Ulam-Hyers and Lyapunov Stability for Some Classes of Fractional Differential Equations and Difference Equations

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A Thesis

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the requirements for the degree of

Doctor of Philosophy

by

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Dedicated

to the loving and grateful memory of

my late mother Smt. Mangalmati Shankar (Aamaa)

and

my late father Shri Jiwan Shankar (Baba).

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Declaration

I do hereby declare that this thesis entitled **"Ulam-Hyers and Lyapunov Stability for Some Classes of Fractional Differential Equations and Difference Equations"** is a presentation of my original research work carried out under the supervision of **Dr. Swaroop Nandan Bora**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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Certificate

It is to certify that the work contained in this thesis entitled **"Ulam-Hyers and Lyapunov Stability for Some Classes of Fractional Differential Equations and Difference Equations"** has been carried out by **Mr. Matap Shankar**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under my supervision for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

November 1, 2023

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Abstract

Studying the behavior of a dynamical system and the dependency of its solution on the initial state or initial condition began in the late 1880s. Describing the dynamics of dynamical systems as a function of time on the state space can generate a differential equation. Thus, the theory of dynamical systems may be said to be a special and important topic in the theory of differential equations. It falls under the qualitative theory which is mainly concerned with properties which are not quantified. The study of qualitative theory leads to a better understanding of the dynamical systems.

This thesis mainly focuses on the stability analysis of dynamical and control systems. We basically deal with two types of stability treatment: (i) the Ulam-Hyers stability of the solution of an impulsive fractional-order integro-differential equation involving Caputo fractional-order derivatives, and also the Ulam-Hyers stability of finite difference equations corresponding to first- and second-order differential equations, (ii) the Lyapunov stability for a class of fractional-order differential equations involving a non-singular kernel fractional-order derivative called Caputo-Fabrizio derivative and discuss the various concepts such as the existence of a periodic solution, stabilization, and asymptotic stability of Caputo-Fabrizio fractional-order semilinear evolution equations.

In the beginning of this study, we establish the existence of Ulam-Hyers and generalized Ulam-Hyers-Rassias stability results of the mild solutions of Caputo fractional non-instantaneous impulsive integro-differential equations in the form of two separate problems. The main results are established by using Banach fixed point theorem under appropriate assumptions. For the second problem, we use our results to estimate the bound for the difference between the fractional-order and the integer-order non-instantaneous impulsive RLC circuit current and show that the bound mainly depends on the bandwidth of the RLC circuit.

In the next part of this thesis, we consider the qualitative analysis as introduced by Lyapunov for different classes of fractional-order differential equations involving Caputo-Fabrizio fractional-order derivatives containing a non-singular kernel. First, we revisit the Lyapunov stability of an equilibrium point of an autonomous Caputo-Fabrizio fractional-order system and show that all isolated equilibrium points of an autonomous system are asymptotically stable, and we find that only constant solutions exist for autonomous systems. Further, we study the Lyapunov stability for the intermediate value Caputo-

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Fabrizio linear and nonlinear autonomous systems and derive the condition required for the equilibrium point for such systems to be asymptotically stable. A suitable example is presented at the end to illustrate the result of the existence of such a stability. In another problem, the existence of a periodic solution of the Caputo-Fabrizio fractional-order system is considered. Under a similar assumption as the one for an integer-order differential system, and by using the properties of the Caputo-Fabrizio derivative, the existence of a periodic solution of a non-autonomous Caputo-Fabrizio fractional-order differential system is established. As an application, we derive a periodic solution of a fractional-order Gunn diode oscillator under a periodic input voltage, and observe that the diameter of the periodic orbit keeps reducing as the fractional-order continuously increases. Further, by constructing a suitable linear feedback control, the solution of a linear non-homogeneous fractional-order system is stabilized to a periodic solution, and an example is presented to support the obtained result.

In another problem, we discuss the asymptotic stability of fractional-order linear and semilinear evolution equations involving a Caputo-Fabrizio fractional derivative for fractional-order $\alpha \in (0,1)$, and introduce a new concept of a global solution for the Caputo-Fabrizio system. Laplace transform and Grönwall inequality are used to derive the local and global asymptotic stability conditions. By constructing a suitable linear feedback control and using our main results, we stabilize the Caputo-Fabrizio fractional-order linear and semilinear evolution equations. In the end, by using the stabilization result, we stabilize a fractional-order chaotic system to support the obtained results.

The last problem of the thesis is considered in a different direction. By using the Ulam-Hyers stability results for the linear recurrence relation, we establish the Ulam-Hyers stability of a second-order convergent finite difference equation corresponding to the first- and second-order non-homogeneous linear differential equations with constant coefficients. Further, as per the location of the roots of the characteristic polynomial of the equivalent recurrence relation, the minimum Ulam-Hyers constant is determined, and a suitable example is presented to support the obtained result.

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CHAPTER 1

Introduction

1.1 Background

Almost all physical phenomena in nature are governed by mathematical laws that can be formulated by differential equations. Newton (1642–1727) was the first one to realize this fact to formulate the laws of mechanics and describe the motion of the planets. The study of differential equations is of the highest interest since differential equations arise in nearly all disciplines of science, medicine, engineering, economics, demography, and biocenology. A differential equation is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables. If there is only one independent variable in the equation, then we call it an ordinary differential equation. Here, the derivatives are represented by the terms like $\frac{du}{dt}, \frac{d^2u}{dt^2}$ (according to Leibniz notation) which we call integer-order derivatives of a dependent variable u with respect to an independent variable t. In this thesis, we use the generalization of the derivative terms by $\frac{d^\alpha u}{dt^\alpha}$, where $\alpha \in \mathbb{R}^+$, which we call as *fractional derivatives*. This means that, if $\alpha \in \mathbb{N}$, then the corresponding differential equation involving such derivatives is called an integer-order differential equation, and for $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, we call it a fractional-order differential equation. In this thesis, we consider both types of derivatives under consideration for various classes of differential equations. The branch of mathematics that deals with the fractional-order derivatives and integrations of functions and studies the relationship between them is called fractional calculus. The original query that led to the name *fractional calculus* is: Can the meaning of a derivative $\frac{d^n u}{dt^n}$ of integer order be extended to have some meaning when n is a fraction? Later, the question became: Can n be any number - fractional, irrational, or complex? Because the latter question was answered affirmatively, the name fractional calculus has become a misnomer and might better be called integration and differentiation to an arbitrary order. Fractional calculus is as old as

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classical calculus, i.e., around the time (17th century) when Newton and Leibnitz developed conventional differential and integral calculus. A list of mathematicians, who have provided important contributions up to the middle of the 20th century includes P. S. Laplace (1812), J. B. J. Fourier (1822), N. H. Abel (1823–1826), J. Liouville (1832–1873), B. Riemann (1847), H. Holmgren (1865–67), A. K. Grünwald (1867–1872), A. V. Letnikov (1868–1872), H. Laurent (1884), P. A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892–1912), S. Pincherle (1902), G. H. Hardy and J. E. Littlewood (1917–1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H. T. Davis (1924–1936), A. Zygmund (1935–1945), E. R. Love (1938–1996), A. Erdélyi (1939–1965), H. Kober (1940), D. V. Widder (1941), M. Riesz (1949), etc. However, the first use of fractional operations was made by Niels Henrik Abel in 1823. Abel applied fractional calculus to solve an integral equation that arose in formulating the tautochrone problem. Fractional calculus may be considered an old yet novel topic. The recognition of the first monograph is credited to K.B. Oldham and J. Spanier [100] who, after a joint collaboration that started in 1968, published a book devoted to fractional calculus in 1974. In this context, the encyclopedic treatises by Kilbas et al. [70] and Podlubny [112] are considered the most prominent ones. A reasonable number of comprehensive books on fractional calculus and fractional differential equations have come up since then.

With the progress of fractional calculus, many different versions of the definitions of fractional derivatives have been proposed by mathematicians, researchers, scientists, and others. However, the definitions proposed by Riemann-Liouville, Grönwall-Letnikov, and Caputo are the most popular ones in the history of fractional calculus. Of late, Hilfer fractional derivative and ψ-Hilfer derivative have also received immense attention from the researchers. The various definitions of fractional-order derivative and its properties can be found in several books and papers, e.g., [112, 89, 25, 9, 146, 145, 148]. The main advantage of fractional calculus over classical calculus lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, and make the fractional-order models more realistic than the integer-order models, for instance, waves [42, 55], visco-elasticity [87, 112], control theory [90, 71, 88], chaos theory [110, 51], electrical circuits [7, 8, 118], biology [31, 127, 41]. It is worth mentioning that an integer-order differential operator is a local operator because, in order to calculate integer-order derivatives of a function, it is required to know its properties in an infinitesimal neighborhood of the considered point, whereas a fractional differential operator is a non-local one which means that the next state of a system depends not only on its current state but also upon all its past states, which makes the geometry and analytical solution more complicated.

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1.2 Fractional-order Operators with Singular and Non-singular Kernels

In the history of fractional calculus so far, depending upon the requirement for modeling real-world problems and taking into account some mathematical difficulty, various fractional-order differential or integral operators have been introduced, and these operators are defined by using a kernel. So far, two types of fractional-order operators have been defined involving singular and non-singular kernels. The most fundamental fractional-order operators are the Caputo and Riemann-Liouville operators with singular kernel $k(t,\varsigma) = \frac{(t-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)}$, $0 < \alpha < 1$. Recently, Caputo and Fabrizio [25] defined a new fractional-order differential operator comprising the non-singular kernel $k(t,\varsigma)) = \exp\Big(-\frac{\alpha(t-\varsigma)}{1-\alpha}\Big)$, $0 < \alpha < 1$, which is now termed a Caputo-Fabrizio derivative. Atangana and Baleanu [13] generalized the definition of Caputo-Fabrizio derivative by replacing the exponential kernel with a Mittag-Leffler kernel $k(t,\varsigma) = E_\alpha\Big(-\alpha\frac{(t-\varsigma)^{-\alpha}}{1-\alpha}\Big)$. Some other researchers also introduced new definitions of the fractional derivatives with non-singular kernels [9, 148, 146, 145]. Such a non-singular kernel fractional-order operator treats certain phenomena related to material heterogeneities that cannot be appropriately modeled by Caputo or Riemann-Liouville fractional-order operator. Thus, from above, we observe that every possible fractional-order operator can be defined with a singular or non-singular kernel. In this direction, fractional differential operators involving non-singular kernels provide some relaxation in solving a fractional differential equation with a non-singular kernel numerically and analytically. On the other hand, using non-singular kernel fractional derivative in modeling a problem has several drawbacks. Diethelm [36] observed that the fractional differential operator with a non-singular kernel failed to satisfy the fundamental theorem of fractional calculus. Furthermore, Zhang [151] pointed out that the value of the derivative at the initial time was always zero, which put restrictions on choosing the initial data for a fractional differential equation.

1.3 Impulsive Differential Equations

Many evolution processes in applied sciences are represented by differential equations. Many real-world problems can be modeled with the help of differential equations. However, there are some problems that need different types of handling, such as mechanical systems with impact, population dynamics, industrial robotics, natural disasters, the intravenous introduction of drugs in blood stream, and so on, in each of which there is an abrupt change in the state. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of the entire process. Such a problem should be modeled through a different class of equations, which we call an impulsive differential equation. The study of the theory of impulsive differential equations has assumed importance for long, but its development reached a high point only in the 1980s, and further development has been going on even now. The advancement in research on such problems allows some real-world phenomena to be modeled more accurately in terms of impulsive differential equations. As such, they have found a significant place in modeling physical problems in several areas of science and engineering [12, 48, 94, 130, 152, 153, 39, 30, 61, 19]. In most of the works, the use of two familiar impulses is found: instantaneous impulses and non-instantaneous impulses. In the case of instantaneous impulses, the time interval of the changes is relatively shorter in comparison to the total duration of the whole process, while in the non-instantaneous case, an impulsive action starts at an arbitrary fixed point and remains active for a finite time interval. The following may serve as an impetus to study such systems involving impulsive differential equations. Consider the simplified situation with respect to the hemodynamical equilibrium of a human being. In the event of the occurrence of a de-compensation, e.g., high or low levels of glucose, some intravenous drugs (insulin) may be prescribed. It is obvious that the introduction of drugs in the bloodstream and the consequent absorption by the body are gradual and continuous processes. In this context, the situation can be interpreted as an impulsive action that starts abruptly and stays active for a finite time interval. It can be clearly observed that this situation can be modeled with the help of a non-instantaneous impulsive differential equation.

1.4 Application of Fractional Calculus

Although fractional calculus is as old as classical calculus, it has not been considered a fully-developed subject from the mathematical and application point of view compared to integer-order calculus. This is probably due to its high complexity and the lack of an acceptable physical and geometrical interpretation. During the early stage of the history of fractional calculus, the subject progressed slowly and focused mainly on the development of its mathematical theory in an abstract sense. Thus, at that time, the subject had less application and was consequently considered less useful in the applied field of science and engineering.

However, in the last few decades, substantial number of works have been carried out by scientists, engineers, and researchers on the application of fractional calculus to various areas of science, engineering, biology, medicine, and interdisciplinary areas. As the fractional-order derivatives and integrals operators are nonlocal, the fractional-order models are able to describe the memory and hereditary properties of the state of the system. This fact is the most significant advantage of the fractional-order models over the integer-order models, in which such effects are neglected. This was adequtely demonstrated, for instance, in [96, 44, 24]. A large number of applications can be found in topics like visco-elasticity, seismology, electromagnetism, control theory, electrical circuits, biology,

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and so forth [8, 7, 118, 112, 31, 55, 14, 52, 50, 99]. In fact, one can find the important explicit applications of fractional calculus on various fields [125, 92, 91, 112, 132, 60].

1.5 Some Important Definitions and Results

In this section, we present the definitions of singular and non-singular kernel fractional-order derivatives and integrals, special functions, and some of their properties.

1.5.1 Some functional spaces

(i) $L^p(J,X)$ **Space:** Let $J=[a,b]$ be a finite interval on $\mathbb{R}$ and $(X,\|.\|_X)$ be a Banach space. We denote by $L^p(J,X)$, $1\le p\le\infty$, the space of all measurable functions $u:J\to X$ with a norm

$$\|u\|_{L^p}=\begin{cases}\left(\int_J \|u(\varsigma)\|_X^p\,d\varsigma\right)^{\frac{1}{p}}, & 1\le p<\infty,\\ ess\sup\limits_{t\in J}\|u(t)\|_X, & p=\infty.\end{cases} \tag{1.1}$$

With this norm, $L^p(J,X)$ is a Banach space.

(ii) $C(J,X)$ **Space:** Let $C(J,X)$ denote the space of all continuous functions $u:J\to X$ with the supremum norm $\|.\|_C$

$$\|u\|_C=\sup_{t\in J}\|u(t)\|_X. \tag{1.2}$$

(iii) $AC(J,\mathbb{C})$ **Space:** Let $AC(J,\mathbb{C})$ denote the space of all absolutely continuous functions $u:J\to\mathbb{C}$. Here, the space $AC(J,\mathbb{C})$ coincides with the space of primitives of Lebesgue summable functions, i.e.,

$$u(t)\in AC(J,\mathbb{C})\Leftrightarrow u(t)=c+\int_a^x v(\varsigma)\,d\varsigma,\ \forall t\in J, \tag{1.3}$$

where c is some constant and v is a Lebesgue summable function, i.e., $\int_a^b v(\varsigma)\,d\varsigma<\infty$. Thus, if a function $u\in AC(J,\mathbb{C})$, then it has a summable derivative $u'(t)$ almost everywhere.

Let us denote by $AC^n(J,\mathbb{C})$, where $n=1,2,...$, the space of those functions u which have continuous derivatives up to order $(n-1)$ on J with $u^{(n-1)}\in AC(J,\mathbb{C})$.

Lemma 1.5.1. (*Grönwall's Inequality*) *Suppose that $u(t)$ and $v(t)$ are continuous real-valued functions defined on $0\le t<T$ with $u(t)\ge 0$. Assume that u and v satisfy*

$$u(t)\le k_1+k_2\int_0^t u(\varsigma)v(\varsigma)\,d\varsigma \tag{1.4}$$

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on $0 \leq t < T$, where k_1 and k_2 are constants with $k_2 \geq 0$. Then,

$$u(t) \leq k_1 \exp\left(k_2 \int_0^t v(\varsigma)\, d\varsigma\right) \tag{1.5}$$

on $0 \leq t < T$.

Lemma 1.5.2. *[109] Let A be an $n \times n$ scalar matrix. Then, the following statements are equivalent:*

(i) *For all $x_1 \in \mathbb{R}^n$, $\lim_{t\to\infty} \exp(At)x_1 = 0$, and for $x_1 \neq 0$, $\lim_{t\to-\infty} \|\exp(At)x_1\| = \infty$.*

(ii) *All eigenvalues of A have negative real parts.*

(ii) *There are positive constants a, c, m and m_1 such that, for all $x_1 \in \mathbb{R}^n$,*

$$\|\exp(At)x_1\| \leq m \exp(-ct)\|x_1\|, \quad \forall\, t \geq 0, \tag{1.6}$$

and

$$\|\exp(At)x_1\| \geq m_1 \exp(-at)\|x_1\|, \quad \forall\, t \leq 0,$$

where $\|.\|$ denotes any vector norm or induced matrix norm.

1.5.2 Gamma function

One of the basic functions of fractional calculus is Euler's Gamma function $\Gamma(z)$, which allows z to take non-integer and even complex values. The Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty \exp(-\varsigma)\varsigma^{z-1}\, d\varsigma \tag{1.7}$$

which converges in the right half of the complex plane $\text{Re}(z) > 0$. The Gamma function $\Gamma(z)$ satisfies the following relations:

$$\Gamma(z+1) = z\Gamma(z), \ \text{Re}(z) > 0, \tag{1.8}$$

$$\Gamma(z+1) = z!, \ z \in \mathbb{N}. \tag{1.9}$$

1.5.3 Mittag-Leffler function

The Mittag-Leffler functions have important roles in the theory of fractional calculus, which are also generalizations of exponential functions. The one-parameter (α) Mittag-Leffler function is given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad z \in \mathbb{C}, \ \text{Re}(\alpha) > 0. \tag{1.10}$$

The two-parameter (α, β) Mittag-Leffler function is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad z \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0. \tag{1.11}$$

1.5.4 Riemann-Liouville fractional integrals

Now, we define the most popular fractional-order integral of order $\alpha > 0$ defined by Riemann and Liouville. This fractional-order integral is the generalization of the Cauchy formula for an n-fold integral where $n \in \mathbb{N}$.

Definition 1.5.1. *[35] The Riemann-Liouville fractional integral ${}_a\mathcal{I}_t^\alpha$ of order $\alpha > 0$ of the function $u \in L^1([a,b],\mathbb{R})$ is defined as*

$${}_a\mathcal{I}_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\varsigma)^{\alpha-1} u(\varsigma)\, d\varsigma, \quad t > a. \tag{1.12}$$

1.5.5 Riemann-Liouville fractional-order derivatives

Here, we define the fractional-order derivative of order $\alpha > 0$ defined by Riemann and Liouville. This definition is motivated by the repeated integer-order differentiation operations in classical calculus.

Definition 1.5.2. *[35] The Riemann-Liouville fractional-order derivative ${}_a\mathcal{D}_t^\alpha$ of order α with a singular kernel of the function $u \in AC^n([a,b],\mathbb{R})$, where $n = \lfloor \alpha \rfloor + 1$, is defined as*

$${}_a\mathcal{D}_t^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t (t-\varsigma)^{n-\alpha-1} u(\varsigma)\, d\varsigma, & n-1 < \alpha < n, \\ \dfrac{d^n u(t)}{dt^n}, & \alpha = n. \end{cases} \tag{1.13}$$

Modeling real-world problems arising in various areas of science and engineering with a differential equation involving Riemann-Liouville fractional-order derivative requires an initial condition as limiting values of a certain Riemann-Liouville fractional-order derivative at the lower limit $t = a$, which makes the initial condition physically very difficult to interpret, and finding such initial conditions is practically a difficult task. So, from the application point of view, the Riemann-Liouville fractional-order derivative is not found suitable for the formulation of real-world problems.

1.5.6 Caputo fractional-order derivatives

In order to overcome the issue of an initial condition for a fractional-order differential equation, Caputo introduced a modified version of Riemann-Liouville fractional-order derivative such that a system containing Caputo fractional-order derivative conveys a clear physical meaning of initial conditions such as $u(a), u'(a)$ and so on.

Definition 1.5.3. *[112] The Caputo fractional-order derivative ${}_a^C\mathcal{D}_t^\alpha$ of order α with a singular kernel of the function $u \in AC^n([a,b],\mathbb{R})$, where $n = \lfloor \alpha \rfloor + 1$, is defined as*

$$ {}_a^C\mathcal{D}_t^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-\varsigma)^{n-\alpha-1} u^{(n)}(\varsigma)\, d\varsigma, & n-1 < \alpha < n, \\ \dfrac{d^n u(t)}{dt^n}, & \alpha = n, \end{cases} \tag{1.14} $$

where $u^{(n)}(t) = \dfrac{d^n u(t)}{dt^n}$.

1.5.7 Properties of singular kernel fractional-order derivatives and integrals

Here, we present some important properties of the singular kernel fractional-order derivatives in Riemann-Liouville and Caputo sense.

Lemma 1.5.3. *[35] The Riemann-Liouville fractional integral satisfies the commutative property over the order of integral, i.e., for $\alpha > 0, \beta > 0$ and $u \in L^1([a,b],\mathbb{R})$, we have*

$$ {}_a\mathcal{I}_t^\alpha \, {}_a\mathcal{I}_t^\beta u(t) = {}_a\mathcal{I}_t^\beta \, {}_a\mathcal{I}_t^\alpha u(t) = {}_a\mathcal{I}_t^{\alpha+\beta} u(t). \tag{1.15} $$

Lemma 1.5.4. *[112] Let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. Assume that the function $u : [a,b] \to \mathbb{R}$ is such that $u \in AC^n([a,b],\mathbb{R})$. Then*

$$ {}_a^{RL}\mathcal{I}_t^\alpha \left({}_a^{RL}\mathcal{D}_t^\alpha u(t) \right) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(n-k)} \, {}_a^{RL}\mathcal{D}_t^{\alpha-k-1} u(t) \,|_{t=a^+}, \tag{1.16} $$

$$ {}_a^{RL}\mathcal{I}_t^\alpha \left({}_a^{C}\mathcal{D}_t^\alpha u(t) \right) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k}}{k!} u^{(k)}(a), \tag{1.17} $$

where $u^{(k)}(a) = \dfrac{d^k u(t)}{dt^k} \,|_{t=a}$.

Lemma 1.5.5. *[35] Suppose $\alpha > 0, n-1 < \alpha \leq n \ (n \in \mathbb{N})$. Then Laplace transforms of the Riemann-Liouville and Caputo fractional derivatives are, respectively, given by*

$$ \mathscr{L}\left\{ {}_0^{RL}\mathcal{D}_t^\alpha u(t) \right\}(s) = s^\alpha U(s) - \sum_{k=0}^{n-1} s^k \left[{}_0^{RL}\mathcal{D}_t^{\alpha-k-1} u(t) \right]_{t=0}, \tag{1.18} $$

$$ \mathscr{L}\left\{ {}_0^{C}\mathcal{D}_t^\alpha u(t) \right\}(s) = s^\alpha U(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} u^{(k)}(0), \tag{1.19} $$

where $U(s) = \mathscr{L}\{u(t)\}(s)$.

Relation between the Riemann-Liouville and Caputo fractional derivative

There are some differences between Riemann-Liouville and Caputo fractional derivatives, such as

(i) the Riemann-Liouville derivative of a constant function is non-zero, but the Caputo derivative of a constant function is zero,

(ii) one advantage of Caputo fractional derivative over Riemann-Liouville derivative is the fact that a system containing the Caputo operator includes the clear physical meaning of initial conditions such as $u(a), u'(a)$, and so on, at the initial time $t = a$.

Corollary 1.5.1. *[112] Suppose $\alpha > 0$ and $u \in C^n([a, b], \mathbb{R})$ where, $n = \lfloor \alpha \rfloor + 1$. Then, the following relation between the Riemann-Liouville and Caputo fractional derivatives holds:*

$$ {}_{a}^{C}\mathcal{D}_{t}^{\alpha} u(t) = {}_{a}^{RL}\mathcal{D}_{t}^{\alpha} \Big(u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} u^{(k)}(a) \Big). \tag{1.20} $$

From the above relation (1.20), we notice that, if the function $u(t)$ be such that $u^{(k)}(a) = 0$, for $k = 0, 1, \ldots, n-1$, then, the Riemann-Liouville and Caputo fractional derivatives coincide, i.e., ${}_{a}^{C}\mathcal{D}_{t}^{\alpha} u(t) = {}_{a}^{RL}\mathcal{D}_{t}^{\alpha} u(t)$.

1.5.8 Non-singular kernel fractional-order derivatives and their properties

Caputo-Fabrizio derivative through a non-singular kernel is already discussed in Section 1.2. In this thesis, we mainly focus on systems involving Caputo-Fabrizio derivatives and integrals.

Definition 1.5.4. *[25] The Caputo-Fabrizio fractional derivative of order α, $0 < \alpha \leq 1$, for a function $u \in H^1(a, b)$, $a < b$, with a non-singular kernel $\exp\Big(-\frac{\alpha(t-\varsigma)}{1-\alpha}\Big)$ is denoted by ${}_{a}^{CF}\mathcal{D}_{t}^{\alpha}$ and is defined by*

$$ {}_{a}^{CF}\mathcal{D}_{t}^{\alpha} u(t) = \begin{cases} \frac{M(\alpha)}{1-\alpha} \int_a^t u'(\varsigma) \exp\Big(-\frac{\alpha(t-\varsigma)}{1-\alpha}\Big) d\varsigma, & 0 < \alpha < 1, \\ u'(t), & \alpha = 1, \end{cases} \tag{1.21} $$

where $M(\alpha)$ is a normalization function with $M(0) = 1 = M(1)$ and $u'(t) = \frac{du(t)}{dt}$.

Such type of non-singular kernel fractional-order operator treats certain phenomena related to material heterogeneities that cannot be appropriately modeled by Caputo or Riemann-Liouville fractional-order operator [25, 13]. Moreover, later in Chapter 4, it is shown that translation of the lower terminal point to some intermediate point is possible for the system involving Caputo-Fabrizio derivative which is not possible for other fractional-order derivatives such as Caputo or Riemann-Liouville. Furthermore, we know that

Caputo or Riemann-Liouville fractional-order systems do not possess non-constant periodic solution; but if we change the fractional-order derivative of the systems by a Caputo-Fabrizio one, the picture is entirely different. Later, in Chapter 5, it is shown that, under suitable assumptions, a Caputo-Fabrizio fractional-order system possesses a non-constant periodic solution.

Definition 1.5.5. *[84] The Caputo-Fabrizio fractional integral of order* $\alpha,\ 0<\alpha<1$, *for a function* $u \in L^1[a,b]$ *is denoted by* ${}^{CF}_{a}\mathcal{I}^{\alpha}_{t}$ *and is defined by*

$$ {}^{CF}_{a}\mathcal{I}^{\alpha}_{t}u(t) = \frac{(1-\alpha)}{M(\alpha)}u(t) + \frac{\alpha}{M(\alpha)}\int_a^t u(\varsigma)\,d\varsigma, \quad t \geq a. \tag{1.22}$$

Lemma 1.5.6. *[123] Let* $u \in AC[a,b]$ *and* $0<\alpha<1$. *Then,*

$$ {}^{CF}_{a}\mathcal{I}^{\alpha}_{t}\left({}^{CF}_{a}\mathcal{D}^{\alpha}_{t}u(t)\right) = u(t) - u(a). \tag{1.23}$$

The fundamental theorem of fractional calculus does not hold good in the case of the Caputo-Fabrizio derivative, i.e., ${}^{CF}_{a}\mathcal{D}^{\alpha}_{t}\left({}^{CF}_{a}\mathcal{I}^{\alpha}_{t}u(t)\right) \neq u(t)$. This will be true if $u(a)=0$. For example, let us consider a fractional initial value problem as follows:

$$ {}^{CF}_{a}\mathcal{D}^{\alpha}_{t}u(t) = 1, \quad 0<\alpha<1, \tag{1.24}$$

$$ u(a) = u_a, \quad v_a \in \mathbb{R}. \tag{1.25}$$

Now, invoking ${}^{CF}_{a}\mathcal{I}^{\alpha}_{t}$ to both sides of this system (i.e., carrying out Caputo-Fabrizio integration), we get

$$\begin{aligned} u(t) &= u_a + \frac{(1-\alpha)}{M(\alpha)} + \frac{\alpha}{M(\alpha)}\int_a^t 1.\,ds \\ &= u_a + \frac{(1-\alpha)}{M(\alpha)} + \frac{\alpha}{M(\alpha)}(t-a). \end{aligned} \tag{1.26}$$

But, if we compute the Caputo-Fabrizio derivative ${}^{CF}_{a}\mathcal{D}^{\alpha}_{t}$ of the above function, we get

$$ {}^{CF}_{a}\mathcal{D}^{\alpha}_{t}u(t) = 1 - \exp\left(-\frac{\alpha(t-a)}{1-\alpha}\right) \neq 1, \quad \forall\, t \geq a. \tag{1.27}$$

In fact, by using Laplace transform, it can be established that this problem does not have a solution. From this simple example, we notice that even a simple system that includes a Caputo-Fabrizio derivative may not have a solution, which is a major drawback of the Caputo-Fabrizio derivative that comes to immediate notice.

Lemma 1.5.7. *[36] Consider* $\alpha,\ 0<\alpha<1$, *and* $v \in AC[a,b]$. *Then,*

$$ {}^{CF}_{a}\mathcal{D}^{\alpha}_{t}\left({}^{CF}_{a}\mathcal{I}^{\alpha}_{t}v(t)\right) = v(t) - v(a)\exp\left(-\frac{\alpha(t-a)}{1-\alpha}\right), \quad t \geq a. \tag{1.28}$$

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It is noticed that the fundamental theorem holds in the case of a Caputo-Fabrizio operator if $v(a) = 0$.

Lemma 1.5.8. *[25] Consider* $\alpha,\ 0 < \alpha < 1$. *Then the Laplace transform of the Caputo-Fabrizio derivative can be written as*

$$\mathscr{L}\Big[{}_0^{CF}\mathcal{D}_t^{\alpha}u(t)\Big](s) = \frac{M(\alpha)}{\Big(s+\alpha(1-s)\Big)}\Big(s\mathscr{L}[u(t)](s) - u(0)\Big), \tag{1.29}$$

where $\mathscr{L}[u(t)](s)$ *is the Laplace transform of the function* $u(t)$.

1.5.9 Fixed point theorems

Another important constituent in the area of fractional differential equations is the fixed point theorems, without which it is very difficult (actually almost impossible) to study the existence and uniqueness of nonlinear differential equations. Fixed point theorems are nowadays the most widely used tool in the area of fractional differential equations. The most frequently used fixed point theorems are Banach fixed point theorem, the nonlinear alternative of Leray-Schauder type, Krasnoselskii's fixed point theorem, Schaefer's fixed point theorem, Schauder's fixed point theorem, Burton-Kirk fixed point theorem, etc. Below, we state those fixed point theorems which are essential for obtaining the results in our works.

Theorem 1.5.1. *[21] (Banach Fixed Point Theorem) Let* (M, d_M) *be a complete metric space. Let* $\Upsilon : M \to M$ *be a contraction map with the Lipschitz constant* $L < 1$. *If there exists a non-negative integer* k *such that* $d_M(\Upsilon^{k+1}y, \Upsilon^k y) < +\infty$ *for some* $y \in M$, *then*
(i) *the sequence* $\Big\{\Upsilon^n y\Big\}$ *converges to a fixed point* x^* *of* Υ,
(ii) x^* *is the unique fixed point of* Υ *in* $M^* = \Big\{z \in M \mid d_M(\Upsilon^k y, z) < \infty\Big\}$,
(iii) *if* $z \in M^*$, *then* $d_M(z, x^*) \leq \frac{1}{1-L} d_M(\Upsilon z, z)$.

Theorem 1.5.2. *[21] (Krasnoselskii's Fixed Point Theorem) Let* $N(\neq \emptyset)$ *be a closed, convex subset of a Banach space* M, *and* $\Upsilon_1, \Upsilon_2 : M \to M$ *be two operators satisfying*
$(i)\Upsilon_1 u + \Upsilon_2 v \in N$, *whenever* $u, v \in N$,
$(ii)\Upsilon_1$ *is continuous and compact,*
$(iii)\Upsilon_2$ *is a contraction operator.*
Then, there exists $w^* \in N$ *such that* $w^* = \Upsilon_1 w^* + \Upsilon_2 w^*$.

1.6 Dynamical Systems

A system consists of interrelated components through a particular set of variables called the states of the system, and at any given time, these states completely determine the behavior of the system. A dynamical system is a system whose state changes with time. To describe the dynamics of the dynamical system as a function of time on the state space can generate a differential equation. The dynamical system theory is used to study various systems in science and engineering, for instance, biological systems, ecological systems, economic systems, control systems, etc. Given an initial state, obtaining the rule or dynamics that defines the state of physical systems is the main problem of interest in science and engineering. The study of differential equations in dynamical and control systems is mainly divided into two parts: quantitative theory and qualitative theory. The quantitative theory deals with finding the explicit analytic closed-form solutions of differential equations, and once we obtain this solution, we will have complete information about the state of the dynamical system. But, finding explicit analytic solutions is itself a challenging task, and sometimes, it is not possible at all to find them. Thus, quantitative theory does not help much to study the behavior of dynamical systems. On the other hand, the qualitative theory deals with the study of dynamical systems and the behavior of its states without obtaining its explicit analytical solution. Thus, to study dynamical systems and to design control in numerous complex engineering problems, the qualitative theory is found much more effective than the quantitative theory. The qualitative theory is mainly concerned with the topological property and stability of the solution of the dynamical systems. In this thesis, we mainly focus on the stability analysis of dynamical and control systems. In the literature, depending upon the requirement to handle the mathematical difficulty and from an application point of view to analyze the behavior of some of the physical quantities connected to the dynamical systems, various stability concepts have been introduced. The qualitative theory of dynamical systems deals with various topics along with their physical existence, such as stability, unbounded orbit, asymptotic stability, periodic orbit, limit cycle, and chaos.

1.6.1 Ulam-Hyers stability

In almost all areas of mathematical analysis, we can raise the following fundamental question: When is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly? If we turn our attention to the case of functional equations, we can particularly ask the question: when must the solution of an equation differing slightly from a given one be close to the solution of the given equation? The stability problem of functional equations originates from such a fundamental question. In 1940, Ulam [139] introduced a problem regarding the stability of functional equations. He presented conditions so that a linear function exists near an approximately linear function. In the following year, Hyers [56] provided a response to Ulam's problem for additive functions on Banach spaces as follows:

For real Banach spaces $(X, || \, . \, ||_X)$ and $(Y, || \, . \, ||_Y)$, $\varepsilon > 0$ and for every function $g : X \to Y$ which satisfies

$$|| \, g(u+v) - g(u) - g(v) \, ||_Y \leq \varepsilon \tag{1.30}$$

for all $u, v \in X$, there exists a unique additive function $B : X \to Y$ satisfying

$$|| \, g(u) - B(u) \, ||_Y \leq \varepsilon, \quad \forall u \in X. \tag{1.31}$$

Such a type of stability concept is known as Ulam-Hyers stability. After the result of Hyers, many mathematicians have extended Ulam-Hyers stability results to other functional equations such as for integer-order differential equations [62, 63, 80, 103], fractional-order differential equations [142] and linear difference equations [114, 113, 115, 15, 16]. For instance, Alsina and Ger [10] considered Ulam-Hyers stability of the linear differential equation on an open interval I given by

$$x'(t) - x(t) = 0, \ \forall t \in I.$$

They proved that, given an arbitrary number $\varepsilon > 0$ and a differentiable function $\phi : I \to \mathbb{R}$ satisfying

$$|\phi'(t) - \phi(t)| \leq \varepsilon, \ \forall t \in I,$$

there exists a solution $x : I \to \mathbb{R}$ of $x'(t) - x(t) = 0$ such that

$$|\phi(t) - x(t)| \leq 3\varepsilon, \forall t \in I.$$

1.6.2 Lyapunov stability

In the late 1880s, researchers began studying the behavior of a dynamical system and the dependency of its solutions on the initial state or initial condition. It is well known that, except for a few special cases, obtaining the explicit analytical solutions of nonlinear dynamical systems is not possible. French mathematician Henri-Poincaré (1854–1912) handled the above problem through a qualitative approach with the help of a combination of analysis and geometry. Even though Poincaré was the first one to analyze the qualitative properties of the solution of the dynamical systems, Isaac Newton (1642–1727) was the first to use mathematical dynamical system theory to model the motion of physical systems with differential equations. So, in the earlier days, a dynamical system was considered a part of mathematical physics. However, the qualitative theory approach introduced by Poincaré was restricted to mathematicians only because the theory needed a rigorous analysis and differential geometry concepts. One of the oldest and the most powerful tools to analyze the theories of dynamical and control systems is the concept of stability introduced by the Russian mathematician A.M. Lyapunov (1857–1918) in his seminal work entitled *“The General Problem of the Stability of Motion”* in the late nineteenth century. Lyapunov stability is concerned with the investigation of the motion or the behavior of the states of dynamical systems under small deviations from the state of motion of a system. As it stands now, Lyapunov theory is a very matured subject for integer-order dynamical systems and has a rich mathematical background [67, 129, 140]. Lyapunov stability theory for integer-order dynamical systems has a huge number of applications in science, biology, climatology, and engineering problems [67, 140, 108, 105, 65, 27]. For example, due to the complex nature of the dynamical systems emerging from engineering systems and to obtain a desired behavior of the system, we require a feedback control. To design and analyze such a feedback controller that manipulates system inputs to obtain a desired effect on the system outputs, the concept of Lyapunov stability plays an important role.

However, for fractional-order dynamical systems, depending upon the requirement to handle the mathematical difficulty and from the application point of view, the scientific community developed the subject; this may be due to fractional-order derivatives involving higher degree of complexity and the lack of an acceptable physical and geometric interpretation. But, in the last few decades, a huge quantity of research has been carried out on the applications of Lyapunov stability for fractional-order dynamical systems to diverse areas of science, engineering, control theory, and biology [14, 31, 55, 112, 138, 135, 73].

1.7 Fractional-order Intermediate Value Systems

An intermediate value problem in differential equations is a special class of problems where the information of the initial condition is not known at the initial point; rather, the information is known at an intermediate point. For integer-order differential equations, the intermediate value problem can be converted to an initial value problem by shifting the initial point to the intermediate point. Let us consider the first-order initial value problem

$$\frac{du}{dt} = f(t,u), \ \forall t \geq t_0 \tag{1.32}$$

with an initial condition

$$u(t_0) = u_0. \tag{1.33}$$

Consider $u(t;t_0,u_0)$ to be the solution of the above initial value problem (1.32) – (1.33).

Next, consider the first-order intermediate value problem

$$\frac{du}{dt} = f(t,u), \ \forall t \geq t_0 \tag{1.34}$$

with an information given at the intermediate point $t_1 > t_0$,

$$u(t_1) = u_1. \tag{1.35}$$

Choose an initial data u_0 such that the solution of the intermediate value problem (1.34) – (1.35) satisfies $u(t_1;t_0,u_0) = u_1$. Thus, the solution $u(t) = u(t;t_0,u_0)$ of the intermediate value problem (1.34) – (1.35) is given by

$$u(t) = u_0 + \int_{t_0}^{t} f(\varsigma, u(\varsigma))\, d\varsigma. \tag{1.36}$$

Now, using the condition $u(t_1; t_0, u_0) = u_1$ and eliminating u_0, we have, from the above equation,

$$\begin{aligned} u(t) &= u_1 - \int_{t_0}^{t_1} f(\varsigma, u(\varsigma; t_0, u_0))\, d\varsigma + \int_{t_0}^{t} f(\varsigma, u(\varsigma; t_0, u_0))\, d\varsigma \\ &= u_1 + \int_{t_1}^{t} f(\varsigma, u(\varsigma; t_0, u_0))\, d\varsigma, \ \forall t \geq t_1. \end{aligned} \tag{1.37}$$

For $t \geq t_1$, the solution $u(t)$ of the above integral equation (1.37) is the solution of the following initial value problem:

$$\frac{du}{dt} = f(t, u), \ \forall t \geq t_1, \tag{1.38}$$

$$u(t_1) = u_1. \tag{1.39}$$

Thus, from the above analysis, we observe that the intermediate value problem (1.34) – (1.35) is converted to the initial value problem (1.38) – (1.39).

Remark 1.7.1. *Assume that the differential equation (1.32) has a unique solution for any given initial value, then from equation (1.37), we have one of the basic properties of an integer-order dynamical system:* $u(t; t_1, u(t_1; t_0, u_0)) = u(t; t_0, u_0)$ *for all* $t \geq t_1$.

Next, we consider the following fractional-order intermediate value problem involving the Caputo fractional derivative:

$${}_{a}^{C}\mathcal{D}_{t}^{\alpha} u(t) = f(t, u), \ 0 < \alpha \leq 1, \ \forall t \geq a, \tag{1.40}$$

$$u(t_1) = u_1, \ t_1 > a. \tag{1.41}$$

Here the information $u(t_1)$ is known at the intermediate point $t = t_1$, not at the initial point $t = a$. This situation often appears in real-world applications where we only know the current value rather than its past values.

