On the Inverse of Bipartite Graphs with Unique Perfect Matchings and Reciprocal Eigenvalue Properties

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Declaration

I do hereby declare that the work contained in this thesis entitled **"On the Inverse of Bipartite Graphs with Unique Perfect Matchings and Reciprocal Eigenvalue Properties "** has done by me, under the supervision of **Prof. Sukanta Pati**, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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Certificate

It is certified that the work contained in this thesis entitled **"On the Inverse of Bipartite Graphs with Unique Perfect Matchings and Reciprocal Eigenvalue Properties "** by **Swarup Kumar Panda**, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

June, 2016

Prof. Sukanta Pati

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Dedicated to

My Father *Gnanranjan Panda*

and

My Mother *Kalyani Panda*

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Abstract

The study of graph structures via different properties of its adjacency matrix is a widely studied subjects. Sometimes different structural properties of a graph get characterized by different properties of the eigenvalues and the eigenvectors of the associated adjacency matrix. For example, it is well-known that 'a (simple) graph G is connected if and only if $\rho(G)$ (spectral radius) is a simple eigenvalue with an eigenvector whose coordinates are nonzero and of the same sign'. There are many interesting results exhibiting the relationship of the graph structure with the eigenvalues and eigenvectors. This thesis aims to establish such relationships with regard to the concepts *inverse graph* and *reciprocal eigenvalue properties*. Both of these play important roles in quantum chemistry.

The notion of *inverse graph* was introduced by C. D. Godsil in (*Inverses of trees*, Combinatorica, 5(1):33–39, 1985). A nonsingular graph G (all graphs are assumed to be simple, finite, undirected) is said to be invertible if $SA(G)^{-1}S \geq 0$ for some signature matrix S (a diagonal matrix with diagonal entries ± 1). The graph with adjacency matrix $SA(G)^{-1}S$ is called the *inverse graph* of the graph G.

A nonsingular graph G is said to satisfy the *property* (SR) if $1/\lambda$ is an eigenvalue of G whenever λ is an eigenvalue of G and both have the same multiplicity. When the multiplicity condition is relaxed, we say G has the *property* (R).

In many ways, these two concepts are related to each other. It was proved in (*Inverses of trees*, Combinatorica, 5(1):33–39, 1985) that 'nonsingular trees possess inverses'. Note that nonsingular trees are bipartite graphs with unique perfect matchings. And any bipartite graph with a unique perfect matching must be nonsingular. Of course, there are such bipartite graphs which do not have inverse graphs. In any case, a class of bipartite graphs with unique perfect matchings which possess inverses was supplied by Godsil in 1985. This class $\mathcal{H}_g$ was much larger than the nonsingular trees. It was natural to ask to *characterize the connected bipartite graphs with unique perfect matchings which possess inverses.* This has remained the main open problem which attracts the researchers to the subject. In (*Directed intervals and dual of a graph*, Linear Algebra and its Applications, 431:792–807, 2009), Tifenbach and Kirkland supplied necessary and sufficient conditions for a connected bipartite graph with a unique perfect matching to possess an inverse (they have used the term 'dual' to mean the 'inverse graph'), utilizing constructions derived from the graph itself. However, there are many connected bipartite graphs with unique perfect matchings for which these conditions cannot help. Thus, the search for a

characterization of bipartite graphs with unique perfect matchings which possess inverses is still open. In this thesis, we introduce the 'even'ness of a nonmatching edge and show that under certain conditions a connected bipartite graph with a unique perfect matching has an inverse. Our results extend some known results, providing us with a larger class of graphs possessing inverses. Our results can also be seen to complement the results by Tifenbach and Kirkland, thus advancing further towards the main goal.

We need another related concept of *simple corona graph*. A graph G is called a simple corona graph if it is obtained from another graph H by adding a new pendant vertex to each vertex of H. A *simple corona tree* is a simple corona graph which is a tree.

The study of different properties of the inverse graphs of the graphs which possess inverses is also an interesting subject. For example, characterizing connected bipartite graphs G with a unique perfect matching for which G is isomorphic to its inverse graph, is another challenging open question. This question, for the class $\mathcal{H}_g$ was asked by Godsil in 1985. At the same time, Godsil showed that $1/\rho(G)$ is the least positive eigenvalue of G in $\mathcal{H}_g$ if and only if $G \cong G^+$, where $\rho(G)$ is the spectral radius of G. By this time, it was known (from the works of Godsil and Mackay 1978, and Cvetkovic, Gutman and Simic 1978) that if a nonsingular tree T has property (SR), then it must be a simple corona tree. In 1989, Simion and Cao came up with a surprising result establishing a deeper relationship of the 'simple corona graphs' with the 'graphs G which are isomorphic to G^+'. They showed that for a graph $G \in \mathcal{H}_g$ (this is the class constructed by Godsil) we have '$G \cong G^+$ if and only if G is a simple corona graph'.

In 2006, the concept of 'reciprocal eigenvalue property' *property (R)* was introduced and in a surprising find it was shown that the property (R) is also an equally important subject of study. The findings of all these articles combined together for a nonsingular tree, reads as follows.

Theorem 0.0.1. *Let T be a nonsingular tree with spectral radius $\rho(T)$. Then the following are equivalent.*

i) $1/\rho(T)$ *is the smallest positive eigenvalue.*

ii) $T \cong T^+$.

iii) T *has property (R).*

iv) T *has property (SR).*

v) T is a simple corona.

Remark 0.0.2. In the above theorem, the tree T can be replaced with a graph from $\mathcal{H}_g$, with appropriate arguments.

There are many natural questions now.
Q1. For a bipartite graph G with a unique perfect matching, if $G \cong G^+$ holds then items i), iii) and iv) hold. Does v) follow in general?
Q2. Do we have a class larger than the class $\mathcal{H}_g$, where all the four conditions i)–iv) are equivalent? If so, what is an appropriate modification of item v), in that case?
Q3. For a graph G, if $G \cong G^+$ holds then items i), iii) and iv) hold. Do we have counter examples for the converses?
We supply answer to all these in the thesis at appropriate places.

A *strongly self-dual* graph is a graph which is isomorphic to its dual/inverse via a particular isomorphism. In (*Strongly self-dual graphs*, Linear Algebra and its Applications, 435, 3151–3167, 2011), Tifenbach introduced the concept of strongly self-dual graph. He supplied necessary and sufficient conditions for a connected bipartite graph with a unique perfect matching to be strongly self-dual, utilizing constructions derived from the graph itself. However, there are many connected bipartite graphs with unique perfect matchings for which these conditions cannot help. We felt that there is a necessity of studying the different properties of the inverse graphs of different classes of bipartite graphs with unique perfect matchings keeping the original graph in mind. In this thesis, we supply a class of connected bipartite graphs with unique perfect matchings containing $\mathcal{H}_g$ in which $G \cong G^+$ can be characterized.

In 2013, M. Neumman and S. Pati supplied a constructive characterization of the inverse graphs of nonsingular trees. As the Theorem 0.0.1 is valid for $\mathcal{H}_g$, one naturally wonders if a characterization of the inverse graphs of the graphs in $\mathcal{H}_g$ can be done. We supply a constructive structures of the inverse graphs of a class of bipartite graphs with unique perfect matchings. We also extend the notion of inverse graph to weighted graphs and use it to settle an open problem posed by them, involving property (R) for weighted trees.

Contents

Chapter 1

Introduction

Throughout this thesis, all graphs are assumed to be simple, undirected and finite. For a graph G, the vertex set and the edge set are denoted by $V(G)$ and $E(G)$, respectively. An edge between vertices i and j is denoted by $[i, j]$. The notations $i \sim j$ (resp. $i \nsim j$) mean 'i is adjacent j' (resp. i is not adjacent to j). The study of graph structures via different properties of matrices associated with it, is a very old and rich area of research; see for example the books [1, 2, 3, 4, 5, 6, 7]. Among the various matrices associated with a graph, the adjacency matrix is probably the most popular and widely investigated one. The adjacency matrix $A(G)$ of G is the symmetric matrix of size n whose (i, j)th entry a_{ij} is given by

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

We say λ is an eigenvalue of G to mean that it is an eigenvalue of $A(G)$. The eigenvalues of G are all real, as $A(G)$ is a symmetric matrix. Let $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ be the eigenvalues of G. The spectrum $\sigma(G)$ of G is the multiset of eigenvalues of G. The largest eigenvalue $\lambda_n(G)$ of G is called the *spectral radius* of G and it is denoted by $\rho(G)$. The spectral radius of G is a widely studied subject; see for example the book [8]. If G is connected, then $\rho(G)$ is simple with an unique (up to a scalar multiple) positive unit eigenvector called a *Perron vector*.

Let G be a graph. Consider the function $\mathrm{w} : E(G) \to (0, \infty)$. The function w may be viewed as associating positive weights to each edge of G. We call w a *weight function*. We use G_{w} to denote the positively weighted graph obtained from G by using the weight function w. An unweighted graph G may be viewed as a weighted graph where each edge has weight 1. The adjacency matrix $A(G_{\mathrm{w}})$ of G_{w} is defined in a natural way with

$$a_{ij} = \begin{cases} \mathrm{w}([i,j]) & \text{if } i \sim j \\ 0 & \text{otherwise} \ . \end{cases}$$

Note that $A(G_{\mathrm{w}})$ is a nonnegative symmetric matrix for a positive weight function w. We remark here that there are documents where the adjacency matrix of graphs with complex weight functions have been studied; see [9, 10]. However, we shall restrict ourselves to positive weight functions only.

We need to use a few well-known terms from graph theory. We shall quickly recall them. We refer the reader to the texts [11, 12, 13] for further clarification.

- Let G and H be two graphs. A mapping $f : V(G) \to V(H)$ is said to be an isomorphism if f is bijective and '$[u,v] \in E(G)$ if and only if $[f(u), f(v)] \in E(H)$ holds for each pair of vertices in $V(G)$'. If an isomorphism exists between two graphs, then the graphs are called isomorphic and we write $G \cong H$.
- A graph G is bipartite if the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 in such a way that no two vertices from the same set are adjacent. It is well-known that 'a graph G is bipartite if and only if G has no cycle of odd length'.
- A set of edges in a graph G is called *matching* if no two edges have a common end vertex. A perfect matching of a graph G is a matching which covers every vertex of G. It is well-known that 'a tree T has a perfect matching if and only if it has a unique perfect matching.
- In general, a graph can have more than one perfect matchings. For example, consider the graph G shown in Figure 1.1. The set of edges $\{[1,2],[3,4],[5,6]\}$, $\{[1,4],[2,3],[5,6]\}$ and $\{[3,5],[2,6],[1,4]\}$ are three different perfect matchings in G.

Figure 1.1: A graph G with three perfect matchings and a graph H with a unique perfect matching.

- If a graph has unique perfect matching, then we shall always denote it by $\mathcal{M}$. The graph H shown in Figure 1.1 has a unique perfect matching. Here the solid edges denote the matching edges. Henceforth, we shall follow this convention in our figures.

1.1 Nonsingular graphs

The adjacency matrix of a graph may or may not be nonsingular. For example, the adjacency matrix of a tree with an odd number of vertices is singular and the adjacency matrix of a tree with a perfect matching is nonsingular. For brevity, we call a graph *nonsingular* (resp. *singular*) if its adjacency matrix $A(G)$ is nonsingular (resp. *singular*). A combinatorial description of $\det(A(G))$ for any graph G, was given by Harary in 1962 and H. Sachs in 1964, independently.

Let G be a graph. A subgraph H is called a *linear subgraph* if each component of H is either a P_2 (path on two vertices) or a cycle. A linear subgraph involving all the vertices of G is called a *spanning linear subgraph* or an $(1,2)$-*factor*.

Theorem 1.1.1. *Let G be a graph of order n. Then*

$$\det(A(G)) = \sum_H (-1)^{n-p(H)-c(H)} 2^{c(H)},$$

where the sum is over all spanning linear subgraphs of G, $p(H)$ is the number of components in H which are paths and $c(H)$ is the number of components in H which are cycles.

Let G be a bipartite graph with a unique perfect matching and H be a spanning linear subgraph of G. If H has a cycle component, it must be an even cycle and that will lead to the existence of more than one perfect matchings. Hence, the only spanning linear subgraph of G is the perfect matching. Therefore, one has the following well-known observation.

Corollary 1.1.2. *Let G be a bipartite graph with a unique perfect matching. Then* $\det(A(G)) = \pm 1$. *Hence, $A(G)^{-1}$ has integer entries.*

Remark 1.1.3. The following are some useful facts.

- Note that the converse of the above corollary is not true in general. For example, consider the graph G in Figure 1.1. One can easily calculate the $\det(A(G)) = -1$ by using Theorem 1.1.1 and it has three perfect matchings.

- But the converse is true for trees.
- If $\det(A(G)) \neq \pm 1$, then G cannot have a unique perfect matching.
- If G is bipartite graph and $\det(A(G)) \neq 0$, then G has perfect matchings.

1.2 Inverses of graphs

A *signature* matrix is a diagonal matrix with diagonal entries from $\{1, -1\}$. Note that for any signature matrix S, we have $S^{-1} = S$, so that two matrices A and B are signature similar if there is a signature matrix S such that $B = SAS$.

The following definition of an *inverse graph* was given by Harary and Minc [14] in 1976.

Definition 1.2.1. Let G be a nonsingular graph. The graph G is said to be invertible if $B = A(G)^{-1}$ is the matrix with entries from $\{0, 1\}$. The graph H with adjacency matrix B is called the inverse graph of G.

Using this definition, Harary and Minc [14] proved the following result.

Theorem 1.2.2. *Let G be a connected nonsingular graph. Then G is invertible if and only if G is P_2.*

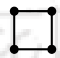
Unfortunately, we have only one connected graph which is invertible. So, the inverse graph H of a graph G has to be defined differently, such that H satisfies the following properties. Inverse graph H is unique and $1/\lambda \in \sigma(H)$ whenever $\lambda \in \sigma(G)$ with both having the same multiplicity. These two are desirable properties of an inverse graph. Almost a decade after, in 1985, Godsil [15] supplied another notion of inverse graph, which is quite similar to the notion of inverse graph given by Harary and Minc.

Definition 1.2.3. [15] Let G be a nonsingular graph. We say G has an inverse G^+ if the matrix $A(G)^{-1}$ is signature similar to a nonnegative matrix. That is, if $SA(G)^{-1}S \geq 0$ for some signature matrix S, then we define G^+ to be the weighted graph with adjacency matrix equal to $SA(G)^{-1}S$. In some literature, 'inverse graph' is known as 'dual'; see [16, 17].

Remark 1.2.4. Let G be a graph. Suppose that G^+ exists. Then $A(G^+)_{i,j} = |A(G)^{-1}_{i,j}|$.

There are some other notions of inverse graph proposed in the different pieces of literature. But those do not satisfy the desirable properties of inverse graph. We list some of them.

- In 1978, Cvetkovic, Gutman and Simic [18] have introduced the *pseudo-inverse* graph of a graph. Let G be a graph. The pseudo-inverse graph $PI(G)$ of G is a graph, defined on the same vertex set as G, and in which the vertices x and y are adjacent if and only if $G - x - y$ has a perfect matching.

  For example the graph  for which $PI(G) = G$ and $\sigma(G) = \sigma(PI(G)) = \{-2, 0, 0, 2\}$, but $1/\lambda \in \sigma(PI(G))$ whenever $\lambda \in \sigma(G)$ is not true.

- In 1988, Buckley, Doty and Harary [19] have introduced the *signed inverse* graph of a graph. A *signed graph* is a graph in which each edge has a positive or negative sign, see [20]. An adjacency matrix of a signed graph is symmetric and each entry is 0, 1, or -1. Let G be a nonsingular graph. The graph G has a signed inverse if $A(G)^{-1}$ is the adjacency matrix of some signed graph H.

- In 1990, Pavlikova and Jediny [21] have introduced another notion of inverse graph of a graph. The *inverse* graph of a nonsingular graph with the spectrum $\lambda_1, \ldots, \lambda_n$ is a graph with the spectrum $1/\lambda_1, \ldots, 1/\lambda_n$. This type of inverse graph of a graph need not be unique. One can construct a class of graphs which have more than one inverse graphs. Consider the nonisomorphic cospectral graphs G and H (shown in Figure 1.2) supplied by Dam and Haemers in [22, Figure 5]. Now obtain G_1 and H_1 from G and H by adding a new pendant vertex to each vertex of G and H, respectively. Using Equation (9) of [23] or Lemma 2.1 of [24], we see that the $\sigma(G_1) = \sigma(H_1)$ and $1/\lambda \in \sigma(G_1)$ whenever $\lambda \in \sigma(G_1)$ with both having the same multiplicity. Therefore, both G_1 and H_1 are inverse graphs of G_1. They are also nonisomorphic and cospectral. Continuing the process on G_1 and H_1, we get G_2 and H_2. In this way, we can get a class of graphs which have more than one inverse graphs.

Figure 1.2: The graphs G and H are cospectral but not isomorphic.

Conventions:

- Henceforth, we follow the notion of inverse graph given by Godsil. We say G has an inverse G^+ if the matrix $A(G)^{-1}$ is signature similar to a nonnegative matrix.
- We denote the class of connected bipartite graphs with a unique perfect matching by $\mathcal{H}$.
- Usually, we shall talk about bipartite graphs with unique perfect matchings. In such cases, we denote the unique perfect matching by $\mathcal{M}$.
- Let G be a bipartite graph with a unique perfect matching. If $v \in V(G)$, then we use v' to denote the vertex that matches to v, that is, $[v, v'] \in \mathcal{M}$.
- An edge in $\mathcal{M}$ is called a *matching* edge and an edge not in $\mathcal{M}$ is called a *nonmatching* edge.
- This terminology can be found in [15]. Let $G \in \mathcal{H}$. By $G/\mathcal{M}$ we denote the graph obtained from G by contracting each matching edge to a single vertex. For an example, see Figure 1.3.
- We shall use $\mathcal{H}_g$ to denote the class $\{G \in \mathcal{H} \mid G/\mathcal{M} \text{ is bipartite}\}$. Thus $\mathcal{H}_g$ is the class of connected bipartite graphs G with unique matchings $\mathcal{M}$ for which $G/\mathcal{M}$ is bipartite.

Example 1.2.5. The graph G shown in Figure 1.3, for which $G/\mathcal{M}$ is bipartite. Hence, $G \in \mathcal{H}_g$. The graph H shown in Figure 1.3, for which $H/\mathcal{M}$ is not bipartite. Hence, $H \notin \mathcal{H}_g$.

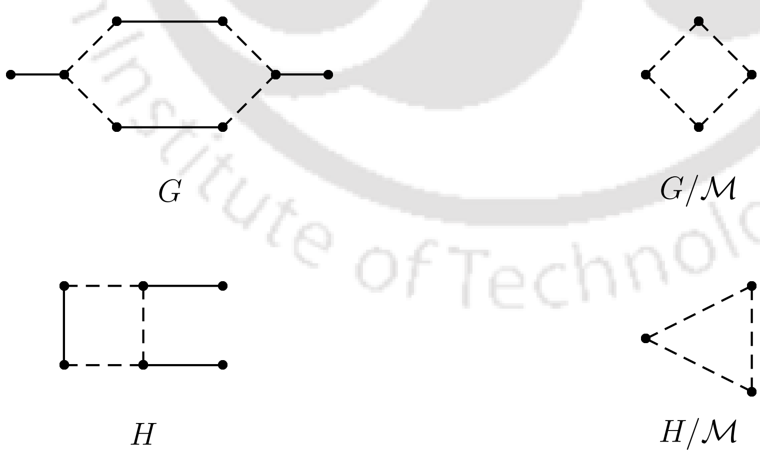
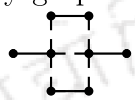
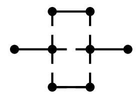


Figure 1.3: The graph G is in $\mathcal{H}_g$ and the graph H is not in $\mathcal{H}_g$. Here the solid edges are the matching edges.

In [15], Godsil proved that each graph in $\mathcal{H}_g$ possesses an inverse and posed the problem to *characterize the graphs in $\mathcal{H}$ which possess inverses*. The class of unicyclic graphs in $\mathcal{H}$ is not a subclass of the class $\mathcal{H}_g$. For example, the graph H shown in Figure 1.3 is a unicyclic graph in $\mathcal{H}$ but not in $\mathcal{H}_g$. In [25], Akbari and Kirkland characterized the unicyclic graphs in $\mathcal{H}$ which possess inverses. In [16], Tifenbach and Kirkland supplied necessary and sufficient conditions for graphs in $\mathcal{H}$ to possess inverses, utilizing constructions derived from the graph itself. However, there are many graphs in $\mathcal{H}$ for which these conditions cannot help. For example, for the graph  these necessary and sufficient conditions cannot help. (An explanation is supplied in the Preliminaries section of Chapter 2.) This keeps 'the search for a characterization of bipartite graphs with unique perfect matchings which possess inverses' open. We consider the problem to supply a class of graphs in $\mathcal{H}$ containing $\mathcal{H}_g$ and the unicyclic graphs, which possess inverses. The first part of the thesis is in this direction. In order to supply this larger class, we realize that one has to make some distinction between the nonmatching edges. We see that there are roughly three types of nonmatching edges, one which can be called *even*, another which can be called *odd* and another which can be called *mixed*. (See Chapter 2 for definitions.) With an appropriate introduction of many new terms and techniques, we finally obtain the class $\mathcal{H}_{nmc}$. We show that $\mathcal{H}_{nmc}$ contains $\mathcal{H}_g$ and the unicyclic graphs, and we characterize the graphs in this class which possess inverses.

There is another reason behind this study and introduction of the new terms and techniques. We are interested in characterizing graphs in $\mathcal{H}$ which are isomorphic to their inverses. This question, for the class $\mathcal{H}_g$ asked by Godsil in 1985 and has already been answered by Simion and Cao in [26]. They showed that 'for a $G \in \mathcal{H}_g$, the graph $G^+ \cong G$ if and only if G is obtained from another connected bipartite graph by adding a new pendant vertex to each vertex of H'. In [16], the author studied the inverse graphs of the unicyclic graphs in $\mathcal{H}$. Satisfactory answers to the following questions were obtained using some very different and new techniques.
Q1. Let G be a unicyclic graph in $\mathcal{H}$ for which G^+ exists. When is G^+ unicyclic?
Q2. Let G be a unicyclic graph in $\mathcal{H}$ for which G^+ exists. When is $G^+ \cong G$?
A *strongly self-dual* graph is a graph which is isomorphic to its dual/inverse via a particular isomorphism. In [17], Tifenbach introduced the concept of strongly self-dual graph and supplied necessary and sufficient conditions for the existence of strongly self-dual graph of a graph in $\mathcal{H}$, utilizing constructions derived from the graph itself. However, there are many graphs in $\mathcal{H}$ for which these conditions

cannot help. For example, for the graph  these conditions cannot help. (An explanation is given in the Preliminaries section of Chapter 3.) We consider the problem of supplying a class of graphs in $\mathcal{H}$ containing $\mathcal{H}_g$ in which $G \cong G^+$ can be characterized. The third chapter of the thesis is in this direction. We supply the class $\mathcal{H}_{nmcs}$ as a solution to this notion. This class contains $\mathcal{H}_g$ and we characterize the graphs in $\mathcal{H}_{nmcs}$ which are isomorphic to their inverses. This characterization is combinatorial in nature and gives us a better understanding of the answer to the same question on $\mathcal{H}_g$. The class $\mathcal{H}_{nmcs}$ does not contain all the unicyclic graphs in $\mathcal{H}$. However, using the tools developed we give another characterization of the distinct classes of unicyclic graphs G in $\mathcal{H}$ for which G^+ is unicyclic and $G \cong G^+$, individually, which are different from the characterization provided by Tifenbach and Kirkland in [16]. We see that there are unicyclic graphs G in $\mathcal{H}$ for which G^+ is unicyclic but $G \ncong G^+$.

Having characterized the unicyclic graphs G in $\mathcal{H}$ for which G^+ is unicyclic, a natural question follows. *Is it necessary that the inverse of a graph in $\mathcal{H}_g$ should be in $\mathcal{H}_g$?* The general answer to this question is in the negative. Note that the underlying unweighted graph of the inverse graph of a graph in $\mathcal{H}$ is also in $\mathcal{H}$. In fact, the same matching $\mathcal{M}$ is the unique perfect matching for both G and G^+. So, an example of a graph $G \in \mathcal{H}_g$ for which $G^+ \notin \mathcal{H}_g$ must have the property that $G^+/\mathcal{M}$ is not bipartite. For example, $G = P_6$ is such a graph. This brings in another natural question. *Can we characterize the class of graphs in $\mathcal{H}_g$ whose inverses are also in $\mathcal{H}_g$?* Notice that, this consideration, in a way compliments the findings of Godsil and of Simion and Cao.
Godsil: which graph in $\mathcal{H}_g$ has an inverse? Answer: all.
Simion and Cao: which graph in $\mathcal{H}_g$ is isomorphic to its own inverse? Answer: a specific class.
Our question: which graph in $\mathcal{H}_g$ has an inverse in $\mathcal{H}_g$ (the case where $G \cong G^+$ is included here)? Our answer must be a subclass of $\mathcal{H}_g$ which contains the answer class of Simion and Cao. Surprisingly, it turns out that, it is the same class. In the process supplying the answer, we supply an alternate proof of the result of Simion and Cao in [26]. These are the contents of Chapter 4.

In [21], Pavlikova and Jediny had given a way to construct T^+ for a nonsingular tree T. Note that for a nonsingular tree T, the inverse T^+ is not always a tree. *Which graphs can occur as the inverse of a nonsingular tree?* An answer was supplied by Neumann and Pati in [27], where the authors supplied a constructive characterization of the class of graphs that are inverse graphs of some nonsingular trees. As graphs in $\mathcal{H}_g$ are very naturally close to trees, one naturally wonders, *whether a characterization of the inverse graphs of graphs in* $\mathcal{H}_g$ is possible. In Chapter 5, we supply such a combinatorial characterization which appropriately generalizes the same for nonsingular trees and gives us a better understanding of all the classes considered.

To proceed further, we extend the notion of inverse graph to weighted graphs with positive weight functions. The following is a natural question. *Let* $G \in \mathcal{H}$ *and* w *be a positive weight function. When does* G^+_{w} *exist?* Note that this is a more general question than its unweighted counterpart. **Let us keep ourselves confined to positive weight functions for which weights of each matching edge is 1.** Here are a few questions we considered along with their answers.

- *Let* $G \in \mathcal{H}$. *Assume that* G^+ *exists. Is it necessary that* G^+_{w} *will exist for some weight function which is not a multiple of* $\mathbb{1}$, *all ones weight function?* Answer: Yes.
- *Let* $G \in \mathcal{H}_g$. *We know that* G^+ *exists. Is it necessary that* G^+_{w} *will exist for* <u>*each*</u> *weight function* w*?* Answer: Yes.
- *Let* $G \in \mathcal{H}$. *Assume that* G^+_{w} *exists for each* w. *Is it necessary that* $G \in \mathcal{H}_g$*?* Answer: Yes.
- *Under which conditions the existence of* G^+_{w} *for one* w *will force* G *to be inside* $\mathcal{H}_g$*?*
- *Let* $G \in \mathcal{H}$. *Assume that* G^+ *does not exist. Is it necessary that* G^+_{w} *will never exist for any weight function* w*?* Answer: No.
- *Supply a class of graphs* G *in* $\mathcal{H}$ *for which* G^+_{w} *never exists.*
- *Supply a class of graphs* G *in* $\mathcal{H}$ *for which* G^+ *exists and* G^+_{w} *exists for some* w *which is not a multiple of* $\mathbb{1}$, *but not for all* w. *Obviously such a graph is outside* $\mathcal{H}_g$.
- *Supply a class of graphs* G *in* $\mathcal{H}$ *for which* G^+ *does not exist but* G^+_{w} *exists for some* $\mathrm{w} \in \mathcal{W}_G$.

These is the content of Chapter 6.

1.3 Reciprocal eigenvalue properties

Spectra of the adjacency matrix may be used to characterize a graph or obtain information about the graph. Graph spectra finds its application in other fields such as chemistry. We refer the reader to a classical book by Cvetkovic, Doob and Sachs [5]. It is well-known that 'a connected graph G is bipartite if and only if $-\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$'.

A contrasting question is to characterize graphs which satisfy a reciprocal eigenvalue property. A nonsingular graph G is said to satisfy the *property* (SR) (*strong reciprocal eigenvalue property*) if $1/\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$ and both have the same multiplicity. When the multiplicity condition is relaxed, we say G has the *property* (R) (*reciprocal eigenvalue property*). It is clear that graph with property (R) or property (SR) must be nonsingular.

The study of property (SR) has been started in 1978 with different names. The class of nonsingular trees with property (SR) has been characterized in [18, 23], independently. In [18], Cvetkovic, Gutman and simic used the term 'property C' to mean the 'property (SR)'. In [23], Godsil and Mckay used the term 'symmetric property' to mean the 'property (SR)'.

In 2006, Barik, Neumann and Pati[28] have restated this eigenvalue property and called it property (SR). The authors also introduced the property (R). In [28], the authors supplied a complete characterization of nonsingular trees with property (SR) and nonsingular trees with property (R). Surprisingly, it turns out that these two properties are equivalent. In [29], Barik, Pati, Nath and Sharma have characterized the unicyclic graphs with property (SR). In [30], Joseph D. Lagrange have characterized the boolean rings such that the zero divisors graph has property (SR).

The following definition is taken from Frucht and Harary [31].

Definition 1.3.1. Let G_1 and G_2 be two graphs on disjoint sets of n and m vertices, respectively. The corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. The corona $G_1 \circ G_2$ has $n(m+1)$ vertices and $|E(G_1)| + n(|E(G_2)| + m)$ edges. The corona $G \circ K_1$ is sometimes called a *simple corona*, where K_p denotes the complete graph on p vertices. If T is a tree, then we call $T \circ K_1$ a *corona tree*. For an example, see Figure 1.4.

In many ways, the concepts 'inverse graph' and 'reciprocal eigenvalue properties' are related to each other. The following theorem is one of those. This theorem has been proved collectively in [18, 23, 24, 28].

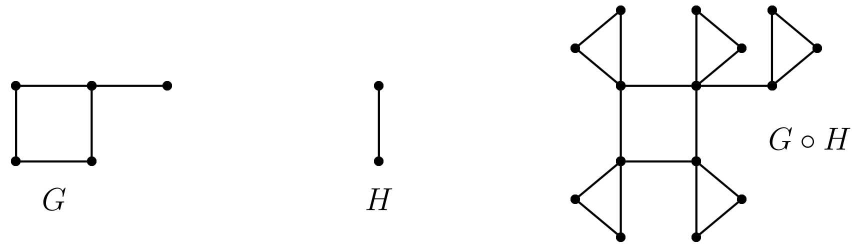


Figure 1.4: Corona of two graphs.

Theorem 1.3.2. *Let T be a nonsingular tree and $\rho(T)$ be the spectral radius of T. Then the following are equivalent.*

i) $1/\rho(T)$ *is the smallest positive eigenvalue of* T.

ii) $T \cong T^+$ *(T is isomorphic to its inverses graph T^+).*

iii) The graph T has property (R).

iv) The graph T has property (SR).

v) T is a corona tree.

In [27], Neumann and Pati extended the notion of reciprocal eigenvalue properties to weighted graphs and supplied a complete characterization of the weighted trees with property (R) under some restrictions of weight. They have shown that the nonsingular weighted tree T_{w} has property (R) if and only if $T = T_1 \circ K_1$ where $\mathrm{w} \in \mathcal{W}_T$ with $\mathrm{w}(e) \geq 1$ for each edge in T and posed the question of *whether this result is true even when one allows the weights of the nonmatching edges to be any positive number.*

In [10], Kalita and Pati have characterized the unicyclic weighted directed graphs with weights from $\{\pm 1, \pm i\}$ satisfying the property (SR).

A natural question here is to supply a class of graphs in $\mathcal{H}$ (larger than the nonsingular trees) where the statements i)–iv) of Theorem 1.3.2 are equivalent. As we have already characterized graphs $G \in \mathcal{H}_{nmcs}$ which are isomorphic to their inverses, we readily have an answer. Note that, as in Theorem 1.3.2, item v) supplied a constructive characterization of the graphs which satisfy the items i)–iv), we also have an equivalent constructive characterization of the graphs. We supply examples to show that no two of i)–iv) are equivalent, in general. However, we are able to show their equivalence, for the class of weighted graphs G_{w} where $G \in \mathcal{H}_g$ and w is any weight function. Furthermore, in such a case, we show that G must have a special structure. These results generalize Theorem 4.2 of [27] while supplying an answer to the problem which was posed in the same article.

1.4 Why do we study the inverse graph and the reciprocal eigenvalue properties?

The smallest positive eigenvalue of a nonsingular graph has an important role in Quantum Chemistry. Sometimes, the smallest positive eigenvalue of a graph is known as the dual index of that graph, see[21]. Theoretically its magnitude is expected to be correlated with the amount of energy needed to remove an electron from the hydrocarbon molecule represented by G, see [32]. Information about the magnitude of this eigenvalue is not easily gained in general. But the information about the magnitude of the largest is available. We know that the least positive eigenvalue of an invertible graph G is the largest eigenvalue of G^+ and if G satisfies the reciprocal eigenvalue properties then reciprocal of the least positive eigenvalue of G is the largest eigenvalue of G. This is one of the motivations to study the inverse graph and reciprocal eigenvalue properties.

1.5 Why do we consider the class of bipartite graphs with unique perfect matchings?

Given a molecule, consider it's skeletal which can be regarded as a graph (with multiple edges); neglect all the hydrogen atoms and their bonds to the carbon, and finally discard all the double bonds. Thus, one finds the simple subgraph induced by only those nodes corresponding to the carbon atoms and it is called a Hückel graph. See Figure 1.5. The Hückel graph is used to model the molecular orbital energies of hydrocarbon. It has been shown that many families of Hückel graphs are bipartite graphs with unique perfect matchings. See Yates[33]. This is one of the motives for considering connected bipartite graphs with unique perfect matchings. The following motivation can be found in Godsil[15]. Suppose G is a bipartite graph with adjacency matrix $A(G)$. In a simple model in Quantum Chemistry, the eigenvalues of $A(G)$ have a physical meaning and so the relation between the graph theoretic properties of G and the eigenvalues of $A(G)$ are of some interest. If G is bipartite then the eigenvalues of $A(G)$ are symmetrically placed about the origin.

In the cases of most chemical importance, $A(G)$ must be nonsingular.

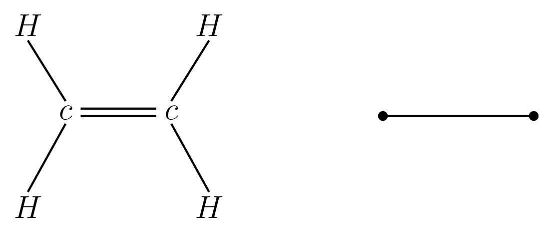


Figure 1.5: An ethylene molecule and its Hückel graph

1.6 Organization of the Thesis

The thesis is organized as follows. There are eight chapters in the thesis. Chapter 1 contains a brief introduction of the thesis and a few lines for motivation. Chapter 2 is devoted to supplying a class of graphs in $\mathcal{H}$ which possess inverses. Chapter 3 contains a study of characterizing a class of graphs in $\mathcal{H}$ satisfying $G \cong G^+$. Chapter 4 is devoted to characterizing the graphs $G \in \mathcal{H}_g$ such that $G^+ \in \mathcal{H}_g$. Chapter 5 provides a constructive characterization of the class of inverse graphs of the graphs in $\mathcal{H}_g$. Chapter 6 describes the existence of inverse graphs of weighted graphs. Chapter 7 supplies the relations between the reciprocal eigenvalue properties of graphs in $\mathcal{H}$ and its inverse graphs. Chapter 8 outlines some future directions of this thesis.

