Fractal Dimensions and Approximations of α-Fractal Interpolation Functions

by

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بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

Bismillahi Rahmani Rahim

In the name of Allah, Most Gracious, Most Merciful

CERTIFICATE

It is certified that the work contained in the thesis titled **"Fractal Dimensions and Approximations of α-Fractal Interpolation Functions"** by **Md Nasim Akhtar (11612310)**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

November, 2016

Dr. M. Guru Prem Prasad
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Dedicated

To

My Parents

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ٱلْحَمْدُ لِلَّهِ رَبِّ ٱلْعَٰلَمِينَ

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Abstract

A fractal set is a union of many smaller copy of itself and it has a highly irregular structure. Using Hutchinson's operator, Barnsley [6], introduced Fractal Interpolation Function (FIF) via certain Iterated Function System (IFS). The FIF is continuous and self-affine in nature. By defining IFS suitably, one can construct various form of fractal functions including non-self-affine and partially self-affine (and partially non-self-affine) FIFs. For any continuous function f, the corresponding fractal analogue f^α is non-self-affine, continuous, nowhere differentiable function [62,63]. The function f^α, known as α-fractal interpolation function, generates a family of continuous functions corresponding to the original function f for different scale vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in (-1, 1)^N$. The fractal function f^α interpolates and approximates f. Kolomogorov [51], introduced the notion of fractal dimension which later demonstrated by Mandelbrot [56] to quantify irregular patterns. The fractal dimension or box dimension measures the complexity or irregularity of sets. It is seen in the literature that the graphs of FIFs have non-integer Hausdorff-Besicovitch dimensions. The present work is mainly devoted to the study pertaining to fractal dimensions and approximations of α-fractal interpolation functions. It is organized in five chapters.

Chapter 1, is the introductory chapter reviewing the basics and some useful results on fractals related to our work. At the end, it also includes two preliminary works carried out by us. (i) Study on the perturbed bivariate fractal interpolation function (BFIF) arising from perturbed IFS and upper bound for the error estimation between the original BFIF and the perturbed BFIF [79]. (ii) Considering the graph-directed iterated function system (GDIFS) for a finite number of generalized data sets, it is shown that the projection of the attractors on $\mathbb{R}^2$ is the graph of the coalescence hidden variable fractal interpolation functions (CHFIFs) interpolating the corresponding data sets [2].

In Chapter 2, the α-fractal interpolation function f^α for an IFS where the original function f and the base function b are Hölderian with exponents β_1 and β_2 respectively, is considered. In classical case, there is a result on dimension of the graph of a function in Hölder class. It states that if a function f is Hölder continuous with exponent β on a compact subset of $\mathbb{R}$, then the dimension of the graph of f is bounded above by $2 - \beta$ [12]. Also different type of FIFs and non-integer box dimensions of their corresponding graphs can be found in the literature [9, 10, 24, 37, 40, 41, 58]. In the present work, the box dimension of the graph G of non-self-affine FIF f^α is settled as follows. In the first approach, for equally spaced nodes, consider $\gamma = \sum_{i=1}^{N} |\alpha_i|$, where the scaling factors $\alpha_i \neq 0$ and $\beta = \min\{\beta_1, \beta_2\}$. It is seen that (a) for $\gamma \leq 1$, $1 \leq \dim_B G \leq 2 - \beta$, (b) for $\gamma > 1$ with $\gamma N^{\beta-1} \leq 1$, $1 \leq \dim_B G \leq 2 - \beta + \log_N \gamma$, (c) for $\gamma > 1$ with $\gamma N^{\beta-1} > 1$, $1 \leq \dim_B G \leq 1 + \log_N \gamma$. To get a non-trivial lower bound, if the interpolation points are not collinear then for $\gamma = \sum_{i=1}^{N} \alpha_i > 1$, where $\alpha_i > 0$ and $\beta = 1$, $\dim_B G \geq 1 + \log_N \gamma$, whenever the original function f is concave and b is affine. In particular, for an IFS with positive scale vector if f is concave and Lipschitz, b is affine and the interpolation points are not collinear then $\dim_B G = 1 + \log_N \gamma$ for $\gamma > 1$ and $\dim_B G = 1$ otherwise. In the second approach, for arbitrary nodes, let D be the solution of $\sum_{i=1}^{N} |\alpha_i| a_i^{D-1} = 1$ and $\gamma = \sum_{i=1}^{N} |\alpha_i|$. It is proved that (a) for $\gamma > 1$,

$1 \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq 1 - \beta + D$, (b) for $\gamma \leq 1$, $1 \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq 2 - \beta$. To get a non-trivial lower bound, if the interpolation points are not collinear then for $\gamma = \sum_{i=1}^{N} \alpha_i > 1$, where $\alpha_i \geq 0$ and $\beta = 1$, $\dim_B G \geq D$, whenever the original function f is concave and b is affine. In particular, for an IFS with non-negative scale vector if f is concave and Lipschitz, b is affine and the interpolation points are not collinear, then $\dim_B G = D$ for $\gamma > 1$ and $\dim_B G = 1$ otherwise. By choosing the parameter $\alpha = 0$, the obtained results coincide with the existing classical one with $f^0 = f$. The research work contained in this chapter is published as [3].

In Chapter 3, the α-fractal function f^α with variable scale vector is considered for f and b in Hölderian class with exponent β_1 and β_2 respectively. The box dimension of the graph G of f^α is estimated in $I = [0, 1]$. The results matches with the results in Chapter 2 for constant scale vector. The fractal function f^α is non-self-affine. So the behavior of the graph of f^α is non-uniform on the domain. Keeping this in mind, the box dimension of the graph of f^α in subintervals is also estimated here. Let $\alpha_{\min} = \min\{\widetilde{\alpha}_i : i = 1, 2, \ldots, N\}$ and $\alpha_{\max} = \max\{\overline{\alpha}_i : i = 1, 2, \ldots, N\}$, where $\widetilde{\alpha}_i = \min_{x \in I} |\alpha_i(x)|$ and $\overline{\alpha}_i = \max_{x \in I} |\alpha_i(x)|$. Define $\overline{\gamma} = \sum_{i=1}^{N} \overline{\alpha}_i$. Let for fixed $i^* \in \{1, 2, \ldots, N\}$, $G_{i^*} = graph(f^\alpha|_{I_{i^*}})$ be the graph of the function f^α on the subinterval I_{i^*}. It is proved that (a) if $\alpha_{\min} > \frac{1}{N^\beta}$ ($\Rightarrow \overline{\gamma} N^{\beta - 1} > 1$), then $1 \leq \dim_B(G_{i^*}) \leq 1 + \log_N \overline{\gamma}$, (b) if $\overline{\gamma} N^{\beta-1} > 1$, then $1 \leq \dim_B(G_{i^*}) \leq 2 + \log_N \alpha_{\max}$, (c) if $\alpha_{\max} \leq \frac{1}{N^\beta}$, then $\dim_B(G_{i^*}) \leq 2 - \beta$ and (d) if $\overline{\alpha}_{i^*} = 0$, then $1 \leq \dim_B G_{i^*} \leq 2 - \beta$. To get a non-trivial lower bound for constant non-negative scale vector, let $\gamma = \sum_{i=1}^{N} \alpha_i > 1$ with $\alpha_{i^*} \neq 0$. If the interpolation points are not collinear, then $\dim_B(G_{i^*}) \geq 1 + \log_N \gamma$, whenever f is concave and b is affine. In particular, for an IFS with constant non-negative scale vector if f is concave and Lipschitz, b is affine and the interpolation points are not collinear then $\dim_B G_{i^*} = 1 + \log_N \gamma$ for $\alpha_{\min} > \frac{1}{N}$ and $\dim_B G_{i^*} = 1$ for $\alpha_{\max} \leq \frac{1}{N}$.

Jacobi system which forms a complete system in the weighted square integrable functions space $\mathcal{L}^2_\rho(-1, 1)$, where $\rho^{(r,s)}(x) = (1-x)^r(1+x)^s; r > -1, s > -1$ is the weight function is considered in Chapter 4. Legendre system, Chebyshev system, Gegenbauer system, etc., are special cases of Jacobi system for certain choices of the parameters r and s. By taking different type of classical functions systems, fractal analogues is defined and an alternative family of Hilbert bases are provided by Navascués [63, 66, 70, 71]. In the present work, the Jacobi system is fractalized by means of a linear and bounded operator which maps the original function to its fractal analogue. The Weierstrass type theorem providing an approximation for square integrable function in terms of α-fractal Jacobi sums is derived. A fractal basis for the space of weighted square integrable functions $\mathcal{L}^2_\rho(-1, 1)$ is found. The convergence analysis of Fourier-Jacobi expansion for affine FIF as well as non-affine smooth FIF for certain IFS is carried out in uniform norm and weighted-mean square norm. The research work contained in this chapter is published as [4].

Functions on the sphere $S \subseteq \mathbb{R}^3$ has important applications in the area of oceanography, environment, metrology, etc. In [64, 69], Navascués constructed fractal versions of the spherical harmonics. Navascués used the linearity and boundedness of the operator $\mathcal{S}^\alpha$ which takes the original function on $\mathcal{L}^2(S)$ to its fractal analogue, to prove fractal version of many classical results in functional analysis/operator theory. In Chapter 5, a family of continuous functions on the unit sphere $S \subseteq \mathbb{R}^3$ generalizing the spherical harmonics, is considered. Using fractal methodology, fractal version of this family of continuous functions on the sphere S is constructed. To do this, an iterated function system and a linear bounded operator that maps classical functions to its fractal analogues is defined. Some approximation properties of fractal functions on the sphere is investigated. Restricting the scale vector involved in the IFS, a fractal Hilbert basis is established for the functions on the sphere.

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Chapter 1

Introduction

An Iterated Function System (IFS) is the core to define the self-referential function, namely, Fractal Interpolation Function (FIF). A brief review on the basic concepts to define a fractal interpolation function is presented below.

1.1 Iterated function systems

Let $\mathcal{X} \subset \mathbb{R}^n$ and $(\mathcal{X}, d)$ be a complete metric space. Consider

$$\mathcal{H}(\mathcal{X}) = \{S \subset \mathcal{X} : S \text{ is a non-empty compact set in } \mathcal{X}\}$$

with the Hausdorff metric $d_{\mathcal{H}}(A, B)$, defined by

$$d_{\mathcal{H}}(A, B) = \max\{d_{\mathcal{X}}(A, B), d_{\mathcal{X}}(B, A)\} ,$$

where $d_{\mathcal{X}}(A, B) = \max_{x \in A} \min_{y \in B} d(x, y)$ for any two sets A and B in $\mathcal{H}(\mathcal{X})$. The completeness of the metric space $(\mathcal{X}, d)$ implies that $(\mathcal{H}, d_{\mathcal{H}})$ is complete [34]. Let for $i = 1, 2, \ldots, N$, $w_i : \mathcal{X} \to \mathcal{X}$ be continuous maps. Then $\{\mathcal{X}; w_i : i = 1, 2, \ldots, N\}$ is called an iterated function system (IFS). For each i, if the map w_i is contractive, that is,

$$d(w_i(x), w_i(y)) \leq \zeta_i d(x, y) \text{ for all } x \text{ and } y \text{ in } X,\ 0 \leq \zeta_i < 1 ,$$

then the IFS $\{\mathcal{X}; w_i : i = 1, 2, \ldots, N\}$ is known as a hyperbolic iterated function system with contractivity factor $\zeta = \max\{\zeta_i : i = 1, 2, \ldots, N\}$. The set valued

Hutchinson operator $W : \mathcal{H}(\mathcal{X}) \to \mathcal{H}(\mathcal{X})$ is defined by

$$W(B) = \bigcup_{i=1}^{N} w_i(B) \text{ for all } B \in \mathcal{H}(\mathcal{X}) \ ,$$

where $w_i(B) := \{w_i(b) : b \in B\}$ and is also contraction with the contractivity factor ζ [44]. Banach fixed point theorem ensures that there exists a unique set $A \in \mathcal{H}(\mathcal{X})$ such that

$$A = W(A) = \bigcup_{i=1}^{N} w_i(A) \ .$$

The set A is called the attractor associated with the IFS $\{\mathcal{X}; w_i : i = 1,\ 2,\ \ldots,\ N\}$. It is also called the invariant set of the IFS.

Definition 1.1.1. *Let $\{\mathcal{X}; w_i : i = 1,\ 2,\ \ldots,\ N\}$ be a hyperbolic IFS with contractivity factor ζ. A set $A \in \mathcal{H}(\mathcal{X})$ is said to be an attractor of the IFS or a deterministic fractal if $\lim_{n\to\infty} W^n(B) = A$ for every $B \in \mathcal{H}(\mathcal{X})$.*

1.2 Fractal interpolation functions

The fractal interpolation is an alternative of classical interpolation such as polynomial interpolation, spline interpolation, etc., in approximation theory generally for irregular curves. It effectively interpolates some experimental data by a non-smooth curve. In 1986, Barnsley [6], introduced the concept of fractal interpolation function via an iterated function system on a compact subset of $\mathbb{R}$. The construction is mainly by defining a suitable IFS such that the attractor of the IFS is the graph of a FIF. The FIF is the unique fixed point of Read-Bajraktarević (RB) operator acting on a suitable functions space. This operator was first studied by Read [82] and followed by Bajraktarević [5]. The general construction of an IFS and a FIF is as follows.

Let a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} \ : \ i = 0,\ 1,\ \ldots,\ N\}$ be given, where $\Delta : x_0 < x_1 < \cdots < x_N$ is a partition of the closed interval $I = [x_0,\ x_N]$ and $y_i \in [h_1, h_2] \subset \mathbb{R}$, $i = 0,\ 1,\ \ldots,\ N$. Set $I_i = [x_{i-1}, x_i]$ for $i = 1,\ 2,\ \ldots,\ N$ and

$K = I \times [h_1, h_2]$. Let $L_i : I \to I_i, i = 1,\ 2,\ \ldots, N$, be contraction homeomorphisms such that

$$L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i, \tag{1.1}$$

$$|L_i(c_1) - L_i(c_2)| \leq d|c_1 - c_2| \text{ for all } c_1 \text{ and } c_2 \text{ in } I, \tag{1.2}$$

for some $0 \leq d < 1$. Furthermore, let $F_i : K \to \mathbb{R}$, $i = 1,\ 2,\ \ldots, N$, be given continuous functions such that

$$F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i, \tag{1.3}$$

$$|F_i(x, y) - F_i(x^*, y)| \leq \gamma|x - x^*|,\ x, x^* \in I, y \in [h_1, h_2], \tag{1.4}$$

$$|F_i(x, \xi_1) - F_i(x, \xi_2)| \leq |\alpha_i||\xi_1 - \xi_2|,\ x \in I, \xi_1, \xi_2 \in [h_1, h_2] \tag{1.5}$$

for a constant γ and for $\alpha_i \in (-1, 1)$, $i = 1,\ 2,\ \ldots, N$. Define the mappings $W_i : K \to I_i \times \mathbb{R}$, $i = 1,\ 2,\ \ldots, N$ by

$$W_i(x, y) = \big(L_i(x), F_i(x, y)\big) \text{ for all } (x, y) \in K\ .$$

Then

$$\big\{K;\ W_i(x, y)\ :\ i = 1,\ 2,\ \ldots, N\big\} \tag{1.6}$$

constitutes an IFS. It is clear from above that for $i = 1,\ 2,\ \ldots, N$, W_i satisfies $W_i(x_0, y_0) = (x_{i-1}, y_{i-1})$, $W_i(x_N, y_N) = (x_i, y_i)$.

Theorem 1.2.1. *(see [6]). The IFS $\big\{K; W_i : i = 1,\ 2,\ \ldots,\ N\big\}$ defined by (1.6) has a unique attractor G, where G is the graph of a continuous function $g : I \to \mathbb{R}$ which obeys $g(x_i) = y_i$ for $i = 0,\ 1,\ \ldots,\ N$.*

Definition 1.2.1. *The function g defined in Theorem 1.2.1 is called a fractal interpolation function (FIF) or simply fractal function for the interpolation data $\big\{(x_i, y_i) : i = 1,\ 2,\ \ldots,\ N\big\}$.*

Let $\mathcal{C}_*$ be the space of continuous functions $g^* : I \to \mathbb{R}$ such that $g^*(x_0) = y_0$, $g^*(x_N) = y_N$ endowed with the uniform metric $d_\infty(g_1^*, g_2^*) = \max_{x \in I} |g_1^*(x) - g_2^*(x)|$.

Then $(\mathcal{C}_*, d_\infty)$ is a complete metric space. The Read-Bajraktarević operator $T : \mathcal{C}_* \to \mathcal{C}_*$ is defined by

$$Tg^*(x) = F_i\big(L_i^{-1}(x), g^* \circ L_i^{-1}(x)\big),\ x \in I_i, i = 1,\ 2,\ \ldots, N$$

and it is contractive with the contractivity factor $|\alpha|_\infty = \max\{|\alpha_i|; i = 1,\ 2,\ \ldots, N\}$. The fractal interpolation function g is the unique function satisfying the following fixed point equation

$$g(x) = F_i\big(L_i^{-1}(x), g(L_i^{-1}(x))\big) \text{ for all } x \in I_i,\ i = 1,\ 2,\ \ldots, N. \tag{1.7}$$

The widely studied FIFs so far are defined by the mappings

$$L_i(x) = a_i\, x\ +\ d_i,\ \ F_i(x, y) = \alpha_i\, y\ + q_i(x),\ \ i = 1,\ 2,\ \ldots, N, \tag{1.8}$$

where the real constants a_i and d_i are determined by the condition (1.1) as

$$a_i = \frac{(x_i - x_{i-1})}{(x_N - x_0)} \text{ and } d_i = \frac{(x_N x_{i-1} - x_0 x_i)}{(x_N - x_0)} \tag{1.9}$$

and $q_i(x)$'s are suitable continuous functions such that the condition (1.3) holds. If all $q_i(x)$ are taken as affine then the corresponding FIF is known as affine FIF (AFIF). If the FIF is not-affine then it is known as non affine FIF. The polynomial IFS, obtained by taking $q_i(x),\ i = 1,\ 2,\ \ldots, N$ as suitable polynomials, are investigated in [17, 73]. Recently, the rational IFS that is obtained by taking $q_i(x),\ i = 1,\ 2,\ \ldots, N$ as suitable rational functions, are investigated in [91–93].

For each i, α_i is a free parameter with $|\alpha_i| < 1$ and is called a vertical scaling factor of the transformation W_i. Then the vector $\alpha = (\alpha_1, \alpha_2,\ \ldots,\ \alpha_N)$ is called the scale vector of the IFS. It may be noted that for an IFS, W_i maps a line parallel to the y-axis into a line parallel to the y-axis. Further, if L denotes a line segment parallel to the y-axis, then the ratio of the length of $W_i(L)$ to the length of L is $|\alpha_i|$. This property has an important role while finding the fractal dimension of a FIF.

Instead of constant vertical scaling factors, one may take function vertical scaling factors to define iterated function system for more flexibility. In [97], Wang and Yu, considered $F_i(x,y) = \alpha_i(x)y + q_i(x)$ in (1.8) of the IFS, where $\alpha_i(x)$ is a Lipschitz function defined on I. Then the corresponding FIF g satisfies

$$g\big(L_i(x)\big) = F_i\big(x, g(x)\big) = \alpha_i(x)g(x) + q_i(x) \text{ for all } x \in I.$$

The actual interpolant is obtained by iterating indefinitely the IFS. Initially, there are $N+1$ data points. In the first iteration, the injective maps $L_i, i = 1, 2, \ldots, N$, gives $N-1$ points in each N subintervals. Consequently, at the end of the first iteration, one can have $N(N-1) + N + 1 = N^2 + 1$ distinct points. Similarly, at the end of second iteration, one can have $N(N^2 - 1) + N + 1 = N^3 + 1$ distinct points. By induction, it follows that the FIF g interpolates $N^{r+1}+1$ distinct points after r-th iteration [91]. Note that the computation is very fast and the interpolant can be viewed quickly as a whole. So the main differences of a FIF from traditional interpolant is that (i) the construction via IFS uses a functional equation for the interpolant and in small scale graph implies a self-similarity, (ii) instead of using an analytic formula, the FIF is constructed by iteration of the interpolant, (iii) the scaling factors offers more flexibility in the choice of an interpolant and the fractal dimension of the graph of FIF is strongly related to scaling factors which can be seen in the present work for certain types of FIF.

1.3 Smooth fractal interpolation functions

In general, the FIF is not differentiable and such type of FIFs can be found in [65]. However one can define the generating IFS by choosing the parameters in such a way that the corresponding fractal function exhibits the $\mathcal{C}^r$-continuity. Barnsley and Harrington [8], first observed that the integration of any number of terms of a FIF is also a FIF, albeit for a different set of interpolation points. The following theorem gives the smoother FIF.

Theorem 1.3.1. *(see [8], Theorem 1). Let g be the FIF associated with the IFS* $\{I \times$

$\mathbb{R}; \big(L_i(x), F_i(x,y)\big) : i = 1,\ 2,\ \ldots, N\}$ *and* $\widehat{g}(x) = \widehat{y}_0 + \int_{x_0}^{x} g(t)dt$. *Then,* $\widehat{g}$ *is the FIF associated with the IFS* $\{I \times \mathbb{R}; \big(L_i(x), \widehat{F}_i(x,y)\big) : i = 1,\ 2,\ \ldots, N\}$, *where* $\widehat{F}_i(x,y) = a_i\alpha_i y + \widehat{q}_i(x)$. *For* $i = 1,\ 2,\ \ldots, N$, $\widehat{y}_i$ *and* $\widehat{q}_i(x)$ *are given by*

$$\widehat{y}_i = \widehat{y}_0 + \sum_{j=1}^{i} a_j \left[\alpha_j(\widehat{y}_N - \widehat{y}_0) + \int_{x_0}^{x_N} q_j(t)dt\right],\ i = 1,\ 2,\ \ldots, N-1\,,$$

$$\widehat{y}_N = \widehat{y}_0 + \frac{\sum_{j=1}^{N} a_j \int_{x_0}^{x} q_j(t)dt}{1 - \sum_{j=1}^{N} a_j\alpha_j}\,,$$

$$\widehat{q}_i(x) = \widehat{y}_{i-1} - a_i\alpha_i\widehat{y}_0 + a_i \int_{x_0}^{x} q_j(t)dt\,.$$

The following proposition which provides a relation between the IFS of g and the IFS of $\widehat{g}$, is an immediate consequence of the above theorem.

Corollary 1.3.1. *(see [8], Corollary). Let* g *be the FIF corresponding to the IFS* $\{I \times \mathbb{R}; \big(L_i(x), F_i(x,y)\big) : i = 1,\ 2,\ \ldots, N\}$, *where* L_i, F_i *are given in (1.8). If* $\widehat{g}(x) = \widehat{y}_0 + \int_{x_0}^{x} g(t)dt$, *then* $\widehat{g}$ *is the FIF associated with the IFS* $\{I \times \mathbb{R}; \big(L_i(x), \widehat{F}_i(x,y)\big) : i = 1,\ 2,\ \ldots, N\}$, *where* $\widehat{F}_i(x,y) = a_i\alpha_i y + \widehat{q}_i(x)$ *and the function* $\widehat{q}_i$ *satisfies* $\widehat{q}_i^{(1)} = a_i q_i$ *for all* $i = 1,\ 2,\ \ldots, N$.

The following theorem by Barnsley, guarantees that by suitably choosing the scaling factors α_i and the functions q_i in the IFS (1.8), one can get the corresponding FIF g in $\mathcal{C}^r$ class.

Theorem 1.3.2. *(see [8], Theorem 2). For a given data set* $x_0 < x_1 < \ldots < x_N$, *let* $L_i(x) = a_i x + b_i$ *be such that it satisfies (1.1), (1.2) and* $F_i(x,y) = \alpha_i y + q_i(x)$ *satisfies (1.3), (1.5) for* $i = 1,\ 2,\ \ldots, N$. *Suppose for some* $r > 0$, $|\alpha_i| < s a_i^r$, $0 < s < 1$ *and* $q_i \in \mathcal{C}^r[x_0, x_N]$, $i = 1,\ 2,\ \ldots, N$. *Let* $F_{i,k}(x,y) = \frac{\alpha_i y + q_i^{(k)}(x)}{a_i^k}$, $y_{0,k} = \frac{q_1^{(k)}(x_0)}{a_1^k - \alpha_1}$, $y_{N,k} = \frac{q_N^{(k)}(x_N)}{a_N^k - \alpha_N}$, $k = 1,\ 2,\ \ldots, r$. *If* $F_{i-1,k}(x_N, y_{N,k}) = F_{i,k}(x_0, y_{0,k})$ *for* $i = 2,\ 3,\ \ldots, N$ *and* $k = 1,\ 2,\ \ldots, r$, *then* $\big\{\big(L_i(x), F_i(x,y)\big)\big\}_{i=1}^{N}$ *determines a FIF* $g \in \mathcal{C}^r[x_0, x_N]$ *and* $g^{(k)}$ *is the FIF determined by* $\big\{\big(L_i(x), F_{i,k}(x,y)\big)\big\}_{i=1}^{N}$ *for* $k = 1,\ 2,\ \ldots, r$.

1.4 α-fractal interpolation functions

The α-fractal interpolation function f^α defined below is continuous, nowhere differentiable, non-affine fractal function which can be viewed as a perturbation of continuous function f. For different choice of $\alpha \in (-1,1)^N$, one can get a family of continuous functions f^α associated to f. It is worth mentioning here that the Hausdorff-Besicovitch and box-counting dimension of the graph of the function f^α depends on the scale vector α.

