CONNECTEDNESS AND SPECTRAL PROPERTIES OF POWER GRAPHS OF FINITE GROUPS

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Connectedness and Spectral Properties of Power Graphs of Finite Groups

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To

My family

Certificate

This is to certify that the thesis entitled *Connectedness and Spectral Properties of Power Graphs of Finite Groups* submitted by *Mr. Ramesh Prasad Panda* to the Indian Institute of Technology Guwahati, for the award of the Degree of Doctor of Philosophy, is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Dr. K. V. Krishna
Supervisor

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Abstract

The power graph of a group G is the graph whose vertex set is G and two distinct vertices are adjacent if one is a power of the other. In this thesis, a study on connectedness and spectral properties of power graphs of finite groups is presented. Several characterizations of minimal separating sets of power graphs of groups in terms of their quotient power graphs are given, and some minimal separating sets of power graphs of finite cyclic groups are obtained. Consequently, two upper bounds of vertex connectivity of power graphs of finite cyclic groups and their actual value for some orders are determined. Some structural properties of components of power graphs of p-groups are found and the number of components of that of abelian p-groups is deduced. Moving forward, it is ascertained that the minimum degree and the edge connectivity of power graphs of finite groups are equal. Then the minimum degree of power graphs of finite cyclic groups (for some orders), dihedral groups, dicyclic groups and abelian p-groups are computed and its equality with the vertex connectivity of the respective power graphs is characterized. Laplacian spectra of power graphs of finite cyclic groups, dicyclic groups and p-groups are investigated. Multiplicity of the Laplacian spectral radius and some bounds of the algebraic connectivity of these power graphs are supplied. A characterization such that the power graph of a dicyclic group is Laplacian integral is given. It is shown that the power graph of any p-group is Laplacian integral. Some characterizations for the equality of vertex connectivity and algebraic connectivity of power graphs of finite cyclic groups, dicyclic groups and p-groups is supplied. Then critical and minimal connectivity of power graphs of finite groups are investigated. Certain characterizations for power graphs that are minimally vertex (edge) connected have been obtained.

Contents

Introduction

This thesis investigates connectedness, structural and spectral properties of power graphs of finite groups. The work presented in this thesis lies in the areas of groups, graphs and matrices.

Graphs associated to groups have a long history. Cayley graphs are first such notion introduced by Cayley [1878]. Later, various other graphs were constructed on groups and semigroups, e.g., intersection graphs (cf. [Zelinka, 1975]), commuting graphs (cf. [Bertram, 1983]) and prime graphs (cf. [Williams, 1981]).

Kelarev and Quinn [2000, 2002] introduced the *directed power graph* of a semigroup S as the directed graph $\vec{\mathcal{G}}(S)$ with vertex set S and there is an arc from a vertex u to another vertex v if $v = u^\alpha$ for some positive integer $\alpha \in \mathbb{N}$. Following this, Chakrabarty et al. [2009] defined (*undirected*) *power graph* $\mathcal{G}(S)$ of a semigroup S by ignoring the direction in $\vec{\mathcal{G}}(S)$, that is, with vertex set S and distinct vertices u and v are adjacent in $\mathcal{G}(S)$ if $v = u^\alpha$ for some $\alpha \in \mathbb{N}$ or $u = v^\beta$ for some $\beta \in \mathbb{N}$.

Power graph has been the topic of interest of many researchers and there are notable contributions on it in literature. Chakrabarty et al. [2009] showed that the power graph of a group is connected if and only if all its elements have finite order. They further showed that the power graph of a finite group is complete if and only if the group is trivial or a cyclic group of prime power order. For an exhaustive collection of results and open problems on power graphs till 2013, one may refer to [Abawajy et al., 2013]. Certain contributions on characterization of finite groups using their power graphs have drawn the attention of researchers towards studying various properties of power graphs. For instance, Cameron and Ghosh [2011] proved that two finite abelian groups with isomorphic power graphs are isomorphic. Moreover, Cameron [2010] proved that if two finite groups have isomorphic power graphs, then their directed power graphs are also isomorphic. He further ascertained that if two finite groups have isomorphic power graphs, then they have same numbers of elements of each order. In the following result, Mirzargar et al. [2012] showed that several classes of finite groups can be uniquely determined by their power graphs. In particular, they proved that if G_1 is among a simple group, cyclic group, symmetric group, dihedral group or generalized quaternion group, and G_2 is a finite group such that $\mathcal{G}(G_1)$ is isomorphic to $\mathcal{G}(G_2)$, then G_1 is isomorphic to G_2. It was shown by Cameron et al. [2017] that if G is a group such that $\mathcal{G}(G)$ isomorphic to $\mathcal{G}(\mathbb{Z})$, then G is isomorphic to $\mathbb{Z}$. Curtin and Pourgholi [2016, 2014] proved that among the power graphs of all finite groups of a given order, that of cyclic group has the maximum number of edges and has the largest clique in its power graph. The *proper power graph* $\mathcal{G}^*(G)$ of a group G is obtained by removing its identity element from $\mathcal{G}(G)$. Doostabadi et al. [2015] showed that the proper power graph of a finite group is regular if and only if the group is cyclic group of prime power order or its exponent is a prime number. The number of components and diameter of some proper power graphs were determined in [Bubboloni et al., 2017a,b] and [Curtin et al., 2015], respectively. The commuting graph of a group G is the graph whose vertex set is G and two distinct vertices u, v are adjacent if $uv = vu$. Aalipour et al. [2017] characterized the finite groups whose power graph is same as their commuting graph. In [Ashrafi et al., 2017; Feng et al., 2016; Hamzeh and Ashrafi, 2017], the automorphism groups of power graphs of various finite groups were obtained. Ma et al. [2015, 2018] computed the metric dimension and the strong metric dimension of the power graphs of certain finite groups. Laplacian spectra and (adjacency) spectra of power graphs of finite groups were investigated in [Chattopadhyay and Panigrahi, 2015] and [Mehranian et al., 2017], respectively. In the literature, some other variants of power graphs have also been studied, viz., strong power graph [Suresh Singh and Manilal, 2010], reduced power graph [Rajkumar and Anitha, 2017], Normal subgroup based power graph [Bhuniya and Bera, 2017] and enhanced power graph [Aalipour et al., 2017].

Chattopadhyay and Panigrahi [2014] investigated the vertex connectivity $\kappa(\mathcal{G}(\mathbb{Z}_n))$ of $\mathbb{Z}_n$. They obtained $\kappa(\mathcal{G}(\mathbb{Z}_n))$ when n is a prime power. When n is not a prime power, they gave a lower bound of $\kappa(\mathcal{G}(\mathbb{Z}_n))$ and showed that equality holds when n is a product of two distinct primes. Further, Chattopadhyay and Panigrahi [2015] supplied upper bounds of $\kappa(\mathcal{G}(\mathbb{Z}_n))$ when n has two prime factors or n is a product of three primes. In this thesis, we establish that these upper bounds are in fact the actual values of $\kappa(\mathcal{G}(\mathbb{Z}_n))$. Moreover, extending the above result from [Chattopadhyay and Panigrahi, 2015], we present two upper bounds of $\kappa(\mathcal{G}(\mathbb{Z}_n))$ for all n.

Mirzargar et al. [2012] studied some combinatorial properties of power graphs of p-groups. Moghaddamfar et al. [2014] presented some group theoretic characterizations for which the proper power graph of a finite p-group is connected. Doostabadi et al. [2015] classified the p-groups whose proper power graph is regular. Contributing to the study power graphs of p-groups further, in this thesis, we present some structural properties of power graphs of p-groups.

Along with the vertex connectivity $\kappa(\Gamma)$, edge connectivity $\kappa'(\Gamma)$ and minimum degree $\delta(\Gamma)$ also serve as measures of connectedness of any graph Γ. For any finite simple graph Γ, the inequality $\kappa(\Gamma) \leqslant \kappa'(\Gamma) \leqslant \delta(\Gamma)$ due to Whitney [1932] is well known. For power graphs of finite groups, the edge connectivity and minimum degree are always equal. We compute the minimum degree of power graphs of $\mathbb{Z}_n$, abelian p-groups, dihedral group D_n and dicyclic group Q_n in this thesis. We then characterize the equality of vertex connectivity and minimum degree for these power graphs.

Chattopadhyay and Panigrahi [2015] studied the Laplacian spectra of power graphs of $\mathbb{Z}_n$ and D_n. They showed that the Laplacian spectral radius of power graph of any finite group is its order. Additionally, when the group is cyclic, they computed its multiplicity. They supplied upper bounds of the algebraic connectivity of power graph of $\mathbb{Z}_n$ for some n and a lower bound for all n. Further, they gave the algebraic connectivity of the power graph of D_n and obtained the complete Laplacian spectrum of power graphs of $\mathbb{Z}_n$ and D_n for some values of n. In [Chattopadhyay, 2015], the Laplacian spectum of power graph of dicyclic group Q_n when n is a power of 2 was computed. Kirkland et al. [2002] obtained a necessary and sufficient condition for equality of vertex connectivity and algebraic connectivity of graphs that are non-complete and connected. In this thesis, we study Laplacian spectra of power graphs of $\mathbb{Z}_n$, Q_n and p-groups. In fact, we improve and generalize some of the above results from [Chattopadhyay, 2015; Chattopadhyay and Panigrahi, 2015]. Further, we characterize the equality of vertex connectivity and algebraic connectivity of power graphs of certain groups.

Graphs having the property that the deletion of a vertex or an edge decreases their vertex connectivity or edge connectivity have been objects of interest for researchers. Halin [1969] proved that the minimum degree of a minimally k-vertex connected graph is k. Lick [1969] gave a necessary condition for a graph to be critically k-vertex connected in terms of upper bound of its minimum degree. Lick [1972] further showed that the minimum degree of a minimally k-edge connected graph is k. For more interesting results on these graph parameters, we can refer to [Chartrand, 1966; Li et al., 2017; Li, 2008]. In this thesis, we investigate these parameters for power graphs of finite groups and obtain some characterizations.

In this thesis, the necessary preliminaries and related works are presented in Chapter 1. Starting with Chapter 2, the contributions of the thesis have been organized into five chapters as follows:

Chapter 2: Vertex Connectivity

Chapter 3: p-Groups

Chapter 4: Minimum Degree and Connectivity

Chapter 5: Laplacian Spectrum

Chapter 6: Critical and Minimal Connectivity

Chapter 2. In this chapter, we study minimal separating sets and vertex connectivity of power graphs of finite groups. We first recall the notion of a quotient power graph of a group from [Bubboloni et al., 2017b]. Then we obtain some characterizations of minimal separating sets of power graph of a finite group in terms of its quotient power graph. We show that $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1$ if and only if n is a product of two distinct primes (cf. Proposition 2.3.3). We obtain some minimal separating sets of $\mathcal{G}(\mathbb{Z}_n)$, and consequently supply two upper bounds for its vertex connectivity (cf. Theorem 2.3.6, 2.3.16). Moreover, we determine values of the vertex connectivity of $\mathcal{G}(\mathbb{Z}_n)$ when n has two prime factors or n is a product of three distinct primes (cf. Theorem 2.3.20, 2.3.22). Through these, we ascertain that one of the upper bounds obtained above is sharp.

Chapter 3. In this chapter, we study some structural properties of power graphs of p-groups. We first obtain some properties of components of proper power graphs of p-groups (cf. Proposition 3.1.2, 3.1.4) . We then find the number of components of the proper power graph of an abelian p-group (cf. Theorem 3.1.5). Moreover, we study the structure of power graphs of p-groups (cf. Theorem 3.2.4, Remark 3.2.5) and present the complete structure of power graphs of groups that are direct product of two cyclic p-groups.

Chapter 4. In this chapter, for power graphs of finite groups, we first ascertain that the edge connectivity and minimum degree are always equal (cf. Theorem 4.1.2). We then obtain the minimum degree of power graphs of various finite groups. We derive some inequalities involving degrees of vertices of $\mathcal{G}(\mathbb{Z}_n)$ (cf. Proposition 4.2.6), and apply them to compute the minimum degree of $\mathcal{G}(\mathbb{Z}_n)$ when n has two prime factors or n is a product of at most four distinct primes (cf. Theorem 4.2.7). Followed by this, we determine the minimum degree of power graphs of abelian p-groups, dihedral groups and dicyclic groups (cf. Theorem 4.2.12, 4.2.14, 4.2.15). For these groups, we also obtain minimum disconnecting sets of their power graphs.

Further, we investigate the equality of vertex connectivity and minimum degree of power graphs in this chapter. If G is a finite group such that the vertex connectivity and minimum degree of $\mathcal{G}(G)$ are equal, then we show that G is of even order (cf. Theorem 4.3.1). Additionally, when G is cyclic, we find the minimum degree of $\mathcal{G}(G)$ (cf. Theorem 4.3.3). We then present a characterization of the equality of vertex connectivity and minimum degree of power graph of $\mathbb{Z}_n$ (cf. Theorem 4.3.5) and abelian p-groups (cf. Theorem 4.3.6). Furthermore, we address the equality for D_n and also for Q_n (cf. Theorem 4.3.7).

Chapter 5. In this chapter, we investigate Laplacian spectra of power graphs of $\mathbb{Z}_n$, Q_n and p-groups. We first provide alternative and shorter proofs of some existing results and obtain some results that are useful for our study of Laplacian spectra of power graphs (cf. Section 5.1). We study the multiplicity of Laplacian spectral radius of power graphs of the above groups and obtain it completely for Q_n and p-groups (cf. Theorem 5.2.2, 5.3.3, 5.4.2). We determine some upper and lower bounds of algebraic connectivity of power graphs of $\mathbb{Z}_n$ and Q_n (cf. Theorem 5.2.5, 5.2.6, 5.3.6). We further determine the value of algebraic connectivity of power graphs of $\mathbb{Z}_n$ and Q_n for some values of n (cf. Theorem 5.2.3, 5.3.7). We show that the power graph of Q_n is Laplacian integral if and only if it is generalized quaternion (cf. Theorem 5.3.7), and that the power graph of a p-group is always Laplacian integral (cf. Theorem 5.4.8). We supply some characterizations for the equality of vertex connectivity and algebraic connectivity of power graphs of $\mathbb{Z}_n$

and Q_n and p-groups (cf. Theorem 5.2.4, 5.3.7, 5.4.3). We express the Laplacian characteristic polynomial of power graph of Q_n as a determinant of sum of Laplacian matrix of $\mathcal{G}(\mathbb{Z}_{2n})$ and a rational matrix (cf. Theorem 5.3.4). As a result, we give some Laplacian eigenvalues of power graph of Q_n and find lower bounds for their multiplicity (cf. Corollary 5.3.5). We determine all possible Laplacian eigenvalues of power graphs of p-groups and explore their multiplicity (cf. Theorem 5.4.8, 5.4.9, 5.4.9, 5.4.11).

Chapter 6. In this chapter, we first examine whether power graphs of $\mathbb{Z}_n$ (for some values of n), D_n, Q_n and p-groups are critically vertex connected (cf. Subsection 6.1.1) and critically edge connected (cf. Subsection 6.1.2). In fact, we obtain a necessary condition such that the power graph of a finite group is critically edge connected (cf. Theorem 6.1.9). Followed by this, we supply some necessary criterion for minimally vertex connected graphs (cf. Theorem 6.2.6). Subsequently, we characterize the finite groups whose power graphs are minimally vertex connected (cf. Theorem 6.2.9). Moreover, we characterize the finite groups of odd order such that their power graphs are minimally edge connected (cf. Theorem 6.2.15, 6.2.16). We further address the question that whether the power graphs of aforementioned finite groups are minimally vertex (edge) connected (cf. Section 6.2).

The results presented in Chapters 2 and 3 have been published in *Journal of Algebra and Its Applications* (cf. [Panda and Krishna, 2018a]). The results presented in Chapter 4 have been published in the journal *Communications in Algebra* (cf. [Panda and Krishna, 2018b]).

Epilogue. The present thesis studies various connectivity parameters and Laplacian spectra of power graphs of finite groups. While many general results are presented, the focus on power graphs of following four finite groups, viz., finite cyclic group, dihedral group, dicyclic group and p-group. This thesis shows the scope for further research on various aspects of power graphs under consideration. Some details on this has been presented with a concluding section at the end of each contributory chapter.

1

Preliminaries and Related Works

In this chapter, we recall various definitions and results from the literature that are necessary for the thesis. We first present fundamentals of graphs and groups in Section 1.2 and Section 1.3, respectively. The Laplacian spectrum of a graph and some useful results are stated in Section 1.4. Section 1.5 is devoted to the notion of power graph of a group. Along with the definitions, we also recall certain fundamental results on power graphs. Moreover, this chapter also reviews various results on connectivity and Laplacian spectra of power graphs. This chapter also fixes various notations used in the thesis. A more detailed review of further results from the literature that are relevant to specific chapters, is shelved for later.

1.1 Numbers and sets

In this section, we state some fundamental notions on numbers and sets that are useful for the thesis.

Notation 1.1.1. In this thesis,

(i) $\mathbb{N}$ is the set of positive integers;

(ii) n always denotes a positive integer;

(iii) p always denotes a prime number.

Definition 1.1.2. The number of positive integers that do not exceed n and are relatively prime to n is denoted by $\phi(n)$ and is known as *Euler's phi function*.

The following formula of $\phi(n)$ can be found in any standard text on number theory, for example, see [Burton, 2006].

Theorem 1.1.3. *If an integer $n > 1$ has the prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, then* $\phi(n) = \prod_{i=1}^{r} (p_i^{\alpha_i} - p_i^{\alpha_i - 1}) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$.

Notation 1.1.4. For any set A (finite or infinite), its number of elements are denoted by $|A|$.

Definition 1.1.5. An irreflexive and transitive binary relation $\prec$ on a set A is called *well-founded* if for every non-empty subset B of A, there exists $b \in B$ such that there is no $a \in B$ with $a \prec b$.

Remark 1.1.6. For any $n \in \mathbb{N}$, usual $<$ relation on the subset $\{1, 2, \dots, n\}$ of $\mathbb{N}$ is well-founded.

The following theorem is known as *principle of well-founded induction.*

Theorem 1.1.7 ([Jech, 2003])**.** *Let $\prec$ be a well-founded relation on a set A and let $\mathcal{P}$ be a property defined on elements of A. Then $\mathcal{P}$ holds for all elements of A if and only if the following holds:*

given any $a \in A$, if $\mathcal{P}$ hold for all $b \in A$ with $b \prec a$, then $\mathcal{P}$ holds for a.

1.2 Graphs

This section deals with basic concepts of graphs and can be found in any standard text on graph theory, for example, see [Bondy and Murty, 2008; West, 2001].

Definition 1.2.1. A *graph* Γ is an ordered pair $(V(\Gamma), E(\Gamma))$ of sets together with a function which assigns each element of $E(\Gamma)$ an unordered pair of elements (not necessarily distinct) of $V(\Gamma)$. An element of $V(\Gamma)$ is called a *vertex* and that of $E(\Gamma)$ is called an *edge*.

In this thesis, Γ (with or without subscript) denotes a graph.

Definition 1.2.2. Two vertices u and v are *adjacent* if they are assigned to the same edge ε. Additionally, in this case, u and v are called the *endpoints* of ε.

Definition 1.2.3. If a vertex v is an endpoint of an edge ε, then v is said to be *incident* to ε and vice versa.

Remark 1.2.4. Graphs are depicted graphically as a dot for each vertex, a line or a curve for an edge; and dots representing vertices u and v are joined by a line or a curve representing an edge ε if u and v are assigned to ε in the graph.

Definition 1.2.5. A graph Γ_1 is called a *subgraph* of Γ if $V(\Gamma_1) \subseteq V(\Gamma)$ and $E(\Gamma_1) \subseteq E(\Gamma)$, and if vertices u, v are assigned to an edge ε in Γ_1, then they are assigned to ε in Γ.

Definition 1.2.6. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same endpoints. A *simple graph* is a graph with no loops or multiple edges. In a simple graph an edge can be viewed as just an unordered pair of vertices.

Notation 1.2.7. For a simple graph Γ, its *complement* $\overline{\Gamma}$ is the simple graph whose vertex set is $V(\Gamma)$ and edges are the pairs of non-adjacent vertices of Γ.

Definition 1.2.8.

(i) A graph with no vertices (and hence no edges) is called a *null graph*, whereas, a graph with at least one vertex is called a *non-null graph*.

(ii) A graph with one vertex and no edges is called a *trivial graph*.

(iii) A *complete graph* is a simple graph in which every pair vertices are adjacent.

(iv) A is a graph having both finite vertex set and finite edge set.

Notation 1.2.9. The subgraph of Γ obtained by deleting a set U of vertices (along with the incident edges) is denoted by $\Gamma - U$. For $u \in V(\Gamma)$, $\Gamma - \{u\}$ is simply written as $\Gamma - u$. In fact, for any arbitrary set A with an element x, we write $A - x$ instead of $A - \{x\}$.

Definition 1.2.10. For $A \subseteq V(\Gamma)$, the graph $\Gamma - \overline{A}$, where $\overline{A} = V(\Gamma) - A$, is called the subgraph of Γ *induced by A*. A subgraph induced by some vertex set is called an *induced subgraph*.

Definition 1.2.11. A *path* in a graph is sequence of distinct vertices such that two vertices are adjacent if they are consecutive in the sequence. If P is a path with the sequence $v_0, v_1, \ldots, v_n$ of vertices, then P is called a v_0, v_n-*path*. Here, we also simply say that v_0 and v_n are *connected by a path* or simply *connected*.

Definition 1.2.12. A graph is said to be *connected* if every pair of distinct vertices are connected by a path; otherwise, it is *disconnected*. A *component* of Γ is a maximal connected subgraph of Γ.

Definition 1.2.13. A *separating set* of Γ is a set of vertices whose deletion increases the number of components of Γ. A separating set is *minimal* if none of its proper subsets is a separating set of Γ. A separating set of Γ with least cardinality is called a *minimum separating set* of Γ.

Definition 1.2.14. The *vertex connectivity* of a graph Γ, denoted by $\kappa(\Gamma)$, is the minimum number of vertices whose deletion results in a disconnected or trivial graph.

Definition 1.2.15. A *disconnecting set* of Γ is a set of edges whose deletion increases the number of components of Γ. A disconnecting set is *minimal* if none of its proper subsets is a disconnecting set of Γ. A disconnecting set of Γ with least cardinality is called a *minimum disconnecting set* of Γ.

Definition 1.2.16. The *edge connectivity* of Γ, denoted by $\kappa'(\Gamma)$, is the minimum number of edges whose deletion results in a disconnected or trivial graph.

Remark 1.2.17. The vertex connectivity and edge connectivity of a disconnected graph or the trivial graph are always 0.

Definition 1.2.18. A *cut-vertex* of Γ is a vertex v such that $\{v\}$ is a separating set Γ. Analogously, *cut-edge* can also be defined.

Definition 1.2.19. The *degree* of vertex v in Γ, denoted by $\deg_\Gamma(v)$ or simply $\deg(v)$, is the number of edges incident with v and with each loop counted as two edges. The minimum degree of Γ, that is the minimum of degrees of all vertices of Γ, is denoted by $\delta(\Gamma)$.

Definition 1.2.20. The *length* of a path is its number of edges. The *distance* between pair of distinct vertices u, v is the least length of a u, v-path. The *diameter* of a graph is the maximum of distance between every pair of distinct vertices.

Having defined the above graph parameters, we now state the relation between them.

Theorem 1.2.21 ([Whitney, 1932]). *For any finite simple graph Γ, $\kappa(\Gamma) \leqslant \kappa'(\Gamma) \leqslant \delta(\Gamma)$.*

Theorem 1.2.22 ([Plesník, 1975]). *If Γ is a graph with diam$(\Gamma) \leqslant 2$, then $\kappa'(\Gamma) = \delta(\Gamma)$.*

Definition 1.2.23. A graph is called *k-regular* if all vertices have degree k. A *regular* graph is a graph which is k-regular for some k.

Definition 1.2.24. The *neighbourhood* $N_\Gamma(v)$ of a vertex v in Γ is the set of vertices which are adjacent to v. Extending this to a vertex set A of Γ, we define the neighbourhood of A in Γ as $N_\Gamma(A) = \bigcup_{v \in A} N(v) - A$. When the context involves just one graph, we omit the graph subscripts.

Notation 1.2.25. For $A, B \subseteq V(\Gamma)$, the set of all edges having one end in A and the other in B is denoted by $E[A, B]$. If $A = \{v\}$, we write $E[v, B]$ instead of $E[A, B]$.

Definition 1.2.26. Given two simple graphs Γ_1 and Γ_2, an *isomorphism* ψ from Γ_1 to Γ_2 is a bijection from $V(\Gamma_1)$ to $V(\Gamma_2)$ such that u and v are adjacent in Γ_1 if and only if $\psi(u)$ and $\psi(v)$ are adjacent in Γ_2.

When there is an isomorphism from Γ_1 to Γ_2, we say that Γ_1 and Γ_2 are *isomorphic* and write $\Gamma_1 \cong \Gamma_2$.

Remark 1.2.27. The isomorphism of graphs is an equivalence relation.

Remark 1.2.28. If two graphs are isomorphic, then they share same graph properties and only can differ by the labels of their vertices and edges. So, sometimes we may omit labels of graphs and refer any member of an equivalence class of isomorphic graphs by one chosen representative of the class. Up to isomorphism, the complete graph on n vertices is denoted as K_n.

Definition 1.2.29. Let Γ_1 and Γ_2 be two simple graphs.

(i) The *union* of Γ_1 and Γ_2, denoted by $\Gamma_1 \cup \Gamma_2$, is the graph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1) \cup E(\Gamma_2)$. Evidently, union of graphs is associative, so that union of any finite number of graphs can be defined accordingly.

(ii) If Γ_1 and Γ_2 are disjoint (i.e., they have no common vertices), we refer to their union as *disjoint union*, and denote it by $\Gamma_1 + \Gamma_2$. For pairwise disjoint graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$, we denote their union by $\sum_{i=0}^{r} \Gamma_i$.

(iii) If Γ_1 and Γ_2 are disjoint, their *join* $\Gamma_1 \vee \Gamma_2$ is the graph obtained by taking $\Gamma_1 + \Gamma_2$ and adding all edges with one endpoint in $V(\Gamma_1)$ and the other in $V(\Gamma_2)$.

Notation 1.2.30. For any graph Γ, upto isomorphism, $r\Gamma$ denotes the graph obtained by taking disjoint union of r copies of Γ.

1.3 Groups

Relevant fundamentals of groups are recalled in this section. One may refer to [Coxeter and Moser, 1980; Dummit and Foote, 2004; Gallian, 2013] for the material covered in this section.

Definition 1.3.1. A pair consisting of a set G along with a binary operation, that assigns $g, h \in G$ an element gh of G, is called a *group* if it satisfies the following:

(i) $(gh)k = g(hk)$ for all $g, h, k \in G$.

(ii) there exists $e \in G$ such that $eg = ge = g$ for all $g \in G$.

(iii) for each $g \in G$, there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

Moreover, if $gh = hg$ for all $g, h \in G$, then it is called an *abelian group*. The element e in the definition of group is called the *identity* element of G.

Notation 1.3.2. In this thesis, G denotes a group and e denotes the identity element of G.

Definition 1.3.3. A subset H of G is said to be a *subgroup* of G if H is itself a group under the binary operation of G.

Notation 1.3.4. For any $A \subseteq G$, we denote $A^* = A - e$.

Definition 1.3.5. For $g \in G$, the set $\langle g \rangle$ consists of *powers* of g, viz. $g^0 = e$, $g^k = gg \dots g$ (k times) for $k > 0$, and $g^k = (g^{-1})^{-k}$ for $k < 0$. If $G = \langle g \rangle$, then G is called the *cyclic group* generated by g and g is called a *genarator* of G.

Definition 1.3.6. For any $g \in G$, the set $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ is a subgroup of G and is called the *cyclic subgroup* generated by g. A group G is *cyclic* if $G = \langle g \rangle$ for some $g \in G$.

Definition 1.3.7. The *order* of a group G is its number of elements. The order of an element g in G, denoted by $\mathrm{o}(g)$, is defined by $\mathrm{o}(g) = |\langle g \rangle|$.

Definition 1.3.8. A *finite group* is a group of finite order. A *p-group* is a finite group whose order is power of p.

Definition 1.3.9. The *exponent* of any finite group G, denoted by $\exp(G)$, is the least common multiple of orders of all its elements.

Definition 1.3.10. An abelian group G having exponent p is called an *elementary abelian group* or more specifically *elementary abelian p-group*.

Notation 1.3.11. Let $n \in \mathbb{N}$. The additive group of integers modulo n is denoted by $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$.

Definition 1.3.12. For $\overline{a}, \overline{b} \in \mathbb{Z}_n$, we write $\overline{a}\,|\,\overline{b}$ and say that $\overline{b}$ is a *multiple* of $\overline{a}$, if there exists an integer c such that $\overline{b} = c\overline{a}$.

Definition 1.3.13. For any integer $n \geqslant 3$, the *dihedral group* D_n is a finite group of order $2n$ having presentation

$$D_n = \left\langle a, b \mid a^n = b^2 = e, ab = ba^{-1} \right\rangle, \tag{1.1}$$

where e is the identity element of D_n.

Remark 1.3.14. The group D_n satisfies the following properties.

(i) For any $0 \leqslant i < n$, $(a^i b)^2 = e$, so that $\langle a^i b \rangle = \{e, a^i b\}$.

(ii) $D_n = \langle a \rangle \cup \bigcup_{i=0}^{n-1} \langle a^i b \rangle$.

(iii) $\langle a \rangle \cap \langle a^i b \rangle = \{e\}$ for all $0 \leqslant i \leqslant n-1$.

Definition 1.3.15. For any integer $n \geqslant 2$, the *dicyclic group* Q_n is a finite group of order $4n$ having presentation

$$Q_n = \left\langle a, b \mid a^{2n} = e, a^n = b^2, ab = ba^{-1} \right\rangle, \tag{1.2}$$

where e is the identity element of Q_n. When n is a power of 2, Q_n is called a *generalized quaternion group* of order $4n$.

Remark 1.3.16. The group Q_n satisfies the following properties.

(i) $(a^i b)^2 = a^n$ for all $0 \leqslant i \leqslant 2n-1$.

(ii) For any $0 \leqslant i \leqslant n-1$, $(a^i b)^3 = a^n a^i b = a^{n+i} b$ and $(a^{n+i} b)^3 = a^n a^{n+i} b = a^i b$. Hence $\langle a^i b \rangle = \langle a^{n+i} b \rangle = \{e, a^i b, a^n, a^{n+i} b\}$ for all $0 \leqslant i \leqslant n-1$.

(iii) Any element of $Q_n - \langle a \rangle$ can be written as $a^i b$ for some $0 \leqslant i \leqslant 2n-1$.

(iv) $Q_n = \langle a \rangle \cup \bigcup_{i=0}^{n-1} \langle a^i b \rangle$.

(v) $\langle a \rangle \cap \langle a^i b \rangle = \{e, a^n\}$ for all $0 \leqslant i \leqslant 2n-1$.

Definition 1.3.17. A map ψ from a group G_1 to a group G_2 is called a *isomorphism* from G_1 to G_2 if it is a bijection and satisfies $\psi(gh) = \psi(g)\psi(h)$ for all $g, h \in G_1$. Moreover, G_1 and G_2 are said to be *isomorphic* and written $G_1 \cong G_2$, if there is an isomorphism from G_1 to G_2.

Definition 1.3.18. The *direct product* $G_1 \times G_2 \times \ldots \times G_r$ of groups $G_1, G_2, \ldots, G_r$ is the set of n-tuples with k^{th} component is an element of G_k $(1 \leqslant k \leqslant r)$ and operation defined component-wise.

Definition 1.3.19. We define an equivalence relation $\approx$ on G by $g \approx h$ if $\langle g \rangle = \langle h \rangle$ for $g, h \in G$. An equivalence class under $\approx$ is referred to as a $\approx$-*class* and the $\approx$-class of any $g \in G$ is denoted by $[g]$.

We now recall some standard results on group theory that are relevant for the thesis. Though these results can be found in any standard book on group theory, we will particularly refer to [Gallian, 2013].

Theorem 1.3.20. *Any finite cyclic group of order $n \in \mathbb{N}$ is isomorphic to $\mathbb{Z}_n$.*

Theorem 1.3.21. *For any element g of infinite order in G, $|[g]| = 2$.*

Theorem 1.3.22. *Let a be an element of order n in a group and let k be a positive integer. Then $\langle a^k \rangle = \langle a^{\gcd(k,n)} \rangle$ and $\mathrm{o}(a^k) = \dfrac{n}{\gcd(k,n)}$.*

The following corollary, which is consequence of Theorem 1.3.22, identifies all the generators in a cyclic subgroup.

Corollary 1.3.23. *For any $g \in G$ of finite order, $\langle g \rangle = \langle g^k \rangle$ if and only if $\gcd(k, \mathrm{o}(g)) = 1$. Consequently, $|[g]| = \phi(\mathrm{o}(g))$.*

Theorem 1.3.24. *If d is a positive divisor of n, the number of elements of order d in a cyclic group of order n is $\phi(d)$.*

The next theorem shows that the order of an element in a direct product of groups can be obtained in terms of the orders of its components.

Theorem 1.3.25. *For finite groups $G_1, G_2, \ldots, G_r$, and $g \in G_1 \times G_2 \times \ldots \times G_r$,*

$$\mathrm{o}(g) = \mathrm{lcm}(\mathrm{o}(g_1), \mathrm{o}(g_2), \ldots, \mathrm{o}(g_r)),$$

where $g = (g_1, g_2, \ldots, g_r)$.

The following theorem, which is known as the fundamental theorem of finite abelian groups, describes any abelian group in terms of cyclic groups.

Theorem 1.3.26. *Every finite abelian group is a direct product of cyclic groups of prime power order. Moreover, the number of groups in the product and the orders of the cyclic groups are uniquely determined by the group.*

Remark 1.3.27. If a group G has exponent 2, it can be observed that G is abelian. Hence by Theorem 1.3.26, G is a direct product of finite copies of $\mathbb{Z}_2$, i.e., G is an elementary abelian 2-group.

1.4 Laplacian spectra

The Laplacian spectrum of a graph plays important role in studying the structure of graph. In this section, the definition of Laplacian spectrum of a graph along with some of its properties are recalled from the literature.

For the rest of the thesis, by graph we mean a simple graph. Moreover, in this section, we consider only finite graph.

Definition 1.4.1. For a graph Γ with ordered vertex set $\{v_1, v_2, \ldots, v_n\}$, the *Laplacian matrix* $L(\Gamma)$ of Γ is defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix whose (i,i)th entry is the degree of v_i and $A(\Gamma)$ is the *adjacency matrix* of Γ whose (i,j)th entry is 1 if v_i is adjacent to v_j and 0 otherwise.

Definition 1.4.2. For a graph Γ, the characteristic polynomial $\det(xI - L(\Gamma))$ of $L(\Gamma)$ is denoted by $\Theta(\Gamma, x)$ and called the *Laplacian characteristic polynomial* of Γ. If Γ is a null graph, for both convenience and consistency, we write $\Theta(\Gamma, x) = 1$.

Lemma 1.4.3 ([Cvetković et al., 2010; Mohar, 1991])**.** *For any graph Γ, the matrix $L(\Gamma)$ have the following fundamental properties.*

(i) *$L(\Gamma)$ is symmetric, positive semidefinite, so that its eigenvalues are real and non-negative.*

(ii) *The sum of each row (column) of $L(\Gamma)$ is zero, so that it is singular and consequently, its smallest eigenvalue is* 0.

Definition 1.4.4. The eigenvalues of $L(\Gamma)$ are called the *Laplacian eigenvalues* of Γ and are denoted as $\lambda_1(\Gamma) \geqslant \lambda_2(\Gamma) \geqslant \ldots \geqslant \lambda_n(\Gamma) = 0$ arranged in non-increasing order. The *Laplacian spectrum* of Γ consists of the Laplacian eigenvalues of Γ along with their multiplicities. If $\lambda_{n_1}(\Gamma) > \lambda_{n_2}(\Gamma) > \ldots > \lambda_{n_r}(\Gamma) = 0$ are the distinct Laplacian eigenvalues of Γ with multiplicities $m_1, m_2, \ldots, m_r$, respectively, then the Laplacian spectrum of Γ is presented as

$$\begin{pmatrix} \lambda_{n_1}(\Gamma) & \lambda_{n_2}(\Gamma) & \cdots & \lambda_{n_r}(\Gamma) \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

Theorem 1.4.5 ([Fiedler, 1973])**.** *For any graph Γ, $\lambda_{n-1}(\Gamma) > 0$ if and only if Γ is connected.*

Due to above result, Fiedler [1973] gave the following definition.