Choose an initial condition u_0 such that the solution of the fractional-order intermediate value problem (1.40) – (1.41) satisfies $u(t_1; a, u_0) = u_1$. Thus, the solution $u(t) = u(t; a, u_0)$ of the intermediate value problem (1.40) – (1.41) is given by

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - \varsigma)^{\alpha - 1} f(\varsigma, u(\varsigma; a, u_0))\, d\varsigma, \ \forall t \geq a. \tag{1.42}$$

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Eliminating u_0 from the above equation by using the condition $u(t_1; a, u_0) = u_1$, we have

$$\begin{aligned} u(t) =& u_1 - \frac{1}{\Gamma(\alpha)} \int_a^{t_1} (t_1 - \varsigma)^{\alpha-1} f(\varsigma, u(\varsigma; a, u_0))\, d\varsigma \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^{t} (t - \varsigma)^{\alpha-1} f(\varsigma, u(\varsigma; a, u_0))\, d\varsigma, \forall t \geq a. \end{aligned} \tag{1.43}$$

Now, we compute $u(t; t_1, u_1)$ as

$$u(t; t_1, u_1) = u_1 + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t - \varsigma)^{\alpha-1} f(\varsigma, u(\varsigma; t_1, u_1))\, d\varsigma. \tag{1.44}$$

The above fractional integral equation (1.44) is equivalent to the following initial value Caputo fractional-order differential equation:

$${}_{t_1}^{C}\mathcal{D}_t^{\alpha} u(t) = f(t, u), \ 0 < \alpha \leq 1, \ \forall\, t \geq t_1, \tag{1.45}$$

with an initial condition $u(t_1) = u_1$.

Remark 1.7.2. *Comparing with Remark 1.7.1 for an integer-order dynamical system, and from equations (1.43) and (1.44), we observe that one of the basic properties of dynamical system is lost:* $u(t; t_1, u(t_1, a, u_0)) \neq u(t; a, u_0)$ *for all* $t > t_1$.

Thus, in a fractional-order system, a memory term gets involved, which makes the intermediate value problem more complicated to analyze since the current dynamics of the system depends upon all the values of the past states, and the basic properties of the dynamical system do not hold. Therefore, a special treatment has to be given to analyze the qualitative behavior of the fractional-order intermediate value system. In this direction, in this thesis, we present a stability analysis of the fractional-order intermediate value system involving a non-singular kernel fractional derivative.

1.8 Literature Review

In the literature, depending upon the requirement to handle the mathematical difficulty, and from an application point of view, various stability concepts, such as Ulam-Hyers and Ulam-Hyers-Rassias stability, Lyapunov stability, etc., have been introduced to analyze the behavior of some of the physical state connected to the dynamical systems. For instance, Jung [62, 63], and Onitsuka [103] established the Ulam-Hyers stability result for the first-order linear differential equation with a real-valued coefficient. Baias et al. [17] obtained the minimum Ulam-Hyers constant for the second-order linear differential operator in terms of the roots of the characteristic polynomial of the differential operator. Li and Shen [80] proved the Ulam-Hyers stability for a second-order linear differential equation. For detailed Lyapunov stability analysis for an integer-order differential equation, one can refer to [67, 140].

Impulsive differential equations have found a very formidable place for modeling physical problems in several areas of science and engineering. In most of the works, two familiar impulses are found: instantaneous and non-instantaneous impulses (for details, one can refer to [18]). In this context, Hernández et al. [54] introduced a new concept on a class of abstract differential equations with non-instantaneous impulses, and they investigated the existence of mild and classical solutions. Wang et al. [144] initiated the new concept of piecewise(PC) mild impulsive Cauchy problem for which they defined the notation of a mild solution and presented some comparison between weak and classical solutions of the problem under consideration. A substantial number of works have been carried out on such problems, and the existence and qualitative properties of the solutions have been examined from various angles. Recently, significant amount of work has been carried out on the existence and stability in the sense of Ulam-Hyers for the solutions of the class of non-instantaneous differential equations involving both integer- and fractional-order derivatives. For a detailed survey on non-instantaneous impulses on Caputo fractional differential equation, the reader is referred to the monograph by Agarwal et al. [1]. Rus [122] derived a basic result for Ulam-Hyers stability for ordinary differential equations. Ding [37] investigated the existence, uniqueness, and Ulam-Hyers stability results for the Caputo fractional delay impulsive differential equation. For a detailed concept for stability in the sense of Ulam-Hyers for functional equations, one can refer to the works by Ulam [139], Rassias [120], Hyers [56] and Jung [64]. Carrying out the stability analysis for various classes of fractional differential equations with a non-singular differential operator, Khan et al. [68] established the existence of a solution for a nonlinear fractional differential equation with a Mittag-Leffler kernel differential operator and the associated Ulam-Hyers stability. Martínez et al. [45], by using Laplace transform and Lyapunov functions, analyzed the dynamics and Lyapunov stability of a general class of fractional-order differential system with a non-singular kernel. With the help of the comparison theorem, they established some inequalities for the solution of the system.

Wang et al. [142] considered a nonlinear non-instantaneous impulsive problem involving a Caputo fractional differential equation and studied the Ulam stability for the following problem in a finite interval $J = [0, T]$:

$$ {}_0^C\mathcal{D}_t^{\alpha} u(t) = f(t, u(t)), \quad t \in J' = J \setminus \{t_1, t_2, \ldots, t_m\}, \tag{1.46}$$

$$ \Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), \quad k = 1, 2, \ldots, m, \tag{1.47}$$

$$ u(0) = u_0, \tag{1.48}$$

where $\alpha \in (0,1)$, $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous, $I_k : \mathbb{R} \to \mathbb{R}$ and t_k, $k = 1, 2, \ldots, m$, satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$.

Zhou et al. [143] considered the Caputo fractional linear differential equation with a periodic boundary condition, i.e., the periodic BVP in the following form:

$$ {}_0^C\mathcal{D}_t^\alpha x(t) = f(t, x(t)), \quad t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \ldots, N, \tag{1.49}$$
$$ x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \ldots, N, \tag{1.50}$$
$$ x(0) = x(a). \tag{1.51}$$

Lin et al. [82] proved the existence and generalized Ulam-Hyers-Rassias stability result for a new class of non-instantaneous impulsive integro-differential equations as follows:

$$ x'(t) = f\Big(t, x(t), \int_0^t k(s, x(s))\,\mathrm{d}s\Big), \quad t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \ldots, m, \tag{1.52}$$
$$ x(t) = g_i\Big(t, x(t), \int_0^t l(s, x(s))\,\mathrm{d}s\Big), \quad t \in (t_i, s_i], \quad i = 1, 2, 3, \ldots, m, \tag{1.53}$$
$$ x(0) = x_0. \tag{1.54}$$

They used the Banach fixed point theorem and assumed suitable conditions on the functions.

When a particular problem is solved numerically, then the problem is converted to an algebraic difference equation. Therefore, studying the stability of a difference equation in the sense that Ulam and Hyers introduced will be helpful to analyze numerical problems. In this context, Popa [114] studied the Ulam-Hyers stability of a first-order difference equation in a Banach space. He proved the Ulam-Hyers stability result for a p-order linear difference equation with constant coefficients [113]. Onitsuka [102] studied the influence of the step size h on the Ulam-Hyers stability of a first-order homogeneous linear difference equation, where he showed that the Ulam-Hyers constant of a linear difference equation was less than the minimum of all the Ulam-Hyers constant of the same analytical equation, which can be considered as a significant result. In [101], Onitsuka established the Ulam-Hyers stability result for a second-order nonhomogeneous linear difference equation with a constant step size. In [11], Anderson et al. found the minimum Ulam-Hyers constant for a second-order h-difference equation with constant coefficients.

With regard to the relatively new concept of Caputo-Fabrizio fractional differential equations, some notable works have been accomplished. Losada and Nieto [84] studied the local existence and uniqueness of the solution of the Caputo-Fabrizio differential equation of order $\alpha \in (0, 1)$. Roscani et al. [121] derived some basic properties related to the Caputo-Fabrizio derivative and proved the existence of a global solution of a nonlinear Caputo-Fabrizio fractional differential equation. Zhang [151] analyzed the uniqueness of the solution for some classes of fractional-order initial value problems involving the Caputo-Fabrizio derivative. Fractional differential operators involving non-singular kernels provide some relaxation in solving a fractional differential equation with a non-singular kernel numerically and analytically. However, using a non-singular kernel fractional derivative in modeling a problem has several drawbacks. Diethelm [36] observed that the fractional differential operator with a non-singular kernel failed to satisfy the fundamental theorem of fractional calculus. Many have studied various fractional-order models involving Caputo-Fabrizio fractional derivatives [133, 4, 38, 69, 41, 119, 3, 148]. These works mainly focused on the application of Caputo-Fabrizio derivative in addressing real-life problems such as Covid-19, cancerous cell growth, hepatitis B, etc. Iqbal et al. [59] analyzed the fuzzy fractional acoustic waves model in terms of the Caputo-Fabrizio operator, and considered the evaluation of regularized long-wave equation via Caputo and Caputo-Fabrizio fractional derivatives [58] . Alesemi et al. [6] investigated the fractional-order Cauchy reaction-diffusion equation involving the Caputo-Fabrizio operator.

Recently, studying the stability of fractional-order systems, where the fractional derivatives contain singular and non-singular kernels, has evoked significant interest due to immense applications in fractional dynamical systems and control theory [135, 88, 138, 73]. For a detailed discussion on various stability results related to fractional-order systems, one can refer to the survey [81]. In the case of a fractional-order system with a singular kernel, there is a reasonable amount of literature available concerning the stability results for linear systems [5, 33, 98, 117, 136] and nonlinear systems [76, 77]. Lenka and Bora [74, 72] derived the asymptotic stability condition for the linear time-varying Caputo fractional-order system. Cheng et al. [78] investigated the stability analysis of a fractional-order linear system involving a Caputo-Fabrizio derivative. Lenka and his co-researchers [71, 73] stabilized the chaotic and unbounded solution to an asymptotically stable solution by using the suitable linear feedback control. Salahshour et al. [123] analyzed the Lyapunov stability of a fractional-order system with a non-singular kernel derivative. In fact, they established the result for a Caputo-Fabrizio derivative. In the above stability investigations, the major drawback of handling the non-singular kernel derivative, as pointed out by Diethelm [36], has not been addressed.

In the qualitative theory for dynamical systems, the analysis of oscillatory behavior of the solution is one of the main objectives. Since, in a dynamical system, a periodic motion is a very important and special phenomenon [43], the existence of periodic solutions is often considered a desirable property in dynamical systems. This constitutes one of the widely regarded research areas in the theory of dynamical systems, with applications ranging from celestial mechanics to biology, engineering, finance, etc. It is known that an integer-order system may have a periodic solution [43], which is, however, not true in the case of a fractional-order system with Caputo or Reimann-Liouville fractional derivative. Tavazoei [137, 134] proved that the time-invariant fractional-order systems involving Caputo fractional derivative do not possess any non-constant periodic solution. However, Yazdani and Salarieh [149] proved that a periodic solution might exist if the lower terminal point of the fractional derivative tends to $-\infty$. Bourafa [22] proved the existence of a periodic solution for a fractional-order differential system involving fractional derivative with a fixed length of sliding memory. However, these two results are not applicable in practice, i.e., it is difficult to construct such systems. Alternatively, different periodic solution concepts have been introduced for a fractional-order system, e.g., Heriique [53], Wang [141] who introduced the concepts of an asymptotic periodic solution and proved the existence result. Further, El-Borai [40] and Cabada [26] elaborated the concepts of almost periodic and positive periodic solutions for some nonlinear fractional differential equations.

1.9 Outline of the Thesis

The thesis is organised in eight chapters with Chapters $2-7$ describing the problems with every detail. Chapter 8 summarizes the contribution made by the thesis and opens the door for research in the future.

- In Chapter 2 and Chapter 3, we investigate the existence and stability results of the mild solution of the Caputo fractional non-instantaneous impulsive integro-differential equation in the sense of Ulam-Hyers and generalized Ulam-Hyers-Rassias stability, respectively, and present an application of our result through RLC circuit.

- In Chapter 4, we revisit the Lyapunov stability of an equilibrium point of an autonomous Caputo-Fabrizio fractional-order system and prove some important results, such as only constant solutions exist for autonomous systems, all isolated equilibrium points of an autonomous system are asymptotically stable. Further, we study the Lyapunov stability for the intermediate value Caputo-Fabrizio linear and nonlinear autonomous systems.

- In Chapter 5, we discuss the existence of a periodic solution of the Caputo-Fabrizio fractional-order system. We use our result to derive a periodic solution of a fractional-order Gunn diode oscillator under a periodic input voltage. In the end, by constructing a suitable linear feedback control, we stabilize the solution of the linear non-homogeneous fractional-order system to a periodic solution.

- In Chapter 6, we discuss the asymptotic stability of the fractional-order linear and semilinear evolution equations involving a Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ and derive the conditions for its asymptotic stability. We stabilize the Caputo-Fabrizio fractional-order linear and semilinear evolution equations by using a suitable linear feedback control. In the end, we stabilize some well-known fractional-order chaotic systems to support the obtained results.

- In Chapter 7, we establish the Ulam-Hyers stability of the second-order convergent difference equations corresponding to the first- and second-order non-homogeneous linear differential equations with constant coefficients, and as an application, we apply our result to the perturbed second-order nonlinear difference equation.

- The conclusions of the thesis and possible future scopes are presented in Chapter 8.

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CHAPTER 2

Existence and Ulam-Hyers stability of the solution for the Caputo fractional non-instantaneous impulsive integro- differential equation

In this chapter, we study the existence and stability of the mild solution of a Caputo fractional non-instantaneous impulsive integro-differential equation in the sense of Ulam-Hyers. Utilization of Banach fixed point theorem establishes the existence of a solution as well as the Ulam-Hyers stability under appropriate assumptions.

2.1 Introduction

Motivated by the works by Wang et al. [142], Zhou et al. [143] and Lin et al. [82] on Ulam-Hyers stability, we wish to examine the existence, uniqueness and Ulam-Hyers stability of the mild solution of the following Caputo fractional non-instantaneous impulsive integro-differential equation:

$$ {}_{0}^{C}\mathcal{D}_t^{\alpha} u(t) = \mathcal{F}\Big(t, u(t), \int_0^t f(\varsigma, u(\varsigma))\,\mathrm{d}\varsigma\Big), \quad t \in (\varsigma_i, t_{i+1}], \quad 0 \le i \le m, \tag{2.1}$$

$$ u(t) = \mathcal{G}_i\Big(t, u(t), \int_0^t g(\varsigma, u(\varsigma))\,\mathrm{d}\varsigma\Big), \quad t \in (t_i, \varsigma_i], \quad 1 \le i \le m, \tag{2.2}$$

$$ u(0) = u_0, \tag{2.3}$$

where $0 < \alpha < 1$, $0 = t_0 = \varsigma_0 < t_1 \le \varsigma_1 \le t_2 < \cdots < t_m \le \varsigma_m < t_{m+1} = T$ are pre-fixed numbers, $J = [0,T]$, $\mathcal{G}_i \in C\Big([t_i, \varsigma_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\Big)$, $i = 1, 2 \cdots m$, $\mathcal{F} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are given functions.

The physical significance of studying the stability in the sense of Ulam-Hyers for such model (2.1) − (2.3) not only gives a new direction to analyse the model qualitatively

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but also helps to estimate the bound for the solution of the considered systems. On the other hand, finding the actual applications of the Ulam-Hyers stability in the dynamical systems is still an open problem. However, we deem it suitable to present an example concerning an RLC circuit in the direction of a potential application. This can be found toward the end of this chapter.

2.2 Preliminary Results

Define $PC(J,\mathbb{R}) = \Big\{\psi : J \to \mathbb{R} : \psi \in C\Big((t_i, t_{i+1}], \mathbb{R}\Big), i = 0, 1, 2, \dots, m;$ $\psi(t_i^+), \psi(t_i^-)$ exist with $\psi(t_i) = \psi(t_i^-)\Big\}$. Here $PC(J,\mathbb{R})$ is a Banach space with the norm

$$\|\psi\|_{PC} = \sup_{t\in J} |\psi(t)|.$$

Lemma 2.2.1. *[21] Let $\widetilde{H} \in C(J,\mathbb{R})$. Then, a function $v \in C(J,\mathbb{R})$ is a solution of the integral equation*

$$v(t) = v_b - \frac{1}{\Gamma(\alpha)} \int_0^b (b-\varsigma)^{\alpha-1} \widetilde{H}(\varsigma)\,\mathrm{d}\varsigma + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \widetilde{H}(\varsigma)\,\mathrm{d}\varsigma \tag{2.4}$$

if and only if v is a solution of the following fractional Cauchy problem:

$$\begin{aligned} {}_0^C\mathcal{D}_t^{\alpha} v(t) &= \widetilde{H}(t), \quad t \in J, \\ v(b) &= v_b, \quad b > 0. \end{aligned} \tag{2.5}$$

2.3 Main Result

2.3.1 Required background

Here we introduce some definitions, lemmas and theorems connected to problem (2.1) − (2.3) which will be required for establishing the Ulam-Hyers stability for the problem undertaken.

Definition 2.3.1. *A function $u \in PC(J,\mathbb{R})$ is called a mild solution of problem (2.1) − (2.3) if*

$$u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^{\varsigma} f(\rho, u(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma, t \in [0, t_1], \\ \mathcal{G}_i\Big(t, u(t), \int_0^t g(\varsigma, u(\varsigma))\,\mathrm{d}\varsigma\Big), \quad t \in (t_i, \varsigma_i], \quad i = 1, 2, 3, \dots, m, \\ \mathcal{G}_i\Big(\varsigma_i, u(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, u(\varsigma))\,\mathrm{d}\varsigma\Big) - \\ \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^{\varsigma} f(\rho, u(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^{\varsigma} f(\rho, u(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma, t \in (\varsigma_i, t_{i+1}]. \end{cases} \tag{2.6}$$

Let us now consider the following inequalities:

$$\begin{cases} |{}_0^C\mathcal{D}_t^\alpha v(t) - \mathcal{F}\Big(t, v(t), \int_0^t f(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big)| \le \varepsilon, & t \in (\varsigma_i, t_{i+1}], 0 \le i \le m, \\ |v(t) - \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big)| \le \varepsilon, & t \in (t_i, \varsigma_i], 1 \le i \le m. \end{cases} \tag{2.7}$$

Theorem 2.3.1. *[2] A function $v \in PC(J,\mathbb{R})$ is a solution of the inequalities (2.7) if and only if there is a function $\widetilde{H} \in PC(J,\mathbb{R})$ and a sequence $\{H_i\}$, $i = 1, 2, \ldots, m$ (which depends on v) such that*
(i) $|\widetilde{H}(t)| \le \varepsilon, \quad t \in J$, and $|H_i| \le \varepsilon, \quad i = 1, 2, \ldots, m$,
(ii) ${}_0^C\mathcal{D}_t^\alpha v(t) = \mathcal{F}\Big(t, v(t), \int_0^t f(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big) + \widetilde{H}(t), \quad t \in (\varsigma_i, t_{i+1}]$,
(iii) $v(t) = \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big) + H_i, \quad t \in (t_i, \varsigma_i], \quad 1 \le i \le m$.

Lemma 2.3.1. *Suppose $\mathcal{F} \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\mathcal{G}_i \in C([t_i, \varsigma_i]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ $i = 1, 2, \ldots, m$, and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous. If $v \in PC(J,\mathbb{R})$ is a solution of the inequalities (2.7), then it satisfies the following integral inequalities:*

$$\begin{cases} |v(t) - v(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma| \\ \quad \le \frac{\varepsilon t_1^\alpha}{\Gamma(\alpha+1)}, \quad t \in (0, t_1], \\ \\ \quad |v(t) - \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big)| \le \varepsilon, \quad t \in (t_i, \varsigma_i], \quad 1 \le i \le m, \\ \\ |v(t) - \mathcal{G}_i\Big(\varsigma_i, v(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big) \\ +\frac{1}{\Gamma(\alpha)}\int_0^{\varsigma_i}(\varsigma_i-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma \\ -\frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma| \le \Big(1 + \frac{\varsigma_i^\alpha + t_{i+1}^\alpha}{\Gamma(\alpha+1)}\Big)\varepsilon, \\ t \in (\varsigma_i, t_{i+1}], \quad i = 1, 2, 3, \ldots, m. \end{cases} \tag{2.8}$$

Proof. From Lemma 2.2.1 and by Theorem 2.3.1, we have, for $t \in [0, t_1]$,

$$\begin{aligned} v(t) =& v(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\widetilde{H}(\varsigma)\,d\varsigma \end{aligned}$$

$$\begin{aligned} \Rightarrow |v(t) - v(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma| \\ \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}|\widetilde{H}(\varsigma)|\,d\varsigma \le \frac{\varepsilon t_1^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

For $t \in (t_i, \varsigma_i], \quad i = 1, 2, \ldots, m,$

$$|v(t) - \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\, \mathrm{d}\varsigma\Big)| \leq |H_i| \leq \varepsilon, \quad t \in (t_i, \varsigma_i] \quad 1 \leq i \leq m.$$

For $t \in (\varsigma_i, t_{i+1}]$, $i = 1, 2, \ldots, m,$

$$\begin{aligned} v(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma + +H_i \\ &- \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \widetilde{H}(\varsigma)\, d\varsigma \\ &+ \mathcal{G}_i\Big(\varsigma_i, v(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, v(\varsigma))\, \mathrm{d}\varsigma\Big) - \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} \widetilde{H}(\varsigma)\, \mathrm{d}\varsigma \\ &\Rightarrow |v(t) - \mathcal{G}_i\Big(\varsigma_i, v(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, v(\varsigma))\, \mathrm{d}\varsigma\Big) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(s), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma| \leq \Big(1 + \frac{\varsigma_i^\alpha + t_{i+1}^\alpha}{\Gamma(\alpha+1)}\Big)\varepsilon. \end{aligned}$$

Hence, the result is established. □

Definition 2.3.2. *The problem described by equations (2.1) − (2.3) is said to be stable in the sense of Ulam-Hyers if there exists some constant $c_{f,\alpha} > 0$ such that, for each $\varepsilon > 0$ and for every solution $v \in PC(J, \mathbb{R})$ of (2.7), there exists a solution $u \in PC(J, \mathbb{R})$ (mild solution) of (2.1) − (2.3) with*

$$|v(t) - u(t)| \leq c_{f,\alpha}\varepsilon, \quad \forall\, t \in J.$$

2.3.2 Stability of the problem

To establish our result on the existence and stability of the solution, we take into account some hypotheses as follows:

(A1) $\mathcal{F} \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\exists\, L_{\mathcal{F}} > 0$ such that

$$|\mathcal{F}(t, x_1, y_1) - \mathcal{F}(t, x_2, y_2)| \leq L_{\mathcal{F}}(|x_1 - x_2| + |y_1 - y_2|)$$

for each $t \in J$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(A2) $\mathcal{G}_i \in C\Big([t_i, \varsigma_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\Big)$ and $\exists\, L_{\mathcal{G}_i} > 0$, $i = 1, 2, \ldots, m$, such that

$$|\mathcal{G}_i(t, x_1, y_1) - \mathcal{G}_i(t, x_2, y_2)| \leq L_{\mathcal{G}_i}(|x_1 - x_2| + |y_1 - y_2|)$$

for each $t \in (t_i, \varsigma_i]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
(A3) $f, g \in C(J \times \mathbb{R}, \mathbb{R})$ and $\exists\, F_f > 0$, $G_g > 0$ such that

$$|f(t,x_1) - f(t,x_2)| \le F_f|x_1 - x_2|, \quad |g(t,y_1) - g(t,y_2)| \le G_g|y_1 - y_2|$$

for each $t \in J$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
Now, we discuss the existence and stability of problem (2.1) − (2.3) by using the concept of Ulam-Hyers.

Theorem 2.3.2. *Assume that assumptions* **(A1), (A2)** *and* **(A3)** *hold and*

$$\Theta = \max_{1\le i\le m}\left\{\frac{L_{\mathcal{F}} t_1^\alpha}{\Gamma(1+\alpha)}\Big(1 + \frac{t_1 F_f}{1+\alpha}\Big), L_{\mathcal{G}_i}\Big(1 + G_g\varsigma_i\Big) + \frac{L_{\mathcal{F}}}{\Gamma(1+\alpha)}\Big(\varsigma_i^\alpha + t_{i+1}^\alpha\Big) + \frac{F_f L_{\mathcal{F}}}{\Gamma(2+\alpha)}\Big(\varsigma_i^{\alpha+1} + t_{i+1}^{\alpha+1}\Big)\right\} < 1.$$

Then, the problem (2.1) − (2.3) *is Ulam-Hyers stable, i.e., there exists a unique mild solution* $u^* \in PC(J, \mathbb{R})$ *for it such that*

$$u^*(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, u^*(\varsigma), \int_0^\varsigma f(\rho, u^*(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma, & t \in [0, t_1] \\ \mathcal{G}_i\Big(t, u^*(t), \int_0^t g(\varsigma, u^*(\varsigma))\,\mathrm{d}\varsigma\Big), \quad t \in (t_i, \varsigma_i], & i = 1, 2, 3, \ldots, m, \\ \mathcal{G}_i\Big(\varsigma_i, u^*(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, u^*(\varsigma))\,\mathrm{d}\varsigma\Big) \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, u^*(\varsigma), \int_0^\varsigma f(\rho, u^*(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma \\ -\frac{1}{\Gamma(\alpha)}\int_0^{\varsigma_i} (\varsigma_i-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, u^*(\varsigma), \int_0^\varsigma f(\rho, u^*(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma, \\ \quad t \in (\varsigma_i, t_{i+1}], \end{cases} \tag{2.9}$$

and for each $v \in PC(J, \mathbb{R})$ *satisfying* (2.7)*, we have*

$$|v(t) - u^*(t)| \le \frac{r\varepsilon}{1-\Theta},$$

where $r = \max_{1\le i\le m}\left\{\frac{t_1^\alpha}{\Gamma(1+\alpha)}, 1 + \frac{\varsigma_i^\alpha + t_{i+1}^\alpha}{\Gamma(1+\alpha)}\right\}$.

Proof. We consider

$$M = PC(J, \mathbb{R}) \tag{2.10}$$

with the metric

$$d_M(g, h) = ||g - h||_{PC} = \sup_{t\in J}|g(t) - h(t)|. \tag{2.11}$$

Define an Operator $\Upsilon : M \to M$ by

$$\Upsilon u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma, & t \in [0, t_1], \\ \mathcal{G}_i\Big(t, u(t), \int_0^t g(\varsigma, u(\varsigma))\, \mathrm{d}\varsigma\Big), \quad t \in (t_i, \varsigma_i], & i = 1, 2, 3, \ldots, m, \\ \mathcal{G}_i\Big(\varsigma_i, u(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, u(\varsigma))\, \mathrm{d}\varsigma\Big) - \\ \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma, & t \in (\varsigma_i, t_{i+1}]. \end{cases}$$

Now, we prove that the operator Υ is a contraction.
Let $h_1, h_2 \in PC(J, \mathbb{R})$.
For $t \in [0, t_1]$,

$$\begin{aligned} |\Upsilon h_1(t) - \Upsilon h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} |\mathcal{F}\Big(\varsigma, h_1(\varsigma), \int_0^\varsigma f(\rho, h_1(\rho))\, d\rho\Big) \\ &\qquad - \mathcal{F}\Big(\varsigma, h_2(\varsigma), \int_0^\varsigma f(\rho, h_2(\rho))\, d\rho\Big)|\, \mathrm{d}\varsigma \\ &\leq \frac{L_{\mathcal{F}} t_1^\alpha}{\Gamma\alpha + 1}\Big(1 + \frac{F_f t_1}{\alpha + 1}\Big) d_M(h_1, h_2) \end{aligned}$$

$$\Rightarrow ||\Upsilon h_1 - \Upsilon h_2||_{(C[0,t_1],\mathbb{R})} \leq \frac{L_{\mathcal{F}} t_1^\alpha}{\Gamma\alpha + 1}\Big(1 + \frac{F_f t_1}{\alpha + 1}\Big) ||h_1 - h_2||_{PC}. \tag{2.12}$$

For $t \in (t_i, \varsigma_i], \quad i = 1, 2, \ldots, m,$

$$\begin{aligned} |\Upsilon h_1 - \Upsilon h_2| &= |\mathcal{G}_i\Big(t, h_1(t), \int_0^t g(\varsigma, h_1(\varsigma))\, \mathrm{d}\varsigma\Big) - \mathcal{G}_i\Big(t, h_2(t), \int_0^t g(\varsigma, h_2(\varsigma))\, \mathrm{d}\varsigma\Big)| \\ &\leq L_{\mathcal{G}_i}\Big(1 + G_g \varsigma_i\Big) d_M(h_1, h_2). \end{aligned}$$

Therefore,

$$||\Upsilon h_1 - \Upsilon h_2||_{(C(t_i,\varsigma_i],\mathbb{R})} \leq L_{\mathcal{G}_i}\Big(1 + G_g \varsigma_i\Big) ||h_1 - h_2||_{PC}. \tag{2.13}$$

For $t \in (\varsigma_i, t_{i+1}], \quad i = 1, 2, \ldots, m,$

$$\begin{aligned} |\Upsilon h_1 - \Upsilon h_2| &\leq |\mathcal{G}_i\Big(\varsigma_i, h_1(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, h_1(\varsigma))\, \mathrm{d}\varsigma\Big) - \mathcal{G}_i\Big(\varsigma_i, h_2(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, h_2(\varsigma))\, \mathrm{d}\varsigma\Big)| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} |\mathcal{F}\Big(\varsigma, h_1(\varsigma), \int_0^\varsigma f(\rho, h_1(\rho))\, \mathrm{d}\rho\Big) \\ &\quad - \mathcal{F}\Big(\varsigma, h_2(\varsigma), \int_0^\varsigma f(\rho, h_2(\rho))\, \mathrm{d}\rho\Big)|\, \mathrm{d}\varsigma + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \\ &\quad |\mathcal{F}\Big(\varsigma, h_1(\varsigma), \int_0^\varsigma f(\rho, h_1(\rho))\, \mathrm{d}\rho\Big) - \mathcal{F}\Big(\varsigma, h_2(\varsigma), \int_0^\varsigma f(\rho, h_2(\rho))\, \mathrm{d}\rho\Big)|\, \mathrm{d}\varsigma \end{aligned}$$

$$\Rightarrow |\Upsilon h_1 - \Upsilon h_2| \leq \Bigg[L_{\mathcal{G}_i}\Big(1 + \varsigma_i G_g\Big) + \frac{L_{\mathcal{F}}}{\Gamma(1+\alpha)}\Big(\varsigma_i^\alpha + t_{i+1}^\alpha\Big)$$

$$+\frac{F_f L_{\mathcal{F}}}{\Gamma(2+\alpha)}\Big(\varsigma_i^{\alpha+1}+t_{i+1}^{\alpha+1}\Big)\Bigg] d_M(h_1,h_2)$$

$$\Rightarrow ||\Upsilon h_1-\Upsilon h_2||_{(C[\varsigma_i,t_{i+1}],\mathbb{R})}\le\Bigg[L_{\mathcal{G}_i}\Big(1+G_g\varsigma_i\Big)+\frac{L_{\mathcal{F}}}{\Gamma(1+\alpha)}\Big(\varsigma_i^{\alpha}+t_{i+1}^{\alpha}\Big)$$
$$+\frac{F_f L_{\mathcal{F}}}{\Gamma(2+\alpha)}\Big(\varsigma_i^{\alpha+1}+t_{i+1}^{\alpha+1}\Big)\Bigg]\times||h_1-h_2||_{PC}. \quad (2.14)$$

Thus, we observe that

$$d_M(\Upsilon h_1,\Upsilon h_2)\le\Theta d_M(h_1,h_2),\quad \forall\, h_1,h_2\in M. \quad (2.15)$$

Therefore, by assumption $\Theta<1$, it implies that Υ is a contraction map. Hence, by Banach fixed point theorem, there exists a unique mild solution $u^*\in M$ to the problem (2.1) – (2.3) as defined in (2.9).

Stability:

We consider a function $v\in PC(J,\mathbb{R})$ satisfying (2.7). Then, by using Lemma 2.3.1, we have

For $t\in[0,t_1]$,

$$|\Upsilon v(t)-v(t)|\le|v(t)-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma,v(\varsigma),\int_0^{\varsigma}f(\rho,v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma-u_0|$$
$$\le\frac{\varepsilon t_1^{\alpha}}{\Gamma(\alpha+1)}.$$

Thus,

$$||\Upsilon v-v||_{(C[0,t_1],\mathbb{R})}\le\frac{\varepsilon t_1^{\alpha}}{\Gamma(\alpha+1)}. \quad (2.16)$$

Similarly, for $t\in(t_i,\varsigma_i],\quad i=1,2,\ldots,m,$

$$|\Upsilon v(t)-v(t)|\le|v(t)-\mathcal{G}_i\Big(t,v(t),\int_0^t g(\varsigma,v(\varsigma))\,\mathrm{d}\varsigma\Big)|\le\varepsilon$$
$$\Rightarrow ||\Upsilon v-v||_{(C(t_i,\varsigma_i],\mathbb{R})}\le\varepsilon. \quad (2.17)$$

For $t\in(\varsigma_i,t_{i+1}],\quad i=1,2,3,\ldots,m,$

$$|\Upsilon v(t)-v(t)|\le|v(t)-\mathcal{G}_i\Big(\varsigma_i,v(\varsigma_i),\int_0^{\varsigma_i}g(\varsigma,v(\varsigma))\,\mathrm{d}\varsigma\Big)$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{\varsigma_i}(\varsigma_i-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma,v(\varsigma),\int_0^{\varsigma}f(\rho,v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma,v(s),\int_0^{\varsigma}f(\rho,v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma|\le\Big(1+\frac{\varsigma_i^{\alpha}+t_{i+1}^{\alpha}}{\Gamma(\alpha+1)}\Big)\varepsilon.$$

Therefore, we obtain

$$||\Upsilon v - v||_{(C(\varsigma_i, t_{i+1}], \mathbb{R})} \leq \Big(1 + \frac{\varsigma_i^\alpha + t_{i+1}^\alpha}{\Gamma(\alpha+1)}\Big)\varepsilon. \tag{2.18}$$

Hence, from above inequalities (2.16) − (2.18), we have

$$d_M(\Upsilon v, v) \leq r\varepsilon < +\infty. \tag{2.19}$$

Here, if the integer $k = 0$, we look at the space

$$M^* = \Big\{z \in M | d_M(v, z) < \infty\Big\}. \tag{2.20}$$

Taking $z(t) = v(t)$ itself, then $d_M(v, v) = 0 < \infty \Rightarrow v \in M^*$.
Hence, by Theorem 1.5.1 (iii), if u^* is a unique fixed point of the operator Υ, we obtain

$$|v(t) - u^*(t)| \leq \frac{d_M(\Upsilon v, v)}{1 - \Theta} \leq \frac{r\varepsilon}{1 - \Theta}, \quad \forall t \in J. \tag{2.21}$$

Therefore, the problem (2.1) − (2.3) is Ulam-Hyers stable. □

2.4 Examples

In this section, we present some examples to illustrate the results obtained in the preceding sections.

Example 2.4.1. *We consider the following Caputo fractional differential equation with a non-instantaneous impulse:*

$$\begin{cases} {}^C_0\mathcal{D}_t^{\frac{1}{3}} u(t) = \frac{1}{5+3t^2}\Big(|u(t)| + \int_0^t \frac{|u(\varsigma)|}{10+7\varsigma^2}\,\mathrm{d}\varsigma\Big), & t \in (0,1] \cup (2,3], \\ u(t) = \frac{1}{(5+3(t-1)^2)(1+|u(t)|)}\Big(|u(t)| + \int_0^t \frac{|u(\varsigma)|}{15+11\varsigma^2}\,\mathrm{d}\varsigma\Big), & t \in (1,2], \end{cases} \tag{2.22}$$

and the corresponding non-instantaneous impulsive fractional differential inequalities with impulse interval (1,2], and for $\varepsilon > 0$*, as*

$$\begin{cases} |{}^C_0\mathcal{D}_t^{\frac{1}{3}} v(t) - \frac{1}{5+3t^2}\Big(|v(t)| + \int_0^t \frac{|v(\varsigma)|}{10+7\varsigma^2}\,\mathrm{d}\varsigma\Big)| \leq \varepsilon, & t \in (0,1] \cup (2,3], \\ |v(t) - \frac{1}{(5+3(t-1)^2)(1+|v(t)|)}\Big(|v(t)| + \int_0^t \frac{|v(\varsigma)|}{15+11\varsigma^2}\,\mathrm{d}\varsigma\Big)| \leq \varepsilon, & t \in (1,2]. \end{cases} \tag{2.23}$$

Here, $J = [0,3]$ *and* $0 = t_0 = \varsigma_0 < t_1 = 1 < \varsigma_1 = 2 < t_2 = 3$, $f(t, u(t)) = \frac{|u(t)|}{10+7t^2}$ *with* $F_f = \frac{1}{10}$ *and*

$$\mathcal{F}\Big(t, u(t), \int_0^t f(\varsigma, u(\varsigma))\,\mathrm{d}\varsigma\Big) = \frac{1}{5+3t^2}\Big(|u(t)| + \int_0^t \frac{|u(\varsigma)|}{10+7\varsigma^2}\,\mathrm{d}\varsigma\Big)$$

with $L_{\mathcal{F}} = \frac{1}{5}$.
Also, $g(t,u(t)) = \frac{|u(t)|}{15+11t^2}$ *with* $G_g = \frac{1}{15}$, *and*

$$\mathcal{G}_1\Big(t,u(t),\int_0^t g(\varsigma,u(\varsigma))\,\mathrm{d}\varsigma\Big) = \frac{1}{(5+3(t-1)^2)(1+|u(t)|)}\Big(|u(t)| + \int_0^t \frac{|u(\varsigma)|}{15+11\varsigma^2}\,\mathrm{d}\varsigma\Big),$$

with $L_{\mathcal{G}_1} = \frac{1}{5}$ *for* $t \in (1,2]$.
Now,

$$\begin{aligned}\Theta &= \max\Big\{\frac{t_1^\alpha L_{\mathcal{F}}}{\Gamma(1+\alpha)}\Big(1+\frac{t_1F_f}{1+\alpha}\Big), L_{\mathcal{G}_1}\big(1+\varsigma_1 G_g\big) + \frac{L_{\mathcal{F}}}{\Gamma(1+\alpha)}\big(\varsigma_1^\alpha + t_2^\alpha\big)\\ &\quad + \frac{F_f L_{\mathcal{F}}}{\Gamma(2+\alpha)}\big(\varsigma_1^{\alpha+1}+t_2^{\alpha+1}\big)\Big\}\\ \Rightarrow \Theta &= \max\Big\{\frac{\frac{1}{5}}{\Gamma(\frac{4}{3})}\Big(1+\frac{3}{40}\Big), \frac{1}{5}\Big(1+\frac{2}{15}\Big) + \frac{\frac{1}{5}}{\Gamma(\frac{4}{3})}\big(\sqrt[3]{2}+\sqrt[3]{3}\big)\\ &\quad + \frac{\frac{1}{50}}{\Gamma(\frac{7}{3})}\big(2\sqrt[3]{2}+3\sqrt[3]{3}\big)\Big\}\\ \Rightarrow \Theta &= \max\Big\{0.240767, 0.946876905\Big\} = 0.946876905 \cong 0.947 < 1.\end{aligned}$$

Here, $r = \max\Big\{\frac{1}{\Gamma(\frac{4}{3})}, 1+\frac{\sqrt[3]{2}+\sqrt[3]{3}}{\Gamma(\frac{4}{3})}\Big\} = \max\Big\{0.89297951, 4.02601637\Big\}$,
i.e., $r = 4.02601637 \cong 4.03$.
By Theorem 2.3.2, there exists a unique solution $u^* : [0,3] \to \mathbb{R}$ *such that*

$$u^*(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\frac{1}{3})}\int_0^t (t-\varsigma)^{-\frac{2}{3}}\frac{1}{5+3\varsigma^2}\Big(|u^*(\varsigma)| + \int_0^\varsigma \frac{|u^*(\rho)|}{10+7\rho^2}\,d\rho\Big)\,d\varsigma, & t\in[0,t_1],\\ \frac{1}{(5+3(t-1)^2)(1+|u^*(t)|)}\Big(|u^*(t)| + \int_0^t \frac{|u^*(\varsigma)|}{15+11\varsigma^2}\,\mathrm{d}\varsigma\Big), & t\in(1,2],\\ \frac{1}{8(1+|u^*(2)|)}\Big(|u^*(2)| + \int_0^2 \frac{|u^*(\varsigma)|}{15+11\varsigma^2}\,d\varsigma\Big) & \\ -\frac{1}{\Gamma(\frac{1}{3})}\int_0^2 (2-\varsigma)^{\frac{-2}{3}}\frac{1}{5+3\varsigma^2}\Big(|u^*(\varsigma)| + \int_0^\varsigma \frac{|u^*(\rho)|}{10+7\rho^2}\,d\rho\Big)\,d\varsigma & \\ +\frac{1}{\Gamma(\frac{1}{3})}\int_0^t (t-\varsigma)^{-\frac{2}{3}}\frac{1}{5+3\varsigma^2}\Big(|u^*(\varsigma)| + \int_0^\varsigma \frac{|u^*(\rho)|}{10+7\rho^2}\,d\rho\Big)\,d\varsigma, & t\in(2,3],\end{cases} \tag{2.24}$$

with

$$|v(t)-u^*(t)| \leq \frac{r\varepsilon}{1-\Theta} = 76.04\varepsilon, \quad \forall t \in J = [0,3],$$

where $v(t)$ *is the solution of the fractional differential inequalities (2.23).*

Example 2.4.2. *Consider the following problem:*

$$\begin{cases} {}_0^C\mathcal{D}_t^{\frac{1}{4}}u(t) = \frac{u(t)}{5} + \frac{1}{8}\int_0^t \frac{u(\varsigma)}{3+\varsigma}\,d\varsigma + F(t), & t \in (0,1] \cup (2,3], \\ u(t) = \frac{\sin(u(t))}{5} + \frac{1}{6}\int_0^t \frac{u(\varsigma)}{2+\varsigma}\,ds + G(t), & t \in (1,2], \\ u(0) = 1, \end{cases} \tag{2.25}$$

where $G(t)$ *and* $F(t)$ *are functions of* t *chosen such that*

$$u(t) = \begin{cases} (t+3)(\frac{t}{10} - \frac{t^{0.25}}{4} + 1), & t \in [0,1], \\ (t+2)(t^{0.25}+1), & t \in (1,2], \\ (t+3)(\frac{t}{10} - \frac{t^{0.25}}{4} + 1) + \frac{21}{2^{1.75}} - 2, & t \in (2,3] \end{cases} \tag{2.26}$$

is the solution of the problem (2.25)*. Clearly, the function in equation* (2.26) *is a unique*

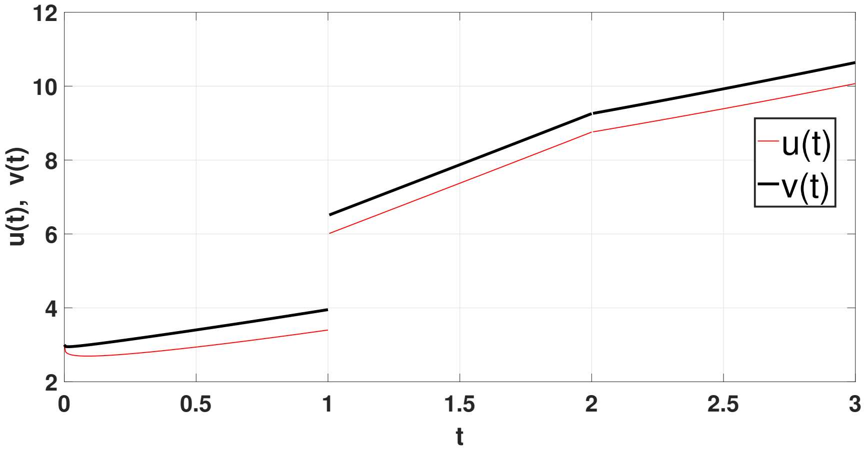

Figure 2.1: Solution $u(t)$ of the problem (2.25) and solution $v(t)$ of the problem (2.28) for $\varepsilon = 0.5$

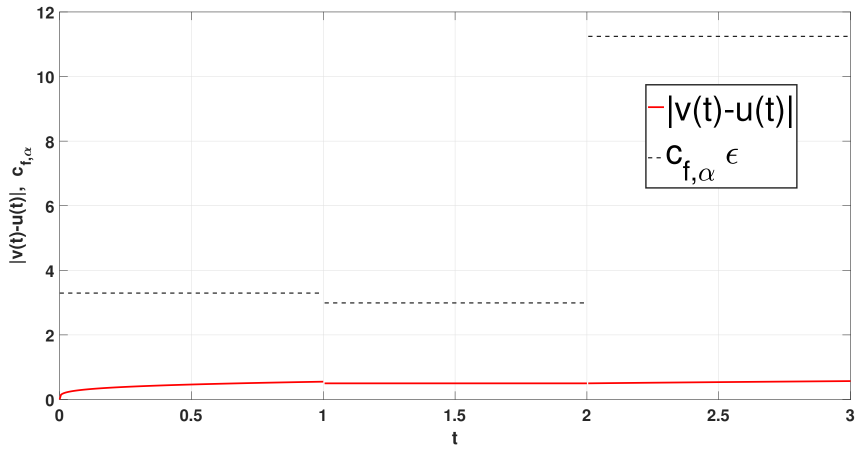

Figure 2.2: The difference function $|v(t) - u(t)|$ and upper bound $c_{f,\alpha}\varepsilon$ for $\varepsilon = 0.5$

solution for the problem by Theorem 2.3.2*, since* $\Theta = 0.833 < 1$*, and all the conditions*

are satisfied.
Further,

$$c_{f,\alpha} = \begin{cases} 6.59, & t \in [0,1], \\ 5.98, & t \in (1,2], \\ 22.49, & t \in (2,3]. \end{cases} \tag{2.27}$$

Thus, by Theorem 2.3.2, the problem (2.25) is Ulam-Hyers stable which means that the inequality holds for all $t \in J = [0,3]$, $|v(t) - u(t)| \leq c_{f,\alpha}\varepsilon$ for any solution $v(t)$ satisfying inequalities (2.7), and according to Theorem 2.3.1, it is the solution of

$$\begin{cases} {}_0^C\mathcal{D}_t^{\frac{1}{4}} v(t) = \frac{v(t)}{5} + \frac{1}{8}\int_0^t \frac{v(\varsigma)}{3+\varsigma}\, d\varsigma + F(t) + \widetilde{H}(t), & t \in (0,1] \cup (2,3], \\ v(t) = \frac{\sin(v(t))}{5} + \frac{1}{6}\int_0^t \frac{v(\varsigma)}{2+\varsigma}\, d\varsigma + G(t) + H_1 & t \in (1,2], \end{cases} \tag{2.28}$$

where $\widetilde{H} \in PC(J,\mathbb{R})$ and $H_1 \in \mathbb{R}$ with $|\widetilde{H}(t)| \leq \varepsilon$, $t \in J$, $|H_1| \leq \varepsilon$. In particular, for $\varepsilon = 0.5$, take $\widetilde{H}(t) = \varepsilon$, $H_1 = \varepsilon$. We plot the graphs of the solution $u(t)$ of the problem (2.25) and the solution $v(t)$ of the problem (2.28) in Figure 2.1. We also present the difference function $|v(t) - u(t)|$ for $\varepsilon = 0.5$ and the upper bound $c_{f,\alpha}\varepsilon$ graphically in Figure 2.2. It points towards a difference which is almost uniform.

