PRICING AND HEDGING OF DERIVATIVES IN MARKOV-MODULATED MARKETS THROUGH BENCHMARK APPROACH

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CERTIFICATE

It is certified that the work contained in the thesis entitled **"PRICING AND HEDGING OF DERIVATIVES IN MARKOV-MODULATED MARKETS THROUGH BENCHMARK APPROACH"** by **I. VENKAT APPAL RAJU** (Roll Number: 05612301), a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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Dr. N. Selvaraju
Thesis Supervisor

To

My Parents

and Sisters

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ABSTRACT

The aim of the thesis is to study the pricing and hedging problems for contingent claims for various Markov-modulated models through the benchmark approach. This approach is based on a specific benchmark portfolio known as the growth optimal portfolio (GOP). GOP has been obtained for different market models using the stochastic control method. When used as a numeraire, GOP ensures that all the benchmarked price processes are supermartingales. Using this supermartingale nature of benchmarked price, a fair price has been defined. Since the GOP is closely related to a martingale density or state-price density, it can be used as a tool to price derivatives in complete or incomplete markets under the real world probability measure itself. The major advantage of this method is that it can be used in situations where the conventional methods do not work.

We consider in this thesis Markov-modulated jump-diffusion models, where the stock price processes are modulated by irreducible continuous-time Markov processes. A Markov-modulated market model incorporates the stochastic volatility in a simple and empirically tractable way. Fair prices for the contingent claims are derived using the benchmark approach. For the case of complete markets, the hedging strategies are determined using the martingale representation theorem. We also derive the Föllmer-Schweizer decomposition for incomplete market models to get the risk-minimizing hedging strategy. For each model, it has been shown that the pricing under the minimal-martingale measure, if it exists, is equivalent to the benchmarked fair price.

We analyze the models under both the scenarios, one where all the information are available to the investor and the one where the investor can only observe the stock price processes (not the underlying uncertainties). In a market with incomplete (or partial or imperfect) information, the investor has to filter the unobservable processes from the available information. The markets with incomplete information is tackled in two ways. In one, the Föllmer-Schweizer decomposition assuming complete information is derived first

and its projection to the smaller filtration (with incomplete information) is constructed to derive the hedging strategy. In another, we convert the market with incomplete information to a complete information case by using the innovation process method in filtering theory and derive the hedging strategy using the Föllmer-Schweizer decomposition.

The derivative securities (or contingent claims) that we consider are of two types: default-free and defaultable, mostly European-style derivatives. A Markov-modulated defaultable Brownian market is considered and the defaultable contingent claims are priced with the intensity-based methodology. The recovery processes are assumed to have random payments at the default time as well as at the maturity of the claims. We consider two classes of models, one where the parameters are modulated by an observable finite state Markov process and another where the parameters are modulated by an unobservable Ornstein-Uhlenbeck process. The representation theorems for defaultable claims are established and, using this representation, the locally-risk-minimizing hedging strategies for the defaultable contingent claims are derived under the benchmark approach.

Contents

ABBREVIATIONS and SYMBOLS

GOP	Growth Optimal Portfolio
SDE	Stochastic Differential Equation
PDE	Partial Differential Equation
OU	Ornstein-Uhlenbeck
HJB	Hamilton-Jacobi-Bellman
$\mathbb{R}$	Set of real numbers
$\mathbf{B}(\mathbb{R})$	Set of bounded functions from $\mathbb{R}$ to $\mathbb{R}$
I	Indicator function
$\mathcal{C}^n$	Set of n times continuously differentiable functions
$L^2(S)$	$= \left\{ \delta \,\middle\vert\, E\left[\int_0^T \delta_s^2 d[S]_s\right] < \infty \right\}$
$\mathbb{L}^2(S)$	$= \left\{ \delta \,\middle\vert\, E\left[\int_0^T (\delta_t)^2 \cdot d[\mathbf{S}]_t + (\int_0^T \delta_t \cdot \mathbf{S}_t \mu_t dt)^2\right] < \infty \right\}$
$\mathbf{C}$	Cost process
$\mathbb{C}$	Consumption process
P	Real world probability mesaure
F	Filtration containing complete information about the market
$F^{X,Y,Z}$	A filtration generated by the processes X, Y and Z
$F^X \vee F^Y$	The smallest filtration containing F^X and F^Y
$\mathcal{E}$	Stochastic exponential
Ω	An abstract set containing all possible states of a financial market model.
$()^\top$	Transpose (of a vector or a matrix)
$t \wedge s$	Minimum of two quantities t and s
$U(T)$	Set of all admissible strategies during the time period $[0,\ T]$
$U(t,T)$	Set of all admissible strategies during the time period $[t,\ T]$
$\mathcal{S}$	State space for a finite state Markov process

Chapter 1

Introduction

1.1 A perspective

Without losing its application to practical aspects of trading and regulation, the theory of finance has become increasingly mathematical, to the point that problems in finance are now driving research in mathematics. Mathematical finance is a field of applied mathematics that works with financial situations to determine pricing models and resource values. Generally, it derives, and extends, the mathematical or numerical models suggested by financial economics. Furthermore, it attempts to tackle the problems in financial markets more efficiently and accurately, by using stochastic calculus.

Financial mathematics came into limelight after 1990 Nobel Prize in economics to Markowitz, Sharpe, and Merton and 1997 Nobel Prize in economics to Merton and Scholes. But it was Bachelier (1900), who created a model of Brownian motion for stock prices in financial markets. Wiener (1923) gave the mathematical construction of Brownian motion, which makes it more powerful to use in financial mathematics. In 1951, Itô refined and greatly improved Wiener's idea (Itô, 1951a, b) by introducing "multiple Wiener integral" . In the second half of 20th century, after Itô's innovative works in stochastic calculus (also known as Itô calculus), stochastic calculus became an area of interest for many mathematicians to analyze the financial markets (for more details on Itô's work, see Itô (1987)).

In his efforts to model a Markov processes $Y = \{Y_t\}_{t\geq 0}$, Itô constructed an stochastic differential equation (SDE) of the form

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t,$$

where t indicates time, μ and σ are deterministic functions and, $W = \{W_t\}_{t\geq 0}$ represents a standard Brownian motion (or Wiener process). He had two objectives: one was to make sense of the stochastic differential $\sigma(Y_t)dW_t$ which he accomplished in the article (Itô, 1944). The second problem was to connect Kolmogorov's work on Markov processes with his interpretation. He has shown that the distribution of Y solves Kolmogorov's forward equation. In Itô (1951a), he proved what is now known as Itô's formula given by

$$f(Y_t) = f'(Y_t)dY_t + \frac{1}{2}f''(Y_t)d[Y_t], \tag{1.1}$$

where f is a twice continuously differentiable real valued function and " $'$ " indicates the differentiation with respect to Y and $[\cdot]$ indicates the quadratic covariation of two stochastic processes. In 1965, Samuelson published two papers of ground breaking work. Samuelson (1965a) showed that a good model for stock price movements is a geometric Brownian motion. Samuelson (1965b) gave his economics arguments that prices must fluctuate randomly. He explains that Bachelier's model failed to ensure that stock prices always be positive, and that this model leads to absurd inconsistencies with economic principles, whereas a geometric Brownian motion model avoids these pitfalls. These developments helped mathematicians to work in different directions of finance more accurately, e.g., asset pricing: derivative securities, hedging strategies for derivatives, risk management of portfolios, portfolio optimization and model choice and calibration. One can see in Markowitz (1952) that handling of these problems are interlinked with each other.

Risk management is particularly important for financial corporations such as banks and insurance companies, since they face uncertain events in various lines of their business. There exists several characterizations of risk which do not always have to describe issues.

This could not only be market risk, credit risk or operational risk, but also model risk or liquidity risk. Insurance companies additionally have to deal with underwriting risk, for example. The aim of integrated risk management is to manage some or all kinds of potential risk simultaneously. According to McNeil et al. (2005), the more general notion of risk management describes a "discipline for living with possibility that future events may cause adverse effects". The methods of dealing with risks go back to the mean-variance optimization problems. These were first considered by De Finetti (1940), whereas the issue became famous with the work of Nobel laureate Markowitz (1952).

Asset pricing and hedging of contingent claims have been the most important problem for the researchers over last few decades. The modern asset pricing concept is considered to have started from Arrow's paper in 1953, where he considered a financial security as a series of commodities in various future states with different values. Markowitz, in his seminal work in 1952, proposed that investors need to balance between the return and risk and he used the mean to describe the expected return and the variance to describe the risk. Within this framework, he developed a mean-variance efficient frontier in which given the risk level, an optimal portfolio with the highest expected return is obtained, or equivalently given the expected return level, an optimal portfolio with the minimum risk is obtained. Based on this efficient frontier, the well known two-fund separation theorem was developed to help rational investors to develop an optimal investment strategy. This idea influenced later many classical theories. Of major influence has been the Capital Asset Pricing Model (CAPM), which is a mean-variance equilibrium model (Sharp, 1964; Lintner, 1965; Merton, 1973) and the Arbitrage Pricing Theory (APT), which uses the arbitrage concept in financial markets (Ross, 1976; Harrison and Kreps, 1979; Harrison and Pliska, 1981; Föllmer and Sondermann, 1986). The Markowitz portfolio model provides an algebraic condition on asset weights in mean-variance efficient portfolios. The CAPM turns this algebraic statement into a testable prediction about the relation between risk and expected return by identifying a portfolio that must be efficient. The attraction of CAPM is that it offers powerful and intuitively pleasing predictions about how to measure risk and the relation between expected return and risk. Unfortunately, the empirical record of the model is poor enough to invalidate the way it is used in applications. In 1976, Ross proposed the Arbitrage Pricing Theory by giving a financial market spanned by a number of factors, based on the results of the factor premiums and factor sensitivities.

The APT describes a mechanism by which investors can identify an asset which is incorrectly priced and bring that price back into alignment with its actual value. It is often viewed as an alternative to the CAPM since the APT has more flexible assumption requirements. While the CAPM formula requires the market's expected return, APT uses the risky asset's expected return and the risk premium of a number of macro-economic factors. Arbitrageurs use the APT model to make profit by taking advantage of mispriced securities. A mispriced security will have a price that differs from the theoretical price predicted by the model. By going short on an over priced security (and vise-versa), while concurrently going long (short) on the portfolio the APT calculations were based on, the arbitrageur is in a position to make a theoretically riskfree profit. By arbitrage we mean making money out of nothing without risk. The formalization of this notion and its path-breaking application to finance were accomplished in the 1970s by Black and Scholes (1973) using a PDE approach, Merton (1973, 1976) using a risk-neutral method, Cox and Ross (1976), Harrison and Kreps (1979) and Harrison and Pliska (1981) using a martingale approach. The revolutionary contribution of the theory of arbitrage was the realization that the absence of arbitrage implies a unique price for the claims (securities) that can be replicated in the market. Later the theory and practice of derivative pricing under the APT was developed in an extensive literature (see, Föllmer and Sondermann, 1986; Föllmer and Schweizer, 1991; Delbaen and Schachermayer, 1994a, b).

Under the APT, it was in a model with exponential Brownian (geometric Brownian) assets that Black and Scholes (1973) constructed a replicating portfolio and with it proposed a 'fair' price for a financial derivative. Their ideas were quickly advanced by Merton (1973). The key insight was that if it was possible to replicate the payoff of the derivative as the gains from trade from a dynamic, self-financing hedging strategy, then the initial fortune required to finance that strategy was exactly the arbitrage-free price for the option. Furthermore, since all the risks associated with the option were removed by hedging, the price is independent of the risk preferences of the agent. The Black, Scholes and Merton approaches are reviewed in sections 1.4 and 1.5.

The argument given above was developed into a martingale theory by Harrison and Kreps (1979) and Harrison and Pliska (1981). The martingale approach for option pricing is reviewed in section 1.6. These authors emphasized the central role of probability theory and martingale in APT. Following similar mathematics to Samuelson (1965a), instead of invoking the postulate that discounted option payoffs follow a martingale, they derived this postulate as an implication of a utility maximizing investor's optimization decision. Herein, they showed that the option's price could be viewed as its discounted expected value, where instead of using the actual probabilities to compute the expectation, one uses utility or risk-adjusted probabilities. These risk-adjusted probabilities later became known as "risk-neutral" or "equivalent martingale" probabilities.

The risk-neutral analysis of the Black-Scholes-Merton model relied on two results from the theory of stochastic processes and Brownian motions. Firstly, the Girsanov theorem that guarantees the existence of an equivalent martingale measure Q, through a Radon-Nikodým derivative which is also known as a state-price density, under which the discounted price processes are martingales. Secondly, the Brownian martingale representation theorem says that any random variable whose value is known at some terminal time can be written as its expected value plus a stochastic integral against Brownian motion. In the Black-

Scholes market setting, this translates into the result that any option payoff can be written as the price plus the gains from trade from a dynamic investment strategy in the underlying asset.

In 1967, Kunita and Watanabe developed the ideas on orthogonality of martingales pioneered by Meyer (1963), and Mottoo and Watanabe (1965), and they developed a theory of stable space of martingales which has been proved fundamental to the theory of martingale representation, known in financial mathematics as “market completeness”. The key result in financial mathematics is due to Delbaen and Schachermayer (1994 a, b) which states that “if the asset price processes is a locally bounded semimartingale, then there exists an equivalent local martingale measure if and only if the model satisfies no-free-lunch with vanishing-risk (NFLVR) condition”, which is known as the first fundamental theorem of asset pricing for continuous-time markets. Here, a local martingale is a stochastic process, satisfying the martingale property in a local sense and a stochastic process is called a semimartingale if it can be decomposed as the sum of a local martingale and an adapted finite-variation process. Using Fatou’s lemma, it can be shown that a non-negative local martingale is a supermartingale (see, Klebaner (2005)). A strict local martingale is a local martingale which is not martingale. Hence, the non-negative strict local martingale is a true supermartingale. Delbaen and Schachermayer (1994) also proposed the condition under which replication strategies can be found for all options and can be related to a condition on the equivalent martingale measures which is known as the second fundamental theorem of asset pricing. It states that “every bounded claim can be replicated if and only if there is unique equivalent local martingale measure”. Jarrow et. al. (2007) extends the mathematical finance literature on bubbles in complete markets.

In mathematical finance, there are models for which it is not necessary that the martingale measure will be unique and all the contingent claims can be perfectly hedged. Such a market scenario is called incomplete financial markets. Incompleteness can arise from many sources, for example transaction costs (Hodges and Neuberger, 1989; Davis et al., 1993), jump models (Merton, 1976; Bardhan and Chao, 1996), constraints on the trading strategies (Cvitanic and Karatzas, 1996; Soner and Touzi, 2000) or stochastic volatility (Hull and White, 1987; Heston, 1993; Fouque et al., 2000).

In a complete market, the fair prices of options are uniquely determined by the replication price. These prices can be calculated as the discounted expected values under the equivalent martingale measure. In an incomplete market, there is no unique fair price and no universal pricing algorithm. Instead, there are several alternative methodologies which have been proposed as pricing mechanisms. The first approach is to finesse the problem by writing down the dynamics of assets under a pricing measure. This approach bypasses the physical market measure. A second and related idea (Heston, 1993) is to chose a market price of risk for the non-traded assets. For example, the Föllmer-Schweizer (1991) minimal-martingale measure corresponds to a choice of a zero market price of risk for the non-traded risk. Föllmer and Sondermann (1986) have proposed the risk-minimization approach when the risky asset is represented by a martingale. Successively, it was extended to the general semimartingale case by Schweizer (1988, 1991) and by Föllmer and Schweizer (1991) where they called it a local-risk-minimizing approach. Local-risk-minimizing hedging strategy is reviewed in section 1.8. Another idea which has sometimes been exploited in the stochastic volatility literature (Scott, 1987) is to assume that there are call options which are liquidly traded. The introduction of other traded assets completes the market model. Hence, given the traded price of a call option, it is possible to price and hedge any other contingent claim. Of course this approach does not explain the price of the original traded call option. This idea has been extended by Dupire (1994) to create an elegant theory for the pricing of exotic options. Suppose that calls with all possible maturities and strikes are traded on the market. Then, under the assumption that the price process possess the Markov property, it is possible to infer the dynamics of the underlying process. In this approach, prices for vanilla options are taken from the market and then used to compute prices for path-dependent exotic options (Brown et al., 2001).

In incomplete markets, Föllmer and Sondermann (1986) suggested pricing through hedging criteria. Since perfect hedging is impossible, they considered the minimizing expected value of the square of the final difference between the contingent claim and the self-financing hedging strategy. In locally-risk-minimizing hedging criteria, introduced by Föllmer and Schweizer (1991), the mean square error has been tried to minimize between the price and the self-financing hedging strategy at each time. Another idea of hedging in incomplete markets is super-replication (El Karoui and Quenez, 1995; Delbaen and Schachermayer, 1994a, b). Rather than choosing a state-price density arbitrarily, Goll and Rüschendorf (2001) has given an approach to choose the state-price density which is smallest in an appropriate sense. For incomplete financial market models, another approach which is mostly found in literature is the utility indifference pricing (see, Hodges and Neuberger, 1989).

Apart from these problems, another classical problem in mathematical finance is the computation of optimal portfolios, where optimal refers to maximization of expected utility from terminal wealth or consumption. The portfolio problem of an investor trading in different assets is to choose an optimal investment and consumption strategy. To be more precise, an investor endowed with a given initial capital has to decide how many shares of which asset he should hold at each time instant to maximize his utility of consumption during the given time interval and of his total wealth at the given time horizon. This is also known as the portfolio management theory. The earliest approach to solving the portfolio problem is the so called mean-variance approach. It was pioneered by Markowitz (1952, 1959) and Tobin (1965). Merton (1969, 1971) determined optimal strategies in a Markovian Itô process setting using the dynamic programming approach. The Hamilton-Jacobi-Bellman equation from stochastic control theory leads to a non-linear partial differential equation (PDE) for the optimal expected utility as a function of time and current wealth. If one can solve this PDE, the optimal portfolio is immediately obtained. In multiperiod discrete-time models, a similar approach leads to a recursive equation instead of a single PDE (Mossin 1968; Samuelson, 1969; Hakansson, 1971a,b). One of the by-product of the utility maximization is the growth optimal portfolio (GOP). In the literature, it has been applied in as diverse connections as portfolio theory and gambling, utility theory, information theory, game theory, theoretical and applied asset pricing, insurance, capital structure theory and event studies. Kelly (1956) is the one who first introduced the GOP motivated from information theory. Breiman (1960, 1961) expanded the analysis of Kelly and discussed applications for long term investments and gambling in a more general mathematical setting. For more recent developments, one may refer to Korn (1997).

In discrete time, GOP has been treated by Bellman and Kalaba (1957), Elton and Gruber (1974) and Maier et al. (1977). In the continuous time case, the problem is much easier and was solved in Merton (1969). This problem along with a general study of the properties of the GOP has been studied for decades and is still being studied even today. The use of the GOP became referred to as the growth optimum theory and it was introduced as an alternative to the expected utility and the mean-variance approaches to asset pricing. It was argued that a theory for portfolio selection and asset pricing based on the GOP would have properties which are more appealing than those implied by the mean-variance approach developed by Markowitz (1952).

GOP is a myopic strategy (Christensen, 2005) which means short sighted and implies that the GOP strategy in a given period depends only on the distribution of returns in the next period. Hence the strategy is independent of the time horizon. Breiman (1960) has proved that when an asset price denominated in terms of the GOP, it becomes supermartingales, i.e., GOP has a numeraire property. Long (1990) has given a more thorough treatment to GOP. Though Long suggested this as a method for measuring abnormal returns in event studies and this approach has been followed by Santis et al. (2000) and Hentschel and Long (2004), the consequences of the numeraire property stretches much further. They suggested that a change of numeraire technique for asset pricing under which a change of probability measure would be unnecessary. In late nineties, this was treated explicitly in Bajeux-Besnainou and Portait (1997). Over the past one decade, this idea became developed further in the benchmark framework of Platen and his co-authors, who emphasize the applicability of this idea in the absence of a risk-neutral probability measure. In the seminal paper on the benchmark approach, Platen (2002) has gone beyond the traditional approaches for pricing derivatives in financial markets and obtained the key features of the CAPM and APT but without restrictive equilibrium assumption or assumptions on the existence of an equivalent risk-neutral measure. The use of the GOP as a tool for derivative pricing is reviewed in section 1.7. After Kelly (1956), GOP is further developed in a stream of literature including Long (1990), Artzner (1997), Bajeux-Besnainou and Portait (1997), Karatzas and Shreve (1998), Kramkov and Schachermayer (1999), and Becherer (2001). A survey of the benchmark approach can be found in Platen (2006).

The Black-Scholes model and its extensions comprise one of the major developments in modern finance under the APT. Since then, different methodologies have been introduced in literature using APT, like pricing via a hedging criteria, risk-neutral measure transformation method, martingale approach for option pricing in continuous-time financial market (see, Black-Scholes, 1973; Merton, 1973, 1976; Harrison and Pliska, 1981). The risk-neutral approach is still most applicable and powerful technique for option pricing in continuous-time financial market. But the existence of the risk-neutral martingale measure is not guaranteed. Heath and Platen (2002) have shown that the equivalent risk-neutral pricing measure will not exist for a constant elasticity of variance (CEV) model and therefore the classical risk-neutral pricing methodology fails.

The classical risk-neutral approach to the pricing of derivatives under the CEV model is described in Beckers (1980) and Schroder (1989), under some assumptions. It should be emphasized that these classical formulations typically describe real world dynamic which reach zero with strictly positive probability (see, Delbaen and Shirakawa, 1997, 2002). This leads to certain subtleties in how the pricing of derivatives should be formulated. Lewis (1998) uses some utility based equilibrium arguments to overcome difficulties in applying the risk-neutral pricing methodology for a class of stochastic volatility models which includes the CEV. Andersen and Andreasen (2000) take account of some of these subtleties when they apply it in the context of certain interest rate term-structure models.

Using Girsanov transformation in risk-neutral method, Merton (1976) has transferred the drifted stock price process to a martingale process under a risk-neutral probability measure. In cases where an equivalent risk-neutral measure does not exist, the Girsanov transformation cannot be applied and the standard risk-neutral pricing methodology breaks down. However, Heath and platen (2002) demonstrated how to overcome this problem by using the concept of fair pricing under the benchmark approach proposed in Platen (2002). Fair prices are the minimal prices which for a complete market permit perfect hedging for the contingent claim. Under the benchmark approach, the existence of an equivalent risk-neutral probability measure is not required.

GOP can be applied to cases where traditional approaches may require extra conditions to satisfy which may not allow to get the derivative prices for certain financial models. Whenever the GOP exists, one can always use the real world probability measure for pricing using the benchmark approach. Under the APT, conditional expectations have been used to define the no-arbitrage concept, which require participants to have perfect knowledge of the probabilistic dynamics of asset prices. To achieve this, they would need to have a correct model that is always exactly calibrated. The advantage of the benchmark methodology is that derivative pricing, value-at-risk analysis, portfolio optimization, calibration, estimation, filtering and other risk management tasks can be performed under one and the same probability measure, the real world probability measure. The benchmark approach can be extended to include asset price dynamics that are modelled as semimartingales incorporating both predictable and inaccessible jumps (Platen, 2004a). Under such a model, the benchmark approach is in a better position to analyze and manage the combinations of market, credit, operational, liquidity, insurance and other risks in an integrated framework. Even in incomplete financial markets, there will exist a unique benchmarked fair price whereas in the risk-neutral method there will be more then one martingale measures and one has to choose an appropriate one. Indeed, in incomplete financial markets, benchmarked option price is the same as the price under the minimal-martingale measure, if it exists (Christensen, 2005). Damir and Platen (2007), have provided a complete characterization of market extensions which are consistent with the existence of a GOP.

In case of market model with incomplete information, where all the random factors can not be observed directly, one has to use filtering theory for derivative pricing and hedging. Filtering theory deals with the real world probability measure to extract information from the observations via filter estimation for the hidden factors. So, it is therefore highly important to explore methods that are purely based on the real world probability measure and allow consistent filtering under partial information.

1.2 Markov-modulated models

In the famous Black-Scholes model of option pricing, it is assumed that the dynamics of stocks are given by a linear stochastic differential equation where the drift term μ and the volatility σ are constants. However, it has been observed that many financial assets do not have a constant volatility σ and the basic assumptions of the Black-Scholes model fails. There are attempts to extend the model by describing the evolution of parameters by stochastic differential equations (Hull and White, 1987; Wiggins, 1987) and find more or less explicit formulae for the hedging strategies. Now the question is which model of stochastic volatility should one choose? A good model should be tractable, realistic and it should be straightforward to estimate the parameters. Moreover, apart from providing a better fit to historic price data, the model should also have the ability to explain option price smiles, both over strike and over maturity. Finally, the model should give superior hedging performance to the Black-Scholes model.

Though the Black-Scholes-Merton model is inadequate to reflect the real market, due to its simplicity and direct insight of the financial markets, it is most useful model in financial market. It is based on a geometric Brownian motion and the normal distribution, but two empirical phenomena have received much attention recently: the first one is the asymmetric leptokurtic features of returns, i.e. the return distribution is skewed to the left, and has higher peak and two heavier tails then those of the normal distribution, and the second is the volatility smile. A jump-diffusion model solves the problem of asymmetric leptokurtic feature (Kou, 2002). To capture volatility smiles, one has to consider stochastic volatility models. Stochastic volatility models where the parameters are modulated by a Markov process with finite state space are studied by Di Masi et al. (1994), Deshpande and Ghosh (2008), among others. A market model modulated by a continuous-time diffusion process is studied by Hull and White (1987).

The use of Markov models is motivated by significant empirical evidence from the literature that favours and endorses Markov-switching models in the study of many macroeconomic variables. This provides more flexibility to financial models and incorporates stochastic volatility in a simple way. Earlier developments in this area during the late 80s within the time series context was pioneered by Hamilton (1988), among others, in a work where the unobserved regime follow a Markov process. Di Masi et al. (1994) obtained option pricing formula for stochastic volatility driven by a Markov chain in continuous time. Elliott and Swichuk (2004) derived option pricing formula for Markov-modulated Brownian and fractional Brownian markets with jumps. Elliott et al. (2005) consider the option pricing problems when the risky underlying assets are driven by a Markov-modulated geometric Brownian motion. They adopt a regime switching random Esscher transform to determine an equivalent martingale pricing measure in the considered incomplete financial market. Deshpande and Ghosh (2008) have obtained the minimal-martingale measure for the regime switching model and express the risk-minimizing strategy. We derive the risk-minimizing strategy in the same way, but under the real world probability measure with numeraire change.

To price American and European options for a regime switching model, Bollen (1998) employs lattice method and simulation, whereas Buffington and Elliott (2002) use partial differential equations. Guo (1999) use regime switching to model the finance market with insider information. Duan et al. (2002) establish a class of GARCH market models under regime switching for option pricing. For the important issue of fitting the regime switching model parameters, Hardy (2001) develops maximum likelihood estimation using real data from the S&P 500 and TSE 300 indices. In addition to option pricing, regime switching models have also been formulated and investigated for other problems; see Zhang (2001) for the developments of an optimal stock selling rule, Zhang and Yin (2004) for application in portfolio management, and Yin and Zhou (2004) for a dynamic Markowitz problem. Rieder and Bäuerle (2005) have studied portfolio optimization problems, where the drift rate of the stock price process is Markov-modulated and the driving factors cannot be observed by the investor. Rieder and Bäuerle (2007) have considered a jump-diffusion stock price process where the jump intensity rate is considered to be a Markov-modulated process and studied the portfolio optimization problems with unobservable driving factors. For a more detailed study of Markov-modulated models, we refer Brémaud (1981), Elliott et al. (1994), and Mamon and Elliott (2007).

In practice, not all quantities which determine the dynamics of security prices can be fully observed. Some of the factors that characterize the evolution of the market, like Markov process, are hidden. So, it is more realistic to assume that the investor has only partial information since prices and interest rates are published and available to the public, but the drift and the path of Brownian motion which are not observable are mere mathematical tools for modelling. Föllmer and Schweizer (1991) studied hedging under incomplete information using the minimal-martingale measure. Pham and Quenez (2001) have addressed the maximization problem of expected utility from terminal wealth under the incomplete information about the stochastic volatility.

1.3 Defaultable markets

Over the last thirty years, credit risk modelling and its pricing has been rapidly expanding field of finance, which is one of the fundamental factors of financial risk. Indeed, we can note a tremendous acceleration in research efforts aimed at a better understanding, modelling and hedging this kind of risk. But what does credit risk mean exactly? A default risk is the possibility that a counterparty in a financial contract will not fulfill a contractual commitment to meet his obligations stated in the contract. If this happens, we say that the party defaults, or that a default event occurs. More generally, by credit risk we mean the risk associated with any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads and default events (bankruptcy, insolvency, missed payments). It is important to make a clear distinction between the reference credit risk and the counterparty credit risk. The first term refers to the situation where both parties involved in a contract are supposed to be default-free, but the underlying assets are defaultable and the second refers where parties are defaultable. Credit derivatives are recently developed financial instruments that allow to trade and transfer the reference credit risk, either completely or partially, between the counterparties. The counterparty risk emerges in a clear way in such contracts as defaultable claims. These derivatives are contingent agreements that are traded over-the-counter between default-prone parties. Each side of contract is exposed to the counterparty risk of the other party but the underlying assets are assumed to be insensitive to credit risk (for an extensive survey of this subject, see Bielecki and Rutkowski (2002)). A classical example of defaultable claim is a European defaultable option, that is an option contract in which the payoff at maturity depends on whether a default event, associated with the option's writer, has occurred before maturity or not. Bélanger et al. (2004) and Blanchet-Scalliet and Jeanblanc (2004) have priced defaultable claims using the martingale representation theorem. The mean-variance hedging method has been extensively used in the context of defaultable markets by Bielecki et al. (2004a), Bielecki et al. (2004b), and Bielecki and Jeanblanc (2009). For instance, Bielecki et al. (2004b) provide an explicit formula for the optimal trading strategy which solves the mean-variance hedging problem. Moreover, they compare the results obtained using strategies adapted to the Brownian filtration, to the ones obtained using strategies based on the enlarged filtration, which encompasses also the observation of the default time.

The credit risk models can be characterize into two broad classes: structural models and reduced form models (also known as the intensity-based models). In the former approach, the total value of the firm's assets is directly used to determine the default event, which occurs when the firm's value falls through some boundary. The random time of default is announced by an increasing sequence of stopping times. By contrast, in the latter approach, the firm's value process either is not modelled at all, or it plays only an auxiliary role of a state variable. The default time is modelled as a stopping time that is not predictable; the default event thus arrives as a total surprise. They are called reduced form since they are abstracted from the explicit economic reasons for the default, i.e., they do not induce the asset-liability structure of the firm to explain the default as in a structural model approach.

Rather, reduced form models use debt prices as a main input to model the bankruptcy process.

Structural models began with Black and Scholes (1973), Merton (1974) and Black and Cox (1976), and continued with Longstaff and Schwartz (1995), Taurén (1999) and Collin-Dufresne, Goldstein (2001) and Eom et al. (2004). Duffie and Singleton (2003) show that any default-free term-structure model can be used to price bonds with default risk. One simply models the spot interest rate to include an instantaneous default spread. Affine term-structure models can then be tweaked to produce closed-form corporate bond prices. Reduced form models are studied by Blanchet-Scalliet and Jeanblanc (2004), Bélanger et al. (2004), Bielecki et al. (2004a), Bielecki et al. (2004b) and Bielecki and Rutkowski (2002), Bielecki and Jeanblanc (2009). For pricing and hedging credit risk, the information set observed by the market is the relevant one. This is the information set used by the market, in equilibrium, to determine prices. Given that belief, a reduced form model should be employed. Reduced-form models have proved to be a useful tool for analyzing the dynamics of credit spreads (Elizalde, 2005). Jarrow and Protter (2004) argue further that reduced form models are more appropriate in an information theoretic context given that we are unlikely to have complete information about the default point and expected recovery.

Bruti-Liberati et al. (2009) have considered defaultable term-structure models in a general setting under the benchmark approach. They demonstrated how a class of tractable defaultable term-structure models can be obtained. More specifically, by assuming independence between the discounted GOP and the default-adjusted short rate, they obtained closed form expressions for the prices of defaultable bonds. In case of a default market, even if an appropriate risk-neutral martingale measure exists, the H-hypothesis may not be stable under the measure transformation. Here, the H-hypothesis means invariance of martingale property under the extension of the filtration. So, it is better to use the benchmark approach for defaultable financial markets.

Collin-Dufresne et al. (2009) were the first to point out that the successive updating of the conditional expectation of the unobservable process in reaction to incoming default observation has the potential to generate contagion effects. Frey and Runggaldier (2010) concentrated on the mathematical analysis of filtering problems in reduced form credit risk models. Credit risk models with incomplete information have been considered by Kusuoka (1999), Duffie and Lando (2001), Giesecke and Goldberg (2004), Jarrow and Protter (2004), Coculescu et al. (2008) and Frey and Runggaldier (2010). They are concerned with structural models where the value of assets and liabilities is not directly observable. None of these contributions addresses the dynamics of credit derivative prices under the benchmark approach.

Here, we consider only the default event for our modelling of credit derivatives, i.e., the counterparty risk. If default occurs, the creditor will only receive the amount recovered from the debtor, called recovery payment. The recovery payment is frequently specified by the recovery rate i.e., which is a fraction of the payoff of the contingent claim in case of default. There are two kinds of recovery payoffs: one recovery payoff at the time of default if default occurs prior to or at the maturity date, and the recovery payoff at maturity if default occurs prior to or at the maturity date. Naturally, one of the recovery payoffs occur in the real market. In a general setting, we consider simultaneously both kinds of recovery payoffs. We derive price and hedging strategies of the default claims under the benchmark approach. Craddock and Platen (2001) makes use of an integrated benchmark modelling framework that allows us to model credit risk and demonstrated how to price contingent claims by taking expectation under the real world probability measure in a benchmarked world.

In the following sections, we give brief descriptions of various option pricing methodologies given under the APT, where we describe the Black-Scholes PDE, the Black-Scholes-Merton risk-neutral approach, the martingale approach and the benchmark approach.

1.4 Black-Scholes PDE

In 1973, Black and Scholes ushered in the modern era of derivative securities with a seminal paper on the pricing and hedging of (European) call and put options. In their work, the famous Black-Scholes formula made its debut, and the Itô calculus was unleashed upon the world of finance.

In its simplest form, the Black-Scholes model involves a frictionless market with two underlying assets, a riskless asset (bank account or cash bond) and a risky asset (stock). The bank account appreciates at the short rate, or riskless rate of return r, which is assumed to be constant. Thus, the price B_t of the riskless asset at time t is assumed to satisfy the differential equation

$$dB_t = B_t r dt. \tag{1.2}$$

The price S_t of one unit of risky asset at time t is assumed to follow a stochastic differential equation (SDE) of the form

$$dS_t = S_t(\mu_t dt + \sigma dW_t), \tag{1.3}$$

where $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion, μ_t is a bounded, deterministic Borel function of t called the drift, and $\sigma > 0$ is a constant called the volatility of the stock. The SDE (1.3) has a unique solution with initial value $S_0 > 0$ (Ch-V Theorem 6 Protter, 2004) and it is given by

$$S_t = S_0 \exp\Big(\sigma W_t - \sigma^2 \frac{t}{2} + \int_0^t \mu_s ds\Big). \tag{1.4}$$

We write $\mathbf{S}_t = (B_t, S_t)^\top$. Here, "$(\cdot)^\top$" indicates the transpose of a matrix or a vector. Throughout the thesis, we consider a complete filtered probability space $(\Omega, \mathfrak{F}, F, P)$, where Ω denotes an abstract set containing all possible states of the financial market under study, $F = \{\mathfrak{F}_t\}_{t\in[0,\,T]}$ is an increasing sequence of σ-fields over Ω (containing information about the market at time $t \in [0,\ T]$) with $\mathcal{F} = \mathcal{F}_T$, for a fixed time horizon $T > 0$. Throughout the thesis, we assume that the filtration F satisfying the usual conditions, i.e. F is right continuous and any set which is a subset of a set of zero probability is $\mathcal{F}_0$-measurable. The vector process $\mathbf{S} = \{\mathbf{S}_t\}_{t\in[0,\ T]}$ is assumed to be adapted to the filtration F. In words, the current price of each asset is known at each time t, given the information $\mathcal{F}_t$. Throughout the thesis, we suppose that all securities are discounted by the riskless asset, and hence $B_t = 1$ almost surely for all $t \in [0,\ T]$. We have chosen to work with discounted price processes so that, although there is a bank account in the model, it does not appear in the analysis. For this Black-Scholes model, the investor needs to choose a strategy, represented by the two-dimensional process

$$\delta = \{\delta_t = (\delta_t^0, \delta_t^1)\}_{t\in[0,\ T]},$$

where δ_t^i denotes the number of units of riskfree and risky assets respectively, for $i \in \{0, 1\}$, that is being held at time $t \in [0,\ T]$. In the following definition, we give a notion of admissible strategy for an investor.

Definition 1.4.1. *An* ***admissible trading strategy*** δ *satisfies the conditions:*

a. δ *is* F*-predictable process and belongs to the set* $\mathbb{L}^2(S)$*, where*

$$\mathbb{L}^2(\mathbf{S}) = \left\{ \delta \;\middle|\; E\left[\int_0^T (\delta_t)^2 \cdot d[\mathbf{S}]_t + \left(\int_0^T \delta_t \cdot \mathbf{S}_t \mu_t dt \right)^2 \right] < \infty \right\}.$$

b. The resulting portfolio value $V_t^\delta = \delta_t \cdot \mathbf{S}_t$ *is non-negative.*

In the above, "$\cdot$" indicates the dot product of two vectors, $(\delta_t)^2$ *indicates a vector whose elements are square of elements of* δ_t *and* $[\cdot]$ *indicates the quadratic variation of the process.*

In the above definition, predictability can be loosely interpreted as left-continuity, but more precisely, it means that the strategy is adapted to the filtration generated by all left-continuous F-adapted processes. In economic terms, it means that the investor cannot change his portfolio to guard against jumps that might occur randomly. The first condition for the admissible trading strategy is a required condition to define a wealth process V^δ corresponding to δ to derive a local-risk-minimizing hedging strategy. The second requirement is important in order to rule out simple, but unrealistic, strategies leading to arbitrage (for instance, doubling strategies). Throughout the thesis we make the following assumption.

Assumption 1.4.1. *All trading strategies are admissible.*

Due to this assumption we shall simply omit the phrase "admissible" in the remainder of the thesis.

Definition 1.4.2. *A trading strategy is called **self-financing** if the corresponding portfolio value satisfies the following equation, with $V_0^\delta \geq 0$,*

$$V_t^\delta = V_0^\delta + \int_0^t \delta_s \cdot d\mathbf{S}_s, \quad \text{for } t \in [0, T].$$

The self-financing strategy definition states that the investor does not withdraw or add any funds throughout the contract period. From here onwards we will not be using the "$\cdot$" symbol between two vectors or matrices for their dot products.

Definition 1.4.3. *We say that the market model does not allow **arbitrage opportunities** if for any strategy δ, the following condition is satisfied:*

$$\text{If } V_0^\delta = 0, \text{ then } V_\tau^\delta \geq 0 \text{ a.s. implies } V_\tau^\delta = 0 \text{ a.s. for any stopping time } \tau \in [0,\, T].$$

Before going further, we require the following definition.

Definition 1.4.4. *A portfolio value process V^δ is called **replicating or hedging** a contingent claim H_T, where H_T is an $\mathfrak{F}_T$-measurable, non-negative square-integrable random variable, if*

$$V_T^\delta = H_T, \quad a.s.$$

If this portfolio is self-financing then we call it a ***perfectly replicating or perfectly hedging strategy****.*

Let $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+)$ be the set of real valued functions on $([0,T]\times\mathbb{R}_+)$ such that the function is of class $\mathcal{C}^1$ with respect to the first argument and $\mathcal{C}^2$ with respect to second argument and $C=\{C(t,S_t)\}_{t\in[0,\,T]}\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}_+)$ be the price process of a European-style derivative whose value depends on both the stock price and time. Consider a trading strategy V^δ (in this thesis, interchangeably we use the term "trading strategy" for both δ and V^δ) consisting of the bank account B and the stock S which perfectly replicates the derivative, i.e.,

$$V_T^\delta = C(T, S_T).$$

Now, from the no-arbitrage condition, we have

$$V_t^\delta = C(t, S_t),$$

for all $t\in[0,\ T]$. In differential form,

$$dV_t^\delta = dC(t, S_t). \tag{1.5}$$

From Itô's formula,

$$\begin{aligned} dC(t,S_t) &= \frac{\partial C(t,S_t)}{\partial t}dt + \frac{\partial C(t,S_t)}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 C(t,S_t)}{\partial S_t^2}(dS_t)^2 \\ &= \Big(\frac{\partial C(t,S_t)}{\partial t} + \mu_t S_t\frac{\partial C(t,S_t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C(t,S_t)}{\partial S_t^2}\Big)dt + \sigma S_t\frac{\partial C(t,S_t)}{\partial S_t}dW_t. \end{aligned}$$

From the self-financing nature of the portfolio, we have

$$dV_t^\delta = \delta_t d\mathbf{S}_t = \delta_t^1 dS_t = \delta_t^1 S_t(\mu_t dt + \sigma dW_t).$$

If we compare dt and dW_t terms in (1.5), we get

$$\delta_t^1 = \frac{\partial C(t,S_t)}{\partial S_t},$$

which is the required hedging strategy to perfectly replicate the contingent claim, and

$$\frac{\partial C(t,S_t)}{\partial t} + \mu_t S_t \frac{\partial C(t,S_t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t,S_t)}{\partial S_t^2} = \delta_t^1 S_t \mu_t,$$

that is,

$$\frac{\partial C(t,S_t)}{\partial t} + \mu_t S_t \frac{\partial C(t,S_t)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t,S_t)}{\partial S_t^2} = \frac{\partial C(t,S_t)}{\partial S_t} S_t \mu_t$$

$$\frac{\partial C(t,S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t,S_t)}{\partial S_t^2} = 0.$$

Hence, the Black-Scholes PDE can be read as

$$\frac{\partial C(t,\mathrm{S})}{\partial t} + \frac{1}{2}\sigma^2 \mathrm{S}^2 \frac{\partial^2 C(t,\mathrm{S})}{\partial \mathrm{S}^2} = 0,$$

for $\mathrm{S} \in (0,\infty)$ with terminal condition $C(T,\mathrm{S}) = H_T$.