Definition 1.4.6. The Laplacian eigenvalue $\lambda_{n-1}(\Gamma)$ is called the *algebraic connectivity* of Γ.

Definition 1.4.7. The largest Laplacian eigenvalue $\lambda_1(\Gamma)$ is called *Laplacian spectral radius* of Γ.

Definition 1.4.8. A graph is *Laplacian integral* if its Laplacian spectrum consists of integers.

We now review some necessary results on Laplacian eigenvalues of a graph from the literature.

Theorem 1.4.9 ([Cvetković et al., 2010])**.** *For any graph Γ, the multiplicity of 0 as an eigenvalue of $L(\Gamma)$ is equal to the number of components of Γ.*

Theorem 1.4.10 ([Mohar, 1991]). *For a graph Γ on n vertices, $\lambda_n\left(\overline{\Gamma}\right) = 0$, and $\lambda_k\left(\overline{\Gamma}\right) = n - \lambda_{n-k}(\Gamma)$ for $1 \leqslant k \leqslant n-1$.*

Theorem 1.4.11 ([Fiedler, 1973]). *For any graph Γ, $\lambda_1(\Gamma) = \max\limits_{1\leqslant i\leqslant r} \lambda_1(\Gamma_i)$, where $\Gamma_1, \ldots, \Gamma_r$ are the components of Γ.*

Theorem 1.4.12 ([Mohar, 1991]). *If Γ is a graph on n vertices, then $\lambda_1(\Gamma) \leqslant n$. Equality holds if and only if $\overline{\Gamma}$ is not connected.*

The following inequalities due Fiedler [1973] are essential for our study of algebraic connectivity of power graphs in Section 5.

Theorem 1.4.13. *For any non-complete graph Γ on n vertices, $\lambda_{n-1}(\Gamma) \leqslant \kappa(\Gamma)$.*

Theorem 1.4.14. *For any graph Γ on n vertices, $\lambda_{n-1}(\Gamma) \geqslant 2\left(1 - \cos\dfrac{\pi}{n}\right)\kappa'(\Gamma)$.*

Theorem 1.4.13 due to Fiedler raises the question that for which graphs the vertex connectivity and algebraic connectivity are equal. In the following theorem, Kirkland et al. [2002] provides a characterization those graphs.

Theorem 1.4.15. *Let Γ be a non-complete, connected graph on n vertices. Then $\kappa(\Gamma) = \lambda_{n-1}(\Gamma)$ if and only if Γ can be written as $\Gamma_1 \vee \Gamma_2$, where Γ_1 is a disconnected graph on $n - \kappa(\Gamma)$ vertices and Γ_2 is a graph on $\kappa(\Gamma)$ vertices with $\lambda_{n-1}(\Gamma_2) \geqslant 2\kappa(\Gamma) - n$.*

They also found the following necessary condition, which is essentially a consequence of Theorem 1.4.15.

Lemma 1.4.16. *For any connected graph Γ on n vertices, if $\overline{\Gamma}$ is connected, then $\kappa(\Gamma) \neq \lambda_{n-1}(\Gamma)$.*

1.5 Power graphs

In this section, we first present the fundamentals of power graph of a group. We then review some necessary results and previous works from the literature.

Definition 1.5.1. The *power graph* $\mathcal{G}(G)$ of a group G is a graph with vertex set G and distinct vertices u and v are adjacent if $v = u^\alpha$ for some $\alpha \in \mathbb{N}$ or $u = v^\beta$ for some $\beta \in \mathbb{N}$.

Example 1.5.2. Consider the power graph of $\mathbb{Z}_6$. As $6\overline{a} = \overline{0}$ for all $\overline{a} \in \mathbb{Z}_6$, $\overline{0}$ is adjacent to all other vertices. Since $\overline{1}$ and $\overline{5}$ are generators, they are adjacent to all other vertices. Neither $\overline{2}$ is a multiple of $\overline{3}$ nor $\overline{3}$ is a multiple of $\overline{2}$, so they are not adjacent. The same goes for $\overline{3}$ and $\overline{4}$ as well. The graph $\mathcal{G}(\mathbb{Z}_6)$ is depicted in Figure 1.1.

Remark 1.5.3. Since an element g of a group G is also a vertex of $\mathcal{G}(G)$, depending upon the context, we may refer g as an element or as a vertex.

The next remarks follows directly from the definition of power graphs.

Remark 1.5.4. For any finite group G, e is adjacent to all other vertices in $\mathcal{G}(G)$.

Remark 1.5.5. If two finite groups are isomorphic, their corresponding power graphs are isomorphic.

Remark 1.5.6. In view of Remark 1.5.5 and since any cyclic group of order n is isomorphic to $\mathbb{Z}_n$, instead of studing the graph theoretic properties of power graph of an arbitrary finite cyclic group of order n, we study it for $\mathbb{Z}_n$.

Theorem 1.5.7 ([Chakrabarty et al., 2009])**.** *Let G be a finite group.*

(i) *The power graph $\mathcal{G}(G)$ is connected.*

(ii) *The power graph $\mathcal{G}(G)$ is complete if and only if G is a cyclic group of order* 1 *or p^α, for some prime p and $\alpha \in \mathbb{N}$.*

(iii) *If H is a subgroup of G, then $\mathcal{G}(H)$ is an induced subgraph of $\mathcal{G}(G)$.*

Notation 1.5.8. For $A \subseteq G$, the subgraph of $\mathcal{G}(G)$ induced by A is denoted by $\mathcal{G}_G(A)$. However, when there is only one group in question, we simply write $\mathcal{G}(A)$.

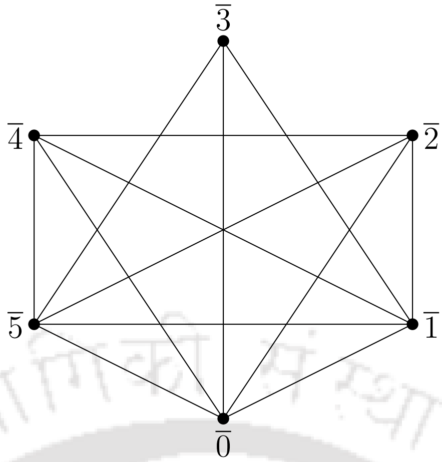

FIGURE 1.1: $\mathcal{G}(\mathbb{Z}_6)$

Remark 1.5.9. For any $A \subseteq G$, $\mathcal{G}(G - A) = \mathcal{G}(G) - A$.

Definition 1.5.10. The *proper power graph* $\mathcal{G}^*(G)$ of a group G is obtained by removing its identity element from $\mathcal{G}(G)$.

Remark 1.5.11. Let G be a finite group and Γ be a connected subgraph of $\mathcal{G}(G)$ having the identity element e of G as one of its vertices. Then, as e is adjacent to all other vertices in Γ, $\kappa(\Gamma - e) = \kappa(\Gamma) - 1$.

Remark 1.5.12. For any $g \in G$, $\mathcal{G}([g])$ is a complete subgraph of $\mathcal{G}(G)$. In particular, if $\mathrm{o}(g) = n$, then $\mathcal{G}([g]) = K_{\phi(n)}$.

Notation 1.5.13. For $n \in \mathbb{N}$, let $\mathcal{S}(\mathbb{Z}_n)$ consist of the identity element and generators of $\mathbb{Z}_n$, i.e., $\mathcal{S}(\mathbb{Z}_n) = \{\overline{a} \in \mathbb{Z}_n : 1 \leqslant a < n, \gcd(a, n) = 1\} \cup \{\overline{0}\}$. We write $\mathbb{Z}'_n = \mathbb{Z}_n - \mathcal{S}(\mathbb{Z}_n)$ and $\mathcal{G}'(\mathbb{Z}_n) = \mathcal{G}(\mathbb{Z}_n) - \mathcal{S}(\mathbb{Z}_n)$. Note that with the above notations, $V(\mathcal{G}'(\mathbb{Z}_n)) = \mathbb{Z}'_n$.

Remark 1.5.14. For any $n \in \mathbb{N}$, each element of $\mathcal{S}(\mathbb{Z}_n)$ is adjacent to every other element of $\mathcal{G}(\mathbb{Z}_n)$.

In particular, we have the following remark.

Remark 1.5.15. For $n \in \mathbb{N}$, $\mathcal{S}(\mathbb{Z}_n) \cong K_{\phi(n)+1}$.

The following results gives some structural properties and connectedness of power graphs of p-groups.

Theorem 1.5.16 ([Mirzargar et al., 2012])**.** *For any p-group G, $\mathcal{G}(G) \cong K_1 \vee nK_{p-1}$ for some n if and only if* $\exp(G) = p$.

Theorem 1.5.17 ([Moghaddamfar et al., 2014])**.** *Let G be a p-group. Then $\mathcal{G}^*(G)$ is connected if and only if G is cyclic or generalized quaternion.*

Theorem 1.5.18 ([Doostabadi et al., 2015])**.** *Let G be a finite group. Then $\mathcal{G}^*(G)$ is regular if and only if G is cyclic p-group or* $\exp(G) = p$.

We now review the existing results on vertex connectivity of power graphs of finite groups from [Chattopadhyay and Panigrahi, 2014, 2015]. We begin with power graphs of finite cyclic groups.

Theorem 1.5.19. *Let n be a positive integer.*

(i) $\kappa(\mathcal{G}(\mathbb{Z}_n)) = n - 1$ *when $n = 1$ or p^α for some prime p and positive integer α.*

(ii) $\kappa(\mathcal{G}(\mathbb{Z}_n)) \geqslant \phi(n) + 1$ *when n is not a prime power. The equality holds for $n = pq$ where p, q are distinct primes.*

Theorem 1.5.20. *Let n be a positive integer.*

(i) *If $n = p_1^{\alpha_1} p_2^{\alpha_2}$ for some primes p_1, p_2, and $\alpha_1, \alpha_2 \in \mathbb{N}$, then* $\kappa(\mathcal{G}(\mathbb{Z}_n)) \leqslant \phi(n) + p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1}$.

(ii) *If $n = p_1 p_2 p_3$ for some primes $p_1 < p_2 < p_3$, then* $\kappa(\mathcal{G}(\mathbb{Z}_n)) \leqslant \phi(n) + p_1 + p_2 - 1$.

The next two theorems give vertex connectivity of power graphs of D_n and Q_n.

Theorem 1.5.21. *For any integer $n \geqslant 3$, e is a cut-vertex of $\mathcal{G}(D_n)$. Consequently,* $\kappa(\mathcal{G}(D_n)) = 1$.

Theorem 1.5.22. *For any integer* $n \geqslant 2$, $\kappa(\mathcal{G}(Q_n)) = 2$. *In fact,* $\{e, a^n\}$ *is minimum separating set of* $\mathcal{G}(Q_n)$.

Chattopadhyay and Panigrahi [2015] obtained many interesting results the Laplacian spectrum of power graphs finite groups. We next recall some of them. In fact, in Chapter 5, we extend and generalize some of these results.

The following theorem ascertains that the Laplacian spectral radius of power graph of any finite group is its order.

Theorem 1.5.23. *For any finite group* G *of order* $n \geqslant 2$, $\lambda_1(\mathcal{G}(G)) = n$.

Theorem 1.5.24. *For each non-prime integer* $n > 3$, *the multiplicity of* n *as a Laplacian eigenvalue of* $\mathcal{G}(\mathbb{Z}_n)$ *is at least* $\phi(n) + 1$.

Chattopadhyay and Panigrahi [2015] obtained the following result upper bounds of algebraic connectivity of power graphs some finite cyclic groups as a consequence of Theorem 1.5.20.

Corollary 1.5.25. *Let* G *be a finite cyclic group of order* n.

(i) *If* $n = p_1^{\alpha_1} p_2^{\alpha_2}$ *for some primes* p_1, p_2, *and* $\alpha_1, \alpha_2 \in \mathbb{N}$, *then* $\lambda_{n-1}(\mathcal{G}(G)) \leqslant \phi(n) + p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1}$.

(ii) *If* $n = p_1 p_2 p_3$ *for some primes* $p_1 < p_2 < p_3$, *then* $\lambda_{n-1}(\mathcal{G}(G)) \leqslant \phi(n) + p_1 + p_2 - 1$.

Theorem 1.5.26. *If* n *is a positive integer having more than one prime factors, then* $\overline{\mathcal{G}'(\mathbb{Z}_n)}$ *is connected.*

In the following theorem, Chattopadhyay and Panigrahi [2015] supplied a strict lower bound of algebraic connectivity $\mathcal{G}(\mathbb{Z}_n)$.

Theorem 1.5.27. *For every integer* $n \geqslant 2$, *the algebraic connectivity* $\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n))$ *of* $\mathcal{G}(\mathbb{Z}_n)$ *satisfies the inequality* $\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) \geqslant \phi(n) + 1$. *Equality holds if* n *is either a prime or product of two distinct primes.*

The following result is in fact a consequence of Theorem 1.5.7(ii).

Theorem 1.5.28. *If n is a prime power, then the Laplacian spectrum of $\mathcal{G}(\mathbb{Z}_n)$ is given by* $\begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}$.

2

Vertex Connectivity

In this chapter, we study minimal separating sets and vertex connectivity of power graphs of finite groups. In Section 2.1, we first recall some existing notions and then obtain some results on quotient of a graph and in particular, on quotient power graph. In Section 2.2, we characterize the minimal separating sets of power graph of a finite group in terms of its quotient power graph. In Section 2.3, we obtain some minimal separating sets of $\mathcal{G}(\mathbb{Z}_n)$ and provide two upper bounds for its vertex connectivity. Followed by this, we make a comparison of these bounds. Then we determine the values of vertex connectivity of $\mathcal{G}(\mathbb{Z}_n)$ when n has two prime factors or n is a product of three distinct primes.

2.1 Quotient graphs

To study the number of components of a graph, Bubboloni [2017] considered the notion of quotient graphs. Subsequently, Bubboloni et al. [2017b] computed the number of components of proper power graphs of some finite groups. In this section, we first recall some of these results and then obtain some new results that are useful for our study of connectivity of power graphs.

In this chapter, G denotes a group with identity element e.

Definition 2.1.1. Let ϱ be an equivalence relation on a set A. The equivalence class of any $x \in A$ under ϱ is called a *ϱ-class* and denoted by x/ϱ. For any $B \subseteq A$, we write $B/\varrho = \{x/\varrho : x \in B\}$.

The following remark follows directly from Definition 2.1.1.

Remark 2.1.2. Let A be a set and $B \subseteq A$. Let ϱ be an equivalence relation on A and ϱ_1 be the restriction of ϱ to $A-B$. Then the following statements are equivalent.

(i) $A/\varrho - B/\varrho = (A - B)/\varrho_1$.

(ii) B is a union of ϱ-classes.

(iii) $x/\varrho = x/\varrho_1$ for all $x \in A - B$.

Definition 2.1.3. Let Γ be a graph with an equivalence relation ϱ on $V(\Gamma)$. The *quotient graph* of Γ with respect to ϱ, denoted by Γ/ϱ, is the graph with the set of all ϱ-classes of $V(\Gamma)$ as the vertex set and two distinct ϱ-classes C_1 and C_2 are adjacent if there exist $x_1 \in C_1$ and $x_2 \in C_2$ such that x_1 and x_2 are adjacent in Γ.

Definition 2.1.4. An equivalence relation ϱ on $V(\Gamma)$ is called *tame* if for every pair $x, y \in V(\Gamma)$, $x \,\varrho\, y$ implies they belong to same component of Γ. If ϱ is tame, then Γ/ϱ is called a *tame quotient graph*.

The following result due to Bubboloni [2017] shows that connectedness is preserved between a graph and its quotient graph under a tame equivalence relation.

Theorem 2.1.5 ([Bubboloni, 2017]). *Let Γ be a graph and ϱ be a tame equivalence relation on $V(\Gamma)$. Then Γ is connected if and only if Γ/ϱ is connected.*

Lemma 2.1.6. *Let Γ be a graph with an equivalence relation ϱ on $V(\Gamma)$. Let $A \subsetneq V(\Gamma)$ and ϱ_1 be the restriction of ϱ to $V(\Gamma) - A$. Then $\Gamma/\varrho - A/\varrho = (\Gamma - A)/\varrho_1$ if and only if A is a union of ϱ-classes.*

Proof. From Remark 2.1.2, $V(\Gamma)/\varrho - A/\varrho = V(\Gamma - A)/\varrho_1$ if and only if and A is a union of ϱ-classes. Moreover, C_1, C_2 are adjacent in $(\Gamma - A)/\varrho_1$ if and only if C_1, C_2 are adjacent in $\Gamma/\varrho - A/\varrho$. Hence the graphs $\Gamma/\varrho - A/\varrho$ and $(\Gamma - A)/\varrho_1$ are equal. □

Now we focus on quotient graph of a power graph. As already defined in Section 1.3, we write $g \approx h$ if $\langle g\rangle = \langle h\rangle$ for all $g, h \in G$. Further, $g/\!\approx\, = [g]$ for any $g \in G$.

Notation 2.1.7. For any $A \subseteq G$, we denote $[A] = A/\!\approx$. The restriction of $\approx$ to any $A \subseteq G$ is denoted by $\approx$ with a subscript, say $\approx_i$. Accordingly, the $\approx_i$-class of any $g \in A$ is denoted by $[g]_i$ and for any $B \subseteq A$, $[B]_i = \{[g]_i : g \in B\}$.

Definition 2.1.8. The quotient graph $\mathcal{G}(G)/\!\approx$ of $\mathcal{G}(G)$ is called the *quotient power graph* of G and is denoted by $\widetilde{\mathcal{G}}(G)$. More generally, for any $A \subseteq G$, $\widetilde{\mathcal{G}}_G(A)$ or simply $\widetilde{\mathcal{G}}(A)$ denotes the quotient graph $\mathcal{G}(A)/\!\approx_1$, where $\approx_1$ is the restriction of $\approx$ to A.

Remark 2.1.9. For any $g \in G$, $\mathcal{G}([g])$ is a complete subgraph of $\mathcal{G}(G)$.

Bubboloni et al. [2017b] observed that $\widetilde{\mathcal{G}}(G)$ is tame. In fact, more generally the following holds.

Remark 2.1.10. The quotient graph $\widetilde{\mathcal{G}}(A)$ is tame for any $A \subseteq G$.

As we will see in the next result, adjacency is preserved when passing from the power graph of a group to its quotient power graph.

Lemma 2.1.11 ([Bubboloni et al., 2017b]). *Let $g, h \in G$ and $g \not\approx h$. Then $[g]$ and $[h]$ are adjacent in $\widetilde{\mathcal{G}}(G)$ if and only if g and h are adjacent in $\mathcal{G}(G)$.*

Remark 2.1.12. Lemma 2.1.11 holds even when G is replaced by any $A \subseteq G$.

If g is adjacent to h, then by definition of quotient power graph, $[g]$ is adjacent to $[h]$. Moreover, for any $g_1 \in [g]$ and $h_1 \in [h]$, we have $[g_1] = [g]$ and $[h_1] = [h]$, so it follows from Lemma 2.1.11 that g_1 is adjacent to h_1. Therefore, Lemma 2.1.11 can be restated as the following remark.

Remark 2.1.13. Let $g, h \in G$ and $g \not\approx h$. Then in $\mathcal{G}(G)$, each element of $[g]$ is adjacent to every element of $[h]$ if and only if g is adjacent to h.

2.2 Minimal separating sets

In this section, we use the concept quotient power graph as a tool to characterize some properties of minimal separating sets of power graphs of finite groups. We then find some separating sets of power graphs.

Theorem 2.2.1. *Let $T \subsetneq G$ be a union of $\approx$-classes. Then T is a separating set of $\mathcal{G}(G)$ if and only if $[T]$ is a separating set of $\widetilde{\mathcal{G}}(G)$.*

Proof. By Remark 2.1.10, $\widetilde{\mathcal{G}}(G - T)$ is tame. Thus by Lemma 2.1.5, $\mathcal{G}(G) - T$ is disconnected if and only if $\widetilde{\mathcal{G}}(G - T)$ is disconnected. Whereas, by Lemma 2.1.6, $\widetilde{\mathcal{G}}(G - T) = \widetilde{\mathcal{G}}(G) - [T]$. Hence the proof follows. □

Theorem 2.2.2. *If T is a minimal separating set of $\mathcal{G}(G)$, then T is a union of $\approx$-classes.*

Proof. If possible, let T be not a union of $\approx$-classes. Then there exists $g \in T$ and $h \in G - T$ such that $g \approx h$. Since T is a minimal separating set of $\mathcal{G}(G)$ and $\widetilde{\mathcal{G}}(G - T)$ is tame, it follows from Lemma 2.1.5 that $\widetilde{\mathcal{G}}(G - T)$ is disconnected and $\widetilde{\mathcal{G}}(G - (T - g))$ is connected.

Let $\approx_1$, $\approx_2$ be the restrictions of $\approx$ to $G - T$ and $G - (T - g)$, respectively. We define the map $\psi : [G - T]_1 \to [G - (T - g)]_2$ by $\psi([x]_1) = [x]_2$. Then it follows from Remark 2.1.12 that ψ is an isomorphism from $\widetilde{\mathcal{G}}(G-T)$ to $\widetilde{\mathcal{G}}(G-(T-g))$. This is a contradiction. So T is a union of $\approx$-classes. □

The following theorem shows that the converse of Theorem 2.2.2 holds when $[T]$ is also a minimal separating set of $\widetilde{\mathcal{G}}(G)$.

Theorem 2.2.3. *For $T \subsetneq G$, T is a minimal separating set of $\mathcal{G}(G)$ if and only if $[T]$ is a minimal separating set of $\widetilde{\mathcal{G}}(G)$ and T is a union of $\approx$-classes.*

Proof. Let T be a minimal separating set of $\mathcal{G}(G)$. Then by Theorem 2.2.1, $[T]$ is a separating set of $\widetilde{\mathcal{G}}(G)$ and by Theorem 2.2.2, T is a union of $\approx$-classes.

Let $g \in T$. Since $T-[g]$ is a union of $\approx$-classes, it follows from Lemma 2.1.6 that $\widetilde{\mathcal{G}}(G)-[T-[g]] = \widetilde{\mathcal{G}}(G-(T-[g]))$. Thus, using the fact that $[T-[g]] = [T]-[g]$, we get $\widetilde{\mathcal{G}}(G)-([T]-[g]) = \widetilde{\mathcal{G}}(G-(T-[g]))$. Since T is a minimal separating set of $\mathcal{G}(G)$, by Lemma 2.1.5, $\widetilde{\mathcal{G}}(G-(T-[g]))$ is connected. So $\widetilde{\mathcal{G}}(G)-([T]-[g])$ is also connected. As a result, $[T]$ is a minimal separating set of $\widetilde{\mathcal{G}}(G)$.

To prove the converse, let $[T]$ be a minimal separating set of $\widetilde{\mathcal{G}}(G)$ and T be a union of $\approx$-classes. Then by Theorem 2.2.1, T is a separating set of $\mathcal{G}(G)$.

Let $h \in T$. Since $T-[h]$ is also a union of $\approx$-classes, by Lemma 2.1.6, $\widetilde{\mathcal{G}}(G) - [T-[h]] = \widetilde{\mathcal{G}}(G-(T-[h]))$. Moreover, as $[T-[h]] = [T]-[h]$, we have

$$\widetilde{\mathcal{G}}(G)-([T]-[h]) = \widetilde{\mathcal{G}}(G-(T-[h])). \tag{2.1}$$

Let $\approx_1$, $\approx_2$ be the restrictions of $\approx$ to $G-(T-[h])$ and $G-(T-h)$, respectively. We define the map $\theta : [G-(T-[h])]_1 \to [G-(T-h)]_2$ by $\theta([x]_1) = [x]_2$. Then it follows from Remark 2.1.12 that θ is a isomorphism from $\widetilde{\mathcal{G}}(G-(T-[h]))$ to $\widetilde{\mathcal{G}}(G-(T-h))$. From this and (2.1), we get

$$\widetilde{G}(G)-([T]-[h]) \cong \widetilde{\mathcal{G}}(G-(T-h)). \tag{2.2}$$

Now, $[T]$ being minimal, $\widetilde{\mathcal{G}}(G)-([T]-[h])$ is connected. Consequently, by (2.2),

$\widetilde{\mathcal{G}}(G-(T-h))$ is also connected. Then by Lemma 2.1.5, $\mathcal{G}(G)-(T-h)$ is connected. Hence T is a minimal separating set of $\mathcal{G}(G)$. $\square$

Theorem 2.2.4. *Let T be a separating set of $\mathcal{G}(G)$. Then T is a minimal separating set of $\mathcal{G}(G)$ if and only if $[T]$ is a minimal separating set of $\widetilde{\mathcal{G}}(G)$.*

Proof. If T is a minimal separating set of $\mathcal{G}(G)$, then it follows from Theorem 2.2.3 that $[T]$ is also a minimal separating set of $\widetilde{\mathcal{G}}(G)$.

To prove the converse, let $[T]$ be a minimal separating set of $\widetilde{\mathcal{G}}(G)$. Since T is a separating set of $\mathcal{G}(G)$, there exists $S \subseteq T$ such that S is a minimal separating set of $\mathcal{G}(G)$. Then again by Theorem 2.2.3, $[S]$ is a minimal separating set of $\widetilde{\mathcal{G}}(G)$ and S is a union of $\approx$-classes. So, if $S \neq T$, then there exists $g \in T$ such that $[g] \cap S = \varnothing$. This in turn implies that $[S] \subsetneq [T]$, contradicting the minimality of $[T]$. Hence $S = T$ and the proof follows. $\square$

Notation 2.2.5. For the rest of the section, for $g \in G$, $N_{\mathcal{G}(G)}(g)$ and $N_{\mathcal{G}(G)}([g])$ (cf. Definition 1.2.24) are denoted simply by $N(g)$ and $N([g])$, respectively. Moreover, $N_{\widetilde{\mathcal{G}}(G)}([g])$ is denoted by $\widetilde{N}([g])$.

The following remark is a consequence of Remark 2.1.13.

Remark 2.2.6. If $g \in G$, then for any $y \in [g]$, $N([g]) = N(y) - [g]$. Moreover, $N(g) = N([g]) \cup ([g] - g)$.

Lemma 2.2.7. *For $g \in G$, the following statements are equivalent.*

(i) *$N(g)$ is a separating set of $\mathcal{G}(G)$.*

(ii) *$N([g])$ is a separating set of $\mathcal{G}(G)$.*

(iii) *$\widetilde{N}([g])$ is a separating set of $\widetilde{\mathcal{G}}(G)$.*

(iv) *There exists $y \in G$ such that g is not adjacent to y.*

Proof. Observe that $\mathcal{G}(G) - N(g)$ is disconnected if and only if there exists $y \in G$ such that g is not adjacent to y. Hence (i) and (iv) are equivalent. Since $[N([g])] = \widetilde{N}([g])$, by Theorem 2.2.1, (ii) and (iii) are equivalent.

We now prove that (iii) and (iv) are equivalent. The quotient graph $\widetilde{\mathcal{G}}(G) - \widetilde{N}([g])$ is disconnected if and only if there exists $[y] \in [G]$ such that $[g]$ is not adjacent to $[y]$. So the proof follows from Lemma 2.1.11. □

Remark 2.2.8. Let $g \in G$. If $o(g) = 1$, then $g = e$, and $N(e) = G - e$ is not a separating set of $\mathcal{G}(G)$. Furthermore, if $o(g) = 2$, then $N(g) = N([g])$.

Lemma 2.2.9. *If $g \in G$ with $o(g) \geqslant 3$, then $N(g)$ is not a minimal separating set of $\mathcal{G}(G)$.*

Proof. If $o(g)$ is infinite, then $|[g]| = 2$ (cf. Lemma 1.3.21). Whereas, if $o(g)$ is finite, since $o(g) \geqslant 3$, then $|[g]| \geqslant 2$ (cf. Corollary 1.3.23). So there exists $y \in [g]$, $y \neq g$. Hence, as $[g] \not\subset N(g)$, it follows from Theorem 2.2.2 that $N(g)$ is not a minimal separating set of $\mathcal{G}(G)$. □

Remark 2.2.8 and Lemma 2.2.9 combinedly ascertain that to study vertex connectivity of power graphs of finite groups, we should focus on neighbourhoods of $\approx$-classes instead of neighbourhoods of vertices.

2.3 Vertex connectivity of $\mathcal{G}(\mathbb{Z}_n)$

In this section, we obtain different minimal separating sets of power graph of $\mathbb{Z}_n$. We provide two upper bounds for the vertex connectivity of power graph of $\mathbb{Z}_n$, and present a comparison among them in terms of n. Further, we obtain the actual values of the vertex connectivity of power graph of $\mathbb{Z}_n$ when n has two prime factors or n is a product of three primes. This will ascertain that one of the aforementioned bounds is sharp.

We begin this section with the following lemma that provides some basic and useful properties of $\mathbb{Z}_n$.

Lemma 2.3.1. *For $n \in \mathbb{N}$, the following statements hold for $\approx$-classes of $\mathbb{Z}_n$.*

(i) *For each $\overline{a} \in \mathbb{Z}_n^*$, there exists a positive divisor d of n such that $\overline{a} \approx \overline{d}$.*

(ii) *For $\overline{a} \in \mathbb{Z}_n^*$, $|[\overline{a}]| = \phi\left(\dfrac{n}{\gcd(n,a)}\right)$.*

(iii) *If $a, b \in \mathbb{N}$ is such that $a\,|\,n$, $b\,|\,n$ and $a \neq b$, then $\overline{a} \not\approx \overline{b}$.*

Proof. (i) Consider $d = \gcd(a,n)$. Then $\langle\overline{a}\rangle = \langle\overline{d}\rangle$ so that $\overline{a} \approx \overline{d}$.

(ii) Since $o(\overline{a}) = \dfrac{n}{\gcd(n,a)}$, the proof follows from Corollary 1.3.23.

(iii) Suppose $\langle\overline{a}\rangle = \langle\overline{b}\rangle$. Then $o(\overline{a}) = o(\overline{b})$. That is, $\dfrac{n}{\gcd(n,a)} = \dfrac{n}{\gcd(n,b)}$. As $a\,|\,n$ and $b\,|\,n$, we in turn get $a = b$; which is a contradiction. Hence $\overline{a} \not\approx \overline{b}$.

□

It is known that $\mathcal{G}(\mathbb{Z}_n)$ is a complete graph when n is a prime power (cf. Theorem 1.5.7(ii)). We now focus on connectedness of $\mathcal{G}(\mathbb{Z}_n)$ when n is not a prime power.

Lemma 2.3.2. *If $n > 1$ is not a prime number, then the following statements hold.*

(i) *If n is not a prime power, then every separating set of $\mathcal{G}(\mathbb{Z}_n)$ contains $\mathcal{S}(\mathbb{Z}_n)$.*

(ii) $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1 + \kappa(\mathcal{G}'(\mathbb{Z}_n))$.

(iii) *If $p_1 < p_2 < \cdots < p_r$ are the prime factors of n, then $\mathbb{Z}_n' = \bigcup_{i=1}^{r}\langle\overline{p_i}\rangle^*$.*

Proof. When n is not a prime power, $\mathcal{G}(\mathbb{Z}_n)$ is not complete (cf. Theorem 1.5.7). So (i) follows from Remark 1.5.14. Since n is not a prime number, $\mathcal{G}'(\mathbb{Z}_n)$ is a non-null graph. So (ii) follows from Remark 1.5.14 and the fact that $|\mathcal{S}(\mathbb{Z}_n)| = \phi(n) + 1$. Moreover, if $\overline{a} \in \mathbb{Z}_n'$, then a is divided by at least one prime factor of n. Hence (iii) follows. □

If n is a product of two distinct primes, then $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1$ (cf. Theorem 1.5.19(ii)). In the next result, we show that the converse holds as well.

Proposition 2.3.3. *For $n \in \mathbb{N}$, the following statements are equivalent.*

(i) *n is a product of two distinct primes.*

(ii) *$\mathcal{S}(\mathbb{Z}_n)$ is a separating set of $\mathcal{G}(\mathbb{Z}_n)$.*

(iii) *$\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1$.*

Proof. It follows from Theorem 1.5.19(ii) that (i) implies (iii). Whereas, if (iii) holds, since $|\mathcal{S}(\mathbb{Z}_n)| = \phi(n) + 1$, (ii) follows from Lemma 2.3.2(i).

We now prove that (ii) implies (i). Let $\mathcal{S}(\mathbb{Z}_n)$ be a separating set of $\mathcal{G}(\mathbb{Z}_n)$. Then $\mathcal{G}(\mathbb{Z}_n)$ is not a complete graph and hence by Theorem 1.5.7(ii), n has at least two distinct prime factors. Since $\mathcal{G}'(\mathbb{Z}_n)$ is disconnected, there exist $\overline{a}, \overline{b} \in \mathbb{Z}'_n$ such that there is no path from $\overline{a}$ to $\overline{b}$ in $\mathcal{G}'(\mathbb{Z}_n)$. In view of Lemma 2.3.2(iii), every element of $\mathbb{Z}'_n$ is in $\langle\overline{p}\rangle^*$ for some prime factor p of n. Let $\overline{a}, \overline{b} \in \langle\overline{p}\rangle$ for some prime factor p of n. Then $\overline{a}, \overline{p}, \overline{b}$ is an $\overline{a}, \overline{b}$-path in $\mathcal{G}'(\mathbb{Z}_n)$, which is a contradiction. Hence $\overline{a} \in \langle\overline{p}\rangle$ and $\overline{b} \in \langle\overline{q}\rangle$ for some distinct prime factors p and q of n. If possible, let $pq < n$, so that $\overline{pq} \in \mathbb{Z}'_n$. Consequently, $\overline{a}$ and $\overline{b}$ are connected by the path $\overline{a}, \overline{p}, \overline{pq}, \overline{q}, \overline{b}$ in $\mathcal{G}'(\mathbb{Z}_n)$, which is again a contradiction. Hence $n = pq$. □

Theorem 2.3.4. *Suppose n is not a product of two primes and has prime factors $p_1 < p_2 < \cdots < p_r$ with $r \geqslant 2$. Then, for any $1 \leqslant k \leqslant r$, $\bigcup_{\substack{i=1 \\ i\neq k}}^{r} \langle\overline{p_i p_k}\rangle^*$ is a minimal separating set of $\mathcal{G}'(\mathbb{Z}_n)$.*

Proof. Let $\Gamma = \mathcal{G}'(\mathbb{Z}_n)$. Then by Proposition 2.3.3, Γ is connected and by Lemma 2.3.2(iii), $V(\Gamma) = \bigcup_{i=1}^{r} \langle\overline{p_i}\rangle^*$.

Let $T = \bigcup_{\substack{i=1 \\ i\neq k}}^{r} \langle\overline{p_i p_k}\rangle^*$. For $i = 1, 2, \ldots, r$, let $T_i = \langle\overline{p_i}\rangle^* - T$ and $U = \bigcup_{\substack{i=1 \\ i\neq k}}^{r} T_i$. Then

$V(\Gamma - T) = \bigcup_{i=1}^{r} T_i = T_k \cup U$. We prove that $\Gamma - T$ is disconnected by showing that no element of T_k is adjacent to any element of U.

If possible, let there exist $x \in T_k$ and $y \in U$ such that they are adjacent. So $y \in T_l$ for some $1 \leqslant l \leqslant r$, $l \neq k$. Then there exist non-zero integers a and b such that $x = a\overline{p_k}$ and $y = b\overline{p_l}$. Since x is adjacent to y, one of them is a multiple of the other. Let $a\overline{p_k} = cb\overline{p_l}$ for some non-zero integer c. Then $ap_k = cbp_l + c_1 n$ for some non-zero integer c_1. This implies that $p_l \,|\, ap_k$. Since $p_l \nmid p_k$, we have $p_l \,|\, a$. Consequently, $a\overline{p_k} \in \langle \overline{p_k p_l} \rangle \subseteq T$. This is a contradiction, as $T_k \cap T = \varnothing$. Similarly, if $b\overline{p_l}$ is a multiple of $a\overline{p_k}$, then also we get a contradiction. Hence $\Gamma - T$ is disconnected. Consequently, T is a separating set of Γ.