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Chapter 2

On some graphs which possess inverses

2.1 Preliminaries

Let us recall some notations that we repeatedly use.

$\mathcal{H}$	:	The class of connected bipartite graphs with unique perfect matchings.
$\mathcal{M}$	:	The perfect matching in a graph with a unique perfect matching.
$G/\mathcal{M}$	:	The graph obtained from G by contracting each matching edge to a vertex, where G is assumed to be in $\mathcal{H}$.
$\mathcal{H}_g$	:	The class of graphs in $\mathcal{H}$ for which $G/\mathcal{M}$ is bipartite.
Use of apostrophe ($'$)	:	For a $G \in \mathcal{H}$, for a $v \in V(G)$, we use v' to denote the vertex that matches to v, that is, $[v, v'] \in \mathcal{M}$.
Signature matrix	:	A signature matrix is a diagonal matrix with diagonal entries from $\{1, -1\}$.
$A(G)$	:	The adjacency matrix of a graph G.

The following definition has been given by Godsil [15] in 1985.

Definition 2.1.1. Let $G \in \mathcal{H}$. We say G has an *inverse* G^+ if the matrix $A(G)^{-1}$ is signature similar to a nonnegative matrix. That is, if $SA(G)^{-1}S \geq 0$ for some signature matrix S, then we define G^+ to be the weighted graph with adjacency matrix equal to $SA(G)^{-1}S$.

The following theorem has been proved by Godsil in [15, Theorem 2.2]

Theorem 2.1.2. *Let $G \in \mathcal{H}_g$. Then G^+ exists.*

In [15], Godsil posed the following problem.

Problem 2.1.3. *Characterize the graphs in $\mathcal{H}$ which possess inverses.*

In 2007, Akbari and Kirkland[25] characterized the unicyclic graphs in $\mathcal{H}$ which possess inverses. Below we mention their main result recalling the necessary definition.

Definition 2.1.4. [25] Let G be a unicyclic graph in $\mathcal{H}$. A matching edge of G is called a *peg* if it is incident with exactly one vertex on the cycle in G.

Theorem 2.1.5. [25, Theorem 12] *Let G be a unicyclic graph in $\mathcal{H}$. Let the cycle be of length $2m$, and suppose that there are $2k$ pegs.*

(a) $A(G)^{-1}$ is signature similar to a $(0,1)$ matrix if and only if one of the following holds:

(i) $k \geq 2$ and $m-k$ is even,

(ii) $k = 1$, m is even and the vertices on the cycle incident with pegs are adjacent.

(b) $A(G)^{-1}$ is signature similar to a nonnegative matrix if and only if either of conditions (i) and (ii) hold, or $k = 1$ and m is odd.

The following lemma is essentially contained in [15].

Lemma 2.1.6. *Let $G \in \mathcal{H}$. Then the adjacency matrix of G can be expressed as*

$$A(G) = \begin{bmatrix} 0 & B_G \\ B_G^t & 0 \end{bmatrix},$$

where B_G is a lower-triangular, square $(0,1)$-matrix with every diagonal entry equal to 1.

In 2009, Tifenbach and Kirkland supplied necessary and sufficient conditions for a graph in $\mathcal{H}$ to possess an inverse. These necessary and sufficient conditions required some novel constructions involving the directed graphs and the undirected interval graphs. An understanding of their main result will be useful. Below we mention that result recalling the necessary definitions while supplying an illustration.

The following definition of an 'alternating path' will only be used in this section. Later we shall adopt a more general one.

Definition 2.1.7. [25, 28] A path $P(i,j) = [i = i_1, i_2, \ldots, i_k = j]$ from i to j is called an *alternating* path if the edges $[i_1, i_2], [i_3, i_4], \ldots, [i_{k-1}, i_k] \in \mathcal{M}$. It follows that the remaining edges of the path are not from $\mathcal{M}$.

Definition 2.1.8. [16] Let $G \in \mathcal{H}$ and B_G be the matrix as mentioned in Lemma 2.1.6. By D_G, denote the directed graph with adjacency matrix $B_G - I$. Let Γ_G be the directed subgraph of D_G such that the arc $x \to y$ is in Γ_G if $x \to y$ is in D_G and there is no directed x-y-path of length more than 1.

Fix vertices i and j. The undirected interval $G[i,j]$ is the subgraph of graph G induced by the vertices x such that there is an alternating i-j-path in G which contains x.

The following is an illustration of Definition 2.1.8.

Example 2.1.9. Consider the graph G shown in Figure 2.1. Notice that $G \in \mathcal{H}$. The matrix

$$B_G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The digraphs D_G and Γ_G are shown in Figure 2.1. There are only three alternating 1-4′-paths, namely, the paths $[1, 1', 4, 4']$, $[1, 1', 2, 2', 4, 4']$ and $[1, 1', 3, 3', 4, 4']$. Therefore, by Definition 2.1.8, we have $V(G[1, 4']) = \{1, 2, 3, 4, 1', 2', 3', 4'\} = V(G)$. Hence, $G[1, 4'] = G$.

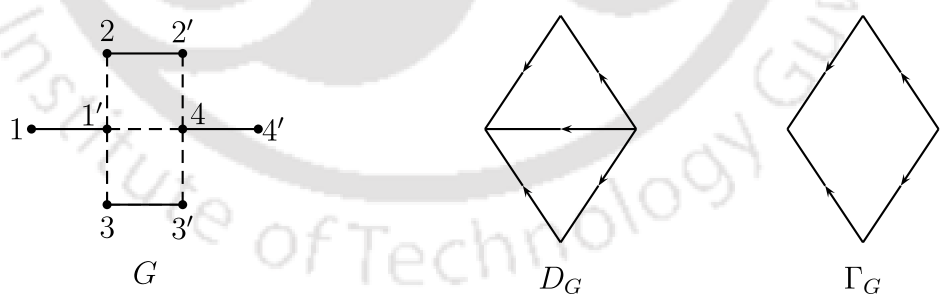


Figure 2.1: A graph $G \in \mathcal{H}$ and its corresponding digraphs D_G and Γ_G.

Theorem 2.1.10. *[16, Theorem 2.6] Let $G \in \mathcal{H}$. Then G^+ exists if and only if both the following conditions hold:*

(i) Each nonempty undirected interval $G[i,j]$ in G possesses an inverse.

(ii) *The digraph* Γ_G *is bipartite.*

Remark 2.1.11. Let $G \in \mathcal{H}$ with $G[i,j] = G$ for some $i, j \in V(G)$. Even if we assume that the digraph Γ_G is bipartite and that each undirected interval $G[u,v] \neq G$ possesses an inverse, we cannot use Theorem 2.1.10 to show that G^+ exists. In fact, with these conditions G^+ may or may not exist, as shown below.

A) Consider the graph G shown in Figure 2.1. The undirected interval $G[1,4'] = G$. Any undirected interval $G[u,v] \neq G[1,4']$ is a tree in $\mathcal{H}$. Hence $G[u,v]$ is invertible. Notice that G is invertible with the signature matrix $S = diag[1,1,-1,-1,1,1,-1,-1]$.

B) Consider the graph shown in Figure 2.2. The undirected interval $H[1,4'] = H$. Any undirected interval $H[u,v] \neq H[1,4']$ is either a tree in $\mathcal{H}$ or a unicyclic graph in $\mathcal{H}$ satisfying condition (ii) of Theorem 2.1.5. Hence $H[u,v]$ is invertible. In Example 2.3.7, we will see that the graph H is not invertible.

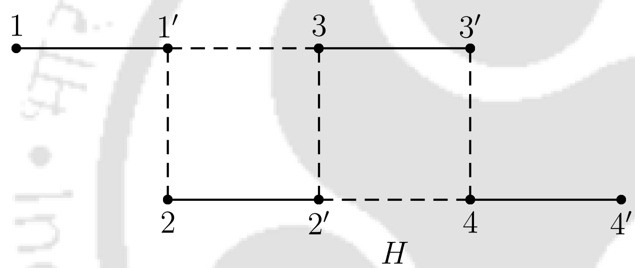


Figure 2.2: A noninvertible graph H in $\mathcal{H}$ with undirected interval $H[1,4'] = H$

The graph G shown in Figure 2.1 is nonunicyclic such that $G/\mathcal{M}$ is not bipartite. So, at this point of thesis we cannot explain 'why the graph has an inverse' by using the known results. (Of course, we can give a signature matrix by trial and error.) We will give a satisfactory explanation in Remark 2.3.16, where we show that this graph is contained in a subclass of $\mathcal{H}$ in which each graph possesses an inverse. It turns out that our class contains the classes $\mathcal{H}_g$ and the unicyclic graphs. Our results can also be seen to complement the results by Tifenbach and Kirkland, thus advancing further towards the main goal. To do that a distinction between nonmatching edges is necessary.

2.2 Types of nonmatching edges

Definition 2.2.1. Let $G \in \mathcal{H}$. A path $P = [u_1, u_2, \ldots, u_k]$ is called an *alternating path* if the edges on P are alternately matching and nonmatching edges, that is, for each i, if $[u_i, u_{i+1}]$ is a matching (resp. nonmatching) edge and $[u_{i+1}, u_{i+2}] \in E(G)$, then $[u_{i+1}, u_{i+2}]$ is a nonmatching (resp. matching) edge.

Let $P = [u_1, u_2, \ldots, u_k]$ be an alternating path. We say P is an *mm-alternating path* (matching-matching-alternating path) if the edges $[u_1, u_2], [u_{k-1}, u_k] \in \mathcal{M}$. We say P is an *nn-alternating path* (nonmatching-nonmatching-alternating path) if the edges $[u_1, u_2], [u_{k-1}, u_k] \notin \mathcal{M}$. The terms *mn-alternating path* and *nm-alternating path* are defined similarly. We note that the mm-alternating paths have been termed as alternating paths in [25, 28].

Remark 2.2.2. The length of each mm-alternating path (resp. nn-alternating path) is odd and the length of each nm-alternating path (resp. mn-alternating path) is even.

Example 2.2.3. Consider the graph G shown in Figure 2.3. The paths $[1, 1', 2, 2', 3, 3']$, $[2', 4, 4', 5, 5', 3]$, $[1', 2, 2', 3, 3']$ and $[1, 1', 2, 2', 3]$ in G are examples of mm-alternating path, nn-alternating path, nm-alternating path and mn-alternating path, respectively.

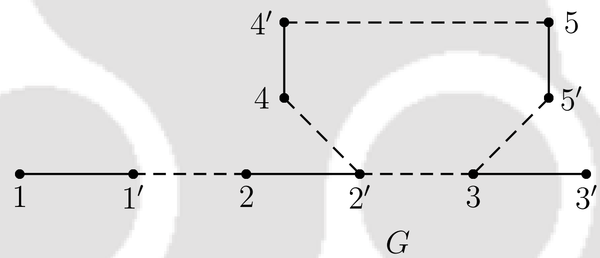


Figure 2.3: . Here the solid edges are the matching edges.

Godsil in [15] has considered the class of graphs $G \in \mathcal{H}$ such that the graph $G/\mathcal{M}$ (obtained by contracting each matching edge to a vertex) is bipartite. We shall consider a larger subclass of graphs in $\mathcal{H}$. In order to describe that subclass we need the following definition.

Definition 2.2.4. Let $G \in \mathcal{H}$ and $[u, v]$ be a nonmatching edge in G. An *extension* at $[u, v]$ is an nn-alternating u-v-path other than $[u, v]$ itself.

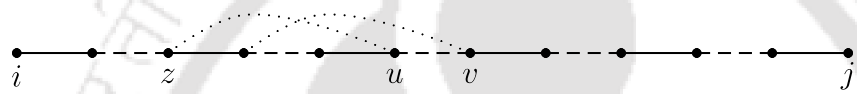
Example 2.2.5. For example, in the graph G shown in Figure 2.3, the path $[2', 4, 4', 5, 5', 3]$ is an extension at $[2', 3]$.

The following is a crucial observation.

Lemma 2.2.6. *Let $G \in \mathcal{H}$ and $P(i,j)$ be an mm-alternating i-j-path. Let $[u,v]$ be a nonmatching edge on $P(i,j)$ and $Q(u,v)$ be an extension at $[u,v]$. Then $Q(u,v)$ contains no vertex of $P(i,j)$ other than u and v. That is, $V(P(i,j)) \cap V(Q(u,v)) = \{u,v\}$.*

Proof: Suppose that $V(P(i,j)) \cap V(Q(u,v)) \supsetneq \{u,v\}$. Imagine traversing $Q(u,v)$ starting from u. Let $z \neq u$ be the first vertex of $P(i,j)$ that we reach. It follows that $z \neq v$. Let $Q(u,z)$ be the u-z-subpath of $Q(u,v)$. Observe that $Q(u,z)$ is an nn-alternating path, because each vertex on $P(i,j)$ is matched to a vertex on $P(i,j)$. Hence the length of $Q(u,z)$ is odd. We have two possibilities.

CASE I. The vertex z is closer to u on $P(i,j)$ than to v. In this case, consider the z-u-subpath $P(z,u)$ of $P(i,j)$. Notice that $\Delta = [Q(u,z), P(z,u)]$ is a cycle.



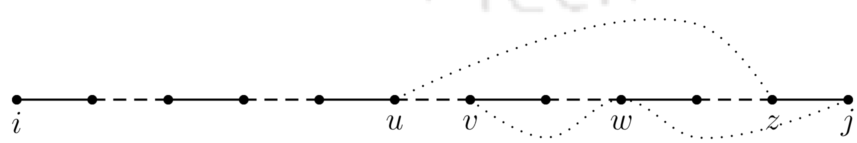
If $P(z,u)$ is an nm-alternating path, then $P(z,u)$ is an even path and hence Δ is an odd cycle, which is not possible. Thus, $P(z,u)$ must be an mm-alternating path. But, in this case Δ contains an equal number of matching edges and nonmatching edges. This implies the existence of more than one perfect matchings, which is a contradiction.

CASE II. The vertex z is closer to v on $P(i,j)$ than to u. In this case, consider the subpath $P(v,z)$ of $P(i,j)$ from v to z. For a path $A = [u_1, u_2, \ldots, u_k]$ let us denote the path $[u_k, u_{k-1}, \ldots, u_1]$ by $\overleftarrow{A}$.

If $P(v,z)$ is an odd path, then $[Q(u,z), \overleftarrow{P(v,z)}, [v,u]]$ gives us a closed cycle of odd length, which is not possible.

Thus $P(v,z)$ must be an even path and therefore an mn-alternating path. Let $Q(z,v)$ be the subpath of $Q(u,v)$ from z to v.

Imagine traversing $Q(z,v)$ from z. Let w be the first vertex on $\overleftarrow{P(v,z)}$ we meet while traversing $Q(z,v)$. Such a vertex exists, as we must reach v eventually.



Now consider the w-z-subpath $P(w,z)$ of $P(v,z)$ and the z-w-subpath $Q(z,w)$ of $Q(z,v)$. Observe that $Q(z,w)$ is an mn-alternating path. Hence its length is even. So the length of $P(w,z)$ must be even, otherwise we have a cycle of odd length. As we know that $P(v,z)$ is an mn-alternating path, it now follows that $P(w,z)$ is an mn-alternating path. Hence the cycle $\Delta = [P(w,z), Q(z,w)]$ contains an equal number of matching and nonmatching edges. That implies the existence of more than one perfect matchings, which is a contradiction. The proof is complete. ■

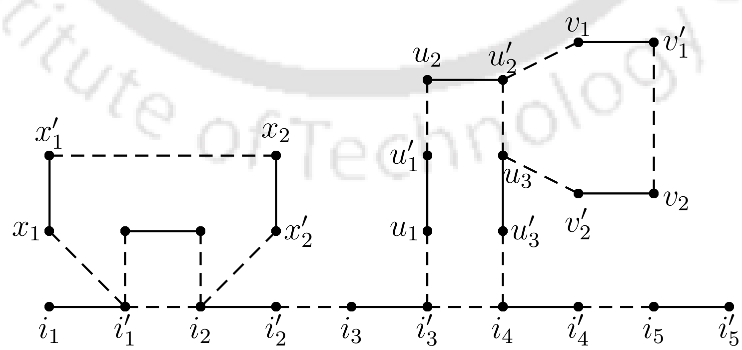
The following is an immediate corollary of Lemma 2.2.6.

Corollary 2.2.7. *Let $G \in \mathcal{H}$ and $P(i,j)$ be an mm-alternating i-j-path. Let $e = [u,v]$ and $f = [x,y]$ be two nonmatching edges on $P(i,j)$. Let Q_e and Q_f be two extensions at e and f, respectively. Then Q_e and Q_f have no vertices in common. That is, $V(Q_e) \cap V(Q_f) = \emptyset$.*

Proof: Assume that the path $P(i,j) = [i, \ldots, u, v, \ldots, x, y, \ldots, j]$ and the extension at $e = [u,v]$ is $Q_e = [u, u_1, \ldots, u_k, v]$. By Lemma 2.2.6, $u_1, \ldots, u_k$ are not on $P(i,j)$. Hence $\hat{P} = [i, \ldots, u, u_1, \ldots, u_k, v, \ldots, x, y, \ldots, j]$ is an mm-alternating i-j-path. Notice that $\hat{P}$ contains the edge $f = [x,y]$ and Q_f is an extension at f. Applying Lemma 2.2.6 we see that $V(Q_f) \cap V(\hat{P}) = \{x, y\}$; that is, Q_f contains none of $u, u_1, \ldots, u_k, v$. Hence, $V(Q_e) \cap V(Q_f) = \emptyset$. ■

The following is a natural question. Let $G \in \mathcal{H}$ and $P(i,j)$ be an mm-alternating i-j-path. Let e be a nonmatching edge on $P(i,j)$ and let Q_1 and Q_2 be two extensions at e. Is it possible that Q_1 and Q_2 have some vertices or edges in common? The answer is 'yes' as shown in the Example 2.2.8.

Example 2.2.8. Consider the graph G shown in Figure 2.4. It is in $\mathcal{H}$. Consider the nonmatching edge $e = [i_3', i_4]$ on the mm-alternating i_3-i_5'-path. Take $Q_1 = [i_3', u_1, u_1', u_2, u_2', u_3, u_3', i_4]$ and $Q_2 = [i_3', u_1, u_1', u_2, u_2', v_1, v_1', v_2, v_2', u_3, u_3', i_4]$. They have an edge $[u_1', u_2]$ in common.



G

Figure 2.4: Here the solid edges are the matching edges.

Using the extensions we now can differentiate between the nonmatching edges of a graph in $\mathcal{H}$.

Definition 2.2.9. a) Let $G \in \mathcal{H}$ and $[u, v]$ be a nonmatching edge. An extension at $[u, v]$ is called *even type* (resp. *odd type*) if the number of nonmatching edges on that extension is even (resp. odd).

b) Let $[u, v]$ be a nonmatching edge. We say $[u, v]$ is an *odd type* edge, if either there are no extensions at $[u, v]$ or each extension at $[u, v]$ is of odd type. An odd type edge $[u, v]$ is said to be a *simple odd type* edge if there are no extensions at $[u, v]$.

c) Let $[u, v]$ be a nonmatching edge. We say $[u, v]$ is an *even type* edge, if each extension at $[u, v]$ is of even type.

d) Let $[u, v]$ be a nonmatching edge. We say $[u, v]$ is a *mixed type* edge, if there is an even type extension and an odd type extension at $[u, v]$.

Example 2.2.10. Consider the graph G shown in Figure 2.4. In the graph G, the extensions $[i'_3, u_1, u'_1, u_2, u'_2, u_3, u'_3, i_4]$ and $[i'_3, u_1, u'_1, u_2, u'_2, v_1, v'_1, v_2, v'_2, u_3, u'_3, i_4]$ are two even type extensions at $[i'_3, i_4]$. These are the only extensions at $[i'_3, i_4]$. Hence, $[i'_3, i_4]$ is an even type edge. The edges $[i'_1, i_2]$ and $[u'_2, u_3]$ are mixed type and odd type, respectively. Every other nonmatching edge is of simple odd type.

We have the following motivating remark.

Remark 2.2.11. Consider the class of graphs $\mathcal{H}_g$, namely, the graphs $G \in \mathcal{H}$ such that $G/\mathcal{M}$ is bipartite. Then it is easy to see that G does not have an even type extension. We shall consider a class where we allow some even type extensions.

Just for curiosity, one may ask 'if $G \in \mathcal{H}$ in which we do not have any even type extension, is $G \in \mathcal{H}_g$?' The answer is 'no' as can be seen in Figure 2.5. In Section 2.3, we will see that there is no invertible graph $G \in \mathcal{H}$ without even type extensions such that $G/\mathcal{M}$ is not bipartite.

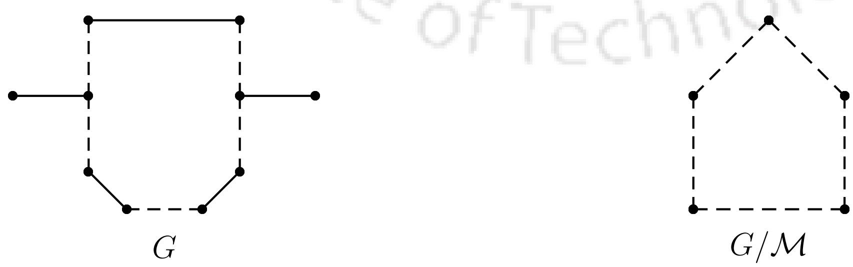


Figure 2.5: A graph G without an even type extension for which $G/\mathcal{M}$ is not bipartite. Here the solid edges are the matching edges.

The following is another crucial observation.

Lemma 2.2.12. *Let $G \in \mathcal{H}$ and $[u, v]$ be an odd type or even type edge in G. Let $Q(u, v)$ be an extension at $[u, v]$. Then each nonmatching edge on $Q(u, v)$ is odd type.*

Proof: First we assume that $[u, v]$ is odd type. Then $Q(u, v)$ is an odd type extension. As $[u, v]$ is a nonmatching edge, it can be viewed as an edge on an mm-alternating path $[u', u, v, v']$. Hence the path $[u', Q(u, v), v']$ is an mm-alternating u'-v'-path, by Lemma 2.2.6.

Let $[u_1, v_1]$ be a nonmatching edge on $Q(u, v)$. Suppose that $[u_1, v_1]$ is even type. Then there is an even type extension $Q(u_1, v_1)$ at $[u_1, v_1]$. As $[u', Q(u, v), v']$ is an mm-alternating u'-v'-path, we see that $Q(u, v) - [u_1, v_1] \cup Q(u_1, v_1)$ is an extension at $[u, v]$ and the total number of nonmatching edges in this extension is sum of the total number of nonmatching edges in $Q(u, v) - [u_1, v_1]$ and $Q(u_1, v_1)$ which is even. This says that the edge $[u, v]$ is mixed type, a contradiction. Therefore $[u_1, v_1]$ must be an odd type edge.

A similar argument works if $[u, v]$ is even type. ∎

We have the following useful corollary.

Corollary 2.2.13. *Let $G \in \mathcal{H}$ and let $\mathcal{E}$ be the set of all even type edges in G. Then $G - \mathcal{E}$ is connected.*

Definition 2.2.14. A *minimal path* in a graph $G \in \mathcal{H}$ from a vertex u to a vertex v is an mm-alternating u-v-path which does not contain an even type extension (at some nonmatching edge in G).

Example 2.2.15. In the graph G shown in Figure 2.4, $[i_1, i'_1, i_2, i'_2, i_3, i'_3, i_4, i'_4, i_5, i'_5]$ is an example of a minimal path from i_1 to i'_5 and $[i_1, i'_1, i_2, i'_2, i_3, i'_3, u_1, u'_1, u_2, u'_2, u_3, u'_3, i_4, i'_4, i_5, i'_5]$ is an example of an mm-alternating i-j path which is not a minimal path.

Note that there can be more than one minimal path from a vertex to another vertex. For example, in the graph shown in Figure 2.4, $[i_1, i'_1, i_2, i'_2, i_3, i'_3, i_4, i'_4, i_5, i'_5]$ and $[i_1, i'_1, x_1, x'_1, x_2, x'_2, i_2, i'_2, i_3, i'_3, i_4, i'_4, i_5, i'_5]$ are two minimal paths from i_1 to i'_5.

Definition 2.2.16. Let $\mathcal{H}_{nm}$ be the class of all graphs $G \in \mathcal{H}$ such that G has no mixed type edges. Here 'nm' is an abbreviation of 'no mixed type edges'.

Example 2.2.17. In the following table we list the graphs used in different figures till now and mention whether they are in the class $\mathcal{H}_{nm}$.

Graphs G	in $\mathcal{H}_{nm}$?
Figure 2.1	Yes
Figure 2.2	Yes
Figure 2.3	Yes
Figure 2.4	No (mixed type edge $[i_1', i_2]$)
Figure 2.5	Yes

Remark 2.2.18. If $G \in \mathcal{H}_{nm}$, then by Lemma 2.2.12, the nonmatching edges of any extension in G are always odd type. Henceforth, we shall consider graphs from this class.

The following result which says that an mm-alternating path is created from a minimal path will be used in the sequel.

Lemma 2.2.19. *Let $G \in \mathcal{H}_{nm}$. Let $P(i,j)$ be an mm-alternating i-j-path. Then there exists a minimal i-j-path $P_m(i,j)$ and a set F of even type edges on $P_m(i,j)$ such that $P(i,j)$ is created from $P_m(i,j)$ by replacing each edge $f \in F$ with an extension Q_f at f.*

Proof: If $P(i,j)$ does not contain an even type extension then $P(i,j)$ itself is a minimal path and the statement is vacuously true. Suppose that $P(i,j)$ contains some even type extensions. Traverse the path $P(i,j)$ starting from i. Assume that $P(i,j) = [v_0 = i, v_1, \ldots, v_k = j]$. Suppose that $[v_l, \ldots, v_p]$ is the first even type extension we meet. Then consider the path $P_1(i,j) = [v_0, \ldots, v_l, v_p, \ldots, v_k]$. If $P_1(i,j)$ is a minimal path, then take $F = \{[v_l, v_p]\}$. The statement holds true in this case.

Assume that $P_1(i,j)$ is not a minimal path. Observe that the subpath $[v_0, \ldots, v_l]$ of $P_1(i,j)$ cannot contain an even type extension as $[v_l, \ldots, v_p]$ was the first even type extension we met while traversing $P(i,j)$. Furthermore, as the edge $[v_l, v_p]$ is even type, it cannot be in any extension, by Lemma 2.2.12.

Now traverse $P_1(i,j)$ starting form i. By the argument in the previous paragraph, the first even type extension we meet on $P_1(i,j)$ must be a subpath of $P(v_p, v_k) = [v_p, \ldots, v_k]$.

Notice that $P(v_p, v_k)$ is an mm-alternating v_p-v_k-path. Applying induction on the length, we see that there is a minimal path $P_m(v_p, v_k)$ and a set F' of even type edges on $P_m(v_p, v_k)$ such that $P(v_p, v_k)$ is created from $P_m(v_p, v_k)$ by replacing each edge $f \in F'$ with one of its extensions Q_f.

Put $F = F' \cup \{[v_l, v_p]\}$ and $P_m(i,j) = [v_0 = i, \ldots, v_l, P_m(v_p, v_k)]$. The proof is complete by induction. ∎

The following is a natural question. Let $G \in \mathcal{H}_{nm}$ and $P(i,j)$ be an mm-alternating i-j-path. Is it possible that $P(i,j)$ is created by two different minimal paths? The answer is 'yes' as shown in the following example.

Example 2.2.20. This example shows that a minimal path of an mm-alternating path need not be unique. Consider the graph G shown in Figure 2.2. The graph G is in $\mathcal{H}_{nm}$. The mm-alternating path $[1, 1', 2, 2', 3, 3', 4, 4']$ can be created from the minimal path $[1, 1', 3, 3', 4, 4']$ by replacing the even type edge $[1', 3]$ with its extension $[1', 2, 2', 3]$. It can also be created from the minimal path $[1, 1', 2, 2', 4, 4']$ by replacing the even type edge $[2', 4]$ with its extension $[2', 3, 3', 4]$. Notice that the even type extension $[1', 2, 2', 3]$ at $[1', 3]$ and the even type extension $[2', 3, 3', 4]$ at $[2', 4]$ have an odd type edge in common, namely, the edge $[2', 3]$.

Remark 2.2.21. Let $G \in \mathcal{H}_{nm}$. Suppose that $P(u,v)$ can be created from the minimal path $P_m(u,v)$ by replacing a set F of even type edges on it with one extension at each edge. Assume that $F = \{[u_1, v_1], \ldots, [u_k, v_k]\}$. Then taking $P(u,v)$ and deleting the subpaths from u_i to v_i (keeping the endvertices u_i and v_i) for each $i = 1, \ldots, k$, we get $P_m(u,v) - F$.

The following result supplies a sufficient condition for the uniqueness of a minimal path of an mm-alternating path.

Lemma 2.2.22. *Let $G \in \mathcal{H}_{nm}$. Assume that 'for each pair of even type edges e and f, and for each extensions Q_e at e and Q_f at f, we have no odd type edge common to Q_e and Q_f'. Let $P(i,j)$ be an mm-alternating i-j-path. Then a minimal path from which $P(i,j)$ is created as described in the statement of Lemma 2.2.19, is unique.*

Proof: Suppose that $P(i,j)$ can be created from the minimal path $P_m(i,j)$ by replacing a set F of even type edges on it with one extension at each edge. Suppose also that $P(i,j)$ can be created from another minimal path $P'_m(i,j)$ by replacing a set F' of even type edges on it with one extension at each edge.

Assume that $F \neq F'$. So, without loss, let $g \in F \setminus F'$ and Q_g be the extension used at g. By construction, $g \notin E(P(i,j))$ and Q_g is a subpath of $P(i,j)$. Notice that all of the nonmatching edges of Q_g cannot be on $P'_m(i,j)$, otherwise, Q_g will be a subpath of $P'_m(i,j)$, contradicting the minimality of $P'_m(i,j)$. Let h be a nonmatching edge on Q_g which is not on $P'_m(i,j)$. As h is on $P(i,j)$, we must have an edge $e' \in F'$ whose extension $Q_{e'}$ contains h. Hence the odd type edge h is common to Q_g and $Q_{e'}$, contrary to our assumption.

Thus $F = F' = \{[u_1, v_1], \ldots, [u_k, v_k]\}$ (say). By Remark 2.2.21, we see that $P_m(i,j) - F = P'_m(i,j) - F'$. So $P'_m(i,j) = P_m(i,j)$, a contradiction. ∎

Remark 2.2.23. The converse of Lemma 2.2.22 is not true. There are graphs $G \in \mathcal{H}_{nm}$ in which mm-alternating paths are created from unique minimal paths but extensions at two distinct even type edges have an odd type edge in common, as shown in Figure 2.6. Here e and f are the only even type edges.

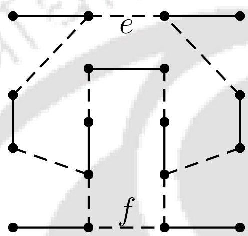


Figure 2.6: Here the solid lines are the matching edges.

Now we are in a position to describe the class of graphs which we wanted to consider.

Definition 2.2.24. By $\mathcal{H}_{nmc}$, we denote the class of graphs G in $\mathcal{H}$ such that

i) G has no mixed type edges,

ii) G satisfies the condition 'C',

condition C: *The extensions at two distinct even type edges never have an odd type edge in common.*
Here 'nmc' is an abbreviation of 'no mixed type edges and a condition'. Thus

$$\mathcal{H}_{nmc} = \{G \in \mathcal{H} \mid G \text{ has no mixed type edges and } G \text{ satisfies condition C}\}.$$

Example 2.2.25. In the following table we list the graphs used in different figures till now and mention whether they are in the class $\mathcal{H}_{nmc}$.

Graphs G	in $\mathcal{H}_{nmc}$?	Justifications
Figure 2.1	Yes	Exactly two edge disjoint extensions which are even type
Figure 2.2	No	The graph H does not satisfy condition 'C'
Figure 2.3	Yes	The graph G has exactly one extension which is odd type
Figure 2.4	No	There is a mixed type edge $[i_1', i_2]$
Figure 2.5	Yes	No even type extensions
Figure 2.6	No	The graph G does not satisfy condition 'C'

2.3 Inverses of graphs in $\mathcal{H}_{nmc}$

The following description of the inverse of the adjacency matrix of a graph in $\mathcal{H}$ was given in [25, 28]. We follow the convention that sum over an empty class is zero.

Lemma 2.3.1. *Let $G \in \mathcal{H}$. Let $B = [b_{ij}]$, where*

$$b_{ij} = \sum_{P(i,j)\ \mathrm{mm-alternating}} (-1)^{(\|P(i,j)\|-1)/2},$$

where $\|P(i,j)\|$ is the number of edges in the i-j-path $P(i,j)$. Then $B = A(G)^{-1}$.

The following is an immediate observation.

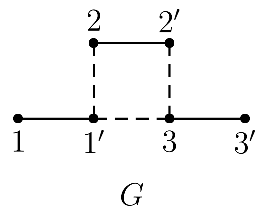
Lemma 2.3.2. *Let $G \in \mathcal{H}$ such that $SA(G)^{-1}S \geq 0$ for some signature matrix S. Let s_i mean the diagonal entry of S for the vertex i. The following statements are true.*
a) If $[u,v] \in \mathcal{M}$, then $s_u = s_v$.
b) If $[u,v]$ is an odd type edge, then $s_u = -s_v$.

Proof: a) First assume $[u,v] \in \mathcal{M}$. Then it is the only *mm*-alternating u-v-path. Hence by Lemma 2.3.1, $A(G)^{-1}_{u,v} = 1$. As $SA(G)^{-1}S \geq 0$, it follows that $s_u A(G)^{-1}_{u,v} s_v = 1$. So $s_u s_v = 1$.

b) Suppose $[u,v]$ is an odd type edge. Let u' and v' be the vertices such that $[u',u],[v,v'] \in \mathcal{M}$. Since $[u,v]$ is odd type, by use of Lemma 2.3.1, we see that $A(G)^{-1}_{u',v'} < 0$. As $s_{u'}A(G)^{-1}_{u',v'}s_{v'} > 0$, we get $s_{u'}s_{v'} < 0$. By Item a), we get $s_u s_v < 0$. ■

Definition 2.3.3. Let $G \in \mathcal{H}_{nm}$ and $\mathcal{E}$ be the set of all even type edges. Then by $(G-\mathcal{E})/\mathcal{M}$ denote the graph obtained by deleting all the even type edges and then contracting each matching edge to a single vertex.

Example 2.3.4. Consider the graph G shown in Figure 2.7 which has exactly one even type edge $[1', 3]$, that is, $\mathcal{E} = \{[1', 3]\}$. The graph $(G - [1', 3])/\mathcal{M}$ is the path with three vertices which is shown in Figure 2.7.



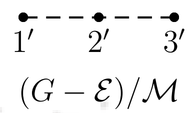


Figure 2.7: Example of a graph $G \in \mathcal{H}$ and the graph $(G - \mathcal{E})/\mathcal{M}$.

The following is a necessary condition for a graph $G \in \mathcal{H}_{nm}$ to have an inverse.