2.5 An Application to Fractional-order RLC

RLC circuit is one of the most basic and fundamental circuits in various electronic devices and there is a huge literature describing the RLC circuit concerning integer-order differential equations. Fractional-order RLC circuit model is the generalization of the classical integer-order RLC circuit. This fractional model has a number of advantages over its integer-order counterpart because of the fractional order. This provides substantial flexibility in design and control of circuit which enhances the performance and the novel behavior. Radwan et al. [118] presented a broad view of the fractional-order RLC circuit model and illustrated that it was not possible to observe some novel phenomena in the absence of the fractional-order model. They also established that the fractional-order impedance was purely imaginary, and it enabled the modeling of a huge capacitance/inductance by considering a very small fractional order. For complete information on fractional-order RLC circuit, the readers are referred to the works in [7, 126, 150].

Dhaneliya et al. [34] computed the analytical solution of the fractional-order RLC circuit in terms of an infinite series in which each term was a generalized Mittag-Leffler function. Hence, it is a challenging task to estimate the bound for the solution; and computation the solution may increase the roundoff error. Therefore, to overcome such problems, we propose to find some bound estimate for the solution.

The following non-instantaneous fractional impulsive RLC circuit under a given input voltage $V(t)$ is considered:

$$L_\alpha\, {}_0^C D_t^\alpha I_\alpha(t) + RI_\alpha(t) + \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma = V(t), \quad t \in \left(0, \frac{1}{4}\right] \cup \left(\frac{1}{2}, 1\right], \tag{2.29}$$

$$RI_\alpha(t) + \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma = V(t), \quad t \in \left(\frac{1}{4}, \frac{1}{2}\right], \tag{2.30}$$

$$I_\alpha(0) = 0, \tag{2.31}$$

where L_α, R and C are inductor, resistor and capacitor, respectively. Here $0 = \varsigma_0 < t_1 = \frac{1}{4} < \varsigma_1 = \frac{1}{2} < t_2 = 1$ and we also assume that $R^2 = 4L_\alpha/C$. In the above circuit, for the time interval $\left(\frac{1}{4}, \frac{1}{2}\right]$, the effect of the inductor is negligible but its effect is clearly visible when $\left(\frac{1}{2}, 1\right]$ is considered.

Theorem 2.5.1. *Consider the non-instantaneous impulsive fractional-order RLC circuit (2.29) – (2.31) and assume that the input voltage $V(t)$ is bounded, i.e., there exists $B > 0$ such that $|V(t)| < B,\ \forall\, t \in [0, 1]$. Let the bandwidth $a = \dfrac{R}{2L_\alpha}$ of the RLC circuit satisfy*

$$\frac{a}{2}\left[1 + \frac{a}{4} + \frac{2}{\Gamma(1+\alpha)}\left\{\frac{a^2}{1+\alpha}\left(1 + \frac{1}{2^{\alpha+1}}\right) + \frac{1}{2^\alpha} + 1\right\}\right] = \Theta < 1. \tag{2.32}$$

Then, the circuit current I_α of the system (2.29) – (2.31) is bounded, i.e.,

$$|I_\alpha(t)| \le L_U B, \quad \forall\, t \in [0, 1], \tag{2.33}$$

where, $L_U = \dfrac{1 + \dfrac{(1 + \frac{1}{2^\alpha})}{\Gamma(1+\alpha)}}{1 - \Theta}.$

Proof. Here,

$$\mathcal{F}\left(t, I_\alpha(t), \int_0^t g(\varsigma, I_\alpha(\varsigma))\, d\varsigma\right) = \frac{1}{L_\alpha}\left(V(t) - RI_\alpha(t) - \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma\right)$$

with $L_{\mathcal{F}} = \max\left\{\frac{R}{L_\alpha}, \frac{1}{CL_\alpha}\right\}$. By using the given condition $\Theta < 1 \Rightarrow a < 1$ and under the given assumption $R^2 = 4L_\alpha/C$, we get $CL_\alpha = \frac{1}{a^2}$. Thus, we have

$$L_{\mathcal{F}} = \max\{2a, a^2\} = 2a. \tag{2.34}$$

Also,

$$\mathcal{G}_1\left(t, I_\alpha(t), \int_0^t h(\varsigma, I_\alpha(\varsigma))\, d\varsigma\right) = \frac{1}{R}\left(V(t) - \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma\right)$$

with $L_{\mathcal{G}_1} = \frac{1}{RC} = \frac{a}{2}$. We also take $F_f = \frac{1}{CL_\alpha} = a^2$ and $G_g = \frac{1}{RC} = \frac{a}{2}$.
Here, we observe that

$$\begin{aligned}\Theta &= \max\left\{\frac{L_{\mathcal{F}} t_1^\alpha}{\Gamma(1+\alpha)}\left(1+\frac{t_1 F_f}{1+\alpha}\right), L_{\mathcal{G}_1}\left(1+\varsigma_1 G_g\right) + \frac{L_{\mathcal{F}}}{\Gamma(1+\alpha)}\left(\varsigma_1^\alpha + t_2^\alpha\right)\right. \\ &\qquad\qquad \left. + \frac{F_f L_{\mathcal{F}}}{\Gamma(2+\alpha)}\left(\varsigma_1^{\alpha+1} + t_2^{\alpha+1}\right)\right\} \\ &= \max\left\{\frac{2a}{4^\alpha\Gamma(1+\alpha)}\left(1+\frac{a^2}{4(1+\alpha)}\right), \frac{a}{2}\left(1+\frac{a}{4}\right) + \frac{2a}{\Gamma(1+\alpha)}\left(1+\frac{1}{2^\alpha}\right)\right. \\ &\qquad\qquad \left. + \frac{2a^3}{\Gamma(2+\alpha)}\left(1+\frac{1}{2^{\alpha+1}}\right)\right\} \\ &= \frac{a}{2}\left[1+\frac{a}{4}+\frac{2}{\Gamma(1+\alpha)}\left\{\frac{a^2}{1+\alpha}\left(1+\frac{1}{2^{\alpha+1}}\right)+\frac{1}{2^\alpha}+1\right\}\right]. \end{aligned} \tag{2.35}$$

By given condition (2.32), we have $\Theta < 1$. Thus, by Theorem 2.3.2, the fractional-order RLC system (2.29) − (2.31) is Ulam-Hyers stable with a Ulam-Hyers constant

$$L_U = \frac{1 + \frac{(1+\frac{1}{2^\alpha})}{\Gamma(1+\alpha)}}{1-\Theta}. \tag{2.36}$$

Now, take $I_\alpha(t) = I(t) = 0, \ \forall\, t \geq 0$, in the system (2.29) − (2.31) to get

$$|L_\alpha\, {}_0^C D_t^\alpha I_\alpha(t) + RI_\alpha(t) + \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma - V(t)| = |V(t)| \leq B, \quad t \in \left(0, \frac{1}{4}\right] \cup \left(\frac{1}{2}, 1\right], \tag{2.37}$$

and

$$|RI_\alpha(t) + \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma - V(t)| = |V(t)| \leq B, \quad t \in \left(\frac{1}{4}, \frac{1}{2}\right]. \tag{2.38}$$

Thus, we take $\varepsilon = B$, and from equations (2.37), (2.38), we observe that the function $I(t) = 0$ satisfies the inequality (2.7). Thus, by Theorem 2.3.2, we have the following bound for the fractional-order RLC circuit current $I_\alpha(t)$:

$$|I_\alpha(t)| = |I_\alpha(t) - I(t)| \leq L_U B, \quad \forall\, t \in [0, 1], \tag{2.39}$$

where L_U is a Ulam-Hyers constant given in equation (2.36). □

Remark 2.5.1. *From the numerical computation, we observe that, for* $a \leq 0.3621315$, *the condition* (2.32) *of the Theorem* 2.5.1 *holds for all* $\alpha \in [0, 1]$. *So, we call* $a_0 = 0.3621315$ *the threshold value of the bandwidth* a *in the sense that, for* $a \leq a_0$, *the condition* (2.32) *of the Theorem* 2.5.1 *holds for all* $\alpha \in [0, 1]$.

First, we plot the graph of the Ulam-Hyers constant L_U with $\alpha \in [0, 1]$ and for $a = a_0$. From Fig. 2.3, we observe the graph of L_U attains its maximum point at $\alpha = 0.119612$,

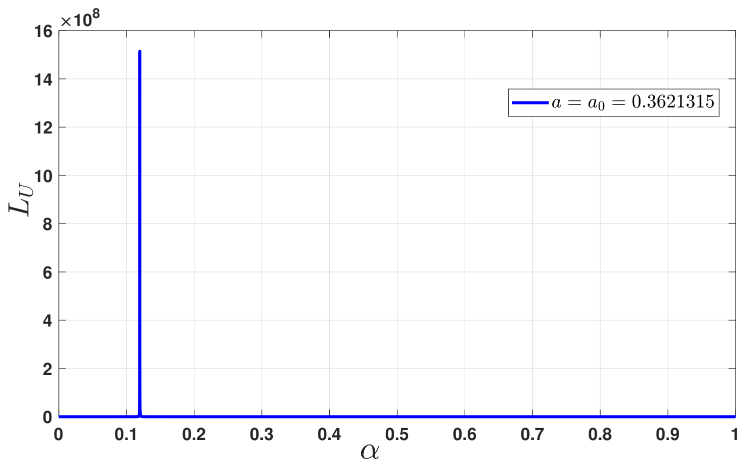

Figure 2.3: Graph of Ulam-Hyers constant L_U given by (2.36) for $a = a_0 = 0.3621315$ (Threshold value)

which means that, if the bandwidth a of the given RLC system $(2.29) - (2.31)$ is equal to the threshold value $a_0 = 0.3621315$, then from inequality (2.33) and under the given input voltage, the absolute value of the circuit current is maximum for $\alpha = 0.119612$, compared to the fractional order $\alpha \in [0, 1] \setminus \{0.119612\}$ of the system.

Next, we study the behaviour of the Ulam-Hyers constant for the bandwidth a closer to the threshold value a_0. From Fig. 2.4, we observe that the variation of the Ulam-Hyers

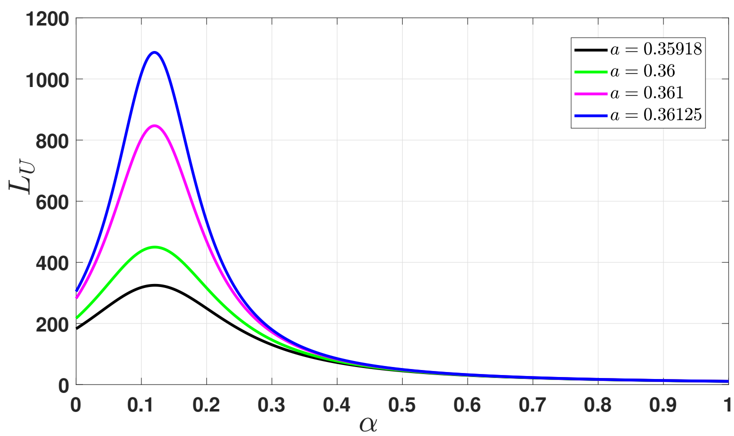

Figure 2.4: Graph of Ulam-Hyers constant L_U given by (2.36) for the bandwidth close to the threshold value a_0

constant is sensitive for the bandwidth a closer to the threshold value a_0. For each value of a, L_U attains its maximum value for the fractional order between $0.1 < \alpha < 0.2$.

Next, we plot the graph of the Ulam-Hyers constant L_U for different values of $a < a_0$. From Fig. 2.5, it is clear that the maximum value of L_U is not as high as compared to the

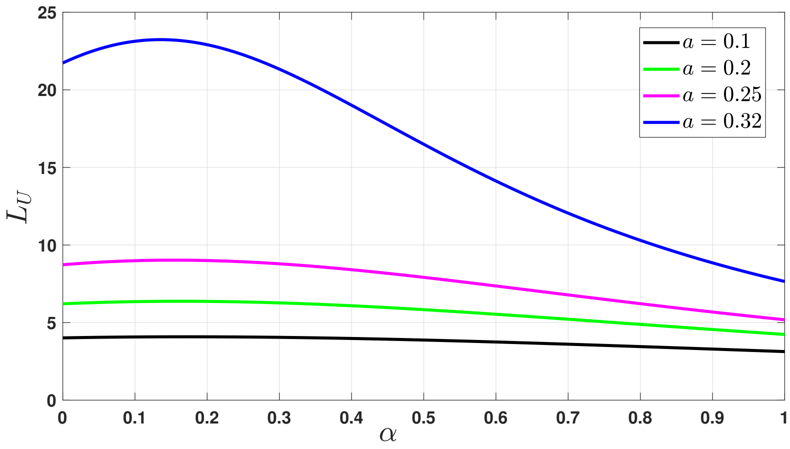

Figure 2.5: Graph of Ulam-Hyers constant L_U given by (2.36) for the bandwidth $a < a_0$

case when the values of the bandwidth are near the threshold value a_0. In comparison to the larger values of the bandwidth, the values of L_U on $0 \leq \alpha \leq 1$, are smaller corresponding to the smaller values of the bandwidth. In other words, if we denote $L_U(a, \alpha)$ as the Ulam-Hyers constant for a given bandwidth a, on $0 \leq \alpha \leq 1$, then $L_U(a_1, \alpha) < L_U(a_2, \alpha)$ for all $0 \leq \alpha \leq 1$ with $a_1 < a_2 \leq a_0$. Here, we also observe that, for smaller values of $a(<< a_0)$, L_U remains almost constant, which means that, for smaller values of the bandwidth, the absolute values of the circuit current I_α are bounded with a bound independent of the fractional-order α.

Finally, from the above observations, we conclude that the absolute values of the circuit current I_α of the fractional-order RLC $(2.29) - (2.31)$ system are large for the larger values of the bandwidth $a(< a_0)$, and for each value of $a(\leq a_0)$, the absolute value of the circuit current I_α is the maximum for the fractional order between $0.1 < \alpha < 0.2$. In particular, for $a = a_0$ and $\alpha = 0.119612$, under a given input voltage, we can generate the circuit current with the largest absolute value for the fractional order $\alpha = 0.119612$, compared to the fractional order $\alpha \in [0, 1] \setminus \{0.119612\}$. This shows that the fractional-order RLC circuit model has many advantages over its integer-order counterpart because of the inclusion of the order α included in the model which affects the performance and enhances the novel behavior lending more flexibility in the design and control of the circuit. Further, this shows the practical applicability of the Ulam-Hyers constant in some important problems. We firmly believe that this finding has the potential to occupy an important place in fractional-order RLC circuit problems as well as in the application of Ulam-Hyers stability.

2.6 Conclusions

In this chapter, we have established the existence and stability results of the mild solution of the Caputo fractional non-instantaneous impulsive integro-differential equation in the sense of Ulam-Hyers stability. The main result is established by using Banach fixed point theorem under appropriate assumptions. Two examples have been presented in order to ascertain the applicability of our results. The main result was used to estimate the bound for the non-instantaneous impulsive fractional-order RLC circuit current I_α, and it is found that the bound mainly depends on the bandwidth and fractional-order of the system. Further, by understanding the behaviour of the Ulam-Hyers constant L_U numerically, we determined that the absolute values of the circuit current I_α of the fractional-order RLC system are larger for larger values of the bandwidth $a(< a_0)$, and for each value of $a(\leq a_0)$, the absolute value of the circuit current I_α is the maximum for the fractional order between $0.1 < \alpha < 0.2$. In particular, for $a = a_0$ and $\alpha = 0.119612$, under a given input voltage, we can generate the circuit current with the largest absolute value for the fractional order $\alpha = 0.119612$, in comparison to the fractional-order $\alpha \in [0,1] \setminus \{0.119612\}$.

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CHAPTER 3

Generalized Ulam-Hyers-Rassias stability of the solution for the Caputo fractional non-instantaneous impulsive integro-differential equation and its application to fractional RLC circuit

In this chapter, we generalize our previous problem discussed in Chapter 2 by replacing the arbitrary given number ε with a non-negative function $\varphi(t)$, and establish a similar result as the previous one under some suitable assumptions on φ. At the end, we use our results to estimate the bound for the difference between fractional-order and integer-order non-instantaneous impulsive RLC circuit currents. We basically wish to examine the existence, uniqueness and generalized Ulam-Hyers-Rassias stability of the mild solution of the following Caputo fractional non-instantaneous impulsive integro-differential equation:

$$ {}^{C}_{0}\mathcal{D}^{\alpha}_{t}u(t) = \mathcal{F}\Big(t, u(t), \int_0^t f(\varsigma, u(\varsigma))\, d\varsigma\Big), \quad t \in (\varsigma_i, t_{i+1}],\ i = 0, 1, \dots, m, \tag{3.1} $$

$$ u(t) = \mathcal{G}_i\Big(t, u(t), \int_0^t g(\varsigma, u(\varsigma))\, d\varsigma\Big),\ t \in (t_i, \varsigma_i], \quad i = 1, 2, \dots, m, \tag{3.2} $$

$$ u(0) = u_0, \tag{3.3} $$

where $0 < \alpha < 1$, $0 = t_0 = \varsigma_0 < t_1 \leq \varsigma_1 \leq t_2 < \dots < t_m \leq \varsigma_m < t_{m+1} = T$ are pre-fixed numbers, $J = [0, T]$, $\mathcal{G}_i \in C\Big([t_i, \varsigma_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\Big)$ $\ i = 1, 2, \dots, m$, $\mathcal{F} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are given functions.

From above, it may be noted that the fractional-order system is of pseudo-state nature and that the states are infinite dimensional. The specification of initial conditions to the considered problem remains a mystery in many situations of the current state-of-the-art

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theory of fractional- order systems. Inspired by the nature of the Caputo derivative operator, we pose the initial condition to the considered system at the initial time 0. One may pose the initial condition at any other initial time, say t_0, not equal to 0. But for the mentioned case, the possible behavior of the considered system is not known by our theory.

Further, after establishing the theoretical results for the system (3.1) – (3.3), we are motivated by a number of works such as [118, 34, 150, 7, 8, 131, 126] to estimate the bound for the solution corresponding to the initial value problem (3.1) – (3.3) and its integer-order counterpart. To the best of the authors' knowledge, generalized Ulam-Hyers-Rassias stability has not been established till date for such class of fractional integro-differential equations. On another note, we also provide a RLC circuit problem in terms of a fractional differential equation to estimate the bound for the difference in the solution, i.e., the current, due to both fractional and conventional set-up. This may be considered a very significant step in providing extra information, and it points towards the scope of applications of such a study.

3.1 Main Result

3.1.1 Required background

Definition 3.1.1. *A function $u \in PC(J, \mathbb{R})$ is called a mild solution of the problem* (3.1) – (3.3) *if*

$$u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma, \;\; t \in [0, t_1], \\ \mathcal{G}_i\Big(t, u(t), \int_0^t g(\varsigma, u(\varsigma))\, \mathrm{d}\varsigma\Big), \quad t \in (t_i, \varsigma_i], \quad i = 1, 2, 3, \ldots, m, \\ \mathcal{G}_i\Big(\varsigma_i, u(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, u(\varsigma))\, \mathrm{d}\varsigma\Big) - \\ \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma, \; t \in (\varsigma_i, t_{i+1}]. \end{cases} \tag{3.4}$$

(This is the same as Definition 2.3.1 in Chapter 2).

Let $\varepsilon > 0$ and $\varphi \in PC(J, \mathbb{R}_+)$ be nondecreasing. Consider the following fractional non-instantaneous impulsive differential inequalities:

$$\begin{cases} |{}_0^C\mathcal{D}_t^\alpha v(t) - \mathcal{F}\Big(t, v(t), \int_0^t f(\varsigma, v(\varsigma))\, d\varsigma\Big)| \le \varphi(t), \;\; t \in (\varsigma_i, t_{i+1}], \;\; 0 \le i \le m, \\ |v(t) - \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\, d\varsigma\Big)| \le \varepsilon, \quad t \in (t_i, \varsigma_i], \quad i = 1, 2, \ldots, m. \end{cases} \tag{3.5}$$

Lemma 3.1.1. *[114] A function $v \in PC(J, \mathbb{R})$ is a solution of the inequalities (3.5) if and only if there is a function $\widetilde{H} \in PC(J, \mathbb{R})$ and a sequence $\{H_i\}$, $i = 1, 2, \ldots, m$ (which depends on v) such that*

(i) $|\widetilde{H}(t)| \le \varphi(t), \quad t \in J$, and $|H_i| \le \varepsilon, \quad i = 1, 2, \ldots, m,$

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(ii) ${}_{0}^{C}\mathcal{D}_{t}^{\alpha}v(t) = \mathcal{F}\Big(t, v(t), \int_0^t f(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big) + \widetilde{H}(t),\ t \in (\varsigma_i, t_{i+1}],\ i = 0, 1, \ldots, m,$

(iii) $v(t) = \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big) + H_i, \quad t \in (t_i, \varsigma_i], \quad i = 1, 2, \ldots, m.$

Definition 3.1.2. *[114] (Ulam-Hyers-Rassias stability) The problem described by equations (3.1) − (3.3) is said to be generalized Ulam-Hyers-Rassias stable with respect to the couple (φ, ε) if there exists a constant $C_{f,\alpha,g} > 0$ such that, for each solution $v \in PC(J, \mathbb{R})$ of the inequalities (3.5), there exists a solution $u \in PC(J, \mathbb{R})$ of (3.1) − (3.3) with*

$$|v(t) - u(t)| \leq C_{f,\alpha,g}\Big(\varphi(t) + \varepsilon\Big) \quad \forall t \in J. \tag{3.6}$$

Lemma 3.1.2. *(Technique to be followed) Let $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ be convergent series with $y_n \geq 0$ and $x_n \leq y_n$ for all n. Then,*

$$\sum_{n=0}^{\infty} x_n \leq \sum_{n=0}^{\infty} y_n. \tag{3.7}$$

Proof. Let $X_n = \sum_{k=0}^{n} x_k$ and $Y_n = \sum_{k=0}^{n} y_k$. Clearly $\{X_n\}$ and $\{Y_n\}$ are convergent sequences since both the series are convergent.

Now, $y_k \geq 0 \Rightarrow \{Y_n\}$ is an increasing sequence and it is given that the series $\sum_{n=0}^{\infty} y_n$ is convergent.

Let $l_y = \sum_{n=0}^{\infty} y_n$. Then, $l_y = \sup_{n \in \mathbb{N}_0} \Big\{Y_n\Big\}$.

We also have

$$\begin{aligned} & X_n \leq Y_n, \quad \forall n \in \mathbb{N}_0 \\ & \Rightarrow X_n \leq Y_n \leq l_y, \quad \forall n \in \mathbb{N}_0 \\ & \Rightarrow \lim_{n \to \infty} X_n \leq l_y \\ & \Rightarrow \sum_{n=0}^{\infty} x_n \leq \sum_{n=0}^{\infty} y_n. \end{aligned}$$

□

Corollary 3.1.1. *If $\sum_{n=0}^{\infty} y_n$ is a convergent series with $0 \leq x_n \leq y_n$, for all n, then the series $\sum_{n=0}^{\infty} x_n$ converges and $\sum_{n=0}^{\infty} x_n \leq \sum_{n=0}^{\infty} y_n$.*

Proof. Since $X_n \leq l_y,\ n \in \mathbb{N}_0$ and X_n is an increasing sequence which is bounded above by l_y, therefore

$$\lim_{n \to \infty} X_n \leq l_y$$

$$\Rightarrow \sum_{n=0}^{\infty} x_n \leq \sum_{n=0}^{\infty} y_n.$$

□

Theorem 3.1.1. *Let* $a \geq 0$ *and* $t \geq 0$. *Then,*
(i) $\frac{1}{\Gamma(\alpha)} e^{at} \leq \mathbb{E}_{1,\alpha}(at)$, *for all* $0 < \alpha \leq 1$,
(ii) $\mathbb{E}_{1,\alpha}(at) \leq \frac{1}{\Gamma(\alpha)} e^{at}$ *and* $\mathbb{E}_{1,\alpha}(-at) \leq \frac{1}{\Gamma(\alpha)} e^{at}$, *for all* $\alpha \geq 1$.

Proof. (i) We know that $\Gamma(k+\alpha) = (k+\alpha-1)\cdot(k+\alpha-2)\cdots(1+\alpha)\cdot\alpha\cdot\Gamma(\alpha)$.
Since it is given that $0 < \alpha \leq 1$, we have
$k \geq k-(1-\alpha)$, $k-1 \geq k-1-(1-\alpha), \ldots, 2 \geq 1+\alpha$, $1 \geq \alpha$. Then,

$$\begin{aligned} & k\cdot(k-1)\cdots 2\cdot 1 \geq (k+\alpha-1)\cdot(k+\alpha-2)\cdots(1+\alpha)\cdot\alpha \\ & \Rightarrow \Gamma(k+1) \geq \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \\ & \Rightarrow \frac{\Gamma(k+1)}{\Gamma(k+\alpha)} \geq \frac{1}{\Gamma(\alpha)}, \quad \forall k, \end{aligned} \tag{3.8}$$

since for each $t > 0$, $\frac{(ak)^k}{\Gamma(k+1)} > 0$. We have the following from (3.8):

$$\frac{(ak)^k}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k+\alpha)} \geq \frac{1}{\Gamma(\alpha)} \frac{(ak)^k}{\Gamma(k+1)}. \tag{3.9}$$

Using Corollary 3.1.1, we get $\frac{1}{\Gamma(\alpha)} e^{at} \leq \mathbb{E}_{1,\alpha}(at)$.

(ii) Since it is given that $\alpha \geq 1$, we have
$k \leq k+\alpha-1$, $k-1 \leq k+\alpha-2, \ldots, 2 \leq 1+\alpha$, $1 \leq \alpha$. Then,

$$\frac{\Gamma(k+1)}{\Gamma(k+\alpha)} \leq \frac{1}{\Gamma(\alpha)}, \quad \forall k. \tag{3.10}$$

Similarly, as in case (i) above, we have $\mathbb{E}_{1,\alpha}(at) \leq \frac{1}{\Gamma(\alpha)} e^{at}$.
Since

$$\frac{(-1)^k (at)^k}{\Gamma(k+\alpha)} \leq \frac{(at)^k}{\Gamma(k+\alpha)} \leq \frac{1}{\Gamma(\alpha)} \frac{(at)^k}{\Gamma(k+1)},$$

we have

$$\frac{(-1)^k (at)^k}{\Gamma(k+\alpha)} \leq \frac{1}{\Gamma(\alpha)} \frac{(at)^k}{\Gamma(k+1)}.$$

Now by using Lemma 3.1.2, we get

$$\mathbb{E}_{1,\alpha}(-at) \leq \frac{1}{\Gamma(\alpha)} e^{at}.$$

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□

3.1.2 Existence and stability

We introduce the following assumptions:

(H1) $\mathcal{F} \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\exists\, L_{\mathcal{F}} > 0$ such that

$$|\mathcal{F}(t, x_1, y_1) - \mathcal{F}(t, x_2, y_2)| \leq L_{\mathcal{F}}\Big(|x_1 - x_2| + |y_1 - y_2|\Big) \tag{3.11}$$

for each $t \in J$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(H2) $\mathcal{G}_i \in C\Big([t_i, \varsigma_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\Big)$ and $\exists\, L_{\mathcal{G}_i} > 0$, $i = 1, 2, \ldots, m$ such that

$$|\mathcal{G}_i(t, x_1, y_1) - \mathcal{G}_i(t, x_2, y_2)| \leq L_{\mathcal{G}_i}\Big(|x_1 - x_2| + |y_1 - y_2|\Big) \tag{3.12}$$

for each $t \in (t_i, \varsigma_i]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(H3) $f, g \in C(J \times \mathbb{R}, \mathbb{R})$ and $\exists\, F_f > 0$, $G_g > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq F_f |x_1 - x_2|, \quad |g(t, y_1) - g(t, y_2)| \leq G_g |y_1 - y_2| \tag{3.13}$$

for each $t \in J$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(H4) The function $\varphi \in C(J, \mathbb{R})$ is a nondecreasing function. There exist $C_\varphi > 0$ and $0 < p < \alpha < 1$ such that

$$\left(\int_0^t \big(\varphi(\varsigma)\big)^{\frac{1}{p}}\, d\varsigma\right)^p \leq C_\varphi \varphi(t), \quad \forall t \in J. \tag{3.14}$$

Lemma 3.1.3. *Suppose hypothesis **(H4)** holds, $\mathcal{F} \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\mathcal{G}_i \in C([t_i, \varsigma_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $i = 1, 2, \ldots, m$, and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous. If $v \in PC(J, \mathbb{R})$ is a solution of the inequalities (3.5), then it satisfies the following integral inequalities:*

$$\begin{cases} \Big|v(t) - v(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma\Big| \\ \qquad\qquad \leq C_\varphi \frac{t_1^{\alpha-p}}{\Gamma(\alpha)} \Big(\frac{1-p}{\alpha-p}\Big)^{1-p} \varphi(t), \ t \in (0, t_1], \\ \Big|v(t) - \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\, \mathrm{d}\varsigma\Big)\Big| \leq \varepsilon, \quad t \in (t_i, \varsigma_i] \quad i = 1, 2, \ldots, m, \\ \Big|v(t) - \mathcal{G}_i\Big(\varsigma_i, v(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, v(\varsigma))\, \mathrm{d}\varsigma\Big) \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ \quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma\Big| \\ \qquad\qquad \leq \varepsilon + \frac{C_\varphi}{\Gamma(\alpha)} \Big(\frac{1-p}{\alpha-p}\Big)^{1-p} \Big(\varsigma_i^{\alpha-p} + t_{i+1}^{\alpha-p}\Big) \varphi(t), \ t \in (\varsigma_i, t_{i+1}], \ 1 \leq i \leq m. \end{cases} \tag{3.15}$$

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Proof. From Lemma 2.2.1 and by Lemma 3.1.1, we have, for $t \in [0, t_1]$,

$$v(t) = v(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, d\rho\Big)\, d\varsigma + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \widetilde{H}(\varsigma)\, d\varsigma$$

$$\begin{aligned} \Rightarrow |v(t) - v(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, d\rho\Big)\, d\varsigma| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} |\widetilde{H}(\varsigma)|\, d\varsigma \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \varphi(\varsigma)\, d\varsigma \\ &\leq \frac{1}{\Gamma(\alpha)} \Big(\int_0^t \varphi(\varsigma)^{\frac{1}{p}}\, d\varsigma\Big)^p \Big(\int_0^t \big(t-\varsigma\big)^{\frac{\alpha-1}{1-p}}\, d\varsigma\Big)^{1-p}. \end{aligned}$$

Now, using Hölder's inequality, we obtain

$$\begin{aligned} |v(t) - v(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, d\rho\Big)\, d\varsigma| \\ \leq C_\varphi \frac{t_1^{\alpha-p}}{\Gamma(\alpha)} \Big(\frac{1-p}{\alpha-p}\Big)^{1-p} \varphi(t). \end{aligned} \tag{3.16}$$

For $t \in (t_i, \varsigma_i], \quad i = 1, 2, \ldots, m,$

$$|v(t) - \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\, d\varsigma\Big)| \leq |H_i| \leq \varepsilon,\ t \in (t_i, \varsigma_i],\ i = 1, 2, \ldots, m. \tag{3.17}$$

For $t \in (\varsigma_i, t_{i+1}]$, $i = 1, 2, \ldots, m$,

$$\begin{aligned} v(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ &- \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ &+ \mathcal{G}_i\Big(\varsigma_i, v(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, v(\varsigma))\, \mathrm{d}\varsigma\Big) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \widetilde{H}(\varsigma)\, d\varsigma \\ &+ H_i - \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i-\varsigma)^{\alpha-1} \widetilde{H}(\varsigma)\, \mathrm{d}\varsigma \\ \Rightarrow |v(t) &+ \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ &- \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ &- \mathcal{G}_i\Big(\varsigma_i, v(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, v(\varsigma))\, \mathrm{d}\varsigma\Big)| \\ &\leq |H_i| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} |\widetilde{H}(\varsigma)|\, d\varsigma + \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i-\varsigma)^{\alpha-1} |\widetilde{H}(\varsigma)|\, \mathrm{d}\varsigma \end{aligned}$$

$$\leq \varepsilon + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1}\varphi(\varsigma)\, d\varsigma + \frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i-\varsigma)^{\alpha-1}\varphi(\varsigma)\, \mathrm{d}\varsigma$$
$$\leq \varepsilon + \frac{C_\varphi}{\Gamma(\alpha)}\Big(\frac{1-p}{\alpha-p}\Big)^{1-p}\Big(\varsigma_i^{\alpha-p} + t_{i+1}^{\alpha-p}\Big)\varphi(t)\ t \in (\varsigma_i, t_{i+1}],\ i = 1, 2, \ldots, m.$$

Hence, the result is established. □

Now, we discuss the existence and stability of problem (3.1) − (3.3) in the sense of generalized Ulam-Hyers-Rassias stability.