In literature, this equation can be seen as, for nondiscounted processes, i.e., for $r \neq 0$,

$$\frac{\partial C(t,\mathrm{S})}{\partial t} + \frac{1}{2}\sigma^2 \mathrm{S}^2 \frac{\partial^2 C(t,\mathrm{S})}{\partial \mathrm{S}^2} + rS\frac{\partial C(t,\mathrm{S})}{\partial \mathrm{S}} - rC(t,\mathrm{S}) = 0,$$

which is the usual Black-Scholes PDE for $\mathrm{S} \in (0,\infty)$, also known as the Black-Scholes equation. The solution of the above second order parabolic PDE with terminal condition $C(T,S_T) = (S_T - K)^+$ will be the price for the European call option at time $t \in [0,\ T]$, with the strike price K. There are several ways of solving the Black-Scholes equation, like using the principle of risk-neutral valuation or using the quantum mechanical formulation.

1.5 Black-Scholes-Merton risk-neutral approach

One of the most remarkable properties of the Black-Scholes equation is that all of the variables that do appear in the equation are independent of risk preferences. In other words, the equation does not involve the expected return μ_t on the stock. Merton has assumed this fact, and used risk-neutral pricing to derive the Black-Scholes equation with just a simple argument as follows: In a world where investors are risk-neutral, the expected return on all investment assets is the riskfree rate of interest 0 (it would be r if we do not consider all price processes are discounted ones). Let Q be the risk-neutral probability measure. Then, under Q, the dynamics of the stock is

$$dS_t = \sigma S_t dW_t^Q,$$

where $W_t^Q = W_t + \int_0^t \frac{\mu_s}{\sigma} ds$ is a standard Brownian motion under the measure Q and from Itô's formula, the dynamics of the call price is

$$dC(t,S_t) = \Big(\frac{\partial C(t,S_t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(t,S_t)}{\partial S_t^2}\Big)dt + \sigma S_t \frac{\partial C(t,S_t)}{\partial S_t} dW_t^Q. \tag{1.6}$$

On the other hand, from the no-arbitrage argument, under Q, the discounted price $C(t,S_t)$ should also be driftless, that is, the dt term of $dC(t,S_t)$ must coincide with zero. Therefore, with this fact and (1.6), it follows that

$$\frac{\partial C(t,\mathrm{S})}{\partial t} + \frac{1}{2}\sigma^2 \mathrm{S}^2 \frac{\partial^2 C(t,\mathrm{S})}{\partial \mathrm{S}^2} = 0,$$

and this is precisely the Black-Scholes equation with terminal condition $C(T,\mathrm{S}) = H_T$.

The market price of risk process θ is given by

$$\theta_t = \frac{\mu_t - r}{\sigma} = \frac{\mu_t}{\sigma},$$

also called risk premium or relative risk. This quantity compares the rate of returns of the risky asset and the riskfree asset, normalized by the volatility of the risky asset, i.e., it is defined as a measure of excess return that investors demand to bear risk. We also recall the risk-neutral density process or the continuous-time Arrow-Debreu price process (state-price density):

$$\mathcal{Z}_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right). \tag{1.7}$$

Itô's formula implies

$$d\mathcal{Z}_t = -\theta_t \mathcal{Z}_t dW_t, \quad \mathcal{Z}_0 = 1. \tag{1.8}$$

In particular, as a stochastic integral, $\mathcal{Z}$ is a local martingale, with expectation equal to one. From the Girsanov theorem, we have for a given process θ, adapted to the information generated by the Wiener process W_t, the process

$$W_t^Q = W_t + \int_0^t \theta_s ds$$

is a standard Brownian motion in a measure defined by the relationship

$$dQ = \mathcal{Z}_T dP,$$

where $\mathcal{Z}_T$ is the Radon-Nikodým derivative.

In the risk-neutral world, one can see easily that any discounted value process is a martingale. It follows that the expected value of any discounted value process at a future time is its current value and thus the value of the call option with strike K and maturity T is given by

$$C(t, S_t) = E^Q[(S_T - K)^+ \big| \mathcal{F}_t],$$

where $E^Q[\cdot|\mathcal{F}_t]$ indicates the conditional expectation with respect to the risk-neutral measure Q for the given information up to time t. The above expression can be further simplified to (Merton, 1973)

$$C(t, S_t) = S_t \mathbf{N}(d) - K\mathbf{N}(d - \sigma\sqrt{T-t}), \tag{1.9}$$

where $d = \frac{ln(S_t/K)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$ and $\mathbf{N}$ is the standard normal distribution function. The equation (1.9) also indicates the solution of the Black-Scholes PDE and known as Black-Scholes formula.

1.6 Martingale approach for option pricing

In this section, we present the option pricing through martingale representation theorem as in Harrison and Kreps (1979), and Harrison and Pliska (1981). In order to show the pricing formula, we need to recall a deep result from the theory of stochastic analysis called the martingale representation theorem (Elliott and Kopp, 2005).

Theorem 1.6.1. *If a process $M = \{M_t\}_{t\geq 0}$ is a square-integrable continuous martingale with respect to the filtration generated by the Brownian motion W, then it has a representation of the form*

$$M_t = E(M_T|\mathcal{F}_t) = E(M_T) + \int_0^t \phi_s dW_s,$$

for some square-integrable predictable process ϕ. In particular,

$$M_T = E(M_T) + \int_0^T \phi_s dW_s.$$

There are different forms of martingale representation theorems and for more details, we refer to Elliott and Kopp, (2005). But we will be using only the mentioned above predictable representation. Now, we state and prove a result on the possibility of replicating a contingent claim with random payoff C at maturity T.

Theorem 1.6.2. *An European contingent claim with a discounted, square-integrable, positive, random payoff C at maturity T can be perfectly replicated, starting with initial wealth V_0, if and only if*

$$E^Q(C(T, S_T)) = V_0.$$

Proof. From the no-arbitrage condition, if we have a self-financing portfolio such that $V_T = C(T, S_T)$ then $V_0 = E^Q(C(T, S_T)) = C(0, S_0)$. Now, let us prove the converse, i.e. existence of a self-financing strategy that replicates C starting with initial amount $V_0 = C(0, S_0) = E^Q(C(T, S_T))$. Since $E^Q(C(T, S_T)|\mathcal{F}_t)$ is a square-integrable, continuous martingale with respect to the filtration F which is generated by the Brownian motion W. From the martingale representation theorem, there exists a predictable process ϕ, with $E\int_0^T \phi_s^2 ds < \infty$, such that

$$E^Q(C(T, S_T)|\mathcal{F}_t) = E^Q(C(T, S_T)) + \int_0^t \phi_s dW_s^Q, \tag{1.10}$$

and its differential form is

$$dE^Q(C(T,S_T)|\mathcal{F}_t) = \phi_t dW_t^Q. \tag{1.11}$$

Under the transformed risk-neutral probability measure Q, a self-financing portfolio V^δ from definition 1.4.2 can be written as

$$dV_t^\delta = S_t\delta_t^1\sigma dW_t^Q. \tag{1.12}$$

If we compare (1.11) and (1.12), we get the required replicating portfolio

$$\delta_t^1 = \frac{\phi_t}{S_t\sigma},$$

with initial wealth V_0. □

The put-call parity in financial mathematics is

$$P(t,S_t) = C(t,S_t) + K - S_t.$$

That is, a put option can be replicated by buying a bond and a call and short selling the stock. Here, P indicates the price process of a European put option, and its value can be derived if the price of call option is known.

1.7 The benchmark approach

The use of the growth optimal portfolio (GOP) has been recently a point of great interest in finance. Kelly has shown the existence of the GOP as an optimal gambling strategy which collects more wealth than any other strategy over a long period of time. This portfolio has become known as the one that maximizes its growth rate for any time horizon. It has been shown in Platen and Heath (2006) and in some other papers such as Breiman (1961) that the GOP outperforms any other strategy as the time horizon increases. This characteristic of the GOP leads to the so-called benchmark approach which consists of taking the GOP as the benchmark or numeraire and using it in asset pricing.

In this section, we review the main results from Platen's theory of benchmark approach for pricing and hedging financial instruments. We define and determine the growth optimal portfolio in a general continuous-time financial market model for a generalized Black-Scholes model. The aim of this section is to show the role of the GOP in option pricing, which leads to the use of real world probability measure.

Here, we suppose that the market is composed of $d+1$ assets. The d risky security processes are denoted by $S^j = \{S_t^j\}_{t\in[0,\,T]}$, $j \in \{1, 2, \cdots, d\}$. Uncertainties are given by a d-dimensional Wiener process. We assume that for the j-th security, the drift process $\mu^j = \{\mu_t^j\}_{t\in[0,\,T]}$ and the volatility process corresponding to the kth Wiener process $\sigma^{j,k} = \{\sigma_t^{j,k}\}_{t\in[0,\,T]}$ are predictable processes and satisfy the integrability conditions

$$E\Big[\int_0^T \sum_{j=1}^d |\mu_s^j| ds\Big] < \infty,$$

and

$$E\Big[\int_0^T \sum_{j,k=1}^d (\sigma_s^{j,k})^2 ds\Big] < \infty.$$

We model the discounted security processes as follows:

$$B_t = 1,$$

$$dS_t^j = S_t^j(\mu_t^j dt + \sum_{k=1}^d \sigma_t^{j,k} dW_t^k), \tag{1.13}$$

for $j \in \{1, 2, \cdots, d\}$ and $t \in [0,\ T]$. For the strong unique solution for the SDE (1.13), we assume that

$$|\mu_t^j| + \sum_{k=1}^d |\sigma_t^{j,k}| \leq \epsilon$$

for all $t \in [0, T]$, $j \in \{1, \cdots, d\}$ and for some constant ϵ, which implies that the parameters of the above SDE are satisfying the Lipschitz and the growth conditions for all $i, j \in \{1, 2, \cdots, d\}$ (see P-47, Theorem 1.5.5.1, Jeanblanc et al., 2009). The solution is given by

$$S_t^j = S_0^j \exp \int_0^t \Big(\mu_s^j ds - \frac{1}{2} \sum_{k=1}^d (\sigma_s^{j,k})^2 ds + \sum_{k=1}^d \sigma_s^{j,k} dW_s^k \Big). \tag{1.14}$$

Let us introduce the market price of risk once again, whose importance can be seen later. First, we introduce the vectors $W_t = (W_t^1, \cdots, W_t^d)^\top$, $\mathbf{S}_t = (B_t, S_t^1, \cdots, S_t^d)^\top$ and the appreciation rate vector $\mu_t = (\mu_t^1, \cdots, \mu_t^d)^\top$. We assume that the matrix $\sigma_t = [\sigma_t^{j,k}]_{j,k\in\{1,\cdots,d\}}$ is invertible almost surely, for $t \in [0,\ T]$, with inverse σ_t^{-1}.

Definition 1.7.1. *The* ***market price of risk*** *is given by the vector*

$$\theta_t = (\theta_t^1, \cdots, \theta_t^d)^\top,$$

such that

$$\theta_t = \sigma_t^{-1} \mu_t.$$

The equation (1.13) can be written as

$$dS_t^j = S_t^j \sum_{k=1}^d \sigma_t^{j,k} (\theta_t^k dt + dW_t^k), \tag{1.15}$$

where θ_t^k is the k-th market price of risk.

A portfolio strategy δ at time $t \in [0,\ T]$ is given by the $\mathbb{R}^{d+1}$-valued predictable vector process $\delta_t = (\delta_t^0, \delta_t^1, \cdots, \delta_t^d)$. The value process V_t^δ corresponding to the strategy δ is given by

$$V_t^\delta = \delta_t \mathbf{S}_t.$$

The portfolio corresponding to a self-financing strategy satisfies the differential form

$$dV_t^\delta = \delta_t d\mathbf{S}_t,$$

for $t \in [0,\ T]$. This means that all the changes in the portfolio value are due to changes in the primary security accounts. Assuming that V_t^δ is positive, we can introduce the j-th fraction π_t^j of V_t^δ invested in the j-th security account. It is given by the j-th component of the vector process

$$\pi_t = \frac{\delta_t * \mathbf{S}_t}{V_t^\delta},$$

where '$*$' indicates the pointwise multiplication of two vectors. Let $U(T)$ be the collection of portfolios π such that its corresponding strategy δ is admissible through out period $[0,\ T]$ and $U(t,T)$ for period $[t,T]$.

The value of a self-financing portfolio δ is given by the SDE

$$dV_t^\pi = V_t^\pi \sigma_t^\pi (dW_t + \theta_t dt), \tag{1.16}$$

where $\sigma_t^\pi = (\pi_t^1, \cdots, \pi_t^d)\sigma_t$.

Remark 1.7.1. *For the above multi-asset discounted model, the Black-Scholes PDE can be read as*

$$\frac{\partial C(t,\mathbf{S})}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} \sigma^{i,j} S^i S^j \frac{\partial^2 C(t,\mathbf{S})}{\partial S^i \partial S^j} = 0.$$

Here $C(t,\mathbf{S}_t)$ indicates an European option. The risk-neutral density process or the continuous-time Arrow-Debreu price process is

$$\mathcal{Z}_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t ||\theta_s||^2 ds\right), \tag{1.17}$$

where $||\cdot||$ indicates the Euclidean norm. In particular $\mathcal{Z}$ is a local martingale. If it satisfies the Novikov condition i.e. $E(\exp(\frac{1}{2}\int_0^T ||\theta_s||^2 ds)) < \infty$, with $E(\mathcal{Z}_T) = 1$, then the process $\mathcal{Z}$ is a martingale, and $Q(A) = E(I_A \mathcal{Z}_T)$ defines a probability measure on $(\Omega, \mathcal{F}_T)$ such that $Q(A) = E(I_A \mathcal{Z}_t)$ for $A \in \mathcal{F}_t$ (see page-348 Platen and Heath, 2006). The transformed risk neutral measure is defined as

$$dQ = \mathcal{Z}_T dP.$$

Then the price of a European option at any time $t \in [0,T]$ is given by

$$C(t,\mathbf{S}_t) = E^Q[C(T,\mathbf{S}_T)|\mathcal{F}_t].$$

For any strategy π, from Itô's formula, $\log(V_t^\pi)$ satisfies

$$d\log(V_t^\pi) = g_t^\pi dt + \sigma_t^\pi dW_t, \tag{1.18}$$

with the expected logarithmic growth rate $g_t^\pi = \sigma_t^\pi \theta_t - \frac{1}{2}\sigma_t^\pi \sigma_t^{\pi\top}$. Here, V_t^δ and V_t^π are different notations but represents the same value. The GOP is defined by the following.

Definition 1.7.2. *In a continuous-time financial market, a positive self-financing trading strategy π^* with value process V^{π^*} is called a **GOP** if, for any positive self-financing trading strategy π with value process V^π, the corresponding growth rates satisfy the inequality*

$$g_t^{\pi^*} \geq g_t^\pi,$$

almost surely for all $t \in [0,\ T]$.

The dynamics of the wealth process corresponding to the GOP is given by the following theorem.

Theorem 1.7.1. *The GOP fraction is*

$$\pi_t^* = \theta_t^\top \sigma_t^{-1},$$

and the corresponding wealth process satisfies the SDE

$$dV_t^{\pi^*} = V_t^{\pi^*} \theta_t^\top (dW_t + \theta_t dt), \tag{1.19}$$

for all $t \in [0,\ T]$.

The proof of the theorem is a simple consequence of Itô's formula, and the first and second order conditions of maximization. Without loss of generality, we assume $V_0^{\pi^*} = 1$ and this holds throughout the thesis. We see that prices can be obtained under the benchmark approach by taking the GOP as reference portfolio and without the need of a risk-neutral measure. We will see that benchmarked portfolios (i.e with GOP as a numeraire) are supermartingales. Further, we prove that the supermartingale property of the benchmarked prices does not allow arbitrage opportunities in the market.

Definition 1.7.3. *Any price process* $\mathbf{U}$ *expressed in units of the GOP is called the* ***benchmarked*** *price process and denoted by* $\widehat{\mathbf{U}}$*. The model where GOP is used as a numeraire is called a* ***benchmarked model****.*

Theorem 1.7.2. *In a continuous-time financial market, the benchmarked portfolio* $\widehat{V}^\pi$*, where* $\widehat{V}_t^\pi = \frac{V_t^\pi}{V_t^{\pi^*}}$*, is an* (F, P)*-supermartingale for any trading strategy* π*.*

Proof. Applying Itô's formula, we get the dynamics of the benchmarked portfolio as

$$d\widehat{V}_t^\pi = \widehat{V}_t^\pi(\sigma_t^\pi - \theta_t)dW_t. \tag{1.20}$$

Equation (1.20) means that the process $\widehat{V}^\pi$ which is the benchmarked portfolio value is driftless. It follows that the process $\widehat{V}^\pi$ is an (F, P)-local martingale. Due to Fatou's lemma, a non-negative local martingale is a supermartingale. The portfolio V^π is non-negative, hence $\widehat{V}^\pi$ is an (F, P)-supermartingale. □

The benchnarked price processes is a positive local martingale. If the benchmarked price process forms a strict local martingale not a martingale then we do not have, in general, an equivalent risk-neutral measure (see Platen, 2004a). This is the case, for instance, if we take a squared Bessel processes (see Heath and Platen, 2002).

Theorem 1.7.3. *In a continuous-time financial market, if the GOP exists then there are no admissible arbitrage opportunities.*

Proof. This follows directly from the supermartingale property of the benchmarked portfolio. If $V_0^\pi = 0$, we have

$$0 = \widehat{V}_0^\pi \geq E(\widehat{V}_\tau^\pi) \geq 0,$$

for any stopping time $\tau \in [0,\ T]$. And since the benchmarked value is non-negative and V^{π^*} is positive, we have

$$P(\widehat{V}_\tau^\pi > 0) = 0.$$

□

Fair pricing formula

Apart from the growth property, there is another property of the GOP, the numeraire property, which is important in order to understand the role of the GOP in derivative pricing. In order to be able to write a pricing formula, we state the following definition (Platen and Heath, 2006).

Definition 1.7.4. *A security price process V is called **fair** if its benchmarked value $\widehat{V}$ is an (F, P)-martingale.*

Intuitively, the martingale property of benchmarked prices relates to the notion of fair price. Due to the supermartingale nature of any benchmarked process, the minimal portfolio value whose benchmarked value replicates the benchmarked contingent claim gives a fair price for the contingent claim. The fair pricing idea generalizes the standard risk-neutral pricing, as we will see later. In practice, the fair pricing is appropriate for determining the competitive price of a contingent-claim. From the definition, it is clear that the actual benchmarked value of a fair wealth process is at any time the best forecast of it's future benchmarked value. The existence of fair price for a contingent claim is equivalent to the no unbounded profit with bounded risk (NUPBR) condition introduced in Kardaras and Karatzas (2007). And the NUPBR condition implies the no-arbitrage (NA) condition which is necessary for the existence of a risk-neutral measure but the converse need not be true.

Definition 1.7.5. *We define a non-negative **contingent claim** H_T that matures at time T to be a random payoff which is non-negative, $\mathfrak{F}_T$-measurable with*

$$E\left(\frac{|H_T|}{V_T^{\pi^*}}\Big|\mathfrak{F}_t\right) < \infty,$$

almost surely for $t \in [0,\ T]$.

Now we state the definition of the complete market taken from Platen (2004a).

Definition 1.7.6. *A benchmarked model is called* ***complete*** *if there exists a fair, perfectly replicating trading strategy for all non-negative contingent claims.*

Theorem 1.7.4. *The fair price of a derivative security at time t having a terminal value H_T is given by*

$$H_t = V_t^{\pi^*} E(\widehat{H}_T | \mathcal{F}_t). \tag{1.21}$$

The proof of the above proposition is straightforward from the fair price definition (see, Platen and Heath (2006)).

Since there is no assumption on the completeness of the market for the existence of the fair price, this formula works both in complete and incomplete market models. The idea is to define the fair price in such a way that the numeraire property of the GOP is undisturbed. In other words, the GOP remains a GOP after the payoff H_T is introduced in the market. There are two primary motivations for this methodology. Firstly, the market may not be complete, in which case there may not be a perfectly replicating portfolio for the payoff H_T. Second, the market may be complete, but there need not exist an equivalent risk-neutral measure, which is usually used for pricing. In the case of complete markets which have an equivalent risk-neutral measure, the fair pricing concept is equivalent to pricing using the standard risk-neutral method (Platen and Heath, 2006). The Radon-Nikodým derivative process $\mathcal{Z}^Q = \{\mathcal{Z}_t^Q\}_{t\in[0,\, T]}$ for the candidate martingale measure Q can be obtained as inverse of the discounted GOP

$$\mathcal{Z}_t^Q = E\Big(\frac{dQ}{dP}|\mathcal{F}_t\Big) = \frac{\widehat{B}_t}{\widehat{B}_0} = \frac{1}{V_t^{\pi^*}},$$

for $t \in [0,\ T]$, where $\mathcal{Z}_t^Q$ satisfies the SDE

$$d\mathcal{Z}_t^Q = -\mathcal{Z}_t^Q \theta_t dW_t, \tag{1.22}$$

for $t \in [0,\ T]$ with $\mathcal{Z}_0^Q = 1$, by the Itô's formula and (1.19). This demonstrates that $\mathcal{Z}^Q$ is an (F, P)-martingale. Furthermore, it follows that $\frac{V_t^\pi}{B_t}\mathcal{Z}_t^Q = \frac{V_t^\pi}{V_t^{\pi^*}} = \widehat{V}_t^\pi$ and $\widehat{V}^\pi =$

$\{\widehat{V}_t^{\pi}\}_{t\in[0,\,T]} = \left\{\frac{V_t^{\pi}}{B_t}\mathcal{Z}_t^Q\right\}_{t\in[0,\,T]}$ is an (F, P)-local martingale for any portfolio π. Due to this result, we can state the following lemma.

Lemma 1.7.1. *Suppose a complete market has an equivalent martingale measure, that is, a probability measure Q equivalent to P and discounted asset prices are Q-local martingales. Then the risk-neutral price given by*

$$\overline{H}_t = E^Q(H_T|\mathcal{F}_t)$$

is identical to the fair price, i.e. $\overline{H}_t = H_t$ almost surely, for all $t \in (0,\, T)$.

In the sequel, we require the following definition.

Definition 1.7.7. *A semimartingale Y is called* ***stochastic exponential*** *of a semimartingale X, and vice-versa, X is called* ***stochastic logarithmic*** *of Y, if they satisfy the SDE*

$$dY_t = Y_{t-}dX_t, \tag{1.23}$$

with $Y_t, Y_{t-} > 0$ and initial condition $Y_0 = 1$. It is denoted by $Y = \mathcal{E}(X)$ and $X = \mathcal{L}og(Y)$.

1.8 Local-risk-minimizing method

So far, we have seen different methods used in literature for option pricing and hedging in continuous-time complete financial markets. But, it is not necessary that the financial markets will always be complete. An incomplete financial market is a situation where all the non-negative contingent claims can not be perfectly hedged. So the hedger has to take some intrinsic risk by introducing a cost process during the contract period to hedge the contingent claim. Different "quadratic" methods have been introduced in financial mathematics to reduce the intrinsic risk in incomplete financial markets, like the mean-variance hedging (Föllmer and Sondermann, 1986) and the local-risk-minimizing hedging

(Schweizer, 1991). In this section, we provide a review of the main results of the theory of local-risk-minimization introduced by Schweizer and his co-authors, since it is closely related to the GOP. For an extensive survey of the approach, we refer to Schweizer (2001).

The main feature of the local-risk-minimization approach is the fact that one has to work with strategies which are not self-financing and the purpose becomes to minimize the riskiness in a suitable way. If we consider a non-attainable (i.e. not perfectly hedgeable) contingent claim H_T, in a Markov-modulated market or in a defaultable market for instance, then according to this method, we look for a hedging strategy with minimal cost that replicates H_T.

The risk-minimization criterion for measuring the riskiness of a strategy was first introduced by Föllmer and Sondermann in 1986 where the risky asset is represented by a martingale. Successively, it was extended to the general semimartingale case by Schweizer (1988, 1991) and by Föllmer and Schweizer (1991). First, we discuss the simple special case where S is a P-martingale. We then give the generalized result for the semimartingale case. Before that we give the following definitions.

Definition 1.8.1. *For any strategy π, the **cost process** is defined by*

$$\mathbf{C}_t^\pi = V_t^\pi - \int_0^t \delta_s d\mathbf{S}_s, \quad t \in [0, T]. \tag{1.24}$$

The quantity $\mathbf{C}_t^\pi$ describes the total costs incurred by π over the interval $[0, t]$. The risk process of π is defined by

$$\mathbf{R}_t^\pi = E((\mathbf{C}_T^\pi - \mathbf{C}_t^\pi)^2 | \mathcal{F}_t), \quad t \in [0, T]. \tag{1.25}$$

Definition 1.8.2. *A strategy π is called **risk-minimizing** if for any other strategy $\overline{\pi}$ such that $V_T^{\overline{\pi}} = V_T^\pi$ P-a.s., we have*

$$\mathbf{R}_t^\pi \le \mathbf{R}_t^{\overline{\pi}},$$

P-a.s. for every $t \in [0, T]$.

Definition 1.8.3. *A strategy π is called **mean-self-financing** if its cost process* $\mathbf{C}$ *is a P-martingale.*

Definition 1.8.4. *Two square-integrable martingales M and L, such that $M_0 = L_0 = 0$, are said to be P-**orthogonal** if $E(M_t N_t) = 0$ for all $t \in [0, T]$. And, two square-integrable martingales are said to be **strongly** P-**orthogonal** if their product is a P-martingale.*

To analyze in simple form we assume only one risky asset S here. We consider $L^2(S)$, the space of all F-predictable processes δ such that

$$E\left[\int_0^T (\delta_s^1)^2 d[S]_s\right] < \infty,$$

and let $\mathfrak{M}_0^2(P)$ be the space of square-integrable P-martingale null at 0. If S is a P-martingale, the risk-minimization problem is always solvable by applying the Galtchouk-Kunita-Watanabe decomposition (see, Schweizer (2001)). Since the set $I^2(S) = \{\int \delta^1 dS \mid \delta \in L^2(S)\}$ is a stable subspace of $\mathfrak{M}_0^2(P)$, any non-negative $H_T \in L^2(\mathfrak{F}_T, P)$, i.e., an $\mathfrak{F}_T$-measurable square-integrable random variable, can be uniquely written as

$$H_T = E(H_T) + \int_0^T \delta_s^1(H_T) dS_s + L_T^H \qquad P - a.s. \tag{1.26}$$

for some $\delta^1 \in L^2(S)$ and some $L^H = \{L_t^H = E(L_T^H | \mathfrak{F}_t)\}_{t \in [0,\, T]} \in \mathfrak{M}_0^2(P)$ strongly orthogonal to $I^2(S)$, which is known as Galtchouk-Kunita-Watanabe decomposition. The following result was obtained by Föllmer and Sondermann (1986) for the one-dimensional case under the assumption that S is a square-integrable P-martingale. Schweizer (1995) has extended this result for a general local P-martingale S.

Theorem 1.8.1. *If S is a P-martingale, then every non-negative contingent claim $H_T \in L^2(\mathfrak{F}_T, P)$ admits a unique risk-minimizing strategy $\widetilde{\pi}$ such that $V_T^{\widetilde{\pi}} = H_T$. In terms of* *decomposition (1.26), the risk-minimizing strategy $\widetilde{\pi}$ is explicitly given by*

$$\begin{aligned}\widetilde{\delta} &= \delta(H_T),\\ V_t^{\widetilde{\pi}} &= E(H_T|\mathcal{F}_t), \;\; t \in [0,\, T],\\ \mathbf{C}_T^{\widetilde{\pi}} &= E[H_T] + L_T^H. \end{aligned} \tag{1.27}$$

When S is in general a semimartingale, the notion of risk-minimization is not directly applicable. However, Föllmer and Schweizer (1991) extended this notion to risk-minimization in a local sense for a semimartingale case by considering small perturbations to the strategies. Interestingly, locally-risk-minimizing strategies are shown to be also mean-self-financing. They have proved that, to get locally-risk-minimizing strategies it is sufficient to find the Föllmer-Schweizer decomposition.

Proposition 1.8.1. *A non-negative contingent claim $H_T \in L^2(\mathcal{F}_T, P)$ admits a locally-risk-minimizing strategy $\widetilde{\pi}$ if only if H admits the Föllmer-Schweizer decomposition*

$$H_T = H_0 + \int_0^T \delta_s^1(H)dS_s + L_T^H \;\; P-a.s. \tag{1.28}$$

with $H_0 \in \mathbb{R}$, $\delta^1(H) \in L^2(S)$, $L_T^H = \{L_t^H = E(L_T^H|\mathcal{F}_t)\}_{t\in[0,\,T]} \in \mathfrak{M}_0^2(P)$ strongly P-orthogonal to martingale part of S. The locally-risk-minimizing strategy is given by

$$\widetilde{\delta}_t^1 = \delta_t^1(H), \;\; t \in [0,\, T],$$

with minimal cost

$$\mathbf{C}_t^{\widetilde{\pi}} = H_0 + L_t^H, \;\; t \in [0,\, T].$$

If (1.28) holds, then the optimal portfolio value at time t is

$$V_t^{\widetilde{\pi}} = \mathbf{C}_t^{\widetilde{\pi}} + \int_0^t \widetilde{\delta}_s^1 dS_s = H_0 + \int_0^t \delta_s^1(H)dS_s + L_t^H. \tag{1.29}$$

In the martingale case, Föllmer-Schweizer decomposition coincides with the Galtchouk-Kunita-Watanabe-decomposition. From Föllmer and Schweizer (1991), one can obtain the local-risk-minimizing hedging strategy using the Galtchouk-Kunita-Watanabe-decomposition by choosing a convenient martingale measure for the semimartingale S.

Definition 1.8.5. *A martingale measure Q equivalent to P with a square-integrable state-price density is called* ***minimal martingale measure*** *if Q is equivalent to P on $\mathcal{F}_0$ and if any square-integrable P-local martingale which is strongly orthogonal to the martingale part of S under P remains a local martingale under Q.*

The minimal martingale measure is the equivalent martingale measure that modifies the martingale structure as little as possible. The existence of the minimal martingale measure, for the case where the stock price process is a right continuous with left limit process, is discussed in Schweizer (1995). He states that the minimal martingale measure exist if the corresponding state price density is a martingale.

Theorem 1.8.2. *Suppose S is a semimartingale and*

$$\mathcal{Z}_t^Q = E\Big(\frac{dQ}{dP}|\mathcal{F}_t\Big) = \mathcal{E}\Big(-\int_0^t \theta_s dW_s\Big)$$

is a square-integrable martingale and define the process $\overline{H}$ as follows:

$$\overline{H}_t = E^Q(H_T|\mathcal{F}_t), \quad t \in [0,\ T],$$

where $E^Q(\cdot|\mathcal{F}_t)$ denote the conditional expectation under the minimal-martingale measure Q. Let

$$\overline{H}_T = E^Q(H_T|\mathcal{F}_T) = \overline{H}_0 + \int_0^T \overline{\delta}_s^1(H)dS_s + \overline{L}_T^H \tag{1.30}$$

be the Galtchouk-Kunita-Watanabe-decomposition of $\overline{H}$ with respect to S under Q. If either $\overline{H}$ admits a Föllmer-Schweizer decomposition under measure P or Galtchouk-Kunita-Watanabe-decomposition under the minimal-martingale measure Q, both give the same locally-risk-minimizing strategy for H_T.

The proof of the above theorem is little tricky and can be found in Föllmer and Schweizer (1991).

1.9 The outline of the thesis

This thesis is devoted to use the benchmark approach farther for option pricing and hedging in complete and incomplete Markov-modulated financial markets with perfect and imperfect information. In this introductory chapter we have given a brief description of developments in financial mathematics and an outline of different methodologies of option pricing in financial markets.

In each chapter, we derive the growth optimal portfolio by stochastic control method. Using this GOP, the benchmarked fair prices have been derived and the concept of local-risk-minimizing hedging strategy has been used for hedging.

In *Chapter 2,* we consider consumption process $\mathbb{C} = \{\mathbb{C}_t\}_{t\in[0,\,T]}$. Asset prices are considered to be driven by multidimensional Brownian motion processes and multidimensional Poisson processes. We consider European and American contingent claims with payoff rates g_t for $t \in [0,\ T]$ in addition to a terminal payoff.

Chapter 3 deals with an incomplete financial market where the stock price process is modulated by an irreducible continuous-time Markov process with finite state space. From this chapter onwards, we consider the pricing and hedging problems for European contingent claims only without payoff rates. It has been shown the pricing through risk-neutral method using the minimal-martingale measure for the considered model is same as the benchmark fair price.

Chapter 4 studies a financial market model with two stock price processes, which are modelled as a Markov-modulated jump-diffusion processes. First, We have derived the GOP under the complete information. By the help of martingale representation theorem the option has been hedged in complete information. Then, the hedging strategy has been derived in incomplete information by taking the optimal projection of the strategy, derived complete information case, towards the smaller filtration generated by the stock price processes.

In *Chapter 5* we consider a financial market where the stock price process has a Markov-modulated drift process and has jumps with Markov-modulated intensity rate. The stock price process is the only process which an investor can observe in this market. Through filtering theory we derive the non-linear filtering equation for the unobservable process, using innovation process method. Then the incomplete information market has been converted to a complete information one by the help of the filtering equation. Since it is not possible to get the GOP in the considered scenario, we derive it in a fictitious market.

The last two chapters deal with defaultable financial markets. In *Chapter 6* we consider a Markov-modulated defaultable Brownian market and price the defaultable contingent claims with the intensity-based methodology. We establish the representation theorems for the defaultable claims. The recovery processes are assumed to have random payments at default time as well as at the maturity of the claims.

Chapter 7 handles a defaultable Brownian market, where the drift process and the default intensity are modulated by an Ornstein-Uhlenbeck (OU) process. In this chapter we consider that the investor has incomplete information about the market. Here, the OU process is unobservable. By the help of filtering equation we convert in incomplete information market to complete information market. Then we derive the pricing and hedging strategy as in the previous chapter.

Chapter 2

A Jump-Diffusion Model with Stochastic Parameters and Consumption Process

Financial economists achieved unprecedented success over the last thirty-five years using simple diffusion models to approximate the stochastic process for returns on financial assets. The so-called "volatility smiles" computed using the volatility implied by the venerable Black-Scholes model reveal, however, that a simple geometric Brownian motion process misses some important features of the data. High frequency returns data display excess kurtosis (fat-tailed distributions), skewness, and volatility clustering. To strike a balance between reality and tractability, in this chapter, we consider a jump-diffusion model with stochastic parameters. The objective of this chapter is to price and hedge the American and European contingent claims with payoff rates, for which one has to consider portfolios with consumption processes $\mathbb{C} = \{\mathbb{C}_t\}_{t\in[0,\,T]}$. Elliott and Kopp (2005) have constructed the hedging strategies using an increasing process in the Doob decomposition of the given contingent claim to define a consumption process and self-financing strategy. Consumption process is also considered during the contract period.

SDEs of jump-diffusion type received much attention in financial and economic modelling (see, Merton (1976) and Cont and Tankov (2004)). Platen (2004a) has considered a class of models with intensity-based jumps in which he performed pricing and hedging for complete markets with jump-diffusions and without the measure transformation. Initially for the case of deterministic parameters, we derive GOP simply by using the first and second order condition to the growth rate. Using the GOP as a numeraire and from the fair price concept, we derive the price and the hedging strategy for both European and American contingent claims under the real world probability measure. While the case of deterministic parameters is relatively easy to deal with, the random behaviour of parameters turns out to be a major issue in option pricing. In Hull and White (1987), option price is determined in a series form for the case in which the stochastic volatility is independent of the stock price. Stein and Stein (1991) used analytic techniques to derive an explicit closed-form solution for the case where volatility is driven by an arithmetic Ornstein-Uhlenbeck process. So, in this chapter, next we handle a general case where all the parameters are modulated by a stochastic process adapted to the given filtration itself. For the case of stochastic parameters, we use the Hamilton-Jacobi-Bellman (HJB) equation to derive the GOP. Then, the case where the stochastic parameters are independent of the stock price process has been considered. In this case, the market turns out to be an incomplete one. By the help of GOP, we derive the Föllmer-Schweizer decomposition of a European contingent claim (ECC). Finally, we prove that the fair price of the ECC is a solution to a partial differential equation (PDE).

2.1 The market model

We consider a complete financial market subject to both diffusive uncertainty as well as jump uncertainty with d risky assets and one riskfree asset for some finite time horizon $T < \infty$. Continuously evolving uncertainty enters the model by means of m independent standard Wiener processes $W^k = \{W_t^k\}_{t\in[0,\,T]}$, for $k \in \{1, 2, \cdots, m\}$, where $m \in \{1, 2, \cdots, d\}$. Event driven uncertainty, for instance the unexpected default of a company, is modelled by $d-m$ independent Poisson processes $N^l = \{N_t^l\}_{t\in[0,\,T]}$, for $l \in \{m+1, \cdots, d\}$. The processes $W = (W^1, \cdots, W^m)^\top$ and $N = (N^{m+1}, \cdots, N^d)^\top$ are defined on a probability space $(\Omega, \mathcal{F}, F, P)$, all are mutually independent with each other, where the filtration $F = \{\mathcal{F}_t\}_{t\in[0,\,T]}$ is taken to be P-augmented of the filtration generated by the vector processes W and N. Events of the lth type are counted by the lth counting process N^l whose intensity $\lambda^l = \{\lambda_t^l\}_{t\in[0,\,T]}$ is a given, predictable process with

$$\lambda_t^l > 0 \quad \text{and} \quad \int_0^t \lambda_s^l ds < \infty$$

almost surely, for $t \in [0,\ T]$ and $l \in \{m+1, \cdots, d\}$. Furthermore, we introduce lth jump martingale $M^l = \{M_t^l\}_{t\in[0,\,T]}$ with SDE

$$dM_t^l = dN_t^l - \lambda_t^l dt,$$

for $l \in \{m+1, \cdots, d\}$ and $t \in [0,\ T]$. The above jump martingales are assumed not to jump at the same time. They represent the compensated source of event driven uncertainty.

The price B_t of the riskless asset at time t is assumed to satisfy the differential equation

$$dB_t = B_t r dt. \tag{2.1}$$

Since we have assumed that all the price processes are discounted one, the riskless asset $B_t = 1$, for all $t \in [0,\ T]$. The discounted price S_t^j, at time $t \in [0,\ T]$, for one unit of the jth stock is modelled by the SDE

$$dS_t^j = S_{t-}^j \left(\mu_t^j dt + \sum_{i=1}^m \sigma_t^{j,i} dW_t^i + \sum_{k=m+1}^d \rho_t^{j,k} dM_t^k \right), \tag{2.2}$$

with initial value $S_0^j > 0$, for $j \in \{1, \cdots, d\}$ and $t \in [0,\ T]$. Hence, the vector process $\mathbf{S}_t = (B_t,\ S_t^1, \cdots,\ S_t^d)^\top$, $t \in [0,\ T]$, characterizes the evolution of all primary security accounts. And μ^j, $\sigma^{j,i} \geq 0$, and $\rho^{j,k} > -1$, are assumed to be uniformly bounded and square-integrable, deterministic functions of time for $j \in \{1, 2, \cdots, d\}$, $i \in \{1, 2, \cdots, m\}$

and $k \in \{m+1, \cdots, d\}$, to get the strong solution of (2.2). The restriction on ρ is to keep the stock price processes positive. The matrix $\sigma_t \sigma_t^\top$ is assumed to be nonsingular and the matrix $[\sigma_t, \rho_t]$ is assumed to satisfy the following inequality

$$[\sigma_t, \rho_t][\sigma_t, \rho_t]^\top \geq cI, \quad \forall t \in [0, T],$$

for some $c > 0$, where I is a d-dimensional unit matrix. It is clear that matrix $[\sigma_t, \rho_t]$ is nonsingular for every $t \in [0, T]$.

The self-financing wealth process $V^\delta = \{V_t^\delta\}_{t\in[0,\, T]}$ satisfies the SDE

$$dV_t^\delta = \sum_{j=0}^{d} \delta_t^j dS_t^j, \tag{2.3}$$

for all $t \in [0,\ T]$.

We denote the appreciation rate vector $\mu_t = (\mu_t^1, \cdots, \mu_t^d)^\top$, and the invertible volatility matrix $b_t = [\sigma_t, \rho_t]_{d\times d}$, with $\sigma_t = [\sigma_t^{j,i}]_{j\in\{1,\cdots,d\}, i\in\{1,\cdots,m\}}$, $\rho_t = [\rho_t^{j,k}]_{j\in\{1,\cdots,d\}, k\in\{m+1,\cdots,d\}}$, for $t \in [0,\ T]$, which allow us to introduce the market price of risk vector

$$\theta_t = (\theta_t^1, \cdots,\ \theta_t^d)^\top = b_t^{-1}\mu_t. \tag{2.4}$$

The SDE for the self-financing positive wealth process (i.e. $V_t > 0, \forall t \in [0,\ T]$) can be written as

$$dV_t^\delta = V_{t-}^\delta \sum_{j=0}^{d} \pi_t^j \frac{dS_t^j}{S_{t-}^j}, \tag{2.5}$$

where π_t^j denotes the proportion of the wealth that is invested at time $t \in [0,\ T]$ in the jth primary security account, i.e.,

$$\pi_t^j = \delta_t^j \frac{S_{t-}^j}{V_{t-}^\delta}, \tag{2.6}$$

for $t \in [0,\ T]$ and $j \in \{0, \cdots, d\}$. These proportions always sum to 1. We call the vector processes of proportions $\pi = \{\pi_t = (\pi_t^0, \pi_t^1, \cdots,\ \pi_t^d)\}_{t\in[0,\, T]}$ a portfolio defined by (2.6).

