ρ_{θ} preserves $\neg$ and $\wedge$. In order to prove that $\rho_{\theta}(\alpha \wedge \beta) = \rho_{\theta}(\alpha) \wedge \rho_{\theta}(\beta)$ we proceed by considering the following cases.

Case I: $\alpha, \beta \notin \{T, F\}$. We have the following subcases:

Subcase 1: $\alpha \wedge \beta \notin \{T, F\}$. Then $\rho_{\theta}(\alpha \wedge \beta) = (A_{\theta}^{\alpha\wedge\beta}, B_{\theta}^{\alpha\wedge\beta})$, $\rho_{\theta}(\alpha) = (A_{\theta}^{\alpha}, B_{\theta}^{\alpha})$ and $\rho_{\theta}(\beta) = (A_{\theta}^{\beta}, B_{\theta}^{\beta})$. Now $(A_{\theta}^{\alpha}, B_{\theta}^{\alpha}) \wedge (A_{\theta}^{\beta}, B_{\theta}^{\beta}) = (A_{\theta}^{\alpha} \cap A_{\theta}^{\beta}, B_{\theta}^{\alpha} \cup (A_{\theta}^{\alpha} \cap B_{\theta}^{\beta}))$. Thus we have to show that

$$(A_{\theta}^{\alpha\wedge\beta}, B_{\theta}^{\alpha\wedge\beta}) = (A_{\theta}^{\alpha} \cap A_{\theta}^{\beta}, B_{\theta}^{\alpha} \cup (A_{\theta}^{\alpha} \cap B_{\theta}^{\beta})).$$

We show that the pairs of sets are equal componentwise.

Let $\overline{q} \in A_{\theta}^{\alpha\wedge\beta}$. Then $q \circ (\alpha \wedge \beta) \in \overline{T}^{\theta}$

$$\begin{array}{lll} \Rightarrow & ((q \circ \alpha) \wedge (q \circ \beta), T) \in \theta & \text{(using (5.6))} \\ \Rightarrow & ((q \circ \alpha) \wedge ((q \circ \alpha) \wedge (q \circ \beta)), (q \circ \alpha) \wedge T) \in \theta & \text{(since } \theta \text{ is a congruence)} \\ \Rightarrow & ((q \circ \alpha) \wedge (q \circ \beta), q \circ \alpha) \in \theta & \text{(using the properties of } \wedge) \\ \Rightarrow & (q \circ \alpha, T) \in \theta & \text{(by transitivity of } \theta) \end{array}$$

so that $\overline{q} \in A_{\theta}^{\alpha}$. Along similar lines one can observe that

$$((((q \circ \alpha) \wedge (q \circ \beta)) \wedge (q \circ \beta)), T \wedge (q \circ \beta)) \in \theta.$$

Consequently $(q \circ \beta, T) \in \theta$ so that $\overline{q} \in A_{\theta}^{\beta}$. Hence $A_{\theta}^{\alpha\wedge\beta} \subseteq A_{\theta}^{\alpha} \cap A_{\theta}^{\beta}$.

For reverse inclusion let $\overline{q} \in A_\theta^\alpha \cap A_\theta^\beta$. Then $(q \circ \alpha, T), (q \circ \beta, T) \in \theta$. Since θ is a congruence we have $((q\circ\alpha)\wedge(q\circ\beta), T\wedge T) = ((q\circ\alpha)\wedge(q\circ\beta), T) = ((q\circ(\alpha\wedge\beta), T) \in \theta$ and so $\overline{q} \in A_\theta^{\alpha\wedge\beta}$. Hence $A_\theta^{\alpha\wedge\beta} = A_\theta^\alpha \cap A_\theta^\beta$.

In order to show that $B_\theta^{\alpha\wedge\beta} \subseteq B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$ consider $\overline{q} \in B_\theta^{\alpha\wedge\beta}$ that is $(q \circ (\alpha \wedge \beta), F) \in \theta$. Since θ is a maximal congruence consider the following three possibilities:

$(q \circ \alpha, F) \in \theta$ Then clearly $\overline{q} \in B_\theta^\alpha$ and so $\overline{q} \in B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

$(q \circ \alpha, T) \in \theta$ Then we have $\overline{q} \in A_\theta^\alpha$. We show that $(q \circ \beta, F) \in \theta$. If this is not the case then either $(q \circ \beta, T) \in \theta$ or $(q \circ \beta, U) \in \theta$. If $(q \circ \beta, T) \in \theta$ then since $(q \circ \alpha, T) \in \theta$ we have $(q \circ (\alpha \wedge \beta), T \wedge T) = (q \circ (\alpha \wedge \beta), T) \in \theta$ using (5.6) and the fact that θ is a congruence. However since $(q \circ (\alpha \wedge \beta), F) \in \theta$ we obtain a contradiction that $(T, F) \in \theta$ (cf. Proposition 5.2.2). Along similar lines if $(q \circ \beta, U) \in \theta$ then as $(q \circ \alpha, T) \in \theta$ we have $(q \circ (\alpha \wedge \beta), U) \in \theta$ and so $(F, U) \in \theta$ a contradiction to Proposition 5.2.2. Hence $(q \circ \beta, F) \in \theta$ so that $\overline{q} \in (A_\theta^\alpha \cap B_\theta^\beta) \subseteq B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

$(q \circ \alpha, U) \in \theta$ Since $(q\circ\beta, q\circ\beta) \in \theta$ we have $(q\circ(\alpha\wedge\beta), U\wedge q\circ\beta) = (q\circ(\alpha\wedge\beta), U) \in \theta$ using (5.6) and the fact that θ is a congruence. However since $(q\circ(\alpha\wedge\beta), F) \in \theta$ we have $(F, U) \in \theta$ a contradiction to Proposition 5.2.2. Thus this case cannot occur.

To show the reverse inclusion let $\overline{q} \in B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$ that is $\overline{q} \in B_\theta^\alpha$ or $\overline{q} \in A_\theta^\alpha \cap B_\theta^\beta$. If $\overline{q} \in B_\theta^\alpha$ then $(q \circ \alpha, F) \in \theta$

$$\begin{aligned}
&\Rightarrow \quad ((q \circ \alpha) \wedge (q \circ \beta), F \wedge (q \circ \beta)) \in \theta && \text{(since } \theta \text{ is a congruence)} \\
&\Rightarrow \quad (q \circ (\alpha \wedge \beta), F \wedge (q \circ \beta)) \in \theta && \text{(using (5.6))} \\
&\Rightarrow \quad (q \circ (\alpha \wedge \beta), F) \in \theta && \text{(since } F \text{ is a left-zero for } \wedge)
\end{aligned}$$

from which it follows that $\overline{q} \in B_\theta^{\alpha\wedge\beta}$. In the case where $\overline{q} \in A_\theta^\alpha \cap B_\theta^\beta$ that is $(q \circ \alpha, T), (q \circ \beta, F) \in \theta$, along similar lines it follows that $(q \circ (\alpha \wedge \beta), T \wedge F) =$

$(q \circ (\alpha \wedge \beta), F) \in \theta$ and so $\overline{q} \in B_\theta^{\alpha\wedge\beta}$. Therefore $B_\theta^{\alpha\wedge\beta} = B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

Subcase 2: $\alpha \wedge \beta \in \{T, F\}$. Using the fact that $M \leq \mathfrak{Z}^X$ for some set X it is easy to see that if $\alpha, \beta \notin \{T, F\}$ then $\alpha \wedge \beta \neq T$. It follows that the only possibility in this case is that $\alpha \wedge \beta = F$. Therefore $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(F) = (\emptyset, S_\theta)$ and $\rho_\theta(\alpha) \wedge \rho_\theta(\beta) = (A_\theta^\alpha \cap A_\theta^\beta, B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta))$ and so we have to show that

$$(\emptyset, S_\theta) = (A_\theta^\alpha \cap A_\theta^\beta, B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)).$$

We first show that $A_\theta^\alpha \cap A_\theta^\beta = \emptyset$. If $A_\theta^\alpha \cap A_\theta^\beta \neq \emptyset$ then let $\overline{q} \in A_\theta^\alpha \cap A_\theta^\beta$ so that $(q \circ \alpha, T) \in \theta, (q \circ \beta, T) \in \theta$

$$\begin{aligned}
&\Rightarrow ((q \circ \alpha) \wedge (q \circ \beta), T \wedge T) = ((q \circ \alpha) \wedge (q \circ \beta), T) \in \theta && \text{(since } \theta \text{ is a congruence)}\\
&\Rightarrow (q \circ (\alpha \wedge \beta), T) \in \theta && \text{(using (5.6))}\\
&\Rightarrow (q \circ F, T) \in \theta && \text{(since } \alpha \wedge \beta = F)\\
&\Rightarrow (\neg(q \circ F), \neg T) = (\neg(q \circ F), F) \in \theta && \text{(since } \theta \text{ is a congruence)}\\
&\Rightarrow (q \circ \neg F, F) = (q \circ T, F) \in \theta && \text{(using (5.5))}
\end{aligned}$$

which is a contradiction to Proposition 5.2.3(ii). Hence $A_\theta^\alpha \cap A_\theta^\beta = \emptyset$.

In order to show that $B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta) = S_\theta$ consider $\overline{q} \in S_\theta$ that is $\overline{q} \neq \overline{\perp}$ which gives $(q, \perp) \notin E_\theta$. We proceed by considering the following three cases:

$(q \circ \alpha, F) \in \theta$ Then it is clear that $\overline{q} \in B_\theta^\alpha \subseteq B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

$(q \circ \alpha, T) \in \theta$ Then we have $\overline{q} \in A_\theta^\alpha$. We show that $(q \circ \beta, F) \in \theta$. Suppose that this is not the case. Since θ is a maximal congruence it implies that either $(q \circ \beta, T) \in \theta$ or $(q \circ \beta, U) \in \theta$. If $(q \circ \beta, T) \in \theta$ then since $(q \circ \alpha, T) \in \theta$ it follows that $(q \circ (\alpha \wedge \beta), T \wedge T) = (q \circ F, T) \in \theta$ so that $(q \circ T, F) \in \theta$. This is a contradiction to Proposition 5.2.3(ii). In the case that $(q \circ \beta, U) \in \theta$ proceeding as earlier we have $(q \circ (\alpha \wedge \beta), T \wedge U) = (q \circ F, U) \in \theta$ so that

$(q \circ T, U) \in \theta$. It follows from Proposition 5.2.3(iii) that $(q, \perp) \in E_\theta$ which is a contradiction to the assumption that $\overline{q} \in S_\theta$. Consequently it must be the case that $(q \circ \beta, F) \in \theta$ so that $\overline{q} \in A_\theta^\alpha \cap B_\theta^\beta \subseteq B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta)$.

$(q \circ \alpha, U) \in \theta$ Since θ is a congruence we have $(q \circ \beta, q \circ \beta) \in \theta$

$$\begin{aligned}
&\Rightarrow ((q \circ \alpha) \wedge (q \circ \beta), U \wedge (q \circ \beta)) \in \theta && \text{(since } \theta \text{ is a congruence)} \\
&\Rightarrow ((q \circ (\alpha \wedge \beta), U) = (q \circ F, U) \in \theta && \text{(since } U \text{ is a left-zero for } \wedge \text{ and using (5.6))} \\
&\Rightarrow (\neg(q \circ F), \neg U) \in \theta && \text{(since } \theta \text{ is a congruence)} \\
&\Rightarrow (q \circ \neg F, \neg U) = (q \circ T, U) \in \theta && \text{(using (5.5))}
\end{aligned}$$

Thus using Proposition 5.2.3(iii) we have $(q, \perp) \in E_\theta$ which is a contradiction to the assumption that $\overline{q} \in S_\theta$. Hence this case cannot occur.

Thus $B_\theta^\alpha \cup (A_\theta^\alpha \cap B_\theta^\beta) = S_\theta$ which completes the proof in the case where $\alpha, \beta \notin \{T, F\}$.

Case II: $\alpha \in \{T, F\}$. The verification is straightforward by considering $\alpha = T$ and $\alpha = F$ casewise.

Subcase 1: $\alpha = T$. Then $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(T \wedge \beta) = \rho_\theta(\beta) = (S_\theta, \emptyset) \wedge \rho_\theta(\beta) = \rho_\theta(T) \wedge \rho_\theta(\beta) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$.

Subcase 2: $\alpha = F$. Then $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(F \wedge \beta) = \rho_\theta(F) = (\emptyset, S_\theta) = (\emptyset, S_\theta) \wedge \rho_\theta(\beta) = \rho_\theta(F) \wedge \rho_\theta(\beta) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$.

Case III: $\beta \in \{T, F\}$. We have the following subcases:

Subcase 1: $\beta = T$. The proof follows along the same lines as *Case II* above since T is the left and right-identity for $\wedge$. Thus $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(\alpha \wedge T) = \rho_\theta(\alpha) = \rho_\theta(\alpha) \wedge (S_\theta, \emptyset) = \rho_\theta(\alpha) \wedge \rho_\theta(T) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$.

Subcase 2: $\beta = F$. If $\alpha \in \{T, F\}$ then this reduces to *Case II* proved above and consequently we have $\rho_\theta(\alpha \wedge \beta) = \rho_\theta(\alpha) \wedge \rho_\theta(\beta)$ in this case. Thus it remains to consider the case where $\alpha \notin \{T, F\}$. We then have the following subcases depending on $\alpha \wedge \beta$:

$\alpha \wedge \beta \notin \{T, F\}$: Then $\rho_\theta(\alpha\wedge\beta) = \rho_\theta(\alpha\wedge F) = (A_\theta^{\alpha\wedge F}, B_\theta^{\alpha\wedge F})$ while $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha)$ and $\rho_\theta(\beta) = \rho_\theta(F) = (\emptyset, S_\theta)$. Thus $\rho_\theta(\alpha) \wedge \rho_\theta(F) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$. We show that

$$(A_\theta^{\alpha\wedge F}, B_\theta^{\alpha\wedge F}) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$$

as earlier by proving that the pairs of sets are equal componentwise.

We show that $A_\theta^{\alpha\wedge F} = \emptyset$ by contradiction. If $A_\theta^{\alpha\wedge F} \neq \emptyset$ then consider $\overline{q} \in A_\theta^{\alpha\wedge F}$. It follows that $(q \circ (\alpha \wedge F), T) \in \theta$

$$\begin{array}{lll}
\Rightarrow & ((q \circ (\alpha \wedge F)) \wedge (q \circ F), T \wedge q \circ F) \in \theta & \text{(since } \theta \text{ is a congruence)} \\
\Rightarrow & ((q \circ (\alpha \wedge F)) \wedge (q \circ F), q \circ F) \in \theta & \text{(since } T \text{ is a left-identity for } \wedge) \\
\Rightarrow & (((q \circ \alpha) \wedge (q \circ F)) \wedge (q \circ F), q \circ F) \in \theta & \text{(using (5.6))} \\
\Rightarrow & ((q \circ \alpha) \wedge (q \circ F), q \circ F) \in \theta & \text{(using the properties of } \wedge) \\
\Rightarrow & (q \circ F, T) \in \theta & \text{(since } \theta \text{ is a congruence)} \\
\Rightarrow & (q \circ T, F) \in \theta & \text{(from (5.5) and since } \theta \text{ is a congruence)}
\end{array}$$

which is a contradiction to Proposition 5.2.3(ii). Hence $A_\theta^{\alpha\wedge F} = \emptyset$.

We show that $B_\theta^{\alpha\wedge F} = A_\theta^\alpha \cup B_\theta^\alpha$ using standard set theoretic arguments. Let $\overline{q} \in B_\theta^{\alpha\wedge F}$ and so $(q \circ (\alpha \wedge F), F) \in \theta$ so that $((q \circ \alpha) \wedge (q \circ F), F) \in \theta$. In view of the maximality of θ it suffices to consider three cases. If either $(q \circ \alpha, T) \in \theta$ or $(q \circ \alpha, F) \in \theta$ then $\overline{q} \in A_\theta^\alpha \cup B_\theta^\alpha$. If $(q \circ \alpha, U) \in \theta$ then $((q \circ \alpha) \wedge ((q \circ \alpha) \wedge (q \circ F)), U \wedge F) = ((q \circ \alpha) \wedge (q \circ F), U) \in \theta$.Thus $(F, U) \in \theta$ which is a contradiction to Proposition 5.2.2. Hence this case cannot occur and so $B_\theta^{\alpha\wedge F} \subseteq A_\theta^\alpha \cup B_\theta^\alpha$.

For the reverse inclusion consider $\overline{q} \in A_\theta^\alpha \cup B_\theta^\alpha$ so that $\overline{q} \in A_\theta^\alpha$ or $\overline{q} \in B_\theta^\alpha$. If $\overline{q} \in A_\theta^\alpha$ then $(q \circ \alpha, T) \in \theta$. Since $\overline{q} \in A_\theta^\alpha \subseteq S_\theta$ using Proposition 5.2.3(iv) we have $(q, \perp) \notin E_\theta \Rightarrow (q \circ F, F) \in \theta$. Consequently $(q \circ (\alpha \wedge F), (T \wedge F)) = (q \circ (\alpha \wedge F), F) \in \theta$ and so $\overline{q} \in B_\theta^{\alpha \wedge F}$. Along similar lines if $\overline{q} \in B_\theta^\alpha$ we have $(q \circ (\alpha \wedge F), F) \in \theta$ so that $\overline{q} \in B_\theta^{\alpha \wedge F}$. Hence $(A_\theta^{\alpha\wedge F}, B_\theta^{\alpha\wedge F}) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$.

$\alpha \wedge \beta \in \{T, F\}$: Using the fact that $M \leq 3^X$ for some set X we have $\alpha \wedge F \neq T$ from which it follows that the only case is $\alpha \wedge \beta = \alpha \wedge F = F$. Thus $\rho_\theta(\alpha \wedge F) = \rho_\theta(F) = (\emptyset, S_\theta)$ while $\rho_\theta(\alpha) \wedge \rho_\theta(F) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$. We show that

$$(\emptyset, S_\theta) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha).$$

In order to show that $A_\theta^\alpha \cup B_\theta^\alpha = S_\theta$ consider $\overline{q} \in S_\theta$. If $(q \circ \alpha, T) \in \theta$ or $(q \circ \alpha, F) \in \theta$ then the proof is complete. If $(q \circ \alpha, U) \in \theta$ then since $\overline{q} \neq \overline{\perp}$ that is $(q, \perp) \notin E_\theta$ by Proposition 5.2.3(iv) we have $(q \circ F, F) \in \theta$. Thus $(q \circ (\alpha \wedge F), U \wedge F) = (q \circ F, U) \in \theta$. Consequently from the transitivity of θ it follows that $(F, U) \in \theta$ which is a contradiction to Proposition 5.2.2. Hence $(\emptyset, S_\theta) = (\emptyset, A_\theta^\alpha \cup B_\theta^\alpha)$.

Thus ρ_θ is a homomorphism of C-algebras with T, F, U. □

Lemma 5.2.6. *The pair $(\phi_\theta, \rho_\theta)$ is a C-monoid homomorphism from $(S_\perp, M)$ to the functional C-monoid $\left(\mathcal{T}_o(S_{\theta_\perp}), 3^{S_\theta}\right)$.*

Proof. In view of Proposition 5.2.4 and Proposition 5.2.5 it suffices to show that $\phi_\theta(\alpha[s,t]) = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$ and $\rho_\theta(s \circ \alpha) = \phi_\theta(s) \circ \rho_\theta(\alpha)$ hold. In order to show that $\phi_\theta(\alpha[s,t]) = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$ we proceed casewise depending on the value of α as per the following:

Case I: $\alpha \in \{T, F\}$. If $\alpha = T$ then $\phi_\theta(\alpha[s,t]) = \phi_\theta(T[s,t]) = \phi_\theta(s) = (S_\theta, \emptyset)[\phi_\theta(s), \phi_\theta(t)] = \rho_\theta(T)[\phi_\theta(s), \phi_\theta(t)] = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$. Along similar lines if $\alpha = F$ then $\phi_\theta(\alpha[s,t]) = \phi_\theta(F[s,t]) = \phi_\theta(t) = (\emptyset, S_\theta)[\phi_\theta(s), \phi_\theta(t)] = \rho_\theta(F)[\phi_\theta(s), \phi_\theta(t)] = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$.