Lemma 2.3.5. *Let $G \in \mathcal{H}_{nm}$ for which G^+ exists. Then $(G - \mathcal{E})/\mathcal{M}$ is a connected, bipartite graph.*

Proof: By Corollary 2.2.13, $G - \mathcal{E}$ is a connected graph containing the matching edges and the odd type nonmatching edges of G. Thus, in view of Lemma 2.3.2, any edge $[u, v] \in G - \mathcal{E}$ with $s_u = s_v$ must be a matching edge (because over odd type edges s_x changes sign). Thus contracting all the matching edges, we see that each edge $[u, v]$ in $(G - \mathcal{E})/\mathcal{M}$ has $s_u s_v = -1$. Taking $X = \{v \in (G - \mathcal{E})/\mathcal{M} \mid s_v > 0\}$ and $Y = \{v \in (G - \mathcal{E})/\mathcal{M} \mid s_v < 0\}$, we get a bipartition. ■

Example 2.3.6. In the following table we list the graphs used in different figures till now and mention whether their corresponding $(G - \mathcal{E})/\mathcal{M}$ graph is bipartite.

Graph G	in $\mathcal{H}_{nm}$?	$(G - \mathcal{E})/\mathcal{M}$
Figure 2.1	Yes	Bipartite
Figure 2.2	Yes	Bipartite
Figure 2.3	Yes	Bipartite
Figure 2.4	No	Nonbipartite
Figure 2.5	Yes	Nonbipartite
Figure 2.6	Yes	Bipartite

The converse of Lemma 2.3.5 is not true as shown in the following example.

Example 2.3.7. Consider the graph H shown in Figure 2.2. The edges $[1', 3]$ and $[2', 4]$ are the even type edges, whose deletion makes the graph a tree. Hence $H - \mathcal{E}$ and so $(H - \mathcal{E})/\mathcal{M}$ is bipartite. However, for this graph H^+ does not exist. To see this put $B = A(H)^{-1}$. Suppose that H^+ exists. Then there is a signature matrix S such that $SBS \geq 0$. Notice that $B(2,3') = -1 = B(3,4') = B(1,2') = B(2,3')$ and $B(1,4') = 1$. Then we have $s_2s'_3 = s_3s'_4 = s_2s'_4 = -1$ and $s_1s'_4 = 1$. Therefore $s_2^2s_3^2s_{3'}^2s_4^2s_{4'}^2 = -1$ which is not possible. Hence H^+ does not exist.

Remark 2.3.8. Let $G \in \mathcal{H}_{nm}$ be such that $SA(G)^{-1}S \geq 0$ for some signature matrix S. Let $\mathcal{E}$ be the set of all even type edges in G. In view of Corollary 2.2.13 and Lemma 2.3.5, we see that $G-\mathcal{E}$ is connected and $(G-\mathcal{E})^+$ exists. Thus, in view of Lemma 2.3.2, we see that S and $-S$ are the only signature matrices for which $SA(G)^{-1}S \geq 0$. This has been observed by Tifenbach and Kirkland in [16].

The following result which gives some structural information about graphs $G \in \mathcal{H}_{nm}$ for which $(G-\mathcal{E})/\mathcal{M}$ is bipartite, will be used later.

Lemma 2.3.9. *Let $G \in \mathcal{H}_{nm}$ such that $(G-\mathcal{E})/\mathcal{M}$ is bipartite. Then G does not contain a cycle which has an odd number of odd type edges. In particular, if one path from u to v (some vertices) contains an odd (resp. even) many odd type edges, then each path from u to v must contain an odd (resp. even) many odd type edges.*

Proof: First suppose that there is a cycle which contains an odd number of odd type edges, an odd number of matching edges and no even type edges. Then $(G-\mathcal{E})/\mathcal{M}$ contains an odd cycle, which is not possible.

Now suppose that there is a cycle C_G which contains odd number of odd type edges, some matching edges and some even type edges. We replace each even type edge on C_G by one of its extensions. (Recall that by Lemma 2.2.12, each nonmatching edge on the extension is odd type). Then we get a closed walk with an odd number of odd type edges and some matching edges. This closed walk gives us a cycle which contains an odd number of odd type edges and some matching edges. This is not possible, by the first part of the proof. The next statement is easy to deduce. ∎

In [15] a class of graphs $G \in \mathcal{H}$ is supplied for which G^+ exists and posed a question to characterize the graphs $G \in \mathcal{H}$ for which G^+ exists. In [25], Akbari and Kirkland have given a characterization of unicyclic graphs $G \in \mathcal{H}$ for which G^+ exists.

In this section, we answer the question for the class $\mathcal{H}_{nmc}$ (graphs in $\mathcal{H}$ such that each nonmatching edge is either odd type or even type and such that the extensions of two distinct even type edges never have an odd type edge in common). In view of Lemma 2.3.5, graphs G for which G^+ exists, must satisfy that '$(G-\mathcal{E})/\mathcal{M}$ is bipartite' (recall that $\mathcal{E}$ is the set of all even types edges in G).

The following is another motivation to investigate the class $\mathcal{H}_{nmc}$.

Remark 2.3.10. The class $\mathcal{H}_{nmc}$ contains the unicyclic graphs in $\mathcal{H}$. This can be seen as follows. Let G be a unicyclic graph in $\mathcal{H}$. It is clear that each nonmatching edge in G is either odd type or even type, otherwise G has at least two cycles. Next, as G has only one cycle, it can have at most one even type extension. Therefore we do not have two distinct even type extensions at two distinct even type edges having an odd type edge in common. Hence $G \in \mathcal{H}_{nmc}$.

The following is an extension of Theorem 2.2 of [15] and Theorem 12 of [25]. It is our main result for this chapter.

Theorem 2.3.11. *Let $G \in \mathcal{H}_{nmc}$ such that $(G-\mathcal{E})/\mathcal{M}$ is bipartite. Then the inverse G^+ exists.*

Proof: As $(G-\mathcal{E})/\mathcal{M}$ is bipartite, take the vertex 1, define $s_1 = 1$. Now to define s_u, take any path from 1 to u. If it has odd many odd type edges define $s_u = -1$, otherwise define $s_u = 1$. By Lemma 2.3.9, the matrix S is well defined.

Now we want to show that $SA(G)^{-1}S \geq 0$. Suppose, by the way of contradiction, that $SA(G)^{-1}S \ngeq 0$. That is, there exist i and j such that $s_i A(G)^{-1}_{i,j} s_j < 0$.

We have two possibilities.

CASE I. The entry $A(G)^{-1}_{i,j} < 0$. Then $s_i = s_j$. So the parity of the number of odd type edges on any path from 1 to i must be the same with that of any path from 1 to j. It follows that any path from i to j must contain an even number of odd type edges.

By Lemma 2.3.1, $A(G)^{-1}_{i,j}$ is obtained from the mm-alternating i-j-paths. By Lemmas 2.2.19 and 2.2.22, each mm-alternating i-j- path can be created uniquely from a minimal path. Let $P^1_m(i,j), P^2_m(i,j), \ldots, P^M_m(i,j)$ be the minimal paths from i to j. Let $\mathcal{P}^r(i,j)$ be the set of all mm-alternating i-j-paths which are created from $P^r_m(i,j)$, for $r = 1, \ldots, M$. Using Lemmas 2.2.19 and 2.2.22, we have $|\mathcal{P}(i,j)| = \sum_{r=1}^{M} |\mathcal{P}^r(i,j)|$. Using Lemma 2.3.1, we have

$$A(G)^{-1}_{i,j} = \sum_{r=1}^{M} \sum_{P(i,j)\in\mathcal{P}^r(i,j)} (-1)^{\frac{\|P(i,j)\|-1}{2}}, \tag{2.1}$$

where $\sum_{P(i,j)\in\mathcal{P}^r(i,j)} (-1)^{\frac{\|P(i,j)\|-1}{2}}$ is the contribution to $A(G)^{-1}_{i,j}$ coming from the minimal path $P^r_m(i,j)$. Now there are two further possibilities.

CASE I(A). The minimal path $P_m^r(i,j)$ contains an odd number of nonmatching edges. We already know that the number of odd type edges on $P_m^r(i,j)$ is even. So there is an odd number of even type edges on $P_m^r(i,j)$. Let them be $e_1, e_2, \ldots, e_k$, where k is odd. Let $m_l \geq 1$ be the number of extensions (these are even type) at the edge e_l, for $l = 1, \ldots, k$.

Suppose that we choose the even type edges $e_{i_1}, \ldots, e_{i_p}$ from $e_1, e_2, \ldots, e_k$ and create an mm-alternating i-j-path by using one extension for each of the chosen even type edges. Then we can create $m_{i_1} \cdots m_{i_p}$ many such mm-alternating i-j-paths and each such path has an odd (resp. even) number of nonmatching edges if p is even (resp. odd).

Recall that the pth elementary symmetric function of the k numbers $m_1, m_2, \ldots, m_k$, $p \leq k$, is $S_p(m_1, m_2, \ldots, m_k) = \sum_{1 \leq i_1 \leq \cdots \leq i_p \leq k} \prod_{j=1}^{p} m_{i_j}$.

Thus the total contribution of $\mathcal{P}^r(i,j)$, the set of mm-alternating i-j-paths that are created from $P_m^r(i,j)$, to $A(G)^{-1}_{i,j}$ is

$$\begin{aligned} &-1 \text{ (for } p=0) \ + S_1(m_1, m_2, \ldots, m_k) \text{ (for } p=1) \\ &-S_2(m_1, m_2, \ldots, m_k) \text{ (for } p=2) \ + \cdots + (-1)^{k+1} S_k(m_1, m_2, \ldots, m_k) \text{ (for } p=k) \\ &= -(1-m_1)(1-m_2)\cdots(1-m_k) \geq 0 \text{ (as } k \text{ is odd).} \end{aligned}$$

CASE I(B). The minimal path $P_m^r(i,j)$ contains an even number of nonmatching edges. We already know that $P_m^r(i,j)$ has an even number of odd type edges. So $P_m^r(i,j)$ has an even number of even type edges. Let them be $e_1, \ldots, e_k$, where k is even. Let $m_l \geq 1$ be the number of extensions at e_l, $l = 1, 2, \ldots, k$. Following an argument similar to that used in CASE I(A), we see that the total contribution of $\mathcal{P}^r(i,j)$ to $A(G)^{-1}_{i,j}$ is

$$(1-m_1)(1-m_2)\cdots(1-m_k) \geq 0 \text{ (as } k \text{ is even).}$$

Hence $A(G)^{-1}_{i,j} \geq 0$, by (2.1). This contradicts the hypothesis that $A(G)^{-1}_{i,j} < 0$.

CASE II. The entry $A(G)^{-1}_{i,j} > 0$. Using similar argument as in CASE I, we arrive at a contradiction to this hypothesis that $A(G)^{-1}_{i,j} > 0$.

Hence we conclude that $SA(G)^{-1}S \geq 0$. The proof is complete. ∎

In view of Lemma 2.3.5 and Theorem 2.3.11, we have the following corollary.

Corollary 2.3.12. *Let $G \in \mathcal{H}_{nmc}$. Then G^+ exists if and only if $(G - \mathcal{E})/\mathcal{M}$ is bipartite.*

Example 2.3.13. Here we give an example of a graph $G \in \mathcal{H}_{nmc}$ and its inverse G^+. The graph G shown in Figure 2.8 is in $\mathcal{H}_{nmc}$. It is clear from the picture that

$(G-\mathcal{E})/\mathcal{M}$ is bipartite. Then by Corollary 2.3.12, G^+ exists and the signature matrix is $S = diag[1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1]$. The inverse graph G^+ is a weighted graph. In the inverse graph G^+, the weights $\text{w}([1,4']) = 2$ and $\text{w}([x,y]) = 1$ for any edge in G^+ other than $[1,4']$. It is clear that G^+ is a connected, weighted, bipartite graph with the same unique perfect matching $\mathcal{M}$.

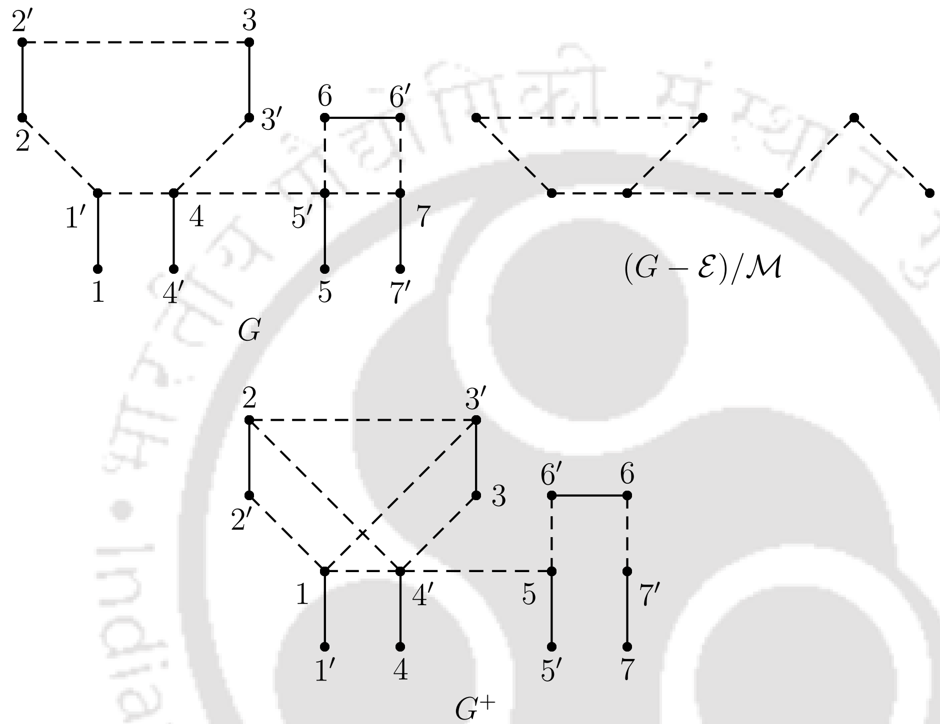


Figure 2.8: Here the solid edges are the matching edges.

Remark 2.3.14. Let G be a graph which is obtained by taking union of finite number of vertex disjoint graphs $G_1, \ldots, G_k$, where $G_i \in \mathcal{H}_{nmc}$ for $i = 1, \ldots, k$. One can easily show that G^+ exists if and only if G_i^+ exists for all $i = 1, \ldots, k$.

The following result is a complete characterization of unicyclic graphs in $\mathcal{H}$ that possess inverses.

Corollary 2.3.15. *Let G be a unicyclic graph in $\mathcal{H}$. Then G^+ exists if and only if either G^+ has exactly one even type edge or $G \in \mathcal{H}_g$.*

Proof: Using Remark 2.3.10, the graph G has atmost one even type extension and G must be in $\mathcal{H}_{nmc}$. Using Corollary 2.3.12, we see that G^+ exists if and only if either G has exactly one even type edge or $G \in \mathcal{H}_g$. ∎

Remark 2.3.16. Now we can explain why the graph G shown in Figure 2.1 has an inverse. It is clear that the graph $G \in \mathcal{H}_{nmc}$ and the nonmatching edge $[2,3]$ is the only even type edge and $(G - [2,3])/\mathcal{M}$ is bipartite. By using Corollary 2.3.12, G^+ exists.

The following is an example of a graph $G \in \mathcal{H}$ such that G^+ exists but $G \notin \mathcal{H}_{nmc}$.

Example 2.3.17. Consider the graph G shown in Figure 2.9. The extensions $[1', 2, 2', 3]$ at $[1', 3]$ and $[2', 3, 3', 4]$ at $[2', 4]$ have an odd type edge $[2', 3]$ in common, so $G \notin \mathcal{H}_{nmc}$. The graph G has an inverse where $S = diag[1, 1, -1, -1, 1, 1, -1, -1]$. The graph itself is an undirected interval graph $G[1, 4']$. At this point of thesis, we cannot explain why the graph has an inverse by using the known results.

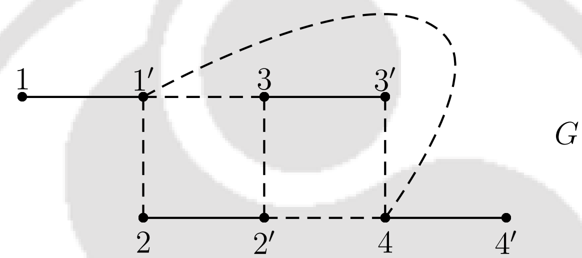


Figure 2.9: Here the solid edges are the matching edges.

2.4 Conclusion

In [15], Godsil introduced the notion of graph inverse and supplied a class of graphs in $\mathcal{H}$ which possess inverses. He posed the problem of characterizing graphs in $\mathcal{H}$ which possess inverses. In [25], the authors provided a complete characterization of unicyclic graphs in $\mathcal{H}$ which possess inverses. In [16], the authors supplied necessary and sufficient conditions for a graph in $\mathcal{H}$ to possess an inverse, utilizing constructions derived from the graph itself. However, there are many graphs in $\mathcal{H}$ for which these conditions cannot help; for example, the graph in Figure 2.1. We have noticed that a nonmatching edge of a graph in $\mathcal{H}$ can be of three types: odd type, even type and mixed type. We have defined $\mathcal{H}_{nm}$ is the class of graphs with no mixed type edges and $\mathcal{H}_{nmc}$ is the class of graphs with no mixed type edges and satisfy a technical condition 'condition C'. We proved that $G \in \mathcal{H}_{nmc}$ is invertible if and only if $(G - \mathcal{E})/\mathcal{M}$ is bipartite. This result extended some known results in [15, 25], providing us with a larger class of graphs possessing inverses. In the process, we have obtained an explanation for the invertibility of the graph shown in Figure 2.1. However, there are yet graphs in $\mathcal{H} \setminus \mathcal{H}P_{nmc}$ which possess inverses; for example, the graph in Figure 2.9. Further characterizations of invertible graphs in $\mathcal{H}$ remain to be provided.

Chapter 3

Self-inverse graphs

3.1 Preliminaries

Let $G \in \mathcal{H}$, if G^+ exists and is isomorphic to G, we say that G is self-inverse. We are interested in characterizing self-inverse graphs in $\mathcal{H}$. This question, for the class $\mathcal{H}_g$ was asked by Godsil in 1985 and has already been answered by Simion and Cao in [26]. The following theorem was proved.

Theorem 3.1.1. [26] *Let $G \in \mathcal{H}_g$. Then $G \cong G^+$ if and only if G is a simple corona, that is, $G = G_1 \circ K_1$ where G_1 is a connected bipartite graph.*

In this chapter we aim to enlarge the subclass of self-inverse graphs within $\mathcal{H}$ containing $\mathcal{H}_g$.

In the literature, one can find other instances where $G \cong G^+$ have been studied. For example, take the class of unicyclic graphs in $\mathcal{H}$, which is not a subclass of the class $\mathcal{H}_g$. In 2009, Tifenbach and Kirkland[16] characterized the unicyclic graphs $G \in \mathcal{H}$ which satisfy $G \cong G^+$. They utilized some constructions derived from the graph itself. We note here that in their article they have used the term *self-dual* instead of *self-inverse*. Let us recall some terminologies used by them. Let $G \in \mathcal{H}$. Then with a renaming of the vertices of G, we can have the following expression of the adjacency matrix

$$A(G) = \begin{bmatrix} 0 & B_G \\ B_G^t & 0 \end{bmatrix},$$

where B_G is a lower triangular matrix with each diagonal entry 1. Let D_G be directed graph with adjacency matrix $B_G - I$ and Γ_G be the spanning subgraph of D_G which contains those arcs $x \to y$, for which there is no directed x-y-path of length more than one in D_G. The following results were obtained.

Theorem 3.1.2. [16] *Let G be a unicyclic graph in $\mathcal{H}$ that is not the corona of a bipartite graph. Then, G^+ is unicyclic if and only if:*

1. *the digraph $\Gamma = \Gamma_G$ is a directed tree that contains exactly two directed paths of length 2; and*
2. *the digraph $D = D_G$ is obtained from Γ by adding a single new arc $\alpha \to \beta$ where $\alpha \rightsquigarrow \beta$ is one of the directed paths of length 2.*

Theorem 3.1.3. [16] *Let G be a unicyclic graph in $\mathcal{H}$ which is not a corona. Let $D = D_G$ and $\Gamma = \Gamma_G$. Then, G is self-dual if and only if, in addition to G^+ being unicyclic, there is a graph isomorphism from Γ to Γ or from Γ to Γ^R, that exchanges the two directed paths of length 2, where Γ^R is the Hasse diagram of D^R and D^R is obtained by reversing the orientation of every arc in D.*

Later on in 2011, Tifenbach[17] introduced a refinement of the concept of self-duality, namely, the strongly self-duality. A graph $G \in \mathcal{H}$ is *strongly self-dual* if $B_G^+ = B_G$. Necessary and sufficient conditions for a graph $G \in \mathcal{H}$ were supplied for G to be strongly self-dual. The approach utilized innovative constructions based on the graph itself. Before we list some results (of our interest) obtained in this article, we have the following remark.

Remark 3.1.4. Let $G \in \mathcal{H}$ and suppose that G^+ exists. In [16], the authors have shown that under a suitable permutation similarity the form of the adjacency matrix of the inverse G^+ has the following form.

$$A(G^+) = \begin{bmatrix} 0 & B_G^+ \\ B_G^{+t} & 0 \end{bmatrix}$$

The following result can be found in [17].

Theorem 3.1.5. [17, Theorem 3.2] *Let $G \in \mathcal{H}$. Then G is strongly self-dual if and only if both the following conditions hold:*

(i) Each nonempty undirected interval $G[i, j]$ in G is strongly self-dual.

(ii) The digraph Γ_G is bipartite.

Remark 3.1.6. Like in Remark 2.1.11, we have a problem here. For an undirected interval graph G for which digraph Γ_G is bipartite and that each undirected interval $G[u, v] \neq G$ is strongly self-dual, we cannot use Theorem 3.1.5 to show that G is strongly self-dual. In fact, with these conditions G may or may not be strongly self-dual, as shown below.

A) Consider the graph G shown in Figure 2.1. The undirected interval $G[1, 4'] = G$. Any undirected interval $G[u, v] \neq G$ is a corona tree. Hence $G[u, v]$ is strongly self-dual. The matrix

$$B_G^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = B_G.$$

Hence G is strongly self-dual graph.

B) Consider the graph H shown in Figure 2.7. The undirected interval $H[1, 3'] = H$. Any undirected interval $H[u, v] \neq H$ is a corona tree. Hence $H[u, v]$ is strongly self-dual. Notice that H^+ is a path of length 5. Hence, H is not strongly self-dual.

Remark 3.1.7. The concept of self-duality is more general than the concept of strongly self-duality. There are many self-dual graphs which are not strongly self-dual. For example, consider the graph G shown in Figure 3.1. The matrices B_G and B_G^+ are given below

$$B_G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}; B_G^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

It is clear that $B_G \neq B_G^+$. Hence, G is not strongly-self dual. Using Lemma 2.3.1, we can easily construct the inverse graph G^+ of the graph G. The inverse graph G^+ is shown in Figure 3.1. It is clear from Figure 3.1 that $G \cong G^+$. The mapping $f : V(G) \to V(G)$ defined by $f(1) = 4, f(4) = 1$; $f(1') = 4', f(4') = 1'$; $f(2) = 3, f(3) = 2$; $f(2') = 3', f(3') = 2'$ is an isomorphism from G to G^+.

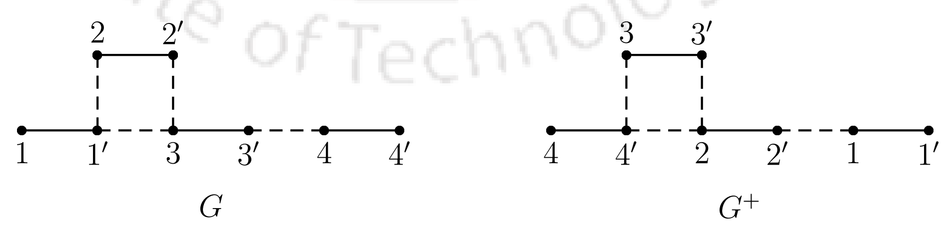


Figure 3.1: A graph $G \in \mathcal{H}$ which is not strongly self-dual but self-dual.

Henceforth, we use the term self-inverse instead of self-dual. To remind, a graph $G \in \mathcal{H}$ is self-inverse if $G \cong G^+$. At this point of thesis, we cannot explain 'why the graph G shown in Figure 2.1 is a self-inverse graph' by using the known results. Of course, we can construct the inverse graph G^+ by using Lemma 2.3.1. We shall supply a satisfactory explanation in Remark 3.2.19, where we show that this graph is contained in a special subclass of $\mathcal{H}$.

A careful examination of the graph G in Figure 2.1 leads us to the class $\mathcal{H}_{nmcs}$ (see definition below). In the next section, we identify those graphs in $\mathcal{H}_{nmcs}$ which are self-inverse. In order to define the class $\mathcal{H}_{nmcs}$, we need the following definition.

Definition 3.1.8. Let G be a graph with a unique perfect matching. An even type edge $[u, v]$ is said to be a *strict even type edge* if there are more than one even type extensions at $[u, v]$. For example, in the graph shown in Figure 2.4, the edge $[i_3', i_4]$ is a strict even type edge and in the graph G shown in Figure 3.1, the edge $[1', 3]$ is not a strict even type edge.

Definition 3.1.9. By $\mathcal{H}_{nmcs}$, we denote the class of graphs G in $\mathcal{H}$ such that

i) G has no mixed type edges,

ii) G satisfies the condition 'C' ,

iii) each even type edge of G is strict.

C: *The extensions at two distinct even type edges never have an odd type edge in common.*

Here 'nmcs' is an abbreviation of 'no mixed type edges, condition C and strict even type edges'. Thus

$$\mathcal{H}_{nmcs} = \{G \in \mathcal{H}_{nmc} \mid \text{ each even type edge of } G \text{ is strict } \}.$$

Example 3.1.10. In the following table we list the graphs used in different figures in Chapter 2 and mention whether they are in the class $\mathcal{H}_{nmcs}$.

Graph G	in $\mathcal{H}_{nmcs}$?	Justifications
Figure 2.1	Yes	Exactly one even type edge with two extensions.
Figure 2.2	No	The graph G does not satisfy condition 'C'.
Figure 2.3	Yes	The graph G has only odd type extensions.
Figure 2.4	No	There is a mixed type edge $[i_1', i_2]$.
Figure 2.5	Yes	No mixed type and no even type edges.
Figure 2.6	No	The graph G does not satisfy condition C.
Figure 2.7	No	The edge $[1', 3]$ has exactly one even type extension.
Figure 2.8	No	None of the even type edges is strict.
Figure 2.9	No	The graph G does not satisfy condition 'C'.

Remark 3.1.11. Notice that $\mathcal{H}_g \subsetneq \mathcal{H}_{nmcs} \subsetneq \mathcal{H}_{nmc} \subsetneq \mathcal{H}_{nm}$. The following table illustrates that with the graphs used in different figures in Chapter 2.

Graph G	$\mathcal{H}_g$?	$\mathcal{H}_{nmcs}$?	$\mathcal{H}_{nmc}$?	$\mathcal{H}_{nm}$?
Figure 2.1	**No**	**Yes**	Yes	Yes
Figure 2.2	No	No	**No**	**Yes**
Figure 2.3	Yes	Yes	Yes	yes
Figure 2.4	No	No	No	No
Figure 2.5	No	Yes	Yes	Yes
Figure 2.6	No	No	No	Yes
Figure 2.7	No	No	Yes	Yes
Figure 2.8	No	**No**	**Yes**	Yes
Figure 2.9	No	No	No	Yes

3.2 Self-inverse graphs in $\mathcal{H}_{nmcs}$

The following observation tells us about the number of extensions at a nonmatching edge for a self-inverse graph in $\mathcal{H}_{nmcs}$.

Lemma 3.2.1. *Let $G \in \mathcal{H}_{nmcs}$ be a self-inverse graph. Let $[u, v]$ be a nonmatching edge in G. If $[u, v]$ is odd type, then it has no extensions. If $[u, v]$ is even type, then it has exactly two extensions.*

Proof: Suppose that there is an odd type extension $Q(u, v)$ at an odd type edge $[u, v]$. Then there is at least two mm-alternating paths from u' to v' and each path contains odd number of nonmatching edges. Then by using Lemma 2.3.1, we get $A(G^+)_{u',v'} \geq 2$ which is a contradiction to the fact that $G \cong G^+$. Similarly, we can show that there are exactly two extensions at each even type edge. ■

As each even type edge in a graph G in $\mathcal{H}_{nmcs}$ is strict, one would expect the edge $[i, j]$ to be in G^+ (if it exists) whenever there is an mm-alternating i-j-path in G. We prove that below.

Lemma 3.2.2. *Let $G \in \mathcal{H}_{nmcs}$ for which G^+ exists. If there is an mm-alternating path $P(i, j)$ in G, then $[i, j] \in E(G^+)$.*

Proof: First we assume that $P(i, j)$ contains an even number of odd type edges. By using Lemmas 2.3.5 and 2.3.9 we see that, each mm-alternating path from i to j

has an even number of odd type edges. By Lemma 2.3.1, $A(G)^{-1}_{i,j}$ is obtained from the mm-alternating i-j-paths. By Lemma 2.2.22, each mm-alternating i-j-path can be created uniquely from a minimal path. Let $P^1_m(i,j), P^2_m(i,j), \ldots, P^M_m(i,j)$ be the minimal paths from i to j. Let $\mathcal{P}^r(i,j)$ be the set of all mm-alternating i-j-paths which are created from $P^r_m(i,j)$, for $r = 1, \ldots, M$. Using Lemma 2.2.22, we have $|\mathcal{P}(i,j)| = \sum_{r=1}^{m} |\mathcal{P}^r(i,j)|$. Since each even type edge in G is strict, by arguing in a way similar to that in the proof of Theorem 2.3.11, we see that the contribution of $\mathcal{P}^r(i,j)$ to $A(G)^{-1}_{i,j}$ is positive, for each $r = 1, \ldots, M$. Then $A(G)^{-1}_{i,j} > 0$. Hence, $[i,j] \in E(G^+)$.

The proof is similar if $P(i,j)$ contains an odd number of odd type edges. ∎

To proceed further, we need the following definitions.

Definition 3.2.3. Let G be a graph with a unique perfect matching $\mathcal{M} = \{[u_i, u'_i] : i = 1, 2, \ldots, n\}$. Consider the mapping $f : V(G) \to V(G)$, defined by $f(u_i) = u'_i$ and $f(u'_i) = u_i$ for $i = 1, 2, \ldots, n$. We call f the *matching mapping* and denote it by $f_{\mathcal{M}}$.

Definition 3.2.4. Let G be a graph with a unique perfect matching $\mathcal{M}$. Consider the permutation matrix $P = [p_{ij}]$ given by the matching, that is, $P_{ij} = 1$ if $[i,j] \in \mathcal{M}$ and 0, otherwise. We call P the *matching permutation matrix* and denote it by $P_{\mathcal{M}}$.

A matrix A is said to be dominant a matrix B if $a_{ij} \geq b_{ij}$ for all $i, j = 1 \ldots, n$. Godsil [15] showed that if $G \in \mathcal{H}_g$, then $A(G^+)$ dominates $A(G)$, under the permutation $P_{\mathcal{M}}$. We now give a similar result for the class $\mathcal{H}_{nmcs}$.

Theorem 3.2.5. *Let $G \in \mathcal{H}_{nmcs}$ for which G^+ exists, then $P^{-1}_{\mathcal{M}} A(G^+) P_{\mathcal{M}}$ dominates $A(G)$. (Note that G^+ may also have some additional edges other than those in G as can be seen from the path P_6 on 6 vertices.)*

Proof: To prove the assertion, assume that the unique perfect matching is $\mathcal{M} = \{[u_k, u'_k] : k = 1, \ldots, n\}$. Recall that the matching mapping $f_{\mathcal{M}}$ maps u_k to u'_k and u'_k to u_k. Using the description of $A(G)^{-1}$ from Lemma 2.3.1, we see that, for each matching edge $[u_k, u'_k] \in E(G)$, the edge $[f_{\mathcal{M}}(u_k), f_{\mathcal{M}}(u'_k)] = [u'_k, u_k] \in E(G^+)$. In fact, $A(G^+)_{u'_k, u_k} = 1$. That is, $A(G^+)_{f_{\mathcal{M}}(u_k), f_{\mathcal{M}}(u'_k)} = A(G)_{u_k, u'_k}$.

For any nonmatching edge $[u,v] \in E(G)$, this $[u,v]$ is either even type or odd type.

If $[u,v]$ is an odd type edge, then all the mm-alternating paths from $f_{\mathcal{M}}(u) = u'$ to $f_{\mathcal{M}}(v) = v'$ contain an odd number of nonmatching edges, by Lemma 2.3.9. Using the description of $A(G)^{-1}$ from Lemma 2.3.1, we see that $[u', v'] = [f_{\mathcal{M}}(u), f_{\mathcal{M}}(v)] \in E(G^+)$ and $A(G^+)_{u',v'}$ is the number of mm-alternating paths from u' to v' which is at least 1. Thus $A(G^+)_{f_{\mathcal{M}}(u),f_{\mathcal{M}}(v)} \geq A(G)_{u,v}$ and the equality happens if and only if we do not have any odd type extensions at $[u, v]$.

If $[u, v]$ is a strict even type edge, then there are at least three mm-alternating paths from $f_{\mathcal{M}}(u) = u'$ to $f_{\mathcal{M}}(v) = v'$, as there are at least two even type extensions at the edge $[u, v]$. Except for the mm-alternating path $[u', u, v, v']$ all other mm-alternating paths from u' to v' contain an even number of nonmatching edges. Now using the description of $A(G)^{-1}$ in Lemma 2.3.1, we see that, $[u', v'] = [f_{\mathcal{M}}(u), f_{\mathcal{M}}(v)] \in E(G^+)$ and $A(G^+)_{u',v'}$ is the number of mm-alternating paths from u' to v' minus two. Thus $A(G^+)_{f_{\mathcal{M}}(u),f_{\mathcal{M}}(v)} \geq A(G)_{u,v}$ and the equality happens if and only if we have exactly two even type extensions at $[u, v]$. Hence $P_{\mathcal{M}}^{-1} A(G^+) P_{\mathcal{M}}$ dominates $A(G)$. ∎

As an application, we have the following observation.

Corollary 3.2.6. *Let $G \in \mathcal{H}_{nmcs}$ be a self-inverse graph. Then the matching mapping $f_{\mathcal{M}}$ is an isomorphism from G to G^+.*

Proof: Note that the sum of the entries of $A(G)$ is $2|E(G)|$, that is, $\mathbb{1}^t A(G) \mathbb{1} = 2|E(G)|$, where $\mathbb{1}$ denotes the all ones vector. So for any permutation matrix P, we have $\mathbb{1}^t P^{-1} A(G) P \mathbb{1} = 2|E(G)|$. In particular, if we are given that $G \cong G^+$, then

$$\mathbb{1}^t P_{\mathcal{M}}^{-1} A(G^+) P_{\mathcal{M}} \mathbb{1} = \mathbb{1}^t A(G^+) \mathbb{1} = \mathbb{1}^t A(G) \mathbb{1}.$$

However, as G^+ exists, we have $A(G)$ is dominated by $P_{\mathcal{M}}^{-1} A(G^+) P_{\mathcal{M}}$. As both these matrices have nonnegative entries and equal sum of the entries, we must have that $A(G) = P_{\mathcal{M}}^{-1} A(G^+) P_{\mathcal{M}}$. That is, the matching mapping $f_{\mathcal{M}}$ is an isomorphism from G to G^+. ∎

The following is an immediate corollary.

Corollary 3.2.7. *Let $G \in \mathcal{H}_{nmcs}$ be a self-inverse graph. Let $P = [i, i', \ldots, j, j']$ be an mm-alternating path in G. Then $[i', j] \in E(G)$.*

Proof: Using Lemma 3.2.2, we see that $[i, j'] \in E(G^+)$. Since $G \cong G^+$, using Corollary 3.2.6, we see that $f_{\mathcal{M}} : G \to G^+$ is an isomorphism. Hence $[i', j] = [f_{\mathcal{M}}^{-1}(i), f_{\mathcal{M}}^{-1}(j')] \in E(G)$. ∎

The following a very crucial necessary condition for a self-inverse graph in $\mathcal{H}_{nmcs}$. In literature, we have instances (see) where 'mm-alternating paths of length 5' have been used (explicitly or implicitly) as a tool to draw some conclusions about the graphs. Here is another instance of that.