The equations (2.2) and (2.5) allow us to obtain the SDE for positive wealth process

$$dV_t^\pi = V_{t-}^\pi \Big\{ \sum_{j,i=1}^{d,m} \pi_t^j \sigma_t^{j,i} (\theta_t^i dt + dW_t^i) + \sum_{j=1,k=m+1}^{d} \pi_t^j \rho_t^{j,k} (\theta_t^k dt + dM_t^k) \Big\} \tag{2.7}$$

which has the solution

$$\begin{aligned} V_t^\pi = & V_0^\pi \exp \left\{ \int_0^t \left\{ \sum_{j,i=1}^{d,m} \pi_s^j \sigma_s^{j,i} \theta_s^i ds + \sum_{j=1,k=m+1}^{d} \pi_s^j \rho_s^{j,k} (\theta_s^k - \lambda_s^k) ds \right. \right. \\ & \left. - \frac{1}{2} \sum_{i=1}^{m} \left(\sum_{j}^{d} \pi_s^j \sigma_s^{j,i} \right)^2 ds + \sum_{j,i=1}^{d,m} \pi_s^j \sigma_s^{j,i} dW_s^i \right\} + \\ & \left. \sum_{k=m+1}^{d} \int_0^t \log \left(\sum_{j=1}^{d} \pi_s^j \rho_s^{j,k} + 1 \right) dN_s^k \right\}. \end{aligned} \tag{2.8}$$

Now we give a general definition of GOP which is equivalent to the definition 1.7.2, since it is myopic.

Definition 2.1.1. *A* ***growth optimal portfolio (GOP)*** *is a portfolio, say π^*, which maximizes the expected logarithmic growth rate of wealth over a given time horizon T. That is, π^* is the solution of the problem*

$$\sup_{\pi \in U(T)} E \log(V_T^\pi),$$

where E denotes the expectation under the (market) probability measure P.

It has been proved in Christensen (2005) that the GOP strategy is myopic and the invested fractions are independent of wealth. So, we can define the GOP π^* as a solution of the problem

$$\sup_{\pi \in U(t,T)} E(d \log(V_t^\pi) | \mathcal{F}_{t-}), \tag{2.9}$$

for all $t \in [0,\ T]$, under the probability measure P and $U(t,T)$ indicates the positive admissible strategies during the planing period $[t,T]$. By using Itô calculus and from equation (2.7), we get

$$E(d\log(V_t^\pi)|\mathcal{F}_{t-}) = \Bigg(\sum_{i=1,j=1}^{d,m} \pi_t^i \sigma_t^{i,j} \theta_t^j - \frac{1}{2} \sum_{j=1}^{m} \Big(\sum_{i=1}^{d} \pi_t^i \sigma_t^{i,j} \Big)^2 + \sum_{i=1,j=m+1}^{d} \pi_t^i \rho_t^{i,j} (\theta_t^j - \lambda_t^j) + \sum_{j=m+1}^{d} \log \Big(\sum_{i=1}^{d} \pi_t^i \rho_t^{i,j} + 1 \Big) \lambda_t^j \Bigg) dt. \tag{2.10}$$

Now, we use the first order condition to maximize the above expression with respect to the admissible strategies, taking the derivatives with respect to π^i and setting them equal to zero. We then get

$$\sum_{j=1}^{m} \sigma_t^{i,j} \Big[\theta_t^j - \Big(\sum_{k=1}^{d} \pi_t^k \sigma_t^{k,j} \Big) \Big] + \sum_{j=m+1}^{d} \rho_t^{i,j} \Big[(\theta_t^j - \lambda_t^j) + \frac{\lambda_t^j}{1 + \sum_{k=1}^d \pi_t^k \rho_t^{k,j}} \Big] = 0, \tag{2.11}$$

for $i \in \{1, \cdots, d\}$, which is equivalent to solving the set of linear equations

$$\theta_t^i - \Big(\sum_{k=1}^{d} \pi_t^k \sigma_t^{k,i} \Big) = 0 \tag{2.12a}$$

$$(\theta_t^j - \lambda_t^j) + \frac{\lambda_t^j}{1 + \sum_{k=1}^d \pi_t^k \rho_t^{k,j}} = 0, \tag{2.12b}$$

for $i \in \{1, \cdots, m\}$ and $j \in \{m+1, \cdots, d\}$, i.e.,

$$\sum_{k=1}^{d} \pi_t^k \sigma_t^{k,i} = \theta_t^i \tag{2.13a}$$

$$\sum_{k=1}^{d} \pi_t^k \rho_t^{k,j} = \frac{\theta_t^j}{\lambda_t^j - \theta_t^j}, \tag{2.13b}$$

for $i \in \{1, \cdots, m\}$ and $j \in \{m+1, \cdots, d\}$. The solution, say π^*, of the system of equations (2.13) satisfies the equation

$$\pi_t^{*\top} [\sigma_t, \rho_t] = \Big[\theta_t^1, \cdots, \theta_t^m, \frac{\theta_t^{m+1}}{\lambda_t^{m+1} - \theta_t^{m+1}}, \cdots, \frac{\theta_t^d}{\lambda_t^d - \theta_t^d} \Big] \tag{2.14}$$

at time $t \in [0,\ T]$. If we check the second order condition for optimization, we get that the extracted π^* maximizes the expression (2.10). The corresponding wealth process is

$$dV_t^{\pi^*} = V_{t-}^{\pi^*} \Big\{ \sum_{i=1}^{m} \theta_t^i (\theta_t^i dt + dW_t^i) + \sum_{k=m+1}^{d} \frac{\theta_t^k}{\lambda_t^k - \theta_t^k} (\theta_t^k dt + dM_t^k) \Big\}, \tag{2.15}$$

where we assume that

$$\lambda_t^k > \theta_t^k,$$

a.s., which ensures that $\frac{\theta_t^k}{\lambda_t^k - \theta_t^k} > -1$, a.s., for the positivity of V^{π^*}, for all $t \in [0,\ T]$ and $k \in \{m+1, \cdots, d\}$. This assumption is necessary to keep the wealth process V^{π^*} positive. When the market price of risk θ_t^k for the kth event-driven uncertainty is zero, it follows by (2.15) that $V_t^{\pi^*}$ does not exhibit jumps of the kth type. On the other hand, when θ_t^k approaches λ_t^k, the impact of jumps becomes extreme. First we show the most important result for our analysis, that is the benchmarked wealth processes are supermartingales.

Proposition 2.1.1. *The benchmarked wealth process $\widehat{V}^\pi$ corresponding to a self-financing portfolio π is a supermartingale.*

Proof. Recall that the benchmarked value of V^π is denoted by $\widehat{V}_t^\pi = \frac{V_t^\pi}{V_t^{\pi^*}}$. By Itô's formula, we have

$$d\widehat{V}_t^\pi = \widehat{V}_{t-}^\pi \left[\left\{ \sum_{j,i=1}^{d,m} \pi_t^j \sigma_t^{j,i} - \theta_t^i \right\} dW_t^i + \left\{ \sum_{j=1,k=m+1}^{d} \pi_t^j \rho_t^{j,k} - \frac{\theta_t^k}{\lambda_t^k - \theta_t^k} \right\} \left\{ 1 - \frac{\theta_t^k}{\lambda_t^k} \right\} dM_t^k \right],$$

which is a local martingale. Then, the proof follows from the fact that a non-negative local martingale is a supermartingale. □

Kardaras and Karatzas (2007) state that a portfolio $\pi' \in U(0,T)$ is called a numeraire portfolio, if for every wealth process V^π the relative wealth process defined as $\frac{V^\pi}{V^{\pi'}}$ is a supermartingale. Therefore, π^* is a numeraire portfolio. Christensen (2005) has proved that a portfolio is numeraire portfolio if and only if it is a GOP. Hence π^* in (2.14) is a GOP. Now we recall the definition of the complete financial market from Chapter 1. We call a market is complete if every contingent claim can be perfectly hedged with respect to a fair, self-financing portfolio. From Platen (2004a), the model considered above is a complete financial market model.

Now, we consider X_t^π to be the value of the amount invested in $V_t^{\pi^*}$, S_t^1, $\cdots$, S_t^d at time t with cumulative consumption $\mathbb{C}_t$ and portfolio π. Note that we are investing in GOP in place of riskfree asset. The riskfree asset is inherited inside the GOP.

Definition 2.1.2. *A* ***consumption process*** $\mathbb{C} = \{\mathbb{C}_t\}_{t\in[0,\,T]}$ *is a square-integrable, progressively measurable process with respect to F, taking values in $[0,\,\infty)$, and satisfying the conditions*

(i) $\mathbb{C}_0 = 0$,

(ii) the path $t \to \mathbb{C}_t$ is non-decreasing and right-continuous almost surely and does not have common jump with N.

The wealth process X^π satisfies the SDE

$$dX_t^\pi = X_{t-}^\pi\left\{\sum_{i=1}^{d}\pi_t^i\frac{dS_t^i}{S_{t-}^i} + \pi_t^0\frac{dV_t^{\pi^*}}{V_{t-}^{\pi^*}}\right\} - d\mathbb{C}_t.$$

Using equation (2.2) and (2.15), it can be rewritten as

$$\begin{aligned} dX_t^\pi =& X_t^\pi\left[\sum_{j=1}^{d}\pi_t^j\left\{\sum_{i=1}^{m}\sigma^{j,i}(dW_t^i + \theta_t^i dt) + \sum_{k=m+1}^{d}\rho_t^{j,k}(dM_t^k + \theta_t^k dt)\right\}\right. \\ &\left. + \pi_t^0\left\{\sum_{i=1}^{m}\theta_t^i(\theta_t^i dt + dW_t^i) + \sum_{k=m+1}^{d}\frac{\theta_t^k}{\lambda_t^k - \theta_t^k}(\theta_t^k dt + dM_t^k)\right\}\right] - d\mathbb{C}_t. \end{aligned} \tag{2.16}$$

2.1.1 The benchmark model

We use the GOP V^{π^*} as a benchmark or reference unit and we call the prices when expressed in units of V^{π^*} benchmarked prices. Furthermore, we call a model of the above prescribed form a benchmark model. By Itô's formula, the benchmarked wealth process $\widehat{X}^\pi = \{\widehat{X}_t^\pi\}_{t\in[0,\,T]}$, with

$$\widehat{X}_t^\pi = \frac{X_t^\pi}{V_t^{\pi^*}},$$

for $t \in [0,\,T]$, satisfies the SDE

$$d\widehat{X}_t^\pi = \widehat{X}_{t-}^\pi\left[\sum_{j,i=1}^{d,m}\pi_t^j\{\sigma_t^{j,i} - \theta_t^i\}dW_t^i + \sum_{j=1,k=m+1}^{d}\pi_t^j\left\{\rho_t^{j,k} - \frac{\theta_t^j}{\lambda_t^k - \theta_t^k}\right\}\left\{1 - \frac{\theta_t^k}{\lambda_t^k}\right\}dM_t^k\right] - \frac{d\mathbb{C}_t}{V_t^{\pi^*}}.$$

That is,

$$\widehat{X}_t^{\pi} + \int_0^t \frac{d\mathbb{C}_s}{V_s^{\pi^*}} = \widehat{x} + \int_0^t \Bigg[\sum_{j,i=1}^{d,m} \varrho_s^j \{\sigma_s^{j,i} - \theta_s^i\} dW_s^i + \sum_{j=1,k=m+1}^{d} \varrho_s^j \left\{ \rho_s^{j,k} - \frac{\theta_s^j}{\lambda_s^k - \theta_s^k} \right\} \left\{ 1 - \frac{\theta_s^k}{\lambda_s^k} \right\} dM_s^k \Bigg], \tag{2.17}$$

where $\pi_t^j \widehat{X}_{t-}^{\pi} = \varrho_t^j$, for $j \in \{1, \cdots, d\}$, and $\widehat{x} = \frac{x}{V_0^{\pi^*}} = x$.

The right-hand-side of (2.17) is a P-local martingale on $[0,\ T]$, whereas the left-hand side is, for every $(\pi, \mathbb{C}) \in \mathcal{A}(T, x)$, a non-negative process. Here, $\mathcal{A}(T, x)$ is the set of all $(\pi, \mathbb{C})$ such that the corresponding δ is an admissible trading strategy with initial endowment $x \geq 0$, in time horizon $[0,\ T]$. Therefore, the right-hand-side of the equation (2.17) is a supermartingale on $[0,\ T]$ for which the optional sampling theorem yields

$$E\left[\widehat{X}_\tau + \int_0^\tau \frac{d\mathbb{C}_s}{V_s^{\pi^*}} \right] \leq x, \tag{2.18}$$

for all stopping times $\tau \in \mathfrak{S}[0,\ T]$, where $\mathfrak{S}[0,\ T]$ is set a of all F-adapted stopping times on the interval $[0,\ T]$. A necessary condition for $(\pi,\ \mathbb{C}) \in \mathcal{A}(T, x)$ is

$$E\left[\int_0^\tau \frac{d\mathbb{C}_s}{V_s^{\pi^*}} \right] \leq x. \tag{2.19}$$

Proposition 2.1.2. *Let $\mathbb{C}$ be a consumption process satisfying (2.19) for a given $x \in [0,\ \infty)$. Then there exists a portfolio π such that $(\pi, \mathbb{C}) \in \mathcal{A}(T, x)$.*

Proof. With $D = \int_0^T \frac{d\mathbb{C}_s}{V_s^{\pi^*}}$ and from (2.19), we define the non-negative process ξ such that

$$\xi_t = E\left(\int_t^T \frac{V_t^{\pi^*}}{V_s^{\pi^*}} d\mathbb{C}_s \Big| \mathcal{F}_t \right) + (x - ED) V_t^{\pi^*}. \tag{2.20}$$

This implies that

$$\begin{aligned} \frac{\xi_t}{V_t^{\pi^*}} + \int_0^t \frac{d\mathbb{C}_s}{V_s^{\pi^*}} &= E\left(D | \mathcal{F}_t \right) - ED + x \\ &= \mathbf{M}_t + x, \end{aligned} \tag{2.21}$$

where $\mathbf{M}_t = E(D|\mathcal{F}_t) - ED$ is a square-integrable martingale. We will now use the fundamental martingale representation theorem to represent $\mathbf{M}$ in terms of Brownian motion and compensated jump process (see, Bardhan and Chao (1995)). For some square-integrable, F-predictable processes, ψ^j and η^k, for $j \in \{1, \cdots, m\}$ and $k \in \{m+1, \cdots, d\}$,

$$\mathbf{M}_t = \sum_{j=1}^{m} \int_0^t \psi_s^j dW_s^j + \sum_{k=m+1}^{d} \int_0^t \eta_s^k dM_s^k$$

Substituting in (2.21), we get

$$\frac{\xi_t}{V_t^{\pi^*}} + \int_0^t \frac{d\mathbb{C}_s}{V_s^{\pi^*}} = \sum_{j=1}^{m} \int_0^t \psi_s^j dW_s^j + \sum_{k=m+1}^{d} \int_0^t \eta_s^k dM_s^k + x. \tag{2.22}$$

Now, comparing it with equation (2.17), we see that

$$\psi_t^i = \sum_{j=1}^{d} \varrho_t^j \{\sigma_t^{j,i} - \theta_t^i\}, \qquad \text{for } i \in \{1, \cdots, m\}, \tag{2.23a}$$

$$\eta_t^k = \sum_{j=1}^{d} \varrho_t^j \left\{ \rho_t^{j,k} - \frac{\theta_t^j}{\lambda_t^k - \theta_t^k} \right\} \left\{ 1 - \frac{\theta_t^k}{\lambda_t^k} \right\}, \qquad \text{for } k \in \{m+1, \cdots, d\}. \tag{2.23b}$$

The above equation is a linear system of equations for ϱ^j, for $j \in \{1, \cdots, d\}$. By solving the linear equation for ϱ, we can find X^π from equation (2.17) and then the required π. $\square$

2.2 Pricing and hedging of contingent claims

Sometimes the holder of the option may have to pay premium. So, here we consider generalized form of contingent claims from Karatzas (1988).

Definition 2.2.1. *A **European contingent claim** (ECC) (T, f_T, g) is a financial instrument consisting of*

i) a maturity date $T \in (0,\ \infty)$

ii) a payoff rate of g_t per unit time on $(0,\ T)$, and

iii) a terminal payoff f_T at maturity.

Here the processes $\{f_t\}_{t\in[0,T]}$ *and* $\{g_t\}_{t\in[0,T]}$ *are assumed to be non-negative, progressively measurable, and satisfy*

$$E(\sup_{s\in[0,T]} f_s + \int_0^t g_s ds)^2 < \infty,$$

where f_t *indicates the intrinsic value of the payoff at time* $t \in [0, T]$.

Definition 2.2.2. *Let* $x \geq 0$ *and* $T > 0$ *be two given finite numbers. A pair* $(\pi, \mathbb{C}) \in \mathcal{A}(T, x)$, *with a corresponding wealth process* X^π, *is called a* ***hedging strategy*** *against the ECC if*

(i) $\mathbb{C}_t = \int_0^t g_s ds$, $t \in [0,\ T]$, *and*

(ii) $X_T^\pi = f_T$ *holds almost surely.*

The hedging strategy duplicates the payoff from the ECC by managing a portfolio consisting of the stocks and the GOP, and by appropriate absolutely continuous consumption as in (i) of Definition 2.2.2.

Definition 2.2.3. *The smallest value* $x \geq 0$ *for which there exist* $(\pi, \mathbb{C}) \in \mathcal{A}(T, x)$ *against the ECC is called the* ***fair price*** *at* $t = 0$ *of the ECC.*

To find the fair prices of the ECC and American contingent claim (ACC), we need to introduce a non-negative process

$$Q_t = \frac{f_t}{V_t^{\pi^*}} + \int_0^t \frac{g_s}{V_s^{\pi^*}} ds, \tag{2.24}$$

where f_t is the value of payoff function at time t.

Theorem 2.2.1. *The fair price for the ECC* (T, f_T, g) *is given by*

$$E(Q_T) = E\left[\frac{f_T}{V_T^{\pi^*}} + \int_0^T \frac{g_s}{V_s^{\pi^*}} ds\right].$$

Moreover, there exists a hedging strategy $(\pi, \mathbb{C}) \in \mathcal{A}(T, x)$ *whose corresponding wealth process* X^π *satisfies*

$$X_t^\pi = E\left[f_T \frac{V_t^{\pi^*}}{V_T^{\pi^*}} + \int_t^T g_s \frac{V_t^{\pi^*}}{V_s^{\pi^*}} ds \,\middle|\, \mathcal{F}_t\right]. \tag{2.25}$$

Proof. For every $x \geq 0$ for which there exists a hedging strategy $(\pi, \mathbb{C}) \in \mathcal{A}(T, x)$ against the ECC, we have, from (2.18), $E(Q_T) \leq x$. Therefore, the fair price cannot be smaller than the number $E(Q_T)$. Now we find a hedging strategy with initial endowment $E(Q_T)$. For this, we consider a process ζ,

$$\zeta_t = V_t^{\pi^*}\left[E(Q_T) + E(Q_T \mid \mathcal{F}_t) - E(Q_T) - \int_0^t \frac{g_s}{V_s^{\pi^*}} ds\right],$$

so that

$$\frac{\zeta_t}{V_t^{\pi^*}} + \int_0^t \frac{g_s}{V_s^{\pi^*}} ds = E(Q_T) + M_t, \tag{2.26}$$

where M is a right-continuous version of the square-integrable martingale $E(Q_T \mid \mathcal{F}_t) - E(Q_T)$ with $M_0 = 0$ almost surely. From the construction it is clear that, for every fixed $t \in [0,\ T]$, ζ_t agrees almost surely with the right-hand-side of (2.25).

From martingale representation theorem (see, Bardhan and Chao (1995)), there will exist some square-integrable F-predictable process ϕ^j and η^k, for $j \in \{1, \cdots, m\}$ and $k \in \{m+1, \cdots, d\}$, and

$$\frac{\zeta_t}{V_t^{\pi^*}} + \int_0^t \frac{g_s}{V_s^{\pi^*}} ds = E(Q_T) + \sum_{j=1}^{m} \int_0^t \psi_s^j dW_s^j + \sum_{j=m+1}^{d} \int_0^t \eta_s^k dM_s^k.$$

The quantities ψ^j and η^k, for $j \in \{1, \cdots, m\}$ and $k \in \{m+1, \cdots, d\}$, can be obtained from the Radon-Nikodým derivatives

$$\psi^j = \frac{d[\frac{\zeta_t}{V_t^{\pi^*}}, W_j]}{d[W_j]}, \quad \eta^k = \frac{d[\frac{\zeta_t}{V_t^{\pi^*}}, m_k]}{d[m_k]}.$$

Now the hedging strategy can be derived as in Proposition 2.1.2 and $\mathbb{C}_t = \int_0^t g_s ds$, and we have to replace $\xi = \zeta,\ x = E(Q_T)$. That is, with initial endowment $E(Q_T)$, there is a hedging strategy $(\pi, \mathbb{C}) \in \mathcal{A}(T, x)$ against the ECC. Hence, $E(Q_T)$ is the fair price. □

Now, we define American contingent claims and derive the price and hedging strategy for them.

Definition 2.2.4. *An* ***American contingent claim*** *(ACC)* (T, f, g) *is a financial instrument consisting of*

i) a maturity date $T \in (0,\ \infty)$

ii) the selection of an exercise time $\tau \in \mathfrak{S}[0,\ T]$

iii) a payoff rate of g_t *per unit time on* $(0,\ \tau)$ *and*

iv) a terminal payoff f_τ *at exercise time.*

Here, f *and* g *satisfies the conditions assumed in ECC definition.*

Contrary to ECC, an ACC can be exercised at any time before the maturity. ACC gives the extra right to the buyer than the ECC, so it's price process must be greater then equal to the price process of a corresponding ECC. The derivation of the ACC price process is not that easy as the ECC price process is. The hedging strategy against ACC is defined in Karatzas (1988) as

Definition 2.2.5. *For the given finite horizon* $T > 0$ *and initial wealth* $x > 0$*, consider a pair* $(\pi, \mathbb{C}) \in \mathcal{A}(T,\ x)$ *and let* X^π *denote the corresponding wealth process. We say that* $(\pi, \mathbb{C})$ *is a* ***hedging strategy*** *against the ACC* $(T,\ f,\ g)$ *and write* $(\pi, \mathbb{C}) \in \mathcal{H}(T,\ x)$*, if for almost surely, the following requirements hold:*

i) $A_t = \mathbb{C}_t - \int_0^t g_s ds, \quad t \in [0,\ T]$*, is a non-decreasing function.*

ii) $X_t^\pi \geq f_t, \quad \forall\, t \in [0, T]$.

iii) $X_T^\pi = f_T$.

iv) $A_t = A_{\tau_t}$ *for every fixed number* $t \in [0,\ T]$*, where* $\tau_t = \inf\{s \in [t,\ T];\ X_s^\pi = f_s\}$.

The fair price at $t = 0$ for the ACC is the value P_0, where

$$P_0 = \inf\{x \geq 0 :\ \exists\, (\pi, \mathbb{C}) \in \mathcal{H}(T,\ x)\}.$$

Before stating the next lemma, we give the definition of essential supremum.

Definition 2.2.6. *The essential supremum* $\overline{f} = ess\ sup_{t \in [0,\ T]} f_t$ *of a stochastic process* f *over a time interval* $[0,\ T]$ *is defined by the properties* *(i) $\overline{f}$ is measurable;*

(ii) $\overline{f} \geq f$ almost surely, for each $t \in [0,\ T]$;

(iii) for any h that satisfies (i) and (ii), $h \geq \overline{f}$ almost surely.

Lemma 2.2.1. *Consider the process*

$$Y_t = ess\ sup_{\tau\in\mathfrak{S}[t,\ T]} E\left[f_\tau \frac{V_t^{\pi^*}}{V_\tau^{\pi^*}} + \int_t^\tau g_s \frac{V_t^{\pi^*}}{V_s^{\pi^*}} ds \,\Big|\, \mathfrak{F}_t \right]. \tag{2.27}$$

Then, there is an admissible strategy π and a consumption process $\mathbb{C}$, with the corresponding wealth process X^π, such that

$$X_t^\pi = Y_t,$$

for $t \in [0,\ T]$.

Proof. Define

$$J_t = ess\ sup_{\tau\in\mathfrak{S}[t,\ T]} E(Q_\tau | \mathfrak{F}_t).$$

It is straightforward to see that J is a supermartingale, and in fact, J is the smallest supermartingale dominating the discounted reward Q. The process is called the Snell envelope.

Since J is a supermartingale, it has a Doob-Meyer decomposition as the difference of a martingale M and a predictable non-decreasing process A, i.e.,

$$J_t = M_t - A_t,$$

$$J_t + A_t = M_t. \tag{2.28}$$

Here, M is a martingale and A is a unique, predictable continuous non-decreasing process with $A_0 = 0$. Now we consider

$$\zeta_t = V_t^{\pi^*} \left[J_t - \int_0^t \frac{g_s}{V_s^{\pi^*}} ds \right],$$

$$\zeta_t = V_t^{\pi^*} \left[M_t - A_t - \int_0^t \frac{g_s}{V_s^{\pi^*}} ds \right].$$

The strategies can be derived, by proceeding exactly as in the proof of Theorem2.2.1, with the substitution $\mathbb{C}_t = \int_0^t g_s ds + \int_0^t V_s^{\pi^*} dA_s$. Since ζ_t has the same value as in the right-hand-side of (2.27), the lemma follows. □

In the next theorem, we derive a fair price of an ACC.

Theorem 2.2.2. *The fair price at $t = 0$ for the ACC is given by*

$$V_0 = sup_{\tau \in \mathfrak{S}[0,\, T]} E \left[\frac{f_\tau}{V_\tau^{\pi^*}} + \int_0^\tau \frac{g_s}{V_s^{\pi^*}} ds \right]. \tag{2.29}$$

Moreover, there exists a strategy $(\pi, \mathbb{C}) \in \mathcal{H}(T,\ V_0)$ with corresponding wealth process $X^\pi = \{X_t^\pi\}_{t \in [0,\, T]}$ satisfying

$$X_t^\pi = ess\ sup_{\tau \in \mathfrak{S}[t,\, T]} E \left[f_\tau \frac{V_t^{\pi^*}}{V_\tau^{\pi^*}} + \int_t^\tau g_s \frac{V_t^{\pi^*}}{V_s^{\pi^*}} ds \;\Big|\; \mathcal{F}_t \right] \ a.s. \tag{2.30}$$

for every fixed $t \in [0,\ T]$.

Proof. Let $x \geq 0$ be any number for which there exists a hedging strategy $(\pi, \mathbb{C}) \in \mathcal{H}(T,\ x)$. The optional sampling theorem applied to the non-negative supermartingale of (2.17) then gives, in conjunction with properties $(i), (ii)$ of the Definition 2.2.5,

$$EQ_T = E \left[\frac{f_T}{V_T^{\pi^*}} + \int_0^T \frac{g_s}{V_s^{\pi^*}} ds \right] \leq E \left[\widehat{X}_\tau^\pi + \int_0^\tau \frac{d\mathbb{C}_t}{V_t^{\pi^*}} \right] \leq x, \tag{2.31}$$

for every $\tau \in \mathfrak{S}[0,\ T]$. Therefore, with the notation $V_t = \sup_{\tau \in \mathfrak{S}[t,\, T]} E(Q_\tau),\quad t \in [0,\ T]$, we have $V_0 \leq x$. The second part of the theorem follows from Lemma 2.2.1. So, V_0 is the fair price at $t = 0$ for the ACC. □

Until now, we have considered the parameters of the model as deterministic functions of time, but many empirical studies suggest that the parameters estimated from stock price returns exhibit time-varying and random characteristics. It is hard to analyze the ACC pricing and hedging in stochastic parameter models. Hence, in the next two sections, we consider the cases of stochastic parameters and derive the fair price and hedging strategy only for ECCs.

2.3 The model with stochastic parameters

In this section, we assume that market parameters are driven by the stochastic processes adapted to the filtration F. The price S_t^j, at time t, for one share of the jth stock satisfies the SDE

$$dS_t^j = S_{t-}^j \left(\mu_t^j dt + \sum_{i=1}^{m} \sigma_t^{j,i} dW_t^i + \sum_{k=m+1}^{d} \rho_t^{j,k} dM_t^k \right), \tag{2.32}$$

for $t \in [0,\ T]$, with initial value $S_0^j > 0$, for $j \in \{1, \cdots, d\}$. Here, μ^j, $\sigma^{j,i}$ and $\rho^{j,k}$, are considered to be deterministic functions, satisfying the same conditions as in Section 2.1, of a stochastic process Y adapted to F, and having a bounded derivative with respect to Y, for all $j \in \{1, \cdots, d\}$, $i \in \{1, \cdots, m\}$ and $k \in \{m+1, \cdots, d\}$. Here, $\mu_t^j = \mu^j(Y_t)$, $\sigma_t^{j,i} = \sigma^{j,i}(Y_t)$ and $\rho_t^{j,k} = \rho^{j,k}(Y_t)$. We assume that the process Y is given by

$$dY_t = \alpha_t dt + \sum_{i=1}^{m} \xi_t^i dW_t^i + \sum_{k=m+1}^{d} \zeta_{t-}^k dN_t^k, \tag{2.33}$$

where α, ξ^i, and ζ^k are considered to be uniformly bounded, square-integrable deterministic functions of time, and have bounded derivative with respect to time, for $i \in \{1, \cdots, m\}$ and $k \in \{m+1, \cdots, d\}$. The matrix $\xi_t \xi_t^\top$ is assumed to be nonsingular and the matrix $[\xi_t, \zeta_t]$ is assumed to satisfy the following inequality

$$[\xi_t, \zeta_t][\xi_t, \zeta_t]^\top \geq \epsilon I, \quad \forall t \in [0, T]$$

for some $\epsilon > 0$, where I is a d-dimensional unit matrix. We assume that Y has continuously differentiable and bounded density function $\tilde{f}$. Note that (V^π, Y) is a Markov process. Due to the nonpredictable nature of the parameters, deriving GOP is not as straightforward as before. We use stochastic control methods to solve the problem. The value function depends on the state variable and wealth, and is given by

$$\varsigma(t,v,y) = \sup_{\pi\in U(t,T)} E[\log V_T^\pi \mid V_t^\pi = v, Y_t = y],$$

where $E[\cdot \mid V_t^\pi = v, Y_t = y]$ denotes the conditional expectation given complete information up to time t. For the model we have considered in this section, the value function ς satisfies the Hamilton-Jacobi-Bellman (HJB) PDE

$$\begin{aligned}
&\varsigma_t + \sup_{\pi\in U(T)} \Bigg[\varsigma_v v\Big\{\sum_{i=1,j=1}^{d,m} \pi_t^i\sigma_t^{i,j}\theta_t^j + \sum_{i=1,j=m+1}^{d} \pi_t^i\rho_t^{i,j}(\theta_t^j - \lambda_t^j)\Big\} \\
&+ \frac{1}{2}\varsigma_{vv}vv\sum_{j=1}^{m}\Big(\sum_{i=1}^{d}\pi_t^i\sigma_t^{i,j}\Big)^2 + \varsigma_y\alpha_t + \frac{1}{2}\varsigma_{yy}\sum_{j=1}^{m}(\xi_t^j)^2+ \\
&\varsigma_{vy}vy\sum_{j=1}^{m}\Big(\sum_{i=1}^{d}\pi_t^i\sigma_t^{i,j}\xi_t^j\Big)+ \\
&\sum_{j=m+1}^{d}\Big\{\varsigma(t, v + v\sum_{i=1}^{d}\pi_{t-}^i\rho_t^{i,j}, y + \zeta_{t-}^j) - \varsigma(t,v,y)\Big\}\lambda_t^j\Bigg] = 0.
\end{aligned} \tag{2.34}$$

From (2.8), ς can be written as the sum of two different functions $\varsigma(t,v,y) = \log v + \mathcal{D}(t,y)$. Then, the equation (2.34) boils down to

$$\begin{aligned}
&\mathcal{D}_t + \sup_{\pi\in U(t,T)} \Bigg[\Big\{\sum_{i=1,j=1}^{d,m} \pi_t^i\sigma_t^{i,j}\theta_t^j + \sum_{i=1,j=m+1}^{d} \pi_t^i\rho_t^{i,j}(\theta_t^j - \lambda_t^j)\Big\} \\
&- \frac{1}{2}\sum_{j=1}^{m}\Big(\sum_{i=1}^{d}\pi_t^i\sigma_t^{i,j}\Big)^2 + \mathcal{D}_y\alpha_t + \frac{1}{2}\mathcal{D}_{yy}\sum_{j=1}^{m}(\xi_t^j)^2 \\
&+ \sum_{j=m+1}^{d}\Big\{\mathcal{D}(t, y + \zeta_{t-}^j) - \mathcal{D}(t,y) + \log(1 + \sum_{i=1}^{d}\pi_t^i\rho_t^{i,j})\Big\}\lambda_t^j\Bigg] = 0.
\end{aligned} \tag{2.35}$$

Using the first order condition to get the supremum, taking the derivatives with respect to π^i and setting them equal to zero, we get

$$\begin{aligned}
&\sum_{j=1}^{m}\sigma_t^{i,j}\Big[\theta_t^j - \Big(\sum_{k=1}^{d}\pi_t^k\sigma_t^{k,j}\Big)\Big]+ \\
&\sum_{j=m+1}^{d}\rho_t^{i,j}\Big[(\theta_t^j - \lambda_t^j) + \frac{\lambda_t^j}{1 + \sum_{k=1}^{d}\pi_t^k\rho_t^{k,j}}\Big] = 0,
\end{aligned} \tag{2.36}$$

for $i \in \{1, \cdots, d\}$, which is the same as (2.11). Hence, GOP will have the same expression as before, but with F-adapted stochastic parameters.

Using (2.36) the HJB equation (2.35) can be written as

$$\begin{aligned}&\mathcal{D}_t + \Big\{\frac{1}{2}\sum_{j=1}^{m}(\theta_t^j)^2 - \sum_{j=m+1}^{d}\theta_t^j\Big\} + \mathcal{D}_y\alpha_t + \frac{1}{2}\mathcal{D}_{yy}\sum_{j=1}^{m}(\xi_t^j)^2 \\ &+ \sum_{j=m+1}^{d}\Big\{\mathcal{D}(t, y+\zeta_{t-}^j) - \mathcal{D}(t,y) + \log(\frac{\lambda_t^j}{\lambda_t^j - \theta_t^j})\Big\}\lambda_t^j = 0. \qquad (2.37)\end{aligned}$$

Theorem 2.3.1. *The HJB equation (2.37) has a unique classical solution of the form*

$$\mathcal{D}(t,y) = E\Big[\int_t^T \Big(\frac{1}{2}\sum_{j=1}^{m}(\theta_s^j)^2 - \sum_{j=m+1}^{d}\theta_s^j + \sum_{j=m+1}^{d}\lambda_s^j\log\Big(\frac{\lambda_s^j}{\lambda_s^j - \theta_s^j}\Big)\Big)ds|\mathcal{F}_t\Big].$$

Proof. First we show that the HJB equation (2.37) has a unique solution in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$. The HJB equation can be written as

$$\begin{aligned}&\mathcal{D}_t + \mathcal{D}_y\alpha_t + \frac{1}{2}\mathcal{D}_{yy}\sum_{j=1}^{m}(\xi_t^j)^2 + \mathbf{D}(t,y) + \\ &\Big\{\frac{1}{2}\sum_{j=1}^{m}(\theta_t^j)^2 - \sum_{j=m+1}^{d}\theta_t^j\Big\} + \sum_{j=m+1}^{d}\Big\{\log(\frac{\lambda_t^j}{\lambda_t^j - \theta_t^j})\Big\}\lambda_t^j = 0,\end{aligned}$$

where $\mathbf{D}(t,y) = \sum_{j=m+1}^{d}\Big\{\mathcal{D}(t,y+\zeta_{t-}^j) - \mathcal{D}(t,y)\Big\}$. Now since $\varsigma(t,v,y) = \log v + \mathcal{D}(t,y)$ and using the definition of ς and (2.8), $D(t,y)$ can be written as

$$\begin{aligned}\mathcal{D}(t,y) &= E\Big[\int_t^T \Big(\frac{1}{2}\sum_{j=1}^{m}(\theta_s^j)^2 - \sum_{j=m+1}^{d}\theta_s^j + \sum_{j=m+1}^{d}\lambda_s^j\log\Big(\frac{\lambda_s^j}{\lambda_s^j - \theta_s^j}\Big)\Big)ds|\mathcal{F}_t\Big] \\ &= \int_{-\infty}^{\infty}\int_t^T \Big(\frac{1}{2}\sum_{j=1}^{m}(\theta_s^j(z))^2 - \sum_{j=m+1}^{d}\theta_s^j(z) + \sum_{j=m+1}^{d}\lambda_s^j\log\Big(\frac{\lambda_s^j}{\lambda_s^j - \theta_s^j(z)}\Big)\Big)\tilde{f}(z|Y_t = y)dsdz\end{aligned} \qquad (2.38)$$

From the assumptions we have made in the beginning of this section, $\mathbf{D} \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$. Now using the results from Fleming and Rishel (1975) (Chapter VI and Appendix-E) and Davis and Lleo (2011), we deduce that the HJB equation (2.37) has a unique solution in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$.

Suppose $\mathcal{D}(t,y)$ solves the PDE (2.37). An application of Ito's lemma for $t<T$ yields

$$\mathcal{D}(T,Y_T)=\mathcal{D}(t,y)+\int_t^T\Big(\mathcal{D}_s(s,Y_s)ds+\mathcal{D}_y(s,Y_s)(\alpha_s ds+\sum_{i=1}^m\xi_s^i dW_s^i)$$
$$+\frac{1}{2}\mathcal{D}_{yy}(s,Y_s)\sum_{i=1}^m(\xi_s^i)^2ds\Big)+\sum_{t\le s\le T}[\mathcal{D}(s,Y_s)-\mathcal{D}(s,Y_{s-})]. \tag{2.39}$$

From (2.37), replacing $\mathcal{D}_s$ by

$$-\Big\{\frac{1}{2}\sum_{j=1}^m(\theta_s^j)^2+\sum_{j=m+1}^d\theta_s^j\Big\}-\mathcal{D}_y(s,Y_s)\alpha_s-\frac{1}{2}\mathcal{D}_{yy}(s,Y_s)\sum_{j=1}^m(\xi_s^j)^2$$
$$-\sum_{j=m+1}^d\Big\{\mathcal{D}(s,Y_{s-}+\zeta_{s-}^j)-\mathcal{D}(s,Y_{s-})+\log\Big(\frac{\lambda_s^j}{\lambda_s^j-\theta_s^j}\Big)\Big\}\lambda_s^j,$$

and taking conditional expectation at time t yields

$$E[\mathcal{D}(T,Y_T)|\mathcal{F}_t]=\mathcal{D}(t,y)-E\Big[\int_t^T\Big(\frac{1}{2}\sum_{j=1}^m(\theta_s^j)^2-\sum_{j=m+1}^d\theta_s^j+\sum_{j=m+1}^d\lambda_s^j\log\Big(\frac{\lambda_s^j}{\lambda_s^j-\theta_s^j}\Big)\Big)ds|\mathcal{F}_t\Big]. \tag{2.40}$$

Since $D(T,Y_T)=0$ the statement follows.

□

In the case of stochastic parameters model that is under consideration till now, the pricing and hedging of the contingent claims in this market will be similar as in the previous market. This will not be the case if we consider the stochastic parameters to be independent of the stock price processes. Due to the extra randomness, the market will be incomplete and there will exist some intrinsic risk while hedging a contingent claim. In the next section, we carry out the pricing and hedging of ECCs in such an incomplete market.

2.4 Incomplete market model

In this section, we consider the filtration $G=\{\mathcal{G}_t\}_{t\in[0,\,T]}$, where $\mathcal{G}_t=\mathcal{F}_t\vee\sigma(\widetilde{W}_s,\widetilde{N}_s,\ s\le t)$ as augmentation of the previous filtration F with $\sigma(\widetilde{W},\widetilde{N})$, where $\widetilde{W}$ is a standard Wiener process and $\widetilde{N}$ is a Poisson process, independent with each other and independent with all the other random processes that generates F. The $\sigma(\widetilde{W}, \widetilde{N})$ is the σ-field generated by $\widetilde{W}, \widetilde{N}$. Here, we suppose that Y, on which market parameters are dependent as in previous section, given by

$$dY_t = \alpha_t dt + \xi_t d\widetilde{W}_t + \zeta_t d\widetilde{M}_t, \tag{2.41}$$

where α, $\xi > 0$, and ζ are considered to be uniformly bounded, square-integrable deterministic functions of time and have bounded derivative with respect to time. $\widetilde{M}_t = \widetilde{N}_t - \int_0^t \widetilde{\lambda}_s ds$ is a compensated counting process with intensity $\widetilde{\lambda}_t$.The matrix $[\xi_t, \zeta_t]$ is assumed to satisfy the following inequality

$$[\xi_t, \zeta_t][\xi_t, \zeta_t]^\top \geq \epsilon I, \quad \forall t \in [0, T]$$

for some $\epsilon > 0$, where I is a two-dimensional unit matrix. We assume that Y has continuously differentiable and bounded density function $\bar{f}$.

For the case considered in this section, the HJB equation reads as

$$\begin{aligned}
&\mathcal{D}_t + \sup_{\pi \in U(t,T)} \Bigg[\Bigg\{ \sum_{i=1,j=1}^{d,m} \pi_t^i \sigma_t^{i,j} \theta_t^j + \sum_{i=1,j=m+1}^{d} \pi_t^i \rho_t^{i,j} (\theta_t^j - \lambda_t^j) \Bigg\} \\
&- \frac{1}{2} \sum_{j=1}^{m} \Bigg(\sum_{i=1}^{d} \pi_t^i \sigma_t^{i,j} \Bigg)^2 + \mathcal{D}_y \Big\{ \alpha_t - \zeta_{t-} \widetilde{\lambda}_t \Big\} + \frac{1}{2} \mathcal{D}_{yy} \xi_t^2 + \\
&\sum_{j=m+1}^{d} \Bigg\{ \log(1 + \sum_{i=1}^{d} \pi_t^i \rho_t^{i,j}) \Bigg\} \lambda_t^j + \Big\{ \mathcal{D}(t, y + \zeta_t) - \mathcal{D}(t, y) \Big\} \widetilde{\lambda}_t \Bigg] = 0.
\end{aligned} \tag{2.42}$$

Using similar arguments as in the HJB equation (2.37), it can be shown that the above HJB equation also has a unique solution. Using the first order condition in the above HJB equation, we will get the same expression for the GOP. But in this case market is incomplete due to unhedgeable random variables $\widetilde{W}$ and $\widetilde{N}$. There will be infinitely many risk-neutral martingale measures for this case. But, in the benchmark model, there is a unique GOP, and hence a unique fair price for the ECC. We can derive the fair price of the ECC in a similar way as in Section 2.2. In this case, the fair price is the same as the price given by a minimal-martingale measure in the risk-neutral approach.