1.4.1 Construction of α-fractal interpolation functions

Let $\mathcal{C}(I)$ denote the normed space of real valued continuous functions on I endowed with the uniform norm $\|f\|_\infty = \sup\{|f(x)| \,:\, x \in I\}$. Let $f \in \mathcal{C}(I)$. Consider

$$q_i(x) = f(L_i(x)) \;-\; \alpha_i\, b(x), \tag{1.10}$$

where $b(x)$ is a continuous function such that $b(x_0) = f(x_0)$, $b(x_N) = f(x_N)$ and $b \not\equiv f$.

Definition 1.4.1. *(see [62, 63]). Let f^α be the continuous function whose graph is the attractor of the IFS (1.8), (1.9) and (1.10). Then, the function f^α is called the α-fractal function associated to f with respect to the base function $b(x)$ and the partition Δ.*

Here the Read-Bajraktarević operator is defined as

$$Tg(x) = F_i\big(L_i^{-1}(x), g \circ L_i^{-1}(x)\big) = f(x) + \alpha_i (g-b)(L_i^{-1}(x)),\; x \in I_i,\; i = 1,\, 2,\, \ldots, N\,.$$

Thus f^α satisfies the following fixed point equation

$$f^\alpha(x) \;=\; f(x) \;+\; \alpha_i\, (f^\alpha - b) \circ L_i^{-1}(x) \text{ for all } x \in I_i,\; i = 1,\, 2,\, \ldots, N. \tag{1.11}$$

From (1.11), it is easy to deduce the following

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - b\|_\infty\,, \tag{1.12}$$

where $|\alpha|_\infty = \max\{|\alpha_i| : i = 1, 2, \ldots, N\}$. For $\alpha = 0$, the fractal function f^α is same as the classical one f.

1.4.2 Calculus of α-fractal interpolation functions

Recall that $\mathcal{C}_* = \{g : I \to \mathbb{R} \mid g$ is continuous and $g(x_0) = y_0; g(x_N) = y_N\}$. Let the operator $\mathcal{B}^\alpha = \mathcal{B}^\alpha_{\Delta,b}$ on $\mathcal{C}_*$ be given by

$$\mathcal{B}^\alpha : \mathcal{C}_* \to \mathcal{C}_*$$

$$f \hookrightarrow f^\alpha .$$

The operator $\mathcal{B}^\alpha = \mathcal{B}^\alpha_{\Delta,b}$ depends on both b and Δ.

Proposition 1.4.1. *(see [62], Proposition 2.3). For fixed Δ and b, $\mathcal{B}^\alpha$ satisfies the following Lipschitz condition*

$$\|\mathcal{B}^\alpha(f) - \mathcal{B}^\alpha(g)\|_\infty \le \frac{1}{1-|\alpha|_\infty}\|f-g\|_\infty \textit{ for all } f \textit{ and } g \textit{ in } \mathcal{C}_* .$$

If the continuous function b given in (1.10), depends linearly on f, that is $b_{\lambda g_1 + \mu g_2} = \lambda b_{g_1} + \mu b_{g_2}$, then the operator

$$\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I) \tag{1.13}$$

$$f \hookrightarrow f^\alpha$$

is linear. For instance, let us choose $b = f \circ c$, where c is a continuous, increasing function such that $c(x_0) = x_0$ and $c(x_N) = x_N$. Then from (1.11),

$$f^\alpha(x) = f(x) + \alpha_i \left(f^\alpha - f \circ c\right) \circ L_i^{-1}(x) ,$$

$$g^\alpha(x) = g(x) + \alpha_i \left(g^\alpha - g \circ c\right) \circ L_i^{-1}(x)$$

for all $x \in I_i$. Multiplying the first equation by λ and the second equation by μ, we obtain

$$(\lambda f^\alpha + \mu g^\alpha)(x) = (\lambda f + \mu g)(x) + \alpha_i\big(\lambda f^\alpha + \mu g^\alpha - (\lambda f + \mu g) \circ c\big) \circ L_i^{-1}(x) .$$

From this equation, it can be seen that the function $\lambda f^\alpha + \mu g^\alpha$ is the fixed point of the Read-Bajraktarević operator

$$Th(x) := (\lambda f + \mu g)(x) + \alpha_i\big(h - (\lambda f + \mu g) \circ c\big) \circ L_i^{-1}(x) .$$

The uniqueness of the fixed point shows that

$$(\lambda f + \mu g)^\alpha = \lambda f^\alpha + \mu g^\alpha \text{ for all } \lambda, \mu \in \mathbb{R} .$$

Therefore the operator $\mathcal{F}^\alpha$ is linear. From (1.12), for $b = f \circ c$, one can have

$$\|\mathcal{F}^\alpha(f) - f\|_\infty \leq \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\|f\|_\infty .$$

That is,

$$\|\mathcal{F}^\alpha(f)\|_\infty \leq \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}\|f\|_\infty$$

and as a consequence, it follows that

$$\|\mathcal{F}^\alpha\|_\infty \leq \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty} .$$

Therefore the operator $\mathcal{F}^\alpha$ is linear and bounded. If we define b as

$$b = Lf, \tag{1.14}$$

where $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is a linear and bounded operator with respect to the uniform norm on $\mathcal{C}(I)$, such that $Lf(x_0) = f(x_0)$ and $Lf(x_N) = f(x_N)$, then for any $f \in \mathcal{C}(I)$ and its fractal function satisfies [72]

$$\|f^\alpha - f\|_\infty \leq |\alpha|_\infty \|f^\alpha - Lf\|_\infty, \tag{1.15}$$

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|I - L\|_\infty \|f\|_\infty, \tag{1.16}$$

where $\|I - L\|_\infty$ represents the corresponding operator norm as well. If we consider $\mathcal{L}^p$-norm $(1 \leq p < \infty)$

$$\|f\|_{\mathcal{L}^p} = \left(\int_I |f|^p dt\right)^{1/p} ,$$

such that the operator $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is linear and bounded with respect to $\mathcal{L}^p$-norm on $\mathcal{C}(I)$, then the following results can be found in [72].

For every $f \in \mathcal{C}(I)$ and b defined as (1.14)

$$\|f^\alpha - f\|_{\mathcal{L}^p} \leq |\alpha|_\infty \|f^\alpha - Lf\|_{\mathcal{L}^p} , \tag{1.17}$$

$$\|f^\alpha - f\|_{\mathcal{L}^p} \le \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|f - Lf\|_{\mathcal{L}^p} \,. \tag{1.18}$$

The results have been extended in [71] for any $f \in \mathcal{L}^p(I)$, $1 \le p < \infty$ as

$$\|\overline{\mathcal{F}}^\alpha(f) - f\|_{\mathcal{L}^p} \le |\alpha|_\infty \|\overline{\mathcal{F}}^\alpha(f) - \overline{L}f\|_{\mathcal{L}^p} \,, \tag{1.19}$$

$$\|\overline{\mathcal{F}}^\alpha(f) - f\|_{\mathcal{L}^p} \le \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|f - \overline{L}f\|_{\mathcal{L}^p} \,, \tag{1.20}$$

where $\overline{\mathcal{F}}^\alpha : \mathcal{L}^p(I) \to \mathcal{L}^p(I)$ and $\overline{L} : \mathcal{L}^p(I) \to \mathcal{L}^p(I)$ are the corresponding extensions of $\mathcal{F}^\alpha$ and L with $\|\overline{\mathcal{F}}^\alpha\|_p = \|\mathcal{F}^\alpha\|_p$ and $\|\overline{L}\|_p = \|L\|_p$, where $\|.\|_p$ represents the norm of the operator with respect to the $\mathcal{L}^p$-norm. The theory of α-fractal functions for different choices of $b(x)$ can be found in [65, 67, 71, 72] and references therein.

1.4.3 α-fractal interpolation functions with variable scaling factors

Instead of constant vertical scaling factors, one may take function vertical scaling factors to define IFS for more flexibility [97]. Consider the IFS with mappings, for $i = 1, 2, \dots, N$

$$L_i(x) = a_i x + d_i \; ; F_i(x, y) = \alpha_i(x) y + f(L_i(x)) \; - \; \alpha_i(x)\, b(x) \,, \tag{1.21}$$

where $\alpha_i(x)$'s are continuous functions on I satisfying $\|\alpha\|_\infty = \max\big\{\|\alpha_i\|_\infty : i = 1, 2, \dots, N\big\} < 1$. Then the operator T takes the following form

$$Tg(x) = F_i\big(L_i^{-1}(x), g\circ L_i^{-1}(x)\big) = f(x) + \alpha_i(L_i^{-1}(x))(g-b)(L_i^{-1}(x)), x \in I_i, i = 1,\ 2,\ \dots, N \,.$$

Thus the corresponding α-fractal interpolation function f^α with function scaling factors satisfies the following fixed point equation

$$f^\alpha(x) \;=\; f(x) \;+\; \alpha_i(L_i^{-1}(x))\; (f^\alpha - b)(L_i^{-1}(x)) \text{ for all } x \in I_i, i = 1,\ 2,\ \dots, N. \tag{1.22}$$

1.4.4 Smooth α-fractal interpolation functions

Let $\mathcal{C}^k(I)$ be the Banach space of real valued functions having k-continuous derivatives with the norm

$$\|f\|_k := \max\left\{\|f^{(r)}\|_\infty : r = 0, 1, \ldots, k\right\} .$$

To define a smooth α-fractal interpolation function g^α corresponding to a smooth function g, Navascués and Sebastián [74], defined an IFS satisfying Theorem 1.3.2, in which it is assumed that the constant scale factors $\alpha_i = \alpha_1$ for all $i = 1, 2, \ldots, N$, that is, the scale vector $\alpha = (\alpha_1, \alpha_1, \ldots, \alpha_1)$ and the base function b as Hermite interpolating polynomial such that the k-th derivative of b satisfies

$$b^{(k)}(x_0) = g^{(k)}(x_0) ,$$

$$b^{(k)}(x_N) = g^{(k)}(x_N)$$

for $k = 0, 1, \ldots, N$. Then

$$q_i(x) = g(L_i(x)) \; - \; \alpha_1 \, b(x)$$

and the IFS consisting of,

$$L_i(x) = a_i \, x \; + \; d_i, \;\; F_i(x, y) = \alpha_i \, y \; + g\big(L_i(x)\big) \; - \; \alpha_i \, b(x) \tag{1.23}$$

for $x \in I$ and $i = 1, \; 2, \; \ldots, N$. The fractal function g^α corresponding to IFS (1.23), preserves the smoothness of g and satisfies

$$\|g^\alpha - g\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|g - b\|_\infty . \tag{1.24}$$

In [92], Viswanathan and Chand relaxed the equality condition on the scale vector α and considered a finite collection of base functions

$$B = \left\{b_i \in \mathcal{C}(I) \mid \; b_i(x_0) = g(x_0), b_i(x_N) = g(x_N), b_i \neq g, i = 1, \; 2, \; \ldots, N\right\} ,$$

instead of taking a single base function b and defined a smooth α-fractal g^α corresponding to a smooth function g, via the IFS consisting of,

$$L_i(x) = a_i \, x \; + \; d_i, \;\; F_i(x, y) = \alpha_i \, y \; + g(L_i(x)) \; - \; \alpha_i \, b_i(x) \tag{1.25}$$

for $x \in I$ and $i = 1,\ 2,\ \ldots, N$. The following theorem can be read in [92].

Theorem 1.4.1. *(see [92], Theorem 3.3). Suppose that for some $p \geq 0$, we have $|\alpha_i| \leq k a_i^p$ for all $i = 1,\ 2,\ \ldots, N$ and $0 < k < 1$. Let $|\alpha|_\infty = \max\{|\alpha_i| : i = 1,\ 2,\ \ldots, N\}$, $g \in \mathcal{C}^p$ and the family $B = \{b_i : i = 1,\ 2,\ \ldots, N\}$ be such that the derivatives up to the p-th order of each of its members agrees with that of g at the end points of the interval. Then the α-fractal function $g^\alpha \in \mathcal{C}^p(I)$ of g with respect to the partition Δ and the family B satisfies*

$$\|g^\alpha - g\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \max\{\|g - b_i\|_\infty : i = 1,\ 2,\ \ldots, N\}\ . \tag{1.26}$$

Recently, Navascués et al. [76], provided condition so that the α-fractal function f^α with variable scalings is k-times continuously differentiable whenever the original function f is k-times continuously differentiable and it is stated below.

Theorem 1.4.2. *(see [76], Theorem 3.2). For a given data set $x_0 < x_1 < \ldots < x_N$, let $L_i(x) = a_i x + b_i$ be such that it satisfies (1.1), (1.2) and $F_i(x, y) = F_{i,0}(x, y) = \alpha_i(x) y + q_i(x)$ satisfies (1.3), (1.5) for $i = 1,\ 2,\ \ldots, N$. Suppose for $p \in \{0, 1, \ldots, k\}$, $y_{0,p}$ and $y_{N,p}$ are arbitrarily chosen real numbers except $y_{0,0} = y_0$ and $y_{N,0} = y_N$. For $i = 1,\ 2,\ \ldots, N$, assume that there exist functions α_i and q_i in $\mathcal{C}^k(I)$ such that $\|\alpha_i\|_k < (a_i/2)^k$ and for $i = 2,\ 3,\ \ldots, N-1$, $p \in \{1, 2, \ldots, k\}$,*

$$\frac{\sum_{j=0}^{p} \binom{p}{j} y_{0,j} \alpha_i^{(p-j)}(x_0) + q_i^{(p)}(x_0)}{a_i^p} = \frac{\sum_{j=0}^{p} \binom{p}{j} y_{N,j} \alpha_{i-1}^{(p-j)}(x_N) + q_{i-1}^{(p)}(x_N)}{a_{i-1}^p}\ ,$$

$$q_1^{(p)}(x_0) = y_{0,p} a_1^p - \sum_{j=0}^{p} \binom{p}{j} y_{0,j} \alpha_1^{(p-j)}(x_0)\ ,$$

$$q_N^{(p)}(x_N) = y_{N,p} a_N^p - \sum_{j=0}^{p} \binom{p}{j} y_{N,j} \alpha_N^{(p-j)}(x_N)\ .$$

Then the corresponding FIF $g \in \mathcal{C}^k(I)$ and for $p \in \{1, 2, \ldots, k\}$,

$$g^{(p)}(L_i(x)) = a_i^{-p} \left[\sum_{j=0}^{p} \binom{p}{j} \alpha_i^{(p-j)}(x) g^{(j)}(x) + q_i^{(p)}(x) \right],\ x \in I$$

for $i = 1, 2, \ldots, N$.

It can be observed that the Barnsley-Harrington theorem given in Theorem 1.3.2 is a special case of Theorem 1.4.2. The following theorem ensures that if $f \in \mathcal{C}^k(I)$ then the corresponding α-fractal function $f^\alpha \in \mathcal{C}^k(I)$.

Theorem 1.4.3. *(see [76], Theorem 3.4). Let the original function $f \in \mathcal{C}^k(I)$. Suppose that for $i = 1, 2, \dots, N$, the scaling functions α_i and the base function b in $\mathcal{C}^k(I)$ such that $\|\alpha_i\|_k < (a_i/2)^k$ and*

$$b^{(p)}(x_0) = f^{(p)}(x_0), b^{(p)}(x_N) = f^{(p)}(x_N),\ p \in \{0, 1, \dots, k\}\ .$$

Then the corresponding f^α given in (1.22) belongs to $\mathcal{C}^k(I)$. Further, for $i = 1, 2, \dots, N$,

$$(f^\alpha)^{(p)}(x_i) = f^{(p)}(x_i),\ p \in \{0, 1, \dots, k\}$$

and $(f^\alpha)^{(p)}$ satisfies

$$(f^\alpha)^{(p)}(L_i(x)) = f^{(p)}(L_i(x)) + a_i^{-p}\left(\sum_{j=0}^{p} \binom{p}{j} \alpha_i^{(p-j)}(x)(f^\alpha - b)^{(j)}(x)\right).$$

1.5 Fractal dimension

In fractal geometry, the index/notion of dimension characterizes the irregularity of a set or an object. There are different kinds of dimensions like topological dimension, fractal dimension, etc., associated with a set. Since the fractal dimension gives a non-integer values to the dimension, it is best suited in the study of irregular objects 'fractals'. Fractal dimensions are also of two kinds: those related to the measure theory, namely, Hausdorff dimension, packing dimension and those obtained by counting methods or integration methods. Kolomogorov [51], introduced the notion of fractal dimension which later demonstrated by Mandelbrot [56] to quantify irregular patterns. For different types of dimensions, one can see [33, 87].

1.5.1 Hausdorff dimension

Let $(\mathcal{X}, d)$ be a metric space and $S \in \mathcal{H}(\mathcal{X})$. Set the diameter of S as $|S| = \sup\{d(x,y) : x, y \in S\}$. Let s be any non-negative real number. Then for any $\delta > 0$, define

$$H_\delta^s(S) = \inf\left\{\sum_{n=1}^{\infty} |U_n|^s : S \subseteq \cup_{n=1}^{\infty} U_n \text{ and } 0 < |U_n| \le \delta \text{ for } n = 1, 2, 3, \dots\right\},$$

where the infimum is taken over all countable covers $\{U_n\}$ of S. Define

$$H^s(S) = \lim_{\delta \to 0} H_\delta^s(S).$$

This limit exists for any subset of $\mathbb{R}^n$ though the limiting value can be 0 or ∞ and known as s-dimensional Hausdorff outer measure. The value of s for which $H^s(S)$ jumps from ∞ to 0 is called the Hausdorff-Besicovitch dimension or simply Hausdorff dimension of S. That is, there exists a unique non-negative number s_0 such that

$$H^s(S) = \begin{cases} \infty, & \text{if} \quad 0 \le s < s_0 \\ 0, & \text{if} \quad s > s_0 . \end{cases}$$

The number s_0 is called the Hausdorff dimension of S and is denoted by $\dim_H S$. Though the Hausdorff dimension has the advantage of being defined for any set, it is rather difficult to calculate in many cases. For easy computations, box dimension is more suitable for fractals.

1.5.2 Box-counting dimension

Box-counting dimension or simply box dimension is largely popular due to its relative ease of mathematical calculation and empirical estimation. Barnsley [7], refers the box dimension as fractal dimension. In this thesis, we deal only with box dimension.

Definition 1.5.1. *(see [33]). Let G be any non-empty bounded subset of $\mathbb{R}^n$ and $\mathcal{N}_\delta(G)$ be the minimum number of closed balls of radius $\delta > 0$ which can cover G. The lower and upper box-counting dimension of G respectively are defined as*

$$\overline{\dim}_B G = \limsup_{\delta \to 0} \frac{\log \mathcal{N}_\delta(G)}{-\log \delta}$$

and

$$\underline{\dim}_B G = \liminf_{\delta\to 0} \frac{\log \mathcal{N}_\delta(G)}{-\log\delta} \ .$$

If both limits exist and are equal then the box-counting dimension of G is denoted by $\dim_B G$ and defined by

$$\dim_B G = \lim_{\delta\to 0} \frac{\log \mathcal{N}_\delta(G)}{-\log\delta} \ .$$

There are equivalent definitions of box-counting dimension by taking $\mathcal{N}_\delta(G)$ as any of the following

(i) the minimum number of square boxes of sides δ parallel to axes;

(ii) the minimum number of δ-mesh cubes that intersect G;

(iii) the maximum number of disjoint balls of radius δ with centers in G.

The following proposition is an alternative for easy calculation of box-counting dimension.

Proposition 1.5.1. *(see [7], Theorem 1.2). Let $S \in \mathcal{H}(\mathbb{R}^m)$ and $\mathcal{N}_n(S)$ be the minimum number of closed boxes of sides length $\frac{1}{2^n}$ which intersects S. Then the box-counting dimension is*

$$\dim_B S = \lim_{n\to\infty} \frac{\log \mathcal{N}_n(S)}{-\log 2^n} \ .$$

In general for any set S, $\dim_H S \le \dim_B S$.

Definition 1.5.2. *A function $g : S \subseteq \mathbb{R} \to \mathbb{R}$ is said to be Hölderian with exponent $\beta \in (0,1]$ if there exists a positive constant C such that*

$$|g(x) - g(y)| \le C|x-y|^\beta \text{ for all } \ x, y \in S \ .$$

Here C is called the Hölder constant of f.

Theorem 1.5.1. *(Besicovitch and Ursell) (see [12]). The Hausdorff dimension of the graph of any Hölderian function with exponent $\beta \in (0,1]$ is d satisfying $1 \le d \le 2-\beta$.*

The above result shows that for smooth function the fractal dimension of its graph is one. The following result gives a non-trivial dimension for the graph of non-smooth fractal function.

Theorem 1.5.2. *(Barnsley). (see [7], Theorem 3.1). Let G be the graph of an affine FIF associated with the data $\{(x_i, y_i) \in \mathbb{R}^2 : i = 1, 2, \ldots, N\}$. If $\sum_{i=1}^{N} |\alpha_i| > 1$ and the interpolation points are not collinear, then the fractal dimension of G is the unique real solution D of*

$$\sum_{i=1}^{N} |\alpha_i| a_i^{D-1} = 1 \, .$$

Otherwise, the fractal dimension of G is 1.

1.6 Functions on the sphere

Spherical harmonics have interesting applications in modeling of the gravitational field, meteorology, computer graphics, etc. The use of 'Laplace series' or 'Fourier-Laplace series' is an old idea, going back to P. Laplace and A. Legendre in the late 1700s. It generalizes the idea of Fourier series representation of a periodic univariate function by using spherical harmonics for a function on the sphere. There is a rich literature on approximation of functions on the sphere, most notably, Gronwall [39], Newman and Shapiro [77], Ragozin [81], Rustamov [85], Ditzian [29, 30] and Dai and Xu [21].

An n-th degree homogeneous polynomial V in the variables x, y, z satisfying the Laplace equation

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \, ,$$

is called a Laplace or harmonic polynomial of degree n. The following result can be read in ([86], Section 18).

Theorem 1.6.1. *There exists only $2n + 1$ linearly independent Laplace polynomials of degree n.*

Consider the spherical coordinates (ρ, φ, θ) for $P \in \mathbb{R}^3$ and

$$\xi(P) = \sin\varphi\cos\theta;\ \eta(P) = \sin\varphi\sin\theta;\ \zeta(P) = \cos\varphi;\ 0 \leqq \varphi \leqq \pi;\ 0 \leqq \theta \leqq 2\pi\ ,$$

where θ is the longitude and φ is the colatitude. Then

$$V_n(x, y, z) = \rho^n V(\xi, \eta, \zeta)\ .$$

The function

$$Y_n(\varphi, \theta) = V_n(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

is called the n-th order Laplace function or spherical harmonic. Let $P_n(x)$ be the n-th Legendre polynomial of degree n. The fundamental system of $2n+1$ spherical harmonics of the point $P \equiv (\varphi, \theta)$ of the unit sphere S is

$$\left.\begin{array}{rcl} U_n^0(\varphi,\theta) &=& P_n(\cos\varphi), \\ U_n^m(\varphi,\theta) &=& P_n^m(\cos\varphi)\cos(m\theta), \\ V_n^m(\varphi,\theta) &=& P_n^m(\cos\varphi)\sin(m\theta), \end{array}\right\} \tag{1.27}$$

$m = 1, 2, \ldots, n$ and P_n^m is the (n, m)-Ferrer's associated Legendre polynomial defined as

$$P_n^m(x) = (1 - x^2)^{\frac{m}{2}} P_n^{(m)}(x). \tag{1.28}$$

Here $P_n^{(m)}(x)$ denotes the derivative of the m-th order of $P_n(x)$ with respect to x. Two spherical harmonics of different order are orthogonal over the sphere S in the sense that

$$\int_S Y_n(P) Y_m(P) dS = 0;\quad n \neq m\ ,$$

where dS is the element of area of the sphere S. The set of spherical harmonics $\mathcal{H}_n$, of order n, is a linear subspace of continuous functions on the sphere with dimension $2n + 1$. The polynomials given in (1.28) satisfy the following [86]

$$\int_{-1}^{1} P_n^{(m)}(x) P_r^{(m)}(x) dx = 0;\quad n \neq r;$$

$$\int_{-1}^{1} (P_n^{(m)}(x))^2 dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}\ .$$

The family

$$\{U_n^0, U_n^m, V_n^m; n = 0, 1, \ldots, m = 1, 2, \ldots, n\}$$

is an orthogonal and complete system in $\mathcal{L}^2(S)$, the space of square integrable functions on the sphere S.

1.7 Motivation of present work

The concept of fractal interpolation function that opened a new door of research in the field of approximation theory was proposed by Barnsley [6]. The graph of a fractal interpolation function $f : I \subset \mathbb{R} \to \mathbb{R}$ which interpolates a data set $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$, is an attractor of some iterated function system. The function is generally self-affine in nature. It has been generalized for higher dimensions as self-affine fractal interpolation surfaces whose graphs are attractors of IFSs associated with given data sets [13, 14, 23, 57]. The non-self-affine fractal function, namely hidden variable fractal interpolation function is defined for generalized data sets in [9]. More flexible fractal functions which are self-affine as well as non-self-affine, can be found in [16, 49].