Case II: $\alpha \notin \{T, F\}$. If $\alpha \notin \{T, F\}$ then using (5.9) we have $\phi_\theta(\alpha[s,t]) = \psi_\theta^{\alpha[s,t]}$ where $\psi_\theta^{\alpha[s,t]}(\overline{v}) = \overline{v \cdot (\alpha[s,t])} = \overline{(v \circ \alpha)[v \cdot s, v \cdot t]}$. Consider $\rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)] = (A_\theta^\alpha, B_\theta^\alpha)[\psi_\theta^s, \psi_\theta^t]$, where

$$(A_\theta^\alpha, B_\theta^\alpha)[\psi_\theta^s, \psi_\theta^t](\overline{v}) = \begin{cases} \overline{v \cdot s}, & \text{if } \overline{v} \in A_\theta^\alpha, \text{ that is } (v \circ \alpha) \in \overline{T}^\theta; \\ \overline{v \cdot t}, & \text{if } \overline{v} \in B_\theta^\alpha \text{ that is } (v \circ \alpha) \in \overline{F}^\theta; \\ \overline{\bot}, & \text{otherwise.} \end{cases}$$

It suffices to consider the following three cases:

Subcase 1: $(v \circ \alpha) \in \overline{T}^\theta$. using Proposition 3.1.6(i) we have $((v \circ \alpha)[v \cdot s, v \cdot t], v \cdot s) \in E_\theta$. Consequently $\overline{(v \circ \alpha)[v \cdot s, v \cdot t]} = \overline{v \cdot s}$.

Subcase 2: $(v \circ \alpha) \in \overline{F}^\theta$. Along similar lines if $(v \circ \alpha) \in \overline{F}^\theta$ then $((v \circ \alpha)[v \cdot s, v \cdot t], v \cdot t) \in E_\theta$, by Proposition 3.1.6(ii) and so $\overline{(v \circ \alpha)[v \cdot s, v \cdot t]} = \overline{v \cdot t}$.

Subcase 3: $(v \circ \alpha) \in \overline{U}^\theta$. Then $((v \circ \alpha)[v \cdot s, v \cdot t], \bot) \in E_\theta$, by Proposition 3.1.6(iii) which gives $\overline{(v \circ \alpha)[v \cdot s, v \cdot t]} = \overline{\bot}$.

Thus we have $\psi_\theta^{\alpha[s,t]}(\overline{v}) = (A_\theta^\alpha, B_\theta^\alpha)[\psi_\theta^s, \psi_\theta^t](\overline{v})$ for every $\overline{v} \in S_{\theta_\bot}$ and so $\phi_\theta(\alpha[s,t]) = \rho_\theta(\alpha)[\phi_\theta(s), \phi_\theta(t)]$.

We show that $\rho_\theta(s \circ \alpha) = \phi_\theta(s) \circ \rho_\theta(\alpha)$ by proceeding casewise depending on the value of α and $s \circ \alpha$.

Case I: $\alpha \notin \{T, F\}, s \circ \alpha \notin \{T, F\}$. Then $\rho_\theta(s \circ \alpha) = (A_\theta^{s \circ \alpha}, B_\theta^{s \circ \alpha})$ and $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha)$. Then $\phi_\theta(s) \circ (A_\theta^\alpha, B_\theta^\alpha) = \psi_\theta^s \circ (A_\theta^\alpha, B_\theta^\alpha) = (C, D)$, where $C = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in A_\theta^\alpha\}$ and $D = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in B_\theta^\alpha\}$. We have to show that

$$(A_\theta^{s \circ \alpha}, B_\theta^{s \circ \alpha}) = (C, D).$$

It is clear that $\overline{q} \in C$

$$\begin{aligned}
&\Leftrightarrow \psi_\theta^s(\overline{q}) \in A_\theta^\alpha\\
&\Leftrightarrow \overline{q \cdot s} \in A_\theta^\alpha\\
&\Leftrightarrow ((q \cdot s) \circ \alpha, T) \in \theta\\
&\Leftrightarrow (q \circ (s \circ \alpha), T) \in \theta \quad \text{(using (5.7))}\\
&\Leftrightarrow \overline{q} \in A_\theta^{s\circ\alpha}
\end{aligned}$$

Along similar lines we have $\overline{q} \in D \Leftrightarrow \overline{q} \in B_\theta^{s\circ\alpha}$.

Case II: $\alpha \in \{T, F\}, s \circ \alpha \notin \{T, F\}$. If $\alpha = T$ then $\rho_\theta(s \circ \alpha) = \rho_\theta(s \circ T) = (A_\theta^{s\circ T}, B_\theta^{s\circ T})$. On the other hand $\phi_\theta(s) \circ \rho_\theta(\alpha) = \phi_\theta(s) \circ \rho_\theta(T) = \psi_\theta^s \circ (S_\theta, \emptyset) = (C, D)$ where $C = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in S_\theta\}$ and $D = \emptyset$. We have to show that

$$(A_\theta^{s\circ T}, B_\theta^{s\circ T}) = (C, \emptyset).$$

We show that $B_\theta^{s\circ T} = \emptyset$ by contradiction. If $B_\theta^{s\circ T} \neq \emptyset$ then let $\overline{q} \in B_\theta^{s\circ T}$

$$\begin{aligned}
&\Rightarrow (q \circ (s \circ T), F) \in \theta\\
&\Rightarrow ((q \cdot s) \circ T, F) \in \theta \quad \text{(using (5.7))}
\end{aligned}$$

which is a contradiction to Proposition 5.2.3(ii). Thus $B_\theta^{s\circ T} = \emptyset$.

We now show that $A_\theta^{s\circ T} = C$. It is clear that $\overline{q} \in C$

$$\begin{aligned}
&\Leftrightarrow \psi_\theta^s(\overline{q}) \in S_\theta\\
&\Leftrightarrow \overline{q \cdot s} \in S_\theta\\
&\Leftrightarrow (q \cdot s, \perp) \notin E_\theta\\
&\Leftrightarrow ((q \cdot s) \circ T, T) \in \theta \quad \text{(using Proposition 5.2.3(iv))}\\
&\Leftrightarrow (q \circ (s \circ T), T) \in \theta \quad \text{(using (5.7))}\\
&\Leftrightarrow \overline{q} \in A_\theta^{s\circ T}
\end{aligned}$$

In the case where $\alpha = F$ the proof follows along similar lines.

Case III: $\alpha \notin \{T, F\}, s \circ \alpha \in \{T, F\}$. We have the following subcases:

Subcase 1: $s \circ \alpha = T$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(T) = (S_\theta, \emptyset)$. On the other hand $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi^s_\theta \circ (A^\alpha_\theta, B^\alpha_\theta) = (C, D)$ where $C = \{\overline{q} \in S_\theta : \psi^s_\theta(\overline{q}) \in A^\alpha_\theta\}$ and $D = \{\overline{q} \in S_\theta : \psi^s_\theta(\overline{q}) \in B^\alpha_\theta\}$. We have to show that

$$(C, D) = (S_\theta, \emptyset).$$

We first show by contradiction that $D = \emptyset$. If $D \neq \emptyset$ consider $\overline{q} \in D$

$$\begin{aligned}
&\Rightarrow \psi^s_\theta(\overline{q}) \in B^\alpha_\theta \\
&\Rightarrow \overline{q \cdot s} \in B^\alpha_\theta \\
&\Rightarrow ((q \cdot s) \circ \alpha, F) \in \theta \\
&\Rightarrow (q \circ (s \circ \alpha), F) \in \theta \quad \text{(using (5.7))} \\
&\Rightarrow (q \circ T, F) \in \theta
\end{aligned}$$

which is a contradiction to Proposition 5.2.3(ii).

In order to show that $C = S_\theta$ consider $\overline{q} \in S_\theta$ that is $(q, \perp) \notin E_\theta$

$$\begin{aligned}
&\Rightarrow (q \circ T, T) \in \theta \qquad \text{(using Proposition 5.2.3(iv))} \\
&\Rightarrow (q \circ (s \circ \alpha), T) \in \theta \\
&\Rightarrow ((q \cdot s) \circ \alpha, T) \in \theta \quad \text{(using (5.7))} \\
&\Rightarrow \overline{q \cdot s} \in A^\alpha_\theta \\
&\Rightarrow \psi^s_\theta(\overline{q}) \in A^\alpha_\theta \\
&\Rightarrow \overline{q} \in C.
\end{aligned}$$

Subcase 2: $s \circ \alpha = F$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(F) = (\emptyset, S_\theta)$ while $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi^s_\theta \circ (A^\alpha_\theta, B^\alpha_\theta) = (C, D)$ where $C = \{\overline{q} \in S_\theta : \psi^s_\theta(\overline{q}) \in A^\alpha_\theta\}$ and $D = \{\overline{q} \in S_\theta : \psi^s_\theta(\overline{q}) \in B^\alpha_\theta\}$. We have to show that

$$(C, D) = (\emptyset, S_\theta).$$

We first show $C = \emptyset$ by contradiction. If $C \neq \emptyset$ consider $\overline{q} \in C$

$$\begin{aligned}
&\Rightarrow \psi^s_\theta(\overline{q}) \in A^\alpha_\theta \\
&\Rightarrow \overline{q \cdot s} \in A^\alpha_\theta \\
&\Rightarrow ((q \cdot s) \circ \alpha, T) \in \theta \\
&\Rightarrow (q \circ (s \circ \alpha), T) \in \theta \quad \text{(using (5.7))} \\
&\Rightarrow (q \circ F, T) \in \theta \\
&\Rightarrow (q \circ T, F) \in \theta
\end{aligned}$$

which is a contradiction to Proposition 5.2.3(ii).

In order to show that $D = S_\theta$ consider $\overline{q} \in S_\theta$ that is $(q, \perp) \notin E_\theta$.

$$\begin{aligned}
&\Rightarrow (q \circ F, F) \in \theta \qquad \text{(using Proposition 5.2.3(iv))} \\
&\Rightarrow (q \circ (s \circ \alpha), F) \in \theta \\
&\Rightarrow ((q \cdot s) \circ \alpha, F) \in \theta \quad \text{(using (5.7))} \\
&\Rightarrow \overline{q \cdot s} \in B^\alpha_\theta \\
&\Rightarrow \psi^s_\theta(\overline{q}) \in B^\alpha_\theta \\
&\Rightarrow \overline{q} \in D
\end{aligned}$$

which completes the proof for the case where $\alpha \notin \{T, F\}$ and $s \circ \alpha \in \{T, F\}$.

Case IV: $\alpha \in \{T, F\}, s \circ \alpha \in \{T, F\}$. Note that $s \circ T \neq F$ as a consequence of Proposition 5.2.3(ii). If $s \circ T = F$ then as θ is a congruence, $(F, F) \in \theta \Rightarrow (s \circ T, F) \in \theta$, a contradiction to Proposition 5.2.3(ii). Similarly we have $s \circ F \neq T$. In view of the above it suffices to consider the following cases:

Subcase 1: $\alpha = T, s \circ \alpha = T$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(T) = (S_\theta, \emptyset)$ and $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi^s_\theta \circ (S_\theta, \emptyset) = (C, D)$ where $C = \{\overline{q} \in S_\theta : \psi^s_\theta(\overline{q}) \in S_\theta\}$ and $D = \emptyset$. Thus it suffices to show that $C = S_\theta$. Let $\overline{q} \in S_\theta$ that is $(q, \perp) \notin E_\theta$

$$\begin{aligned}
&\Rightarrow (q \circ T, T) \in \theta \qquad \text{(using Proposition 5.2.3(iv))} \\
&\Rightarrow (q \circ (s \circ T), T) \in \theta
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow ((q \cdot s) \circ T, T) \in \theta && \text{(using (5.7))}\\
&\Rightarrow (q \cdot s, \perp) \notin E_\theta && \text{(using Proposition 5.2.3(iv))}\\
&\Rightarrow \overline{q \cdot s} \in S_\theta \\
&\Rightarrow \psi_\theta^s(\overline{q}) \in S_\theta \\
&\Rightarrow \overline{q} \in C.
\end{aligned}
$$

Thus $C = S_\theta$.

Subcase 2: $\alpha = F, s \circ \alpha = F$. Then $\rho_\theta(s \circ \alpha) = \rho_\theta(F) = (\emptyset, S_\theta)$ and $\phi_\theta(s) \circ \rho_\theta(\alpha) = \psi_\theta^s \circ (\emptyset, S_\theta) = (C, D)$ where $C = \emptyset$ and $D = \{\overline{q} \in S_\theta : \psi_\theta^s(\overline{q}) \in S_\theta\}$. The proof follows along similar lines as above. In order to show that $D = S_\theta$ consider $\overline{q} \in S_\theta$ that is $(q, \perp) \notin E_\theta$

$$
\begin{aligned}
&\Rightarrow (q \circ F, F) \in \theta && \text{(using Proposition 5.2.3(iv))}\\
&\Rightarrow (q \circ (s \circ F), F) \in \theta \\
&\Rightarrow ((q \cdot s) \circ F, F) \in \theta && \text{(using (5.7))}\\
&\Rightarrow (q \cdot s, \perp) \notin E_\theta && \text{(using Proposition 5.2.3(iv))}\\
&\Rightarrow \overline{q \cdot s} \in S_\theta \\
&\Rightarrow \psi_\theta^s(\overline{q}) \in S_\theta \\
&\Rightarrow \overline{q} \in D.
\end{aligned}
$$

Hence $D = S_\theta$ which completes the proof.

Thus $(\phi_\theta, \rho_\theta)$ is a homomorphism of C-monoids. □

Proposition 5.2.7. *For all $\alpha \in M$ the following statements hold:*

(i) $\rho_\theta(\alpha) = (S_\theta, \emptyset) \Rightarrow (\alpha, T) \in \theta$.

(ii) $\rho_\theta(\alpha) = (\emptyset, S_\theta) \Rightarrow (\alpha, F) \in \theta$.

Proof.

(i) If $\alpha = T$ then the result is obvious. Suppose that $\alpha \neq T$ and $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha) = (S_\theta, \emptyset)$. It follows that $(t \circ \alpha, T) \in \theta$ for all $\overline{t} \in S_\theta$. Using Proposition 5.2.3(vi) and (5.4) we have $\overline{1} \in S_\theta$ and so $(1 \circ \alpha, T) = (\alpha, T) \in \theta$.

(ii) Along similar lines if $\alpha \neq F$ then $\rho_\theta(\alpha) = (A_\theta^\alpha, B_\theta^\alpha) = (\emptyset, S_\theta)$ gives $(t \circ \alpha, F) \in \theta$ for all $\overline{t} \in S_\theta$. Using Proposition 5.2.3(vi) and (5.4) we have $\overline{1} \in S_\theta$ and so $(1 \circ \alpha, F) = (\alpha, F) \in \theta$.

□

Lemma 5.2.8. *For every $s, t \in S_\perp$ where $s \neq t$ there exists a maximal congruence θ on M such that $\phi_\theta(s) \neq \phi_\theta(t)$.*

Proof. Using Lemma 3.2.2 we have $\bigcap E_\theta = \Delta_{S_\perp}$ and so since $s \neq t$ there exists a maximal congruence θ on M such that $(s, t) \notin E_\theta$, i.e., $\overline{s} \neq \overline{t}$. For this θ, consider $\phi_\theta : S_\perp \to \mathcal{T}_o(S_{\theta_\perp})$. Then $\phi_\theta(s) = \psi_\theta^s$, $\phi_\theta(t) = \psi_\theta^t$. For $\overline{1} \in S_{\theta_\perp}$ we have $\psi_\theta^s(\overline{1}) = \overline{1 \cdot s} = \overline{s}$ while $\psi_\theta^t(\overline{1}) = \overline{1 \cdot t} = \overline{t}$. Since $\overline{s} \neq \overline{t}$ it follows that $\phi_\theta(s) \neq \phi_\theta(t)$. □

Lemma 5.2.9. *For every $\alpha, \beta \in M$ where $\alpha \neq \beta$ there exists a maximal congruence θ on M such that $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$.*

Proof. Using Remark 3.2.3 since $\alpha \neq \beta$ there exists a maximal congruence θ on M such that $(\alpha, \beta) \notin \theta$. We show that $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$. If α or β is in $\{T, F\}$ but $\rho_\theta(\alpha) = \rho_\theta(\beta)$ then using Proposition 5.2.7 we have $(\alpha, \beta) \in \theta$, a contradiction. In the case where $\alpha, \beta \notin \{T, F\}$ we show that

$$(A_\theta^\alpha, B_\theta^\alpha) \neq (A_\theta^\beta, B_\theta^\beta)$$

by showing that either $A_\theta^\alpha \neq A_\theta^\beta$ or that $B_\theta^\alpha \neq B_\theta^\beta$. Owing to Proposition 5.2.2 it suffices to consider the following three cases:

Case I: $(\alpha, T) \in \theta$. Note that Proposition 5.2.3(vi) gives $\overline{1} \in S_\theta$. Thus we have $\overline{1} \in S_\theta$ for which $1 \circ \alpha = \alpha \in \overline{T}^\theta$ and so $\overline{1} \in A_\theta^\alpha$. However $\overline{1} \notin A_\theta^\beta$ since $(\alpha, \beta) \notin \theta$.

Case II: $(\alpha, F) \in \theta$. Along similar lines for $\overline{1} \in S_\theta$ we have $1 \circ \alpha = \alpha \in \overline{F}^\theta$ and so $\overline{1} \in B_\theta^\alpha$. It is clear that $\overline{1} \notin B_\theta^\beta$ since $(\alpha, \beta) \notin \theta$.

Case III: $(\alpha, U) \in \theta$. In view of Proposition 5.2.2 it suffices to consider the following cases:

Subcase 1: $(\beta, T) \in \theta$. As earlier we have $\overline{1} \in A_\theta^\beta \setminus A_\theta^\alpha$.

Subcase 2: $(\beta, F) \in \theta$. It is clear that $\overline{1} \in B_\theta^\beta \setminus B_\theta^\alpha$.

Thus $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$ which completes the proof. □

5.2.3 Embedding into a functional C-monoid

Let $\{\theta\}$ be the collection of all maximal congruences of M. Define the set X to be the disjoint union of S_θ taken over all maximal congruences of M, written

$$X = \bigsqcup_\theta S_\theta \tag{5.11}$$

Set $X_\perp = X \cup \{\perp\}$ with base point $\perp \notin X$. For notational convenience we use the same symbol $\perp$ in $X_\perp$ as well as in $S_\perp$. Which $\perp$ we are referring to will be clear from the context of the statement.

In this subsection we obtain monomorphisms $\phi : S_\perp \to \mathcal{T}_o(X_\perp)$ and $\rho : M \to \mathfrak{Z}^X$, using which we establish that $(S_\perp, M)$ can be embedded into the functional C-monoid $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$.

Remark 5.2.10.

(i) Let $q \in S$ be fixed. For different θ's the representation of classes $\overline{q}^{E_\theta}$'s are different in the disjoint union X of S_θ's.

(ii) Let $\{A_\lambda\}, \{B_\lambda\}$ be two families of sets indexed over Λ. Then $\bigsqcup_\lambda (A_\lambda \cap B_\lambda) = \left(\bigsqcup_\lambda A_\lambda\right) \cap \left(\bigsqcup_\lambda B_\lambda\right)$ and $\bigsqcup_\lambda (A_\lambda \cup B_\lambda) = \left(\bigsqcup_\lambda A_\lambda\right) \cup \left(\bigsqcup_\lambda B_\lambda\right)$.

Notation 5.2.11.

(i) For the pair of sets (A, B), we denote by $\pi_1(A, B)$ the first component A, and by $\pi_2(A, B)$ the second component B.

(ii) For a family of pairs of sets (A_λ, B_λ) where $\lambda \in \Lambda$ we denote by $\bigsqcup_\lambda (A_\lambda, B_\lambda)$ the pair of sets $\left(\bigsqcup_\lambda A_\lambda, \bigsqcup_\lambda B_\lambda\right)$.

Lemma 5.2.12. *Consider $\phi : S_\perp \to \mathcal{T}_o(X_\perp)$ given by*

$$(\phi(s))(x) = \begin{cases} (\phi_\theta(s))(\overline{q}^{E_\theta}), & \text{if } x = \overline{q}^{E_\theta} \in S_\theta \text{ and } (\phi_\theta(s))(\overline{q}^{E_\theta}) \neq \overline{\perp}^{E_\theta}; \\ \perp, & \text{otherwise.} \end{cases}$$

Then ϕ is a monoid monomorphism that maps the zero (and base point) of $S_\perp$ to that of $\mathcal{T}_o(X_\perp)$, that is $\perp \mapsto \zeta_\perp$.

Proof. It is clear that ϕ is well-defined and that $\phi(s) \in \mathcal{T}_o(X_\perp)$ since $(\phi(s))(\perp) = \perp$.

Claim: ϕ is injective. Let $s \neq t \in S_\perp$. Using Lemma 5.2.8 there exists a maximal congruence θ on M such that $\phi_\theta(s) \neq \phi_\theta(t)$. Hence there exists a $\overline{q}^{E_\theta} (\neq \overline{\perp}^{E_\theta})$ such that $(\phi_\theta(s))(\overline{q}) \neq (\phi_\theta(t))(\overline{q})$. By extrapolation it follows that $(\phi(s))(\overline{q}) \neq (\phi(t))(\overline{q})$ and so $\phi(s) \neq \phi(t)$.