Lemma 3.2.8. *Let $G \in \mathcal{H}_{nmcs}$. If G is self-inverse, then there is no minimal path of length 5 in G.*

Proof: Suppose that there is such a path, say, $P(u_1, u_3') = [u_1, u_1', u_2, u_2', u_3, u_3']$. Then, by Corollary 3.2.7, $[u_1', u_3] \in E(G)$. Hence $[u_1', u_2, u_2', u_3]$ becomes an even type extension at $[u_1', u_3]$. But the path $P(u_1, u_3') = [u_1, u_1', u_2, u_2', u_3, u_3']$ contains this extension. This is a contradiction to the fact that $P(u_1, u_3')$ is minimal. ∎

The following is another necessary condition for a self-inverse graph in $\mathcal{H}_{nmcs}$.

Lemma 3.2.9. *Let $G \in \mathcal{H}_{nmcs}$. If G is self-inverse, then each even type extension in G has length 3.*

Proof: Assume, on the contrary, that there is an even type extension of length more than 3 at the edge $[u, v]$ in G. Let it be $[u, u_1, u_1', u_2, u_2', \ldots, u_{2k-1}, u_{2k-1}', v]$, where $k > 1$. Then the path $[u_1, u_1', u_2, u_2', \ldots, u_{2k-1}, u_{2k-1}']$ is an mm-alternating path. As G is self-inverse, using Corollary 3.2.7, the edge $[u_1', u_{2k-1}] \in E(G)$. Then the path $[u, u_1, u_1', u_{2k-1}, u_{2k-1}', v]$ is an odd type extension at $[u, v]$, which implies that $[u, v]$ is a mixed type edge. This contradicts the fact that $G \in \mathcal{H}_{nm}$. ∎

The following result further analyzes the matching edge on an even type extension in a self-inverse graph in $\mathcal{H}_{nmcs}$. By $d_G(v)$, we denote the degree of a vertex v in G.

Lemma 3.2.10. *Let $G \in \mathcal{H}_{nmcs}$ be a self-inverse graph. Let $[u, v, v', w]$ be an even type extension at $[u, w]$. Then the degrees $d_G(v) = d_G(v') = 2$, that is, the degrees of the endvertices of a matching edge on an extension must be 2.*

Proof: Suppose that $d_G(v) > 2$. Let $t \neq u, v'$ be adjacent to v. If t is adjacent to w, then we have two even type extensions $[u, v, v', w]$ and $[t, v, v', w]$, which have an odd type edge $[v', w]$ in common. This contradicts that $G \in \mathcal{H}_{nmcs}$. So t is not adjacent to w. But, in that case, the path $[t', t, v, v', w, w']$ is a minimal path of length 5. This contradicts Lemma 3.2.8. Proving $d_G(v') = 2$ is similar. ∎

The following is an immediate corollary. It shows that a self-inverse graph in $\mathcal{H}_{nmcs}$ satisfies a more strict condition than the condition 'C'.

Corollary 3.2.11. *If $G \in \mathcal{H}_{nmcs}$ be a self-inverse graph, then the extensions at distinct even type edges never have an edge (neither odd type nor matching edges) in common.*

Proof: Follows, as the degrees if the internal vertices in these paths are 2. ∎

The following result gives us the final piece of structural information about self-inverse graphs G in $\mathcal{H}_{nmcs}$.

Proposition 3.2.12. *Let $G \in \mathcal{H}_{nmcs}$ be a self-inverse graph. Let H be the graph obtained from G by deleting all the even type extensions while keeping the endvertices of the extensions. Then H is a corona of a connected bipartite graph.*

Proof: Note that, as we are deleting extensions, on which the internal vertices have degree 2, the resulting graph H remains connected. As G is bipartite, so is H. We claim that each matching edge in H is a leaf. Assume, if possible that there is a matching edge $[v, v']$ in H which is not a leaf, that is, $d_H(v), d_H(v') \geq 2$. So, we can find a path $[u, v, v', w]$ in H and hence the path $[u', u, v, v', w, w']$ is an mm-alternating path of length 5 in G. Hence by Corollary 3.2.7, $[u, w] \in E(G)$. Hence $[u, v, v', w]$ is an even type extension at $[u, v]$. But, as H is obtained by deleting all the internal vertices on the even type extensions, the vertices v, v' should not be in H. This is a contradiction and so our claim is justified. Hence H must be a corona graph. ∎

Recall that by Lemma 2.3.9, if G^+ exists for a graph $G \in \mathcal{H}_{nmc}$, then each cycle in G must contain an even number of odd type edges. Keeping that in mind we define the following class of graphs.

Definition 3.2.13. Let H be a connected bipartite corona graph. Let S be a subset of nonmatching edges of H such that each cycle in H has an even number of edges from S. Let *boxminus corona* $H_S^{\boxminus}$ be the graph created from H by adding two even type extensions of length 3 at each edge $e \in S$. This is same as replacing each $[u, v] \in S$ with the *boxminus graph*. For boxminus graph, see Figure 3.2.

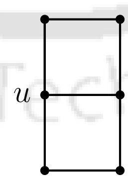


Figure 3.2: The boxminus graph

Example 3.2.14. Let H be the bipartite corona graph shown in Figure 3.3. We consider the subset $S = \{[1, 2], [3, 4], [5, 6]\}$ of nonmatching edges of H. Notice that each cycle in H contains even number of edges from H. Now we replace the edges

$[1,2]$, $[3,4]$ and $[5,8]$ by the boxminus graph shown in Figure 3.2. The final graph is $H_S^{\boxminus}$.

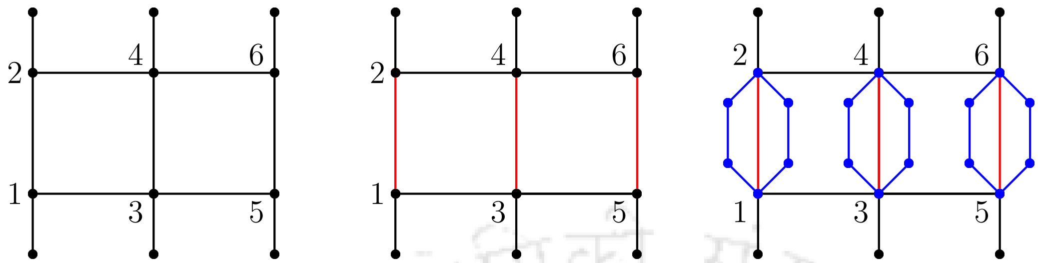


Figure 3.3: Example of a boxminus corona graph.

Remark 3.2.15. Let G be a bipartite graph and $S \subset E(G)$. How do we check whether each cycle of G contains an even number of edges from S? Contract each edge in S to a vertex. Call the new graph H. Then each cycle in G contains an even number of edges from S if and only if H does not have any odd cycle.

The following result gives us the precise structural information about a self-inverse graph in $\mathcal{H}_{nmcs}$.

Proposition 3.2.16. *Let $G \in \mathcal{H}_{nmcs}$ be a self-inverse graph. Let S be the set of all even type edges in G. Let H be the (corona) graph obtained from G by deleting all the even type extensions while keeping the endvertices of the extensions. Then $G = H_S^{\boxminus}$.*

Proof: Let S be the set of all even type edges in G. Note that, by Lemma 2.2.12, each nonmatching edge on an even type extension is odd type. Hence $S \subset E(H)$. By Proposition 3.2.12, H is a connected bipartite corona. Let Δ be a cycle in H. Then Δ has an even number of edges and each edge is odd type in H. Out of these, the edges which are in S, were even type in G. As Δ was available in G itself, by Lemma 2.3.9, Δ contains an even number of edges from S. By Lemmas 3.2.1 and 3.2.9, there are no extensions at the odd type edges and there are exactly two extensions of length 3 at an even type edge. It follows that $G = H_S^{\boxminus}$. ■

Now we prove the converse, as one would have expected.

Proposition 3.2.17. *Let H be a connected bipartite corona graph, S be a subset of nonmatching edges of H such that each cycle in H has an even number of edges from S. From H, create the graph $G = H_S^{\boxminus}$ by adding two even type extensions of length 3 at each edge $e \in S$. Then $G \in \mathcal{H}_{nmcs}$ and $G \cong G^+$.*

Proof: The following are true by construction of $G = H_S^{\boxminus}$.

1. Each edge in S is strict even type in G.
2. All other nonmatching edges in G are odd type.
3. Extensions at two distinct even type edges never have an edge in common.
4. The mm-alternating paths in $G = H_S^{\boxminus}$ can have lengths 1 or 3 or 5. Those which have lengths 1 or 3 are minimal and those which have length 5 are not minimal.

By the first three items, we have, $G \in \mathcal{H}_{nmc}$.

To show that G^+ exists, it is enough to show that $(G - \mathcal{E})/\mathcal{M}$ is bipartite, by Theorem 2.3.11. Towards that, note that $\mathcal{E} = S$ and suppose that $(G-S)/\mathcal{M}$ is not bipartite. Then it has an odd cycle, say, Δ. Notice that, each boxminus is reduced to a diamond C_4 in $(G - \mathcal{E})/\mathcal{M}$ and Δ contains exactly 2 consecutive edges or no edges from each diamond. Since Δ is an odd cycle, it follows that, Δ contains an odd number of edges apart from the edges from the diamonds. Consider the cycle Δ' obtained from Δ by doing the following:

'for each pair of consecutive edges that Δ contains from the diamond corresponding an edge $[u, v] \in S$, we replace these edges by $[u, v]$'.

Then Δ' is a cycle in H. By construction, Δ' has an even number of edges from S. Hence, the length of Δ' is odd, contradicting the bipartiteness of H. Hence G^+ exists.

Now, we show that $G \cong G^+$ via the isomorphism $f_{\mathcal{M}}$. Going through the proof of Theorem 3.2.5, we have the following observations.

1. For a matching edge $[u_k, u_k']$, we have $A(G^+)_{f_{\mathcal{M}}(u_k), f_{\mathcal{M}}(u_k')} = A(G)_{u_k, u_k'} = 1$.
2. Any nonmatching edge $[u, v] \in G$, this $[u, v]$ is either even type or odd type.
3. For any odd type edge $[u, v]$, we have $A(G^+)_{f_{\mathcal{M}}(u), f_{\mathcal{M}}(v)} \geq A(G)_{u,v}$ and the equality happens if and only if we do not have any odd type extensions at $[u, v]$. Here, the equality holds.
4. For any strict even type edge $[u, v]$, we have $A(G^+)_{f_{\mathcal{M}}(u), f_{\mathcal{M}}(v)} \geq A(G)_{u,v}$ and the equality happens if and only if we have exactly two even type extensions at $[u, v]$. Here, the equality holds.

Now, we only need to show that G^+ does not contain a new edge. Towards that, suppose that $A(G^+)_{f_{\mathcal{M}}(u),f_{\mathcal{M}}(v)} \neq 0$ but $[u,v] \notin E(G)$. This means there is an mm-alternating path P from u' to v' in G. The length of P cannot be 3, otherwise P must be $[u',u,v,v']$ making $[u,v] \in E(G)$. As there are no mm-alternating paths of length more than 5 in G, we conclude that P must have length 5. So let $P = [u',u,x,x',v,v']$. But then, as the edge $[u,v] \notin G$, the path P is a minimal path of length 5 in G, a contradiction to the observation made in the beginning of the proof. ∎

Combining the previous two propositions, we present our main result of this section.

Theorem 3.2.18. *Let $G \in \mathcal{H}_{nmcs}$. Then the following are equivalent.*

i) The graph $G \cong G^+$.

ii) The graph $G = H_S^{\boxminus}$, where H is a connected bipartite corona graph and S is a subset of nonmatching edges of H such that each cycle in H has an even number of nonmatching edges from S.

Remark 3.2.19. Now we can explain why the graph G shown in Figure 2.1 is self-inverse. The graph G is $H_S^{\boxminus}$, where $H = P_4 = [1,1',4,4']$ and $S = \{[1',4]\}$. Hence by Theorem 3.2.18, $G \cong G^+$.

3.3 Inverses of unicyclic graphs in $\mathcal{H}$

A connected graph G is said to be a *unicyclic* graph if G has the same number of edges and vertices. In this section, we consider the class $\mathcal{H}_u = \{G \in \mathcal{H} \mid G$ is unicyclic$\}$. We already have noticed that $\mathcal{H}_u$ is not a subclass of the class $\mathcal{H}_{nmcs}$. Now that we have a complete characterization of the self-inverse graphs in $\mathcal{H}_{nmcs}$, a natural question is as follows. *Can we supply a complete characterization of the self-inverse unicyclic graphs in $\mathcal{H}$ by using the developed tools?* In this section, we supply such a characterization. This characterization is completely different from the characterization provided by Tifenbach and Kirkland in [16]. Tifenbach and Kirkland supplied necessary and sufficient conditions for which an invertible unicyclic graph $G \in \mathcal{H}_u$ to be self-inverse. This necessary and sufficient conditions required some constructions involving the directed graphs and the undirected interval graphs, but the tools which we use to supply a complete characterization of the self-inverse unicyclic graphs in $\mathcal{H}$ do not require any directed or undirected interval graphs.

In [25], Akbari and Kirkland presented a complete characterization of graphs in $\mathcal{H}_u$ that possess inverses, see Theorem 2.1.5. In Chapter 2, we have presented another characterization of graphs in $\mathcal{H}_u$ that possess inverses, see Corollary 2.3.15, where we proved that a graph G in $\mathcal{H}_u$ possesses an inverse G^+ if and only if either G has exactly one even type edge or G is in $\mathcal{H}_g$.

It is known that if $G \in \mathcal{H}_g$, then $G \cong G^+$ if and only if G is a simple corona graph. So to characterize self-inverse graphs in $\mathcal{H}_u$ is sufficient to have a characterization of self-inverse graphs in $\mathcal{H}_u \setminus \mathcal{H}_g$. If $G \in \mathcal{H}_u$ with $G \cong G^+$, then G^+ is unicyclic. So, as a first step we ask the following question. *Suppose that $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ possesses an inverse G^+. Is it necessary that G^+ is unicyclic?* The following example shows that 'it is not'.

Example 3.3.1. Consider the graph $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ shown in Figure 3.4. Using Corollary 2.3.15, G^+ exists. Using Lemma 2.3.1, we construct the inverse graph G^+ which is shown in Figure 3.4. The inverse graph G^+ is a tree.

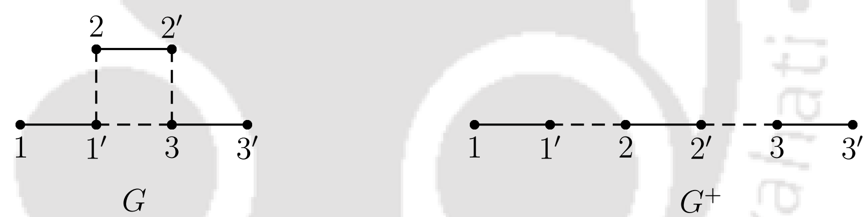


Figure 3.4:

So, the next question is the following. *Characterize the graphs in $\mathcal{H}_u \setminus \mathcal{H}_g$ with unicyclic inverses.* This we do in the Subsection 3.3.1. Then among the graphs in $\mathcal{H}_u \setminus \mathcal{H}_g$ with unicyclic inverses, we identify those that are self-inverse graphs. This is done in the Subsection 3.3.2. In the process we observe that there are many unicyclic graphs in $\mathcal{H}_u$ with unicyclic inverses which are not self-inverse graphs. In order to do this we need the following definition.

Definition 3.3.2. [29] Let $G \in \mathcal{H}_u$ and u be a vertex in G. Then a component T of $G - u$ not containing any vertex of the cycle in G is called a *tree-branch* at u. A tree-branch T at some vertex in G is called a *nontrivial tree-branch* if T has at least two vertices.

Now we mention a few notations and a few immediate observations.

Observation 3.3.3. 1. Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ possess an inverse. Then by Corollary 2.3.15, G has exactly one even type edge. As G is unicyclic, we must have exactly one even type extension.

Convention: We shall always fix this edge to be $[1', 3]$ and we use $Q(1', 3)$ to denote the even type extension at $[1', 3]$. The cycle in G is denoted by $C_G = Q(1', 3) \cup [1', 3]$.

2. Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ possess an inverse and x, y be two vertices. Then exactly one of the following happens.

   i) There are no mm-alternating x-y-paths. In this case, $A(G^+)_{x,y} = 0$.

   ii) There is exactly one mm-alternating x-y-path $[x, x', \ldots, y', y]$. Then this path cannot contain the edge $[1', 3]$, otherwise we will have one more alternating path due to $Q(1', 3)$. This path cannot contain the extension $Q(1', 3)$ completely, otherwise we will have more than one mm-alternating x-y-paths. Thus, we have exactly one minimal x-y-path not containing the edge $[1', 3]$. In this case, $A(G^+)_{x,y} = 1$.

   iii) There are exactly two mm-alternating paths from x to y. One of them contains the edge $[1', 3]$. The other one contains the extension $Q(1', 3)$. In this case, $A(G^+)_{x,y} = 0$.

   That is, $A(G^+)(x, y) = 1$ if and only if there is exactly one minimal x-y-path not containing $[1', 3]$ and $A(G^+)_{x,y} = 0$, otherwise. Hence, G^+ is an unweighted graph.

3. Recall that when $G \in \mathcal{H}$, then G^+, when exists is a weighted, connected, bipartite graph with a unique perfect matching which is nothing but the perfect matching $\mathcal{M}$ of G. Thus, if $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ and G^+ exists, then $G^+ \in \mathcal{H}$.

4. Let $G \in \mathcal{H}$ and H be a subgraph of G. By $f_{\mathcal{M}}(H)$ we denote the graph

$$\Big(f_{\mathcal{M}}(V(H)),\ \big\{[f_{\mathcal{M}}(u), f_{\mathcal{M}}(v)] \mid [u, v] \in E(H)\big\}\Big).$$

5. Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ for which G^+ exists. Then the following hold true.

   i) If $[u, v] \neq [1', 3]$ is a nonmatching edge in G, then $A(G^+)_{u',v'} = 1$.

   ii) If $[u, u']$ is a matching edge in G, then $A(G^+)_{u,u'} = 1$.

   iii) We also have that $A(G^+)_{1,3'} = 0$.

   Thus $f_{\mathcal{M}}(G - [1', 3])$ is a subgraph of G^+.

Lemma 3.3.4. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ possess an inverse. Assume that G has a minimal path of length 7. Then G has two minimal paths of length 5 not containing the (only even type) edge $[1', 3]$.*

Proof: Let $P(i,j) = [i, i', i_1, i_1', i_2, i_2', j', j]$ be a minimal path of length 7 in G. There are two cases.

CASE I. The path $P(i,j)$ does not contain the edge $[1', 3]$. Then the paths $[i, i', i_1, i_1', i_2, i_2']$ and $[i_1, i_1', i_2, i_2', j', j]$ are two minimal paths of length 5 not containing $[1', 3]$.

CASE II. The path $P(i,j)$ contains the edge $[1', 3]$. There are three subcases.

CASE II(A). The edge $[1', 3] = [i', i_1]$. Let $Q(1', 3) = [1', x_1, x_1' \ldots, x_{2k-1}, x_{2k-1}', 3]$ be the even type extension at $[1', 3]$, where $k \geq 1$. Then we consider the path $[x_1, x_1' \ldots, x_{2k-1}, x_{2k-1}', i_1, i_1', i_2, i_2', j', j]$ which is a minimal path of length at least 7 not containing the edge $[1', 3]$. Then G has two minimal paths of length 5 not containing $[1', 3]$, this is followed by CASE I.

CASE II(B). The edge $[1', 3] = [i_1', i_2]$. Let $Q(1', 3) = [1', x_1, x_1' \ldots, x_{2k-1}, x_{2k-1}', 3]$ where $k \geq 1$. Then the paths $[i, i', i_1, i_1', x_1, x_1']$ and $[x_{2k-1}, x_{2k-1}', i_2, i_2', j', j]$ are two minimal paths of length 5 not containing $[1', 3]$.

CASE III(C). The edge $[1', 3] = [i_2', j']$. Arguing in a similar manner to the CASE II(A). The proof is complete. ∎

Lemma 3.3.5. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ possess an inverse. Then G^+ has a nonmatching edge $[x, y]$ such that $[x', y']$ is not in G if and only if G has a minimal x-y-path of length at least 5 not containing $[1', 3]$.*

Proof: First we assume that there is a nonmatching edge $[x, y]$ in G^+ such that $[x', y']$ is not in G. By using Observation 3.3.3, G has a minimal x-y-path not containing $[1', 3]$. The length of this minimal path cannot be 3, otherwise P must be $[x, x', y', y]$ making $[x', y'] \in E(G)$.

Now, we assume that G has a minimal x-y-path of length at least 5 not containing $[1', 3]$. Then by Observation 3.3.3, $A(G^+) = 1$. Hence $[x, y]$ in G^+ but $[x', y']$ not in G. ∎

Lemma 3.3.6. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ possess an inverse. Suppose that $[i, i', \ldots, j', j]$ and $[x, x', \ldots, y', y]$ are two minimal paths in G of length 5 not containing the edge $[1', 3]$. Then $[i', j']$ and $[x', y']$ are two distinct edges that are not in G such that the edges $[i, j]$ and $[x, y]$ are in G^+. Thus G^+ has at least $n + 1$ edges.*

Proof: Let $P_{ij} = [i, i', \ldots, j', j]$ and $P_{xy} = [x, x', \ldots, y', y]$. First assume that the edges $[i', j']$ and $[x', y']$ are equal. Then the subpath $[i', \ldots, j']$ of P_{ij} along with the subpath $[x', \ldots, y']$ of P_{xy} must contain a cycle, as P_{ij} is different from P_{xy}. As the graph is unicyclic, one of these two subpaths must contain $[1', 3]$, which is not possible by the hypothesis. So, the edges $[i', j']$ and $[x', y']$ are two distinct edges. Since these two paths are minimal, we have that the edges $[i', j']$ and $[x', y']$ are not in G. Using Lemma 3.3.5, the edges $[i, j]$ and $[x, y]$ are in G^+. Using Observation 3.3.3, we see that G^+ has at least $n + 1$ edges. ∎

Lemma 3.3.7. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ possess an inverse. Assume G^+ has at most n edges. Then the length of the even type extension $Q(1', 3)$ is 3.*

Proof: Suppose that the length of $Q(1', 3)$ is more than 3. Let $Q(1', 3) = [1', x_1, x_1' \ldots, x_{2k-1}, x_{2k-1}', 3]$, where $k > 1$. Then the paths $[1, 1', x_1, x_1', x_2, x_2']$ and $[x_{2k-2}, x_{2k-2}', x_{2k-1}, x_{2k-1}', 3, 3']$ are two minimal paths of length 5 not containing $[1', 3]$. Then by using Observation 3.3.3 and Lemma 3.3.5, G^+ has at least $n + 1$ edges. A contradiction to the fact that G^+ is unicyclic. ∎

Convention: Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ and $G^+ \in \mathcal{H}_u$. By Lemma 3.3.7, the length of $Q(1', 3)$ is 3. We shall always assume it to be $Q(1', 3) = [1', 2, 2', 3]$.

3.3.1 Unicyclic graphs in $\mathcal{H}$ with unicyclic inverses

The following theorem supplies a necessary and sufficient condition for a graph $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ to have an unicyclic inverse.

Theorem 3.3.8. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ possess an inverse. Then $G^+ \in \mathcal{H}_u$ if and only if G has exactly one minimal path of length 5 not containing $[1', 3]$.*

Proof: Suppose the number of minimal paths in G of length 5 not containing $[1', 3]$ is 0. By using Observation 3.3.3 and Lemma 3.3.5, G^+ has exactly $n-1$ edges. A contradiction to the fact that $G^+ \in \mathcal{H}_u$. Hence, G has a minimal path of length 5 not containing $[1', 3]$.

Now we show that G has exactly one minimal path of length 5 not containing $[1', 3]$. Suppose that G has two minimal paths of length 5 not containing $[1', 3]$. Then by Lemma 3.3.6, G^+ has at least $n + 1$ edges. A contradiction to the fact that $G^+ \in \mathcal{H}_u$. Hence G has exactly one minimal path of length 5 not containing $[1', 3]$.

The converse is true by using Observation 3.3.3 and Lemma 3.3.5. ∎

The following is an immediate corollary of Theorem 3.3.8.

Corollary 3.3.9. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ and $G^+ \in \mathcal{H}_u$. Then there are no minimal paths of length 7.*

Proof: The proof follows by Lemma 3.3.4 and Theorem 3.3.8. ■

Lemma 3.3.10. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ and $G^+ \in \mathcal{H}_u$. Then there are no mm-alternating paths of length more than 7 in G.*

Proof: Suppose that G has an mm-alternating path of length more than 7, say, $P(i,j)$. Let $P(i,j) = [i, i', i_1, i'_1, i_2, i'_2, i_3, i'_3, j', j]$. By using Corollary 3.3.9, $P(i,j)$ is not a minimal path. So, $P(i,j)$ contains the extension $Q(1',3)$. By using Lemma 3.3.7, the length of the extension is 3. By replacing the extension $Q(1',3)$ with $[1',3]$ we get another mm-alternating i-j-path which contains $[1',3]$ and which is minimal. Then the length of this minimal path is 7, which is not possible by Corollary 3.3.9.■

Lemma 3.3.11. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ and $G^+ \in \mathcal{H}_u$. Then there is at most one mm-alternating path of length 7 in G.*

Proof: Suppose that G has two mm-alternating paths of length 7, say P_1 and P_2. By Corollary 3.3.9, P_1 and P_2 are not minimal paths. So both the paths P_1 and P_2 contain the even type extension $[1', 2, 2', 3]$ and the path $[1, 1', 2, 2', 3, 3']$ is subpath of P_1 and P_2. Then either the degree $d_G(1) = 1$ or $d_G(3') = 1$, otherwise we have an mm-alternating path of length 9. We get two minimal paths P_1^m and P_2^m of length 5 not containing $[1',3]$ such that P_1^m and P_2^m are subpaths of P_1 and P_2, respectively. The paths P_1^m and P_2^m are distinct, otherwise $P_1 = P_2$. A contradiction to the fact that G has exactly one minimal paths of length 5 not containing $[1',3]$.■

A *pendant* vertex of a graph is a vertex of degree 1 and a *quasi pendant* vertex of a graph is a vertex adjacent to a pendant vertex.

Remark 3.3.12. Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ and $G^+ \in \mathcal{H}_u$. Suppose that G has an mm-alternating path P of length 7. Then the following statements hold.

1. By using Lemma 3.3.7, the length of $Q(1',3)$ is 3 and $Q(1',3) = [1', 2, 2', 3]$.
2. By using Theorem 3.3.8, the path P is not minimal. Then P contains the extension $[1', 2, 2', 3]$.
3. Let $P = [1, 1', 2, 2', 3, 3', 4, 4']$. Notice that $[2, 2', 3, 3', 4, 4']$ is the minimal path of length 5 not containing $[1',3]$. Then the degrees $d_G(1) = 1 = d_G(4')$, $d_G(2) = d_G(2') = d_G(3') = 2$ and $d_G(3) = 3$, otherwise G has more than one minimal paths of length 5 not containing $[1',3]$.

4. Each nontrivial tree-branch at $1'$ (resp. 4) is a corona tree and a quasi pendant vertex of that tree-branch is adjacent to $1'$ (resp. 4), otherwise G has more than one minimal paths of length 5 not containing $[1', 3]$.

Theorem 3.3.13. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$. Suppose that G has an mm-alternating path of length 7. Then $G^+ \in \mathcal{H}_u$ if and only if G has the structure shown in Figure 3.5, where*

1. *F_1 and F_2 are forests of simple corona trees,*
2. *a quasi pendant vertex of each tree in F_1 is adjacent to $1'$, and*
3. *a quasi pendant vertex of each tree in F_2 is adjacent to 4.*

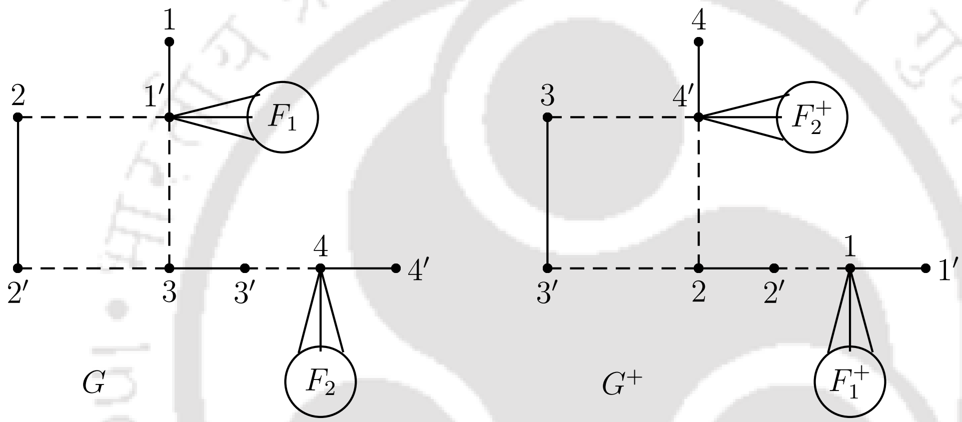


Figure 3.5: The graph G and its inverse G^+

Proof: First assume that G has the structure shown in Figure 3.5. By using Observation 3.3.3, we see that the graph $f_{\mathcal{M}}(G - [1', 3])$ is a subgraph of G^+. Since G has a minimal path $[2, 2', 3, 3', 4, 4']$ of length 5 not containing $[1', 3]$, by using Lemma 3.3.5, the edge $[2, 4']$ is in G^+ but $[2', 4] \notin E(G)$. The graph G has exactly one minimal path of length 5 not containing $[1', 3]$. By using Lemma 3.3.5, G^+ has no extra edges $[x, y]$ other than $[2, 4']$ such that $[x', y']$ not in G. Hence, G^+ has n number of edges with n number of vertices. Hence, G^+, being connected, is unicyclic.

Now assume that $G^+ \in \mathcal{H}_u$. By using Lemma 3.3.7, the length of $Q(1', 3)$ is 3 and $Q(1', 3) = [1', 2, 2', 3]$. Using Theorem 3.3.8, G has exactly one minimal path of length 5 not containing $[1', 3]$. Using Corollary 3.3.9, length of each minimal path in G is at most 5. Using Lemmas 3.3.10, 3.3.11, G has no mm-alternating paths of length more than 7 and G has at most one mm-alternating path of length 7. By hypothesis, G has an mm-alternating path of length 7. Then G has exactly one mm-alternating path of length 7 which contains the extension $Q(1', 3) = [1', 2, 2', 3]$. Using Remark 3.3.12, the graph G must have the structure shown in Figure 3.5. ∎

Remark 3.3.14. Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ and $G^+ \in \mathcal{H}_u$. Suppose that G has no mm-alternating paths of length 7. Then the following statements are true.

1. By using Lemma 3.3.7, the length of $Q(1', 3)$ is 3 and $Q(1', 3) = [1', 2, 2', 3]$.
2. The length of each mm-alternating path in G is at most 5. By Theorem 3.3.8, G has exactly one minimal path of length 5 not containing $[1', 3]$.
3. Let $[1, 1', 2, 2', 3, 3']$ be an mm-alternating path containing the extension $Q(1', 3)$.
4. Notice that the degrees $d_G(1) = 1 = d_G(3')$, otherwise G has an mm-alternating path of length 7.
5. Notice that the degrees $2 \leq d_G(2), d_G(2') \leq 3$, otherwise G has two minimal paths of length 5 not containing $[1', 3]$, which is not possible by Theorem 3.3.8. But the degree of 2 and $2'$ can never be more than 2 simultaneously, otherwise, we have two minimal paths of length 5 not containing $[1', 3]$. So without loss of generality we assume that $d_G(2) = 2$ and $d_G(2') \leq 3$. Then there are two cases either $d_G(2') = 2$ or $d_G(2') = 3$.
6. Assume that $d_G(2') = 3$. The path $[1, 1', 2, 2']$ (or $[2, 2', 3, 3']$, which is similar) is a subpath of the minimal path of length 5 not containing $[1', 3]$. Let $[1, 1', 2, 2', 4, 4']$ be the minimal path of length 5 not containing $[1', 3]$. Then $d_G(4') = 1$. Each nontrivial tree-branch at $1'$ (resp. 3 and 4) is a corona tree and a quasi pendant vertex of that tree-branch is adjacent to $1'$ (resp. 3 and 4), otherwise G has at least two minimal paths of length 5 not containing $[1', 3]$.
7. Assume that $d_G(2') = 2$. The minimal path of length 5 not containing $[1', 3]$ and the path $[1, 1', 2, 2', 3, 3']$ have some vertices in common or these two paths are vertex disjoint.
8. Suppose that the minimal path of length 5 not containing $[1', 3]$ and the path $[1, 1', 2, 2', 3, 3']$ have some vertices in common. Since $d_G(2') = 2 = d_G(2)$, we have either the edge $[1, 1']$ (or $[3, 3']$, which is similar) is common. Then 1 is the one end vertex of that minimal path. Let $[1, 1', 4, 4', 5, 5']$ be the minimal path of length 5 not containing $[1', 3]$. Then $d_G(4) = 2 = d_G(4')$ and $d_G(5') = 1$, otherwise G has more than one minimal paths of length 5. Each nontrivial tree-branch at $1'$ (resp. 3 and 4) is a corona tree and a quasi pendant vertex

of that tree-branch (except the nontrivial tree-branch containing the vertex 4) is adjacent to $1'$ (resp. 3 and 5), otherwise G has at least two minimal paths of length 5 not containing $[1', 3]$.

9. Suppose that the minimal path of length 5 not containing $[1', 3]$ and the path $[1, 1', 2, 2', 3, 3']$ are vertex disjoint. Then any nontrivial tree-branch at 1 (or 3 which is similar) will give us the minimal path of length 5 not containing $[1', 3]$. Let T be the such nontrivial tree-branch at $1'$ and $[6, 6', 7, 7', 8, 8']$ be the the minimal path of length 5 not containing $[1', 3]$. Then $d_G(6) = 1 = d_G(8')$ and $d_G(7) = d_G(7') = 2$ and $T - 7 - 7'$ is a forest of corona trees, otherwise G has at least two minimal path of length 5 not containing $[1', 3]$. Each nontrivial tree-branch (except T) at $1'$ (resp. 3 and 8) is a corona tree and a quasi pendant vertex of that tree-branch is adjacent to $1'$ (resp. 3 and 5), otherwise G has at least two minimal paths of length 5 not containing $[1', 3]$.