Theorem 3.1.2 (Main Result)**.** *Assume that the hypotheses* ***(H1)-(H4)*** *hold and*

$$\Theta = \max_{1\leq i\leq m}\Bigg\{L_{\mathcal{G}_i}\Big(C_\varphi T^{1-p}G_g + TG_g + 1\Big) + \frac{2L_{\mathcal{F}}}{\Gamma(\alpha)}\Bigg\{\frac{T^\alpha}{\alpha}\Big(1 + \frac{TF_f}{\alpha+1}\Big) + C_\varphi T^{\alpha-p}\Big(\frac{1-p}{\alpha-p}\Big)^{1-p} + \frac{C_\varphi T^{\alpha-p+1}F_f}{\alpha}\Big(\frac{1-p}{\alpha+1-p}\Big)^{1-p}\Bigg\}\Bigg\} < 1.$$

Then, there exists a unique mild solution $u^* \in PC(J,\mathbb{R})$ for problem (3.1) − (3.3) *such that*

$$u^*(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, u^*(\varsigma), \int_0^\varsigma f(\rho, u^*(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma, & t\in[0,t_1],\\ \mathcal{G}_i\Big(t, u^*(t), \int_0^t g(\varsigma, u^*(\varsigma))\,\mathrm{d}\varsigma\Big), \quad t\in(t_i,\varsigma_i], \quad i=1,2,\ldots,m, & \\ \mathcal{G}_i\Big(\varsigma_i, u^*(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, u^*(\varsigma))\,\mathrm{d}\varsigma\Big) & \\ -\frac{1}{\Gamma(\alpha)}\int_0^{\varsigma_i}(\varsigma_i-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, u^*(\varsigma), \int_0^\varsigma f(\rho, u^*(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma & \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, u^*(\varsigma), \int_0^\varsigma f(\rho, u^*(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma, & t\in(\varsigma_i, t_{i+1}]. \end{cases} \tag{3.18}$$

For each $v \in PC(J,\mathbb{R})$ satisfying (3.5)*, and $v(0) = u(0)$, we have*

$$|v(t) - u^*(t)| \leq \frac{1+\mathcal{C}}{1-\Theta}(\varphi(t)+\varepsilon), \quad t\in J, \tag{3.19}$$

where

$$\mathcal{C} = \max_{1\leq i\leq m}\Bigg\{\frac{C_\varphi t_1^{\alpha-p}}{\Gamma(\alpha)}\Big(\frac{1-p}{\alpha-p}\Big)^{1-p}, \frac{C_\varphi}{\Gamma(\alpha)}\Big(\frac{1-p}{\alpha-p}\Big)^{1-p}\Big(\varsigma_i^{\alpha-p} + t_{i+1}^{\alpha-p}\Big)\Bigg\}. \tag{3.20}$$

Proof. Consider the following space of piecewise continuous functions:

$$M = \Big\{v : J \to \mathbb{R}|\ v \in PC(J,\mathbb{R})\Big\} \tag{3.21}$$

endowed with the generalized metric on M defined by

$$d_M(g,h) = \inf\Big\{K_1 + K_2 \in [0,+\infty) :$$

$$|g(t) - h(t)| \leq (K_1 + K_2)(\varphi(t) + \varepsilon), \forall t \in J\Big\}, \tag{3.22}$$

where

$$K_1 \in \Big\{\mathcal{K} \in [0, +\infty) \ : \ |g(t) - h(t)| \leq \mathcal{K}\varphi(t) \quad \forall t \in (\varsigma_i, t_{i+1}], \quad i = 0, 1, \dots, m\Big\}$$

and

$$K_2 \in \Big\{\mathcal{K} \in [0, +\infty) \ : \ |g(t) - h(t)| \leq \mathcal{K}\varepsilon \quad \forall t \in (t_i, \varsigma_i], \quad i = 1, 2, \dots, m\Big\}.$$

It is easy to verify that (M, d_M) is a complete metric space.
We define an operator $\Upsilon : M \to M$ as

$$\Upsilon u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, d\rho\Big)\, \mathrm{d}\varsigma, \quad t \in [0, t_1], \\ \mathcal{G}_i\Big(t, u(t), \int_0^t g(\varsigma, u(\varsigma))\, \mathrm{d}\varsigma\Big), \quad t \in (t_i, \varsigma_i], \quad i = 1, 2, \dots, m, \\ \mathcal{G}_i\Big(\varsigma_i, u(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, u(\varsigma))\, \mathrm{d}\varsigma\Big) \\ -\frac{1}{\Gamma(\alpha)} \int_0^{\varsigma_i} (\varsigma_i - \varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma \\ +\frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, u(\rho))\, \mathrm{d}\rho\Big)\, \mathrm{d}\varsigma, \quad t \in (\varsigma_i, t_{i+1}]. \end{cases}$$

First, we prove that the operator Υ is a contraction.
Take $h_1, h_2 \in M$. From the definition of metric space (M, d_M), it is possible to find the constants $K_1, K_2 \geq 0$ such that

$$|h_1(t) - h_2(t)| \leq \begin{cases} K_1\varphi(t), \quad t \in (\varsigma_i, t_{i+1}], \quad i = 0, 1, 2, \dots, m, \\ K_2\varepsilon \quad t \in (t_i, \varsigma_i], \quad i = 1, 2, \dots, m. \end{cases} \tag{3.23}$$

For $t \in [0, t_1]$,

$$\begin{aligned} |\Upsilon h_1(t) - \Upsilon h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} |\mathcal{F}\Big(\varsigma, h_1(\varsigma), \int_0^\varsigma f(\rho, h_1(\rho))\, d\rho\Big) \\ &\qquad - \mathcal{F}\Big(\varsigma, h_2(\varsigma), \int_0^\varsigma f(\rho, h_2(\rho))\, d\rho\Big)|\, \mathrm{d}\varsigma \\ &\leq \frac{L_{\mathcal{F}}}{\Gamma(\alpha)} \Big[K_1 \int_0^t (t-\varsigma)^{\alpha-1} \varphi(\varsigma) d\varsigma + K_1 F_f \int_0^t (t-\varsigma)^{\alpha-1} \Big(\int_0^\varsigma \varphi(\rho)\, d\rho\Big) d\varsigma\Big] \\ &\leq \frac{K_1 L_{\mathcal{F}}}{\Gamma(\alpha)} \Big[\int_0^t (t-\varsigma)^{\alpha-1} \varphi(\varsigma) d\varsigma + F_f \int_0^t \varphi(\rho) \Big(\int_\rho^t (t-\varsigma)^{\alpha-1} d\varsigma\Big)\, d\rho\Big] \\ &\leq \frac{K_1 L_{\mathcal{F}}}{\Gamma(\alpha)} \Big[\Big(\int_0^t \varphi(\varsigma)^{\frac{1}{p}} d\varsigma\Big)^p . \Big(\int_0^t (t-\varsigma)^{\frac{\alpha-1}{1-p}} d\varsigma\Big)^{1-p} \end{aligned}$$

$$+\frac{F_f}{\alpha}\Big(\int_0^t \varphi(\rho)^{\frac{1}{p}}\, d\rho\Big)^p.\Big(\int_0^t (t-\rho)^{\frac{\alpha}{1-p}}\, d\rho\Big)^{1-p}\Bigg]$$

$$\leq \frac{K_1 L_{\mathcal{F}}}{\Gamma(\alpha)}\Bigg[\frac{C_\varphi T^{\alpha-p+1}F_f}{\alpha}\Big(\frac{1-p}{\alpha+1-p}\Big)^{1-p}+C_\varphi T^{\alpha-p}\Big(\frac{1-p}{\alpha-p}\Big)^{1-p}\Bigg]\varphi(t). \quad (3.24)$$

For $t \in (t_i, \varsigma_i]$, $i = 1, 2, \ldots, m$,

$$|\Upsilon h_1(t)-\Upsilon h_2(t)| \leq |\mathcal{G}_i\Big(t, h_1(t), \int_0^t g(\varsigma, h_1(\varsigma))\, \mathrm{d}\varsigma\Big) - \mathcal{G}_i\Big(t, h_2(t), \int_0^t g(\varsigma, h_2(\varsigma))\, \mathrm{d}\varsigma\Big)|$$

$$\leq L_{\mathcal{G}_i}\Bigg[|h_1(t)-h_2(t)| + G_g\int_0^t |h_1(\varsigma)-h_2(\varsigma)|d\varsigma\Bigg]$$

$$\leq L_{\mathcal{G}_i}\Bigg[K_2\varepsilon + G_g\Big(K_1+K_2\Big)\int_0^t (\varepsilon+\varphi(\varsigma))d\varsigma\Bigg]$$

$$\leq L_{\mathcal{G}_i}\Bigg[K_2\varepsilon + G_g\Big(K_1+K_2\Big)\Big(T\varepsilon+\int_0^t \varphi(\varsigma)d\varsigma\Big)\Bigg]$$

$$\leq L_{\mathcal{G}_i}\Bigg[K_2\varepsilon + G_g\Big(K_1+K_2\Big)\Big\{T\varepsilon+\Big(\int_0^t \varphi(\varsigma)^{\frac{1}{p}}d\varsigma\Big)^p.\Big(\int_0^t d\varsigma\Big)^{1-p}\Big\}\Bigg]$$

$$\Rightarrow |\Upsilon h_1(t)-\Upsilon h_2(t)| \leq L_{\mathcal{G}_i}\Bigg[K_2\varepsilon + G_g T\Big(K_1+K_2\Big)\varepsilon + G_g T^{1-p}\Big(K_1+K_2\Big)C_\varphi\varphi(t)\Bigg]. \quad (3.25)$$

For $t \in (\varsigma_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$|\Upsilon h_1(t)-\Upsilon h_2(t)| \leq I_1 + I_2 + I_3, \quad (3.26)$$

where

$$I_1 = |\mathcal{G}_i(\varsigma_i, h_1(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, h_1(\varsigma))\, \mathrm{d}\varsigma) - \mathcal{G}_i(\varsigma_i, h_2(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, h_2(\varsigma))\, \mathrm{d}\varsigma)|,$$

$$I_2 = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}|\mathcal{F}(\varsigma, h_1(\varsigma), \int_0^\varsigma f(\rho, h_1(\rho))\, d\rho) - \mathcal{F}(\varsigma, h_2(\varsigma), \int_0^\varsigma f(\rho, h_2(\rho))\, d\rho)|\, \mathrm{d}\varsigma,$$

$$I_3 = \frac{1}{\Gamma(\alpha)}\int_0^{\varsigma_i} (\varsigma_i-\varsigma)^{\alpha-1}|\mathcal{F}(\varsigma, h_1(\varsigma), \int_0^\varsigma f(\rho, h_1(\rho))\, d\rho) - \mathcal{F}(\varsigma, h_2(\varsigma), \int_0^\varsigma f(\rho, h_2(\rho))\, d\rho)|\, \mathrm{d}\varsigma.$$

Now,

$$I_1 \leq L_{\mathcal{G}_i}\Bigg[|h_1(\varsigma_i)-h_2(\varsigma_i)| + G_g\int_0^{\varsigma_i} |h_1(\varsigma)-h_2(\varsigma)|d\varsigma\Bigg]$$

$$\leq L_{\mathcal{G}_i}\Big[K_2\varepsilon + G_g\Big(K_1+K_2\Big)\int_0^{\varsigma_i}(\varepsilon+\varphi(\varsigma))d\varsigma\Big]$$
$$\leq L_{\mathcal{G}_i}\Big[K_2\varepsilon + TG_g\Big(K_1+K_2\Big)\varepsilon + G_gT^{1-p}\Big(K_1+K_2\Big)C_\varphi\varphi(t)\Big],\ t\in(\varsigma_i,t_{i+1}]. \tag{3.27}$$

Similarly, for $t\in(\varsigma_i,t_{i+1}]$, $i=1,2,\dots,m$,

$$\begin{aligned}
I_2 &\leq \frac{L_{\mathcal{F}}}{\Gamma(\alpha)}\int_0^t(t-\varsigma)^{\alpha-1}\Big[|h_1(\varsigma)-h_2(\varsigma)| + F_f\int_0^\varsigma|h_1(\rho)-h_2(\rho)|\,d\rho\Big]d\varsigma\\
&\leq \frac{L_{\mathcal{F}}}{\Gamma(\alpha)}\int_0^t(t-\varsigma)^{\alpha-1}\Big[\Big(K_1+K_2\Big)\Big(\varepsilon+\varphi(\varsigma)\Big)\\
&\qquad\qquad + F_f\Big(K_1+K_2\Big)\int_0^\varsigma\Big(\varepsilon+\varphi(\rho)\Big)\,d\rho\Big]d\varsigma\\
&\leq \frac{L_{\mathcal{F}}\Big(K_1+K_2\Big)}{\Gamma(\alpha)}\Big[\int_0^t(t-\varsigma)^{\alpha-1}\varphi(\varsigma)d\varsigma + \varepsilon\int_0^t(t-\varsigma)^{\alpha-1}d\varsigma\\
&\qquad + F_f\varepsilon\int_0^t(t-\varsigma)^{\alpha-1}s\,d\varsigma + F_f\int_0^t(t-\varsigma)^{\alpha-1}\Big(\int_0^\varsigma\varphi(\rho)\,d\rho\Big)\Big]\\
&\leq \frac{L_{\mathcal{F}}\Big(K_1+K_2\Big)}{\Gamma(\alpha)}\Big[\frac{T^\alpha\varepsilon}{\alpha} + \frac{F_fT^{\alpha+1}\varepsilon}{\alpha(\alpha+1)} + C_\varphi T^{\alpha-p}\Big(\frac{1-p}{\alpha-p}\Big)^{1-p}\varphi(t)\\
&\qquad + F_fC_\varphi\frac{T^{\alpha-p+1}}{\alpha}\Big(\frac{1-p}{\alpha-p+1}\Big)^{1-p}\varphi(t)\Big].
\end{aligned} \tag{3.28}$$

For $(\varsigma_i,t_{i+1}]$, $i=1,2,\dots,m$,

$$\begin{aligned}
I_3 \leq \frac{L_{\mathcal{F}}\Big(K_1+K_2\Big)}{\Gamma(\alpha)}\Big[\frac{T^\alpha\varepsilon}{\alpha} + \frac{T^{\alpha+1}F_f\varepsilon}{\alpha(\alpha+1)} + C_\varphi T^{\alpha-p}\Big(\frac{1-p}{\alpha-p}\Big)^{1-p}\varphi(t)\\
+ F_fC_\varphi\frac{T^{\alpha-p+1}}{\alpha}\Big(\frac{1-p}{\alpha-p+1}\Big)^{1-p}\varphi(t)\Big].
\end{aligned} \tag{3.29}$$

Thereafter, by using (3.27) – (3.29), we get from (3.26), for $t\in(\varsigma_i,t_{i+1}]$, $i=1,2,\dots,m$,

$$\begin{aligned}
|\Upsilon h_1(t)-\Upsilon h_2(t)| \leq \Big[\Big\{L_{\mathcal{G}_i}K_2 + G_gL_{\mathcal{G}_i}T(K_1+K_2)\\
+\frac{2L_{\mathcal{F}}T^\alpha}{\alpha\Gamma(\alpha)}\Big(1+\frac{TF_f}{\alpha+1}\Big)(K_1+K_2)\Big\}\varepsilon + \Big\{L_{\mathcal{G}_i}G_gT^{1-p}C_\varphi\\
+\frac{2L_{\mathcal{F}}C_\varphi T^{\alpha-p}}{\Gamma(\alpha)}\Big(\Big(\frac{1-p}{\alpha-p}\Big)^{1-p} + \frac{TF_f}{\alpha}\Big(\frac{1-p}{\alpha-p+1}\Big)^{1-p}\Big)\Big\}(K_1+K_2)\varphi(t)\Big]
\end{aligned}$$

which implies

$$|\Upsilon h_1(t)-\Upsilon h_2(t)| \leq \Big[L_{\mathcal{G}_i}\Big(G_gC_\varphi T^{1-p} + TG_g + 1\Big) + \frac{2L_{\mathcal{F}}}{\Gamma(\alpha)}\Big\{\frac{T^\alpha}{\alpha}\Big(1+\frac{TF_f}{\alpha+1}\Big) +$$

$$+ C_\varphi T^{\alpha-p}\left(\frac{1-p}{\alpha-p}\right)^{1-p} + \frac{C_\varphi T^{\alpha-p+1}F_f}{\alpha}\left(\frac{1-p}{\alpha+1-p}\right)^{1-p}\Bigg\}\Bigg](K_1+K_2)(\varphi(t)+\varepsilon). \quad (3.30)$$

Therefore, from above, we have

$$|\Upsilon h_1(t) - \Upsilon h_2(t)| \leq \Theta\Big(K_1+K_2\Big)(\varphi(t)+\varepsilon), \quad t \in J.$$

Thus, we get

$$d_M(\Upsilon h_1, \Upsilon h_2) \leq \Theta d_M(h_1,h_2), \quad \forall h_1, h_2 \in M. \quad (3.31)$$

By assumption $\Theta < 1$ and hence, (3.31) implies that Υ is a contraction map. Subsequently, by Banach fixed point theorem (Theorem 1.5.1), there exists a unique solution to the problem (3.1) $-$ (3.3) given by equation (3.18).

Stability:

Take a function $v \in PC(J,\mathbb{R})$ satisfying (3.5) and $v(0) = u(0)$, and by using Lemma 3.1.3, we have

For $t \in [0,t_1]$,

$$|\Upsilon v(t) - v(t)| \leq |v(t) - x_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\,\mathrm{d}\rho)\,\mathrm{d}\varsigma|$$

$$\Rightarrow |\Upsilon v(t) - v(t)| \leq C_\varphi \frac{t_1^{\alpha-p}}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p}\varphi(t) \leq C_\varphi \frac{t_1^{\alpha-p}}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p}\Big(\varphi(t)+\varepsilon\Big). \quad (3.32)$$

Similarly, for $t \in (t_i, \varsigma_i]$, $i = 1,2,\ldots,m$,

$$|\Upsilon v(t) - v(t)| \leq |v(t) - \mathcal{G}_i\Big(t, v(t), \int_0^t g(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big)| \leq \varepsilon \leq \Big(\varphi(t)+\varepsilon\Big). \quad (3.33)$$

For $t \in (\varsigma_i, t_{i+1}]$, $i = 1,2,\ldots,m$,

$$\begin{aligned}|\Upsilon v(t) - v(t)| \leq |v(t) &- \mathcal{G}_i\Big(\varsigma_i, v(\varsigma_i), \int_0^{\varsigma_i} g(\varsigma, v(\varsigma))\,\mathrm{d}\varsigma\Big)\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{\varsigma_i}(\varsigma_i-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, u(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma\\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}\mathcal{F}\Big(\varsigma, v(\varsigma), \int_0^\varsigma f(\rho, v(\rho))\,\mathrm{d}\rho\Big)\,\mathrm{d}\varsigma|\end{aligned}$$

$$\Rightarrow |\Upsilon v(t) - v(t)| \leq \varepsilon + \frac{C_\varphi}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p}\Big(\varsigma_i^{\alpha-p} + t_{i+1}^{\alpha-p}\Big)\varphi(t), \quad t \in (\varsigma_i, t_{i+1}]. \quad (3.34)$$

Therefore, from (3.32) − (3.34), we have

$$|\Upsilon v(t) - v(t)| \leq \Big(1 + \mathcal{C}\Big)\Big(\varphi(t) + \varepsilon\Big), \quad \forall t \in J, \tag{3.35}$$

where $\mathcal{C} = \max\limits_{1\leq i\leq m}\left\{ \frac{C_\varphi t_1^{\alpha-p}}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p}, \frac{C_\varphi}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p}\left(\varsigma_i^{\alpha-p} + t_{i+1}^{\alpha-p}\right)\right\}$.

Thus, we get

$$d_M(\Upsilon v, v) \leq 1 + \mathcal{C} < +\infty. \tag{3.36}$$

Here, for the integer $k = 0$, we look at the space, $M^* = \Big\{z \in M | d_M(v, z) < \infty\Big\}$.
Taking $z(t) = v(t)$ itself, we have $d_M(v, v) = 0 < \infty \Rightarrow v \in M^*$.
Hence, by Banach fixed point theorem, if u^* is a fixed point for an operator Υ, we get

$$|v(t) - u^*(t)| \leq \frac{d_M(\Upsilon y, y)}{1-\Theta} \leq \frac{1+\mathcal{C}}{1-\Theta}\Big(\varphi(t) + \varepsilon\Big), \quad \forall t \in J. \tag{3.37}$$

Hence, the problem (3.1) − (3.3) is generalized Ulam-Hyers-Rassias stable. This completes the proof. □

3.2 Example: An Application to Fractional RLC Circuit

Fractional-order RLC circuit model has many advantages over the integer-order one because of an extra parameter included in the model. This extra parameter affects the performance and enhances the novel behavior which gives more flexibility in circuit design and control. Some novel phenomena can be observed only in the case of fractional-order model, as illustrated by Radwan and Salama [118]. They presented a broad view of the RLC circuit in terms of the fractional-order sense. They showed that the fractional-order impedance was purely imaginary and it enabled one to model a huge capacitance/ inductance based on a very small fractional-order. Numerical study of the fractional-order RLC circuits under the variant and non-variant voltage source was carried out in [66], and the solution of the fractional-order circuit was compared graphically with the corresponding integer-order circuit. Sene [125] and Ali et al. [34] derived the analytical solutions of RL, RC and RLC circuits with the generalized fractional derivatives and compared the obtained results with the corresponding integer-order circuit for RL and RC circuits. In the case of RLC circuit, they did not compare the result with the integer-order RLC circuit since the analytical solution was in the form of an infinite series where each term was a generalized Mittag-Leffler function which might increase the roundoff error. Thus, in general computing the fractional-order solution numerically may increase the roundoff error. In this context, we wish to propose some bound estimate for the solution with the help of the corresponding solution of the integer-order circuit. Analysis of various fractional-order electric circuit models with a non-singular kernel differential operator can be found in [97, 92, 91]. For more literature on fractional RLC circuit, one can refer to the works in [150, 7, 8, 131, 126].

In the governing equation of the conventional RLC circuit, we cannot directly replace the integer-order derivatives by the fractional-order derivatives because we have to take care of the dimension of the physical quantity like resistor, inductor and capacitor. Gómez-Aguilar et al. [70, 2] overcame this difficulty by introducing an auxiliary parameter having the dimension of time and they experimentally analysed the fractional-order RLC circuit. They compared the Nyquist diagram among the integer-order equation, the spectroscopy result and fractional-order equation, and found that the fractional-order system fitted the experimental data as the best fit which established the existence of the fractional RLC in the physical world.

We wish to overcome the difficulty faced by Ali et al. [34] for comparing the solutions of fractional-order and integer-order RLC circuits. Thus, we make an attempt to establish the bound for $|I_\alpha(t) - I(t)|$ in a certain interval where $I_\alpha(t)$ is a solution of the fractional-order non-instantaneous impulsive RLC circuit whereas $I(t)$ is a solution of the same non-instantaneous impulsive RLC circuit but with an integer-order.

Example 3.2.1. *We consider the following non-instantaneous impulsive RLC circuit with fractional-order under the unit step input voltage $H(t)$ (Heaviside function) as*

$$L_\alpha \, {}_0^C D_t^\alpha I_\alpha(t) + RI_\alpha(t) + \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma = H(t), \quad t \in (0,1] \cup (2,3], \tag{3.38}$$

$$RI_\alpha(t) + \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma = H(t), \quad t \in (1,2], \tag{3.39}$$

$$I_\alpha(0) = 0, \tag{3.40}$$

and the corresponding integer order non-instantaneous impulsive RLC circuit as

$$L_\alpha \frac{dI(t)}{dt} + RI(t) + \frac{1}{C}\int_0^t I(\varsigma)d\varsigma = H(t), \quad t \in (0,1] \cup (2,3], \tag{3.41}$$

$$RI(t) + \frac{1}{C}\int_0^t I(\varsigma)d\varsigma = H(t), \quad t \in (1,2], \tag{3.42}$$

$$I(0) = 0, \tag{3.43}$$

where L_α, R and C are the inductor, resistor and capacitor, respectively. Here $0 = \varsigma_0 = t_0 < t_1 = 1 < \varsigma_1 = 2 < t_2 = 3$, and we also assume that $R^2 = 4L_\alpha/C$. In the above circuit, for the time interval $(1,2]$, the effect of the inductor is neglected but its effect is noticeable when $(2,3]$ is considered.

In this example, we first find the solution $I(t)$ of the integer-order system (3.41) – (3.43) and then we put this solution $I(t)$ in the original problem, i.e., in the fractional-order system (3.38) – (3.40). After doing so, we get some function bound to the system

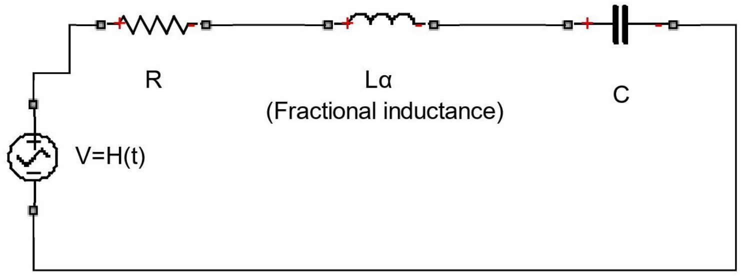

Figure 3.1: Fractional-order RLC circuit under unit step input voltage $H(t)$ with fractional order influence on the inductance

(3.38) − (3.40) *in the interval* $(0,1] \cup (2,3]$ *via the function* $\varphi(t)$ *(as defined in* (3.5)*). Therefore, before applying our main theorem, i.e., Theorem 3.1.2, we have to identify the function* $\varphi(t)$.
The solution $I(t)$ *of the integer order non-instantaneous impulsive* RLC *circuit* (3.41) − (3.43) *is given by*

$$I(t) = \begin{cases} \dfrac{te^{-at}}{L_\alpha}, & t \in (0,1], \\ \dfrac{e^{-at/2}}{R}, & t \in (1,2], \\ \Big[a(3+2a-2e^{a/2})(t-2)+1\Big]\dfrac{e^{-a(t-1)}}{R}, & t \in (2,3], \end{cases} \tag{3.44}$$

where $a = \dfrac{R}{2L_\alpha}$ *is a bandwidth of the RLC circuit. Putting this solution in the fractional* RLC *system* (3.38)-(3.40) *and using the result of Theorem 3.1.1, we get the bound as given below.*
For $t \in (0,1]$,

$$|L_\alpha \, {}_0^C D_t^\alpha I(t) + RI(t) + \frac{1}{C}\int_0^t I(\varsigma)d\varsigma - H(t)| \le (1+a)\left(1 + \frac{1}{\Gamma(2-\alpha)}\right)\frac{e^{at}}{L_\alpha}. \tag{3.45}$$

Similarly, for $t \in (1,2]$,

$$|RI(t) + \frac{1}{C}\int_0^t I(\varsigma)d\varsigma - H(t)| = 0. \tag{3.46}$$

For $t \in (2,3]$,

$$|L_\alpha \, {}_0^C D_t^\alpha I(t) + RI(t) + \frac{1}{C}\int_0^t I(\varsigma)d\varsigma - H(t)|$$

$$\leq \left(1 + \frac{3^{1-\alpha}}{\Gamma(2-\alpha)}\right)\left[(1+a)(3+2a-2e^{a/2})+1\right]\frac{ae^{a(t+1)}}{R}. \tag{3.47}$$

By observing the above inequalities, we can take our function $\varphi(t)$ as

$$\varphi(t) = \left(1 + \frac{3^{1-\alpha}}{\Gamma(2-\alpha)}\right)\left[(1+a)(3+2a-2e^{a/2})+1\right]\frac{ae^{a(t+1)}}{R}. \tag{3.48}$$

Thus, we get the function $\varphi(t)$ as required in (3.5). Therefore, we now proceed to substantiate our main result by using Theorem 3.1.2.
Here,

$$\mathcal{F}(t, I_\alpha(t), \int_0^t f(\varsigma, I_\alpha(\varsigma))\, d\varsigma) = \frac{1}{L_\alpha}\left(H(t) - RI_\alpha(t) - \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma\right)$$

with $L_{\mathcal{F}} = a\max\{a, 2\}$,
and

$$\mathcal{G}_1(t, I_\alpha(t), \int_0^t g(\varsigma, I_\alpha(\varsigma)\, d\varsigma) = \frac{1}{R}\left(H(t) - \frac{1}{C}\int_0^t I_\alpha(\varsigma)d\varsigma\right)$$

with $L_{\mathcal{G}_1} = \frac{a}{2}$.
We also take $G_g = \frac{a}{2}$ and $F_f = a^2$ and then we observe that

$$\left(\int_0^t \left(\varphi(\varsigma)\right)^{\frac{1}{p}} d\varsigma\right)^p \leq C_\varphi \varphi(t) \tag{3.49}$$

is true for any $p > 0$ (hence it will be true for $0 < p < \alpha < 1$), where $C_\varphi = \left(\frac{p}{a}\right)^p$.
Here,

$$\begin{aligned}\Theta =&\left(3^{1-p}\left(\frac{p}{a}\right)^p a + 3a + 2\right)\frac{a}{4} + \frac{4a}{\Gamma(\alpha)}\Bigg\{\frac{3^\alpha}{\alpha}\left(1 + \frac{3a^2}{1+\alpha}\right)\\ &+ 3^{\alpha-p}\left(\frac{p}{a}\right)^p\left(\frac{1-p}{\alpha-p}\right)^{1-p} + 3^{\alpha-p+1}\left(\frac{p}{a}\right)^p a^2\left(\frac{1-p}{\alpha-p+1}\right)^{1-p}\Bigg\},\end{aligned} \tag{3.50}$$

and

$$\mathcal{C} = \frac{\left(\frac{p}{a}\right)^p}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p}\left(2^{\alpha-p} + 3^{\alpha-p}\right). \tag{3.51}$$

Now, we choose different a, α, p such that the conditions of Theorem 3.1.2 are satisfied, i.e., $\Theta < 1$, and $0 < \alpha < p < 1$. Then, by Theorem 3.1.2, we have that, for the solution $I(t)$ of integer-order RLC (3.41) – (3.43) defined by (3.44), there exists a unique solution $I_\alpha(t)$ of fractional-order RLC system given by (3.38) – (3.40) such that

$$|I_\alpha(t) - I(t)| \leq C_a \frac{e^{at}}{L_\alpha}, \quad \forall t \in [0, 3], \tag{3.52}$$

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where $C_a = \left(\frac{1+\mathcal{C}}{1-\Theta}\right)\left(1+\frac{3^{1-\alpha}}{\Gamma(2-\alpha)}\right)\left[(1+a)(3+2a-2e^{a/2})+1\right]\frac{e^a}{2}$.
We then go ahead to compute the values of bound coefficient C_a *for different values of* a *and* α *such that conditions of Theorem 3.1.2 hold.*

Table 3.1: Values of C_a for $a = 0.05$, $p = 0.000005$ and for different order α.

α	0.125	0.25	0.5	0.75	0.99
Θ	0.4878	0.5429	0.6484	0.7454	0.8285
C_a	40.8699	47.4719	63.3356	84.7377	116.3921

Table 3.2: Values of C_a for $a = 0.03$, $p = 0.000005$ and for different order α.

α	0.125	0.25	0.5	0.75	0.99
Θ	0.2906	0.3236	0.3869	0.4452	0.4950
C_a	28.3640	30.8401	34.9166	37.3695	37.9956

Table 3.3: Values of C_a for $a = 0.01$, $p = 0.000005$ and for different order α.

α	0.125	0.25	0.5	0.75	0.99
Θ	0.0964	0.1073	0.1284	0.1479	0.1645
C_a	21.3985	22.4568	23.6035	23.3822	22.0682

Hence, from Tables 3.1 − 3.3, for different values of order α *and using the concepts of generalized Ulam-Hyers stability, it is observed that we establish a bound for* $|I_\alpha(t) - I(t)|$ *as given by (3.52). For a given inductor* L_α *and a bandwidth* $a = 0.05$*, we observe from Table 3.1 that the bound coefficient* C_a *strictly increases as the value of* α *increases, whereas, for smaller value of bandwidth* $a = 0.01$ *in Table 3.3, we observe that the* C_a *does not change much as compared to the case for higher values of the bandwidth. Therefore, the bound for* $|I_\alpha(t) - I(t)|$ *is larger in the case for the larger bandwidth compared to the smaller bandwidth even for an order* $\alpha = 0.99(\approx 1)$ *- very close to an integer order. In other words, the bound mainly depends on the bandwidth of the RLC system. This finding is expected to find an important place in fractional RLC circuit problems.*

3.3 Conclusions

In this chapter, we have established the existence and the generalized Ulam-Hyers-Rassias stability results of the mild solution of the Caputo fractional non instantaneous impulsive integro-differential equation. For establishing the unique solution, we considered some appropriate assumptions and a specific space of continuous functions. By showing a map from it into it to be a contraction map, uniqueness has been established. The same map along with a piecewise continuous function was utilized for proving the generalized Ulam-Hyers-Rassias stability with the help of Banach fixed point theorem. With the help of our result, we estimated the bound for the difference $|I_\alpha(t) - I(t)|$ for different fractional order α. Three tables have been presented in which the bound coefficient and the bound have been computed for different fractional orders. The bound for $|I_\alpha(t) - I(t)|$ is found to be larger in the case for the larger bandwidth as compared to the smaller bandwidth even for an order $\alpha = 0.99(\approx 1)$ which is very close to the integer order 1. In other words, the bound mainly depends on the bandwidth of the RLC system. Sometimes it is very difficult to compute the solution for the fractional-order RLC circuit compared to integer-order system. Therefore, our last result will give some estimate for the bound of the solution of the fractional-order RLC system.

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CHAPTER 4

Lyapunov stability analysis of the Caputo-Fabrizio fractional-order intermediate value systems

4.1 Introduction

The study of stability of fractional-order systems contaning singular and non-singular kernel fractional-order derivatives has evoked significant interest due to huge number of applications [135, 88, 138, 73]. In this context, several researchers have studied Lyapunov stability with respect to Caputo-Fabrizio derivative [78, 78].

Subsequently, in this work, we revisit the concept of Lyapunov stability of an autonomous fractional-order system with Caputo-Fabrizio deivative by taking care of the point suggested by Diethelm [36] and Zhang [151]. We first show that (i) only a constant solution exists for an initial value autonomous Caputo-Fabrizio fractional-order system, (ii) a non-constant solution exists only in case of a non-autonomous system, and (iii) all the isolated equilibrium points of an autonomous system are asymptotically stable. Secondly, we show that the existence of a non-constant solution is possible for an autonomous system only in the case of an intermediate value problem. In this context, we analyse the Lyapunov stability of an intermediate value autonomous Caputo-Fabrizio fractional-order system given by

$$ {}_{0}^{CF}\mathcal{D}_t^{\alpha}u(t) = F(u(t)), \quad 0 < \alpha < 1,\ t \geq 0, \tag{4.1} $$

$$ u(t_0) = u_0,\ t_0 > 0, \tag{4.2} $$

where $u \in \mathbb{R}^n,\ \ F : \Omega\ (\subset \mathbb{R}^n) \to \mathbb{R}^n$. Here, the information $u(t_0)$ is known at the intermediate point $t = t_0 > 0$, not at the initial point $t = 0$. This type of situation often appears in real-world applications in which we only know the current value rather

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than its past values. In a fractional-order system, a memory term is involved which makes it more complicated since the current dynamics of the system depends upon all the values of the past states. In this work, we investigate the stability of an equilibrium point of the above intermediate value problem for linear and nonlinear systems in a neighborhood of the point $u(t_0)$, i.e., we are interested in the stability for future time progress $t \geq t_0$. For $t \in [0, t_0]$, the above system is called a terminal value problem. For more detailed information regarding intermediate and terminal value problem for fractional-order differential equations, one can refer to [147] and [35], respectively.

4.2 Preliminaries

Definition 4.2.1. *[140] A function $\gamma : [0, \infty) \to [0, \infty)$ is called a class-K function if it is continuous and strictly increasing with $\gamma(0) = 0$.*

Lemma 4.2.1. *[36] Let us consider the following Caputo-Fabrizio initial value fractional differential equation:*

$$ {}_{a}^{CF}\mathcal{D}_t^{\alpha} u(t) = h(t, u(t)), \quad 0 < \alpha < 1, \ t \geq a, \tag{4.3} $$

$$ u(a) = u_a, \tag{4.4} $$

where $h : [a, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$. Then, the system (4.3) − (4.4) can have a solution only if $h(a, u(a)) = h(a, u_a) = 0$.

Thus, in the system (4.3) − (4.4), in order to have a solution to exist, one is forced to choose the initial condition $u(a)$ such that $h(a, u(a)) = 0$, which is a major drawback of using a non-singular kernel fractional-order operator.

Lemma 4.2.2. *[123] (Comparison Theorem) Let us consider α, $0 < \alpha < 1$ and ${}_{0}^{CF}\mathcal{D}_t^{\alpha} u(t) \geq {}_{0}^{CF}\mathcal{D}_t^{\alpha} v(t)$ with $u(0) = v(0)$. Then $u(t) \geq v(t)$.*

4.3 Main Results

Before establishing and analyzing the main results, we present some definitions for the following non-autonomous Caputo-Fabrizio initial value system:

$$ {}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t) = F(t, u), \tag{4.5} $$

$$ u(0) = u_0, \tag{4.6} $$

where ${}_{0}^{CF}\mathcal{D}_t^{\alpha} u(t) = \left({}_{0}^{CF}\mathcal{D}_t^{\alpha} u_1(t), \ldots, {}_{0}^{CF}\mathcal{D}_t^{\alpha} u_n(t) \right)$, with $0 < \alpha < 1$, and $F : [0, \infty) \times \Omega \ (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ is Lipschitz continuous in u on a domain Ω.

Definition 4.3.1. *A point* $\bar{u} \in \Omega$ $(\subseteq \mathbb{R}^n)$ *is said to be an equilibrium point for the system* (4.5) $-$ (4.6) *if* $F(t, \bar{u}) = 0, \ \forall\, t \geq 0$. *Define an equilibrium point set as* $S_E = \{\ \bar{u} \in \Omega \ \mid F(t, \bar{u}) = 0, \ \forall\, t \geq 0\}$.

Definition 4.3.2. *A point* $u_0 \in \Omega$ $(\subseteq \mathbb{R}^n)$ *is said to be a solution existence point if, for the system* (4.5) $-$ (4.6)*, the solution* $u(t, u_0)$ *exists for all* $t \geq 0$ *with initial condition* u_0. *Define a solution existence point set as* $S_s = \{\ u_0 \in \Omega \ \mid$ *solution* $u(t, u_0)$ *exists* $\forall\, t \geq 0\}$.

Definition 4.3.3. *A point* $u_0 \in \Omega$ $(\subseteq \mathbb{R}^n)$ *is said to be a possible solution existence point for the system* (4.5) $-$ (4.6) *if* $F(0, u_0) = 0$. *Define a possible solution existence point set as* $S_{ps} = \{\ u_0 \in \Omega \ \mid \ F(0, u_0) = 0\}$. *Here, if* $u_0 \in S_{ps}$*, then the solution* $u(t, u_0)$ *may or may not exist.*

Corollary 4.3.1. *For the non-autonomous Caputo-Fabrizio initial value system* (4.5) $-$ (4.6)*, we have* $S_E \subset S_s \subset S_{ps}$.

Proof. Let $\bar{u} \in S_E$ which gives $F(t, \bar{u}) = 0, \ \forall\, t \geq 0\, implying F(0, \bar{u}) = 0$, and starting with initial condition $u(0) = \bar{u}$, $u(t) = \bar{u}$ is a solution for the system (4.5) $-$ (4.6) which means $\bar{u} \in S_s \Rightarrow S_E \subset S_s$.
Next, take $u_0 \in S_s$ which means that the solution $u(t, u_0)$ exists for all $t \geq 0$. Then, by Lemma 4.2.1, we have $F(0, u_0) = 0$ which means $u_0 \in S_{ps} \Rightarrow S_s \subset S_{ps}$. Hence, the result $S_E \subset S_s \subset S_{ps}$ follows. □

The above subset relation is strict and we will later show that the equality holds for the autonomous systems.

4.3.1 Caputo-Fabrizio autonomous initial value systems (Revisited):

We consider the following initial value Caputo-Fabrizio fractional autonomous system:

$$ {}_{0}^{CF}\mathcal{D}_t^{\alpha} u(t) = F(u(t)), \quad 0 < \alpha < 1, \ t \geq 0, \tag{4.7} $$

$$ u(0) = u_0, \tag{4.8} $$

where $F : \Omega \ (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ is continuous and Lipschitz in u on a domain $\Omega(\subseteq \mathbb{R}^n)$.

Definition 4.3.4. *The equilibrium point* $\bar{u} \in S_E$ *of the system* (4.7) $-$ (4.8) *is*
(i) *Lyapunov stable if, for each* $\varepsilon > 0$*, there exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$ \|u(0) - \bar{u}\| < \delta \ \Rightarrow \|u(t) - \bar{u}\| < \varepsilon, \ \forall\, t \geq 0. \tag{4.9} $$

(ii) *Asymptotically stable if it is stable and there exists* $\delta_1 > 0$ *such that*

$$ \|u(0) - \bar{u}\| < \delta_1 \ \Rightarrow \lim_{t \to \infty} u(t) = \bar{u}. \tag{4.10} $$

Corollary 4.3.2. *For the autonomous system* (4.7) − (4.8)*, we have* $S_E = S_s = S_{ps}$.

Proof. Let $u_1 \in S_{ps}$. Then, for the system (4.7) − (4.8), we have $F(u_1) = 0 \ \Rightarrow u_1 \in S_E$, and $u(t) = u_1$ is a solution for the system which means $u_1 \in S_s \ \Rightarrow S_{ps} \subset S_s$. Similarly, one can show that $S_s \subset S_E$ and consequently, we have $S_{ps} \subset S_s \subset S_E$. Subsequently, by using Corollary 4.3.1, it is obvious that $S_E = S_s = S_{ps}$. □

Theorem 4.3.1. *A Caputo-Fabrizio autonomous system possesses only a constant solution, and the non-constant solution exists only in the case of a non-autonomous system.*

Proof. The proof directly follows from Corollary 4.3.2. Moreover, $S_E = S_s$, i.e., the set of all solutions is equal to the set of all equilibrium points which means that only a constant solution exists for a Caputo-Fabrizio autonomous system. □

Theorem 4.3.2. *All the isolated equilibrium points of the autonomous system* (4.7) − (4.8) *are asymptotically stable.*

Proof. Let $\bar{u} \in S_E$ be an isolated equilibrium point. Then, there exists a neighborhood $N_\delta(\bar{u})$ such that $N_\delta(\bar{u}) \cap S_E = \{\bar{u}\}$, and by using Corollary 4.3.2 in the neighbourhood $N_\delta(\bar{u})$, the solution exists only for an initial condition $u(0) = \bar{u} \in N_\delta(\bar{u})$. The solution is given by $u(t) = \bar{u}$, for all $t \geq 0$. Consequently, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $\|u(0) - \bar{u}\| < \delta$, we have $\|u(t) - u(0)\| < \varepsilon$. Moreover, $\lim_{t\to\infty} u(t) = \bar{u}$. □

4.3.2 Caputo-Fabrizio autonomous intermediate value systems:

From the previous subsection, it may be realized that there does not lie much interest in studying the stability of the equilibrium point for the initial value autonomous system (4.7)−(4.8). Subsequently, now we want to study the stability of the following intermediate value autonomous system:

$$ {}_0^{CF}\mathcal{D}_t^\alpha u(t) = F(u(t)), \quad 0 < \alpha < 1,\ t \geq 0, \tag{4.11} $$

$$ u(t_0) = u_0, \quad t_0 > 0, \tag{4.12} $$

where $u \in \mathbb{R}^n,\ F : \Omega\ (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$.

Theorem 4.3.3. *Let* $F : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function. A function* $u \in C(J, \mathbb{R}^n)$ *is a solution of the above intermediate value system* (4.11) − (4.12) *if and only if* u *is a solution of the following integral equation:*

$$ u(t) = u_0 + \frac{(1-\alpha)}{M(\alpha)}\Big[F(u(t)) - F(u(t_0))\Big] + \frac{\alpha}{M(\alpha)} \int_{t_0}^{t} F(u(s))\, ds, \tag{4.13} $$

where J *is the maximal interval of existence of the solution containing* t_0.