Claim: $\phi(\perp) = \zeta_\perp$. Using Proposition 5.2.4 we have $\phi_\theta(\perp) = \zeta_{\overline{\perp}^{E_\theta}}$ for all θ and so by definition $(\phi(\perp))(x) = \perp$ for all $x \in X_\perp$.

Claim: $\phi(1) = id_{X_\perp}$. It is clear that $(\phi(1))(\perp) = \perp$. Consider $\overline{q} \in X$ that is $\overline{q}^{E_\theta} \in S_\theta$ for some θ. Then by Proposition 5.2.4 we have $(\phi(1))(\overline{q}^{E_\theta}) = (\phi_\theta(1))(\overline{q}^{E_\theta}) = \overline{q}^{E_\theta}$ and hence $\phi(1) = id_{X_\perp}$.

Claim: $\phi(s \cdot t) = \phi(s) \cdot \phi(t)$. Clearly $(\phi(s \cdot t))(\perp) = \perp = (\phi(s) \cdot \phi(t))(\perp)$. Let $\overline{q} \in X$ that is $\overline{q}^{E_\theta} \in S_\theta$ for some θ. Suppose that $(\phi(s \cdot t))(\overline{q}) = \perp$ so that $(\phi_\theta(s \cdot t))(\overline{q}) = \overline{\perp}$

$$\begin{aligned} &\Rightarrow \ ((\phi_\theta(s) \cdot \phi_\theta(t))(\overline{q}) = \overline{\perp} \quad \text{(using Proposition 5.2.4)} \\ &\Rightarrow \ \phi_\theta(t)(\phi_\theta(s)(\overline{q})) = \overline{\perp} \\ &\Rightarrow \ \phi(t)(\phi_\theta(s)(\overline{q})) = \perp \end{aligned}$$

Noting that there are only two possibilities for $\phi(s)(\overline{q})$ we see that if $\phi(s)(\overline{q}) = \phi_\theta(s)(\overline{q})$ then we are through. On the other hand if $\phi(s)(\overline{q}) = \perp$ that is $\phi_\theta(s)(\overline{q}) = \overline{\perp}$ then we have $(\phi(s \cdot t))(\overline{q}) = \perp = (\phi(s) \cdot \phi(t))(\overline{q})$ which completes the proof in this case.

Consider the case where $(\phi(s \cdot t))(\overline{q}) \neq \perp$. Using Proposition 5.2.4 it follows that $(\phi(s \cdot t))(\overline{q}) = (\phi_\theta(s \cdot t))(\overline{q}) = (\phi_\theta(s) \cdot \phi_\theta(t))(\overline{q}) = \phi_\theta(t)(\phi_\theta(s)(\overline{q}))$ and so $(\phi_\theta(s))(\overline{q}) \neq \overline{\perp}$. Consequently $\phi(t)(\phi(s)(\overline{q})) = \phi_\theta(t)(\phi_\theta(s)(\overline{q}))$ since $(\phi_\theta(s))(\overline{q}) \neq \overline{\perp}$. It follows that $(\phi(s \cdot t))(\overline{q}) = (\phi(s) \cdot \phi(t))(\overline{q})$ which completes the proof. □

Lemma 5.2.13. *The function $\rho : M \to \mathfrak{Z}^X$ defined by*

$$\rho(\alpha) = \sqcup_\theta \rho_\theta(\alpha)$$

is a monomorphism of C-algebras with T, F, U.

Proof. *Claim: ρ is well defined.* Let $\alpha \in M$. Using Remark 5.2.10(i) we have $\pi_1(\rho(\alpha)) \cap \pi_2(\rho(\alpha)) = \emptyset$ due to the distinct representation of equivalence classes. Also by Proposition 5.2.5 we have $\pi_1(\rho_\theta(\alpha)), \pi_2(\rho_\theta(\alpha)) \subseteq S_\theta$ and so $\perp \notin \pi_1(\rho(\alpha)) \cup \pi_2(\rho(\alpha))$ that is $\rho(\alpha)$ is can be identified with a pair of sets over X.

Claim: ρ is injective. Let $\alpha \neq \beta \in M$. By Lemma 5.2.9 there exists a θ such that $\rho_\theta(\alpha) \neq \rho_\theta(\beta)$. Without loss of generality we infer that there exists a $\overline{q}^{E_\theta} \in \pi_1(\rho_\theta(\alpha)) \setminus \pi_1(\rho_\theta(\beta))$. Since $\rho(\alpha)$ is formed by taking the disjoint union of the individual images under $\rho_\theta(\alpha)$, using Remark 5.2.10(i) we can say that $\overline{q} \in \pi_1(\rho(\alpha)) \setminus \pi_1(\rho(\beta))$ that is $\rho(\alpha) \neq \rho(\beta)$.

Claim: ρ preserves the constants T, F, U. It follows easily from Proposition 5.2.5 that $\rho(T) = (X, \emptyset)$, $\rho(F) = (\emptyset, X)$ and $\rho(U) = (\emptyset, \emptyset)$.

Claim: $\rho(\neg\alpha) = \neg(\rho(\alpha))$. If $\alpha \in \{T, F\}$ then the result is obvious. If $\alpha \notin \{T, F\}$ then $\neg\alpha \notin \{T, F\}$. Using Proposition 5.2.5 we have $\rho(\neg\alpha) = (\sqcup A_\theta^{\neg\alpha}, \sqcup B_\theta^{\neg\alpha}) = (\sqcup B_\theta^{\alpha}, \sqcup A_\theta^{\alpha})$. Thus $\rho(\neg\alpha) = (\sqcup B_\theta^{\alpha}, \sqcup A_\theta^{\alpha}) = \neg(\rho(\alpha))$.

Claim: $\rho(\alpha \wedge \beta) = \rho(\alpha) \wedge \rho(\beta)$. In view of Remark 5.2.10(ii) we have $\sqcup((A_\lambda, B_\lambda) \wedge (C_\lambda, D_\lambda)) = (\sqcup A_\gamma, \sqcup B_\gamma) \wedge (\sqcup C_\gamma, \sqcup D_\gamma)$ for the family of pairs of sets $(A_\lambda, B_\lambda), (C_\lambda, D_\lambda)$ where $\lambda \in \Lambda$ over X. In view of the above and Proposition 5.2.5 we have $\sqcup\rho_\theta(\alpha \wedge \beta) = \sqcup(\rho_\theta(\alpha) \wedge \rho_\theta(\beta)) = (\sqcup\rho_\theta(\alpha)) \wedge (\sqcup\rho_\theta(\beta)) = \rho(\alpha) \wedge \rho(\beta)$ which completes the proof. □

Lemma 5.2.14. *The pair (ϕ, ρ) is a C-monoid monomorphism from $(S_\perp, M)$ to the functional C-monoid $\left(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\right)$.*

Proof. In view of Lemma 5.2.12 and Lemma 5.2.13 it suffices to show $\phi(\alpha[s, t]) = (\rho(\alpha))[\phi(s), \phi(t)]$ and $\rho(s \circ \alpha) = \phi(s) \circ \rho(\alpha)$.

In order to show that $\phi(\alpha[s, t]) = (\rho(\alpha))[\phi(s), \phi(t)]$ we show that $\phi(\alpha[s, t])(x) = (\rho(\alpha))[\phi(s), \phi(t)](x)$ for all $x \in X_\perp$. Thus we have the following cases:

Case I: $x = \perp$. It is clear that $\phi(\alpha[s, t])(\perp) = \perp = (\rho(\alpha))[\phi(s), \phi(t)](\perp)$ since $\pi_1(\rho(\alpha)), \pi_2(\rho(\alpha)) \subseteq X$ and $\perp \notin X$.

Case II: $x \in X$. Consider $\overline{q} \in X$ that is $\overline{q}^{E_\theta} \in S_\theta$ for some θ. We have the following subcases:

Subcase 1: $\phi(\alpha[s,t])(\overline{q}) = \bot$. then $\phi_\theta(\alpha[s,t])(\overline{q}) = \overline{\bot}$ and so using Lemma 5.2.6 we have $\phi_\theta(\alpha[s,t])(\overline{q}) = \overline{\bot} = (\rho_\theta(\alpha))[\phi_\theta(s), \phi_\theta(t)](\overline{q})$. It follows that either $\overline{q} \notin \pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha))$ or that $\overline{q} \in \pi_1(\rho_\theta(\alpha))$ and $\phi_\theta(s)(\overline{q}) = \overline{\bot}$ or, similarly, that $\overline{q} \in \pi_2(\rho_\theta(\alpha))$ and $\phi_\theta(t)(\overline{q}) = \overline{\bot}$. Thus we have the following:

$\overline{q} \notin \pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha))$: In view of Remark 5.2.10(i) it follows that $\overline{q} \notin \pi_1(\rho(\alpha)) \cup \pi_2(\rho(\alpha))$ and so $(\rho(\alpha))[\phi(s), \phi(t)](\overline{q}) = \bot$.

$\overline{q} \in \pi_1(\rho_\theta(\alpha))$ and $\phi_\theta(s)(\overline{q}) = \overline{\bot}$: Then $\overline{q} \in \pi_1(\rho(\alpha))$ and $\phi(s)(\overline{q}) = \bot$ and so $(\rho(\alpha))[\phi(s), \phi(t)](\overline{q}) = \bot$.

$\overline{q} \in \pi_2(\rho_\theta(\alpha))$ and $\phi_\theta(t)(\overline{q}) = \overline{\bot}$: Along similar lines we have $(\rho(\alpha))[\phi(s), \phi(t)](\overline{q}) = \bot$.

Subcase 2: $\phi(\alpha[s,t])(\overline{q}) \neq \bot$. Then $\phi(\alpha[s,t])(\overline{q}) = \phi_\theta(\alpha[s,t])(\overline{q})$ and so using Lemma 5.2.6 we have $\phi(\alpha[s,t])(\overline{q}) = (\rho_\theta(\alpha))[\phi_\theta(s), \phi_\theta(t)](\overline{q})$. It follows that

$$\phi(\alpha[s,t])(\overline{q}) = (\rho_\theta(\alpha))[\phi_\theta(s), \phi_\theta(t)](\overline{q}) = \begin{cases} \phi_\theta(s)(\overline{q}), & \text{if } \overline{q} \in \pi_1(\rho_\theta(\alpha)); \\ \phi_\theta(t)(\overline{q}), & \text{if } \overline{q} \in \pi_2(\rho_\theta(\alpha)); \\ \bot, & \text{otherwise.} \end{cases}$$

$\overline{q} \in \pi_1(\rho_\theta(\alpha))$: It follows that $\overline{q} \in \pi_1(\rho(\alpha))$ and so $(\rho(\alpha))[\phi(s), \phi(t)](\overline{q}) = \phi(s)(\overline{q})$. Note that $\phi_\theta(s)(\overline{q}) \neq \overline{\bot}$ else $\phi(\alpha[s,t])(\overline{q}) = \bot$, a contradiction. Thus $\phi(s)(\overline{q}) = \phi_\theta(s)(\overline{q})$ so that $\phi(\alpha[s,t])(\overline{q}) = (\rho(\alpha))[\phi(s), \phi(t)](\overline{q})$.

$\overline{q} \in \pi_2(\rho_\theta(\alpha))$: The proof follows along similar lines as above.

$\overline{q} \notin (\pi_1(\rho_\theta(\alpha)) \cup \pi_2(\rho_\theta(\alpha)))$: This case cannot occur since we assumed that $\phi(\alpha[s,t])(\overline{q}) \neq \bot$.

Thus $\phi(\alpha[s,t]) = (\rho(\alpha))[\phi(s), \phi(t)]$.

We now show that $\rho(s \circ \alpha) = \phi(s) \circ \rho(\alpha)$. In order to prove this we proceed by showing that

$$\pi_i(\rho(s \circ \alpha)) = \pi_i(\phi(s) \circ \rho(\alpha))$$

for $i \in \{1, 2\}$.

Let $\overline{q} \in \pi_1(\rho(s \circ \alpha)) = \sqcup \pi_1(\rho_\theta(s \circ \alpha))$. Then $\overline{q}^{E_\theta} \in S_\theta$ for some θ and $\overline{q}^{E_\theta} \in \pi_1(\rho_\theta(s \circ \alpha))$

$$\begin{aligned}
&\Rightarrow \quad \overline{q}^{E_\theta} \in \pi_1(\phi_\theta(s) \circ \rho_\theta(\alpha)) && \text{(using Lemma 5.2.6)}\\
&\Rightarrow \quad \phi_\theta(s)(\overline{q}^{E_\theta}) \in \pi_1(\rho_\theta(\alpha)) \subseteq S_\theta\\
&\Rightarrow \quad \phi_\theta(s)(\overline{q}^{E_\theta}) \neq \overline{\bot}\\
&\Rightarrow \quad \phi(s)(\overline{q}^{E_\theta}) = \phi_\theta(s)(\overline{q}^{E_\theta})\\
&\Rightarrow \quad \phi(s)(\overline{q}^{E_\theta}) \in \sqcup \pi_1(\rho_\theta(\alpha))\\
&\Rightarrow \quad \phi(s)(\overline{q}^{E_\theta}) \in \pi_1(\rho(\alpha))\\
&\Rightarrow \quad \overline{q}^{E_\theta} \in \pi_1(\phi(s) \circ \rho(\alpha))
\end{aligned}$$

and so $\pi_1(\rho(s \circ \alpha)) \subseteq \pi_1(\phi(s) \circ \rho(\alpha))$.

For the reverse inclusion assume that $\overline{q} \in \pi_1(\phi(s) \circ \rho(\alpha))$. Consequently we have $\overline{q}^{E_\theta} \in S_\theta$ for some θ and $\phi(s)(\overline{q}^{E_\theta}) \in \pi_1(\rho(\alpha)) \subseteq X$

$$\begin{aligned}
&\Rightarrow \quad \phi(s)(\overline{q}^{E_\theta}) \neq \bot\\
&\Rightarrow \quad \phi(s)(\overline{q}^{E_\theta}) = \phi_\theta(s)(\overline{q}^{E_\theta})(\neq \overline{\bot}^{E_\theta})\\
&\Rightarrow \quad \phi_\theta(s)(\overline{q}^{E_\theta}) \in \pi_1(\rho_\theta(\alpha)) && \text{(using Remark 5.2.10(i))}\\
&\Rightarrow \quad \overline{q}^{E_\theta} \in \pi_1(\phi_\theta(s) \circ \rho_\theta(\alpha))\\
&\Rightarrow \quad \overline{q}^{E_\theta} \in \pi_1(\rho_\theta(s \circ \alpha)) && \text{(using Lemma 5.2.14)}\\
&\Rightarrow \quad \overline{q}^{E_\theta} \in \sqcup \pi_1(\rho_\theta(s \circ \alpha)) = \pi_1(\rho(s \circ \alpha))
\end{aligned}$$

from which it follows that $\pi_1(\phi(s) \circ \rho(\alpha)) \subseteq \pi_1(\rho(s \circ \alpha))$. Proceeding along exactly the same lines we can show that $\pi_2(\rho(s \circ \alpha)) = \pi_2(\phi(s) \circ \rho(\alpha))$ which completes the proof. □

5.2.4 Proof of Theorem 5.2.1

Let $\{\theta\}$ be the collection of all maximal congruences of M. Consider the set X as in (5.11). The functions $\phi : S_\perp \to \mathcal{T}_o(X_\perp)$ and $\rho : M \to \mathfrak{Z}^X$ as defined in Lemma 5.2.12 and Lemma 5.2.13, respectively, are monomorphisms. Further, by Lemma 5.2.14, the pair (ϕ, ρ) is a monomorphism from $(S_\perp, M)$ to the functional C-monoid $\big(\mathcal{T}_o(X_\perp), \mathfrak{Z}^X\big)$. From the construction of X it is also evident that if M and $S_\perp$ are finite then there are only finitely many maximal congruences θ on M and finitely many equivalence classes E_θ on $S_\perp$ and so X must be finite.

Corollary 5.2.15. *An identity (quasi-identity) is satisfied in every C-monoid $(S_\perp, M)$ where M is an ada if and only if it is satisfied in all functional C-monoids.*

Though the following identity is derivable from the axioms (5.2) – (5.10) in a straightforward manner, we can alternatively achieve the following result in view of Corollary 5.2.15 and (5.1).

Corollary 5.2.16. *In every C-monoid $(S_\perp, M)$ where M is an ada we have* $(f \circ T)[f, f] = f$.

5.3 Conclusion

The axiomatization of systems based on various program constructs is extremely useful in the study of program semantics in general, and in establishing program equivalence in particular. While many authors have studied the axiomatization of the `if-then-else` construct, we have considered the case where the programs and tests could possibly be non-halting. In this connection, we introduce the notion of C-sets to axiomatize the systems of `if-then-else` in which the tests are drawn from an abstract C-algebra. When the C-algebra is an ada, we obtain a subdirect representation of C-sets through basic C-sets. This in turn establishes the completeness of the axiomatization and paves the way for determining the equivalence of programs under consideration through basic C-sets. Further, in order to axiomatize `if-then-else` systems with the equality test, we have extended the concept of C-sets to agreeable C-sets and have obtained similar results.

We have then extended the notion of C-sets to that of C-monoids which include the composition of programs as well as composition of programs with tests. For the class of C-monoids where the C-algebra is an ada we obtain a Cayley-type theorem which exhibits the embedding of such C-monoids into functional C-monoids. Using this, we obtain a mechanism to determine the equivalence of programs through functional C-monoids.

As future work, one may investigate a complete axiomatization of the systems under consideration in which all the axioms are equations. It is also desirable to extend the results to the general case of C-sets without restricting the C-algebra to be an ada. Further we would like to achieve such a representation for the general class of C-monoids with no restriction on the C-algebra, which can be considered as future work. Note that the term $f \circ T$ in the standard functional model of a C-monoid represents the aspect of the domain of the function, as used in Jackson and Stokes [2015] and in Desharnais et al. [2009]. It is interesting to see what relation these two concepts might have with one another.

Part II

Structure of C-algebras

6

Applications of if-then-else

In Section 6.1 we introduce a notion of annihilators in C-algebras through the `if-then-else` action. The notion of Galois connection yields a closure operator in terms of annihilator, which in turn, yields closed sets. In Section 6.2 we characterise the collection of closed sets in the C-algebra $\mathfrak{3}^X$ (cf. Theorem 6.2.1) and show that this collection forms a complete Boolean algebra (cf. Theorem 6.2.5). We also obtain a classification of the elements of $\mathfrak{3}^X$ where the elements of the Boolean algebra $\mathfrak{2}^X$ form a distinct class (cf. Theorem 6.2.6). In Section 6.3 we define a notion of idempotent elements and that of idempotent operations through the `if-then-else` action and study their properties. We conclude this chapter with Section 6.4 by listing various unanswered problems.

In this chapter, unless stated otherwise, M is a C-algebra with T, F, U.

6.1 Annihilators

Rao [2013] introduced the concept of annihilator ideals in a C-algebra M by defining the annihilator of an element $a \in M$ as the set $\{b \in M : b \wedge a \text{ is a left-zero of } M\}$. In this section we show that the presence of the `if-then-else` action on the C-algebra M delineates a mechanism to define a notion of annihilators akin to the concept of annihilators in modules.

Henceforth we consider the C-set (M, M) where M is a C-algebra with T, F, U with the action $\alpha[\![\beta, \gamma]\!] = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma)$. Since $\alpha[\![\_\,,\_\,]\!]$ can be treated as a binary operation for each $\alpha \in M$ we define the notion of the annihilator of an element $a \in M$ to be all the binary operations $\alpha[\![\_\,,\_\,]\!]$ which map the pair (a, a) to U. We state the definition explicitly in Definition 6.1.1. Hereafter we use $\alpha, \beta, \gamma, \delta$ to denote elements of M treated as binary operations while a, b, c is used otherwise. The elements of the C-algebra $\mathfrak{Z}^X$ will also be denoted by $\alpha, \beta, \gamma, \delta$. Recall that the constants T, F, U of $M \leq \mathfrak{Z}^X$ are denoted by $\mathbf{T}, \mathbf{F}, \mathbf{U}$ respectively (cf. Notation 1.4.8).

Definition 6.1.1. For $a \in M$,

$$Ann(a) = \{\alpha \in M : \alpha[\![a, a]\!] = U\}.$$

We overload the notation of Ann in a natural manner to subsets of M.

Definition 6.1.2. The operator $Ann : \wp(M) \to \wp(M)$ is given by

$$Ann(S) = \bigcap_{a \in S} Ann(a).$$

In other words $Ann(S) = \{\alpha \in M : \text{ for all } a \in S, \alpha[\![a, a]\!] = U\}$.