Theorem 3.3.15. *Let $G \in \mathcal{H}_u \backslash \mathcal{H}_g$ and assume that G has no mm-alternating paths of length 7. Let $d_G(2') = 3$. Then $G^+ \in \mathcal{H}_u$ if and only if G has the structure shown in Figure 3.6, where*

1. *F_1, F_2 and F_3 are forests of simple corona trees,*
2. *a quasi pendant vertex of each tree in F_1 is adjacent to $1'$,*
3. *a quasi pendant vertex of each tree in F_2 is adjacent to 4, and*
4. *a quasi pendant vertex of each tree in F_3 is adjacent to 3.*

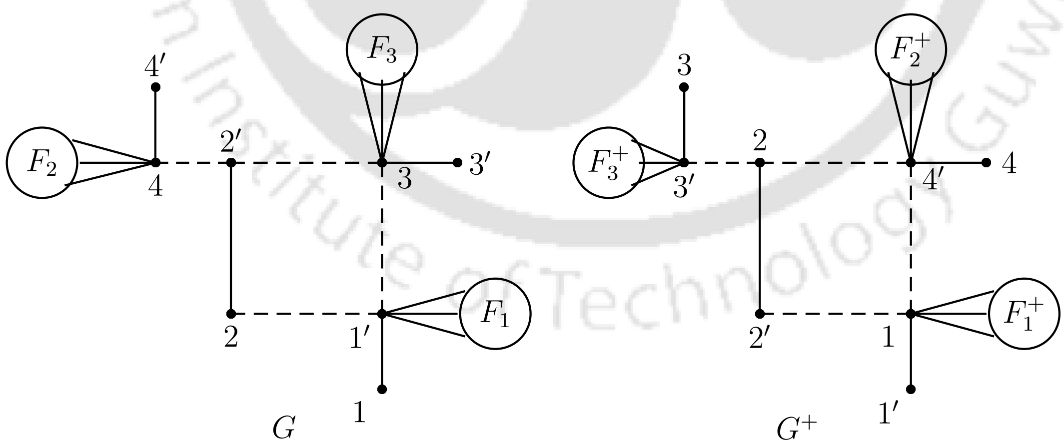


Figure 3.6: The graph G and its inverse G^+

Proof: First assume that G has the structure shown in Figure 3.6. By using Observation 3.3.3, we see that the graph $f_{\mathcal{M}}(G - [1', 3])$ is a subgraph of G^+. Since

G has a minimal path $[1, 1', 2, 2', 4, 4']$ of length 5 not containing $[1', 3]$, by using Lemma 3.3.5, the edge $[1, 4']$ is in G^+ but $[1', 4] \notin E(G)$. The graph G has exactly one minimal path of length 5 not containing $[1', 3]$. By using Lemma 3.3.5, G^+ has no extra edges $[x, y]$ other than $[1, 4']$ such that $[x', y']$ not in G. Hence, G^+ has n number of edges with n number of vertices. Hence, G^+ is unicyclic.

Now assume that $G^+ \in \mathcal{H}_u$. By using Lemma 3.3.7, the length of $Q(1', 3)$ is 3 and $Q(1', 3) = [1', 2, 2', 3]$. Using Theorem 3.3.8, G has exactly one minimal path of length 5 not containing $[1', 3]$. Using Corollary 3.3.9, length of each minimal path in G is at most 5. Using Lemmas 3.3.10, 3.3.11, G has no mm-alternating paths of length more than 7 and G has at most one mm-alternating path of length 7. By hypothesis, G has no mm-alternating paths of length 7 and $d_G(2') = 2$. By using Remark 3.3.14, the graph G must have the structure shown in Figure 3.6. ■

Theorem 3.3.16. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ and suppose that G has no mm-alternating paths of length 7. Let $d_G(2') = 2$. Then $G^+ \in \mathcal{H}_u$ if and only if G has one the structures shown in Figures 3.7 and 3.8, where*

1. *F_1, F_2 and F_3 are forests of simple corona trees in Figure 3.7,*
2. *a nonpendant vertex of each tree in F_1 is adjacent to $1'$,*
3. *a quasi pendant vertex of each tree in F_2 is adjacent to 3,*
4. *a quasi pendant vertex of each tree in F_3 is adjacent to 5,*
5. *F_1, F_2, F_3, F_4, F_5 and F_6 are forests of simple corona trees in Figure 3.8,*
6. *a quasi pendant vertex of each tree in F_1 is adjacent to $1'$,*
7. *a quasi pendant vertex of each tree in F_2 is adjacent to 3,*
8. *a quasi pendant vertex of each tree in F_3 is adjacent to 8,*
9. *a quasi pendant vertex of each tree in F_4 is adjacent to 4,*
10. *a quasi pendant vertex of each tree in F_5 is adjacent to 5,*
11. *a quasi pendant vertex of each tree in F_6 is adjacent to 6, and*
12. *the vertices y and z are quasi pendant vertices of F_4 and F_5, respectively.*

Proof: First assume that G has the structure shown in Figure 3.7. By using Observation 3.3.3, we see that the graph $f_{\mathcal{M}}(G - [1', 3])$ is a subgraph of G^+. Since G has a minimal path $[1, 1', 4, 4', 5, 5']$ of length 5 not containing $[1', 3]$, by using Lemma 3.3.5, the edge $[1, 5']$ is in G^+ but $[1', 5] \notin E(G)$. The graph G has exactly

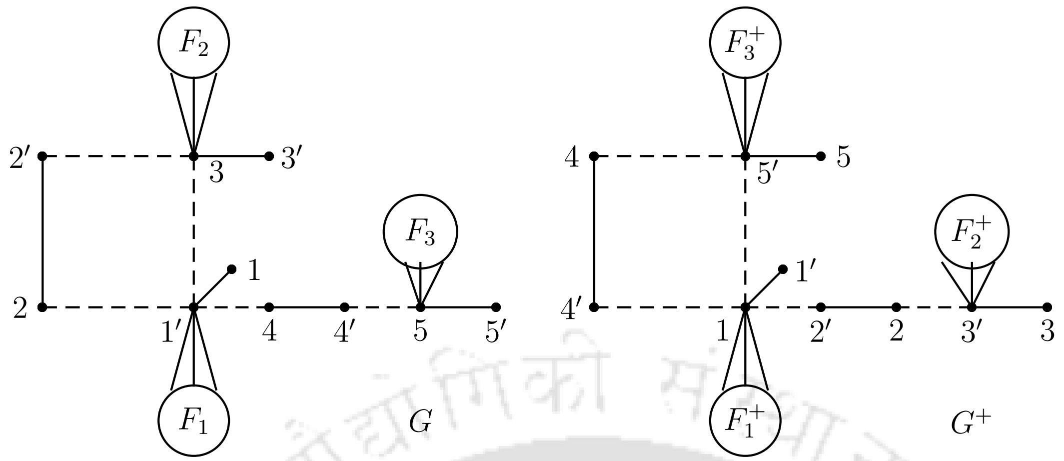


Figure 3.7: The graph G and its inverse G^+

one minimal path of length 5 not containing $[1', 3]$. By using Lemma 3.3.5, G^+ has no extra edges $[x, y]$ other than $[1, 5']$ such that $[x', y']$ not in G. Hence, G^+ has n number of edges with n number of vertices. Hence, G^+ is unicyclic. Similar arguments work if G has the structure shown in Figure 3.8

Now assume that $G^+ \in \mathcal{H}_u$. By using Lemma 3.3.7, the length of $Q(1', 3)$ is 3 and $Q(1', 3) = [1', 2, 2', 3]$. Using Theorem 3.3.8, G has exactly one minimal path of length 5 not containing $[1', 3]$. Using Corollary 3.3.9, length of each minimal path in G is at most 5. Using Lemmas 3.3.10, 3.3.11, G has no mm-alternating paths of length more than 7 and G has at most one mm-alternating path of length 7. By hypothesis, G has no mm-alternating paths of length 7. Using Remark 3.3.14, the path $[1, 1', 2, 2', 3, 3']$ is an mm-alternating path of length 5 and the degrees $d_G(1) = 1 = d_G(3')$, $d_G(2) = 2$ and $2 \le d_G(2') \le 3$. By the hypothesis the degree $d_G(2') = 2$. Using Remark 3.3.14, the graph G must have one of the structures shown in Figures 3.7 and 3.8. ∎

3.3.2 Self-inverse unicyclic graphs in $\mathcal{H}$

In the Subsection 3.3.1, we have characterized the unicyclic graphs $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ for which G^+ is unicyclic. It is clear that if $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ such that $G \cong G^+$, then $G^+ \in \mathcal{H}_u$. So to characterize self-inverse graphs in $\mathcal{H}_u \setminus \mathcal{H}_g$ it is sufficient to have a characterization of self-inverse graphs in $\{G \in \mathcal{H}_u \setminus \mathcal{H}_g \mid G^+ \textit{ is unicyclic}\}$.

Theorem 3.3.17. *Let $G \in \mathcal{H}_u$ have the structure shown in Figure 3.5. Let G_1 and G_2 be two subgraphs of G such that*

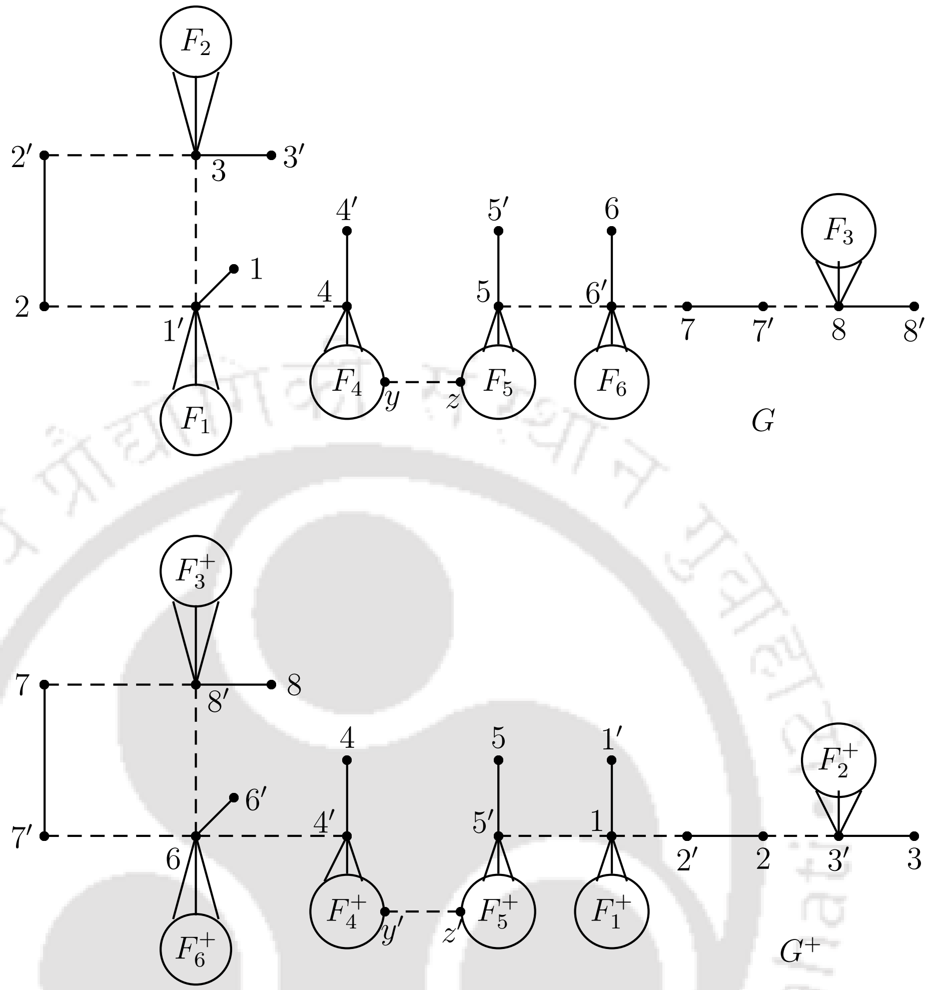


Figure 3.8: The graph G and its inverse G^+

1. *the subgraph G_1 is induced by the vertices $V(F_1) \cup \{1, 1'\}$, and*

2. *the subgraph G_2 is induced by the vertices $V(F_2) \cup \{4, 4'\}$.*

Then $G \cong G^+$ if and only if there is an isomorphism $f : G_1 \to G_2$ with $f(1') = 4$.

Proof: The inverse graph G^+ of the graph G has been constructed in Theorem 3.3.13 and G^+ shown in Figure 3.5. Since G_1 and G_2 are corona trees, using Theorem 3.1.1 $G_1 \cong G_1^+$ and $G_2 \cong G_2^+$. By using Corollary 3.2.6, the matching mapping $f_{\mathcal{M}}$ is an isomorphism form G_i to G_i^+ for $i = 1, 2$. The subgraphs G_1^+ and G_2^+ of G^+ are induced by the vertices $V(F_1^+) \cup \{1', 1\}$ and $V(F_2^+) \cup \{4', 4\}$, respectively. Let H be the graph obtained from G^+ by replacing G_1^+ with G_1 and G_2^+ with G_2 in G^+. The structures of H and G^+ are same. The graph H shown in Figure 3.9. The structures of the graph G and the graph H say that $G \cong G^+$ if and only if there is an isomorphism $f : G_1 \to G_2$ with $f(1') = 4$. ∎

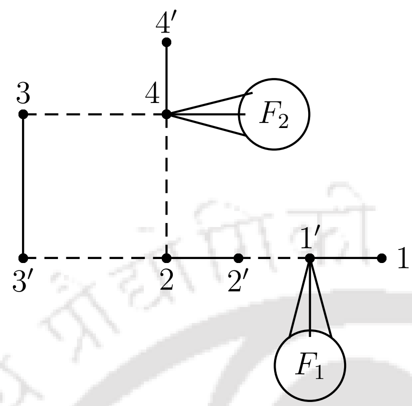


Figure 3.9: The graph H obtained from the graph G^+ shown in Figure 3.5

Theorem 3.3.18. *Let $G \in \mathcal{H}_u$ have the structure shown in Figure 3.6. Let G_1 and G_2 be two subgraphs of G such that*

1. *the subgraph G_1 is induced by the vertices $V(F_3) \cup \{3, 3'\}$, and*
2. *the subgraph G_2 is induced by the vertices $V(F_2) \cup \{4, 4'\}$.*

Then $G \cong G^+$ if and only if there is an isomorphism $f : G_1 \to G_2$ with $f(3) = 4$.

Proof: The inverse graph G^+ of the graph G has been constructed in Theorem 3.3.13 and G^+ shown in Figure 3.6. Let G_3 be a subgraph of G induced by the vertices $V(F_1) \cup \{1, 1'\}$. Since G_1, G_2 and G_3 are corona trees, using Theorem 3.1.1, we have $G_1 \cong G_1^+$, $G_2 \cong G_2^+$ and $G_3 \cong G_3^+$. By using Corollary 3.2.6, the matching mapping $f_{\mathcal{M}}$ is an isomorphism form G_i to G_i^+ for $i = 1, 2, 3$. The subgraphs G_1^+, G_2^+ and G_3^+ of G^+ are induced by the vertices $V(F_3^+) \cup \{3', 3\}$, $V(F_2^+) \cup \{4', 4\}$ and $V(F_1^+) \cup \{1', 1\}$, respectively. Let H be the graph obtained from G^+ by replacing G_1^+ with G_1, G_2^+ with G_2 and G_3^+ with G_3 in G^+. The structures of H and G^+ are same. The graph H shown in Figure 3.10. The structures of the graph G and the graph H say that $G \cong G^+$ if and only if there is an isomorphism $f : G_1 \to G_2$ with $f(3) = 4$. ∎

Theorem 3.3.19. *Let $G \in \mathcal{H}_u$ have the structure shown in Figure 3.7. Let G_1 and G_2 be the subgraphs of G such that*

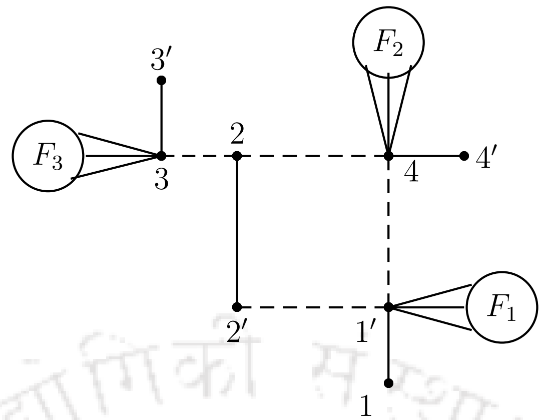


Figure 3.10: The graph H obtained from the graph G^+ shown in Figure 3.6

1. *the subgraph G_1 is induced by the vertices $V(F_2) \cup \{3, 3'\}$, and*
2. *the subgraph G_2 is induced by the vertices $V(F_3) \cup \{5, 5'\}$.*

Then $G \cong G^+$ if and only if there is an isomorphism $f : G_1 \to G_2$ with $f(5) = 3$.

Proof: The inverse graph G^+ of the graph G has been constructed in Theorem 3.3.13 and G^+ shown in Figure 3.7. Let G_3 be a subgraph of G induced by the vertices $V(F_1) \cup \{1, 1'\}$. Since G_1, G_2 and G_3 are corona trees, using Theorem 3.1.1, we have $G_1 \cong G_1^+$, $G_2 \cong G_2^+$ and $G_3 \cong G_3^+$. By using Corollary 3.2.6, the matching mapping $f_{\mathcal{M}}$ is an isomorphism form G_i to G_i^+ for $i = 1, 2, 3$. The subgraphs G_1^+, G_2^+ and G_3^+ of G^+ are induced by the vertices $V(F_2^+) \cup \{3', 3\}$, $V(F_3^+) \cup \{5', 5\}$ $V(F_1^+) \cup \{1', 1\}$, respectively. Let H be the graph obtained from G^+ by replacing G_1^+ with G_1, G_2^+ with G_2 and G_3^+ with G_3 in G^+. The structures of H and G^+ are same. The graph H shown in Figure 3.11. The structures of the graph G and the graph H say that $G \cong G^+$ if and only if there is an isomorphism $f : G_1 \to G_2$ with $f(5) = 3$. ■

The proof of the following theorem is same as the proof of Theorems 5.2.7, 6.2.9 and 3.3.19.

Theorem 3.3.20. *Let $G \in \mathcal{H}_u$ have the structure shown in Figure 3.8. Let G_1, G_2, G_3, G_4 and G_5 be the subgraphs of G such that*

1. *the subgraph G_1 is induced by the vertices $V(F_2) \cup \{3, 3'\}$,*
2. *the subgraph G_2 is induced by the vertices $V(F_3) \cup \{8, 8'\}$,*
3. *the subgraph G_3 is induced by the vertices $V(F_1) \cup \{1, 1'\}$,*

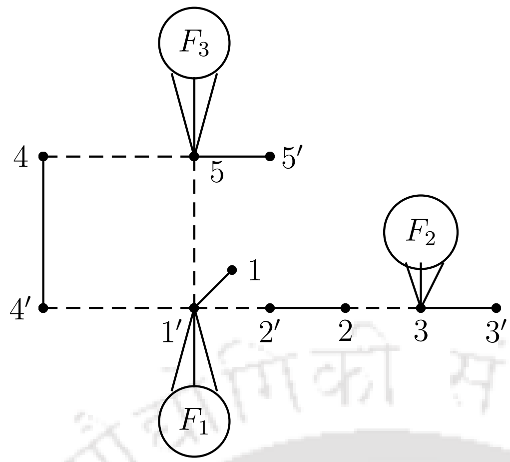


Figure 3.11: The graph H obtained from the graph G^+ shown in Figure 3.7

4. *the subgraph* G_4 *is induced by the vertices* $V(F_4) \cup \{4, 4'\}$,
5. *the subgraph* G_5 *is induced by the vertices* $V(F_5) \cup \{5, 5'\}$, *and*
6. *the subgraph* G_6 *is induced by the vertices* $V(F_6) \cup \{6, 6'\}$.

Then $G \cong G^+$ *if and only if there are isomorphisms* f_1, f_2 *and* f_3 *such that*

1. $f_1 : G_1 \to G_2$ *with* $f_1(3) = 8$,
2. $f_2 : G_4 \to G_5$ *with* $f_2(4) = 5$ *and* $f(y) = z$, *and*
3. $f_3 : G_3 \to G_6$ *with* $f_3(1') = 6'$.

Proof: The inverse graph G^+ of the graph G has been constructed in Theorem 3.3.13 and G^+ shown in Figure 3.8. Since G_i is a corona tree for $i = 1, \ldots, 6$, using Theorem 3.1.1, we have $G_i \cong G_i^+$ for $i = 1, \ldots, 6$. By using Corollary 3.2.6, the matching mapping $f_{\mathcal{M}}$ is an isomorphism form G_i to G_i^+ for $i = 1, \ldots, 6$. The subgraphs G_1^+, G_2^+, G_3^+, G_4^+, G_5^+ and G_6^+ of G^+ are induced by the vertices $V(F_2^+) \cup \{3', 3\}$, $V(F_3^+) \cup \{8', 8\}$, $V(F_1^+) \cup \{1', 1\}$, $V(F_4^+) \cup \{4', 4\}$, $V(F_5^+) \cup \{5', 5\}$ and $V(F_6^+) \cup \{6', 6\}$, respectively. Let H be the graph obtained from G^+ by replacing G_i^+ with G_i for $i = 1, \ldots, 6$. The structures of H and G^+ are same. The graph H shown in Figure 3.12. The structures of the graph G and the graph H say that $G \cong G^+$ if and only if there are isomorphisms f_1, f_2 and f_3 such that $f_1 : G_1 \to G_2$ with $f_1(3) = 8$ $f_2 : G_4 \to G_5$ with $f_2(4) = 5$ and $f(y) = z$ and $f_3 : G_3 \to G_6$ with $f_3(1') = 6'$. ∎

Remark 3.3.21. There are many unicyclic graphs G in $\mathcal{H}_u$ such that $G^+ \in \mathcal{H}_u$ but G is not isomorphic to G^+. For example, consider a graph G which has the

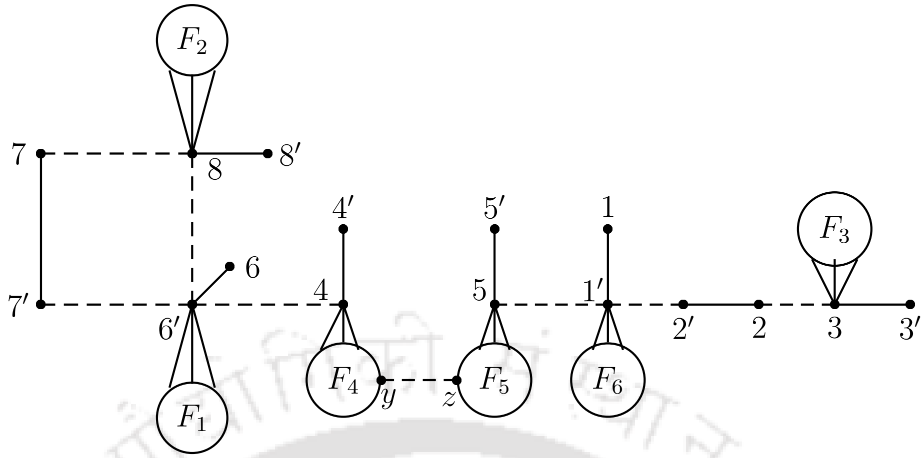


Figure 3.12: The graph H obtained from the graph G^+ shown in Figure 3.8

structure shown in Figure 3.5, where F_1 and F_2 are corona trees of different number of vertices.

3.4 Conclusion

This chapter has been devoted to the study of self-inverse graphs. In the previous chapter we have supplied a class of graphs which are invertible. Once we have a class of invertible graphs, then we are interested in characterizing self-inverse graphs from that class. This question, for the class $\mathcal{H}_g$ was asked by Godsil in 1985 and has already been answered by Simion and Cao in [26]. In [17], the author introduced the notion of strongly self-dual graph and supplied necessary and sufficient conditions for a graph in $\mathcal{H}$ to have strongly self-dual, utilizing constructions derived from the graph itself. However, there are many graphs in $\mathcal{H}$ for which these conditions cannot help; for example, the graph in Figure 2.1. We have seen that the notion of strongly self-dual is a particular case of the notion of self-inverse. Here we used the notion of self-inverse and characterized a class of graphs in $\mathcal{H}$ containing $\mathcal{H}_g$ in which $G \cong G^+$. Using the tools developed we supplied a characterization of self-inverse unicyclic graphs in $\mathcal{H}$.

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Chapter 4

Some structural properties of inverse graphs of a class of graphs

4.1 Preliminaries

In [26], Simion and Cao proved that for a graph $G \in \mathcal{H}_g$, the graph $G \cong G^+$ if and only if G is a simple corona. In the previous chapter, we have seen that there are many graphs $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ with $G^+ \in \mathcal{H}_u$ but G is not isomorphic to G^+. Once we have this much information, we ask ourselves the following question. Does there exist a noncorona graph in $\mathcal{H}_g$ such that G^+ in $\mathcal{H}_g$? It is not necessary that the inverse of a graph in $\mathcal{H}_g$ should be in $\mathcal{H}_g$. For example, we consider the graph $G = P_6$. By using Lemma 2.3.1, we can easily construct the graph G^+. The graph $G^+/\mathcal{M} = C_3$ (cycle with three vertices), so G^+ is not in $\mathcal{H}_g$. For the graphs $G = P_6$, G^+ and $G^+/\mathcal{M}$, see Figure 4.1.

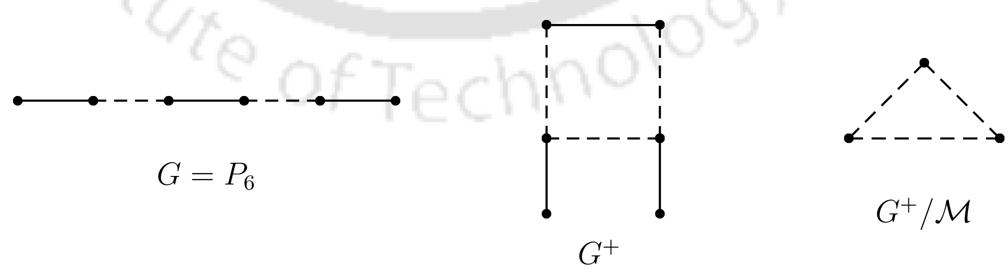

Figure 4.1: The graphs P_6, G^+ and $G^+/\mathcal{M}$

Here are a few questions we considered.

Q1 Does there exist a noncorona graph in $\mathcal{H}_g$ such that G^+ in $\mathcal{H}_g$ (that is, G^+ is unweighted and $G^+/\mathcal{M}$) is bipartite)?

Q2. Are there graphs in $\mathcal{H}_g$ for which $G^+ \in \mathcal{H}_{nm} \setminus \mathcal{H}_g$ (that is, G^+ is unweighted, $G^+/\mathcal{M}$ is not bipartite and G^+ has no mixed type edges)?

Q3. If there are such graphs, is it necessary that the inverse of a noncorona graph in $\mathcal{H}_g$ should be in $\mathcal{H}_{nm} \setminus \mathcal{H}_g$?

Q4. If it is not necessary, can we characterize the class of graphs in $\mathcal{H}_g$ whose inverses are in $\mathcal{H}_{nm} \setminus \mathcal{H}_g$?

Q5. Are there graphs G in $\mathcal{H}_g$ for which $G^+ \in \mathcal{H}_{nmc} \setminus \mathcal{H}_g$ (that is, G^+ is unweighted, $G^+/\mathcal{M}$ is not bipartite, G^+ has no mixed type edges and no two even type extensions at two distinct even type edges have an odd type edge in common)?

Q6. Are there graphs G in $\mathcal{H}_g$ for which $G^+ \in \mathcal{H}_{nmcs} \setminus \mathcal{H}_g$ (that is, G^+ is unweighted, $G^+/\mathcal{M}$ is not bipartite, G^+ has no mixed type edges, no two even type extensions at two distinct even type edges have an odd type edge in common and each even type edge of G^+ is strict)?

This chapter is devoted to answering all these questions. Note that the **underlying unweighted graph** of the inverse graph of a graph in $\mathcal{H}$ is also in $\mathcal{H}$. In fact, the same matching $\mathcal{M}$ is the unique perfect matching for both G and G^+.

4.2 Characterizing the graphs in $\mathcal{H}_g$ for which G^+ in $\mathcal{H}_g$

In this section, we shall supply the answer to question Q1. The following result essentially contained in [15, Theorem 2].

Lemma 4.2.1. *Let $G \in \mathcal{H}_g$. Then G^+ dominates G.*

The following remark which gives some structural information about the graphs $G \in \mathcal{H}_g$

Remark 4.2.2. Let $G \in \mathcal{H}_g$.

i) As G and $G/\mathcal{M}$ are bipartite, each cycle in G must contain an even number of matching edges.

ii) A path from i to j contains an odd (resp. even) number of nonmatching edges, then each path from i to j must contain an odd (resp. even) number of nonmatching edges.

iii) The inverse graph G^+ is either weighted or unweighted. Let $[x, y] \in E(G^+)$. Then $\mathrm{w}([x, y]) = A(G^+)_{x,y} = |A(G)^{-1}_{x,y}|$ = total number of mm-alternating x-y-path in G.

Let $\mathcal{P}_G$ be the set of mm-alternating paths in G.

Corollary 4.2.3. *Let $G \in \mathcal{H}_g$. Then $|\mathcal{P}_G| \geq |E(G^+)| \geq |E(G)|$.*

Proof: If $[i, j] \in E(G^+)$, then there is an mm-alternating i-j-path in G and in view of Remark 4.2.2, we get the first inequality. The second one follows by Theorem 2.3.11. ∎

Using Theorem 2.3.1 and Remark 4.2.2, one gets the following conclusion.

Lemma 4.2.4. *Let $G \in \mathcal{H}_g$ and $[i, j] \in E(G^+)$. Then the weight of the edge $[i, j]$ in G^+ is the total number of mm-alternating i-j-paths in G. Hence, G^+ is an unweighted graph if and only if the number of mm-alternating i-j-paths is at most one.*

The following theorem says that there are no noncorona graphs $G \in \mathcal{H}_g$ such that $G^+ \in \mathcal{H}_g$. This theorem is an extension of [28, Theorem 2.1].

Theorem 4.2.5. *Let $G \in \mathcal{H}_g$. Then the following are equivalent.*

i) $|\mathcal{P}_G| = |E(G)|$.

ii) $G \cong G^+$.

iii) $G^+ \in \mathcal{H}_g$

iv) $G = G_1 \circ K_1$ *for some connected bipartite graph* G_1.

Proof: i)⇒ii). It follows that $|E(G^+)| = |E(G)|$. As $G \in \mathcal{H}_g$ using Lemmas 2.3.1 and 4.2.4 and Theorem 2.3.11, we see that $G \cong G^+$. The proofs of ii)⇒iii) and iv)⇒i) are trivial.

iii)⇒iv). Suppose that G is not a corona. Then there exists a matching edge $[u_i, u_i']$ which is not a leaf. This can be extended to an mm-alternating path $[u_k, u_k', u_i, u_i', u_m, u_m']$ in G. Hence the cycle $[u_k, u_i', u_i, u_m', u_k] \in G^+$. As this cycle contains only one matching edge, it cannot be in $\mathcal{H}_g$, a contradiction. ∎

Remark 4.2.6. The equivalence of items ii) and iv) in Theorem 4.2.5 has previously been observed in [26, Theorem 2], but we supplied an alternate easy proof.

4.3 Characterizing the graphs in $\mathcal{H}_g$ for which G^+ in $\mathcal{H}_{nm} \setminus \mathcal{H}_g$

In this section, we shall supply answers to questions Q2–Q6. Using Remark 2.2.11, we see that a graph $G \in \mathcal{H}_g$ does not have an even type extension and for a graph $G \in \mathcal{H}$ in which we do not have any even type extension, then G may not be in $\mathcal{H}_g$. Using this fact, we define the following class.

Definition 4.3.1. Let $\mathcal{H}_{odd}$ be the class of graphs $G \in \mathcal{H}$ such that each nonmatching edge in G is odd type. That is, if $G \in \mathcal{H}_{odd}$, then G does not have an even type extension.

Example 4.3.2. In the following table we list the graphs used in different figures in Chapter 2 and mention whether they are in the class $\mathcal{H}_{odd}$.

Graph G	$\mathcal{H}_{odd}$?	Justifications
Figure 2.1	No	The graph G has an even type extension
Figure 2.2	No	The graph G has an even type extensions
Figure 2.3	Yes	The graph G has no even type extensions
Figure 2.4	No	The graph G has an even type extension
Figure 2.5	Yes	The graph G does not have even type extensions
Figure 2.6	No	The graph G has an even type extension
Figure 2.7	No	The graph G has an even type extension
Figure 2.8	No	The graph G has an even type extension
Figure 2.9	No	The graph G has an even type extension

Remark 4.3.3. Notice that $\mathcal{H}_g \subsetneq \mathcal{H}_{odd} \subsetneq \mathcal{H}_{nmcs} \subsetneq \mathcal{H}_{nmc} \subsetneq \mathcal{H}_{nm}$. By Remark 3.1.11, we have that $\mathcal{H}_g \subsetneq \mathcal{H}_{nmcs} \subsetneq \mathcal{H}_{nmc} \subsetneq \mathcal{H}_{nm}$. Figures 2.5 and 2.1 say that $\mathcal{H}_g \subsetneq \mathcal{H}_{odd}$ and $\mathcal{H}_{odd} \subsetneq \mathcal{H}_{nmcs}$, respectively.

The answer to question Q2 is affirmative. Consider the graph $G = P_6$, we have already seen that $G^+ \in \mathcal{H}_{nm} \setminus \mathcal{H}_g$. For details see Figure 4.1.

Lemma 4.3.4. *Let $G \in \mathcal{H}_g$. Let $P = [1, 1', 2, 2', 3, 3', 4, 4']$ be an mm-alternating path. Then the following are true.*

i) The path $[1, 2', 2, 3', 3, 4']$ is an odd type extension at $[1, 4']$ in G^+ .

ii) The path $[1, 2', 2, 4']$ is an even type extension at $[1, 4']$ in G^+.

iii) The edge $[1,4']$ is mixed type edge in G^+.

Proof: The proof follows by using Lemma 2.3.1 and Remark 4.2.2. ∎

The following remark answers question Q3.

Remark 4.3.5. There are graphs $G \in \mathcal{H}_g$ for which $G^+ \notin \mathcal{H}_{nm}$. For example, we consider the graph $G = P_8$. By using Lemma 2.3.1, we can construct the inverse graph G^+. The graphs G and G^+ shown in Figure 4.2. In G^+, the edge $[1,4']$ is mixed type. The paths $[1,2',2,3',3,4']$ and $[1,2',2,4']$ are odd type and even type extensions at $[1,4']$ in G^+, respectively.

Figure 4.2: The path P_8 and its inverse graph

The following lemma answers question Q4.

Theorem 4.3.6. *Let $G \in \mathcal{H}_g$ be a noncorona graph. Then $G^+ \in \mathcal{H}_{nm} \setminus \mathcal{H}_g$ if and only if*

i) the graph G has at most one mm-alternating path in G from one vertex to another vertex;

ii) there are no mm-alternating paths in G of length 7.

Proof: First we consider the graph $G^+ \in \mathcal{H}_{nm} \setminus \mathcal{H}_g$. Then G^+ is an unweighted graph. By using Remark 4.2.2, G has at most one mm-alternating path in G from one vertex to another vertex. By using Lemma 4.3.4, Item ii) is true.

The converse follows by Lemma 4.3.4. ∎

Remark 4.3.7. The above lemma proves more. In particular, if $G \in \mathcal{H}_g$ satisfies the hypothesis of Theorem 4.3.6, then $G^+ \in \mathcal{H}_{nmc} \setminus \mathcal{H}_g$. So, the answer to Q5 is affirmative.

The following lemma answers question Q6.

Lemma 4.3.8. *Let $G \in \mathcal{H}_g$ be a noncorona graph. Then $G^+ \notin \mathcal{H}_{nmcs} \setminus \mathcal{H}_g$.*

Proof: Suppose that $G^+ \in \mathcal{H}_{nmcs} \setminus \mathcal{H}_g$. The graph G^+ is noncorona, otherwise $G^+ \in \mathcal{H}_g$. Then G^+ has an mm-alternating path of length 5, say, $[u_1, u_1', u_2, u_2', u_3, u_3']$. Let $H = G^+$. By using Lemma 3.2.2, the path $[u_1, u_2', u_2, u_3']$ and the edge $[u_1, u_3']$ are in H^+. Since $H^+ = G$, the graph G has an even type extension $[u_1, u_2', u_2, u_3']$ at $[u_1, u_3']$. A contradiction to the fact that $G \in \mathcal{H}_g$. ∎

The following is an immediate corollary of Lemma 4.3.8.