Proof. Let $u(t) = u(t; 0, c)$ be a solution of the system (4.11) with an initial condition $u(0) = c$ such that $u(t_0; 0, c) = u_0$. Then $u(t; 0, c)$ satisfies the integral equation

$$u(t) = c + \frac{(1-\alpha)}{M(\alpha)} F(u(t)) + \frac{\alpha}{M(\alpha)} \int_0^t F(u(s))\, ds. \tag{4.14}$$

Since $u(t_0; 0, c) = u_0$, we have

$$\begin{aligned} u_0 &= c + \frac{(1-\alpha)}{M(\alpha)} F(u(t_0)) + \frac{\alpha}{M(\alpha)} \int_0^{t_0} F(u(s))\, ds \\ \Rightarrow c &= u_0 - \frac{(1-\alpha)}{M(\alpha)} F(u(t_0)) - \frac{\alpha}{M(\alpha)} \int_0^{t_0} F(u(s))\, ds. \end{aligned} \tag{4.15}$$

Using the above value of c in equation (4.14), we get (following [36])

$$u(t) = u_0 + \frac{(1-\alpha)}{M(\alpha)} \Big[F(u(t)) - F(u(t_0))\Big] + \frac{\alpha}{M(\alpha)} \int_{t_0}^t F(u(s))\, ds. \tag{4.16}$$

Conversely, let $u \in C(J, \mathbb{R}^n)$ satisfy the integral equation (4.16). Then, u is clearly differentiable and thus, we get

$$u'(t) = \frac{(1-\alpha)}{M(\alpha)} \frac{dF(u(t))}{dt} + \frac{\alpha}{M(\alpha)} F(u(t)). \tag{4.17}$$

Now, multiplying both sides by $\exp\left(\frac{\alpha t}{1-\alpha}\right)$, we obtain

$$\begin{aligned} \exp\left(\frac{\alpha t}{1-\alpha}\right) u'(t) &= \exp\left(\frac{\alpha t}{1-\alpha}\right) \left(\frac{(1-\alpha)}{M(\alpha)} \frac{dF(u(t))}{dt} + \frac{\alpha}{M(\alpha)} F(u(t))\right) \\ &= \frac{(1-\alpha)}{M(\alpha)} \frac{d}{dt} \left(\exp\left(\frac{\alpha t}{1-\alpha}\right) F(u(t))\right). \end{aligned} \tag{4.18}$$

Integrating both sides from 0 to t where $t > 0$, we have

$$\int_0^t \exp\left(\frac{\alpha \tau}{1-\alpha}\right) u'(\tau)\, d\tau = \frac{(1-\alpha)}{M(\alpha)} \left[\exp\left(\frac{\alpha t}{1-\alpha}\right) F(u(t)) - F(u(0))\right]. \tag{4.19}$$

By Lemma 4.2.1, we have $F(u(0)) = 0$, and from the above equation, we get

$${}_0^{CF}\mathcal{D}_t^{\alpha} u(t) = F(u(t)). \tag{4.20}$$

Hence, $u \in C(J, \mathbb{R}^n)$ satisfying the integral equation (4.16) is a solution of the intermediate value system (4.11) – (4.12). □

Definition 4.3.5. *A point $\bar{u} \in \Omega$ is said to be an equilibrium point for the intermediate value system (4.11) − (4.12) if $F(\bar{u}) = 0$. Define an equilibrium point set as $E = \{\, \bar{u} \in \Omega \mid F(\bar{u}) = 0\}$.*

Remark 4.3.1. *Let $\bar{u} \in E$. If we take the intermediate value $u(t_0) = \bar{u}$, then from the equivalent integral equation relation (4.13), we get*

$$u(t) = \bar{u} + \frac{(1-\alpha)}{M(\alpha)} F(u(t)) + \frac{\alpha}{M(\alpha)} \int_{t_0}^{t} F(u(s))\, ds. \tag{4.21}$$

Clearly, the solution $u(t)$ of the above integral equation (4.21) is the solution of the following system:

$$ {}_{t_0}^{CF}\mathcal{D}_t^{\alpha} u(t) = F(u(t)), \quad 0 < \alpha < 1, \tag{4.22}$$

$$ u(t_0) = \bar{u}. \tag{4.23}$$

Since $F(\bar{u}) = 0$ and the above system (4.22) − (4.23) has a unique solution, it means that the solution of the integral equation (4.21) is given by $u(t) = \bar{u}$. Hence, the given intermediate value system (4.11) − (4.12) has a constant solution $u(t) = \bar{u}$.

Theorem 4.3.4. *The above intermediate value autonomous system (4.11) − (4.12) can be transformed to the following initial value non-autonomous system:*

$$ {}_{0}^{CF}\mathcal{D}_t^{\alpha} v(t) = F(v(t)) - \exp\Big(-\frac{\alpha t}{1-\alpha}\Big) F(v(0)), \quad 0 < \alpha < 1,\ t \geq 0, \tag{4.24}$$

$$ v(0) = u(t_0), \tag{4.25}$$

where $v(t) = u(t + t_0)$, for all $t \geq 0$.

Proof. Let us translate the origin by the transformation $\tau = t - t_0$. Then,

$$ {}_{0}^{CF}\mathcal{D}_\tau^{\alpha} u(\tau + t_0) = \frac{M(\alpha)}{1-\alpha} \int_0^{\tau} u'(\tau + t_0) \exp\Big(-\frac{\alpha(\tau - s)}{1-\alpha}\Big)\, ds. \tag{4.26}$$

Thereafter, upon evaluating the derivatives at $t = \tau + t_0$, we have

$$\begin{aligned} {}_{0}^{CF}\mathcal{D}_t^{\alpha} u(t)\,|_{t=\tau+t_0} &= \frac{M(\alpha)}{1-\alpha} \int_0^{\tau+t_0} u'(s) \exp\Big(-\frac{\alpha(\tau + t_0 - s)}{1-\alpha}\Big)\, ds \\ &= \frac{M(\alpha) e^{-\frac{\alpha\tau}{1-\alpha}}}{1-\alpha} \int_0^{t_0} u'(s) \exp\Big(-\frac{\alpha(t_0 - s)}{1-\alpha}\Big)\, ds \\ &+ \frac{M(\alpha)}{1-\alpha} \int_{t_0}^{\tau+t_0} u'(s) \exp\Big(-\frac{\alpha(\tau + t_0 - s)}{1-\alpha}\Big)\, ds. \end{aligned} \tag{4.27}$$

Now, by using the system (4.7), we get

$$ F(u(\tau + t_0)) = \exp\Big(-\frac{\alpha\tau}{1-\alpha}\Big) F(u(t_0)) + {}_{0}^{CF}\mathcal{D}_\tau^{\alpha} u(\tau + t_0),$$

$$\Rightarrow {}_{0}^{CF}\mathcal{D}_{\tau}^{\alpha}u(\tau+t_0) = F(u(\tau+t_0)) - \exp\Big(-\frac{\alpha\tau}{1-\alpha}\Big)F(u(t_0)),\ \tau \geq 0. \tag{4.28}$$

Denoting $v(t) = u(t+t_0),\ t \geq 0$, the above equation can be rewritten as

$${}_{0}^{CF}\mathcal{D}_{t}^{\alpha}v(t) = F(v(t)) - \exp\Big(-\frac{\alpha t}{1-\alpha}\Big)F(v(0)), \quad t \geq 0, \tag{4.29}$$

with an initial condition $v(0) = u(t_0)$. □

Corollary 4.3.3. *If $v(t)\ \forall\, t \geq 0$ is a solution of the system (4.24) – (4.25), then $u(t) = v(t-t_0)$ is a solution of the intermediate value system (4.11) – (4.12) for $t \geq t_0$, and conversely, if $u(t)$ is a solution of the system (4.11) – (4.12) for $t \geq t_0$, then $v(t) = u(t+t_0),\ \ \forall\, t \geq 0$, is a solution for the system (4.24) – (4.25).*

Corollary 4.3.4. *By Theorem 4.3.1, the above transformed non-autonomous system (4.24) – (4.25) has a non-constant solution which means that an autonomous intermediate value system (4.11) – (4.12) can have a non-constant solution.*

This is an interesting property of the intermediate value system since, by Theorem 4.3.1, it is known that an autonomous initial value system possesses only a constant solution.

Theorem 4.3.5. *A point $\bar{u} \in \Omega$ is an equilibrium point of the system (4.11) – (4.12) if and only if it is an equilibrium point of the system (4.24) – (4.25).*

Proof. Let us consider $\bar{u} \in E$ to be an equilibrium point for the system (4.11) – (4.12). Then, $F(\bar{u}) = 0$, and we observe that, for $v = \bar{u}$,

$$g(t,\bar{u}) = F(\bar{u}) - \exp\Big(-\frac{\alpha t}{1-\alpha}\Big)F(\bar{u}) = 0,\ t \geq 0 \tag{4.30}$$

which implies that $\bar{u}$ is an equilibrium point for the system (4.24) – (4.25). Conversely, let $\bar{u}$ be an equilibrium point for the system (4.24) – (4.25). Then, we have $g(t,\bar{u}) = 0$. Therefore,

$$\begin{aligned} &F(\bar{u}) - \exp\Big(-\frac{\alpha t}{1-\alpha}\Big)F(\bar{u}) = 0 \\ \Rightarrow &\Big(1 - \exp\Big(-\frac{\alpha t}{1-\alpha}\Big)\Big)F(\bar{u}) = 0,\ \forall\, t \geq 0, \end{aligned}$$

which is true only if $F(\bar{u}) = 0$. Thus, $\bar{u}$ is also an equilibrium point of the system (4.11) – (4.12). □

By considering Remark 4.3.1, we are interested in studying the Lyapunov stability of the equilibrium point of the intermediate value problem in a neighborhood of the point $u(t_0)$, i.e., we wish to study the stability in time progress for $t \geq t_0$. Therefore, we analyse the stability of the equilibrium points of the above intermediate value autonomous problem for both linear and nonlinear cases. Before that, we quickly define the Lyapunov stability of the equilibrium point of the intermediate value system (4.11) − (4.12).

Definition 4.3.6. *An equilibrium point $\bar{u} \in E$ of the intermediate value problem* (4.11) − (4.12) *is*
(i) *Lyapunov stable if, for each $\varepsilon > 0$, there exists some $\delta > 0$ such that*

$$\|u(t_0) - \bar{u}\| < \delta \ \Rightarrow \|u(t) - \bar{u}\| < \varepsilon, \ \forall\, t \geq t_0. \tag{4.31}$$

(ii) *Asymptotically stable if it is stable and there exists some $\delta_1 > 0$ satisfying*

$$\|u(t_0) - \bar{u}\| < \delta_1 \ \Rightarrow \lim_{t\to\infty} u(t) = \bar{u}. \tag{4.32}$$

4.3.3 Part I: Intermediate value Caputo-Fabrizio linear autonomous systems

Here, we discuss the stability of the following linear system:

$$ {}_0^{CF}\mathcal{D}_t^{\alpha} u(t) = Au(t), \ t \geq 0, \tag{4.33}$$

$$ u(t_0) = u_0, \ t_0 > 0, \tag{4.34}$$

where A is an $n \times n$ non-singular scalar matrix.

Remark 4.3.2. *If $t_0 = 0$, then by Theorem 4.3.2, the equilibrium point $\bar{u} = 0$ of the above linear system* (4.33) − (4.34) *is globally asymptotically stable.*

Theorem 4.3.6. *The equilibrium point $\bar{u} = 0$, of the system* (4.33) − (4.34) *is asymptotically stable if and only if the eigenvalues of the matrix A do not lie inside the disc $S_1(\alpha) = \Big\{(x, y) \in \mathbb{R}^2 \ | \ \Big(x - \frac{1}{2(1-\alpha)}\Big)^2 + y^2 \leq \frac{1}{4(1-\alpha)^2}\Big\}$.*

Proof. By using the translation $\tau = t - t_0$, the system (4.33) − (4.34) is transformed to the following initial value non-autonomous system:

$$ {}_0^{CF}\mathcal{D}_t^{\alpha} v(t) = Av(t) - \exp\Big(-\frac{\alpha t}{1-\alpha}\Big)Av(0), \ t \geq 0, \tag{4.35}$$

$$ v(0) = u(t_0). \tag{4.36}$$

Now, taking Laplace transform on both sides of equation (4.35) and $M(\alpha) = 1$, we have

$$\frac{1}{\big(p + \alpha(1-p)\big)}\Big(p\mathscr{L}[v(t)](p) - v(0)\Big) = A\mathscr{L}[v(t)](p) - \frac{1}{p + \frac{\alpha}{1-\alpha}}Av(0), \tag{4.37}$$

$$\Big[pN - \alpha A\Big]\mathscr{L}[v(t)](p) = Nv(0), \tag{4.38}$$

where $N = I - (1-\alpha)A$. Since the eigenvalues $\lambda(A)$ of the matrix A do not lie inside the disc $S_1(\alpha)$, i.e., $\lambda(A) \notin S_1(\alpha)$, it implies that $\frac{1}{1-\alpha} \in \rho(A)$ (the resolvent set of A). Therefore, the matrix $N\left(= -(1-\alpha)\big(A - \frac{1}{1-\alpha}I\big)\right)$ is invertible. Subsequently, we get

$$\left[sI - \alpha N^{-1}A\right]\mathscr{L}[v(t)](p) = v(0). \tag{4.39}$$

Thus, the solution of the system (4.35) − (4.36) is given by

$$v(t) = \exp(\alpha N^{-1}At)v(0), \tag{4.40}$$

and the solution of the intermediate value system (4.33) − (4.34), for $t \geq t_0$, is given by

$$u(t) = \exp(\alpha N^{-1}A(t-t_0))u(t_0), \tag{4.41}$$

which means that $u(t) \to 0$ as $t \to \infty$ if and only if all the eigenvalues of the matrix $N^{-1}A$ have negative real parts. So, we only need to show that the eigenvalue $\lambda(N^{-1}A)$ of the matrix $N^{-1}A$ has a negative real part. From the well-known result of the invertible matrices that

$$\big(I - (1-\alpha)A\big)^{-1} = I + (1-\alpha)\big(I - (1-\alpha)A\big)^{-1}A,$$

we get

$$\alpha N^{-1}A = \alpha\big(I - (1-\alpha A)\big)^{-1}A = \frac{\alpha}{1-\alpha}\left(\big(I-(1-\alpha)A\big)^{-1} - I\right). \tag{4.42}$$

Suppose $\lambda(A) = \lambda_1 + \mathrm{i}\lambda_2$. Then, we have

$$\lambda(\alpha N^{-1}A) = \frac{\alpha}{1-\alpha}\left(\frac{1}{1-(1-\alpha)\lambda(A)} - 1\right), \tag{4.43}$$

from which we get $\mathrm{Re}(\lambda(\alpha N^{-1}A)) = \dfrac{\alpha\big(\lambda_1 - (1-\alpha)(\lambda_1^2+\lambda_2^2)\big)}{(1-(1-\alpha)\lambda_1)^2 + (1-\alpha)^2\lambda_2^2}$. Therefore, if $(\lambda_1, \lambda_2) \notin S_1(\alpha)$, then clearly $\mathrm{Re}(\lambda(\alpha N^{-1}A)) < 0$. □

4.3.4 Part II: Intermediate value Caputo-Fabrizio nonlinear autonomous systems

Now, we discuss the Lyapunov stability for the nonlinear autonomous intermediate value system (4.11)−(4.12). Since for $t \geq t_0$, the intermediate value system (4.11)−(4.12) is equivalent to the initial value system (4.24) − (4.25), thus we will establish the stability result for the initial value system (4.24) − (4.25).

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Theorem 4.3.7. *The equilibrium point* $\bar{v} = 0$ *of the system* (4.24) − (4.25) *is stable if there exists a continuously differentiable function* $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ *with respect to all of its arguments such that*

(i) $V(t, 0) = 0, \ \forall\, t \geq 0,$

(ii) there exist a function γ *of class K and a positive real number* $s > 0$ *such that*

$$\gamma(\|v\|) \leq V(t, v), \ \ \forall\, v \in B_s, \ \forall\, t \geq 0,$$

(iii) there exists a positive constant $r > 0$ *such that*

$${}_{0}^{CF}\mathcal{D}_t^{\alpha} V(t, v(t)) \leq 0, \ \ v \in B_r, \ \forall\, t \geq 0.$$

Proof. Let $\varepsilon > 0$, and define a function $\beta(\delta_1) = \sup_{\|v\| \leq \delta_1} V(0, v)$. Then $\delta_1 \to 0$ implies $\beta(\delta_1) \to 0$.

Let $\varepsilon_1 = \min\{\varepsilon, r, s\}$. Since $\gamma(\varepsilon_1) > 0$, by definition of limit, there exists a $\delta > 0$ such that

$$\beta(\delta) < \gamma(\varepsilon_1). \tag{4.44}$$

Claim: $\|v(0)\| < \delta \ \Rightarrow \|v(t)\| < \varepsilon_1, \ \ \forall\, t \geq 0.$

Suppose the above statement is not true so that there exists some T such that

(i) $\|v(t)\| < \varepsilon_1, \ \forall\, t \in [0, T)$, and

(ii) $\|v(T)\| = \varepsilon_1$.

Since $\varepsilon_1 \leq r$, then by the given condition (iii), we have

$$\begin{aligned} &{}_{0}^{CF}\mathcal{D}_t^{\alpha} V(t, v(t)) \leq 0, \ \ \forall\, t \in [0, T] \\ &\Rightarrow V(T, v(T)) \leq V(0, v(0)). \end{aligned}$$

Thus, if $\|v(0)\| < \delta$ and using inequality (4.44), we have

$$V(T, v(T)) \leq V(0, v(0)) < \gamma(\varepsilon_1). \tag{4.45}$$

Also, from the condition (ii), as $\varepsilon_1 \leq s$, we obtain

$$\begin{aligned} &\gamma(\|v(T)\|) \leq V(T, v(T)) \\ &\Rightarrow V(T, v(T)) \geq \gamma(\varepsilon_1). \end{aligned} \tag{4.46}$$

The inequalities (4.45) and (4.46) invite a contradiction which implies that there does not exist any such T. Hence, $\|v(0)\| < \delta \ \Rightarrow \|v(t)\| < \varepsilon_1 \leq \varepsilon, \ \forall\, t \geq 0.$ □

Lemma 4.3.1. *[123] Let $V : [0,\infty) \times \Omega \to \mathbb{R}$ be continuously differentiable with respect to all of its arguments and convex over the domain $\Omega \subset \mathbb{R}^n$. Then, for $0 < \alpha < 1$, we have*

$$ {}_0^{CF}\mathcal{D}_t^{\alpha} V(t, v(t)) \leq \left(\frac{\partial V(t,v)}{\partial v}\right)^T {}_0^{CF}\mathcal{D}_t^{\alpha} v(t), \quad \forall t \geq 0. \tag{4.47} $$

Corollary 4.3.5. *Let the assumption and conditions (i) and (ii) of Theorem 4.3.7 be true on V which is supposed to be convex. Then, the equilibrium point $\bar{v} = 0$ of the system $(4.24) - (4.25)$ is stable if*

$$ \left(\frac{\partial V(t,v)}{\partial v}\right)^T . \left(F(v(t)) - \exp(-\frac{\alpha t}{1-\alpha})F(v(0))\right) \leq 0, \quad \forall t \geq 0. \tag{4.48} $$

4.4 Example

In this section, we present a suitable example which obeys the conditions of our obtained result Theorem 4.3.6 and authenticates it.

Example 4.4.1. *Consider the intermediate value Caputo-Fabrizio linear autonomous system as follows:*

$$ {}_0^{CF}\mathcal{D}_t^{\alpha} u(t) = Au(t), \ t \geq 0, \tag{4.49} $$

$$ u(2) = u_0, \tag{4.50} $$

where

$$ A = \begin{bmatrix} 3\sqrt{2} & -1 \\ 10 - 5\sqrt{2} & 4 \end{bmatrix}, \qquad u_0 \in \mathbb{R}^2. \tag{4.51} $$

The eigenvalues of the matrix A are $\lambda_1 = (2+\frac{3}{\sqrt{2}})+i(1+\frac{1}{\sqrt{2}})$ and $\lambda_2 = (2+\frac{3}{\sqrt{2}})-i(1+\frac{1}{\sqrt{2}})$. Now, for $\alpha = \frac{1}{\sqrt{2}}$, clearly the eigenvalues λ_1 and λ_2 do not lie inside the disc $S_1(\frac{1}{\sqrt{2}})$. Hence, by Theorem 4.3.6, the equilibrium point $\bar{u} = 0$ of the above intermediate value system is asymptotically stable. Figure 4.1 depicts this fact. Moreover, the solution of the above system for $t \geq 2$ is given by

$$ u(t) = \exp\left(N^{-1}A\Big(\frac{t-2}{\sqrt{2}}\Big)\right)u_0, \ t \geq 2, \tag{4.52} $$

where $N = I - (1 - \frac{1}{\sqrt{2}})A$. On the other hand, for $\alpha = \frac{5}{6}$, the eigenvalues λ_1 and λ_2 lie inside the disc $S_1\left(\frac{5}{6}\right)$ and hence, the equilibrium point $\bar{u} = 0$ of the above linear system is unstable. This can be ascertained from Figure 4.2.

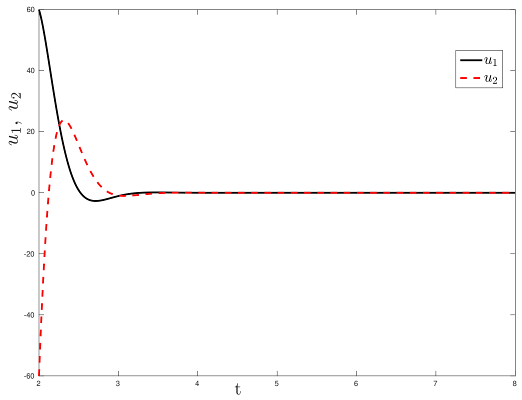

Figure 4.1: Solution of the system (4.49) for fractional order $\alpha = \frac{1}{\sqrt{2}}$, with $\lambda_1, \lambda_2 \notin S_1(\frac{1}{\sqrt{2}})$

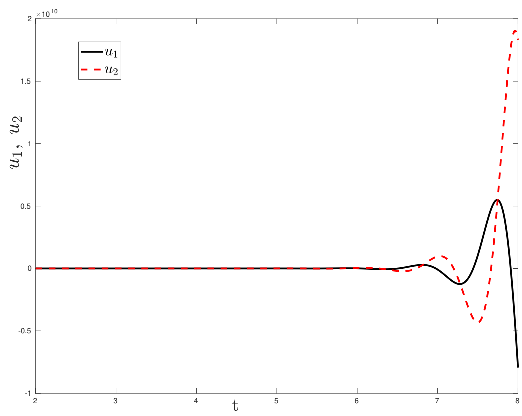

Figure 4.2: Solution of the system (4.49) for fractional order $\alpha = \frac{5}{6}$, with $\lambda_1, \lambda_2 \in S_1(\frac{5}{6})$

4.5 Conclusions

In this chapter, by using the properties of Caputo-Fabrizio derivative and the concept of the equilibrium point of the Caputo-Fabrizio fractional-order systems, we have found that only a constant solution exists for an autonomous Caputo-Fabrizio system, and a non-constant solution exists only in the case of a non-autonomous system. Based on this finding, we revisited the Lyapunov stability of an equilibrium point of an autonomous Caputo-Fabrizio fractional-order system and established that all isolated equilibrium points of a Caputo-Fabrizio autonomous system are asymptotically stable. We have discussed Caputo-Fabrizio fractional-order intermediate value problems for both linear and nonlinear autonomous systems as introduced by Yang [147] and Diethelm [35]. Further, we have shown that a non-constant solution exists in the case of an autonomous intermediate value fractional-order system and studied the Lyapunov stability for the intermediate value Caputo-Fabrizio problems for both linear and nonlinear autonomous systems in time progress for $t \geq t_0$. At the end, an example is presented in order to ascertain the applicability of one of our results (Theorem 4.3.6). The obtained results are expected to show a new direction with respect to Lyapunov stability of intermediate value Caputo-Fabrizio fractional-order systems and to find applications in some areas.

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CHAPTER 5

Caputo-Fabrizio fractional-order systems: Periodic solution and stabilization of non-periodic solution with application to Gunn diode oscillator

5.1 Introduction

In the last few decades, the dynamical behavior of fractional-order systems has captured a growing attention of mathematicians, scientists and engineers due to its immense applications. A large number of applications can be found in topics like visco-elasticity [14, 107, 112] (control of viscoelastically damped structures [14], connecting the grain-shearing mechanism of wave propagation in marine sediments to fractional-order wave equations [107], viscoelasticity [112]), electromagnetism [52, 55] (electromagnetic theory [52], wave theory [55]), control theory [50, 71, 73, 88] (dynamics and control of initialized fractional-order systems [50], asymptotic stability and stabilization of fractional-order systems [71], global asymptotic stability conditions for nonlinear fractional-order systems [73], stability results for fractional differential equations with application to control processing [88]), electrical circuits [9] (application of generalized Caputo-Fabrizio fractional derivative to electrical circuits), biology [31, 86, 127] (neurodynamics of the vestibulo-ocular reflex-VOR fractional-order model [31], fractional calculus in bioengineering [86], abundant bursting patterns of a fractional-order Morris-Lecar neuron model [127]) and so forth. For a collection of real-world applications of fractional calculus in science and engineering, one can refer to the review paper [132] and the reference therein. In this context, analysis of oscillatory behavior of the solutions of dynamical systems is one

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of the main issues. Since periodic motion is a very important and special phenomenon in a dynamical system [43], the existence of periodic solutions is often considered a desirable property in dynamical systems. This constitutes one of the widely regarded research areas in the theory of dynamical systems with the applications ranging from celestial mechanics to biology, engineering, finance etc. It is known that an integer-order system may have a periodic solution [43] which is, however, not true in the case of a fractional-order system with Caputo or Reimann-Liouville fractional derivative. This is due to the fact that the fractional-order derivative are non-local operators which results in the fractional-order derivative of a periodic function not be a function with the same period. As a consequence, no time-invariant fractional-order system has a non-constant periodic solution. In this direction, several rsearchers have carried out studies. Tavazoei [137, 134] proved that the time invariant fractional-order systems involving Caputo fractional derivative do not possess any non-constant periodic solution. However, Yazdani and Salarieh [149] proved that a periodic solution may exist if the lower terminal point of the fractional derivative tends to $-\infty$. Bourafa [22] proved the existence of the periodic solution of a fractional-order differential system with a fixed length of sliding memory. But these two results are not applicable in practice, i.e., it is difficult to construct such systems. Alternatively, different periodic solution concepts have been introduced for a fractional-order system, e.g., Henríque [53], Wang [141] introduced the concepts of asymptotic periodic solution and proved the existence result. El-Borai [40] and Cabada [26] also elaborated the concepts of an almost periodic solution and a positive periodic solution, respectively, for some nonlinear fractional differential equations.

Thus, from the above notable works, we observe that the fractional-order autonomous systems do not possess any non-constant periodic solutions, and to the best of our knowledge, there are no existing results regarding the existence of a periodic solution for fractional-order non-autonomous systems. Subsequently, in this work, an attempt is made to fill the above gap by trying to establish the existence of a periodic solution of the Caputo-Fabrizio fractional-order system, by making similar assumptions as the ones for an integer-order system, and by taking care of the shortcomings mentioned by Diethelm [36], Guadalupe [49] and Zhang [151] that the Caputo-Fabrizio fractional derivative with a non-singular kernel fails to satisfy the fundamental theorem of fractional calculus due to which the value of the derivative at the initial time is always zero. This puts a restriction in choosing the initial data for a Caputo-Fabrizio fractional differential equation. With respect to the novelty of the present work, the following may be stated: (i) establishing the periodic solution of a non-autonomous Caputo-Fabrizio fractional differential equation; (ii) the subsequent construction of a periodic solution to a linear non-homogeneous Caputo-Fabrizio fractional-order system; (iii) using the obtained results, the derivation of a periodic solution of a fractional-order Gunn diode oscillator under a periodic input voltage;

and (iv) the construction of a suitable linear feedback control to stabilize the solution to a linear non-homogeneous fractional-order system to a periodic solution.

5.2 Main Results

In this section, we first prove that an autonomous Caputo-Fabrizio system cannot admit a non-constant periodic solution, and under a suitable assumption, we discuss the existence of a periodic solution for a non-autonomous Caputo-Fabrizio system.

Autonomous Caputo-Fabrizio initial value systems:

First, let us begin with a discussion on the existence of a periodic solution for an autonomous Caputo-Fabrizio system

$$ {}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t) = F(u), \tag{5.1}$$

$$ u(0) = u_0, \tag{5.2}$$

where $F : \Omega(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$.

Theorem 5.2.1. *An autonomous Caputo-Fabrizio system cannot have a non-constant periodic solution.*

Proof. By Theorem 4.3.1, we know that only a constant solution exists for the autonomous Caputo-Fabrizio system (5.1) $-$ (5.2). Therefore, the proof follows directly. □

Non-autonomous Caputo-Fabrizio initial value systems:

Here, we make an effort to establish the existence of a periodic solution of a non-autonomous Caputo-Fabrizio fractional-order system under some suitable assumptions.

Theorem 5.2.2. *Consider the following Caputo-Fabrizio fractional-order initial value problem:*

$$ {}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t) = F(t,u), \quad t \geq 0, \tag{5.3}$$

$$ u(0) = u_0, \tag{5.4}$$

where ${}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} = \left({}_{0}^{CF}\mathcal{D}_t^{\alpha_1} u_1(t), \ldots, {}_{0}^{CF}\mathcal{D}_t^{\alpha_n} u_n(t)\right)$, $0 < \alpha_i < 1$, $i = 1, 2, \ldots, n$ *and* $F : [0,\infty) \times \Omega\ (\subset \mathbb{R}^n) \to \mathbb{R}^n$. *Let us assume the existence of a positive constant* $T > 0$ *satisfying*

$$ (i)\ F(t+T,u) = F(t,u),\ \forall\, t \geq 0, \quad (ii)\ u(T) = u_0. \tag{5.5}$$

Consequently, the solution u *is also periodic with period* T, *i.e.,* $u(t+T) = u(t),\ \forall\, t \geq 0$.

Proof. Let $v(t) = u(t+T)$. Then

$$ {}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} v(t) = \frac{M(\tilde{\alpha})}{1-\tilde{\alpha}} \int_0^t u'(\varsigma + T) \exp\Big(- \frac{\tilde{\alpha}(t-\varsigma)}{1-\tilde{\alpha}}\Big) d\varsigma, \tag{5.6} $$

and evaluating the derivative above at $t = t + T$, we have

$$ \begin{aligned} {}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t)\mid_{t=t+T} &= \frac{M(\tilde{\alpha})}{1-\tilde{\alpha}} \int_0^{t+T} u'(\varsigma) \exp\Big(- \frac{\tilde{\alpha}(t-\varsigma)}{1-\tilde{\alpha}}\Big) d\varsigma \\ &= \frac{M(\tilde{\alpha})\exp(-\frac{\tilde{\alpha}t}{1-\tilde{\alpha}})}{1-\tilde{\alpha}} \int_0^{T} u'(\varsigma) \exp\Big(- \frac{\tilde{\alpha}(T-\varsigma)}{1-\tilde{\alpha}}\Big) d\varsigma \\ &\quad + \frac{M(\tilde{\alpha})}{1-\tilde{\alpha}} \int_T^{t+T} u'(\varsigma) \exp\Big(- \frac{\tilde{\alpha}(t+T-\varsigma)}{1-\tilde{\alpha}}\Big) d\varsigma. \end{aligned} \tag{5.7} $$

Now, using the system (5.3), we have

$$ \begin{aligned} F(t+T, u(t+T)) &= \exp(-\frac{\tilde{\alpha}t}{1-\tilde{\alpha}}) F(T, u(T)) + {}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t+T), \\ \text{that is, } \; {}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} v(t) &= F(t, v) - \exp(-\frac{\tilde{\alpha}t}{1-\tilde{\alpha}}) F(T, u(T)), \; t \geq 0, \end{aligned} \tag{5.8} $$

with an initial condition $v(0) = u(T) = u_0$. Since, for the existence of solution by Lemma 4.2.1, we have $F(0, u(0)) = 0$, then assumption (i) gives $F(T, u(T)) = F(0, u(0)) = 0$. Thus, the above system (5.8) becomes

$$ {}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} v(t) = F(t, v), \; t \geq 0, \tag{5.9} $$

$$ v(0) = u_0. \tag{5.10} $$

Since the system (5.3) – (5.4) possesses a unique solution, we have $v(t) = u(t)$, i.e., $u(t+T) = u(t)$ for all $t \geq 0$. □

Next, we prove the above result in a more general form by considering the lower terminal point of the fractional derivative as an arbitrary time but before that, we establish the transformed system under the translation of the initial point.

Theorem 5.2.3. *Consider the following system:*

$$ {}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t) = F(t, u(t)), \quad t \geq t_0, \tag{5.11} $$

$$ u(t_0) = u_0. \tag{5.12} $$

Then, under the translation $t = t - T$, where $T > 0$, the system gets transformed to the following system:

$$
{}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t+T) = F(t+T, u(t+T)) - \exp(-\frac{\tilde{\alpha}(t-t_0)}{1-\tilde{\alpha}}) F(t_0+T, u(t_0+T)), \quad t \geq t_0. \tag{5.13}
$$

Proof. Let us apply the translation $t - T$ on time. Then, we have

$$
{}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t+T) = \frac{M(\tilde{\alpha})}{1-\tilde{\alpha}} \int_{t_0}^{t} u'(\varsigma+T) \exp(-\frac{\tilde{\alpha}(t-\varsigma)}{1-\tilde{\alpha}})\, d\varsigma. \tag{5.14}
$$

Evaluating the derivatives in (5.11) at $t = t + T$, we have

$$
\begin{aligned}
{}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t)\, |_{t=t+T} = \frac{M(\tilde{\alpha}) \exp\left(-\frac{\tilde{\alpha}(t-t_0)}{1-\tilde{\alpha}}\right)}{1-\alpha} \int_{t_0}^{t_0+T} & u'(\varsigma) \exp\Big(-\frac{\tilde{\alpha}(t_0+T-\varsigma)}{1-\tilde{\alpha}}\Big)\, d\varsigma \\
+ \frac{M(\tilde{\alpha})}{1-\tilde{\alpha}} \int_{t_0}^{t} u'(\varsigma+T) \exp\Big(-\frac{\tilde{\alpha}(t-\varsigma)}{1-\tilde{\alpha}}\Big)\, d\varsigma. &
\end{aligned} \tag{5.15}
$$

Now, using the system (5.11), the following can be obtained:

$$
\begin{aligned}
F(t+T, u(t+T)) &= \exp\left(-\frac{\tilde{\alpha}(t-t_0)}{1-\tilde{\alpha}}\right) F(t_0+T, u(t_0+T)) + {}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t+T), \\
{}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t+T) &= F(t+T, u(t+T)) - \exp\left(-\frac{\tilde{\alpha}(t-t_0)}{1-\tilde{\alpha}}\right) F(t_0+T, u(t_0+T)),\ t \geq t_0.
\end{aligned} \tag{5.16}
$$

Taking $v(t) = u(t+T)$, we find that v satisfies the following system:

$$
{}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} v(t) = F(t+T, v) - \exp\left(-\frac{\tilde{\alpha}(t-t_0)}{1-\tilde{\alpha}}\right) F(t_0+T, v(t_0)),\ t \geq t_0, \tag{5.17}
$$

$$
v(t_0) = u(t_0+T). \tag{5.18}
$$

□

Theorem 5.2.4. *Consider the following Caputo-Fabrizio fractional-order initial value problem:*

$$
{}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} u(t) = F(t, u(t)), \quad t \geq t_0, \tag{5.19}
$$

$$
u(t_0) = u_0. \tag{5.20}
$$

Let us assume that there exists a positive constant $T > 0$ such that

$$
(i)\ F(t+T, u) = f(t, u),\ \forall\, t \geq t_0,\ \ (ii)\ u(T+t_0) = u_0. \tag{5.21}
$$

Then, the solution x is also periodic with period T, i.e., $u(t+T) = u(t),\ \forall\, t \geq t_0$.

Proof. Let $v(t) = u(t+T)$. Then, by Theorem 5.2.3, v satisfies the following system:

$$ {}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} v(t) = F(t+T, v) - \exp(-\frac{\tilde{\alpha}(t-t_0)}{1-\tilde{\alpha}})F(t_0+T, v(t_0)),\ t \geq t_0, \tag{5.22}$$

$$ v(t_0) = u(t_0+T). \tag{5.23}$$

By using assumptions (i) and (ii), we have

$$ {}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} v(t) = F(t, v) - \exp(-\frac{\tilde{\alpha}(t-t_0)}{1-\tilde{\alpha}})F(t_0, v(t_0)), \tag{5.24}$$

and due to the existence of solution by Lemma 4.2.1, $F(t_0, u_0) = 0$. Thus, we have, on the right hand side of above equation, $F(t_0+T, v(t_0)) = F(t_0, u_0) = 0$. Then, the above system becomes

$$ {}_{t_0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} v(t) = F(t, v),\ t \geq t_0, \tag{5.25}$$

$$ v(t_0) = u_0. \tag{5.26}$$

By the uniqueness of solution of the system (5.19) − (5.20), we get $v(t) = u(t)$, i.e., $u(t+T) = u(t)$, for all $t \geq t_0$. □

Example 5.2.1. *Consider the following fractional-order Caputo-Fabrizio system:*

$$ {}_{0}^{CF}\mathcal{D}_t^{\alpha} x(t) = \frac{1}{1-\alpha}x + \exp(-\sin^2(t)), \quad 0 < \alpha < 1, \tag{5.27}$$

$$ x(0) = -(1-\alpha). \tag{5.28}$$

Since, all the conditions of Theorem 5.2.2 hold, therefore, the above system has a periodic solution which is given by

$$ x(t) = \frac{(1-\alpha)}{\alpha}\Big[(1-\alpha)\sin(2t) - \alpha\Big]\exp(-\sin^2(t)). \tag{5.29}$$

From Figure 5.1, we observe that the amplitude of the periodic solution decreases with respect to the order of the Caputo-Fabrizio derivative, i.e., compared to large values of the order α, the amplitude of the periodic solution is larger for smaller value of α.

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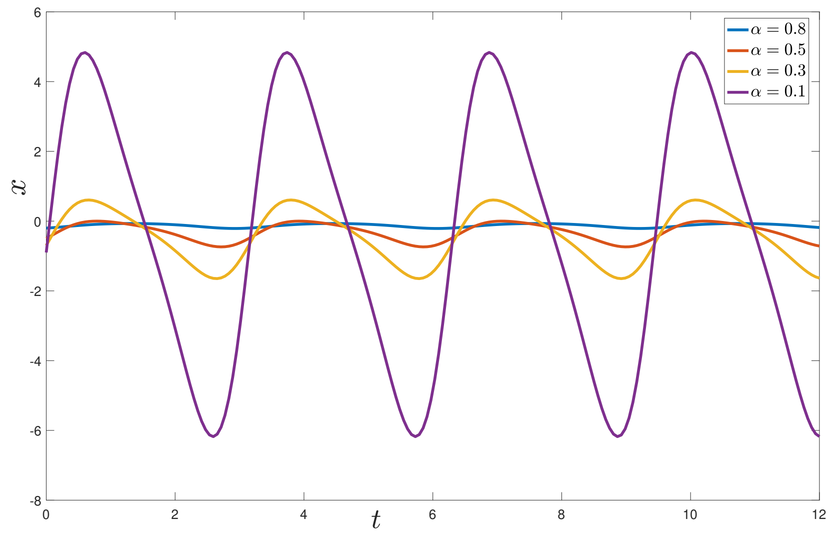

Figure 5.1: Solution of the system (5.27) − (5.28) for various values of order α

5.3 Periodic Solution of a Linear Non-homogeneous System

Now we try to find the periodic solution of a linear non-homogeneous differential system involving Caputo-Fabrizio derivative by using our main result obtained earlier:

$$ {}_0^{CF}\mathcal{D}_t^{\tilde{\alpha}} X(t) = AX + G(t), \tag{5.30} $$

$$ X(0) = -A^{-1}G(0), \tag{5.31} $$

where, $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n),\ 0 < \alpha_i < 1,\ G : [0,\infty) \to \mathbb{R}^n,\ X \in \mathbb{R}^n$ and A is a $n \times n$ scalar matrix. Here, by Lemma 4.2.1, there arises a compulsion to choose an initial condition $X(0) = -A^{-1}F(0)$, which means that, if we choose the initial condition other than this value, then the solution does not exist. Li et al. [78] studied the stability of a linear Caputo-Fabrizio fractional-order system $(0 < \alpha < 1)$ and showed that the linear system is asymptotically stable if and only if the eigenvalues $\lambda(A)$ of the matrix A lie inside the circle $C(\alpha) = \left\{(x,y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{2(1-\alpha)}\right)^2 + y^2 = \frac{1}{4(1-\alpha)^2}\right\}$, and the system is unstable if the eigenvalues $\lambda(A)$ lie outside the circle $C(\alpha)$. Thus, we observe that the circle $C(\alpha)$ acts as a splitting curve that divides the entire $\mathbb{R}^2$-plane into an asymptotically stable region and an unstable region, but no conclusion can be made if the eigenvalues $\lambda(A) \in C(\alpha)$. Therefore, we may expect a periodic solution if the eigenvalues $\lambda(A)$ lie on the circle $C(\alpha)$, as in the case of the integer-order linear systems where if the eigenvalues $\alpha(A)$ lie on the splitting curve $y = 0$ on the $\mathbb{R}^2$-plane, then a periodic solution exists for the linear system. Thus, for the linear non-homogeneous Caputo-Fabrizio system (5.30) − (5.31), we take the eigenvalues $\lambda(A)$ of the matrix to be $\dfrac{1}{1-\alpha_i} \in C(\alpha_i),\ i = 1, 2, \ldots, n$, in order to obtain the possible periodic solution.

Theorem 5.3.1. *Let G be a continuously differentiable periodic function with $G'(0) = 0$, and A be a diagonalizable matrix with eigenvalues, $\dfrac{1}{1-\alpha_i}$, $i = 1, 2, \ldots, n$. Then, the above system has a periodic solution given by*

$$X(t) = \Big[A\big(I - A\big)\Big]^{-1} G'(t) - A^{-1}G(t). \tag{5.32}$$

Proof. By hypothesis of the theorem, we observe that the condition of Theorem 5.2.2 holds and therefore, the given system (5.30) − (5.31) has a periodic solution. Now, we proceed to find this periodic solution. Since the matrix A is diagonalizable, there exists an invertible matrix P such that $A = PBP^{-1}$, where

$$B = \begin{bmatrix} \frac{1}{1-\alpha_1} & & \\ & \ddots & \\ & & \frac{1}{1-\alpha_n} \end{bmatrix}. \tag{5.33}$$

Let $Y(t) = P^{-1}X(t)$. Then, Y satisfies

$${}_{0}^{CF}\mathcal{D}_t^{\tilde{\alpha}} Y(t) = BY(t) + P^{-1}G(t). \tag{5.34}$$

The solution of the above system (5.34) satisfies the following integral equation:

$$Y(t) = Y(0) + B^{-1}\Big[BY(t) + P^{-1}G(t)\Big] + B_{\tilde{\alpha}} \int_0^t \Big[BY(\varsigma) + P^{-1}G(\varsigma)\Big]\, d\varsigma, \tag{5.35}$$

where $B_{\tilde{\alpha}} = \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{bmatrix}$.