It is desirable that the operator Ann has some fundamental properties. For instance every element $\alpha \in M$ should annihilate U. Conversely every element must be annihilated by U. In the following we ascertain these and other properties of the operator Ann which may be deemed natural.

Proposition 6.1.3. *The following hold in any C-algebra with T, F, U:*

(i) $Ann(U) = M$.

(ii) *For any* $a \in M$, $U \in Ann(a)$.

(iii) *For any* $a \in M_{\#}$, $Ann(a) = \{U\}$.

(iv) $Ann(M) = \{U\}$.

(v) $b \in Ann(a) \Leftrightarrow a \in Ann(b)$.

(vi) $B \subseteq Ann(A) \Leftrightarrow A \subseteq Ann(B)$.

(vii) $A \subseteq B \Rightarrow Ann(B) \subseteq Ann(A)$.

Proof.

(i) Using Proposition 2.2.1(i) for the C-set (M, M) we have $\alpha[\![U, U]\!] = U$ for each $\alpha \in M$ so that $Ann(U) = M$.

(ii) Using (2.1) on the C-set (M, M) we have $U[\![a, a]\!] = U$ so that $U \in Ann(a)$ for all $a \in M$.

(iii) Using Proposition 6.1.3(ii) it is clear that $\{U\} \subseteq Ann(a)$. For the reverse inclusion since $M \leq \mathfrak{Z}^X$ for some set X, for $a \in M_{\#}$ we have $a(x) \in \{T, F\}$ for all $x \in X$. Suppose that $\alpha(x_o) \in \{T, F\}$ for some $x_o \in X$ so that $(\alpha(x_o) \wedge a(x_o)) \vee (\neg\alpha(x_o) \wedge a(x_o)) \in \{T, F\}$. However $Ann(a) = \{\alpha \in M : (\alpha(x) \wedge a(x)) \vee (\neg\alpha(x) \wedge a(x)) = \mathbf{U}$ for all $x \in X\}$, a contradiction. Consequently $\alpha(x) = U$ for all $x \in X$ hence $\alpha = \mathbf{U}$.

(iv) Since $Ann(M) = \bigcap_{a \in M} Ann(a)$, using Proposition 6.1.3(iii) we have $Ann(M) = \{U\}$.

(v) Consider $M \leq \mathfrak{Z}^X$ for some set X and $b \in Ann(a)$ so that $(b(x) \wedge a(x)) \vee (\neg b(x) \wedge a(x)) = U$ for all $x \in X$. For $x \in X$ we have the following cases for $(a(x) \wedge b(x)) \vee (\neg a(x) \wedge b(x))$:

$b(x) = T$: Since $(b(x) \wedge a(x)) \vee (\neg b(x) \wedge a(x)) = U$ we have $(T \wedge a(x)) \vee (F \wedge a(x)) = a(x) \vee F = a(x) = U$. Thus $(a(x) \wedge b(x)) \vee (\neg a(x) \wedge b(x)) = (U \wedge b(x)) \vee (U \wedge b(x)) = U$.

$b(x) = F$: Along similar lines since $(b(x) \wedge a(x)) \vee (\neg b(x) \wedge a(x)) = U$ we have $(F \wedge a(x)) \vee (T \wedge a(x)) = F \vee a(x) = a(x) = U$. Hence $(a(x) \wedge b(x)) \vee (\neg a(x) \wedge b(x)) = (U \wedge b(x)) \vee (U \wedge b(x)) = U$.

$b(x) = U$: Then $(a(x) \wedge b(x)) \vee (\neg a(x) \wedge b(x)) = (a(x) \wedge U) \vee (\neg a(x) \wedge U) = U$ for $a(x) \in \{T, F, U\}$.

Hence $a[\![b, b]\!] = U$ so that $a \in Ann(b)$. The converse follows along similar lines.

(vi) This follows as a direct consequence of Proposition 6.1.3(v).

(vii) Let $A \subseteq B$, $\beta \in Ann(B)$ and $a \in A$. Since $\beta \in Ann(b)$ for each $b \in B$ and $a \in A \subseteq B$ we have $\beta \in Ann(a)$. Thus $Ann(B) \subseteq Ann(A)$.

□

We have the following result which follows from Proposition 6.1.3(vi) and Proposition 6.1.3(vii).

Proposition 6.1.4. *The pair (Ann, Ann) is an antitone Galois connection.*

The following are consequences of Theorem 1.2.8 and Proposition 6.1.4.

Corollary 6.1.5. *The function $Ann^2 : \wp(M) \to \wp(M)$ is a closure operator.*

Corollary 6.1.6. *Consider* $Ann^3 : \wp(M) \to \wp(M)$*. Then* $Ann^3 = Ann$*.*

Using Theorem 1.2.3 and Corollary 6.1.5 we have the following.

Corollary 6.1.7. *The collection of closed sets* $\mathfrak{I} = \{I \subseteq M : Ann^2(I) = I\}$ *forms a complete lattice.*

The collection of closed sets $\mathfrak{I}$ has a distinct property.

Proposition 6.1.8. *Let* $I \in \mathfrak{I}$ *such that* $I \neq M$*. Then* $I \cap M_{\#} = \emptyset$*.*

Proof. Suppose that there exists $a \in I \cap M_{\#}$ where $I \in \mathfrak{I}$ such that $I \neq M$. For any $\alpha \in Ann(a)$ we have $\alpha = U$ using Proposition 6.1.3(iii) since $a \in M_{\#}$. Further, since $Ann(I) = \bigcap_{a \in I} Ann(a)$ we have $Ann(I) = \{U\}$. Using Proposition 6.1.3(i) we have $Ann^2(I) = Ann(U) = M$. It follows that $I = Ann^2(I) = M$ since $I \in \mathfrak{I}$, which is a contradiction. Thus $I \cap M_{\#} = \emptyset$. □

Remark 6.1.9. We established that Ann^2 is a closure operator. In fact, the following shows that, in general, Ann^2 need not be an algebraic closure operator.

Let $M = \mathbf{3}^{\mathbb{N}}$ where $\mathbb{N}$ is the set of natural numbers, viz., $\mathbb{N} = \{1, 2, 3, \ldots\}$. Consider the subset $A \subseteq M$ given by

$$A = \{(T, U, U, U, \ldots), (U, T, U, U, \ldots), (U, U, T, U, \ldots), \ldots\}.$$

Then

$$\begin{aligned} Ann(A) &= \{(U, x_2, x_3, x_4, \ldots) : x_i \in \{T, F, U\} \text{ for } i \in \mathbb{N} \setminus \{1\}\} \\ &\cap \{(y_1, U, y_3, y_4, \ldots) : y_i \in \{T, F, U\} \text{ for } i \in \mathbb{N} \setminus \{2\}\} \\ &\cap \{(z_1, z_2, U, z_4, \ldots) : z_i \in \{T, F, U\} \text{ for } i \in \mathbb{N} \setminus \{3\}\} \cap \ldots \\ &= \{(U, U, U, U, \ldots)\} \end{aligned}$$

Thus $Ann^2(A) = M$.

If Ann^2 is an algebraic closure operator then for $i \in I$ where I is some index set, consider $B_i \subseteq A$ where B_i is finite for each i and $\bigcup Ann^2(B_i) = Ann^2(A) = M$. Then for each i, B_i comprises elements of the form $\beta(j) = \begin{cases} T, & \text{for fixed } j = k_\beta; \\ U, & \text{otherwise.} \end{cases}$
Along similar lines as above it follows that $Ann(B_i)$ will have elements whose coordinates do not take value U at infinitely many places. Thus all the elements in $Ann^2(B_i)$ will have infinitely many coordinates that take value U. If $\bigcup Ann^2(B_i) = M$ then the element $\mathbf{T} = (T, T, T, T, \ldots)$ must belong in some $Ann^2(B_i)$, a contradiction since $\mathbf{T}$ does not take value U in infinitely many coordinates. Thus Ann^2 is not an algebraic closure operator.

6.2 Closed sets of $\mathfrak{Z}^X$

In this section we consider the C-algebra $\mathfrak{Z}^X$ and give a characterisation of the closed sets in $\mathfrak{I}$ with respect to operator Ann^2. To that aim in this section we consider the C-algebra in question to be precisely $\mathfrak{Z}^X$ for some set X.

Theorem 6.2.1. *Let $I \subseteq \mathfrak{Z}^X$. $I \in \mathfrak{I} \Leftrightarrow$ there exists $Y \subseteq X$ such that*

(P1) *for all $\alpha \in I$, for all $y \in Y$, $\alpha(y) = U$*

(P2) *for all $f : Y^c \to \mathfrak{Z}$ there exists $\alpha \in I$ such that $\alpha|_{Y^c} = f$.*

Proof. ($\Leftarrow$) Let $I \subseteq \mathfrak{Z}^X$ such that there exists $Y \subseteq X$ which satisfies both the given conditions (P1) and (P2). We show that $Ann^2(I) = I$. In view of the fact that Ann^2 is extensive it suffices to show that $Ann^2(I) \subseteq I$.

Let $\beta \in Ann^2(I)$. For $\beta|_{Y^c} : Y^c \to \mathfrak{Z}$ there exists $\alpha \in I$ such that $\alpha|_{Y^c} = \beta|_{Y^c}$. Moreover, $\alpha(y) = U$ for all $y \in Y$. We show that $\beta(y) = U$ for all $y \in Y$ so that $\beta = \alpha$ from which it follows that $\beta \in I$.

Suppose if possible, that $\beta(y_o) \in \{T, F\}$ for some $y_o \in Y$. Since $\beta \in Ann^2(I)$ we have $(\beta[\![\gamma, \gamma]\!])(y_o) = U$ for all $\gamma \in Ann(I)$ and so $\gamma(y_o) = U$ for all $\gamma \in Ann(I)$.

Consider $\delta \in \mathfrak{Z}^X$ given by

$$\delta(x) = \begin{cases} T, & \text{if } x \in Y, \\ U, & \text{otherwise.} \end{cases}$$

Due to the fact that $\alpha(y) = U$ for all $y \in Y$, for all $\alpha \in I$, we infer that $\delta[\![\alpha, \alpha]\!] = \mathbf{U}$ so that $\delta \in Ann(I)$. However $\delta(y_o) = T \neq U$, a contradiction. Hence $\beta \in I$ and so $I \in \mathfrak{I}$.

($\Rightarrow$) Let $I \in \mathfrak{I}$. Consider the following.

$$A = \{x \in X : \alpha(x) = U \text{ for all } \alpha \in I\}$$
$$B = \{x \in X : \alpha(x) \in \{T, F\} \text{ for some } \alpha \in I\} = X \setminus A.$$

We show that $Y = A$ is the required set. It is clear that $\alpha(y) = U$ for all $\alpha \in I$ and for all $y \in A$. Let $f : B \to \mathfrak{Z}$. Consider its extension $\hat{f} : X \to \mathfrak{Z}$ given by the following:

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \in B; \\ U, & \text{if } x \in A. \end{cases}$$

Thus $\hat{f}|_B = \hat{f}|_{A^c} = f$. Let $\beta \in \mathfrak{Z}^X$. It is clear that

$$\beta \in Ann(I) \Leftrightarrow \beta(z) = U \text{ for all } z \in B.$$

Consider $\beta \in Ann(I)$. It follows that $(\hat{f}[\![\beta, \beta]\!])(z) = U$ for all $z \in B$. Also since $\hat{f}(y) = U$ for all $y \in A$ we have $(\hat{f}[\![\beta, \beta]\!])(y) = U$ and so $\hat{f}[\![\beta, \beta]\!] = U$ from which it follows that $\hat{f} \in Ann^2(I) = I$. This completes the proof. $\square$

Theorem 6.2.1 equips us with a mechanism to identify the collection of closed sets in $\mathfrak{I}$ with respect to Ann^2.

Definition 6.2.2. For $A \subseteq X$ define $I_A \subseteq \mathfrak{Z}^X$ by

$$I_A = \{\alpha \in \mathfrak{Z}^X : \alpha(y) = U \text{ for all } y \in A\}. \tag{6.1}$$

Proposition 6.2.3. $\mathfrak{I} = \{I_A : A \subseteq X\}$.

Proof. For $A \subseteq X$ consider I_A as defined by (6.1). It follows in a straightforward manner that I_A satisfies (P1) and (P2) of Theorem 6.2.1 for $Y = A$ so that $I_A \in \mathfrak{I}$.

Conversely for $I \in \mathfrak{I}$ using Theorem 6.2.1 we have $Y \subseteq X$ such that (P1) and (P2) are satisfied. We show that $I = I_Y$. Clearly $I \subseteq I_Y$ due to (P1). Conversely assume that $\alpha \in I_Y$ that is $\alpha(y) = U$ for all $y \in A$. Using (P2) of Theorem 6.2.1 we have $\beta|_{Y^c} = \alpha|_{Y^c}$ for some $\beta \in I$. Property (P1) of Theorem 6.2.1 ascertains that $\beta(y) = U$ for all $y \in Y$. It follows that $\alpha = \beta$ so that $\alpha \in I$. □

We make use of the following result to prove Theorem 6.2.5.

Lemma 6.2.4. *For $A \subseteq X$ and $I_A \in \mathfrak{I}$ the following hold:*

(i) $Ann(I_A) = I_{A^c}$.

(ii) $I_A \cap I_B = I_{A \cup B}$.

(iii) $Ann(Ann(I_A) \cap Ann(I_B)) = I_{A \cap B}$.

Proof.

(i) Let $\alpha \in Ann(I_A)$. In view of Definition 6.2.2 and Proposition 6.2.3 it suffices to show that $\alpha(y) = U$ for all $y \in A^c$. For each $y \in A^c$ consider $\beta_y \in \mathfrak{Z}^X$ given by

$$\beta_y(x) = \begin{cases} T, & \text{if } x = y; \\ U, & \text{otherwise.} \end{cases}$$

It is straightforward to see that $\beta_y \in I_A$ for all $y \in A^c$. Thus $\alpha[\![\beta_y, \beta_y]\!] = U$ for all $y \in A^c$ and so $(\alpha[\![\beta_y, \beta_y]\!])(y) = U$ for all $y \in A^c$. Since $\beta_y(y) = T$ it follows that $\alpha(y) = U$ for all $y \in A^c$.

For the reverse inclusion consider $\alpha \in I_{A^c}$ and $\beta \in I_A$. Using Definition 6.2.2 and Proposition 6.2.3 we have $\alpha(y) = U$ for all $y \in A^c$, so that $(\alpha[\![\beta, \beta]\!])(y) = U$ for all $y \in A^c$. Thus $\beta(y) = U$ for all $y \in A$ so that $(\alpha[\![\beta, \beta]\!])(y) = U$ for all $y \in A$. Thus $\alpha \in Ann(I_A)$ and consequently $Ann(I_A) = I_{A^c}$.

(ii) Consider $\alpha \in I_A \cap I_B$ and $y \in A \cup B$. It suffices to show that $\alpha(y) = U$. If $y \in A$ then $\alpha(y) = U$ since $\alpha \in I_A$. Along similar line $\alpha(y) = U$ if $y \in B$ so that $\alpha \in I_{A \cup B}$.

For the reverse inclusion consider $\alpha \in I_{A \cup B}$. For $y \in A \subseteq A \cup B$ we have $\alpha(y) = U$ so that $\alpha \in I_A$. Proceeding along similar lines we can show that $\alpha \in I_B$ from which the result follows.

(iii) Using Lemma 6.2.4(i) and Lemma 6.2.4(ii) we have $Ann(Ann(I_A) \cap Ann(I_B)) = Ann(I_{A^c} \cap I_{B^c}) = Ann(I_{A^c \cup B^c}) = I_{(A^c \cup B^c)^c} = I_{A \cap B}$.

$\square$

Theorem 6.2.5. *The set $\mathfrak{I}$ of closed sets of $\mathfrak{Z}^X$ with respect to Ann^2 is a Boolean algebra with respect to the operations*

$$\neg I = Ann(I)$$

$$I_1 \wedge I_2 = I_1 \cap I_2$$

$$I_1 \vee I_2 = Ann(Ann(I_1) \cap Ann(I_2))$$

and $\{\mathbf{U}\}$ and $\mathfrak{Z}^X$ as the constants 0 and 1 respectively. Moreover, $\mathfrak{I} \cong 2^X$ and is therefore complete.

Proof. We rely on the representation of $\mathfrak{I}$ as given in Proposition 6.2.3. In view of Lemma 6.2.4 we show that the operations given as follows define a Boolean algebra on $\mathfrak{I}$:

$$\neg I_A = I_{A^c}$$
$$I_A \wedge I_B = I_{A\cup B}$$
$$I_A \vee I_B = I_{A\cap B}$$

The verification is straightforward and involves set theoretic arguments.

Let $A, B \subseteq X$. Using Lemma 6.2.4 we have $I_A \wedge I_B = I_{A\cup B} = I_{B\cup A} = I_B \wedge I_A$ and similarly $I_A \vee I_B = I_{A\cap B} = I_{B\cap A} = I_B \vee I_A$. Along similar lines the axioms of associativity, idempotence, absorption and distributivity can be verified so that $\langle \mathfrak{I}, \vee, \wedge \rangle$ is a distributive lattice.

Note that $I_X = \{\mathbf{U}\}$ while $I_\emptyset = \mathfrak{3}^X$. Using Lemma 6.2.4 we have $I_A \wedge I_X = I_{A\cup X} = I_X$ and $I_A \vee I_\emptyset = I_{A\cap\emptyset} = I_\emptyset$ for all $A \subseteq X$. Also $I_A \wedge Ann(I_A) = I_A \wedge I_{A^c} = I_{A\cup A^c} = I_X$ and $I_A \vee Ann(I_A) = I_A \vee I_{A^c} = I_{A\cap A^c} = I_\emptyset$.

Hence $\langle \mathfrak{I}, \vee, \wedge, \neg, I_X, I_\emptyset \rangle$ is a Boolean algebra. It is a straightforward verification to ascertain that the assignment given by $I_A \mapsto A^c$ from $\mathfrak{I}$ to $\mathfrak{2}^X$ is a Boolean algebra isomorphism. □

We now give a classification of elements of $M = \mathfrak{3}^X$ which segregates the elements of $\mathfrak{2}^X (= M_\#)$ into one class.

Theorem 6.2.6. *For each $A \subseteq X$ define $S_A = \{\alpha \in \mathfrak{3}^X : Ann(\alpha) = I_A\}$. The collection $\{S_A : A \subseteq X\}$ forms a partition of $\mathfrak{3}^X$ in which all the elements of $\mathfrak{2}^X$ form a single equivalence class.*

Proof. We first show that $S_A \neq \emptyset$ for any $A \subseteq X$. To that aim consider $\alpha \in \mathfrak{3}^X$ given by

$$\alpha(x) = \begin{cases} U, & \text{if } x \in A^c; \\ T, & \text{otherwise.} \end{cases}$$

Using Definition 6.2.2 and Proposition 6.2.3 it is straightforward to verify that $Ann(\alpha) = I_A$. Consequently $S_A \neq \emptyset$.

It is self-evident that $\alpha \in S_A \cap S_B$ is a violation of the well-definedness of $Ann(\alpha)$ from which it follows that $S_A \cap S_B = \emptyset$ for $A, B \subseteq X$ where $A \neq B$.

Note that for any $\alpha \in \mathfrak{Z}^X$ we have $Ann(\alpha) \in \mathfrak{I}$ that is $Ann(\alpha) = I_A$ for some $A \subseteq X$, since $Ann^2(Ann(\alpha)) = Ann^3(\alpha) = Ann(\alpha)$ using Corollary 6.1.6. Thus $Ann(\alpha) = I_A$ for some $A \subseteq X$ so that $\alpha \in S_A$. Therefore $\bigcup\limits_{A \subseteq X} \{S_A : A \subseteq X\} = \mathfrak{Z}^X$ and hence the collection $\{S_A : A \subseteq X\}$ forms a partition of $\mathfrak{Z}^X$.

Further, for $\alpha \in 2^X$ we have $Ann(\alpha) = \{\mathbf{U}\} = I_X$ so that $\alpha \in S_X$. Conversely any $\alpha \in S_X$ would satisfy $Ann(\alpha) = I_X = \{\mathbf{U}\}$. If $\alpha(x_o) = U$ for some $x_o \in X$ then it follows that $\beta \in \mathfrak{Z}^X$ given by

$$\beta(x) = \begin{cases} T, & \text{if } x = x_o; \\ U, & \text{otherwise.} \end{cases}$$

satisfies $\beta[\![\alpha, \alpha]\!] = \mathbf{U}$ and so $\mathbf{U} \neq \beta \in Ann(\alpha)$ which is a contradiction. Thus $\alpha(x) \in \{T, F\}$ for all $x \in X$ from which it follows that $\alpha \in 2^X$. Hence the equivalence class $S_X = 2^X$. □

We conclude this section with some remarks on annihilators.