Due to the unhedgeable randomness in the model considered in this section, every contingent claim cannot have fair, perfectly trading strategy. Hence, there will be some intrinsic risk which cannot be ignored but it can be reduced. In literature, two quadratic approaches have been introduced to reduce the risk: one is mean-variance hedging and the second is locally-risk-minimizing hedging (see, Schweizer (2001)). We use the locally-risk-minimizing hedging strategy introduced in Schweizer (1995). It has been shown that finding the local risk-minimizing strategy is equivalent to finding the Föllmer-Schweizer decomposition (Schweizer, 2001).

Definition 2.4.1. *A ECC* $(T,\ f_T,\ g)$ *is said to admit a* ***Föllmer-Schweizer decomposition*** *if it can be expressed as*

$$f_T + \int_0^T g_t dt = f_0 + \sum_{i=1}^{d} \int_0^T \xi_t^i dS_t^i + L_T, \tag{2.43}$$

where $f_0 \in \mathbb{R}$, $\xi^i \in \mathbb{L}^2(S)$, *for* $i \in \{1, 2, \cdots, d\}$, *are* F*-predictable processes, and* $L = \{L_t = E(L_T|\mathcal{F}_t)\}_{t\in[0,\,T]}$ *is a square-integrable martingale strongly orthogonal to the martingale parts of* S^i, *for* $i \in \{1, 2, \cdots, d\}$.

Since the market is incomplete, we have to consider cost process along with the trading strategy. To derive the Föllmer-Schweizer decomposition, we consider a wealth process X^π consisting of GOP in place of riskless asset, with consumption process $\mathbb{C}$ and cost process $\mathbf{C}$. The dynamics of X^π is given by the SDE

$$dX_t^\pi = X_{t-}^\pi \left\{ \sum_{i=1}^{d} \pi_t^i \frac{dS_t^i}{S_{t-}^i} + \pi_t^0 \frac{dV_t^{\pi^*}}{V_{t-}^{\pi^*}} \right\} - d\mathbb{C}_t + d\mathbf{C}_t, \tag{2.44}$$

where the cost process $\mathbf{C}$ is defined by

$$\mathbf{C}_t = X_t^\pi - \int_0^t X_{s-}^\pi \left\{ \sum_{i=1}^{d} \pi_t^i \frac{dS_s^i}{S_{s-}^i} + \pi_s^0 \frac{dV_s^{\pi^*}}{V_{s-}^{\pi^*}} \right\} + \mathbb{C}_t.$$

Using (2.2) and (2.15), (2.44) can be rewritten as

$$\begin{aligned} dX_t^\pi =& X_{t-}^\pi \Bigg\{ \sum_{j=1}^{d} \pi_t^j \left[\sum_{i=1}^{m} \sigma^{j,i}(dW_t^i + \theta_t^i dt) + \sum_{k=m+1}^{d} \rho_t^{j,k}(dM_t^k + \theta_t^k dt) \right] \\ &+ \pi_t^0 \Bigg[\sum_{j,i=1}^{d,m} \theta_t^i(\theta_t^i dt + dW_t^i) \\ &+ \sum_{j=1,k=m+1}^{d} \frac{\theta_t^k}{\lambda_t^k - \theta_t^k}(\theta_t^k dt + dM_t^k) \Bigg] \Bigg\} - d\mathbb{C}_t + d\mathbf{C}_t. \end{aligned} \tag{2.45}$$

$$\begin{aligned} \widehat{X}_t^\pi + \int_0^t \frac{d\mathbb{C}_s}{V_s^{\pi^*}} =& x + \int_0^t \Bigg[\sum_{j,i=1}^{d,m} \varrho_s^j \{\sigma_s^{j,i} - \theta_s^i\} dW_s^i \\ &+ \sum_{j=1,k=m+1}^{d} \varrho_s^j \left\{ \rho_s^{j,k} - \frac{\theta_s^j}{\lambda_s^k - \theta_s^k} \right\} \left\{ 1 - \frac{\theta_s^k}{\lambda_s^k} \right\} dM_s^k \Bigg] \\ &+ \int_0^t \frac{d\mathbf{C}_s}{V_s^{\pi^*}}. \end{aligned} \tag{2.46}$$

Under the filtration $\mathcal{G}$, using equation (2.26) and from the martingale representation theorem, there exist some square-integrable $\mathcal{G}$-predictable processes, ψ, η, ψ^j and η^k, for $j \in \{1, \cdots, m\}$ and $k \in \{m+1, \cdots, d\}$, such that

$$\begin{aligned} \frac{\zeta_t}{V_T^{\pi^*}} + \int_0^t \frac{g_s}{V_s^{\pi^*}} ds = E(Q_T) + \sum_{j=1}^{j=m} \int_0^t \psi_s^j dW_s^j + \sum_{j=m+1}^{j=d} \int_0^t \eta_s^k dM_s^k \\ + \int_0^t \psi_s d\widetilde{W}_s + \int_0^t \eta_s d\widetilde{M}_s. \end{aligned} \tag{2.47}$$

Comparing with (2.46), we get

$$\psi_t^i = \sum_{j=1}^{d} \varrho_t^j \{\sigma_t^{j,i} - \theta_t^i\} \qquad \text{for} \quad j \in \{1, \cdots, m\}$$

$$\eta_t^k = \sum_{j=1}^{d} \varrho_t^j \left\{ \rho_t^{j,k} - \frac{\theta_t^j}{\lambda_t^k - \theta_t^k} \right\} \left\{ 1 - \frac{\theta_t^k}{\lambda_t^k} \right\} \qquad \text{for} \quad k \in \{m+1, \cdots, d\} \tag{2.48}$$

$\mathbb{C}_t = \int_0^t g_s ds$ and $\mathbf{C}_t = E(Q_T) + \int_0^t V_s^{\pi^*} \psi_s d\widetilde{W}_s + \int_0^t V_s^{\pi^*} \eta_s d\widetilde{M}_s$.

$$\zeta_t + \int_0^t g_s ds = \int_0^t \left\{ \sum_{i=1}^{d} \varrho_s^i \frac{dS_s^i}{S_{s-}^i} + \varrho_s^0 \frac{dV_s^{\pi^*}}{V_{s-}^{\pi^*}} \right\} + \mathbf{C}_t,$$

more precisely,

$$\zeta_T + \int_0^T g_s ds = \int_0^T \left\{ \sum_{i=1}^d \varrho_s^i \frac{dS_s^i}{S_{s-}^i} + \varrho_s^0 \frac{dV_s^{\pi^*}}{V_{s-}^{\pi^*}} \right\} + \mathbf{C}_T.$$

For some predictable processes ν^i, for $i \in \{1, \cdots, d\}$, the above equation can be written as

$$\zeta_T + \int_0^T g_s ds = \int_0^T \sum_{i=1}^d \nu_s^i dS_s^i + \mathbf{C}_T$$

which is the required decomposition (2.43) of f_T with $f_T = \zeta_T$, $\xi_t^i = \nu_t^i$, and $L_T = \mathbf{C}_T$.

2.5 The benchmarked PDE for ECC

To get the option pricing one has to compute the expectation that has been derived. Direct analytical evaluation of the expectation is not possible for the models we have considered here. Numerical computation of the expectation by simulation will be expensive to implement. Transformation of the expectation into a PDE allows us to resort to the vast field of numerical analysis applied to parabolic PDEs. The Feynman-Kac theorem states that we can find the (time-dependent) expectation of a function of a Markovian stochastic process by solving a partial differential equation, subject to appropriate boundary conditions.

We show for the deterministic parameter model with payoff rate zero, like in the case of Black-Scholes, that the benchmarked pricing function of ECC

$$\widehat{V} : [0,\ T] \times (0,\ \infty) \times (0,\ \infty) \to [0,\ \infty)$$

obtained as $\widehat{V}(t,\ S_t,\ V_t^{\pi^*}) = \widehat{V}(t)$ can be expressed as a solution to a partial differential equation (PDE).

We determine whether there exists a sufficiently often differentiable benchmarked pricing function such that

$$\widehat{V}(t) = \widehat{V}(t,\ S_t,\ V_t^{\pi^*}) = E\left(\frac{f_T}{V_T^{\pi^*}} \,\middle|\, \mathcal{F}_t \right),$$

where the column vector $S_t = (S_t^1, \cdots, S_t^d)^\top$, for $t \in [0, T]$. Application of Itô's formula to the function

$$d\widehat{V}(t, S_t, V_t^{\pi^*}) = L\widehat{V}(t, S_t, V_t^{\pi^*})dt + \sum_{i=1}^{m}\left(\sum_{j=1}^{d}\widehat{V}_{S_j}(t)S_t^j\sigma_t^{j,i} + \widehat{V}_{V^{\pi^*}}(t)V_t^{\pi^*}\theta_t^i\right)dW_t^i +$$
$$\sum_{k=m+1}^{d}\left(\widehat{V}\left(t, S_{t-} + \rho_t^k, V_{t-}^{\pi^*} + \frac{\theta_t^k}{\lambda_t^k - \theta_t^k}\right) - \widehat{V}(t, S_{t-}, V_{t-}^{\pi^*})\right)dM_t^k, \tag{2.49}$$

where $\rho_k = (\rho_{1,k}, \cdots, \rho_{d,k})$, for $k \in [m+1, \cdots, d]$, and with operator

$$L\widehat{V}(t, S_t, V_t^{\pi^*}) = \frac{\partial \widehat{V}(t, S_t, V_t^{\pi^*})}{\partial t} + \sum_{j=1}^{d} S_t^j(\mu_t^j - \sum_{k=m+1}^{d}\rho^{j,k}\lambda_t^k)\frac{\partial \widehat{V}(t, S_t, V_t^{\pi^*})}{\partial S_j} +$$
$$\frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{m}\sigma_t^{i,k}\sigma_t^{j,k}S_t^iS_t^j\frac{\partial^2\widehat{V}(t, S_t, V_t^{\pi^*})}{\partial S_t^i\partial S_t^j} + (\sum_{i=1}^{m}(\theta_t^i)^2)V_t^{\pi^*}\frac{\partial \widehat{V}(t, S_t, V_t^{\pi^*})}{\partial V_t^{\pi^*}} +$$
$$\frac{1}{2}\sum_{i=1}^{m}(\theta_t^i)^2V_{\pi^*}^2\frac{\partial^2\widehat{V}(t, S_t, V_t^{\pi^*})}{\partial V_t^{\pi^*}\partial V_t^{\pi^*}} + \sum_{i,k=1}^{d,m}S_t^iV^{\pi^*}\sigma_t^{i,k}\theta_t^k\frac{\partial^2\widehat{V}(t, S_t, V_t^{\pi^*})}{\partial S_t^i\partial V_t^{\pi^*}} +$$
$$\sum_{k=m+1}^{d}\left(\widehat{V}(t, S_{t-} + \rho_t^k, V_{t-}^{\pi^*} + \frac{\theta_t^k}{\lambda_t^k - \theta_t^k}) - \widehat{V}(t, S_{t-}, V_{t-}^{\pi^*})\right)\lambda_t^k, \tag{2.50}$$

for $t \in [0, T)$. Since the process $\widehat{V} = \{\widehat{V}(t, S_t, V_T^{\pi^*})\}_{t\in[0, T]}$, is an (F, P)-martingale,

$$L\widehat{V}(t, S_t, V_t^{\pi^*}) = 0, \tag{2.51}$$

for $(t, S_t, V_t^{\pi^*}) \in (0, T) \times (0, \infty) \times (0, \infty)$, with benchmarked terminal condition

$$\widehat{V}(T, S_T, V_T^{\pi^*}) = \frac{f_T}{V_T^{\pi^*}},$$

for $(S_T, V_T^{\pi^*}) \in (0, \infty) \times (0, \infty)$.

Hence, by Feynman-Kac theorem, applying Itô's rule to the process $\widehat{V}(t, S_t, V_t^{\pi^*})$ and using the above equation, we arrive at

$$d\widehat{V}(t, S_t, V_t^{\pi^*}) = \sum_{i=1}^{m}(\sum_{j=1}^{d}V_{S_t^j}S_t^j\sigma^{j,i} + V_{V^{\pi^*}}(t)V_t^{\pi^*}\theta_t^i)dW_t^i +$$
$$\sum_{k=m+1}^{d}\left(\widehat{V}(t, S_{t-} + \rho_t^k, V_{t-}^{\pi^*} + \frac{\theta_t^k}{\lambda_t^k - \theta_t^k}) - \widehat{V}(t, S_{t-}, V_{t-}^{\pi^*})\right)dM_t^k. \tag{2.52}$$

A comparison with (2.17) will give the required portfolio to hedge the given ECC.

For lower dimensions, numerical techniques can be implemented to solve the PDE with the above initial conditions, like finite difference techniques can be implemented to solve the PDE with initial condition (Ghosh and Goswami, 2009). In the similar way, we can derive the PDE for option pricing in the remaining chapters also, but we will not be concentrating on deriving the PDEs.

Chapter 3

A Markov-Modulated Brownian Market Model

In the previous chapter, we have considered three different forms of jump-diffusion models with stochastic parameters. But, perhaps the simplest way to introduce additional randomness into the standard geometric Brownian model is to let the volatility and rate of return be functions of a finite state Markov chain. We can imagine that such a model might describe regime switching behavior of some kind, perhaps related to the business cycle, or other economic indicators. The terms regime switching and Markov-modulated dynamics are used to describe such models, and there are already many interesting contributions here, such as applications to option pricing (Guo and Zhang, 2004; Buffington and Elliott, 2002; Guo, 1999), portfolio optimization (Yin and Zhou, 2003), and optimal trading strategies (Zhang, 2001). Di Masi et.al. (1994) considered the problem of hedging an European call option for a diffusion model where the drift and the volatility are functions of a Markov jump process. To derive a hedging strategy, they followed the approach based on the idea of hedging under a mean-variance criterion as suggested by Föllmer and Sondermann (1986) and Schweizer (1988, 1991) in an incomplete financial market. Deshpande and Ghosh (2008) considered option pricing in a regime switching market with a finite state continuous-time Markov chain. They have computed the minimal-martingale measure for the regime switching model and expressed the risk-minimizing strategy under the minimal-martingale measure.

The model considered in this chapter is incomplete due to the non-tradeable Markov process X. We will follow the same strategy as in Deshpande and Ghosh (2008) for the risk-minimizing strategy, but under the real world probability measure with numeraire change. In the present model, we prove that the fair pricing value will be the same as the locally-risk-minimizing price derived in Di Masi et al. (1994).

3.1 The market model

Our objective in this chapter is to study the pricing and hedging of options in a regime switching market with the benchmark approach. We assume that the state of the market is governed by a finite state Markov process $X = \{X_t\}_{t\geq 0}$ taking values in $\mathcal{S} = \{1, 2, \cdots, N\}$. We assume that a single discounted stock price process $S = \{S_t\}_{t\geq 0}$ evolves according to the SDE, with $S_0 > 0$,

$$dS_t = S_t(\mu(X_t)dt + \sigma(X_t)dW_t), \tag{3.1}$$

where the drift term $\mu(\cdot)$ and the diffusion term $\sigma(\cdot)(> 0)$ are real valued, uniformly bounded, square-integrable, deterministic functions on $\mathcal{S}$. and $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion. Under the given probability measure, the Brownian motion W and the continuous-time Markov chain X are assumed to be independent. The filtration $F = \{\mathcal{F}_t\}_{t\in[0,\,T]}$ is the augmentation under P of the natural filtration $F^{W,X}$, generated by the two sources of uncertainties W and X completed with null sets. So, we have the complete information about the market at any time $t \in [0,\ T]$.

We consider $X = \{X_t\}_{t\geq 0}$ be an irreducible continuous-time Markov chain describing an observable exogenous quantity (the state of the market) whose evolution is described in the sequel. Let, for $A \subseteq \mathcal{S}$, $n(t, A)$ counts the number of transitions of X to a state in A during the time interval $(0,\ t]$, including the transitions taking place within A. For example, the quantity $n(t, u)$ counts the number of occurrences of $s \in (0,\ t]$ such that $X_s = u$ and $X_{s-} = i \neq u$, for $i, u \in \mathcal{S}$. We also call n Markov jump process. Note that

n is a random counting measure. Let ν be the usual counting measure on $\mathcal{S}$. Then ν has the following two properties: for $A \subseteq \mathcal{S}$, $\nu(A) = \int I_A \nu(du)$ (i.e. $\nu(A)$ counts the number of elements in A) and $\int_A f(u)\nu(du) = \sum_{u\in A} f(u)$.

A marked point process or a random measure is uniquely characterized by its stochastic intensity kernel. The stochastic intensity kernel of $n(t, \cdot)$ can be defined as

$$\gamma_n(dt,\ du) = h(u; X_t)\nu(du)dt, \tag{3.2}$$

where $h(u; X_t)$ is the conditional regime-shift (from regime X_t to u) intensity at time t (we assume that $h(u; X_t)$ is bounded). Heuristically, $\gamma_n(dt,\ du)$ can be thought of as the conditional probability of shifting from regime X_t to u during $(t, t + dt)$ given X_t. Then $\gamma_n(t, A)$, the compensator of $n(t, A)$, can be written as (Wu and Zeng, 2006)

$$\gamma_n(t, A) = \int_0^t \int_A h(u; X_s)\nu(du)ds = \sum_{u\in A} \int_0^t h(u; X_s)ds.$$

The process $m = \{m(t, \mathcal{S})\}_{t\in[0,\, T]}$ where $\{m(t, A) = n(t, A) - \gamma_n(t, A)\}_{t\geq 0}$, for each fixed $A \subseteq \mathcal{S}$, is a martingale under P. We are now in a position to present the integral and differential forms for the evolution of regime X_t at time t, using the random measure n defined above. First, the integral form is

$$X_t = X_0 + \int_{[0,\, t]\times\mathcal{S}} (u - X_{s-})n(ds, du). \tag{3.3}$$

Note that $n(ds, du)$ is zero most of the times and only assumes the value one at regime switching times t_i, with $u = X_{t_i}$, the new regime at time t_i. Observe that the above expression is nothing but a telescoping sum: $X_t = X_0 + \sum_{t_i\leq t}(X_{t_i} - X_{t_{i-1}})$. The differential form then is

$$dX_t = \int_{\mathcal{S}} (u - X_{t-})n(dt, du). \tag{3.4}$$

In the following subsections, we define a trading strategy with cost process and derive the GOP.

3.1.1 Growth optimal portfolio

Suppose that a cumulative cost process is defined by

$$\mathbf{C}_t = V_t^\pi - \int_0^t \delta_s^1 dS_s.$$

Then the positive wealth process is characterized by

$$dV_t^\pi = \delta_t^1 dS_t + d\mathbf{C}_t. \tag{3.5}$$

Note that, for a self-financing portfolio, by definition, the cost process will be constant and changes in the value of a discounted self-financing portfolio are exactly matched by the corresponding gains from trade. Here, $\pi = (\pi^0, \pi^1)$ indicates the fraction of wealth invested in the riskless and risky assets.

Now we define the GOP according to the market scenario and use the corresponding wealth process as a numeraire for pricing contingent claims.

Definition 3.1.1. *A positive self-financing trading strategy π^* is called a **GOP** if it has wealth process V^{π^*} such that*

$$E_{t,x,v} \log V_T^{\pi^*} \geq E_{t,x,v} \log V_T^\pi \quad a.s.$$

for all positive self-financing wealth process V^π corresponding to a admissible portfolio π and for all $t \in [0,\ T]$, where $E_{t,x,v}[\cdot] = E[\cdot|X_t = x, V_t^\pi = v]$ indicates the conditional expectation under P given complete information till time t.

We now have an optimization problem of finding an investment strategy that maximizes the expected logarithmic utility from terminal wealth. In this regime switching model, we solve this problem via stochastic control method. The SDE for a positive, wealth process corresponding to a self-financing portfolio is

$$\begin{aligned} dV_t^\pi &= V_t^\pi \pi_t^1 \frac{dS_t}{S_t} \\ &= V_t^\pi \pi_t^1 \sigma(X_t)(dW_t + \theta(X_t)dt), \end{aligned} \tag{3.6}$$

where, $V_0^\pi = v_0 > 0$ and $\theta(X_t) = \frac{\mu(X_t)}{\sigma(X_t)}$ is the market price of diffusion risk, which is also regime dependent. This linear stochastic differential equation can be solved explicitly and the solution is given by

$$V_t^\pi = v \exp\left\{\int_0^t \left[\pi_s^1\sigma(X_s)\theta(X_s) - \frac{1}{2}\sigma^2(X_s)(\pi_s^1)^2\right]ds + \int_0^t \sigma(X_s)\pi_s^1 dW_s\right\}. \tag{3.7}$$

If we take the expectation of the logarithmic value of the above equation, it gives

$$E_{0,x,v_0}\log(V_T^\pi) = \log(v_0) + E_{0,x}\left(\int_0^T \left[\pi_s^1\sigma(X_s)\theta(X_s) - \frac{1}{2}\sigma^2(X_s)(\pi_s^1)^2\right]ds\right). \tag{3.8}$$

Finally, we have a stochastic control problem of maximizing the objective function $\mathcal{J}(0, x, v_0, \pi) = E_{0,x,v_0}\log(V_T^\pi)$ which is a concave, increasing, logarithmic utility function with π^1 as a controller. That is, our aim is to solve

$$\sup_{\pi\in U(T)} \mathcal{J}(0, x, v_0, \pi),$$

where the supremum is taken over all admissible trading strategies.

3.1.2 Dynamic programming

In our optimization problem, the time evolution of V^π is actively influenced by another stochastic process $\pi^1 = \{\pi_t^1\}_{t\in[0,\,T]}$ called the control process. Here, we consider the finite time horizon $s \in [t,\ T]$. Since the control is allowed to observe the states V_t^π of the process being controlled, we may as well assume that the initial state $X_t = x$ is known ($x \in \mathcal{S}$) and $V_t^\pi = v$ at time t. The starting point for dynamic programming is to regard the supremum of the quantity $\mathcal{J}$ being maximized as a function $\varsigma(t, x, v)$ of the initial data. The optimization problem can then be written as

$$\varsigma(t, x, v) := \sup_{\pi\in U(t,T)} \mathcal{J}(t, x, v, \pi).$$

Here, $\mathcal{J}(t, x, v, \pi) = E_{t,x,v}\log(V_T^\pi)$.

Now, from the Bellman's principle of dynamic programming, for $t+l \in [t,\ T]$,

$$\varsigma(t,x,v) = \sup_{\pi \in U(t,t+l)} E_{t,x,v}\varsigma(t+l, X_{t+l}, V^{\pi}_{t+l}). \tag{3.9}$$

Speaking intuitively, the expression inside the expectation represents the maximum expected cost obtained by proceeding on $[t+l,\ T]$ with $(t+l, X_{t+l}, V^{\pi}_{t+l})$ as initial data.

The dynamic programming equation is obtained formally from (3.9) by the following derivations. If we take any control $\pi_s \in U(t, t+l)$, then

$$\varsigma(t,x,v) \geq E_{t,x,v}\varsigma(t+l, X_{t+l}, V^{\pi}_{t+l}).$$

We subtract $\varsigma(t,x,v)$ from both sides, divide by l and let $l \to 0$. The right-hand-side then becomes

$$\begin{aligned} &\lim_{l\to 0+}\frac{1}{l}[E_{t,x,v}\varsigma(t+l, X_{t+l}, V^{\pi}_{t+l}) - \varsigma(t,x,v)] \\ &= \lim_{l\to 0+}\frac{1}{l}E_{t,x,v}\int_t^{t+l}\Big(\varsigma_s + V^{\pi}_s[\pi^1_s\sigma(X_s)\theta(X_s)]\varsigma_v + \frac{1}{2}(V^{\pi}_s)^2(\pi^1_s)^2\sigma^2(X_s)\varsigma_{vv}\Big)ds \\ &\qquad + \sum_{y\in\mathcal{S}} h(y;x)[\varsigma(t,y,v) - \varsigma(t,x,v)]. \end{aligned} \tag{3.10}$$

Hence, for all $\pi \in U(T)$,

$$\begin{aligned} 0 \ \geq\ & \varsigma_t + v[\pi^1_t\sigma(x)\theta(x)]\varsigma_v + \frac{1}{2}v^2(\pi^1_t)^2\sigma^2(x)\varsigma_{vv} + \sum_{y\in\mathcal{S}} h(y;x)[\varsigma(t,y,v) - \varsigma(t,x,v)] \\ & = A^{\pi}\varsigma(t,x,v), \end{aligned} \tag{3.11}$$

where

$$A^{\pi\varsigma(t,x,v)} = \varsigma_t + v[\pi^1_t\sigma(x)\theta(x)]\varsigma_v + \frac{1}{2}v^2(\pi^1_t)^2\sigma^2(x)\varsigma_{vv} + \sum_{y\in\mathcal{S}} h(y;x)[\varsigma(t,y,v) - \varsigma(t,x,v)]$$

is a partial differential operator. If π^* is an optimal Markov control policy, we should have

$$\varsigma(t,x,v) = E_{t,x,v}\varsigma(t+l, X_{t+l}, V^{\pi^*}_{t+l}),$$

where V^{π^*} is the Markov diffusion process generated by A^{π^*}. A similar argument gives

$$0 = A^{\pi^*}\varsigma(t,x,v). \tag{3.12}$$

The inequality (3.11) together with (3.12) is equivalent to the dynamic programming equation

$$0 = \max_{\pi \in U(t,T)} A^{\pi} \varsigma(t, x, v). \tag{3.13}$$

The above equation is to be considered on $[0,\ T] \times \mathcal{S} \times [0,\ \infty)$ with the terminal condition $\varsigma(T, x, v) = \log v$. Thus, we have

$$\sup_{\pi^1 \in U(t,T)} \Big\{ \varsigma_t + v[\pi_t^1 \sigma(x)\theta(x)]\varsigma_v + \frac{1}{2} v^2 (\pi_t^1)^2 \sigma^2(x) \varsigma_{vv} + \sum_{y \in \mathcal{S}} h(y; x)[\varsigma(t, y, v) - \varsigma(t, x, v)] \Big\} = 0 \tag{3.14}$$

which is HJB partial differential equation subject to the terminal condition

$$\varsigma(T, x, v) = \log v. \tag{3.15}$$

From Bäuerle and Rieder (2004) we state the following two results, where the first one states that if G is a classical solution of the HJB equation, then $G(t, x, v)$ equals the maximum conditionally expected logarithmic growth of any portfolio belonging to $U(t, T)$. In the second theorem, the GOP strategy and the corresponding wealth process are derived. Using these results in the following section, we introduce the benchmark model.

Theorem 3.1.1 (Verification Theorem). *Suppose that $G \in \mathcal{C}^{1,2}$ is a solution of the HJB equation with $G(t, x, v)| \le K(1 + |v|^k)$ for some constant $K > 0$, $k \in \mathbb{N}$, $\forall\ x \in \mathcal{S}$ and $t \in [0,\ T]$. Then the following holds:*

a) $G(t, x, v) \ge \mathcal{J}(t, x, v, \pi)$, for all $0 \le t \le T$, $v > 0$, $x \in \mathcal{S}$ and $\pi \in U(t, T)$.

b) If π^* is a maximizer of the HJB, i.e., π_s^* maximizes

$$\pi \longrightarrow A^{\pi} G(s, x, V_s^{\pi})$$

for all $s \in [t,\ T]$, where V^{π^*}, π^* and X solve (3.6), then $G(t, x, v) = \mathcal{J}(t, x, v, \pi^*)$, for all $v > 0$, $x \in \mathcal{S}$. In particular, π^* is the growth optimal portfolio strategy.

Theorem 3.1.2. *The GOP strategy is given by*

$$\pi_t^* = \frac{\theta(X_t)}{\sigma(X_t)},$$

a fraction to be invested in risky asset, and the optimal value at time $t \in [0,\ T]$ *is given by*

$$\varsigma(t, x, v) = \log(v) + \mathcal{D}(t, x),$$

where x, v *are the values of Markov process and the wealth process at given time* t *and* $\mathcal{D}(t, x)$ *is the unique solution of the following linear differential equation*

$$\mathcal{D}_t(t, x) + \frac{1}{2}\theta^2(x) + \sum_{y\in\mathcal{S}} h(x, y)[\mathcal{D}(t, y) - \mathcal{D}(t, x)] = 0 \tag{3.16}$$

with boundary condition $\mathcal{D}(T, x) = 0$ *for* $x \in \mathcal{S}$.

From equation (3.8), $\mathcal{D}(t, x)$ can be written as

$$\mathcal{D}(t, x) = \sup_{\pi\in U[t,T]} E_{t,x}\Big(\int_t^T \Big[\pi_s^1\sigma(X_s)\theta(X_s) - \frac{1}{2}\sigma^2(X_s)(\pi_s^1)^2\Big]ds\Big). \tag{3.17}$$

From the above expression it is clear that $\mathcal{D} \in \mathcal{C}^1[0, T]$ with respect to time t which implies that the linear differential equation (3.16) has a unique classical solution. The solution is given explicitly by

$$\mathcal{D}(t, x) = E_{t,x}\Big(\int_t^T \frac{1}{2}\theta^2(X_s)ds\Big). \tag{3.18}$$

Finally, we conclude this section by writing the discounted wealth process corresponding to the GOP which we get by substituting π^* from Theorem 3.1.2 in (3.6)

$$dV_t^{\pi^*} = V_t^{\pi^*}\theta(X_t)(dW_t + \theta(X_t)dt). \tag{3.19}$$

3.2 The benchmark model

Using the GOP derived in the previous section, we introduce the benchmark model and derive the fair price. We take the V^{π^*}, the wealth process corresponding to the GOP, as the benchmark or the reference unit. The benchmarked value of the wealth process corresponding to the portfolio π, is given by $\widehat{V}_t^{\pi} = \frac{V_t^{\pi}}{V_t^{\pi^*}}$, for $t \in [0,\ T]$. By Itô's formula, the benchmarked value process satisfies the SDE

$$d\widehat{V}_t^{\pi} = \widehat{V}_t^{\pi}(\pi_t^1 \sigma(X_t) - \theta(X_t))dW_t. \tag{3.20}$$

Note that $\widehat{V}^{\pi} = \{\widehat{V}_t^{\pi}\}_{t\in[0,\ T]}$ is driftless and thus an (F, P)-local martingale. The SDE for the primary security account $\widehat{S}$ can be obtained by making the substitutions $\pi_t^1 = 1$ for $t \in [0,\ T]$.

3.2.1 Fair pricing

First, let us recall the definitions of fair price.

Definition 3.2.1. *We call a price process $\Lambda = \{\Lambda_t\}_{t\in[0,\ T]}$* ***fair*** *if the corresponding benchmarked process*

$$\widehat{\Lambda} = \left\{\widehat{\Lambda}_t = \frac{\Lambda_t}{V_t^{\pi^*}}\right\}_{t\in[0,\ T]}.$$

forms an (F, P)-martingale, i.e., it satisfies the conditions

$$E(|\widehat{\Lambda}_t|) < \infty \text{ and } \widehat{\Lambda}_t = E[\widehat{\Lambda}_T|\mathcal{F}_t].$$

As stated in Chapter 1, the fair pricing idea generalizes the standard risk-neutral pricing for the Markov-modulated model. In practice, the fair pricing is appropriate for determining the competitive price of a contingent claim (see, Platen and Heath, 2006).

Proposition 3.2.1. *The fair price Λ_t^H at time t for a given non-negative contingent claim H_T is given by the fair pricing formula*

$$\Lambda_t^H = V_t^{\pi^*} E\left(\frac{H_T}{V_T^{\pi^*}}\middle|\mathcal{F}_t\right), \tag{3.21}$$

for $t \in [0,\ T]$.

The proof of the above proposition is straightforward from the fair price definition. If the minimal-martingale measure exists then the fair price defined above coincides with the corresponding risk-neutral price. The Radon-Nikodým derivative process $\mathcal{Z}^Q = \{\mathcal{Z}_t^Q\}_{t\in[0,\,T]}$ for the candidate minimal-martingale measure Q can be obtained as inverse of the discounted GOP

$$\mathcal{Z}_t^Q = E\Big[\frac{dQ}{dP}\Big|\mathcal{F}_t\Big] = \frac{\widehat{B}_t}{\widehat{B}_0} = \frac{1}{V_t^{\pi^*}},$$

for $t \in [0,\ T]$ (Di Masi et al., 1994, Deshpande and Ghosh, 2008). From the above relation we can see that $\mathcal{Z}_t^Q$ will satisfy the SDE

$$d\mathcal{Z}_t^Q = -\mathcal{Z}_t^Q \theta(X_t) dW_t, \tag{3.22}$$

for $t \in [0,\ T]$ with $\mathcal{Z}_0^Q = 1$, by the Itô's formula and (3.19). Furthermore, by (3.20), it follows that $\frac{V_t^\pi}{B_t}\mathcal{Z}_t^Q = \frac{V_t^\pi}{V_t^{\pi^*}} = \widehat{V}_t^\pi$ and $\widehat{V}^\pi = \{\widehat{V}_t^\pi\}_{t\in[0,\,T]}$ is an F-local martingale for any portfolio π.

Corollary 3.2.1. *If the minimal-martingale measure Q exists and $\widehat{V}^\pi$ is an (F, P)-martingale, the corresponding risk-neutral pricing formula*

$$V_t^\pi = V_t^{\pi^*} E(\widehat{V}_s^\pi|\mathcal{F}_t) = E\left(\frac{\mathcal{Z}_t^Q}{\mathcal{Z}_s^Q}\frac{B_t}{B_s}V_s^\pi\Big|\mathcal{F}_t\right) = E_Q\left(\frac{B_t}{B_s}V_s^\pi\Big|\mathcal{F}_t\right) \tag{3.23}$$

holds, for all $t \in [0,\ T]$ and $s \in [t,\ T]$. Here, E_Q denotes expectation under the minimal-martingale measure Q.

The above corollary implies that pricing through standard minimal-martingale measure is a particular case of fair pricing whenever the minimal-martingale measure exists. In general, in a benchmark model, the candidate minimal-martingale measure Q may not be equivalent to the real world probability measure P. Also the Radon-Nikodým $\mathcal{Z}^Q$ may not be an (F, P)-martingale. In Platen and Heath (2006), for the case of complete markets, an example of three-dimensional Bessel process is given, where the Radon-Nikodým derivative process is not a martingale. Since a benchmark model does not require the existence of an equivalent risk-neutral martingale measure it provides a more general modelling framework than the standard risk-neutral setup.

3.2.2 Hedging

It is of great practical importance to identify a wealth process that permits the hedging of a given non-negative contingent claim. But in the model, considered in this chapter, additional uncertainty arising due to the regime switching leads to incompleteness of the market. Due to which the writer of the option cannot hedge himself perfectly. In other words, every contingent-claim in such market will have an intrinsic risk. We obtain the optimal mean-self-financing strategy and the residual risk.

Definition 3.2.2. *For a given non-negative contingent claim with payoff H_T at time T, a strategy π^H is called **replicating** if the corresponding wealth process $V_T^{\pi^H}$ is such that*

$$V_T^{\pi^H} = H_T.$$

The strategy is sometimes called as a H_T-admissible trading strategy. From definitions 3.2.1 and 3.2.2, and the martingale property of a benchmarked, fair, replicating wealth process, we directly obtain the following result.

Proposition 3.2.2. *For a given non-negative contingent claim H_T, V^{π^H} is a fair replicating wealth process if*

$$V_t^{\pi^H} = \Lambda_t^H$$

at time $t \in [0,\ T]$, with Λ_t^H satisfying the fair pricing formula (3.21).

The question now is how to reduce the intrinsic risk $\mathbf{R}$ defined in (1.25), which arises due to the incompleteness of the market, from among the H_T-admissible portfolios.

3.2.3 Locally-risk-minimizing hedging strategy

Here, P is itself a martingale measure for the benchmark model. In this section, we follow the approach by Föllmer and Sondermann (1986).

Let a non-negative contingent claim with payoff H_T time T be such that

$$\widehat{H}_T \in L^2(\Omega, \mathcal{F}, P).$$

We consider a strategy δ which yields the terminal payoff H_T with a cost process $\mathbf{C}$. Then, the value of the portfolio is

$$V_t^\pi = \delta_t^0 B_t + \delta_t^1 S_t$$

which satisfies the SDE

$$dV_t^\pi = V_t^\pi \pi_t^1(\sigma(X_t)\theta(X_t)dt + \sigma(X_t)dW_t)] + d\mathbf{C}_t.$$

Then, the SDE for the benchmarked wealth is

$$d\widehat{V}_t^\pi = \widehat{V}_t^\pi(\pi_t^1\sigma(X_t) - \theta(X_t))dW_t + \frac{d\mathbf{C}_t}{V_t^{\pi^*}}. \tag{3.24}$$

Theorem 3.2.1. *The optimal strategy to hedge the non-negative contingent claim H_T in the incomplete market under consideration is given by*

$$dV_t^\pi = V_t^\pi[\pi_t^1(\sigma(X_t)\theta(X_t)dt + \sigma(X_t)dW_t)] + d\mathbf{C}_t,$$

where $\pi_t^1 = \dfrac{1}{\sigma(X_t)}\Big(\dfrac{\xi_t}{\widehat{V}_t^\pi} + \theta(X_t)\Big)$ *and* $\mathbf{C}_t = \displaystyle\int_0^t \int_{\mathcal{S}} V_s^{\pi^*}\eta_s m(ds, du)$, *with*

$$\xi_t = \frac{d[\widehat{\Lambda}^H, W]_t}{d[W]_t}, \quad \eta_t = \frac{d[\widehat{\Lambda}^H, m]_t}{d[m]_t},$$

with $[\cdot]$ indicating the quadratic covariation between two processes.

Proof. Under the benchmark model, the fair price process is a martingale with respect to (F, P). Therefore, by the help of the martingale representation theorem (see Elliott et. al. 2007b), we can write the representation

$$\widehat{H}_T = \widehat{\Lambda}_T^H = \widehat{\Lambda}_0^H + \int_0^T \xi_s dW_s + \int_0^T \int_{\mathcal{S}} \eta_s m(ds, du), \tag{3.25}$$

where

$$E\int_0^T(\xi_s^2+\eta_s^2)ds<\infty,$$

for a unique pair of predictable stochastic processes $\{\xi_s, \eta_s\}$. The quantities ξ and η can now be obtained from the Radon-Nikodým derivative

$$\xi_t=\frac{d[\widehat{\Lambda}^H,W]_t}{d[W]_t},\quad \eta_t=\frac{d[\widehat{\Lambda}^H,m]_t}{d[m]_t}.$$

Now, by comparing (3.24) and (3.25), we can find π^1 and the required optimal cost process as

$$\pi_t^1=\frac{1}{\sigma(X_t)}\Big(\frac{\xi_t}{\widehat{V}_t^\pi}+\theta(X_t)\Big)$$

and

$$\mathbf{C}_t^\pi=H_0+\int_0^t\int_U V_s^{\pi^*}\eta_s m(ds,du).$$

This completes the proof. □

We have derived the locally-risk-minimizing hedging strategy, using the GOP as a reference portfolio, for a Markov-modulated Brownian market model .

Chapter 4

A Markov-Modulated Jump-Diffusion Market Model

In Platen (2004a), pricing and hedging are performed for complete markets with jump-diffusions and without the measure transformation. Elliott et al. (2007) considered the pricing of options when the dynamics of the risky underlying asset is driven by a Markov-modulated jump-diffusion model. They supposed that the market interest rate, the drift and the volatility of the underlying risky asset switch over time according to the state of an economy, which is modelled by a continuous-time Markov chain. They employed the generalized regime-switching Esscher transform to determine an equivalent-martingale-measure in the incomplete market setting.

The insufficient information in financial modelling is not uncommon in practice, some of the factors that characterize the evolution of the financial markets may be hidden. Platen and Runggaldier (2004) have generalized the Platen (2004a) model by using the GOP for pricing and hedging in incomplete markets when there are unobserved factors to be filtered. Föllmer and Schweizer (1991) discussed the case where the market incompleteness is due to incomplete information. They assumed that the claim H_T admits an Itô representation as a stochastic integral with respect to some large filtration. But in constructing a dynamic strategy, they use a smaller filtration. They show that the unique optimal strategy can be constructed by projecting the Itô integrand down to the smaller filtration. In this chapter, we have followed the same methodology to derive the hedging strategy for the incomplete information case under the benchmark approach.

In this chapter, we consider a Markov-modulated jump-diffusion model that can not only incorporate both the leptokurtic feature and volatility smile but also present the economic features of volatility clustering and empirical study (see, Elliott et al. 2005). First, we derive a GOP for the case of complete information by using the Hamilton-Jacobi-Bellman equation. After deriving the optimal strategy with complete information, we take up the optimal filtration of the strategy with incomplete information where the hidden process is not possible to retrieve from the given set of information.

4.1 The market model

We consider a mathematical model of the financial market, which has two stock price processes S^1 and S^2, satisfying the following SDEs

$$\begin{aligned} dS_t^1 &= S_{t-}^1(\mu_1(X_t)dt + \sigma_1(X_t)dW_t + \rho_1(X_t)dN_t), \\ dS_t^2 &= S_{t-}^2(\mu_2(X_t)dt + \sigma_2(X_t)dW_t + \rho_2(X_t)dN_t), \end{aligned} \tag{4.1}$$

where $X = \{X_t\}_{t\geq 0}$ is an irreducible continuous-time Markov chain and follows the same representation as in Chapter 3 and $S_0^i > 0, i = 1, 2$. And, $\mu_i(\cdot)$, $\sigma_i(\cdot) \geq 0$, $\rho_i(\cdot) > -1$, for $i = 1, 2$, are uniformly bounded, square-integrable, deterministic functions of X. In the above stochastic differential equations, $W = \{W_t\}_{t\geq 0}$ indicates a standard Brownian motion and $N = \{N_t\}_{t\geq 0}$ indicates a Poisson process with deterministic intensity $\lambda = \{\lambda_t\}_{t\geq 0}$. We indicate by $M_t = N_t - \int_0^t \lambda_s ds$ the compensated Poisson process. All the processes are adapted to the filtration F, and W, N and X are independent with each other. The investor knows the initial distribution of X_0. We consider two filtered probability spaces. The first one is $(\Omega, \mathcal{F}, F, P)$ which has complete information about all the randomness in the market at any time $t \in [0,\ T]$. In practice, such a filtration is possible to generate by an insider. The second one is $(\Omega, \mathcal{F}^S, F^S, P)$ which is the filtered probability space generated by the stock price processes $S = \{S^1, S^2\}$ alone. This filtration can be observed by any investor at any time $t \in [0,\ T]$. Intuitively, it is clear that $\mathcal{F}_t^S \subseteq \mathcal{F}_t$ for all $t \in [0,\ T]$.