The above equation can be rewritten as

$$Y(0) + B^{-1}P^{-1}G(t) + B_{\tilde{\alpha}} \int_0^t \Big[BY(\varsigma) + P^{-1}G(\varsigma)\Big]\, d\varsigma = 0,$$
$$\text{that is,}\quad B^{-1}P^{-1}G'(t) + B_{\tilde{\alpha}}BY(t) + B_{\tilde{\alpha}}P^{-1}G(t) = 0.$$

Using the fact that $B^{-1} = I - B_{\tilde{\alpha}}$, we get

$$\begin{aligned} Y(t) &= -\Big[(B_{\tilde{\alpha}}B)^{-1}B^{-1}P^{-1}G'(t) + (B_{\tilde{\alpha}}B)^{-1}B_{\tilde{\alpha}}P^{-1}G(t)\Big] \\ &= [I - B]^{-1}B^{-1}P^{-1}G'(t) - B^{-1}P^{-1}G(t) \\ &= \Big[I + B + B^2 + \cdots\Big]B^{-1}P^{-1}G'(t) - B^{-1}P^{-1}G(t) \end{aligned}$$

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$$= \Big[B^{-1}P^{-1} + P^{-1} + BP^{-1} + \cdots\Big]G'(t) - B^{-1}P^{-1}G(t). \tag{5.36}$$

Finally, the solution of the given system can be obtained as

$$\begin{aligned} X(t) &= PY(t) \\ &= \Big[PB^{-1}P^{-1} + I + PBP^{-1} + \cdots\Big]G'(t) - PB^{-1}P^{-1}G(t) \\ &= \Big[A^{-1} + I + A + A^2 + \cdots\Big]G'(t) - A^{-1}G(t) \\ &= \Big[A^{-1} + (I-A)^{-1}\Big]G'(t) - A^{-1}G(t) \\ &= \Big[A\big(I-A\big)\Big]^{-1}G'(t) - A^{-1}G(t). \end{aligned} \tag{5.37}$$

Here, the matrix $A\big(I-A\big)$ is invertible since its eigenvalues are given by $-\frac{\alpha_i}{(1-\alpha_i)^2}$, $i = 1, 2, \ldots, n$. □

Now, to illustrate Theorem 5.3.1, we present the following example.

Example 5.3.1. *We consider the linear non-homogeneous system* (5.30) − (5.31) *in three-dimensional space with*

$$A = \frac{1}{4}\begin{bmatrix} 5d_1 + d_2 - 2d_3 & 2d_3 - 2d_2 & 5d_1 - d_2 - 4d_3 \\ 3d_1 - d_2 - 2d_3 & 2d_2 + 2d_3 & 3d_1 + d_2 - 4d_3 \\ 2d_3 - d_1 - d_2 & 2d_2 - 2d_3 & d_2 - d_1 + 4d_3 \end{bmatrix},$$

where $d_i = \frac{1}{1-\alpha_i}$, $i = 1, 2, 3$. *Take* $G(t) = \begin{bmatrix} 30 + 10e^{\sin^2(t)} \\ 17\sin^2(t) \\ 11 + 13\cos^2(t) \end{bmatrix}$. *Here, the matrix* A *is diagonalizable with diagonal entries* d_i, $i = 1, 2, 3$. *The periodic solutions are plotted in Figure* 5.2 *and Figure* 5.3 *in the three-dimensional context.*

5.4 Application to Fractional-order Gunn Diode Oscillator

Now, we look for examples which depict some physical situation. In this context, we consider a fractional-order Gunn diode oscillator which finds an important place in electronics. Gunn diode is an n-type semi-conductor slab of one of the compounds, namely, Ga As, InP, InAs, InSb and CdTd. This diode exhibits a negative resistance dynamics when it is biased to a potential gradient more than a certain value known as the threshold field E_{th} due to the phenomenon known as Gunn Effect or Transferred Electron Effect (TEE). A Gunn diode is mainly used in high-frequency electronics and can produce some of the highest output power of any semiconductor device at these frequencies [116]. Gunn diode oscillators are used in microwave amplifiers to amplify signals in various electronic devices such as

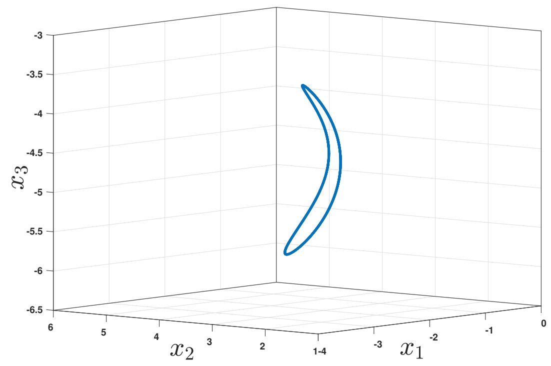

Figure 5.2: Periodic solution of the three-dimensional system for $\alpha_1 = 0.9,\ \alpha_2 = 0.7,\ \alpha_3 = 0.8$

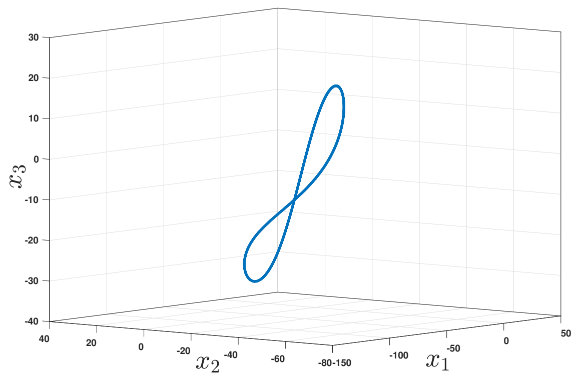

Figure 5.3: Periodic solution of the three-dimensional system for $\alpha_1 = 0.1,\ \alpha_2 = 0.2,\ \alpha_3 = 0.3$

airborne collision avoidance radar, anti-lock brakes, sensors to avoid derailment of trains, car radar detectors, traffic signal controllers, slow-speed sensors, submillimeter-wave radio astronomy receivers and many more.

We know that, in the case of an integer-order Gunn diode oscillator, the oscillations grow exponentially in energy and amplitude. Here, we want to analyse the fractional-order Gunn diode oscillator. Under suitable assumptions on the parameter, we make an attempt to oscillate the circuit current of the Gunn diode oscillator periodically.

Example 5.4.1. *Let us consider the fractional-order Gunn diode oscillator for* $0 < \alpha_1 < 1,\ 0 < \alpha_2 < 1$*:*

$$ {}_{0}^{CF}\mathcal{D}_{t}^{\alpha_1+\alpha_2} i(t) - \left(\frac{r-R}{L}\right) {}_{0}^{CF}\mathcal{D}_{t}^{\alpha_1} i(t) + \frac{1}{LC} i(t) = v_b(t), \tag{5.38} $$

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where R, L, C *are the equivalent resistance, inductance, capacitance, respectively, and* r *is the diode's negative resistance,* $v_b : [0, \infty) \to \mathbb{R}$ *is an input voltage of the Gunn diode circuit as shown in Figure 5.4. Such type of systems is used to model a negative resistance circuit [20], where the Ohm's law does not hold. Here, under the condition that the diode's negative resistance obeys* $r > R$ *with*

$$r - R = \frac{2 - (\alpha_1 + \alpha_2)}{C}, \quad L = \frac{C}{(1 - \alpha_1)(1 - \alpha_2)}, \tag{5.39}$$

and under the assumption that the function input voltage $v_b(t)$ *is a periodic function with*

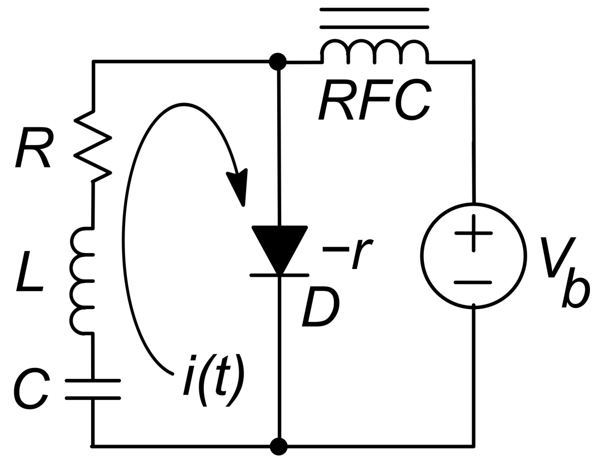

Figure 5.4: Fractional-order Gunn diode oscillator circuit diagram

$v_b'(0) = 0$, *we show that the fractional-order Gunn diode oscillator circuit current* $i(t)$ *(as given in (5.38)) oscillates periodically. By denoting* $x_1(t) = i(t),\ x_2(t) = {}_0^{CF}\mathcal{D}_t^{\alpha_1} i(t)$, *the above system gets converted to the following linear system of equations:*

$$ {}_0^{CF}\mathcal{D}_t^{\alpha_1} x_1(t) = x_2, \tag{5.40}$$

$$ {}_0^{CF}\mathcal{D}_t^{\alpha_2} x_2(t) = \left(\frac{r - R}{L}\right) x_2 - \frac{1}{LC} x_1(t) + v_b(t), \tag{5.41}$$

which has the matrix form

$$ {}_0^{CF}\mathcal{D}_t^{\tilde{\alpha}} X(t) = AX(t) + V_b(t), \tag{5.42}$$

where $\tilde{\alpha} = (\alpha_1, \alpha_2),\ X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & \frac{r-R}{L} \end{bmatrix}$ *and*

$V_b(t) = \begin{bmatrix} 0 \\ v_b(t) \end{bmatrix}.$

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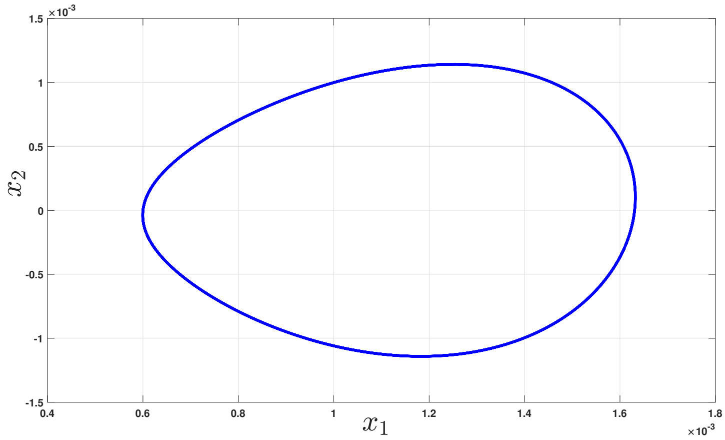

Figure 5.5: Periodic solution of the system (5.40) − (5.41) for $\alpha_1 = 0.99,\ \alpha_2 = 0.98$

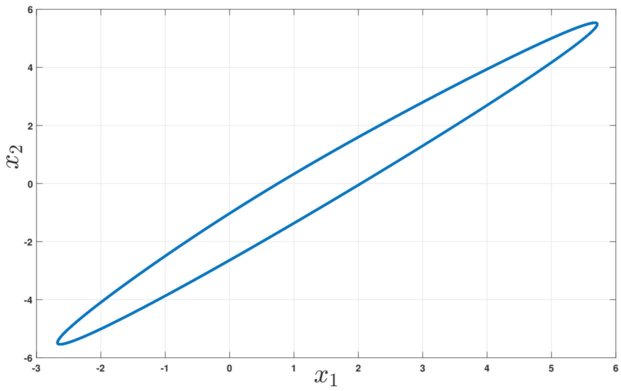

Figure 5.6: Periodic solution of the system (5.40) − (5.41) for $\alpha_1 = 0.5,\ \alpha_2 = 0.5$

Under the assumption (5.39), the eigenvalues of the matrix A are $\frac{1}{1-\alpha_1}, \frac{1}{1-\alpha_2}$, and if we take $v_b(t) = 3e^{\cos^2(t)}$, then by Theorem 5.3.1, we have the periodic solution for the system (5.40) − (5.41) given by the expression (5.37). The solution $X = (x_1, x_2)$ is discussed by plotting x_2 versus x_1 by considering higher values of the fractional-order in Figure 5.5 and middle values in Figure 5.6. From these figures, it can be clearly observed that, as the orders $\alpha_1,\ \alpha_2$ decrease, a shrinking elliptic periodic solution arises. Compared to the larger values of $\alpha_1,\ \alpha_2$, the diameter of the periodic solution is much larger corresponding to smaller values of $\alpha_1,\ \alpha_2$. If the values of $\alpha_1,\ \alpha_2$ are reduced further, the elliptic solution will shrink further and may look like an inclined line.

It is interesting to observe that, if the diode's negative resistance obeys $r > R$, then in the case of an integer-order Gunn diode oscillator, the oscillations grow exponentially *in energy and amplitude whereas the solution oscillates periodically and is bounded for fractional-order Gunn diode oscillator.*

5.5 Stabilization of a Solution of Linear Non-homogeneous Fractional-order Systems to a Periodic Solution

Controlling the behavior of the solution of a given dynamical system by using a controller is an interesting problem. Lenka and Banerjee [71] and Lenka and Bora [73] stabilized the chaotic and unbounded solution to an asymptotically stable solution by using a suitable linear feedback control. Here, we attempt to stabilize the solution of a given linear non-homogeneous fractional-order system to a periodic solution. We are going to construct a suitable linear state feedback gain matrix to control the linear non-homogeneous Caputo-Fabrizio fractional-order system, i.e., we seek to stabilize the solution of the linear non-homogeneous fractional-order system

$$ {}_{0}^{CF}\mathcal{D}_{t}^{\tilde{\alpha}} X(t) = AX + C(t), \tag{5.43} $$

$$ X(0) = X_0, \tag{5.44} $$

to a periodic solution by using the linear state feedback control. For doing this, it is required to determine the feedback gain matrix B of the state feedback controller $u(t) = BX(t)$ such that the closed-loop system

$$ \begin{aligned} {}_{0}^{CF}\mathcal{D}_{t}^{\tilde{\alpha}} X(t) &= AX + C(t) + u(t) \\ &= (A+B)X(t) + C(t), \end{aligned} \tag{5.45} $$

with an initial condition $X(0) = X_0$, will have a periodic solution for fractional-order $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n),\ 0 < \alpha_i < 1,\ i = 1, 2, \ldots, n.$

Theorem 5.5.1. *Let $C : [0, \infty) \to \mathbb{R}^n$ be a continuously differentiable periodic function with period $T > 0$ and $C'(0) = 0$. Suppose the feedback gain matrix B be chosen such that following conditions are satisfied:*

(i) *$A + B$ is a diagonalizable matrix with eigenvalues $\dfrac{1}{1-\alpha_i},\ i = 1, 2, \ldots, n.$*

(ii) *Initial data X_0 belongs to the null space of B, i.e., $BX_0 = 0$.*

Then, the controlled system (5.45) with initial condition $X(0) = X_0$ will have a periodic solution with period $T > 0$.

Proof. The proof follows from Theorem 5.3.1 provided the initial data X_0 satisfies

$$ (A+B)X_0 + C(0) = 0. \tag{5.46} $$

Since the given linear non-homogeneous fractional order system (5.43) − (5.44) has a solution with an initial data X_0, so we must have $AX_0 + C(0) = 0$. Thus, using this fact and the given condition (ii), i.e., $BX_0 = 0$, equation (5.46) holds. □

Example 5.5.1. *Let us consider the following linear non-homogeneous fractional-order system:*

$$
\begin{aligned}
{}_0^{CF}\mathcal{D}_t^{\alpha} x_1(t) &= -x_2 + a\exp(\log(1 + 10\sin^2(t))),\\
{}_0^{CF}\mathcal{D}_t^{\alpha} x_1(t) &= \frac{k^2}{(1-\alpha)^2} x_1(t) + \frac{2k}{1-\alpha} x_2(t) + b\cos^2(t),
\end{aligned}
\tag{5.47}
$$

with an initial condition $X_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{a(3k-1)(1-\alpha)}{2k^2} \\ a \end{bmatrix}$, *where* $k(\neq 1) \in \mathbb{R}$, *and the*

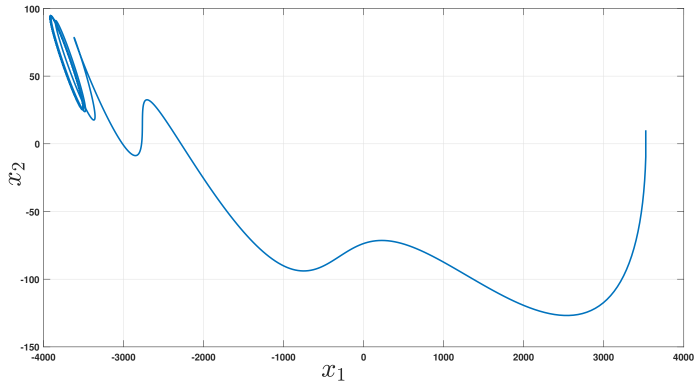

Figure 5.7: Non-periodic solution of the system (5.47) for $\alpha = 0.7,\ a = 10,\ k = \frac{1}{50}$

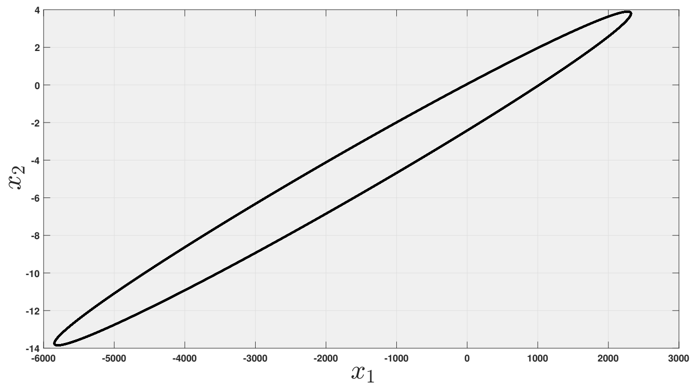

Figure 5.8: Periodic solution of the controlled system (5.48) for $\alpha = 0.7,\ a = 10,\ k = \frac{1}{50}$

real numbers a *and* b *are related by* $b = \left(\frac{k+1}{1-\alpha}\right)\frac{a}{2}$. *For the fractional-order* $\alpha = 0.7,\ a =$

$10,\ k = \frac{1}{50}$, *the above system* (5.47) *possesses a non-periodic solution as shown in Figure* 5.7. *In order to stabilize this non-periodic solution of the above system* (5.47) *to a periodic solution, we introduce a linear state feedback controller* $u(t) = BX(t)$ *to system* (5.47) *such that the controlled fractional-order system is*

$$\begin{aligned} {}_{0}^{CF}\mathcal{D}_t^{\alpha} X(t) &= AX + C(t) + u(t) \\ &= (A+B)X(t) + C(t), \end{aligned} \tag{5.48}$$

where

$$A = \begin{bmatrix} 0 & -1 \\ \frac{k^2}{(1-\alpha)^2} & \frac{2k}{1-\alpha} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{2(1-k)}{1-\alpha} & \frac{(1-k)(3k-1)}{k^2} \\ 0 & 0 \end{bmatrix},$$
$$C(t) = \begin{bmatrix} a\exp(\log(1+10\sin^2(t))) \\ b\cos^2(t) \end{bmatrix}.$$

Here, the feedback gain matrix B *and the external force term* C *satisfy all the conditions of Theorem* 5.5.1. *Thus, for a given initial data* X_0, *the above controlled fractional-order system* (5.48) *should have a periodic solution. This periodic solution can be visualized in Figure* 5.8.

5.6 Conclusions

In this work, by using the properties of Caputo-Fabrizio derivative and the concept of the equilibrium point of the Caputo-Fabrizio fractional-order systems, it has been established that only a constant solution exists for an autonomous Caputo-Fabrizio system, and a non-constant solution exists only in the case of a non-autonomous system. Based on this finding, we have proved that an autonomous Caputo-Fabrizio fractional-order system cannot have a non-constant periodic solution. By making a similar assumption as the one for an integer-order differential system, and using the properties of the Caputo-Fabrizio derivative given by Diethelm [36], we have established the existence of a periodic solution of the non-autonomous Caputo-Fabrizio fractional-order differential system. By applying our main results, the periodic solution of the Caputo-Fabrizio linear fractional-order system has been found. Further, we have applied the result in establishing the existence of a periodic solution of the fractional-order Gunn diode oscillator model and observed that the diameter of the periodic orbit reduces very rapidly as the orders $\alpha_1,\ \alpha_2$ continuously increase. By using the result on a linear system, a non-periodic solution of a Caputo-Fabrizio linear non-homogeneous system has been shown to stabilize to a periodic solution by constructing a suitable linear feedback control.

The present result acts as an alternative in using the Caputo-Fabrizio fractional-order derivative to model a system with an oscillatory behavior. A limit cycle is defined as an isolated periodic orbit, and it is a very special phenomenon in dynamical systems because the existence of a stable limit cycle in a system assures the self-sustainable systems which means that there exists a domain such that if the initial state starts from this domain, then the state always moves along a periodic orbit in all future time. Thus, once we know the existence of a periodic solution, there is a possibility of having a limit cycle for a given dynamical system. Thus, the present work may help in proving the existence of a limit cycle for a given dynamical system. Further, we can get an idea to convert the chaotic oscillations to regular oscillations in a Caputo-Fabrizio fractional-order system. Thus, one can work on more complex chaotic systems and stabilize the chaotic oscillations to a regular oscillation. Therefore, finding answers to such questions or some more intriguing queries concerned with the Caputo-Fabrizio dynamical systems opens the door for further relevant research in the future.

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CHAPTER 6

Stabilization and asymptotic stability of the Caputo-Fabrizio fractional-order linear and semilinear evolution equations

6.1 Introduction

Studying the long term behavior of the state of dynamical systems is an important problem. For example, in the modelling of a biological species, one may have an interest in knowing the sustainability of species in future. In case of a control problem, by using a control, one may want to know under what condition a system will settle down to its stable position in future time when it goes through a certain disturbance. Therefore, mathematically speaking, one can have an interest in studying the asymptotic stability of a system or deriving its asymptotic stability conditions. Subsequently, in this work, we mainly focus in analysing the asymptotic stability of fractional-order linear and semilinear evolution equations involving a Caputo-Fabrizio fractional operator with a non-singular kernel and try to find the conditions for the asymptotic stability of the system under consideration.

6.2 Main Results

Before we establish and analyze our main result, we present a definition and a Lemma for the following non-autonomous Caputo-Fabrizio initial value system:

$$
{}_0^{CF}\mathcal{D}_t^{\alpha} v(t) = G(t, v), \tag{6.1}
$$

$$
v(0) = v_0, \tag{6.2}
$$

where ${}_0^{CF}\mathcal{D}_t^{\alpha} v(t) = \left({}_0^{CF}\mathcal{D}_t^{\alpha_1} v_1(t), \ldots, {}_0^{CF}\mathcal{D}_t^{\alpha_n} v_n(t)\right)$, with $0 < \alpha_i < 1,\ i = 1, 2, \ldots, n$ and $G : [0, \infty) \times \Omega\ (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ is Lipschitz continuous in v on a domain Ω.

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Definition 6.2.1. *[28] The equilibrium point $\bar{v} = 0$ of the system (6.1) is said to be*

(i) *stable if, for any initial value v_0, there exists $\varepsilon > 0$ such that the solution $v(t)$ with an initial condition $v(0) = v_0$ satisfies $\|v(t)\| \leq \varepsilon$, for all $t \geq 0$,*

(ii) *asymptotically stable if, in addition to being stable, $||v(t)|| \to 0$ as $t \to \infty$.*

Now we are in a position to study the asymptotic stability of the solution of a Caputo-Fabrizio fractional-order evolution equation

$$ {}_0^{CF}\mathcal{D}_t^{\alpha} v(t) = Av(t) + F(t, v), \quad 0 < \alpha < 1, \tag{6.3}$$

$$ v(0) = v_0, \tag{6.4}$$

where A is an $n \times n$ scalar matrix and $F : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 6.2.1. *Suppose $0 < \alpha < 1$ and the eigenvalues of matrix A do not lie inside the disc $D(\alpha) = \Big\{(x, y) \in \mathbb{R}^2 \mid \Big(x - \frac{1}{2(1-\alpha)}\Big)^2 + y^2 \leq \frac{1}{4(1-\alpha)^2}\Big\}$. Then, the matrix $N = I - (1 - \alpha)A$ is invertible and the eigenvalues of the matrix $N^{-1}A$ have negative real parts.*

Proof. Since the eigenvalues $\lambda(A)$ of matrix A do not lie inside the disc $D(\alpha)$, i.e., $\lambda(A) \notin D(\alpha)$, it implies that $\frac{1}{1-\alpha} \in \rho(A)$ (the resolvent set of A). Therefore, the matrix $N \Big(= -(1 - \alpha)\big(A - \frac{1}{1-\alpha} I\big)\Big)$ is invertible. Next, from the well-known result of the invertible matrices that

$$\big(I - (1 - \alpha)A\big)^{-1} = I + (1 - \alpha)\big(I - (1 - \alpha)A\big)^{-1} A,$$

we get

$$N^{-1}A = \big(I - (1 - \alpha A)\big)^{-1} A = \frac{1}{1 - \alpha}\Big(\big(I - (1 - \alpha)A\big)^{-1} - I\Big). \tag{6.5}$$

Suppose $\lambda(A) = \lambda_1 + \mathrm{i}\lambda_2$. Then, we have

$$\lambda(N^{-1}A) = \frac{1}{1 - \alpha}\left(\frac{1}{1 - (1 - \alpha)\lambda(A)} - 1\right), \tag{6.6}$$

from which we get $\mathrm{Re}(\lambda(N^{-1}A)) = \dfrac{\big(\lambda_1 - (1 - \alpha)(\lambda_1^2 + \lambda_2^2)\big)}{(1 - (1 - \alpha)\lambda_1)^2 + (1 - \alpha)^2\lambda_2^2}$. Therefore, if $(\lambda_1, \lambda_2) \notin D(\alpha)$, then clearly $\mathrm{Re}(\lambda(N^{-1}A)) < 0$. □

6.2.1 Linear case

Here, we discuss the asymptotic stability of the zero solution of the linear Caputo-Fabrizio fractional-order evolution equation

$$ {}_0^{CF}\mathcal{D}_t^{\alpha} v(t) = Av(t) + B(t)v(t), \quad 0 < \alpha < 1, \tag{6.7} $$
$$ v(0) = v_0, \tag{6.8} $$

where $v \in \mathbb{R}^n$, A is an $n \times n$ scalar matrix, and $B : [0, \infty) \to \mathbb{R}^{n \times n}$ is a continuously differentiable function.

Theorem 6.2.1. *The equilibrium point $\bar{v} = 0$ of the system (6.7) – (6.8) is asymptotically stable if the following conditions are satisfied:*

(i) *the eigenvalues of the matrix A do no lie inside the disc $D(\alpha) = \Big\{(x, y) \in \mathbb{R}^2 \mid \Big(x - \frac{1}{2(1-\alpha)}\Big)^2 + y^2 \le \frac{1}{4(1-\alpha)^2}\Big\}$,*

(ii) *there exists a constant $M > 0$ such that $\|B(t)\| \le M$, for all $t \ge 0$, and $M < \dfrac{1}{(1-\alpha)\|N^{-1}\|}$, where $N = I - (1-\alpha)A$,*

(iii) $c = -\max\{Re(\lambda(N^{-1}A))\} > \dfrac{m\|N^{-1}\|^2 M}{1-(1-\alpha)\|N^{-1}\|M}$,

where c and m satisfy $\|\exp(N^{-1}At)\| \le m\exp(-ct)$.

Proof. By using Laplace transform and using the fact that if the solution exists, we must have $Av_0 + B(0)v_0 = 0$, and the solution of the above system (6.7) – (6.8) is given by

$$ v(t) = \exp(N^{-1}At)N^{-1}v_0 + (1-\alpha)N^{-1}B(t)v(t) + \alpha N^{-1}\int_0^t \exp(N^{-1}A(t-\tau))N^{-1}B(\tau)v(\tau)\, d\tau, \tag{6.9} $$

where $N = I - (1-\alpha)A$, and by Lemma 6.2.1, the matrix N is invertible. Therefore, from (6.9), we have

$$ \|v(t)\| \le \|\exp(\alpha N^{-1}At)\|\|N^{-1}v_0\| + (1-\alpha)\|N^{-1}\|\|B(t)\|\|v(t)\| + \alpha\|N^{-1}\|\int_0^t \|\exp(\alpha N^{-1}A(t-\tau))\|\|N^{-1}\|\|B(\tau)\|\|v(\tau)\|\, d\tau. \tag{6.10} $$

Under the given condition (i), by Lemma 6.2.1, each of the eigenvalues $\lambda(N^{-1}A)$ of the matrix $N^{-1}A$ has a negative real part. Thus, by Lemma 1.5.2, there exists a constant $m > 0$ such that $\|\exp(N^{-1}At)\| \le m\exp(-ct)$ for all $t \ge 0$, where

$c = -\max\{\text{Re}(\lambda(N^{-1}A))\}$, and by condition (ii), we have

$$\|v(t)\| \leq m\|N^{-1}v_0\|\exp(-\alpha ct) + (1-\alpha)M\|N^{-1}\|\|v(t)\| + \alpha m M\|N^{-1}\|^2 \int_0^t \exp(-\alpha c(t-\tau))\|v(\tau)\|\,d\tau. \tag{6.11}$$

Now, multiplying both sides of the above inequality by $\exp(\alpha ct)$, we get

$$\|v(t)\|\exp(\alpha ct) \leq m\|N^{-1}v_0\| + (1-\alpha)M\|N^{-1}\|\|v(t)\|\exp(\alpha ct) + \alpha m M\|N^{-1}\|^2 \int_0^t \exp(\alpha\tau))\|v(\tau)\|\,d\tau,$$

that is,

$$\Big(1-(1-\alpha)M\|N^{-1}\|\Big)\|v(t)\|\exp(\alpha ct) \leq \alpha m M\|N^{-1}\|^2 \int_0^t \exp(\alpha\tau))\|v(\tau)\|\,d\tau + m\|N^{-1}v_0\| \tag{6.12}$$

By using Lemma 1.5.1, one can obtain the following from the above inequality:

$$\|v(t)\| \leq \frac{m\|N^{-1}v_0\|}{1-(1-\alpha)M\|N^{-1}\|}\exp\left(\left[\frac{mM\|N^{-1}\|^2}{1-(1-\alpha)M\|N^{-1}\|} - c\right]\alpha t\right). \tag{6.13}$$

Under the conditions (ii) and (iii), using (6.13), we have $\|v(t)\| \to 0$ as $t \to \infty$ which means that the zero solution of the system (6.7) is asymptotically stable. □

6.2.2 Semilinear Case

Now, we discuss the asymptotic stability of the zero solution of the semilinear Caputo-Fabrizio fractional-order evolution equation

$${}_0^{CF}\mathcal{D}_t^{\alpha}v(t) = Av(t) + F(t,v), \quad 0 \leq \alpha < 1, \tag{6.14}$$

$$v(0) = v_0, \tag{6.15}$$

where $v \in \mathbb{R}^n$, A is an $n \times n$ scalar matrix and $F : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function.

Theorem 6.2.2. *The equilibrium point $\bar{v} = 0$ of the fractional-order system (6.14) – (6.15) is locally asymptotically stable if, in addition to the condition (i) of Theorem 6.2.1, the following conditions are also satisfied:*

(i) $\lim_{v\to 0}\dfrac{\|F(t,v)\|}{\|v\|} = 0$ *uniformly in* $t \in [0,\infty)$,

(ii) $c = -\max\{\textit{Re}(\lambda(N^{-1}A))\} > \frac{m}{1-\alpha}\|N^{-1}\|$,

where c and m satisfy $\|\exp(N^{-1}At)\| \leq m\exp(-ct)$.

Proof. First, we find the solution of the system (6.14) – (6.15) by using Laplace transform. So, applying Laplace transform to (6.14) and denoting $V(s) = \mathscr{L}\{v(t)\}$, we get

$$\frac{sV(s) - v(0)}{s + \alpha(1-s)} = AV(s) + F(s, V(s))$$
$$\Rightarrow \Big[s\{I - (1-\alpha)A\} - \alpha A\Big]V(s) = v_0 + \Big(\alpha + s(1-\alpha)\Big)F(s, V(s)). \tag{6.16}$$

Denote $N = I - (1-\alpha)A$. Then, by the above assumption of the theorem and by Lemma 6.2.1, matrix N is invertible. Thus from the above equation, we get

$$\begin{aligned}\Big[I - \alpha N^{-1}A\Big]V(s) &= N^{-1}v_0 + \Big(\alpha + s(1-\alpha)\Big)N^{-1}F(s, V(s)) \\ &= N^{-1}v_0 + \alpha N^{-1}F(s, V(s)) + (1-\alpha)N^{-1}\Big(sF(s, V(s))\Big) \\ &= N^{-1}v_0 + \alpha N^{-1}F(s, V(s)) + (1-\alpha)N^{-1}\left[\mathscr{L}\left\{\frac{dF(t, v(t))}{dt}\right\} + F(0, v(0))\right].\end{aligned} \tag{6.17}$$

Since the initial condition v_0 satisfies $Av_0 + F(0, v_0) = 0$, then from the above, we get

$$\begin{aligned}\Big[I - \alpha N^{-1}A\Big]V(s) &= N^{-1}v_0 + \alpha N^{-1}F(s, V(s)) + (1-\alpha)N^{-1}\left[\mathscr{L}\left\{\frac{dF(t, v(t))}{dt}\right\} - Av_0\right] \\ &= N^{-1}\Big[I - (1-\alpha)A\Big]v_0 + \alpha N^{-1}F(s, V(s)) + (1-\alpha)N^{-1}\left[\mathscr{L}\left\{\frac{dF(t, v(t))}{dt}\right\}\right] \\ &= v_0 + \alpha N^{-1}F(s, V(s)) + (1-\alpha)N^{-1}\left[\mathscr{L}\left\{\frac{dF(t, v(t))}{dt}\right\}\right].\end{aligned} \tag{6.18}$$

Using the Laplace convolution theorem and applying inverse Laplace transform, we get

$$\begin{aligned}v(t) = \exp(\alpha N^{-1}At)v_0 &+ \alpha \int_0^t \exp(\alpha N^{-1}A(t-\tau))N^{-1}F(\tau, v(\tau))\, d\tau \\ &+ (1-\alpha)\int_0^t \exp(\alpha N^{-1}A(t-\tau))N^{-1}\frac{dF(\tau, v(\tau))}{d\tau}\, d\tau\end{aligned}$$

$$\begin{aligned}&= \exp(\alpha N^{-1}At)v_0 + \alpha \int_0^t \exp(\alpha N^{-1}A(t-\tau))N^{-1}F(\tau, v(\tau))\, d\tau \\ &\quad + (1-\alpha)\left[\exp(\alpha N^{-1}A(t-\tau))N^{-1}F(\tau, v(\tau))\Big|_{\tau=0}^{t} \right. \\ &\qquad \left. + \alpha N^{-1}A\int_0^t \exp(\alpha N^{-1}A(t-\tau))N^{-1}F(\tau, v(\tau))\, d\tau\right] \\ &= \exp(\alpha N^{-1}At)\Big[I + (1-\alpha)N^{-1}A\Big]v_0 + (1-\alpha)N^{-1}F(t, v(t)) \\ &\quad + \alpha\Big[I + (1-\alpha)N^{-1}A\Big]\int_0^t \exp(\alpha N^{-1}A(t-\tau))N^{-1}F(\tau, v(\tau))\, d\tau.\end{aligned} \tag{6.19}$$

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From the well-known result of invertible matrices, we have $N^{-1} = I + (1-\alpha)N^{-1}A$. Then, from (6.19), one can obtain the solution of the system (6.14) − (6.15) as

$$v(t) = \exp(\alpha N^{-1}At)v_0 + (1-\alpha)N^{-1}F(t, v(t)) \\ + \alpha N^{-1}\int_0^t \exp(\alpha N^{-1}A(t-\tau))N^{-1}F(\tau, v(\tau))\,d\tau. \tag{6.20}$$

Under the given condition (ii), we have

$$\|v(t)\| \leq m\|N^{-1}v_0\|\exp(-\alpha ct) + (1-\alpha)\|N^{-1}\|\|F(t, v(t))\| \\ + \alpha m\|N^{-1}\|^2\int_0^t \exp(-\alpha c(t-\tau))\|F(\tau, v(\tau))\|\,d\tau. \tag{6.21}$$

Since $\bar{v} = 0$ is an equilibrium point of the system (6.14), it implies that $F(t, 0) = 0$, for all $t \geq 0$. Using this fact and condition (i), there exists a $\delta > 0$ such that

$$\|F(t, v(t))\| \leq \frac{1}{2(1-\alpha)\|N^{-1}\|}\|v(t)\| \quad as \quad \|v(t)\| \leq \delta. \tag{6.22}$$

Using inequality (6.22) in (6.20) and after simplifying, we get

$$\|v(t)\|\exp(\alpha ct) \leq 2m\|N^{-1}v_0\| + \frac{\alpha m\|N^{-1}\|}{1-\alpha}\int_0^t \exp(\alpha c\tau)\|v(\tau)\|\,d\tau. \tag{6.23}$$

Now, by Lemma 1.5.1, we obtain

$$\|v(t)\| \leq 2m\|N^{-1}v_0\|\exp\left(\left[\frac{m}{1-\alpha}\|N^{-1}\| - c\right]\alpha t\right). \tag{6.24}$$

By condition (ii), we have $\|v(t)\| \to 0$ as $t \to \infty$ which means that the system (6.14) is locally asymptotically stable. □

The notion of the global solution for the Caputo-Fabrizio fractional-order system (6.1) − (6.2) has some complication. Every solution $v(t, v_0)$ of the initial value problem (6.1)−(6.2) with an initial condition $v(0) = v_0$ must satisfy $F(0, v_0) = 0$, i.e., the possible solution set S_{ps} may be a discrete set or even $\mathbb{R}^n$. However, the set $\mathbb{R}^n \setminus S_{ps}$ is a non-existent solution set, i.e., if $v_0 \in \mathbb{R}^n \setminus S_{ps}$, the Caputo-Fabrizio system (6.1) − (6.2) will have no solution with an initial condition $v(0) = v_0$. So, here we observe that there is a clear region $\mathbb{R}^n \setminus S_{ps}$ in $\mathbb{R}^n$ in which the solution does not exist. Thus, to fill this gap, we present an alternative concept of the global solution for a Caputo-Fabrizio system (6.1) − (6.2), since by Corollary 4.3.1, we know that the solution existence point set S_s is a subset of

S_{ps}. Therefore, the maximum possible domain for S_s is S_{ps} and subsequently, we have the following definition.

Definition 6.2.2. *The Caputo-Fabrizio fractional-order system* $(6.1)-(6.2)$ *has a global solution if the solution existence point set* S_s *is equal to the possible solution point set* S_{ps} *i.e.,* $S_s = S_{ps}$.

Theorem 6.2.3. (*Globally asymptotically stable*) : *The equilibrium point* $\bar{v} = 0$, *of the system* (6.14) *is globally asymptotically stable if, in addition to condition* (i) *of Theorem 6.2.1, the following conditions are also satisfied:*

(i) $F(t, v)$ *is uniformly globally Lipschitz with a Lipschitz constant* $L > 0$ *such that* $L < \frac{1}{(1-\alpha)\|N^{-1}\|}$, *i.e.,* $\|F(t, v_1) - F(t, v_2)\| \le L\|v_1 - v_2\|$ *for all* $v_1,\ v_2 \in \mathbb{R}^n$, $\forall\, t \in [0, \infty)$,

(ii) $c = -\max\{Re(\lambda(N^{-1}A))\} > \frac{mL\|N^{-1}\|^2}{1-(1-\alpha)L\|N^{-1}\|}$,

where c *and* m *satisfy* $\|\exp(N^{-1}At)\| \le m\exp(-ct)$.

Proof. From (6.20), the solution of the system $(6.14)-(6.15)$ is given by

$$v(t) = \exp(\alpha N^{-1}At)v_0 + (1-\alpha)N^{-1}F(t, v(t)) + \alpha N^{-1}\int_0^t \exp(\alpha N^{-1}A(t-\tau))N^{-1}F(\tau, v(\tau))\, d\tau. \tag{6.25}$$

Under the given conditions $(i),\ (ii)$ and using the fact that $F(t, 0) = 0$, for all $t \ge 0$, we get

$$\|v(t)\| \le m\|N^{-1}v_0\|\exp(-\alpha ct) + (1-\alpha)L\|N^{-1}\|\|v(t)\| + \alpha mL\|N^{-1}\|^2\int_0^t \exp(-\alpha c(t-\tau))\|v(\tau)\|\, d\tau. \tag{6.26}$$

Multiplying both sides by $\exp(\alpha ct)$ and after simplification, we obtain

$$\Big(1-(1-\alpha)L\|N^{-1}\|\Big)\|v(t)\|\exp(\alpha ct) \le m\|N^{-1}v_0\| + \alpha mL\|N^{-1}\|^2\int_0^t \exp(\alpha c\tau))\|v(\tau)\|\, d\tau. \tag{6.27}$$

Now, by using Lemma 1.5.1, we get

$$\|v(t)\| \le \frac{m\|N^{-1}v_0\|}{1-(1-\alpha)L\|N^{-1}\|}\exp\left(\left[\frac{\alpha mL\|N^{-1}\|^2}{1-(1-\alpha)L\|N^{-1}\|} - c\right]\alpha t\right). \tag{6.28}$$

Therefore, under the condition (ii), we have $\|v(t)\| \to 0$ as $t \to \infty$ which means that the system (6.14) is globally asymptotically stable. □

6.3 Stabilization of a Caputo-Fabrizio Fractional-order Evolution Equation

Controlling the behavior of the solution to a given dynamical system by using a controller is an interesting problem. Lenka and his co-researchers [71, 73] stabilized the chaotic and unbounded solution to an asymptotically stable solution by using a suitable linear feedback control. By using control, Chen et al. [28, 29] stabilized the classes of nonlinear fractional-order systems containing Caputo derivative. Li et al. [75] stabilized the chaotic fractional-order Chen systems using a linear feedback control. We are going to construct an appropriate linear state feedback gain matrix for controlling the Caputo-Fabrizio fractional-order system, i.e., to find a mean to stabilize the Caputo-Fabrizio fractional-order system

$$ {}_{0}^{CF}\mathcal{D}_t^{\alpha} v(t) = Av(t) + F(t,v), \tag{6.29} $$

$$ v(0) = v_0, \tag{6.30} $$

by using the linear state feedback control. To achieve this, it is required to find the feedback gain matrix B of the state feedback controller $u(t) = Bv(t)$ in order that the closed-loop system

$$ \begin{aligned} {}_{0}^{CF}\mathcal{D}_t^{\alpha} v(t) &= Av(t) + F(t,v) + u(t), \\ &= (A+B)v(t) + F(t,v), \end{aligned} \tag{6.31} $$

with an initial condition $v(0) = v_0$, for fractional-order $0 < \alpha < 1$, is asymptotically stable.