Remark 6.2.7.

(i) The statement $Ann(\alpha) = \{U\} \Leftrightarrow \alpha \in M_\#$ holds in $\mathfrak{Z}^X$ but need not be true in general.

Consider $M = \{(T,T), (F,F), (U,U), (F,U), (T,U)\} \leq \mathfrak{Z}^2$ and $(T,U) \in M$. Then $Ann(T,U) = \{\beta \in M : \beta[\![(T,U),(T,U)]\!] = (U,U)\}$. Hence for $(x,y) \in Ann(T,U)$ we have $((x \wedge T) \vee (\neg x \wedge T), (y \wedge U) \vee (\neg y \wedge U)) = (U,U)$ and so $(x \vee \neg x, U) = (U,U)$. It follows that $x = U$ and so $Ann(T,U) = \{(U,U)\}$. However $(T,U) \notin M_\#$.

(ii) The only closed sets of M are M and $\{(U,U)\}$. Again note that the collection of closed sets is a Boolean algebra.

(iii) For $I \subseteq M$ where $M \leq \mathfrak{Z}^X$ we have $Ann_M(I) = Ann_{\mathfrak{Z}^X}(I) \cap M$.

Let $\alpha \in Ann_M(I)$. Clearly $\alpha \in M$ and $\alpha \in Ann_{\mathfrak{Z}^X}(I)$. Conversely suppose $\alpha \in Ann_{\mathfrak{Z}^X}(I) \cap M$. Then it is clear that $\alpha \in Ann_M(I)$.

(iv) Thus on applying Ann to the previous statement and making appropriate substitutions we have $Ann^2_M(I) = Ann_{\mathfrak{Z}^X}(Ann_{\mathfrak{Z}^X}(I) \cap M) \cap M$.

6.3 Idempotents

In this section we consider the notion of idempotence on C-algebras by involving the `if-then-else` action. First we study the notion of idempotent elements relative to the binary operation $\alpha[\![\_,\_]\!]$ for $\alpha \in M$ followed by idempotent binary operations $\alpha[\![\_,\_]\!]$ defined in a natural manner. For most of this section we distinguish elements of the C-algebra M when being treated as operations by denoting them by α, β, γ and as elements by a, b, c.

Definition 6.3.1. For $\alpha \in M$ define *the idempotent elements with respect to* α as $E_\alpha = \{a \in M : \alpha[\![a,a]\!] = a\}$.

Definition 6.3.2. Let $\alpha \in M$. Then $\alpha[\![\_,\_]\!]$ is an idempotent operation if every $a \in M$ is an idempotent element with respect to this operation, i.e.,

$$\alpha[\![a,a]\!] = a \text{ for all } a \in M.$$

The set of all idempotent operations, denoted by O, is therefore given by $O = \{\alpha \in M : \alpha[\![a,a]\!] = a \text{ for all } a \in M\}$.

It is natural to ask what the set of all idempotent operations is. In this section we show that the set of all idempotent operations is precisely the Boolean algebra

$M_{\#}$. In order to achieve this, we first consider idempotent operations relative to a fixed element $a \in M$.

Definition 6.3.3. For $a \in M$ define *the set of idempotent operations with respect to* a as $O_a = \{\alpha \in M : \alpha[\![a, a]\!] = a\}$.

Remark 6.3.4. It is easy to observe that therefore, $O = \bigcap_{a \in M} O_a$.

We make use of this notion to obtain the following result.

Theorem 6.3.5. $O = M_{\#}$.

Proof. We first make the following observations.

(i) For all $a \in M$ we have $M_{\#} \subseteq O_a$.

(ii) $M_{\#} = O_a \Leftrightarrow a \in M_{\#}$.

The proof of each observation follows in a straightforward manner as given in the following.

(i) Using Proposition 2.2.3 we have $\alpha[\![a, a]\!] = a$ for all $\alpha \in M_{\#}$. It follows that $M_{\#} \subseteq O_a$.

(ii) Let $a \in M_{\#}$. Clearly $M_{\#} \subseteq O_a$. Let $\alpha \in O_a$. Then $\alpha[\![a, a]\!] = (\alpha \wedge a) \vee (\neg\alpha \wedge a) = a$. Consider $M \leq 3^X$ for some set X. Then $a(x) \in \{T, F\}$ for all $x \in X$. Thus if $\alpha(x) = U$ for some $x \in X$ we have $(\alpha[\![a, a]\!])(x) = (\alpha(x) \wedge a(x)) \vee (\neg\alpha(x) \wedge a(x)) = U \vee U = U$ which is a contradiction. Hence $\alpha \in M_{\#}$. Conversely suppose $M_{\#} = O_a$ for some $a \in M$. If $a \notin M_{\#}$ then using the previous point we have $a \in O_a = M_{\#}$ which is a contradiction. Thus $a \in M_{\#}$.

In view of the aforementioned observations (i) and (ii), and Remark 6.3.4 we have $O = \bigcap_{a \in M} O_a = M_{\#}$. □

It is natural to enquire about the relation between the two notions of idempotent elements with respect to a fixed operation and idempotent operations with respect to a fixed element. We make some observations on the notions of E_α and O_α for $\alpha \in M$.

Proposition 6.3.6. *For all $\alpha \in M$ we have $\alpha \in O_\alpha$ and $U \in E_\alpha$.*

Proof. Consider the identity $\alpha \vee (\neg\alpha \wedge \alpha) = \alpha$ in the C-algebra $\mathfrak{Z}$:

$\alpha = T$**:** $T \vee (\neg T \wedge T) = T \vee (F \wedge T) = T \vee F = T$.

$\alpha = F$**:** $F \vee (\neg F \wedge F) = F \vee (T \wedge F) = F \vee F = F$.

$\alpha = U$**:** $U \vee (\neg U \wedge U) = U \vee (U \wedge U) = U \vee U = U$.

Thus the identity $\alpha[\![\alpha, \alpha]\!] = \alpha \vee (\neg\alpha \wedge \alpha) = \alpha$ holds in $\mathfrak{Z}$, and consequently holds in all C-algebras. Hence $\alpha \in O_\alpha$.

Further $\alpha[\![U, U]\!] = U$ by Proposition 2.2.1(i) and so $U \in E_\alpha$. □

Proposition 6.3.7. $O_U = M$ *and* $E_U = \{U\}$.

Proof. Proposition 2.2.1(i) applied to the C-set (M, M) gives $\alpha[\![U, U]\!] = U$ for all $\alpha \in M$. Consequently, since $O_U = \{\alpha \in M : \alpha[\![U, U]\!] = U\}$ we have $O_U = M$.

Using Proposition 6.3.6 we have $\{U\} \subseteq E_U$. For $a \in E_U$ we have $U[\![a, a]\!] = a$ and so $(U \wedge a) \vee (\neg U \wedge a) = U \vee U = U = a$ and hence $E_U = \{U\}$. □

Along similar lines as observation (ii) of Theorem 6.3.5 we have the following result.

Proposition 6.3.8.

(i) *$U \in O_a \Leftrightarrow a = U$.*

(ii) *For all $\alpha \in M$ we have $E_\alpha = M \Leftrightarrow \alpha \in M_\#$.*

Proof.

(i) Clearly if $a = U$ then using Proposition 6.3.6 we have $U \in O_a$. Conversely suppose that $U \in O_a$ for some $a \in M$. Then $U[\![a, a]\!] = a$ and so $(U \wedge a) \vee (\neg U \wedge a) = U \vee U = U = a$.

(ii) Suppose that $\alpha \in M_{\#}$. Then $E_\alpha = \{a \in M : \alpha[\![a, a]\!] = a\}$. However Proposition 2.2.3 applied to the C-set (M, M) gives $\alpha[\![a, a]\!] = a$ for all $a \in M$ so that $E_\alpha = M$. Conversely suppose that $E_\alpha = M$ for some $\alpha \in M$. Then $\alpha[\![T, T]\!] = T$ and so $(\alpha \wedge T) \vee (\neg\alpha \wedge T) = \alpha \vee \neg\alpha = T$ so that $\alpha \in M_{\#}$.

□

The following result pertains to the relations between the idempotent elements (and operations) with respect to α and those with respect to $\neg\alpha$.

Proposition 6.3.9. *For all $\alpha \in M$ we have $O_\alpha = O_{\neg\alpha}$ and $E_\alpha = E_{\neg\alpha}$*

Proof. Let $\beta \in O_\alpha$. Then $\beta[\![\alpha, \alpha]\!] = \alpha$. Consider $M \leq \mathfrak{Z}^X$ for some set X. The various cases of $(\beta(x) \wedge \neg\alpha(x)) \vee (\neg\beta(x) \wedge \neg\alpha(x))$ are listed below:

$\beta(x) = T$**:** $(T \wedge \neg\alpha(x)) \vee (\neg T \wedge \neg\alpha(x)) = \neg\alpha(x) \vee F = \neg\alpha(x)$.

$\beta(x) = F$**:** $(F \wedge \neg\alpha(x)) \vee (\neg F \wedge \neg\alpha(x)) = F \vee \neg\alpha(x) = \neg\alpha(x)$.

$\beta(x) = U$**:** $(U \wedge \neg\alpha(x)) \vee (\neg U \wedge \neg\alpha(x)) = U \vee U = U$.

Note that if $\beta(x) = U$ then $(\beta[\![\alpha, \alpha]\!])(x) = U = \alpha(x)$ and so $\neg\alpha(x) = U$. Thus we have $(\beta[\![\neg\alpha, \neg\alpha]\!])(x) = (\neg\alpha)(x)$ for all $x \in X$ so that $\beta \in O_{\neg\alpha}$. The converse is clear since $\neg\neg\alpha = \alpha$.

Let $a \in E_\alpha$. Then $\alpha[\![a, a]\!] = a$. Consider the identity $(\neg\alpha \wedge a) \vee (\alpha \wedge a) = (\alpha \wedge a) \vee (\neg\alpha \wedge a)$ in the C-algebra $\mathfrak{Z}$:

$\alpha = T$**:** $(\neg T \wedge a) \vee (T \wedge a) = F \vee a = a \vee F = (T \wedge a) \vee (\neg T \wedge a)$.

$\alpha = F$**:** $(\neg F \wedge a) \vee (F \wedge a) = a \vee F = F \vee a = (F \wedge a) \vee (\neg F \wedge a)$.

$\alpha = U$**:** $(\neg U \wedge a) \vee (U \wedge a) = U \vee U = (U \wedge a) \vee (\neg U \wedge a)$.

Thus the identity $(\neg\alpha \wedge a) \vee (\alpha \wedge a) = (\alpha \wedge a) \vee (\neg\alpha \wedge a)$ holds in $\mathfrak{3}$ and therefore in all C-algebras. Since $\alpha[\![a, a]\!] = (\alpha \wedge a) \vee (\neg\alpha \wedge a) = a$ it follows that $\neg\alpha[\![a, a]\!] = (\neg\alpha \wedge a) \vee (\neg\neg\alpha \wedge a) = (\neg\alpha \wedge a) \vee (\alpha \wedge a) = (\alpha \wedge a) \vee (\neg\alpha \wedge a) = a$ so that $a \in E_{\neg\alpha}$. Hence $E_\alpha \subseteq E_{\neg\alpha}$. The reverse inclusion follows along similar lines since $\neg\neg\alpha = \alpha$. □

We make the following observation on the subset O_a for $a \in M$.

Proposition 6.3.10. *For $a \in M$ the set O_a is a C-algebra under the induced operations of M.*

Proof. It suffices to show that O_a is closed under $\neg, \wedge, \vee$.

Let $\alpha, \beta \in O_a$. Then $(\alpha \wedge \beta)[\![a, a]\!] = ((\alpha \wedge \beta) \wedge a) \vee (\neg(\alpha \wedge \beta) \wedge a)$. Consider $M \leq \mathfrak{3}^X$ for some set X. Then using the fact that $\alpha, \beta \in O_a$ the expression evaluates to the following:

$\alpha(x) = T$**:** $((T \wedge \beta(x)) \wedge a(x)) \vee (\neg(T \wedge \beta(x)) \wedge a(x)) = (\beta(x) \wedge a(x)) \vee (\neg\beta(x) \wedge a(x)) = a(x)$.

$\alpha(x) = F$**:** $((F \wedge \beta(x)) \wedge a(x)) \vee (\neg(F \wedge \beta(x)) \wedge a(x)) = (F \wedge a(x)) \vee (\neg F \wedge a(x)) = F \vee a(x) = a(x)$.

$\alpha(x) = U$**:** $((U \wedge \beta(x)) \wedge a(x)) \vee (\neg(U \wedge \beta(x)) \wedge a(x)) = (U \wedge a(x)) \vee (\neg U \wedge a(x)) = U = a(x)$ since $\alpha \in O_a$.

Thus $((\alpha \wedge \beta)[\![a, a]\!])(x) = a(x)$ for all $x \in X$ as a consequence of which we have $\alpha \wedge \beta \in O_a$.

Consider $(\neg\alpha)[\![a, a]\!] = (\neg\alpha \wedge a) \vee (\neg\neg\alpha \wedge a) = (\neg\alpha \wedge a) \vee (\alpha \wedge a)$. Consider the following cases:

$\alpha(x) = T$: $(\neg T \wedge a(x)) \vee (T \wedge a(x)) = F \vee a(x) = a(x)$.

$\alpha(x) = F$: $(\neg F \wedge a(x)) \vee (F \wedge a(x)) = a(x) \vee F = a(x)$.

$\alpha(x) = U$: $(\neg U \wedge a(x)) \vee (U \wedge a(x)) = U \vee U = U = a(x)$ since $\alpha \in I_a$.

Thus $((\neg\alpha)[\![a, a]\!])(x) = a(x)$ for all $x \in X$ so that $\neg\alpha \in O_a$. It is easy to see that, therefore, $\alpha \vee \beta \in O_a$. The result follows. □

Remark 6.3.11. Note that since $M_\# \subseteq O_a$ (from observation (i) of Theorem 6.3.5) we have $T, F \in O_a$. However using Proposition 6.3.8(i) $U \in O_a$ only when $a = U$. Thus O_a is always closed under T and F, and is closed under U if and only if $a = U$. Further, in this case using Proposition 6.3.7 we have $O_a = O_U = M$.

In the following result we see that the set of idempotent elements with respect to $\alpha \in M$ is contained in the closure of the idempotent operations with respect to α.

Theorem 6.3.12. *For $\alpha \in M$ we have $E_\alpha \subseteq Ann^2(O_\alpha)$.*

Proof. In order to prove this we shall show that $Ann(O_\alpha) \subseteq Ann(E_\alpha)$. Consequently, using Proposition 6.1.3(vii) we have $Ann^2(E_\alpha) \subseteq Ann^2(O_\alpha)$. Finally, using Corollary 6.1.5 we have Ann^2 is extensive, so that $E_\alpha \subseteq Ann^2(E_\alpha) \subseteq Ann^2(O_\alpha)$ from which the result follows. We now show that $Ann(O_\alpha) \subseteq Ann(E_\alpha)$.

Let $\beta \in Ann(O_\alpha)$ and $a \in E_\alpha$. Consider $M \leq \mathbf{3}^X$ for some set X. We show that $\beta[\![a, a]\!](x) = U$ for all $x \in X$. If $\beta(x) = U$ then $\beta[\![a, a]\!](x) = (\beta(x) \wedge a(x)) \vee (\neg\beta(x) \wedge a(x)) = U$. Let $\beta(x) \in \{T, F\}$. We show that $a(x) = U$. Let $\gamma \in O_\alpha$. Then $\beta[\![\gamma, \gamma]\!](x) = U$ and since $\beta(x) \in \{T, F\}$ we have $\gamma(x) = U$. Also $\gamma \in O_\alpha$ and so it follows that $\gamma[\![\alpha, \alpha]\!](x) = \alpha(x)$. Since $\gamma(x) = U$ it follows that $\alpha(x) = U$. Since $a \in E_\alpha$ we have $\alpha[\![a, a]\!](x) = a(x)$ and so $a(x) = U$ which completes the proof. □

6.4 Conclusion

In this chapter we have delved into the algebraic structure of C-algebras with the help of the augmented `if-then-else` action. We introduce a notion of annihilators in C-algebras with T, F, U and obtain a closure operator in terms of the annihilator. In the C-algebra of transformations $\mathbb{3}^X$, we have obtained a characterisation of closed sets and showed that the collection of closed sets forms the Boolean algebra $\mathbb{2}^X$. Further, this characterisation allows us to achieve a classification of elements of $\mathbb{3}^X$ such that the elements of the Boolean algebra $\mathbb{2}^X$ form one distinct class. It remains to be seen what characterisation may be achieved for the closed sets of an arbitrary C-algebra with T, F, U, and whether the closed sets in such a C-algebra always form a Boolean algebra. We also defined the notions of idempotent elements and idempotent operations and obtain a relating factor between the two through the closure operator.

7

Atomicity

In this chapter we adopt the notion of atoms in Boolean algebras to C-algebras. First, in Section 7.1 a partial order is given on the C-algebra M, following which the notions of atoms and atomic C-algebras are introduced. We state various properties related to atomicity in Section 7.2 while a characterisation of atoms in $\mathfrak{3}^X$ is given in Section 7.3 (cf. Theorem 7.3.1). Subsequently, we present some necessary or sufficient conditions for the atomicity of C-algebras in Section 7.5 (cf. Theorems 7.5.1, 7.5.2, 7.5.4). Finally in Section 7.6 we obtain a characterisation of finite atomic C-algebras and establish that they are precisely adas (cf. Theorem 7.6.3). We conclude by proposing some avenues for further study in Section 7.7.

7.1 Atoms and atomicity

In this chapter we assume that M is a C-algebra with T, F, U unless mentioned otherwise. We denote elements of M by a, b, c and α, β, γ. The elements of the C-algebra $\mathfrak{3}^X$ will also be denoted by $\alpha, \beta, \gamma, \delta$. We continue to denote constants T, F, U of $M \leq \mathfrak{3}^X$ by $\mathbf{T}, \mathbf{F}, \mathbf{U}$ respectively. We begin with a partial order defined on C-algebras and follow the notion of the partial order given in Chang [1958] regarding MV-algebras.

Proposition 7.1.1. *The relation $\leq$ on M defined by $a \leq b$ if $a \vee b = b$ is a partial order on M.*

Proof. Let $a, b, c \in M$. Since $a \vee a = a$ we have $a \leq a$ from which it follows that $\leq$ is reflexive.

Suppose that $a \leq b$ and $b \leq a$ so that $a \vee b = b$ and $b \vee a = a$. Using the fact that $M \leq \mathfrak{3}^X$ for some set X we have $a(x) \vee b(x) = b(x)$ and $b(x) \vee a(x) = a(x)$ for all $x \in X$. It suffices to consider the following three cases:

$b(x) = T$**:** Then $b(x) \vee a(x) = a(x)$ gives $T \vee a(x) = a(x)$ that is $a(x) = T$.

$b(x) = F$**:** Then $a(x) \vee b(x) = b(x)$ and so $a(x) \vee F = F$ so that $a(x) = F$.

$b(x) = U$**:** Then $b(x) \vee a(x) = a(x)$ that is $U \vee a(x) = a(x)$ and so $a(x) = U$.

In all three cases $a(x) = b(x)$ and so $a = b$. Hence $\leq$ is antisymmetric.

In order to show that $\leq$ is transitive consider $a \leq b$ and $b \leq c$. Then $a \vee b = b$ and $b \vee c = c$. It is clear that $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c$ and so $a \leq c$. This completes the proof. □

Example 7.1.2. In the C-algebra $\mathfrak{3}$ we have $F \leq T$ and $F \leq U$ while $T \nleq U$ and $U \nleq T$.

Remark 7.1.3. In fact $F \leq a$ for all $a \in M$. This partial order does not induce a lattice structure on M.

With this partial order we define the notion of an atom in M below.

Definition 7.1.4. An element $a \in M$ where $a \neq F$ is said to be an *atom* if for all $b \in M$ if $F \leq b \leq a$ and $b \neq a$ then $b = F$. We denote the set of atoms of M by $\mathscr{A}(M)$.

For $A \subseteq M$ where $\{F\} \subseteq A$ define the *atoms relative to* A as those elements $a \in A$ such that for all $b \in A$ if $F \leq b \leq a$ and $b \neq a$ then $b = F$. We denote the set of atoms relative to A as $\mathscr{A}(A)$.

Example 7.1.5. In $\mathfrak{Z}$ we have $\mathscr{A}(\mathfrak{Z}) = \{T, U\}$.

Example 7.1.6. In $\mathfrak{Z} \times \mathfrak{Z} = \mathfrak{Z}^2$ we have $\mathscr{A}(\mathfrak{Z}^2) = \{(T, F), (F, T), (F, U), (U, F)\}$.