The matrix $\sigma_t \sigma_t^\top$ is assumed to be a 2×2 nonsingular matrix and the 2×2 matrix $[\sigma_t, \rho_t]$ is assumed to satisfy the following inequality

$$[\sigma_t, \rho_t][\sigma_t, \rho_t]^\top \geq cI, \quad \forall t \in [0, T]$$

for some $c > 0$, where I is a d-dimensional unit matrix. It is clear that matrix $[\sigma_t, \rho_t]$ is nonsingular for every $t \in [0, T]$

We denote a portfolio strategy by $\pi_t = \{\pi_t^0, \pi_t^1, \pi_t^2\}$, where π_t^i the fraction of the wealth invested in risk-free $(i = 0)$ and in risky $(i = 1, 2)$ assets, respectively, at time $t \in [0,\ T]$.

The wealth process corresponding to an admissible trading strategy $\pi \in U(T)$ having positive value is given by

$$\begin{aligned} dV_t^\pi = \quad & V_{t-}^\pi((\pi_t^1 \mu_1(X_t) + \pi_t^2 \mu_2(X_t))dt + (\pi_t^1 \sigma_1(X_t) + \pi_t^2 \sigma_2(X_t))dW_t + \\ & (\pi_t^1 \rho_1(X_t) + \pi_t^2 \rho_2(X_t))dN_t), \end{aligned} \tag{4.2}$$

with initial wealth $V_0^\pi = v_0 > 0$. It can be written as

$$\begin{aligned} dV_t^\pi = \quad & V_{t-}^\pi((\pi_t^1 \sigma_1(X_t) + \pi_t^2 \sigma_2(X_t))(dW_t + \theta_1(X_t)dt) + \\ & (\pi_t^1 \rho_1(X_t) + \pi_t^2 \rho_2(X_t))(dN_t + \theta_2(X_t)dt)), \end{aligned} \tag{4.3}$$

where $\theta_1(x)$ and $\theta_2(x)$ are the market price of risks characterized by the equations

$$\sigma_1(x)\theta_1(x) + \rho_1(x)\theta_2(x) = \mu_1(x)$$

$$\sigma_2(x)\theta_1(x) + \rho_2(x)\theta_2(x) = \mu_2(x).$$

From the above assumptions, the system of linear equations has a unique solution. Equation (4.3) has the solution

$$V_t^\pi = \quad V_0^\pi \exp\left\{\int_0^t \left\{(\pi_s^1\sigma_1(X_s)+\pi_s^2\sigma_2(X_s))(dW_s+\theta_1(X_s)ds)-\frac{1}{2}(\pi_s^1\sigma_1(X_s)+\pi_s^2\sigma_2(X_s))^2ds\right.\right.$$
$$\left.\left.+(\pi_s^1\rho_1(X_s)+\pi_s^2\rho_2(X_s))\theta_2(X_s)ds+\log(\pi_s^1\rho_1(X_s)+\pi_s^2\rho_2(X_s)+1)dN_s\right\}\right\}. \qquad (4.4)$$

4.2 Growth optimal portfolio

In this section, we derive the GOP under the filtration F. So the optimization problem is now to maximize the expected log utility from the terminal wealth at any time $t \in [0,\ T]$ given complete information till time t

$$\varsigma(t,v,x) = \sup_{\pi\in U[t,T]} E_{t,v,x}[\log(V_T^\pi)] = \sup_{\pi\in U[t,T]} \mathcal{J}(t,x,v,\pi). \qquad (4.5)$$

We solve the above problem by the stochastic control method. Using the HJB equation, the above optimization problem can be written as

$$\sup_{\pi\in U[t,T]}\{\varsigma_t + v((\pi_t^1\mu_1(x)+\pi_t^2\mu_2(x)))\varsigma_v + \tfrac{1}{2}v^2(\pi_t^1\sigma_1(x)+\pi_t^2\sigma_2(x))^2\varsigma_{vv})$$
$$+(\varsigma(t,x,v(1+(\pi_t^1\rho_1(x)+\pi_t^2\rho_2(x)))) - \varsigma(t,x,v))\lambda_t + \textstyle\sum_{y\in\mathcal{S}} h(y,x)[\varsigma(t,y,v)$$
$$-\varsigma(t,x,v)]\} = 0 \qquad (4.6)$$

with the boundary condition $\varsigma(T,v,x) = \log v$.

The verification theorem, i.e., if $G(t,x,v)$ is a classical solution of the HJB equation then $G(t,x,v)$ equals the maximum conditionally expected logarithmic growth of any portfolio belonging to $U(t,T)$, follows from Theorem1 in Bäuerle and Rieder (2004).

We can write from (4.4)

$$\varsigma(t,v,x) = \log(v) + \mathcal{D}(t,x), \qquad (4.7)$$

for some function $\mathcal{D}$ of t and x. Now, the HJB equation (4.6) can be written as

$$\sup_{\pi\in U[t,T]}\{\mathcal{D}_t + ((\pi_t^1\mu_1(x)+\pi_t^2\mu_2(x))) - \tfrac{1}{2}(\pi_t^1\sigma_1(x)+\pi_t^2\sigma_2(x))^2$$
$$+\log(1+(\pi_t^1\rho_1(x)+\pi_t^2\rho_2(x)))\lambda_t + \textstyle\sum_{y\in\mathcal{S}} h(y,x)[\mathcal{D}(t,y)-\mathcal{D}(t,x)]\} = 0. \qquad (4.8)$$

Using this HJB equation in the following theorem, we derive the GOP and the corresponding optimal wealth process.

Theorem 4.2.1. *The GOP strategy is given by*

$$\pi_t^* = \begin{bmatrix} \sigma_1(X_t) & \sigma_2(X_t) \\ \rho_1(X_t) & \rho_2(X_t) \end{bmatrix}^{-1} \Big[\theta_1(X_t), \frac{\lambda_t - \theta_2(X_t)}{\theta_2(X_t)}\Big]^\top.$$

and the optimal value is given by $\varsigma(t,v,x) = \log(v) + \mathcal{D}(t,x)$, *where* $\mathcal{D}(t,x)$ *is the unique solution of the following linear differential equation*

$$\begin{aligned} &\mathcal{D}_t + \tfrac{1}{2}(\theta_1(x))^2 + (\lambda_t - \theta_2(x)) + \log(\tfrac{\lambda_t}{\theta_2(x)})\lambda_t \\ &+ \textstyle\sum_{y\in\mathcal{S}} h(y,x)[\mathcal{D}(t,y) - \mathcal{D}(t,x)] = 0, \end{aligned} \tag{4.9}$$

with boundary condition $\mathcal{D}(T,x) = 0$ *for* $x \in \mathcal{S}$.

Proof. Applying the first order condition to the left hand side of the (4.8), we will get the maximizer π^* of the HJB equation as

$$\pi_t^* = \begin{bmatrix} \sigma_1(X_t) & \sigma_2(X_t) \\ \rho_1(X_t) & \rho_2(X_t) \end{bmatrix}^{-1} \Big[\theta_1(X_t), \frac{\lambda_t - \theta_2(X_t)}{\theta_2(X_t)}\Big]^\top.$$

Insertion of the value of π^* in the HJB equation (4.8) leaves us with

$$\mathcal{D}_t + \frac{1}{2}(\theta_1(x))^2 + (\lambda_t - \theta_2(x)) + \log(\frac{\lambda_t}{\theta_2(x)})\lambda_t + \sum_{y\in\mathcal{S}} h(y,x)[\mathcal{D}(t,y) - \mathcal{D}(t,x)] = 0, \tag{4.10}$$

and the boundary condition $\mathcal{D}(T,x) = 0$ for $x \in \mathcal{S}$. From equation (4.2), $\mathcal{D}$ can be written as

$$\begin{aligned} \mathcal{D}(t,x) = \sup_{\pi \in U_{[t,T]}} E_{t,x}\Big(\int_t^T (\pi_s^1\mu_1(X_s) + \pi_s^2\mu_2(X_s))ds - \frac{1}{2}(\pi_t^1\sigma_1(X_t) + \pi_t^2\sigma_2(X_t))^2 ds \\ + \log(\pi_t^1\rho_1(X_t) + \pi_t^2\rho_2(X_t) + 1)\lambda_s ds\Big) \end{aligned} \tag{4.11}$$

From the above expression, it is clear that $\mathcal{D} \in \mathcal{C}^1(0,T)$ with respect to time t which implies that the first-order linear differential equation (4.10) has a unique classical solution. □

The wealth process corresponding to the GOP, which we get by substituting π^* from Theorem4.2.1 in (4.3), is given by

$$dV_t^{\pi^*} = V_{t-}^{\pi^*}(\theta_1(X_t)(dW_t + \theta_1(X_t)dt) + \frac{\lambda_t - \theta_2(X_t)}{\theta_2(X_t)}(\theta_2(X_t) + dN_t)), \tag{4.12}$$

where we assume that

$$\frac{\lambda_t - \theta_2(X_t)}{\theta_2(X_t)} > 0,$$

a.s. for all $t \in [0, T]$.

The benchmarked value of the wealth process at time t corresponding to the portfolio π is $\widehat{V}_t^{\pi} = \frac{V_t^{\pi}}{V_t^{\pi^*}}$ for $t \in [0,\ T]$. By Itô's formula, the benchmarked value process satisfies the SDE

$$d\widehat{V}_t^{\pi} = \widehat{V}_{t-}^{\pi}\Big((\pi_t^1\sigma_1(X_t) + \pi_t^2\sigma_2(X_t) - \theta_1(X_t))dW_t + \\ \Big[\tfrac{(\pi_t^1\rho_1(X_t)+\pi_t^2\rho_2(X_t)+1)\theta_2(X_t)}{\lambda_t} - 1\Big]dM_t\Big).$$

Note that $\widehat{V}^{\pi}$ is driftless and thus an $(F,\ P)$-locally martingale. Since a non-negative local martingale is a supermartingale, we deduce that the above process is a supermartingale. The proof of the next proposition is straightforward from the definition of fair prices.

Proposition 4.2.1. *The fair price Λ_t^H at time t for a given European contingent claim H_T under the complete information case is given by the fair pricing formula*

$$\Lambda_t^H = V_t^{\pi^*} E\left(\frac{H_T}{V_T^{\pi^*}}\Big|\mathcal{F}_t\right) \tag{4.13}$$

for $t \in [0,\ T]$.

If a minimal-martingale measure exists, then the fair price defined above coincides with the corresponding risk-neutral price in incomplete markets. The Radon-Nikodým derivative process $\mathcal{Z}^Q = \{\mathcal{Z}_t^Q\}_{t\in[0,\ T]}$ for the candidate minimal-martingale measure Q can be obtained as the inverse of the discounted GOP, as derived in the Chapter 3.

4.3 Hedging

Due the unhedgeable risk generated by the Markov process there does not exist a fair, perfectly replicating trading strategy for all non-negative contingent claims in this market. To replicate H_T in the incomplete market with complete information, we have to have a cost process $\mathbf{C} = \{\mathbf{C}_t\}_{t\in[0,\, T]}$ such that

$$V_T^{\pi^H} = H_T,$$

where $V_t^{\pi^H}$ for $t \in [0,\, T]$ satisfies the SDE

$$\begin{aligned} dV_t^{\pi^H} \;=\; & V_{t-}^{\pi^H}((\pi_t^1\sigma_1(X_t) + \pi_t^2\sigma_2(X_t))(dW_t + \theta_1(X_t)dt) + \\ & (\pi_t^1\rho_1(X_t) + \pi_t^2\rho_2(X_t))(dN_t + \theta_2(X_t)dt)) + d\mathbf{C}_t. \end{aligned} \tag{4.14}$$

Minimizing the intrinsic risk arising due to the incompleteness of the market is equivalent to finding a mean-self-financing strategy, where the cost process $\mathbf{C}$ is strongly orthogonal to the martingale part of S, which is called a locally-risk-minimizing strategy.

Lemma 4.3.1. *The optimal strategy to hedge the contingent claim H_T in the incomplete market with complete information is given by*

$$\begin{aligned} dV_t^{\pi^H} \;=\; & V_{t-}^{\pi^H}((\pi_t^1\sigma_1(X_t) + \pi_t^2\sigma_2(X_t))(dW_t + \theta_1(X_t)dt) + \\ & (\pi_t^1\rho_1(X_t) + \pi_t^2\rho_2(X_t))(dN_t + \theta_2(X_t)dt)) + d\mathbf{C}_t, \end{aligned}$$

where π is characterized by

$$\pi_t^1\sigma_1(X_t) + \pi_t^2\sigma_2(X_t) = \xi_t^1 + \theta_1(X_t),$$
$$\pi_t^1\rho_1(X_t) + \pi_t^2\rho_2(X_t) = (\xi_t^2 + 1)\frac{\lambda_t}{\theta_2(X_t)} - 1,$$

and $\mathbf{C}_t = \int_0^t \int_{\mathbb{S}} V_s^{\pi^*}\eta_s m(ds, du)$, with

$$\xi_t^1 = \frac{d[\widehat{\Lambda}^H, W]_t}{d[W]_t}, \quad \xi_t^2 = \frac{d[\widehat{\Lambda}^H, N]_t}{d[N]_t},$$

and

$$\eta_t = \frac{d[\widehat{\Lambda}^H, m]_t}{d[m]_t}.$$

Here m is the compensated Markov jump process, as considered in Chapter 3.

Using the martingale representation theorem in Bardhan and Chao (1995) we have the representation

$$E[H_T|\mathcal{F}_t] = E[H_T|\mathcal{F}_0] + \int_0^t \xi_s^1 dW_s + \int_0^t \xi_s^2 dM_s + \int_0^t \int_{\mathbb{S}} \eta_s dm(ds, du),$$

for some square integrable, F-predictable processes ξ^1, ξ^2 and η. The proof of Lemma 4.3.1 is simple consequence of the above martingale representation of the fair price and comparing with equation (4.14). From the Lemma (4.3.1), H_T can be written as

$$H_T = H_0 + \int_0^T \delta_t^1 dS_t^1 + \int_0^T \delta_t^2 dS_t^2 + L_T, \tag{4.15}$$

for some F-predictable processes $\delta \in \mathbb{L}^2(\mathbf{S})$ and $L = \{L_t = E(L_T|\mathcal{F}_t)\}_{t\in[0,\,T]}$ is a square-integrable martingale strongly orthogonal to the martingale part of S which is the Föllmer-Schweizer decomposition of H_T.

We have derived the risk-minimizing hedging strategy in the case of complete information. Now let us consider the incomplete information case for an investor who can see only the stock price process. Now we have the filtration F^S. But for a financial model where the stock price process satisfies the equation (4.1), we can reduce the partial information case to the complete one with the following standing assumption.

Assumption 4.3.1. *At least one of the functions $\rho_1(\cdot)$ or $\rho_2(\cdot)$ and $\sigma_1(\cdot)$ or $\sigma_2(\cdot)$ are continuous and invertible and the inverse are denoted by $\varpi(\cdot)$ and $\iota(\cdot)$, respectively.*

Without lost of generality, we assume that $\rho_1(\cdot)$ and $\sigma_1(\cdot)$ are continuous and invertible.

Lemma 4.3.2. *Under assumption (4.3.1), the filtration F^S is augmented with filtration of (X, W, N).*

Proof. Let $F^{X,W,N}$ be the augmented filtration of (X, W, N). Obviously, $F^S \subseteq F^{X,W,N}$. From (4.1), we have

$$\Delta S_t^1 = \rho_1(X_t) I_{\{\Delta S_t^1 \neq 0\}}$$

where $I_{\{\Delta S_t^1 \neq 0\}}$ is a random variable with value 1 if $\Delta S_t^1 \neq 0$, and 0 otherwise for all $t \in [0,\ T]$. Here, $\Delta E_t = E_t - E_{t-}$. It is clear that process $\{\Delta S_t^1\}$ and $\{I_{\{\Delta S_t^1 \neq 0\}}\}$ are F^S-adapted so is $\rho_1(X_t)$. Moreover, from the assumption (4.3.1), for the jump time τ of N, the process X satisfies

$$X_\tau = \varpi(\rho_1(X_\tau)), \qquad \tau \in [0,\ T],$$

which implies that it is also $\{\mathcal{F}_\tau^S\}$-adapted. Now for the case when there are no jumps, the quadratic variation of the stock price process

$$\langle S^1 \rangle_t = \int_0^t (S_s^1)^2 \sigma_1^2(X_s) ds$$

can be observed and so is σ. From the assumption (4.3.1), X can be observed when there is no jump. Hence, X is F^S-adapted. Now we can write

$$N_t = \sum_{s \leq t} \rho_1^{-1}(X_s) \Delta S_s^1.$$

So, N and, from (4.1), W are F^S-adapted as well, which proves the relation $F^{X,W,N} \subseteq F^S$. Therefore, we get $F^{X,W,N} = F^S$. □

But if ρ_1 and ρ_2 or σ_1 and σ_2 are not invertible then it is not possible to retrieve the complete information of the market. The strategy we have derived in (4.15) is not F^S-adapted. But to get a strategy under the incomplete information it must be F^S-adapted. Therefore, for this case, we consider optimal projection towards F^S by

$$\bar{\delta}_t = E(\delta_t \mid \mathcal{F}_t^S).$$

Theorem 4.3.1. *H_T admits the Föllmer-Schweizer decomposition under the incomplete information*

$$H_T = H_0 + \int_0^T \overline{\delta}_s^1 dS_1(s) + \int_0^T \overline{\delta}_s^2 dS_2(s) + \overline{L}_T, \tag{4.16}$$

where $\overline{\delta}_t^i = E(\delta_t^i \mid \mathcal{F}_t^S) \in \mathbb{L}^2(S_i)$ are F^S-adapted, for $i \in \{1, 2\}$, and $\overline{L} = \{\overline{L}_t\}_{t \in [0,\, T]}$ is the square-integrable F^S-martingale strongly orthogonal to the martingale part of S given by

$$\overline{L}_t = E(L_t \mid \mathcal{F}_t^S). \tag{4.17}$$

Proof. From (4.15), we can write

$$dH_t = \delta_t^1 dS_t^1 + \delta_t^2 dS_t^2 + dL_t$$

where $L_t = E(L_T|\mathcal{F}_t)$. If we take conditional expectation with respect to the filtration F^S, we get

$$\begin{aligned} E(dH_t|F_t^S) &= E(\delta_t^1 dS_t^1|F_t^S) + E(\delta_t^2 dS_t^2|\mathcal{F}_t^S) + E(dL_t|\mathcal{F}_t^S) \\ dH_t &= E(\delta_t^1|\mathcal{F}_t^S)dS_t^1 + E(\delta_t^2|\mathcal{F}_t^S)dS_t^2 + E(dL_t|\mathcal{F}_t^S) \end{aligned}$$

since H and S are F^S-adapted. We can write this in integral form as

$$H_T = H_0 + \int_0^T \overline{\delta}_s^1 dS_1(s) + \int_0^T \overline{\delta}_s^2 dS_2(s) + \overline{L}_T,$$

where $\overline{\delta}_t^i = E(\delta_t^i \mid \mathcal{F}_t^S)$ are clearly F^S-adapted for $i = 1, 2$. Now it remains to prove that $\overline{L}_t = E(L_t \mid \mathcal{F}_t^S)$ is F^S-martingale and strongly orthogonal to the martingale part of S with respect to F^S. For $t \in [s,\, T]$

$$\begin{aligned} E(\overline{L}_t|\mathcal{F}_s^S) &= E(E(L_t \mid \mathcal{F}_t^S)|\mathcal{F}_s^S) \\ &= E(L_t \mid \mathcal{F}_s^S) \\ &= E(E(L_t \mid \mathcal{F}_s) \mid \mathcal{F}_s^S) \qquad \text{(from tower property)} \\ &= E(L_s \mid \mathcal{F}_s^S) \\ &= \overline{L}_s \end{aligned}$$

which proves that $\overline{L}_t = E(L_t \mid \mathcal{F}_t^S)$ is an F^S-martingale. Since $\overline{L}$ is still independent of W and N, it's covariance with the martingale part of S is zero, which shows that it is strongly orthogonal to the martingale part of S. The square-integrability of $\overline{\delta}$ and $\overline{L}$ follows from the square-integrability of δ and L. □

The above theorem gives the optimal strategy for an investor who can only observe the stock price process as a piece of information to minimize the intrinsic risk arising due to the untradeable random process X. But without knowing the dynamics of the estimated unknown value it is very difficult to get the option price. In the next chapter, filtering theory has been used to derived the non-linear filtering equation, for an incomplete information Markov-modulated model.

Chapter 5

An Unobservable Markov-Modulated Model

In practice, not all quantities which determine the dynamics of security prices can be fully observed. Some of the factors that characterize the evolution of the market are hidden. For instance, processes that drive the market like the Markov process in previous chapters cannot be observed. However, these unobserved factors and their correct calibration are essential to reflect correctly the market dynamics in a financial market model, that one empirically observes. This leads naturally to a stochastic filtering problem. Given the available information, corresponding filter methods determine the distribution, called filter distribution, of the unobserved factors. This distribution allows then to compute the expectation of quantities that are dependent on unobserved factors, including derivative prices, optimal portfolio strategies and risk measures.

Rieder and Bäuerle (2005) have studied portfolio optimization problems, where the drift rate of the stock price process is Markov-modulated and the driving factors cannot be observed by the investor. Rieder and Bäuerle (2007) have considered a jump-diffusion stock price process where the jump intensity rate is considered to be a Markov-modulated process and studied the portfolio optimization problems with unobservable driving factors. In contrast, we have taken a model where both the drift process and the jump intensity rate process as Markov-modulated. We do the portfolio optimization for the logarithmic utility, and use the corresponding portfolio for pricing and hedging of European contingent claims.

In order to solve the pricing and hedging problems in a market with incomplete information, the technique is to use the well-established filter theory to reduce the market model with partial observation to one with complete observation. In filtering, one has to deal with the real world probability measure to extract from the observations via filter estimates for the hidden factors, which suggests us to work under the real world probability measure. In this chapter, we assume that the investor is only able to observe the stock price process and have to base his decision only on this observation. In particular, the investor is neither informed about the state of the drift process and the intensity rate of the jump process nor he can retrieve the hidden information from the given set of information. So, the problem is to find the filter process of X (i.e. the dynamics of the conditional expectation of X for given information) on the observed path of stock price process S. We shall call this situation the case of partial information.

5.1 The market model

We consider a market model consisting of a discounted risky asset which are traded continuously in the market for the given time period $[0,\ T]$. The discounted stock price process follows the SDE

$$dS_t = S_{t-}(\mu(X_t)dt + \sigma_t dW_t + \rho_t dN_t), \tag{5.1}$$

with $S_0 = s_0 > 0$. $\mu(i)$ is the growth rate of the stock in regime $i \in \mathcal{S}$. $\sigma_t > 0$ and ρ_t are the volatility and jump amplitude of the stock respectively at time $t \in [0,\ T]$ which are uniformly bounded, deterministic functions of time, with bounded derivative. We assume that $\rho_t > -1$ and ρ_t^{-1} exist for all $t \in [0,\ T]$. Here, $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion and $N = \{N_t\}_{t\geq 0}$ is a non-homogeneous Poisson process with intensity rate $\lambda = \{\lambda(X_t)\}_{t\geq 0}$ driven by the Markov chain X. The drift rate of S is also driven by the Markov chain X.

We consider the representation of Markov process as given in Bain and Crisan (2009), with the generating matrix $Q = \{q_{i,j}(t), i, j \in \mathcal{S}, t \geq 0\}$, where $\mathcal{S}$ indicates the state space of the Markov process. The martingale representation of the Markov process $X = \{X_t\}_{t\in[0,\ T]}$ is

$$X_t = X_0 + \int_0^t Q(s, X_s)ds + M_t^X, \qquad t \geq 0 \tag{5.2}$$

where $M^X = \{M_t^X\}_{t\geq 0}$ is an F-adapted right-continuous martingale. Here, we assume that the filtration $F = F^{W,X,N}$ is generated by the Brownian motion, the Markov process and the jump diffusion process, that is, it has complete information about the market. In the representation (5.2), $Q : [0,\ T] \times \mathcal{S} \to \mathbb{R}$ is defined in a natural way as

$$Q(s, i) = \sum_{j\in\mathcal{S}} q_{i,j}(s)j,$$

for $(s, i) \in [0,\ T] \times \mathcal{S}$. This representation of the Markov process is similar to the representation considered in Chapter 3. We assume that M^X and the compensated Poisson process $M = N - \lambda$ are orthogonal and W is independent of N and X.

Here, we work with the filtration $F^S = \{\mathcal{F}_t^S\}_{t\in[0,\ T]}$ which is generated by the stock price process S. That is, we consider that our investor is only able to observe the stock price process and knows the initial distribution of X_0. In particular, the investor is not informed about the current state of the Markov chain. Here the investor has insufficient information about the market, which we call an incomplete information case.

First, we derive the GOP for the incomplete information case by introducing a non-tradable stock price process and then we use the corresponding wealth process as numeraire for pricing and hedging European contingent claims. We now give the definition of growth optimal portfolio (GOP) for the given market model with incomplete information.

Definition 5.1.1. *A portfolio* $\pi^* \in U[t,T]$ *is called GOP if it has a positive wealth process* V^{π^*} *such that*

$$E_{t,v} \log V_T^{\pi^*} \geq E_{t,v} \log V_T^{\pi} \qquad a.s.$$

for all portfolio $\pi \in U[t,T]$ *with positive wealth process* V_s^π *for all* $s \in [t,T]$*, where* $E_{t,v}(\cdot) = E(\cdot \big| V_t^\pi = v)$ *indicates the conditional expectation under* P*, for given information* $\mathcal{F}_t^S$*.*

In the following subsection, we reduce the incomplete information model to a complete information case.

5.1.1 The reduction to complete information case

Let

$$dY_t = \mu(X_t)dt + \sigma_t dW_t + \rho_t dN_t, \tag{5.3}$$

that is, Y is the stochastic logarithm of S. Then, it is clear that $F^Y = F^S$. By stochastic filtering theory we estimate the unobservable stochastic process, i.e., the Markov process, based on the observable information, i.e., the stock price process. The filtering problem consists in determining the conditional distribution Π_t of the unobservable X at time t given the information accumulated from observing Y in the interval $[0,\ T]$, that is, for $\phi \in \mathbb{B}(\mathcal{S})$, where $\mathbb{B}$ the space of bounded Borel functions from $\mathcal{S}$ to $\mathbb{R}$, deriving the dynamics of

$$\Pi_t(\phi) = E[\phi(X_t) \mid \mathcal{F}_t^Y]. \tag{5.4}$$

We have to choose a suitable regularization of the process $\Pi(\phi) = \{\Pi_t(\phi)\}_{t\geq 0}$. By Theorem2.24 of Bain and Crisan (2009), we can do this such that Π_t is an optional, $\mathcal{F}_t^Y$-adapted process. We now deduce the evolution equation for Π. First, we prove lemmas required to evaluate the dynamics of Π_t for $t \in [0,\ T]$.

Lemma 5.1.1. *Let, for all* $t \geq 0$*,* $(X_t,\ S_t)$ *be an* $\mathcal{F}_t$*-adapted solution of (5.1) and (5.2). Then* $\widetilde{W}_t = \int_0^t (dW_s + (\frac{\mu(X_s) - \Pi_s(\mu)}{\sigma_s}))ds$ *is a Brownian motion and* $\widetilde{N}_t = N_t - \int_0^t \Pi_s(\lambda)ds$ *is a compensated Poisson process with intensity* $\Pi_t(\lambda)$ *on the probability space* $(\Omega,\ \mathcal{F}^Y,\ F^Y,\ P)$*.*

Proof. The equation (5.3) can be written as

$$dY_t = \widetilde{\mu}_t dt + \sigma_t d\widetilde{W}_t + \rho_t d\widetilde{N}_t, \tag{5.5}$$

where $\widetilde{\mu}_t = \Pi_t(\mu) + \Pi_t(\lambda)\rho_t$.

So, N can be written as

$$\Delta N_t = \rho_t^{-1}\Delta Y_t.$$

From this representation, it is clear that N is F^Y-adapted and so is $\widetilde{N}$ as $\Pi_t(\lambda)$ is F^Y-adapted. From (5.5), $\widetilde{W}$ is F^Y-adapted as both N and $\Pi_t(\mu)$ are. First, we show that $\widetilde{W}$ is a continuous F^Y-martingale. As a consequence of assumptions made, $\widetilde{W}_t$ is integrable. Hence taking conditional expectation for the given information up to time $s \leq t$,

$$\begin{aligned} E(\widetilde{W}_t \mid \mathcal{F}_s^Y) &= E(\widetilde{W}_t - \widetilde{W}_s + \widetilde{W}_s \mid \mathcal{F}_s^Y) \qquad\qquad (\text{as } \widetilde{W} \text{ is } F^Y\text{-adapted}) \\ &= E\left(W_t - W_s + \int_s^t \Big(\frac{\mu(X_u) - \Pi_u(\mu)}{\sigma_u}\Big) du \,\Big|\, \mathcal{F}_s^Y\right) + \widetilde{W}_s \\ &= \widetilde{W}_s. \end{aligned} \tag{5.6}$$

Here from the definition of Brownian motion and tower property,

$$E(W_t - W_s \mid \mathcal{F}_s^Y) = E(W_t - W_s) = 0.$$

From tower property, we can write

$$E\Big(\int_s^t \Big(\frac{\mu(X_u) - \Pi_u(\mu)}{\sigma_u}\Big) du \mid \mathcal{F}_s^Y\Big) = E\Big(\int_s^t E\Big(\Big(\frac{\mu(X_u) - \Pi_u(\mu)}{\sigma_u}\Big) \mid \mathcal{F}_u^Y\Big) du \mid \mathcal{F}_s^Y\Big) = 0.$$

So, $\widetilde{W}$ is a continuous F^Y-martingale and its quadratic variation is given by

$$[\widetilde{W}]_t = [W]_t = t.$$

Hence $\widetilde{W}$ is a F^Y-Brownian motion, by Lévy's characterization of a Brownian motion. □

We call $\widetilde{W} = \{\widetilde{W}_t\}_{t\in[0,\,T]}$ and $\widetilde{N} = \{\widetilde{N}_t\}_{t\in[0,\,T]}$ the innovation processes for W and N, respectively (see, Bain and Crisan, 2009). The equation (5.1) can be rewritten on the probability space $(\Omega, \mathcal{F}^Y, F^Y, P)$ as

$$dS_t = S_{t-}(\widetilde{\mu}_t dt + \sigma_t d\widetilde{W}_t + \rho_t d\widetilde{N}_t), \tag{5.7}$$

where $\widetilde{\mu}_t = \Pi_t(\mu) + \Pi_t(\lambda)\rho_t$.

Denoting by $F^{\widetilde{W},\widetilde{N}} = \{\mathcal{F}_t^{\widetilde{W},\widetilde{N}}\}_{t\in[0,\,T]}$ the filtration generated by the innovation processes, we have $\mathcal{F}_t^{\widetilde{W},\widetilde{N}} = \sigma(\widetilde{W}_s, \widetilde{N}_s, s \in [0,\ t])$. Since $\widetilde{W}$ and $\widetilde{N}$ are F^Y-adapted from Lemma 5.1.1, we have $\mathcal{F}_t^{\widetilde{W},\widetilde{N}} \subseteq \mathcal{F}_t^Y$. But the converse relationship is not true in general. Next we reproduce the result from Fujisaki at al. (1972) useful to prove the main results of the work.

Lemma 5.1.2. *If $\zeta = \{\zeta_t\}_{t\in[0,T]}$ is a right continuous with left limit square-integrable F^Y-martingale, then it admits a representation of the form*

$$d\zeta_t = f(t,w)d\widetilde{W}_t + g(t,w)d\widetilde{N}_t,$$

where $\{f(t,w)\}$ and $\{g(t,w)\}$ are $\mathfrak{F}^Y$-predictable processes, such that

$$E\int_0^t (f(s,w))^2 ds < \infty, \text{ and } E\int_0^t (g(s,w))^2 ds < \infty$$

for $t \in [0,T]$ and $w \in \Omega$.

Proof. We have $dY_t = (\widetilde{\mu}_t dt + \sigma_t d\widetilde{W}_t + \rho_t d\widetilde{N}_t)$, where $\widetilde{W}$ is an F^Y-Brownian motion and $\widetilde{N}$ is a compensated F^Y-Poisson process under the probability measure P. Now we consider a measure transformation under which $\overline{W}_t = \int_0^t \frac{\widetilde{\mu}_s}{\sigma_s} ds + \widetilde{W}_t$ will be an F^Y-Brownian motion. Let $\overline{P}$ be the required measure, then

$$\left.\frac{d\overline{P}}{dP}\right|_{\mathcal{F}_t^Y} = Z_t, \quad t \in [0,\ T],$$

where $Z_t = \exp\{-\int_0^t \frac{\widetilde{\mu}_s}{\sigma_s} d\widetilde{W}_s - \frac{1}{2}\int_0^t |\frac{\widetilde{\mu}_s}{\sigma_s}|^2 ds\}$ a continuous process. So, $dY_t = \sigma_t d\overline{W}_t + \rho_t d\widetilde{N}_t$ is F^Y-martingale under the measure $\overline{P}$. From previous arguments, it can be shown that $F^Y = F^{\overline{W},\widetilde{N}}$.

Let us show that $\overline{\zeta} = \{\overline{\zeta}_t = \frac{\zeta_t}{Z_t}\}_{t\in[0,\, T]}$ will be F^Y-locally square integrable martingale under $\overline{P}$. Indeed, for all $A \in \mathcal{F}^Y_s,\; s \leq t$

$$\int_A \overline{\zeta}_t d\overline{P} = \int_\Omega I_A \zeta_t E\Big(\frac{Z_T}{Z_t}|\mathcal{F}^Y_t\Big)dP = \int_A \zeta_s dP = \int_A \overline{\zeta}_s d\overline{P} \tag{5.8}$$

This shows that $\overline{\zeta}$ is a F^Y-martingale under $\overline{P}$. Let

$$\sigma_N = inf\{t \in [0,\; T] : |Z_t^{-1}| + |\zeta_t| > N\}.$$

Then $\sigma_N \uparrow \infty$, as $N \uparrow \infty$, since $(Z)^{-1}$ is a continuous process, and ζ is a right continuous with left limit process. Now since ζ is a square integrable $(F^Y, P)-$martingale, we have

$$\sup_{t\leq T} \overline{E}|\overline{\zeta}_{t\wedge\sigma_N}|^2 \leq N \sup_{t\leq T} \overline{E}[|\zeta_{t\wedge\sigma_N}|^2 (Z_{t\wedge\sigma_N})^{-1}] = N \sup_{t\leq T} E|\zeta_{t\wedge\sigma_N}|^2 \leq N \sup_{t\leq T} E|\zeta_t|^2 < \infty. \tag{5.9}$$

This shows that $\overline{\zeta}$ is locally $\overline{P}$-square integrable. Therefore $\overline{\zeta}$ is a locally square integrable $(F^Y, \overline{P})$-martingale. Now from martingale representation theorem (see Bardhan and Chao, 1995), there exist processes $\{\overline{f}_t\}_{t\in[0,\, T]}$ and $\{\overline{g}_t\}_{t\in[0,\, T]}$ which are F^Y-predictable and $\overline{E}\int_0^t (\overline{f}(s,w))^2 ds < \infty$, and $\overline{E}\int_0^t (\overline{g}(s,w))^2 ds < \infty$, such that

$$d\overline{\zeta}_t = \overline{f}(t,w)d\overline{W}_t + \overline{g}(t,w)d\widetilde{N}_t.$$

Notice that $\zeta = \overline{\zeta}Z$. Applying Itô's formula, one verifies for all $t \in [0,\; T]$

$$\begin{aligned} d\zeta_t &= Z_t\overline{f}_t\frac{\widetilde{\mu}_t}{\sigma_t}dt + Z_t\overline{f}_t d\widetilde{W}_t + Z_t\overline{g}_t d\widetilde{N}_t - \zeta_t\frac{\widetilde{\mu}_t}{\sigma_t}d\widetilde{W}_t - Z_t\overline{f}_t\frac{\widetilde{\mu}_t}{\sigma_t}dt \\ d\zeta_t &= f_t d\widetilde{W}_t + g_t d\widetilde{N}_t, \end{aligned}$$

where $f_t = Z_t\overline{f}_t - \zeta_t\frac{\widetilde{\mu}_t}{\sigma_t}$ and $g_t = Z_t\overline{g}_t$. Moreover, it is evident that $\{f_t\}_{t\in[0,\, T]}$ and $\{g_t\}_{t\in[0,\, T]}$ are F^Y-predictable. And from Fujisaki at al. (1972) it can be shown that f and g are (F^Y, P)-square integrable. □

Lemma 5.1.3. *1.* $\Pi_t(X) - \Pi_0(X) - \int_0^t \Pi_s(Q(s,X_s))ds$ *is a square-integrable* F^Y*-martingale.*

2. *There exist square-integrable F^Y-predictable processes $\{f(t,w)\}$ and $\{g(t,w)\}$, such that*

$$\Pi_t(X) = \Pi_0(X) + \int_0^t \Pi_s(Q(s,X_s))ds + \widetilde{U}_t, \tag{5.10}$$

where $\widetilde{U}_t = \int_0^t f(s,w)d\widetilde{W}_s + \int_0^t g(s,w)d\widetilde{N}_s$.

Here $\Pi_t(Q(t,X_t)) = E[Q(t,X_t) \mid \mathfrak{F}_t^Y]$ for all $t \in [0,\ T]$.

Proof. We have

$$X_t = X_0 + \int_0^t Q(s,X_s)ds + M_t^X.$$

If we take the conditional expectation with respect to the observable filtration, we have

$$\Pi_t(X) = E(X_0 \mid \mathfrak{F}_t^Y) + E\Big(\int_0^t Q(s,X_s)ds \mid \mathfrak{F}_t^Y\Big) + E(M_t^X \mid \mathfrak{F}_t^Y).$$

Now, if we subtract $\Pi_0(X) + \int_0^t \Pi_s(Q(s,X_s))ds$ on both sides, we get

$$\begin{aligned}\Pi_t(X) - \Pi_0(X) - \int_0^t \Pi_s(Q(s,X_s))ds =& E(X_0 \mid \mathfrak{F}_t^Y) - \Pi_0(X) + E\Big(\int_0^t Q(s,X_s)ds \mid \mathfrak{F}_t^Y\Big) \\ & - \int_0^t \Pi_s(Q(s,X_s))ds + E(M_t^X \mid \mathfrak{F}_t^Y).\end{aligned} \tag{5.11}$$

First we show that $E(\int_0^t Q(s,X_s)ds \mid \mathfrak{F}_t^Y) - \int_0^t \Pi_s(Q(s,X_s))ds$ and $E(M_t^X \mid \mathfrak{F}_t^Y)$ are F^Y-martingales. For $u \le t$,

$$\begin{aligned}&E\Big(E\Big(\int_0^t Q(s,X_s)ds \mid \mathfrak{F}_t^Y\Big) - \int_0^t \Pi_s(Q(s,X_s))ds \;\Big|\; \mathfrak{F}_u^Y\Big) \\ =&E\Big(E\Big(\int_u^t Q(s,X_s)ds \mid \mathfrak{F}_t^Y\Big) - \int_u^t \Pi_s(Q(s,X_s))ds \;\Big|\; \mathfrak{F}_u^Y\Big) + \\ &E\Big(E\Big(\int_0^u Q(s,X_s)ds \mid \mathfrak{F}_t^Y\Big) - \int_0^u \Pi_s(Q(s,X_s))ds \;\Big|\; \mathfrak{F}_u^Y\Big) \\ =&E\Big(\int_u^t Q(s,X_s)ds \mid \mathfrak{F}_u^Y\Big) - E\Big(\int_u^t \Pi_s(Q(s,X_s))ds \;\Big|\; \mathfrak{F}_u^Y\Big) + \\ &E\Big(\int_0^u Q(s,X_s)ds \mid \mathfrak{F}_u^Y\Big) - E\Big(\int_0^u \Pi_s(Q(s,X_s))ds \;\Big|\; \mathfrak{F}_u^Y\Big) \\ =&E\Big(\int_0^u Q(s,X_s)ds \mid \mathfrak{F}_u^Y\Big) - \int_0^u \Pi_s(Q(s,X_s))ds,\end{aligned}$$

where we have used the tower property of the conditional expectation. Hence, it is an F^Y-martingale. Again, for $u \leq t$,

$$\begin{aligned} E\Big(E(M_t^X \mid \mathcal{F}_t^Y) \mid \mathcal{F}_u^Y\Big) &= E(M_t^X \mid \mathcal{F}_u^Y) \\ &= E(M_u^X \mid \mathcal{F}_u^Y) + E(M_t^X - M_u^X \mid \mathcal{F}_u^Y) \\ &= E(M_u^X \mid \mathcal{F}_u^Y) + E\Big(E(M_t^X - M_u^X \mid \mathcal{F}_u) \mid \mathcal{F}_u^Y\Big) \\ &= E(M_u^X \mid \mathcal{F}_u^Y), \end{aligned}$$

where we have used the martingale property of M with respect to the complete information filtration F and the tower property of the conditional expectation. Hence it is an F^Y-martingale. This implies that the right-hand-side of (5.11) is F^Y-martingale that starts from zero. Hence, from Lemma 5.1.2, there exist square-integrable, F^Y-predictable processes $\{f(t,w)\}_{t\in[0,\,T]}$ and $\{g(t,w)\}_{t\in[0,\,T]}$, such that

$$\Pi_t(X) - \Pi_0(X) - \int_0^t \Pi_s(Q(s,X_s))ds = \int_0^t f(s,w)d\widetilde{W}_s + \int_0^t g(s,w)d\widetilde{N}_s \tag{5.12}$$

for all $t \in [0,\ T]$. Hence the lemma follows. □

We now have sufficient ingredients to derive the non-linear filtering equation for the partially observed system.