We now show the minimality of T. First of all, $\mathcal{G}(T_k)$ is connected because all of its vertices are adjacent to $\overline{p_k}$. Moreover, note that $\mathcal{G}(U)$ is connected. For instance, let $u, v \in U$. Then $u \in T_i$, $v \in T_j$ for some $1 \leqslant i, j \leqslant r$. If $i = j$, then u and v are connected by the path $u, \overline{p_i}, v$, and if $i \neq j$, then u and v are connected by the path $u, \overline{p_i}, \overline{p_i p_j}, \overline{p_j}, v$.

Now let $z \in T$. So $z = d\overline{p_k p_l}$ for some non-zero integer d and $1 \leqslant l \leqslant r$, $l \neq k$. Since both $\overline{p_k} \in T_k$ and $\overline{p_l} \in U$ are adjacent to z, $\Gamma - (T - z)$ is connected. Consequently, T is a minimal separating set of $\mathcal{G}'(\mathbb{Z}_n)$. $\square$

Lemma 2.3.5. *Suppose n is not a product of two primes and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_i \in \mathbb{N}$ for all $1 \leqslant i \leqslant r$. Then for any $1 \leqslant k \leqslant r$,*

$$\left| \bigcup_{\substack{i=1 \\ i \neq k}}^{r} \langle \overline{p_i p_k} \rangle \right| = \frac{n}{p_k} - p_k^{\alpha_k - 1} \phi\left(\frac{n}{p_k^{\alpha_k}}\right)$$

and for any $1 \leqslant j < k \leqslant r$,

$$\left| \bigcup_{\substack{i=1 \\ i \neq k}}^{r} \langle \overline{p_i p_k} \rangle \right| \leqslant \left| \bigcup_{\substack{i=1 \\ i \neq j}}^{r} \langle \overline{p_i p_j} \rangle \right|. \tag{2.3}$$

Proof. Observe that $\bigcup\limits_{\substack{i=1\\ i\neq k}}^{r}\langle\overline{p_ip_k}\rangle$ consists of those elements of $\langle\overline{p_k}\rangle$ which are divisible by some $\overline{p_i}$, $1\leqslant i\leqslant r, i\neq k$. Therefore, we get $\bigcup\limits_{\substack{i=1\\ i\neq k}}^{r}\langle\overline{p_ip_k}\rangle$ by deleting those elements from $\langle\overline{p_k}\rangle$ which are relatively prime to $\overline{p_i}$, $1\leqslant i\leqslant r, i\neq k$. Hence $\bigcup\limits_{\substack{i=1\\ i\neq k}}^{r}\langle\overline{p_ip_k}\rangle = A-B$, where $A=\left\{a\overline{p_k} : a\in\mathbb{N}, 0\leqslant a<\frac{n}{p_k}\right\}$, and
$B=\left\{a\overline{p_k} : a\in\mathbb{N}, 0\leqslant a<\frac{n}{p_k}, (a,p_i)=1\ \forall\ 1\leqslant i\leqslant r, i\neq k\right\}$.

Take $n_1=\prod_{j=1,j\neq k}^{r}p_j$ and $n_2=\dfrac{n}{p_kn_1}=\dfrac{n}{p_1\dots p_r}$. For $0\leqslant m\leqslant n_2-1$, let $P_m=\{a\overline{p_k} : a\in\mathbb{N}, mn_1\leqslant a<(m+1)n_1, (a,n_1)=1\}$. Trivially $P_l\cap P_m=\varnothing$ for $l\neq m$ and

$$B=\bigcup_{m=0}^{n_2-1}P_m. \tag{2.4}$$

Observe that $a\overline{p_k}\in P_m$ if and only if $(a-mn_1)\overline{p_k}\in P_0$. Thus $|P_m|=|P_0|$ for all $0\leqslant m\leqslant n_2-1$. Further, $|P_0|=|\{a\overline{p_k} : a\in\mathbb{N}, 0\leqslant a<n_1, (a,n_1)=1\}|=\phi(n_1)$. So for all $0\leqslant m\leqslant n_2-1$,

$$|P_m|=\phi(n_1). \tag{2.5}$$

From (2.4) and (2.5), we have

$$|B|=\sum_{m=0}^{n_2-1}|P_m|=n_2\phi(n_1)=\frac{n}{p_k}\prod_{\substack{i=1\\ i\neq k}}^{r}\left(1-\frac{1}{p_i}\right)=p_k^{\alpha_k-1}\phi\left(\frac{n}{p_k^{\alpha_k}}\right).$$

As $|A|=\dfrac{n}{p_k}$ and $B\subseteq A$, we finally have

$$\left|\bigcup_{\substack{i=1\\ i\neq k}}^{r}\langle\overline{p_ip_k}\rangle\right|=|A|-|B|=\frac{n}{p_k}-p_k^{\alpha_k-1}\phi\left(\frac{n}{p_k^{\alpha_k}}\right).$$

Now we prove (2.3).

$$\begin{aligned}\left|\bigcup_{\substack{i=1\\i\neq j}}^{r}\langle\overline{p_ip_j}\rangle\right|-\left|\bigcup_{\substack{i=1\\i\neq k}}^{r}\langle\overline{p_ip_k}\rangle\right| &= \frac{n}{p_j}-p_j^{\alpha_j-1}\phi\left(\frac{n}{p_j^{\alpha_j}}\right)-\left\{\frac{n}{p_k}-p_k^{\alpha_k-1}\phi\left(\frac{n}{p_k^{\alpha_k}}\right)\right\}\\
&= \frac{n}{p_j}-\frac{n}{p_j}\prod_{\substack{i=1\\i\neq j}}^{r}\left(1-\frac{1}{p_i}\right)-\left\{\frac{n}{p_k}-\frac{n}{p_k}\prod_{\substack{i=1\\i\neq k}}^{r}\left(1-\frac{1}{p_i}\right)\right\}\\
&= \frac{n}{p_jp_k}\left[p_k-p_k\prod_{\substack{i=1\\i\neq j}}^{r}\left(1-\frac{1}{p_i}\right)-\left\{p_j-p_j\prod_{\substack{i=1\\i\neq k}}^{r}\left(1-\frac{1}{p_i}\right)\right\}\right]\\
&= \frac{n}{p_jp_k}\left\{p_k-p_j-\{p_k-1-(p_j-1)\}\prod_{\substack{i=1\\i\neq j,k}}^{r}\left(1-\frac{1}{p_i}\right)\right\}\\
&= \frac{n(p_k-p_j)}{p_jp_k}\left\{1-\prod_{\substack{i=1\\i\neq j,k}}^{r}\left(1-\frac{1}{p_i}\right)\right\}\geqslant 0.\end{aligned}$$

□

In the following theorem, we shall give an upper bound for the vertex connectivity of power graphs of $\mathbb{Z}_n$. This generalizes Theorem 1.5.20 due to Chattopadhyay and Panigrahi [2014] to all $n \in \mathbb{N}$.

Theorem 2.3.6. *If $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$, where $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_j \in \mathbb{N}$ for $1 \leqslant j \leqslant r$, then*

$$\kappa(\mathcal{G}(\mathbb{Z}_n)) \leqslant \phi(n)+\frac{n}{p_r}-p_r^{\alpha_r-1}\phi\left(\frac{n}{p_r^{\alpha_r}}\right). \tag{2.6}$$

Proof. If n is a product of two primes, the inequality follows from Proposition 2.3.3. Now suppose n is not a product of two primes. By Theorem 2.3.4 and Lemma 2.3.5, we have

$$\kappa(\mathcal{G}'(\mathbb{Z}_n)) \leqslant \left|\bigcup_{i=1}^{r-1}\langle\overline{p_ip_r}\rangle^*\right| = \frac{n}{p_r}-p_r^{\alpha_r-1}\phi\left(\frac{n}{p_r^{\alpha_r}}\right)-1.$$

Hence the proof follows from Lemma 2.3.2(ii). □

Remark 2.3.7. We prove in Theorem 2.3.20 and Theorem 2.3.22 that the equality holds in (2.6) if n has exactly two prime factors or n is a product of three distinct primes. This will show that the upper bound given in Theorem 2.3.6 is sharp.

We now concentrate on minimal separating sets of $\mathcal{G}(\mathbb{Z}_n)$ that arise from neighborhoods of $\approx$-classes. Subsequently, we obtain an alternate upper bound for $\kappa(\mathcal{G}(\mathbb{Z}_n))$, and ascertain the conditions on n for which this bound is an improvement to that of Theorem 2.3.6.

The following remark is a consequence of Lemma 2.2.7.

Remark 2.3.8. If $\overline{a} \in \mathcal{S}(\mathbb{Z}_n)$, then $N(\overline{a})$ and $N([\overline{a}])$ are not separating sets of $\mathcal{G}(\mathbb{Z}_n)$.

The next remark is immediate from Lemma 2.1.6.

Remark 2.3.9. $[\mathcal{S}(\mathbb{Z}_n)] = \{[\overline{0}], [\overline{1}]\}$. Notice that for any $\overline{a} \in \mathbb{Z}_n'$, we have $\mathcal{S}(\mathbb{Z}_n) \subseteq N(\overline{a})$ and $[\mathcal{S}(\mathbb{Z}_n)] \subseteq \widetilde{N}([\overline{a}])$.

Notation 2.3.10. We denote $\widetilde{\mathcal{G}}'(\mathbb{Z}_n) = \widetilde{\mathcal{G}}(\mathbb{Z}_n) - [\mathcal{S}(\mathbb{Z}_n)]$, $N'(\overline{a}) = N(\overline{a}) - \mathcal{S}(\mathbb{Z}_n)$, $N'([\overline{a}]) = N([\overline{a}]) - \mathcal{S}(\mathbb{Z}_n)$ and $\widetilde{N}'([\overline{a}]) = \widetilde{N}([\overline{a}]) - [\mathcal{S}(\mathbb{Z}_n)]$.

Remark 2.3.11. For any $n \in \mathbb{N}$, $\widetilde{\mathcal{G}}'(\mathbb{Z}_n) = \widetilde{\mathcal{G}}(\mathbb{Z}_n - \mathcal{S}(\mathbb{Z}_n))$.

Lemma 2.3.12. *Suppose $n \in \mathbb{N}$ is neither a prime power nor a product of two distinct primes. Then for any $\overline{a} \in \mathbb{Z}_n'$,*

(i) *$N'([\overline{a}])$ is a separating set of $\mathcal{G}'(\mathbb{Z}_n)$.*

(ii) *$\widetilde{N}'([\overline{a}])$ is a separating set of $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$.*

(iii) *$N'([\overline{a}])$ is a minimal separating set of $\mathcal{G}'(\mathbb{Z}_n)$ if and only if $\widetilde{N}'([\overline{a}])$ is a minimal separating set of $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$.*

Proof. By Theorem 1.5.7, $\mathcal{G}(\mathbb{Z}_n)$ is not complete and by Proposition 2.3.3, $\mathcal{G}'(\mathbb{Z}_n)$ is connected. Thus by Lemma 2.2.7, $N([\overline{a}])$ is a separating set of $\mathcal{G}(\mathbb{Z}_n)$ and $\widetilde{N}([\overline{a}])$ is a separating set of $\widetilde{\mathcal{G}}(\mathbb{Z}_n)$. Consequently, (i) and (ii) follow from Remark 1.5.14. Finally, (iii) follows from (i) and Theorem 2.2.4. □

Lemma 2.1.11 and Lemma 2.3.1 give the following observation, which will be useful in proving Theorem 2.3.14.

Lemma 2.3.13. *For $n \in \mathbb{N}$, $\overline{a} \in \mathbb{Z}'_n$ and $b = \gcd(a, n)$, the following holds in $\mathcal{G}(\mathbb{Z}_n)$:*

$$N'([\overline{a}]) = \bigcup_{\substack{c\,|\,b \\ 1<c<b}} [\overline{c}] \;\cup \bigcup_{\substack{b\,|\,d,d\,|\,n \\ b<d<n}} [\overline{d}]. \tag{2.7}$$

The following theorem further narrows down our study of minimal separating sets of $\mathcal{G}(\mathbb{Z}_n)$.

Theorem 2.3.14. *Suppose $n \in \mathbb{N}$ is not a product of two primes and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_i \in \mathbb{N}$ for all $1 \leqslant i \leqslant r$. Then for any $1 \leqslant k \leqslant r$, the following statements hold.*

(i) $N'\left(\left[\overline{p_k^{\alpha_k}}\right]\right)$ *is a minimal separating set of $\mathcal{G}'(\mathbb{Z}_n)$.*

(ii) *If $\alpha_k > 1$ and $1 \leqslant \beta_k < \alpha_k$, $N'\left(\left[\overline{p_k^{\beta_k}}\right]\right)$ is not a minimal separating set of $\mathcal{G}'(\mathbb{Z}_n)$.*

Proof. (i) Let $T = \widetilde{N}'\left(\left[\overline{p_k^{\alpha_k}}\right]\right)$ and $\Gamma = \widetilde{\mathcal{G}}'(\mathbb{Z}_n) - T$. By Lemma 2.3.12(ii), Γ is disconnected. Moreover, by Lemma 2.1.11, $V(\Gamma) = \left\{\left[\overline{p_k^{\alpha_k}}\right]\right\} \bigcup \{[\overline{a}] : a\,|\,n, 1 < a < n, p_k^{\alpha_k} \nmid a, a \nmid p_k^{\alpha_k}\}$. Let $T_1 = \left\{\left[\overline{p_k^{\alpha_k}}\right]\right\}$ and $T_2 = \{[\overline{a}] : a\,|\,n, 1 < a < n, p_k^{\alpha_k} \nmid a, a \nmid p_k^{\alpha_k}\}$. Then the subgraph of Γ induced by T_1 is complete and hence connected. Furthermore, $[\overline{p_i}] \in T_2$ for all $1 \leqslant i \leqslant r, i \neq k$ and every other $[\overline{b}] \in T_2$ is adjacent to some $[\overline{p_j}]$ for $1 \leqslant j \leqslant r, j \neq k$ in Γ. Additionally, $[\overline{p_i}], [\overline{p_j}] \in T_2$, $i \neq j$, then both are adjacent to $[\overline{p_i p_j}] \in T_2$ in Γ. Thus the subgraph of Γ induced by T_2 is also connected. So Γ consists of exactly two components - the subgraphs induced by T_1 and T_2.

Therefore, to show that T is a minimal separating set of $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$, it is enough to show that every element of T is adjacent to some element of T_1 and some element of T_2. Let $C \in T$. Then $\left[\overline{p_k^{\alpha_k}}\right]$ is adjacent to C. We next show that C is adjacent to some element of T_2.

By Lemma 2.3.1(i), $C = [\overline{c}]$ for some $1 < c < n$, $c \,|\, n$. Since C is adjacent to $\left[\overline{p_k^{\alpha_k}}\right]$, it follows again from Lemma 2.1.11 that either $\overline{p_k^{\alpha_k}} \,|\, \overline{c}$ or $\overline{c} \,|\, \overline{p_k^{\alpha_k}}$. As both c and $p_k^{\alpha_k}$ are factors of n, either $p_k^{\alpha_k} \,|\, c$ or $c \,|\, p_k^{\alpha_k}$. First let $p_k^{\alpha_k} \,|\, c$, so that $c = a p_k^{\alpha_k}$ for some integer a. If $\gcd\left(a, \dfrac{n}{p_k^{\alpha_k}}\right) = 1$, then $C = \left[\overline{p_k^{\alpha_k}}\right]$, which is not possible. So, as $\dfrac{n}{p_k^{\alpha_k}} = \prod_{i=1, i \neq k}^{r} p_i^{\alpha_i}$, there exist $1 \leqslant l \leqslant r, l \neq k$ such that $p_l \,|\, a$. Then $\overline{p_l} \,|\, \overline{a}$ and hence $\overline{p_l} \,|\, \overline{c}$. So C is adjacent to $[\overline{p_l}]$ and $[\overline{p_l}] \in T_2$. Now let $c \,|\, p_k^{\alpha_k}$. Then $c = p_k^{\beta}$ for some $1 \leqslant \beta < \alpha_k$. So C is adjacent to $\left[\overline{p_k^{\beta} p_m}\right]$ for all $1 \leqslant m \leqslant r, m \neq k$ and $\left[\overline{p_k^{\beta} p_m}\right] \in T_2$. Thus T is a minimal separating set of $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$ and hence the proof follows from Lemma 2.3.12(iii).

(ii) Observe that $\left[\overline{p_k^{\alpha_k}}\right] \in \widetilde{N}'\left(\left[\overline{p_k^{\beta_k}}\right]\right)$. So that by Lemma 2.3.13, we have

$$\widetilde{N}'\left(\left[\overline{p_k^{\alpha_k}}\right]\right) \subseteq \widetilde{N}'\left(\left[\overline{p_k^{\beta_k}}\right]\right) \cup \left\{\left[\overline{p_k^{\beta_k}}\right]\right\}. \tag{2.8}$$

By Lemma 2.1.11, $\left[\overline{p_k^{\beta_k}}\right]$ is adjacent to $\left[\overline{p_k^{\alpha_k}}\right]$, and by (2.8), none of $\left[\overline{p_k^{\alpha_k}}\right]$ and $\left[\overline{p_k^{\beta_k}}\right]$ is adjacent to any other vertex of $\widetilde{\mathcal{G}}'(\mathbb{Z}_n) - \left(\widetilde{N}'\left(\left[\overline{p_k^{\beta_k}}\right]\right) - \left[\overline{p_k^{\alpha_k}}\right]\right)$. Hence $\widetilde{N}'\left(\left[\overline{p_k^{\beta_k}}\right]\right) - \left[\overline{p_k^{\alpha_k}}\right]$ is a separating set of $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$ and as a result, $\widetilde{N}'\left(\left[\overline{p_k^{\beta_k}}\right]\right)$ is not a minimal separating set of $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$. From this and Lemma 2.3.12(iii), the proof follows. □

The next result shows that minimal separating sets of $\mathcal{G}(\mathbb{Z}_n)$ obtained in Theorem 2.3.4 and Theorem 2.3.14 are same when the largest prime dividing n has power one.

Proposition 2.3.15. *Suppose $n \in \mathbb{N}$ is not a product of two primes and $n = p_1^{\alpha_1} \ldots p_{r-1}^{\alpha_{r-1}} p_r$, where $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_i \in \mathbb{N}$ for $1 \leqslant i \leqslant r$. Then $N'([\overline{p_r}]) = \bigcup_{i=1}^{r-1} \langle \overline{p_i p_r} \rangle^*$.*

Proof. $N'([\overline{p_r}]) = \langle\overline{p_r}\rangle^* - [\overline{p_r}]$

$$= \langle\overline{p_r}\rangle^* - \left\{ap_r : 1 \leqslant a < p_1^{\alpha_1}\dots p_{r-1}^{\alpha_{r-1}}, \gcd(a, p_1^{\alpha_1}\dots p_{r-1}^{\alpha_{r-1}}) = 1\right\}$$
$$= \bigcup_{i=1}^{r-1} \langle\overline{p_ip_r}\rangle^*$$

□

We now provide an upper bound for $\kappa(\mathcal{G}(\mathbb{Z}_n))$ in the following theorem.

Theorem 2.3.16. *Suppose n is not a product of two primes and $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$, where $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_i \in \mathbb{N}$ for $1 \leqslant i \leqslant r$, then*

$$\kappa(\mathcal{G}(\mathbb{Z}_n)) \leqslant \phi(n) + \frac{n}{p_r^{\alpha_r}} + (p_r^{\alpha_r-1} - 2)\phi\left(\frac{n}{p_r^{\alpha_r}}\right).$$

Proof. From Theorem 2.3.14(i), $\kappa(\mathcal{G}'(\mathbb{Z}_n)) \leqslant |N'([\overline{p_r^{\alpha_r}}])|$, and by Lemma 2.3.13, we have

$$\begin{aligned}|N'([\overline{p_r^{\alpha_r}}])| &= \left|\langle\overline{p_r^{\alpha_r}}\rangle^*\right| - |[\overline{p_r^{\alpha_r}}]| + \sum_{j=1}^{\alpha_r}\left|\left[\overline{p_r^{j}}\right]\right| - |[\overline{p_r^{\alpha_r}}]| \\
&= \frac{n}{p_r^{\alpha_r}} - 1 - \phi\left(\frac{n}{p_r^{\alpha_r}}\right) + \sum_{j=1}^{\alpha_r}\phi\left(\frac{n}{p_r^{j}}\right) - \phi\left(\frac{n}{p_r^{\alpha_r}}\right) \\
&= \frac{n}{p_r^{\alpha_r}} - 1 + \phi\left(\frac{n}{p_r^{\alpha_r}}\right)\sum_{j=1}^{\alpha_r}\phi\left(p_r^{\alpha_r-j}\right) - 2\phi\left(\frac{n}{p_r^{\alpha_r}}\right) \\
&= \frac{n}{p_r^{\alpha_r}} + (p_r^{\alpha_r-1} - 2)\phi\left(\frac{n}{p_r^{\alpha_r}}\right) - 1.\end{aligned}$$

Hence by Lemma 2.3.2(ii), the proof follows. □

Notation 2.3.17. We denote the upper bound of $\kappa(\mathcal{G}(\mathbb{Z}_n))$ obtained in Theorem 2.3.6 and Theorem 2.3.16 by $\xi_1(n)$ and $\xi_2(n)$, respectively.

In the following theorem, we compare the upper bounds $\xi_1(n)$ and $\xi_2(n)$ of $\kappa(\mathcal{G}(\mathbb{Z}_n))$ in the following theorem.

Theorem 2.3.18. *Suppose n is not a product of two primes and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_i \in \mathbb{N}$ for $1 \leqslant i \leqslant r$.*

(i) *$\xi_2(n) = \xi_1(n)$ if and only if $\alpha_r = 1$, or $r = 2$ and $p_1 = 2$.*

(ii) *$\xi_2(n) < \xi_1(n)$ if and only if $\alpha_r \geqslant 2$ and $\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i}\right) < \frac{1}{2}$.*

(iii) *$\xi_2(n) > \xi_1(n)$ if and only if $\alpha_r \geqslant 2$ and $\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i}\right) > \frac{1}{2}$.*

Proof. Note that

$$\begin{aligned}
\xi_2(n) - \xi_1(n) &= \frac{n}{p_r^{\alpha_r}} + (p_r^{\alpha_r - 1} - 2)\phi\left(\frac{n}{p_r^{\alpha_r}}\right) - \left\{\frac{n}{p_r} - p_r^{\alpha_r - 1}\phi\left(\frac{n}{p_r^{\alpha_r}}\right)\right\} \\
&= \left(1 - p_r^{\alpha_r - 1}\right)\frac{n}{p_r^{\alpha_r}} + 2(p_r^{\alpha_r - 1} - 1)\phi\left(\frac{n}{p_r^{\alpha_r}}\right) \\
&= (p_r^{\alpha_r - 1} - 1)\left\{2\phi\left(\frac{n}{p_r^{\alpha_r}}\right) - \frac{n}{p_r^{\alpha_r}}\right\} \\
&= (p_r^{\alpha_r - 1} - 1)\frac{n}{p_r^{\alpha_r}}\left\{2\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i}\right) - 1\right\}. \qquad (2.9)
\end{aligned}$$

The right hand side of (2.9) equals 0 if and only if $\alpha_r = 1$, or

$$2\prod_{i=1}^{r-1}\left(\frac{p_i - 1}{p_i}\right) = 1. \qquad (2.10)$$

We show that (2.10) holds if and only if $r = 2$ and $p_1 = 2$.

If $p_1 > 2$, then $\prod_{i=1}^{r-1} p_i$ is odd. Since $2\prod_{i=1}^{r-1}(p_i - 1)$ is always even, (2.10) does not hold. Thus $p_1 = 2$. If $r > 2$, then

$$2\prod_{i=1}^{r-1}\left(\frac{p_i - 1}{p_i}\right) = \prod_{i=2}^{r-1}\left(\frac{p_i - 1}{p_i}\right) \neq 1$$

as the numerator is even and denominator is odd. So we must have $r = 2$. Conversely, if $r = 2$ and $p_1 = 2$, then (2.10) holds. This proofs (i).

Since $p_r^{\alpha_r - 1} - 1 > 0$ if and only if $\alpha_r \geqslant 2$, (ii) and (iii) follow from (2.9). □

Remark 2.3.19. The *lexicographic order* $\prec$ on $\mathbb{N} \times \mathbb{N}$, defined by $(a_1, b_1) \prec (a_2, b_2)$ if

$$a_1 < a_2, \text{ or } a_1 = a_2 \text{ and } b_1 < b_2,$$

is a well-founded relation.

We now give the actual values of $\kappa(\mathcal{G}(\mathbb{Z}_n))$ when n has two prime factors (cf. Theorem 2.3.20) and n is a product of three distinct primes (cf. Theorem 2.3.22).

Theorem 2.3.20. *If $n = p^\alpha q^\beta$, where p, q are distinct primes and $\alpha, \beta \in \mathbb{N}$, then*

$$\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + p^{\alpha-1}q^{\beta-1}. \tag{2.11}$$

In fact, for $n \neq pq$, $\langle \overline{pq} \rangle^*$ is a minimum separating set of $\mathcal{G}'(\mathbb{Z}_n)$.

Proof. We consider the lexicographic order $\prec$ on $\mathbb{N} \times \mathbb{N}$, and prove by applying the principle of well-founded induction that (2.11) holds for all $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$. Note that, as n is not a prime power, $\mathcal{G}(\mathbb{Z}_n)$ and hence $\mathcal{G}'(\mathbb{Z}_n)$ are not complete graphs.

Since $\kappa(\mathcal{G}(\mathbb{Z}_{pq})) = \phi(pq) + 1$ (cf. Theorem 1.5.19), the statement holds for $(\alpha, \beta) = (1, 1)$.

Now take $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ such that $(1, 1) \prec (\alpha, \beta)$. Suppose that (2.11) holds for all $(a, b) \prec (\alpha, \beta)$. Then $n \neq pq$ and hence by Proposition 2.3.3, $\Gamma := \mathcal{G}'(\mathbb{Z}_n)$ is connected. Further, by Theorem 2.3.4, $\langle \overline{pq} \rangle^*$ is a minimal separating set of Γ. We show that $\langle \overline{pq} \rangle^*$ is a minimum separating set of Γ.

Let T be a minimal separating set of Γ. We show that $|\langle \overline{pq} \rangle^*| \leqslant |T|$. If $\langle \overline{pq} \rangle^* \subseteq T$, we are done. So let $\langle \overline{pq} \rangle^* \nsubseteq T$. Then there exists an element $\overline{a} \in \langle \overline{pq} \rangle^*$ such that $\overline{a} \notin T$. Let $\Gamma_1 = \mathcal{G}(\langle \overline{p} \rangle^*)$ and $\Gamma_2 = \mathcal{G}(\langle \overline{q} \rangle^*)$. Then observe that $\Gamma = \Gamma_1 \cup \Gamma_2$, and hence $\Gamma - T = (\Gamma_1 - T) \cup (\Gamma_2 - T)$. Further, $\overline{a} \in V(\Gamma_1 - T) \cap V(\Gamma_2 - T)$. Hence as $\Gamma - T$ is disconnected, at least one of $\Gamma_1 - T$ or $\Gamma_2 - T$ is disconnected.

Case 1: Let $\Gamma_1 - T$ be disconnected. If $\alpha = 1$, then $|\langle \overline{p} \rangle| = q^\beta$ and hence Γ_1 is a complete graph. So $\Gamma_1 - T$ cannot be disconnected. So $\alpha \geqslant 2$. Then,

$$\begin{aligned}
|T| - |\langle \overline{pq} \rangle^*| &\geqslant \kappa(\Gamma_1) - |\langle \overline{pq} \rangle^*| \\
&= \kappa(\mathcal{G}(\langle \overline{p} \rangle^*)) - |\langle \overline{pq} \rangle^*| \\
&= \kappa(\mathcal{G}(\langle \overline{p} \rangle) - \overline{0}) - |\langle \overline{pq} \rangle^*| \\
&= \kappa(\mathcal{G}(\langle \overline{p} \rangle)) - 1 - (|\langle \overline{pq} \rangle| - 1) \text{ (by Remark 1.5.11)} \\
&= \kappa(\mathcal{G}(\mathbb{Z}_{p^{\alpha-1}q^{\beta}})) - |\langle \overline{pq} \rangle| \\
&= \phi(p^{\alpha-1}q^{\beta}) + p^{\alpha-2}q^{\beta-1} - \left(p^{\alpha-1}q^{\beta-1}\right) \text{ (by induction hypothesis)} \\
&= p^{\alpha-2}q^{\beta-1}(p-1)(q-1) + p^{\alpha-2}q^{\beta-1} - p^{\alpha-1}q^{\beta-1} \\
&= p^{\alpha-2}q^{\beta-1}\{(p-1)(q-1) + 1 - p\} \\
&= p^{\alpha-2}q^{\beta-1}(p-1)(q-2) \geqslant 0. \qquad (2.12)
\end{aligned}$$

Case 2: Let $\Gamma_2 - T$ be disconnected. Proceeding as in Case 1, we have $\beta \geqslant 2$, and

$$\begin{aligned}
|T| - |\langle \overline{pq} \rangle^*| &\geqslant \kappa(\Gamma_2) - |\langle \overline{pq} \rangle^*| \\
&= \kappa(\mathcal{G}(\langle \overline{q} \rangle^*)) - |\langle \overline{pq} \rangle^*| \\
&= p^{\alpha-1}q^{\beta-2}\{(p-1)(q-1) + 1 - q\} \\
&\geqslant p^{\alpha-1}q^{\beta-2}\{(q-1) + 1 - q\} = 0. \qquad (2.13)
\end{aligned}$$

So for $(1,1) \prec (\alpha, \beta)$, $\langle \overline{pq} \rangle^*$ is a minimum separating set of $\mathcal{G}'(\mathbb{Z}_n)$ and hence $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + p^{\alpha-1}q^{\beta-1}$. Therefore by the principle of well-founded induction, (2.11) holds for all $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$. □

The following is a simple consequence of Theorem 2.3.20.

Corollary 2.3.21. *If $n = 2^{\alpha}p^{\beta}$, where p is an odd prime and $\alpha, \beta \in \mathbb{N}$, then $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \frac{n}{2}$.*

Proof. $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 2^{\alpha-1}p^{\beta-1} = 2^{\alpha-1}p^{\beta-1}((2-1)(p-1)+1) = 2^{\alpha-1}p^{\beta} = \frac{n}{2}$. □

In the following result, we obtain the vertex connectivity of $\mathcal{G}(\mathbb{Z}_n)$ when n is a product of three distinct primes.

Theorem 2.3.22. *If $n = pqr$, where $p < q < r$ are primes, then $[\overline{pr}] \cup [\overline{qr}]$ is a minimum separating set of $\mathcal{G}'(\mathbb{Z}_n)$. Consequently,*

$$\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + p + q - 1.$$

Proof. It follows from Lemma 2.3.1 that the equivalence classes of $\mathbb{Z}'_n$ with respect to $\approx$ are $[\overline{p}]$, $[\overline{q}]$, $[\overline{r}]$, $[\overline{pq}]$, $[\overline{pr}]$ and $[\overline{qr}]$.

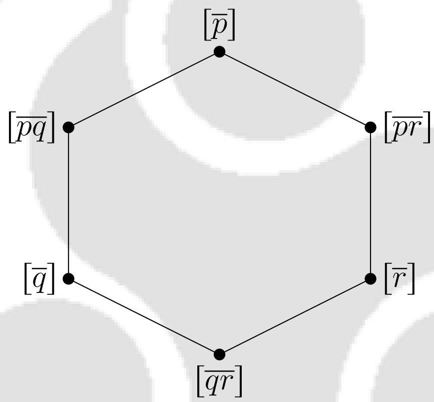

FIGURE 2.1: $\widetilde{\mathcal{G}}'(\mathbb{Z}_{pqr})$

Using Remark 2.1.12, $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$ is depicted in Figure 2.1. It is evident from Figure 2.1 that $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$ does not become disconnected by deletion of any one vertex ($\approx$-class), whereas, deletion of any two non-adjacent vertices disconnects $\widetilde{\mathcal{G}}'(\mathbb{Z}_n)$. Hence by Lemma 2.3.12(iii), a minimal separating set of $\mathcal{G}'(\mathbb{Z}_n)$ is precisely the union of any two non-adjacent $\approx$-classes (in Figure 2.1). By Lemma 2.3.1(ii), we have $|[\overline{p}]| = (q-1)(r-1)$, $|[\overline{q}]| = (p-1)(r-1)$, $|[\overline{r}]| = (p-1)(q-1)$, $|[\overline{pq}]| = r-1$, $|[\overline{pr}]| = q-1$ and $|[\overline{qr}]| = p-1$. We thus have the following inequalities:

$$|[\overline{p}]| > |[\overline{q}]| > |[\overline{r}]|, |[\overline{pq}]| > |[\overline{pr}]| > |[\overline{qr}]|, |[\overline{r}]| > |[\overline{pr}]|. \tag{2.14}$$

Consequently, $[\overline{pr}] \cup [\overline{qr}]$ is of minimum cardinality among the union of non-adjacent pairs of $\approx$-classes. Hence $[\overline{pr}] \cup [\overline{qr}]$ is a minimum separating set of $\mathcal{G}'(\mathbb{Z}_n)$. Since $|[\overline{pr}] \cup [\overline{qr}]| = p+q-2$, by Lemma 2.3.2(ii), $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n)+p+q-1$. □

2.4 Conclusion

In this chapter, we obtained certain minimum separating sets of $\mathcal{G}(\mathbb{Z}_n)$ and supplied two upper bounds $\xi_1(n)$ and $\xi_2(n)$ of $\kappa(\mathcal{G}(\mathbb{Z}_n))$. We further proved that $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \xi_1(n)$ when n has two prime factors or n is a product of three primes, thus by establishing that it is a sharp bound.

Recently, we found that Chattopadhyay et al. [2018b] have further shown that for n having prime factors $p_1 < p_2 < \cdots < p_r$ $(r \geqslant 2)$, if $\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i}\right) \geqslant \frac{1}{2}$, then $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \xi_1(n)$. Moreover, if $r = 3$ and $2\phi(p_1p_2) < p_1p_2$, then $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \xi_2(n)$. Hence computation of $\kappa(\mathcal{G}(\mathbb{Z}_n))$ when $r \geqslant 4$ and $\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i}\right) < \frac{1}{2}$ is still open for study.

In addition to the above, investigation of vertex connectivity of $\mathcal{G}(G)$ when G is finite non-cyclic group is also an interesting problem. To this end, we refer the recent work of Chattopadhyay et al. [2018a] on finite and non-cyclic nilpotent groups.

3

p-Groups

This chapter investigates structural properties of power graphs of p-groups. In Section 3.1, we show that in every component of proper power graph of a p-group, elements of order p are adjacent to all other elements. Consequently, we show that these components have exactly $p-1$ elements of order p. Applying this, we find the number of components of proper power graph of an abelian p-group. We then describe the structure of power graph of a p-group G in Section 3.2 and show that $\mathcal{G}(G)$ can be presented iteratively as join and disjoint union of some of its subgraphs. Finally, we derive the complete structure of power graphs of groups that are direct product of two cyclic p-groups.

3.1 Components

Throughout this chapter, G denotes a p-group. In this section, we study some properties of components of proper power graphs of p-groups. If C is one such component, we show that C has exactly $p-1$ elements of order p and that all other vertices in C are adjacent to them. We then compute the number of components of proper power graph of an abelian p-group.

Notation 3.1.1. Let G be a group and $g \in G$. We denote $U(g) = \{h \in G : g \in \langle h \rangle\}$ and let $\Gamma(g)$ be the subgraph of $\mathcal{G}(G)$ induced by the set of vertices $U(g)$. Moreover, we denote the component of $\mathcal{G}^*(G)$ containing g by $C(g)$.