Corollary 6.3.1. *Suppose that the feedback gain matrix B be chosen such that following conditions are satisfied:*

(i) *The eigenvalues of the matrix $A+B$ do not lie inside the disc* $D(\alpha) = \left\{(x,y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{2(1-\alpha)}\right)^2 + y^2 \leq \frac{1}{4(1-\alpha)^2}\right\}$.

(ii) $\lim_{v \to 0} \dfrac{\|F(t,v)\|}{\|v\|} = 0$ *uniformly in* $t \in [0,\infty)$.

(iii) *Initial condition v_0 belongs to the null space of B, i.e., $Bv_0 = 0$.*

(iv) $c = -\max\{Re(\lambda(N_B^{-1}(A+B)))\} > \frac{m}{1-\alpha}\|N_B^{-1}\|$,
where $N_B = I - (1-\alpha)(A+B)$, *$c$ and m satisfy* $\|\exp(N_B^{-1}(A+B)t)\| \leq m\exp(-ct)$.

Then, the controlled system (6.31) with the same initial condition $v(0) = v_0$ is locally asymptotically stable.

Proof. The proof follows from Theorem 6.2.2 provided the initial condition v_0 satisfies

$$(A+B)v_0 + F(0, v_0) = 0. \tag{6.32}$$

Since the given linear non-homogeneous fractional-order system (6.29) − (6.30) has a solution with an initial data v_0, so we must have $Av_0 + F(0, v_0) = 0$. Thus, using this fact and the given condition (ii), i.e., $Bv_0 = 0$, equation (6.32) holds. □

Corollary 6.3.2. *Suppose that the feedback gain matrix B be chosen such that the following conditions are satisfied:*

(i) *The eigenvalues of the matrix $A+B$ do no lie inside the disc $D(\alpha) = \Big\{(x, y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{2(1-\alpha)}\right)^2 + y^2 \leq \frac{1}{4(1-\alpha)^2}\Big\}$.*

(ii) *$F(t, v)$ is uniformly globally Lipschitz with a Lipschitz constant $L_{_B} > 0$ such that $L_{_B} < \frac{1}{(1-\alpha)\|N_B^{-1}\|}$, i.e., $\|F(t, v_1) - F(t, v_2)\| \leq L_{_B}\|v_1 - v_2\|$, for all $v_1,\ v_2 \in \mathbb{R}^n$, $\forall\, t \in [0, \infty)$.*

(iii) *Initial condition v_0 belongs to the null space of B, i.e., $Bv_0 = 0$.*

(iv) *$c = -\max\{Re(\lambda(N_{_B}^{-1}(A+B)))\} > \frac{mL_B\|N_B^{-1}\|^2}{1-(1-\alpha)L_B\|N_B^{-1}\|}$,*

where $N_{_B} = I - (1-\alpha)(A+B)$, c and m satisfy $\|\exp(N_{_B}^{-1}(A+B)t)\| \leq m\exp(-ct)$.

Then, the controlled system (6.31) with the same initial condition $v(0) = v_0$ is globally asymptotically stable.

Proof. By proceeding similarly as above and using Theorem 6.2.3, we can establish the result. □

6.4 Examples

In this section, we present some examples which obey our main results. For numerical simulation, we use the three-step Adams-Bashforth method for the Caputo-Fabrizio fractional-order sytem proposed by Owolabi and Atangana [106].

Example 6.4.1. *Let us consider the linear Caputo-Fabrizio fractional-order evolution equation*

$$\begin{aligned} {}^{CF}_{0}\mathcal{D}^{\alpha}_{t} x_1(t) &= a x_1(t) + 1.2\exp(-3t)x_1(t) + 1.2\cos^2(3t)x_2(t), \\ {}^{CF}_{0}\mathcal{D}^{\alpha}_{t} x_2(t) &= 1.2\exp(-t^2)x_1(t) + b x_2(t) + 1.2\cos(2t^2)x_2(t), \end{aligned} \tag{6.33}$$

for which we choose $a = 1,\ b = 1$, and $\alpha = 0.8$. In this case, the eigenvalues of matrix

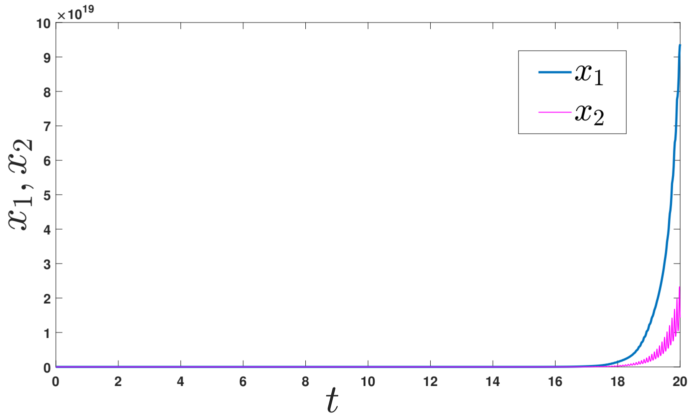

Figure 6.1: Unbounded solution of the system (6.33) for fractional-order $\alpha = 0.8$ and $a = 1,\ b = 1$

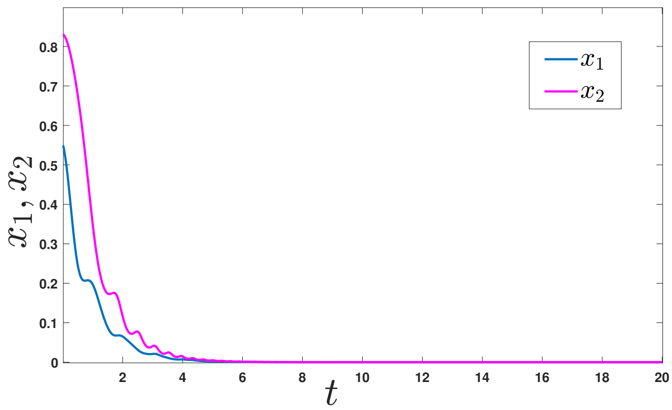

Figure 6.2: Solution of the system (6.33) for fractional-order $\alpha = 0.8$ and $a = -3,\ b = -2$

A are $\lambda_1 = 1 = \lambda_2$ which lies inside the disc $D(\alpha)$. Here, condition (i) of Theorem 6.2.1 does not hold for these values of a and b. The numerical solution of the above system (6.33) is plotted in Figure 6.1 from which it is observed that the zero solution is unstable. Here,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1.2\exp(-3t) & 1.2\cos^2(3t) \\ 1.2\exp(-t^2) & 1.2\cos(2t^2) \end{bmatrix}, \tag{6.34}$$

and $m = 1$, $\|B(t)\|_1 \leq M = 1.2$.

Next, we choose $a = -3,\ b = -2$. Then the eigenvalues of matrix A do not lie inside the disc $D(\alpha)$. Moreover, $\|N^{-1}\|_1 = \frac{5}{7}$, $M = 1.2 < \frac{1}{(1-\alpha)\|N^{-1}\|_1} = 7$ and

$c = -\max\{\mathrm{Re}(\lambda(N^{-1}A))\} = \min\{\frac{2}{1+2(1-\alpha)}, \frac{3}{1+3(1-\alpha)}\} = \frac{10}{7} > \frac{m\|N^{-1}\|_1^2 M}{1-(1-\alpha)\|N^{-1}\|_1 M} = 0.7389$. *Thus, for the above values of a and b, all the conditions of Theorem 6.2.1 are satisfied. Consequently, the equilibrium point $\bar{v} = 0$ of the fractional-order system (6.33) is asymptotically stable. The numerical solution of the system (6.33) for $a = -3,\ b = -2$ is plotted in Figure 6.2 with an initial condition $x(0) = (x_1(0), x_2(0))$ satisfying $Ax(0) + B(0) = 0$.*

In integer-order dynamical systems, it is well known that chaos cannot occur in autonomous systems of integer order less than three according to the Poincare-Bendixon theorem [124]. The order of the system can be defined as the sum of the orders of all involved derivatives. Proposed Examples 6.4.2 – 6.4.3 below deal with the fractional-order chaotic systems. To the best of our knowledge, no general theory is available in the literature to analyze fractional-order chaotic systems. The existence of chaos in fractional-order systems has been investigated by considering the order of the fractional derivative as one of the bifurcation parameters and analyzing the dynamics of the fractional-order chaotic systems through numerical simulations. For instance, Hartley and Lorenzo [51] showed that the fractional-order Chua's circuit could produce a chaotic attractor with the system order as low as 2.7. Li and Chen [75], through numerical results, established that the chaos could exist in the fractional-order Chen systems with the system order as low as 2.1. In another work [85], by varying the parameter value, it was shown that chaos could exist in fractional-order Chen systems with the order as low as 0.3. In Examples 6.4.2 – 6.4.3, we try to stabilize a fractional-order chaotic system, but varying the order α of the Caputo-Fabrizio fractional derivative as low as in $0 < \alpha < 0.5$ may not generate a chaotic attractor. Therefore, by observing the existing literature [51, 75, 85] and taking a safer side, we consider the value of the fractional-order α close to 1.

Example 6.4.2. *Let us consider the Caputo-Fabrizio fractional-order Lorenz system [83]*

$$\begin{aligned}
{}_0^{CF}\mathcal{D}_t^{\alpha} x_1(t) &= a\big(x_2(t) - x_1(t)\big) + 0.5\exp(-7t)x_1^2(t),\\
{}_0^{CF}\mathcal{D}_t^{\alpha} x_2(t) &= x_1(t)\big(c - x_3(t)\big) - x_2(t),\\
{}_0^{CF}\mathcal{D}_t^{\alpha} x_3(t) &= x_1(t)x_2(t) - bx_3(t),
\end{aligned} \tag{6.35}$$

where a is the Prandtl number and c is the Rayleigh number, all the parameters a, b and c are strictly positive. However, for computational purpose, the values $a = 10,\ b = \frac{8}{3},\ c = 28$ are taken here. We perturb the Lorentz system by the quantity $0.5\exp(-7t)x_1(t)$ in the first equation in order to obtain a non-autonomous system. Otherwise, we have only a constant solution for the conventional autonomous Lorenz system. We take the fractional-order $\alpha = 0.9997$, and the numerical solution is plotted in Figure 6.3 from which it is observed that the system (6.35) exhibits a chaotic behavior.

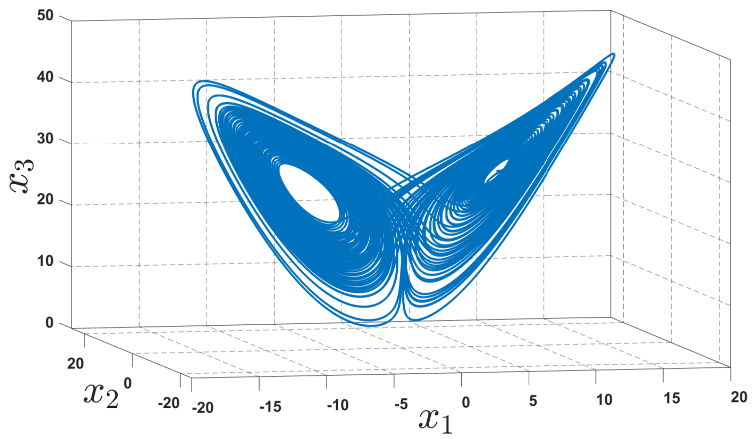

Figure 6.3: Chaotic behavior of the fractional-order Lorentz system (6.35) for fractional-order $\alpha = 0.9997$ and $a = 10,\ b = \frac{8}{3},\ c = 28$

Next, we try to stabilize this chaotic fractional-order Lorenz system (6.35) *by introducing a linear state feedback controller* $u(t) = Bx(t)$ *to it where* B *is a* 3×3 *scalar matrix such that the controlled system*

$$\begin{aligned} {}_0^{CF}\mathcal{D}_t^{\alpha} x(t) &= Ax(t) + F(t,x) + u(t) \\ &= (A+B)x(t) + F(t,x) \end{aligned} \tag{6.36}$$

becomes asymptotically stable. Here,

$$A = \begin{bmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{bmatrix},\ F(t,x) = \begin{bmatrix} 0.5\exp(-7t)x_1 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix},$$

with $\dfrac{\|F(t,x)\|_1}{\|x\|_1} = \dfrac{|0.5\exp(-7t)x_1| + |x_1x_3| + |x_1x_2|}{|x_1|+|x_2|+|x_3|} \leq |x_1| \ \Rightarrow \lim\limits_{x\to 0} \dfrac{\|F(t,x)\|_1}{\|x\|_1} = 0$ *uniformly in* $t \in [0,\infty)$. *Thus,* $F(t,x)$ *satisfies condition* (ii) *of Corollary* 6.3.1. *Next, the following feedback gain matrix is considered:*

$$B = \begin{bmatrix} 5 & -5 & 0 \\ -31 & 0 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}. \tag{6.37}$$

Then, the eigenvalues of the matrix $A+B$ *do not lie inside the disc* $D(\alpha)$ *and all other conditions of Corollary* 6.3.1 *are satisfied. Subsequently, the controlled system* (6.36) *is locally asymptotically stable. The numerical solution of the controlled system* (6.36) *for the fractional-order* $\alpha = 0.98$ *is shown in Figure* 6.4. *Here, the feedback gain matrix* B

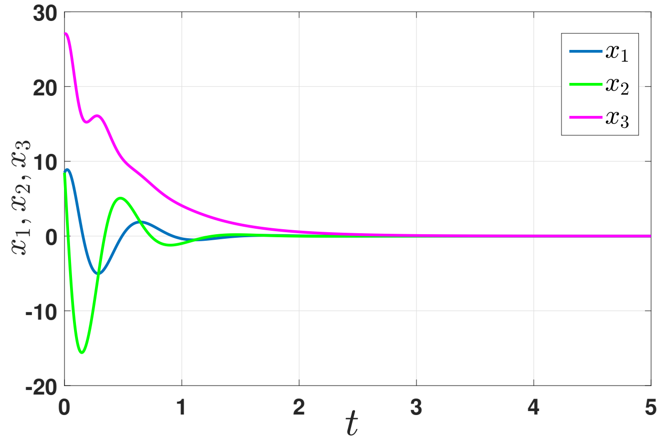

Figure 6.4: Stabilized solution of the fractional-order Lorenz system (6.35) using state feedback controller for fractional-order $\alpha = 0.9997$

is not unique since, depending upon the conditions of Corollary 6.3.1, we may vary the parameters in matrix B.

Example 6.4.3. *Let us consider the Caputo-Fabrizio fractional-order Shimizu-Morioka system [128]*

$$\begin{aligned} {}_0^{CF}\mathcal{D}_t^{\alpha} x_1(t) &= a x_2(t) + 0.001 \sin(2\pi t) x_1(t) x_3(t), \\ {}_0^{CF}\mathcal{D}_t^{\alpha} x_2(t) &= x_1(t)\big(1 - x_2(t)\big) - c x_2(t), \\ {}_0^{CF}\mathcal{D}_t^{\alpha} x_3(t) &= x_1^2(t) - b x_3(t), \end{aligned} \tag{6.38}$$

where $a = 1,\ b = 0.3758,\ c = 0.81$. Here, we perturb the Shimizu-Morioka system by the quantity $0.001 \sin(2\pi t) x_1(t) x_3(t)$ in the first equation in order to obtain a non-autonomous system. Otherwise, we have only a constant solution for the conventional autonomous Shimizu-Morioka system. We consider the fractional-order $\alpha = 0.98$, and the numerical solution is plotted in Figure 6.5 with an initial condition $x(0) = (x_1(0), x_2(0), x_3(0)) = (-0.61237, 0, 1)$. We observe that the system (6.38) exhibits a chaotic behavior.

Next, we try to stabilize this chaotic fractional-order Shimizu-Morioka system (6.38) by introducing a linear state feedback controller $u(t) = Bx(t)$ to it where B is a 3×3 scalar matrix such that the controlled system

$$\begin{aligned} {}_0^{CF}\mathcal{D}_t^{\alpha} x(t) &= Ax(t) + F(t, x) + u(t) \\ &= (A + B)x(t) + F(t, x) \end{aligned} \tag{6.39}$$

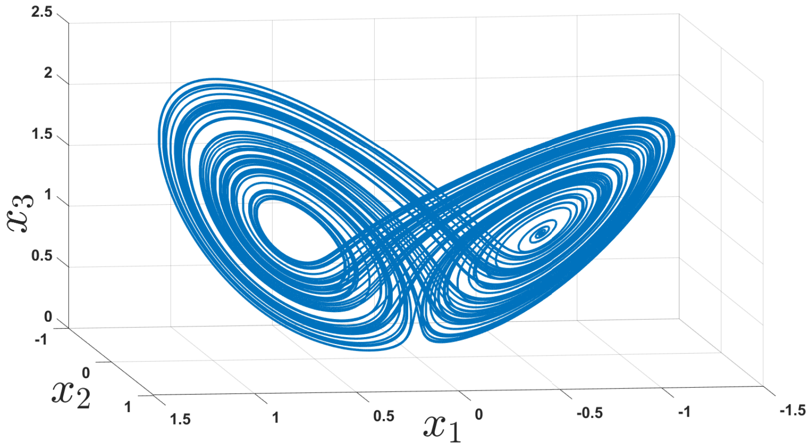

Figure 6.5: Chaotic behavior of the fractional-order Shimizu-Morioka system (6.38) for fractional-order $\alpha = 0.98$ and $a = 1,\ b = 0.3758,\ c = 0.81$

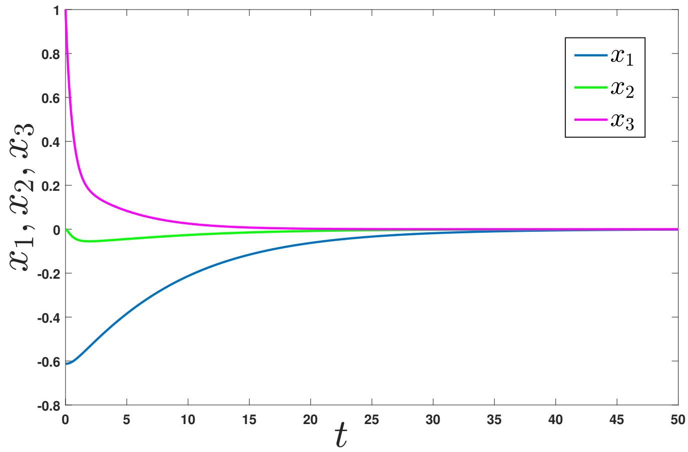

Figure 6.6: Stabilized solution of the fractional-order Shimizu-Morioka system (6.38) using state feedback controller for fractional-order $\alpha = 0.98$

becomes asymptotically stable. Here,

$$A = \begin{bmatrix} 0 & a & 0 \\ 1 & -c & 0 \\ 0 & 0 & -b \end{bmatrix},\ F(t,x) = \begin{bmatrix} 0.001\sin(2\pi t)x_1(t)x_3(t) \\ -x_1(t)x_3(t) \\ x_1^2(t) \end{bmatrix}, \qquad (6.40)$$

with $\dfrac{\|F(t,x)\|_1}{\|x\|_1} = \dfrac{|0.001\sin(2\pi t)x_1x_3| + |x_1(t)x_3(t)| + |x_1^2|}{|x_1| + |x_2| + |x_3|} \leq 2|x_1|$
$\Rightarrow \lim\limits_{x\to 0} \dfrac{\|F(t,x)\|_1}{\|x\|_1} = 0$ *uniformly in* $t \in [0,\infty)$. *Thus,* $F(t,x)$ *satisfies condition* (ii) *of*

Corollary 6.3.1. Then, the following feedback gain matrix is considered:

$$B = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -7.19 & 0 \\ 0 & 0 & -1.6242 \end{bmatrix}.$$

The eigenvalues of the matrix $A + B$ do not lie inside the disc $D(\alpha)$ and all other conditions of Corollary 6.3.1 are satisfied. Hence, the controlled system (6.39) is locally asymptotically stable. The numerical solution of the controlled system (6.39) for the fractional-order $\alpha = 0.98$ is shown in Figure 6.6. Here, the feedback gain matrix B is not unique and one can choose other forms of B in order that the conditions of Corollary 6.3.1 get satisfied.

Example 6.4.4. *Let us consider the following Caputo-Fabrizio fractional-order system:*

$$\begin{aligned} {}^{CF}_{0}\mathcal{D}^{\alpha}_{t} x_1(t) &= 2x_1 + \frac{\cos(t^2)}{5} x_2 - \frac{\sin(2x_1)}{4}, \\ {}^{CF}_{0}\mathcal{D}^{\alpha}_{t} x_2(t) &= 2x_2 - \frac{\cos(2\pi t)}{4} x_1 + \frac{\sin(x_2)}{5}. \end{aligned} \tag{6.41}$$

Here, we take the fractional-order $\alpha = 0.98$, and the numerical solution of the system

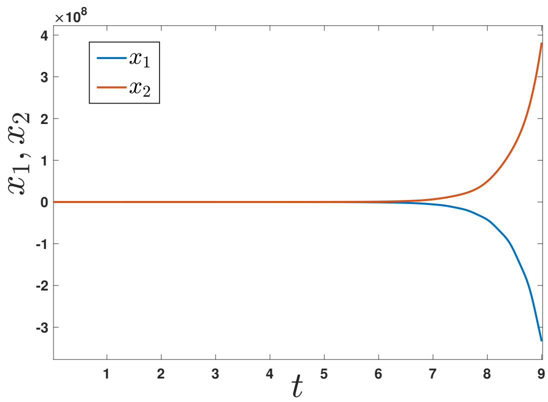

Figure 6.7: Unstable solution of the system (6.41) for fractional-order $\alpha = 0.98$

(6.41) is plotted in Figure 6.7 from which it is observed that the equilibrium point $\bar{v} = 0$ of the system (6.41) is unstable.

Next, we try to stabilize the system (6.41) by introducing a linear state feedback controller $u(t) = Bx(t)$ to it where B is a 3×3 scalar matrix such that the controlled system

$$\begin{aligned} {}^{CF}_{0}\mathcal{D}^{\alpha}_{t} x(t) &= Ax(t) + F(t,x) + u(t) \\ &= (A + B)x(t) + F(t,x) \end{aligned} \tag{6.42}$$

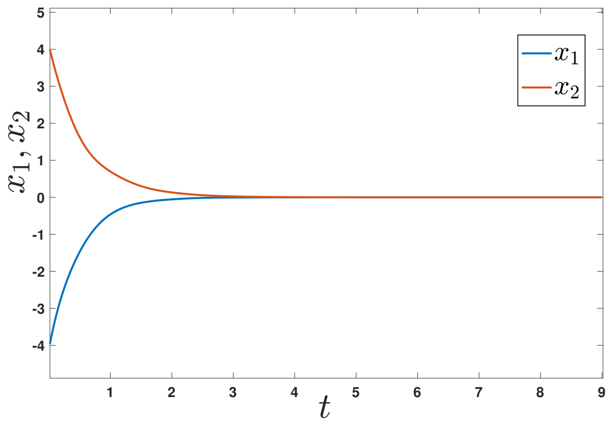

Figure 6.8: Stabilized solution of the fractional-order system (6.41) for fractional-order $\alpha = 0.98$

becomes asymptotically stable. Here,

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \ F(t,x) = \begin{bmatrix} \frac{\cos(t^2)x_2}{5} - \frac{\sin(2x_1)}{4} \\ \frac{\sin(x_2)}{5} - \frac{\cos(2\pi t)x_1}{4} \end{bmatrix},$$

with $F(t,x)$ uniformly globally Lipschitz with a Lipschitz constant $L_B = 1$. Thus, $F(t,x)$ satisfies condition (ii) of Corollary 6.3.2. Next, the following feedback gain matrix is considered:

$$B = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}.$$

Then, the eigenvalues of the matrix $A+B$ are $-2, -2$ which lies outside the disc $D(\alpha)$, $\|N_B^{-1}\|_1 = \frac{25}{26}$ and $c = -\max\{Re(\lambda(N_B^{-1}(A+B)))\} = \frac{25}{13}$. Here, clearly $m = 1$ and we get $c = 2 \times \frac{25}{26} > \frac{mL_B\|N_B^{-1}\|^2}{1-(1-\alpha)L_B\|N_B^{-1}\|} = 0.98 \times \frac{25}{26}$. Thus, all the conditions of Corollary 6.3.2 are satisfied. Subsequently, by Corollary 6.3.2, the controlled system (6.42) is globally asymptotically stable in the sense of Definition 6.2.2. The numerical solution of the controlled system (6.42) is shown in Figure 6.8.

Remark 6.4.1. *We look forward to finding an explicit application that will fit with our proposed problem and examples in future. Under the current objectives, we deal with the theory first and foremost which is considered immensely important before any applications can be found. We are hopeful that the proposed theory may provide useful information to tackle an application in real-time scenario in experiment labs. However, as per our understanding, Examples 6.4.1 and 6.4.2 involving Lorenz system may be applicable in meteorology while studying the small-scale thermal convection in the lithosphere of the Earth. In a similar manner, we believe that the other examples, namely Examples 6.4.3* *and 6.4.4, may find application mainly in electrical engineering in connection with chaos and bifurcation.*

6.5 Conclusions

In this work, by using the properties of Caputo-Fabrizio derivative and the concept of the equilibrium point of the Caputo-Fabrizio fractional-order systems, it has been established that only a constant solution exists for an autonomous Caputo-Fabrizio system with the introduction of a new concept of global solution. By using Laplace transform and Grönwall inequality, we have derived the local and global asymptotic stability conditions for the fractional-order linear and semilinear evolution equations involving Caputo-Fabrizio fractional derivative of fractional-order $\alpha \in (0, 1)$. We have stabilized such systems by constructing a suitable linear feedback control and using our main results. At the end, by using the obtained result, we have stabilized some nonlinear chaotic fractional-order system.

From this work, we have observed that one has to be very careful in choosing an initial condition for the system involving a non-singular kernel operator and the initial data set S_s, where the solution exists, which may not be an open connected set. Since the possible solution set S_{ps} may not equal the whole $\mathbb{R}^n$ and $S_s \subset S_{ps}$, there is no concept of the global solution of the system. From the above finding, although a fractional-order differential operator with a non-singular kernel has several drawbacks, it may be inferred that it is sometimes easier to analyse systems involving a non-singular kernel operator compared to other operators with singular kernels.

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CHAPTER 7

Ulam-Hyers stability of second-order convergent finite difference scheme for first- and second-order nonhomogeneous linear differential equations with constant coefficients

7.1 Introduction

Motivated by the works in [11, 102, 101, 103], we want to study the Ulam-Hyers stability of second-order convergent finite difference equations corresponding to first- and second-order nonhomogeneous linear differential equations with constant coefficients by using previous Ulam-Hyers stability results [114, 113, 115, 15] for the linear recurrence relation. Instead of using some rigorous mathematical analysis to prove the Ulam-Hyers stability for a difference equation, we can easily prove the stability by studying the behavior of the roots of the characteristic polynomial of the equivalent recurrence relation of the corresponding difference equations. Therefore, first we convert the difference equation to its equivalent recurrence relation, and then by studying the location of the roots of the characteristic polynomial associated with the equivalent recurrence relation, we establish the Ulam-Hyers stability result for the difference equation. Hence, in this work, we proceed to examine the Ulam-Hyers stability of the second-order convergent difference equations corresponding to the following first- and second-order non-homogeneous linear differential equations:

$$x'(t) - bx(t) = f(t), \quad t \in I = [a, \infty), \tag{7.1}$$

$$x''(t) + \alpha x'(t) + \beta x(t) = g(t), \quad t \in I, \tag{7.2}$$

where $a, b, \alpha, \beta \in \mathbb{R}$ and $f, g : I \to \mathbb{R}$ are given functions.

7.2 Preliminaries

Before we proceed, we state some theorems and results concerning Ulam-Hyers stability for the linear recurrence equation of p-order. Consider the p-order linear recurrence equation with constant coefficients as

$$x_{n+p} = c_p x_{n+p-1} + c_{p-1} x_{n+p-2} + \cdots + c_2 x_{n+1} + c_1 x_n + d_n, \; n \geq 0, \tag{7.3}$$

where $(d_n)_{n\geq 0}$ is a sequence of real numbers $x_0, x_1, \ldots, x_{p-1}$.

Definition 7.2.1. *[113] The p-order linear recurrence equation (7.3) is said to be Ulam-Hyers stable if, for a given arbitrary $\varepsilon > 0$ and a sequence $(x_n)_{n\geq 0}$ of real numbers satisfying*

$$|x_{n+p} - c_p x_{n+p-1} - c_{p-1} x_{n+p-2} - \cdots - c_2 x_{n+1} - c_1 x_n - d_n| \leq \varepsilon, \quad n \geq 0,$$

there exist a constant $L > 0$ and a sequence $(y_n)_{n\geq 0}$ of real numbers satisfying the recurrence relation

$$y_{n+p} = c_p y_{n+p-1} + c_{p-1} y_{n+p-2} + \cdots + c_2 y_{n+1} + c_1 y_n + d_n, \; n \geq 0$$

such that

$$|x_n - y_n| \leq L\varepsilon, \quad n \geq 0.$$

The constant L is called a Ulam-Hyers constant of equation (7.3). Let us denote by L_U the infimum of all such Ulam-Hyers constants L, but in general the infimum of all the Ulam-Hyers constants of an equation may not be an Ulam-Hyers constant of that equation. So, if L_U exists and it is a Ulam-Hyers constant of an equation, then we call L_U the *best Ulam-Hyers constant.* There is a reasonable amount of literature available for finding the Ulam-Hyers constant for various types of problems but only few of them deal with finding the *best Ulam-Hyers constant L_U*. Now we state an important theorem dealing with the stability of the linear recurrence in the Ulam-Hyers sense.

Theorem 7.2.1. *[23] The p-order linear recurrence equation (7.3) is Ulam-Hyers stable if and only if the roots of the characteristic polynomial*

$$\chi^p - c_p \chi^{p-1} - \cdots - c_2 \chi - c_1 = 0$$

do not lie on the unit circle $S = \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}$.

Theorem 7.2.2. *[113] Let $\chi_1, \chi_2, \ldots, \chi_p$ be the roots of the characteristic polynomial*

$$\chi^p - c_p \chi^{p-1} - \cdots - c_2 \chi - c_1 = 0$$

with $|\chi_i| \neq 1, 1 \leq i \leq p$. Then the p-order linear recurrence equation (7.3) is Ulam-Hyers stable, and in this case, the Ulam-Hyers constant is given by $L = \frac{1}{|(|\chi_1|-1)\cdots(|\chi_p|-1)|}$. Moreover, if $|\chi_i| > 1, \quad \forall i = 1, 2, \ldots, p$, then the sequence $(y_n)_{n\geq 0}$ of Definition 7.2.1 is unique and if $|\chi_i| < 1$, then one can find an infinite number of sequences $(y_n)_{n\geq 0}$. Indeed, one can choose $y_k = x_k$, for all $0 \leq k \leq p-1$, that satisfies Definition 7.2.1.

Theorem 7.2.3. *[16] If the roots of the characteristic polynomial associated with the recurrence relation (7.3) satisfy $\chi_i \in \mathbb{R}$, $\chi_i > 1$ for $1 \leq i \leq p$, then the best Ulam-Hyers constant of the recurrence relation (7.3) is given by*

$$L_U = \frac{1}{\prod_{i=1}^{p}(\chi_i - 1)}.$$

Remark 7.2.1. *[16] For $p = 2$, if the roots of the second-order linear recurrence relation $x_{n+2} = c_2 x_{n+1} + c_1 x_n + d_n, n \geq 0$, are distinct and satisfy $|\chi_1| > 1, |\chi_2| > 1$, then the best Ulam-Hyers constant in this particular case is given by*

$$L_U = \frac{1}{|\chi_1 - \chi_2|} \sum_{n=1}^{\infty} \left| \frac{1}{\chi_1^n} - \frac{1}{\chi_2^n} \right|.$$

7.3 Main Results

7.3.1 Ulam-Hyers stability for difference equation: first-order ODE

We want to study the Ulam-Hyers stability of the difference equation corresponding to the homogeneous first-order linear differential equation (7.1). Here, we want to establish the Ulam-Hyers stability of the following second-order convergent difference equation corresponding to the first-order differential equation (7.1):

$$\hat{\Delta}_h x(t) - bx(t) = f(t), \quad t \in \mathbb{T}, \tag{7.4}$$

where $\hat{\Delta}_h x(t) = \dfrac{-x(t+2h) + 4x(t+h) - 3x(t)}{2h}$ is the second-order approximation to $x'(t)$, $\mathbb{T} = \left\{ a + nh \;\middle|\; n \in \mathbb{N}_0 \right\}$, $a \in \mathbb{R}$ is an initial point, $h > 0$ is a step size of discretization and at the n-th node $t_n = a + nh \in \mathbb{T}$, we denote $x(t_n) = x_n$, $f(t_n) = f_n$ for all $n \geq 0$. The above finite difference equation (7.4) can be written in its equivalent linear recurrence relation as

$$x_{n+2} = 4x_{n+1} - (3 + 2hb)x_n - 2hf_n, \quad n \geq 0. \tag{7.5}$$

If $b = -\frac{3}{2h}$, then the order of the recurrence relation reduces to 1 and hence, it does not satisfy the finite difference relation (7.4), and we must have $b \neq -\frac{3}{2h}$.

Definition 7.3.1. *[102] The finite difference equation (7.4) is said to be Ulam-Hyers stable if, for a given arbitrary $\varepsilon > 0$ and a function $x : \mathbb{T} \to \mathbb{R}$ satisfying*

$$|\hat{\Delta}_h x(t) - bx(t) - f(t)| \leq \varepsilon, \quad \forall\, t \in \mathbb{T},$$

there exist a constant $L > 0$ and a solution $y : \mathbb{T} \to \mathbb{R}$ of (7.4) such that

$$|x(t) - y(t)| \leq L\varepsilon, \quad \forall\, t \in \mathbb{T}.$$

Theorem 7.3.1. *Let $\varepsilon > 0$ be a given arbitrary number. Suppose a function $x : \mathbb{T} \to \mathbb{R}$ satisfies the finite difference inequality*

$$|\hat{\Delta}_h x(t) - bx(t) - f(t)| \leq \varepsilon, \quad t \in \mathbb{T}, \tag{7.6}$$

where $b \neq -\frac{3}{2h}$. Then, one of the following holds:

(i) If $0 < b < \frac{1}{2h}$, then there exists a unique solution $y : \mathbb{T} \to \mathbb{R}$ of the finite difference equation (7.4) such that

$$|x(t) - y(t)| \leq \frac{\varepsilon}{b}, \quad \forall t \in \mathbb{T}. \tag{7.7}$$

In this case, $L_U = \frac{1}{b}$ is the best Ulam-Hyers constant for equation (7.4).

(ii) If $b > \frac{1}{2h}$, then there exists a unique solution $y : \mathbb{T} \to \mathbb{R}$ of the finite difference equation (7.4) such that

$$|x(t) - y(t)| \leq L_U\varepsilon, \quad \forall t \in \mathbb{T}, \tag{7.8}$$

where $L_U = 2h \sum_{n=0}^{\infty} \Big|U_n(\cos\theta)\Big| \frac{1}{r^{n+2}}$, $U_n(x)$ are Chebyshev polynomials of the second kind, $r = \sqrt{3 + 2bh}$ and $\theta = \tan^{-1}\left(\frac{\sqrt{2bh-1}}{2}\right)$. In this case, this L_U is the best Ulam-Hyers constant for equation (7.4).

(iii) If $-\frac{3}{2h} < b < 0$, then any solution $y : \mathbb{T} \to \mathbb{R}$ of (7.4) with initial value $|x(a) - y(a)| \leq \dfrac{\varepsilon}{|b|}$ satisfies

$$|x(t) - y(t)| \leq \frac{\varepsilon}{|b|}, \quad \forall\, t \in \mathbb{T}. \tag{7.9}$$

(iv) If $-\frac{4}{h} < b < -\frac{3}{2h}$, then any solution $y : \mathbb{T} \to \mathbb{R}$ with the initial value $|x(a) - y(a)| \leq \dfrac{h\varepsilon}{1 + ah + \sqrt{1 - 2bh}}$ satisfies

$$|x(t) - y(t)| \leq \frac{h\varepsilon}{1 + ah + \sqrt{1 - 2bh}}, \quad \forall\, t \in \mathbb{T}. \tag{7.10}$$

(v) *If* $b < -\frac{4}{h}$*, then there exists a unique solution* $y : \mathbb{T} \to \mathbb{R}$ *of the finite difference equation* (7.4) *such that*

$$|x(t) - y(t)| \leq L_U \varepsilon, \quad \forall\, t \in \mathbb{T}. \tag{7.11}$$

Here, $L_U = \frac{h}{\sqrt{1-2bh}} \sum_{n=1}^{\infty} \left| \left(2 + \sqrt{1-2bh}\right)^{-n} - \left(2 - \sqrt{1-2bh}\right)^{-n} \right|$ *is the best Ulam-Hyers constant for equation* (7.4).

Proof. Let $\varepsilon > 0$ be arbitrarily given and suppose the function $x : \mathbb{T} \to \mathbb{R}$ satisfies

$$|\hat{\Delta}_h x(t) - bx(t) - f(t)| \leq \varepsilon, \quad t \in \mathbb{T}. \tag{7.12}$$

At the n-th node $t_n = a + nh \in \mathbb{T}$, by denoting $x(t_n) = x_n,\ f(t_n) = f_n$, we get

$$|x_{n+2} - 4x_{n+1} + (3 + 2bh)x_n + 2hf_n| \leq 2h\varepsilon, \quad \forall n \geq 0. \tag{7.13}$$

Now we wish to study the Ulam-Hyers stability of the equivalent linear recurrence equation (7.5) in connection with inequality (7.13). Let r_1 and r_2 be the roots of the characteristic polynomial of the recurrence equation (7.5). Then,

$$r_1 = 2 + \sqrt{1 - 2bh}, \quad r_2 = 2 - \sqrt{1 - 2bh}. \tag{7.14}$$

(i) If $0 < b < \frac{1}{2h}$, both roots are positive with $r_1 > 1,\ r_2 > 1$. Then, by Theorem 7.2.2, there exists a unique solution $(y_n)_{n \geq 0}$ of the recurrence equation (7.5) such that

$$|x_n - y_n| \leq \frac{2h\varepsilon}{(r_1 - 1)(r_2 - 1)} = \frac{\varepsilon}{b}, \quad n \geq 0. \tag{7.15}$$

Now, define a function $y : \mathbb{T} \to \mathbb{R}$ such that at the n-th node $t_n = a + nh \in \mathbb{T}$, we denote $y(t_n) = y_n,\ n \geq 0$. Then, clearly y is a unique solution of the finite difference equation (7.4) such that

$$|x(t) - y(t)| \leq \frac{\varepsilon}{b}, \quad \forall\, t \in \mathbb{T}. \tag{7.16}$$

By Theorem 7.2.3, it follows that $L_U = \dfrac{1}{b}$ is the best Ulam-Hyers constant for equation (7.4).