Navascués [62, 63, 68, 70–72], defined and studied a family of non-smooth fractal functions f^α known as α-fractal interpolation functions corresponding to a continuous function f. The functions f^α approximate and interpolate f simultaneously in a compact subset of $\mathbb{R}$. The scale vector $\alpha = (\alpha_1,\ \alpha_2,\ \ldots\ \alpha_N) \in \mathbb{R}^N$ gives a family of fractal functions f^α. On the other hand, fractal dimension which is the heart of fractal geometry helps to understand the visual complexity of an object. Mandelbrot [56], demonstrated that fractal dimension is useful to quantify irregular patterns such as fractals. Many works have been carried out on different types of fractal functions and calculation of box dimensions of their graphs [8, 10, 18, 24, 36, 40, 55, 60, 80, 83]. It is seen in the literature that the scale vector of the IFS has an important role for the estimation of the fractal dimension of the graph of a fractal function. So far, there is no existing result or formula for box dimension of the graph of α-fractal function f^α. In the present work, an estimation for the box dimension of the graph of a α-fractal function f^α corresponding to Hölder function f is calculated.

For more flexibility, function scale vector is considered in the IFS and box dimension of the graph of the corresponding α-fractal function f^α is estimated in the present work. The α-fractal function f^α is non-self-affine in nature. So the behavior of the graph of f^α is not self-similar in the interval. By looking at the graph of f^α, it is not clear that how the dimension of the graph differs in subintervals. In the present work, this issue has been investigated in detail in subintervals for different possible form of scale vector.

Several properties of the operator $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$ defined by $f \mapsto f^\alpha$ have been explored and also been extended to more general spaces like Lebesgue spaces $\mathcal{L}^p(I)(1 \leq p < \infty)$. Navascués proposed a general procedure to define non-smooth fractal versions of classical trigonometric approximants and generalized some classical results namely Dini-Lipschitz's theorem on $\mathcal{C}(2\pi)$ [67]. In this case, the fractal maps are defined using IFS theory generalizing the classical 2π-periodic continuous maps on the unit circle and the existence of a Hilbert basis of fractal maps on the circle is proved [66]. The Lipschitz properties of the original function guarantee a good approximation of the represented variable with truncated Chebyshev fractal object for high frequency sample [68]. In [75], the Legendre expansion of a real sampled function by means of fractal methods is considered and its pointwise, uniform and mean-square convergence for suitable choice of scale vector are established. In the present work, the class of orthonormal polynomials, namely, Jacobi polynomials [53] $P_k^{(r,s)}(x) = \gamma_k(r,s)x^k +$lower degree terms, where r and s are real parameters > -1 and $\gamma_k(r,s) > 0$ is considered. The system of Jacobi polynomials forms a Schauder basis/complete orthonormal basis for $\mathcal{L}^2_\rho(-1,1)$ [43]. Using the completeness of this classical basis, a fractal Schauder basis for the space of weighted square integrable functions $\mathcal{L}^2_\rho(-1,1)$ is found. The domain of uniform convergence of the Fourier-Jacobi expansion of a function depends on the parameters r and s. In the present work, an attempt is made to approximate continuous function with the Fourier-Jacobi expansion of affine FIF. To approximate continuously p-differentiable function, Fourier-Jacobi expansion of non-affine p-differentiable FIF is used.

Navascués [64], fractalizes the functions on the unit sphere in $\mathbb{R}^3$ by means of a linear and bounded operator. Navascués studied the fractal version of classical results on functional analysis in a parallel way and provided fractal Hilbert bases for square integrable functions on the unit sphere. In the present work, an attempt is made to construct fractal version of a family of continuous functions on the unit sphere $S \subseteq \mathbb{R}^3$, generalizes the spherical harmonics.

1.8 Organization of present work

The thesis is organized as follows.

The first chapter is the introductory one, consisting of definitions and the basics related to our work. The concept of iterated function system, construction of fractal interpolation function, smooth fractal interpolation function, α-fractal interpolation function are reviewed here. The calculus of α-FIF and some basic results of the fractal operator $\mathcal{F}^\alpha$ are given followed by the smooth α-FIF. The definitions of Hausdorff dimension and box dimension, and some existing results on fractal dimension are presented. Further, a brief review of functions on the sphere is presented. Two new work, namely, results on perturbations on vertical scaling factors of fractal interpolation surfaces are presented here, followed by graph-directed coalescence hidden variable FIF.

In Chapter 2, the box dimension of the graph of α-FIF for Hölder function is estimated. This is done for equally spaced data set as well as arbitrary data set. It is also observed that our results generalize the existing result for classical one with $\alpha = 0$.

In Chapter 3, the box dimension of the graph of α-FIF with function scale vector for Hölder function is calculated but the approach is different from the previous chapter. The box dimension in the subintervals for the graph of α-FIF is also estimated. Finally, the box dimension of the graph of f^α in subintervals is compared with the box dimension of the graph of f^α in whole interval.

In Chapter 4, an orthonormal system of Jacobi polynomials in $\mathcal{L}^2(-1,1)$ with respect to the weight function $\rho^{(r,s)}(x) = (1-x)^r(1+x)^s$, $r > -1$ and $s > -1$ is considered. A fractal Jacobi system which is fractal analogous of Jacobi polynomials is defined. The Weierstrass type theorem providing an approximation for a square integrable function in terms of α-fractal Jacobi sums is derived. A fractal basis for the space of weighted square integrable functions $\mathcal{L}^2_\rho(-1,1)$ is found. The Fourier-Jacobi expansion corresponding to an affine FIF interpolating certain data set is considered and its convergence in uniform norm and weighted-mean square norm are established. The closeness of the original function to the Fourier-Jacobi expansion of the affine FIF is proved for certain scale vector. Finally, the Fourier-Jacobi expansion corresponding to a non-affine smooth FIF interpolating certain data set is considered and its convergence in uniform norm and weighted-mean square norm are investigated as well.

Chapter 5 is of two fold. Here a family of continuous functions on the unit sphere $S \subseteq \mathbb{R}^3$ generalizing the spherical harmonics, is considered. Using fractal methodology, fractal version of the first type of this family on the sphere S is constructed. To do this, an iterated function system and a linear bounded operator that maps classical functions to its fractal analogues is defined. Some approximation properties of fractal functions on the sphere are investigated. Restricting the scale vectors involved in the IFS, a fractal Hilbert basis is established for the functions on the sphere. Again fractalizing the family of continuous functions on the sphere S by means a linear and bounded operator, a fractal version of the second type of this family is constructed. For different values of the scale vector in the IFS, Bessel sequence, frames and Riesz bases are established for the space $\mathcal{L}^2(S)$ of square integrable functions on the sphere S.

1.9 Fractal interpolation surfaces and perturbations on vertical scaling factors

This section contains a research work which is carried out by us related to perturbation on vertical scaling factors of a bivariate fractal interpolation function [79].

1.9.1 Fractal interpolation surfaces

In [23], Dalla gave a method of construction of a bivariate fractal interpolation function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ that interpolates given points $\{ (x_0,\ y_0,\ z_{00}),\ \cdots,\ (x_N,\ y_M,\ z_{NM}) \}$ in $\mathbb{R}^3$. The graph of f is known as the bivariate fractal interpolation surface (FIS) which is the attractor of the bivariate IFS.

H-Y Wang and his coworkers studied the perturbation and error analysis for bivariate FIFs by introducing perturbation functions in the variables x and y in [96] and for one variable FIFs in [95]. On the other hand, there are studies on how the vertical scaling factors affect the bounds of the affine fractal interpolation function [84] and the bounds of the attractor of the iterated function system with double vertical scaling factor [94]. Xu and Feng [98], studied the problem of perturbing the vertical scaling factors of one variable FIF and gave a condition under which the perturbed IFS meet the fractal interpolation continuous condition.

In the present work, we study the problem of perturbing the vertical scaling factors of bivariate FIFs. We obtained conditions for which the new perturbed IFS to satisfy the continuous condition. Finally, an upper bound for the error estimation between the two FIFs are also found.

Let a set of data point $T = \{(x_i,\ y_j,\ z_{ij})\ :\ i = 0,\ 1,\ \ldots,\ N;\ \ j = 0,\ 1,\ \ldots,\ M\}$ be given in $\mathbb{R}^3$ with $N > 1$ and $M > 1$. Let $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ and $c = y_0 < y_1 < y_2 < \cdots < y_M = d$. Set $I = [a,\ b]$, $J = [c,\ d]$, $I_i = [x_{i-1},\ x_i]$, $J_j = [y_{j-1},\ y_j]$ and $D_{ij} = I_i \times J_j$ for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Assume that the interpolation points are such that each of the sets

$$\begin{aligned}
\{(x_0,\ y_j,\ z_{0j}) &\ :\ j = 0,\ 1,\ \ldots,\ M\},\\
\{(x_N,\ y_j,\ z_{Nj}) &\ :\ j = 0,\ 1,\ \ldots,\ M\},\\
\{(x_i,\ y_0,\ z_{i0}) &\ :\ i = 0,\ 1,\ \ldots,\ N\},\\
\{(x_i,\ y_M,\ z_{iM}) &\ :\ i = 0,\ 1,\ \ldots,\ N\}
\end{aligned}$$

is collinear.

Let $D = I \times J$ and define $w_{ij} : D \times \mathbb{R} \to \mathbb{R}^3$ by

$$w_{ij}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_i x + b_i \\ c_j y + d_j \\ e_{ij}x + f_{ij}y + g_{ij}xy + s_{ij}z + k_{ij} \end{pmatrix} = \begin{pmatrix} u_i(x) \\ v_j(y) \\ F_{ij}(x, y, z) \end{pmatrix}, \tag{1.29}$$

where the constants a_i, b_i, c_j, d_j, e_{ij}, f_{ij}, g_{ij}, k_{ij} are defined by the equations

$$\left.\begin{aligned}
w_{ij}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix} &= \begin{pmatrix} x_{i-1} \\ y_{j-1} \\ z_{(i-1)(j-1)} \end{pmatrix},\\
w_{ij}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} &= \begin{pmatrix} x_i \\ y_{j-1} \\ z_{i(j-1)} \end{pmatrix},\\
w_{ij}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix} &= \begin{pmatrix} x_{i-1} \\ y_j \\ z_{(i-1)j} \end{pmatrix},\\
w_{ij}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} &= \begin{pmatrix} x_i \\ y_j \\ z_{ij} \end{pmatrix},
\end{aligned}\right\} \tag{1.30}$$

as

$$\begin{aligned}
a_i &= \frac{x_i - x_{i-1}}{x_N - x_0},\ b_i = \frac{x_{i-1}x_N - x_i x_0}{x_N - x_0},\\
c_j &= \frac{y_j - y_{j-1}}{y_M - y_0},\ d_i = \frac{y_{j-1}y_M - y_j y_0}{y_M - y_0},\\
e_{ij} &= \frac{z_{i-1,j-1} - z_{i,j-1} - s_{ij}(z_{00} - z_{N0}) - g_{ij}y_0(x_0 - x_N)}{x_0 - x_N},\\
f_{ij} &= \frac{z_{i-1,j-1} - z_{i-1,j} - s_{ij}(z_{00} - z_{0M}) - g_{ij}x_0(y_0 - y_M)}{y_0 - y_M},\\
g_{ij} &= \frac{z_{ij} + z_{i-1,j-1} - z_{i-1,j} - z_{i,j-1} - s_{ij}(z_{NM} + z_{00} - z_{0j} - z_{N0})}{(x_N - x_0)(y_M - y_0)},
\end{aligned}$$

$$k_{ij} = z_{ij} - e_{ij}x_N - f_{ij}y_M - s_{ij}z_{NM} - g_{ij}x_N y_M \, .$$

Set $\phi_{ij}(x,\ y) = e_{ij}x + f_{ij}y + g_{ij}xy + k_{ij}$ for $(x,\ y) \in D$. Then, $F_{ij}(x,\ y,\ z) = s_{ij}z + \phi_{ij}(x,\ y)$ for $(x,\ y,\ z) \in D \times \mathbb{R}$. The functions ϕ_{ij}'s are bivariate Lipschitz functions. The constants s_{ij}'s are called the vertical scaling factors. Let $0 < |s_{ij}| < 1$. Then

$$\{D \times \mathbb{R},\ w_{ij} \equiv (u_i,\ v_j,\ F_{ij})\ :\ i = 1,\ \ldots,\ N;\ \ j = 1,\ \ldots,\ M\} \tag{1.31}$$

constitutes an IFS.

The IFS has a unique attractor G which is the graph of a continuous function $f : D \to \mathbb{R}$ that interpolates the given data [23]. The function f is called the bivariate fractal interpolation function (BFIF) for the IFS (1.31) and f satisfies the fixed point equation

$$f(x,\ y)\ =\ s_{ij}\ f(u_i^{-1}(x),\ v_j^{-1}(y))\ +\ \phi_{ij}(u_i^{-1}(x),\ v_j^{-1}(y)) \qquad \text{for all } (x,\ y) \in D_{ij}, \tag{1.32}$$

for $i = 1,\ 2,\ \ldots,\ N$ and $j = 1,\ 2,\ \ldots,\ M$.

1.9.2 Perturbation in scaling factors

We introduce a perturbation in the scaling factors of the IFS (1.31) as follows. Define $T_{ij}\ :\ D \times \mathbb{R} \to \mathbb{R}$ for $i = 1,\ 2,\ \ldots,\ N$ and $j = 1,\ 2,\ \ldots,\ M$ by

$$T_{ij}(x,\ y,\ z)\ =\ (s_{ij} + \delta_{ij})z\ +\ \phi_{ij}(x,\ y)\ ,$$

where δ_{ij}'s satisfies the condition $0 < |s_{ij} + \delta_{ij}| < 1$.

Then, we construct a new IFS

$$\{D \times \mathbb{R},\ (u_i,\ v_j,\ T_{ij})\ :\ i = 1,\ \ldots,\ N;\ \ j = 1,\ \ldots,\ M\}\ . \tag{1.33}$$

We call the IFS (1.33) as the perturbed IFS of the given IFS (1.31).

In the sequel, we denote the matrix consisting of the scaling factors as $S = (s_{ij})$ and the matrix consisting of the perturbations added to the scaling factors as $\Delta = (\delta_{ij})$.

Example 1.1. *Let f be a fractal interpolation function whose graph is the attractor of the IFS $\{D \times \mathbb{R},\ w_{ij} \equiv (u_i,\ v_j,\ F_{ij})\ :\ i = 1,\ \ldots,\ 3;\ \ j = 1,\ \ldots,\ 3\}$ that interpolates points given in the following table*

y/x	*0*	*64*	*128*	*192*
0	*100*	*100*	*100*	*100*
64	*100*	*130*	*130*	*140*
128	*100*	*140*	*130*	*140*
192	*100*	*100*	*130*	*140*

For this IFS, the vertical scaling factors are taken as

$$S = \begin{pmatrix} 0.45 & 0.4 & -0.45 \\ 0.35 & -0.15 & 0.15 \\ -0.3 & 0.25 & -0.45 \end{pmatrix}.$$

The computed functions F_{ij}'s are given by

$$\begin{aligned}
F_{11}(x,\ y,\ z) &= 0.45z + 0.0003255xy + 55, \\
F_{12}(x,\ y,\ z) &= 0.4z + 0.1563x - 0.000434xy + 60, \\
F_{13}(x,\ y,\ z) &= -0.45z + 0.1563x + 0.0007595xy + 145, \\
F_{21}(x,\ y,\ z) &= 0.35z + 0.1563y - 0.0001085xy + 65, \\
F_{22}(x,\ y,\ z) &= -0.15z + 0.0521x - 0.0001085xy + 145, \\
F_{23}(x,\ y,\ z) &= 0.15z + 0.0521y - 0.0001628xy + 115, \\
F_{31}(x,\ y,\ z) &= -0.3z + 0.2083y - 0.0007595xy + 130, \\
F_{32}(x,\ y,\ z) &= 0.25z - 0.2083x - 0.0521y + 0.0008138xy + 115, \\
F_{33}(x,\ y,\ z) &= -0.45z + 0.0521y + 0.0004883xy + 175.
\end{aligned}$$

Then, it is easy to check that the continuity condition (1.30) are satisfied for the given data. For example, $F_{33}(0, 0, 100) = F_{22}(192, 192, 140) = F_{23}(192, 0, 100) = F_{32}(0, 192, 100) = 130 = z_{22}$ *and so on.*

In the above IFS, the vertical scaling factors S are perturbed and the new vertical *scaling factors* S^* *are obtained as follows.*

$$S^* = S + \Delta$$
$$= \begin{pmatrix} 0.45 & 0.4 & -0.45 \\ 0.35 & -0.15 & 0.15 \\ -0.3 & 0.25 & -0.45 \end{pmatrix} + \begin{pmatrix} -0.15 & 0.1 & 0.15 \\ 0.15 & -0.15 & 0.05 \\ -0.1 & 0.05 & 0.15 \end{pmatrix}$$
$$= \begin{pmatrix} 0.3 & 0.5 & -0.3 \\ 0.5 & -0.3 & 0.2 \\ -0.4 & 0.3 & -0.3 \end{pmatrix}.$$

Then the perturbed IFS $\{D \times \mathbb{R},\ (u_i,\ v_j,\ T_{ij}) \ :\ i = 1,\ \dots,\ 3;\quad j = 1,\ \dots,\ 3\}$ *is constructed where,*

$$\begin{aligned}
T_{11}(x,\ y,\ z) &= 0.3z + 0.0003255xy + 55, \\
T_{12}(x,\ y,\ z) &= 0.5z + 0.1563x - 0.000434xy + 60, \\
T_{13}(x,\ y,\ z) &= -0.3z + 0.1563x + 0.0007595xy + 145, \\
T_{21}(x,\ y,\ z) &= 0.5z + 0.1563y - 0.0001085xy + 65, \\
T_{22}(x,\ y,\ z) &= -0.3z + 0.0521x - 0.0001085xy + 145, \\
T_{23}(x,\ y,\ z) &= 0.2z + 0.0521y - 0.0001628xy + 115, \\
T_{31}(x,\ y,\ z) &= -0.4z + 0.2083y - 0.0007595xy + 130, \\
T_{32}(x,\ y,\ z) &= 0.3z - 0.2083x - 0.0521y + 0.0008138xy + 115, \\
T_{33}(x,\ y,\ z) &= -0.3z + 0.0521y + 0.0004883xy + 175.
\end{aligned}$$

Observe that $T_{33}(0,0,100) = 145,\ T_{22}(192,192,140) = 109$ *and hence* $T_{33}(0,0,100) \neq T_{22}(192,192,140)$. *Therefore, the continuity condition is not satisfied by the perturbed IFS.*

In the following, we find conditions so that the perturbed IFS will satisfy the continuity condition. The main aim of the following theorem is that instead of changing all the coefficients of IFS, only modification at constant term is sufficient to ensure the existence of fractal function.

Theorem 1.9.1. *Consider the perturbed IFS (1.33) of the IFS (1.31). Let* $M_{ij}(x,\ y,\ z) = T_{ij}(x,\ y,\ z)+\lambda_{ij}$ *for* $i = 1, 2, \ldots, N$ *and* $j = 1, 2, \ldots, M$*, where* λ_{ij}*'s are real constants. Assume that* $\lambda_{11} = 0$.

1. *If* λ_{ij} *satisfies*
$$M_{ij}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} = M_{(i+1)j}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix}$$
then
$$\lambda_{ij} = \left(\sum_{k=1}^{i-1}\delta_{kj}\right) z_{N0} - \left(\sum_{k=2}^{i}\delta_{kj}\right) z_{00} + \lambda_{1j}\,. \tag{1.34}$$

2. *If* λ_{ij} *satisfies*
$$M_{ij}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix} = M_{i(j+1)}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix}$$
then
$$\lambda_{ij} = \left(\sum_{k=1}^{j-1}\delta_{ik}\right) z_{0M} - \left(\sum_{k=2}^{j}\delta_{ik}\right) z_{00} + \lambda_{i1}\,. \tag{1.35}$$

3. *If* λ_{ij} *satisfies*
$$M_{ij}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} = M_{i(j+1)}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix}$$
then
$$\lambda_{ij} = \left(\sum_{k=1}^{j-1}\delta_{ik}\right) z_{NM} - \left(\sum_{k=2}^{j}\delta_{ik}\right) z_{N0} + \lambda_{i1}\,. \tag{1.36}$$

4. *If* λ_{ij} *satisfies*
$$M_{ij}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} = M_{(i+1)j}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix}$$
then
$$\lambda_{ij} = \left(\sum_{k=1}^{i-1}\delta_{kj}\right) z_{NM} - \left(\sum_{k=2}^{i}\delta_{kj}\right) z_{0M} + \lambda_{1j}\,. \tag{1.37}$$

5. *If* λ_{ij} *satisfies*
$$M_{i(j+1)}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} = M_{(i+1)j}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix}$$

then

$$\lambda_{(i+1)j} \;=\; \delta_{i(j+1)} z_{N0} \;-\; \delta_{(i+1)j}\; z_{0M} \;+\; \lambda_{i(j+1)} \;. \tag{1.38}$$

6. If λ_{ij} satisfies

$$M_{ij}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} = M_{(i+1)(j+1)}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix}$$

then

$$\lambda_{(i+1)(j+1)} \;=\; \delta_{ij}\; z_{NM} \;-\; \delta_{(i+1)(j+1)}\; z_{00} \;+\; \lambda_{ij} \;. \tag{1.39}$$

Then the changed IFS

$$\{D \times \mathbb{R},\; (u_i,\; v_j,\; M_{ij}) \;\; : \; i = 1,\; \ldots,\; N; \;\; j = 1,\; \ldots,\; M\} \tag{1.40}$$

satisfies the fractal interpolation continuous condition. We denote the function whose graph is the attractor of the IFS (1.40) by $f_\delta(x,\; y)$.

Proof. 1. The fractal interpolation continuous condition

$$M_{ij}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} = M_{(i+1)j}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix}$$

gives

$$\begin{aligned} \delta_{ij}\; z_{N0} \;+\; \lambda_{ij} \;&=\; \delta_{(i+1)j}\; z_{00} \;+\; \lambda_{(i+1)j} \;, \\ \lambda_{(i+1)j} \;&=\; \delta_{ij}\; z_{N0} \;-\; \delta_{(i+1)j}\; z_{00} \;+\; \lambda_{ij} \;. \end{aligned}$$

Now,

$$\begin{aligned} \lambda_{ij} \;&=\; \delta_{(i-1)j}\; z_{N0} \;-\; \delta_{ij}\; z_{00} \;+\; \lambda_{(i-1)j} \;, \\ \lambda_{(i-1)j} \;&=\; \delta_{(i-2)j}\; z_{N0} \;-\; \delta_{(i-1)j}\; z_{00} \;+\; \lambda_{(i-2)j} \;, \\ \cdots \;&=\; \cdots \\ \lambda_{2j} \;&=\; \delta_{1j}\; z_{N0} \;-\; \delta_{2j}\; z_{00} \;+\; \lambda_{1j} \;. \end{aligned}$$

Substituting the values of $\lambda_{1j}, \ldots, \lambda_{(i-1)j}$ in λ_{ij} we get

$$\lambda_{ij} = \left(\sum_{k=1}^{i-1} \delta_{kj} \right) z_{N0} - \left(\sum_{k=2}^{i} \delta_{kj} \right) z_{00} + \lambda_{1j} .$$

The proofs of the remaining other cases are similar to the case 1. □

Note:

1. Putting $j = 1$ in (1.37) and substituting λ_{i1} in (1.36), we get

$$\lambda_{ij} = \left(\sum_{k=1}^{j-1} \delta_{ik} + \sum_{k=1}^{i-1} \delta_{k1} \right) z_{NM} - \left(\sum_{k=2}^{j} \delta_{ik} \right) z_{N0} - \left(\sum_{k=2}^{i} \delta_{k1} \right) z_{0M} . \quad (1.41)$$

   Putting $i = 1$ in (1.36) and substituting λ_{1j} in (1.37), we get

$$\lambda_{ij} = \left(\sum_{k=1}^{i-1} \delta_{kj} + \sum_{k=1}^{j-1} \delta_{1k} \right) z_{NM} - \left(\sum_{k=2}^{i} \delta_{kj} \right) z_{0M} - \left(\sum_{k=2}^{j} \delta_{1k} \right) z_{N0} . \quad (1.42)$$

2. For the boundary points $\{(x_i, y_0, z_{i0}) : i = 0, 1, \ldots, N\}$, (1.34) is used to check the continuity of M_{ij}'s $(1 \le i \le N,\ j = 1)$.

3. For the boundary points $\{(x_0, y_j, z_{0j}) : j = 0, 1, \ldots, M\}$, (1.35) is used to check the continuity of M_{ij}'s $(i = 1,\ 1 \le j \le M)$.

4. For the boundary points $\{(x_N, y_j, z_{Nj}) : j = 0, 1, \ldots, M\}$, (1.41) is used to check the continuity of M_{ij}'s.

5. For the boundary points $\{(x_i, y_M, z_{iM}) : i = 0, 1, \ldots, N\}$, (1.42) is used to check the continuity of M_{ij}'s.

6. For other fractal interpolation points, (1.38) or (1.39) or (1.41) is used to determine λ_{ij}.

Example 1.2. *Consider the IFS and the perturbed IFS given in Example 1.1. Using Theorem 1.9.1, we compute λ_{ij}'s, M_{ij}'s and show that the fractal interpolation continuity conditions are satisfied.*

Since $\lambda_{11} = 0$, *we have* $M_{11}(192, 192, 140) = 109 + \lambda_{11} = 109 + 0 = 109$.

Using (1.41), we get $\lambda_{12} = -31$ *and* $M_{12}(192, 0, 100) = 140 + \lambda_{12} = 109$.

Using (1.38), we get $\lambda_{21} = -36$ *and* $M_{21}(0, 192, 100) = 145 + \lambda_{21} = 109$.

Using (1.39), we get $\lambda_{22} = -6$ *and* $M_{22}(0, 0, 100) = 115 + \lambda_{22} = 109$.