Proposition 3.1.2. *If $x \in G^*$ is an element of order p, then x is adjacent to every other vertex of $C(x)$.*

Proof. Let C be the component of $\mathcal{G}^*(G)$ that contains x. Consider $y \in V(C)$, $y \neq x$. We show that x is adjacent to y. Note that there exists at least one x, y-path in $\mathcal{G}^*(G)$; say $x = x_0, x_1, \ldots, x_m = y$. We claim that for all $1 \leqslant i \leqslant m$,

$$x \in \langle x_i \rangle. \tag{3.1}$$

As x and x_1 are adjacent, either $x \in \langle x_1 \rangle$ or $x_1 \in \langle x \rangle$. If $x \in \langle x_1 \rangle$, then (3.1) holds for $i = 1$. Now let $x_1 \in \langle x \rangle$. Since $o(x) = p$, we have $\langle x \rangle = \langle x_1 \rangle$. So, again (3.1) holds for $i = 1$.

Suppose $x \in \langle x_k \rangle$ for some $1 \leqslant k \leqslant m-1$. We show that $x \in \langle x_{k+1} \rangle$. From adjacency of x_k and x_{k+1}, we have $x_k \in \langle x_{k+1} \rangle$ or $x_{k+1} \in \langle x_k \rangle$. If $x_k \in \langle x_{k+1} \rangle$, we get $x \in \langle x_{k+1} \rangle$, by induction hypothesis. Thus (3.1) holds for $i = k+1$.

Now take $x_{k+1} \in \langle x_k \rangle$. Then $x_{k+1} = x_k^{c_{k+1} p^{\alpha_{k+1}}}$ for some $c_{k+1} \in \mathbb{N}$, $(c_{k+1}, p) = 1$ and $\alpha_{k+1} \in \mathbb{N} \cup \{0\}$. Hence

$$\langle x_{k+1} \rangle = \left\langle x_k^{p^{\alpha_{k+1}}} \right\rangle \tag{3.2}$$

If $\alpha_{k+1} = 0$, then $\langle x_{k+1} \rangle = \langle x_k \rangle$. So $x \in \langle x_{k+1} \rangle$ follows from induction hypothesis.

Now let $\alpha_{k+1} > 0$. As $x \in \langle x_k \rangle$, $x = x_k^{c_k p^{\alpha_k}}$ for some $c_k \in \mathbb{N}$, $(c_k, p) = 1$ and $\alpha_k \in \mathbb{N} \cup \{0\}$. Hence $\langle x \rangle = \langle x_k^{p^{\alpha_k}} \rangle$. If $\alpha_k = 0$, then $\langle x \rangle = \langle x_k \rangle$. This along with (3.2) imply that $\langle x_{k+1} \rangle = \langle x^{p^{\alpha_{k+1}}} \rangle$. Because $\mathrm{o}(x) = p$ and $\alpha_{k+1} > 0$, we get $\langle x_{k+1} \rangle = \langle e \rangle$, which is a contradiction. Thus $\alpha_k > 0$. Since $\mathrm{o}(x) = p$ and $\langle x \rangle = \langle x_k^{p^{\alpha_k}} \rangle$, we get $\mathrm{o}(x_k) = p^{\alpha_k + 1}$. Moreover, if $\mathrm{o}(x_{k+1}) = p^{\beta}$, from (3.2) we get $\mathrm{o}(x_k) = p^{\alpha_{k+1} + \beta}$. Hence, using the fact that $\beta \geqslant 1$, we get $\alpha_k \geqslant \alpha_{k+1}$. Then $\langle x \rangle = \langle x_k^{p^{\alpha_k}} \rangle \subseteq \langle x_k^{p^{\alpha_{k+1}}} \rangle = \langle x_{k+1} \rangle$, so that $x \in \langle x_{k+1} \rangle$. Therefore, (3.1) holds for $i = k + 1$.

We thus conclude that $x \in \langle x_i \rangle$ for all $1 \leqslant i \leqslant n$. In particular, $x \in \langle y \rangle$, so that x is adjacent to y. □

Proposition 3.1.3. *If g is an element of order p in G, then $C(g) = \Gamma(g)$.*

Proof. Since both $\Gamma(g)$ and $C(g)$ are induced subgraphs of $\mathcal{G}(G)$, we only need to show that their vertex sets are equal. Clearly $U(g) \subseteq V(C(g))$. To show the reverse inclusion, let $h \in V(C(g))$. By Proposition 3.1.2, g is adjacent to every other vertex of $C(g)$. So g is adjacent to h and since $\mathrm{o}(g)$ is prime, we have $g \in \langle h \rangle$. Hence $h \in U(g)$. □

Proposition 3.1.4. *In $\mathcal{G}^*(G)$, each component has exactly $p - 1$ elements of order p.*

Proof. Let C be a component of $\mathcal{G}^*(G)$. Let $x \in V(C)$ and $\mathrm{o}(x) = p^{\gamma}$ for some $\gamma \in \mathbb{N}$. Then $y = x^{p^{\gamma-1}}$ is an element of order p in $V(C)$. Hence by Proposition 3.1.3, C has at least $p - 1$ vertices of order p and for any $z \in V(C)$, $z \neq y$ of order p, $y \in \langle z \rangle$. Since both y and z have prime orders, we get $\langle y \rangle = \langle z \rangle$. Hence C has exactly $p - 1$ vertices of order p. □

Before calculating the number of components of $\mathcal{G}^*(G)$ for any abelian p-group G, we note that G is essentially an unique direct product of cyclic p-groups (cf. Theorem 1.3.26).

Theorem 3.1.5. *Let G be an abelian p-group. If G is isomorphic to a direct product of r cyclic groups, then the number of components of $\mathcal{G}^*(G)$ is $p^{r-1} + p^{r-2} + \ldots + 1$.*

Proof. Let G be isomorphic to $H := H_1 \times H_2 \times \ldots \times H_r$, where $H_1, H_2, \ldots, H_r$ are cyclic p-groups. Then it is enough to prove the above statement for $\mathcal{G}^*(H)$.

For any $1 \leqslant i \leqslant r$, H_i has $p-1$ elements of order p (cf. Theorem 1.3.24), and if $(x_1, x_2, \ldots, x_r) \in H$, then $\mathrm{o}((x_1, x_2, \ldots, x_r)) = \mathrm{lcm}(\mathrm{o}(x_1), \mathrm{o}(x_2), \ldots, \mathrm{o}(x_r))$ (cf. Theorem 1.3.25). So H has $p^r - 1$ elements of order p. Hence by Proposition 3.1.4, the number of components of $\mathcal{G}^*(H)$ is $\dfrac{p^r-1}{p-1} = p^{r-1} + p^{r-2} + \ldots + 1$. □

It follows from Theorem 3.1.5 that the proper power graph of a non-cyclic abelian p-group has more than one component. Thus we have the following corollary.

Corollary 3.1.6. *If G is a non-cyclic abelian p-group, then $k(\mathcal{G}(G)) = 1$.*

3.2 Structure

In this section, we investigate the structure of power graphs of p-groups by introducing the notion of a primitive class. We first present a method to obtain the structure of power graphs of abelian p-groups. We then present the complete structure of power graphs of groups that are product of two cyclic p-groups. We begin by introducing the notion of a primitive class.

Definition 3.2.1. For $g, h \in G$, $[g]$ is called a *primitive class* of $[h]$ if $g \neq e$ and $[h] = [g^p]$. Equivalently, we also say that $[g]$ is a primitive class of h.

Notation 3.2.2. We denote the number of primitive classes of g and $[g]$ by $\pi(g)$ and $\pi([g])$, respectively.

The proof of the following lemma is straightforward.

Lemma 3.2.3. *If $g \in G$ and $\pi(g) = 0$, then $\Gamma(g) = \mathcal{G}([g]) \cong K_{\phi(\mathrm{o}(g))}$.*

The following theorem iteratively presents the structure of power graph of a p-group.

Theorem 3.2.4. *Let $g \in G$ and $\pi(g) > 0$. If the distinct primitive classes of g are $[g_1], [g_2], \ldots, [g_{\pi(g)}]$, then*

$$\Gamma(g) \cong K_{\phi(\mathrm{o}(g))} \vee \left\{\Gamma(g_1) + \Gamma(g_2) + \ldots + \Gamma(g_{\pi(g)})\right\}. \tag{3.3}$$

In particular, for $g = e$,

$$\mathcal{G}(G) \cong K_1 \vee \left\{\Gamma(g_1) + \Gamma(g_2) + \ldots + \Gamma(g_{\pi(e)})\right\}. \tag{3.4}$$

Proof. Let $\mathrm{o}(g) = p^k$ and h be a vertex in $\Gamma(g)$. Then $p^k | \mathrm{o}(h)$. If $\mathrm{o}(h) = p^k$, then $h \in [g]$. Now let $\mathrm{o}(h) = p^l$ for some $l \geqslant k+1$. Then $h_1 := h^{p^{l-k-1}}$ has order p^{k+1}, so that $[h_1]$ is a primitive class of $[g]$. As a result, $[h_1] = [g_i]$ for some $1 \leqslant i \leqslant \pi(g)$, and hence $h \in U(g_i)$. So we have $U(g) = [g] \cup U(g_1) \cup U(g_2) \cup \ldots \cup U(g_{\pi(g)})$.

Let $\pi(g) \geqslant 2$ and $1 \leqslant i, j \leqslant \pi(g)$, $i \neq j$. If $g_i, g_j \in \langle h \rangle$ for some $h \in G$, then $[g_i] = [g_j]$; a contradiction. Thus $\Gamma(g_i)$ and $\Gamma(g_j)$ have disjoint vertices. Now, if possible, suppose $u_i \in U(g_i)$ is adjacent to $u_j \in U(g_j)$ in $\mathcal{G}(G)$. Then $u_i \in \langle u_j \rangle$ or $u_j \in \langle u_i \rangle$. Without loss of generality, taking $u_i \in \langle u_j \rangle$, we get $u_j \in U(g_i)$. Since $\Gamma(g_i)$ and $\Gamma(g_j)$ have disjoint vertices, this is not possible.

In addition to the above, each element of $[g]$ is adjacent to every other element of $\Gamma(g)$, so that $\Gamma(g) = \mathcal{G}([g]) \vee \left\{\Gamma(g_1) + \Gamma(g_2) + \ldots + \Gamma(g_{\pi(g)})\right\}$. Hence (3.3) follows from this and the fact that $\mathcal{G}([g]) \cong K_{\phi(\mathrm{o}(g))}$. Additionally, since $\mathcal{G}(G) = \Gamma(e)$, (3.4) follows. □

Remark 3.2.5. The structure of $\mathcal{G}(G)$ can be obtained (up to isomorphism) as described below. Starting with (3.4), for every $1 \leqslant i \leqslant \pi(e)$, we substitute for $\Gamma(g_i)$ from (3.3) and Lemma 5.4.4 when $\pi(g_i) > 0$ and $\pi(g_i) = 0$, respectively. Further, if $\pi(g_i) > 0$ for any $1 \leqslant i \leqslant \pi(e)$, we do the similar substitution for $\Gamma(h_{i,j})$ for every primitive class $[h_{i,j}]$ of g_i. We continue this process till the substitution for $\Gamma(h)$ for every $\approx$-class $[h]$, $h \neq e$ is done.

For the rest of this chapter, let α and β be positive integers. Since $\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta} \cong \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^\alpha}$, without loss of generality, let $\alpha \geqslant \beta$. For notational convenience, for any $(a, b) \in \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}$, we denote $\Gamma((a, b))$ simply by $\Gamma(a, b)$. In what follows, we determine the complete structure of $\mathcal{G}(\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta})$.

For the rest of this section, the underlying group of the results presented is $\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}$.

Lemma 3.2.6. *For integers $0 \leqslant k \leqslant \alpha$, $0 \leqslant l \leqslant \beta$ and a, b satisfying $\gcd(a, p) = \gcd(b, p) = 1$, $[(\overline{ap^k}, \overline{b})]$ and $[(\overline{a}, \overline{bp^l})]$ have no primitive class.*

Proof. If $p\overline{c} = \overline{b}$ for some $\overline{c} \in \mathbb{Z}_{p^\beta}$, then $p \,|\, (pc - b)$. This implies $p|b$, which contradicts the fact that $\gcd(b, p) = 1$. So $[(\overline{ap^k}, \overline{b})]$ has no primitive class. Similarly, $[(\overline{a}, \overline{bp^l})]$ also has no primitive class. □

Lemma 3.2.7. *For integers $1 \leqslant k \leqslant \alpha$ and $1 \leqslant l \leqslant \beta$, the primitive classes (not necessarily distinct) of $[(\overline{p^k}, \overline{p^l})]$ are given by $[((ap^{\alpha-k} + 1)\overline{p^{k-1}}, (bp^{\beta-l} + 1)\overline{p^{l-1}})]$, $a = 0, \ldots, p-1$, $b = 0, \ldots, p-1$ excluding $[(\overline{0}, \overline{0})]$.*

Proof. Let $(\overline{p^k}, \overline{p^l}) = p(\overline{w}, \overline{z})$. Then $p^\alpha | (pw - p^k) \Rightarrow p^{\alpha-1} | (w - p^{k-1})$, so that $w = ap^{\alpha-1} + p^{k-1}$ for some integer a. Hence the distinct solutions of $p\overline{w} = \overline{p^k}$ (in variable $\overline{w}$) are $\overline{w} = (ap^{\alpha-k} + 1)\overline{p^{k-1}}$, $a = 0, \ldots, p-1$. Analogously, the distinct solutions of $p\overline{z} = \overline{p^l}$ (in variable $\overline{z}$) are $\overline{z} = (bp^{\beta-l} + 1)\overline{p^{l-1}}$, $b = 0, \ldots, p-1$. Thus the proof follows. □

Lemma 3.2.8. *Consider integers $0 \leqslant k \leqslant \alpha$ and $0 \leqslant l \leqslant \beta$ satisfying $\alpha - k \geqslant \beta - l$. For integers $0 \leqslant b_1, b_2 \leqslant p-1$, $b_1 \neq b_2$, the classes $[(\overline{p^{k-1}}, (b_1 p^{\beta-l} + 1)\overline{p^{l-1}})]$ and $[(\overline{p^{k-1}}, (b_2 p^{\beta-l} + 1)\overline{p^{l-1}})]$ are distinct.*

Proof. If possible, let

$$(\overline{p^{k-1}}, (b_1 p^{\beta-l} + 1)\overline{p^{l-1}}) = c(\overline{p^{k-1}}, (b_2 p^{\beta-l} + 1)\overline{p^{l-1}}) \tag{3.5}$$

for some integer c satisfying $\gcd(c,p)=1$.

Then equating first components, $c = c_1p^{\alpha-k+1}+1$ for some integer c_1. Moreover, $\alpha-k+l \geqslant \beta-l+l=\beta$. So that in $\mathbb{Z}_{p^\beta}$,

$$\begin{aligned} c(b_2p^{\beta-l}+1)\overline{p^{l-1}} &= c_1\overline{p^{\alpha-k+l}} + (b_2p^{\beta-l}+1)\overline{p^{l-1}} \\ &= (b_2p^{\beta-l}+1)\overline{p^{l-1}}. \end{aligned} \tag{3.6}$$

From this and equating second components of (3.5), we get $b_1\overline{p^{\beta-1}} = b_2\overline{p^{\beta-1}}$. Without loss of generality, let $b_1 \geqslant b_2$, so that $p|(b_1-b_2)$. Since $0 \leqslant b_1, b_2 \leqslant p-1$, we get $b_1=b_2$. This being a contradiction, the given classes are distinct. □

Lemma 3.2.8, is not necessarily true when $\alpha-k<\beta-l$. For example, in $\mathbb{Z}_{16}\times\mathbb{Z}_8$, $5(\overline{4},\overline{1}) = (\overline{4},\overline{5})$, so that $[(\overline{4},\overline{1})] = [(\overline{4},\overline{5})]$.

Lemma 3.2.9. *The distinct primitive classes of* $[(\overline{0},\overline{0})]$ *are given by* $[(\overline{0},\overline{p^{\beta-1}})]$ *and* $[(\overline{p^{\alpha-1}},\overline{bp^{\beta-1}})]$, $b=0,1,\ldots,p-1$.

Proof. It follows from Lemma 3.2.7 that the primitive classes (not necessarily distinct) of $[(\overline{0},\overline{0})]$ are given by $[(\overline{ap^{\alpha-1}},\overline{bp^{\beta-1}})]$, $a=0,1,\ldots,p-1$, $b=0,1,\ldots,p-1$, $a+b\neq 0$. For $a=0$, $[(\overline{0},\overline{bp^{\beta-1}})] = [(\overline{0},\overline{p^{\beta-1}})]$ for all $1\leqslant b\leqslant p-1$. Now let $1\leqslant a\leqslant p-1$. Then there exist integers c_1, c_2 such that $ac_1+c_2p=1$. Here, since $(c_1,p)=1$ as well,

$$\begin{aligned} [(\overline{ap^{\alpha-1}},\overline{bp^{\beta-1}})] &= [c_1(\overline{ap^{\alpha-1}},\overline{bp^{\beta-1}})] \\ &= [(\overline{p^{\alpha-1}},\overline{dp^{\beta-1}})], \text{ where } d = bc_1 (\mathrm{mod}\ p). \end{aligned}$$

So the primitive classes of $[(\overline{0},\overline{0})]$ are given by $[(\overline{0},\overline{p^{\beta-1}})]$ and $[(\overline{p^{\alpha-1}},\overline{bp^{\beta-1}})]$, $b=0,1,\ldots,p-1$. Now by Lemma 3.2.8, the classes $[(\overline{p^{\alpha-1}},\overline{bp^{\beta-1}})]$, $b=0,1,\ldots,p-1$ are all distinct. Moreover, it is easy to show that for any $0\leqslant b\leqslant p-1$, $[(\overline{0},\overline{p^{\beta-1}})]$ and $[(\overline{p^{\alpha-1}},\overline{bp^{\beta-1}})]$ are distinct. Hence the proof follows. □

The proof of following theorem is analogous to that of Lemma 3.2.9.

Lemma 3.2.10. *The distinct primitive classes of* $[(\overline{0},\overline{0})]$ *are given by* $[(\overline{p^{\alpha-1}},\overline{0})]$ *and* $[(\overline{ap^{\alpha-1}},\overline{p^{\beta-1}})]$, $a=0,1,\ldots,p-1$.

Lemma 3.2.11. *Suppose* $\alpha>1$ *and consider integers* $1\leqslant k<\alpha$ *and* $1\leqslant l\leqslant\beta$ *satisfying* $\alpha-k\geqslant\beta-l$. *Then the distinct primitive classes of* $[(\overline{p^k},\overline{p^l})]$ *are* $[(\overline{p^{k-1}},(bp^{\beta-l}+1)\overline{p^{l-1}})]$, $b=0,1,\ldots,p-1$.

In particular, the distinct primitive classes of $[(\overline{p^k},\overline{0})]$ *are given by* $[(\overline{p^{k-1}},b\overline{p^{\beta-1}})]$, $b=0,1,\ldots,p-1$.

Proof. We first show that for integers $0\leqslant a_1,a_2\leqslant p-1$,

$$[((a_1p^{\alpha-k}+1)\overline{p^{k-1}},(a_2p^{\beta-l}+1)\overline{p^{l-1}})]=[(\overline{p^{k-1}},(bp^{\beta-l}+1)\overline{p^{l-1}})] \tag{3.7}$$

for some integer $0\leqslant b\leqslant p-1$.

Since $k<\alpha$, for any integer $0\leqslant b\leqslant p-1$,

$$\begin{aligned}(a_1p^{\alpha-k}+1)&\big(\overline{p^{k-1}},(bp^{\beta-l}+1)\overline{p^{l-1}}\big)\\&=\big((a_1p^{\alpha-k}+1)\overline{p^{k-1}},(a_1p^{\alpha-k}+bp^{\beta-l}+1)\overline{p^{l-1}}\big).\end{aligned} \tag{3.8}$$

Thus, if $\alpha-k>\beta-l$, then by setting $b=a_2$,

$$\begin{aligned}(a_1p^{\alpha-k}+1)&(\overline{p^{k-1}},(bp^{\beta-l}+1)\overline{p^{l-1}})\\&=((a_1p^{\alpha-k}+1)\overline{p^{k-1}},(a_2p^{\beta-l}+1)\overline{p^{l-1}}).\end{aligned} \tag{3.9}$$

Now let $\alpha-k=\beta-l$. If $a_2\geqslant a_1$, take $b=a_2-a_1$, otherwise $b=a_2-a_1+p$. Then by (3.8),

$$\begin{aligned}(a_1p^{\alpha-k}+1)&(\overline{p^{k-1}},(bp^{\beta-l}+1)\overline{p^{l-1}})\\&=((a_1p^{\alpha-k}+1)\overline{p^{k-1}},\{(a_1+b)p^{\beta-l}+1\}\overline{p^{l-1}})\\&=((a_1p^{\alpha-k}+1)\overline{p^{k-1}},(a_2p^{\beta-l}+1)\overline{p^{l-1}}).\end{aligned} \tag{3.10}$$

Since $k < \alpha$, we have $\gcd(a_1 p^{\alpha-k} + 1, p) = 1$. Consequently, (3.9) and (3.10) together yield (3.7).

By applying Lemma 3.2.7 and (3.7), the primitive classes of $[(\overline{p^k}, \overline{p^l})]$ are $[(\overline{p^{k-1}}, (bp^{\beta-l} + 1)\overline{p^{l-1}})]$, $b = 0, \ldots, p-1$, and by Lemma 3.2.8, they are all distinct. Hence the proof follows. The particular case follows by taking $l = \beta$. □

By arguments analogous to that of Lemma 3.2.11, we have the following lemma.

Lemma 3.2.12. *Suppose $\beta > 1$ and consider integers $1 \leqslant k \leqslant \alpha$ and $1 \leqslant l < \beta$ satisfying $\alpha - k \leqslant \beta - l$. Then the distinct primitive classes of $[(\overline{p^k}, \overline{p^l})]$ are $[((ap^{\alpha-k} + 1)\overline{p^{k-1}}, \overline{p^{l-1}})]$, $a = 0, 1, \ldots, p-1$.*

In particular, the distinct primitive classes of $[(\overline{0}, \overline{p^l})]$ are given by $[(\overline{ap^{\alpha-1}}, \overline{p^{l-1}})]$, $a = 0, 1, \ldots, p-1$.

Lemma 3.2.13. *For integers $0 \leqslant k \leqslant \alpha - 1$ and $0 \leqslant l \leqslant \beta - 1$, and a, b satisfying $\gcd(a, p) = \gcd(b, p) = 1$,*

(i) $\Gamma(\overline{ap^k}, \overline{bp^l}) = \Gamma(\overline{cp^k}, \overline{p^l}) = \Gamma(\overline{p^k}, \overline{dp^l})$ *for some c, d satisfying* $\gcd(c, p) = \gcd(d, p) = 1$;

(ii) $\Gamma(\overline{ap^k}, \overline{0}) = \Gamma(\overline{p^k}, \overline{0})$ *and* $\Gamma(\overline{0}, \overline{bp^l}) = \Gamma(\overline{0}, \overline{p^l})$.

Proof. We first notice that if $[g] = [h]$, then $\Gamma(g) = \Gamma(h)$.

(i) As $\gcd(a, p) = 1$, there exist integers a_1 and a_2 such that $a_1 a + a_2 p^{\alpha-k} = 1$. So $a_1(\overline{ap^k}, \overline{bp^l}) = (\overline{p^k}, \overline{dp^l})$, where $d = a_1 b (\text{mod } p^{\beta-l})$. Additionally, since $\gcd(a_1, p) = 1$, we have $[(\overline{ap^k}, \overline{bp^l})] = [(\overline{p^k}, \overline{dp^l})]$. Similarly, $[(\overline{ap^k}, \overline{bp^l})] = [(\overline{cp^k}, \overline{p^l})]$ for some integer c satisfying $\gcd(c, p) = 1$. Hence the proof follows.

(ii) The proof follows from the fact that $[(\overline{ap^k}, \overline{0})] = [(\overline{p^k}, \overline{0})]$ and $[(\overline{0}, \overline{bp^l})] = [(\overline{0}, \overline{p^l})]$. □

Proposition 3.2.14. *For integers $0 \leqslant k \leqslant \alpha - 1$ and $0 \leqslant l \leqslant \beta - 1$, and a, b satisfying $\gcd(a, p) = \gcd(b, p) = 1$, we have*

$$\Gamma(\overline{ap^k}, \overline{bp^l}) \cong \Gamma(\overline{p^k}, \overline{p^l}).$$

Proof. We prove the proposition for $k \geqslant l$ and $\alpha - k \geqslant \beta - l$. By similar procedure, other cases can also be shown. In view of Lemma 3.2.13(i), it is enough to show that $\Gamma(\overline{p^k}, \overline{bp^l}) \cong \Gamma(\overline{p^k}, \overline{p^l})$.

In view of Lemma 3.2.3 and Lemma 3.2.6,

$$\Gamma(\overline{p^{k-l}}, \overline{1}) \cong K_{\phi(p^{\alpha-k+l})} \cong \Gamma(\overline{p^{k-l}}, \overline{b}). \tag{3.11}$$

Thus, if $l = 0$, then the proof is complete.

Now suppose $l > 0$. By applying well-founded induction on $0 \leqslant m \leqslant l$, we show that

$$\Gamma(\overline{p^{k-l+m}}, \overline{p^m}) \cong \Gamma(\overline{p^{k-l+m}}, \overline{bp^m}). \tag{3.12}$$

It follows from (3.11) that (3.12) holds for $m = 0$. So let (3.12) hold for $m = j-1$ for some $1 \leqslant j \leqslant l$, that is,

$$\Gamma(\overline{p^{k-l+j-1}}, \overline{p^{j-1}}) \cong \Gamma(\overline{p^{k-l+j-1}}, \overline{bp^{j-1}}).$$

We now show it for $m = j$.

Using Theorem 3.2.4 and Lemma 3.2.11, we get

$$\Gamma(\overline{p^{k-l+j}}, \overline{bp^j}) \cong K_{\phi(p^{\alpha-k+l-j})} \vee \Big\{ \sum_{c=0}^{p-1} \Gamma(\overline{p^{k-l+j-1}}, \overline{b(cp^{\beta-j}+1)p^{j-1}}) \Big\}.$$

By induction hypothesis, we have

$$\Gamma(\overline{p^{k-l+j}}, \overline{bp^j}) \cong K_{\phi(p^{\alpha-k+l-j})} \vee p\Gamma(\overline{p^{k-l+j-1}}, \overline{p^{j-1}}).$$

Since this holds for any b satisfying $(b,p) = 1$, we have $\Gamma(\overline{p^{k-l+j}}, \overline{p^j}) \cong \Gamma(\overline{p^{k-l+j}}, \overline{bp^j})$.

Hence $\Gamma(\overline{p^{k-l+m}}, \overline{bp^m}) \cong \Gamma(\overline{p^{k-l+m}}, \overline{p^m})$ for all $0 \leqslant m \leqslant l$. Finally, for $m = l$, we have $\Gamma(\overline{p^k}, \overline{bp^l}) \cong \Gamma(\overline{p^k}, \overline{p^l})$. $\square$

Theorem 3.2.15. *For any integer* $0 \leqslant l \leqslant \beta - 1$, $\Gamma(\overline{p^{\alpha-\beta+k_l}}, \overline{p^l}) \cong \Gamma(\overline{0}, \overline{p^l})$ *for all integers* k_l *satisfying* $l \leqslant k_l \leqslant \beta - 1$. *In particular,* $\Gamma(\overline{p^{\alpha-1}}, \overline{p^{\beta-1}}) \cong \Gamma(\overline{0}, \overline{p^{\beta-1}})$.

Proof. Observe that $\Gamma(\overline{p^{\alpha-\beta+k_0}}, \overline{1}) \cong \Gamma(\overline{0}, \overline{1}) \cong K_{p^\beta}$. So, if $\beta = 1$, then the proof is complete. Now suppose that $\beta > 1$.

We prove the theorem by applying well-founded induction on $0 \leqslant l \leqslant \beta - 1$. As shown above, the statement holds for $l = 0$.

Suppose that it holds for some $l = m$, $0 \leqslant m < \beta - 1$, that is, $\Gamma(\overline{p^{\alpha-\beta+k_m}}, \overline{p^m}) \cong \Gamma(\overline{0}, \overline{p^m})$ for all k_m, $m \leqslant k_m \leqslant \beta - 1$. We next show it for $l = m + 1$. Using Theorem 3.2.4, Lemma 3.2.11 and Proposition 3.2.14 we have

$$\Gamma(\overline{p^{\alpha-\beta+k_{m+1}}}, \overline{p^{m+1}}) \cong K_{\phi(p^{\beta-m-1})} \vee p\,\Gamma(\overline{p^{\alpha-\beta+k_{m+1}-1}}, \overline{p^m}). \tag{3.13}$$

Further, using Theorem 3.2.4, Lemma 3.2.12 and Proposition 3.2.14, we have

$$\Gamma(\overline{0}, \overline{p^{m+1}}) \cong K_{\phi(p^{\beta-m-1})} \vee \{\Gamma(\overline{0}, \overline{p^m}) + (p-1)\,\Gamma(\overline{p^{\alpha-1}}, \overline{p^m})\}.$$

By induction hypothesis, $\Gamma(\overline{p^{\alpha-1}}, \overline{p^m}) \cong \Gamma(\overline{0}, \overline{p^m})$, so that

$$\Gamma(\overline{0}, \overline{p^{m+1}}) \cong K_{\phi(p^{\beta-m-1})} \vee p\,\Gamma(\overline{0}, \overline{p^m}). \tag{3.14}$$

Since $k_{m+1} - 1 \geqslant m$, it follows from induction hypothesis that

$$\Gamma(\overline{p^{\alpha-\beta+k_{m+1}-1}}, \overline{p^m}) = \Gamma(\overline{0}, \overline{p^m}).$$

Therefore, we conclude from (3.13) and (3.14) that

$$\Gamma(\overline{p^{\alpha-\beta+k_{m+1}}}, \overline{p^{m+1}}) = \Gamma(\overline{0}, \overline{p^{m+1}}).$$

Consequently, the proof of first part of the theorem follows. Now, setting $l = k_l = \beta - 1$, we get $\Gamma(p^{\alpha-1}, p^{\beta-1}) \cong \Gamma(\overline{0}, \overline{p^{\beta-1}})$. □

We now present three theorems which together give the complete structure of $\mathcal{G}(\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta})$.

Theorem 3.2.16. $\mathcal{G}(\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}) \cong K_1 \vee \left\{\Gamma\left(\,\overline{p^{\alpha-1}},\overline{0}\,\right) + p\Gamma\left(\,\overline{p^{\alpha-1}},\overline{p^{\beta-1}}\,\right)\right\}.$

Proof. Using Theorem 3.2.4 and Lemma 3.2.9,

$$\mathcal{G}(\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}) \cong K_1 \vee \left\{\sum_{b=0}^{p-1}\Gamma\left(\,\overline{p^{\alpha-1}},\overline{bp^{\beta-1}}\,\right) + \Gamma\left(\,\overline{0},\overline{p^{\beta-1}}\,\right)\right\}.$$

Hence the proof follows from Proposition 3.2.14 and Theorem 3.2.15. □

Theorem 3.2.17. *For any* $1 \leqslant k \leqslant \alpha - 1$,

$$\Gamma\left(\,\overline{p^k},\overline{0}\,\right) \cong K_{\phi(p^{\alpha-k})} \vee \left\{\Gamma\left(\,\overline{p^{k-1}},\overline{0}\,\right) + (p-1)\Gamma\left(\,\overline{p^{k-1}},\overline{p^{\beta-1}}\,\right)\right\} \tag{3.15}$$

and $\Gamma\left(\,\overline{1},\overline{0}\,\right) \cong K_{\phi(p^\alpha)}$.

Proof. By Theorem 3.2.4 and Lemma 3.2.11,

$$\Gamma\left(\,\overline{p^k},\overline{0}\,\right) \cong K_{\phi(p^{\alpha-k})} \vee \left\{\sum_{b=0}^{p-1}\Gamma\left(\,\overline{p^{k-1}},\overline{bp^{\beta-1}}\,\right)\right\}.$$

Hence (3.15) follows from Proposition 3.2.14 and Theorem 3.2.15. Moreover, by Lemma 3.2.3 and Lemma 3.2.6, $\Gamma\left(\,\overline{1},\overline{0}\,\right) \cong K_{\phi(p^\alpha)}$. □

Theorem 3.2.18. *Let* $1 \leqslant k \leqslant \alpha - 1$, $1 \leqslant l \leqslant \beta - 1$ *and* $\alpha - k \geqslant \beta - l$.

$$\Gamma\left(\,\overline{p^k},\overline{p^l}\,\right) \cong K_{\phi(p^{\alpha-k})} \vee p\Gamma\left(\,\overline{p^{k-1}},\overline{p^{l-1}}\,\right), \tag{3.16}$$

and

$$\Gamma\left(\,\overline{p^{k-m}},\overline{p^{l-m}}\,\right) \cong K_{\phi(p^{\alpha-k+m})}, \tag{3.17}$$

where $m = \min\{k, l\}$.

Proof. By Theorem 3.2.4 and Lemma 3.2.11,

$$\Gamma(\,\overline{p^k}, \overline{p^l}\,) \cong K_{\phi(p^{\alpha-k})} \vee \left\{ \sum_{b=0}^{p-1} \Gamma(\,\overline{p^{k-1}}, (bp^{\beta-l}+1)\overline{p^{l-1}}\,) \right\}.$$

Hence (3.21) follows from Proposition 3.2.14. In view of Lemma 3.2.3 and Lemma 3.2.6, $\Gamma(\,\overline{p^{k-m}}, \overline{p^{l-m}}\,) \cong K_{\phi(p^{\alpha-k+m})}$. □

Remark 3.2.19. We obtain the structure of $\Gamma(\,\overline{p^{\alpha-1}}, \overline{p^{\beta-1}}\,)$ by recursive application of Theorem 3.2.18. Further, we get the structure of $\Gamma(\,\overline{p^{\alpha-1}}, \overline{0}\,)$ by recursive application of Theorem 3.2.17 and Theorem 3.2.18. Consequently, using Theorem 3.2.16, we obtain the structure of $\mathcal{G}(\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta})$.

Theorem 3.2.20. *For any integer* $0 \leqslant k \leqslant \beta - 1$, $\Gamma(\overline{p^k}, \overline{p^l}) \cong \Gamma(\overline{p^k}, \overline{0})$ *for all* l *satisfying* $k \leqslant l \leqslant \beta - 1$. *In particular, for any* $0 \leqslant k \leqslant \beta - 1$, $\Gamma(\overline{p^k}, \overline{p^{\beta-1}}) \cong \Gamma(\overline{p^k}, \overline{0})$.

Proof. We show that $\Gamma(\overline{p^k}, \overline{p^{l_k}}) \cong \Gamma(\overline{p^k}, \overline{0})$ for all l_k satisfying $k \leqslant l_k \leqslant \beta - 1$. We first observe that $\Gamma(\overline{1}, \overline{p^{l_0}}) \cong \Gamma(\overline{1}, \overline{0}) \cong K_{p^\alpha}$ for all $0 \leqslant l_0 \leqslant \beta - 1$. So, if $\beta = 1$, the proof is complete. So, suppose $\beta \geqslant 2$.

We prove by applying well-founded induction on $0 \leqslant k \leqslant \beta - 1$. As shown above, the statement holds for $k = 0$. Suppose the assertion holds for $k < m$, where $1 \leqslant m \leqslant \beta - 1$. We next show that the statement holds for $k = m$.

Observe that,

$$\Gamma(\overline{p^m}, \overline{p^{l_m}}) = K_{\phi(p^{\alpha-m})} \vee \{p\,\Gamma(\overline{p^{m-1}}, \overline{p^{l_m-1}})\}. \tag{3.18}$$

and

$$\Gamma(\overline{p^m}, \overline{0}) = K_{\phi(p^{\alpha-m})} \vee \{\Gamma(\overline{p^{m-1}}, \overline{0}) + (p-1)\,\Gamma(\overline{p^{m-1}}, \overline{p^{\beta-1}})\}.$$

By induction hypothesis, $\Gamma(\overline{p^{m-1}}, \overline{0}) \cong \Gamma(\overline{p^{m-1}}, \overline{p^{\beta-1}})$. Hence

$$\Gamma(\overline{p^m}, \overline{0}) \cong K_{\phi(p^{\alpha-m})} \vee \{p\,\Gamma(\overline{p^{m-1}}, \overline{0})\} \tag{3.19}$$

Moreover, since $m-1 \leqslant l_m - 1$, $\Gamma(\overline{p^{m-1}}, \overline{p^{l_m-1}}) \cong \Gamma(\overline{p^{m-1}}, \overline{0})$. Hence we conclude from (3.18) and (3.19) that $\Gamma(\overline{p^m}, \overline{p^{l_m}}) \cong \Gamma(\overline{p^m}, \overline{0})$. This concludes the proof. □

By applying Theorem 3.2.20, we have the following corollary of Theorem 3.2.16.