Example 7.1.7. Consider $M = \mathfrak{Z}^2 \backslash \{(T, F), (F, T)\}$. Then $\mathscr{A}(M) = \{(T, T), (F, U), (U, F)\}$.

Remark 7.1.8. The representation of elements as join of atoms need not be unique. Consider $M = \mathfrak{Z}^2 \setminus \{(T, F), (F, T)\}$ as in Example 7.1.7. Then $(T, T) = (T, T) \vee (F, U)$ while also $(T, T) = (T, T) \vee (U, F)$.

Definition 7.1.9. Let $\{a_i : 1 \leq i \leq N\}$ be a finite set of atoms of M such that for every rearrangement of $(a_i)_{i=1}^N$ the join of these elements remain unchanged. More precisely, if for every bijection $\sigma : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ we have

$$a_{\sigma(1)} \vee a_{\sigma(2)} \vee \cdots \vee a_{\sigma(N)} = a_1 \vee a_2 \vee \cdots \vee a_N = a_o \text{ (say)}$$

then define

$$\bigoplus_{i=1}^{N} a_i = a_o.$$

Remark 7.1.10. Thus $\bigoplus_{i=1}^N a_i$ exists when the a_i's commute under $\vee$.

Example 7.1.11. Let $M = \mathfrak{Z}^2$. Then $(T, U) = (T, F) \oplus (F, U)$.

Definition 7.1.12. Let M be a C-algebra with T, F, U. We say that M is *atomic* if for every $(F \neq)\ a \in M$ we have

$$a = \bigoplus_{i=1}^{N} a_i$$

for a finite set of atoms $\{a_i : 1 \leq i \leq N\} \subseteq \mathscr{A}(M)$.

Example 7.1.13. The C-algebra $M = \mathfrak{Z}^2$ is atomic.

Example 7.1.14. Consider $M = \mathfrak{Z}^2 \setminus \{(T, F), (F, T)\}$ for which $\mathscr{A}(M) = \{(T, T), (F, U), (U, F)\}$. Then (T, U) cannot be written as $\oplus$ of atoms. Thus M is not atomic.

7.2 Properties of atoms

In this section we list some properties that are satisfied by the set of atoms of M.

Proposition 7.2.1. *Let M be a finite C-algebra with T, F, U. Then for each $a \in M$ $(a \neq F)$ there exists $a_o \in \mathscr{A}(M)$ such that $a_o \leq a$.*

Proof. If $a \in \mathscr{A}(M)$ then we are done since $a \leq a$. If $a \notin \mathscr{A}(M)$ then there exists $a_1 \in M$ such that $F \lneq a_1 \lneq a$. If $a_1 \in \mathscr{A}(M)$ then we are done. If not, there exists $a_2 \in M$ such that $F \lneq a_2 \lneq a_1 \lneq a$. Proceeding along similar lines if there is no atom in the list then there exists an infinite strictly descending chain of elements in M which is a contradiction, since M is finite. The result follows. $\square$

This result suggests an immediate corollary. Note that M is atomless if $\mathscr{A}(M) = \emptyset$.

Corollary 7.2.2. *Let M be a finite C-algebra with T, F, U. Then $\mathscr{A}(M) \neq \emptyset$. Thus no finite C-algebra with T, F, U is atomless.*

The following result relates the effect of the partial ordering on elements of $M_\#$.

Proposition 7.2.3. *If $a, b \in M$ such that $a \leq b$ and $b \in M_\#$ then $a \in M_\#$.*

Proof. Since the identity $a \vee \neg a = a \vee T$ holds in $\mathfrak{Z}$, it holds in all C-algebras. We have $a \vee b = b$ since $a \leq b$. Further, $b \in M_\#$ gives $b \vee \neg b = T$. Thus $a \vee \neg a = a \vee T = a \vee (b \vee \neg b) = (a \vee b) \vee \neg b = b \vee \neg b = T$ which completes the proof. □

We have the following corollary, which can also be proved independently.

Corollary 7.2.4. *If $a \in M$ such that $a \leq T$ then $a \in M_\#$.*

We now list some properties which are useful in establishing the characterisation of atomic C-algebras.

Proposition 7.2.5. *The following hold for all $\alpha, \gamma, \delta \in M$:*

(i) $\alpha \wedge F \leq \alpha$.

(ii) $\alpha \wedge F \leq U$.

(iii) $\alpha \wedge F = U \Leftrightarrow \alpha = U$.

(iv) $\alpha \wedge F = F \Leftrightarrow \alpha \in M_\#$.

(v) $\alpha \wedge F = \alpha \Leftrightarrow \alpha \wedge \beta = \alpha$ *for all* $\beta \in M$.

(vi) $\alpha \leq \gamma \Rightarrow \alpha \wedge \gamma = \alpha$.

(vii) $\alpha \leq \alpha \vee \beta$ *for all* $\beta \in M$.

(viii) $\alpha \leq \delta$ *and* $\gamma \leq \delta \Rightarrow \alpha \vee \gamma \leq \delta$.

Proof.

(i) In the C-algebra $\mathfrak{Z}$ consider the identity $(\alpha \wedge F) \vee \alpha = \alpha$:

$\alpha = T$**:** $(T \wedge F) \vee T = F \vee T = T$.

$\alpha = F$**:** $(F \wedge F) \vee F = F \vee F = F$.

$\alpha = U$: $(U \wedge F) \vee U = U \vee U = U$.

Thus this identity holds in $\mathfrak{Z}$ and so it holds in all C-algebras. It follows that $\alpha \wedge F \leq \alpha$.

(ii) In the C-algebra $\mathfrak{Z}$ consider the identity $(\alpha \wedge F) \vee U = U$:

$\alpha = T$: $(T \wedge F) \vee U = F \vee U = U$.

$\alpha = F$: $(F \wedge F) \vee U = F \vee U = U$.

$\alpha = U$: $(U \wedge F) \vee U = U \vee U = U$.

Since this identity holds in $\mathfrak{Z}$ it therefore holds in all C-algebras. It follows that $\alpha \wedge F \leq U$.

(iii) Clearly $U \wedge F = U$. Suppose that $\alpha \wedge F = U$. Since $M \leq \mathfrak{Z}^X$ for some set X we have $\alpha(x) \wedge F = U$ for all $x \in X$. If $\alpha(x_o) \in \{T, F\}$ for some $x_o \in X$ then $\alpha(x_o) \wedge F = F$, a contradiction. Hence $\alpha(x) = U$ for all $x \in X$ so that $\alpha = U$ in M.

(iv) Clearly if $\alpha \in M_{\#}$ then $\alpha \wedge F = F$. Note that the identities $\alpha \wedge F = \alpha \wedge \neg\alpha$ and $\neg\alpha \vee \alpha = \alpha \vee \neg\alpha$ hold in all C-algebras since they hold in $\mathfrak{Z}$. Thus $\alpha \wedge \neg\alpha = F$. Using (1.26) we have $\neg\alpha \vee \alpha = T$ so that $\alpha \vee \neg\alpha = T$. Consequently $\alpha \in M_{\#}$.

(v) It is clear that $\alpha \wedge \beta = \alpha$ for all $\beta \in M \Rightarrow \alpha \wedge F = \alpha$. Suppose that $\alpha \wedge F = \alpha$. Then for $\beta \in M$ we have $\alpha \wedge \beta = (\alpha \wedge F) \wedge \beta = \alpha \wedge (F \wedge \beta) = \alpha \wedge F = \alpha$.

(vi) Since $\alpha \leq \gamma$ we have $\alpha \vee \gamma = \gamma$. Thus $\alpha \wedge \gamma = \alpha \wedge (\alpha \vee \gamma) = \alpha$ using (1.25),(1.26) and (1.30).

(vii) Consider $\alpha \vee (\alpha \vee \beta) = (\alpha \vee \alpha) \vee \beta = \alpha \vee \beta$. Thus $\alpha \leq \alpha \vee \beta$.

(viii) Consider $(\alpha \vee \gamma) \vee \delta = \alpha \vee (\gamma \vee \delta) = \alpha \vee \delta = \delta$. Thus $\alpha \vee \gamma \leq \delta$.

□

Remark 7.2.6. Note that the converse of Proposition 7.2.5(vi) is not true in general. For instance $U \wedge F = U$, however $U \not\leq F$.

Proposition 7.2.7. *Let* $a \in M$ *be such that* $a = \bigoplus_{i=1}^{N} a_i$ *where* $a_i \in \mathscr{A}(M)$ *for all* $1 \leq i \leq N$. *Then* $a_i \leq a$ *for all* $1 \leq i \leq N$.

Proof. Consider $a_i \vee a = a_i \vee (\bigoplus_{j=1}^{N} a_j) = a_i \vee (a_i \vee \bigvee_{j \neq i} a_j) = (a_i \vee a_i) \vee \bigvee_{j \neq i} a_j = a_i \vee \bigvee_{j \neq i} a_j = \bigoplus_{j=1}^{N} a_j = a$. □

Proposition 7.2.8. $\mathscr{A}(M) \cap M_\# = \mathscr{A}(M_\#)$. *Moreover,* $\mathscr{A}(M) \cap (M_\#)^c \subseteq \{a \in M : a \wedge b = a$ *for all* $b \in M\}$.

Proof. Let $a \in \mathscr{A}(M) \cap M_\#$. Suppose that there exists $b \in M_\#$ such that $F \lneq b \lneq a$. It follows that $b \in M$ such that $F \lneq b \lneq a$ which is a contradiction to the fact that $a \in \mathscr{A}(M)$. Conversely if $a \in \mathscr{A}(M_\#)$ then clearly $a \in M_\#$. If there exists $b \in M$ such that $F \lneq b \lneq a$ then using Proposition 7.2.3 we have $b \in M_\#$ which is a contradiction to the fact that $a \in \mathscr{A}(M_\#)$. The result follows.

Let $a \in \mathscr{A}(M) \cap (M_\#)^c$. In order to show that a is a left-zero for $\wedge$, using Proposition 7.2.5(v) it suffices to show that $a \wedge F = a$. Suppose if possible that $a \wedge F \neq a$. Using Proposition 7.2.5(i) we have $a \wedge F \lneq a$ and so since $a \in \mathscr{A}(M)$ it must follow that $a \wedge F = F$. Consider $M \leq \mathfrak{Z}^X$ for some set X. Then $a \wedge \mathbf{F} = \mathbf{F}$. If $a(x_o) = U$ for some $x_o \in X$ then $(a \wedge \mathbf{F})(x_o) = a(x_o) \wedge F = U \wedge F = U \neq F$, a contradiction. Thus $a(x) \in \{T, F\}$ for all $x \in X$ and so $a \in M_\#$ which is a contradiction to our assumption that $a \in (M_\#)^c$. Hence $a \wedge F = a$ so that a is a left-zero for $\wedge$. □

The following result gives a necessary condition for a to be an atom of M.

Proposition 7.2.9. *If* $a \in \mathscr{A}(M)$ *then* $a \wedge b \leq b$ *or* $a \wedge b = a$ *for all* $b \in M$.

Proof. Let $a \in \mathscr{A}(M)$ and $b \in M$. If $a \wedge b \leq b$ then we are through. Suppose not. If $a \in \mathscr{A}(M) \cap M_\#^c$ then using Proposition 7.2.8 we have a is a left-zero for $\wedge$ from which the result follows. If $a \in \mathscr{A}(M) \cap M_{\#}$ then consider $M \leq \mathfrak{Z}^X$ for some set X. Thus $a = a_{T,A}$ for some $\emptyset \neq A \subseteq X$ so that

$$(a \wedge b)(x) = \begin{cases} b(x), & \text{if } x \in A; \\ F, & \text{otherwise.} \end{cases}$$

Hence $((a \wedge b) \vee b)(x) = b(x)$ for all $x \in X$ so that $a \wedge b \leq b$. □

Remark 7.2.10. The converse of Proposition 7.2.9 need not be true, i.e., if $a \wedge b \leq b$ or $a \wedge b = a$ for all $b \in M$ then a need not be in $\mathscr{A}(M)$. Consider $M = \mathfrak{Z}^4$ and $a = (U, U, F, F) \in \mathfrak{Z}^4$. This is a left-zero for $\wedge$ but is not an atom since $(F, F, F, F) \leq (U, F, F, F) \leq (U, U, F, F)$.

Remark 7.2.11. (i) For $a \in \mathscr{A}(M)$ and $b \in M$ either $a \leq b$ or $a \wedge b \leq b$ need not, in general, hold. Consider $M = \{(T, T, T, T), (F, F, F, F), (U, U, U, U), (T, T, F, F), (F, F, T, T), (U, U, F, F), (U, U, T, T), (F, F, U, U), (T, T, U, U)\} \leq \mathfrak{Z}^4$. Take $a = (F, F, U, U) \in \mathscr{A}(M)$ and $b = (U, U, T, T) \in M$. However $a = (F, F, U, U) \not\leq (U, U, T, T) = b$ and $a \wedge b = (F, F, U, U) \not\leq (U, U, T, T) = b$. Note that in this case $a \wedge b = a$.

(ii) For $a \in \mathscr{A}(M)$ and $b \in M$ it need not be true that $a \wedge b \in \mathscr{A}(M)$. Consider $M = \{(T, T), (F, F), (U, U), (F, U), (T, U)\} \leq \mathfrak{Z}^2$. Take $a = (T, T) \in \mathscr{A}(M)$ and $b = (T, U) \in M$. Then $a \wedge b = (T, U) \notin \mathscr{A}(M)$.

(iii) For $a, b \in M$ it need not be true that $b \leq a \vee b$. For instance in $\mathfrak{Z}$ we have $T \not\leq U \vee T = U$.

(iv) For $a, b \in M$ we need not have $a \wedge b \leq a$ nor $a \wedge b \leq b$ in general. Consider $M = \mathfrak{Z}^3$, $a = (T, U, F)$ and $b = (U, T, F)$. Then $a \wedge b = (U, U, F) \not\leq (T, U, F) = a$ and $a \wedge b = (U, U, F) \not\leq (U, T, F) = b$.

(v) For $a \in \mathscr{A}(M)$ it need not hold that $a \wedge U \in \mathscr{A}(M)$. Consider $M =$

$\{(T,T),(F,F),(U,U),(F,U),(T,U)\} \leq \mathfrak{Z}^2$ and $a = (T,T) \in \mathscr{A}(M)$ (since $\mathscr{A}(M) = \{(T,T),(F,U)\}$). However $a \wedge \mathbf{U} = (T,T) \wedge (U,U) = (U,U) \notin \mathscr{A}(M)$.

Let $\hat{M}$ be the enveloping ada of M as defined in Remark 1.4.11. We have the following properties in $\hat{M}$.

Proposition 7.2.12. *The following are equivalent for all $\beta \in M$:*

(i) *β is a left-zero for $\wedge$.*

(ii) *$\beta \wedge F = \beta$.*

(iii) *$\beta^{\downarrow} = F$ in $\hat{M}$.*

Proof. ((i) $\Leftrightarrow$ (ii)) This is shown in Proposition 7.2.5(v).
((ii) $\Rightarrow$ (iii)) Let $\beta \wedge F = \beta$. Consider $\hat{M} \leq \mathfrak{Z}^X$ for some set X. Then $(\beta \wedge \mathbf{F})(x) = \beta(x)$ gives $\beta(x) \in \{F,U\}$ for all $x \in X$. Thus $(\beta^{\downarrow})(x) = (\beta(x))^{\downarrow} = F$ for all $x \in X$. Hence $\beta^{\downarrow} = F$ in $\hat{M}$.
((iii)) $\Rightarrow$ (ii)) Let $\beta^{\downarrow} = F$ in $\hat{M}$. Consider $\hat{M} \leq \mathfrak{Z}^X$ for some set X. Then $(\beta^{\downarrow})(x) = (\beta(x))^{\downarrow} = F$ for all $x \in X$. It follows that $\beta(x) \in \{F,U\}$ for all $x \in X$ and so $(\beta \wedge \mathbf{F})(x) = \beta(x)$ for all $x \in X$. Hence $\beta \wedge F = \beta$ in M. $\square$

The left-zeros of M play an important role in understanding the atomicity of M.

Notation 7.2.13. For $\varphi \in \mathfrak{Z}^X$ denote by $\varphi_{T,A}$ the element represented by the pair of sets (A, A^c) and $\varphi_{U,A}$ the element represented by the pair of sets $(\emptyset, A^c)$. If $A = \{x\}$ then we simply use the notation $\varphi_{T,x}$ and $\varphi_{U,x}$.

We now establish a relation between atoms of $M_{\#}$ and those of $M^c_{\#}$ for an ada M.

Theorem 7.2.14. *Let M be an ada. There exists a bijection between the sets $\mathscr{A}(M) \cap M^c_{\#}$ and $\mathscr{A}(M) \cap M_{\#}$.*

Proof. Consider the function $G : \mathscr{A}(M)\cap M_{\#}^{c} \to \mathscr{A}(M)\cap M_{\#}$ given by the following:

$$G(\alpha) = \neg((\neg\alpha)^{\downarrow}).$$

Let $\alpha \in \mathscr{A}(M)\cap M_{\#}^{c}$. It is straightforward to deduce that $G(\alpha) \in M_{\#}$. Consider $M \leq \mathfrak{Z}^X$ for some set X. Since α is a left-zero for $\wedge$ we have $\alpha = \alpha_{_{U,A}}$ for some $\emptyset \neq A \subseteq X$. It follows that $G(\alpha) = \neg((\neg\alpha)^{\downarrow}) = \delta_{_{T,A}}$. If $G(\alpha)$ is not an atom of $M_{\#}$ then there exists $\gamma = \gamma_{_{T,B}}$ where $\emptyset \neq B \subsetneq A$ and $\mathbf{F} \lneq \gamma \lneq \delta$. Thus $\beta = \gamma \wedge \mathbf{U} = \beta_{_{U,B}}$ and $\mathbf{F} \lneq \beta \lneq \alpha$ which is a contradiction to the fact that $\alpha \in \mathscr{A}(M)\cap M_{\#}^{c}$. It follows that G is well-defined.

Suppose that $\neg((\neg\alpha)^{\downarrow}) = \neg((\neg\beta)^{\downarrow})$ for some $\alpha, \beta \in \mathscr{A}(M)\cap M_{\#}^{c}$. Then $(\neg\alpha)^{\downarrow} = (\neg\beta)^{\downarrow} \in M_{\#}$. Consider $M \leq \mathfrak{Z}^X$ for some set X. Then $(\neg\alpha)^{\downarrow} = (\neg\beta)^{\downarrow} = \gamma_{_{T,A}}$ for some $A \subseteq X$. It follows that $\neg\alpha$ and $\neg\beta$ can be represented by the pairs of sets (A, B_α) and (A, B_β) where $B_\alpha, B_\beta \subseteq A^c$. Thus α and β can be represented by the pairs of sets (B_α, A) and (B_β, A) where $B_\alpha, B_\beta \subseteq A^c$. Since $\alpha, \beta \in \mathscr{A}(M)\cap M_{\#}^{c}$ we have $\alpha = \alpha_{_{U,C}}$ and $\beta = \beta_{_{U,D}}$ for some $C, D \subseteq X$. Hence in the representation for α and β that is (B_α, A) and (B_β, A) respectively we must have $B_\alpha = \emptyset = B_\beta$. It follows that $\alpha = \beta$ and so G is injective.

Let $\beta \in \mathscr{A}(M)\cap M_{\#}$. Consider $M \leq \mathfrak{Z}^X$ for some set X. It follows that $\beta = \beta_{_{T,A}}$ for some $\emptyset \neq A \subseteq X$. Consider $\alpha = \beta \wedge \mathbf{U} = \alpha_{_{U,A}} \in M_{\#}^{c}$. Along similar lines as in the proof for the well-definedness of G, we show that $\alpha \in \mathscr{A}(M)\cap M_{\#}^{c}$. Further, $G(\alpha) = \beta$ so that G is surjective.

$\square$

Corollary 7.2.15. *Let M be a finite ada. Then $|\mathscr{A}(M)|$ is even.*

7.3 Atomicity of $\mathfrak{Z}^X$

In this section we consider the C-algebra $\mathfrak{Z}^X$ and first establish a characterisation for its atoms.