Using (1.41), we get $\lambda_{12} = -31$ *and* $M_{12}(192, 192, 140) = 144 + \lambda_{12} = 113$.

Using (1.41), we get $\lambda_{13} = -32$ *and* $M_{13}(192, 0, 100) = 145 + \lambda_{13} = 113$.

Using (1.38), we get $\lambda_{22} = -2$ *and* $M_{22}(0, 192, 100) = 115 + \lambda_{22} = 113$.

Using (1.39), we get $\lambda_{23} = -22$ *and* $M_{23}(0, 0, 100) = 135 + \lambda_{23} = 113$

Using (1.41), we get $\lambda_{21} = -36$ *and* $M_{21}(192, 192, 140) = 161 + \lambda_{21} = 125$.

Using (1.41), we get $\lambda_{22} = 0$ *and* $M_{22}(192, 0, 100) = 125 + \lambda_{22} = 125$.

Using (1.38), we get $\lambda_{31} = -5$ *and* $M_{31}(0, 192, 100) = 130 + \lambda_{31} = 125$.

Using (1.39), we get $\lambda_{32} = -20$ *and* $M_{32}(0, 0, 100) = 145 + \lambda_{32} = 125$.

Using (1.41), we get $\lambda_{22} = 0$ *and* $M_{22}(192, 192, 140) = 109 + \lambda_{22} = 109$.

Using (1.41), we get $\lambda_{23} = -26$ *and* $M_{23}(192, 0, 100) = 135 + \lambda_{23} = 109$.

Using (1.38), we get $\lambda_{32} = -26$ *and* $M_{32}(0, 192, 100) = 135 + \lambda_{32} = 109$.

Using (1.39), we get $\lambda_{33} = -36$ *and* $M_{33}(0, 0, 100) = 145 + \lambda_{33} = 109$.

It can be easily checked that the continuity condition of M_{ij}*'s in (1.40) are satisfied on the boundary points of the rectangular domain by using the value of* λ_{ij} *in (1.40) computed from (1.34) or (1.35) or (1.41) or (1.42).*

Therefore, it is concluded that the IFS (1.40) satisfies the fractal interpolation continuous condition when adding corresponding λ_{ij}.

1.9.3 Error estimation

In this section, the estimate on error between the FIF $f(x,\ y)$ of the IFS (1.31) and the FIF $f_\delta(x,\ y)$ of the perturbed IFS (1.40) is computed.

Set $\mathcal{N} = \{1,\ 2,\ \ldots,\ N\}$ and $\mathcal{M} = \{1,\ 2,\ \ldots,\ M\}$. For $i_k \in \mathcal{N}$, $k = 1,\ 2,\ \ldots,$ n, define $u_{i_1 i_2 \cdots i_n}(x) = u_{i_1} \circ u_{i_2} \circ \cdots u_{i_n}(x)$ and for $j_k \in \mathcal{M}$, $k = 1,\ 2,\ \ldots,\ n$ define $v_{j_1 j_2 \cdots j_n}(y) = v_{j_1} \circ v_{j_2} \circ \cdots \circ v_{j_n}(y)$.

Lemma 1.9.1. *Let $f(x,y)$ and $f_\delta(x,y)$ be the bivariate FIFs generated by the IFS (1.31) and (1.40) respectively. Then for any $(x,y) \in D$, $i_k \in \mathcal{N}$, $j_k \in \mathcal{M}$, $k = 1,\ 2,\ \ldots,\ n$, we have,*

$$u_{i_1 i_2 \cdots i_n}(x) = \left(\prod_{k=1}^{n} a_{i_k}\right)\ x\ + \sum_{r=1}^{n}\left(\prod_{k=1}^{r-1} a_{i_k}\right)\ b_{i_r}, \tag{1.43}$$

$$v_{j_1 j_2 \cdots j_n}(y) = \left(\prod_{k=1}^{n} c_{j_k}\right)\ y\ + \sum_{r=1}^{n}\left(\prod_{k=1}^{r-1} c_{j_k}\right)\ d_{j_r}, \tag{1.44}$$

$$\begin{aligned}
f_\delta(u_{i_1 i_2 \cdots i_n}(x),\ v_{j_1 j_2 \cdots j_n}(y)) \ &= \ \left(\prod_{k=1}^{n} (s_{i_k j_k} + \delta_{i_k j_k})\right)\ f_\delta(x,y) \\
&+ \left(\prod_{k=1}^{n-1} (s_{i_k j_k} + \delta_{i_k j_k})\right)\ \phi_{i_n j_n}(x,\ y) \\
&+ \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r-1} (s_{i_k j_k} + \delta_{i_k j_k})\right)\ \phi_{i_r j_r}(\overline{x},\ \overline{y}) \\
&+ \sum_{r=1}^{n}\left(\prod_{k=1}^{r-1} (s_{i_k j_k} + \delta_{i_k j_k})\right)\ \lambda_{i_r j_r},
\end{aligned}$$

where

$$\overline{x} = u_{i_{r+1} \ldots i_n}(x) = \left(\prod_{k=1}^{n-r} a_{i_{r+k}}\right) x + \sum_{l=1}^{n-r}\left(\prod_{k=1}^{l-1} a_{i_{r+k}}\right) b_{i_{r+l}}\ , \tag{1.45}$$

and

$$\overline{y} = v_{j_{r+1} \ldots j_n}(y) = \left(\prod_{k=1}^{n-r} c_{j_{r+k}}\right) y + \sum_{l=1}^{n-r}\left(\prod_{k=1}^{l-1} c_{j_{r+k}}\right) d_{j_{r+l}}\ . \tag{1.46}$$

Proof.

$$
\begin{aligned}
u_{i_1 i_2 \ldots i_n}(x) &= u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_n}(x) \\
&= u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_{n-1}}(a_{i_n} x + b_{i_n}) \\
&= u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_{n-2}}(a_{i_{n-1}}(a_{i_n} x + b_{i_n}) + b_{i_{n-1}}) \\
\vdots \;\; \vdots \;\; \vdots & \quad\; \vdots \\
&= \left(\prod_{k=1}^{n} a_{i_k} \right) x + \sum_{r=1}^{n} \left(\prod_{k=1}^{r-1} a_{i_k} \right) b_{i_r}.
\end{aligned}
$$

Similarly,

$$v_{j_1 j_2 \cdots j_n}(y) = \left(\prod_{k=1}^{n} c_{j_k} \right) x + \sum_{r=1}^{n} \left(\prod_{k=1}^{r-1} c_{j_k} \right) d_{j_r} .$$

Now

$$f(x,y) = s_{ij} f(u_i^{-1}(x), v_j^{-1}(y)) + \phi_{ij}(u_i^{-1}(x), v_j^{-1}(y)) \text{ for all } (x,y) \in D_{ij},$$

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$.

Then

$$f(u_{i_1}(x), v_{j_1}(y)) = s_{i_1 j_1} f(x,y) + \phi_{i_1 j_1}(x,y)$$

$$
\begin{aligned}
f(u_{i_1 i_2}(x), v_{j_1 j_2}(y)) &= s_{i_1 j_1} f(u_{i_2}(x), v_{j_2}(y)) + \phi_{i_1 j_1}(u_{i_2}(x), v_{j_2}(y)) \\
&= s_{i_1 j_1}(s_{i_2 j_2} f(x,y) + \phi_{i_2 j_2}(x,y)) + \phi_{i_1 j_1}(u_{i_2}(x), v_{j_2}(y)) \\
&= s_{i_1 j_1} s_{i_2 j_2} f(x,y) + s_{i_1 j_1} \phi_{i_2 j_2}(x,y) + \phi_{i_1 j_1}(u_{i_2}(x), v_{j_2}(y))
\end{aligned}
$$

$$
\begin{aligned}
f(u_{i_1 i_2 i_3}(x), v_{j_1 j_2 i_3}(y)) &= s_{i_1 j_1} s_{i_2 j_2} f(u_{i_3}(x), v_{j_3}(y)) + s_{i_1 j_1} \phi_{i_2 j_2}(u_{i_3}(x), v_{j_3}(y)) \\
&+ \phi_{i_1 j_1}(u_{i_2 i_3}(x), v_{j_2 j_3}(y)) \\
&= s_{i_1 j_1} s_{i_2 j_2} s_{i_3 j_3} f(x,y) + s_{i_1 j_1} s_{i_2 j_2} \phi_{i_3 j_3}(x,y) \\
&+ s_{i_1 j_1} \phi_{i_2 j_2}(u_{i_3}(x), v_{j_3}(y)) + \phi_{i_1 j_1}(u_{i_2 i_3}(x), v_{j_2 j_3}(y)).
\end{aligned}
$$

In general,

$$\begin{aligned} f(u_{i_1i_2\dots i_n}(x), v_{j_1j_2\dots j_n}(y)) &= \left(\prod_{k=1}^{n} s_{i_kj_k}\right) f(x,y) + \left(\prod_{k=1}^{n-1} s_{i_kj_k}\right) \phi_{i_nj_n}(x,y) \\ &+ \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r-1} s_{i_kj_k}\right) \phi_{i_rj_r}(u_{i_{r+1}\dots i_n}(x), v_{j_{r+1}\dots j_n}(y)). \end{aligned}$$

Therefore

$$\begin{aligned} f(u_{i_1i_2\dots i_n}(x), v_{j_1j_2\dots j_n}(y)) &= \left(\prod_{k=1}^{n} s_{i_kj_k}\right) f(x,y) + \left(\prod_{k=1}^{n-1} s_{i_kj_k}\right) \phi_{i_nj_n}(x,y) \\ &+ \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r-1} s_{i_kj_k}\right) \phi_{i_rj_r}(\overline{x}, \overline{y}). \end{aligned} \tag{1.47}$$

Now

$$\begin{aligned} f_\delta(u_{i_1i_2\dots i_n}(x), v_{j_1j_2\dots j_n}(y)) &= \prod_{k=1}^{n} (s_{i_kj_k} + \delta_{i_kj_k})\, f_\delta(x,y) \\ &+ \left(\prod_{k=1}^{n-1} (s_{i_kj_k} + \delta_{i_kj_k})\right) (\phi_{i_nj_n}(x,y) + \lambda_{i_nj_n}) \\ &+ \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r-1} s_{i_kj_k} + \delta_{i_kj_k}\right) (\phi_{i_rj_r}(\overline{x},\ \overline{y}) + \lambda_{i_rj_r}) \\ &= \prod_{k=1}^{n} (s_{i_kj_k} + \delta_{i_kj_k})\, f_\delta(x,y) + \left(\prod_{k=1}^{n-1} (s_{i_kj_k} + \delta_{i_kj_k})\right) \phi_{i_nj_n}(x,y) \\ &+ \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r-1} (s_{i_kj_k} + \delta_{i_kj_k})\right) \phi_{i_rj_r}(\overline{x},\ \overline{y}) \\ &+ \sum_{r=1}^{n}\left(\prod_{k=1}^{r-1} (s_{i_kj_k} + \delta_{i_kj_k})\right) \lambda_{i_rj_r}\ . \end{aligned} \tag{1.48}$$

□

Theorem 1.9.2. *Let $f(x,y)$ and $f_\delta(x,y)$ be the bivariate FIFs generated with the IFSs (1.31) and (1.40) respectively. For any given $(x,y) \in D$, let $\{i_k\}$, $i_k \in \mathcal{N}$ be the sequence such that x satisfies*

$$x = \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1} a_{i_k}\right) b_{i_r} \tag{1.49}$$

and let $\{j_k\}$, $j_k \in \mathcal{M}$ be another sequence such that y satisfies

$$y = \sum_{r=1}^{\infty}(\prod_{k=1}^{r-1} c_{j_k})d_{j_r} \ . \tag{1.50}$$

Then

$$\begin{aligned} f_\delta(x,y) - f(x,y) &= \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_kj_k}+\delta_{i_kj_k}) - \prod_{k=1}^{r-1} s_{i_kj_k}\right)\phi_{i_rj_r}(x',y') \\ &+ \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_kj_k}+\delta_{i_kj_k})\right)\lambda_{i_rj_r} \ , \end{aligned} \tag{1.51}$$

where $x' = \sum_{l=1}^{\infty}(\prod_{k=1}^{l-1} a_{i_{r+k}})b_{i_{r+l}}$, $y' = \sum_{l=1}^{\infty}(\prod_{k=1}^{l-1} c_{j_{r+k}})d_{j_{r+l}}$.

Proof. Since $u_i : I \to I_i,\ i = 1,\ 2\ldots, N$ given by

$$u_i(x) = a_i x + b_i \ ,$$

where $a_i = \frac{x_i - x_{i-1}}{x_N - x_0}$, is contractive on the closed interval I, the sequence of sets $\{u_{i_1i_2\cdots i_n}(I)\}$ is monotonically decreasing with diameter tends to zero as $n \to \infty$. Hence by the Cantor's intersection theorem, $\bigcap_{n=1}^{\infty} u_{i_1i_2\cdots i_n}(I)$ consists of a single point in I for any sequence $\{i_k\}, i_k \in \mathcal{N}$. Then for any given $x \in I$, there exists a sequence $\{i_k\}, i_k \in \mathcal{N}$ such that

$$\{x\} = \bigcap_{n=1}^{\infty} u_{i_1i_2\cdots i_n}(I) = \lim_{n\to\infty} u_{i_1i_2\cdots i_n}(I) \ .$$

Since each a_{i_k} in (1.43) obeys $0 < a_{i_k} < 1$, x can be expressed as

$$x = \lim_{n\to\infty} u_{i_1i_2\cdots i_n}(x^*) = \sum_{r=1}^{\infty}(\prod_{k=1}^{r-1} a_{i_k})b_{i_r} \ ,$$

where x^* is arbitrarily chosen point in I. Similarly for any given $y \in J$, there exists another sequence $\{j_k\}$, $j_k \in \mathcal{M}$, such that y satisfies (1.50) .

Using (1.45), (1.46) and (1.47), $f(x,y)$ can be expressed for any $(x,y) \in D$ as

$$\begin{aligned} f(x\ ,\ y) &= \lim_{n\to\infty} f(u_{i_1i_2\cdots i_n}(x^*), v_{j_1j_2\cdots j_n}(y^*)) \\ &= \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1} s_{i_kj_k}\right)\phi_{i_rj_r}\left(\sum_{l=1}^{\infty}\left(\prod_{k=1}^{l-1} a_{i_{r+k}}\right)b_{i_{r+l}}, \sum_{l=1}^{\infty}\left(\prod_{k=1}^{l-1} c_{j_{r+k}}\right)d_{j_{r+l}}\right) \\ &= \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1} s_{i_kj_k}\right)\phi_{i_rj_r}(x',y'), \end{aligned} \tag{1.52}$$

where $x' = \sum_{l=1}^{\infty}\left(\prod_{k=1}^{l-1} a_{i_{r+k}}\right) b_{i_{r+l}}$, $y' = \sum_{l=1}^{\infty}\left(\prod_{k=1}^{l-1} c_{j_{r+k}}\right) d_{j_{r+l}}$ and $(x^*,\ y^*)$ are chosen arbitrarily in D.

From (1.45), (1.46), it is clear that $(x',\ y')$ belongs to D.

Similarly, by (1.48), we can get

$$f_\delta(x\ ,\ y) = \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_kj_k}+\delta_{i_kj_k})\right)\phi_{i_rj_r}(x',y') + \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_kj_k}+\delta_{i_kj_k})\right)\lambda_{i_rj_r}\ . \quad (1.53)$$

Therefore by (1.52) and (1.53)

$$\begin{aligned} f_\delta(x,y) - f(x,y) &= \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_kj_k}+\delta_{i_kj_k}) - \prod_{k=1}^{r-1} s_{i_kj_k}\right)\phi_{i_rj_r}(x',y') \\ &+ \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_kj_k}+\delta_{i_kj_k})\right)\lambda_{i_rj_r}, \end{aligned}$$

which completes the proof. □

Corollary 1.9.1. *Let $f(x,y)$ and $f_\delta(x,y)$ be the bivariate FIFs generated by the IFS (1.31) and (1.40) respectively. Let $s = \max\left\{|s_{ij}| : i \in \mathcal{N}, j \in \mathcal{M}\right\} < 1$, $0 < \delta = \max\left\{|\delta_{ij}|, i \in \mathcal{N}, j \in \mathcal{M}\right\}$ such that $s+\delta < 1$ and $A = \max\left\{\|\phi_{ij}\| : i \in \mathcal{N}, j \in \mathcal{M}\right\}$ where $\|\phi_{ij}\| = \max\left\{|\phi_{ij}(x,y)| : (x,y) \in D\right\}$. Then*

$$\left|f_\delta(x,y) - f(x,y)\right| \le \frac{\delta\left\{A + (1-s)(N+M)\left(|z_{NM}|+|z_{N0}|+|z_{0M}|+|z_{00}|\right)\right\}}{(1-s)(1-s-\delta)}\ .$$

Proof. (1.34), (1.35), (1.38), (1.39), (1.41) and (1.42) together imply,

$$\max_{n\in\mathcal{N}, m\in\mathcal{M}} |\lambda_{nm}| \le (N+M)\delta\left(|z_{NM}|+|z_{N0}|+|z_{0M}|+|z_{00}|\right)\ .$$

Therefore, by (1.51),

$$\begin{aligned} |f_\delta(x,y) - f(x,y)| &\le \frac{A}{1-s-\delta} - \frac{A}{1-s} + \frac{(N+M)\delta\left(|z_{NM}|+|z_{N0}|+|z_{0M}|+|z_{00}|\right)}{1-s-\delta} \\ &= \frac{\delta\left\{A + (1-s)(N+M)\left(|z_{NM}|+|z_{N0}|+|z_{0M}|+|z_{00}|\right)\right\}}{(1-s)(1-s-\delta)}\ . \end{aligned}$$

This completes the proof. □

1.10 Graph-directed coalescence hidden variable fractal interpolation functions

In the present section, a research work carried out by us related to graph-directed coalescence hidden variable FIF, which generalizes the self-affine graph-directed FIF in [2]. The idea of self-affine hidden variable fractal interpolation function has been extended to a generalized data set in $\mathbb{R}^3$ such that the projection of the graph of the corresponding FIF onto $\mathbb{R}^2$ provides a non-self-affine interpolation function namely hidden variable FIF for a given data set $\{(x_i, y_i) : i = 0, 1, \ldots, N\}$ [9]. Chand and Kapoor [18], introduced the concept of coalescence hidden variable FIF which is both self-affine and non-self-affine for generalized IFS. The extra degree of freedom is useful to adjust the shape and fractal dimension of the interpolation function. For coalescence hidden variable fractal interpolation surface, one can see [49,50]. In [28], Deniz et al. considered graph-directed iterated function system (GDIFS) for finite number of data sets and proved the existence of fractal functions interpolating corresponding data sets with graphs as the attractors of the GDIFS.

In the present work, we considered generalized GDIFS for generalized interpolation data sets in $\mathbb{R}^3$. Corresponding to the data sets, it is proved that there exist coalescence hidden variable fractal interpolation functions (CHFIFs) whose graphs are the projections of the attractors of the GDIFS on $\mathbb{R}^2$.

1.10.1 Coalescence hidden variable FIF

To construct a coalescence hidden variable FIF, a set of real parameters z_i for $i = 1, 2, \ldots, N$ is introduced and the generalized interpolation data $\{(x_i, y_i, z_i) \in \mathbb{R}^3 : i = 0, 1, \ldots, N\}$ is considered. Then, define the maps $w_i : I \times \mathbb{R}^2 \to I_i \times \mathbb{R}^2$, $i = 1, 2, \ldots, N$ by

$$w_i(x, y, z) = \big(L_i(x), F_i(x, y, z)\big) ,$$

where $L_i : I \to I_i,\ i = 1,\ 2,\ \ldots, N$ are given in (1.8) and the functions $F_i : I \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_i(x, y, z) = \big(F_i^1(x, y, z), F_i^2(x, y, z)\big) = (\alpha_i y + \beta_i z + c_i x + d_i, \gamma_i z + e_i x + f_i)$ satisfies the join-up conditions

$$F_i(x_0, y_0, z_0) = (y_{i-1}, z_{i-1}) \text{ and } F_i(x_N, y_N, z_N) = (y_i, z_i)\ .$$

Here α_i and γ_i are free variables with $|\alpha_i| < 1$, $|\gamma_i| < 1$ and β_i is a constrained variable such that $|\beta_i| + |\gamma_i| < 1$. Then the generalized IFS

$$\big\{I \times \mathbb{R}^2;\ w_i(x, y, z)\ :\ i = 1,\ 2,\ \ldots, N\big\}$$

has an attractor G such that

$$G = \bigcup_{i=1}^{N} w_i(G) = \bigcup_{i=1}^{N} \{w_i(x, y, z)\ :\ (x, y, z) \in G\}\ .$$

The attractor G is the graph of a vector valued function $f : I \to \mathbb{R}^2$ such that $f(x_i) = (y_i, z_i)$ for $i = 0,\ 1,\ \ldots,\ N$ and $G = \{(x, f(x))\ :\ x \in I, f(x) = (y(x), z(x))\}$. If $f = (f_1, f_2)$, then the projection of the attractor G on $\mathbb{R}^2$ is the graph of the function f_1 which satisfies $f_1(x_i) = y_i$ and is of the form

$$f_1(L_i(x)) = F_i^1\big(x, f_1(x), f_2(x)\big) = \alpha_i f_1(x) + \beta_i f_2(x) + c_i x + d_i,\ x \in I$$

also known as CHFIF corresponding to the data $\big\{(x_i, y_i) \in I \times \mathbb{R}\ :\ i = 0,\ 1,\ \ldots,\ N\big\}$ [18].

1.10.2 Graph-directed iterated function systems

Let $G = (V, E)$ be a directed graph where V is the set of vertices and E is the set of edges. For all $u, v \in V$, let E^{uv} denote the set of edges from u to v with elements $e_i^{uv},\ i = 1,\ 2,\ \ldots,\ K^{uv}$, where K^{uv} is the number of elements of E^{uv}. An iterated function system realizing the graph G is given by a collection of metric spaces $(X^v, \rho^v),\ v \in V$ with contraction mapping $w_i^{uv} : X^v \to X^u$ corresponding to the edge e_i^{uv} in the opposite direction of e_i^{uv}. An attractor (or invariant list) for such an iterated function system is a list of non-empty compact sets $A^u \subset X^u$ such that for all $u \in V$,

$$A^u = \bigcup_{v \in V} \bigcup_{i=1}^{K^{uv}} w_i^{uv}(A^v) \ .$$

Then, $(X^u; w_i^{uv})$ is the graph directed iterated function system (GDIFS) realizing the graph G [32, 59].

Example 1.3. *An example of GDIFS can be seen in [27, 28].*

1.10.3 Graph-directed coalescence hidden variable FIF

In this section, for a finite number of data sets, generalized graph-directed iterated function system is defined so that the projection of each attractor on $\mathbb{R}^2$ is the graph of a CHFIF which interpolates the corresponding data set and call it as graph-directed coalescence hidden variable fractal interpolation function (GDCHFIF). For simplicity, only two sets of data are considered. Let the two data sets be

$$D^1 = \left\{(x_0^1, y_0^1), (x_1^1, y_1^1), \ldots, (x_N^1, y_N^1)\right\} ,$$

$$D^2 = \left\{(x_0^2, y_0^2), (x_1^2, y_1^2), \ldots, (x_M^2, y_M^2)\right\} ,$$

where $N, M \geq 2$ with

$$\frac{x_i^1 - x_{i-1}^1}{x_M^2 - x_0^2} < 1 \text{ and } \frac{x_j^2 - x_{j-1}^2}{x_N^1 - x_0^1} < 1 \tag{1.54}$$

for all $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. By introducing two sets of real parameters z_i^1, z_j^2 for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$, consider the two generalized data sets

$$\mathcal{D}^1 = \left\{(x_0^1, y_0^1, z_0^1), (x_1^1, y_1^1, z_1^1), \ldots, (x_N^1, y_N^1, z_N^1)\right\} ,$$

$$\mathcal{D}^2 = \left\{(x_0^2, y_0^2, z_0^2), (x_1^2, y_1^2, z_1^2), \ldots, (x_M^2, y_M^2, z_M^2)\right\}$$

corresponding to D^1 and D^2 respectively. Also consider the directed graph $G = (V, E)$ with $V = \{1, 2\}$ such that

$$K^{11} + K^{12} = N \text{ and } K^{21} + K^{22} = M \ .$$

To construct a generalized GDIFS associated with the data $\mathcal{D}^r, (r = 1, 2)$ and realizing the graph G, consider the functions $w_i^{rs} : \mathbb{R}^3 \to \mathbb{R}^3$, defined as

$$w_i^{rs}(x, y, z) = \big(L_i^{rs}(x), F_i^{rs}(x, y, z)\big),\ i = 1, 2, \ldots, K^{rs},\ r, s \in \{1, 2\} \tag{1.55}$$

are such that

- $\begin{cases} w_i^{11}(x_0^1, y_0^1, z_0^1) = (x_{i-1}^1, y_{i-1}^1, z_{i-1}^1) \\ w_i^{11}(x_N^1, y_N^1, z_N^1) = (x_i^1, y_i^1, z_i^1) \end{cases}$ for $i = 1, 2, \ldots, K^{11}$,

- $\begin{cases} w_{i-K^{11}}^{12}(x_0^2, y_0^2, z_0^2) = (x_{i-1}^1, y_{i-1}^1, z_{i-1}^1) \\ w_{i-K^{11}}^{12}(x_M^2, y_M^2, z_M^2) = (x_i^1, y_i^1, z_i^1) \end{cases}$ for $i = K^{11} + 1, \ldots, K^{11} + K^{12} = N$,

- $\begin{cases} w_i^{21}(x_0^1, y_0^1, z_0^1) = (x_{i-1}^2, y_{i-1}^2, z_{i-1}^2) \\ w_i^{21}(x_N^1, y_N^1, z_N^1) = (x_i^2, y_i^2, z_i^2) \end{cases}$ for $i = 1, 2, \ldots, K^{21}$,

- $\begin{cases} w_{i-K^{21}}^{22}(x_0^2, y_0^2, z_0^2) = (x_{i-1}^2, y_{i-1}^2, z_{i-1}^2) \\ w_{i-K^{21}}^{22}(x_M^2, y_M^2, z_M^2) = (x_i^2, y_i^2, z_i^2) \end{cases}$ for $i = K^{21} + 1, \ldots, K^{21} + K^{22} = M$.