Corollary 3.2.21. $\mathcal{G}(\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\alpha}) \cong K_1 \vee (p+1)\Gamma\left(\overline{p^{\alpha-1}}, \overline{p^{\alpha-1}}\right)$.

Additionally, The following is a trivial consequence of Theorem 3.2.18.

Corollary 3.2.22. *For any integer* $1 \leqslant k \leqslant \alpha - 1$,

$$\Gamma\left(\overline{p^k}, \overline{p^k}\right) \cong K_{\phi(p^{\alpha-k})} \vee p\Gamma\left(\overline{p^{k-1}}, \overline{p^{k-1}}\right), \tag{3.20}$$

and

$$\Gamma\left(\overline{1}, \overline{1}\right) \cong K_{\phi(p^\alpha)}, \tag{3.21}$$

where $m = \max\{k, l\}$.

By procedure similar to that of Remark 3.2.19, Corollary 3.2.21 and 3.2.22 give the complete structure of $\mathcal{G}(\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\alpha})$.

3.3 Conclusion

For any p-group G, when G is neither cyclic nor generalized quaternion, $\mathcal{G}^*(G)$ is disconnected (cf. Lemma 1.5.17). Hence finding the number of components of $\mathcal{G}^*(G)$ becomes an interesting problem. In this chapter, we addressed this problem for all abelian p-groups. Thus one may target to obtain it for the rest of the groups. Further, we presented the complete structure of power graphs of groups that are product of two cyclic p-groups. So one may aim to generalize the above study to product of three or more cyclic p-groups. In fact, considering Theorem 1.3.26, this would mean that finding it for all abelian p-groups.

4

Minimum Degree and Connectivity

In this chapter, we study the minimum degree of power graphs of finite groups and determine its connection with connectivity. In Section 4.1, we ascertain that the edge connectivity and the minimum degree of power graphs of finite groups are equal. In Section 4.2, we first compute the minimum degree of power graph of $\mathbb{Z}_n$ when n has two prime factors or n is a product of at most four distinct primes. Followed by this, we determine the minimum degree of power graphs of abelian p-groups, D_n and Q_n. Along with minimum degree, we also obtain minimum disconnecting sets of power graphs of these groups. In Section 4.3, we characterize the equality of vertex connectivity and minimum degree of power graphs of $\mathbb{Z}_n$ and abelian p-groups. We further address the equality for D_n and Q_n.

4.1 Edge connectivity

In this section, we show that edge connectivity and minimum degree of power graphs of finite groups are equal. As a result, we present a simple way to find the minimum disconnecting sets of power graphs in terms of the neighbourhoods of the vertices having minimum degree.

In view of Theorem 1.2.21, we have the following inequality for power graphs of finite groups.

Lemma 4.1.1. *If G is a finite group, then $\kappa(\mathcal{G}(G)) \leqslant \kappa'(\mathcal{G}(G)) \leqslant \delta(\mathcal{G}(G))$.*

Let G be a finite group with identity element e. If $|G| \leqslant 2$, then trivially $\kappa'(\mathcal{G}(G)) = \delta(\mathcal{G}(G))$. If $|G| \geqslant 3$, then every pair g, h of distinct vertices is connected by the path g, e, h of length two in $\mathcal{G}(G)$. Combining this fact with Theorem 1.2.22, we obtain the following.

Theorem 4.1.2. *If G is a finite group, then $\kappa'(\mathcal{G}(G)) = \delta(\mathcal{G}(G))$.*

Let G be a finite group, $|G| > 1$ and $g \in G$ be such that $\delta(\mathcal{G}(G)) = \deg(g)$. Then observe that $E[g, N(g)]$ is a disconnecting set of $\mathcal{G}(G)$. Moreover, by Theorem 4.1.2, we have $|E[g, N(g)]| = \deg(g) = \kappa'(\mathcal{G}(G))$. Thus we have the following lemma.

Lemma 4.1.3. *Let G be a finite group, $|G| > 1$ and $g \in G$ be such that $\delta(\mathcal{G}(G)) = \deg(g)$. Then $E[g, N(g)]$ is a minimum disconnecting set of $\mathcal{G}(G)$.*

4.2 Minimum degree

In this section, we find the minimum degree and minimum disconnecting sets of power graphs of $\mathbb{Z}_n$, abelian p-groups, D_n and Q_n in respective subsections.

We begin with the following lemma which gives a sufficient condition for equality of degrees of elements of power graphs of finite groups.

Lemma 4.2.1. *Let G be a finite group and $g_1, g_2 \in G$. If $\langle g_1 \rangle = \langle g_2 \rangle$, then* $\deg(g_1) = \deg(g_2)$ *in $\mathcal{G}(G)$.*

Proof. Note that, if $\langle g_1 \rangle = \langle g_2 \rangle$, then g_1 and g_2 are powers of each other in G. In particular, g_1 and g_2 are adjacent in $\mathcal{G}(G)$. Now, any $h \in G$, distinct from both g_1 and g_2, is adjacent to g_1 if and only if it is adjacent to g_2 in $\mathcal{G}(G)$. Hence $\deg(g_1) = \deg(g_2)$. $\square$

The converse of Lemma 4.2.1 need not be true though. For example, in $\mathcal{G}(\mathbb{Z}_{12})$, $\deg(\overline{2}) = \deg(\overline{6}) = 9$, but $\langle \overline{2} \rangle \neq \langle \overline{6} \rangle$.

We deduce the following lemma by applying Theorem 1.5.7(ii).

Lemma 4.2.2. *For a finite group G, $\delta(\mathcal{G}(G)) = |G| - 1$ if and only if G is a cyclic group of order 1 or p^α for some prime number p and $\alpha \in \mathbb{N}$.*

4.2.1 Finite cyclic group

In this subsection, We observe that $\delta(\mathcal{G}(\mathbb{Z}_n))$ is the degree of one of the proper divisors of n (cf. Lemma 4.2.3). We show that $\phi(n) + 1$ is a sharp lower bound for $\delta(\mathcal{G}(\mathbb{Z}_n))$. Further, we obtain some inequalities involving degrees of various elements of $\mathbb{Z}_n$ (cf. Proposition 4.2.6). Subsequently, we determine $\delta(\mathcal{G}(\mathbb{Z}_n))$ when n has two prime factors or n is a product of at most four distinct primes (cf. Theorem 4.2.7). We conclude this subsection by giving two sharp upper bounds of $\delta(\mathcal{G}(\mathbb{Z}_n))$.

Lemma 4.2.3. *If $n \in \mathbb{N}$ is a composite number, then there exists a proper divisor $c > 1$ of n such that $\delta(\mathcal{G}(\mathbb{Z}_n)) = \deg(\overline{c})$.*

Proof. Since n is a composite number, $\mathbb{Z}_n'$ is non-empty. Note that $\deg(\overline{a}) \leqslant n - 1$ for all $\overline{a} \in \mathbb{Z}_n$. Moreover, by Remark 1.5.14, $\deg(\overline{b}) = n - 1$ for all $\overline{b} \in \mathcal{S}(\mathbb{Z}_n)$. Thus there exists $\overline{a} \in \mathbb{Z}_n'$ such that $\delta(\mathcal{G}(\mathbb{Z}_n)) = \deg(\overline{a})$. Now take $c = \gcd(a, n)$. Then $c \mid n$, $1 < c < n$ and from Theorem 1.3.22, we have $\langle \overline{c} \rangle = \langle \overline{a} \rangle$. Hence by Lemma 4.2.1, $\delta(\mathcal{G}(\mathbb{Z}_n)) = \deg(\overline{c})$. $\square$

Theorem 4.2.4. *For any integer $n > 1$,*

(i) *If n is a composite number, then $\delta(\mathcal{G}(\mathbb{Z}_n)) = \phi(n)+1+\delta(\mathcal{G}'(\mathbb{Z}_n))$. Consequently, $\delta(\mathcal{G}(\mathbb{Z}_n)) \geqslant \phi(n) + 1$.*

(ii) *$\delta(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1$ if and only if $n = 2p$ for some prime $p \geqslant 3$.*

Proof. (i) Let $n \in \mathbb{N}$ be a composite number. In view of Lemma 4.2.3, it is enough to consider only elements of $\mathbb{Z}'_n$. Since each $\overline{a} \in \mathbb{Z}'_n$ is adjacent to all elements of $\mathcal{S}(\mathbb{Z}_n)$ and $|\mathcal{S}(\mathbb{Z}_n)| = \phi(n) + 1$, we have $\deg_{\mathcal{G}(\mathbb{Z}_n)}(\overline{a}) = \deg_{\mathcal{G}'(\mathbb{Z}_n)}(\overline{a}) + \phi(n) + 1$. Thus the proof follows.

(ii) Let $n = 2p$ for some prime $p \geqslant 3$. Then by Proposition 2.3.3, $\mathcal{G}'(\mathbb{Z}_n)$ is disconnected, and its component induced by $\langle\overline{p}\rangle^*$ has $\overline{p}$ as its only vertex. Consequently, $\delta(\mathcal{G}'(\mathbb{Z}_n)) = 0$, and hence by (i), $\delta(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1$.

Conversely, let $\delta(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1$. If n is a prime number, then $\delta(\mathcal{G}(\mathbb{Z}_n)) = n - 1 \neq \phi(n) + 1$. So n is a composite number. Then by (i), $\delta(\mathcal{G}'(\mathbb{Z}_n)) = 0$, and hence $\mathcal{G}'(\mathbb{Z}_n)$ is disconnected. Accordingly, by Proposition 2.3.3, n is a product of two distinct primes; say $n = pq$. It is easy to see that subgraphs induced by $\langle p\rangle^*$ and $\langle q\rangle^*$ are the only components of $\mathcal{G}'(\mathbb{Z}_n)$. If both p and q are odd primes, then $|\langle p\rangle^*|, |\langle q\rangle^*| \geqslant 2$ and hence $\delta(\mathcal{G}'(\mathbb{Z}_n)) \geqslant 1$. As this is a contradiction, exactly one of p or q is 2. Hence taking $q = 2$, we get $n = 2p$. □

The following lemma gives us a formula to compute degrees of vertices of $\mathbb{Z}'_n$ in $\mathcal{G}(\mathbb{Z}_n)$.

Lemma 4.2.5 ([Moghaddamfar et al., 2014])**.** *Suppose the integer $n > 1$ is not a prime power and $\overline{a} \in \mathbb{Z}'_n$. If $b = \gcd(a, n)$, then in $\mathcal{G}(\mathbb{Z}_n)$,*

$$\deg(\overline{a}) = \frac{n}{b} + \sum_{d|b, d\neq b} \phi\left(\frac{n}{d}\right) - 1.$$

We now establish some inequalities involving degrees of vertices of $\mathcal{G}(\mathbb{Z}_n)$.

Proposition 4.2.6. *Suppose* $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ *where* $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ *are primes and* $\alpha_i \in \mathbb{N}$ *for* $1 \leqslant i \leqslant r$. *Then the following inequalities hold in* $\mathcal{G}(\mathbb{Z}_n)$.

(i) $\deg\left(\overline{p_1^{\alpha_1}}\right) \geqslant \deg\left(\overline{p_r^{\alpha_r}}\right)$.

(ii) $\deg\left(\overline{p_i^{\gamma}}\right) \geqslant \deg\left(\overline{p_i^{\beta}}\right)$ *for any* $1 \leqslant i \leqslant r$ *and* $1 \leqslant \gamma < \beta \leqslant \alpha_i$.

(iii) $\deg\left(\overline{p_i^{\beta}}\right) \geqslant \deg\left(\overline{p_j^{\beta}}\right)$ *for any* $1 \leqslant i < j \leqslant r$ *and* $1 \leqslant \beta \leqslant \min\{\alpha_i, \alpha_j\}$.

(iv) $\deg\left(\overline{p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r}}\right) \geqslant \deg\left(\overline{p_2^{\beta_2} \dots p_r^{\beta_r}}\right)$ *for* $\sum_{i=1}^{r} \beta_i < \sum_{i=1}^{r} \alpha_i$, *where* $1 \leqslant \beta_i \leqslant \alpha_i$ *for any* $1 \leqslant i \leqslant r$.

Proof. (i) Let $m = \dfrac{n}{p_1^{\alpha_1} p_r^{\alpha_r}}$.

$$\begin{aligned}
\deg\left(\overline{p_1^{\alpha_1}}\right) - \deg\left(\overline{p_r^{\alpha_r}}\right) &= \frac{n}{p_1^{\alpha_1}} + \sum_{k=0}^{\alpha_1-1} \phi\left(\frac{n}{p_1^k}\right) - \left\{\frac{n}{p_r^{\alpha_r}} + \sum_{k=0}^{\alpha_r-1} \phi\left(\frac{n}{p_r^k}\right)\right\} \\
&= \phi(m)\left\{\phi(p_r^{\alpha_r}) \sum_{k=0}^{\alpha_1-1} \phi\left(p_1^{\alpha_1-k}\right) - \phi(p_1^{\alpha_1}) \sum_{k=0}^{\alpha_r-1} \phi\left(p_r^{\alpha_r-k}\right)\right\} \\
&\qquad + m\left(p_r^{\alpha_r} - p_1^{\alpha_1}\right) \\
&= \phi(m)\left\{(p_r^{\alpha_r} - p_r^{\alpha_r-1})(p_1^{\alpha_1} - 1) - (p_1^{\alpha_1} - p_1^{\alpha_1-1})(p_r^{\alpha_r} - 1)\right\} \\
&\qquad + m\{p_1^{\alpha_1}(p_r^{\alpha_r} - 1) - p_r^{\alpha_r}(p_1^{\alpha_1} - 1)\} \\
&= (p_r^{\alpha_r} - 1)\left[p_1^{\alpha_1-1}\phi(m) + p_1^{\alpha_1}\{m - \phi(m)\}\right] \\
&\qquad - (p_1^{\alpha_1} - 1)\left[p_r^{\alpha_r-1}\phi(m) + p_r^{\alpha_r}\{m - \phi(m)\}\right] \\
&\geqslant (p_r^{\alpha_r} - 1)\left[p_1^{\alpha_1-1}\phi(m) + p_1^{\alpha_1}\{m - \phi(m)\}\right] \\
&\qquad - p_1^{\alpha_1}\left[p_r^{\alpha_r-1}\phi(m) + p_r^{\alpha_r}\{m - \phi(m)\}\right] \\
&= (p_r^{\alpha_r} p_1^{\alpha_1-1} - p_1^{\alpha_1} p_r^{\alpha_r-1})\phi(m) - \left[p_1^{\alpha_1-1}\phi(m) + p_1^{\alpha_1}\{m - \phi(m)\}\right] \\
&= p_1^{\alpha_1-1}\left[p_r^{\alpha_r-1}(p_r - p_1)\phi(m) - [\phi(m) + p_1\{m - \phi(m)\}]\right] \\
&\geqslant p_1^{\alpha_1-1}\left[(p_r - p_1)\phi(m) - [\phi(m) + p_1\{m - \phi(m)\}]\right] \\
&= p_1^{\alpha_1-1}\left\{(p_r - 1)\phi(m) - p_1 m\right\}. \qquad (4.1)
\end{aligned}$$

Now, if $r = 2$, then $m = \dfrac{n}{p_1^{\alpha_1} p_2^{\alpha_2}} = 1$. Hence from (4.1), we get

$$\deg\left(\overline{p_1^{\alpha_1}}\right) - \deg\left(\overline{p_r^{\alpha_r}}\right) \geqslant p_1^{\alpha_1 - 1}\left(p_r - 1 - p_1\right) \geqslant 0.$$

If $r > 2$, then $m = \dfrac{n}{p_1^{\alpha_1} p_r^{\alpha_r}} = \prod_{i=2}^{r-1} p_i^{\alpha_i}$. Again, from (4.1), we get

$$\deg\left(\overline{p_1^{\alpha_1}}\right) - \deg\left(\overline{p_r^{\alpha_r}}\right) \geqslant \prod_{i=1}^{r-1} p_i^{\alpha_i - 1}\left\{\prod_{i=2}^{r}(p_i - 1) - \prod_{i=1}^{r-1} p_i\right\} \geqslant 0,$$

since $p_{i+1} - 1 \geqslant p_i$ for all $1 \leqslant i \leqslant r-1$.

(ii) $\deg\left(\overline{p_i^{\gamma}}\right) - \deg\left(\overline{p_i^{\beta}}\right)$

$$\begin{aligned}
&= \frac{n}{p_i^{\gamma}} + \sum_{k=0}^{\gamma-1} \phi\left(\frac{n}{p_i^k}\right) - \frac{n}{p_i^{\beta}} - \sum_{k=0}^{\beta-1} \phi\left(\frac{n}{p_i^k}\right) \\
&= \frac{n}{p_i^{\gamma}} - \frac{n}{p_i^{\beta}} - \sum_{k=\gamma}^{\beta-1} \phi\left(\frac{n}{p_i^k}\right) \\
&= \frac{n}{p_i^{\alpha_i}}\left(p_i^{\alpha_i-\gamma} - p_i^{\alpha_i-\beta}\right) - \phi\left(\frac{n}{p_i^{\alpha_i}}\right)\sum_{k=\gamma}^{\beta-1} \phi\left(p_i^{\alpha_i-k}\right) \\
&= \frac{n}{p_i^{\alpha_i}}\left(p_i^{\alpha_i-\gamma} - p_i^{\alpha_i-\beta}\right) - \phi\left(\frac{n}{p_i^{\alpha_i}}\right)\left(p_i^{\alpha_i-\gamma} - p_i^{\alpha_i-\beta}\right) \\
&= \left(p_i^{\alpha_i-\gamma} - p_i^{\alpha_i-\beta}\right)\left\{\frac{n}{p_i^{\alpha_i}} - \phi\left(\frac{n}{p_i^{\alpha_i}}\right)\right\} \geqslant 0, \text{ since } \gamma < \beta.
\end{aligned}$$

(iii) $\deg\left(\overline{p_i^{\beta}}\right) - \deg\left(\overline{p_j^{\beta}}\right) = \dfrac{n}{p_i^{\beta}} - \dfrac{n}{p_j^{\beta}} + \sum_{k=0}^{\beta-1}\left\{\phi\left(\dfrac{n}{p_i^k}\right) - \phi\left(\dfrac{n}{p_j^k}\right)\right\}.$

Since $p_i < p_j$, we have $\dfrac{n}{p_i^{\beta}} > \dfrac{n}{p_j^{\beta}}$. Further $\alpha_i, \alpha_j \geqslant \beta$, so that for all $0 \leqslant k \leqslant \beta - 1$, we have $\phi\left(\dfrac{n}{p_i^k}\right) - \phi\left(\dfrac{n}{p_j^k}\right) = \left(\dfrac{n}{p_i^k} - \dfrac{n}{p_j^k}\right)\prod_{l=1}^{r}\left(1 - \dfrac{1}{p_l}\right) \geqslant 0$. Hence the proof follows.

(iv) $\deg\left(\overline{p_1^{\beta_1}p_2^{\beta_2}\dots p_r^{\beta_r}}\right) - \deg\left(\overline{p_2^{\beta_2}\dots p_r^{\beta_r}}\right)$

$$= \frac{n}{p_1^{\beta_1}p_2^{\beta_2}\dots p_r^{\beta_r}} + \sum_{\substack{0\leqslant \sum_{i=1}^{r}\gamma_i < \sum_{i=1}^{r}\beta_i,\\ 0\leqslant\gamma_i\leqslant\beta_i\,\forall\,1\leqslant i\leqslant r}} \phi\left(\frac{n}{p_1^{\gamma_1}p_2^{\gamma_2}\dots p_r^{\gamma_r}}\right)$$
$$-\left\{\frac{n}{p_2^{\beta_2}\dots p_r^{\beta_r}} + \sum_{\substack{0\leqslant \sum_{i=2}^{r}\gamma_i < \sum_{i=2}^{r}\beta_i,\\ 0\leqslant\gamma_i\leqslant\beta_i\,\forall\,2\leqslant i\leqslant r}} \phi\left(\frac{n}{p_2^{\gamma_2}\dots p_r^{\gamma_r}}\right)\right\}$$
$$= \frac{n}{p_1^{\beta_1}p_2^{\beta_2}\dots p_r^{\beta_r}} + \sum_{0\leqslant\gamma_i\leqslant\beta_i\,\forall\,1\leqslant i\leqslant r} \phi\left(\frac{n}{p_1^{\gamma_1}p_2^{\gamma_2}\dots p_r^{\gamma_r}}\right) - \phi\left(\frac{n}{p_1^{\beta_1}p_2^{\beta_2}\dots p_r^{\beta_r}}\right)$$
$$-\left\{\frac{n}{p_2^{\beta_2}\dots p_r^{\beta_r}} + \sum_{0\leqslant\gamma_i\leqslant\beta_i\,\forall\,2\leqslant i\leqslant r} \phi\left(\frac{n}{p_2^{\gamma_2}\dots p_r^{\gamma_r}}\right) - \phi\left(\frac{n}{p_2^{\beta_2}\dots p_r^{\beta_r}}\right)\right\}$$
$$= \sum_{\substack{1\leqslant\gamma_1\leqslant\beta_1,\\ 0\leqslant\gamma_i\leqslant\beta_i\,\forall\,2\leqslant i\leqslant r}} \phi\left(\frac{n}{p_1^{\gamma_1}p_2^{\gamma_2}\dots p_r^{\gamma_r}}\right) + \phi(m)\left\{\phi(p_1^{\alpha_1}) - \phi(p_1^{\alpha_1-\beta_1})\right\}$$
$$+ m\left(p_1^{\alpha_1-\beta_1} - p_1^{\alpha_1}\right)\ \left(\text{by setting } m = p_2^{\alpha_2-\beta_2}\dots p_r^{\alpha_r-\beta_r}\right)$$
$$\geqslant \left\{\sum_{0\leqslant\gamma_i\leqslant\beta_i\,\forall\,2\leqslant i\leqslant r} \phi\left(p_2^{\alpha_2-\gamma_2}\dots p_r^{\alpha_r-\gamma_r}\right)\right\} \sum_{1\leqslant\gamma_1\leqslant\beta_1} \phi\left(p_1^{\alpha_1-\gamma_1}\right) + m\left(p_1^{\alpha_1-\beta_1} - p_1^{\alpha_1}\right)$$
$$\geqslant \phi\left(p_2^{\alpha_2}\dots p_r^{\alpha_r}\right) \sum_{1\leqslant\gamma_1\leqslant\beta_1} \phi\left(p_1^{\alpha_1-\gamma_1}\right) + m\left(p_1^{\alpha_1-\beta_1} - p_1^{\alpha_1}\right)$$
$$= \prod_{i=2}^{r}\left\{p_i^{\alpha_i-1}(p_i-1)\right\} \sum_{1\leqslant\gamma_1\leqslant\beta_1} \phi\left(p_1^{\alpha_1-\gamma_1}\right) - m\left(p_1^{\alpha_1} - p_1^{\alpha_1-\beta_1}\right)$$
$$\geqslant m\left\{\prod_{i=2}^{r}(p_i-1) \sum_{1\leqslant\gamma_1\leqslant\beta_1} \phi\left(p_1^{\alpha_1-\gamma_1}\right) - \left(p_1^{\alpha_1} - p_1^{\alpha_1-\beta_1}\right)\right\}.$$

Thus it is enough to show that

$$\rho := \prod_{i=2}^{r}(p_i-1) \sum_{1\leqslant\gamma_1\leqslant\beta_1} \phi\left(p_1^{\alpha_1-\gamma_1}\right) - \left(p_1^{\alpha_1} - p_1^{\alpha_1-\beta_1}\right) \geqslant 0.$$

We have $\beta_1 \leqslant \alpha_1$. First take $\alpha_1 = \beta_1$. Then $\sum_{1\leqslant\gamma_1\leqslant\beta_1} \phi\left(p_1^{\alpha_1-\gamma_1}\right) = p_1^{\alpha_1-1}$, so that

$$\rho = \prod_{i=2}^{r}(p_i - 1)p_1^{\alpha_1 - 1} - (p_1^{\alpha_1} - 1)$$
$$\geqslant (p_2 - 1)p_1^{\alpha_1 - 1} - p_1^{\alpha_1} + 1$$
$$\geqslant p_1^{\alpha_1} - p_1^{\alpha_1} + 1 > 0.$$

Now we take $\alpha_1 > \beta_1$. In this case, $\sum\limits_{1 \leqslant \gamma_1 \leqslant \beta_1} \phi\left(p_1^{\alpha_1 - \gamma_1}\right) = p_1^{\alpha_1 - 1} - p_1^{\alpha_1 - \beta_1 - 1}$, so that

$$\rho = (p_2 - 1)\dots(p_r - 1)\left(p_1^{\alpha_1 - 1} - p_1^{\alpha_1 - \beta_1 - 1}\right) - \left(p_1^{\alpha_1} - p_1^{\alpha_1 - \beta_1}\right)$$
$$= \left(p_1^{\alpha_1 - 1} - p_1^{\alpha_1 - \beta_1 - 1}\right)\{(p_2 - 1)\dots(p_r - 1) - p_1\} \geqslant 0.$$

□

Theorem 4.2.7. *Let $n \in \mathbb{N}$ and $p_1 < p_2 < p_3 < p_4$ be prime numbers.*

(i) *If $n = p_1^{\alpha_1} p_2^{\alpha_2}$, $\alpha_1, \alpha_2 \in \mathbb{N}$, then $\overline{p_2^{\alpha_2}}$ has the minimum degree among all vertices in $\mathcal{G}(\mathbb{Z}_n)$, and $\delta(\mathcal{G}(\mathbb{Z}_n)) = (p_2^{\alpha_2} - 1)\phi(p_1^{\alpha_1}) + p_1^{\alpha_1} - 1$.*

(ii) *If $n = p_1p_2p_3$, then $\overline{p_3}$ has the minimum degree among all vertices in $\mathcal{G}(\mathbb{Z}_n)$, and $\delta(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + p_1p_2 - 1$.*

(iii) *Let $n = p_1p_2p_3p_4$. If n is odd or $p_4 \geqslant p_3 + \dfrac{2(p_3 - 1)}{p_2 - 1}$, then $\overline{p_4}$ has the minimum degree among all vertices in $\mathcal{G}(\mathbb{Z}_n)$, and $\delta(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + p_1p_2p_3 - 1$. Otherwise, $\overline{p_3p_4}$ has the minimum degree among all vertices in $\mathcal{G}(\mathbb{Z}_n)$, and $\delta(\mathcal{G}(\mathbb{Z}_n)) = (p_2 - 1)(p_3p_4 + 1) + 1$.*

Proof. In view of Lemma 4.2.3, to determine $\delta(\mathcal{G}(\mathbb{Z}_n))$, it is sufficient to compare the degrees of vertices of the form $\overline{c}$, where $c > 1$ is a proper divisor of n.

(i) Consider $\beta_1, \beta_2 \in \mathbb{N}$ satisfying $1 \leqslant \beta_i \leqslant \alpha_i$ for $i = 1, 2$. Using Proposition 4.2.6, we have

(a) $\deg\left(\overline{p_1^{\beta_1}}\right) \geqslant \deg\left(\overline{p_1^{\alpha_1}}\right) \geqslant \deg\left(\overline{p_2^{\alpha_2}}\right)$.

(b) $\deg\left(\overline{p_2^{\beta_2}}\right) \geqslant \deg\left(\overline{p_2^{\alpha_2}}\right)$.

(c) $\deg\left(\overline{p_1^{\beta_1}p_2^{\beta_2}}\right) \geqslant \deg\left(\overline{p_2^{\beta_2}}\right) \geqslant \deg\left(\overline{p_2^{\alpha_2}}\right)$.

Hence $\overline{p_2^{\alpha_2}}$ has the minimum degree among all vertices in $\mathcal{G}(\mathbb{Z}_n)$ and by Lemma 4.2.5, $\delta(\mathcal{G}(\mathbb{Z}_n)) = (p_2^{\alpha_2} - 1)\phi(p_1^{\alpha_1}) + p_1^{\alpha_1} - 1$.

(ii) If i, j, k is a permutation of $1, 2, 3$ with $i < j$, we have

$$\begin{aligned}
\deg(\overline{p_ip_j}) - \deg(\overline{p_j}) &= p_k + \phi(p_ip_k) + \phi(p_jp_k) - p_ip_k \\
&= (p_i - 1)(p_k - 1) + (p_j - 1)(p_k - 1) - p_k(p_i - 1) \\
&= (p_j - 1)(p_k - 1) - (p_i - 1) \geqslant 0, \text{ since } p_i < p_j.
\end{aligned}$$

Further, by Proposition 4.2.6(iii), $\deg(\overline{p_1}) \geqslant \deg(\overline{p_2}) \geqslant \deg(\overline{p_3})$. Hence $\overline{p_3}$ has the minimum degree among all vertices in $\mathcal{G}(\mathbb{Z}_n)$. Consequently, by Lemma 4.2.5, $\delta(\mathcal{G}(\mathbb{Z}_n)) = \deg(\overline{p_3}) = \phi(n) + p_1p_2 - 1$.

(iii) Let i, j, k, l be a permutation of $1, 2, 3, 4$.

For $i < j < k$, we have

$$\begin{aligned}
&\deg(\overline{p_ip_jp_k}) - \deg(\overline{p_jp_k}) \\
&= p_l + \sum_{\substack{d \mid p_ip_jp_k \\ d \neq p_ip_jp_k}} \phi\left(\frac{n}{d}\right) - \left\{ p_ip_l + \sum_{\substack{d \mid p_jp_k \\ d \neq p_jp_k}} \phi\left(\frac{n}{d}\right) \right\} \\
&= p_l + \phi\left(\frac{n}{p_ip_j}\right) + \phi\left(\frac{n}{p_ip_k}\right) + \phi\left(\frac{n}{p_jp_k}\right) + \phi\left(\frac{n}{p_i}\right) - p_ip_l
\end{aligned}$$

$$= (p_l - 1)\{(p_k - 1) + (p_j - 1) + (p_i - 1) + (p_j - 1)(p_k - 1)\} - p_l(p_i - 1)$$
$$= (p_l - 1)\{(p_k - 1) + (p_j - 1) + (p_j - 1)(p_k - 1)\} - (p_i - 1)$$
$$\geqslant (p_l - 1)(p_j - 1) - (p_i - 1) \geqslant 0, \text{ since } p_i < p_j.$$

Now take $i < j$ with no condition on k and l.

$$\begin{aligned}\deg(\overline{p_i p_j}) - \deg(\overline{p_j}) &= p_k p_l + \phi(p_i p_k p_l) + \phi(p_j p_k p_l) - p_i p_k p_l \\ &= (p_i - 1)(p_k - 1)(p_l - 1) + (p_j - 1)(p_k - 1)(p_l - 1) - p_l p_k (p_i - 1) \\ &= (p_j - 1)(p_k - 1)(p_l - 1) - (p_i - 1)(p_k + p_l - 1). \qquad (4.2)\end{aligned}$$

Since k and l can be interchanged in (4.2), without loss of generality, let $p_k < p_l$.

If n is odd, then $p_k > 2$, so that

$$\begin{aligned}&\deg(\overline{p_i p_j}) - \deg(\overline{p_j}) \\ &\geqslant (p_i - 1)\{(p_k - 1)(p_l - 1) - (p_k + p_l - 1)\} \text{ (since } p_i < p_j) \\ &= (p_i - 1)\{(p_k - 2)(p_l - 1) - p_k\} \\ &\geqslant (p_i - 1)\{(p_l - 1) - p_k\} \geqslant 0, \text{ since } p_k > 2 \text{ and } p_k < p_l.\end{aligned}$$

Now let n be even, that is, $p_1 = 2$. If $p_k > 2$, then from (4.2), $\deg(\overline{p_i p_j}) \geqslant \deg(\overline{p_j})$ as shown above. So take $p_k = 2$. From (4.2), we have

$$\deg(\overline{p_i p_j}) - \deg(\overline{p_j}) = (p_j - 1)(p_l - 1) - (p_i - 1)(p_l + 1). \qquad (4.3)$$

In (4.3), let $p_i \neq p_3$. Since $i < j$, p_i cannot be p_4. Moreover, $p_k = p_1 = 2$, so we have $p_i = p_2$. As a result, $p_l > p_2$. Then from (4.3),

$$\begin{aligned}
\deg(\overline{p_i p_j}) - \deg(\overline{p_j}) &= (p_l - 1)\{(p_j - 1) - (p_2 - 1)\} - 2(p_2 - 1) \\
&\geqslant 2(p_l - 1) - 2(p_2 - 1) \text{ (since } p_j - p_2 \geqslant 2) \\
&= 2(p_l - p_2) > 0, \text{ since } p_l > p_2.
\end{aligned}$$

Now take $p_i = p_3$ in (4.3). Then $p_j = p_4$. We already have $p_k = 2$, and since $p_k < p_l$, we have $p_l = p_2$. Then from (4.3),
$\deg(\overline{p_i p_j}) - \deg(\overline{p_j}) = \deg(\overline{p_3 p_4}) - \deg(\overline{p_4}) = (p_4 - 1)(p_2 - 1) - (p_3 - 1)(p_2 + 1)$, and hence

$$\deg(\overline{p_3 p_4}) \geqslant \deg(\overline{p_4}) \Leftrightarrow p_4 \geqslant p_3 + \frac{2(p_3 - 1)}{p_2 - 1}. \tag{4.4}$$

Case 1: n is odd or $p_4 \geqslant p_3 + \dfrac{2(p_3 - 1)}{p_2 - 1}$

As shown above, for all $1 \leqslant i < j < k \leqslant 4$,

$$\deg(\overline{p_i p_j p_k}) \geqslant \deg(\overline{p_j p_k}) \geqslant \deg(\overline{p_k}). \tag{4.5}$$

Further, it follows from Proposition 4.2.6(iii) that

$$\deg(\overline{p_1}) \geqslant \deg(\overline{p_2}) \geqslant \deg(\overline{p_3}) \geqslant \deg(\overline{p_4}). \tag{4.6}$$

So we conclude that $\overline{p_4}$ has the minimum degree among all vertices in $\mathcal{G}(\mathbb{Z}_n)$. Consequently, by Lemma 4.2.5, we get $\delta(\mathcal{G}(\mathbb{Z}_n)) = \deg(\overline{p_4}) = \phi(n) + p_1 p_2 p_3 - 1$.

Case 2: n is even and $p_4 < p_3 + \dfrac{2(p_3 - 1)}{p_2 - 1}$

Then from (4.4), $\deg(\overline{p_3 p_4}) < \deg(\overline{p_4})$, whereas all other inequalities in (4.5) and (4.6) hold. Thus $\overline{p_3 p_4}$ has the minimum degree among all vertices in $\mathcal{G}(\mathbb{Z}_n)$. Thus by Lemma 4.2.5, we get

$$\begin{aligned}\delta(\mathcal{G}(\mathbb{Z}_n)) &= 2p_2 + (p_2-1)\{(p_3-1)(p_4-1) + (p_3-1) + (p_4-1)\} - 1\\ &= (p_2-1)(p_3p_4+1)+1.\end{aligned}$$

□

Corollary 4.2.8. *Let* $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$, $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ *be prime numbers and* $\alpha_i \in \mathbb{N}$ *for* $1 \leqslant i \leqslant r$. *Let*

$$\eta_1(n) = \frac{n}{p_r^{\alpha_r}} + (p_r^{\alpha_r}-1)\phi\left(\frac{n}{p_r^{\alpha_r}}\right) - 1, \tag{4.7}$$

and

$$\eta_2(n) = \frac{n}{p_{r-1}p_r} + \phi(n) + \phi\left(\frac{n}{p_r}\right) + \phi\left(\frac{n}{p_{r-1}}\right) - 1. \tag{4.8}$$

Then $\eta_1(n)$ *and* $\eta_2(n)$ *are sharp upper bounds of* $\delta(\mathcal{G}(\mathbb{Z}_n))$.