Theorem 5.1.1. *The conditional distribution Π_t of the Markov process satisfies the following evolution equation*

$$\begin{aligned} \Pi_t(X) =& \Pi_0(X) + \int_0^t \Pi_s(Q(s,X_s))ds + \int_0^t \Pi_s(X_s\frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s})d\widetilde{W}_s \\ &+ \int_0^t \frac{\Pi_s(X_s(\lambda(X_s) - \Pi_s(\lambda)))}{\Pi_t(\lambda)}d\widetilde{N}_s. \end{aligned} \tag{5.13}$$

Proof. Let $\widetilde{Z}_t = \int_0^t \gamma_s d\widetilde{W}_s + \int_0^t \psi_s d\widetilde{N}_s$, where $\{\gamma_t\}_{t\in[0,\,T]}$ is any given bounded F^Y-adapted random process and $\{\psi_t\}_{t\in[0,\,T]}$ is any given bounded F^Y-predictable random process.

Then, $\widetilde{Z}$ is an F^Y-martingale process. Using the value of $\widetilde{U}$ from (5.10),

$$\begin{aligned} E(\widetilde{U}_t\widetilde{Z}_t) &= E\int_0^t f_s\gamma_s ds + E\int_0^t g_s\psi_s dN_s \\ &= E\int_0^t f_s\gamma_s ds + E\int_0^t \Pi_s(\lambda)g_s\psi_s ds. \end{aligned} \tag{5.14}$$

From tower property, $E(dN_t) = E(\Pi_t(\lambda)dt)$. Again, from (5.10),

$$\begin{aligned} E(\widetilde{U}_t\widetilde{Z}_t) &= E(\widetilde{Z}_t\Pi_t(X)) - E(\widetilde{Z}_t\Pi_0(X)) - E\Big(\widetilde{Z}_t\int_0^t \Pi_s(Q(s,X_s))ds\Big) \\ &= E(\widetilde{Z}_t\Pi_t(X)) - E\Big(\int_0^t E(\widetilde{Z}_t \mid \mathfrak{F}_s^Y)\Pi_s(Q(s,X_s))ds\Big) \\ &= E(\widetilde{Z}_t\Pi_t(X)) - E\Big(\int_0^t \widetilde{Z}_s\Pi_s Q(s,X_s)ds\Big) \\ &= E(\widetilde{Z}_tX_t) - E\Big(\int_0^t \widetilde{Z}_s Q(s,X_s)ds\Big). \\ &\qquad (\because\ \widetilde{Z}_t\Pi_t(X) = E(\widetilde{Z}_tX_t|\mathfrak{F}_t^Y)) \end{aligned} \tag{5.15}$$

Now,

$$\begin{aligned} \widetilde{Z}_t &= \int_0^t \gamma_s dW_s + \int_0^t \psi_s(dN_s - \lambda(X_s)ds) + \int_0^t \gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds \\ &\quad + \int_0^t \psi_s(\lambda(X_s)-\Pi_s(\lambda))ds \\ \widetilde{Z}_t &= Z_t + \int_0^t \gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds + \int_0^t \psi_s(\lambda(X_s)-\Pi_s(\lambda))ds, \end{aligned} \tag{5.16}$$

where

$$Z_t = \int_0^t \gamma_s dW_s + \int_0^t \psi_s(dN_s - \lambda(X_s)ds).$$

From (5.15) and (5.16), we have

$$
\begin{aligned}
E(\widetilde{U}_t\widetilde{Z}_t) =& E\Big(X_tZ_t - \int_0^t Z_sQ(s,X_s)ds\Big) + \\
& E\Big(X_t\Big(\int_0^t \gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds + \int_0^t \psi_s(\lambda(X_s)-\Pi_s(\lambda))ds\Big)\Big) - \\
& E\Big(\int_0^t Q(u,X_u)\Big(\int_0^u \gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds + \int_0^u \psi_s(\lambda(X_s)-\Pi_s(\lambda))ds\Big)du\Big) \\
=& E\Big(\int_0^t X_s\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds + \int_0^t X_s\psi_s(\lambda(X_s)-\Pi_s(\lambda))ds\Big) + \\
& E\Big(\int_0^t (X_t-X_s)\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds + \int_0^t (X_t-X_s)\psi_s(\lambda(X_s)-\Pi_s(\lambda))ds\Big) \\
& - E\Big(\int_0^t Q(u,X_u)\Big(\int_0^u \gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds + \int_0^u \psi_s(\lambda(X_s)-\Pi_s(\lambda))ds\Big)du\Big).
\end{aligned} \tag{5.17}
$$

where

$$
\begin{aligned}
E\Big(X_tZ_t - \int_0^t Z_sQ(s,X_s)ds\Big) &= E\Big(X_tZ_t - \int_0^t E(Z_t|\mathfrak{F}_s)Q(s,X_s)ds\Big) \\
&= E\Big(X_tZ_t - Z_tX_0 - \int_0^t Z_tQ(s,X_s)ds\Big) \\
&\qquad (\because\ E(Z_tX_0)=0 \text{ and from tower property}) \\
&= E\Big[Z_t\Big(X_t - X_0 - \int_0^t Q(s,X_s)ds\Big)\Big] \\
&= 0. \qquad (\because\ X \text{ and } Z \text{ are orthogonal})
\end{aligned}
$$

Now

$$
\begin{aligned}
& E\Big(\int_0^t (X_t-X_s)\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds + \int_0^t (X_t-X_s)\psi_s(\lambda(X_s)-\Pi_s(\lambda))ds\Big) \\
&= E\Big(\int_0^t \Big(\int_s^t Q(u,X_u)du + M_t^X - M_s^X\Big)[\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s} + \psi_s(\lambda(X_s)-\Pi_s(\lambda))]ds\Big) \\
&= E\Big(\int_0^t\int_s^t Q(u,X_u)du[\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s} + \psi_s(\lambda(X_s)-\Pi_s(\lambda))]ds\Big) \\
&\qquad (\because\ E(M_t^X - M_s^X|\mathfrak{F}_s)=0) \\
&= E\Big(\int_0^t Q(u,X_u)\int_0^u[\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s} + \psi_s(\lambda(X_s)-\Pi_s(\lambda))]dsdu\Big).
\end{aligned}
$$

Hence,

$$E(\widetilde{U}_t\widetilde{Z}_t) = E\Big(\int_0^t X_s\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds + \int_0^t X_s\psi_s(\lambda(X_s)-\Pi_s(\lambda))ds\Big). \tag{5.18}$$

Subtracting (5.18) from (5.14), we get

$$E\Big(\int_0^t \gamma_s(f_s - X_s\frac{(\mu(X_s)-\Pi_s(\mu)))}{\sigma_s}ds + \int_0^t \psi_s(g_s\Pi_s(\lambda) - X_s(\lambda(X_s)-\Pi_s(\lambda)))ds\Big) = 0.$$

$$\begin{aligned} E\Big(\int_0^t \gamma_s\Big(f_s - E(X_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s} \mid \mathfrak{F}_s^Y)\Big)ds + \\ \int_0^t \psi_s\Big(g_s\Pi_s(\lambda) - E(X_s(\lambda(X_s)-\Pi_s(\lambda))ds \mid \mathfrak{F}_s^Y)\Big)\Big) = 0. \end{aligned} \tag{5.19}$$

Since the above equation is true for any F^Y-adapted random process γ and F^Y-predictable random process ψ, we have

$$\begin{aligned} f_s &= E(X_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s} \mid \mathfrak{F}_s^Y) \\ g_s &= E\Big(\frac{X_s(\lambda(X_s)-\Pi_s(\lambda))}{\Pi_s(\lambda)} \mid \mathfrak{F}_s^Y\Big). \end{aligned}$$

Hence the theorem follows. □

The model with complete observation is now characterized by

$$dS_t = S_t(\widetilde{\mu}_t dt + \sigma_t d\widetilde{W}_t + \rho_t d\widetilde{N}_t) \tag{5.20}$$

with $S_0 = s_0$,

$$\begin{aligned} \Pi_t(X) = &\Pi_0(X) + \int_0^t \Pi_s(Q(s,X_s))ds + \int_0^t \Pi_s(X_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s})d\widetilde{W}_s \\ &+ \int_0^t \frac{\Pi_s(X_s(\lambda(X_s)-\Pi_s(\lambda))}{\Pi_s(\lambda)}d\widetilde{N}_s \end{aligned} \tag{5.21}$$

and $\Pi_0(X) = x$. The control process for $\pi \in U[0,\ T]$ is characterized by

$$d\widetilde{V}_t^\pi = \widetilde{V}_t^\pi(\pi_t^1\widetilde{\mu}_t dt + \pi_t^1\sigma_t d\widetilde{W}_t + \pi_t^1\rho_t d\widetilde{N}_t). \tag{5.22}$$

Notice that $\widetilde{V} = V$. Now we will redefine the GOP according to the reduced complete information model.

Definition 5.1.2. *A self-financing portfolio $\pi^* \in U[t,T]$ is called GOP if it has a positive wealth process V^{π^*} such that*

$$E_{t,\varrho,v} \log V_T^{\pi^*} \geq E_{t,\varrho,v} \log V_T^{\pi} \qquad a.s.$$

for all self-financing $\pi \in U[t,T]$ with positive wealth process $\{V_s^\pi\}$ for all $s \in [t,T]$, where $E_{t,\varrho,v}(\cdot) = E(\cdot \big| V_t^\pi = v,\ \Pi_t(X) = \varrho)$ indicates the conditional expectation under P given complete information till time $t \in [0,\ T]$.

For all $\pi \in U[t,T]$ and $v > 0$ and from the results that have been shown till now, it is straightforward to see that

$$E_{t,\varrho,v} \log \widetilde{V}_T^\pi = E_{t,v} \log V_T^\pi$$

and

$$E_{t,\varrho,v} \log \widetilde{V}_T^{\pi^*} = E_{t,v} \log V_T^{\pi^*}.$$

The reduced model is now with complete information, and so we will solve the optimization problem with the help of the HJB equation.

5.2 Growth optimal portfolio

In this section, we derive the GOP for the complete information case. From (5.22), we can write the SDE for a wealth process corresponding to a self-financing portfolio as

$$dV_t^\pi = V_{t-}^\pi \Big(\pi_t^1 \sigma_t (d\widetilde{W}_t + \theta_t^1 dt) + \pi_t^1 \rho_t (d\widetilde{N}_t + \theta_t^2 dt) \Big), \tag{5.23}$$

where θ_t^1 and θ_t^2 are the market prices of risk that depend on Π_t characterized by

$$\sigma_t \theta_t^1 + \rho_t \theta_t^2 = \widetilde{\mu}_t. \tag{5.24}$$

To get the risk-minimizing hedging strategy we consider the θ of the specific form

$$\theta_t = \Gamma_t^\top (\Sigma_t \Gamma_t^\top)^{-1} \widetilde{\mu}_t, \tag{5.25}$$

where $\theta_t = (\theta_t^1,\ \theta_t^2)^\top$, $\Sigma_t = (\sigma_t,\ \rho_t)$, $\Gamma_t = (\sigma_t,\ \rho_t\Pi_t(\lambda))$. (5.23) can be solved explicitly and the solution is given by

$$\begin{aligned} V_t^\pi =& v\exp\Big(\int_0^t (\pi_s^1\sigma_s\theta_s^1 + \pi_s^1\rho_s(\theta_s^2 - \Pi_s(\lambda))) \\ &- \frac{1}{2}(\pi_s^1\sigma_s)^2)ds + \int_0^t \pi_s^1\sigma_s d\widetilde{W}_s\Big) \prod_{\Delta N_s=1, s\in[0,\,T]} (\pi_s^1\rho_s + 1). \end{aligned}$$

So,

$$\begin{aligned} E_{0,\varrho,v}\log(V_t^\pi) = \log v + E_{0,\varrho,v}\Big(\int_0^t (\pi_s^1\sigma_s\theta_s^1 + \pi_s^1\rho_s(\theta_s^2 - \Pi_s(\lambda))) \\ - \frac{1}{2}(\pi_s^1\sigma_s)^2)ds\Big) + E_{0,\varrho,v}\int_0^t \log(\pi_s^1\rho_s + 1)\lambda(X_s)ds. \end{aligned}$$

Finally, we have a stochastic control problem of maximizing the objective function

$$\mathfrak{J}(0,\varrho,v,\pi) = E_{0,\varrho,v}\log(V_T^\pi) = \log v + \mathfrak{H}(0,\varrho,\pi),$$

where

$$\begin{aligned} \mathfrak{H}(0,\varrho,\pi) =& E_{0,\varrho,v}\Big(\int_0^T (\pi_s^1\sigma_s\theta_s^1 + \pi_s^1\rho_s(\theta_s^2 - \Pi_s(\lambda)) - \frac{1}{2}(\pi_s^1\sigma_s)^2)ds\Big) \\ &+ E_{0,\varrho,v}\int_0^T \log(\pi_s^1\rho_s + 1)\lambda(X_s)ds. \end{aligned}$$

Here the objective function $\mathfrak{J}(0,\varrho,v,\pi)$ is a concave, increasing, logarithmic utility function with π as a control process. That is, our aim is to solve the problem

$$\sup_{\pi\in U[0,\,T]} \mathfrak{J}(0,\varrho,v,\pi), \tag{5.26}$$

where the supremum is taken over all admissible trading strategies.

We consider the time horizon $t \le s \le T$. Since the controller is allowed to observe the state V_t^π of the process being controlled, we may as well assume that the initial state $\Pi_t(X) = \varrho$ is known and $V_t^\pi = v$. The starting point for dynamic programming is then to regard the supremum of the quantity $\mathfrak{J}$ in (5.26) being maximized as a function $\varsigma(t,\varrho,v)$

of the initial data, i.e.,

$$\varsigma(t,\varrho,v)=\sup_{\pi\in U[t,\,T]}\mathfrak{J}(t,\varrho,v,\pi)=\log v+\sup_{\pi\in U[t,\,T]}\mathfrak{H}(t,\varrho,\pi)=\log v+\mathcal{D}(t,\varrho). \tag{5.27}$$

The key equation to solve the problem is the HJB equation. For the reduced model, it can be derived as

$$\begin{aligned}\sup_{\pi\in U[t,\,T]}[\varsigma_t+\varsigma_v v(\pi_t^1\widetilde{\mu}_t-\pi_t^1\rho\Pi_t(\lambda))+\frac{1}{2}\varsigma_{vv}v^2\sigma_t^2(\pi_t^1)^2+\varsigma_\varrho(\Pi_t(Q(t,x))-\\ \Pi_t(x(\lambda(x)-\Pi_t(\lambda))))+\frac{1}{2}\varsigma_{\varrho\varrho}\Pi_t^2(x\frac{(\mu(x)-\Pi_t(\mu))}{\sigma_t})+\\ \varsigma_{v\varrho}\Pi(x\frac{(\mu(x)-\Pi_t(\mu))}{\sigma_t})v\sigma_t\pi_t^1+(\varsigma(t,\Pi,v)-\varsigma(t-,\Pi-,v-))\Pi_t(\lambda)]=0.\end{aligned}$$

From (5.27), $\varsigma(t,\varrho,v)=\log v+\mathcal{D}(t,\varrho)$,

$$\begin{aligned}\sup_{\pi\in U[t,\,T]}[\mathcal{D}_t+(\pi_t^1\widetilde{\mu}_t-\pi_t^1\rho_t\Pi_t(\lambda))-\frac{1}{2}\sigma_t^2(\pi_t^1)^2+\mathcal{D}_\varrho(\Pi_t(Q(t,x))-\Pi_t(x(\lambda(x)-\\ \Pi_t(\lambda))))+\frac{1}{2}\mathcal{D}_{\varrho\varrho}\Pi_t^2(x\frac{(\mu(x)-\Pi_t(\mu))}{\sigma_t})+(\mathcal{D}(t,\Pi)-\mathcal{D}(t-,\Pi-)+\log(\pi_t^1\rho_t+1))\Pi_t(\lambda)]=0\end{aligned} \tag{5.28}$$

with the boundary condition $\mathcal{D}(T,\varrho)=0$. But here, the GOP does not exist in a natural way, as it did in previous chapters. Therefore, we add a non-tradable asset in the model to solve the problem, so that it reflects the need of GOP to price assets, rather than being the investment choice of an investor having log-utility. We consider a fictitious market, which is an extension of the existing market with an additional stock following the SDE

$$d\breve{S}_t=\breve{S}_{t-}(\rho_t\Pi_t(\lambda)d\widetilde{W}_t-\sigma_t d\widetilde{N}) \tag{5.29}$$

as its dynamics. $\breve{S}$ is chosen in such a way that market price of risk θ is not affected in the original market and

$$\begin{bmatrix}\sigma_t & \rho_t\Pi_t(\lambda)\\ \rho_t & -\sigma_t\end{bmatrix}$$

is an invertible matrix. In this market, a portfolio strategy (π_1,π_2) represents the ratio of wealth invested in the stock S and $\breve{S}$ respectively, and $\pi_{0t}=1-\pi_{1t}-\pi_{2t}$ is the ratio of the wealth invested in the riskless asset at time $t \in [0,\ T]$. The positive wealth process of a self-financing strategy satisfies the SDE

$$\begin{aligned} dV_t^{\pi} =& V_{t-}^{\pi}\Big(\pi_{1t}\frac{dS_t}{S_{t-}} + \pi_{2t}\frac{d\breve{S}_t}{\breve{S}_{t-}}\Big) \\ =& V_{t-}^{\pi}\Big((\pi_{1t}\sigma_t + \pi_{2t}\rho_t\Pi_t(\lambda))(d\widetilde{W}_t + \theta_t^1 dt) + (\pi_{1t}\rho_t - \pi_{2t}\sigma_t)(d\widetilde{N}_t + \theta_t^2 dt)\Big), \end{aligned} \tag{5.30}$$

where θ^1 and θ^2 satisfies the following system of linear equations

$$\begin{aligned} \sigma_t\theta_t^1 + \rho_t\theta_t^2 &= \widetilde{\mu}_t \\ \rho_t\Pi_t(\lambda)\theta_t^1 - \sigma_t\theta_t^2 &= 0 \end{aligned} \tag{5.31}$$

which has a unique solution as per our assumptions. Now, in the extended market, the HJB equation will be

$$\begin{aligned} \sup_{\pi\in U(t,T)} & [\mathfrak{D}_t + (\pi_{1t}\widetilde{\mu}_t - (\pi_{1t}\rho_t - \pi_{2t}\sigma_t)\Pi_t(\lambda)) - \frac{1}{2}(\sigma_t\pi_{1t} + \rho_t\Pi_t(\lambda)\pi_{2t})^2 + \\ & \mathfrak{D}_{\varrho}(\Pi_t(Q(t,x)) - \Pi_t(x(\lambda(x) - \Pi_t(\lambda)))) + \frac{1}{2}\mathfrak{D}_{\varrho\varrho}\Pi_t^2(x\frac{(\mu(x) - \Pi_t(\mu))}{\sigma_t}) \\ & + (\mathfrak{D}(t,\Pi) - \mathfrak{D}(t-,\Pi-) + \log(\pi_{1t}\rho_t - \pi_{2t}\sigma_t + 1))\Pi_t(\lambda)] = 0. \end{aligned} \tag{5.32}$$

Theorem 5.2.1. *The self-financing strategy* $\pi_t^{*\top} = \Xi_t^{-1}\Theta_t$ *is the maximizer of the HJB equation in the fictitious market, where* $\Theta_t = (\theta_t^1, \frac{\theta_t^2}{\Pi_t(\lambda)-\theta_t^2})^\top$, *and* $\Xi_t = \begin{bmatrix} \sigma_t & \rho_t\Pi_t(\lambda) \\ \rho_t & -\sigma_t \end{bmatrix}$. *The wealth process* V^{π^*} *of the admissible strategy* π^* *satisfies the SDE*

$$dV_t^{\pi^*} = V_{t-}^{\pi^*}\Big(\theta_t^1(d\widetilde{W}_t + \theta_t^1 dt) + \frac{\theta_t^2}{\Pi_t(\lambda) - \theta_t^2}(d\widetilde{N}_t + \theta_t^2 dt)\Big). \tag{5.33}$$

Proof. We use the first order condition for maximizing the function in (5.32). Partially differentiating with respect to π_1 and π_2, and from (5.31), we get respectively

$$\sigma_t\theta_t^1 + \rho_t(\theta_t^2 - \Pi_t(\lambda)) - \sigma_t(\pi_{1t}\sigma_t + \pi_{2t}\rho_t\Pi_t(\lambda)) + \frac{\rho_t\Pi_t(\lambda)}{\pi_{1t}\rho_t - \pi_{2t}\sigma_t + 1} = 0$$

$$\rho_t\Pi_t(\lambda)\theta_t^1 + \sigma_t(\theta_t^2 - \Pi_t(\lambda)) - \rho_t\Pi_t(\lambda)(\pi_{1t}\sigma_t + \pi_{2t}\rho_t\Pi_t(\lambda)) - \frac{\sigma_t\Pi_t(\lambda)}{\pi_{1t}\rho_t - \pi_{2t}\sigma_t + 1} = 0.$$

From above set of equations, we get

$$\pi_{1t}\sigma_t + \pi_{2t}\rho_t\Pi_t(\lambda) = \theta_t^1$$

$$\pi_{1t}\rho_t - \pi_{2t}\sigma_t = \frac{\theta_t^2}{\Pi_t(\lambda) - \theta_t^2}.$$

Solving these set of equations, we get $\pi_t^* = \Xi_t^{-1}\Theta_t$ with $\Theta_t = (\theta_t^1, \frac{\theta_t^2}{\Pi_t(\lambda)-\theta_t^2})^\top$ and

$$\Xi_t = \begin{bmatrix} \sigma_t & \rho_t\Pi_t(\lambda) \\ \rho_t & -\sigma_t \end{bmatrix}.$$

Using the second order condition, it can be seen that π^* maximizes the HJB equation. After putting the value of π_t^* in (5.30), we get

$$dV_t^{\pi^*} = V_{t-}^{\pi^*}\Big(\theta_t^1(d\widetilde{W}_t + \theta_t^1 dt) + \frac{\theta_t^2}{\Pi_t(\lambda) - \theta_t^2}(d\widetilde{N}_t + \theta_t^2 dt)\Big).$$

$\square$

In the above theorem, we have derived the wealth process of the GOP which is not tradable in the actual market. We use the non-tradable GOP only for pricing and hedging the contingent claim in financial markets.

Using the value of optimal portfolio π^*, the HJB equation (5.32) can be written as

$$\begin{aligned} &\mathcal{D}_t + \mathcal{D}_\varrho(\Pi_t(Q(t,x)) - \Pi_t(x(\lambda(x) - \Pi_t(\lambda)))) + \frac{1}{2}\mathcal{D}_{\varrho\varrho}\Pi_t^2(x\frac{(\mu(X_t) - \Pi_t(\mu))}{\sigma_t}) \\ &+ \Big(\mathcal{D}(t,\Pi) - \mathcal{D}(t-,\Pi-) + \log\Big(\frac{\Pi_t(\lambda)}{\Pi_t(\lambda) - \theta_t^2}\Big)\Big)\Pi_t(\lambda) + \frac{1}{2}(\theta_t^1)^2 - \theta_t^2 = 0. \end{aligned} \tag{5.34}$$

It can be written as

$$\begin{aligned} &\mathcal{D}_t + \mathcal{D}_\varrho(\Pi_t(Q(t,x)) - \Pi_t(x(\lambda(x) - \Pi_t(\lambda)))) + \frac{1}{2}\mathcal{D}_{\varrho\varrho}\Pi_t^2(x\frac{(\mu(x) - \Pi_t(\mu))}{\sigma_t}) \\ &+ \Big(\mathbf{D}(t,\Pi) + \log\Big(\frac{\Pi_t(\lambda)}{\Pi_t(\lambda) - \theta_t^2}\Big)\Big)\Pi_t(\lambda) + \frac{1}{2}(\theta_t^1)^2 - \theta_t^2 = 0, \end{aligned} \tag{5.35}$$

where $\mathbf{D}(t,\Pi) = \mathcal{D}(t,\Pi) - \mathcal{D}(t-,\Pi-)$ and from (5.33), $\mathcal{D}(t,\Pi)$ can be written as

$$\mathcal{D}(t,\Pi) = E_{t,\varrho}\Big(\int_t^T \Big[\frac{1}{2}\theta_s^1 - \theta_s^2 + \Pi_s(\lambda)\log\Big(\frac{\Pi_s(\lambda)}{\Pi_s(\lambda)-\theta_s^2}\Big)\Big]ds\Big).$$

From the assumptions we have made, $\mathbf{D} \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$. Now using the results from Fleming and Rishel (1975) (Chapter VI, Appendix -E) and Davis and Lleo (2011), the HJB equation (5.35) has a unique solution in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$, which is the optimal value of the wealth process.

5.3 The benchmark model

For a portfolio wealth process V^π, its benchmarked value $\hat{V}_t^\pi = \frac{V_t^\pi}{V_t^{\pi^*}}$ at time t satisfies, by the Itô formula, (5.23) and (5.33),

$$d\hat{V}_t^\pi = \hat{V}_{t-}^\pi\Big((\pi_t\sigma_t - \theta_t^1)d\widetilde{W}_t - \Big(\frac{\pi_t\rho_t(\theta_t^2 - \Pi_t(\lambda)) + \theta_t^2}{\Pi_t(\lambda)}\Big)d\widetilde{N}_t\Big) \tag{5.36}$$

for $t \in [0,\ T]$. Since the SDE is driftless, any non-negative portfolio when expressed in units of the GOP forms an (F^S, P)-supermartingale.

Definition 5.3.1. *Let H_T be an $\mathcal{F}_T^S$-measurable non-negative, square-integrable, random variable which is contingent on S_T at maturity T and satisfies $E|\widehat{H}_T| < \infty$ a.s.. Then the* ***fair price*** *process of the contingent claim H_T is given by $\Lambda^H = \{\Lambda_t^H\}_{t\in[0,\,T]}$, where*

$$\Lambda_t^H = V_t^{\pi^*}E\big(\widehat{H}_T \mid \mathcal{F}_t^S\big).$$

5.3.1 Risk-neutral pricing

In this subsection, we show that fair pricing generalizes the established standard risk-neutral pricing methodology. Here the incompleteness of the market is due to two reasons. The first is that we have only one tradable risky asset but it is driven by two independent random processes, i.e., all the driving random processes are not tradable and the second is that the stock price process is modulated by an irreducible Markov process with finite state space. The incompleteness of the first type has been studied in Schweizer (1995) and Arai (2004). They have derived the minimal-martingale measure for the corresponding models. The incompleteness of the second type with the complete information of the Markov process has been studied by Di Masi et al. (1994) and Deshpande and Ghosh (2008), wherein they have derived the minimal-martingale measure and the risk-minimizing hedging strategy.

The fair price Λ_t^H for the payoff H_T is at time t is given by the expression

$$\begin{aligned}\Lambda_t^H &= V_t^{\pi^*} E\big(\widehat{H}_T \mid \mathcal{F}_t^S\big)\\ &= E\Big(\frac{\mathcal{Z}_T}{\mathcal{Z}_t} H_T \mid \mathcal{F}_t^S\Big),\end{aligned} \tag{5.37}$$

with Radon-Nikodým derivative

$$\mathcal{Z}_t = \frac{V_0^{\pi^*}}{V_t^{\pi^*}} \tag{5.38}$$

for $t \in [0,\ T]$. The Radon-Nikodým derivative satisfies the SDE

$$d\mathcal{Z}_t = -\mathcal{Z}_t\Big(\theta_t^1 d\widetilde{W}_t + \frac{\theta_t^2}{\Pi_t(\lambda)} d\widetilde{N}\Big). \tag{5.39}$$

The pricing formula (5.37) gives us general access to fair prices via conditional expectations. If $\mathcal{Z}$ is an (F^S, P)-martingale, then an equivalent risk-neutral measure Q with $\frac{dQ}{dP} = \mathcal{Z}_T$ exists and so is the minimal-martingale measure (see, Arai, 2004). In this case, the well-known risk-neutral pricing formula is obtained from (5.37) by Girsanov's theorem and Bayes' formula. Denoting by E^Q the expectation with respect to Q, we can write the fair pricing formula (5.37) in the common form

$$\Lambda_t^H = \overline{H}_t = E^Q\Big(H_T \mid \mathcal{F}_t^S\Big) \tag{5.40}$$

for $t \in [0,\ T]$. Here, recall that $\overline{H}_t$ indicates the risk neutral price of the claim at time $t \in [0,\ T]$.

5.4 Hedging

So far, we have calculated the value of the fair price of the claim in the reduced complete information case. Now, we try to get the hedging strategy. We need additional cash infusions or withdrawals in order to finance the hedge portfolio. The cost $\mathbf{C}$ induced by the hedging strategy then consist of the initial cost of the hedge portfolio $\mathbf{C}_0 = H_0$, and the additional cash flows during the life of the option, necessary to maintain the hedge portfolio. We consider a non-self-financing strategy V_t^π, with the cost process $\mathbf{C} = \{\mathbf{C}_t\}_{t\in[0,\,T]}$, satisfying the SDE

$$dV_t^\pi = V_{t-}^\pi(\pi_t^1(\widetilde{\mu}_t)dt + \pi_t^1\sigma_t d\widetilde{W}_t + \pi_t^1\rho_t d\widetilde{N}_t) + d\mathbf{C}_t. \tag{5.41}$$

We look for an admissible strategy which minimizes, at each time t, the remaining risk defined by $\mathbf{R}_t^\pi = E[(\mathbf{C}_T^\pi - \mathbf{C}_t^\pi)^2|\mathcal{F}_t]$ over all admissible strategies with the same terminal value.

To get the risk-minimizing strategy, we suppose that the investor is investing in GOP in place of the riskless asset in the fictitious market, i.e., we consider the portfolio consisting the risky asset S_t and GOP $V_t^{\pi^*}$ at time $t \in [0,\ T]$, with cost process $\tilde{\mathbf{C}}$, satisfying the SDE

$$\begin{aligned}
dV_t^{\tilde\pi} =& V_{t-}^{\tilde\pi}\Big(\tilde\pi_t^1\frac{dS_t}{S_{t-}} + \tilde\pi_t^0\frac{dV_t^{\pi^*}}{V_{t-}^{\pi^*}}\Big) + d\tilde{\mathbf{C}}_t\\
=& V_{t-}^{\tilde\pi}\Big(\tilde\pi_t^1\sigma_t(d\widetilde{W}_t + \theta_t^1 dt) + \tilde\pi_t^1\rho_t(d\widetilde{N}_t + \theta_t^2 dt)\\
&+ (1-\tilde\pi_t^1)(\theta_t^1(d\widetilde{W}_t + \theta_t^1 dt) + \frac{\theta_t^2}{\Pi_t(\lambda) - \theta_t^2}(d\widetilde{N}_t + \theta_t^2 dt))\Big) + d\tilde{\mathbf{C}}_t.
\end{aligned} \tag{5.42}$$

Here, $\tilde\pi = \{\tilde\pi^0, \tilde\pi^1\}$, where $\tilde\pi^1$ indicates the ratio of the wealth invested in risky asset and $\tilde\pi^0 = 1 - \tilde\pi^1$ in GOP. The corresponding benchmarked value is

$$d\widehat{V}_t^{\tilde\pi} = \widehat{V}_{t-}^{\tilde\pi}\tilde\pi_t^1\Big((\sigma_t - \theta_t^1)d\widetilde{W} + (\rho_t - \frac{\theta_t^2}{\Pi_t(\lambda) - \theta_t^2})(\Pi_t(\lambda) - \theta_t^2)d\widetilde{N}\Big) + \frac{d\tilde{\mathbf{C}}_t}{V_t^{\pi^*}}. \tag{5.43}$$

Now, we consider

$$dM_t^1 = (\sigma_t - \theta_t^1)d\widetilde{W} + (\rho_t - \frac{\theta_t^2}{\Pi_t(\lambda) - \theta_t^2})(\Pi_t(\lambda) - \theta_t^2)d\widetilde{N} \tag{5.44}$$

and M_t^2 in such a way that it is strongly orthogonal with the martingale part of S with specific structure

$$dM_t^2 = \rho_t \Pi_t(\lambda)d\widetilde{W} - \sigma_t d\widetilde{N}. \tag{5.45}$$

Here, it can be seen that $\{M_t^1\}_{t\in[0,\,T]}$ and $\{M_t^2\}_{t\in[0,\,T]}$ are $F^{\widetilde{W},\widetilde{N}}$-martingales. It is clear that $\mathfrak{F}_t^{M^1,M^2} \subseteq \mathfrak{F}_t^{\widetilde{W},\widetilde{N}}$, where $\mathfrak{F}_t^{M^1,M^2}$ is a σ-algebra generated by M_t^1 and M_t^2 at time $t \in [0,\ T]$. As we can see from equation (5.44) and (5.45), $\widetilde{W}, \widetilde{N}$ can be written in terms of M^1, M^2, which implies that $\mathfrak{F}_t^{\widetilde{W},\widetilde{N}} \subseteq \mathfrak{F}_t^{M^1,M^2}$. Hence, we have the following lemma.

Lemma 5.4.1. *The filtration generated by M^1 and M^2 is the same as the filtration generated by the innovation process, i.e.,*

$$F_t^{M^1,M^2} = F_t^{\widetilde{W},\widetilde{N}}$$

for all $t \in [0,\ T]$.

We know that the benchmarked fair price process $\widehat{\Lambda}^H$ is an F^S-martingale, i.e., F^{M^1,M^2}-martingale. From the martingale representation theorem, there exist square-integrable F^{M^1,M^2}-predictable processes η_1 and η_2 such that the benchmarked fair price process $\widehat{\Lambda}_t^H$ has the representation

$$d\widehat{\Lambda}_t^H = \eta_t^1 dM_t^1 + \eta_t^2 dM_t^2. \tag{5.46}$$

Now by the help of above considerations, we can derive the risk-minimizing hedging trading strategy.

Theorem 5.4.1. *The locally-risk-minimizing hedging trading strategy δ^1 for H_T, the amount invested in stock, is given by the following formula, for $t \in [0,T]$,*

$$\delta_t^1 = \frac{V_t^{\tilde{\pi}}(\tilde{\pi}_t^1 + (1-\tilde{\pi}_t^1)\pi_{1t}^*)}{S_{t-}}$$

and the cost process is given by

$$d\mathbf{C}_t = \left(V_t^{\tilde{\pi}}(1 - \tilde{\pi}_t^1)\pi_{2t}^* + V_t^{\pi^*}\eta_t^2 \right) dM_t^2,$$

where $V_t^{\tilde{\pi}}$ *is the amount of the wealth investing and* $\tilde{\pi}_t^1 = \frac{\eta_t^1}{\hat{V}_t^{\tilde{\pi}}}$. *And* $V_t^{\pi^*}$ *is the wealth of GOP,* π_{1t}^* *and* π_{2t}^* *are given by Theorem 5.2.1.*

Proof. By comparing (5.43) and (5.46), we get

$$\tilde{\pi}_t^1 = \frac{\eta_t^1}{\hat{V}_t^{\tilde{\pi}}}$$

$$d\tilde{\mathbf{C}}_t = V_t^{\pi^*}\eta_t^2 dM_t^2.$$

Now, if we replace the value of $\frac{dV_t^{\pi^*}}{V_t^{\pi^*}}$ in terms of the primary securities, then the equation (5.43) reduces to

$$\begin{aligned}
dV_t^{\tilde{\pi}} =& V_t^{\tilde{\pi}} \left((1 - \tilde{\pi}_t^1)(\pi_{1t}^* \frac{dS_t}{S_{t-}} + \pi_{2t}^* \frac{d\breve{S}_t}{\breve{S}_{t-}}) + \tilde{\pi}_t^1 \frac{dS_t}{S_{t-}} \right) + d\tilde{\mathbf{C}}_t \\
=& V_t^{\tilde{\pi}} \left((\tilde{\pi}_t^1 + (1 - \tilde{\pi}_t^1)\pi_{1t}^*) \frac{dS_t}{S_{t-}} \right) + V_t^{\tilde{\pi}}(1 - \tilde{\pi}_t^1)\pi_{2t}^* \frac{d\breve{S}_t}{\breve{S}_{t-}} + d\tilde{\mathbf{C}}_t \\
=& V_t^{\tilde{\pi}} \left((\tilde{\pi}_t^1 + (1 - \tilde{\pi}_t^1)\pi_{1t}^*) \frac{dS_t}{S_{t-}} \right) + \left(V_t^{\tilde{\pi}}(1 - \tilde{\pi}_t^1)\pi_{2t}^* + V_t^{\pi^*}\eta_t^2 \right) dM_t^2.
\end{aligned}$$

Hence, if we invest

$$\delta_t^1 = \frac{V_t^{\tilde{\pi}}(\tilde{\pi}_t^1 + (1 - \tilde{\pi}_t^1)\pi_{1t}^*)}{S_{t-}}$$

amount of wealth in risky asset with total amount of investment $V_t^{\tilde{\pi}}$ and cost process

$$d\mathbf{C}_t = \left(V_t^{\tilde{\pi}}(1 - \tilde{\pi}_t^1)\pi_{2t}^* + V_t^{\pi^*}\eta_t^2 \right) dM_t^2,$$

we get the required risk-minimizing hedging strategy. We can see that the cost process $\mathbf{C}$ is F^S-martingale and strongly orthogonal to the martingale part of S, so the corresponding strategy is a risk-minimizing strategy. □

We have derived the local risk-minimizing hedging strategy for a contingent claim H_T in an incomplete information financial market. Initially, we have considered a fictitious market, where the investor can invest in GOP. Using the wealth process corresponding to this portfolio, we have derived the local risk-minimizing hedging strategy in the actual market.

Chapter 6

A Defaultable Financial Market with Complete Information

Credit risk is the distribution of financial losses due to unexpected changes in the credit quality of a counter-party in a financial agreement. But there is no clear agreement in the literature about the appropriate approach to the pricing of derivatives subject to credit risk. In the risk-neutral method, there will be more then one martingale measures and there have been different approaches introduced in literature to choose an appropriate martingale measure, like the minimal entropy martingale measure or the minimal-martingale measure. Bielecki et al. (2004b) have presented the mean-variance hedging framework and studied the indifference price approach for pricing defaultable claims in the situation when perfect hedging is not possible. Biagini and Cretarola (2007) has used minimal-martingale measure approach to price the defaultable contingent claims. They applied the local risk-minimization approach to defaultable claims and compared it with intensity-based evaluation formulas and the mean-variance hedging. Biagini and Cretarola (2008) studied the local risk-minimization approach for defaultable claims withe random recovery at default time. It is not clear what the appropriate pricing measure should be, for credit risk, or even whether such a pricing measure can actually be found. Even if an appropriate risk-neutral martingale measure exists, the H-hypothesis may not be stable under the measure transformation. To overcome all these problems, we consider the defaultable contingent claim in a framework that is beyond the standard risk-neutral approach.

In this chapter, our goal is to price and hedge defaultable contingent claims, using benchmark approach under the reduced form methodology of credit risk modelling. Credit risk is the risk associated with any kind of credit-linked event, such as changes in the credit quality, variations of credit spreads and the default event. Here, we consider only the default event for our modelling of credit derivatives, i.e., the default risk. A default risk is a possibility that a counterparty in a financial contract will not fulfill a contractual commitment to meet its obligation stated in the contract. If default occurs, the creditor will only receive the amount recovered from the debtor, called recovery payment. The recovery payment is frequently specified by the recovery rate, i.e., the fraction of the payoff of the contingent claim in case of default. There are two kinds of recovery payoffs: one recovery payoff Z at the time of default if the default occurs prior to or at the maturity date T, and a recovery payoff at time T if default occurs prior to or at the maturity date T. Naturally, only one of the recovery payoffs occur in real markets. To use a more general model, we consider both together. Default is modelled by a stochastic process with an exogenous default intensity as hazard rate which we consider to be modulated by a Markov process X.

6.1 The market model

In this chapter, we consider a financial market with only one discounted stock price process S satisfying the SDE, with $S_0 > 0$,

$$dS_t = S_t(\mu(X_t)dt + \sigma(X_t)dW_t), \tag{6.1}$$

with a Brownian motion $W = \{W_t\}_{t\geq 0}$, independent of X. Here, μ and $\sigma > 0$ are uniformly bounded, square-integrable in $[0,\ T]$, deterministic functions of X. The Markov process X is assumed to have the same representation as in Chapter 5.

We consider the defaultable state of a firm as described by the process $\mathcal{H} = \{\mathcal{H}_t\}_{t\geq 0}$ with $\mathcal{H}_t = I_{\{\tau \leq t\}}$, where the F-stopping time $\tau : \Omega \to \mathbb{R} \vee \{+\infty\}$ is a non-negative random variable indicating the default time of the firm, and is defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Here, I is an indicator function taking the value 1 if the event happens, otherwise 0. For convenience, we assume that $P\{\tau = 0\} = 0$ and $P\{\tau > t\} > 0$ for any $t \in \mathbb{R}_+$. The model is driven by a factor process $X = \{X_t\}_{t\geq 0}$. The default time τ has an F-adapted intensity $\lambda(X_t)$, where $\lambda(\cdot)$ is a given bounded, deterministic function. We suppose that the investor has the filtration $F = F^{S,X,\mathcal{H}}$. Then, $M_t^{\mathcal{H}} = \mathcal{H}_t - \int_0^{t\wedge\tau} \lambda(X_s)ds$ is an F-martingale and assumed to be orthogonal to M^X, the martingale part of X. In this setup, the process $(X, \mathcal{H})$ is jointly Markov. Here S is a F-semimartingale.