From each of the above conditions, the following can be derived respectively.

$$\begin{cases} a_i^{11}x_0^1 + b_i^{11} = x_{i-1}^1 \\ a_i^{11}x_N^1 + b_i^{11} = x_i^1 \\ c_i^{11}x_0^1 + \alpha_i^{11}y_0^1 + \beta_i^{11}z_0^1 + d_i^{11} = y_{i-1}^1 \\ c_i^{11}x_N^1 + \alpha_i^{11}y_N^1 + \beta_i^{11}z_N^1 + d_i^{11} = y_i^1 \\ e_i^{11}x_0^1 + \gamma_i^{11}z_0^1 + f_i^{11} = z_{i-1}^1 \\ e_i^{11}x_N^1 + \gamma_i^{11}z_N^1 + f_i^{11} = z_i^1 \end{cases} \text{for } i = 1, 2, \ldots, K^{11}, \tag{1.56}$$

$$\begin{cases} a_{i-K^{11}}^{12}x_0^2 + b_{i-K^{11}}^{12} = x_{i-1}^1 \\ a_{i-K^{11}}^{12}x_M^2 + b_{i-K^{11}}^{12} = x_i^1 \\ c_{i-K^{11}}^{12}x_0^2 + \alpha_{i-K^{11}}^{12}y_0^2 + \beta_{i-K^{11}}^{12}z_0^2 + d_{i-K^{11}}^{12} = y_{i-1}^1 \\ c_{i-K^{11}}^{12}x_M^2 + \alpha_{i-K^{11}}^{12}y_M^2 + \beta_{i-K^{11}}^{12}z_M^2 + d_{i-K^{11}}^{12} = y_i^1 \\ e_{i-K^{11}}^{12}x_0^2 + \gamma_{i-K^{11}}^{12}z_0^2 + f_{i-K^{11}}^{12} = z_{i-1}^1 \\ e_{i-K^{11}}^{12}x_M^2 + \gamma_{i-K^{11}}^{12}z_M^2 + f_{i-K^{11}}^{12} = z_i^1 \end{cases} \text{for } i = K^{11} + 1, \ldots, N, \tag{1.57}$$

$$\begin{cases} a_i^{21}x_0^1 + b_i^{21} = x_{i-1}^2 \\ a_i^{21}x_N^1 + b_i^{21} = x_i^2 \\ c_i^{21}x_0^1 + \alpha_i^{21}y_0^1 + \beta_i^{21}z_0^1 + d_i^{21} = y_{i-1}^2 \\ c_i^{21}x_N^1 + \alpha_i^{21}y_N^1 + \beta_i^{21}z_N^1 + d_i^{21} = y_i^2 \\ e_i^{21}x_0^1 + \gamma_i^{21}z_0^1 + f_i^{21} = z_{i-1}^2 \\ e_i^{21}x_N^1 + \gamma_i^{21}z_N^1 + f_i^{21} = z_i^2 \end{cases} \text{for } i = 1, 2, \ldots, K^{21}, \tag{1.58}$$

$$\begin{cases} a^{22}_{i-K^{21}} x^2_0 + b^{22}_{i-K^{21}} = x^2_{i-1} \\ a^{22}_{i-K^{21}} x^2_M + b^{22}_{i-K^{21}} = x^2_i \\ c^{22}_{i-K^{21}} x^2_0 + \alpha^{22}_{i-K^{21}} y^2_0 + \beta^{22}_{i-K^{21}} z^2_0 + d^{22}_{i-K^{21}} = y^2_{i-1} \\ c^{22}_{i-K^{21}} x^2_M + \alpha^{22}_{i-K^{21}} y^2_M + \beta^{22}_{i-K^{21}} z^2_M + d^{22}_{i-K^{21}} = y^2_i \\ e^{22}_{i-K^{21}} x^2_0 + \gamma^{22}_{i-K^{21}} z^2_0 + f^{22}_{i-K^{21}} = z^2_{i-1} \\ e^{22}_{i-K^{21}} x^2_M + \gamma^{22}_{i-K^{21}} z^2_M + f^{22}_{i-K^{21}} = z^2_i \end{cases} \text{for } i = K^{21}+1, \ldots, M. \quad (1.59)$$

From the linear system of equations (1.56), (1.57), (1.58) and (1.59) the constants a^{rs}_i, b^{rs}_i, c^{rs}_i, d^{rs}_i, e^{rs}_i and f^{rs}_i for $r, s \in \{1, 2\}$, $i = 1, 2, \ldots, K^{rs}$ are determined as follows.

$$\begin{aligned} a^{11}_i &= \frac{x^1_i - x^1_{i-1}}{x^1_N - x^1_0}, \\ b^{11}_i &= \frac{x^1_N x^1_{i-1} - x^1_0 x^1_i}{x^1_N - x^1_0}, \\ c^{11}_i &= \frac{y^1_i - y^1_{i-1} - \alpha^{11}_i (y^1_N - y^1_0) - \beta^{11}_i (z^1_N - z^1_0)}{x^1_N - x^1_0}, \\ d^{11}_i &= \frac{x^1_N y^1_{i-1} - x^1_0 y^1_i - \alpha^{11}_i (x^1_N y^1_0 - x^1_0 y^1_N) - \beta^{11}_i (x^1_N z^1_0 - x^1_0 z^1_N)}{x^1_N - x^1_0}, \\ e^{11}_i &= \frac{z^1_i - z^1_{i-1} - \gamma^{11}_i (z^1_N - z^1_0)}{x^1_N - x^1_0}, \\ f^{11}_i &= \frac{x^1_N z^1_{i-1} - x^1_0 z^1_i - \gamma^{11}_i (x^1_N z^1_0 - x^1_0 z^1_N)}{x^1_N - x^1_0}, \end{aligned}$$

$$\begin{aligned} a^{12}_i &= \frac{x^1_i - x^1_{i-1}}{x^2_M - x^2_0}, \\ b^{12}_i &= \frac{x^2_M x^1_{i-1} - x^2_0 x^1_i}{x^2_M - x^2_0}, \\ c^{12}_i &= \frac{y^1_i - y^1_{i-1} - \alpha^{12}_i (y^2_M - y^2_0) - \beta^{12}_i (z^2_M - z^2_0)}{x^2_M - x^2_0}, \\ d^{12}_i &= \frac{x^2_M y^1_{i-1} - x^2_0 y^1_i - \alpha^{12}_i (x^2_M y^2_0 - x^2_0 y^2_M) - \beta^{12}_i (x^2_M z^2_0 - x^2_0 z^2_M)}{x^2_M - x^2_0}, \\ e^{12}_i &= \frac{z^1_i - z^1_{i-1} - \gamma^{12}_i (z^2_M - z^2_0)}{x^2_M - x^2_0}, \\ f^{12}_i &= \frac{x^2_M z^1_{i-1} - x^2_0 z^1_i - \gamma^{12}_i (x^2_M z^2_0 - x^2_0 z^2_M)}{x^2_M - x^2_0}, \end{aligned}$$

$$\begin{aligned} a^{21}_i &= \frac{x^2_i - x^2_{i-1}}{x^1_N - x^1_0}, \\ b^{21}_i &= \frac{x^1_N x^2_{i-1} - x^1_0 x^2_i}{x^1_N - x^1_0}, \\ c^{21}_i &= \frac{y^2_i - y^2_{i-1} - \alpha^{21}_i (y^1_N - y^1_0) - \beta^{21}_i (z^1_N - z^1_0)}{x^1_N - x^1_0}, \\ d^{21}_i &= \frac{x^1_N y^2_{i-1} - x^1_0 y^2_i - \alpha^{21}_i (x^1_N y^1_0 - x^1_0 y^1_N) - \beta^{21}_i (x^1_N z^1_0 - x^1_0 z^1_N)}{x^1_N - x^1_0}, \\ e^{21}_i &= \frac{z^2_i - z^2_{i-1} - \gamma^{21}_i (z^1_N - z^1_0)}{x^1_N - x^1_0}, \\ f^{21}_i &= \frac{x^1_N z^2_{i-1} - x^1_0 z^2_i - \gamma^{21}_i (x^1_N z^1_0 - x^1_0 z^1_N)}{x^1_N - x^1_0}, \end{aligned}$$

$$
\begin{aligned}
a_i^{22} &= \frac{x_i^2-x_{i-1}^2}{x_M^2-x_0^2}, \\
b_i^{22} &= \frac{x_M^2x_{i-1}^2-x_0^2x_i^2}{x_M^2-x_0^2}, \\
c_i^{22} &= \frac{y_i^2-y_{i-1}^2-\alpha_i^{22}(y_M^2-y_0^2)-\beta_i^{22}(z_M^2-z_0^2)}{x_M^2-x_0^2}, \\
d_i^{22} &= \frac{x_M^2y_{i-1}^2-x_0^2y_i^2-\alpha_i^{22}(x_M^2y_0^2-x_0^2y_M^2)-\beta_i^{22}(x_M^2z_0^2-x_0^2z_M^2)}{x_M^2-x_0^2}, \\
e_i^{22} &= \frac{z_i^2-z_{i-1}^2-\gamma_i^{22}(z_M^2-z_0^2)}{x_M^2-x_0^2}, \\
f_i^{22} &= \frac{x_M^2z_{i-1}^2-x_0^2z_i^2-\gamma_i^{22}(x_M^2z_0^2-x_0^2z_M^2)}{x_M^2-x_0^2}.
\end{aligned}
$$

The following theorem shows that each map w_i^{rs} is contraction with respect to metric equivalent to the Euclidean metric and ensures the existence of attractors of generalized GDIFS.

Theorem 1.10.1. *Let $\{\mathbb{R}^3; w_i^{rs}, i = 1, 2, \ldots, K^{rs}\}$ be the generalized GDIFS defined in (1.55) realizing the graph $\mathcal{G}_r$ and associated with the data sets $\mathcal{D}^r$, $(r = 1, 2)$ which satisfy (1.54). If $|\alpha_i^{rs}| < 1$, $|\gamma_i^{rs}| < 1$ and β_i^{rs} is chosen such that $|\beta_i^{rs}| + |\gamma_i^{rs}| < 1$ for all $r, s \in \{1, 2\}$ and $i = 1, 2, \ldots, K^{rs}$, then there exists a metric δ on $\mathbb{R}^3$ equivalent to the Euclidean metric such that the GDIFS is hyperbolic with respect to δ. In particular, there exists non-empty compact sets G^r such that*

$$G^r = \bigcup_{s=1}^{2} \bigcup_{i=1}^{K^{rs}} w_i^{rs}(G^s) .$$

Proof. Proof follows in the similar lines of Theorem 2.1.1 of [16] and using the above condition (1.54). □

Following is the main result regarding existence of coalescence hidden variable FIFs for generalized GDIFS.

Theorem 1.10.2. *Let G^r, $r \in V$ be the attractors of the generalized GDIFS as in Theorem 1.10.1. Then G^r, $r \in V$ is the graph of a vector valued continuous function $f^r : I^r \to \mathbb{R}^2$ such that for $r \in V$, $f^r(x_i^r) = (y_i^r, z_i^r)$ for all $i = 1, 2, \ldots, N^r$. If $f^r = (f_1^r, f_2^r)$ then the projection of the attractors G^r, $r \in V$ on $\mathbb{R}^2$ is the graph of the continuous function $f_1^r : I^r \to \mathbb{R}$ known as CHFIF such that for $r \in V$, $f_1^r(x_i^r) = (y_i^r)$. That is $G^r\big|_{\mathbb{R}^2} = \{(x, f_1^r(x)) : x \in I^r\}$.*

Proof. Let $I^1 = [x_0^1, x_N^1]$ and $I^2 = [x_0^2, x_M^2]$. Consider the vector valued function spaces

$$\mathcal{F} = \left\{ f : [x_0^1, x_N^1] \to \mathbb{R}^2 \text{ continuous such that } f(x_0^1) = (y_0^1, z_0^1), f(x_N^1) = (y_N^1, z_N^1) \right\} ,$$

$$\mathcal{H} = \left\{ h : [x_0^2, x_M^2] \to \mathbb{R}^2 \text{ continuous such that } h(x_0^2) = (y_0^2, z_0^2), h(x_M^2) = (y_M^2, z_M^2) \right\}$$

with metrics

$$d_{\mathcal{F}}(f, f^*) = \sup_{x \in [x_0^1, x_N^1]} \left\| f(x) - f^*(x) \right\| ,$$
$$d_{\mathcal{H}}(h, h^*) = \sup_{x \in [x_0^2, x_M^2]} \left\| h(x) - h^*(x) \right\|$$

respectively, where $\|.\|$ denotes a norm on $\mathbb{R}^2$. Since $(\mathcal{F}, d_{\mathcal{F}})$ and $(\mathcal{H}, d_{\mathcal{H}})$ are complete metric spaces, $\big(\mathcal{F} \times \mathcal{H}, d\big)$ is also a complete metric space where

$$d\big((f, h), (f^*, h^*)\big) = \max \left\{ d_{\mathcal{F}}(f, f^*), d_{\mathcal{H}}(h, h^*) \right\} .$$

Following are the affine maps,

$$I_i : [x_0^1, x_N^1] \to [x_{i-1}^1, x_i^1],\ I_i(x) = a_i^{11} x + b_i^{11} \text{ for } i = 1, 2, \ldots, K^{11} ,$$

$$I_i : [x_0^2, x_M^2] \to [x_{i-1}^1, x_i^1],\ I_i(x) = a_{i-K^{11}}^{12} x + b_{i-K^{11}}^{12} \text{ for } i = K^{11} + 1, \ldots, N ,$$

$$J_i : [x_0^1, x_N^1] \to [x_{i-1}^2, x_i^2],\ J_i(x) = a_i^{21} x + b_i^{21} \text{ for } i = 1, 2, \ldots, K^{21} ,$$

$$J_i : [x_0^2, x_M^2] \to [x_{i-1}^2, x_i^2],\ J_i(x) = a_{i-K^{21}}^{22} x + b_{i-K^{21}}^{22} \text{ for } i = K^{21} + 1, \ldots, M .$$

Now define the mapping

$$T : \mathcal{F} \times \mathcal{H} \to \mathcal{F} \times \mathcal{H}$$

$$T(f, h)(x, y) = (\widetilde{f}(x), \widetilde{h}(y)) ,$$

where for $x \in [x_{i-1}^1, x_i^1]$,

$$\widetilde{f}(x) = \begin{cases} \big(c_i^{11} I_i^{-1}(x) + \alpha_i^{11} y_f^1(I_i^{-1}(x)) + \beta_i^{11} z_f^1(I_i^{-1}(x)) + d_i^{11}, \\ \quad \gamma_i^{11} z_f^1(I_i^{-1}(x)) + e_i^{11} I_i^{-1}(x) + f_i^{11}\big) & \text{for } i = 1, 2, \ldots, K^{11} \\ (c_{i-K^{11}}^{12} I_i^{-1}(x) + \alpha_{i-K^{11}}^{12} y_h^2(I_i^{-1}(x)) + \beta_{i-K^{11}}^{12} z_h^2(I_i^{-1}(x)) + d_{i-K^{11}}^{12}, \\ \quad \gamma_{i-K^{11}}^{12} z_h^2(I_i^{-1}(x)) + e_{i-K^{11}}^{12} I_i^{-1}(x) + f_{i-K^{11}}^{12}) & \text{for } i = K^{11} + 1, \ldots, N \end{cases}$$

and for $x \in [x^2_{j-1}, x^2_j]$,

$$\widetilde{h}(x) = \begin{cases} (c^{21}_j J^{-1}_j(x) + \alpha^{21}_j y^1_f(J^{-1}_j(x)) + \beta^{21}_j z^1_f(J^{-1}_j(x)) + d^{21}_j, \\ \quad \gamma^{21}_j z^1_f(J^{-1}_j(x)) + e^{21}_j J^{-1}_j(x) + f^{21}_j) & \text{for } j = 1, \ldots, K^{21} \\ (c^{22}_{j-K^{21}} J^{-1}_j(x) + \alpha^{22}_{j-K^{21}} y^2_h(J^{-1}_j(x)) + \beta^{22}_{j-K^{21}} z^2_h(J^{-1}_j(x)) + d^{22}_{j-K^{21}}, \\ \quad \gamma^{22}_{j-K^{21}} z^2_h(J^{-1}_j(x)) + e^{22}_{j-K^{21}} J^{-1}_j(x) + f^{22}_{j-K^{21}}) & \text{for } j = K^{21}+1, \ldots, M \ . \end{cases}$$

Now using (1.56) − (1.59), it is clear that,

$$\widetilde{f}(x^1_0) = F^{11}_1(I^{-1}_i(x), y^1_f(I^{-1}_i(x)), z^1_f(I^{-1}_i(x))) = (y^1_0, z^1_0) \ ,$$

$$\widetilde{f}(x^1_N) = F^{12}_N(I^{-1}_i(x), y^2_h(I^{-1}_i(x)), z^2_h(I^{-1}_i(x))) = (y^1_N, z^1_N) \ .$$

Similarly, $\widetilde{h}(x^2_0) = (y^2_0, z^2_0)$ and $\widetilde{h}(x^2_M) = (y^2_M, z^2_M)$. It proves that T maps $\mathcal{F} \times \mathcal{H}$ into itself. Since for each $i = 1, 2, \ldots, N$, $I^{-1}_i(x)$ is continuous and therefore, $\widetilde{f}$ is continuous on each subintervals $[x^1_{i-1}, x^1_i]$.

For $i = 1, 2, \ldots, K^{11}$, using (1.56), it follows that $\widetilde{f}(x^{1-}_i) = \widetilde{f}(x^{1+}_i) = (y^1_i, z^1_i)$.

For $i = K^{11} + 1, \ldots, N - 1$, using (1.57), it follows that $\widetilde{f}(x^{1-}_i) = \widetilde{f}(x^{1+}_i) = (y^1_i, z^1_i)$.

For $i = K^{11}$, using (1.56) and (1.57), it follows that $\widetilde{f}(x^{1-}_i) = \widetilde{f}(x^{1+}_i) = (y^1_i, z^1_i)$, since $I^{-1}_i(x^1_i) = x^1_N$ and $I^{-1}_{i+1}(x^1_i) = x^2_M$.

Hence $\widetilde{f}$ is continuous on I. Similarly it can be shown that $\widetilde{h}$ is continuous on J. Consequently T is continuous. To show that T is a contraction map on $\mathcal{F} \times \mathcal{H}$, let $T(f, f^*) = (\widetilde{f}, \widetilde{f^*})$ and $T(h, h^*) = (\widetilde{h}, \widetilde{h^*})$. Now,

$$\begin{aligned} \sup_{x \in [x^1_0, x^1_{K^{11}}]} \{\|\widetilde{f}(x) - \widetilde{f^*}(x)\|\} &= \max_{\substack{i=1,2,\ldots,K^{11} \\ x \in [x^1_{i-1}, x^1_i]}} \{\|\alpha^{11}_i (y^1_f(I^{-1}_i(x)) - y^1_{f^*}(I^{-1}_i(x))) \\ &\quad + \beta^{11}_i (z^1_f(I^{-1}_i(x)) - z^1_{f^*}(I^{-1}_i(x))), \\ &\quad \gamma^{11}_i (z^1_f(I^{-1}_i(x)) - z^1_{f^*}(I^{-1}_i(x)))\|\} \\ &\leq \delta^{11} \max_{\substack{i=1,2,\ldots,K^{11} \\ x \in [x^1_{i-1}, x^1_i]}} \{y^1_f(I^{-1}_i(x)) - y^1_{f^*}(I^{-1}_i(x)) \\ &\quad + z^1_f(I^{-1}_i(x)) - z^1_{f^*}(I^{-1}_i(x)), z^1_f(I^{-1}_i(x)) - z^1_{f^*}(I^{-1}_i(x))\} \\ &\leq \delta^{11} d_{\mathcal{F}}(f, f^*), \end{aligned}$$

$$\sup_{x\in[x^1_{K^{11}},x^1_N]} \left\{\|\widetilde{f}(x)-\widetilde{f}^*(x)\|\right\} = \max_{\substack{i=K^{11}+1,\dots,N\\ x\in[x^1_{i-1},x^1_i]}} \left\{\|\alpha^{12}_{i-K^{11}}(y^2_h(I_i^{-1}(x))-y^2_{h^*}(I_i^{-1}(x)))\right.$$

$$+\beta^{12}_{i-K^{11}}(z^2_h(I_i^{-1}(x))-z^2_{h^*}(I_i^{-1}(x))),$$

$$\left.\gamma^{12}_{i-K^{11}}(z^2_h(I_i^{-1}(x))-z^2_{h^*}(I_i^{-1}(x)))\|\right\}$$

$$\leq \delta^{12}\max_{\substack{i=K^{11}+1,\dots,N\\ x\in[x^1_{i-1},x^1_i]}} \left\{y^2_h(I_i^{-1}(x))-y^2_{h^*}(I_i^{-1}(x))\right.$$

$$\left.+z^2_h(I_i^{-1}(x))-z^2_{h^*}(I_i^{-1}(x)), z^2_h(I_i^{-1}(x))-z^2_{h^*}(I_i^{-1}(x))\right\}$$

$$\leq \delta^{12} d_{\mathcal{H}}(h,h^*),$$

where

$$\delta^{11} = \max_{i=1,2,\dots,K^{11}} \left\{|\alpha^{11}_i|,|\beta^{11}_i|,|\gamma^{11}_i|\right\} < 1$$

and

$$\delta^{12} = \max_{i=K^{11}+1,\dots,N} \left\{|\alpha^{12}_i|,|\beta^{12}_i|,|\gamma^{12}_i|\right\} < 1\ .$$

Therefore

$$d_{\mathcal{F}}(\widetilde{f},\widetilde{f}^*) \leq \max\left\{\delta^{11},\delta^{12}\right\}\max\left\{d_{\mathcal{F}}(f,f^*),d_{\mathcal{H}}(h,h^*)\right\}.$$

Similarly, it follows that

$$d_{\mathcal{H}}(\widetilde{h},\widetilde{h}^*) \leq \max\left\{\delta^{21},\delta^{22}\right\}\max\left\{d_{\mathcal{F}}(f,f^*),d_{\mathcal{H}}(h,h^*)\right\},$$

where

$$\delta^{21} = \max_{i=1,2,\dots,K^{21}} \left\{|\alpha^{21}_i|,|\beta^{21}_i|,|\gamma^{21}_i|\right\} < 1$$

and

$$\delta^{22} = \max_{i=K^{21}+1,\dots,M} \left\{|\alpha^{22}_i|,|\beta^{22}_i|,|\gamma^{22}_i|\right\} < 1\ .$$

Then

$$d\big(T(f,h),T(f^*,h^*)\big) = \max\left\{d_{\mathcal{F}}(\widetilde{f},\widetilde{f}^*),d_{\mathcal{H}}(\widetilde{h},\widetilde{h}^*)\right\} \leq \delta\max\left\{d_{\mathcal{F}}(f,f^*),d_{\mathcal{H}}(h,h^*)\right\},$$

where $\delta=\max\left\{\delta^{11},\delta^{12},\delta^{21},\delta^{22}\right\}<1$ and hence T is a contraction mapping. By Banach fixed point theorem, T possesses a unique fixed point, say (f^1,f^2).