(ii) If $b > \frac{1}{2h}$, then the roots are complex with magnitude $|r_1| = |r_2| = r = \sqrt{3 + 2bh} > 1$. By using Theorem 7.2.2, Theorem 7.2.3 and following a similar argument, there exists a unique solution $y : \mathbb{T} \to \mathbb{R}$ of the finite difference equation (7.4) with the best Ulam-Hyers constant $L_U = \dfrac{2h}{|r_1 - r_2|} \sum_{n=1}^{\infty} \left| \dfrac{1}{r_1^n} - \dfrac{1}{r_2^n} \right|$ such that

$$|x(t) - y(t)| \leq L_U \varepsilon, \quad \forall\, t \in \mathbb{T}. \tag{7.17}$$

Since $r_1 = re^{i\theta},\ r_2 = re^{-i\theta}$, where $\theta = \tan^{-1}\left(\frac{\sqrt{2bh-1}}{2}\right)$, L_U can be written as

$$\begin{aligned}
L_U &= \frac{h}{\sqrt{2bh-1}} \sum_{n=1}^{\infty} |e^{-ni\theta} - e^{in\theta}| \frac{1}{r^n} \\
&= \frac{2h}{\sqrt{2bh-1}} \sum_{n=1}^{\infty} \left| \frac{e^{in\theta} - e^{-in\theta}}{2i} \right| \frac{1}{r^n} = \frac{2h}{\sqrt{2bh-1}} \sum_{n=1}^{\infty} |\sin(n\theta)| \frac{1}{r^n} \\
&= \frac{2h\sin\theta}{\sqrt{2bh-1}} \sum_{n=1}^{\infty} \left| \frac{\sin(n\theta)}{\sin\theta} \right| \frac{1}{r^n} = 2h \sum_{n=0}^{\infty} \left| \frac{\sin(n+1)\theta}{\sin\theta} \right| \frac{1}{r^{n+2}}, \\
&= 2h \sum_{n=0}^{\infty} \left| U_n(\cos\theta) \right| \frac{1}{r^{n+2}}, \quad \left(\text{since} \quad \sin\theta = \frac{\sqrt{2bh-1}}{r}\right).
\end{aligned} \tag{7.18}$$

(iii) If $-\frac{3}{2h} < b < 0$, then both roots are positive with $r_1 > 1,\ 0 < r_2 < 1$. By Theorem 7.2.2, any solution $y : \mathbb{T} \to \mathbb{R}$ with initial value $|x(a) - y(a)| \le \frac{2h\varepsilon}{(r_1-1)(1-r_2)} = \frac{\varepsilon}{|b|}$ satisfies

$$|x(t) - y(t)| \le \frac{\varepsilon}{|b|}, \quad \forall t \ \in \mathbb{T}. \tag{7.19}$$

By following a similar argument, and using Theorem 7.2.2 and Remark 7.2.1, we can establish (iv) and (v). □

Corollary 7.3.1. *If $b = 0, -\frac{4}{h}$, then one of the roots lies on the unit circle and hence, by Theorem 7.2.1, the finite difference equation (7.4) is not Ulam-Hyers stable for $b = 0, -\frac{4}{h}$.*

Lemma 7.3.1. *Let $b > \frac{1}{2h}$. Then the finite difference equation (7.4) is Ulam-Hyers stable and the best Ulam-Hyers constant for (7.4) on $\mathbb{T}$ is at least $\dfrac{1}{b}$.*

Proof. From Theorem 7.3.1 (ii), the best Ulam-Hyers constant in this case is given by $L_U = 2h \sum_{n=0}^{\infty} \left| U_n(\cos\theta) \right| \frac{1}{r^{n+2}}$, and we know that the generating function relation for the Chebyshev polynomials of second kind $U_n(x)$ is given by

$$\sum_{n=0}^{\infty} U_n(x) t^n = \frac{1}{1 - 2xt + t^2}. \tag{7.20}$$

Using this relation, we get

$$\begin{aligned}
L_U &\ge \left| 2h \sum_{n=0}^{\infty} U_n(\cos\theta) \frac{1}{r^{n+2}} \right| = \frac{2h}{r^2} \left| \sum_{n=0}^{\infty} U_n(\cos\theta) \frac{1}{r^n} \right| \\
&\ge \frac{2h}{r^2 - 2r\cos\theta + 1} = \frac{2h}{r^2 - 3} = \frac{1}{b}, \quad \left(\text{since} \quad \cos\theta = \frac{2}{r},\ r^2 = 3 + 2bh\right).
\end{aligned}$$

This shows that the best Ulam-Hyers constant is at least $\dfrac{1}{b}$. □

7.3.2 Ulam-Hyers stability for difference equation: second-order ODE

We want to study the Ulam-Hyers stability of the finite difference equation corresponding to the nonhomogeneous second-order linear differential equation (7.2). Here, based on the idea adopted in [101], we want to establish the Ulam-Hyers stability of the following second-order convergent central finite deference equation corresponding to the second-order differential equation (7.2):

$$\delta_h^2 x(t) + \alpha\Delta_h^c x(t) + \beta x(t) = g(t), \qquad \forall\, t \in \mathbb{T}, \tag{7.21}$$

where

$$\delta_h^2 x(t) = \frac{x(t+h) - 2x(t) + x(t-h)}{h^2}, \quad \Delta_h^c x(t) = \frac{x(t+h) - x(t-h)}{2h}, \tag{7.22}$$

with $\alpha, \beta \in \mathbb{R}$, and $\mathbb{T} = \{a + nh \mid n \in \mathbb{N}_0\}$ a discretized domain for a given step size of discretization $h > 0$ and $g : \mathbb{T} \to \mathbb{R}$ is a given function.
At the n-th node $t_n = a + nh \in \mathbb{T}$, let us denote $x(t_n) = x_n$ and $g(t_n) = g_n$. Then, the above finite difference equation (7.21) can be written in its equivalent second-order linear recurrence relation as

$$\left(1 + \frac{\alpha h}{2}\right)x_{n+1} - (2 - \beta h^2)x_n + \left(1 - \frac{\alpha h}{2}\right)x_{n-1} = h^2 g_n, \quad n \geq 1. \tag{7.23}$$

If $\alpha \in \Lambda$, where $\Lambda = \{-\frac{2}{h}, \frac{2}{h}\}$, then we cannot execute the recurrence relation uniquely because the order of the recurrence relation reduces to 1 but the differential equation is of second-order and we are given two initial inputs. Therefore, in order to execute the recurrence relation (7.23), we must have $\alpha \notin \Lambda$ and the initial data x_0, x_1.

The finite difference scheme (7.21) is equivalent to the linear recurrence relation (7.23) of second-order in the sense that, if the sequence $(x_n)_{n\geq 0}$ satisfies the recurrence relation (7.23), then $x(t_n) = x_n$ satisfies the finite difference equation (7.21) for all $t \in \mathbb{T}$. Hence, we want to study the Ulam-Hyers stability of the linear recurrence relation (7.23) from which we can conclude the stability of the finite difference equation (7.21) in the Ulam-Hyers sense.

Definition 7.3.2. *The finite difference equation (7.21) is said to be Ulam-Hyers stable if, for a given arbitrary $\varepsilon > 0$ and a function $x : \mathbb{T} \to \mathbb{R}$ satisfying*

$$|\delta_h^2 x(t) + \alpha\Delta_h^c x(t) + \beta x(t) - g(t)| \leq \varepsilon, \quad \forall\, t \in \mathbb{T}, \tag{7.24}$$

there exist a constant $L > 0$ and a solution $y : \mathbb{T} \to \mathbb{R}$ of equation (7.21) such that

$$|x(t) - y(t)| \leq L\varepsilon, \quad \forall\, t \in \mathbb{T}. \tag{7.25}$$

Since the finite difference scheme (7.21) is equivalent to the linear recurrence relation (7.23), hence in order to prove the stability result for the finite difference scheme (7.21), we study the Ulam-Hyers stability for the linear recurrence (7.23). Define the region in the $(\alpha, \beta)-$plane as

$$R_I = \left\{ (\alpha,\beta) \in \mathbb{R}^2 \;\middle|\; \alpha \notin \Lambda,\ \beta > \frac{4}{h^2} \right\},$$
$$R_{II} = \left\{ (\alpha,\beta) \in \mathbb{R}^2 \;\middle|\; \alpha \notin \Lambda \cup \{0\},\ 0 < \beta < \frac{4}{h^2} \right\},$$
$$R_{III} = \left\{ (\alpha,\beta) \in \mathbb{R}^2 \;\middle|\; \alpha \notin \Lambda,\ \beta < 0 \right\}.$$

Clearly $R_I \cap R_{II} = \emptyset$, $R_{II} \cap R_{III} = \emptyset$ and $R_I \cap R_{III} = \emptyset$.

Theorem 7.3.2. *If $(\alpha, \beta) \in R_I \cup R_{III}$, then the finite difference equation (7.21) is Ulam-Hyers stable on $\mathbb{T}$. Further, one of the following holds:*

(i) If $(\alpha, \beta) \in R_I \cup R_{III}$ with $|\alpha| > \frac{2}{h}$, then the Ulam-Hyers constant for (7.21) on $\mathbb{T}$ is $\dfrac{|1+\frac{\alpha h}{2}|}{\sqrt{\frac{\alpha^2-4\beta}{h^2}+\beta^2} - \frac{|\alpha|}{h}}$.

(ii) If $(\alpha, \beta) \in R_I$ with $|\alpha| < \frac{2}{h}$, then the Ulam-Hyers constant for (7.21) on $\mathbb{T}$ is $\dfrac{1+\frac{\alpha h}{2}}{\beta - \frac{4}{h^2}}$.

(iii) If $(\alpha, \beta) \in R_{III}$ with $|\alpha| < \frac{2}{h}$, then the Ulam-Hyers constant for (7.21) on $\mathbb{T}$ is $\dfrac{1+\frac{\alpha h}{2}}{|\beta|}$.

Proof. Let $\varepsilon > 0$ be arbitrarily given and suppose the function $x : \mathbb{T} \to \mathbb{R}$ satisfies

$$|\delta_h^2 x(t) + \alpha \Delta_h^c x(t) + \beta x(t) - g(t)| \le \varepsilon, \quad t \in \mathbb{T}. \tag{7.26}$$

At the n-th node $t_n = a + nh \in \mathbb{T}$, we denote $x(t_n) = x_n,\ g(t_n) = g_n$. Then, we get

$$\left| \left(1 + \frac{\alpha h}{2}\right) x_{n+1} - (2 - \beta h^2) x_n + \left(1 - \frac{\alpha h}{2}\right) x_{n-1} - h^2 g_n \right| \le h^2 \varepsilon, \quad \forall\, n \ge 1. \tag{7.27}$$

Now we wish to study the Ulam-Hyers stability of the equivalent linear recurrence equation (7.23) in connection with inequality (7.27). Let r_1 and r_2 be the roots of the characteristic polynomial of the recurrence equation (7.23). Then,

$$r_1 = \frac{2 - \beta h^2 + \sqrt{(2-\beta h^2)^2 + \alpha^2 h^2 - 4}}{2 + \alpha h}, r_2 = \frac{2 - \beta h^2 - \sqrt{(2-\beta h^2)^2 + \alpha^2 h^2 - 4}}{2 + \alpha h}. \tag{7.28}$$

We know from the theory of polynomials that

$$r_1 + r_2 = \frac{2 - \beta h^2}{\left(1 + \frac{\alpha h}{2}\right)}, \quad r_1 r_2 = \frac{2 - \alpha h}{2 + \alpha h}. \tag{7.29}$$

If $(\alpha, \beta) \in R_I \cup R_{III}$, then the roots are real with $r_1 \neq 1,\ r_2 \neq 1$. Hence, by Theorem 7.2.2, the linear recurrence relation (7.23) is Ulam-Hyers stable with Ulam-Hyers constant $L = \frac{1}{|(|r_1|-1)(|r_2|-1)|}$, i.e., there exists a sequence $(y_n)_{n\geq 0}$ which is a solution of the linear recurrence (7.23) such that

$$|x_n - y_n| \leq \frac{h^2 \varepsilon}{|(|r_1| - 1)(|r_2| - 1)|}, \tag{7.30}$$

where $(x_n)_{n\geq 0}$ is a solution of the inequality (7.27).
If we denote $y(t_n) = y_n$, then from the inequality (7.30), the finite difference scheme (7.21) is also Ulam-Hyers stable with Ulam-Hyers constant $L = \frac{h^2}{|(|r_1|-1)(|r_2|-1)|}$.
Now we find out the exact form of L.

(i) If $(\alpha, \beta) \in R_I \cup R_{III}$ with $\beta < 0$ and $\alpha < -\frac{2}{h}$, then $r_1 < -1$ and $0 < r_2 < 1$ which implies $L = \frac{|1+\frac{\alpha h}{2}|}{\sqrt{\frac{\alpha^2-4\beta}{h^2}+\beta^2-\frac{|\alpha|}{h}}}$. On the other hand, if $\alpha > \frac{2}{h}$, then $r_1 > 1$ and $-1 < r_2 < 0$. Hence in this case also, we get the same constant. Combining these two cases, we conclude that, for $(\alpha, \beta) \in R_I \cup R_{III}$, with $\beta < 0$ and $|\alpha| > \frac{2}{h}$, the Ulam-Hyers constant is given by $L = \frac{|1+\frac{\alpha h}{2}|}{\sqrt{\frac{\alpha^2-4\beta}{h^2}+\beta^2-\frac{|\alpha|}{h}}}$. Thus, there exists a solution $y : \mathbb{T} \to \mathbb{R}$, where $y(t_n) = y_n$. From inequality (7.30), we have

$$|x(t) - y(t)| \leq \frac{|1 + \frac{\alpha h}{2}|\, \varepsilon}{\sqrt{\frac{\alpha^2-4\beta}{h^2} + \beta^2 - \frac{|\alpha|}{h}}}, \qquad \forall\, t \in \mathbb{T}. \tag{7.31}$$

Similarly, the other case $(\alpha, \beta) \in R_I \cup R_{III}$, with $\beta > \frac{4}{h^2}$ and $|\alpha| > \frac{2}{h}$, can be studied depending upon the location of the characteristic roots.

(ii) If $(\alpha, \beta) \in R_I$ with $|\alpha| < \frac{2}{h}$ and $\beta > \frac{4}{h^2}$, we have $-1 < r_1 < 0$ and $r_2 < -1$. Hence, in this case, the Ulam-Hyers constant for the finite difference equation (7.21) is given by $L = \frac{1+\frac{\alpha h}{2}}{\beta - \frac{4}{h^2}}$.

(iii) Lastly, if $(\alpha, \beta) \in R_{III}$, with $|\alpha| < \frac{2}{h}$ and $\beta < 0$, then $r_1 > 1,\ 0 < r_2 < 1$. Hence, in this case, the Ulam-Hyers constant for the finite difference equation (7.21) is given by $L = \frac{1+\frac{\alpha h}{2}}{|\beta|}$. □

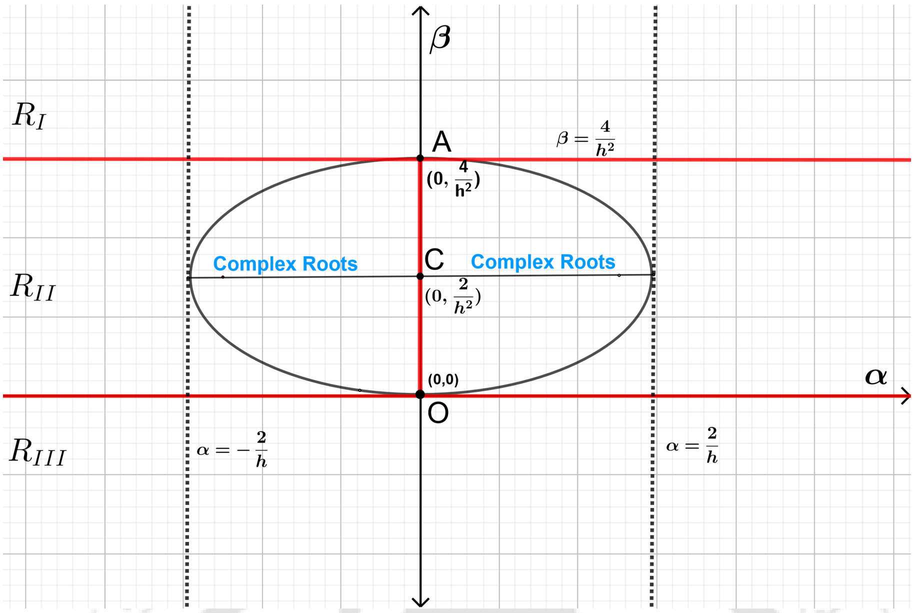


Figure 7.1: If $(\alpha, \beta) \in R_I \cup R_{II} \cup R_{III}$, then (7.21) has Ulam-Hyers stability on $\mathbb{T}$, and if (α, β) lies along the thick (red) lines, then the finite difference equation (7.21) is not Ulam-Hyers stable.

Now we define an elliptic disk $E = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \alpha \neq 0,\ \frac{\alpha^2}{\left(\frac{2}{h}\right)^2} + \frac{\left(\beta - \frac{2}{h^2}\right)^2}{\left(\frac{2}{h^2}\right)^2} < 1 \right\}$ lying inside the region R_{II}, i.e., $E \subset R_{II}$. Note that, if (α, β) lies inside the elliptic disk E, then the roots of the characteristic polynomial are complex. The entire two-dimensional $(\alpha, \beta)-$plane is divided into three infinite strip regions R_I, R_{II} and R_{III} except the thick (red) lines and the dotted lines. In Figure 7.1, we show the different regions R_I, R_{II}, R_{III} and E in the $\alpha\beta-$plane.

Theorem 7.3.3. *If $(\alpha, \beta) \in R_{II}$, then the finite difference equation (7.21) is Ulam-Hyers stable on $\mathbb{T}$. Further, one of the following holds:*

(i) If $(\alpha, \beta) \in E$, then the Ulam-Hyers constant for (7.21) on $\mathbb{T}$ is $\frac{1 + \frac{\alpha h}{2}}{\frac{2}{h^2} - \sqrt{\frac{4}{h^4} - \frac{\alpha^2}{h^2}}}$.

(ii) If $(\alpha, \beta) \in R_{II} \setminus E$ with $|\alpha| < \frac{2}{h}$ and $0 < \beta < \frac{2}{h^2}$, then the Ulam-Hyers constant for (7.21) on $\mathbb{T}$ is $\dfrac{1 + \frac{\alpha h}{2}}{\beta}$.

(iii) If $(\alpha, \beta) \in R_{II} \setminus E$ with $|\alpha| < \frac{2}{h}$ and $\frac{2}{h^2} < \beta < \frac{4}{h^2}$, then the Ulam-Hyers constant for (7.21) on $\mathbb{T}$ is $\dfrac{1 + \frac{\alpha h}{2}}{\frac{4}{h^2} - \beta}$.

(iv) If $(\alpha, \beta) \in R_{II} \setminus E$ with $|\alpha| > \frac{2}{h}$ and $0 < \beta < \frac{4}{h^2}$, then the Ulam-Hyers constant for (7.21) on $\mathbb{T}$ is $\frac{|1 + \frac{\alpha h}{2}|}{\frac{|\alpha|}{h} - \sqrt{\frac{\alpha^2 - 4\beta}{h^2} + \beta^2}}$.

Proof. If $(\alpha, \beta) \in R_{III}$, then $|r_1| \neq 1,\ |r_2| \neq 1$. Hence, by Theorem 7.2.2, the linear recurrence relation (7.23) is Ulam-Hyers stable with Ulam-Hyers constant $L = \frac{1}{|(|r_1|-1)(|r_2|-1)|}$, i.e., there exists a sequence $(y_n)_{n\geq 0}$ which is a solution of the linear recurrence relation (7.23) such that

$$|x_n - y_n| \leq \frac{h^2\varepsilon}{|(|r_1|-1)(|r_2|-1)|}, \tag{7.32}$$

where $(x_n)_{n\geq 0}$ is a solution of the inequality (7.27).
If we denote $y(t_n) = y_n$, then from the inequality (7.30), the finite difference equation (7.21) is also Ulam-Hyers stable with Ulam-Hyers constant $L = \frac{h^2}{|(|r_1|-1)(|r_2|-1)|}$. Hence, the first part of the theorem is proved. Now, we find out the exact form of L.
(i) If $(\alpha, \beta) \in E$, then the roots r_1 and r_2 of the characteristic polynomial of linear recurrence relation (7.23) are complex with equal magnitude $|r_1| = |r_2| = \sqrt{\frac{2-\alpha h}{2+\alpha h}} \neq 1$ on E. Moreover, $|r_1| = |r_2| > 1$ on the left half of the elliptic disk E, i.e., for $(\alpha, \beta) \in E$ with $-\frac{2}{h} < \alpha < 0$, and on the right half $\left((\alpha, \beta) \in E \text{ with } 0 < \alpha < \frac{2}{h}\right)$, we have $|r_1| = |r_2| < 1$. Hence, in this case, the Ulam-Hyers constant for the finite difference equation (7.21) on $\mathbb{T}$ is given by $L = \frac{1+\frac{\alpha h}{2}}{\frac{2}{h^2}-\sqrt{\frac{4}{h^4}-\frac{\alpha^2}{h^2}}}$.
(ii) If $(\alpha, \beta) \in R_{II} \setminus E$ with $-\frac{2}{h} < \alpha < 0$ and $0 < \beta < \frac{2}{h^2}$, then $r_1 > 1,\ r_2 > 1$. On the other side, if $0 < \alpha < \frac{2}{h}$, then $0 < r_1 < 1,\ 0 < r_2 < 1$. Hence, combining these two cases, we have, for $(\alpha, \beta) \in R_{II} \setminus E$ with $|\alpha| < \frac{2}{h}$ and $0 < \beta < \frac{2}{h^2}$, the Ulam-Hyers constant as $L = \frac{1+\frac{\alpha h}{2}}{\beta}$.
(iii) If $(\alpha, \beta) \in R_{II} \setminus E$ with $-\frac{2}{h} < \alpha < 0$ and $\frac{2}{h^2} < \beta < \frac{4}{h^2}$, then $r_1 < -1,\ r_2 < -1$. On the other side, if $0 < \alpha < \frac{2}{h}$, then $-1 < r_1 < 0,\ -1 < r_2 < 0$. Hence, combining these two cases, we have, for $(\alpha, \beta) \in R_{II} \setminus E$ with $|\alpha| < \frac{2}{h}$ and $\frac{2}{h^2} < \beta < \frac{4}{h^2}$, the Ulam-Hyers constant as $L = \frac{1+\frac{\alpha h}{2}}{\frac{4}{h^2}-\beta}$.
In a similar manner, (iv) can be proved by examining the behavior of the roots. □

Remark 7.3.1. *In the $\alpha\beta-$plane, if $(\alpha, \beta) \in \mathbb{R}^2$ except along the thick (red) lines and the dotted lines in Figure 7.2, i.e., $(\alpha, \beta) \in R_I \cup R_{II} \cup R_{III}$, then (7.21) is Ulam-Hyers stable on $\mathbb{T}$. Let $L(h)$ denote the Ulam-Hyers constant. Then,*

$$L(h) = \frac{h^2}{|(|r_1|-1)(|r_2|-1)|}. \tag{7.33}$$

Remark 7.3.2. *If (α, β) belongs to the left half of the region R_{II}, i.e., $(\alpha, \beta) \in R_{II}$ with $\alpha < 0$ and $0 < \beta < \frac{4}{h^2}$ (this region is shaded in Figure 7.2), then in this case, we have $|r_1| > 1,\ |r_2| > 1$. Thus, by Theorem 7.2.3, there exists exactly one solution $y : \mathbb{T} \to \mathbb{R}$ of finite difference equation (7.21) such that*

$$|x(t) - y(t)| \leq \frac{h^2\varepsilon}{|(|r_1|-1)(|r_2|-1)|}, \quad \forall\, t \in \mathbb{T}, \tag{7.34}$$

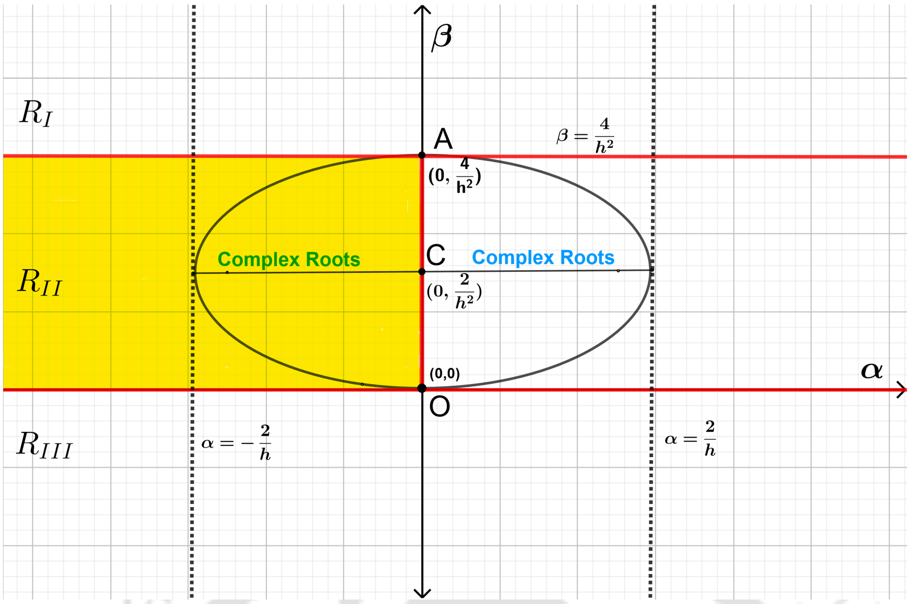


Figure 7.2: If (α, β) lies in the shaded region on the left half of region R_{II} except the thick (red) and the dotted lines, then there exists a unique solution of linear finite difference equation (7.21) that satisfies (7.34). Along thick (red) lines, (7.21) is not Ulam-Hyers stable.

where $x(t)$ satisfies the inequality (7.26). In fact, if the roots are distinct and belong to this yellow-shaded region, then by Remark 7.2.1, the best Ulam-Hyers constant L_U for finite difference equation (7.21) is given by $L_U = \dfrac{h^2}{|r_1 - r_2|} \sum_{k=1}^{\infty} \left| \dfrac{1}{r_1^k} - \dfrac{1}{r_2^k} \right|$.

Corollary 7.3.2. *If (α, β) lies inside the left half of the ellipse,* $E = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \alpha \neq 0,\ \dfrac{\alpha^2}{\left(\frac{2}{h}\right)^2} + \dfrac{\left(\beta - \frac{2}{h^2}\right)^2}{\left(\frac{2}{h^2}\right)^2} < 1 \right\}$ *with $-\frac{2}{h} < \alpha < 0$, then the best Ulam-Hyers constant H_U for the finite difference equation (7.21) is given by*

$$L_U = h^2 \sum_{n=0}^{\infty} \left| U_n(\cos\theta) \right| \frac{1}{r^{n+2}}, \tag{7.35}$$

where $U_n(x)$ are Chebyshev polynomials of the second kind, $r = \sqrt{\dfrac{2 - \alpha h}{2 + \alpha h}}$ *and* $\theta = \tan^{-1}\left(\dfrac{h\sqrt{4\beta - \alpha^2 - \beta^2 h^2}}{2 - \beta h^2} \right)$. *Further, the best Ulam-Hyers constant for the finite difference equation (7.21) on $\mathbb{T}$ is at least* $\dfrac{1}{\beta}\left(1 + \dfrac{\alpha h}{2}\right)$.

7.4 Applications: Perturbed Finite Difference Equations

In this section, we apply our main result in connection with deriving a discrete continuous dependence with the initial data for the second-order perturbed linear difference equation with constant coefficients:

$$\delta_h^2 x(t) + \alpha \Delta_h^c x(t) + \beta x(t) = f(t, x(t), \Delta x(t)), \qquad \forall\, t \in \mathbb{T}, \tag{7.36}$$

where $f : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Delta x(t)$ is some finite difference scheme for $x'(t)$ on $\mathbb{T}$ and we consider the corresponding unperturbed linear difference equation (7.21) with $g(t) = 0$. So, basically after assuming that the function f is bounded on $\mathbb{T} \times \mathbb{R} \times \mathbb{R}$ and the roots of the characteristic polynomial are distinct and satisfy $|r_1| < 1,\ |r_2| < 1$, we estimate some bound for the solution of the perturbed system (7.36) and show that the bound depends on the initial data.

Remark 7.4.1. *If $(\alpha, \beta) \in R_{II}$ with $\alpha > 0$ and (α, β) not lying on the boundary of the elliptic disk E, i.e., on the right half of the region R_{II} except the points on the boundary of E and on the dotted lines, then only $r_1 \neq r_2$, with $|r_1| < 1,\ |r_2| < 1$, is feasible.*

Theorem 7.4.1. *Suppose the roots of the characteristic polynomial of the linear recurrence relation (7.23) are distinct ($r_1 \neq r_2$) and satisfy $|r_1| < 1,\ |r_2| < 1$. If there exists a constant $H_B > 0$ such that $|f(t, x, y)| \leq H_B$ for all $(t, x, y) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R}$, then the solution $x : \mathbb{T} \to \mathbb{R}$ of the perturbed linear difference equation (7.36) is bounded on $\mathbb{T}$. Moreover,*

$$|x(t)| < H_B L(h) + \frac{2\big(|x_0| + |x_1|\big)}{|r_1 - r_2|}, \quad \forall\, t \in \mathbb{T}. \tag{7.37}$$

Proof. Let $x(t)$ be a solution of the perturbed linear difference equation (7.36). Then, we have

$$|\delta_h^2 x(t) + \alpha \Delta_h^c x(t) + \beta x(t)| \leq H_B, \quad \forall\, t \in \mathbb{T}. \tag{7.38}$$

Since the roots r_1 and r_2 of the characteristic polynomial of the linear recurrence relation (7.23) are distinct and satisfy $|r_1| < 1,\ |r_2| < 1$, then by Remark 7.3.1 and Theorem 7.2.2, there exists a solution $y : \mathbb{T} \to \mathbb{R}$ of the unperturbed finite difference equation, $\delta_h^2 y(t) + \alpha \Delta_h^c y(t) + \beta y(t) = 0$ with $y(t_0) = x(t_0),\ y(t_1) = x(t_1)$ and a constant $L(h)$ given by equation (7.33) such that

$$|x(t) - y(t)| \leq H_B L(h), \quad \forall\, t \in \mathbb{T}. \tag{7.39}$$

At the n-th node $t_n = a + nh \in \mathbb{T}$, for all $n \geq 0$, y is of the form

$$y(t_n) = c_1 r_1^{\frac{t_n - a}{h}} + c_2 r_2^{\frac{t_n - a}{h}}, \quad \forall\, n \geq 0, \tag{7.40}$$

where $c_1 = \dfrac{x_1 - x_0 r_2}{r_1 - r_2}$ and $c_2 = \dfrac{x_0 r_1 - x_1}{r_1 - r_2}$ are constants. Here $x_0 = x(t_0),\ x_1 = x(t_1)$. Thus, by using (7.39), we have from (7.40), for all $t \in \mathbb{T}$,

$$|y(t)| \leq |c_1||r_1|^{\frac{t_n - a}{h}} + |c_2||r_2|^{\frac{t_n - a}{h}} < |c_1| + |c_2|$$

$$< \Big[|x_1 - x_0 r_2| + |x_0 r_1 - x_1|\Big] \frac{1}{|r_1 - r_2|} < \frac{2\Big(|x_0| + |x_1|\Big)}{|r_1 - r_2|}. \tag{7.41}$$

Hence, from (7.39) and using the inequality (7.41), we have, for all $t \in \mathbb{T}$,

$$|x(t)| \leq H_B L(h) + |y(t)| < H_B L(h) + \frac{2\Big(|x_0| + |x_1|\Big)}{|r_1 - r_2|}. \tag{7.42}$$

□

Example 7.4.1. *Consider the following initial value problem:*

$$x''(t) + 2x'(t) + \frac{1}{3}x(t) = \sin(t + x(t) + x'(t)), \; t \in I = [0, \infty), \tag{7.43}$$

$$x(0) = 1, \; x'(0) = 10. \tag{7.44}$$

The corresponding finite difference equation for (7.43) on $\mathbb{T} = \{nh \mid n \in \mathbb{N}_0\}$ is given by

$$\delta_h^2 x(t) + 2\Delta_h^c x(t) + \frac{1}{3}x(t) = \sin(t + x(t) + \Delta x(t)), \; t \in \mathbb{T}, \tag{7.45}$$

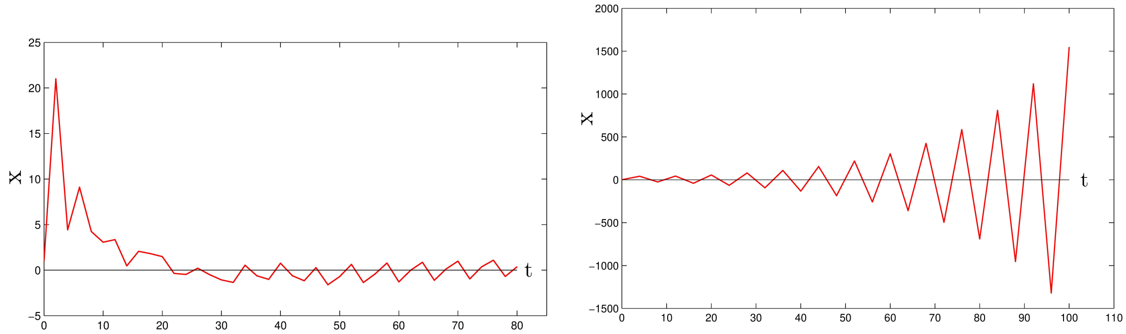
where $h \neq -\frac{2}{3}, \frac{2}{3}$, and $\Delta x(t) = \frac{x(t)-x(t-h)}{h}$ is a backward difference operator on $\mathbb{T}$. Note that $|\sin(t + x(t) + \Delta x(t))| \leq 1$ for all $(t, x, \Delta x) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R}$. Here, we take $H_B = 1$. If we take $h = 2$, then $|r_1| < 1, \; |r_2| < 1$ with $r_1 - r_2 = \frac{4\sqrt{7}}{9}$ and $L(h) = \frac{9(3+\sqrt{7})}{2}$. Now, using Theorem 7.4.1, we observe that any solution $x : \mathbb{T} \to \mathbb{R}$ of the difference equation (7.45) satisfies

$$|x(t)| < \frac{9}{2}(3 + \sqrt{7}) + \frac{9}{2\sqrt{7}}\Big(|x_0| + |x_1|\Big) = \frac{9}{2\sqrt{7}}\Big(|x_0| + |x_1| + 3\sqrt{7} + 7\Big), \; \forall t \in \mathbb{T}. \tag{7.46}$$

In Figure 7.3(a) we plot the solution $x(t)$ for $h = 2$ of the finite difference equation (7.45) with an initial condition (7.44) and observe that it satisfies (7.46), i.e.,

$$|x(t)| < \frac{9(23 + 3\sqrt{7})}{2\sqrt{7}} = 52.62, \; \forall \, t \in \mathbb{T}. \tag{7.47}$$

Next, for $h = 4$, we observe that $|r_1| < 1, \; |r_2| > 1$. Here, the condition of Theorem 7.4.1 does not hold and hence we may expect that the solution $x(t)$ will not be bounded on $\mathbb{T}$. In Figure 7.3(b), we plot the solution $x(t)$ of the difference equation (7.45) for $h = 4$ with the same initial condition (7.44). We observe that this solution oscillates and diverges.



(a) Solution of the difference equation (7.45) for $h = 2$ **(b)** Solution of the difference equation (7.45) for $h = 4$

Figure 7.3: Graph of the solution $x(t)$ of the finite difference equation (7.45), for $h = 2$ and $h = 4$. Solution oscillates and diverges for $h = 4$

7.5 Conclusions

In this chapter, we have established the Ulam-Hyers stability result for second-order convergent difference equations corresponding to first- and second-order nonhomogeneous linear differential equations. Subsequently, converting the finite difference equation to its equivalent recurrence relation, the main result is established by using previous Ulam-Hyers stability results for the linear recurrence relation. For the first-order differential equation, we have computed the best Ulam-Hyers constant. For the second-order differential equation, depending on the location of (α, β) in the two-dimensional region in the $\alpha\beta$-plane, we have determined the Ulam-Hyers constant. It has been observed that the Ulam-Hyers constant gets changed by the influence of the step size. In the end, we use the obtained results to estimate the bound for the solution of the perturbed finite difference equation on an unbounded domain of discretization. Further, an example is presented in order to ascertain the applicability of our result. From the above two results, we finally conclude that, instead of using some rigorous mathematical analysis to prove the Ulam-Hyers stability for a finite difference equation, we can easily prove the stability by studying the behavior of the roots of the characteristic polynomial of the equivalent recurrence relation of the corresponding difference equation.

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CHAPTER 8

Conclusions and future scope

8.1 Conclusions

In this thesis, we have presented a qualitative treatment to various classes of differential equations involving integer- and fractional-order derivatives. We basically considered two types of stability treatment: (i) first, we discussed the Ulam-Hyers stability of solution of the impulsive fractional-order integro-differential equation involving Caputo fractional-order derivative, and we also discussed the Ulam-Hyers stability of difference equations corresponding to first and second-order differential equations, (ii) secondly, we studied the Lyapunov stability analysis for a class of fractional-order differential equations involving a non-singular kernel fractional-order Caputo-Fabrizio derivative and discussed the various concepts such as the existence of periodic solution, stabilization, and asymptotic stability of the Caputo-Fabrizio fractional-order linear and semilinear evolution equations.

Here, we highlight chapter-wise main conclusions.

- In Chapter 2 and Chapter 3, we established the existence and stability results of the mild solution of a Caputo fractional non-instantaneous impulsive integro-differential equation in the sense of Ulam-Hyers and generalized Ulam-Hyers-Rassias stability, respectively. The main result was established by using Banach fixed point theorem under appropriate assumptions. We used our results to estimate the bound for the difference between the fractional-order and the integer-order non-instantaneous impulsive RLC circuit current, and showed that the bound mainly depends on the bandwidth of the RLC circuit. This finding is expected to find an important place in fractional-order RLC circuit problems.

- In Chapter 4, we revisited the Lyapunov stability of an equilibrium point of an autonomous Caputo-Fabrizio fractional-order system and showed that all isolated equilibrium points of an autonomous system are asymptotically stable and found that only constant solutions exist for autonomous systems. Further, we showed that a non-constant solution exists for the case of autonomous intermediate value fractional-order systems and further studied the Lyapunov stability for the intermediate value Caputo-Fabrizio linear and nonlinear autonomous systems. The condition required for the equilibrium point for such systems to be asymptotically stable was derived. A suitable example was presented at the end to illustrate the result of the existence of such a stability.

- In Chapter 5, we discussed the existence of a periodic solution of the Caputo-Fabrizio fractional-order system. First, by using the concepts of an equilibrium point, it was proved that an autonomous Caputo-Fabrizio system cannot admit a non-constant periodic solution. Under a similar assumption as the one for an integer-order differential equation, the existence of a periodic solution of a non-autonomous Caputo-Fabrizio fractional-order differential system was established. The main result was utilized in constructing and finding the periodic solution of the linear non-homogeneous Caputo-Fabrizio system. By using the result on linear systems, we derived a periodic solution of a fractional-order Gunn diode oscillator under a periodic input voltage and observed that the diameter of the periodic orbit kept reducing as the fractional-order continuously increased. In the end, by using the result on a linear non-homogeneous system and by constructing a suitable linear feedback control, the solution of the linear non-homogeneous fractional-order system was stabilized to a periodic solution. An example was presented to support the obtained result.

- In Chapter 6, we discussed the asymptotic stability of fractional-order linear and semilinear evolution equations involving a Caputo-Fabrizio derivative for fractional-order $\alpha \in (0, 1)$ with a non-singular kernel, and a new concept of a global solution for the Caputo-Fabrizio system was introduced. We stabilized the Caputo-Fabrizio fractional-order linear and semilinear evolution equations. In the end, by using the stabilization result, we stabilized a fractional-order chaotic system to support the obtained results. From this work, we have observed that one has to be very careful in choosing an initial condition for the system involving a non-singular kernel operator, and the initial data set S_s, where the solution exists, might not be an open, connected set. Thus, although a fractional-order differential operator with a non-singular kernel has several drawbacks, analyzing the systems involving a non-singular kernel operator is sometimes easier than other singular kernel operators.

- In Chapter 7, we studied Ulam-Hyers stability of the second-order convergent difference equations corresponding to first- and second-order non-homogeneous linear differential equations with constant coefficients. First, we converted the difference equation to its equivalent linear recurrence relation, and then by using existing Ulam- Hyers stability results for the linear recurrence relation, we established the Ulam-Hyers stability for the finite difference equation, and depending upon the location of the roots of the characteristic polynomial of the equivalent recurrence relation, the minimum Ulam-Hyers constant was determined. We applied our result to the perturbed second-order nonlinear difference equation and presented a suitable example to support the result.

8.2 Future scope

- Most physical and engineering systems are inherently nonlinear; however, nonlinear systems can exhibit a very rich dynamical behavior. The simplest way to analyze any nonlinear system is to linearize the systems locally near the steady state. But, in the case of the Caputo-Fabrizio systems, only a constant solution exists for a linearized system. Thus, the linearized method is ineffective in studying the nonlinear Caputo-Fabrizio systems. Hence, instead of linearizing the given nonlinear system with a time-invariant matrix, we can linearize it with a time-variant matrix and use the outcome of Chapter 6.
- A limit cycle is defined as an isolated periodic orbit, and it is a very special phenomenon in dynamical systems because the existence of a stable limit cycle in a system assures the self-sustainable systems which means there exists a domain such that, if the initial state starts from this domain, then the state always moves along a periodic orbit in all future time. Thus, once we know the existence of a periodic solution, there is a possibility of having a limit cycle for a given dynamical system. With the help of the problems in Chapter 4 and Chapter 5, we will work in this direction.
- How to reduce the chaotic oscillations to regular oscillations in chaotic fractional-order systems? We know that fractional-order systems do not have periodic orbit. The problems in Chapter 5 and Chapter 6 may help to answer such a question by replacing the derivative with a Caputo-Fabrizio fractional derivative, and we will work on more complex chaotic systems and try to stabilize the chaotic oscillations to a regular oscillation.
- Controlling the behavior of the solution of a given dynamical system by using a controller is an interesting problem. In this direction, we plan to initiate work in the area of control theory to design the controller to control the states of the fractional-order electrical circuit systems.
- Previously, a periodic solution did not come into picture in the fractional-order systems with Caputo or Riemann-Liouville fractional derivatives, but from the outcome

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of Chapter 5, we know that, under some suitable assumptions, the existence of a periodic solution for a non-autonomous fractional-order systems involving Caputo-Fabrizio fractional derivatives is viable. Thus, we want to revisit the bifurcation theory for fractional-order systems involving Caputo-Fabrizio fractional derivative by including the condition to the parameter of the systems for the existence of a periodic solution.

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Published and Communicated Papers

Based on the works carried out in this thesis, the following articles are published/communicated:

1. Matap Shankar and Swaroop Nandan Bora, Generalized Ulam-Hyers-Rassias stability of solution for the Caputo fractional non-instantaneous impulsive integro-differential equation and its application to fractional RLC circuit. *Circuits, Systems, and Signal Processing,* **42**, 1959-1983 (2023). https://doi.org/10.1007/s00034-022-02217-x

2. Swaroop Nandan Bora and Matap Shankar, Ulam-Hyers stability of second-order convergent finite difference scheme for first- and second-order nonhomogeneous linear differential equations with constant coefficients. *Results in Mathematics,* **78(17)**, 1-18 (2023). https://doi.org/10.1007/s00025-022-01791-5

3. Matap Shankar and Swaroop Nandan Bora, Stabilization and asymptotic stability of the Caputo-Fabrizio fractional-order linear and semilinear evolution equations. *Franklin Open,* **5(100043),** 1-10 (2023). https://doi.org/10.1016/j.fraope.2023.100043

4. Matap Shankar and Swaroop Nandan Bora, Caputo-Fabrizio fractional-order systems: Periodic solution and stabilization of non-periodic solution with application to Gunn diode oscillator. *Physica Scripta*, **98(12)**, 1-15 (2023). https://doi.org/10.1088/1402-4896/ad0c12

5. Matap Shankar and Swaroop Nandan Bora, Lyapunov stability analysis of the Caputo-Fabrizio fractional-order intermediate value systems. (Under Review)

6. Matap Shankar and Swaroop Nandan Bora, Existence and Ulam-Hyers stability of solution for the Caputo fractional non-instantaneous impulsive integro- differential equation. (Under Review)

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