We analyze the market under the consideration that all the factors that drive the market are observable. Let $\mathcal{T}$ stands for the (right-continuous) cumulative conditional distribution function of τ for given information about the stock price process S and Markov process X up to the present time, i.e., $\mathcal{T}_t = P\{\tau \leq t|\mathcal{F}_t^{S,X}\}$ for every $t \in \mathbb{R}_+$. Recall that the filtration $F^{S,X}$ is generated by the stock price process and the Markov process. The survival function G of τ is defined by the formula: $\mathrm{G}_t = 1 - \mathcal{T}_t = P\{\tau > t|\mathcal{F}_t^{S,X}\}$ for every $t \in \mathbb{R}_+$. We also introduce the $\mathbb{R}_+ \vee \{+\infty\}$-valued hazard process $\Gamma_t = -\log\{G_t\}$. We assume that the F-hazard process Γ admits the following representation

$$\Gamma_t = \int_0^t \lambda(X_s)ds, \qquad t \in [0\ T].$$

In the following section, we assume that the default process is independent of W. In the subsequent section, we analyze the market under the consideration that the stock price process and the default time are dependent with each other.

6.2 Pricing and hedging

In this section, we suppose that $W = \{W_t\}_{t\geq 0}$ is independent of $\mathcal{H}$. The market participants can form self-financing portfolios with primary security accounts as constituents. The unique GOP, $V_t^{\pi^*}$, for the model under consideration satisfies the SDE

$$dV_t^{\pi^*} = V_t^{\pi^*}\theta(X_t)(\theta(X_t)dt + dW_t), \tag{6.2}$$

for all $t \geq 0$, with $V_0^{\pi^*} = 1$ (see, Chapter 3).

In the following section, we use the reduced form model for pricing the derivatives in the presence of credit risk, which are also known as hazard rate models or intensity-based models.

6.2.1 Reduced form valuations

Elliott et al. (2000), Frey and McNeil (2001), Bielecki and Rutkowski (2002) and many others have used the reduced form methodology for pricing the contingent claims. The main tool in this approach is an exogenous specification of the conditional probability of default, given that the default has not yet occurred.

We consider the processes which helps to model the defaultable market as given below.

i. The promised F_T^S-measurable non-negative contingent claim H_T representing the firm's liabilities to be redeemed at time T, if there is no default prior to or at time T.

ii. The F_T-adapted recovery claim $\widetilde{H}_T$, which represents the recovery payoff received at time T, if default occurs prior to or at the maturity date T.

iii. The F^S-predictable recovery process $Z = \{Z_t\}_{t\geq 0}$, which specifies the recovery payoff at the time of default, if it occurs prior to or at the maturity date T.

If default occurs after time T, the promised claim H_T is paid in full at time T. Otherwise, depending on the adopted model, either the amount Z_τ is paid at default time τ, or the amount $\widetilde{H}_T$ is paid at the maturity date T or both. Most practical situations deal with only one type of recovery payoff, i.e., either $\widetilde{H}_T = 0$ or $Z = 0$. But, here we consider the general setting, i.e., simultaneously both the kinds of recovery payoffs exist, and thus the defaultable claims formally defined as a quadruple DCT=$(H_T, \widetilde{H}_T, Z, \tau)$.

The financial interpretation of the components of a defaultable claim that we are going to analyze is clear from the definition of total cash flow D_t up to time $t \in [0,\ T]$ given by

$$D_t = H_T I_{\{\tau>T\}} I_{\{t=T\}} + \widetilde{H}_T I_{\{\tau\leq t\}} I_{\{t=T\}} + Z_\tau I_{\{\tau\leq t\}}. \tag{6.3}$$

Definition 6.2.1. *The benchmarked value process* $\widehat{D} = \{\widehat{D}_t\}_{t\geq 0}$ *of the total cash flow* $D = \{D_t\}_{t\geq 0}$, *which is* $\mathcal{F}$*-measurable and at time* $t \in [0,\ T]$ *is given as*

$$\widehat{D}_t = \frac{H_T}{V_T^{\pi^*}} I_{\{\tau>T\}} I_{\{t=T\}} + \frac{\widetilde{H}_T}{V_T^{\pi^*}} I_{\{\tau\leq t\}} I_{\{t=T\}} + \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{\tau\leq t\}}. \tag{6.4}$$

The benchmarked value of the cash flow in the time interval $[t,\ T]$ is given by

$$\widehat{D}_{[t,\ T]} = \widehat{D}_T - \widehat{D}_t = \frac{H_T}{V_T^{\pi^*}} I_{\{\tau>T\}} + \frac{\widetilde{H}_T}{V_T^{\pi^*}} I_{\{\tau\leq t\}} + \int_t^T \frac{Z_u}{V_u^{\pi^*}} d\mathcal{H}_u = \widehat{D}_T - \int_0^t \frac{Z_u}{V_u^{\pi^*}} d\mathcal{H}_u, \tag{6.5}$$

where $\int_t^T \frac{Z_u}{V_u^{\pi^*}} d\mathcal{H}_u = \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{t<\tau\leq T\}} = \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{\tau\leq T\}} - \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{\tau\leq t\}}$ and $t < T$. From Definitions 1.7.4 and 6.2.1, we obtain the fair pricing formula as given below.

Proposition 6.2.1. *For the defaultable claim DCT=*$(H_T, \widetilde{H}_T, Z, \tau)$, *the fair price at time* $t \in [0,\ T]$ *is given by the fair pricing formula*

$$\Lambda_t = V_t^{\pi^*} \widehat{\Lambda}_t, \tag{6.6}$$

where the corresponding fair, benchmarked defaultable claim price process $\widehat{\Lambda} = \{\widehat{\Lambda}_t\}_{t\in[0,\ T]}$ *has the value*

$$\widehat{\Lambda}_t = E(\widehat{D}_{[t,\ T]} | \mathcal{F}_t) = E(\widehat{D}_T | \mathcal{F}_t) - \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{\tau\leq t\}}, \tag{6.7}$$

Proof. If we add the benchmarked fair price process $\widehat{\Lambda} = \{\widehat{\Lambda}_t\}_{t\in[0,\,T]}$ with the cash flow up to the given time, i.e., $\widehat{\Lambda}_t + \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{\tau\le t\}}$, it becomes an F-martingale. This implies that the Λ_t is fair price for the remaining cash flow $D_{[t,\,T]}$ at each time $t \in [0,\ T]$. □

The seller of claim always try to hedge the risk he is going face in future due to the market uncertainty. In the next subsection, we try to hedge the defaultable claim.

6.2.2 Hedging

As all the market factors cannot be traded in our model under consideration, we can not get fair, perfectly replicating trading strategy for every contingent claim in this market. Hence, the market is incomplete. Biagini and Cretarola (2007), for the first time, used the locally-risk-minimizing method to the pricing and hedging of defaultable derivatives under the risk-neutral measure. They considered the particular case of a default put option with random recovery rate and solved explicitly the problem of finding the pseudo-locally-risk-minimizing strategy and the portfolio with minimal cost with the minimal-martingale measure. We try to find a portfolio "with minimal cost" that perfectly replicates DCT according to the locally-risk-minimizing criterion under the real world probability measure.

Definition 6.2.2. *For the payment stream $D = \{D_t\}_{t\in[0,\,T]}$ given by (6.3), the cumulative cost process $\mathbf{C}^D = \{\mathbf{C}_t^D\}_{t\in[0,\,T]}$ of a strategy π is*

$$\mathbf{C}_t^D = D_t + V_t^\pi - \int_0^t \frac{V_s^\pi \pi_s}{S_s} dS_s. \tag{6.8}$$

π is called self-financing if $\mathbf{C}^D$ is constant and mean-self-financing if $\mathbf{C}^D$ is an F-martingale. π is called 0-achieving if $V_T^\pi = 0$. The risk process $R^\pi(D) = \{R_t^\pi(D)\}_{t\in[0,\,T]}$ of π is defined as

$$\mathbf{R}_t^\pi(D) = E((\mathbf{C}_T^D - \mathbf{C}_t^D)^2 | \mathcal{F}_t).$$

Here, 0-achieving means the claim is hedgeable, since the terminal payoff goes inside the cash flow D. From Definition 6.2.2 and (6.7) and the martingale property of the benchmarked, fair, replicating wealth process, we directly obtain the following result.

Proposition 6.2.2. *For the given payment stream D, if $V^{\pi}(D)$ is a replicating wealth process then it has the value*

$$V_t^{\pi}(D) = \Lambda_t,$$

at time $t \in [0,\ T)$, with Λ_t satisfying the fair pricing formula (6.6).

Now the question is how to reduce the intrinsic risk which arises due to the incompleteness of the market. To get such strategy we derive the locally-risk-minimizing hedging strategy. First we define the Föllmer-Schweizer decomposition of the defaultable claim.

Definition 6.2.3. *An $\mathcal{F}_T$-measurable random variable D_T admits a Follmer-Schweizer decomposition if it can be written as*

$$D_T = D_0 + \int_0^T \phi_s^D dS_s + L_T^D, \quad P-a.s., \tag{6.9}$$

where D_0 is $\mathcal{F}_0$-measurable, $\phi^D = \{\phi_t^D\}_{t\geq 0}$ is such that the corresponding $\pi^D = \{\pi_t^D = \frac{\phi_t^D S_t}{V_t^{\pi}(D)}\}_{t\geq 0}$ is in $U(T)$, and the process $L^D = \{L_t^D\}_{t\geq 0}$ is a right-continuous square-integrable martingale null at 0 and strongly orthogonal to W.

Using the above definition and the definition of the 0-achieving strategy, we get the following result.

Proposition 6.2.3. *A payment stream D admits a locally-risk-minimizing strategy if and only if D_T admits a Föllmer-Schweizer decomposition. In that case, the locally-risk-minimizing strategy π is given by*

$$\pi_t^1 = \frac{\phi_t^D S_t}{V_t^{\pi}(D)}, \quad t \in [0,\ T],$$

with

$$V_t^{\pi}(D) = D_0 + \int_0^t \phi_s^D dS_s + L_t^D - D_t \tag{6.10}$$

and the mean-self-financing cost process is

$$\mathbf{C}_t^D = D_0 + L_t^D.$$

In the next proposition, we derive representation for an F-martingale process.

Let us consider a 0-achieving strategy π for D, with cost process $\mathbf{C}^D$ satisfying the SDE

$$dV_t^\pi = V_t^\pi \pi_t^1 \frac{dS_t}{S_t} + d\mathbf{C}_t^D - dD_t.$$

It's benchmarked value is

$$d\widehat{V}_t^\pi = \widehat{V}_t^\pi(\pi_t^1 \sigma(X_t) + \theta(X_t))dW_t + \frac{d\mathbf{C}_t^D}{V_{t-}^{\pi^*}} - \frac{dD_t}{V_{t-}^{\pi^*}}. \tag{6.11}$$

Before getting the Föllmer-Schweizer decomposition, we need the following result.

Proposition 6.2.4. *Let K be an $\mathcal{F}_T$-measurable and square-integrable random variable. Then the F-martingale $M^K = \{M_t^K = E(K|\mathcal{F}_t)\}_{t\in[0,\,T]}$ admits the following representation*

$$M_t^K = M_0^K + \int_0^t \xi_u^K dW_u + \int_0^t \zeta_u^K dM_u^X + \int_0^t \gamma_u^K dL_u,$$

where ξ^K, ζ^K and γ^K are square-integrable, F-predictable stochastic processes and W, M^X and $L_t = (1-\mathcal{H}_t)e^{\Gamma_t}$ are F-martingales. Moreover, M^X and L are strongly orthogonal with W.

Proof. Since $\mathcal{F}_T = \mathcal{F}_T^{\mathcal{H}} \vee \mathcal{F}_T^{S,X}$, from page-160 Bielecki and Rutkowski (2002), we consider the $\mathcal{F}_T$-measurable random variable of the form $K = (1-\mathcal{H}_s)Z$ for some $s \le T$ and some $\mathcal{F}_T^{S,X}$-measurable random variable Z. We can write

$$K = (1-\mathcal{H}_s)Z = (1-\mathcal{H}_s)e^{\Gamma_s}\widetilde{Z} = L_s\widetilde{Z},$$

where $\widetilde{Z} = \frac{Z}{e^{\Gamma_s}}$ is an $\mathcal{F}_T^{S,X}$-measurable, integrable random variable and $L_s = (1-\mathcal{H}_s)e^{\Gamma_s}$. Here $L = \{L_t\}_{t\in[0,T]}$ is F-martingale (see, Bielecki and Rutkowski (2002), P-152). From the martingale representation theorem, the square-integrable $F^{S,X}$-martingale can be written as

$$U_t = E(\widetilde{Z}|\mathcal{F}_t^{S,X}) = E(\widetilde{Z}) + \int_0^t \xi_u dW_u + \int_0^t \zeta_u dM_u^X,$$

for some square-integrable $\mathcal{F}_T^{S,X}$-predictable processes ξ and ζ. Now, from Itô's formula,

$$\begin{aligned} K &= L_0U_0 + \int_0^T L_{t-}dU_t + \int_0^s U_{t-}dL_t + [L,U]_s \\ &= L_0U_0 + \int_0^T L_{t-}\xi_t dW_t + \int_0^T L_{t-}\zeta_t dM_t^X + \int_0^T U_{t-}I_{[0,s]}dL_t. \end{aligned}$$

Here $[L,U]_s = 0$, since L and U are orthogonal. So the asserted formula holds,

$$M_t^K = E(K|\mathcal{F}_t) = M_0^K + \int_0^t \xi_s^K dW_s + \int_0^t \zeta_s^K dM_s^X + \int_0^t \gamma_s^K dL_s,$$

with the square-integrable processes $\xi_t^K = \xi_t L_{t-}$, $\zeta_t^K = \zeta_t L_{t-}$ and $\gamma_t^K = U_{t-}I_{[0,s]}$. The strong orthogonality follows from the independency of M^X and L with W. $\square$

From (6.7) and Preposition 6.2.4, there exist square-integrable, F-predictable processes Φ_1, Φ_2, and Φ_3 such that

$$\widehat{\Lambda}_t + \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{\tau \le t\}} = \widehat{\Lambda}_0 + \int_0^t \Phi_u^1 dW_u + \int_0^t \Phi_u^2 dM_u^X + \int_0^t \Phi_u^3 dL_u. \tag{6.12}$$

That is

$$\widehat{\Lambda}_t = \widehat{\Lambda}_0 + \int_0^t \Phi_u^1 dW_u + \int_0^t \Phi_u^2 dM_u^X + \int_0^t \Phi_u^3 dL_u + \int_0^t \frac{Z_u}{V_u^{\pi^*}} d\mathcal{H}_u \tag{6.13}$$

and

$$d\widehat{\Lambda}_t = \Phi_t^1 dW_t + \Phi_t^2 dM_t^X + \Phi_t^3 dL_t + \frac{Z_t}{V_t^{\pi^*}} d\mathcal{H}_t. \tag{6.14}$$

If we compare (6.11) and (6.14), we get the required values of risk-minimizing hedging strategy π and the mean-self-financing cost process $\mathbf{C}^D$ given by

$$\pi_t^1 = \frac{\frac{\Phi_t^1}{\widehat{V}_t} - \theta(X_t)}{\sigma_t}$$

and

$$d\mathbf{C}_t^D = V_{t-}^{\pi^*}(\Phi_t^2 dM_t^X + \Phi_t^3 dL_t),$$

where C is strongly orthogonal to W.

Now, with some assumptions, we try to derive the predictable terms Φ^i, for $i \in \{1,2,3\}$, more explicitly. From (6.4), we have

$$E(\widehat{D}_T|\mathcal{F}_t) = E(\widehat{H}_T(1-\mathcal{H}_T) + \widehat{\widehat{H}}_T\mathcal{H}_T + \widehat{Z}_\tau\mathcal{H}_T|\mathcal{F}_t), \tag{6.15}$$

where we have assumed the recovery payoff received at time T of the form $\widehat{\widehat{H}}_T = h(\tau\wedge T)\widehat{H}_T$, with $h(\tau\wedge T) \in [0,\ 1)$ is the recovery rate at the maturity and is a fraction of the claim which has to be paid at maturity if there is no default prior to or at maturity T and λ is a deterministic function of time. Then

$$\begin{aligned}
E(\widehat{D}_T|\mathcal{F}_t) &= E(\widehat{H}_T((1-\mathcal{H}_T) + h(\tau\wedge T)\mathcal{H}_T)|\mathcal{F}_t) + E(\widehat{Z}_\tau\mathcal{H}_T|\mathcal{F}_t)\\
&= E(\widehat{H}_T|\mathcal{F}_t)E(1 + (h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t) + E(\widehat{Z}_\tau\mathcal{H}_T|\mathcal{F}_t)\\
&= \Theta_t^1\Theta_t^2 + E(\widehat{Z}_\tau\mathcal{H}_T|\mathcal{F}_t).
\end{aligned}$$

Here, $E(\widehat{H}_T|\mathcal{F}_t)$ is an $F^{S,X}$-adapted martingale and admits the representation for some square integrable $F^{S,X}$-predictable processes ξ and ζ

$$\Theta_t^1 = E(\widehat{H}_T|\mathcal{F}_t) = c + \int_0^t \xi_s dW_s + \int_0^t \zeta_s dM_s^X. \tag{6.16}$$

We can write $1+(h(\tau\wedge T)-1)\mathcal{H}_T = f(\tau)$ for some integrable Borel function $f:\mathbb{R}_+\to[0,1]$. By Proposition 4.3.1 of Bielecki and Rutkowski (2002) and, since λ is considered to be deterministic function of time and default processes is independent of W, we have

$$\Theta_t^2 = E(1+(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t) = E(1+(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t^{\mathcal{H}}) = c_h + \int_0^t \widetilde{f}(s)dM_s^{\mathcal{H}}, \tag{6.17}$$

where $c_h = E(f(\tau))$ and the function $\widetilde{f}:\mathbb{R}_+\to\mathbb{R}$ is given by the formula

$$\widetilde{f}(t) = f(t) - e^{\Gamma_t}E(I_{\{\tau>T\}}f(\tau)).$$

Since

$$d[\Theta^1, \Theta^2]_t = 0,$$

applying Itô's formula, we get

$$\begin{aligned} d\Theta_t^1\Theta_t^2 =& \Theta_t^2 d\Theta_t^1 + \Theta_t^1 d\Theta_t^2 + d[\Theta_t^1, \Theta_t^2] \\ =& \left(c_h + \int_0^t \widetilde{f}(s) dM_s^{\mathcal{H}}\right)(\xi_t dW_t + \zeta_t dM_t^X) \\ &+ \left(c + \int_0^t \xi_s dW_s + \int_0^t \zeta_s dM_s^X\right)(\widetilde{f}(t) dM_t^{\mathcal{H}}) \\ =& \chi_t^1 dW_t + \chi_t^2 dM_t^X + \chi_t^3 dM_t^{\mathcal{H}}, \end{aligned} \tag{6.18}$$

where

$$\chi_t^1 = \left(c_h + \int_0^t \widetilde{f}(s) dM_s^{\mathcal{H}}\right)\xi_t,$$

$$\chi_t^2 = \left(c_h + \int_0^t \widetilde{f}(s) dM_s^{\mathcal{H}}\right)\zeta_t,$$

$$\chi_t^3 = \left(c + \int_0^t \xi_s dW_s + \int_0^t \zeta_s dM_s^X\right)\widetilde{f}(t).$$

It remains to find the representation for $E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t)$. From corollary 5.1.2, Bielecki and Rutkowski (2002), it can be decomposed as

$$\begin{aligned} E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t) &= \mathcal{H}_t E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t^{S,X} \vee \mathcal{F}_T^{\mathcal{H}}) + (1-\mathcal{H}_t)e^{\Gamma_t} E((1-\mathcal{H}_t)\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t^{S,X}) \\ &= \mathcal{H}_t \widehat{Z}_\tau + (1-\mathcal{H}_t)e^{\Gamma_t} E\left(\int_t^T \widehat{Z}_u e^{-\Gamma_u} d\Gamma_u|\mathcal{F}_t^{S,X}\right) \\ &= \mathcal{H}_t \widehat{Z}_\tau + I_{\{\tau>t\}} Q_t, \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} Q_t &= e^{\Gamma_t} E\left(\int_t^T \widehat{Z}_u e^{-\Gamma_u} d\Gamma_u|\mathcal{F}_t^{S,X}\right) \\ &= e^{\Gamma_t}\left(E\left(\int_0^T \widehat{Z}_u e^{-\Gamma_u} d\Gamma_u|\mathcal{F}_t^{S,X}\right) - \int_0^t \widehat{Z}_u e^{-\Gamma_u} d\Gamma_u\right). \end{aligned}$$

Then

$$\begin{aligned} dQ_t &= Q_t d\Gamma_t + e^{\Gamma_t}(dm_t - \widehat{Z}_t e^{-\Gamma_t} d\Gamma_t) \\ &= (Q_t - \widehat{Z}_t) d\Gamma_t + e^{\Gamma_t} dm_t, \end{aligned}$$

with $F^{S,X}$-martingale $m_t = E(\int_0^T \widehat{Z}_u e^{-\Gamma_u} d\Gamma_u | \mathcal{F}_t^{S,X})$ and the quadratic covariation term

$$\left[e^{\Gamma_t}, \left(E\left(\int_0^T \widehat{Z}_u e^{-\Gamma_u} d\Gamma_u | \mathcal{F}_t^{S,X}\right) - \int_0^t \widehat{Z}_u e^{-\Gamma_u} d\Gamma_u\right)\right] = 0,$$

due to the representation of Γ. Therefore, we have

$$Q_t = m_0 + \int_0^t e^{\Gamma_s} dm_s + \int_0^t (Q_s - \widehat{Z}_s) d\Gamma_s.$$

Furthermore, since Q is a right-continuous process with left-limit and with assumption $\Delta Q_\tau = 0$, we have

$$\begin{aligned} I_{\{\tau > t\}} Q_t &= m_0 + \int_0^{t\wedge\tau} dQ_u - I_{\{\tau \le t\}} Q_\tau \\ &= m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s} dm_s + \int_0^{t\wedge\tau} (Q_s - \widehat{Z}_s) d\Gamma_s - \int_0^t Q_{s-} d\mathcal{H}_s \\ &= m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s} dm_s + \int_0^{t\wedge\tau} (Q_{s-} - \widehat{Z}_s) d\Gamma_s - \int_0^t Q_{s-} d\mathcal{H}_s \\ &= m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s} dm_s - \int_0^t Q_{s-} dM_s^{\mathcal{H}} - \int_0^{t\wedge\tau} \widehat{Z}_s d\Gamma_s, \end{aligned}$$

where $M_t^{\mathcal{H}} = \mathcal{H}_t - \int_0^{t\wedge\tau} \lambda_s ds = \mathcal{H}_t - \int_0^{t\wedge\tau} d\Gamma_s$, and we have used the continuity of Γ. To get the predictable representation we need to consider the continuity of Q at default time τ. Consequently, we can rewrite (6.19) as follows:

$$\begin{aligned} E[\widehat{Z}_\tau \mathcal{H}_T | \mathcal{F}_t] &= m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s} dm_s - \int_0^t Q_{s-} dM_s^{\mathcal{H}} - \int_0^{t\wedge\tau} \widehat{Z}_s d\Gamma_s + \mathcal{H}_t \widehat{Z}_\tau \\ &= m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s} dm_s + \int_0^t (\widehat{Z}_s - Q_{s-}) dM_s^{\mathcal{H}}. \end{aligned} \tag{6.20}$$

As m is an $F^{S,X}$-martingale, it will have the decomposition

$$m_t = m_0 + \int_0^t \xi_s' dW_s + \int_0^t \zeta_s' dM_s^X, \tag{6.21}$$

for some $F^{S,X}$-predictable processes ξ' and ζ'. Hence

$$E[\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t] = m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s}\xi'_s dW_s + \int_0^{t\wedge\tau} e^{\Gamma_s}\zeta'_s dM_s^X + \int_0^t (\widehat{Z}_s - Q_{s-})dM_s^{\mathcal{H}}. \tag{6.22}$$

Finally, the decomposition of $E(\widehat{D}|\mathcal{F}_t)$ is

$$\begin{aligned} E(\widehat{D}_T|\mathcal{F}_t) =& \Theta_0^1\Theta_0^2 + m_0 + \int_0^t (\chi_s^1 + I_{\{\tau\geq s\}}e^{\Gamma_s}\xi'_s)dW_s + \int_0^t (\chi_s^2 + I_{\{\tau\geq s\}}e^{\Gamma_s}\zeta'_s)dM_s^X \\ &+ \int_0^t (\chi_s^3 + (\widehat{Z}_s - Q_{s-}))dM_s^{\mathcal{H}}. \end{aligned} \tag{6.23}$$

Theorem 6.2.1. *The 0-achieving locally-risk-minimizing portfolio for DCT is given by*

$$V_t^\pi = \int_0^t \Phi_u^1 dS_u + \mathbf{C}_t^D - D_t, \tag{6.24}$$

where the locally-risk-minimizing strategy is

$$\Phi_t^1 = \frac{V_t^\pi \pi_t^1}{S_t},$$

with

$$\pi_t^1 = \frac{\frac{(\chi_t^1 + I_{\{\tau\geq t\}}e^{\Gamma_t}\xi'_t)}{\widehat{V}_t^\pi} - \theta(X_t)}{\sigma_t},$$

and the mean-self-financing cost is

$$\mathbf{C}_t^D = \mathbf{C}_0^D + \int_0^t V_{s-}^{\pi^*}(\chi_s^2 + I_{\{\tau\leq s\}}e^{\Gamma_s}\zeta'_s)dM_s^X + \int_0^t V_{s-}^{\pi^*}(\chi_s^3 + (\widehat{Z}_s - Q_{s-}))dM_s^{\mathcal{H}}.$$

Proof. If we compare the equations (6.11) and (6.23), we get the required result. □

6.3 The dependent case

In the previous section, we have investigated the locally-risk-minimizing strategy under the assumption that the default time and the stock price process are independent. In this section, we consider the dependent case where the dynamics of the risky asset price may be influenced by the occurring of a default event and also the default time itself may depend on the asset price behavior. To get the locally-risk-minimizing hedging strategy for the defaultable claim under these considerations, again, we provide the Föllmer-Schweizer decomposition of it.

If we enlarge a filtration, it is not necessary that a martingale in the smaller one remains a martingale in the enlarged one. But the representation of the hazard rate Γ implies that an invariance of martingale property (i.e., the H-hypothesis) holds, which means that the dynamics of the asset prices are the same in the default-free world and in the defaultable world. In the case of risk-neutral measure transformation, the H-hypothesis may not be stable. But, as we are using a numeraire method instead of a measure transformation, we are dealing with only the real world probability measure. So, in a benchmark approach, there is no need to check for the stability of the H-hypothesis. Thus, in the present case, we have the H-hypothesis as given below.

(H) Any $F^{S,X}$-square-integrable martingale is an F-square-integrable martingale.

To replicate the fair price process $\widehat{\Lambda}_t = \frac{\Lambda_t}{V_t^{\tau^*}} = E(\widehat{D}_{[t,\,T]}|\mathcal{F}_t) = E(\widehat{D}_T|\mathcal{F}_t) - \frac{Z_\tau}{V_\tau^{\tau^*}} I_{\{\tau \le t\}}$, we try again to derive $E(\widehat{D}_T|\mathcal{F}_t)$ explicitly.

$$\begin{aligned} E(\widehat{D}_T|\mathcal{F}_t) &= E(\widehat{H}_T((1-\mathcal{H}_T) + h(\tau\wedge T)\mathcal{H}_T)|\mathcal{F}_t) + E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t) \\ &= E(\widehat{H}_T|\mathcal{F}_t) + E(\widehat{H}_T(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t) + E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t). \end{aligned} \tag{6.25}$$

Now, we compute $E(\widehat{H}_T(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t)$.

$$\begin{aligned}
E(\widehat{H}_T(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t) =&\mathcal{H}_t E(\widehat{H}_T(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t^{S,X}\vee\mathcal{F}_T^{\mathcal{H}})+\\
&(1-\mathcal{H}_t)e^{\Gamma_t}E((1-\mathcal{H}_t)\widehat{H}_T(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t^{S,X})\\
=&\mathcal{H}_t E(\widehat{H}_T(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t^{S,X}\vee\mathcal{F}_T^{\mathcal{H}})+\\
&(1-\mathcal{H}_t)e^{\Gamma_t}E(I_{\{t<\tau\le T\}}\widehat{H}_T(h(\tau\wedge T)-1)|\mathcal{F}_t^{S,X})\\
=&\mathcal{H}_t(h(\tau\wedge T)-1)\mathcal{H}_T E(\widehat{H}_T|\mathcal{F}_t^{S,X}\vee\mathcal{F}_T^{\mathcal{H}})+\\
&(1-\mathcal{H}_t)e^{\Gamma_t}E(E(I_{\{t<\tau\le T\}}\widehat{H}_T(h(\tau\wedge T)-1)|\mathcal{F}_{\tau-}^{S,X})|\mathcal{F}_t^{S,X})\\
=&\mathcal{H}_t(h(\tau\wedge T)-1)\mathcal{H}_T E(\widehat{H}_T|\mathcal{F}_t^{S,X}\vee\mathcal{F}_T^{\mathcal{H}})+\\
&(1-\mathcal{H}_t)e^{\Gamma_t}E(E(I_{\{t<\tau\le T\}}\widehat{H}_T(h(\tau\wedge T)-1)|\mathcal{F}_{\tau-})|\mathcal{F}_t^{S,X})\\
=&\mathcal{H}_t(h(\tau\wedge T)-1)\mathcal{H}_T E(\widehat{H}_T|\mathcal{F}_t^{S,X}\vee\mathcal{F}_T^{\mathcal{H}})+\\
&(1-\mathcal{H}_t)e^{\Gamma_t}E(I_{\{t<\tau\le T\}}(h(\tau\wedge T)-1)E(\widehat{H}_T|\mathcal{F}_{\tau-})|\mathcal{F}_t^{S,X})
\end{aligned} \tag{6.26}$$

Since $\mathcal{F}_{\tau-}^{S,X}=\mathcal{F}_{\tau-}$ and the F-stopping time τ is $\mathcal{F}_{\tau-}$-measurable by Theorem 5.6 (on page 118) of Dellacherie and Meyer (1978) and Lemma 5.1.3 of Bielecki and Rutkowski (2002). Here

$$\mathcal{F}_{\tau-}^{S,X}=\sigma(A\cap\{\tau>t\},A\in\mathcal{F}_t^{S,X},0\le t\le T).$$

From page-148 Bielecki and Rutkowski (2002), it can be written as

$$(h(\tau\wedge T)-1)E(\widehat{H}_T|\mathcal{F}_{\tau-})=N_\tau,$$

for some $F^{S,X}$-predictable process N and on the event $\{\tau\le t\}$, the following equality holds:

$$\mathcal{H}_tN_\tau=\mathcal{H}_t(h(\tau\wedge T)-1)E(\widehat{H}_T|\mathcal{F}_t^{S,X}\vee\mathcal{F}_T^{\mathcal{H}}). \tag{6.27}$$

Hence (6.26) can be written as

$$E(\widehat{H}_T(h(\tau\wedge T)-1)\mathcal{H}_T|\mathcal{F}_t)=\mathcal{H}_tN_\tau+(1-\mathcal{H}_t)e^{\Gamma_t}E(I_{\{t<\tau\le T\}}N_\tau|\mathcal{F}_t^{S,X}).$$

The conditional expectation $E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t)$ can be decomposed as

$$\begin{aligned} E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t) &= \mathcal{H}_t E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t^{S,X} \vee \mathcal{F}_T^{\mathcal{H}}) + (1-\mathcal{H}_t)e^{\Gamma_t}E((1-\mathcal{H}_t)\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t^{S,X}) \\ &= \mathcal{H}_t\widehat{Z}_\tau + (1-\mathcal{H}_t)e^{\Gamma_t}E((1-\mathcal{H}_t)\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t^{S,X}), \end{aligned}$$

since on the set $\{\tau \leq t\}$, $E(\widehat{Z}_\tau \mathcal{H}_T|\mathcal{F}_t^{S,X} \vee \mathcal{F}_T^{\mathcal{H}}) = \widehat{Z}_\tau$. So, (6.25) can be written as

$$\begin{aligned} E(\widehat{D}|\mathcal{F}_t) =& E(\widehat{H}_T|\mathcal{F}_t) + \mathcal{H}_t N_\tau + \\ & (1-\mathcal{H}_t)e^{\Gamma_t}E(I_{\{t<\tau\leq T\}}N_\tau|\mathcal{F}_t^{S,X}) + \mathcal{H}_t\widehat{Z}_\tau \\ & + (1-\mathcal{H}_t)e^{\Gamma_t}E((1-\mathcal{H}_t)\widehat{Z}_\tau\mathcal{H}_T|\mathcal{F}_t^{S,X}) \\ =& E(\widehat{H}_T|\mathcal{F}_t) + \mathcal{H}_t N_\tau + \mathcal{H}_t\widehat{Z}_\tau \\ & + (1-\mathcal{H}_t)e^{\Gamma_t}E(I_{\{t<\tau\leq T\}}(N_\tau + \widehat{Z}_\tau)|\mathcal{F}_t^{S,X}) \\ =& E(\widehat{H}_T|\mathcal{F}_t) + \mathcal{H}_t N_\tau + \mathcal{H}_t\widehat{Z}_\tau + I_{\{\tau>t\}}Q_t, \end{aligned} \tag{6.28}$$

where

$$\begin{aligned} Q_t &= e^{\Gamma_t}E\left(\int_t^T (N_u + \widehat{Z}_u)e^{-\Gamma_u}d\Gamma_u|\mathcal{F}_t^{S,X}\right) \\ &= e^{\Gamma_t}\left(E\left(\int_0^T (N_u + \widehat{Z}_u)e^{-\Gamma_u}d\Gamma_u|\mathcal{F}_t^{S,X}\right) - \int_0^t (N_u + \widehat{Z}_u)e^{-\Gamma_u}d\Gamma_u\right). \end{aligned}$$

Then

$$\begin{aligned} dQ_t &= Q_t d\Gamma_t + e^{\Gamma_t}(dm_t - (N_t + \widehat{Z}_t)e^{-\Gamma_t}d\Gamma_t) \\ &= (Q_t - (N_t + \widehat{Z}_t))d\Gamma_t + e^{\Gamma_t}dm_t, \end{aligned} \tag{6.29}$$

where m is an $F^{S,X}$-martingale with $m_t = E(\int_0^T (N_u+\widehat{Z}_u)e^{-\Gamma_u}d\Gamma_u|\mathcal{F}_t^{S,X})$, and the quadratic covariation term $\left[e^{\Gamma_t}, \left(E\left(\int_0^T (N_u + \widehat{Z}_u)e^{-\Gamma_u}d\Gamma_u|\mathcal{F}_t^{S,X}\right) - \int_0^t (N_u + \widehat{Z}_u)e^{-\Gamma_u}d\Gamma_u\right)\right] = 0$, due to the representation of Γ considered initially. Therefore, we have

$$Q_t = m_0 + \int_0^t e^{\Gamma_s}dm_s + \int_0^t (Q_s - (N_s + \widehat{Z}_s))d\Gamma_s. \tag{6.30}$$

Again with assumption $\Delta Q_\tau = 0$, we have the following

$$\begin{aligned} I_{\{\tau>t\}}Q_t &= m_0 + \int_0^{t\wedge\tau} dQ_u - I_{\{\tau\le t\}}Q_\tau \\ &= m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s}dm_s + \int_0^{t\wedge\tau}(Q_s - (N_s + \widehat{Z}_s))d\Gamma_s - \int_0^t Q_{s-}d\mathcal{H}_s \\ &= m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s}dm_s + \int_0^{t\wedge\tau}(Q_{s-} - (N_s + \widehat{Z}_s))d\Gamma_s - \int_0^t Q_{s-}d\mathcal{H}_s \\ &= m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s}dm_s - \int_0^t Q_{s-}dM_s^{\mathcal{H}} - \int_0^{t\wedge\tau}(N_s + \widehat{Z}_s)d\Gamma_s. \end{aligned}$$

Consequently, we can rewrite (6.28) as follows:

$$\begin{aligned} E(\widehat{D}_T|\mathcal{F}_t) =& E(\widehat{H}_T|\mathcal{F}_t) + \mathcal{H}_tN_\tau + \mathcal{H}_t\widehat{Z}_\tau + m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s}dm_s \\ & - \int_0^t Q_{s-}dM_s^{\mathcal{H}} - \int_0^{t\wedge\tau}(N_s + \widehat{Z}_s)d\Gamma_s. \\ =& E(\widehat{H}_T|\mathcal{F}_t) + m_0 + \int_0^{t\wedge\tau} e^{\Gamma_s}dm_s + \int_0^t (N_s + \widehat{Z}_s - Q_{s-})dM_s^{\mathcal{H}} \\ =& E(\widehat{H}_T|\mathcal{F}_t) + m_0 + \int_0^t I_{\{\tau\ge s\}}e^{\Gamma_s}dm_s + \int_0^t (N_s + \widehat{Z}_s - Q_{s-})dM_s^{\mathcal{H}}. \end{aligned}$$

Since m is an $F^{S,X}$-martingale, it will have the decomposition

$$m_t = m_0 + \int_0^t \xi_s' dW_s + \int_0^t \zeta_s' dM_s^X, \tag{6.31}$$

for some square-integrable, $F^{S,X}$-predictable processes ξ' and ζ'. As before

$$E(\widehat{H}_T|\mathcal{F}_t) = c + \int_0^t \xi_s dW_s + \int_0^t \zeta_s dM_s^X. \tag{6.32}$$

Finally, we have

$$\begin{aligned} E(\widehat{D}_T|\mathcal{F}_t) =& c + \int_0^t \xi_s dW_s + \int_0^t \zeta_s dM_s^X + m_0 + \int_0^t \xi_s' I_{\tau\ge s}e^{\Gamma_s}dW_s \\ & + \int_0^t I_{\{\tau\ge s\}}e^{\Gamma_s}\zeta_s' dM_s^X + \int_0^t (N_s + \widehat{Z}_s - Q_{s-})dM_s^{\mathcal{H}} \\ =& c + m_0 + \int_0^t (\xi_s + \xi_s' I_{\{\tau\ge s\}}e^{\Gamma_s})dW_s + \int_0^t (\zeta_s + I_{\{\tau\ge s\}}e^{\Gamma_s}\zeta_s')dM_s^X \\ & + \int_0^t (N_s + \widehat{Z}_s - Q_{s-})dM_s^{\mathcal{H}}. \end{aligned} \tag{6.33}$$

We state the following theorem for the locally-risk-minimizing hedging strategy from the above derived decomposition for the defaultable claims.

Theorem 6.3.1. *The 0-achieving locally-risk-minimizing portfolio for DCT is given by*

$$V_t^{\pi} = \int_0^t \Phi_u^1 dS_u + \mathbf{C}_t^D - D_t,$$

where the locally-risk-minimizing hedging strategy is

$$\Phi_t^1 = \frac{V_t^{\pi}\pi_t^1}{S_t},$$

with

$$\pi_t^1 = \frac{\frac{(\xi_t + \xi_t' I_{\{\tau \geq t\}} e^{\Gamma_t})}{\widehat{V}_t^{\pi}} - \theta(X_t)}{\sigma_t},$$

and the mean-self-financing cost process

$$\mathbf{C}_t^D = \mathbf{C}_0^D(\pi) + \int_0^t V_{s-}^{\pi^*}(\zeta_s + I_{\{\tau \geq s\}} e^{\Gamma_s} \zeta_s') dM_s^X + \int_0^t V_{s-}^{\pi^*}(N_s + \widehat{Z}_s - Q_{s-}) dM_s^{\mathcal{H}},$$

which is strongly orthogonal to the martingale part of S.

Proof. If we compare the equations (6.11) and (6.33), we get the required result. □

Thus, we have derived the local risk-minimizing hedging strategy for the case when the default time and the stock price process are dependent.

Chapter 7

A Defaultable Financial Market Under Incomplete Information

In intensity-based models, one specifies the default intensities that affect the default probabilities and the prices of credit derivatives. A critical point in this context is the modelling of the information set available to the market. Some of the information about the market are hidden, like default time which is totally inaccessible. To this effect, we propose a model in an incomplete information framework, that is, we assume that both the default intensities and the drift process of the stock price process are not observable.

Reduced-form models can be viewed as structural models analyzed under different information filtrations: Structural models are based on the information set available to the firm's management,which includes continuous-time observations of both the firm's asset value and liabilities; reduced form models are based on the information set available to the market, typically including only partial observations of both the firm's asset value and liabilities. As shown in examples by Duffie and Lando (2001), Collin-Dufresne et al. (2003), it is possible to transform a structural model with a predictable default time into a reduced form model, with a totally inaccessible default time, by introducing incomplete information. For instance, Duffie and Lando (2001) used a noisy and discretely observed firm asset value in a continuous-time model, while Collin-Dufresne et al. (2003) used a simple form of delayed information in a Brownian motion model.

This chapter is concerned with the pricing of credit derivatives in reduced form portfolio credit risk models under incomplete information. We consider models where the default intensity of the firms in a given portfolio is driven by an Ornstein-Uhlenbeck (OU) process X. We assume that X is not directly observable for investors trading in the market. We derive the non-linear filtering equation of the OU process X through innovation method. Using this continuously updating process about the unobservable process X, we reduce the model to one with complete information case. Finally, we define the fair pricing formula in the reduced model and derive the martingale representation of the claim to get the locally-risk-minimizing hedging strategy in the reduced model.

7.1 The market model

As in the previous chapter, we suppose that the defaultable state of the firm is described by the process $\mathcal{H} = \{\mathcal{H}_t\}_{t\geq 0}$ with $\mathcal{H}_t = I_{\{\tau\leq t\}}$, where $\tau : \Omega \to \mathbb{R} \vee \{+\infty\}$ is a non-negative random variable. For convenience, we assume that $P\{\tau = 0\} = 0$ and $P\{\tau > t\} > 0$ for any $t \in \mathbb{R}_+$. Here, the stopping time τ indicates the default time of the firm. Let $\mathcal{T}$ stands for the (right-continuous) cumulative conditional distribution function of τ for given information about the stock price process S and Ornstein-Uhlenbeck (OU) process X up to the present time, i.e., $\mathcal{T}_t = P\{\tau \leq t|\mathcal{F}_t^{S,X}\}$ for every $t \in \mathbb{R}_+$. The survival function G of τ is defined by the formula: $\mathrm{G}_t = 1 - \mathcal{T}_t = P\{\tau > t|\mathcal{F}_t^{S,X}\}$ for every $t \in \mathbb{R}_+$ and the $\mathbb{R}_+ \vee \{+\infty\}$-valued hazard process $\Gamma_t = -\log\{G_t\}$. The instantaneous rate of return of the stock and the intensity rate of the default is driven by a factor process $X = \{X_t\}_{t\geq 0}$, modelled as an OU process also known as a mean-reverting process.

$$dX_t = \kappa(\alpha - X_t)dt + \rho dW_t^1, \quad t \geq 0, \tag{7.1}$$

where $W^1 = \{W_t^1\}_{t\geq 0}$ is an $F^{S,X}$-standard Brownian motion, $\kappa \geq 0$ is the rate of mean reversion, $\alpha \geq 0$ is the long-term mean of the process and $\rho \geq 0$ is the volatility. The term mean-reverting means that a given process will continue to return to an average value over time, despite fluctuations. The Default time τ has the F-adapted intensity $\lambda(X_t)$, where $\lambda(\cdot)$ is a given deterministic function. Then, $M_t^{\mathcal{H}} = \mathcal{H}_t - \int_0^t \lambda^\tau(X_s)ds$ is an F-martingale, where $F = F^{S,X,\mathcal{H}}$ and $\lambda^\tau(X_t) = \lambda(X_t)I_{\{\tau \geq t\}}$.