Now, for $i = 1, 2, \ldots, K^{11}$,

$$\begin{aligned} f^1(x_i^1) &= \big(c_{i+1}^{11} I_{i+1}^{-1}(x_i^1) + \alpha_{i+1}^{11} y_{f^1}^1(I_{i+1}^{-1}(x_i^1)) + \beta_{i+1}^{11} z_{f^1}^1(I_{i+1}^{-1}(x_i^1)) + d_{i+1}^{11}, \\ &\qquad \gamma_{i+1}^{11} z_{f^1}^1(I_{i+1}^{-1}(x_i^1)) + e_{i+1}^{11} I_{i+1}^{-1}(x_i^1) + f_{i+1}^{11}\big) \\ &= (y_i^1, z_i^1). \end{aligned}$$

For $i = K^{11}+1, \ldots, N-1$,

$$\begin{aligned} f^1(x_i^1) &= \big(c_{i+1-K^{11}}^{12} I_{i+1}^{-1}(x_i^1) + \alpha_{i+1-K^{11}}^{12} y_{f^2}^2(I_{i+1}^{-1}(x_i^1)) + \beta_{i+1-K^{11}}^{12} z_{f^2}^2(I_{i+1}^{-1}(x_i^1)) + d_{i+1-K^{11}}^{12}, \\ &\qquad \gamma_{i+1-K^{11}}^{12} z_{f^2}^2(I_{i+1}^{-1}(x_i^1)) + e_{i+1-K^{11}}^{12} I_{i+1}^{-1}(x_i^1) + f_{i+1-K^{11}}^{12}\big) \\ &= (y_i^1, z_i^1). \end{aligned}$$

This shows that f^1 is the function which interpolates the data $\big\{(x_i^1, y_i^1, z_i^1) : i = 0, 1, \ldots, N\big\}$. Similarly, it can be shown that f^2 is the function which interpolates the data $\big\{(x_i^2, y_i^2, z_i^2) : i = 0, 1, \ldots, M\big\}$. For $x \in [x_0^1, x_N^1]$ and $x \in [x_0^2, x_M^2]$,

$$\begin{aligned} f^1\big(I_i(x)\big) &= \big(c_i^{11} x + \alpha_i^{11} y_{f^1}^1(x) + \beta_i^{11} z_{f^1}^1(x) + d_i^{11}, \\ &\qquad \gamma_i^{11} z_{f^1}^1(x) + e_i^{11} x + f_i^{11}\big) \qquad \text{for } i = 1, 2, \ldots, K^{11}, \end{aligned}$$

$$\begin{aligned} f^1\big(I_i(x)\big) &= \big(c_i^{12} x + \alpha_i^{12} y_{f^2}^2(x) + \beta_i^{12} z_{f^2}^2(x) + d_i^{12}, \\ &\qquad \gamma_i^{12} z_{f^2}^2(x) + e_i^{12} x + f_i^{12}\big) \qquad \text{for } i = 1, 2, \ldots, K^{12}, \end{aligned}$$

and

$$\begin{aligned} f^2\big(J_i(x)\big) &= \big(c_i^{21} x + \alpha_i^{21} y_{f^1}^1(x) + \beta_i^{21} z_{f^1}^1(x) + d_i^{21}, \\ &\qquad \gamma_i^{21} z_{f^1}^1(x) + e_i^{21} x + f_i^{21}\big) \qquad \text{for } i = 1, 2, \ldots, K^{21}, \end{aligned}$$

$$\begin{aligned} f^2\big(J_i(x)\big) &= \big(c_i^{22} x + \alpha_i^{22} y_{f^2}^2(x) + \beta_i^{22} z_{f^2}^2(x) + d_i^{22}, \\ &\qquad \gamma_i^{22} z_{f^2}^2(x) + e_i^{22} x + f_i^{22}\big) \qquad \text{for } i = 1, 2, \ldots, K^{22}. \end{aligned}$$

If F and H are the graphs of f^1 and f^2 respectively, then

$$F = \bigcup_{i=1}^{K^{11}} w_i^{11}(F) \bigcup \bigcup_{i=1}^{K^{12}} w_i^{12}(H) \ ,$$

$$H = \bigcup_{i=1}^{K^{21}} w_i^{21}(F) \bigcup \bigcup_{i=1}^{K^{22}} w_i^{22}(H) \ .$$

The uniqueness of the attractor implies $F = G^1$ and $H = G^2$. That is $G^1 = \big\{(x, f^1(x)) : x \in I\big\}$ and $G^2 = \big\{(x, f^2(x)) : x \in J\big\}$. □

Example 1.4. *Consider the data sets as*

$$D^1 = \big\{(0,5),(1,4),(2,1),(3,1),(4,4),(5,5)\big\} \ ,$$

$$D^2 = \big\{(0,1),(1,2),(2,3),(3,2),(4,1)\big\}$$

realizing the graph with $K^{11} = 3$, $K^{12} = 2$, $K^{21} = 1$, $K^{22} = 3$ *as in Figure 1.1. Take*

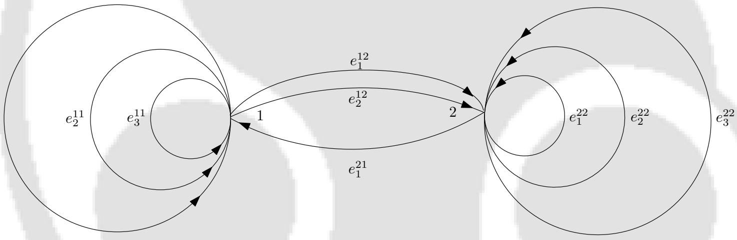

Figure 1.1: Directed graph for Example 1.4.

the first set of generalized data

$$\mathcal{D}^1 = \big\{(0,5,5),(1,4,4),(2,1,1),(3,1,1),(4,4,4),(5,5,5)\big\}$$

and

$$\mathcal{D}^2 = \big\{(0,1,1),(1,2,2),(2,3,3),(3,2,2),(4,1,1)\big\}$$

corresponding to D^1 *and* D^2 *respectively. Here* $y_i = z_i$ *for both the generalized data sets. Choose* $\alpha_i^{rs} = 1/3$, $\beta_i^{rs} = 1/3$, $\gamma_i^{rs} = 1/3$ *for all* $r, s \in \{1,2\}$ *and* $i = 1, 2, \ldots, K^{rs}$. *Then Figure* 1.2 *are the attractors of the corresponding generalized GDIFS.*

Keeping the free variables and constrained variables same, Figure 1.3 are the attractors of the generalized GDIFS associated with the second set of generalized data

$$\mathcal{D}^1 = \{(0,5,3),(1,4,2),(2,1,5),(3,1,2),(4,4,1),(5,5,4)\},$$

$$\mathcal{D}^2 = \{(0,1,2),(1,2,5),(2,3,1),(3,2,3),(4,1,1)\}.$$

Take the third set of generalized data

$$\mathcal{D}^1 = \{(0,5,3),(1,4,2),(2,1,5),(3,1,2),(4,4,1),(5,5,4)\}$$

and

$$\mathcal{D}^2 = \{(0,1,2),(1,5,5),(2,3,1),(3,2,3),(4,4,1)\}$$

corresponding to D^1 *and* D^2 *respectively. For the generalized GDIFS with the free variables and constraints variables given in the following Table 1.1, the attractors are given in Figure* 1.4.

α	α_1^{11}	α_2^{11}	α_3^{11}	α_1^{12}	α_2^{12}	α_1^{21}	α_1^{22}	α_2^{22}	α_3^{22}
	0.8	0.7	0.8	0.7	0.8	0.99	0.99	0.99	0.99
β	β_1^{11}	β_2^{11}	β_3^{11}	β_1^{12}	β_2^{12}	β_1^{21}	β_1^{22}	β_2^{22}	β_3^{22}
	-0.3	-0.4	-0.2	-0.3	-0.4	0.99	0.99	0.99	0.99
γ	γ_1^{11}	γ_2^{11}	γ_3^{11}	γ_1^{12}	γ_2^{12}	γ_1^{21}	γ_1^{22}	γ_2^{22}	γ_3^{22}
	0.5	0.3	0.6	0.5	0.3	0.005	0.005	0.005	0.005

Table 1.1: The generalized GDIFS with the free variables and constraints variables.

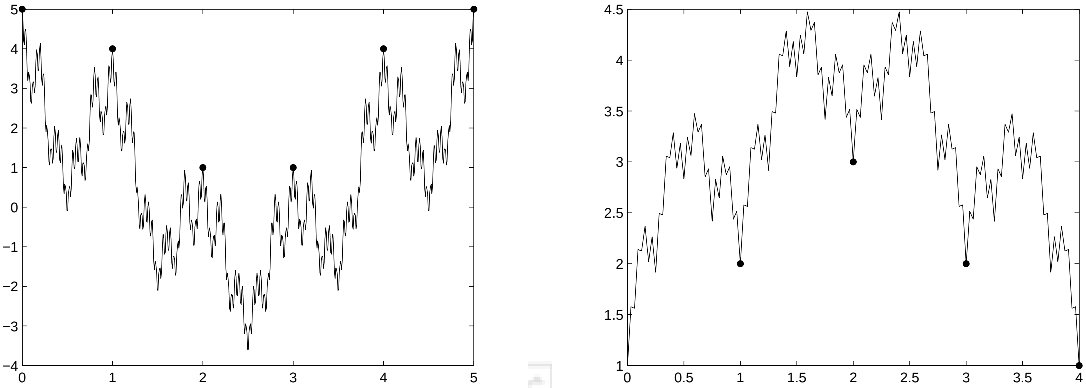

Figure 1.2: Attractors for the first set of generalized data.

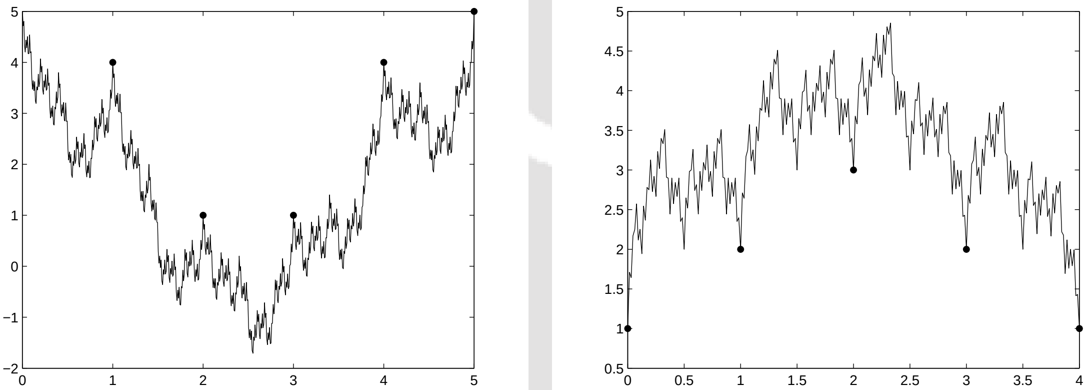

Figure 1.3: Attractors for the second set of generalized data.

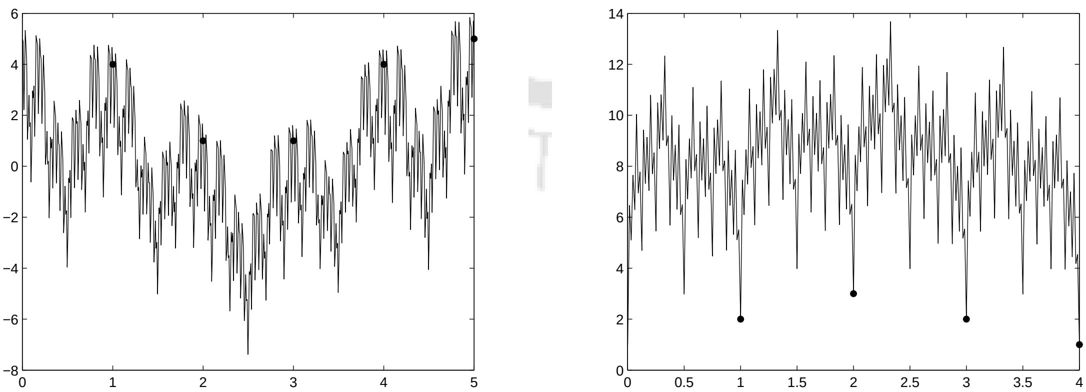

Figure 1.4: Attractors for the third set of generalized data.

Chapter 2

Box dimensions of α-fractal functions with constant scaling factors

In [40], Hardin et al. first calculated box dimension of affine fractal interpolation function for equally spaced data. They also pointed out that their formula can be obtained using the Lyapunov exponents of an associated dynamical system, supporting a conjecture of Yorke's. Barnsely et al. estimated the box dimension of projection of self-affine function in higher dimensions [9]. The result has been generalized by Dalla et al. for certain non-affine fractal interpolation functions [24]. Various researchers [10,14,18,35,36,41,80,83], investigated and found estimates for the box dimensions of different type of fractal interpolation functions as well as fractal interpolation surfaces.

In the present chapter, the box dimension of the graph of nowhere diffrentiable, continuous, non-affine fractal interpolation function f^{α} is investigated. In classical case, if f is uniformly Hölderian function with exponent β, then the box dimension of the graph of f is bounded by $2-\beta$ [12]. In particular, if f is Lipscitz, the dimension is 1. The function f^{α} is defined for any continuous function f. If one can control the increments of a function f, then one can estimate the dimension of its graph. Hölderian function has minimal regularity [26]. Noting this fact, in this chapter, the original function f and the base function b have been taken as Hölderian to calculate the box dimension of the graph of f^α. However it is still open whether one can get exact value or a bound for box dimension of the graph of f^α corresponding to any continuous function f?

The present chapter is organized as follows: Section 2.1 contains some existing results on box dimensions for different type of fractal functions. In Section 2.2, the box dimension of α-fractal interpolation function f^α for an IFS with equally spaced nodes and Hölder continuous functions f and b is estimated. The proofs of the theorems in this section are based on methods first developed in [40] and then later in [10]. The variation of a continuous function and its related basic results are presented in Section 2.3. In Section 2.4, using variation of α-fractal interpolation function f^α, the box dimension of f^α is calculated. In this section, the assumption on equally spaced nodes is relaxed. Here the proofs of the results are similar to that in [35, 36].

2.1 Preliminaries

In this section, we recall some well established results on the box dimensions of the graphs of different type of fractal functions, which are motivation behind our work. Using box counting method, Hardin and Massopust [40], gave an estimate of the box dimension of the graph of an affine fractal interpolation function for equally spaced data.

Theorem 2.1.1. *(see [40]). Let G be the graph of an affine FIF f corresponding to the IFS (1.8) with q_i's as affine maps. If $\gamma = \sum_{i=1}^{N} |\alpha_i| > 1$ and the interpolation points are not collinear, then*

$$dim_B G = \begin{cases} 1 + \log_N \gamma, & \text{if } \gamma > 1 \\ 1 & \text{otherwise} . \end{cases}$$

Later, Barnsley et al. [9], showed how the class of one-dimensional interpolation functions can be usefully widened by considering the projections of the graphs of higher-dimensional self-affine functions. The assumption on equally spaced data set is relaxed and the result is stated as follows.

Theorem 2.1.2. *(see [9], Theorem 4). Let G be the graph of an affine FIF f correspond-*

ing to the IFS (1.8) with q_i's as affine maps. If $\gamma = \sum_{i=1}^{N} |\alpha_i| > 1$ and the interpolation points are not collinear, then $dim_B G$ is the unique real solution D of $\sum_{i=1}^{N} |\alpha_i| a_i^{D-1} = 1$; otherwise $dim_B G = 1$.

This idea has been generalized by Dalla et al. [24], for certain type of non-affine fractal functions. For that, they defined the IFS as

$$L_i(x) = a_i x + d_i, \; M_i(x, y) = c_i g_i(x) + s_i h_i(y) + e_i, \tag{2.1}$$

where L_i is defined in (1.8) and $M_i : K \to \mathbb{R}$. The maps $g_i : I \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}$ are continuous functions such that

$$|g_i(x) - g_i(x^*)| \leq m_i |x - x^*|$$

with $g_i(x_N) \neq g_i(x_0)$ and

$$l_i |y - y^*| \leq |h_i(y) - h_i(y^*)| \leq r_i |y - y^*| \, ,$$

where $x, x^* \in I$, $y, y^* \in \mathbb{R}$ and $l_i > 0$, $r_i > 0$ are constants for $i = 1, 2, \ldots, N$. The real constants c_i and e_i depend on the free real parameters s_i and chosen so that

$$M_i(x_0, y_0) = y_{i-1}, \; M_i(x_N, y_N) = y_i \; .$$

Lemma 2.1.1. *(see [24], Lemma 3). Let $\{(x_i, y_i) : i = 0, 1, 2, \ldots, N\}$ be given points and $V_k = (y_k - y_{k-1}) - (y_{k+1} - y_{k-1})(x_k - x_{k-1})/(x_{k+1} - x_{k-1}) \neq 0$ for some $k \in \{1, 2, \ldots, N-1\}$. Choose $g_i(x) = x$ for all $i = 1, 2, \ldots, N$ and*

(a) if there exists a $k \in \{1, 2, \ldots, N-1\}$ such that $V_k > 0$, choose $s_i h_i$ for $i = 1, 2, \ldots, N$ to be increasing and concave;

(b) if there exists a $l \in \{1, 2, \ldots, N-1\}$ such that $V_l < 0$, choose $s_i h_i$ for $i = 1, 2, \ldots, N$ to be increasing and convex;

(c) if there exist $k, l \in \{1, 2, \ldots, N-1\}$ such that $V_k V_l < 0$, choose $s_i h_i$ for $i = 1, 2, \ldots, N$ to be (all) increasing and concave or (all) increasing and convex.

Then, if $\gamma_1 = \sum_{i=1}^{N} l_i|s_i| > 1$,

$$\lim_{\epsilon \to 0^+} \epsilon \mathcal{N}^*(\epsilon) = \infty ,$$

where $\mathcal{N}^*(\epsilon)$ *is defined in Section 4 of [9].*

Theorem 2.1.3. *(see [24], Theorem 3). Let* G *be the graph of the fractal function* f *corresponding to the IFS (2.1).*

(a) If $\sum_{i=1}^{N} r_i|s_i| > 1$ *and* $(x_i, y_i) \in K$ *for* $i = 0, 1, 2, \ldots, N$ *are not all collinear, then* $\overline{\dim}_B(G) \leq D$*, where* $D \in (1, 2)$ *is the unique real solution of* $\sum_{i=1}^{N} r_i|s_i|a_i^{D-1} = 1$.

(b) If the IFS (2.1) satisfies in addition the conditions of Lemma 2.1.1, then $d \leq \underline{\dim}_B(G)$*, where* $d \in (1, 2)$ *is the unique real solution of* $\sum_{i=1}^{N} l_i|s_i|a_i^{d-1} = 1$.

Recently in [10], Barnsley and Massopust, defined bilinear IFS and estimated the box dimension of the graph of a bilinear fractal function explicitly.

Theorem 2.1.4. *(see [10], Theorem 6). Let* $\mathcal{F}$ *denote the bilinear IFS and let* $\Gamma(f)$ *denote its attractor. Suppose that the knots* $X_j : j = 0, 1, 2, \ldots, N$ *are uniformly spaced on* I*, that is,* $X_j = j/N$ *for all* $j = 0, 1, 2, \ldots, N$*, and suppose that* $s_0 = s_N$*. If* $\gamma := \sum_{i=1}^{N} \frac{s_{i-1}+s_i}{2} > 1$ *and* $\Gamma(f)$ *is not a straight line segment then*

$$dim_B\Gamma(f) = \begin{cases} 1 + \log_N \gamma, & \text{if } \gamma > 1 \\ 1 & \text{otherwise .} \end{cases}$$

In [36], Feng and Sun, defined fractal interpolation surfaces in a rectangular domain with arbitrary nodes. Using the variation of a continuous function, the box dimension of a fractal interpolation surface is estimated.

Theorem 2.1.5. *(see [36], Theorem 5.1). Let* $z = f(x, y)$, $(x, y) \in [a, b] \times [c, d]$ *be a fractal interpolation surface,* $\Lambda = \sum_{i=1}^{m} |s_i| > 1$ *and* D *be the unique solution of* $\sum_{i=1}^{m} |s_i|a_i^{D-1} = 1$*. If there exists a* $y \in [c, d]$ *such that the data set* $\{(x_i, u_i(y) : i = 0, 1, 2, \ldots, m)\}$ *is not collinear and* $D \geq \max\{\overline{\dim}_B\Gamma(u_i; [c, d]) : i = 0, 1, 2, \ldots, m\}$*, then*

$$\dim_B \Gamma\big(f; [a, b] \times [c, d]\big) = D + 1 .$$

Let $f^* : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be the fractal interpolation function for a given IFS. Consider the components of $f^* = (f_1^*, f_2^*, \ldots, f_m^*)$. The graph of f_j^* is the projection of graph of f^* onto $\mathbb{R}^n \times 0 \times \ldots \times \mathbb{R} \times \ldots \times 0$, where the factor $\mathbb{R}$ is in the j-th position [41]. Then the following theorem gives an estimate for the box dimension of the graph of a fractal function in higher dimensions using oscillation of continuous function. For notations see [41].

Theorem 2.1.6. *(see [41], Theorem 6). Assume that*

(a) $d_\nu \neq 0$ *for some* $\nu \in \{1, 2, \ldots, \mu\}$,

(b) $\sum_{i=1}^{n} a_i^{n-1} s > 1$.

Then the box dimension d of the graph of f_j^* is the unique positive solution of

$$\sum_{i=1}^{N} a_i^{d-1} s = 1 \ ,$$

otherwise $d = n$.

These ideas are used to prove our main results in Section 2.4. In [37], Gang considered the IFS

$$\left.\begin{array}{rcl} L_i(x) & = & x_{i-1} + (x_i - x_{i-1})x, \\ F_i(x, y) & = & \alpha_i y + q_i(x), \end{array}\right\} \tag{2.2}$$

where $0 < \alpha_i < 1$ and $q_i(x) \in \text{Lip}\ \beta_i, 0 < \beta_i \leq 1$ to obtain the Hölder exponents of the corresponding fractal interpolation function, and estimated the box dimension of the graph of the FIF as given in the following theorem.

Theorem 2.1.7. *(see [37], Theorem 5.1). Let f be the FIF generated by the IFS (2.2) with the critical condition $C = 1$. Then*

$$1 - \frac{\log \sum_{i=1}^{N} \alpha_i}{\log |I_{\max}|} \leq \dim_B(graph(f)) \leq 1 - \beta - \frac{\log N}{\log |I_{\max}|} \ ,$$

where $\beta = \min\{\beta_i : i = 1, 2, \ldots, N\}$, $C_i = \frac{\alpha_i}{|I|^\beta}$ *and* $C = \max\{C_i : i = 1, 2, \ldots, N\}$.

From the above mentioned results, it is clear that the box dimension depends on the scale vector $\alpha \in \mathbb{R}^N$. Sections 2.2 and 2.4 are devoted to the estimation of the box dimensions of the graphs of α-fractal functions f^α for equally spaced (uniform) data set as well as arbitrary (non-uniform) data set. It is shown that the box dimension of the graph of a α-fractal interpolation function f^α, depends on the scale vector $\alpha \in \mathbb{R}^N$ as well.

2.2 Box dimensions of α-fractal functions

In this section, equally spaced horizontal nodes on $I = [0, 1]$ is considered. That is the partition is $\Delta : 0 = x_0 < x_1 < \cdots < x_N = 1$, where $x_i - x_{i-1} = \frac{1}{N}$ for $i = 1, 2, \ldots, N$. Then the affine maps $L_i(x) : I \to I_i$ are defined as

$$L_i(x) = \frac{1}{N}x + \frac{i-1}{N} \text{ for } i = 1, 2, \ldots, N. \tag{2.3}$$

Let f be Hölderian on I with exponent $\beta_1 \in (0, 1]$. Consider the continuous maps

$$F_i(x, y) = \alpha_i\, y + f(L_i(x)) - \alpha_i\, b(x) \tag{2.4}$$

where $b(x)$ is also Hölderian with exponent $\beta_2 \in (0, 1]$ such that $b(x_0) = f(x_0)$, $b(x_N) = f(x_N)$ and $b \neq f$. Let f^α be the α-fractal interpolation function of f for the corresponding IFS (2.3) and (2.4). Using the techniques developed in [10, 24, 40], the bounds on the box dimension of the graph of α-fractal function f^α is estimated in the following theorem.

Theorem 2.2.1. *Let $G = \{(x, f^\alpha(x)) : x \in I\}$ be the graph of the α-fractal interpolation function f^α corresponding to the IFS (2.3) and (2.4). Let the interpolation points $\{(x_i, y_i)\}_{i=0}^N$ be not collinear and $\gamma = \sum_{i=1}^N |\alpha_i|$, where $\alpha_i \neq 0$ for all i. Let $\beta = \min\{\beta_1, \beta_2\}$. Then*

(a) For $\gamma \leq 1$, $1 \leq \dim_B G \leq 2 - \beta$.

(b) *For* $\gamma > 1$ *with* $\gamma N^{\beta-1} \leq 1$, $1 \leq \dim_B G \leq 2 - \beta + \log_N \gamma$.

(c) *For* $\gamma > 1$ *with* $\gamma N^{\beta-1} > 1$, $1 \leq \dim_B G \leq 1 + \log_N \gamma$.

Proof. To compute the box dimension of G, consider the cover of G whose elements are squares of sides $\frac{1}{N^r}$, $r \in \mathbb{N} \cup \{0\}$ and parallel to the coordinate axes. Let $\mathcal{N}(r)$ be the minimum number of squares of size $\frac{1}{N^r} \times \frac{1}{N^r}$ of the form

$$\left[\frac{k-1}{N^r}, \frac{k}{N^r}\right] \times \left[a, a + \frac{1}{N^r}\right]; r = 0, 1, \ldots; k = 1, 2, \ldots, N^r, a \in \mathbb{R} \tag{2.5}$$

that are needed to cover G. Let $\mathcal{N}_0(r)$ be the smallest number of arbitrary squares of size $\frac{1}{N^r} \times \frac{1}{N^r}$, needed to cover G. Then it follows that $\mathcal{N}_0(r) \leq \mathcal{N}(r)$. Since each arbitrary square of size $\frac{1}{N^r} \times \frac{1}{N^r}$ can be covered by at most two squares of the form (2.5), one can have $\mathcal{N}(r) \leq 2\mathcal{N}_0(r)$. So it is sufficient to consider the cover of the form (2.5) to compute the box dimension of the graph of f^α.