Proof. By Lemma 4.2.5,

$$\deg(\overline{p_r^{\alpha_r}}) = \eta_1(n), \tag{4.9}$$

and

$$\deg(\overline{p_{r-1}p_r}) = \eta_2(n), \tag{4.10}$$

so that $\eta_1(n)$ and $\eta_2(n)$ are upper bounds of $\delta(\mathcal{G}(\mathbb{Z}_n))$. Moreover, as shown in Theorem 4.2.7, $\delta(\mathcal{G}(\mathbb{Z}_n))$ attains $\eta_1(n)$ for some values of n and $\eta_2(n)$ for some other values of n, and hence the bounds are sharp. □

In view of Lemma 4.2.1 and Lemma 4.1.3, we have the following corollary of Theorem 4.2.7.

Corollary 4.2.9. *Let* $n \in \mathbb{N}$ *and* $p_1 < p_2 < p_3 < p_4$ *be prime numbers.*

(i) *If* $n = p_1^{\alpha_1}p_2^{\alpha_2}$, $\alpha_1, \alpha_2 \in \mathbb{N}$, *then for any* $\overline{a} \in \left[\overline{p_2^{\alpha_2}}\right]$,
$E\left[\overline{a}, \left\langle\overline{p_2^{\alpha_2}}\right\rangle \cup \bigcup_{i=0}^{\alpha_2-1}\left[\overline{p_2^{i}}\right] - \overline{a}\right]$ *is a minimum disconnecting set of* $\mathcal{G}(\mathbb{Z}_n)$.

(ii) *If $n = p_1p_2p_3$, then for any $\overline{a} \in [\overline{p_3}]$, $E\left[\overline{a}, \langle\overline{p_3}\rangle \cup [\overline{1}] - \overline{a}\right]$ is a minimum disconnecting set of $\mathcal{G}(\mathbb{Z}_n)$.*

(iii) *Let $n = p_1p_2p_3p_4$. If n is odd or $p_4 \geqslant p_3 + \dfrac{2(p_3-1)}{p_2-1}$, then for any $\overline{a} \in [\overline{p_4}]$, $E\left[\overline{a}, \langle\overline{p_4}\rangle \cup [\overline{1}] - \overline{a}\right]$ is a minimum disconnecting set of $\mathcal{G}(\mathbb{Z}_n)$. Otherwise, for any $\overline{b} \in [\overline{p_3p_4}]$, $E\left[\overline{b}, \langle\overline{p_3p_4}\rangle \cup [\overline{p_3}] \cup [\overline{p_4}] \cup [\overline{1}] - \overline{b}\right]$ is a minimum disconnecting set of $\mathcal{G}(\mathbb{Z}_n)$.*

4.2.2 Abelian p-group

In this subsection, we find the minimum degree and minimum disconnecting sets of power graphs of abelian p-groups.

Notation 4.2.10. We recall that a finite abelian group G is isomorphic to an unique direct product of cyclic groups of prime power order (cf. Theorem 1.3.26). In this product, let $\sigma(G)$ be the number of cyclic groups and $\tau(G)$ be order of the smallest cyclic group.

Lemma 4.2.11. *Consider a prime number p, positive integers r and α_i for all $1 \leqslant i \leqslant r$. For any $1 \leqslant i \leqslant r$, if $u \in H := \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_r}}$ with i^{th} component $\overline{1}$ and all other components are $\overline{0}$, then $\deg(u) = p^{\alpha_i} - 1$.*

Proof. Note that $\mathrm{o}(u) = p^{\alpha_i}$. If possible, let there exist v adjacent to u in $\mathcal{G}(H)$ and $v \notin \langle u\rangle$. Then $u \in \langle v\rangle$, so that $u = cp^kv$ for some $k \in \mathbb{N}$ and integer c, $(c,p) = 1$. Now comparing the i^{th} components, $p \,|\, (cp^ka - 1)$, where $\overline{a}$ is the i^{th} component of v. This implies $p \,|\, 1$, which is not possible. Hence we have $\deg(u) = |\langle u\rangle| - 1 = p^{\alpha_i} - 1$. □

Theorem 4.2.12. *If G is an abelian p-group, then $\delta(\mathcal{G}(G)) = \tau(G) - 1$.*

Proof. Suppose $G \cong H := \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_r}}$ for some $r \in \mathbb{N}$ and $\alpha_i \in \mathbb{N}$ for all $1 \leqslant i \leqslant r$. Take $\alpha_t = \min\limits_{1\leqslant i\leqslant r} \alpha_i$, so that $\tau(G) = p^{\alpha_t}$.

In light of Remark 1.5.5, it is enough to show that $\delta(\mathcal{G}(H)) = p^{\alpha_t} - 1$. If $r = 1$, the proof follows from Lemma 4.2.2. So for the rest of the proof, we assume $r \geqslant 2$.

Let $x = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}) \in H$. For any $1 \leqslant i \leqslant r$, if $\overline{a_i} \neq \overline{0}$, then $a_i = c_i p^{\beta_i}$ for some integers c_i, β_i satisfying $(c_i, p) = 1$ and $\beta_i \geqslant 0$. Take $\beta_s = \min\{\beta_i \mid 1 \leqslant i \leqslant r, \overline{a_i} \neq \overline{0}\}$, and define $y = \left(\overline{b_1}, \overline{b_2}, \ldots, \overline{b_r}\right)$ by

$$b_i = \begin{cases} c_i p^{\beta_i - \beta_s} & \text{if } \overline{a_i} \neq \overline{0}, \\ 0 & \text{if } \overline{a_i} = \overline{0}. \end{cases} \tag{4.11}$$

Then $\overline{b_s} = \overline{c_s}$, and hence $\mathrm{o}(\overline{b_s}) = p^{\alpha_s}$. Since $\mathrm{o}(y) = \mathrm{lcm}(\mathrm{o}(\overline{b_1}), \mathrm{o}(\overline{b_2}), \ldots, \mathrm{o}(\overline{b_r}))$ (cf. Theorem 1.3.25), we get $\mathrm{o}(y) \geqslant p^{\alpha_s}$. Since $\mathrm{o}(y)$ is a prime power, $\langle y \rangle$ is a clique in $\mathcal{G}(H)$ (cf. Theorem 1.5.7(ii)). Thus, as $x \in \langle y \rangle$, we have $\deg(x) \geqslant p^{\alpha_s} - 1 \geqslant p^{\alpha_t} - 1$. Therefore,

$$\delta(\mathcal{G}(H)) \geqslant p^{\alpha_t} - 1. \tag{4.12}$$

Hence we conclude from Lemma 4.2.11 that $\delta(\mathcal{G}(H)) = p^{\alpha_t} - 1$. This completes the proof of the theorem. □

Theorem 4.2.13. *Let G be an abelian p-group and $\psi : G \to \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_r}}$ be an isomorphism and $\tau(G) = p^{\alpha_t}$. If $g \in G$ is such that all components of $\psi(g)$ are $\overline{0}$ except t^{th}, say $\overline{a}$, satisfying $\gcd(a, p) = 1$, then $E[g, \psi^{-1}(\langle \psi(g) \rangle) - g]$ is a minimum disconnecting set of $\mathcal{G}(G)$.*

Proof. Take $\psi(g) = z$. Following the proof of Theorem 4.2.12, $N(z) = \langle z \rangle - z$. Then, ψ being an isomorphism, $N(g) = \psi^{-1}(\langle z \rangle) - g$. Thus by Lemma 4.1.3, $E[g, \psi^{-1}(\langle z \rangle) - g]$ is a minimum disconnecting set of $\mathcal{G}(G)$. □

4.2.3 Dihedral and dicyclic group

In this subsection, we find the minimum degree and minimum disconnecting sets of power graphs of D_n and Q_n.

Theorem 4.2.14. *For $n \geqslant 3$, $\delta(\mathcal{G}(D_n)) = 1$. Moreover, for any $0 \leqslant i < n$, the edge between e and $a^i b$ is a cut-edge of $\mathcal{G}(D_n)$.*

Proof. It follows from Remark 1.3.16 that for any $0 \leqslant i < n$, the only vertex adjacent to a^ib is e and hence $\deg(a^ib) = 1$. As $\mathcal{G}(D_n)$ is connected, $\deg(g) \geqslant 1$ for all $g \in D_n$. Hence $\delta(\mathcal{G}(D_n)) = 1$ and the edge between e and a^ib is a cut-edge of $\mathcal{G}(D_n)$ for all $0 \leqslant i < n$. □

Theorem 4.2.15. *For $n \geqslant 2$, $\delta(\mathcal{G}(Q_n)) = 3$. Moreover, for any $0 \leqslant i \leqslant n-1$, $E[a^ib, \{e, a^n, a^{n+i}b\}]$ and $E[a^{n+i}b, \{e, a^n, a^ib\}]$ are minimum disconnecting sets of $\mathcal{G}(Q_n)$.*

Proof. We follow the presentation of Q_n in (1.2). Let $g \in \langle a \rangle$. Since $o(a) = 2n$, by Theorem 4.2.4, $\deg(g) \geqslant \phi(2n)+1$. For $m > 2$, $\phi(m)$ is an even integer (cf. [Burton, 2006]). So, in particular, $\deg(g) \geqslant \phi(2n) + 1 \geqslant 3$.

By Remark 1.3.16, $N(a^ib) = \{e, a^n, a^{n+i}b\}$ and $N(a^{n+i}b) = \{e, a^n, a^ib\}$ for all $0 \leqslant i \leqslant n-1$. Let $h \in Q_n - \langle a \rangle$. Then from Remark 1.3.16(iii), $h = a^jb$ for some $0 \leqslant j \leqslant 2n-1$. Consequently, $\deg(h) = 3$. Hence we conclude that $\delta(\mathcal{G}(Q_n)) = 3$. Moreover, we deduce that $E[a^ib, \{e, a^n, a^{n+i}b\}]$ and $E[a^{n+i}b, \{e, a^n, a^ib\}]$ are minimum disconnecting sets of $\mathcal{G}(Q_n)$. □

4.3 Equality of vertex connectivity and minimum degree

In this section, we investigate the equality of vertex connectivity and minimum degree of power graphs of finite groups. We first obtain some necessary conditions for the equality to hold for power graphs of finite groups (cf. Theorem 4.3.1). We derive a necessary and sufficient condition for the equality of vertex connectivity and minimum degree of power graph of $\mathbb{Z}_n$ (cf. Theorem 4.3.5). Followed by this, we examine the equality for abelian p-groups, D_n and Q_n (cf. Theorem 4.3.6, 4.3.7).

Let G be a cyclic group of prime power order. Then it follows from Theorem 1.5.7(ii) that $\kappa(\mathcal{G}(G)) = \delta(\mathcal{G}(G)) = n - 1$. The next theorem gives some necessary conditions of the concerned equality for groups that are not cyclic groups of prime power order.

Theorem 4.3.1. *Let* G, $|G| > 1$ *be a finite group and* $\kappa(\mathcal{G}(G)) = \delta(\mathcal{G}(G))$. *If* G *is not a cyclic group of prime power order and* $\delta(\mathcal{G}(G)) = \deg(g)$ *for some* $g \in G$, *then the following statements hold.*

(i) $N(g)$ *is a minimum separating set of* $\mathcal{G}(G)$.

(ii) *The order of* g *is* 2 *in* G. *Consequently,* G *is of even order.*

Proof. (i) Let $|G| = n$. In view of Theorem 1.5.7, $\mathcal{G}(G)$ is not a complete graph. If possible, let $N(g) = G - g$. Then $\delta(\mathcal{G}(G)) = \deg(g) = n - 1$, which implies $\deg(h) = n - 1$ for all $h \in G$. Again, this is possible only when $\mathcal{G}(G)$ is a complete graph. Thus $N(g) \neq G - g$, i.e., there exists at least one vertex g_1 non-adjacent to g in $\mathcal{G}(G)$. Thus there does not exist any path from g_1 to g in $\mathcal{G}(G) - N(g)$, and hence $N(g)$ is a separating set of $\mathcal{G}(G)$. Further, $|N(g)| = \delta(\mathcal{G}(G)) = \kappa(\mathcal{G}(G))$. Hence we conclude that $N(g)$ is a minimum separating set of $\mathcal{G}(G)$.

(ii) By (i), $N(g)$ is a minimal separating set of $\mathcal{G}(G)$. Then in view of Theorem 2.2.2, either $[g] \subseteq N(g)$ or $[g] \cap N(g) = \varnothing$. However, $[g] - \{g\} \subseteq N(g)$ and $g \notin N(g)$. Hence $[g] - \{g\} = \varnothing$, that is, $\big|[g]\big| = 1$. Consequently, $o(g) = 1$ or $o(g) = 2$. If $o(g) = 1$, then g is the identity element of G, so that $N(g) = G - g$. As observed in the proof of (i), is not possible. Thus $o(g) = 2$. Since order of an element divides order of the group for a finite group, $|G|$ is even. □

Lemma 4.3.2. *Let* $n \in \mathbb{N}$, p *be a prime factor of* n *and* α *be the largest integer such that* p^α *divides* n. *Then for any integer* β *satisfying* $1 \leqslant \beta \leqslant \alpha$,

$$\sum_{d \,\big|\, \frac{n}{p^\beta}} \phi\left(\frac{n}{d}\right) = n - \frac{n}{p^{\alpha-\beta+1}}.$$

Proof. Taking $m = \dfrac{n}{p^{\alpha}}$, we have

$$\begin{aligned}
\sum_{d \,\big|\, \frac{n}{p^{\beta}}} \phi\left(\frac{n}{d}\right) &= \sum_{d|m} \phi\left(\frac{n}{d}\right) + \sum_{d|m} \phi\left(\frac{n}{pd}\right) + \ldots + \sum_{d|m} \phi\left(\frac{n}{p^{\alpha-\beta}d}\right) \\
&= \left\{\phi(p^{\alpha}) + \phi(p^{\alpha-1}) + \ldots + \phi(p^{\beta})\right\} \sum_{d|m} \phi\left(\frac{m}{d}\right) \\
&= (p^{\alpha} - p^{\beta-1})m \\
&= n - \frac{n}{p^{\alpha-\beta+1}}.
\end{aligned}$$

□

In the next theorem, when $\kappa(\mathcal{G}(\mathbb{Z}_n))$ and $\delta(\mathcal{G}(\mathbb{Z}_n))$ are equal, we first find their common value. As a consequence of this, we then give a minimum separating set and a minimum disconnecting set of $\mathcal{G}(\mathbb{Z}_n)$.

Theorem 4.3.3. *If the integer $n > 1$ is not a prime power and $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \delta(\mathcal{G}(\mathbb{Z}_n)) = k$ (say), then $k = deg\left(\overline{\frac{n}{2}}\right) = n - \frac{n}{2^{\alpha}}$, where α is the largest integer such that 2^{α} divides n.*

Proof. We first note that since n is even, deg $\left(\overline{\frac{n}{2}}\right)$ is well defined (cf. Theorem 4.3.1). Let $\overline{a}$ be the vertex such that $\deg(\overline{a}) = k$. Then by Theorem 4.3.1(ii), $o(\overline{a}) = 2$. However, o $\left(\overline{\frac{n}{2}}\right) = 2$, and by following Theorem 1.3.24, $\mathbb{Z}_n$ has exactly one element of order 2. Hence $\overline{a} = \overline{\frac{n}{2}}$. Furthermore, from Lemma 4.2.5,

$$\begin{aligned}
k = deg\left(\overline{\frac{n}{2}}\right) &= 1 + \sum_{d \,|\, \frac{n}{2}, d \neq \frac{n}{2}} \phi\left(\frac{n}{d}\right) \\
&= 1 + \sum_{d \,|\, \frac{n}{2}} \phi\left(\frac{n}{d}\right) - \phi(2) \\
&= \sum_{d \,|\, \frac{n}{2}} \phi\left(\frac{n}{d}\right) \\
&= n - \frac{n}{2^{\alpha}}, \text{ by Lemma 4.3.2.}
\end{aligned}$$

□

In view of Lemma 4.1.3 and Theorem 4.3.1(i), we have the following corollary of Theorem 4.3.3.

Corollary 4.3.4. *Suppose the integer $n > 1$ be not a prime power. If $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \delta(\mathcal{G}(\mathbb{Z}_n))$, then $\{\overline{0}\} \cup \bigcup_{a \mid \frac{n}{2}, a \neq \frac{n}{2}} [\overline{a}]$, say A, is a minimum separating set and $E\left[\frac{\overline{n}}{2}, A\right]$ is a minimum disconnecting set of $\mathcal{G}(\mathbb{Z}_n)$.*

We now present a characterization for the equality of vertex connectivity and minimum degree of power graph of $\mathbb{Z}_n$ in terms n.

Theorem 4.3.5. *For $n \in \mathbb{N}$, $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \delta(\mathcal{G}(\mathbb{Z}_n))$ if and only if $n = p^\alpha$ for some prime p and $\alpha \in \mathbb{N}$ or $n = 2q^\beta$ for some prime $q > 2$ and $\beta \in \mathbb{N}$.*

Proof. Let $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \delta(\mathcal{G}(\mathbb{Z}_n))$. By Theorem 1.5.7(ii), if $\mathcal{G}(\mathbb{Z}_n)$ is complete, then $n = p^\alpha$ for some prime p. Now suppose $\mathcal{G}(\mathbb{Z}_n)$ is not complete. So n is not a prime power and hence by Theorem 4.3.1(ii), n is even. Let $n = 2^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, $r \geqslant 2$, $2 < p_2 < \cdots < p_r$ are primes and $\alpha_i \in \mathbb{N}$ for $1 \leqslant i \leqslant r$. Moreover, from Theorem 4.3.3, $\delta(\mathcal{G}(\mathbb{Z}_n)) = deg\left(\frac{\overline{n}}{2}\right)$.

$$\begin{aligned}
&\deg\left(\frac{\overline{n}}{2}\right) - \deg\left(\frac{\overline{n}}{2^{\alpha_1}}\right) \\
&= 2 + \sum_{d \mid \frac{n}{2}} \phi\left(\frac{n}{d}\right) - \left\{2^{\alpha_1} + \sum_{d \mid \frac{n}{2^{\alpha_1}}} \phi\left(\frac{n}{d}\right)\right\} \text{(by Lemma 4.2.5)} \\
&= \left\{2 + n - \frac{n}{2^{\alpha_1}} - \phi(2)\right\} - \left\{2^{\alpha_1} + n - \frac{n}{2} - \phi(2^{\alpha_1})\right\} \text{(by Lemma 4.3.2)} \\
&= \left(1 - \frac{n}{2^{\alpha_1}}\right) - \left(2^{\alpha_1 - 1} - \frac{n}{2}\right) \\
&= \left(\frac{n}{2} - \frac{n}{2^{\alpha_1}}\right) - (2^{\alpha_1 - 1} - 1) \\
&= \left(\frac{n}{2^{\alpha_1}} - 1\right)(2^{\alpha_1 - 1} - 1).
\end{aligned}$$

Since $r \geqslant 2$, we have $\frac{n}{2^{\alpha_1}} > 1$, and if $\alpha_1 > 1$, then $2^{\alpha_1 - 1} > 1$. Hence $\deg\left(\overline{\frac{n}{2}}\right) > \deg\left(\overline{\frac{n}{2^{\alpha_1}}}\right)$. This contradicts the fact that $\delta(\mathcal{G}(\mathbb{Z}_n)) = deg\left(\overline{\frac{n}{2}}\right)$. Thus $\alpha_1 = 1$, i.e., $n = 2p_2^{\alpha_2} \dots p_r^{\alpha_r}$.

Take $m = \frac{n}{2}$. If possible, let $r \geqslant 3$, so that $\frac{n}{2p_2^{\alpha_2}} \neq 1$.

$$
\begin{aligned}
&\deg\left(\overline{\frac{n}{2}}\right) - \deg\left(\overline{\frac{n}{2p_2^{\alpha_2}}}\right) \\
&= 2 + \sum_{d \mid \frac{n}{2}, d \neq \frac{n}{2}} \phi\left(\frac{n}{d}\right) - \left\{ 2p_2^{\alpha_2} + \sum_{d \mid \frac{n}{2p_2^{\alpha_2}}, d \neq \frac{n}{2p_2^{\alpha_2}}} \phi\left(\frac{n}{d}\right) \right\} \text{(by Lemma 4.2.5)} \\
&= 2 + \phi(2) \sum_{d \mid m, d \neq m} \phi\left(\frac{m}{d}\right) - \left\{ 2p_2^{\alpha_2} + \phi(2) \sum_{d \mid \frac{m}{p_2^{\alpha_2}}, d \neq \frac{m}{p_2^{\alpha_2}}} \phi\left(\frac{m}{d}\right) \right\} \\
&= (2 + m - 1) - \left\{ 2p_2^{\alpha_2} + m - \frac{m}{p_2} - \phi(p_2^{\alpha_2}) \right\} \text{(by Lemma 4.3.2)} \\
&= 1 + \frac{m}{p_2} - (p_2^{\alpha_2} + p_2^{\alpha_2 - 1}) \\
&= 1 + p_2^{\alpha_2 - 1} \left\{ \frac{m}{p_2^{\alpha_2}} - (p_2 + 1) \right\}.
\end{aligned}
$$

Since $r \geqslant 3$, we have $\frac{m}{p_2^{\alpha_2}} \geqslant p_3 > p_2 + 1$. So $\deg\left(\overline{\frac{n}{2}}\right) > \deg\left(\overline{\frac{n}{2p_2^{\alpha_2}}}\right)$. This again contradicts the fact that $\delta(\mathcal{G}(\mathbb{Z}_n)) = deg\left(\overline{\frac{n}{2}}\right)$. Thus $r = 2$, i.e., $n = 2p_2^{\alpha_2}$. This completes the proof of forward implication.

We now prove the converse. If $n = p^\alpha$ for some prime p and $\alpha \in \mathbb{N}$, then by Theorem 1.5.7, $\mathcal{G}(\mathbb{Z}_n)$ is complete. Thus $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \delta(\mathcal{G}(\mathbb{Z}_n)) = n - 1$. Now let $n = 2q^\beta$ for some prime $q > 2$ and $\beta \in \mathbb{N}$. Then by Theorem 4.2.7(i) and Theorem 2.3.20, $\kappa(\mathcal{G}(\mathbb{Z}_n)) - \delta(\mathcal{G}(\mathbb{Z}_n)) = \phi(2q^\beta) + q^{\beta-1} - \{2 + (2-1)(q^\beta - 1) - 1\} = q^\beta - q^\beta = 0$ □

We now explore the relation between the vertex connectivity and minimum degree of power graphs of some more groups.

Theorem 4.3.6. *Let G be an abelian p-group. Then $\kappa(\mathcal{G}(G)) = \delta(\mathcal{G}(G))$ if and only if $\sigma(G) = 1$ or $\tau(G) = 2$.*

Proof. From Theorem 1.5.7(ii), G is cyclic if and only if $\kappa(\mathcal{G}(G)) = |G| - 1 = \delta(\mathcal{G}(G))$. Now let G be non-cyclic. Then by Theorem 4.2.12, $\delta(\mathcal{G}(G)) = \tau(G) - 1$ and by Corollary 3.1.6, $\kappa(\mathcal{G}(G)) = 1$. Thus $\kappa(\mathcal{G}(G)) = \delta(\mathcal{G}(G))$ if and only if $\tau(G) = 2$. Hence, if either $\sigma(G) = 1$ or $\sigma(G) > 1$ and $\tau(G) = 2$, then $\kappa(\mathcal{G}(G)) = \delta(\mathcal{G}(G))$. Whereas, when $\sigma(G) > 1$ and $\tau(G) > 2$, then $\kappa(\mathcal{G}(G)) \neq \delta(\mathcal{G}(G))$. □

Theorem 4.3.7. *For $n \in \mathbb{N}$, the following statements hold.*

(i) *For $n \geqslant 3$, $\kappa(\mathcal{G}(D_n)) = \delta(\mathcal{G}(D_n))$.*

(ii) *For $n \geqslant 2$, $\kappa(\mathcal{G}(Q_n)) \neq \delta(\mathcal{G}(Q_n))$.*

Proof. From Theorem 1.5.21 and Theorem 1.5.22, $\kappa(\mathcal{G}(D_n)) = 1$ and $\kappa(\mathcal{G}(Q_n)) = 2$, respectively. On the other hand, from Theorem 4.2.14, $\delta(\mathcal{G}(D_n)) = 1$ and from Theorem 4.2.15, $\delta(\mathcal{G}(Q_n)) = 3$. Hence the result follows. □

4.4 Conclusion

In this chapter, we computed the minimum degree of $\mathcal{G}(\mathbb{Z}_n)$ when n has two prime factors or is a product of at most four distinct primes. This in fact shows that $\eta_1(n)$ and $\eta_2(n)$ are sharp upper bounds of the minimum degree of $\mathcal{G}(\mathbb{Z}_n)$ (cf. Corollary 4.2.8). Thus one may be interested in determining the minimum degree of $\mathcal{G}(\mathbb{Z}_n)$ for all $n \in \mathbb{N}$.

Moreover, we obtained a necessary and sufficient condition for the equality of the vertex connectivity and minimum degree of $\mathcal{G}(\mathbb{Z}_n)$ and examined it for power graphs of abelian p-groups, D_n and Q_n. So characterizing the above equality for power graphs of all groups is still an open problem.

5

Laplacian Spectrum

Recently, various spectral properties of power graphs have been investigated, see [Chattopadhyay and Panigrahi, 2015], [Mehranian et al., 2017], [Chattopadhyay et al., 2017]. In particular, Laplacian spectra of power graphs of $\mathbb{Z}_n$ and D_n was studied by Chattopadhyay and Panigrahi [2015]. In this chapter, we study Laplacian spectra of power graphs of $\mathbb{Z}_n$, Q_n and p-groups in Section 5.2, 5.3 and 5.4, respectively. Kirkland et al. [2002] investigated the equality of vertex connectivity and algebraic connectivity of graphs that are non-complete and connected. Here, we characterize this equality for power graphs of $\mathbb{Z}_n$, Q_n and p-groups. Further, we show that the power graph of Q_n is Laplacian integral if and only if it is generalized quaternion and that power graphs of p-groups are always Laplacian integral.

5.1 Essential results and alternate proofs

In this section, we first recall certain results necessary for this chapter and then give alternative and shorter proofs for some of them. We then obtain some results that are essential for study of Laplacian spectrum of power graphs.

We begin by recalling the following two results from the survey article [Mohar, 1991] on Laplacian spectra of graphs. We note that Theorem 5.1.2 was originally shown in [Kel'mans, 1965, 1966].

Theorem 5.1.1. *If* Γ *is the disjoint union of graphs* $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$, *then*

$$\Theta(\Gamma, x) = \prod_{i=1}^{r} \Theta(\Gamma_i, x).$$

Theorem 5.1.2. *If* Γ_1 *and* Γ_2 *are vertex disjoint graphs on* n_1 *and* n_2 *vertices, respectively, then*

$$\Theta(\Gamma_1 \vee \Gamma_2, x) = \frac{x(x - n_1 - n_2)}{(x - n_1)(x - n_2)} \Theta(\Gamma_1, x - n_2)\Theta(\Gamma_2, x - n_1).$$

By applying Theorem 5.1.1 and Theorem 5.1.2, we now give alternative and shorter proofs for some results on Laplacian spectra of power graphs of $\mathbb{Z}_n$, D_n and Q_n from [Chattopadhyay, 2015; Chattopadhyay and Panigrahi, 2015].

By Remark 1.5.14 and Remark 1.5.15, $\mathcal{G}(\mathbb{Z}_n) \cong K_{\phi(n)+1} \vee \mathcal{G}'(\mathbb{Z}_n)$. Hence by applying Theorem 5.1.2, we get

$$\Theta(\mathcal{G}(\mathbb{Z}_n), x) = \frac{x(x - n)^{\phi(n)+1}}{x - \phi(n) - 1} \Theta(\mathcal{G}'(\mathbb{Z}_n), x - \phi(n) - 1). \tag{5.1}$$

For any matrix M, let $\Phi(M, x)$ denote its characteristic polynomial. Then observe that $\Theta(\mathcal{G}'(\mathbb{Z}_n), x - \phi(n) - 1) = \Phi(L'(\mathcal{G}(\mathbb{Z}_n)), x)$, where $L'(\mathcal{G}(\mathbb{Z}_n))$ is the submatrix of $L(\mathcal{G}(\mathbb{Z}_n))$ obtained by deleting rows and columns corresponding to elements of $\mathcal{S}(\mathbb{Z}_n)$. Consequently, we have the following theorem.

Theorem 5.1.3 ([Chattopadhyay and Panigrahi, 2015]). *For any integer $n \geqslant 2$,*

$$\Theta(\mathcal{G}(\mathbb{Z}_n), x) = \frac{x(x-n)^{\phi(n)+1}}{x-\phi(n)-1}\Phi(L'(\mathcal{G}(\mathbb{Z}_n)), x). \tag{5.2}$$

As observed in [Chattopadhyay and Panigrahi, 2014], for distinct primes p and q,

$$\mathcal{G}(\mathbb{Z}_{pq}) \cong K_{\phi(pq)+1} \vee (K_{p-1} + K_{q-1}).$$

Hence for $n = pq$, we get

$$\Theta(\mathcal{G}(\mathbb{Z}_n), x) = x(x-n)^{\phi(n)+1}(x-\phi(n)-1)(x-n+p-1)^{p-2}(x-n+q-1)^{q-2}.$$

Consequently, we have the following theorem.

Theorem 5.1.4 ([Chattopadhyay and Panigrahi, 2015]). *If $n = pq$ for distinct primes p and q, then the Laplacian spectrum of $\mathcal{G}(\mathbb{Z}_n)$ is given by*

$$\begin{pmatrix} 0 & \phi(n)+1 & n-q+1 & n-p+1 & n \\ 1 & 1 & p-2 & q-2 & \phi(n)+1 \end{pmatrix}.$$

From [Chattopadhyay and Panigrahi, 2014], we have the structure of $\mathcal{G}(D_n)$ and $\mathcal{G}(Q_n)$ as presented below. For any integer $n \geqslant 3$,

$$\mathcal{G}(D_n) \cong K_1 \vee (\mathcal{G}^*(\mathbb{Z}_n) + nK_1), \tag{5.3}$$

and for any integer $\alpha \geqslant 2$,

$$\mathcal{G}(Q_{2^{\alpha-1}}) \cong K_2 \vee (K_{2^\alpha - 2} + 2^{\alpha-1}K_2). \tag{5.4}$$

As a consequence of (5.3), we have the following theorem.

Theorem 5.1.5 ([Chattopadhyay and Panigrahi, 2015]). *For $n \in \mathbb{N}$,*

$$\Theta(L(\mathcal{G}(D_n)), x) = \frac{(x-1)^n(x-2n)}{(x-n)}\Theta(\mathcal{G}(\mathbb{Z}_n), x).$$

Moreover, from (5.4) we obtain

$$\Theta(L(\mathcal{G}(Q_{2^{\alpha-1}})), x) = x\{(x-2)(x-4)\}^{2^{\alpha-1}}(x-2^{\alpha+1})^2(x-2^{\alpha})^{2^{\alpha}-3}.$$

Hence we have the following theorem.

Theorem 5.1.6 ([Chattopadhyay, 2015]). *For any integer $\alpha \geqslant 2$, the Laplacian spectrum of $\mathcal{G}(Q_{2^{\alpha-1}})$ is given by*

$$\begin{pmatrix} 0 & 2 & 4 & 2^{\alpha} & 2^{\alpha+1} \\ 1 & 2^{\alpha-1} & 2^{\alpha-1} & 2^{\alpha}-3 & 2 \end{pmatrix}.$$

We now present some results that are fundamental for Laplacian spectra of power graphs and hold for all finite groups.

Since the identity element of any finite group G is adjacent to all other elements in $\mathcal{G}(G)$, we have $\mathcal{G}(G) \cong K_1 \vee \mathcal{G}^*(G)$. So that by Theorem 5.1.2,

$$\Theta(\mathcal{G}(G), x) = \frac{x(x-|G|)}{x-1}\Theta(\mathcal{G}^*(G), x-1).$$

This results in the following lemma.

Lemma 5.1.7. *If G is a finite group of order $n \geqslant 3$, then*

$$\lambda_i(\mathcal{G}(G)) = \lambda_{i-1}(\mathcal{G}^*(G)) + 1$$

for $2 \leqslant i \leqslant n-1$. In particular, algebraic connectivity of $\mathcal{G}(G)$ satisfies $\lambda_{n-1}(\mathcal{G}(G)) \geqslant 1$.

The following remark is an immediate consequence of Lemma 5.1.7.

Remark 5.1.8. For a finite group G, the algebraic connectivity of $\mathcal{G}(G)$ is 1 if and only if 1 is a Laplacian eigenvalue of $\mathcal{G}(G)$.

Theorem 1.4.9 and Lemma 5.1.7 together yield the following proposition.

Proposition 5.1.9. *Let G be a finite group of order $n \geqslant 3$.*

(i) *The algebraic connectivity of $\mathcal{G}(G)$ is 1 if and only if its vertex connectivity is 1.*

(ii) *If the algebraic connectivity of $\mathcal{G}(G)$ is 1, then its multiplicity is equal to one less than the number of components of $\mathcal{G}^*(G)$.*

Theorem 1.4.14 and Theorem 4.1.2 together yield the following lemma.

Lemma 5.1.10. *If G is a group of order $n \geqslant 2$, then*

$$\lambda_{n-1}(\mathcal{G}(G)) \geqslant 2\left(1 - \cos\frac{\pi}{n}\right)\delta(\mathcal{G}(G)).$$

5.2 Finite cyclic group

In this section, we investigate multiplicity and bounds of Laplacian eigenvalues of $\mathcal{G}(\mathbb{Z}_n)$. We further characterize the equality of the vertex connectivity and algebraic connectivity of $\mathcal{G}(\mathbb{Z}_n)$.

Using (5.1), we have the following lemma.

Lemma 5.2.1. *If the integer $n > 1$ is not a prime number, then*

$$\lambda_i(\mathcal{G}(\mathbb{Z}_n)) = \begin{cases} n & \text{for } 1 \leqslant i \leqslant \phi(n)+1, \\ \lambda_{i-\phi(n)-1}(\mathcal{G}'(\mathbb{Z}_n)) + \phi(n) + 1 & \text{for } \phi(n)+2 \leqslant i \leqslant n-1, \\ 0 & \text{for } i = n. \end{cases}$$

When n is a prime power, the multiplicity of n as a Laplacian eigenvalue of $\mathcal{G}(\mathbb{Z}_n)$ is $n-1$ (cf. Theorem 1.5.28). Moreover, the multiplicity of n is at least $\phi(n)+1$ for any non-prime integer $n > 1$ (cf. Theorem 1.5.24).

Moreover, as recalled in Theorem 1.5.27,

$$\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) \geqslant \phi(n) + 1 \tag{5.5}$$

and equality holds if n is a prime or product of two distinct primes. We observe that the last two results are consequences of Lemma 5.2.1. In the following result, we determine the condition for which the multiplicity of n as a Laplacian eigenvalue of $\mathcal{G}(\mathbb{Z}_n)$ is exactly $\phi(n)+1$. We further show in Theorem 5.2.3 that for equality to hold in (5.5), the aforementioned sufficient condition is necessary as well.

Theorem 5.2.2. *For an integer $n > 1$, the multiplicity of the Laplacian eigenvalue n of $\mathcal{G}(\mathbb{Z}_n)$ is $\phi(n)+1$ if and only if $n = 4$ or n has at least two prime factors.*

Proof. For $n = 4$, the multiplicity of n is $n-1 = 3 = \phi(n)+1$. Now suppose n has at least two prime factors. Then from Lemma 1.5.26, $\overline{\mathcal{G}'(\mathbb{Z}_n)}$ is connected, and hence it follows from Theorem 1.4.12 that $\lambda_1(\mathcal{G}'(\mathbb{Z}_n)) < n - \phi(n) - 1$. This together with Lemma 5.2.1 imply that $\lambda_i(\mathcal{G}(\mathbb{Z}_n)) = n$ for all $1 \leqslant i \leqslant \phi(n)+1$ and $\lambda_j(\mathcal{G}(\mathbb{Z}_n)) < n$ for all $\phi(n)+2 \leqslant j \leqslant n$, and hence the multiplicity of n is $\phi(n)+1$.