Theorem 7.3.1. *Let X be any set. Then $\mathscr{A}(\mathfrak{Z}^X) = \{\alpha \in \mathfrak{Z}^X :$ there exists a unique $x_o \in X$ such that $\alpha(x_o) \in \{T, U\}\}$.*

Proof. Let $M = \mathfrak{Z}^X$ and $A = \{\alpha \in M :$ there exists a unique $x_o \in X$ such that $\alpha(x_o) \in \{T, U\}\}$. Let $\alpha \in A$. If α is not an atom then there exists $\beta \in M$ such that $\mathbf{F} \lneq \beta \lneq \alpha$. Since $\alpha \in A$ we have $\alpha(x) = F$ for all $x \neq x_o$. Thus since $F \leq \beta(x) \leq \alpha(x)$ we must have $\beta(x) = F$ for all $x \neq x_o$. It is clear that since $\beta \lneq \alpha$ we must have $\beta(x_o) \lneq \alpha(x_o)$ and so $\beta(x_o) = F$. This holds as $F \lneq T$ and $F \lneq U$ but $T \nleq U$ and $U \nleq T$. However this gives $\beta = \mathbf{F}$ which is a contradiction. Thus $\alpha \in \mathscr{A}(M)$.

Conversely suppose that $\alpha \in \mathscr{A}(M)$ but $\alpha \notin A$. Then either there exist $x_o, y_o \in X$ where $x_o \neq y_o$ and $\alpha(x_o), \alpha(y_o) \in \{T, U\}$ or we have $\alpha(x) = F$ for all $x \in X$. If $\alpha(x) = F$ for all $x \in X$ then clearly $\alpha = \mathbf{F}$ and so $\alpha \notin \mathscr{A}(M)$ which is a contradiction. If there exist $x_o, y_o \in X$ where $x_o \neq y_o$ and $\alpha(x_o), \alpha(y_o) \in \{T, U\}$ then consider $\beta \in M$ given by the following:

$$\beta(x) = \begin{cases} \alpha(x), & \text{if } x \neq x_o; \\ F, & \text{if } x = x_o. \end{cases}$$

It is easy to see that $F \leq \beta(x) \leq \alpha(x)$ for all $x \in X$ and so $\mathbf{F} \leq \beta \leq \alpha$. Since $\beta(x_o) = F \lneq \alpha(x_o)$ and $\beta(y_o) = \alpha(y_o) \in \{T, U\}$ we have $\mathbf{F} \lneq \beta \lneq \alpha$ which is a contradiction to the assumption that $\alpha \in \mathscr{A}(M)$. The result follows. □

This gives us the following result on the number of atoms in $\mathfrak{Z}^X$.

Corollary 7.3.2. *For $X \neq \emptyset$ we have $|\mathscr{A}(\mathfrak{Z}^X)| = 2 \times |X|$.*

In view of the fact that all finite adas are isomorphic to $\mathfrak{Z}^X$ (cf. Remark 1.4.15) we note that Corollary 7.3.2 is in fact a stronger version of Corollary 7.2.15.

We now study the set of atoms in $\mathfrak{Z}^X$ that have existence of $\oplus$.

Notation 7.3.3. Let $\alpha \in \mathscr{A}(\mathfrak{Z}^X)$. Using Theorem 7.3.1, denote by x_α the unique co-ordinate satisfying $\alpha(x_\alpha) \in \{T, U\}$.

Theorem 7.3.4. *Let $\{\alpha_i : 1 \leq i \leq N\}$ be a finite set of atoms in $\mathfrak{Z}^X$. Then $\bigoplus_{i=1}^{N} \alpha_i$ exists if and only if $x_{\alpha_i} \neq x_{\alpha_j}$ for all $i \neq j$. Further*

$$\bigoplus_{i=1}^{N} \alpha_i(x) = \begin{cases} \alpha_i(x_{\alpha_i}), & \text{if } x = x_{\alpha_i}; \\ F, & \text{otherwise.} \end{cases}$$

Proof. Let $x_{\alpha_i} \neq x_{\alpha_j}$ for all $i \neq j$. Thus using Theorem 7.3.1 for any $x \in X$ there exist at most one α_x in this collection such that $\alpha_x(x) \in \{T, U\}$. In view of the fact that F is a left and right-identity for $\vee$ we have

$$\alpha_1(x) \vee \alpha_2(x) \vee \cdots \vee \alpha_N(x) = \begin{cases} \alpha_i(x_{\alpha_i}), & \text{if } x = x_{\alpha_i} \text{ for some } 1 \leq i \leq N; \\ F, & \text{otherwise.} \end{cases}$$

Hence for any bijection $\sigma : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ we have:

$$\alpha_{\sigma(1)}(x) \vee \alpha_{\sigma(2)}(x) \vee \cdots \vee \alpha_{\sigma(N)}(x) = \begin{cases} \alpha_i(x_{\alpha_i}), & \text{if } x = x_{\alpha_i} \text{ for some } 1 \leq i \leq N; \\ F, & \text{otherwise.} \end{cases}$$

It follows that $\alpha_{\sigma(1)} \vee \alpha_{\sigma(2)} \vee \cdots \vee \alpha_{\sigma(N)} = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_N$ and so $\bigoplus_{i=1}^{N} \alpha_i$ exists.

Conversely suppose that $\bigoplus_{i=1}^{N} \alpha_i$ exists and $x_{\alpha_i} = x_{\alpha_j}$ for some $i \neq j$. Without loss of generality assume that $\alpha_i(x_{\alpha_i}) = T$ while $\alpha_j(x_{\alpha_i}) = U$. It follows that for the bijection $\sigma : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ given by

$$\sigma(n) = \begin{cases} i, & \text{if } n = 1; \\ j, & \text{if } n = 2; \\ n, & \text{otherwise} \end{cases}$$

we have $\alpha_{\sigma(1)}(x_{\alpha_i}) \vee \alpha_{\sigma(2)}(x_{\alpha_i}) \vee \cdots \vee \alpha_{\sigma(N)}(x_{\alpha_i}) = \alpha_i(x_{\alpha_i}) \vee \alpha_j(x_{o_{\alpha_i}}) \vee \cdots \vee \alpha_{\sigma(N)}(x_{o_{\alpha_i}}) = T \vee U \vee \cdots \vee \alpha_{\sigma_1(N)}(x_{\alpha_i}) = T$, since T is a left-zero for $\vee$.

On the other hand for the bijection $\tau : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ where

$$\tau(n) = \begin{cases} j, & \text{if } n = 1; \\ i, & \text{if } n = 2; \\ n, & \text{otherwise} \end{cases}$$

we have $\alpha_{\tau(1)}(x_{\alpha_i}) \vee \alpha_{\tau(2)}(x_{\alpha_i}) \vee \cdots \vee \alpha_{\tau(N)}(x_{\alpha_i}) = \alpha_j(x_{\alpha_i}) \vee \alpha_i(x_{\alpha_i}) \vee \cdots \vee \alpha_{\tau(N)}(x_{\alpha_i}) = U \vee T \vee \cdots \vee \alpha_{\tau(N)}(x_{\alpha_i}) = U$, since U is a left-zero for $\vee$. This is a contradiction to the assumption that $\bigoplus_{i=1}^{N} \alpha_i$ exists and the result follows. The expression for $\bigoplus \alpha_i$ is also clear from the above proof. □

Theorem 7.3.5. *If X is finite then $\mathfrak{Z}^X$ is atomic.*

Proof. Let $\beta \in \mathfrak{Z}^X$ such that $\beta \neq \mathbf{F}$. Using the pairs of sets representation of $\mathfrak{Z}^X$ identify β with the pair of sets (A, B). Since $\beta \neq \mathbf{F}$ it follows that $B^c \neq \emptyset$. Consider the family of elements defined by the following for $y \in B^c$:

$$\alpha_y(x) = \begin{cases} \beta(y), & \text{if } x = y; \\ F, & \text{otherwise.} \end{cases}$$

Using Theorem 7.3.1 since $\alpha_y(y) = \beta(y) \in \{T, U\}$ we have $\alpha_y \in \mathscr{A}(\mathfrak{Z}^X)$ for each $y \in B^c$. Further there are finitely many α_y since X and therefore B^c is finite. Note that $x_{\alpha_y} = y$ and so $x_{\alpha_y} \neq x_{\alpha_z}$ for $y \neq z$. Consequently, using Theorem 7.3.4 $\bigoplus_{y \in B^c} \alpha_y$ exists.

For $x \in B$ we have $\beta(x) = F$. Also $\alpha_y(x) = F$ for all $y \in B^c$ and so $\bigoplus \alpha_y(x) = F = \beta(x)$. For $x \in B^c$ using Theorem 7.3.4 we have $\bigoplus \alpha_y(x) = \alpha_x(x) = \beta(x)$. Thus we have a finite set $\{\alpha_y : y \in B^c\} \subseteq \mathscr{A}(\mathfrak{Z}^X)$ such that $\bigoplus \alpha_y = \beta$. Hence $\mathfrak{Z}^X$ is atomic. □

Remark 7.3.6. Note that $\mathfrak{Z}^X$ where X is infinite will be non-atomic since the element **T** can never be expressed in terms of finitely many atoms.

7.4 g-closed C-algebras

In this section we consider $M \leq \mathfrak{Z}^X$ and try to understand the atomicity of M from information about the atoms of $\mathfrak{Z}^X$. First we justify the feasibility of this approach.

Remark 7.4.1. Let $\phi : M \to \mathfrak{Z}^X$ be a C-algebra embedding. Then ϕ is also order-preserving. Let $x \leq y \in M$. Then $\phi(x) \vee \phi(y) = \phi(x \vee y) = \phi(y)$ and so $\phi(x) \leq \phi(y)$.

Thus we make use of the notion of atoms in $\mathfrak{Z}^X$ to gain an understanding of the same in M where $M \leq \mathfrak{Z}^X$. In this section we assume that $M \leq \mathfrak{Z}^X$.

Remark 7.4.2. It is straightforward to verify that $M \cap \mathscr{A}(\mathfrak{Z}^X) \subseteq \mathscr{A}(M)$. In general the inclusion could be proper.

To illustrate this consider

$$M = \{(T,T), (F,F), (U,U), (F,U), (U,F), (T,U), (U,T)\}$$

where $M \leq \mathfrak{Z}^2$. Then $\mathscr{A}(M) = \{(F,U), (U,F), (T,T)\} \supsetneq M \cap \mathscr{A}(\mathfrak{Z}^2)$ since $(T,T) \notin \mathscr{A}(\mathfrak{Z}^2)$.

Thus not all atoms of M are atoms of $\mathfrak{Z}^X$. The atoms of M that remain atoms in $\mathfrak{Z}^X$ are in some sense *global* atoms. If every atom of M is an atom of $\mathfrak{Z}^X$, and therefore, a global atom, then M is closed with respect to global atoms. Thus we define the following notion.

Definition 7.4.3. *M is said to be closed with respect to global atoms in $\mathfrak{Z}^X$ if $\mathscr{A}(M) \subseteq \mathscr{A}(\mathfrak{Z}^X)$. In short we say that M is g-closed in $\mathfrak{Z}^X$.*

Remark 7.4.4. If M is g-closed in $\mathfrak{Z}^X$ then we have $\mathscr{A}(M) = M \cap \mathscr{A}(\mathfrak{Z}^X)$.

Remark 7.4.5. Consider $M \leq \mathfrak{Z}^2$. The subalgebras of $\mathfrak{Z}^2$ are as follows:

$$\begin{aligned}
M_0 &= \{(T,T),(F,F),(U,U)\},\\
M_1 &= \{(T,T),(F,F),(U,U),(F,U),(T,U)\},\\
M_2 &= \{(T,T),(F,F),(U,U),(U,F),(U,T)\},\\
M_3 &= \{(T,T),(F,F),(U,U),(F,U),(T,U),(U,F),(U,T)\},\\
M_4 &= \mathfrak{Z}^2.
\end{aligned}$$

The set of atoms of each subalgebra are as follows:

$$\begin{aligned}
\mathscr{A}(M_0) &= \{(T,T),(U,U)\},\\
\mathscr{A}(M_1) &= \{(T,T),(F,U)\},\\
\mathscr{A}(M_2) &= \{(T,T),(U,F)\},\\
\mathscr{A}(M_3) &= \{(T,T),(F,U),(U,F)\},\\
\mathscr{A}(M_4) &= \mathfrak{Z}^2.
\end{aligned}$$

Since $(T,T) \in \mathscr{A}(M_i)$ for $0 \leq i \leq 3$ and $(T,T) \notin \mathscr{A}(\mathfrak{Z}^2)$, no proper subalgebra is g-closed in $\mathfrak{Z}^2$.

We ascertain all the globally closed subalgebras of $\mathfrak{Z}^X$. To that aim we first have the following result.

Lemma 7.4.6. *Let $M \leq \mathfrak{Z}^X$ where X is finite. If $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$ then $M = \mathfrak{Z}^X$.*

Proof. Let $\alpha \in \mathfrak{Z}^X$. If $\alpha = \mathbf{F}$ then we are done since $\mathbf{F} \in M$. Suppose that $\alpha \neq \mathbf{F}$. Then α can be represented by the pair of sets (A, B) where $B^c \neq \emptyset$. Consider as earlier for each $y \in B^c$ the family of elements given below:

$$\alpha_y(x) = \begin{cases} \alpha(y), & \text{if } x = y; \\ F, & \text{otherwise.} \end{cases}$$

Using Theorem 7.3.1 we have $\alpha_y \in \mathscr{A}(\mathfrak{Z}^X)$. Further $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$ gives $\alpha_y \in \mathscr{A}(M) \subseteq M$. Note that since X is finite, so is B^c. Consequently there are only finitely many such α_y. Moreover using Theorem 7.3.4 $\bigoplus \alpha_y$ exists and so $\bigoplus \alpha_y \in \mathfrak{Z}^X$ so that $\bigoplus \alpha_y \in M$. It is straightforward to verify that $\bigoplus\limits_{y \in B^c} \alpha_y = \alpha$ so that $\alpha \in M$. Thus $M = \mathfrak{Z}^X$. □

Theorem 7.4.7. *Let M be g-closed in $\mathfrak{Z}^X$ where X is finite. Then $M = \mathfrak{Z}^X$.*

Proof. We describe an algorithmic mechanism to generate all atoms from one. In view of Lemma 7.4.6, on obtaining $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$ we then have $M = \mathfrak{Z}^X$. It suffices to show that $\alpha_{T,x} \in M$ for each $x \in X$, because if $\alpha_{T,x} \in M$ then $\alpha_{T,x} \wedge \mathbf{U} = \alpha_{U,x} \in M$.

Since X is finite we have M is finite. Using Proposition 7.2.1 and Proposition 7.2.3 for $\mathbf{T} \in M$ there exists $\alpha \in \mathscr{A}(M)$ such that $\alpha \leq \mathbf{T}$ so that $\alpha \in M_{\#} \leq 2^X$. Since M is g-closed in $\mathfrak{Z}^X$ we have $\alpha \in \mathscr{A}(\mathfrak{Z}^X)$ so that $\alpha = \alpha_{T,x_1}$ for some $x_1 \in X$.

Define $\beta_1 = \alpha_{T,x_1}$ and so $\neg\beta_1 = \neg\alpha_{T,x_1} = \alpha_{T,X\setminus\{x_1\}}$. If $\neg\beta_1$ is an atom then $X \setminus \{x_1\}$ is a singleton and so $X = \{x_1, x_2\}$ and so the algebra is $\mathfrak{Z}^2$. The only subalgebra g-closed in $\mathfrak{Z}^2$ is itself and we are done.

If $\neg\beta_1$ is not an atom then there exists $\alpha_{T,x_2} \in \mathscr{A}(M)$ such that $\alpha_{T,x_2} \leq \neg\beta_1 \leq \mathbf{T}$. Define $\beta_2 = \alpha_{T,x_2}$ and so $\neg\beta_2 = \neg\alpha_{T,x_2} = \alpha_{T,X\setminus\{x_2\}}$. If $\neg\beta_2$ is an atom then we are through. Else there exists $\alpha_{T,x_3} \in \mathscr{A}(M)$ such that $\alpha_{T,x_3} \leq \neg\beta_2 \leq \mathbf{T}$. Define $\beta_3 = \alpha_{T,x_3}$ and so $\neg\beta_3 = \neg\alpha_{T,x_3} = \alpha_{T,X\setminus\{x_3\}}$ and so on.

This process can take at most $|X|$ steps. Further, as mentioned above if $\alpha_{T,x} \in M$ then $\alpha_{T,x} \wedge \mathbf{U} = \alpha_{U,x} \in M$ so that $\mathscr{A}(M) = \mathscr{A}(\mathfrak{Z}^X)$. Hence $M = \mathfrak{Z}^X$. □

Corollary 7.4.8. *The collection of g-closed subalgebras in $\mathfrak{Z}^X$ where X is finite comprises atomic algebras.*

Remark 7.4.9.

(i) Let $M \leq \mathfrak{Z}^X$ where X is finite and M is not g-closed in $\mathfrak{Z}^X$. Then M may be atomic. For instance, consider $M = M_0 \leq \mathfrak{Z}^2$ as given in Remark 7.4.5. We know that M_0 is not g-closed in $\mathfrak{Z}^2$. However M_0 is clearly atomic.

(ii) Let $M \leq \mathfrak{Z}^X$ where X is finite, M is non-trivial and M is not g-closed in $\mathfrak{Z}^X$. Then M may still be atomic. Consider $M = \{(T,T,T,T),(F,F,F,F),(U,U,U,U),(T,T,F,F),(F,F,T,T),(U,U,F,F),(U,U,T,T),(F,F,U,U),(T,T,U,U)\} \leq \mathfrak{Z}^4$. In this case $\mathscr{A}(M) = \{(T,T,F,F),(F,F,T,T),(U,U,F,F),(F,F,U,U)\}$ and so it is not g-closed in $\mathfrak{Z}^4$. However M is atomic.

7.5 Non-atomic C-algebras

In this section we investigate the relation between the atomicity of $M_\#$ and that of M. It is a straightforward assertion that if M is a finite C-algebra with T, F, U then no such relation holds since $M_\#$ is always atomic but M need not be so. However the question stands in the case where M is infinite. In this section we consider M to be a C-algebra with T, F, U unless otherwise mentioned.

Theorem 7.5.1. *If $M_\#$ is non-atomic then M is non-atomic.*

Proof. If possible let $M_\#$ be non-atomic and M be atomic. Let $a \in M_\# \subseteq M$ then there exist finitely many $a_i \in \mathscr{A}(M)$ such that $a = \bigoplus a_i$. Using Proposition 7.2.7 we have $a_i \leq a$. Moreover, using Proposition 7.2.3 we have $a_i \in M_\#$. Further using Proposition 7.2.8 we have $\mathscr{A}(M) \cap M_\# = \mathscr{A}(M_\#)$ so that $a_i \in \mathscr{A}(M_\#)$ and $a = \bigoplus a_i$. Thus $M_\#$ is atomic, a contradiction. □

The following result relates to atomless adas.

Theorem 7.5.2. *Let M be an ada. If $M_\#$ is atomless then M is atomless.*

Proof. If possible let $M_\#$ be atomless but M not be atomless. Therefore let $\alpha \in \mathscr{A}(M)$. It is clear that $\alpha \notin M_\#$ since otherwise using Proposition 7.2.8 we have $\alpha \in \mathscr{A}(M) \cap M_\# = \mathscr{A}(M_\#)$ which is a contradiction since $M_\#$ is atomless. Thus $\alpha \in M_\#^c$ and so $\alpha^\downarrow \neq \alpha$ (cf. Remark 1.4.13). We have the following cases.

Case I: $\alpha^\downarrow \neq F$: The ada identity $\alpha^\downarrow \vee \alpha = \alpha$ holds in $\mathfrak{Z}$ and therefore in all adas. Thus we have $F \lneq \alpha^\downarrow \lneq \alpha$ which is a contradiction since $\alpha \in \mathscr{A}(M)$.

Case II: $\alpha^\downarrow = F$: Using Proposition 7.2.8 we have $\alpha \in \{a \in M : a \wedge b = a \text{ for all } b \in M\}$. Consider $M \leq \mathfrak{Z}^X$ for some set X. It follows that $\alpha = \alpha_{U,A}$ for some $A \subseteq X$. This is true since if $\alpha(x) = T$ for some $x \in X$ then $\alpha^\downarrow(x) = T$ and so $\alpha^\downarrow \neq \mathbf{F}$. Also $A \neq \emptyset$ since $\alpha \neq \mathbf{F}$. Then

$$\neg\alpha(x) = \neg\alpha_{U,A}(x) = \begin{cases} U, & \text{if } x \in A; \\ T, & \text{otherwise.} \end{cases}$$

Then $(\neg\alpha)^\downarrow \in M$ since M is an ada so that

$$(\neg\alpha)^\downarrow(x) = \begin{cases} F, & \text{if } x \in A; \\ T, & \text{otherwise.} \end{cases}$$

In fact $(\neg\alpha)^\downarrow \in M_\#$. Consider $\neg((\neg\alpha)^\downarrow) \in M_\#$ where in fact $\neg((\neg\alpha)^\downarrow) = \beta_{T,A} \in M_\#$. Since $M_\#$ is atomless it follows that there exists $\beta_{T,B} \in M_\#$ where $\emptyset \neq B \subsetneq A$ and $\mathbf{F} \lneq \beta_{T,B} \lneq \beta_{T,A}$. Consider $\beta_{U,B} = \beta_{T,B} \wedge \mathbf{U} \in M$. Since $\emptyset \neq B \subsetneq A$ we have $\mathbf{F} \lneq \beta_{U,B} \lneq \alpha_{U,A} = \alpha$ which is a contradiction to the fact that $\alpha \in \mathscr{A}(M)$. □

Remark 7.5.3. Theorem 7.5.2 allows us to construct an atomless ada from an atomless Boolean algebra. For an atomless Boolean algebra B, the ada $B^\star$ will also be atomless. For further reading on atomless Boolean algebras refer to Givant and Halmos [2009].