Let us denote $\Lambda(r)$ as the collection of squares of the form (2.5) whose interior is disjoint, with minimum cardinality $\mathcal{N}(r)$ which covers G. Let $\Lambda(r,k)$ be the collection of squares of $\Lambda(r)$ lying in between the lines $x = \frac{k-1}{N^r}$ and $x = \frac{k}{N^r}$ for $k = 1, 2, \ldots N^r$. Let $\mathcal{N}(r,k)$ be the number of squares in $\Lambda(r,k)$ and

$$\Lambda^*(r,k) := \cup\{A| \ A \in \Lambda(r,k)\}.$$

Each member of $\Lambda(r)$ must meet G as it is a minimal cover of G. With the continuity of f^α, it can be shown that $\Lambda^*(r,k)$ is a rectangle of width $\frac{1}{N^r}$ and height $\frac{\mathcal{N}(r,k)}{N^r}$ (see Figure 2.1). Observe that $\mathcal{N}(r) = \sum_{k=1}^{N^r} \mathcal{N}(r,k)$. Now for each $i = 1, 2, \ldots, N$, the image of the rectangle $\Lambda^*(r,k)$ under the mapping $w_i := \big(L_i, F_i\big)$, is contained in

$$\left[\frac{l(k,i)-1}{N^{r+1}}, \frac{l(k,i)}{N^{r+1}}\right] \times \mathbb{R}, \ \text{ where } \ l(k,i) := k + (i-1)N^r\ .$$

Therefore,

$$\mathcal{N}(r+1) = \sum_{i=1}^{N} \sum_{k=1}^{N^r} \mathcal{N}(r+1, l(k,i))\ .$$

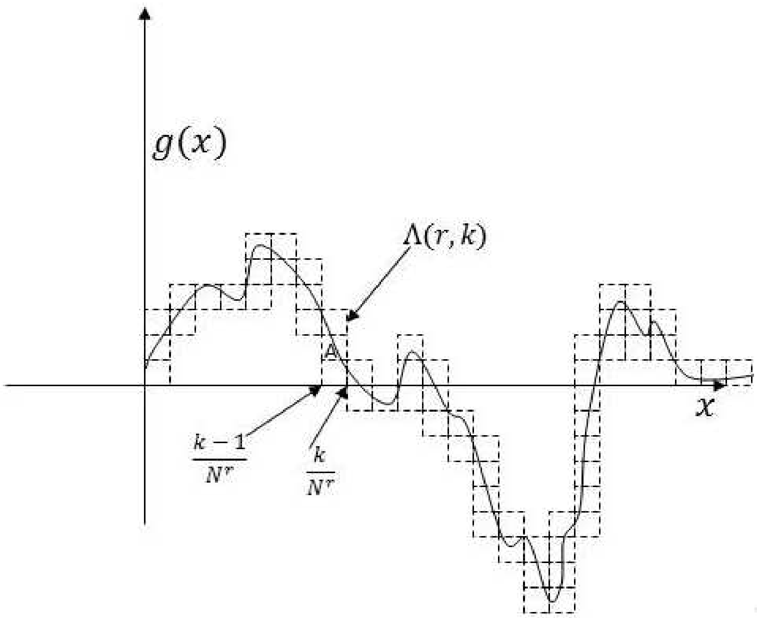

Figure 2.1: Minimal cover of a continuous function using squares sides parallel to axes.

Now,

$$G = \bigcup_{i=1}^{N} w_i(G)$$

implies that

$$G \subseteq \bigcup_{i=1}^{N} w_i\Big(\bigcup_{k=1}^{N^r} \Lambda^*(r,k)\Big), \quad \text{since} \quad G \subseteq \bigcup_{k=1}^{N^r} \Lambda^*(r,k)\ .$$

For $(x, y) \in \Lambda^*(r, k)$,

$$w_i\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{N}x + \frac{i-1}{N} \\ \alpha_i y + f(L_i(x)) - \alpha_i b(x) \end{pmatrix}.$$

Since f is uniformly Hölderian in I with exponent $\beta_1 \in (0, 1]$, there exists $s > 0$ such that

$$|f\big(L_i(x)\big) - f\big(L_i(x')\big)| \leq \frac{s}{N^{(r+1)\beta_1}} \quad \text{whenever} \quad x, x' \in \left[\frac{k-1}{N^r}, \frac{k}{N^r}\right],$$

as

$$L_i\Big(\big[\frac{k-1}{N^r}, \frac{k}{N^r}\big]\Big) = \left[\frac{l(k,i)-1}{N^{r+1}}, \frac{l(k,i)}{N^{r+1}}\right].$$

Similarly, since b is uniformly Hölderian in I with exponent $\beta_2 \in (0, 1]$, there exists

$m > 0$, such that

$$|b(x) - b(x')| \leq \frac{m}{N^{r\beta_2}} \quad \text{whenever} \quad x, x' \in \left[\frac{k-1}{N^r}, \frac{k}{N^r}\right].$$

Hence $w_i(\Lambda^*(r,k))$ is contained in a rectangle of width $\frac{1}{N^{r+1}}$ and height

$$\frac{|\alpha_i|\mathcal{N}(r,k)}{N^r} + \frac{s}{N^{(r+1)\beta_1}} + \frac{m|\alpha_i|}{N^{r\beta_2}},$$

(see Figure 2.2). Thus

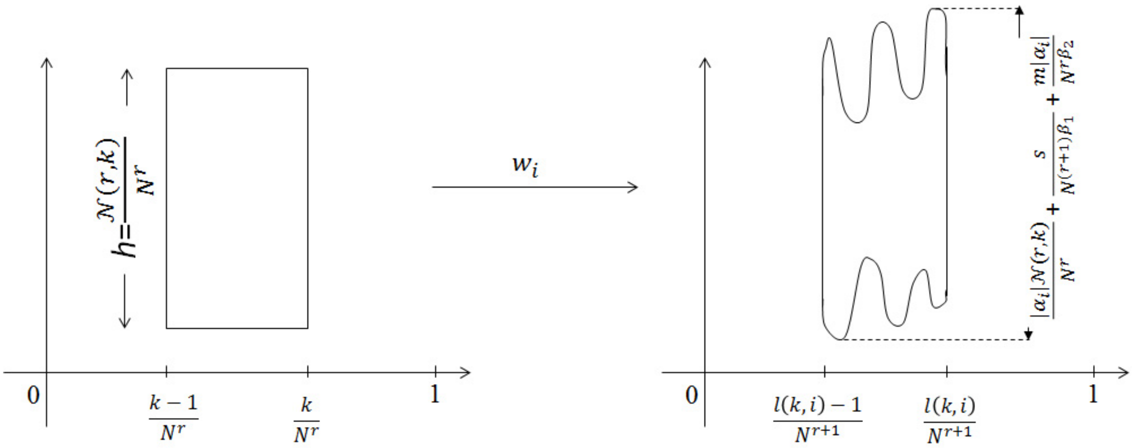

Figure 2.2: Image of the rectangle $\Lambda^*(r,k)$ under the map w_i.

$$\begin{aligned}
\mathcal{N}\big(r+1, l(k,i)\big) &\leq \left[\frac{|\alpha_i|\mathcal{N}(r,k)}{N^r} + \frac{s}{N^{(r+1)\beta_1}} + \frac{m|\alpha_i|}{N^{r\beta_2}}\right] N^{r+1} + 2 \\
&= N|\alpha_i|\mathcal{N}(r,k) + sN^{(r+1)(1-\beta_1)} + m|\alpha_i|N^{r(1-\beta_2)+1} + 2.
\end{aligned}$$

By summing over i,

$$\begin{aligned}
\sum_{i=1}^{N} \mathcal{N}\big(r+1, l(k,i)\big) &\leq \sum_{i=1}^{N} N|\alpha_i|\mathcal{N}(r,k) + \sum_{i=1}^{N} sN^{(r+1)(1-\beta_1)} \\
&+ \sum_{i=1}^{N} m|\alpha_i|N^{r(1-\beta_2)+1} + 2N \\
&= N\gamma\mathcal{N}(r,k) + sN^{(r+1)(1-\beta_1)+1} + m\gamma N^{r(1-\beta_2)+1} + 2N,
\end{aligned}$$

where $\gamma = \sum_{i=1}^{N} |\alpha_i|$. Now,

$$\begin{aligned}
\sum_{k=1}^{N^r}\Big(\sum_{i=1}^{N}\mathcal{N}(r+1,l(k,i))\Big) &\leq N\gamma\mathcal{N}(r) + sN^{(r+1)(1-\beta_1)+r+1} + m\gamma N^{r(1-\beta_2)+r+1}\\
&+ 2N^{r+1}\\
&\leq N\gamma\mathcal{N}(r) + sN^{(r+1)(2-\beta)} + m\gamma N^{(r+1)(2-\beta)}\\
&+ 2N^{(r+1)(2-\beta)}, \qquad (\text{since } 2-\beta \geq 1)\\
&= N\gamma\mathcal{N}(r) + C_1 N^{(r+1)(2-\beta)},
\end{aligned}$$

where $C_1 = s + m\gamma + 2$. Therefore

$$\mathcal{N}(r+1) \leq N\gamma\mathcal{N}(r) + C_1 N^{(r+1)(2-\beta)} .$$

By repeated applications of above inequality on r, we can get

$$\begin{aligned}
\mathcal{N}(r) &\leq N\gamma\mathcal{N}(r-1) + C_1 N^{r(2-\beta)}\\
&\leq N\gamma[N\gamma\mathcal{N}(r-2) + C_1 N^{(r-1)(2-\beta)}] + C_1 N^{r(2-\beta)}\\
&\leq N^2\gamma^2\mathcal{N}(r-2) + C_1 N^{r(2-\beta)}[1+\gamma N^{\beta-1}]\\
&\leq N^2\gamma^2[N\gamma\mathcal{N}(r-3) + C_1 N^{(r-2)(2-\beta)}] + C_1 N^{r(2-\beta)}[1+\gamma N^{\beta-1}]\\
&\leq N^3\gamma^3\mathcal{N}(r-3) + C_1 N^{r(2-\beta)}[1+\gamma N^{\beta-1} + \gamma^2 N^{2(\beta-1)}].
\end{aligned}$$

Continuing the process, we get

$$\mathcal{N}(r) \leq N^r\gamma^r\mathcal{N}(0) + C_1 N^{r(2-\beta)}\big(1+\gamma N^{\beta-1}\cdots + \gamma^{r-1}N^{(r-1)(\beta-1)}\big). \tag{2.6}$$

<u>Case 1</u>: Consider $\gamma \leq 1$. Also $N^{\beta-1} \leq 1$ for $0 < \beta \leq 1$, $N > 1$. Then

$$1+\gamma N^{\beta-1}\cdots + \gamma^{r-1}N^{(r-1)(\beta-1)} \leq r .$$

Therefore from (2.6),

$$\begin{aligned}
\mathcal{N}(r) &\leq N^r\gamma^r\mathcal{N}(0) + C_1 r N^{r(2-\beta)}\\
&\leq N^{r(2-\beta)} r\mathcal{N}(0) + C_1 r N^{r(2-\beta)}, \quad \text{since} \quad \gamma \leq 1, 2-\beta \geq 1\\
&\leq C_2 r N^{r(2-\beta)},
\end{aligned}$$

where $C_2 = (\mathcal{N}(0) + C_1)$. Hence

$$\dim_B G \le \lim_{r\to\infty} \frac{\log C_2 r N^{r(2-\beta)}}{\log N^r} = 2 - \beta \ .$$

The continuity of the fractal function f^α implies that $\dim_B G \ge 1$ and hence for $\gamma \le 1$,

$$1 \le \dim_B G \le 2 - \beta \ .$$

Case 2: Consider $\gamma > 1$ and $\gamma N^{\beta-1} \le 1$. Using (2.6),

$$\begin{aligned} \mathcal{N}(r) &\le N^r \gamma^r \mathcal{N}(0) + C_1 r N^{r(2-\beta)} \\ &\le C_2 r \gamma^r N^{r(2-\beta)}. \end{aligned}$$

Hence

$$\dim_B G \le \lim_{r\to\infty} \frac{\log C_2 r \gamma^r N^{r(2-\beta)}}{\log N^r} = 2 - \beta + \log_N \gamma \ .$$

Since $\dim_B G \ge 1$, we get

$$1 \le \dim_B G \le 2 - \beta + \log_N \gamma \ .$$

Case 3: Consider $\gamma > 1$ and $\gamma N^{\beta-1} > 1$. Then

$$1 + \gamma N^{\beta-1} \cdots + \gamma^{r-1} N^{(r-1)(\beta-1)} = \frac{(\gamma N^{\beta-1})^r - 1}{\gamma N^{\beta-1} - 1} \le \frac{(\gamma N^{\beta-1})^r}{\gamma N^{\beta-1} - 1} \ .$$

Then from (2.6),

$$\begin{aligned} \mathcal{N}(r) &\le \gamma^r N^r \mathcal{N}(0) + C_1 \frac{N^{r(2-\beta)} \gamma^r N^{r(\beta-1)}}{\gamma N^{\beta-1} - 1} \\ &\le C_3 \gamma^r N^r, \end{aligned}$$

where $C_3 = \mathcal{N}(0) + \frac{C_1}{\gamma N^{\beta-1} - 1}$. Therefore

$$\dim_B G \le \lim_{r\to\infty} \frac{\log C_3 \gamma^r N^r}{\log N^r} = 1 + \log_N \gamma \ .$$

□

The following definition can be read in [33].

Definition 2.2.1. *Given a continuous function h, the maximum range R_h of h on I is given by*

$$R_h(I) = \sup_{s,t\in I} |h(s) - h(t)| .$$

To get a non-trivial lower bound for the box dimensions of the graphs of affine fractal functions, the idea of Lemma 2.1.1 of [40] plays an important role [9, 10, 36]. But for non-affine fractal functions, this result is not valid in general and one need to impose extra conditions on the IFS parameters. Also for non-affine fractal function f^α, one can not take the original function f and the base function b as affine together. By imposing certain restrictions on (2.4), a similar type of result of Lemma 2.1.1 of [40], is established as follows.

Lemma 2.2.1. *Let f^α be the α-fractal interpolation function corresponding to the IFS (2.3) and (2.4) with scaling factors $\alpha_i \geq 0$ for all $i = 1, 2, \ldots, N$. Also assume that f is concave and b is affine. Let $\gamma = \sum_{i=1}^{N} |\alpha_i| = \sum_{i=1}^{N} \alpha_i > 1$. Assume that the interpolation points $\{(x_i, y_i)\}_{i=0}^{N}$ are not collinear. Then*

$$\lim_{r\to\infty} \frac{\mathcal{N}(r)}{N^r} = \infty .$$

Proof. Let for some $j \in \{1, 2, \ldots, N-1\}$, $(x_{j-1}, f^\alpha(x_{j-1}))$, $(x_j, f^\alpha(x_j))$ and $(x_{j+1}, f^\alpha(x_{j+1}))$ be three non-collinear points. Let $x_j = (1-\lambda)x_{j-1} + \lambda x_{j+1}$ for some $\lambda \in (0, 1)$. Then

$$f^\alpha(x_j) - (1-\lambda) f^\alpha(x_{j-1}) - \lambda f^\alpha(x_{j+1}) = V_j \neq 0$$

(see Figure 2.3). Clearly,

$$V_j \leq \max\{|f^\alpha(x_j) - f^\alpha(x_{j-1})|, |f^\alpha(x_j) - f^\alpha(x_{j+1})|\} .$$

Therefore,

$$R_{f^\alpha}(I) \geq V_j .$$

Since f is concave,

$$f(x_j) - (1-\lambda) f(x_{j-1}) - \lambda f(x_{j+1}) \geq 0 .$$

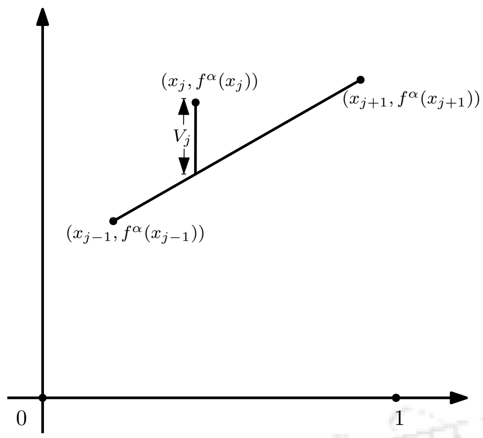

Figure 2.3: Non-collinear interpolation points.

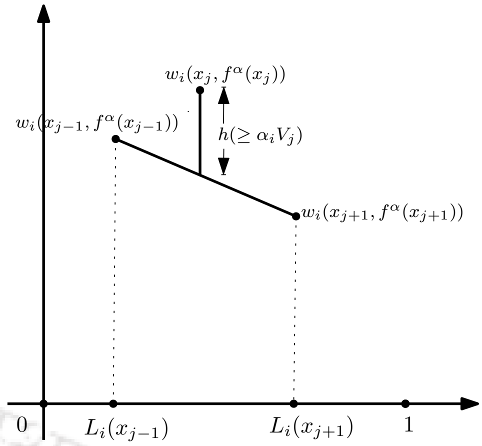

Figure 2.4: Effect of w_i.

As f^α interpolates f on the nodes $x_i, i = j-1, j, j+1$, it follows that

$$f^\alpha(x_j) - (1-\lambda) f^\alpha(x_{j-1}) - \lambda f^\alpha(x_{j+1}) = V_j > 0 \ .$$

Since G is the graph of a continuous function f^α and $(x_{j-1}, f^\alpha(x_{j-1}))$, $(x_j, f^\alpha(x_j))$ and $(x_{j+1}, f^\alpha(x_{j+1})) \in G$, it follows that

$$\mathcal{N}(r) \geq V_j N^r \ .$$

Note that for each $i = 1, 2, \ldots, N$, L_i is affine and therefore

$$L_i(x_j) = (1-\lambda) L_i(x_{j-1}) + \lambda L_i(x_{j+1}) \ .$$

Since for $i = 1, 2, \ldots, N$, $w_i(G) \subset G$, the points

$w_i(x_{j-1}, f^\alpha(x_{j-1})) = (L_i(x_{j-1}), F_i(x_{j-1}, f^\alpha(x_{j-1}))), w_i(x_j, f^\alpha(x_j)) = (L_i(x_j), F_i(x_j, f^\alpha(x_j)))$

and $w_i(x_{j+1}, f^\alpha(x_{j+1})) = (L_i(x_{j+1}), F_i(x_{j+1}, f^\alpha(x_{j+1})))$ belong to G. Now from (2.4), it follows that

$$\begin{aligned}
& F_{i_1}(x_j, f^\alpha(x_j)) - (1-\lambda) F_{i_1}(x_{j-1}, f^\alpha(x_{j-1})) - \lambda F_{i_1}(x_{j+1}, f^\alpha(x_{j+1})) \\
&= \alpha_{i_1} f^\alpha(x_j) + f(L_{i_1}(x_j)) - \alpha_{i_1} b(x_j) - (1-\lambda)\big(\alpha_{i_1} f^\alpha(x_{j-1}) + f(L_{i_1}(x_{j-1})) - \alpha_{i_1} b(x_{j-1})\big) \\
&\quad - \lambda\big(\alpha_{i_1} f^\alpha(x_{j+1}) + f(L_{i_1}(x_{j+1})) - \alpha_{i_1} b(x_{j+1})\big) \\
&= \alpha_{i_1}\big(f^\alpha(x_j) - (1-\lambda) f^\alpha(x_{j-1}) - \lambda f^\alpha(x_{j+1})\big) - \alpha_{i_1}\big(b(x_j) - (1-\lambda) b(x_{j-1}) - \lambda b(x_{j+1})\big) \\
&\quad + \big(f(L_{i_1}(x_j)) - (1-\lambda) f(L_{i_1}(x_{j-1})) - \lambda f(L_{i_1}(x_{j+1}))\big).
\end{aligned} \tag{2.7}$$

Note that for affine FIF, the last two bracketed terms in (2.7) become zero but for non-affine FIF f^α, may not be.

Since f is concave,

$$f(L_{i_1}(x_j)) - (1-\lambda)f(L_{i_1}(x_{j-1})) - \lambda f(L_{i_1}(x_{j+1})) \geq 0 \ .$$

Since b is affine,

$$b(x_j) - (1-\lambda)b(x_{j-1}) - \lambda b(x_{j+1}) = 0 \ .$$

Therefore using these in (2.7), it follows that

$$\begin{aligned} F_{i_1}(x_j, f^\alpha(x_j)) - (1-\lambda)F_{i_1}(x_{j-1}, f^\alpha(x_{j-1})) - \lambda F_{i_1}(x_{j+1}, f^\alpha(x_{j+1})) \\ \geq \alpha_{i_1}\big(f^\alpha(x_j) - (1-\lambda)f^\alpha(x_{j-1}) - \lambda f^\alpha(x_{j+1})\big). \end{aligned}$$

Hence

$$F_{i_1}(x_j, f^\alpha(x_j)) - (1-\lambda)F_{i_1}(x_{j-1}, f^\alpha(x_{j-1})) - \lambda F_{i_1}(x_{j+1}, f^\alpha(x_{j+1})) \geq \alpha_{i_1} V_j, \tag{2.8}$$

(see Figure 2.4). Therefore,

$$R_{f^\alpha}(I_{i_1}) \geq \alpha_{i_1} V_j \ .$$

Hence to cover $G = \bigcup_{i_1=1}^{N} L_{i_1}(I) \times f^\alpha\big(L_{i_1}(I)\big)$, we need

$$\mathcal{N}(r) \geq \sum_{i_1=1}^{N} \alpha_{i_1} V_j N^r \ \text{ for } \ r \geq 1 \ .$$

Similarly,

$$\begin{aligned} &F_{i_2}(L_{i_1}(x_j), f^\alpha(L_{i_1}(x_j)) - (1-\lambda)F_{i_2}(L_{i_1}(x_{j-1}), f^\alpha(L_{i_1}(x_{j-1})) \\ &\quad - \lambda F_{i_2}(L_{i_1}(x_{j+1}), f^\alpha(L_{i_1}(x_{j+1})) \\ &= \alpha_{i_2}(f^\alpha(L_{i_1}(x_j)) - (1-\lambda)f^\alpha(L_{i_1}(x_{j-1})) - \lambda f^\alpha(L_{i_1}(x_{j+1}))) \\ &\quad + \big(f(L_{i_1 i_2}(x_j)) - (1-\lambda)f(L_{i_1 i_2}(x_{j-1})) - \lambda f(L_{i_1 i_2}(x_{j+1}))\big) \\ &\quad - \alpha_{i_2}\big(b(L_{i_1}(x_j)) - (1-\lambda)b(L_{i_1}(x_{j-1})) - \lambda b(L_{i_1}(x_{j+1}))\big). \end{aligned}$$

Using the given hypothesis, the last two terms of the above inequality are greater than or equal to zero. Therefore,

$$\begin{aligned} &F_{i_2}(L_{i_1}(x_j), f^\alpha(L_{i_1}(x_j)) - (1-\lambda)F_{i_2}(L_{i_1}(x_{j-1}), f^\alpha(L_{i_1}(x_{j-1})) \\ &\quad - \lambda F_{i_2}(L_{i_1}(x_{j+1}), f^\alpha(L_{i_1}(x_{j+1})) \\ &\geq \alpha_{i_2}\big(f^\alpha(L_{i_1}(x_j)) - (1-\lambda)f^\alpha(L_{i_1}(x_{j-1})) - \lambda f^\alpha(L_{i_1}(x_{j+1}))\big). \end{aligned}$$

But

$$\begin{aligned}
&f^\alpha(L_{i_1}(x_j)) - (1-\lambda)f^\alpha(L_{i_1}(x_{j-1})) - \lambda f^\alpha(L_{i_1}(x_{j+1}))\\
&= F_{i_1}(x_j, f^\alpha(x_j)) - (1-\lambda)F_{i_1}(x_{j-1}, f^\alpha(x_{j-1})) - \lambda F_{i_1}(x_{j+1}, f^\alpha(x_{j+1}))\\
&\geq \alpha_{i_1} V_j, \qquad \text{using (2.8).}
\end{aligned}$$

Therefore,

$$\begin{aligned}
F_{i_2}(L_{i_1}(x_j), f^\alpha(L_{i_1}(x_j)) - (1-\lambda)F_{i_2}(L_{i_1}(x_{j-1}), f^\alpha(L_{i_1}(x_{j-1}))\\
- \lambda F_{i_2}(L_{i_1}(x_{j+1}), f^\alpha(L_{i_1}(x_{j+1})) \geq \alpha_{i_2}\alpha_{i_1} V_j.
\end{aligned}$$

It shows that

$$R_{f^\alpha}(I_{i_1 i_2}) \geq \alpha_{i_2}\alpha_{i_1} V_j \ .$$

Hence to cover $G = \bigcup_{i_1,i_2=1}^{N} L_{i_1 i_2}(I) \times f^\alpha\big(L_{i_1 i_2}(I)\big)$, we need

$$\mathcal{N}(r) \geq \sum_{i_1,i_2=1}^{N} \alpha_{i_2}\alpha_{i_1} V_j N^r \ \text{ for } \ r \geq 2 \ .$$

Define $I_{i_1 i_2 \ldots i_k} = L_{i_1 i_2 \ldots i_k}(I) = L_{i_k} \circ L_{i_{k-1}} \circ \cdots \circ L_{i_1}(I)$. Then by succession, it follows that

$$\begin{aligned}
&F_{i_k}(L_{i_1 i_2 \ldots i_{k-1}}(x_j), f^\alpha(L_{i_1 i_2 \ldots i_{k-1}}(x_j)) - (1-\lambda)F_{i_k}(L_{i_1 i_2 \ldots i_{k-1}}(x_{j-1}), f^\alpha(L_{i_1 i_2 \ldots i_{k-1}}(x_{j-1}))\\
&\quad - \lambda F_{i_k}(L_{i_1 i_2 \ldots i_{k-1}}(x_{j+1}), f^\alpha(L_{i_1 i_2 \ldots i_{k-1}}(x_{j+1}))\\
&\geq \alpha_{i_k}\alpha_{i_{k-1}} \cdots \alpha_{i_1} V_j.
\end{aligned}$$

Therefore,

$$R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \geq \alpha_{i_k}\alpha_{i_{k-1}} \cdots \alpha_{i_1} V_j. \tag{2.9}$$

Hence,

$$\mathcal{N}(r) \geq \sum_{i_1,\ldots,i_k=1}^{N} \alpha_{i_k}\alpha_{i_{k-1}} \ldots \alpha_{i_1} V_j N^r \ \text{ for } \ r \geq k \ .$$

Therefore,

$$\mathcal{N}(r) \geq \Big(\gamma^r V_j - 1\Big) N^r$$

and since $\gamma > 1$, the result follows. □

In the following theorem, a non-trivial lower bound for the box dimension of the graph G of f^α is found.