Conversely, if n is a prime power and $n \neq 4$, then it folllows from Corollary 1.5.28 that the multiplicity of the Laplacian eigenvalue n of $\mathcal{G}(\mathbb{Z}_n)$ is $n-1$, and it is easy to see that $n-1 \neq \phi(n)+1$. □

Theorem 5.2.3. *For an integer $n > 1$, the algebraic connectivity of $\mathcal{G}(\mathbb{Z}_n)$ is $\phi(n)+1$ if and only if n is a prime or product of two distinct primes.*

Proof. If n is a prime or a product of two distinct primes, then the algebraic connectivity $\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n))$ is $\phi(n)+1$ (cf. Theorem 1.5.27).

For converse, let $\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) = \phi(n)+1$. Observe that $n = \phi(n)+1$ if and only if n is prime. So suppose n is not prime. Then by Lemma 5.2.1, $\lambda_{n-\phi(n)-2}(\mathcal{G}'(\mathbb{Z}_n)) = 0$.

Thus by Theorem 1.4.9, $\mathcal{G}'(\mathbb{Z}_n)$ is disconnected. Finally, applying Proposition 2.3.3, we conclude that n is a product of two distinct primes. □

In the next theorem, we characterize the equality of the vertex connectivity and algebraic connectivity of $\mathcal{G}(\mathbb{Z}_n)$.

Theorem 5.2.4. *For an integer $n > 1$, $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n))$ if and only if n is a product of two distinct primes.*

Proof. If n is a product of two distinct primes, then from Theorem 1.5.19(ii) and Theorem 5.1.4, we have $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1 = \lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n))$.

Now suppose n is not a product of two distinct primes. If n is a prime power, by Theorem 1.5.7(ii), $\kappa(\mathcal{G}(\mathbb{Z}_n)) = n - 1$ and $\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) = n$. So n has at least two distinct prime factors. Then from Proposition 2.3.3 and Lemma 1.5.26, $\mathcal{G}'(\mathbb{Z}_n)$ and $\overline{\mathcal{G}'(\mathbb{Z}_n)}$ are connected. Thus it follows from Lemma 1.4.16 that $\kappa(\mathcal{G}'(\mathbb{Z}_n)) \neq \lambda_{n-\phi(n)-2}(\mathcal{G}'(\mathbb{Z}_n))$. Finally, applying Lemma 2.3.2(ii), and Lemma 5.2.1, we get $\kappa(\mathcal{G}(\mathbb{Z}_n)) \neq \lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n))$. □

In the next two results, we supply upper and lower bounds of the algebraic connectivity of $\mathcal{G}(\mathbb{Z}_n)$ using certain existing results on bounds and values of the vertex connectivity and the minimum degree of $\mathcal{G}(\mathbb{Z}_n)$, respectively. In fact, the first result extends Corollary 1.5.25 to all n.

Theorem 5.2.5. *Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $r \geqslant 2$, $p_1 < p_2 < \cdots < p_r$ are primes, $\alpha_i \in \mathbb{N}$ for $1 \leqslant i \leqslant r$, and n is not a product of two distinct primes. If either $\alpha_r \geqslant 2$ and $\prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i}\right) \geqslant \frac{1}{2}$ or $\alpha_r = 1$, then*

$$\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) < \phi(n) + \frac{n}{p_r} - p_r^{\alpha_r - 1} \phi\left(\frac{n}{p_r^{\alpha_r}}\right), \tag{5.6}$$

otherwise

$$\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) < \phi(n) + \frac{n}{p_r^{\alpha_r}} + \phi\left(\frac{n}{p_r^{\alpha_r}}\right)(p_r^{\alpha_r - 1} - 2). \tag{5.7}$$

Proof. Let the right hand side of (5.6) and (5.7) be $\xi_1(n)$ and $\xi_2(n)$, respectively. Then from Theorem 2.3.6 and Theorem 2.3.16, $\xi_1(n)$ and $\xi_2(n)$ are upper bounds of $\kappa(\mathcal{G}(\mathbb{Z}_n))$. Furthermore, $\xi_1(n) \leqslant \xi_2(n)$ if either $\alpha_r = 1$ or $\alpha_r \geqslant 2$ and $\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i}\right) \geqslant \frac{1}{2}$, and $\xi_1(n) > \xi_2(n)$ otherwise. Hence $\xi_1(n)$ and $\xi_2(n)$ are upper bounds of $\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n))$. Finally, it follows from Theorem 5.2.4 that the upper bounds are strict. □

Theorem 4.2.7 together with Lemma 5.1.10 yield the following theorem.

Theorem 5.2.6. *Let $p_1 < p_2 < p_3 < p_4$ be prime numbers and $\alpha_1, \alpha_2 \in \mathbb{N}$.*

(i) *If $n = p_1^{\alpha_1} p_2^{\alpha_2}$, then*

$$\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) \geqslant 2\left(1-\cos\frac{\pi}{n}\right)\left\{(p_1^{\alpha_1}-p_1^{\alpha_1-1})(p_2^{\alpha_2}-1)+p_1^{\alpha_1}-1\right\}.$$

(ii) *If $n = p_1p_2p_3$, then $\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) \geqslant 2\left(1-\cos\frac{\pi}{n}\right)\left\{\phi(n)+p_1p_2-1\right\}$.*

(iii) *Let $n = p_1p_2p_3p_4$. If n is odd or $p_4 \geqslant p_3 + \frac{2(p_3-1)}{p_2-1}$, then*

$$\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) \geqslant 2\left(1-\cos\frac{\pi}{n}\right)\left\{\phi(n)+p_1p_2-1\right\},$$

otherwise

$$\lambda_{n-1}(\mathcal{G}(\mathbb{Z}_n)) \geqslant 2\left(1-\cos\frac{\pi}{n}\right)\left\{(p_2-1)(p_3p_4+1)+1\right\}.$$

5.3 Dicyclic group

In this section, we first derive an expression for Laplacian characteristic polynomial of $\mathcal{G}(Q_n)$ involving $L(\mathcal{G}(\mathbb{Z}_n))$. We further supply bounds and least value of multiplicity of some Laplacian eigenvalues of $\mathcal{G}(Q_n)$. We then show that the vertex connectivity and algebraic connectivity of $\mathcal{G}(Q_n)$ are equal if and only if the latter is an integer.

Using Theorem 4.2.15 and Lemma 5.1.10, we have the following theorem.

Theorem 5.3.1. *For any integer $n \geqslant 2$, the algebraic connectivity of $\mathcal{G}(Q_n)$ satisfies* $\lambda_{4n-1}(\mathcal{G}(Q_n)) \geqslant 6\left(1 - \cos\frac{\pi}{4n}\right)$.

We next give a characterization of a generalized quaternion group among dicyclic groups in terms of adjacency property of its power graph. As an application of this result, we characterize the same in terms of Laplacian spectrum.

Proposition 5.3.2. *For any integer $n \geqslant 2$, a^n is adjacent to all other vertices of $\mathcal{G}(Q_n)$ if and only if Q_n is generalized quaternion.*

Proof. Let Q_n be generalized quaternion, i.e., n is a power of 2. Additionally, since $\langle a \rangle \cong \mathbb{Z}_{2n}$, it follows from Theorem 1.5.7(ii) that $\langle a \rangle$ is a clique. Hence a^n is adjacent to all other elements of $\langle a \rangle$ in $\mathcal{G}(Q_n)$. Furthermore, from Remark 1.3.16(ii), a^n is adjacent to $a^i b$ for all $0 \leqslant i \leqslant 2n$. Hence in view of Remark 1.3.16(iii), a^n is adjacent to all other vertices of $\mathcal{G}(Q_n)$.

Whereas, if Q_n is not generalized quaternion, then there exists a prime factor $p > 2$ of n. So $a^{\frac{2n}{p}}$ is an element of order p. Further, the order of a^n is 2. Hence the orders of a^n and $a^{\frac{2n}{p}}$ are co-prime and this implies that they are not adjacent in $\mathcal{G}(Q_n)$. □

Theorem 5.3.3. *For any integer $n \geqslant 2$, the Laplacian eigenvalue $4n$ of $\mathcal{G}(Q_n)$ has multiplicity two if Q_n is generalized quaternion and one otherwise.*

Proof. In view of Remark 1.3.16(ii), a^n is adjacent to every element of $Q_n - \langle a \rangle$. Moreover, $\overline{\mathcal{G}(Q_n)} - \{e, a^n\}$ is connected since each element of $\overline{\mathcal{G}(Q_n)} - \langle a \rangle$ is adjacent to every element of $\langle a \rangle - \{e, a^n\}$. Hence it follows from Proposition 5.3.2 that the number of components of $\overline{\mathcal{G}(Q_n)}$ is three if n is a power of 2 and two otherwise. Accordingly, by Theorem 1.4.9, the multiplicity of 0 as a Laplacian eigenvalue of

$\overline{\mathcal{G}(Q_n)}$ is three if n is a power of 2 and two otherwise. By Theorem 1.4.10, the multiplicity of $4n$ as a Laplacian eigenvalue of $\mathcal{G}(Q_n)$ is equal to one less than the multiplicity of 0 as a Laplacian eigenvalue of $\overline{\mathcal{G}(Q_n)}$. Thus the result follows. □

We order the elements of Q_n as

$$(e, a^n, a, a^{n+1}, \ldots, a^{n-1}, a^{2n-1}, b, a^n b, ab, a^{n+1}b, \ldots, a^{n-1}b, a^{2n-1}b)$$

and accordingly index rows and columns of $L(\mathcal{G}(Q_n))$. Also, we index the rows and columns of $L(\mathcal{G}(\mathbb{Z}_{2n}))$ corresponding to the ordering

$$(\overline{0}, \overline{n}, \overline{1}, \overline{n+1}, \ldots, \overline{n-1}, \overline{2n-1}).$$

Theorem 5.3.4. *For any integer $n \geqslant 2$,*

$$\Theta(\mathcal{G}(Q_n), x) = (x-2)^n (x-4)^n \det\Big(xI_{2n} - R_{2n}(x) - L(\mathcal{G}(\mathbb{Z}_{2n}))\Big),$$

where $R_{2n}(x)$ is the $2n \times 2n$ matrix

$$\frac{2n}{(x-2)} \begin{pmatrix} (x-1) & 1 & 0 & \cdots & 0 \\ 1 & (x-1) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Proof. By adjacency relations in $\mathcal{G}(\mathbb{Z}_{2n})$ and $\mathcal{G}(Q_n)$, the Laplacian matrix of $\mathcal{G}(Q_n)$ is

$$L(\mathcal{G}(Q_n)) = \left(\begin{array}{c|c} L(\mathcal{G}(\mathbb{Z}_{2n})) + M_{2n} & N_{2n} \\ \hline N_{2n}^T & P_{2n} \end{array}\right),$$

where M_{2n}, N_{2n} and P_{2n} are $2n \times 2n$ matrices:

(a) The $(1,1)$ and $(2,2)$ entries of M_{2n} are both $2n$, and all other entries are 0,

(b) N_{2n} has all entries -1 in first two rows and the rest of the entries are 0, and

(c) P_{2n} is given by

$$P_{2n} = \begin{pmatrix} 3 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 3 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 3 \end{pmatrix}.$$

It is known that A, B, C and D are square matrices of the same order and D is invertible, then $\det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \det(D)\det(A - BD^{-1}C)$ (cf. [Zhang, 2005]). So, as P_{2n} is invertible, we have

$$\begin{aligned} &\Theta(\mathcal{G}(Q_n), x) \\ &= \det \left(\begin{array}{c|c} xI_{2n} - L(\mathcal{G}(\mathbb{Z}_{2n})) - M_{2n} & -N_{2n} \\ \hline -N_{2n}^T & xI_{2n} - P_{2n} \end{array}\right) \\ &= \det(xI_{2n} - P_{2n}) \\ &\qquad \det\left(xI_{2n} - L(\mathcal{G}(\mathbb{Z}_{2n})) - M_{2n} - N_{2n}(xI_{2n} - P_{2n})^{-1}N_{2n}^T\right). \end{aligned} \tag{5.8}$$

Observe that

$$\det(xI_{2n} - P_{2n}) = \left\{ \det \begin{pmatrix} x-3 & 1 \\ 1 & x-3 \end{pmatrix} \right\}^n = (x-2)^n(x-4)^n. \tag{5.9}$$

Moreover,

$$(xI_{2n} - P_{2n})^{-1} = \frac{1}{(x-3)^2-1}\begin{pmatrix} x-3 & -1 & 0 & 0 & \cdots & 0 \\ -1 & x-3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x-3 & -1 & \cdots & 0 \\ 0 & 0 & -1 & x-3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & x-3 & -1 \\ 0 & \cdots & \cdots & \cdots & -1 & x-3 \end{pmatrix},$$

so that

$$N_{2n}(xI_{2n} - P_{2n})^{-1}N_{2n}^T = \frac{1}{(x-3)^2-1}\begin{pmatrix} 2n(x-4) & 2n(x-4) & 0 & \cdots & 0 \\ 2n(x-4) & 2n(x-4) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \frac{2n}{(x-2)}\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence from (5.8) and (5.9), and by setting $R_{2n}(x) = M_{2n} + N_{2n}(xI_{2n} - P_{2n})^{-1}N_{2n}^T$, the proof follows. □

Corollary 5.3.5. *For any integer $n \geqslant 2$, 2 and 4 are Laplacian eigenvalues of $\mathcal{G}(Q_n)$ with multiplicities at least $n-1$ and n, respectively.*

Proof. In Theorem 5.3.4, let $T_{2n}(x)$ be the matrix obtained by subtracting first row from second row of $xI_{2n} - R_{2n}(x) - L(\mathcal{G}(\mathbb{Z}_{2n}))$. Then $(x-2)\det(T_{2n}(x))$ is a polynomial matrix and $\det(T_{2n}(x)) = \det(xI_{2n} - R_n(x) - L(\mathcal{G}(\mathbb{Z}_{2n})))$. Hence the proof follows. □

Theorem 5.3.6. *For any integer $n \geqslant 2$, the algebraic connectivity of $\mathcal{G}(Q_n)$ satisfies*

$$1 < \lambda_{4n-1}(\mathcal{G}(Q_n)) \leqslant 2.$$

Proof. Since $\mathcal{G}^*(Q_n)$ is connected, it follows from Theorem 1.4.12 that $\lambda_1(\overline{\mathcal{G}^*(Q_n)}) < 4n-1$. Moreover, by applying Theorem 1.4.11 we have

$$\lambda_1(\overline{\mathcal{G}(Q_n)}) = \max\{\lambda_1(\overline{\mathcal{G}^*(Q_n)}), \lambda_1(\mathcal{G}(\{e\}))\} = \lambda_1(\overline{\mathcal{G}^*(Q_n)}).$$

Consequently, $\lambda_1(\overline{\mathcal{G}(Q_n)}) < 4n-1$ and hence by Theorem 1.4.10, $\lambda_{4n-1}(\mathcal{G}(Q_n)) > 1$. □

Theorem 5.3.7. *For any integer $n \geqslant 2$, the following statements are equivalent.*

(i) $\kappa(\mathcal{G}(Q_n)) = \lambda_{4n-1}(\mathcal{G}(Q_n))$.

(ii) *The algebraic connectivity of $\mathcal{G}(Q_n)$ is 2.*

(iii) *The algebraic connectivity of $\mathcal{G}(Q_n)$ is an integer.*

(iv) *$\mathcal{G}(Q_n)$ is Laplacian integral.*

(v) *Q_n is generalized quaternion.*

Proof. Suppose $\kappa(\mathcal{G}(Q_n)) = \lambda_{4n-1}(\mathcal{G}(Q_n))$. We first show that $\{e, a^n\}$ is the only minimum separating set of $\mathcal{G}(Q_n)$. Let S be a minimum separating set of $\mathcal{G}(Q_n)$ and if possible, $a^n \notin S$. Note that $e \in S$, and in view of Lemma 1.5.22, $|S| = 2$. So S contains at most one element of $[a]$, and as $n \geqslant 2$, $|[a]| = \phi(2n) \geqslant 2$. So $\mathcal{G}(\langle a \rangle) - S$ is connected. Additionally, by following Remark 1.3.16(ii), all elements of $(\mathcal{G}(Q_n) - \langle a \rangle) - S$ are adjacent to a^n. Consequently, $\mathcal{G}(Q_n) - S$ is connected; a contradiction. Hence $S = \{e, a^n\}$ and by applying Lemma 1.4.15, we have $\mathcal{G}(Q_n) =$

$\{\mathcal{G}(Q_n) - \{e, a^n\}\} \vee \mathcal{G}(\{e, a^n\})$. As a result, a^n is adjacent to all other vertices of $\mathcal{G}(Q_n)$. Hence applying Proposition 5.3.2, Q_n is generalized quaternion. We thus proved that (i) implies (v).

If Q_n is generalized quaternion, then it follows from Theorem 5.1.6 that $\mathcal{G}(Q_n)$ is Laplacian integral. So (iv) follows from (v). Whereas, it is trivial to see that (iv) implies (iii) . Now, if $\lambda_{4n-1}(\mathcal{G}(Q_n))$ is an integer, then it follows from Theorem 5.3.6 that $\lambda_{4n-1}(\mathcal{G}(Q_n)) = 2$. We thus have (ii). Finally, using Lemma 1.5.22, we see that (i) is a consequence of (ii). □

5.4 p-Group

In this section, we determine the algebraic connectivity and multiplicity of Laplacian spectral radius of power graphs of all p-groups. Moreover, we show that algebraic connectivity and vertex connectivity are always equal for power graphs of p-groups. We show that power graphs of p-groups are Laplacian integral by finding their all possible Laplacian eigenvalues. Followed by this, we discuss their multiplicities.

Since $\Gamma(g)$ is connected for any group G and $g \in G$, we have the following lemma.

Lemma 5.4.1. *For any group G and $g \in G$, the multiplicity of the Laplacian eigenvalue 0 of $\Gamma(g)$ is one.*

For the rest of this section, G denotes a p-group. Accordingly, every element of G^* has prime power order, so that $\exp(G)$ is the largest order of an element in G.

When G is cyclic or generalized quaternion, the algebraic connectivity and multiplicity of Laplacian spectral radius of $\mathcal{G}(G)$ are already known (cf. Corollary 1.5.28, Theorem 5.1.6). In what follows, we determine the above two graph theoretic quantities for the rest of the p-groups and show the dependency among them.

Theorem 5.4.2. *The following statements are equivalent.*

(i) *G is neither cyclic nor generalized quaternion.*

(ii) *The algebraic connectivity of* $\mathcal{G}(G)$ *is* 1.

(iii) *The multiplicity of the Laplacian eigenvalue* n *of* $\mathcal{G}(G)$ *is one.*

Proof. Lemma 1.5.17 and Proposition 5.1.9(i) together ascertain that (i) and (ii) imply each other.

We now show that (i) and (iii) are equivalent. Let G be neither cyclic nor generalized quaternion. Then $\mathcal{G}^*(G)$ has at least two components (cf. Lemma 1.5.17). By Proposition 3.1.3 and the fact that $[g] \subseteq U(g)$ for all $g \in G$, each component of $\mathcal{G}^*(G)$ has at least $p-1$ vertices. This implies that each component of $\mathcal{G}^*(G)$ has at most $n-p$ vertices. Thus by applying Theorem 1.4.11, we conclude that the Laplacian eigenvalues of $\mathcal{G}^*(G)$ are bounded above by $n-p$. Hence using Lemma 5.1.7, $\lambda_i(\mathcal{G}(G)) \leqslant n-p+1 < n$ for all $2 \leqslant i \leqslant n-1$. Consequently, the multiplicity of the Laplacian eigenvalue n of $\mathcal{G}(G)$ is one. For converse, let G be either cyclic or generalized quaternion. Then it follows from Corollary 1.5.28 and Theorem 5.1.6 that the multiplicity of n is at least two.

This completes the proof of the theorem. □

Theorem 5.4.3. *Let* G *is of order* n*. Then* $\kappa(\mathcal{G}(G)) = \lambda_{n-1}(\mathcal{G}(G))$ *if and only if* G *is not cyclic.*

Proof. Let G be cyclic. By Theorem 1.5.19 and Corollary 1.5.28, $\kappa(\mathcal{G}(G)) = n-1$ and $\lambda_{n-1}(\mathcal{G}(G)) = n$, respectively. Hence the above equality does not hold.

Now suppose G is not cyclic. If G is generalized quaternion, Theorem 5.1.6 and Lemma 1.5.22 together yield $\kappa(\mathcal{G}(G)) = 2 = \lambda_{n-1}(\mathcal{G}(G))$. If G is neither cyclic nor generalized quaternion, then it follows from Proposition 5.1.9(i) and Theorem 5.4.2 that $\kappa(\mathcal{G}(G)) = 1 = \lambda_{n-1}(\mathcal{G}(G))$. □

The following lemma is a consequence of Lemma 3.2.3.

Lemma 5.4.4. *For any* $g \in G$ *with* $\pi(g) = 0$, $\Theta(\Gamma(g), x) = x(x - \phi(\mathrm{o}(g)))^{\phi(\mathrm{o}(g))-1}$.

Using Theorem 5.1.1, Theorem 5.1.2 and Theorem 3.2.4, we have the following proposition.

Proposition 5.4.5. *Let $g \in G$ and $\pi(g) > 0$. If the distinct primitive classes of g are $[g_1], [g_2], \ldots, [g_{\pi(g)}]$, then*

$$\Theta(\Gamma(g), x) = \frac{x(x - |U(g)|)^{\phi(\mathrm{o}(g))}}{x - \phi(\mathrm{o}(g))} \prod_{i=1}^{\pi(g)} \Theta\left(\Gamma(g_i), x - \phi(\mathrm{o}(g))\right). \tag{5.10}$$

In particular, for $g = e$,

$$\Theta(\mathcal{G}(G), x) = \frac{x(x-n)}{x-1} \prod_{i=1}^{\pi(e)} \Theta(\Gamma(g_i), x - 1). \tag{5.11}$$

Analogous to Remark 3.2.5 for structure, the following remark describes how Proposition 5.4.5 gives the Laplacian spectrum of power graph a p-group.

Remark 5.4.6. The complete Laplacian characteristic polynomial of $\mathcal{G}(G)$ can be obtained as described below. Starting with (5.11), for every $1 \leqslant i \leqslant \pi(e)$, we substitute for $\Theta(\Gamma(g_i), x-1)$ by following (5.10) and Lemma 5.4.4 when $\pi(g_i) > 0$ and $\pi(g_i) = 0$, respectively. Now if $\pi(g_i) > 0$ for any $1 \leqslant i \leqslant \pi(e)$, we do the similar substitution for $\Theta(\Gamma(h_{i,j}), x - p)$ for every primitive class $[h_{i,j}]$ of g_i. We continue this process till the substitution for $\Theta(\Gamma(h), x - \frac{\mathrm{o}(h)}{p})$ for every $\approx$-class $[h]$, $h \neq e$ is done.

Proposition 5.4.7. *Suppose* $\exp(G) = p^k$ *for some $k \in \mathbb{N}$. Then for any $0 \leqslant l \leqslant k-1$, if g is an element of order p^{k-l}, then every Laplacian eigenvalue of $\Gamma(g)$ is $\frac{\mathrm{o}(h_1)}{p} - p^{k-l-1}$ for some $h_1 \in U(g)$ or $|U(h_2)| + \frac{\mathrm{o}(h_2)}{p} - p^{k-l-1}$ for some $h_2 \in U(g)$.*

Proof. We note that the usual relation $<$ is well-founded on $\{l \in \mathbb{Z} : 0 \leqslant l \leqslant k-1\}$. We prove the proposition by applying the principle of well-founded induction on the above set.

If g is an element of order p^k, then clearly $\pi(g) = 0$. From Lemma 5.4.4, $\Theta(\Gamma(g), x) = x(x - \phi(p^k))^{\phi(p^k)-1}$. Moreover, $\frac{o(g)}{p} - p^{k-1} = 0$ and $|U(g)| + \frac{o(g)}{p} - p^{k-1} = \phi(p^k)$, which are precisely the two Laplacian eigenvalues of $\Gamma(g)$. Hence we have proved for $l = 0$. In fact, if $k = 1$, then we are done. So let $k > 1$. We now assume that the assertion holds for $l = m - 1$, where $1 \leqslant m \leqslant k - 1$, and then show it for $l = m$.

Let g be an element of order p^{k-m}. If $\pi(g) = 0$, then the statement holds for $l = m$ by an argument similar to that of order p^k. So let $\pi(g) > 0$ and the distinct primitive classes of g be $[g_1], [g_2], \ldots, [g_{\pi(g)}]$. Then for each $1 \leqslant i \leqslant \pi(g)$, it follows from induction hypothesis that every root of $\Theta\left(\Gamma(g_i), x - \phi(p^{k-m})\right)$ is $\frac{o(h_1)}{p} - p^{k-m-1}$ for some $h_1 \in U(g_i)$ or $|U(h_2)| + \frac{o(h_2)}{p} - p^{k-m-1}$ for some $h_2 \in U(g_i)$. Furthermore, from Proposition 5.4.5, we have

$$\Theta(\Gamma(g), x) = \frac{x(x - |U(g)|)^{\phi(p^{k-m})}}{x - \phi(p^{k-m})} \prod_{i=1}^{\pi(g)} \Theta\left(\Gamma(g_i), x - \phi(p^{k-m})\right).$$

Hence we conclude that the assertion holds for $l = m$. This completes the proof of the proposition. □

Theorem 5.4.8. *Every Laplacian eigenvalue of $\mathcal{G}(G)$ is among 0, $|G|$, $\frac{o(g)}{p}$ for some $g \in G^*$ or $|U(h)| + \frac{o(h)}{p}$ for some $h \in G^*$. In particular, $\mathcal{G}(G)$ is Laplacian integral.*

Proof. Since $|G| \geqslant p$, we have $\pi(e) > 0$. Let the distinct primitive classes of e be $[g_1], [g_2], \ldots, [g_{\pi(e)}]$. By Proposition 5.4.7, for each $1 \leqslant i \leqslant \pi(e)$, every Laplacian eigenvalue of $\Gamma(g_i)$ is of the form $\frac{o(h_1)}{p} - 1$ for some $h_1 \in U(g_i)$ or $|U(h_2)| + \frac{o(h_2)}{p} - 1$ for some $h_2 \in U(g_i)$. Hence applying (5.11), the proof follows. □

We next explore the multiplicities of the Laplacian eigenvalues obtained in Theorem 5.4.8. For $g, h \in G$, if $[g] = [h]$, then $o(g) = o(h)$ and $|U(g)| + o(g) = |U(h)| + o(h)$. When $[g] \neq [h]$, however, the above equalities may or may not hold.

Hence in this case we treat $\mathrm{o}(g)$ and $\mathrm{o}(h)$ (similarly, $|U(g)|+\mathrm{o}(g)$ and $|U(h)|+\mathrm{o}(h)$) as distinct Laplacian eigenvalues.

Theorem 5.4.9. *For any $g \in G^*$, $\mathrm{o}(g)$ is a Laplacian eigenvalue of $\mathcal{G}(G)$ if and only if either $\pi(g) = 0$ and $\mathrm{o}(g) > 2$ or $\pi(g) \geqslant 2$. If $\mathrm{o}(g)$ is a Laplacian eigenvalue of $\mathcal{G}(G)$, then its multiplicity is*

(i) $\phi(\mathrm{o}(g)) - 1$ *when* $\pi(g) = 0$ *and* $\mathrm{o}(g) > 2$;

(ii) $\pi(g) - 1$ *when* $\pi(g) \geqslant 2$.

Proof. If $\pi(g) = 0$, from Lemma 5.4.4, we have $\Theta(\Gamma(g), x) = x(x - \phi(\mathrm{o}(g)))^{\phi(\mathrm{o}(g))-1}$. Moreover, for $\pi(g) > 0$, it follows from Lemma 5.4.1 and (5.10) that $x - \phi(\mathrm{o}(g))$ is a factor of $\Theta(\Gamma(g), x)$ of exact power $\pi(g) - 1$. Hence, if $\pi(g) = 0$ and $\mathrm{o}(g) > 2$, then $\phi(\mathrm{o}(g))$ is a Laplacian eigenvalue of $\Theta(\Gamma(g), x)$ of multiplicity $\phi(\mathrm{o}(g)) - 1$. If $\pi(g) \geqslant 2$, then $\phi(\mathrm{o}(g))$ is a Laplacian eigenvalue of $\Theta(\Gamma(g), x)$ of multiplicity $\pi(g) - 1$. Whereas, if either $\pi(g) = 0$ and $\mathrm{o}(g) \leqslant 2$ or $\pi(g) = 1$, then $\phi(\mathrm{o}(g))$ is not a Laplacian eigenvalue of $\Theta(\Gamma(g), x)$. Consequently, the proof follows by applying Proposition 5.4.5 and Remark 5.4.6. □

In the following theorem, we apply Theorem 5.4.9 to determine the lower bound of multiplicity of p^k, $k \in \mathbb{N}$ as a Laplacian eigenvalue of power graph of a p-group.

Theorem 5.4.10. *Suppose that in G, the number of distinct $\approx$-classes with representatives of order p^k, $k \in \mathbb{N}$, having no primitive classes is s and those having at least one primitive class is t.*

(i) *If $t = 0$, $s > 0$ and $p^k > 2$, then p^k is a Laplacian eigenvalue of $\mathcal{G}(G)$ with multiplicity at least $s(\phi(p^k) - 1)$.*

(ii) *Suppose $t > 0$ and the distinct $\approx$-classes with representatives of order p^k are $[g_1], [g_2], \ldots, [g_t]$. If either $s > 0$ and $p^k > 2$ or $\pi(g_i) \geqslant 2$ for some $1 \leqslant i \leqslant t$, then p^k is a Laplacian eigenvalue of $\mathcal{G}(G)$ with multiplicity at least $s(\phi(p^k) - 1) + \sum_{i=1}^{t}(\pi(g_i) - 1)$.*

Theorem 5.4.11. *For any $g \in G^*$, $|U(g)| + \frac{\mathrm{o}(g)}{p}$ is a Laplacian eigenvalue of $\mathcal{G}(G)$ if and only if either $\pi(g) = 0$ and $\mathrm{o}(g) > 2$ or $\pi(g) > 0$. If $|U(g)| + \frac{\mathrm{o}(g)}{p}$ is a Laplacian eigenvalue of $\mathcal{G}(G)$, then its multiplicity is*

(i) $\phi(\mathrm{o}(g)) - 1$ *when* $\pi(g) = 0$ *and* $\mathrm{o}(g) > 2$;

(ii) $\phi(\mathrm{o}(g))$ *when* $\pi(g) > 0$.

Proof. If $\pi(g) = 0$, from Lemma 5.4.4, $\Theta(\Gamma(g), x) = x(x - \phi(\mathrm{o}(g)))^{\phi(\mathrm{o}(g))-1}$. Now if $\pi(g) > 0$, then from (5.10), $x - |U(g)|$ is a factor of $\Theta(\Gamma(g), x)$ of exact power $\phi(\mathrm{o}(g))$. Hence, if $\pi(g) = 0$ and $\mathrm{o}(g) > 2$, then $|U(g)| = \phi(\mathrm{o}(g))$ is a Laplacian eigenvalue of $\Theta(\Gamma(g), x)$ of multiplicity $\phi(\mathrm{o}(g)) - 1$. If $\pi(g) > 0$, then $|U(g)|$ is a Laplacian eigenvalue of $\Theta(\Gamma(g), x)$ of multiplicity $\phi(\mathrm{o}(g))$. Whereas, if $\pi(g) = 0$ and $\mathrm{o}(g) \leqslant 2$, then $\phi(\mathrm{o}(g))$ is not a Laplacian eigenvalue of $\Theta(\Gamma(g), x)$. These facts together with Proposition 5.4.5 and Remark 5.4.6 yield the required proof. □

Proposition 5.4.12. *For any $g \in G^*$, the following statements hold.*

(i) $|U(g)| + \frac{\mathrm{o}(g)}{p}$ *is a multiple of* $\mathrm{o}(g)$.

(ii) *If* $|U(g)| + \frac{\mathrm{o}(g)}{p}$ *is a prime power and* $\pi(g) > 0$, *then* $\pi(g) = ap + 1$ *for some integer* $m \geqslant 0$.

Proof. Let $\mathrm{o}(g) = p^k$ for some $k \in \mathbb{N}$. If $\pi(g) = 0$, then

$$|U(g)| + \frac{\mathrm{o}(g)}{p} = \phi(p^k) + p^{k-1} = p^k \text{ (by Lemma 3.2.3(i)).} \tag{5.12}$$

Whereas, if $\pi(g) > 0$, then there exist positive integers $a_1, \ldots, a_l$ with $a_1 = \pi(g)$ such that

$$\begin{aligned} |U(g)| &= \phi(p^k) + a_1\phi(p^{k+1}) + \ldots + a_l\phi(p^{k+l}) \text{ (by Proposition 3.2.4)} \\ \Rightarrow |U(g)| + \frac{\mathrm{o}(g)}{p} &= p^k + a_1\phi(p^{k+1}) + \ldots + a_l\phi(p^{k+l}). \end{aligned} \tag{5.13}$$

Consequently, (i) follows from (5.12) and (5.13). Now let $|U(g)| + \frac{o(g)}{p} = p^m$ for some $m \in \mathbb{N}$. Then it is evident from (5.13) that $m \geqslant k+1$. So comparing both sides of (5.13), we see that $p|(a_1 - 1)$. This completes the proof of (ii). □

Using Theorem 5.4.8 and Proposition 5.4.12(i), we have the following theorem.

Theorem 5.4.13. *Any non-zero Laplacian eigenvalue of $\mathcal{G}(G)$ is 1 or a multiple of p.*

The Laplacian spectra of a group of order p^2 is obtained in the following theorem. This also serves as an illustration for some of the previous results of this section.

Theorem 5.4.14. *If G is a group of order p^2, then the Laplacian spectrum of $\mathcal{G}(G)$ is*

$$\begin{pmatrix} 0 & p^2 \\ 1 & p^2-1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & p & p^2 \\ 1 & p & (p+1)(p-2) & 1 \end{pmatrix}.$$

Proof. If G is cyclic, then by Corollary 1.5.28, the Laplacian spectrum of $\mathcal{G}(\mathbb{Z}_{p^2})$ is $\begin{pmatrix} 0 & p^2 \\ 1 & p^2-1 \end{pmatrix}$. Since any group of order p^2 is abelian (cf. [Gallian, 2013]), if G is not cyclic, then $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In $\mathbb{Z}_p \times \mathbb{Z}_p$, the distinct primitive classes of $(\bar{0}, \bar{0})$ are precisely $[(\bar{1}, \bar{0})]$, $[(\bar{0}, \bar{1})]$ and $[(\bar{1}, \bar{1})], \ldots, [(\bar{1}, \overline{p-1})]$. Moreover, we observe that each of these classes induce subgraphs isomorphic to K_{p-1} in $\mathcal{G}(\mathbb{Z}_p \times \mathbb{Z}_p)$. So by applying (5.11), we get

$$\begin{aligned} \Theta(\mathcal{G}(G), x) &= \frac{x(x-p^2)}{x-1} \prod_{i=1}^{p+1} \Theta(K_{p-1}, x-1) \\ &= x(x-1)^p (x-p)^{(p+1)(p-2)} (x-p^2). \end{aligned}$$

This completes the proof of the theorem. □

5.5 Conclusion

We determined various necessary and sufficient conditions for the equality of vertex connectivity and algebraic connectivity of power graphs of $\mathbb{Z}_n$ and Q_n. We further showed that the equality always holds for p-groups. However, finding a group theoretic characterization for the above equality for all finite groups is still an open problem.

We proved that the power graph of Q_n is Laplacian integral if and only if Q_n is generalized quaternion. Moreover, power graph of a p-group is always Laplacian integral. Based on our observations, we state the following for $\mathbb{Z}_n$.

Conjecture 5.5.1. For any integer $n \geqslant 2$, the following statements are equivalent.

(i) The algebraic connectivity of $\mathcal{G}(\mathbb{Z}_n)$ is an integer.

(ii) $\mathcal{G}(\mathbb{Z}_n)$ is Laplacian integral.

(iii) n is a prime power or product of two primes.