Corollary 4.3.9. *Let $G \in \mathcal{H}_g$. Then $G^+ \notin \mathcal{H}_{odd} \setminus \mathcal{H}_g$.*

Remark 4.3.10. Let $G \in \mathcal{H}_g$ and G^+ is noncorona. Then the following facts hold.

- The inverse graph G^+ is either weighted or unweighted.
- The underlying unweighted graph of G^+ is either in $\mathcal{H}_{nmc}$ or G^+ has mixed type edges.
- The underlying unweighted graph of G^+ never lies in $\mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$.
- There are graphs $G \in \mathcal{H}_g$ such that G^+ is weighted and underlying unweighted graph of G^+ in $\mathcal{H}_{nmcs}$. For example, consider the graph G shown in Figure 1.3. The inverse graph G^+ of G is shown in Figure 4.3. We noticed that $G^+ = G_{\mathrm{w}}$ where G is the graph shown in Figure 2.1 and w is the weight function.

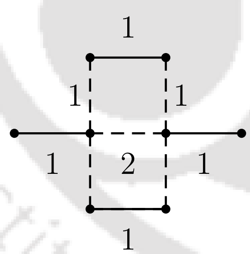

Figure 4.3: The inverse graph G^+ of the graph G

4.4 Conclusion

In this chapter, we have studied different properties of the inverse graphs of the graphs in $\mathcal{H}_g$. In the previous chapter, we saw that the class of self-inverse unicyclic graphs in $\mathcal{H}$ is a proper subclass of the class of unicyclic graphs in $\mathcal{H}$ with unicyclic inverses this fact has raised the following question. Does there exist a graph in

$G \in \mathcal{H}_g$ such that $G^+ \in \mathcal{H}_g$ but $G \ncong G^+$? We have shown that there are no such graphs. Then we discussed a few other properties of the inverse graphs of the graphs in $\mathcal{H}_g$.

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Chapter 5

Constructive structures of the class of inverse graphs of a class of graphs

5.1 Preliminaries

Pavlikova and Jediny [21] had given a way to construct T^+ for a nonsingular tree T. This construction is simply a combinatorial description of the $A(T)^{-1}$. Later on in 2006, Barik, Neuman and Pati supplied the combinatorial description of the $A(G)^{-1}$ of $G \in \mathcal{H}$. Note that for a nonsingular tree T, the inverse T^+ is not always a tree, so a natural question follows. *Which graphs can occur as the inverse of a nonsingular tree?* An answer was supplied by Neumann and Pati in [27], where the authors supplied a constructive characterization of the class of graphs that are inverse graphs of some nonsingular trees. As graphs in $\mathcal{H}_g$ are very naturally close to trees, one naturally wonders, *whether a characterization of the inverse graphs of graphs in* $\mathcal{H}_g$ is possible. This chapter is devoted to supply a constructive characterization of the class of graphs that are inverse graphs of some graphs in $\mathcal{H}_g$.

In Section 5.2, we supply a constructive characterization of the class of weighted graphs H_{w} that can occur as the inverse of some graph $G \in \mathcal{H}_g$, generalizing the result in [27, Theorem 2.6].

5.1.1 Construction of the class of inverse graphs of nonsingular trees

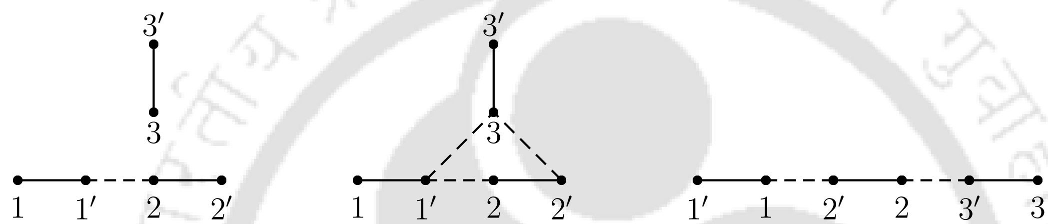
The construction given by Neumann and Pati is the following.

Let $\mathcal{F}_1 = \{P_2\}$ and $\mathcal{F}_2 = \{P_4\}$. Let G_k, $k \geq 3$ be a graph obtained by the following steps:

1. taking the disjoint union of a graph $G_{k-1} \in \mathcal{F}_{k-1}$ and $P_2 = [u, u']$ and
2. by making u adjacent to all neighbors of some vertex $w \in G_{k-1}$.

Theorem 5.1.1. [27, Theorem 2.6] *Let $k \geq 1$. A graph G is in $\mathcal{F}_k$ if and only if G is the inverse of a nonsingular tree on $2k$ vertices.*

Example 5.1.2. Let $P_4 = [1, 1', 2, 2']$. We take the disjoint union of P_4 and $[3, 3']$ and making 3 adjacent to all neighbors of 2. Then the resulting graph is an element in $\mathcal{F}_3$ and the inverse graph of P_6.



5.2 Constructive characterization of the class of inverse graphs of graphs in $\mathcal{H}_g$

Lemma 5.2.1. *[27] Let $G \in \mathcal{H}$. Then G has at least two pendant (degree one) vertices.*

The following theorem gives some structural information on connected, bipartite graphs with unique perfect matchings.

Theorem 5.2.2. *Let $G \in \mathcal{H}$. Then G has a pendant vertex v such that $G - v - v'$ is connected, where v' is the vertex adjacent to v.*

Proof: In view of Lemma 5.2.1, take a pendant vertex u. Let u' be adjacent to u. If $G - u - u'$ is connected, we have nothing to prove. So, assume that $G - u - u'$ is not connected. Let C be a component of $G - u - u'$.

Claim. *The component C has a pendant vertex v which is also a pendant vertex in G.* To see the claim, note that C has pendant vertex (by Lemma 5.2.1), say v. If v is not pendant vertex of G, then $[u', v] \in E(G)$. As C has a perfect matching $\mathcal{M}$, let $[v, v'] \in \mathcal{M}$. Note that $v' \nsim u'$, otherwise we have an odd cycle, which is not possible. Thus, $v' \sim v_1$ for some $v_1 \in C$. Again, as C has a perfect matching $\mathcal{M}$, let

$[v_1, v_1'] \in \mathcal{M}$. Note again that $v_1' \nsim u'$, otherwise we have an odd cycle, which is not possible. Continue the process. As C is finite, we must have a repetition of a vertex, say w, for the first time. Our walk so far must look like $[v, v', v_1, v_1', \ldots, v_{j-1}', v_j, w]$ or $[v, v', v_1, v_1', \ldots, v_{j-1}', w]$. As each vertex on $[v, v', v_1, v_1', \ldots, v_{j-1}']$ have already been matched, our walk so far cannot look like $[v, v', v_1, v_1', \ldots, v_{j-1}', v_j, w]$. So our walk so far looks like $[v, v', v_1, v_1', \ldots, v_{j-1}', w]$. If $w \in \{v, v_1, \ldots, v_{j-2}\}$, then we have an mm-alternating cycle giving us more than one perfect matching. If $w \in \{v', v_1', \ldots, v_{j-2}'\}$, then we have an odd cycle, which is not possible. Thus, the claim is justified.

Now we continue the main proof. Select a pendant vertex v of G which is in C. Let v be adjacent to v'. If $G - v - v'$ is connected, then we have nothing to prove. So, suppose that $G - v - v'$ is disconnected. Consider a component C' which does not contain u'. Then C' is a strict subgraph of C with less number of vertices. So C' has a pendant vertex which is also a pendant vertex in G. As G is finite, this process cannot be continued indefinitely, so we finally get a pendant vertex w of G which is adjacent to w' and $G - w - w'$ has a component which is just an edge $[x, x']$. Assume (in view of Lemma 5.2.1) that x is a pendant vertex of G. Then $d_G(x') = 2$ and $G - x - x'$ is connected. ■

By $N_G(v)$, we denote the neighborhood of a vertex v in G.

Proposition 5.2.3. *Let $G \in \mathcal{H}_g$. Take a pendant vertex u which is adjacent to u' such that $G - u - u'$ is connected. Let $N_G(u') = \{v_1, v_2, \ldots, v_m\}$. Then each path from v_i to v_j contains an even number of matching edges.*

Proof: Suppose that there is a path $P(v_i, v_j)$ from v_i to v_j contains an odd number of matching edges. Then the cycle $\Delta = [u', P(v_i, v_j), u']$ has an odd number of matching edges. This contradicts the fact that $G \in \mathcal{H}_g$. ■

Example 5.2.4. We explain the above lemma by an example. Consider the graph G shown in Figure 5.1. Here $G - 5 - 5'$ is connected and $5'$ adjacent to vertices 1 and 3. There are two paths from 1 to 3 which are $[1, 5', 3]$ and $[1, 1', 2, 2', 3]$. The path $[1, 5', 3]$ does not contain matching edges and the path $[1, 1', 2, 2', 3]$ contains two matching edges which are $[1, 1']$ and $[2, 2']$.

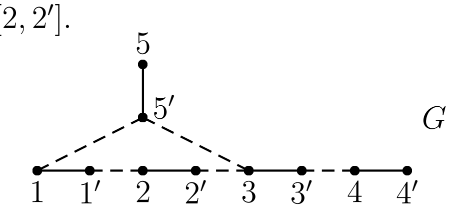


Figure 5.1: A graph $G \in \mathcal{H}_g$

Lemma 5.2.5. *Let $G \in \mathcal{H}_g$ have $n \geq 4$ vertices and let $H = G^+$. Then there exist vertices $u', u, v_1, v_2, \ldots, v_m \in H$ such that*

i) u' is pendant in H;

ii) $u \sim u'$ and for each $i = 1, \ldots, m$, we have $u' \nsim v_i$ in H;

iii) $N_H(u) = \cup_{i=1}^m N_H(v_i) \cup \{u'\}$; and

iv) for each $x \in \cup_{i=1}^m N_H(v_i)$ we have $\mathrm{w}_H([u,x]) = \sum_{v_i \sim x} \mathrm{w}_H([v_i, x])$.

Proof: We use Proposition 5.2.3 and select $u, u', v_1, v_2, \ldots, v_m \in G$ as described in the proof of Proposition 5.2.3. Using Lemma 2.3.1, we see that u' is pendant, $u' \sim u$ and for each $i = 1, \ldots, m$, $u' \nsim v_i$ in H.

Let $x \in \cup_{i=1}^m N_H(v_i)$. Then there is an mm-alternating path from some v_i to x in G. Hence an mm-alternating path from u to x exists in G. Using Remark 4.2.2, we see that $[u, x] \in E(H)$.

Conversely, if $x \sim_H u$ and $x \neq u'$, then there is an mm-alternating path P from u to x in G. As u is pendant and $u \sim_G u'$, the third vertex on P must be some v_i. Then $P - u - u'$ gives an mm-alternating path from v_i to x in G. Thus $x \sim_H v_i$.

The final assertion follows from the fact that $\mathrm{w}_H([u,x])$ is the number of mm-alternating paths from u to x in G and when we delete u and u' from each such path we get an mm-alternating path from some v_i to x in G. ∎

Example 5.2.6. Now we explain the above lemma by considering the graph G shown in Figure 5.1. By using Lemma 2.3.1, we construct the inverse graph $H = G^+$. The graph $H = G^+$ shown in Figure 5.2. Here $N_H(1) = \{1', 2', 3', 4'\}$ and $N_H(3) = \{3', 4'\}$. So, $N_H(5) = \{1', 2', 3', 4'\}$. There is at most one mm-alternating path from one vertex to another vertex in G. So, by using Remark 4.2.2, we have $\mathrm{w}_H([1, 1']) = \mathrm{w}_H([1, 2']) = \mathrm{w}_H([1, 3']) = \mathrm{w}_H([1, 4']) = 1$ and $\mathrm{w}_H([3, 3']) = \mathrm{w}_H([3, 4']) = 1$. Therefore $\mathrm{w}_H([5, 1']) = \mathrm{w}_H([5, 2']) = 1$ and $\mathrm{w}_H([5, 3']) = \mathrm{w}_H([3', 1]) + \mathrm{w}_H([3', 3]) = 1 + 1 = 2$, similarly $\mathrm{w}_H([5, 4']) = 2$.

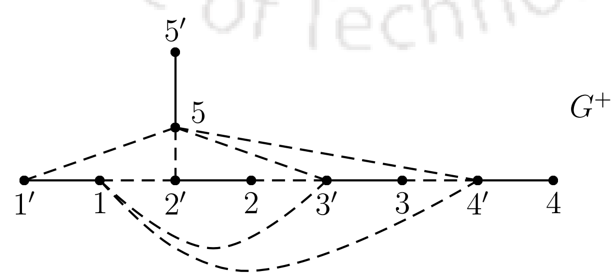


Figure 5.2: The inverse graph G^+ of the graph G in Figure 5.1

We have the following extension of [27, Theorem 2.2].

Theorem 5.2.7. *Let $\mathcal{F}_1 = \{P_2\}$ and $\mathcal{F}_2 = \{P_4\}$. Notice that graphs in $\mathcal{F}_i$ are precisely the inverses of graphs of order $2i$ in $\mathcal{H}_g$. Assume that we have constructed the graph class $\mathcal{F}_{k-1}$ which consists of the graphs which can occur as the inverse of some graph of order $2(k-1)$ in $\mathcal{H}_g$. Now we construct $\mathcal{F}_k$ in the following manner.*

1. *Take a graph $H_{k-1} \in \mathcal{F}_{k-1}$. Take a disjoint union of $[u, u']$ with H_{k-1}.*
2. *Choose a set of vertices $S = \{v_1, v_2, \ldots, v_m\}$ in H_{k-1} such that $dist_{H_{k-1}}(v_i, v_j)$ is even and no path from v_i to v_j in H^+_{k-1} contains an odd number of matching edges for $i, j = 1, 2, \ldots, m$.*
3. *For each $x \in \cup_{i=1}^m N_{H_{k-1}}(v_i)$, add the edge $[u, x]$ and put the weight $\mathrm{w}([u, x]) = \sum_{x\sim v_i} \mathrm{w}_{H_{k-1}}([x, v_i])$ and the weight $\mathrm{w}([u, u']) = 1$.*

Then $\mathcal{F}_k$ consists of the graphs of order $2k$ which occur as the inverses of the graphs in $\mathcal{H}_g$.

Before we supply a proof let us illustrate the theorem by constructing one element from $\mathcal{F}_3$ and one element from $\mathcal{F}_4$.

Construction of a graph in $\mathcal{F}_3$:

Note that $H_2 = P_4 = [x, x', y, y'] \in \mathcal{F}_2$ and $H_2^+ = [x', x, y', y] \in \mathcal{H}_g$. We take a disjoint union of H_2 and $[u, u']$. We cannot choose more than one vertex from H_2 which satisfy Condition 2 in the hypothesis in Theorem 5.2.7. So, let us choose $S = \{y\}$. Now, $N_{H_2}(y) = \{x', y'\}$. Now we add the edges $[u, x']$ and $[u, y']$. Since the weights of the edges $[x', y]$ and $[y, y']$ are 1 in H_2, using Condition 3 in the hypothesis, we set $\mathrm{w}([u, x']) = \mathrm{w}([u, y']) = 1$. This graph H_3 is shown in Figure 5.3. By using Lemma 2.3.1, we see that $H_3^+ = P_6 = [x', x, y', y, u', u]$.

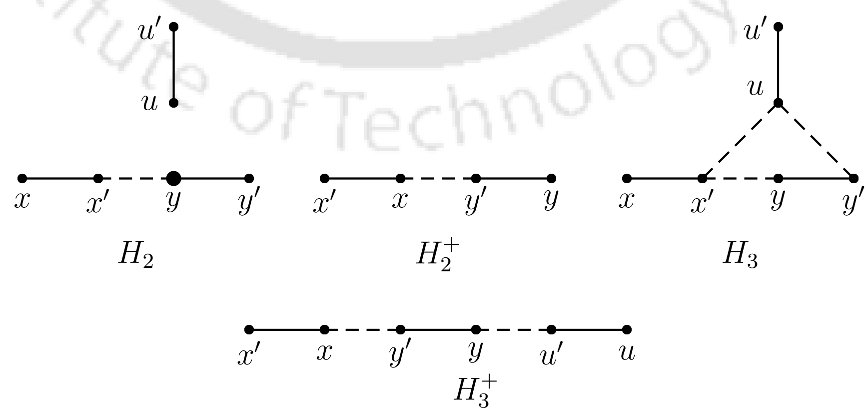


Figure 5.3: Construction of a graph H_3 in $\mathcal{F}_3$.

Construction of a graph in $\mathcal{F}_4$:

Now, let us take a disjoint union of H_3 and $[u_1, u_1']$. Take $S = \{x, u\}$. Then these vertex have even distance among them in H_3 and no path among them in H_3^+ contains an odd number of matching edges. This satisfy Condition 2 in the hypothesis of Theorem 5.2.7. Notice that $N_{H_3}(x) = \{x'\}$ and $N_{H_3}(u) = \{u', x', y'\}$. Now we add the edges $[u_1, x'], [u_1, y']$ and $[u_1, u']$. Using Condition 3 in the hypothesis, we set $\mathrm{w}([u_1, x']) = 2$ because x' is in $N_{H_3}(x) \cap N_{H_3}(u)$. Similarly, we set $\mathrm{w}([u_1, y']) = 1$ and $\mathrm{w}([u_1, u']) = 1$. The new graph H_4 and its inverse H_4^+ are shown in Figure 5.4.

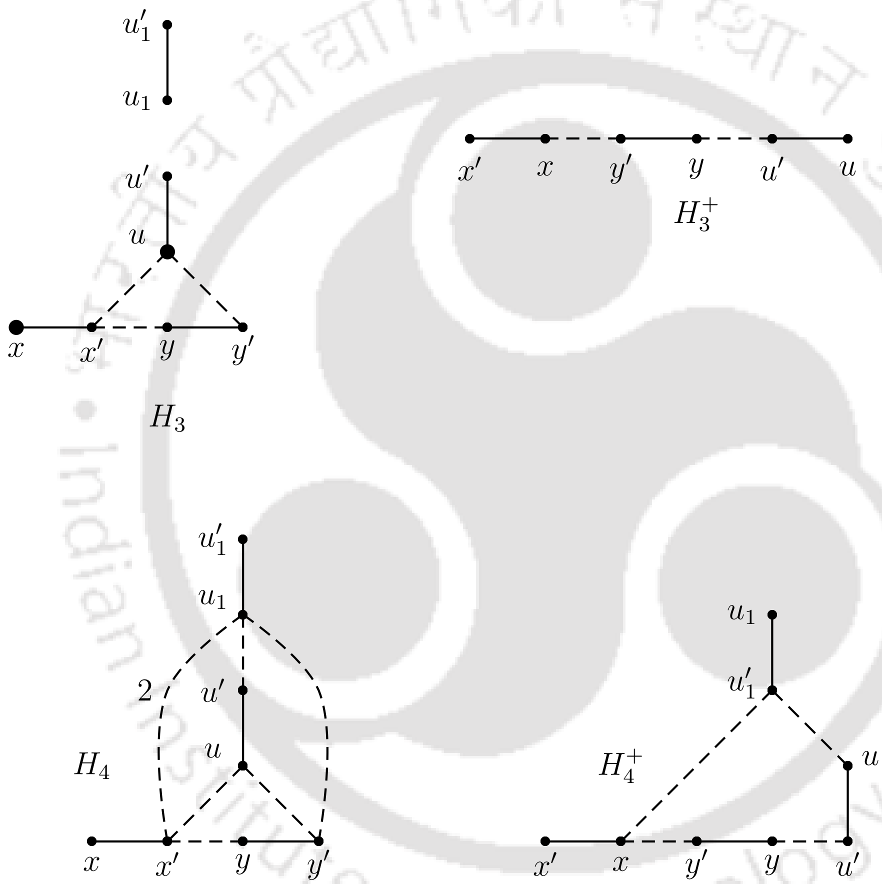


Figure 5.4: Construction of a graph H_4 in $\mathcal{F}_4$.

Now we prove the theorem.

Proof: Let $G \in \mathcal{H}_g$ of order $2k$ and $G^+ = H$. By using Proposition 5.2.3 and Lemma 5.2.5, there exist vertices $u', u, v_1, v_2, \ldots, v_m \in G$ such that

1. u is pendant and $u \sim u'$ in G;

2. $N_G(u') = \{v_1, v_2, \ldots, v_m\}$;

3. u' is pendant in H;

4. $u \sim u'$ and for each $i = 1, \dots, m$, we have $u' \nsim v_i$ in H;

5. $N_H(u) = \cup_{i=1}^m N_H(v_i) \cup \{u'\}$; and

6. for each $x \in \cup_{i=1}^m N_H(v_i)$ we have $\mathrm{w}_H([u,x]) = \sum_{v_i \sim x} \mathrm{w}_H([v_i, x])$.

It is clear that $G - u - u' \in \mathcal{H}_g$. By the hypothesis there is a graph $H' \in \mathcal{F}_{k-1}$ such that $(G - u - u')^+ = H'$. Then the graph $H - u - u' = H'$ and $H \in \mathcal{F}_k$. Therefore the inverses of the graphs $G \in \mathcal{H}_g$ of order $2k$ lie in $\mathcal{F}_k$.

Now we show that each $H_k \in \mathcal{F}_k$ is the inverse of some graph $G \in \mathcal{H}_g$ or order $2k$. Let $H_k \in \mathcal{F}_k$. Then H_k is constructed in the following manner.

1. Take a graph $H_{k-1} \in \mathcal{F}_{k-1}$. Take a disjoint union of $[u, u']$ with H_{k-1};

2. Choose a set of vertices $S = \{v_1, v_2, \dots, v_m\}$ in H_{k-1} such that $dist_{H_{k-1}}(v_i, v_j)$ is even and no path from v_i to v_j in H_{k-1}^+ contains an odd number of matching edges for $i, j = 1, 2, \dots, m$;

3. For each $x \in \cup_{i=1}^m N_{H_{k-1}}(v_i)$, add the edge $[u, x]$ and put the weight $\mathrm{w}_{H_k}([u,x]) = \sum_{x \sim v_i} \mathrm{w}_{H_{k-1}}([x, v_i])$ and the weight $\mathrm{w}_{H_k}([u, u']) = 1$.

By the hypothesis there is a graph $G_{k-1} \in \mathcal{H}_g$ such that $G_{k-1}^+ = H_{k-1}$. Now take the disjoint union of G_{k-1} and $[u, u']$. Adding the edges $[u', v_1], \dots, [u', v_m]$ we get a new graph G_k. It is clear that $G_k \in \mathcal{H}_g$. By Lemma 5.2.5, $G_K^+ = H_k$. Therefore each graph $H_k \in \mathcal{F}_k$ is the inverse of some graph in $\mathcal{H}_g$ of order $2k$. Hence we conclude that the class $\mathcal{F}_k$ consists of the graphs of order $2k$ which occur as the inverses of the graphs in $\mathcal{H}_g$. The proof is complete. ■

5.3 Conclusion

In this chapter, we have supplied a constructive characterization of the class of inverse graphs of graphs $G \in \mathcal{H}_g$. This characterization generalizes the result in [27], where the authors supplied a constructive characterization of the class of inverse graphs of nonsingular trees.

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Chapter 6

Inverses of weighted graphs

6.1 Preliminaries

To aid the reader, we recall some notation that will be used in this chapter.

w	:	A function from $E(G)$ to $(0, \infty)$, we call w a positive weight function.
G_{w}	:	A weighted graph obtained from G by using w.
$\mathcal{W}_G$	:	A class of weight functions w such that $\mathrm{w}(e) = 1$ for each matching edge e in G, where G is assumed to be in $\mathcal{H}$.

In this chapter, we extend the notion of graph inverse to positive weighted graphs. We begin our discussion by considering a couple of graphs H and G shown in Figures 2.2 and 2.7, respectively. We have the following observations about these two graphs.

- By using Example 2.3.7, the graph H is not invertible.
- Consider the weight function w such that $\mathrm{w}(e) = 1$ if $e \neq [2', 3]$ and $\mathrm{w}([2', 3]) = 2$. The weighted graph H_{w} is invertible, where $S = [1, 1, -1, -1, 1, 1, -1, -1]$.
- By using Lemma 2.3.15, the graph G is invertible.
- Consider the weight function w such that $\mathrm{w}(e) = 1$ if $e \neq [1', 3]$ and $\mathrm{w}([1', 3]) = 2$. The weighted graph G_{w} is not invertible. To see this put $B = A(G)^{-1}$. Notice that $B(1, 2') = -1 = B(2, 3') = B(1, 3')$ and $B(2, 2') = 1$. If G^+ exists, then cycle $[1, 2', 2, 3', 1]$ must be present in it. Let S be a signature matrix such that $SBS \geq 0$. Then $s_1^2 s_2^2 s_{2'}^2 s_{3'}^2 < 0$ which is a contradiction. Hence G_{w}^+ does not exist.

These two graphs have ensured that the invertibility of any graph G is not a necessary and sufficient condition for the invertibility of G_{w} and made us feel that there is a necessity of studying the invertibility of weighted graphs in $\mathcal{H}$.

Here are a few questions we considered.

Q1. *Let $G \in \mathcal{H}$. Assume that G^+ exists. Is it necessary that G_{w}^+ will exist for some weight function which not a multiple of $\mathbb{1}$, the all ones weight function?*

Q2. *Let $G \in \mathcal{H}_g$. We know that G^+ exists. Is it necessary that G_{w}^+ will exist for <u>each</u> weight function* w*?*

Q3. *Under which conditions the existence of G_{w}^+ for one* w *will force G to be inside $\mathcal{H}_g$?*

Q4. *Let $G \in \mathcal{H}$. Assume that G_{w}^+ exists for each* w*. Is it necessary that $G \in \mathcal{H}_g$?*

Q5. *Let $G \in \mathcal{H}$. Assume that G^+ does not exist. Is it necessary that G_{w}^+ will never exist for any weight function* w*?*

Q6. *Supply a class of graphs G in $\mathcal{H}$ for which G_{w}^+ never exists.*

Q7. *Supply a class of graphs G in $\mathcal{H}$ for which G_{w}^+ exists for some* w *but not for all* w*. Obviously such a graph is outside $\mathcal{H}_g$.*

Q8. *Supply a class of graphs G in $\mathcal{H}$ for which G^+ does not exist but G_{w}^+ exists for some* w*.*

We supply answer to all these questions in this chapter.

Remark 6.1.1. It is clear that $\mathcal{H}_g \subsetneq \mathcal{H}_{odd} \subsetneq \mathcal{H}_{nmc}$. The graph G shown in Figure 2.5 is in $\mathcal{H}_{odd}$ but not in $\mathcal{H}_g$ and the graph shown in Figure 2.1 is in $\mathcal{H}_{nmc}$ but not in $\mathcal{H}_{odd}$.

6.2 Inverses of weighted graphs in $\mathcal{H}_{odd}$

In this section, we answer questions Q1–Q4. Actually this section deals with the study of invertibility of weighted graphs in $\mathcal{H}_{odd}$. We start the discussion with a decisive answer question Q1. Then we show that for each graph $G \in \mathcal{H}_g$ and for each $\mathrm{w} \in \mathcal{W}_G$, the weighted graph G_{w} is invertible which supplies positive answer question Q2. We also show that there are no graphs G in $\mathcal{H}_{odd} \setminus \mathcal{H}_g$ for which G_{w}^+ exists for some $\mathrm{w} \in \mathcal{W}_G$ which answers question Q3. As a corollary of this result we shall get a class of graphs $G \in \mathcal{H}$ such that G_{w}^+ does not exist for each $\mathrm{w} \in \mathcal{W}_G$.

Finally, we show that the class $\mathcal{H}_g$ is the only subclass of graphs G in $\mathcal{H}$ such that G^+_{w} exists for each $\mathrm{w} \in \mathcal{W}_G$ which answers question Q4. In order accomplish this we need the following definition.

Definition 6.2.1. Let G be a graph. Assume that P is a path in G. We use $\mathrm{w}(P)$ to mean the weight of P, which is the product of the weights of the edges on P.

The following is essentially contained in [34, Theorem 1] and [27, Lemma 2.1]. We note that the mm-alternating paths have been termed as alternating paths in [34, 27].

Lemma 6.2.2. *Consider G_{w}, where $G \in \mathcal{H}$ and $\mathrm{w} \in \mathcal{W}_G$. Let $B = [b_{ij}]$, where*

$$b_{ij} = \sum_{P(i,j) \in \mathcal{P}(i,j)} (-1)^{(\|P(i,j)\|-1)/2} \mathrm{w}(P),$$

where $\mathcal{P}(i,j)$ is the set of mm-alternating i-j-paths in G_{w} and $\|P(i,j)\|$ is the number of edges in the i-j-path $P(i,j)$. Then $B = A(G_{\mathrm{w}})^{-1}$.

To proceed further we need the following lemma which is the generalization of [25, Theorem 5].

Lemma 6.2.3. *Let $G \in \mathcal{H}$ and $\mathrm{w} \in \mathcal{W}_G$. Consider $A(G_{\mathrm{w}})^{-1}$ and construct a weighted graph $\hat{G}$ from G_{w} as follows: for each pair of vertices i, j take i adjacent to j in $\hat{G}$ whenever $\sum_{P \in \mathcal{P}_{ij}} (-1)^{\frac{\|P\|-1}{2}} \mathrm{w}(P) \neq 0$, and let the weight of that edge be 1 or -1 according as $\sum_{P \in \mathcal{P}_{ij}} (-1)^{\frac{\|P\|-1}{2}} \mathrm{w}(P)$ is positive or negative. Then $A(G_{\mathrm{w}})^{-1}$ is diagonally similar to a non-negative matrix if and only if the product of the edge weights on any cycle in $\hat{G}$ is 1.*

Proof: The proof is similar to the proof of Theorem 5 in [25]. ∎

The following proposition answers question Q1.

Proposition 6.2.4. *Let $G \in \mathcal{H}$ and G^+ exists. Then there exists a weight function w, not a multiple of $\mathbb{1}$, such that G^+_{w} exists.*

Proof: By using Lemma 5.2.1, the graph G has a pendant vertex. Let u' be a pendant vertex in G which is adjacent to u. Let $N_G(u) = \{u', u_1, \ldots, u_k\}$. We consider a weight function w defined as.

$$\mathrm{w}([x,y]) = \begin{cases} 1 & \text{if } [x,y] \in E(G) - \cup_{i=1}^{k} \{[u,u_i]\} \\ 2 & \text{otherwise.} \end{cases}$$

By using Lemma 6.2.2, we see that the entries of $A(G_{\mathrm{w}})^{-1}$ are related to the entries of $A(G)^{-1}$ which are given by.

$$A(G_{\mathrm{w}})^{-1}_{x,y} = \begin{cases} A(G)^{-1}_{x,y} & \text{if } x, y \neq u' \\ 2A(G)^{-1}_{x,y} & \text{otherwise.} \end{cases}$$

Then the graph $\hat{G}$ (for $\hat{G}$, see Lemma 6.2.3) corresponding to G_{w} is same as the the graph $\hat{G}$ corresponding to G. Since G^+ exists, by using Lemma 6.2.3, the product of the edge weights on any cycle in $\hat{G}$ is 1. Then by using Lemma 6.2.3, G^+_{w} exists.■

The following is an extension of [15, Theorem 2.2] to the weighted case which also answers question Q2.

Theorem 6.2.5. *Let $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$. Then G^+_{w} exists.*

Proof: First, we define a signature matrix S in the following way. Put $s_1 = 1$. For $i \neq 1$, put $s_i = 1$ if any path from 1 to i contains an even number of nonmatching edges; otherwise we put $s_i = -1$. By Remark 4.2.2, the matrix S is well defined.

Assume that $SA(G_{\mathrm{w}})^{-1}S \not\geq 0$, that is, there exist i and j such that $s_i A(G_{\mathrm{w}})^{-1}_{i,j} s_j < 0$. First suppose that $A(G_{\mathrm{w}})^{-1}_{i,j} < 0$. Then $s_i = s_j$. So the parities of the number of nonmatching edges on any 1-i-path and any 1-j-path are the same. Hence, any i-j-path must contain an even number of nonmatching edges. In that case, $A(G_{\mathrm{w}})^{-1}_{i,j}$ must be nonnegative, by Remark 4.2.2 and Lemma 6.2.2. A similar contradiction is obtained if $A(G_{\mathrm{w}})^{-1}_{i,j} > 0$. Hence $SA(G_{\mathrm{w}})^{-1}S \geq 0$, that is , G^+_{w} exists. ■

From Theorem 6.2.5, could it be the case that if $G \in \mathcal{H}$, then G^+_{w} may exist for each $\mathrm{w} \in \mathcal{W}_G$? As it turns out this is not true in general. That is , the weights do matter in deciding whether the inverse exists or not. The following result addresses this issue.

Theorem 6.2.6. *Let $G \in \mathcal{H} \setminus \mathcal{H}_{odd}$. Then there is a weight function $\mathrm{w} \in \mathcal{W}_G$ such that G^+_{w} does not exist.*

Proof: As $G \in \mathcal{H} \setminus \mathcal{H}_{odd}$, it has a nonmatching edge $[u, v]$ which is either even type or mixed type. We first assume that $[u, v]$ is even type.

Let $Q(u, v) = [u, u_1, u'_1, u_2, u'_2, \ldots, u_{2k-1}, u'_{2k-1}, v]$ be a maximum length even type extension at $[u, v]$. Let $[x, y]$ be a nonmatching edge on $Q(u, v)$.

Claim1. There are no odd type extensions at $[x, y]$.

Proof of the claim. Note that, if there is an odd type extension $Q(x, y)$ at $[x, y]$, then by Lemma 2.2.6, x and y are the only common points on the paths $Q(x, y)$

and $[u', Q(u,v), v']$. In that case, by replacing $[x,y]$ with $Q(x,y)$ in $Q(u,v)$, we get a larger length even type extension at $[u,v]$, which is a contradiction. So the claim is justified.

Claim2. There are no even type extensions at $[x,y]$.

Proof of the claim. Proceeding in a way similar to that of Claim1, we can get an odd type extension at $[u,v]$, which contradicts the fact that $[u,v]$ is an even type edge. So this claim is also justified.

Thus each nonmatching edge on $Q(u,v)$ is simple odd type. Now take a weight function $\mathrm{w} \in \mathcal{W}_G$ such that $\mathrm{w}(e) = 1$ for each edge $e \neq [u,v]$ and $\mathrm{w}([u,v]) = N+1$, where N is the total number of extensions at $[u,v]$. Consider $B = A(G_{\mathrm{w}})^{-1}$. By using Lemma 6.2.2, we see that

i) $b_{u_i,u_i'} = 1$ for all $i = 1, \ldots, 2k-1$;

ii) $-1 = b_{u',u_1'} = b_{v',u_{2k-1}} = b_{u_i,u_{i+1}'} < 0$ for all $i = 1, \ldots, 2k-2$, as each nonmatching edge on $Q(u,v)$ is simple odd type; and

iii) $b_{u',v'} = -1$.

We see that the cycle $[u', u_1', u_1, u_2', u_2, \ldots, u_{2k-1}', u_{2k-1}, v', u']$ is available in $\hat{G}$ with the product of the edge weights is -1. Using Lemma 6.2.3, G_{w}^+ cannot exist.