In this chapter, we analyze a market under the consideration that X is unobservable and the market has a single stock price process S satisfying the SDE

$$dS_t = S_t(\mu(X_t)dt + \sigma_t dW_t), \tag{7.2}$$

with a standard Brownian motion $W = \{W_t\}_{t\geq 0}$, correlated with W^1, and with $S_0 > 0$. We assume that default time τ is independent of W. We assume also that $cov(W, W^1)_t = b_t$ for $t \in [0,\ T]$. Here, $\mu(\cdot)$ is a given square-integrable in $[0,\ T]$, deterministic bounded function of X and $\sigma_t > 0$ is a square-integrable in $[0,\ T]$, deterministic, bounded function of time. The filtration $F = F^{S,X,\mathcal{H}}$ is generated by the stock price process, OU process and the default process. But here, we work under the filtration $F^{S,\mathcal{H}}$, i.e., the filtration generated by the stock price process and the default process.

The process X is not directly observable for investor. Rather, investors' information consists of the default history, and of noisy price observation for traded credit products. For pricing and hedging defaultable claims under incomplete information, we reduce the model as if we are in complete information case. But, to do so, we need to know the dynamics of $\Pi_t(\Psi)$, where $\Psi \in \mathbf{B}(\mathbb{R})$, where $\mathbf{B}(\mathbb{R})$ is the space of bounded functions from $\mathbb{R}$ to $\mathbb{R}$, which is defined as

$$\Pi_t(\Psi) = E(\Psi(X_t)|\mathcal{F}_t^{S,\mathcal{H}}). \tag{7.3}$$

The computation of $\Pi_t(\Psi)$, for the present market model, leads to a non-linear filtering problem.

7.2 Reduction to complete information case

Let

$$dN_t = \mu(X_t)dt + \sigma_t dW_t. \tag{7.4}$$

That is, N is the stochastic logarithm of S. It is clear the $F^N = F^S$. By stochastic filtering theory, we estimate unobservable stochastic process, i.e., OU process, based on the observable information, i.e., from the stock price process and the default process. The filtering problem consists in determining the conditional distribution Π_t of the unobservable X at time t given the information accumulated from observing N and $\mathcal{H}$ in the interval $[0,\ t]$.

We deduce the evolution equation for Π. Before that, we prove lemmas required to evaluate the dynamics of Π_t for $t \in [0,\ T]$.

Lemma 7.2.1. *$\widetilde{W}_t = \int_0^t (dW_s + (\frac{\mu(X_s)-\Pi_s(\mu)}{\sigma_s})ds)$, for $t \in [0,\ T]$, is a Brownian motion and $\widetilde{M}_t^{\mathcal{H}} = \mathcal{H}_t - \int_0^t \Pi_s(\lambda^\tau)ds$, for $t \in [0,\ T]$, is a compensated hazard process with intensity $\Pi_t(\lambda)$ on the probability space $(\Omega,\ \mathcal{F}^{N,\mathcal{H}},\ F^{N,\mathcal{H}},\ P)$.*

Proof. The equation (7.4) can be written as

$$dN_t = \Pi_t(\mu)dt + \sigma_t d\widetilde{W}_t. \tag{7.5}$$

Since $\Pi_t(\mu)$ is $F^{N,\mathcal{H}}$-adapted, from the above equation, $\widetilde{W}$ is also $F^{N,\mathcal{H}}$-adapted. First we show that $\widetilde{W}_t$ is continuous martingale. Taking conditional expectation for the given information up to time $s \leq t$

$$\begin{aligned} E(\widetilde{W}_t \mid \mathcal{F}_s^{N,\mathcal{H}}) &= E(\widetilde{W}_t - \widetilde{W}_s + \widetilde{W}_s \mid \mathcal{F}_s^{N,\mathcal{H}}) \\ &= E(W_t - W_s + \int_s^t \Big(\frac{\mu(X_u) - \Pi_u(\mu)}{\sigma_u}\Big)du \mid \mathcal{F}_s^{N,\mathcal{H}}) + \widetilde{W}_s \\ &\qquad\qquad (\text{as } \widetilde{W} \text{ is } F^{N,\mathcal{H}}\text{-adapted }) \\ &= \widetilde{W}_s. \end{aligned} \tag{7.6}$$

From the definition of the Brownian motion the increment is independent of the past history and from tower property,

$$E(W_t - W_s \mid \mathcal{F}_s^{N,\mathcal{H}}) = E(W_t - W_s) = 0.$$

From tower property, we can write

$$E\Big(\int_s^t (\mu(X_u) - \Pi_u(\mu))du \mid \mathcal{F}_s^{N,\mathcal{H}}\Big) = E\Big(\int_s^t E((\mu(X_u) - \Pi_u(\mu)) \mid \mathcal{F}_u^{N,\mathcal{H}})du \mid \mathcal{F}_s^{N,\mathcal{H}}\Big) = 0.$$

So, $\widetilde{W}$ is a continuous martingale and its quadratic variation is given by

$$[\widetilde{W}]_t = [W]_t = t.$$

Hence, $\widetilde{W}$ is a Brownian motion, by Lévy's characterization of a Brownian motion.

It is now straightforward to see that $\widetilde{M}_t^{\mathcal{H}} = \mathcal{H}_t - \int_0^{t\wedge\tau} \Pi_s(\lambda^\tau)ds$ is a compensated hazard process with intensity $\Pi_t(\lambda^\tau)$ on the probability space $(\Omega,\ \mathcal{F}^{N,\mathcal{H}},\ F^{N,\mathcal{H}},\ P)$. □

We call $\widetilde{W} = \{\widetilde{W}_t\}_{t\in[0,\,T]}$ and $\widetilde{M}^{\mathcal{H}} = \{\widetilde{M}_t^{\mathcal{H}}\}_{t\in[0,\,T]}$ the innovation processes for W and $\mathcal{H}$ respectively (see, Bain and Crisan, 2009). The equation (7.2) can be rewritten on the probability space $(\Omega,\ \mathcal{F}^{N,\mathcal{H}},\ F^{N,\mathcal{H}},\ P)$ as

$$dS_t = S_t(\Pi_t(\mu)dt + \sigma_t d\widetilde{W}_t). \tag{7.7}$$

Denote by $F^{\widetilde{W},\widetilde{M}^{\mathcal{H}}} = \{\mathcal{F}_t^{\widetilde{W},\widetilde{M}^{\mathcal{H}}}\}_{t\in[0,\,T]}$ the filtration generated by the innovation process, i.e., $\mathcal{F}_t^{\widetilde{W},\widetilde{M}^{\mathcal{H}}} = \sigma(\widetilde{W}_s, \widetilde{M}_s^{\mathcal{H}}, s \in [0,\ t])$. Since $\widetilde{W}$ and $\widetilde{M}^{\mathcal{H}}$ are $F^{N,\mathcal{H}}$-adapted, from Lemma 7.2.1, we have $\mathcal{F}_t^{\widetilde{W},\widetilde{M}^{\mathcal{H}}} \subseteq \mathcal{F}_t^{N,\mathcal{H}}$. But the converse relation is not true in general. However, the next lemma is true.

Lemma 7.2.2. *If $\zeta = \{\zeta_t\}_{t\in[0,\,T]}$ is a square-integrable $F^{N,\mathcal{H}}$-martingale, then it admits a representation of the form*

$$d\zeta_t = h_t d\widetilde{W}_t + k_t d\widetilde{M}_t^{\mathcal{H}},$$

where h and k are some square-integrable $F^{N,\mathcal{H}}$-predictable processes.

Proof. We have $dN_t = \Pi_t(\mu)dt + \sigma_t d\widetilde{W}_t$, where $\widetilde{W}$ is a $F^{N,\mathcal{H}}$-Brownian motion and $\widetilde{M}^{\mathcal{H}}$ is a compensated $F^{N,\mathcal{H}}$-hazard process under the probability measure P. Now we consider a measure transformation under which $\overline{W}_t = \int_0^t \frac{\Pi_s(\mu)}{\sigma_s}ds + \widetilde{W}_t$ will be an $F^{N,\mathcal{H}}$-Brownian motion. The Radon-Nikodým derivative of the measure transformation is given by

$$\frac{d\overline{P}}{dP}\bigg|_{\mathcal{F}_t^{N,\mathcal{H}}} = O_t, \quad t \in [0,\ T],$$

where $O_t = \exp\{-\int_0^t \frac{\Pi_s(\mu)}{\sigma_s}d\widetilde{W}_s - \frac{1}{2}\int_0^t |\frac{\Pi_s(\mu)}{\sigma_s}|^2 ds\}$. So $dN_t = \sigma_t d\overline{W}_t$ is an $F^{N,\mathcal{H}}$-martingale under the measure $\overline{P}$. As in Chapter 4, it can be shown that $F^{N,\mathcal{H}} = F^{\overline{W},\mathcal{H}}$. Since ζ is $F^{N,\mathcal{H}}$-martingale under P, $\overline{\zeta} = \frac{\zeta}{O} = \{\frac{\zeta_t}{O_t}\}_{t\in[0,\ T]}$ will be $F^{N,\ mathcalH}$-martingale under $\overline{P}$. From martingale representation theorem, there exist $\overline{f}$ and $\overline{g}$, which are square-integrable and $F^{N,\mathcal{H}}$-predictable processes, such that

$$d\overline{\zeta}_t = \overline{f}_t d\overline{W}_t + \overline{g}_t d\widetilde{M}_t^{\mathcal{H}}.$$

Notice that $\zeta = \overline{\zeta}O$. Applying Itô's formula, one verifies for all $t \in [0,\ T]$

$$\begin{aligned} d\zeta_t =& O_t\overline{f}_t\frac{\Pi_t(\mu)}{\sigma_t}dt + O_t\overline{f}_t d\widetilde{W}_t + O_t\overline{g}_t d\widetilde{M}_t^{\mathcal{H}} - \zeta_t\frac{\Pi_t(\mu)}{\sigma_t}d\widetilde{W}_t - O_t\overline{f}_t\frac{\Pi_t(\mu)}{\sigma_t}dt \\ =& f_t d\widetilde{W}_t + g_t d\widetilde{M}_t^{\mathcal{H}}, \end{aligned}$$

where $h_t = O_t\overline{f}_t - \zeta_t\frac{\Pi_t(\mu)}{\sigma_t}$ and $k_t = O_t\overline{g}_t$. Moreover, it is evident that $f = \{f_t\}_{t\in[0,T]}$ and $g = \{g_t\}_{t\in[0,T]}$ are square-integrable, $F^{N,\mathcal{H}}$-predictable. □

Now by using the above derived results, we prove the following lemma.

Lemma 7.2.3. *1. $\Pi_t(X) - \Pi_0(X) - \int_0^t \Pi_s(\kappa(\alpha - X_s))ds$ is a square-integrable $F^{N,\mathcal{H}}$-martingale.*

2. There exist square-integrable, $F^{N,\mathcal{H}}$-predictable processes f and g such that

$$\Pi_t(X) = \Pi_0(X) + \int_0^t \Pi_s(\kappa(\alpha - X_s))ds + \widetilde{U}_t, \tag{7.8}$$

where $\widetilde{U}_t = \int_0^t f_s d\widetilde{W}_s + \int_0^t g_s d\widetilde{M}_s^{\mathcal{H}}$.

Here, $\Pi_t(\kappa(\alpha - X_t)) = E[\kappa(\alpha - X_t) \mid \mathcal{F}_t^{N,\mathcal{H}}]$ for all $t \in [0,\ T]$.

Proof. We have

$$X_t = X_0 + \int_0^t \kappa(\alpha - X_s)ds + \rho W_t^1.$$

If we take conditional expectation with respect to the observable filtration $F^{N,\mathcal{H}}$

$$\Pi_t(X) = E(X_0 \mid \mathcal{F}_t^{N,\mathcal{H}}) + E\Big(\int_0^t \kappa(\alpha - X_s)ds \mid \mathcal{F}_t^{N,\mathcal{H}}\Big) + E(\rho W_t^1 \mid \mathcal{F}_t^{N,\mathcal{H}}).$$

Now, if we subtract $\Pi_0(X) + \int_0^t \Pi_s(\kappa(\alpha - X_s))ds$ on both sides we get

$$\begin{aligned}\Pi_t(X) - \Pi_0(X) - \int_0^t \Pi_s(\kappa(\alpha - X_s))ds =& E(X_0 \mid \mathcal{F}_t^{N,\mathcal{H}}) - \Pi_0(X) + E(\int_0^t \kappa(\alpha - X_s)ds \mid \mathcal{F}_t^{N,\mathcal{H}}) \\ & - \int_0^t \Pi_s(\kappa(\alpha - X_s))ds + E(\rho W_t^1 \mid \mathcal{F}_t^{N,\mathcal{H}}).\end{aligned} \tag{7.9}$$

First, we show that $E\Big(\int_0^t \kappa(\alpha - X_s)ds \mid \mathcal{F}_t^{N,\mathcal{H}}\Big) - \int_0^t \Pi_s(\kappa(\alpha - X_s))ds$ and $E(\rho W_t^1 \mid \mathcal{F}_t^{N,\mathcal{H}})$ are $F^{N,\mathcal{H}}$-martingales. For $u \le t$,

$$\begin{aligned}& E\Big(E(\int_0^t \kappa(\alpha - X_s)ds \mid \mathcal{F}_t^{N,\mathcal{H}}) - \int_0^t \Pi_s(\kappa(\alpha - X_s))ds \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) \\ =& E\Big(E(\int_u^t \kappa(\alpha - X_s)ds \mid \mathcal{F}_t^{N,\mathcal{H}}) - \int_u^t \Pi_s(\kappa(\alpha - X_s))ds \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) + \\ & E\Big(E(\int_0^u \kappa(\alpha - X_s)ds \mid \mathcal{F}_t^{N,\mathcal{H}}) - \int_0^u \Pi_s(\kappa(\alpha - X_s))ds \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) \\ =& E\Big(\int_u^t \kappa(\alpha - X_s)ds \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) - E\Big(\int_u^t \Pi_s(\kappa(\alpha - X_s))ds \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) + \\ & E\Big(\int_0^u \kappa(\alpha - X_s)ds \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) - E\Big(\int_0^u \Pi_s(\kappa(\alpha - X_s))ds \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) \\ =& E\Big(\int_0^u \kappa(\alpha - X_s)ds \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) - \int_0^u \Pi_s(\kappa(\alpha - X_s))ds,\end{aligned}$$

where we have used the tower property of the conditional expectation and $F^{N,\mathcal{H}}$-martingale nature follows. Again, for $u \le t$,

$$\begin{aligned}E\Big(E(\rho W_t^1 \mid \mathcal{F}_t^{N,\mathcal{H}}) \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) &= E(\rho W_t^1 \mid \mathcal{F}_u^{N,\mathcal{H}}) \\ &= E(\rho W_u^1 \mid \mathcal{F}_u^{N,\mathcal{H}}) + E(\rho W_t^1 - \rho W_u^1 \mid \mathcal{F}_u^{N,\mathcal{H}}) \\ &= E(\rho W_u^1 \mid \mathcal{F}_u^{N,\mathcal{H}}) + E\Big(E(\rho W_t^1 - \rho W_u^1 \mid \mathcal{F}_u) \mid \mathcal{F}_u^{N,\mathcal{H}}\Big) \\ &= E(\rho W_u^1 \mid \mathcal{F}_u^{N,\mathcal{H}}),\end{aligned}$$

where we used the martingale property of W^1 with respect to the complete information filtration $\mathcal{F}$ and the tower property of the conditional expectation. Hence $E(\rho W_t^1|\mathcal{F}_t^{N,\mathcal{H}})$ is an $F^{N,\mathcal{H}}$-martingale. This implies that the right-hand-side of (7.9) is $F^{N,\mathcal{H}}$-martingale that starts from zero. Hence, from Lemma 7.2.2, there exist an $F^{N,\mathcal{H}}$-adapted and an $F^{N,\mathcal{H}}$-predictable processes f and g, respectively, such that

$$\Pi_t(X) - \Pi_0(X) - \int_0^t \Pi_s(\kappa(\alpha - X_s))ds = \int_0^t f_s d\widetilde{W}_s + \int_0^t g_s d\widetilde{M}_s^{\mathcal{H}} \tag{7.10}$$

for all $t \in [0,\ T]$. Hence the lemma follows. □

We have now sufficient ingredients to derive the non-linear filtering equation for the partially observed system.

Theorem 7.2.1. *The process Π_t satisfies the following evolution equation*

$$\begin{aligned}\Pi_t(X) =&\Pi_0(X) + \int_0^t \Pi_s(\kappa(\alpha - X_s))ds + \int_0^t (\rho b_s + \Pi_s(X_s\frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s}))d\widetilde{W}_s \\ &+ \int_0^t \frac{\Pi_s(X_s(\lambda^\tau(X_s) - \Pi_s(\lambda^\tau)))}{\Pi_s(\lambda^\tau)} d\widetilde{M}_s^{\mathcal{H}}.\end{aligned} \tag{7.11}$$

Proof. Let $\widetilde{Z}_t = \int_0^t \gamma_s d\widetilde{W}_s + \int_0^t \psi_s d\widetilde{M}_s^{\mathcal{H}}$, where $\{\gamma_t\}$ is any given bounded $F^{N,\mathcal{H}}$-adapted random process and $\{\psi_t\}$ is any given bounded $F^{N,\mathcal{H}}$-predictable random process. Then $\widetilde{Z}$ is an $F^{N,\mathcal{H}}$-martingale process. Using the value of $\widetilde{U}$ from (7.8),

$$\begin{aligned}E(\widetilde{U}_t\widetilde{Z}_t) &= E\int_0^t f_s\gamma_s ds + E\int_0^t g_s\psi_s d\mathcal{H}_s \\ &= E\int_0^t f_s\gamma_s ds + E\int_0^t \Pi_s(\lambda^\tau)g_s\psi_s ds.\end{aligned} \tag{7.12}$$

Again, from (7.8),

$$\begin{aligned}E(\widetilde{U}_t\widetilde{Z}_t) &= E(\widetilde{Z}_t\Pi_t(X)) - E(\widetilde{Z}_t\Pi_0(X)) - E\Big(\widetilde{Z}_t\int_0^t \Pi_s(\kappa(\alpha - X_s))ds\Big) \\ &= E(\widetilde{Z}_t\Pi_t(X)) - E\Big(\int_0^t E(\widetilde{Z}_t \mid \mathcal{F}_s^{N,\mathcal{H}})\Pi_s(\kappa(\alpha - X_s))ds\Big) \\ &= E(\widetilde{Z}_tX_t) - E\Big(\int_0^t \widetilde{Z}_s\kappa(\alpha - X_s)ds\Big). \qquad (\because\ \widetilde{Z}_t\Pi_t(X) = E(\widetilde{Z}_tX_t|\mathcal{F}_t^{N,\mathcal{H}}))\end{aligned} \tag{7.13}$$

Now

$$\begin{aligned}\widetilde{Z}_t &= \int_0^t \gamma_s dW_s + \int_0^t \psi_s(d\mathcal{H}_s - \lambda^\tau(X_s)ds) + \int_0^t \gamma_s \frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s} ds \\ &\quad + \int_0^t \psi_s(\lambda^\tau(X_s) - \Pi_s(\lambda^\tau))ds \\ &= Z_t + \int_0^t \gamma_s \frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s} ds + \int_0^t \psi_s(\lambda^\tau(X_s) - \Pi_s(\lambda^\tau))ds, \end{aligned} \tag{7.14}$$

where

$$Z_t = \int_0^t \gamma_s dW_s + \int_0^t \psi_s(d\mathcal{H}_s - \lambda^\tau(X_s)ds).$$

From (7.13) and (7.14), we have

$$\begin{aligned}E(\widetilde{U}_t\widetilde{Z}_t) =& E\Big(X_t Z_t - \int_0^t Z_s \kappa(\alpha - X_s)ds\Big) \\ &+ E\Big(X_t\Big(\int_0^t \gamma_s \frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s} ds + \int_0^t \psi_s(\lambda^\tau(X_s) - \Pi_s(\lambda^\tau))ds\Big)\Big) \\ &- E\Big(\int_0^t \kappa(\alpha - X_u)\Big(\int_0^u \gamma_s \frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s} ds + \int_0^u \psi_s(\lambda^\tau(X_s) - \Pi_s(\lambda^\tau))ds\Big)du\Big) \\ =& \int_0^t \gamma_s \rho b_s ds + E\Big(\int_0^t X_s \gamma_s \frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s} ds + \int_0^t X_s \psi_s(\lambda^\tau(X_s) - \Pi_s(\lambda^\tau))ds\Big) \\ &+ E\Big(\int_0^t (X_t - X_s)\gamma_s \frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s} ds + \int_0^t (X_t - X_s)\psi_s(\lambda^\tau(X_s) - \Pi_s(\lambda^\tau))ds\Big) \\ &- E\Big(\int_0^t \kappa(\alpha - X_u)\Big(\int_0^u \gamma_s \frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s} ds + \int_0^u \psi_s(\lambda^\tau(X_s) - \Pi_s(\lambda^\tau))ds\Big)du\Big)\end{aligned} \tag{7.15}$$

where

$$
\begin{aligned}
E\Big(X_tZ_t-\int_0^t Z_s\kappa(\alpha-X_s)ds\Big) &= E\Big(X_tZ_t-Z_t\int_0^t\kappa(\alpha-X_s)ds\Big)\\
&= E\Big(Z_t(X_t-\int_0^t\kappa(\alpha-X_s)ds)\Big)\\
&= E\Big(Z_t(X_t-X_0-\int_0^t\kappa(\alpha-X_s)ds)\Big)\\
&= E(Z_t\rho W_t^1)\\
&= E\Big(\rho\int_0^t\gamma_s dW_s\int_0^t dW_s^1+\int_0^t\psi_s(d\mathcal{H}_s-\lambda^\tau(X_s)ds)\int_0^t dW_s^1\Big)\\
&= E\Big(\rho\int_0^t\gamma_s dW_s\int_0^t dW_s^1\Big) \qquad (\because\ M^{\mathcal{H}} \text{ and } W^1 \text{ are orthogonal})\\
&= E\Big(\rho\int_0^t\gamma_s\frac{d\langle W_s,W_s^1\rangle}{ds}ds\Big)\\
&= E\Big(\int_0^t\gamma_s\rho b_s ds\Big) \qquad (\because\ EZ_tX_0=0),
\end{aligned}
\tag{7.16}
$$

(see, Situ (2005), P-187 for the covariation part). Now,

$$
\begin{aligned}
&E\Big(\int_0^t(X_t-X_s)\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds+\int_0^t(X_t-X_s)\psi_s(\lambda^\tau(X_s)-\Pi_s(\lambda^\tau))ds\Big)\\
&=E\int_0^t\Big(\int_s^t\kappa(\alpha-X_s)ds+\rho(W_t^1-W_s^1)\Big)\Big(\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}+\psi_s(\lambda^\tau(X_s)-\Pi_s(\lambda^\tau))\Big)ds\\
&=E\int_0^t(\int_s^t\kappa(\alpha-X_s)ds)\Big(\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}+\psi_s(\lambda^\tau(X_s)-\Pi_s(\lambda^\tau))\Big)ds\\
&=E\int_0^t\kappa(\alpha-X_u)\int_0^u\Big(\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}+\psi_s(\lambda^\tau(X_s)-\Pi_s(\lambda^\tau))\Big)dsdu,
\end{aligned}
\tag{7.17}
$$

where we have used $E(W_t^1-W_s^1|\mathcal{F}_s)=0$. Therefore,

$$
E(\widetilde{U}_t\widetilde{Z}_t)=E\Big(\int_0^t\gamma_s\rho b_s ds+\int_0^t X_s\gamma_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}ds+\int_0^t X_s\psi_s(\lambda^\tau(X_s)-\Pi_s(\lambda^\tau))ds\Big).
\tag{7.18}
$$

Subtracting (7.18) from (7.12), we get

$$
\begin{aligned}
E\Bigg(\int_0^t\gamma_s\Big(f_s-E\Big(\rho b_s+X_s\frac{(\mu(X_s)-\Pi_s(\mu))}{\sigma_s}\mid\mathcal{F}_s^{N,\mathcal{H}}\Big)\Big)\\
+\int_0^t\psi_s\Big(g_s\Pi_s(\lambda^\tau)-E\Big(X_s(\lambda^\tau(X_s)-\Pi_s(\lambda^\tau)(X_s))\mid\mathcal{F}_s^{N,\mathcal{H}}\Big)\Big)\Bigg)=0.
\end{aligned}
$$

Since the above equality is true for any given bounded $F^{N,\mathcal{H}}$-adapted random process $\{\gamma_s\}$ and any given bounded $F^{N,\mathcal{H}}$-predictable random process $\{\psi_s\}$, we get

$$f_s = E(\rho b_s + X_s \frac{(\mu(X_s) - \Pi_s(\mu))}{\sigma_s} \mid \mathcal{F}_s^{N,\mathcal{H}})$$
$$g_s = E\Big(\frac{X_s(\lambda^\tau(X_s) - \Pi_t(\lambda^\tau)(X_s))}{\Pi_s(\lambda^\tau)} \mid \mathcal{F}_s^{N,\mathcal{H}}\Big).$$

Hence the theorem follows. □

The stock price process with complete observation is now characterized by

$$dS_t = S_t(\Pi_t(\mu)dt + \sigma_t d\widetilde{W}_t), \tag{7.19}$$

with $S_0 = s_0$,

$$\begin{aligned}\Pi_t(X) =&\Pi_0(X) + \int_0^t \Pi_s(\kappa(\alpha - X_s))ds + \int_0^t \int_0^t (\rho b_s + \Pi_s(X_s \frac{(\mu(X_s) - \Pi_s(\mu)))}{\sigma_s})d\widetilde{W}_s \\ &+ \int_0^t \frac{\Pi_s(X_s(\lambda^\tau(X_s) - \Pi_t(\lambda^\tau)))}{\Pi_s(\lambda^\tau)} d\widetilde{M}_s^{\mathcal{H}},\end{aligned} \tag{7.20}$$

and $\Pi_0(X) = x$.

The positive, self-financing portfolio for $\pi \in U[0,\ T]$ is characterized by

$$dV_t^\pi = V_t^\pi(\pi_t^1 \Pi_t(\alpha)dt + \pi_t^1 \sigma_t d\widetilde{W}_t). \tag{7.21}$$

Here $\pi = \{\pi^0, \pi^1\}$, and its components indicates the ratio of wealth invested in the riskfree asset and in the risky asset respectively. Now, we define the GOP according to the model with complete observation.

Definition 7.2.1. *A self-financing portfolio $\pi^* \in U[t,\ T]$ is called GOP if it has a positive wealth process V^{π^*} such that*

$$E_{t,\varrho,v} \log V_T^{\pi^*} \geq E_{t,\varrho,v} \log V_T^{\pi} \qquad a.s.,$$

for all self-financing $\pi \in U[t,\ T]$ with positive wealth process V^π and for all $s \in [t,\ T]$, where $E_{t,\varrho,v}(\cdot) = E(\cdot \big| V_t^\pi = v,\ \Pi_t(X) = \varrho)$ indicates the conditional expectation under P given complete information till time $t \in [0,\ T]$.

The reduced model is now with complete observation. As in Chapter 3, the unique GOP V^{π^*} for the market with incomplete information satisfies the SDE

$$dV_t^{\pi^*} = V_t^{\pi^*}\Pi_t(\theta)(\Pi_t(\theta)dt + d\widetilde{W}_t), \tag{7.22}$$

for all $t \geq 0$ with $V_0^{\pi^*} = 1$. Here $\Pi_t(\theta) = E(\theta(X_t)|\mathcal{F}_t^{N,\mathcal{H}}) = E(\frac{\mu(X_t)}{\sigma_t}|\mathcal{F}_t^{N,\mathcal{H}})$ is the market price of risk under the incomplete information.

Elliott et al. (2000), Frey and McNeil (2001), Bielecki and Rutkowski (2004) and many others have used the reduced form methodology for pricing the contingent claims. The main tool in this approach is an exogenous specification of the conditional probability of default, given that default has not yet occurred.

In this chapter, as in the previous chapter, we assume that both the kinds of recovery payoffs can exist simultaneously, and the defaultable claim is formally defined as a quadruple DCT=$(H, \widetilde{H}_T, Z, \tau)$. The total cash flow D_t up to time $t \in [0,\ T]$ is

$$D_t = H_T I_{\{\tau>T\}} I_{\{t=T\}} + \widetilde{H}_T I_{\{\tau\leq t\}} I_{\{t=T\}} + Z_\tau I_{\{\tau\leq t\}}. \tag{7.23}$$

Definition 7.2.2. *The benchmarked value $\widehat{D} = \{\widehat{D}_t\}_{t\geq 0}$ of the total cash flow $D = \{D_t\}_{t\geq 0}$, which is $F^{N,\mathcal{H}}$-measurable and at time $t \in [0,\ T]$ is given as*

$$\widehat{D}_t = \frac{H_T}{V_T^{\pi^*}} I_{\{\tau>T\}} I_{\{t=T\}} + \frac{\widetilde{H}_T}{V_T^{\pi^*}} I_{\{\tau\leq t\}} I_{\{t=T\}} + \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{\tau\leq t\}}. \tag{7.24}$$

The benchmarked value of the cash flow in the time interval $[t,\ T]$ is given by

$$\widehat{D}_{[t,\ T]} = \widehat{D}_T - \widehat{D}_t = \frac{H_T}{V_T^{\pi^*}} I_{\{\tau>T\}} + \frac{\widetilde{H}_T}{V_T^{\pi^*}} I_{\{\tau\leq t\}} + \int_t^T \frac{Z_u}{V_u^{\pi^*}} d\mathcal{H}_u = \widehat{D}_T - \int_0^t \frac{Z_u}{V_u^{\pi^*}} d\mathcal{H}_u. \tag{7.25}$$

Now we obtain the fair pricing formula for the cash flow.

Corollary 7.2.1. *For the defaultable claim DCT=$(H_T, \widetilde{H}_T, Z, \tau)$, the fair price at time $t \in [0,\ T]$ is given by the fair pricing formula*

$$P_t = V_t^{\pi^*}\widehat{P}_t, \tag{7.26}$$

where the corresponding fair, benchmarked defaultable claim price process $\widehat{P} = \{\widehat{P}_s\}_{s\in[0,\,T]}$ *has at time* $t \in [0,\ T]$ *the value*

$$\widehat{P}_t = \frac{P_t}{V_t^{\pi^*}} = E(\widehat{D}_{[t,\,T]}|\mathcal{F}_t^{N,\mathcal{H}}) = E(\widehat{D}|\mathcal{F}_t^{N,\mathcal{H}}) - \frac{Z_\tau}{V_t^{\pi^*}} I_{\{\tau\le t\}}, \tag{7.27}$$

If we add the benchmarked fair price process $\widehat{P} = \{\widehat{P}_s\}_{s\in[0,\,T]}$ with the cash flow up to the given time, i.e., $\widehat{P}_t + \frac{Z_\tau}{V_t^{\pi^*}} I_{\{\tau\le t\}}$, it becomes an F-martingale. This implies that P is fair price for the remaining cash flow $D_{[t,T]}$ at each time $t \in [0, T]$.

7.3 Hedging

In this section, we find a portfolio "with minimal cost" that perfectly replicates DCT according to the locally-risk-minimizing criterion under the real world probability measure.

Definition 7.3.1. *For the payment stream* $D = \{D_t\}_{t\in[0,\,T]}$ *given by (7.23) the cumulative cost process* $\mathbf{C}^D = \{\mathbf{C}_t^D\}_{t\in[0,\,T]}$ *of a strategy* π *is*

$$\mathbf{C}_t^D = D_t + V_t^\pi - \int_0^t \frac{V_s^\pi \pi_s}{S_s} dS_s. \tag{7.28}$$

π *is called self-financing if* $\mathbf{C}^D$ *is constant and mean-self-financing if* $\mathbf{C}^D$ *is* F*-martingale.* π *is called* 0*-achieving if* $V_T^\pi = 0$*. The risk process* $\mathbf{R}^D(\pi) = \{\mathbf{R}_t^D(\pi)\}_{t\in[0,\,T]}$ *of* π *is defined as*

$$\mathbf{R}_t^D(\pi) = E((\mathbf{C}_T^D - \mathbf{C}_t^D)^2|\mathcal{F}_t^{N,\mathcal{H}}).$$

From Definition 7.3.1 and (7.26) and the martingale property of the benchmarked, fair, replicating wealth process, we directly obtain the following result.

Proposition 7.3.1. *For the given payment stream* D*, if* $V^\pi(D)$ *is the replicating wealth process then*

$$V_t^\pi(D) = P_t,$$

at time $t \in [0,\ T]$*, with* P_t *satisfying the fair pricing formula (7.26).*

The question now is how to reduce the intrinsic risk which arises due to the incompleteness of the market.

The quantity $\mathbf{C}_t^D$ describes the cumulative cost on $[0,\ t]$ with payment stream according to D and trading strategy according to π (for more details, see Schweizer (2008)). Schweizer (2008) has proved that, if the mean-variance tradeoff process is continuous, which is true in this model, then the existence of locally-risk-minimizing strategy for D is equivalent to a strategy π which is 0-achieving and mean-self-financing and the cost process $\mathbf{C}^D$ is strongly orthogonal to $\widetilde{W}$. We know that finding locally-risk-minimizing strategy is equivalent to finding the Föllmer-Schweizer decomposition of the claim, which is defined as follows.

Definition 7.3.2. *An $\mathcal{F}_T^{N,\mathcal{H}}$-measurable random variable D_T admit a Föllmer-Schweizer decomposition if it can be written as*

$$D_T = D_0 + \int_0^T \phi_s^D dS_s + L_T^D, \quad P - a.s., \tag{7.29}$$

where D_0 is $\mathcal{F}_0^{N,\mathcal{H}}$-measurable, $\phi^D = \{\phi_t^D\}_{t\geq 0}$ is such that the corresponding $\pi^D = \{\pi_t^D = \frac{\phi_t^D S_t}{V_t^\pi(D)}\}_{t\geq 0}$ is in U, and the process $L^D = \{L_t^D\}_{t\geq 0}$ is a right-continuous square-integrable martingale null at time 0 and strongly orthogonal to $\widetilde{W}$.

Proposition 7.3.2. *A payment stream D admits a locally-risk-minimizing strategy if only if D_T admits a Föllmer-Schweizer decomposition. In that case, the locally-risk-minimizing strategy π is given by*

$$\pi_t^1 = \frac{\phi_t^D S_t}{V_t^D}, \quad t \in [0,\ T],$$

with

$$V_t^\pi = D_0 + \int_0^t \phi_s^D dS_s + L_t^D - D_t \tag{7.30}$$

and the mean-self-financing cost process is

$$\mathbf{C}_t^D = D_0 + L_t^D.$$

Proof. Let us consider a 0-achieving strategy π for D, with cost process $\mathbf{C}^D$ satisfying the SDE

$$dV_t^\pi = V_t^\pi \pi_t^1 \frac{dS_t}{S_t} + d\mathbf{C}_t^D - dD_t.$$

It's benchmarked value is

$$d\widehat{V}_t^\pi = \widehat{V}_t^\pi(\pi_t^1\sigma_t + \Pi_t(\theta))d\widetilde{W}_t + \frac{dC_t^D}{V_t^{\pi^*}} - \frac{dD_t}{V_t^{\pi^*}}. \tag{7.31}$$

As the reduced form of incomplete information market is incomplete, we derive the locally-risk-minimizing hedging strategy. From (7.26) and Lemma (7.2.2) there exist square integrable, $F^{N,\mathcal{H}}$-predictable processes Φ^1 and Φ^2 such that

$$\widehat{P}_t + \frac{Z_\tau}{V_\tau^{\pi^*}} I_{\{\tau\le t\}} = \int_0^t \Phi_s^1 d\widetilde{W}_s + \int_0^t \Phi_s^2 d\widetilde{M}_s^{\mathcal{H}}. \tag{7.32}$$

That is,

$$\widehat{P}_t = \int_0^t \Phi_s^1 d\widetilde{W}_s + \int_0^t \Phi_s^2 d\widetilde{M}_s^{\mathcal{H}} + \int_0^t \frac{Z_u}{V_u^{\pi^*}} d\mathcal{H}_u \tag{7.33}$$

and

$$d\widehat{P}_t = \Phi_t^1 d\widetilde{W}_t + \Phi_t^2 d\widetilde{M}_t^{\mathcal{H}} + \frac{Z_t}{V_t^{\pi^*}} d\mathcal{H}_t. \tag{7.34}$$

If we compare (7.31) and (7.34), we will get the required values of risk-minimizing hedging strategy π and the mean-self-financing cost process $\mathbf{C}_t$ as given by

$$\pi_t^1 = \frac{\frac{\Phi_t^1}{\widehat{V}_t} - \Pi_t(\theta)}{\sigma_t},$$

$$dC_t^D = V_t^{\pi^*}\Phi_t^2 d\widetilde{M}_t^{\mathcal{H}}.$$

□

We have derived the locally-risk-minimizing hedging strategy for the defaultable contingent claim in an incomplete information market where parameters are modulated by a OU process. The strategies can be derived more explicitly as in the precious chapter, but to avoid the repetition we do not look for that part.

Chapter 8

Conclusion

The benchmark approach introduced by Platen (2002) is a powerful technique for option pricing and hedging in financial mathematics. We have used this technique in various complete and incomplete financial market models with partial as well as full information.

In this thesis, we have considered the Markov-modulated models. For each case, GOP has been derived using the HJB equation and is used as a numeraire in the benchmark approach. For every model, we used the fair pricing concept for pricing the contingent claims and derived the locally-risk-minimizing hedging strategy.

We have considered both European and American contingent claims with payoff rates along with the terminal payoffs. By introducing consumption processes, we derived the hedging strategies for these contingent claims. We have also discussed the cases where the parameters of the jump-diffusion model are modulated by a stochastic process. We derived the benchmarked PDE to make the option price more tractable through numerical methods.

Then, a Markov-modulated Brownian market is considered wherein the parameters are modulated by a finite state Markov process. We have proved the derived benchmarked fair price is the same as the risk-neutral price derived in the Di Masi et al. (1994). Further, a jump-diffusion model is considered with two stock price processes, where the parameters are modulated by an unobservable Markov process. We have obtained the Föllmer-Schweizer decomposition of the claim for case when the hidden information can be retrieved. And, for the case of incomplete information, the optimal projection of the Föllmer-Schweizer decomposition has been taken to get the hedging strategy.

Next, we have considered a jump-diffusion model with a single stock price process where the drift process and the jump-intensity process are modulated by an unobservable Markov process. Only the stock price process is assumed to be observable and it is not possible to retrieve the hidden information. Using the filtering theory, we derived the non-linear filtering equation for the hidden Markov process based on the given information. We can not derive the GOP naturally in the considered model. Hence, we have assumed a fictitious market where we have considered a non-tradable asset such that the market price of risk is not affected. The GOP derived in the fictitious market is then used to hedge the option in the actual market.

We have then taken a defaultable financial market model with a Markov-modulated stock price process and the hazard rate process. Both the recovery payoffs, one at the time of default and another at the maturity, are considered if the default occurs before the maturity date. The benchmarked cash flow has been defined for the defaultable claim. Martingale representation for the claim is derived and used to determine the hedging strategy.

Finally, a defaultable financial market model with incomplete information is considered. The drift parameter of the stock price process and the hazard rate process are modulated by an unobservable Ornstein-Uhlenbeck (OU) process. Again, based on the given information, the non-linear filtering equation is derived for the OU process. And, using the non-linear filtering equation, martingale representation is given for the defaultable claims.

Scope for future work

1. Constraints on portfolio amounts and constraints on number of shares of assets have not been considered under the benchmark approach. Sometimes, it is very difficult to get the growth optimal portfolio in a constraint portfolio. One may use the techniques by Cvitanic and Karatzas (1992, 1996) for the portfolio optimization with constraints to get the GOP.

2. In this thesis, we have not used any numerical techniques to obtain the values for the fair price. We can use the fast fourier transformation to calculate the expectation to get the explicit option prices. The benchmarked PDE can be derived for the each model, like in Chapter 2, and solved by using the numerical techniques like finite difference methods.

3. Except vanilla options, other options has not been analyzed much through the benchmark approach and this remains an area of further study.

4. The benchmark approach can also be extended to the financial markets with transaction costs.

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List of Publications Based on the Thesis

1. *Growth optimal portfolio for the unobservable Markov-modulated markets.* International Journal of Mathematics in Operational Research, 2011 (To appear).

2. *Pricing and hedging of financial derivatives under the real world measure.* Proceedings of International Conference on New Trends in Statistics and Optimization (ICONTSO08), 2008.

3. *Growth optimal portfolio for the unobservable Markov-modulated markets.* Electronic proceedings of International Conference on Operations Research Application in Engineering and Management (ICOREM09), 2009.

4. *A benchmark approach for pricing and hedging in an unobservable Markov-modulated model.* Communicated, 2011.

5. *Option pricing and hedging in a benchmark model with stochastic parameters.* To be communicated, 2011.

6. *Numeraire method for pricing and hedging of defaultable claims.* To be communicated, 2011.

7. *The growth optimal portfolio in a Markov-modulated Brownian market.* To be communicated, 2011.

8. *A benchmark approach for pricing and hedging of defaultable claims under incomplete information.* To be communicated, 2011.