6

Critical and Minimal Connectivity

Critically and minimally connected graphs have been in focus for their applications in communication networks. Certain necessary conditions for graphs exhibiting these properties can be found in [Halin, 1969; Lick, 1969]. Later, some of these notions were further generalized in [Akiyama et al., 2002; Maurer and Slater, 1977/78]. In Section 6.1, we examine whether power graphs of $\mathbb{Z}_n$ (for some orders), D_n, Q_n and p-groups are critically vertex (edge) connected. In Section 6.2, we characterize the finite groups such that their power graphs are minimally vertex connected. For finite groups of odd order, we then provide a characterization such that their power graphs are minimally edge connected. Furthermore, we test whether power graphs of the aforementioned groups are minimally vertex (edge) connected.

6.1 Critical connectivity

In this chapter, all graphs considered are finite and all groups considered are non-trivial, i.e., their order is at least two. Throughout this and the next section, we follow (1.1) and (1.2) for presentation of dihedral and dicyclic groups, respectively. In this section, we examine whether power graphs of some finite groups are critically vertex (edge) connected.

A brief comparison of notion of critically and minimally connected graphs along with some examples can be found in [Halin, 1969]. We begin with the following definitions.

Definition 6.1.1. A graph Γ is said to be *critically k-vertex connected* if $\kappa(\Gamma) = k$ and $\kappa(\Gamma - v) = k - 1$ for every vertex v of Γ. A *critically vertex connected graph* is a graph which is critically k-vertex connected for some k.

Definition 6.1.2. A graph Γ is said to be *critically k-edge connected* if $\kappa'(\Gamma) = k$ and $\kappa'(\Gamma - v) = k - 1$ for every vertex v of Γ. A *critically edge connected graph* is a graph which is critically k-edge connected for some k.

6.1.1 Vertex connectivity

In this subsection, we show that $\mathcal{G}(\mathbb{Z}_n)$ is not critically vertex connected when n is a product of two or three distinct primes. We then show that $\mathcal{G}(D_n)$ and $\mathcal{G}(Q_n)$ are not critically vertex connected. Finally, we provide a necessary and sufficient condition such that the power graph of a p-group is critically vertex connected.

Lemma 6.1.3. *If G is a finite group such that $\mathcal{G}(G)$ is critically vertex connected, then $\kappa(\mathcal{G}(G)) > 1$.*

Proof. Considering Remark 1.5.4, $\kappa(\mathcal{G}(G) - g) \geqslant 1$ for all $g \in G^*$. Thus, if $\kappa(\mathcal{G}(G)) = 1$, then $\mathcal{G}(G)$ is not critically vertex connected. Hence the proof follows. □

Theorem 6.1.4. *If n is a product of two or three distinct primes, $\mathcal{G}(\mathbb{Z}_n)$ is not critically vertex connected.*

Proof. If $n = pq$, where p and q are distinct primes, then $\kappa(\mathcal{G}(\mathbb{Z}_n)) = \phi(n) + 1$ (cf. Theorem 1.5.19). So, in view of Remark 1.5.14, we have $\kappa(\mathcal{G}(\mathbb{Z}_n) - \overline{a}) = \phi(n) + 1$ for every $\overline{a} \in \mathbb{Z}_n'$. Consequently, $\mathcal{G}(\mathbb{Z}_{pq})$ is not critically vertex connected.

Now let $n = pqr$ for primes $p < q < r$. Let $G_{\overline{p}} = \mathbb{Z}_n' - \overline{p}$ and $\approx_{\overline{p}}$ be the restriction of $\approx$ to $G_{\overline{p}}$. Then the equivalence classes of $G_{\overline{p}}$ under $\approx_{\overline{p}}$ are given by

$$[\overline{p}] - \{\overline{p}\}, [\overline{q}], [\overline{r}], [\overline{pq}], [\overline{pr}], [\overline{qr}], \tag{6.1}$$

where $[\overline{a}]$ is the equivalence class of $\overline{a}$ under $\approx$. Following the proof of Theorem 2.3.22, a minimal separating set of $\mathcal{G}(G_{\overline{p}})$ is precisely the union of any two $\approx_{\overline{p}}$ -classes C_1 and C_2 from (6.1) such that no element of C_1 is adjacent to any element of C_2 in $\mathcal{G}(G_{\overline{p}})$. Using the inequalities in (2.14), $[\overline{pr}] \cup [\overline{qr}]$ is a minimum separating set of $\mathcal{G}(G_{\overline{p}})$. Moreover, $[\overline{pr}] \cup [\overline{qr}]$ is also a minimum separating set of $\mathcal{G}'(\mathbb{Z}_n)$ (cf. Theorem 2.3.22). Thus in light of Remark 1.5.9, $\kappa(\mathcal{G}'(\mathbb{Z}_n) - \overline{p}) = \kappa(\mathcal{G}'(\mathbb{Z}_n))$. Finally, by Remark 1.5.14, we conclude that $\mathcal{G}(\mathbb{Z}_n)$ is not critically vertex connected. □

Theorem 6.1.5. *For any integer $n \geqslant 3$, $\mathcal{G}(D_n)$ is not critically vertex connected.*

Proof. Theorem 1.5.21 together with Lemma 6.1.3 give the required proof. □

Theorem 6.1.6. *For any integer $n \geqslant 2$, $\mathcal{G}(Q_n)$ is not critically vertex connected.*

Proof. Consider the presentation (1.2) of Q_n. Let $\Gamma_b = \mathcal{G}(Q_n) - b$. Then using Remark 1.3.16(v), $\Gamma_b - g$ is connected for all $g \in G - b$, whereas $\Gamma_b - \{e, a^n\}$ is disconnected. Hence $\kappa(\Gamma_b) = 2$. Consequently, by Theorem 1.5.22, $\mathcal{G}(Q_n)$ is not critically vertex connected. □

Theorem 6.1.7. *If G is a p-group, then $\mathcal{G}(G)$ is critically vertex connected if and only if G is cyclic.*

Proof. Let $\mathcal{G}(G)$ be critically vertex connected. Then by Theorem 6.1.3, $\mathcal{G}^*(G)$ is connected. So either G is cyclic or generalized quaternion (cf. Lemma 1.5.17). Since it follows from Theorem 6.1.6 that G is not generalized quaternion, we conclude that G is cyclic. Conversely, if G is cyclic then $\mathcal{G}(G)$ is complete (cf. Theorem 1.5.7(ii)), so that it is critically vertex connected. □

Corollary 6.1.8. *If $n \in \mathbb{N}$ is a prime power, then $\mathcal{G}(\mathbb{Z}_n)$ is critically vertex connected.*

6.1.2 Edge connectivity

In this subsection, we first supply a necessary condition for which the power graph of a finite group is critically edge connected. We ascertain that $\mathcal{G}(D_n)$ is not critically edge connected. We then provide characterizations such that power graphs of Q_n and p-groups are critically edge connected.

Theorem 6.1.9. *Let G be a finite group. Suppose that there exists $g \in G$ such that $\deg(g) > \delta(\mathcal{G}(G))$ and g is not adjacent to any $h \in G$ with $\deg(h) = \delta(\mathcal{G}(G))$. Then $\mathcal{G}(G)$ is not critically edge connected.*

Proof. First observe that $g \neq e$ and $\mathcal{G}(G)$ is not complete. Let $\Gamma_g = \mathcal{G}(G) - g$. Since g is not adjacent to any $h_1 \in G$ with $\deg_{\mathcal{G}(G)}(h_1) = \delta(\mathcal{G}(G))$, we have $\deg_{\Gamma_g}(h_1) = \deg_{\mathcal{G}(G)}(h_1)$. On the other hand, $\deg_{\Gamma_g}(h_2) \geqslant \deg_{\mathcal{G}(G)}(h_2) - 1$ for any $h_2 \in G$ with $\deg_{\mathcal{G}(G)}(h_2) > \delta(\mathcal{G}(G))$. Therefore, $\delta(\Gamma_g) = \delta(\mathcal{G}(G))$. Further, as $g \neq e$, we have $\mathrm{diam}(\Gamma_g) \leqslant 2$. Hence in view Theorem 1.2.22, we have $\kappa'(\Gamma_g) = \delta(\Gamma_g)$. These facts along with Theorem 4.1.2 imply that $\kappa'(\Gamma_g) = \kappa'(\mathcal{G}(G))$. As a result, $\mathcal{G}(G)$ is not critically edge connected. □

Theorem 6.1.10. *If G is a finite group and $\mathcal{G}(G)$ is critically edge connected, then $\kappa'(\mathcal{G}(G)) > 1$.*

Proof. Let $\kappa'(\mathcal{G}(G)) = 1$. Suppose there exists $g \in G$ such that $\mathrm{o}(g) > 2$. Then $|[g]| \geqslant 2$ and hence $\mathcal{G}(G) - g$ is connected. Thus $\mathcal{G}(G)$ is not critically edge connected. Now suppose $\mathrm{o}(g) = 2$ for all $g \in G^*$. Then it follows from Theorem 1.5.16 that $\mathcal{G}(G)$ remains connected even after deletion of any $g \in G^*$. So that again $\mathcal{G}(G)$ is not critically edge connected. This gives the proof of the theorem. □

Theorem 6.1.11. *For any integer $n \geqslant 3$, $\mathcal{G}(D_n)$ is not critically edge connected.*

Proof. Theorem 4.2.14 and Theorem 6.1.10 together give the required proof. □

Theorem 6.1.12. *For any integer $n \geqslant 2$, $\mathcal{G}(Q_n)$ is critically edge connected if and only if $n = 2$.*

Proof. By Theorem 4.1.2 and Theorem 4.2.15, $\kappa'(\mathcal{G}(Q_n)) = 3$. First let $n \geqslant 3$. For any $g \in \langle a \rangle$, if $\mathrm{o}(g) \geqslant 3$, then $|[g]| \geqslant 2$. Thus $\deg(g) > \phi(2n) + 1 \geqslant 3$. Whereas, if $\mathrm{o}(g)$ is 1 or 2, then g is e or a^n, respectively. Hence $\deg(g) > 3$ for all $g \in \langle a \rangle$. Now let $h \in Q_n - \langle a \rangle$. By Remark 1.3.16, a is not adjacent to h. Moreover, by following the proof of Theorem 4.2.15, $\delta(\mathcal{G}(Q_n)) = \deg(h) = 3$. Consequently, applying Theorem 6.1.9, we conclude that $\mathcal{G}(Q_n)$ is not critically edge connected.

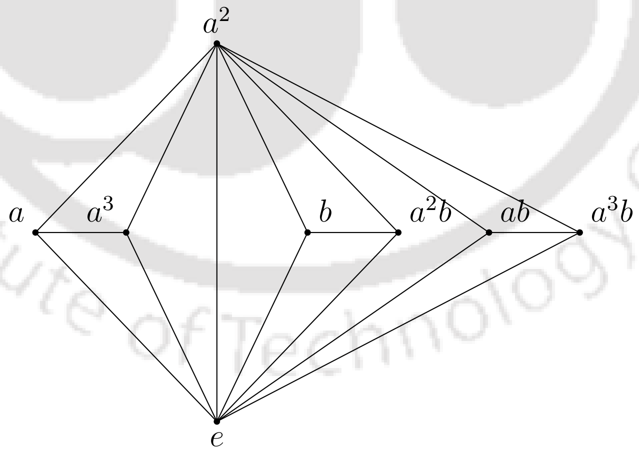

Figure 6.1: $\mathcal{G}(Q_2)$

Now let $n = 2$. Considering Remark 1.3.16, $\mathcal{G}(Q_2)$ is depicted in Figure 6.1. We can deduce that $\mathrm{diam}(\mathcal{G}(Q_2) - g) = 2$ and $\delta(\mathcal{G}(Q_2) - g) = 2$ for any $g \in Q_2 - \{e, a^2\}$.

Consequently, by Theorem 1.2.22, $\kappa'(\mathcal{G}(Q_2)-g) = 2$. As a result, $\mathcal{G}(Q_n)$ is critically edge connected. □

Theorem 6.1.13. *If G is a p-group, then $\mathcal{G}(G)$ is critically edge connected if and only if G is cyclic or $G = Q_2$.*

Proof. Let $\mathcal{G}(G)$ be critically edge connected. If possible, let $\mathcal{G}^*(G)$ be disconnected. That is, $\kappa'(\mathcal{G}(G) - e) = 0$. Thus, as $\mathcal{G}(G)$ is critically edge connected, we have $\kappa'(\mathcal{G}(G)) = 1$. Since this contradicts Theorem 6.1.10, $\mathcal{G}^*(G)$ is connected. Then by Lemma 1.5.17, either G is cyclic or generalized quaternion. Further, if G is generalized quaternion, then it follows from Theorem 6.1.12 that $G = Q_2$.

Conversely, if G is cyclic, then $\mathcal{G}(G)$ is complete (cf. Theorem 1.5.7(ii)). Hence $\mathcal{G}(G)$ be critically edge connected. Whereas, is follows from Theorem 6.1.12 that $\mathcal{G}(Q_2)$ is critically edge connected. □

An immediate corollary of Theorem 6.1.13 is the following.

Corollary 6.1.14. *If $n \in \mathbb{N}$ is a prime power, then $\mathcal{G}(\mathbb{Z}_n)$ is critically edge connected.*

6.2 Minimal connectivity

In this section, we do some study on power graphs of finite groups that are minimally vertex (edge) connected.

Definition 6.2.1. A graph Γ is said to be *minimally k-vertex connected* if $\kappa(\Gamma) = k$ and $\kappa(\Gamma - \varepsilon) = k - 1$ for every edge ε of Γ. A *minimally vertex connected graph* is a graph which is minimally k-vertex connected for some k.

Definition 6.2.2. A graph Γ is said to be *minimally k-edge connected* if $\kappa'(\Gamma) = k$ and $\kappa'(\Gamma - \varepsilon) = k - 1$ for every edge ε of Γ. A *minimally edge connected graph* is a graph which is minimally k-edge connected for some k.

6.2.1 Vertex connectivity

In this subsection, we first present some results on effect of deletion of an edge on the vertex connectivity of a graph. Consequently, we give some necessary conditions for minimally vertex connected graphs. We then apply these results on power graphs and obtain a characterization such that power graph of a finite group is minimally vertex connected. We finally examine this concept on some specific finite groups.

Lemma 6.2.3 ([Halin, 1969]). *The minimum degree of a minimally k-vertex connected graph is k.*

Lemma 6.2.4. *Let Γ be a graph with an edge ε such that $\Gamma - \varepsilon$ is connected. If $\kappa(\Gamma - \varepsilon) = \kappa(\Gamma) - 1$, then no minimum separating set of $\Gamma - \varepsilon$ contains endpoints of ε.*

Proof. Suppose S is a minimum separating set of $\Gamma - \varepsilon$ and ε is incident to u and v. If possible, let S contain at least one of u or v. Then we get $(\Gamma - \varepsilon) - S = \Gamma - S$. So S is a separating set of Γ. Thus $\kappa(\Gamma) \leqslant \kappa(\Gamma - \varepsilon)$; a contradiction. Hence no minimum separating set of $\Gamma - \varepsilon$ contains u and v. □

The next result follows immediately from Lemma 6.2.4.

Lemma 6.2.5. *If Γ is a minimally vertex connected graph, then for any edge ε, no minimum separating set of $\Gamma - \varepsilon$ contains endpoints of ε.*

Theorem 6.2.6. *Let Γ be a non-complete graph with n vertices. If Γ is minimally vertex connected, then it has at most one vertex of degree $n - 1$.*

Proof. We first note that $n \geqslant 3$. If possible, let Γ has two vertices of degree $n - 1$, say u and v. Clearly u and v are adjacent. Let $\Gamma_\varepsilon = \Gamma - \varepsilon$, where ε is the edge incident to u and v and S be a minimum separating set of Γ_ε. From the fact that Γ is non-complete and is minimally vertex connected, $\kappa(\Gamma_\varepsilon) < \kappa(\Gamma) \leqslant n - 2$. Thus $\Gamma_\varepsilon - S$ has at least one vertex not equal to u and v. As both u and v have degree

$n-1$ in Γ, they are adjacent to every vertex w, $w \neq u, v$, in $\Gamma_\varepsilon - S$. This implies that $\Gamma_\varepsilon - S$ is connected; a contradiction. Consequently, the proof follows. □

Theorem 6.2.7. *Let Γ be a non-complete graph with adjacent vertices u and v such that $\kappa(\Gamma-\varepsilon) = \kappa(\Gamma)-1$, where ε is the edge incident to u and v. If $\Gamma-\varepsilon$ is connected and S is a minimum separating set of $\Gamma-\varepsilon$, then $S \cup \{u\}$ or $S \cup \{v\}$ is a minimum separating set of Γ.*

Proof. Let Γ be a graph on n vertices. Since Γ is non-complete, $\kappa(\Gamma) \leqslant n-2$, and hence $\kappa(\Gamma-\varepsilon) \leqslant n-3$. Let S be a minimum separating set of $\Gamma-\varepsilon$. Then $\Gamma - S$ is a connected graph with at least three vertices. In view of Lemma 6.2.4, $u, v \notin S$. Hence, for every vertex w, $w \neq u, v$, in $\Gamma - S$, there exists exactly one path connecting w to u or v and not containing ε. Hence $S \cup \{u\}$ or $S \cup \{v\}$ is a separating set of Γ. Since $\kappa(\Gamma-\varepsilon) = \kappa(\Gamma)-1$, the proof follows. □

We now apply the results obtained above to characterize power graphs of finite groups that are minimally vertex connected.

Proposition 6.2.8. *Let G be a finite group and $g, h \in G$, $g \neq h$. If $\langle g \rangle = \langle h \rangle$ and ε is the edge incident to g and h, then $\kappa(\mathcal{G}(G)-\varepsilon) = \kappa(\mathcal{G}(G))$.*

Proof. Let $\langle g \rangle = \langle h \rangle$. If possible, suppose $\kappa(\mathcal{G}(G)-\varepsilon) \neq \kappa(\mathcal{G}(G))$. Since g is connected to h by the path g, e, h, $\mathcal{G}(G)-\varepsilon$ is connected. Let S be a minimum separating set of $\mathcal{G}(G)-\varepsilon$. Then it follows from Theorem 6.2.7 that $S \cup \{g\}$ or $S \cup \{h\}$, say $S \cup \{g\}$ is a minimum separating set of $\mathcal{G}(G)$. In view of Theorem 2.2.2, $S \cup \{g\}$ is a union of $\approx$-classes. This implies $[g] \subset S \cup \{g\}$, so that $h \in S$. However, by Lemma 6.2.4, $h \notin S$. Hence we conclude that $\kappa(\mathcal{G}(G)-\varepsilon) = \kappa(\mathcal{G}(G))$. □

Theorem 6.2.9. *For a finite group G, $\mathcal{G}(G)$ is minimally vertex connected if and only if G is a cyclic group of prime power order or G is an elementary abelian 2-group.*

Proof. Let $\mathcal{G}(G)$ be minimally vertex connected. If $\mathcal{G}(G)$ is complete, then G a cyclic group of prime power order (cf. Theorem 1.5.7(ii)). Now, suppose $\mathcal{G}(G)$ is non-complete. Let $g \in G^*$. By Proposition 6.2.8, if there exists $h \in [g] - g$, then $\kappa(\mathcal{G}(G) - \varepsilon) = \kappa(\mathcal{G}(G))$, where ε is the edge incident to g and h. However, this contradicts the fact that $\mathcal{G}(G)$ is minimally vertex connected. Thus $[g] = \{g\}$, and since $|[g]| = \phi(\mathrm{o}(g))$, $\mathrm{o}(g) = 2$. Hence in light of Remark 1.3.27, G is an elementary abelian 2-group.

Conversely, if G a cyclic group of prime power order, then $\mathcal{G}(G)$ is complete (cf. Theorem 1.5.7(ii)). Whereas, if G is an elementary abelian 2-group, then $\kappa(\mathcal{G}(G)) = 1$ and $\mathcal{G}(G) - \varepsilon$ is disconnected for every edge ε in $\mathcal{G}(G)$. Consequently, $\mathcal{G}(G)$ is minimally vertex connected. □

We now state some consequences of Theorem 6.2.9.

Corollary 6.2.10. *For $n \geqslant 2$, $\mathcal{G}(\mathbb{Z}_n)$ is minimally vertex connected if and only if n is a prime power.*

Corollary 6.2.11. *For any integer $n \geqslant 3$, $\mathcal{G}(D_n)$ is not critically vertex connected.*

Corollary 6.2.12. *For any integer $n \geqslant 2$, $\mathcal{G}(Q_n)$ is not critically vertex connected.*

Corollary 6.2.13. *For any p-group G, $\mathcal{G}(G)$ is minimally vertex connected if and only if G is a cyclic or an elementary abelian 2-group.*

6.2.2 Edge connectivity

For power graph of a finite group that is minimally edge connected, we give a sufficient condition involving regularity of its proper power graph. We then show that the converse holds for finite groups of odd order. Following this, we present some group theoretic characterizations for power graphs of finite groups of odd order that are minimally edge connected. We then study this notion on power graphs of some classes of groups.

Theorem 6.2.14. *Let G be a finite group. Then $\mathcal{G}(G)$ is minimally edge connected and $\kappa'(\mathcal{G}(G)) = 1$ if and only if G is an elementary abelian 2-group.*

Proof. Suppose $\mathcal{G}(G)$ is minimally edge connected and $\kappa'(\mathcal{G}(G)) = 1$. If possible, let there exist $g \in G$ such that $o(g) > 2$, so that $|[g]| \geqslant 2$. Further, let ε be an edge with endpoints in $[g]$. Then $\mathcal{G}(G) - \varepsilon$ is connected, which contradicts our initial assumption. Hence $o(g) = 2$ for all $g \in G^*$. Consequently, by Remark 1.3.27, G is an elementary abelian 2-group.

For converse, let G be an elementary abelian 2-group. Then it follows from Lemma 1.5.16 that deleting any edge of $\mathcal{G}(G)$ makes it disconnected. Hence $\kappa'(\mathcal{G}(G)) = 1$ and $\mathcal{G}(G)$ is minimally edge connected. □

Theorem 6.2.15. *Let G be a finite group. If $\mathcal{G}^*(G)$ is regular, then $\mathcal{G}(G)$ is minimally edge connected. The converse holds if G is of odd order.*

Proof. Let $\mathcal{G}^*(G)$ be regular. We set $\kappa'(\mathcal{G}(G)) = k$, so that in view of Theorem 4.1.2, all non-identity elements have degree k in $\mathcal{G}(G)$. Let ε be an edge in $\mathcal{G}(G)$. Then at least one endpoint of ε is a non-identity element, say g. Since deletion all edges incident to g in $\mathcal{G}(G)-\varepsilon$ (if any) makes it disconnected, we have $\kappa'(\mathcal{G}(G)-\varepsilon) \leqslant k-1$. Since $\kappa'(\mathcal{G}(G) - \varepsilon) \geqslant k - 1$, we have $\kappa'(\mathcal{G}(G) - \varepsilon) = k - 1$. Consequently, $\mathcal{G}(G)$ is minimally edge connected.

For converse, suppose G is of odd order and $\mathcal{G}^*(G)$ is not regular. Then there exists $g \in G$, $g \neq e$ with $\deg_{\mathcal{G}(G)}(g) > \delta(\mathcal{G}(G))$. Since G is of odd order, $|[g]| \geqslant 2$, so that $h \in [g]$, $h \neq g$. Note that $\deg_{\mathcal{G}(G)}(g) = \deg_{\mathcal{G}(G)}(h)$ (cf. Lemma 4.2.1). Let ε be the edge incident to g and h and $\Gamma_\varepsilon := \mathcal{G}(G) - \varepsilon$. Then $\deg_{\Gamma_\varepsilon}(g) = \deg_{\Gamma_\varepsilon}(h) = \deg_{\mathcal{G}(G)}(g) - 1$ and $\deg_{\Gamma_\varepsilon}(g_1) = \deg_{\mathcal{G}(G)}(g_1)$ for any $g_1 \in G - \{g, h\}$. Hence, as $\deg_{\mathcal{G}(G)}(g) > \delta(\mathcal{G}(G))$, we have $\delta(\Gamma_\varepsilon) = \delta(\mathcal{G}(G))$. Moreover, since $\mathrm{diam}(\Gamma_\varepsilon) \leqslant 2$, we have $\kappa'(\Gamma_\varepsilon) = \delta(\Gamma_\varepsilon)$ (cf. Theorem 1.2.22). These facts along with Theorem 4.1.2 yield $\kappa'(\Gamma_\varepsilon) = \kappa'(\mathcal{G}(G))$. Hence $\mathcal{G}(G)$ is not minimally edge connected. □

By applying Theorem 1.5.18 and Theorem 6.2.15, we have the following theorem.

Theorem 6.2.16. *Let G be a finite group of odd order. Then $\mathcal{G}(G)$ is minimally edge connected if and only if G is a cyclic group of prime power order or* $\exp(G)$ *is a prime.*

The following are two immediate corollaries of Theorem 6.2.16.

Corollary 6.2.17. *For any odd integer $n > 0$, $\mathcal{G}(\mathbb{Z}_n)$ is minimally edge connected if and only if n is a prime power.*

Corollary 6.2.18. *Let G be a p-group and $p \geqslant 3$. Then $\mathcal{G}(G)$ is minimally edge connected if and only if G is cyclic or* $\exp(G) = p$.

Proposition 6.2.19. *Let G be a finite group isomorphic to direct product of finite copies of $\mathbb{Z}_4$. Then $\mathcal{G}(G)$ is minimally edge connected if and only if G is isomorphic to $\mathbb{Z}_4$.*

Proof. Let G be isomorphic to $H := \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$ (t times) for some integer $t \geqslant 2$. We show that $\mathcal{G}(H)$ is not minimally edge connected. We denote $\mathbf{0} = (\overline{0}, \ldots, \overline{0})$. Considering Theorem 1.3.25, all non-identity elements in H have order either two or four. Moreover, by Proposition 3.1.4, no two elements of order two are adjacent in $\mathcal{G}(H)$.

We first claim that if w is an element of order two in H, then w is adjacent to exactly $2t$ elements of order four in $\mathcal{G}(H)$. Clearly every component of w is either $\overline{0}$ or $\overline{2}$. We construct an element $z \in H$ as follows. If the i^{th} component of w is $\overline{0}$, set i^{th} component of z as $\overline{0}$ or $\overline{2}$, and if the i^{th} component of w is $\overline{2}$, set i^{th} component of z as $\overline{1}$ or $\overline{3}$. Then $w = 2z$ and there are $2t$ such z possible for w.

By following Theorem 4.1.2 and Lemma 4.2.11, we have $\kappa'(\mathcal{G}(H)) = 3$. Let $z = (\overline{1}, \overline{0}, \ldots, \overline{0})$ and ε be the edge between $\mathbf{0}$ and $2z$ in $\mathcal{G}(H)$. We denote $\Gamma_\varepsilon = \mathcal{G}(H) - \varepsilon$. Suppose S is a minimum disconnecting set of Γ_ε.

If $|S| = 1$, then $S \cup \{\varepsilon\}$ is a disconnecting set of $\mathcal{G}(H)$. Since $\kappa'(\mathcal{G}(H)) = 3$, this is not possible. Thus $|S| \geqslant 2$. Now let $|S| = 2$ and $S = \{\varepsilon_1, \varepsilon_2\}$. We showed that if w is any element of order two in H, then it is adjacent to at least four vertices of order four in $\mathcal{G}(H)$. Hence w is adjacent to at least one vertex of order four which is adjacent to $\mathbf{0}$ in $\Gamma_\varepsilon - S$. Consequently, w is connected to $\mathbf{0}$ by a path in $\Gamma_\varepsilon - S$. We have the following cases for $\Gamma_\varepsilon - S$.

Case 1: All elements of order four are adjacent to $\mathbf{0}$ in $\Gamma_\varepsilon - S$. Then $\Gamma_\varepsilon - S$ is connected.

Case 2: All except one element, say z_1, of order four are adjacent to $\mathbf{0}$ (in $\Gamma_\varepsilon - S$). Since z_1 has order four, there exists $z_2 \in [z_1]$, $z_2 \neq z_1$. As $|S| = 2$, z_1 is adjacent to z_2 or $2z_1$ in $\Gamma_\varepsilon - S$. If z_1 is adjacent to z_2, then z_1 is connected to $\mathbf{0}$ (by a path in $\Gamma_\varepsilon - S$) because z_2 is adjacent to $\mathbf{0}$. Whereas, if z_1 is adjacent to $2z_1$, being an element of order two, $2z_1$ is connected to $\mathbf{0}$. Hence z_1 is connected to $\mathbf{0}$. Since all other vertices are connected to $\mathbf{0}$, $\Gamma_\varepsilon - S$ is connected.

Case 3: All except two elements, say z_1 and z_2, of order four are adjacent to $\mathbf{0}$ (in $\Gamma_\varepsilon - S$). Then as $|S| = 2$, both z_1 and z_2 are adjacent to $2z_1$ and $2z_2$, respectively, and since $2z_1$ and $2z_2$ are of order two, they are connected to $\mathbf{0}$. Since all other vertices are connected to $\mathbf{0}$, $\Gamma_\varepsilon - S$ is connected.

From the above cases, we conclude that if $|S| = 2$, then $\Gamma_\varepsilon - S$ is connected; a contradiction. So $|S| \geqslant 3$. Since $\kappa'(\mathcal{G}(H)) = 3$, we have $|S| = 3$. Consequently, $\mathcal{G}(H)$ and hence $\mathcal{G}(G)$ is not minimally edge connected.

Conversely, if G is isomorphic to $\mathbb{Z}_4$, then $\mathcal{G}(G)$ is complete (cf. Theorem 1.5.7(ii)). Hence $\mathcal{G}(G)$ is not minimally edge connected. □

Theorem 6.2.20. *Let G be an abelian p-group. Then $\mathcal{G}(G)$ is minimally edge connected if and only if G is a cyclic or or* $\exp(G) = p$.

Proof. Let $\mathcal{G}(G)$ be minimally edge connected. If $p \geqslant 3$, then it follows from Corollary 6.2.18 that G is a cyclic or $\exp(G) = p$.

Now suppose $p = 2$ and G is not cyclic. In view of Theorem 1.3.26, G is isomorphic to $H := \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_t}}$ for some positive integers $t \geqslant 2$ and

$\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_t$. Then considering Remark 1.5.5, $\mathcal{G}(H)$ is minimally edge connected, and it is enough to show that $\exp(H) = p$.

For every $1 \leqslant i \leqslant t$, consider $z_i \in H$ with i^{th} component $\overline{1}$ and all other components are $\overline{0}$. Then by Lemma 4.2.11, $\deg_H(z_i) = p^{\alpha_i}$ for all $1 \leqslant i \leqslant r$.

If possible, let $\alpha_t > \alpha_1$. Let ε be an edge with endpoints in $[z_t]$ and $\Gamma_\varepsilon := \mathcal{G}(H) - \varepsilon$. Then $\deg_{\Gamma_\varepsilon}(u) \geqslant \deg_{\mathcal{G}(H)}(u) - 1$ for all $u \in [z_t]$ and $\deg_{\Gamma_\varepsilon}(v) = \deg_{\mathcal{G}(H)}(v)$ for all $v \in H - [z_t]$. Moreover, by Theorem 4.2.12, we have $\delta(\mathcal{G}(H)) = p^{\alpha_1}$. Since $\alpha_t > \alpha_1$, we get $\delta(\mathcal{G}(H)) = \delta(\Gamma_\varepsilon)$. Since $\operatorname{diam}(\Gamma_\varepsilon) \leqslant 2$, in view of Theorem 1.2.22 and Theorem 4.1.2, we have $\kappa'(\mathcal{G}(H)) = \kappa'(\Gamma_\varepsilon)$. This contradicts the fact that $\mathcal{G}(H)$ is minimally edge connected. So we have $\alpha_1 = \alpha_2 = \ldots = \alpha_t$ ($= \alpha$ say).

Now if possible, let $\alpha \geqslant 2$. Since $\langle z_1 \rangle$ is a clique in $\mathcal{G}(H)$, pz_1 is adjacent to all other vertices in $\langle z_1 \rangle$. Consider an element

$$w = (\overline{a_1 p^{\alpha_1 - 1} + 1}, \overline{a_2 p^{\alpha_2 - 1}}, \ldots, \overline{a_t p^{\alpha_t - 1}}),$$

where $0 \leqslant a_i \leqslant p-1$ for all $1 \leqslant i \leqslant t$. We notice that w is adjacent to $pz_1 = (\overline{p}, \overline{0}, \ldots, \overline{0})$, but if $a_i \neq 0$ for any $2 \leqslant i \leqslant t$, then $w \notin \langle z_1 \rangle$. Thus

$$\deg_{\mathcal{G}(H)}(g) > \deg_{\mathcal{G}(H)}(z_1) \tag{6.2}$$

for all $g \in [pz_1]$.

We now have the following cases.

Case 1: $\alpha \geqslant 3$, or $\alpha = 2$ and $p \geqslant 3$. First let $\alpha \geqslant 3$. By an argument similar to that of (6.2), we have

$$\deg_{\mathcal{G}(H)}(p^2 z_1) > \deg_{\mathcal{G}(H)}(z_1). \tag{6.3}$$

So if ε_1 is the edge between pz_1 and $p^2 z_1$, then by (6.2) and (6.3), we have $\delta(\mathcal{G}(H)) =$

$\delta(\mathcal{G}(H) - \varepsilon_1)$.

Now let $\alpha = 2$ and $p \geqslant 3$. Then $|[pz_1]| = \phi(p) \geqslant 2$. Let ε_2 be an edge with endpoints in $[pz_1]$, then by (6.2), we have $\delta(\mathcal{G}(H)) = \delta(\mathcal{G}(H) - \varepsilon_2)$. By an argument similar to that of ε, we get $\kappa'(\mathcal{G}(H)) = \kappa'(\mathcal{G}(H) - \varepsilon_j)$ for $j = 1, 2$. This is again a contradiction.

Case 2: $\alpha = 2$ and $p = 2$. Then it follows from Proposition 6.2.19 that $\mathcal{G}(H)$ is not minimally edge connected.

Therefore, we conclude that $\alpha = 1$, that is, $\exp(G) = p$. Conversely, if G is cyclic or $\exp(G) = p$, then by Theorem 1.5.18 and Theorem 6.2.15, $\mathcal{G}(G)$ is minimally edge connected. □

Theorem 6.2.21. *For any integer $n \geqslant 3$, $\mathcal{G}(D_n)$ is not minimally edge connected.*

Proof. Since $\exp(D_n) = n$, the proof follows from Theorem 4.2.14 and Theorem 6.2.14. □

Theorem 6.2.22. *For any integer $n \geqslant 2$, $\mathcal{G}(Q_n)$ is not minimally edge connected.*

Proof. In view of Theorem 4.1.2 and Theorem 4.2.15, we note that $\kappa'(\mathcal{G}(Q_n)) = 3$.

First let $n \geqslant 3$. Suppose ε_1 is an edge with endpoints in $[a]$. We denote $\Gamma_1 = \mathcal{G}(Q_n) - \varepsilon_1$. Then $\deg_{\Gamma_1}(g) \geqslant 2n - 2 \geqslant 4$ for all $g \in [a]$. Whereas, all other vertices of Γ_1 and $\mathcal{G}(Q_n)$ have same degree. In particular, $\deg_{\Gamma_1}(h) = \deg_{\mathcal{G}(Q_n)}(h) = 3$ for all $h \in Q_n - \langle a \rangle$. In fact, by following the proof of Theorem 4.2.15, we have $\delta(\Gamma_1) = 3$. Hence, as $\operatorname{diam}(\Gamma_1) = 2$, we have $\kappa'(\Gamma_1) = 3$ (cf. Theorem 1.2.22). We thus conclude that $\mathcal{G}(Q_n)$ is not minimally edge connected.

Now let $n = 2$. Suppose ε_2 is the edge incident to e and a^2. We denote $\Gamma_2 = \mathcal{G}(Q_2) - \varepsilon_2$. In view of Figure 6.1, $\delta(\Gamma_2) = 3$ and $\operatorname{diam}(\Gamma_2) = 2$. Hence application Theorem 1.2.22 yields $\kappa'(\Gamma_2) = 3$. Consequently, $\mathcal{G}(Q_2)$ is not minimally edge connected. □

6.3 Conclusion

In this chapter, we supplied some necessary and sufficient conditions for power graphs of finite groups that are minimally vertex (edge) connected. Moreover, we tested critical connectedness on power graphs of some specific groups. So one may consider supplying a group theoretic characterization for power graphs of finite groups that are critically vertex (edge) connected.

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