Next, we assume that $[u,v]$ is a mixed type edge. Let $Q(u,v) = [u, u_1, u_1', u_2, u_2', \ldots, u_{2k-1}, u_{2k-1}', v]$ be a maximum length even type extension at $[u,v]$. In a similar way as the previous arguments, we can show that if $[x,y]$ is any nonmatching edge on $Q(u,v)$, then there cannot be any odd type extensions at $[x,y]$. If each nonmatching edge on $Q(u,v)$ is simple odd type, then we proceed with the same weight function w as used previously, to show that G_{w}^+ cannot exist. The other possibility is that some nonmatching edge $[x,y]$ on $Q(u,v)$ is even type. In this case, G_{w}^+ cannot exist, as argued in the first part. ∎

Note that Theorem 6.2.6 tells us that a graph $G \in \mathcal{H} \setminus \mathcal{H}_{odd}$ can be given weights w such that G_{w}^+ does not exist. What about the graphs inside $\mathcal{H}_{odd}$? Do some of them also have this property? The next result addresses this question. The next result also characterizes the graphs $G \in \mathcal{H}_{odd}$ for which G_{w}^+ exists for some $\mathrm{w} \in \mathcal{W}_G$.

Theorem 6.2.7. *Let $G \in \mathcal{H}_{odd}$ and $\mathrm{w} \in \mathcal{W}_G$. Then the following are equivalent.*

i) The inverse G_{w}^+ exists.

ii) The graph $G \in \mathcal{H}_g$.

Proof: i)⇒ii) We assume that $G \notin \mathcal{H}_g$. Then, we see that $G/\mathcal{M}$ is not bipartite. Then G has a cycle Δ which contains an odd number of nonmatching edges and these are odd type edges, by hypothesis. As G is bipartite Δ must contain an odd number of matching edges. Assume first that Δ contains only one matching edge. Let $\Delta = [u_1, \ldots, u_m, x_1, x'_1, u_1]$, where m is even. Take any $\mathrm{w} \in \mathcal{W}_G$. Consider $B = A(G_\mathrm{w})^{-1}$. By using Lemma 6.2.2, we see that

i) $b_{x_1,x'_1} = 1$;

ii) $b_{u'_i,u'_{i+1}} < 0$, for $i = 1, \ldots, m-1$, as each nonmatching edge on Δ is odd type. Also $b_{u'_m,x'_1} < 0$ and $b_{x_1,u'_1} < 0$.

Notice that the cycle $[u'_1, u'_2, u'_3, \ldots, u'_m, x', x, u'_1]$ is available in $\hat{G}$ with the product of edge weights is -1. Hence by Lemma 6.2.3, G^+_w cannot exist. A contradiction to the hypothesis. The argument is similar if Δ contains more matching edges.

ii)⇒1) The proof follows by using Theorem 6.2.5. ∎

Remark 6.2.8. By using Theorem 6.2.7, we address question Q3. Let $G \in \mathcal{H}$. Then the existence of G^+_w for one $\mathrm{w} \in \mathcal{W}_G$ will force G to be inside $\mathcal{H}_g$ if and only if G does not contain any even type extension. This is the desired condition.

We are now in a position to supply an answer question Q4.

Theorem 6.2.9. *Let $G \in \mathcal{H}$. Then the following are equivalent.*

i) The inverse G^+_w exists for all $\mathrm{w} \in \mathcal{W}_G$.

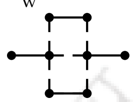
ii) The graph $G \in \mathcal{H}_g$.

Proof: The proof follows by using Theorems 6.2.5, 6.2.6 and 6.2.7. ∎

Remark 6.2.10. We have a complete characterization of graphs $G \in \mathcal{H}$ for which G^+_w exists for all $\mathrm{w} \in \mathcal{W}_G$. This class is nothing but the class $\mathcal{H}_g$. By Lemma 6.2.7, the class $\mathcal{H}_{odd}$ contains the graphs G such that either G^+_w exists for each $\mathrm{w} \in \mathcal{W}_G$ or G^+_w does not exist for any $\mathrm{w} \in \mathcal{W}_G$. That is, there is no graph G in $\mathcal{H}_{odd}$ such that G^+_w exists for some $\mathrm{w} \in \mathcal{W}_G$ but not for all $\mathrm{w} \in \mathcal{W}_G$. So, if we are looking for some such graphs, we must search for them in $\mathcal{H} \setminus \mathcal{H}_{odd}$. That is, such graphs must have an even extension at some nonmatching edge.

6.3 Inverses of weighted graphs in $\mathcal{H} \setminus \mathcal{H}_{odd}$

Now we proceed to find out graphs $G \in \mathcal{H}$ such that G_{w}^{+} exists for one $\mathrm{w} \in \mathcal{W}_G$ and G_{w}^{+} does not exist for another $\mathrm{w} \in \mathcal{W}_G$, if they exist. This will answer our question Q7. As mentioned earlier we shall search such graphs in $\mathcal{H} \setminus \mathcal{H}_{odd}$. In this section, we show that such graphs exist and in the process we give another alternate class of graphs which answers question Q6. Note that, $\mathcal{H}_{odd} \subsetneq \mathcal{H}_{nmc}$ and the later is proper superset. Note that, according to Theorem 2.3.11, $G \in \mathcal{H}_{nmc}$ and $(G - \mathcal{E})/\mathcal{M}$ is bipartite, then G^{+} exists. So, we have a class of graphs $G \in \mathcal{H}_{nmc} \setminus \mathcal{H}_{odd}$ such that G_{w}^{+} exists for the weight function $\mathrm{w} \equiv 1$. This class is nonempty as it contains



It is also true that if $G \in \mathcal{H} \setminus \mathcal{H}_{odd}$, then G_{w}^{+} cannot exist for each $\mathrm{w} \in \mathcal{W}_G$. Naturally, we conclude that $\{G \in \mathcal{H}_{nmc} \setminus \mathcal{H}_{odd} \mid (G - \mathcal{E})/\mathcal{M} \text{ is bipartite}\}$ is a class of graphs such that for each G in this class, G_{w}^{+} exists for some $\mathrm{w} \in \mathcal{W}_G$ but not for all $\mathrm{w} \in \mathcal{W}_G$. Now the question is, given a graph G in this class, what are those weight functions $\mathrm{w} \in \mathcal{W}_G$ for which G_{w}^{+} does not exist? In this section, we give an unexpected combinatorial answer to a little more general question, namely, the following. *Let $\{G \in \mathcal{H}_{nmc} \mid (G - \mathcal{E})/\mathcal{M}$ is bipartite$\}$. What are those weight functions $\mathrm{w} \in \mathcal{W}_G$ for which G_{w}^{+} does not exist?* In order to answer this we need the following definition.

Definition 6.3.1. Recall that for a weighted graph G_{w}, the weight $\mathrm{w}(P)$ of P is the number $\mathrm{w}(P) = \prod_{e \in E(P)} \mathrm{w}(e)$. Let $G \in \mathcal{H}$, $\mathrm{w} \in \mathcal{W}_G$ and e be an even type edge in G. We define $W(e) = \sum_{Q(e)} \mathrm{w}(Q(e))$, where the sum is taken over all extensions at e. That is, $W(e)$ is the sum of the weights of all extensions at e.

The following theorem supplies an answer to the previously asked question Q7.

Theorem 6.3.2. *Let $G \in \mathcal{H}_{nmc}$ for which $(G - \mathcal{E})/\mathcal{M}$ is bipartite and $\mathrm{w} \in \mathcal{W}_G$. Then G_{w}^{+} exists if and only if $\mathrm{w}(e) \leq W(e)$ for each $e \in \mathcal{E}$, where $\mathcal{E}$ is the set of all even type edges.*

Proof: We recall that as $G \in \mathcal{H}_{nmc}$ and $(G - \mathcal{E})/\mathcal{M}$ is bipartite, by Theorem 2.3.11, G^{+} exists. Now suppose that $\mathrm{w} \in \mathcal{W}_G$ is such a weight function that G_{w}^{+} exists. We shall show that $\mathrm{w}(e) \leq W(e)$ holds for each even type edge e in G. Proceeding by the way of contradiction, let if possible, $\mathrm{w}(e) > W(e)$ hold for some even type edge $e = [u, v]$. Now considering a maximum length extension $Q(u, v)$ (it is of course, even type) and following as in the first part of the proof of Theorem

6.2.6, we see that $G_{\rm w}^+$ cannot exist. A contradiction to our hypothesis that $G_{\rm w}^+$ exists. Hence, $\mathrm{w}(e) \leq W(e)$ for all $e \in \mathcal{E}$.

Conversely, let $G \in \mathcal{H}_{nmc}$ with $(G-\mathcal{E})/\mathcal{M}$ bipartite. Take a $\mathrm{w} \in \mathcal{W}_G$ such that $\mathrm{w}(e) \leq W(e)$ holds for each $e \in \mathcal{E}$. We shall show that $G_{\rm w}^+$ exists. Let S be the signature matrix defined by $s_1 = 1$ and $s_i = (-1)^{(\textit{number of odd type edges on a } i\text{-}1\textit{-path})}$. This matrix is well defined, in view of Lemma 2.3.9. Suppose that $SA(G_{\rm w})^{-1}S \not\geq 0$. That is, there exist i and j such that $s_i A(G_{\rm w})^{-1}_{i,j} s_j < 0$. We have two possibilities.

CASE I. The entry $A(G_{\rm w})^{-1}_{i,j} < 0$. Then $s_i = s_j$. By Lemma 2.3.9, the parity of the number of odd type edges on any path from 1 to i is the same with that of any path from 1 to j. It follows that any path from i to j must contain an even number of odd type edges.

Let $P_m^1(i,j), P_m^2(i,j), \ldots, P_m^t(i,j)$ be the minimal paths from i to j. Let $\mathcal{P}^r(i,j)$ be the set of all mm-alternating i-j-paths which are created from $P_m^r(i,j)$, for $r = 1, \ldots, t$. Using Lemma 2.2.22, we have $|\mathcal{P}(i,j)| = \sum_{r=1}^t |\mathcal{P}^r(i,j)|$. Using Lemma 6.2.2, we have

$$A(G_{\rm w})^{-1}_{i,j} = \sum_{r=1}^{t} \sum_{P(i,j)\in\mathcal{P}^r(i,j)} \left[(-1)^{\frac{\|P(i,j)\|-1}{2}} \mathrm{w}(P(i,j))\right], \tag{6.1}$$

where $\sum_{P(i,j)\in\mathcal{P}^r(i,j)} \left[(-1)^{\frac{\|P(i,j)\|-1}{2}} \mathrm{w}(P(i,j))\right]$ is the contribution to $A(G_{\rm w})^{-1}_{i,j}$ coming from the rth minimal path $P_m^r(i,j)$.

Assume first that $P_m^r(i,j)$ contains an odd number of nonmatching edges. As any i-j-path contains an even number of odd type edges, we must have an odd number of even type edges on $P_m^r(i,j)$. Let $e_1, e_2, \ldots, e_k$ be the even type edges on the rth minimal path $P_m^r(i,j)$, where k is odd. Let $m_l \geq 1$ be the number of extensions (these are even type) at the edge e_l, for $l = 1, \ldots, k$. Suppose that we choose the even type edges $e_{i_1}, \ldots, e_{i_p}$ from $e_1, e_2, \ldots, e_k$ and create an mm-alternating i-j-path by using one extension for each of the chosen even type edges. Then we can create $m_{i_1} \cdots m_{i_p}$ many such mm-alternating i-j-paths and each such path has an odd (resp. even) number of nonmatching edges if p is even (resp. odd). Thus the contribution of the mm-alternating paths that are created from $P_m^r(i,j)$ by choosing p many edges out of $e_1, e_2, \ldots, e_k$, to $A(G_{\rm w})^{-1}_{i,j}$ is

$$(-1)^{p+1}\mathrm{w}(P_m^r(i,j)) \sum_{\{e_{i_1},\ldots,e_{i_p}\}\subseteq\{e_1,e_2,\ldots,e_k\}} \frac{W(e_{i_1})W(e_{i_2})\cdots W(e_{i_p})}{\mathrm{w}(e_{i_1})\mathrm{w}(e_{i_2})\cdots \mathrm{w}(e_{i_p})}.$$

Hence the total contribution of $\mathcal{P}^r(i,j)$, the set of mm-alternating i-j-paths that are created from $P^r_m(i,j)$, to $A(G_{\mathrm{w}})^{-1}_{i,j}$ is

$$\begin{aligned} &\sum_{p=0}^{k}(-1)^{p+1}\mathrm{w}(P^r_m(i,j)) \sum_{\{e_{i_1},\ldots,e_{i_p}\}\subseteq\{e_1,e_2,\ldots,e_k\}} \frac{W(e_{i_1})W(e_{i_2})\cdots W(e_{i_p})}{\mathrm{w}(e_{i_1})\mathrm{w}(e_{i_2})\cdots \mathrm{w}(e_{i_p})} \\ = \quad & -\mathrm{w}(P^r_m(i,j))\sum_{p=0}^{k}\sum_{T\subseteq\{e_1,e_2,\ldots,e_k\}}\prod_{e_i\in T}\frac{-W(e_i)}{\mathrm{w}(e_i)} \\ = \quad & -\mathrm{w}(P^r_m(i,j))\prod_{i=1}^{k}\left[1-\frac{W(e_i)}{\mathrm{w}(e_i)}\right] \geq 0, \end{aligned}$$

as k is odd. Similarly, if $P^r_m(i,j)$ contains an even number of nonmatching edges, then also the contribution $P^r_m(i,j)$, to $A(G_{\mathrm{w}})^{-1}_{i,j}$ is nonnegative. Hence $A(G_{\mathrm{w}})^{-1}_{i,j} \geq 0$, by (6.1). This contradicts the hypothesis that $A(G_{\mathrm{w}})^{-1}_{i,j} < 0$.

CASE II. The entry $A(G_{\mathrm{w}})^{-1}_{i,j} > 0$. Carrying the arguments in a way similar to the CASE I, we get a contradiction to our hypothesis that $A(G_{\mathrm{w}})^{-1}_{i,j} > 0$.

Hence we conclude that $SA(G_{\mathrm{w}})^{-1}S \geq 0$. That is, G^+_{w} exists. ∎

Remark 6.3.3. It is clear that Theorems 2.3.11 and 6.2.5 are particular cases of Theorem 6.3.2.

Instead of looking at $\{G \in \mathcal{H}_{nmc} \mid (G-\mathcal{E})/\mathcal{M}$ is bipartite$\}$, let us look at the larger class $\mathcal{H}_{nmc}$ itself. Suppose that for $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$, the inverse G^+_{w} exists. What can be said about such a graph G? The following result says 'in that case the graph G must belong to $\{G \in \mathcal{H}_{nmc} \mid (G-\mathcal{E})/\mathcal{M}$ is bipartite$\}$. In other words, these are the only graphs in $\mathcal{H}_{nmc}$ which have inverses for some weight functions. The others have no inverse with respect to any weight functions.

Proposition 6.3.4. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. If G^+_{w} exist, then $(G-\mathcal{E})/\mathcal{M}$ is bipartite.*

Proof: Let $G \in \mathcal{H}_{nmc}$, $\mathrm{w} \in \mathcal{W}_G$ for which G^+_{w} exists. Let S be the signature matrix such that $SA(G_{\mathrm{w}})^{-1}S \geq 0$. As $G \in \mathcal{H}_{nmc}$, deleting the even type edges, we see that $(G-\mathcal{E}) \in \mathcal{H}_{odd}$. Then by using Lemma 6.2.2, we have

i) $A(G_{\mathrm{w}})^{-1}_{u',v'} < 0$ for any nonmatching edge $[u,v] \in (G-\mathcal{E})$ and

ii) $A(G_{\mathrm{w}})^{-1}_{x,x'} = 1$ for any matching edge $[x,x'] \in (G-\mathcal{E})$.

Let $[u,v] \in (G-\mathcal{E})$ be a nonmatching edge. So $A(G_{\mathrm{w}})^{-1}_{u',v'} < 0$. Since $s_{u'}A(G_{\mathrm{w}})^{-1}_{u',v'}s_{v'} \geq 0$, we have that $s_{u'}s_{v'} = -1$. Let $[x,x']$ be a matching edge in $(G-\mathcal{E})$. By similar arguments, we have $s_x s_{x'} = 1$. Taking $X = \{u \in (G-\mathcal{E})/\mathcal{M} \mid s_u > 0\}$ and $Y = \{u \in (G-\mathcal{E})/\mathcal{M} \mid s_u < 0\}$, we get a bipartition. ∎

We summarize our observation of this section by the following result.

Theorem 6.3.5. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$.*

i) If $(G - \mathcal{E})/\mathcal{M}$ is bipartite, then G^+_{w} exists if and only if $\mathrm{w}(e) \leq W(e)$ for each $e \in \mathcal{E}$.

ii) If $(G - \mathcal{E})/\mathcal{M}$ is not bipartite, then G^+_{w} does not exist.

Our discussion leads to the following open problem.

Problem 6.3.6. *Characterize the class of graphs $G \in \mathcal{H} \setminus \mathcal{H}_{nmc}$ and the weight functions $\mathrm{w} \in \mathcal{W}_G$ for which G^+_{w} exists.*

6.4 A class of graphs $G \in \mathcal{H}$ for which G^+ does not exist but G^+_{w} exists for some w

Now we proceed to find out graphs $G \in \mathcal{H}$ such that G^+ does not exist but G^+_{w} exists for some $\mathrm{w} \in \mathcal{W}_G$. In this section, we construct a class of such graphs $G \in \mathcal{H}$. This will answer our question Q8.

Construction: Let $P(u_1, u'_m) = [u_1, u'_1, \ldots, u_m, u'_m]$ be an mm-alternating path. Let G be a graph obtained from $P(u_1, u'_m)$ by making u'_{k_1} adjacent to u_{k_2} and u'_{l_1} adjacent to u_{l_2} where $1 \leq k_1 < l_1 < k_2 < l_2 \leq m$ and $k_2 - k_1 + 1$ and $l_2 - l_1 + 1$ are multiples of 4.

Let $\mathcal{G}$ be the class of such graphs G.

Example 6.4.1. The graph shown in Figure 6.1 is an example of a graph in $\mathcal{G}$. The graph H shown in Figure 2.2 is an example of a graph in $\mathcal{G}$ which is obtained from the mm-alternating path $[1, 1', 2, 2', 3, 3', 4, 4']$.

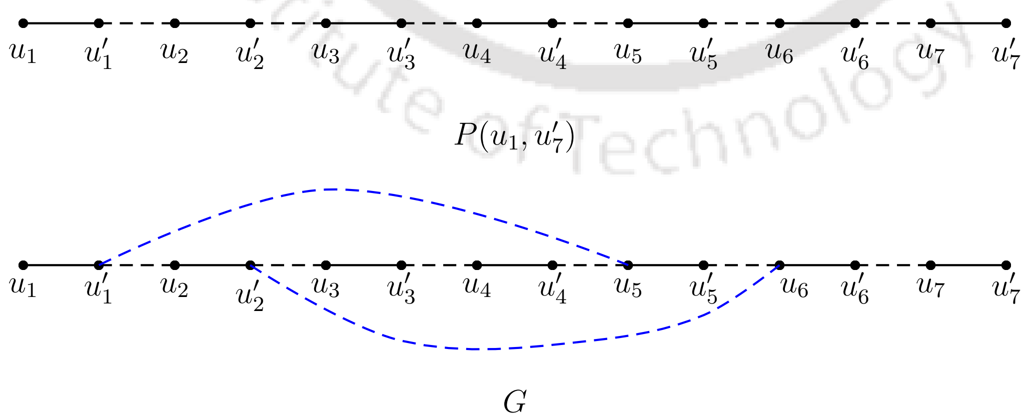


Figure 6.1: An example of a graph $G \in \mathcal{G}$

Remark 6.4.2. Let $G \in \mathcal{G}$. The following are a few observations about G.

1. The graph $G \in \mathcal{H}$.
2. Since $k_2 - k_1 + 1$ and $l_2 - l_1 + 1$ are multiples of 4, the paths $Q(u'_{k_1}, u_{k_2}) = [u'_{k_1}, u_{k_1+1}, \ldots, u_{k_2}]$ and $Q(u'_{l_1}, u_{l_2}) = [u'_{l_1}, u_{l_1+1}, \ldots, u_{l_2}]$ are nn-alternating paths containing an even number of nonmatching edges. Hence $Q(u'_{k_1}, u_{k_2})$ and $Q(u'_{l_1}, u_{l_2})$ are even type extensions at $[u'_{k_1}, u_{k_2}]$ and $[u'_{l_1}, u_{l_2}]$, respectively. There are no other extensions in G. Then the edges $[u'_{k_1}, u_{k_2}]$ and $[u'_{l_1}, u_{l_2}]$ are the only even type edges in G.
3. The graph G has no mixed type edges, so $G \in \mathcal{H}_{nm}$.
4. The extensions $Q(u'_{k_1}, u_{k_2})$ and $Q(u'_{l_1}, u_{l_2})$ have an odd type edge $[u'_{k_2-1}, u_{k_2}]$ in common, so $G \notin \mathcal{H}_{nmc}$.
5. The graph $G - \mathcal{E}$ is a tree.
6. There are no mm-alternating paths in G which contain both the even type edges.
7. There are at most three mm-alternating paths from one vertex two another vertex in G.
8. Let x, y be any two vertices in G. Assume that, there are exactly two mm-alternating x-y paths, say, P_1, P_2 paths. Then one of them contains exactly one even type extension and other one contains the corresponding even type edge. Without loss of generality we assume that P_1 contains an even type extension and P_2 contains the corresponding even type edge. An even type extension has an even number of nonmatching edges, then P_1 contains an even (resp. odd) number of nonmatching edges if and only if P_2 contains an odd (resp. even) number of nonmatching edges.
9. Let x, y be any two vertices in G. Assume that, there are exactly three mm-alternating x-y paths, say, P_1, P_2 and P_3. Then one of them contains both the even type extensions and other two contain exactly one even type edge. Without loss of generality we assume that P_1 contains both the even type extensions, P_1 contains an even type edge $[u'_{k_1}, u_{k_2}]$ and P_2 contains an even type edge $[u'_{l_1}, u_{l_2}]$. An even type extension contains an even number of nonmatching edges, then P_1 contains an even (resp. odd) number of nonmatching edges if and only if P_2 and P_3 contain an odd (resp. even) number of nonmatching edges.

Proposition 6.4.3. *Let $G \in \mathcal{G}$. Then G is not invertible.*

Proof: By Lemma 6.2.2 and Remark 6.4.2, we see

i) $A(G)^{-1}_{u_i,u'_i} = 1$, for $i = 1,\dots,m$

ii) $A(G)^{-1}_{u_i,u'_{i+1}} = -1$ for $i = 1,\dots,m-1$,

iii) $A(G)^{-1}_{u_1,u'_m} = -1$ if $P(u_1,u'_m)$ contains an even number of nonmatching edges, otherwise $A(G)^{-1}_{u_1,u'_m} = 1$

First assume that $P(u_1,u'_m)$ contains an even number of nonmatching edges. The total number of nonmatching edges on $P(u_1,u'_m)$ is $\frac{\|P(u_1,u'_m)\|-1}{2} = \frac{2m-1-1}{2} = m-1$ which is even. Then we see that the cycle $[u_1,u'_2,u_2,\dots,u'_m,u_1]$ is available in $\hat{G}$ (for $\hat{G}$, see Lemma 6.2.3). This cycle contains the m number of nonmatching edges and each nonmatching edge has weight -1. Then the product of the edge weights on $[u_1,u'_2,u_2,\dots,u'_m,u_1]$ is -1, as m is odd. Using Lemma 6.2.3, G^+ cannot exist. Similar arguments work if $P(u_1,u'_m)$ contains an odd number of nonmatching edges.
■

Remark 6.4.4. Let $G \in \mathcal{G}$ and $\mathrm{w} \in \mathcal{W}_G$ such that $\mathrm{w}[u'_{k_2-1},u_{k_2}] = 2$ while the remaning are one. Let P_1 be an mm-alternating path in G. Then the weight $\mathrm{w}(P_1)$ is given below.

$$\mathrm{w}(P_1) = \begin{cases} 1 & \text{if } P_1 \text{ does not contain even type extensions} \\ 2 & \text{otherwise.} \end{cases}$$

Proposition 6.4.5. *Let $G \in \mathcal{G}$ and $\mathrm{w} \in \mathcal{W}_G$ such that $\mathrm{w}([u'_{k_2-1},u_{k_2}]) = 2$ and rest are one. Then G^+_{w} exists.*

Proof: As $(G-\mathcal{E})/\mathcal{M}$ is a tree, take the vertex u_1, define $s_{u_1} = 1$. Now to define s_x, take any path from u_1 to x. If it has odd many odd type edges define $s_x = -1$, otherwise define $s_x = 1$. Let i and j be two any vertices in G. We must show that $s_i A(G_{\mathrm{w}})^{-1}_{i,j} s_j \geq 0$. If there are no mm-alternating i-j-paths in G, then there is nothing to prove. We assume that there are mm-alternating i-j-paths in G. There are three cases.

CASE I There is exactly one mm-alternating i-j-path in G. Let $P_1(i,j)$ be such path. If $P_1(i,j)$ has an odd number of nonmatching edges, then $s_i s_j = -1$ and $A(G_{\mathrm{w}})^{-1}_{i,j} = (-1)^{(\|P_1(i,j)\|-1)/2}\mathrm{w}(P_1) = -\mathrm{w}(P_1)$. Hence $s_i A(G_{\mathrm{w}})^{-1}_{i,j} s_j > 0$. If $P_1(i,j)$ has an even number of nonmatching edges, then $s_i s_j = 1$ and $A(G_{\mathrm{w}})^{-1}_{i,j} = (-1)^{(\|P_1(i,j)\|-1)/2}\mathrm{w}(P_1) = \mathrm{w}(P_1)$. Hence $s_i A(G_{\mathrm{w}})^{-1}_{i,j} s_j > 0$.

CASE II There are exactly two mm-alternating i-j-paths. By using Item 8) of Remark 6.4.2, the paths are P_1 and P_2. First we assume that P_1 has an odd number of nonmatching edges. By Item 8) of Remark 6.4.2, P_2 has an even number of nonmatching edges and $s_i s_j = -1$. By using Lemma 6.2.2 and using Remark 6.4.4, we have $A(G_{\text{w}})^{-1}_{i,j} = (-\text{w}(P_1) + \text{w}(P_2)) = (-2+1) = -1$. Hence $s_i A(G_{\text{w}})^{-1}_{i,j} s_j = 1 > 0$. Similar arguments work if P_1 contains an even number of nonmatching edges.

CASE III There are three mm-alternating i-j-paths. By using Item 9) of Remark 6.4.2, the paths are P_1, P_2 and P_3. First we assume that P_1 has an even number of nonmatching edges. By Item 9) of Remark 6.4.2, the paths P_2 and P_3 have an odd number of nonmatching edges and $s_i s_j = 1$. By using Lemma 6.2.2, we have $A(G_{\text{w}})^{-1}_{i,j} = -\text{w}(P_1) + \text{w}(P_2) + \text{w}(P_3) = -2 + 1 + 1 = 0$. Hence $s_i A(G_{\text{w}})^{-1}_{i,j} s_j = 0$. Similar arguments work if P_3 has an odd number of nonmatching edges.

Therefore G^+_{w} exists. The proof is complete. ■

6.5 Conclusion

In this chapter, we have extended the notion of graph inverse to weighted graphs. Initially, we have supplied a couple of graphs G and H in $\mathcal{H}$ such that G^+ exists but G^+_{w} does not exist for some weight w and H^+ does not exist but H^+_{w} exists for some weight w. These two graphs have ensured that the invertibility of G is not a necessary and sufficient condition for the invertibility of G_{w} and made us feel that there is a necessity of studying the invertibility of weighted graphs in $\mathcal{H}$. The discussion has been started with a positive answer to the following question which is the first step towards this discussion. Let $G \in \mathcal{H}$ for which G^+ exists. Does there exist a weight function w which is not a multiple of $\mathbb{1}$ such that G^+_{w} exists? Then we have shown that G^+_{w} exists for each $G \in \mathcal{H}_g$ and $\text{w} \in \mathcal{W}_G$. We have proved that, there are no graphs $G \in \mathcal{H}_{odd} \setminus \mathcal{H}_g$ such that G^+_{w} exists for some $\text{w} \in \mathcal{W}_G$. Further, we have shown that, there are no graphs $G \in \mathcal{H} \setminus \mathcal{H}_{odd}$ such that G^+_{w} exists for each $\text{w} \in \mathcal{W}_G$. So it is clear that if $G \in \mathcal{H} \setminus \mathcal{H}_{odd}$, then either G^+_{w} does not exist for each $\text{w} \in \mathcal{W}_G$ or G^+_{w} exists for some $\text{w} \in \mathcal{W}_G$, but not for all $\text{w} \in \mathcal{W}_G$. We have supplied a complete characterization of the graphs $G \in \mathcal{H}_{nmc}$ and the weight functions $\text{w} \in \mathcal{W}_G$ such that G^+_{w} exists. Finally, we have constructed a class of graphs $G \in \mathcal{H}$ such that G^+ does not exist but G^+_{w} exists for some w.

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Chapter 7

On reciprocal eigenvalue properties of graphs

7.1 Preliminaries

Let G be a graph. We say λ is an eigenvalue of G to mean that it is an eigenvalue of $A(G)$. Let $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ be the eigenvalues of G. The spectrum $\sigma(G)$ of G is the multiset of eigenvalues of G. The largest eigenvalue $\lambda_n(G)$ of G is called the spectral radius of G and it is denoted by $\rho(G)$.

Our interest lies in a very specific area of studying the structural relationship of a graph G with the eigenvalues and eigenvectors of $A(G)$. For example, it is known that 'a connected graph is bipartite if and only if $-\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$'. A contrasting question is to *characterize graphs which satisfy the property that the reciprocal of each eigenvalue of G is also an eigenvalue of G.*

A nonsingular graph G has *property (R)* if $1/\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$. Further, if $1/\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$ and both have the same multiplicity, then we say that the graph G has *property (SR)*. These two properties are together called *reciprocal eigenvalue properties.*

Example 7.1.1. The graph G in Figure 7.1 satisfies property (SR) and the graph H in Figure 7.1 satisfies property (R). The following two tables give the $\sigma(G)$ and $\sigma(H)$, respectively.

λ	Multiplicity	$1/\lambda$	Multiplicity
$-\frac{\sqrt{3-\sqrt{5}}}{\sqrt{2}}$	1	$-\frac{\sqrt{3+\sqrt{5}}}{\sqrt{2}}$	1
$\frac{\sqrt{3-\sqrt{5}}}{\sqrt{2}}$	1	$\frac{\sqrt{3+\sqrt{5}}}{\sqrt{2}}$	1

λ	Multiplicity	$1/\lambda$	Multiplicity
$\frac{-1-\sqrt{5}}{2}$	1	$\frac{1-\sqrt{5}}{2}$	2
$\frac{-1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	2
$\frac{\sqrt{2(13-\sqrt{41})}-(\sqrt{41}-1)}{4}$	1	$\frac{-\sqrt{2(13-\sqrt{41})}-(\sqrt{41}-1)}{4}$	1
$\frac{\sqrt{2(13+\sqrt{41})}+\sqrt{41}+1}{4}$	1	$\frac{-\sqrt{2(13+\sqrt{41})}+\sqrt{41}+1}{4}$	1

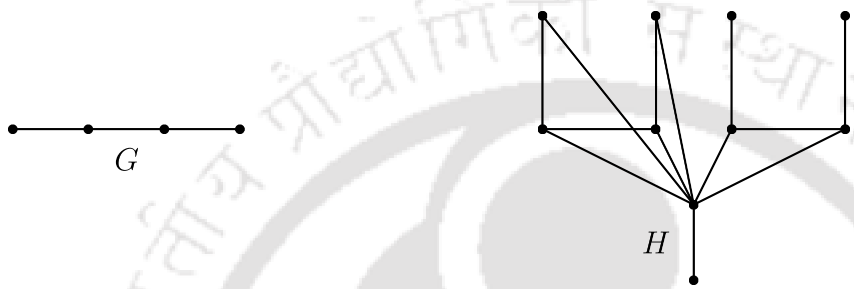

Figure 7.1: Examples of a graph G with property (SR) and a graph H with property (R)

Let T be a nonsingular tree. There is a nice relation among the structure of T^+, property (R), property (SR) and the spectral radius $\rho(T)$. The following theorem tells us about such a relation which has been proved collectively in [18, 23, 24, 28].

Theorem 7.1.2. *Let T be a nonsingular tree and $\rho(T)$ be the spectral radius of T. Then the following are equivalent.*

i) $1/\rho(T)$ is the smallest positive eigenvalue of T.

ii) $T \cong T^+$ (T is isomorphic to its inverses graph T^+).

iii) The graph T has property (R).

iv) The graph T has property (SR).

v) T is a corona tree.

Note that, in the above the tree T can be replaced with a graph from $\mathcal{H}_g$, with appropriate arguments.

There are many natural questions now.

Q1. For a bipartite graph G with a unique perfect matching, if $G \cong G^+$ holds then items i), iii) and iv) also hold. Does v) follow in general?

Q2. Do we have a class larger than the class $\mathcal{H}_g$, where all the four conditions i)–iv) are equivalent? If so, what is an appropriate modification of item v), in that case?

Q3. Do we have counter examples to show that no two of i)–iv) are equivalent, in general? for the converses?

We provide answers to all these questions in this chapter.

Recall that, let $G \in \mathcal{H}$. We shall consider weight functions w such that $\mathrm{w}(e) = 1$ for each $e \in \mathcal{M}$. Let $\mathcal{W}_G$ be the class of such weight functions on G. The notion of reciprocal eigenvalue properties of weighted graphs has been introduced by Neumann and Pati in [27]. The following theorem is essentially contained in [27, Theorem 4.6].

Theorem 7.1.3. *Let T be a nonsingular tree on at least 8 vertices and $\mathrm{w} \in \mathcal{W}_T$ such that $\mathrm{w}(e) \geq 1$ for each edge in T. Then the weighted tree T_w has property (R) if and only if $T = T_1 \circ K_1$, T_1 is a tree.*

The following problem has been posed in [27]

Problem 7.1.4. *Whether Theorem 7.1.3 is true even when one allows the weights of the nonmatching edges to be any positive number.*

It is natural to wonder whether the statements i)–iv) of Theorem 7.1.2 are equivalent for the class of weighted nonsingular trees. In Section 7.5, we address this issue. We show that the statements i)–iv) of Theorem 7.1.2 are equivalent for the weighted graphs G_w where $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$. The class $\mathcal{H}_g$ is the larger class of graphs than the class of nonsingular trees. Moreover, we answer Problem 7.1.4, affirmatively.

7.2 Reciprocal eigenvalue properties of graphs in $\mathcal{H}_{nmcs}$

In Chapter 2, we noticed that $\mathcal{H}_g \subsetneq \mathcal{H}_{nmcs}$. Having characterized the class of graphs G in $\mathcal{H}_{nmcs}$ satisfying $G \cong G^+$, a natural question follows. *Does the statements i)–iv) of Theorem 7.1.2 are equivalent for the class of graphs $G \in \mathcal{H}_{nmcs}$ for which G^+ exists?* The following result addresses this question along with questions Q1 and Q2.

Theorem 7.2.1. *Let $G \in \mathcal{H}_{nmcs}$ for which G^+ exists. Then the following statements are equivalent.*

i) The number $1/\rho(G)$ is the least positive eigenvalue of G.

ii) The graph $G \cong G^+$.

iii) The graph G has property (R).

iv) The graph G has property (SR).

Proof: By using Lemma 3.2.5, we see that $P_{\mathcal{M}}^{-1} A(G^+) P_{\mathcal{M}} \geq A(G)$.

i)⇒ii). Let $1/\rho(G)$ be the least positive eigenvalue of G. Then $\rho(G)$ must be the spectral radius of G^+. As G^+ is connected (if G^+ is disconnected, then G will be disconnected), using Perron-Frobenius theory [35, Sec 8.1], we get $P_{\mathcal{M}}^{-1} A(G^+) P_{\mathcal{M}} = A(G)$. Hence $G \cong G^+$.