Theorem 7.5.4. *Let M be a finite C-algebra with T, F, U such that $|M| > 3$ and $T \in \mathscr{A}(M)$. Then M is not atomic.*

Proof. Since $T \in \mathscr{A}(M)$ it is clear that $M_{\#} = \{T, F\}$. Since $|M| > 3$ there exists $\gamma \in M \setminus \{T, F, U\}$ and since M is finite, using Proposition 7.2.1 there exists $\alpha \in \mathscr{A}(M)$ such that $\alpha \leq \gamma$. Clearly $\alpha \in \mathscr{A}(M) \cap M_{\#}^c$.

Consider $M \leq 3^X$ for some set X. Then $\alpha = \alpha_{U,A}$ for some $\emptyset \neq A \subseteq X$. Suppose that $A = X$. Then $\alpha = \mathbf{U} \in \mathscr{A}(M)$. Hence $M = \{\mathbf{T}, \mathbf{F}, \mathbf{U}\}$ else if there was some $\beta \in M_{\#}^c \setminus \{\mathbf{U}\}$ then using Proposition 7.2.5(ii) we have $\mathbf{F} \lneq \beta \wedge F \lneq \mathbf{U}$ which is a contradiction to the fact that $\mathbf{U} \in \mathscr{A}(M)$. Thus $M = \{\mathbf{T}, \mathbf{F}, \mathbf{U}\}$, a contradiction to our assumption that M is non-trivial. Thus $\alpha = \alpha_{U,A}$ where $\emptyset \neq A \subsetneq X$.

Suppose that M is atomic. Consider $\neg\alpha \in M_{\#}^c$. There exist finitely many $a_i \in \mathscr{A}(M)$ such that $\neg\alpha = \bigoplus a_i$. Clearly $\mathbf{T} \notin \{a_i\}$ since $\mathbf{T} \vee a = \mathbf{T} \neq \neg\alpha$. Since $M_{\#} = \{\mathbf{T}, \mathbf{F}\}$ we have $\mathscr{A}(M) \setminus \{\mathbf{T}\} \subseteq M_{\#}^c$. Thus $a_i = a_{U,A_i}$ for $\emptyset \neq A_i \subseteq X$. However $\neg\alpha = \neg\alpha_{U,A}$ where $\emptyset \neq A \subsetneq X$ and so we have

$$\neg\alpha(x) = \begin{cases} U, & \text{if } x \in A; \\ T, & \text{otherwise.} \end{cases}$$

Moreover $\neg\alpha = \bigoplus a_{U,A_i}$ gives

$$\neg\alpha(x) = \begin{cases} U, & \text{if } x \in A_i \text{ for some i;} \\ F, & \text{otherwise.} \end{cases}$$

This is a contradiction since $A \subsetneq X$ which implies that $\neg\alpha(x_o) = T$ for some $x_o \in X$. Hence M is not atomic. □

Corollary 7.5.5. *Let M be a finite C-algebra with T, F, U such that $|M| > 3$. Then*

$\overline{M_{\#}^c} = M_{\#}^c \cup \{T, F\}$ *is never atomic.*

Proof. Note that $\overline{M_{\#}^c}$ is also a non-trivial finite C-algebra with T, F, U. Since $(\overline{M_{\#}^c})_{\#} = \{T, F\}$ we have $T \in \mathscr{A}(\overline{M_{\#}^c})$. The result follows from Theorem 7.5.4. $\square$

Remark 7.5.6. The converse of Theorem 7.5.4 need not be true. That is, if M be a C-algebra with T, F, U such that M is not atomic then T need not be in $\mathscr{A}(M)$. Consider $M = \{(T,T,T), (F,F,F), (U,U,U), (U,F,F), (U,T,T), (F,F,T), (T,T,F), (F,F,U), (T,T,U), (U,U,F), (U,T,F), (U,F,T), (U,F,U), (U,T,U), (U,U,T)\} \leq \mathfrak{Z}^3$. Then $(T,T,T) \notin \mathscr{A}(M)$ since $\mathscr{A}(M) = \{(U,F,F), (F,F,T), (T,T,F), (F,F,U)\}$. However M is not atomic since (U,T,F) can only be written as join of atoms (U,F,F) and (T,T,F) but the $\oplus$ of these atoms is not defined.

7.6 Finite atomic C-algebras

In this section we establish a characterisation of all finite atomic C-algebras. First we establish some results on the existence of $\oplus$ in M where M is an arbitrary C-algebra with T, F, U.

Proposition 7.6.1. *Consider $M \leq \mathfrak{Z}^X$ for some set X and let $\alpha_i \in M$ for $1 \leq i \leq N$ be represented by the pairs of sets (A_i, B_i) respectively. Then $\bigoplus_{1 \leq i \leq N} \alpha_i$ exists if and only if $A_i \cap (A_j \cup B_j)^c = \emptyset$ for all $i, j \in I$.*

Proof. If $A_i \cap (A_j \cup B_j)^c = \emptyset$ for all $i, j \leq N$ then

$$\alpha_1(x) \vee \alpha_2(x) \vee \cdots \vee \alpha_N(x) = \begin{cases} T, & \text{if } x \in A_1; \\ U, & \text{if } x \in (A_1 \cup B_1)^c; \\ \alpha_2(x) \vee \alpha_3(x) \vee \cdots \vee \alpha_N(x), & \text{otherwise.} \end{cases}$$

$$= \begin{cases} T, & \text{if } x \in A_1; \\ U, & \text{if } x \in (A_1 \cup B_1)^c; \\ T, & \text{if } x \in A_2; \\ U, & \text{if } x \in (A_2 \cup B_2)^c; \\ \alpha_3(x) \vee \cdots \vee \alpha_N(x), & \text{otherwise.} \end{cases}$$

Note that the well-definedness of this expression follows from the fact that $A_i \cap (A_j \cup B_j)^c = \emptyset$ so that we do not have $x \in A_1 \cap (A_2 \cup B_2)^c$ or $x \in A_2 \cap (A_1 \cup B_1)^c$. This process yields the following:

$$\alpha_1(x) \vee \alpha_2(x) \vee \cdots \vee \alpha_N(x) = \begin{cases} T, & \text{if } x \in \bigcup A_i; \\ U, & \text{if } x \in \bigcup (A_i \cup B_i)^c; \\ F, & \text{otherwise} \end{cases}$$

which is well-defined and establishes that the join is independent of the order of the elements. Consequently $\bigoplus_{1 \leq i \leq N} \alpha_i$ exists and can be expressed as follows:

$$\bigoplus_{1 \leq i \leq N} \alpha_i(x) = \begin{cases} T, & \text{if } x \in A_i \text{ for some } 1 \leq i \leq N; \\ U, & \text{if } x \in (A_i \cup B_i)^c \text{ for some } 1 \leq i \leq N; \\ F, & \text{otherwise.} \end{cases}$$

Conversely, suppose if possible that $x \in A_i \cap (A_j \cup B_j)^c$ for some $x \in X$ and some

$i, j \leq N$ where $i \neq j$. Then $(\alpha_i \vee \alpha_j)(x) = T \vee U = T$ while $(\alpha_j \vee \alpha_i)(x) = U \vee T = U$, a contradiction to the fact that $\bigoplus_{1 \leq i \leq N} \alpha_i$ is defined. The result follows. $\square$

Proposition 7.6.2. *Let $\alpha_i \in M$ for $i \in I$ where I is finite, such that $\bigoplus_{i \in I} \alpha_i$ exists. Consider $\emptyset \neq J \subseteq I$. Then $\bigoplus_{j \in J} \alpha_j$ exists.*

Proof. Consider $M \leq 3^X$ for some set X. Let α_i be identified with the pair of sets (A_i, B_i) for each $i \in I$. Since $\bigoplus_{i \in I} \alpha_i$ exists, using Proposition 7.6.1 we have $A_{i_1} \cap (A_{i_2} \cup B_{i_2})^c = \emptyset$ for all $i_1, i_2 \in I$. Thus $A_{j_1} \cap (A_{j_2} \cup B_{j_2})^c = \emptyset$ for all $j_1, j_2 \in J$ so that $\bigoplus_{j \in J} \alpha_j$ exists. $\square$

We now arrive at the main result in this section.

Theorem 7.6.3. *Let M be a finite C-algebra with T, F, U. M is atomic if and only if M is an ada.*

Proof. ($\Leftarrow$) In view of Remark 1.4.15 we have M is isomorphic to 3^X for some finite set X. Using Theorem 7.3.5 we establish that M is atomic.

($\Rightarrow$) If possible let M be atomic and M not be an ada. Then $M \lneq \hat{M}$ where $\hat{M}$ is the enveloping ada of M. Consider $\hat{M} \leq 3^X$ as adas for some finite set X. Thus $M \leq \hat{M} \leq 3^X$ as C-algebras.

Since $M \lneq \hat{M}$ there exists $\gamma \in M$ such that $\gamma^{\downarrow} \notin M$. Therefore there exists $x_1 \in X$ such that $\gamma(x_1) = T$ since otherwise $\gamma^{\downarrow} = \mathbf{F} \in M$. Further, there exists $x_2 \in X$ such that $\gamma(x_2) = U$ since otherwise $\gamma^{\downarrow} = \gamma \in M$, a contradiction. Hence γ can be identified with the pair of sets (A, B) where $A \neq \emptyset \neq (A \cup B)^c$.

Since M is atomic there exist α_i where $i \in I$ (I: finite) and $\alpha_i \in \mathscr{A}(M) \cap M_{\#}$ and β_j where $j \in J$ (J: finite) and $\beta_j \in \mathscr{A}(M) \cap M_{\#}^c$ such that

$$\gamma = (\bigoplus \alpha_i) \oplus (\bigoplus \beta_j).$$

It is clear that each α_i can be identified with the pair of sets (A_i, A_i^c) and that each

β_j can be identified with the pair of sets $(\emptyset, B_j^c)$ where $A_i, B_j \subseteq X$. In other words $\alpha_i = \alpha_{_{T,A_i}}$ and $\beta_j = \beta_{_{U,B_j}}$.

Since we have ascertained that $A \neq \emptyset \neq (A \cup B)^c$ we have $I \neq \emptyset \neq J$. Since $\oplus$ is defined, using Proposition 7.6.1 we have $A_i \cap (\emptyset \cup (B_j)^c)^c = A_i \cap B_j = \emptyset$ for all $i \in I$ and $j \in J$. Further, $\bigcup A_i = A$.

Since I is finite we have $\bigoplus \alpha_i \in M_{\#} \subseteq M$. Also $\bigoplus \alpha_i = \gamma^{\downarrow}$ since $\gamma^{\downarrow}$ is represented by the pair of sets (A, A^c) and $\bigoplus \alpha_i$ is represented by the pair of sets $(\bigcup A_i, (\bigcup A_i)^c)$. Thus $\gamma^{\downarrow} \in M$ which is a contradiction. The result follows. □

7.7 Conclusion

In this chapter we have defined the notions of atoms and atomicity in C-algebras by adopting the partial order of Chang [1958]. After obtaining some properties related to atomicity, we obtain a characterisation of atoms in $\mathfrak{Z}^X$. We have also presented necessary or sufficient conditions for the atomicity of C-algebras and have showed that the class of finite atomic C-algebras is precisely that of finite adas.

A point of interest would be to enquire whether the representation of elements through atoms by $\oplus$ as defined in this chapter is unique. Further, by the definition of atomic C-algebras proposed by us, we note that $\mathfrak{Z}^X$ is not atomic for infinite X. It is therefore desirable to obtain a suitable definition for atomicity so that $\mathfrak{Z}^X$ is atomic for any set X.

Bibliography

Arnold, A.: 1978, Sémantique algébrique de l'appel par valeur, *RAIRO Inform. Théor.* **12**(2), 69–82, iv.

Belnap, Jr., N. D.: 1970, Conditional assertion and restricted quantification, *Noûs* **4**(1), 1–13. Commentators: W. V. Quine and J. Michael Dunn, Symposia to be held at the 68th Annual Meeting of the American Philosophical Association, Western Division (St. Louis, Mo., 1970).

Bergman, G. M.: 1991, Actions of Boolean rings on sets, *Algebra Universalis* **28**(2), 153–187.

Bergstra, J. A., Bethke, I. and Rodenburg, P.: 1995, A propositional logic with 4 values: true, false, divergent and meaningless, *J. Appl. Non-Classical Logics* **5**(2), 199–217.

Birkhoff, G.: 1935, On the structure of abstract algebras, *Proc. Camb. Philos. Soc.* **31**, 433–454.

Birkhoff, G.: 1944, Subdirect unions in universal algebra, *Bull. Amer. Math. Soc.* **50**, 764–768.

Birkhoff, G. and Lipson, J. D.: 1970, Heterogeneous algebras, *J. Combinatorial Theory* **8**, 115–133.

Bloom, S. L. and Tindell, R.: 1983, Varieties of “if-then-else”, *SIAM J. Comput.* **12**, 677–707.

Bochvar, D. A.: 1981, Ob odnom tréhznacnom isčislenii i égo priménénii k analiza paradoksov klassičéskogo răssirénnogo funkcional’nogo isčisléniá (in Russian). *matématičeskij sbornik*, 4: 287–308, 1939. Translated to English by M. Bergmann “On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus”, *History and Philosophy of Logic* **2**, 87–112.

Burris, S. and Sankappanavar, H. P.: 1981, *A course in universal algebra*, Vol. 78 of *Graduate Texts in Mathematics*, Springer-Verlag, New York-Berlin.

Chang, C. C.: 1958, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.* **88**, 467–490.

Climent Vidal, J. and Soliveres Tur, J.: 2015, On the directly and subdirectly irreducible many-sorted algebras, *Demonstr. Math.* **48**(1), 1–12.

Cornish, W. H.: 1973, Annulets and α-ideals in a distributive lattice, *J. Austral. Math. Soc.* **15**, 70–77.

Courcelle, B.: 1979/80, Infinite trees in normal form and recursive equations having a unique solution, *Math. Systems Theory* **13**(2), 131–180.

Courcelle, B. and Nivat, M.: 1976, Algebraic families of interpretations, *17th Annual Symposium on Foundations of Computer Science (Houston, Tex., 1976)*, IEEE Comput. Soc., Long Beach, Calif., pp. 137–146.

de Bakker, J. W.: 1969, Semantics of programming languages, *Advances in Information Systems Science*, Springer, pp. 173–227.

Desharnais, J., Jipsen, P. and Struth, G.: 2009, Domain and antidomain semigroups, *Relations and Kleene algebra in computer science*, Vol. 5827 of *Lecture Notes in Comput. Sci.*, Springer, Berlin, pp. 73–87.

Givant, S. and Halmos, P.: 2009, *Introduction to Boolean Algebras*, Springer Science+Business Media, New York, NY.

Goguen, J. A. and Meseguer, J.: 1985, Completeness of many-sorted equational logic, *Houston J. Math.* **11**(3), 307–334.

Grätzer, G.: 1969, Free Σ-structures, *Trans. Amer. Math. Soc.* **135**, 517–542.

Grätzer, G.: 2008, *Universal Algebra*, Springer, New York, NY.

Guessarian, I. and Meseguer, J.: 1987, On the axiomatization of "if-then-else", *SIAM J. Comput.* **16**(2), 332–357.

Guzmán, F. and Squier, C. C.: 1990, The algebra of conditional logic, *Algebra Universalis* **27**(1), 88–110.

Heyting, A.: 1934, Die formalen regeln der intuitionistischen logik, sitzungsberichte der preuszischen akademie der wissenschaften, physikalischmathematische klasse,(1930), 42–56 57–71 158–169 in three parts, *Sitzungsber. preuss. Akad. Wiss* **42**, 158–169.

Higgins, P. J.: 1963, Algebras with a scheme of operators, *Math. Nachr.* **27**, 115–132.

Igarashi, S.: 1971, Semantics of ALGOL-like statements, *Symposium on Semantics of Algorithmic Languages*, Springer, pp. 117–177.

Jackson, M. and Stokes, T.: 2003, Agreeable semigroups, *J. Algebra* **266**(2), 393–417.

Jackson, M. and Stokes, T.: 2009, Semigroups with if-then-else and halting programs, *Int. J. Algebra Comput.* **19**(7), 937–961.

Jackson, M. and Stokes, T.: 2015, Monoids with tests and the algebra of possibly non-halting programs, *J. Log. Algebr. Methods Program.* **84**(2), 259–275.

Kennison, J. F.: 1981, Triples and compact sheaf representation, *J. Pure Appl. Algebra* **20**(1), 13–38.

Kleene, S. C.: 1938, On notation for ordinal numbers, *The Journal of Symbolic Logic* **3**(4), 150–155.

Kleene, S. C.: 1952, *Introduction to metamathematics*, D. Van Nostrand Co., Inc., New York, N. Y.

Lukasiewicz, J.: 1920, On three-valued logic. Ruch Filozoficzny, 5,(1920), English translation in Borkowski, L.(ed.) 1970. Jan Lukasiewicz: Selected Works.

Mac Lane, S.: 1971, *Categories for the working mathematician*, Springer-Verlag, New York-Berlin. Graduate Texts in Mathematics, Vol. 5.

Mandelker, M.: 1970, Relative annihilators in lattices, *Duke Math. J.* **37**, 377–386.

Manes, E. G.: 1990, A transformational characterization of if-then-else, *Theoret. Comput. Sci.* **71**(3), 413–417.

Manes, E. G.: 1993, Adas and the equational theory of if-then-else, *Algebra Universalis* **30**(3), 373–394.

Manna, Z. and Vuillemin, J.: 1972, Fixpoint approach to the theory of computation, *Comm. ACM* **15**(7), 528–536. Twenty-fifth anniversary of the Association for Computing Machinery.

McCarthy, J.: 1963, A basis for a mathematical theory of computation, *Computer programming and formal systems*, North-Holland, Amsterdam, pp. 33–70.

Mekler, A. H. and Nelson, E. M.: 1987, Equational bases for if-then-else, *SIAM J. Comput.* **16**(3), 465–485.

Panicker, G., Krishna, K. V. and Bhaduri, P.: 2017a, Axiomatization of `if-then-else` over possibly non-halting programs and tests, *Int. J. Algebra Comput.* **27**(3), 273–297.

Panicker, G., Krishna, K. V. and Bhaduri, P.: 2017b, Monoids of non-halting programs with tests, *Algebra Universalis* . Accepted.

Pigozzi, D.: 1991, Equality-test and if-then-else algebras: Axiomatization and specification, *SIAM J. Comput.* **20**(4), 766–805.

Rao, M. S.: 2013, On annihilator ideals of C-algebras, *Asian-Eur. J. Math.* **6**(1), 1350009, 11.

Satyanarayana, A.: 2012, *Algebraic study of certain classes of pre $A^\star$-algebras and C-algebras*, PhD thesis, Acharya Nagarjuna University, India.
URL: *http://hdl.handle.net/10603/8054*

Sethi, R.: 1978, Conditional expressions with equality tests, *J. ACM* **25**(4), 667–674.

Stokes, T.: 1998, Sets with B-action and linear algebra, *Algebra Universalis* **39**(1-2), 31–43.

Stokes, T.: 2010, Comparison semigroups and algebras of transformations, *Semigroup Forum* **81**(2), 325–334.

Swamy, U. M., Rao, G. C. and Ravi Kumar, R. V. G.: 2004, Representation of C-algebras by sections of sheaves, *Southeast Asian Bull. Math.* **28**(6), 1121–1128.

Vali, S. K., Sundarayya, P. and Swamy, U. M.: 2010, Ideals of C-algebras, *Asian-Eur. J. Math.* **3**(3), 501–509.

Vali, S. K., Sundarayya, P. and Swamy, U. M.: 2014, Weak ideals of a C-algebra, *Southeast Asian Bull. Math.* **38**(1), 141–151.

Vali, S. K., Sundarayya, P. and Swamy, U. M.: 2015, Annihilators of C-algebras, *Southeast Asian Bull. Math.* **39**(4), 529–538.

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