The proofs of ii)⇒iv), iv)⇒iii), and iii)⇒i) follow from the definitions. ∎

Theorems 7.2.1 and 3.2.18 lead to the following result, which is our main result of this section.

Theorem 7.2.2. *Let $G \in \mathcal{H}_{nmcs}$ for which G^+ exists. Then the following are equivalent.*

i) The number $1/\rho(G)$ is the least positive eigenvalue of G.

ii) The graph $G \cong G^+$.

iii) The graph G has property (R).

iv) The graph G has property (SR).

v) The graph $G = H_S^{\boxminus}$, where H is a connected bipartite corona graph and S is a subset of nonmatching edges of H such that each cycle in H has an even number of nonmatching edges from S..

Proof: By Theorem 7.2.1, the first four statements are equivalent. By Theorem 3.2.18, we know that ii)⇒v) and v)⇒ii). ∎

The following is an immediate application of Theorem 7.2.2.

Corollary 7.2.3. *Let T be a corona tree. Replace some of the nonmatching edges $[u, v]$ with the boxminus graph (Figure 3.2 in Section 3.2 of Chapter 3), where the other vertices are new vertices. Then the resulting graph has property (SR).*

7.3 Property (SR) of unicyclic graphs in $\mathcal{H}$

In Theorem 7.2.2, we have shown that the inverse graphs and reciprocal eigenvalue properties of graphs in $\mathcal{H}_{nmcs}$ are related to each other. We also have seen that for a unicyclic graph G in $\mathcal{H}_g$ ($\mathcal{H}_g \subsetneq \mathcal{H}_{nmcs}$) has property (SR) if and only if $G \cong G^+$ which is equivalent to G being a simple corona. In this section, we supply a relation among the inverse graph and Property (SR) of graphs in $\mathcal{H}_u \setminus \mathcal{H}_g$, where $\mathcal{H}_u = \{G \in \mathcal{H} \mid G \text{ is unicyclic}\}$. In [29], the authors studied the structure of a noncorona unicyclic graph with property (SR). They supplied three specific structures and showed that a noncorona unicyclic graph with property (SR) has one of that three structures. However, there is one more structure which is also a necessary structure for a graph $G \in \mathcal{H}_u \setminus \mathcal{H}_g$ to have property (SR). First we supply a necessary condition for a graph in $\mathcal{H}_u \setminus \mathcal{H}_g$ to have property (SR). This necessary condition shows that for any graph in $\mathcal{H}_u \setminus \mathcal{H}_g$, the inverse graph and property (SR) are related to each other. In the process, we show that a unicyclic graph in $\mathcal{H}_u \setminus \mathcal{H}_g$ with property (SR) has one of four specific structures. In order to do this, we need the following result.

Lemma 7.3.1. [5, Equation (1.35)″] *Let G_{w} be a weighted connected graph,* w *is allowed to take nonzero real values. Let*

$$P(G_{\mathrm{w}}; x) = x^n + a_1 x^{n-1} + \ldots + a_n$$

be the characteristic polynomial of $A(G_{\mathrm{w}})$. Then

$$a_i = \sum_{H_{\mathrm{w}}} (-1)^{p(H_{\mathrm{w}})} 2^{c(H_{\mathrm{w}})} \prod (H_{\mathrm{w}}),$$

where H_{w} is a linear subgraph (a subgraph in which each component is either an edge or a cycle) of G_{w} of order i; $p(H_{\mathrm{w}})$ and $c(H_{\mathrm{w}})$ are the number of disjoint components and cycles in H_{w}, respectively; and

$$\prod (H_{\mathrm{w}}) = \left[\prod_{e \text{ is on a cycle in } H_{\mathrm{w}}} \mathrm{w}(e) \right] \left[\prod_{e \text{ is not on a cycle in } H_{\mathrm{w}}} \mathrm{w}(e)^2 \right].$$

In particular,

$$\det A(G_{\mathrm{w}}) = \sum_{H_{\mathrm{w}}} (-1)^{n-p(H_{\mathrm{w}})} 2^{c(H_{\mathrm{w}})} \prod (H_{\mathrm{w}})$$

and

$$a_2 = - \sum_{e \in E(G)} \mathrm{w}(e)^2.$$

The following proposition gives a necessary condition for a graph in $\mathcal{H}_u$ to have property (SR).

Proposition 7.3.2. *Let $G \in \mathcal{H}_u$. Suppose that G has property (SR). Then G^+ exists and it is unicyclic.*

Proof: Suppose that G^+ does not exist. Then the graph $(G - \mathcal{E})/\mathcal{M}$ is not bipartite and G does not have any even type extensions, otherwise G^+ exists. The cycle in G must contain matching edges, otherwise $(G - \mathcal{E})/\mathcal{M}$ is bipartite. Let $[x, x']$ be a matching edge on the cycle in G. Then we have an mm-alternating path $[y', y, x, x', z, z']$. Let $P(y', z')$ be a longest mm-alternating y'-z'-path in G and consider the subpath $[y', y, v, v', w, w']$ of $P(y', z')$. This is an mm-alternating path of length 5. Since G is unicyclic, there are no mm-alternating y'-w' paths other than $[y', y, v, v', w, w']$. By using Lemma 2.3.1, we see that

1. $A^{-1}(G)_{u,u'} = 1 = A(G)_{u,u'}$ for any matching edge $[u, u']$ in G,
2. $A(G)^{-1}_{u',v'} \leq -1 = -A(G)_{u,v}$ for any nonmatching edge $[u, v]$ in G,
3. $A(G)^{-1}_{y',w'} = 1$.

Let a_2 and b_2 be the coefficients of the term x^{n-2} in the characteristic polynomials of $A(G)$ and $A(G)^{-1}$, respectively. Since G has property (SR), we have $a_2 = b_2$. Let G^{-1} be a weighted graph associated to the matrix $A(G)^{-1}$. Then by Lemma 7.3.1, $a_2 = -|E(G)|$ and $b_2 \leq -|E(G)| - 1$. Hence $b_2 \neq a_2$. A contradiction to the fact that G has property (SR). Hence G^+ exists. By using Item 2) of Observation 3.3.3, G^+ is unweighted.

Since G has property (SR), the graphs G and G^+ have the same set of eigenvalues. Then G^+ and G have the same number of edges. Hence G^+ is unicyclic. The proof is complete. ■

Remark 7.3.3. Let G be a unicyclic graph in $\mathcal{H}$. We have already seen that if G has property (SR), then G^+ exists and G^+ is unicyclic. But the converse is not true. The graph G shown in Figure 7.2, is an example of a unicyclic graph $G \in \mathcal{H}$ such that G^+ exists and unicyclic, but G does not have property (SR).

The following result is an application of Theorems 3.3.13, 3.3.15 and 3.3.16.

Theorem 7.3.4. *Let $G \in \mathcal{H}_u \setminus \mathcal{H}_g$. Suppose that G has property (SR). Then G has one of the structures shown in Figures 3.5, 3.6, 3.7 and 3.8.*

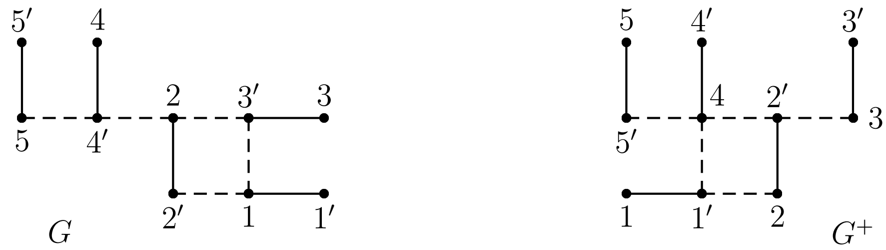

Figure 7.2: A unicyclic graph $G \in \mathcal{H}_u$ with G^+ in $\mathcal{H}_u$ but G does not have property (SR).

Remark 7.3.5. In [29], the authors showed that a noncorona unicyclic graph with property (SR) has one of three specified structures shown in Figures 3.5, 3.6 and 3.7. By using Theorem 7.3.4, we see that there is one more necessary structure shown in Figure 3.8 of a noncorona unicyclic graph to have property (SR). That is there are four specified structures for a noncorona unicyclic graph with property (SR). But the structure shown in Figure 3.8 was not identified in [29]. The techniques used in [29] to get a complete characterization of necessary structures of noncorona unicyclic graphs to have property (SR) is completely different from the techniques which used here. So, Theorem 7.3.4 can also be seen to complement the results in [29].

Our discussion leads the following problem.

Problem 7.3.6. *Does there exist a graph $G \in \mathcal{H}_u$ which has property (SR) but $G \ncong G^+$?*

7.4 Counter examples

In this section, we supply examples to show that no two of the following are equivalent, in general.

i) The number $1/\rho(G)$ is the least positive eigenvalue of G.

ii) The graph $G \cong G^+$.

iii) The graph G has property (R).

iv) The graph G has property (SR).

To further proceed we need the following definition.

Definition 7.4.1. Let $P(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree n, where a_i's are real. Then $P(x)$ is called palindromic if $a_i = a_{n-i}$ for $i = 0, 1, \ldots, n$ and $P(x)$ is called antipalindromic if $a_i = -a_{n-i}$ for $i = 0, 1, \ldots, n$.

Remark 7.4.2. If λ is a root of a polynomial that is either palindromic or antipalindromic, then $1/\lambda$ is also a root and has the same multiplicity.

7.4.1 Graphs with property (SR) but not self-inverses

Example 7.4.3. [i)$\not\Rightarrow$ii) and iv)$\not\Rightarrow$ii)] This is an example of a graph $G \in \mathcal{H}_{nmc}$ which has property (SR). Hence statements i) and iv) hold for G. However, G^+ does not exist for this graph. Consider the graph G shown in Figure 7.3. In G the edges $[u, v]$ and $[x, y]$ are the only even type edges and all other nonmatching edges are odd type. Then $G \in \mathcal{H}_{nmc}$. It is clear that $(G - [u, v] - [x, y])/\mathcal{M}$ is not bipartite. By Theorem 2.3.11, G^+ cannot exist. Hence, G is not isomorphic to G^+. The characteristic polynomial of G is

$$P(G; x) = x^{16} - 19x^{14} + 122x^{12} - 345x^{10} + 482x^8 - 345x^6 + 122x^4 - 19x^2 + 1.$$

The polynomial $P(G; x)$ is palindromic. Hence G has property (SR).

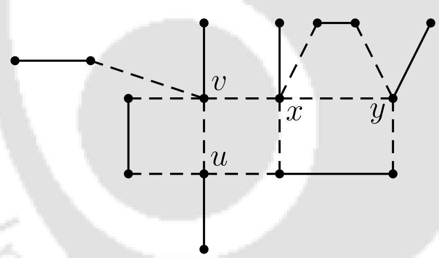

Figure 7.3: A graph $G \in \mathcal{H}_{nmc}$ which has property (SR) but G^+ does not exist. Here the solid lines are the matching edges.

In the previous example we have seen that G has property (SR) but G^+ does not exist, so $G \ncong G^+$. Now we ask ourselves the following question. Is $G \cong G^+$ for any invertible graph G with property (SR)? The Example 7.4.4 says that answer is no.

Example 7.4.4. [iv)$\not\Rightarrow$ii)] This is an example of a graph $G \in \mathcal{H}$ which has property (SR) for which G^+ exists. However, $G \ncong G^+$. Consider the graph G, shown in Figure 7.4. It has an antipalindromic characteristic polynomial

$$P(G; x) = x^{14} - 16x^{12} + 80x^{10} - 172x^8 + 172x^6 - 80x^4 + 16x^2 - 1.$$

Hence G has property (SR). Taking the signature matrix

$$S = diag[1, -1, -1, 1, -1, 1, -1, -1, -1, 1, 1, 1, -1, -1],$$

we note that $A(G)^{-1}$ is signature similar to a nonnegative matrix. Hence G^+ exists. The inverse graph G^+ shown in Figure 7.4. The graphs G and G^+ are not isomorphic because the highest degree vertex 5 in G^+ is adjacent to a pendant vertex 2 but there is no pendant vertex which is adjacent to the highest degree vertex 4 in G.

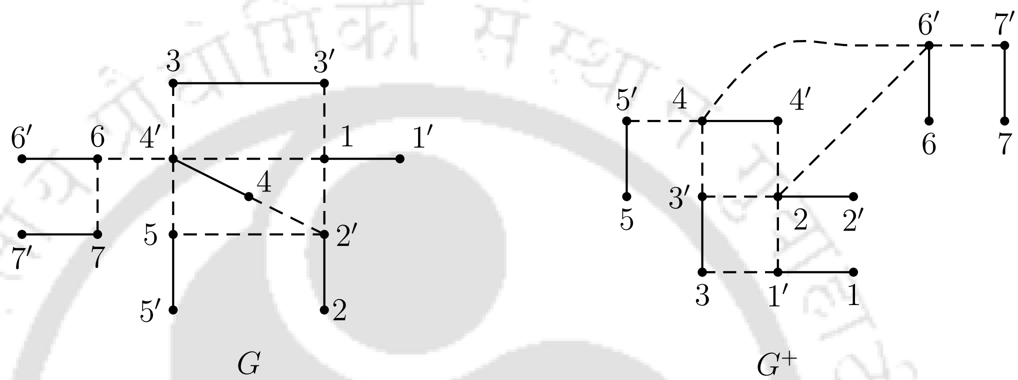

Figure 7.4: A graph $G \in \mathcal{H}$ in which G^+ exists and G has property (SR) but G is not isomorphic to G^+. Here the solid lines are the matching edges.

7.4.2 A class of graphs with property (R) but not property (SR)

Here we construct a class of connected, non-bipartite graphs G which has property (R) but not property (SR) and which is not invertible. This class of graphs itself example of graphs for which i)$\not\Rightarrow$iv) and iii)$\not\Rightarrow$iv).

Definition 7.4.5. Let G_k, $k \geq 2$, be a graph obtained by taking k copies of P_4 and one copy of $P_2 = [u, u']$, and then joining the vertex u' to every vertex in one copy of P_4 and to every degree two vertex in the remaining $k-1$ copies. Let $\mathcal{F} = \{G_k \mid k \in \mathbb{N}, k \geq 2\}$.

We shall use the following two results from the literature.

Lemma 7.4.6. *[5] Let v be a vertex in the graph G and $C(v)$ be the set of all cycles containing v. Then*

$$P(G; x) = xP(G - v; x) - \sum_{u \sim v} P(G - u - v; x) - 2 \sum_{Z \in C(v)} P(G - Z; x).$$

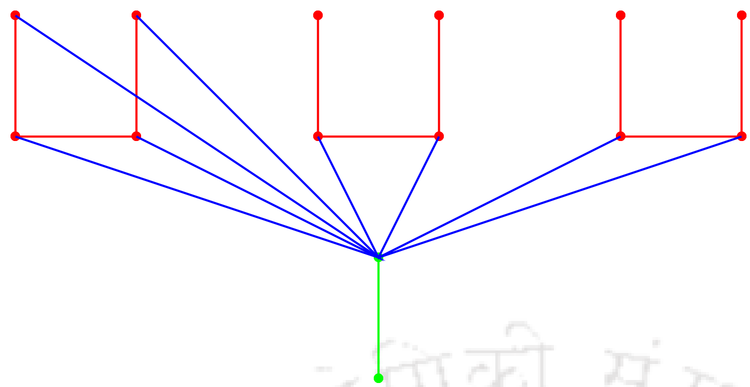

Figure 7.5: A graph G_3 in $\mathcal{F}$.

Lemma 7.4.7. *[5] Let v be a vertex of degree 1 in the graph G and u be the vertex adjacent to v. Then*

$$P(G;x) = xP(G-v;x) - P(G-u-v;x).$$

The next result tells us that each graph in $\mathcal{F}$ satisfies a) and c), but does not satisfy d).

Theorem 7.4.8. *Let $G_k \in \mathcal{F}$. Then $P(G_k;x)$*

$$= (x^2+x-1)^{k-1}(x^2-x-1)^{k-1}(x^4-x^3-(4+2k)x^2-x+1)(x^2+x-1).$$

Hence, G_k has property (R) but not (SR).

Proof: By using Lemma 7.4.7, we see that $P(G_k;x) = xP(G_k-v;x) - P(G_k - v-u)$. In the graph $G_k - v$, all the cycles are containing the vertex u. Then by using Lemma 7.4.6, we see that

$$\begin{aligned}
&P(G_k-v;x)\\
&= xP(G_k-v-u;x) - \sum_{s\sim u} P(G_k-v-s-u;x) - 2\sum_{Z\in C(V)} P(G_k-v-Z;x)\\
&= x[P(P_4;x)]^k - 2kx[P(P_4;x)]^{k-1}P(P_2;x) - 2[P(P_4;x)]^{k-1}P(P_3;x)\\
&- 2kx^2[P(P_4;x)]^{k-1} - 4[P(P_4;x)]^{k-1}P(P_2;x) - 4x[P(P_4;x)]^{k-1} - 2[P(P_4;x)]^{k-1}
\end{aligned}$$

We know that,

$$P(P_4;x) = (x^2+x-1)(x^2-x-1);\quad P(P_2;x) = (x^2-1);\quad P(P_3;x) = x^3-2x.$$

Hence

$$P(G_k - v; x) = (x^2 + x - 1)^{k-1}(x^2 - x - 1)^{k-1}[x(x^2 + x - 1)(x^2 - x - 1) - 2kx(x^2 - 1) - 2(x^3 - 2x) - 2kx^2 - 4(x^2 - 1) - 4x - 2].$$

Therefore,

$$P(G_k; x) = (x^2 + x - 1)^{k-1}(x^2 - x - 1)^{k-1}(x^4 - x^3 - (4 + 2k)x^2 - x + 1)(x^2 + x - 1).$$

Notice that if λ is a zero of the polynomial $(x^2 + x - 1)$, then $1/\lambda$ is a zero of the polynomial $(x^2 - x - 1)$. The polynomial $(x^4 - x^3 - (4+2k)x^2 - x + 1)$ is palindromic. Hence the reciprocal of a zero of this polynomial is also a zero of this polynomial and both have the same multiplicity. Hence G_k has property (R). However, as there is an extra factor $(x^2 + x - 1)$ present in $P(G_k; x)$, we see that the zeros and the reciprocal zeros do not have the same multiplicity. Thus G_k cannot have property (SR). ■

7.4.3 Graph G without property (R) but $1/\rho(G)$ is the least positive eigenvalue

[**i)$\not\Rightarrow$iii) and i)$\not\Rightarrow$iv)**] Consider the graph G shown in Figure 7.6. The table of the $\sigma(G)$ says that $1/\rho(G) = .4142$ is the smallest positive eigenvalue where $\rho(G) = 2.4142$. We notice that $2 \in \sigma(G)$ but $.5 \notin \sigma(G)$. Hence G does not have property (R) and property (SR).

$\sigma(G)$	± 2.4142	± 2	± 1	$\pm .4142$
Multiplicity	1	1	4	1

7.5 Reciprocal eigenvalue properties of weighted graphs in $\mathcal{H}_g$

In this section, we show that the statements i)–iv) of Theorem 7.1.2 are equivalent for the weighted graphs $G_{\rm w}$ in $\{G_{\rm w} \mid G \in \mathcal{H}_g, {\rm w} \in \mathcal{W}_G\}$ where H_g is the larger class than the class of nonsingular trees. We also show that if $G_{\rm w} \in \{G_{\rm w} \mid G \in \mathcal{H}_g, {\rm w} \in \mathcal{W}_G\}$ satisfies one of the statements i)–iv) of Theorem 7.1.2, then G has a special structure which is $G = G_1 \circ K_1$. That is, G is a simple corona.

The following are crucial observations.

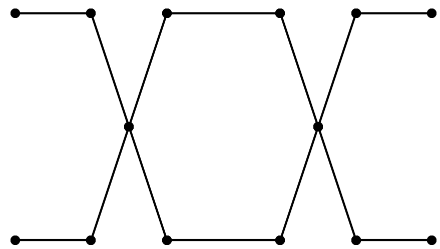

Figure 7.6: A graph G where $1/\rho(G)$ is the smallest positive eigenvalue but G does not have property (R)

Lemma 7.5.1. *Let $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$. Then $P_{\mathcal{M}}^{-1}A(G_{\mathrm{w}}^{+})P_{\mathcal{M}} \geq A(G_{\mathrm{w}})$, where $P_{\mathcal{M}}$ is matching permutation matrix.*

Proof: By using Theorem 6.2.5, G_{w}^{+} exists. To prove the assertion, let $\mathcal{M} = \{[u_k, u_k'] : k = 1, \ldots, 2n\}$. Now define a map $f : V(G_{\mathrm{w}}) \to V(G_{\mathrm{w}})$ such that $f(u_k) = u_k'$ and $f(u_k') = u_k$. Using the description of $A(G_{\mathrm{w}})^{-1}$ from Lemma 6.2.2, we see that, for each matching edge $[u_k, u_k'] \in E(G_{\mathrm{w}})$, the edge $[f(u_k), f(u_k')] = [u_k', u_k] \in E(G_{\mathrm{w}}^{+})$. In fact, $A(G_{\mathrm{w}}^{+})_{u_k', u_k} = 1$. Thus $A(G_{\mathrm{w}}^{+})_{f(u_k), f(u_k')} = A(G_{\mathrm{w}})_{u_k, u_k'}$.

For any nonmatching edge $[u, v] \in E(G_{\mathrm{w}})$, we have an alternating path $P^* = [u', u, v, v']$ of length 3 from u' to v'. Hence, each alternating path from $f(u) = u'$ to $f(v) = v'$ contains an odd number of nonmatching edges. Using of the description of $A(G_{\mathrm{w}})^{-1}$ from Lemma 6.2.2, we see that $[u', v'] = [f(u), f(v)] \in E(G_{\mathrm{w}}^{+})$ and

$$\begin{aligned} A(G_{\mathrm{w}}^{+})_{u',v'} &= \left| \sum_{P(u',v') \in \mathcal{P}(i,j)} \prod_{e \in P(u',v')} g(e) \right| = \sum_{P(u',v') \in \mathcal{P}(i,j)} \mathrm{w}(P(u',v')) \\ &\geq \mathrm{w}(P^*) = \mathrm{w}([u,v]) = A(G_{\mathrm{w}})_{u,v}. \end{aligned}$$

It follows that $P_{\mathcal{M}}^{-1}A(G_{\mathrm{w}}^{+})P_{\mathcal{M}} \geq A(G_{\mathrm{w}})$. ■

Lemma 7.5.2. *Let $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$. Let $\rho(G_{\mathrm{w}})$ be the spectral radius of G_{w}. Then $1/\rho(G_{\mathrm{w}})$ is the smallest positive eigenvalue of G_{w} if and only if $G_{\mathrm{w}} \cong G_{\mathrm{w}}^{+}$.*

Proof: By using Lemma 7.5.1, we see that $P_{\mathcal{M}}^{-1}A(G_{\mathrm{w}}^{+})P_{\mathcal{M}} \geq A(G_{\mathrm{w}})$, where P is the permutation matrix given by the matching. As $1/\rho(G_{\mathrm{w}})$ is the smallest positive eigenvalue of G_{w}, we see that $\rho(G_{\mathrm{w}})$ is the spectral radius of G^{+}. As G^{+} is connected, using Perron-Frobenius theory [13, sec 8.1], we get $P_{\mathcal{M}}^{-1}A(G_{\mathrm{w}}^{+})P_{\mathcal{M}} = A(G_{\mathrm{w}})$. Hence $G_{\mathrm{w}} \cong G_{\mathrm{w}}^{+}$. The converse is straight forward. ■

The Lemmas 7.5.1 and 7.5.2 lead the following theorem.

Theorem 7.5.3. *Let $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$. Let $\rho(G_\mathrm{w})$ be the spectral radius of G_w. Then the following are equivalent.*

i) $1/\rho(G_\mathrm{w})$ *is the smallest positive eigenvalue.*

ii) $G_\mathrm{w} \cong G_\mathrm{w}^+$.

iii) G_w *has property (R).*

iv) G_w *has property (SR).*

Proof: i)⇒ii). It follows by using Lemma 7.5.2.
ii)⇒iv). It is trivial.
iv)⇒iii). It follows from the definition of property (R).

The following question is natural. Is there any special structures of a graph $G \in \mathcal{H}_g$ such that G_w satisfies one of the statements of Theorem 7.5.3? The answer is yes and the following lemma provides the confirmation.

Lemma 7.5.4. *Let $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$ such that $G_\mathrm{w} \cong G_\mathrm{w}^+$. Then G is a simple corona of a connected, bipartite graph.*

Proof: Let $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$ such that $G_\mathrm{w} \cong G_\mathrm{w}^+$. Hence by Remark 4.2.2, each cycle in G_w^+ has an even number of nonmatching edges. Let $\mathcal{M} = \{[u_k, u_k'] : k = 1, \ldots, 2n\}$.

We must show that G is a corona graph. Assume, if possible that G is not a corona graph. Then there is a matching edge, say $[u_i, u_i']$, such that neither of u_i and u_i' is a pendant vertex. Then we can find an alternating path $[v', v, u_i', u_i, w', w]$ of length 5. In view of Lemma 6.2.2 and Remark 4.2.2, we see that $[v', u_i, u_i', w, v']$ is a cycle in G_w^+ and it contains an odd number of nonmatching edges. We have a contradiction. Hence, G is a simple corona of a connected, bipartite graph. ■

The following known result will be required in the sequel.

Lemma 7.5.5. [27] *Let G_w' be a weighted, bipartite graph. Let G_w be obtained from G_w' by adding a new pendant vertex to each vertex of G_w' and by taking the weight of the new edges to be 1. Then G_w has property (SR).*

The following is an extension of Theorem 4.6 in [27]. It is our main result of this section. It answers the question raised in [27] affirmatively.

Theorem 7.5.6. *Let $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$. Let $\rho(G_\mathrm{w})$ be the spectral radius of G_w. Then the following are equivalent.*

i) $1/\rho(G_{\mathrm{w}})$ *is the smallest positive eigenvalue.*

ii) $G_{\mathrm{w}} \cong G_{\mathrm{w}}^{+}$.

iii) G_{w} *has property (R).*

iv) G_{w} *has property (SR).*

v) $G = G_1 \circ K_1$ *where* G_1 *is a connected, bipartite graph.*

Proof: iv)⇒v). Since G_{w} has property (R), so $1/\rho(G_{\mathrm{w}})$ is the smallest positive eigenvalue of G_{w}. Then by using Lemmas 7.5.2 and 7.5.4, we get $G = G_1 \circ K_1$ where G_1 is a connected, bipartite graph.

v)⇒iv). The proof follows by using Lemma 7.5.5. ∎

7.6 Conclusion

Theorem 7.1.2 supplies a nice relation among the structure of T^{+}, property (SR), property (R) and the spectral radius $\rho(T)$. In this chapter, we supplied a larger class of graphs for which the statements i)–iv) of Theorem 7.1.2 are valid. We also supplied a constructive characterization of this class. Then we have shown by examples that the statements i)–iv) of Theorem 7.1.2 are not equivalent in general. Finally, we have shown that for each the statements i)–iv) of Theorem 7.1.2 are equivalent for the weighted graphs G_{w} where $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$ and we have shown that if $G_{\mathrm{w}} \in \{G_{\mathrm{w}} \mid G \in \mathcal{H}_g, \mathrm{w} \in \mathcal{W}_G\}$ which satisfies one of the i)–iv) properties, then G is a simple corona. As an application, we answered that Theorem 7.1.3 is true even when one allows the weights of the nonmatching edges to be any positive number.

Chapter 8

Future work

1. **Inverses of graphs:** Characterizing the graphs $G \in \mathcal{H}$ which possess inverses is an open problem since 1985. In Chapter 2, we noticed that a nonmatching edge of a graph in $\mathcal{H}$ can be of three types: odd type, even type and mixed type. We have defined $\mathcal{H}_{nm}$ is the class of graphs with no mixed type edges and $\mathcal{H}_{nmc}$ is the class of graphs with no mixed type edges and satisfy a technical condition 'condition C'. We proved that $G \in \mathcal{H}_{nmc}$ is invertible if and only if $(G - \mathcal{E})/\mathcal{M}$ is bipartite. We have noticed that there is invertible graphs in $\mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$, but we have not been able to identify those graphs. We shall try to characterize the invertible graphs in $\mathcal{H}_{nm} \setminus \mathcal{H}_{nmc}$.

   The class of bicyclic graphs in $\mathcal{H}$ is not a subclass of the class $\mathcal{H}_{nmc}$. So another possible problem is to identify those bicyclic graphs in $\mathcal{H}$ which possess inverses.

2. **Self-inverse graphs:** Once we have a class of invertible graphs, then the next challenging problem is to identify self-inverse graphs among the graphs from that class. We have identified the self-inverse graphs in $\mathcal{H}_{nmcs}$. But we have not able to identify the self-inverse graphs in $\mathcal{H}_{nmc} \setminus \mathcal{H}_{nmcs}$. The main hope for progress in self-inverse graphs seem to identify the self-inverse graphs in $\mathcal{H}_{nmc} \setminus \mathcal{H}_{nmcs}$. Then we try to identify the self-inverse bicyclic graphs in $\mathcal{H}$.

3. **Reciprocal eigenvalue properties:** The nonsingular trees with property (SR) have been completely characterized. The nonsingular unicyclic graphs with property (SR) have been been characterized. The property (SR) of bicyclic graphs in $\mathcal{H}$ have not been studied. The main hope for progress in reciprocal eigenvalue properties seem to study the property (SR) of bicyclic graphs in $\mathcal{H}$.

4. **Inverses of weighted graphs:** We have already seen that weight is also matter for invertibility. We have shown that for each $G \in \mathcal{H} \setminus \mathcal{H}_{nmc}$ there is a weight function w such that G_{w}^{+} does not exist. We shall try to characterize the graphs $G \in \mathcal{H} \setminus \mathcal{H}_{nmc}$ and weight functions $\mathrm{w} \in \mathcal{W}_G$ such that G_{w}^{+} exists.

Bibliography

[1] R. B. Bapat. *Graphs and Matrices.* TRIM Series-58, Hindustan Book Agency, New Delhi, 2010.

[2] Abraham Berman, Robert J. Plemmons. *Nonnegative Matrices in the Mthematical Sciences.* Siam in Applied Mathematics, 1994.

[3] N. Biggs. *Algebraic graph theory.* Cambridge University Press, NewYork, 1993.

[4] Richard A. Brualdi and Herbert J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, NewYork, 1991.

[5] D. M. Cvetkovic, M. Doob and H. Sachs. *Spectra of Graphs.* Academic Press, NewYork, 1980.

[6] C. Godsil and G. Royal. *Algebraic Graph Theory.* Springer-Verlag, New York, 2001.

[7] H. Minc. *Nonnegative matrices.* Wiley Interscience Publication, (1987).

[8] Dragan Stevanović. *Spectral radius of graphs.* Academic Press is in an imprint of Elsevier, (2015).

[9] R. B. Bapat, D. Kalita and S. Pati. *On weighted directed graphs.* Linear Algebra and its Applications, 436 (2012), pp. 99–111.

[10] D. Kalita and S. Pati. *A reciprocal eigenvalue property for unicyclic weighted directed graphs with weights from* $\{\pm 1, \pm \mathrm{i}\}$. Linear Algebra and its Applications, 449, pp. 417–434.

[11] Armen S. Asratian, Tristan M. J. Denley and Roland. Haggkvist. *Bipartite graphs and their applications.* Cambridge University Press, New York, 1998.

[12] Sebastian M. Cioaba and M. Ram Murty. *A first course in graph theory and combinatorics*. TRIM Series-55, Hindustan Book Agency, New Delhi, 2009.

[13] A. Bondy and U. S. Murty. *Graph theory*. Springer, 2008.

[14] F. Harary and H. Minc. *Which nonnegative matrices are self-inverses?*. Math. Mag, 49 (1976), pp. 91–92.

[15] C. D. Godsil. *Inverses of trees*. Combinatorica, 5(1) (1985), pp. 33–39.

[16] R. M. Tifenbach, S. J. Kirkland. *Directed intervals and dual of a graph*. Linear Algebra and its Applications, 431 (2009), pp. 792–807.

[17] R. M. Tifenbach. *Strongly self-dual graphs*. Linear Algebra and its Applications, 435 (2011), pp. 3151–3167.

[18] Dragos M. Cvetkovic, Ivan Gutman and Slobodan K. Simic. *On self pseudo-inverse graphs*. Univ. Beogard. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No. 602- 633(1978) pp. 111–117.

[19] F. Buckley, L. L. Doty and F. Harary. *On graphs with signed inverses*. Networks. 18 (1988), pp. 151–157.

[20] F. Harary. *On the notion of balance of a signed graph*. Mich. Math. J. 2 (1953), pp. 143-146.

[21] S. Pavlikova and J. K. Jediny. *On the inverse and the dual index of a tree*. Linear and Multilinear Algebra, 28 (1990), pp. 93–109.

[22] Edwin R. van Dam and Willem H. Haemers. *Which graphs are determined by their spectrum?*. Linear Algebra and its Applications, 373 (2003), pp. 241–272.

[23] C. D. Godsil and B. D. Mckay. *A new graph product and its spectrum*. Bull. Austral. Math. Soc., 18(1978), pp. 21–28.

[24] S. Barik, S. Pati and B. K. Sarma. *The spectrum of the corona of two graphs*. SIAM Journal of Discrete Mathematics, 21 (2007), pp. 47–56.

[25] S. Akbari and S.J. Kirkland. *On unimodular graphs*. Linear Algebra and its Applications, 421 (2007), pp. 3–15.

[26] R. Simion and D. S. Cao. *Solution to a problem of C. D. Godsil regarding bipartite graphs with unique perfect matching.* Combinatorica, 9(1) (1989), pp. 85–89.

[27] M. Neumann and S. Pati. *On reciprocal eigenvalue property of weighted trees.* Linear Algebra and its Applications, 438 (2013), pp. 3817–3828.

[28] S. Barik, M. Neumann and S. Pati. *On nonsingular trees and reciprocal eigenvalue property.* Linear and Multilinear Algebra, 54 (2006), pp. 453–465.

[29] S. Barik, M. Nath, S. Pati, B. K. Sarma. *Unicyclic graphs with the strong reciprocal eigenvalue property.* Electronic Journal of Linear Algebra, 17 (2008), pp. 139–153.

[30] Joseph D. Lagrange. *Boolean rings and reciprocal eigenvalue properties.* Linear Algebra and its Applications, 436 (2012), pp. 1863–1871.

[31] R. Frucht and F. Harary. *On the corona of two graphs.* Aequationes Mathematicae, 4 (1970), pp. 322–325.

[32] I. Gutman and D. H. Rouvray. *An approximate topological formula for the homo-lumo separation in alternant hydrocarbons.* Chemical Physics Letters, 62(2) (1979), pp. 384–388.

[33] Keith Yates. *Hückel molecular orbital theory.* Academic press, Inc(1978)

[34] R. B. Bapat, E. Ghorbani. *Inverses of triangular matrices and bipartite graphs.* Linear Algebra and its Application, 447 (2014), pp. 68–73.

[35] Roger A. Horn and Charles R. Johnson. *Matrix Analysis.* Cambridge University Press, 1990.

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List of published papers

1. **S. K. Panda** and S. Pati, *On the inverse of a class of bipartite graphs with unique perfect matchings*, Electronic Journal of Linear Algebra, 29 (2015), pp. 89–101.

2. **S. K. Panda** and S. Pati, *On some graphs which possess inverses*, Linear and Multilinear Algebra, 64(7) (2016), pp. 1445–1459.

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