Theorem 2.2.2. *Let G be the graph of the α-fractal interpolation function f^α corresponding to the IFS (2.3) and (2.4). Also assume that f is concave and b is affine. Let the interpolation points $\{(x_i, y_i)\}_{i=0}^N$ be not collinear and $\gamma = \sum_{i=1}^N |\alpha_i| = \sum_{i=1}^N \alpha_i > 1$, where $\alpha_i > 0$ for all i. If $\beta = \min\{\beta_1, \beta_2\} = 1$, then*

$$\dim_B(G) \geq 1 + \log_N \gamma \, .$$

Proof. If the interpolation points are not collinear, then for $\gamma > 1$, we proceed as follows to get a non-trivial lower bound for $\dim_B G$. Since f and b are continuous functions, for $(x, y) \in \Lambda^*\big(r+1, l(k,i)\big)$,

$$w_i^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Nx - (i-1) \\ \frac{y - f(x) + \alpha_i b(Nx-(i-1))}{\alpha_i} \end{pmatrix} .$$

Then $w_i^{-1}\big(\Lambda^*(r+1, l(k,i))\big)$ is contained in a rectangle of width $\frac{1}{N^r}$ and height

$$\frac{1}{|\alpha_i|}\left[\frac{\mathcal{N}(r+1, l(k,i))}{N^{r+1}} + \frac{s}{N^{(r+1)\beta_1}} + |\alpha_i|\frac{m}{N^{r\beta_2}}\right] ,$$

(see Figure 2.5). Therefore

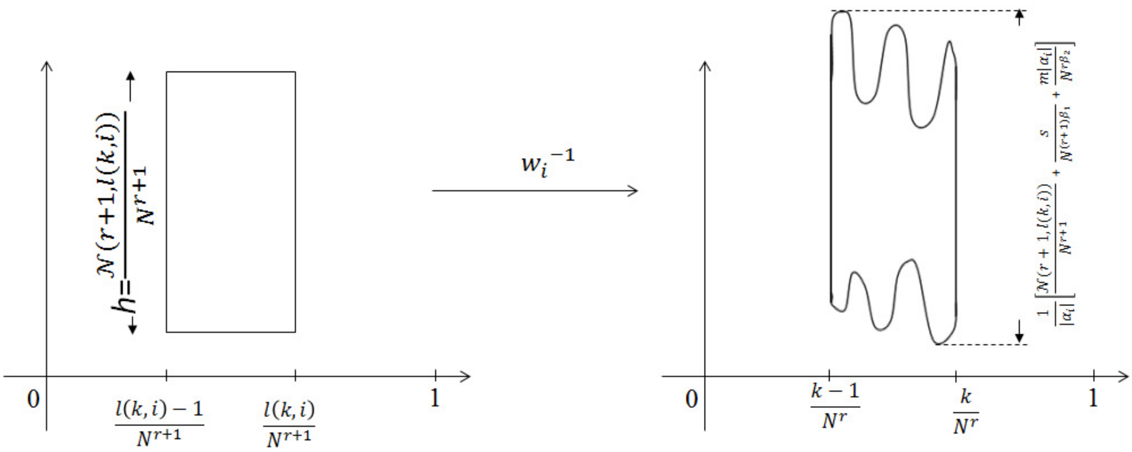

Figure 2.5: Image of the rectangle $\Lambda^*(r+1, l(k,i))$ under the map w_i^{-1}.

$$\mathcal{N}(r,k) \leq \frac{1}{|\alpha_i|}\left[\frac{\mathcal{N}(r+1, l(k,i))}{N^{r+1}} + \frac{s}{N^{(r+1)\beta_1}} + |\alpha_i|\frac{m}{N^{r\beta_2}}\right] N^r + 2$$

and hence

$$\mathcal{N}\big(r+1, l(k,i)\big) \geq N|\alpha_i|\mathcal{N}(r,k) - sN^{(r+1)(1-\beta_1)} - |\alpha_i| m N^{r(1-\beta_2)+1} - 2N|\alpha_i| \ .$$

By summing over i,

$$\begin{aligned}
\sum_{i=1}^{N} \mathcal{N}\big(r+1, l(k,i)\big) &\geq \sum_{i=1}^{N} N|\alpha_i|\mathcal{N}(r,k) - \sum_{i=1}^{N} sN^{(r+1)(1-\beta_1)} - \sum_{i=1}^{N} |\alpha_i| m N^{r(1-\beta_2)+1} \\
&- \sum_{i=1}^{N} 2N|\alpha_i| \\
&= N\gamma\mathcal{N}(r,k) - sN^{(r+1)(1-\beta_1)+1} - m\gamma N^{r(1-\beta_2)+1} - 2N\gamma.
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{k=1}^{N^r} \Big(\sum_{i=1}^{N} \mathcal{N}(r+1, l(k,n))\Big) &\geq N\gamma\mathcal{N}(r) - sN^{(r+1)(1-\beta_1)+(r+1)} - m\gamma N^{r(1-\beta_2)+r+1} \\
&- 2N^{r+1}\gamma \\
&\geq N\gamma\mathcal{N}(r) - sN^{(r+1)(2-\beta)} - m\gamma N^{(r+1)(2-\beta)} \\
&- 2\gamma N^{(r+1)(2-\beta)}.
\end{aligned}$$

Therefore,

$$\mathcal{N}(r+1) \geq N\gamma\mathcal{N}(r) - C_4 N^{(r+1)(2-\beta)} \ ,$$

where $C_4 = s + (m+2)\gamma$. Iterating on r gives,

$$\begin{aligned}
\mathcal{N}(r) &\geq N\gamma\mathcal{N}(r-1) - C_4 N^{r(2-\beta)} \\
&\geq N\gamma[N\gamma\mathcal{N}(r-2) - C_4 N^{(r-1)(2-\beta)}] - C_4 N^{r(2-\beta)} \\
&= N^2\gamma^2\mathcal{N}(r-2) - C_4 N^{r(2-\beta)}[1 + \gamma N^{\beta-1}] \\
&\geq N^2\gamma^2[N\gamma\mathcal{N}(r-3) - C_4 N^{(r-2)(2-\beta)}] - C_4 N^{r(2-\beta)}[1 + \gamma N^{\beta-1}] \\
&= N^3\gamma^3\mathcal{N}(r-3) - C_4 N^{r(2-\beta)}[1 + \gamma N^{\beta-1} + \gamma^2 N^{2(\beta-1)}].
\end{aligned}$$

Continuing the process, we get

$$\mathcal{N}(r) \geq (N\gamma)^{r-j}\mathcal{N}(j) - C_4 N^{r(2-\beta)}\big(1 + \gamma N^{\beta-1} + \ldots + \gamma^{r-j-1} N^{(r-j-1)(\beta-1)}\big) \tag{2.10}$$

for all $1 \le j \le r$.

Let $\gamma > 1$ with $\gamma N^{\beta-1} > 1$. Then from (2.10)

$$\begin{aligned}
\mathcal{N}(r) &\ge (N\gamma)^{r-j}\mathcal{N}(j) - C_4 N^{r(2-\beta)} \frac{(\gamma N^{\beta-1})^{r-j} - 1}{\gamma N^{\beta-1} - 1} \\
&\ge (N\gamma)^{r-j}\mathcal{N}(j) - C_4 N^{r(2-\beta)} \frac{(\gamma N^{\beta-1})^{r-j}}{\gamma N^{\beta-1} - 1} \\
&= \gamma^{r-j} N^{(2-\beta)r} M \text{ for all } 1 \le j \le r \text{ and } \gamma > 1,
\end{aligned}$$

where

$$M = \left(\frac{\mathcal{N}(j)}{N^{(1-\beta)r+j}} - C_4 \frac{\gamma N^{(\beta-1)(r-j)}}{\gamma N^{\beta-1} - 1} \right) .$$

Now

$$\begin{aligned}
N^{(1-\beta)r+j} M &= \mathcal{N}(j) - C_4 \frac{\gamma N^{(\beta-1)(r-j)} N^{r(1-\beta)+j}}{\gamma N^{\beta-1} - 1} \\
&= \mathcal{N}(j) - C_4 \frac{\gamma N^{j(2-\beta)}}{\gamma N^{\beta-1} - 1}.
\end{aligned}$$

Then for $\gamma > 1$ and non-collinear interpolation points, by Lemma 2.2.1

$$\lim_{r\to\infty} \frac{\mathcal{N}(r)}{N^r} = \infty .$$

When $\beta = 1$, using the above expression, it follows that for large j

$$\left(\mathcal{N}(j) - C_4 \frac{\gamma N^{j(2-\beta)}}{\gamma N^{\beta-1} - 1} \right) > 0 ,$$

that is, $M > 0$. Therefore for $\beta = 1$

$$\mathcal{N}(r) \ge M^* \gamma^r N^r \quad \text{for large } r ,$$

where $M^* = M\gamma^{-j} > 0$. Hence

$$\dim_B G \ge \lim_{r\to\infty} \frac{\log M^* \gamma^r N^r}{\log N^r} = 1 + \log_N \gamma .$$

□

Corollary 2.2.1. *Let f be a concave and Lipschitz function, b be an affine function with $b(x_0) = f(x_0)$, $b(x_0) = f(x_0)$ and $b \neq f$. Also assume that the interpolation points $\{(x_i, y_i)\}_{i=0}^N$ are not collinear and $\gamma = \sum_{i=1}^N |\alpha_i| = \sum_{i=1}^N \alpha_i$, where $\alpha_i > 0$ for all i. Then the graph G of the α-fractal interpolation function f^α corresponding to the IFS (2.3) and (2.4) has box dimension*

$$\dim_B G = \begin{cases} 1 + \log_N \gamma, & \text{if } \gamma > 1 \\ 1 & \text{otherwise} . \end{cases}$$

Proof. The result follows immediately from Theorems 2.2.1 and 2.2.2 with $\beta = 1$. □

In the following, we demonstrate the obtained results with an example.

Example 2.1. *Let $I = [0, 1]$ and $\Delta : 0 < 0.25 < 0.5 < 0.75 < 1$ be the partition of I. Let $f(x) = x^{1/2} - 2x^{1/3} + 2$ be the Hölder function with exponent $1/2$ and $b(x) = 2 - x^{1/2}$. The corresponding α-fractal function f^α for the IFS with scale vector $\alpha = (0.3, 0.4, 0.3, 0.2)$ is given in Figure 2.6. Then according to the Theorem 2.2.1, the box dimension of the graph G of the function f^α is*

$$1 \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq 2 - 1/2 + \frac{\log 1.2}{\log 4} = 1.6315 .$$

Let $f(x) = \sin(x)$ be a Lipschitz function which is concave in I and $b(x) = x \sin(1)$ be an affine function. Then by Corollary 2.2.1, the box dimension of the graph G of the corresponding α-fractal function f^α given in Figure 2.7 is $\dim_B G = 1 + \log_4 1.2 = 1.1315$.

2.3 Variation of continuous functions and its properties

Let $h \in \mathcal{C}(I)$. For $\delta \geq 0$ and any $x \in I$, define $I[x; \delta] = I \cap [x - \delta, x + \delta]$. Then the δ-oscillation of h at x on I is defined as

$$O^I_{h;\delta}(x) = \sup_{x' \in I[x;\delta]} h(x') - \inf_{x' \in I[x;\delta]} h(x') = \sup_{x',x'' \in I[x;\delta]} |h(x') - h(x'')| .$$

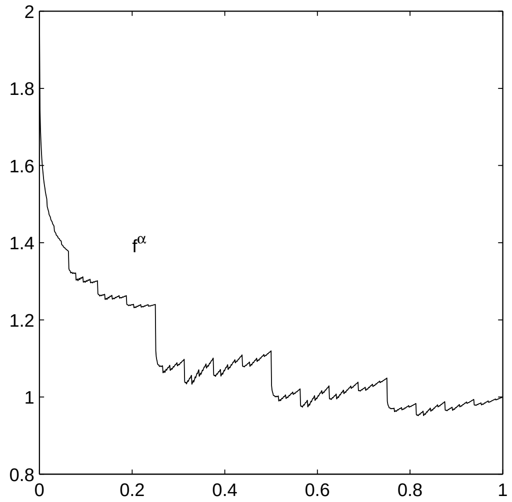

Figure 2.6: The graph of f^α corresponding to $f(x) = x^{1/2} - 2x^{1/3} + 2$ and $b(x) = 2 - x^{1/2}$.

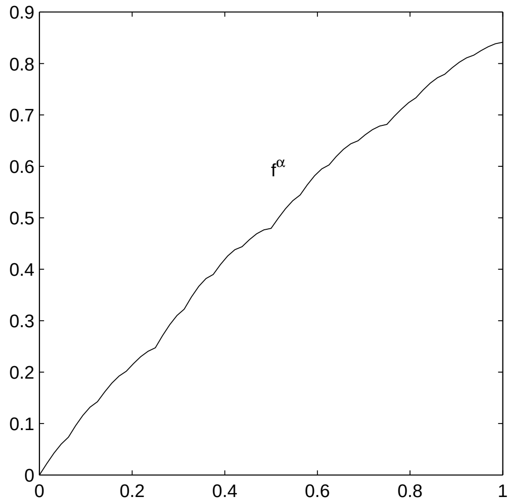

Figure 2.7: The graph of f^α corresponding to $f(x) = \sin(x)$ and $b(x) = x\sin(1)$.

The continuity of h ensures the continuity of $O^I_{h;\delta}$. The δ-variation of h on I is denoted by $V_{h;\delta}(I)$ and defined by [20, 31, 90]

$$V_{h;\delta}(I) = \int_I O^I_{h;\delta}(x)dx \ .$$

If h is a real valued non-constant continuous function then the relation between box dimension of the graph G and the variation of h is given by [31, 33]

$$\overline{\dim}_B G = \limsup_{\delta \to 0^+} \left(2 - \frac{\log V_{h;\delta}(I)}{\log \delta}\right) \tag{2.11}$$

and

$$\underline{\dim}_B G = \liminf_{\delta \to 0^+} \left(2 - \frac{\log V_{h;\delta}(I)}{\log \delta}\right). \tag{2.12}$$

Lemma 2.3.1. *Let f_1 and f_2 be two continuous functions on I. For real constants c_1, c_2, the following statements are true.*

(a) $V_{c_1 f_1 + c_2;\delta}(I) = |c_1| V_{f_1;\delta}(I)$,

(b) $|c_1| V_{f_1;\delta}(I) - |c_2| V_{f_2;\delta}(I) \leq V_{c_1 f_1 + c_2 f_2;\delta}(I) \leq |c_1| V_{f_1;\delta}(I) + |c_2| V_{f_2;\delta}(I)$.

Proof.

(a) By definition, for $\delta \geq 0$

$$\begin{aligned} O^I_{c_1 f_1 + c_2;\delta}(x) &= \sup_{x', x'' \in I[x;\delta]} |c_1 f_1(x') + c_2 - c_1 f_1(x'') - c_2| \\ &= |c_1| \sup_{x', x'' \in I[x;\delta]} |f_1(x') - f_1(x'')| \\ &= |c_1| O^I_{f_1;\delta}(x). \end{aligned}$$

Integrating $O^I_{c_1 f_1 + c_2;\delta}(x)$ over I, the result follows.

(b) Using the fact

$$\begin{aligned} |c_1||f_1(x') - f_1(x'')| - |c_2||f_2(x') - f_2(x'')| &\leq |(c_1 f_1(x') + c_2 f_2(x')) - (c_1 f_1(x'') + c_2 f_2(x''))| \\ &\leq |c_1||f_1(x') - f_1(x'')| + |c_2||f_2(x') - f_2(x'')|, \end{aligned}$$

it follows easily that

$$|c_1| O^I_{f_1;\delta}(x) - |c_2| O^I_{f_2;\delta}(x) \leq O^I_{c_1 f_1 + c_2 f_2;\delta}(x) \leq |c_1| O^I_{f_1;\delta}(x) + |c_2| O^I_{f_2;\delta}(x)\ .$$

Integrating $O^I_{c_1 f_1 + c_2 f_2;\delta}(x)$ over I, the result follows. □

Lemma 2.3.2. *Let* $h \in \mathcal{C}(I)$ *and* $\Delta : 0 = x_0 < x_1 < \cdots < x_N = 1$ *be the partition of* I *with* $I_i = [x_{i-1}, x_i]$*. Then*

$$\sum_{i=1}^{N} V_{h;\delta}(I_i) \le V_{h;\delta}(I) \le \sum_{i=1}^{N} V_{h;\delta}(I_i) + 2(N-1)V_h(I)\delta \ ,$$

where $V_h(I) = \sup_{x\in I} h(x) - \inf_{x\in I} h(x)$*.*

Proof. The proof follows on similar lines of Lemma 5.2 of [36]. But for completeness, it is presented here.

For any $i = 1, 2, \ldots, N$, $I_i[x;\delta] \subseteq I[x;\delta]$ and hence $O^{I_i}_{h;\delta}(x) \le O^{I}_{h;\delta}(x)$. Therefore, it follows that

$$\sum_{i=1}^{N} V_{h;\delta}(I_i) \le V_{h;\delta}(I) \ .$$

For right hand inequality, let $I_1^* = [0, x_1 - \delta]$, $I_i^* = [x_{i-1}+\delta, x_i - \delta]$ for $i = 2, 3, \ldots, N-1$, $I_j^{**} = [x_j - \delta, x_j + \delta] \cap I$ for $j = 1, 2, \ldots, N-1$ and $I_N^* = [x_{N-1} + \delta, 1]$. Clearly,

$$\big(\cup_{i=1}^{N} I_i^*\big) \bigcup \big(\cup_{j=1}^{N-1} I_j^{**}\big) = I \ .$$

For $x \in I_i^*$, $i = 1, 2, \ldots, N$

$$O^{I}_{h;\delta}(x) = O^{I_i}_{h;\delta}(x)$$

and

$$O^{I}_{h;\delta}(x) \le V_h(I)$$

for $x \in I_j^{**}$, $j = 1, 2, \ldots, N-1$. Therefore

$$\begin{aligned} V_{h;\delta}(I) &\le \sum_{i=1}^{N} \int_{I_i^*} O^{I_i}_{h;\delta}(x)dx + \sum_{j=1}^{N-1} \int_{I_j^{**}} V_h(I)dx \\ &\le \sum_{i=1}^{N} V_{h;\delta}(I_i) + (N-1)V_h(I)2\delta. \end{aligned}$$

□

Lemma 2.3.3. *Let* h *be a fractal interpolation function on* I*. If* $L(x) = ax + b$*, where* a *and* b *are constants with* $a > 0$ *such that* $L(I) \subset I$*. Then*

$$V_{h(L^{-1}(.));\delta}(L(I)) = aV_{h;\frac{\delta}{a}}(I) \ .$$

Proof. Denote $J = L(I)$. For $x \in J$,

$$\begin{aligned} O^J_{h(L^{-1}(.));\delta}(x) &= \sup_{x' \in J[x;\delta]} h(L^{-1}(x')) - \inf_{x' \in J[x;\delta]} h(L^{-1}(x')) \\ &= \sup_{p' \in I[p;\frac{\delta}{a}]} h(p') - \inf_{p' \in I[p;\frac{\delta}{a}]} h(p') \\ &= O^I_{h;\frac{\delta}{a}}(p). \end{aligned}$$

Therefore

$$\begin{aligned} V_{h(L^{-1}(.));\delta}(L(I)) &= \int_J O^J_{h(L^{-1}(.));\delta}(x)dx \\ &= a\int_I O^I_{h;\frac{\delta}{a}}(p)dp \\ &= aV_{h;\frac{\delta}{a}}(I). \end{aligned}$$

□

Lemma 2.3.4. *If h is Hölderian on I with exponent $\beta \in (0,1]$ and $\delta > 0$, then there exists some positive constant M such that*

$$V_{h;\delta}(I) \leq M|I|\delta^\beta .$$

Proof. Since h is β Hölderian in I, for $\delta > 0$ there exists a positive constant C such that

$$|h(x') - h(x'')| \leq C|x' - x''|^\beta \quad \text{for all} \quad x', x'' \in I .$$

Therefore $O^I_{h;\delta} \leq 2^\beta C\delta^\beta$ and hence

$$V_{h;\delta}(I) \leq M|I|\delta^\beta, \quad \text{where} \quad M = 2^\beta C .$$

□

2.4 Box dimensions of α-fractal functions using variation

In this section, the restriction on the data set as equally spaced is relaxed. Note that f and b are uniformly Hölderian with exponents β_1 and β_2 respectively. In the present section, the calculation of the box dimension of the graph of f^α is estimated using the techniques developed in [35, 36]. Before proving the main theorem, we first prove the following lemma.

Lemma 2.4.1. *Let f^α be the α-fractal interpolation function corresponding to the IFS (1.8), (1.9) and (1.10). Then there exist positive constants C_1, C_2 and δ such that*

$$\sum_{i=1}^{N} |\alpha_i| a_i V_{f^\alpha;\frac{\delta}{a_i}}(I) - C_1\delta^\beta \le V_{f^\alpha;\delta}(I) \le \sum_{i=1}^{N} |\alpha_i| a_i V_{f^\alpha;\frac{\delta}{a_i}}(I) + C_2\delta^\beta$$

for all $0 < \delta < 1$ and $\beta = \min\{\beta_1, \beta_2\}$.

Proof. Since f^α is continuous on I, the left inequality of Lemma 2.3.2 gives

$$V_{f^\alpha;\delta}(I) \ge \sum_{i=1}^{N} V_{f^\alpha;\delta}(I_i)\,.$$

Since f^α satisfies the fixed point equation (1.11), using Lemma 2.3.1, it follows that

$$\begin{aligned} V_{f^\alpha;\delta}(I) &\ge \sum_{i=1}^{N} \left[|\alpha_i| \big(V_{f^\alpha(L^{-1}(.));\delta}(I_i) - V_{b(L^{-1}(.));\delta}(I_i)\big) - V_{f;\delta}(I_i)\right] \\ &\ge \sum_{i=1}^{N} \left[|\alpha_i| a_i \big(V_{f^\alpha;\frac{\delta}{a_i}}(I) - V_{b;\frac{\delta}{a_i}}(I)\big)\right] - V_{f;\delta}(I), \text{ using Lemma 2.3.3.} \end{aligned}$$

But by Lemma 2.3.4, there exist positive constants M_1, M_2 such that

$$V_{f;\delta}(I) \le M_1|I|\delta^{\beta_1} \le M_1|I|\delta^\beta$$

and

$$V_{b;\frac{\delta}{a_i}}(I) \le M_2|I| \left(\frac{\delta}{a_i}\right)^{\beta_2} \le M_2|I| \left(\frac{\delta}{a_i}\right)^{\beta}.$$

Hence

$$V_{f^\alpha;\delta}(I) \ge \sum_{i=1}^{N} |\alpha_i| a_i V_{f^\alpha;\frac{\delta}{a_i}}(I) - C_1\delta^\beta\,,$$

where $C_1 = \left(M_1 + \sum_{i=1}^{N} |\alpha_i| a_i^{1-\beta} M_2\right)|I|$. On the other hand using Lemma 2.3.2

$$\begin{aligned} V_{f^\alpha;\delta}(I) &\le \sum_{i=1}^{N} \left[|\alpha_i| \big(V_{f^\alpha(L^{-1}(.));\delta}(I_i) + V_{b(L^{-1}(.));\delta}(I_i)\big) + V_{f;\delta}(I_i)\right] + 2(N-1)V_{f^\alpha}(I)\delta \\ &\le \sum_{i=1}^{N} |\alpha_i| a_i V_{f^\alpha;\frac{\delta}{a_i}}(I) + C_2\delta^\beta, \end{aligned}$$

where $C_2 = C_1 + 2(N-1)V_{f^\alpha}(I)$. $\square$

The following theorem is useful to get an upper bound for the box dimension of the graph of f^α.

Theorem 2.4.1. *Let D be the solution of $\sum_{i=1}^{N} |\alpha_i| a_i^{D-1} = 1$ and f^α be the α-fractal interpolation function corresponding to the IFS (1.8), (1.9) and (1.10).*

(a) If $\gamma > 1$, then there exist positive constants B_2 and δ_0 such that $V_{f^\alpha;\delta}(I) \leq B_2\delta^{1+\beta-D}$ for $0 < \delta \leq \delta_0$.

(b) If $\gamma < 1$, then there exist positive constants B_3 and δ_0 such that $V_{f^\alpha;\delta}(I) \leq B_3\delta^\beta$ for $0 < \delta \leq \delta_0$.

(c) If $\gamma = 1$, then there exist positive constants B_4 and δ_0 such that $V_{f^\alpha;\delta}(I) \leq B_4\delta^\beta \log\frac{1}{\delta}$ for $0 < \delta \leq \delta_0$.

Proof.
(a) Since $\gamma > 1$, it follows that $D > 1$ and $\beta > 1 + \beta - D$. From Lemma 2.4.1, for $\delta_0 > 0$ choose $k_2 > 0$ large enough so that

$$V_{f^\alpha;\delta}(I) \leq \frac{C_2\delta^\beta}{1-\gamma} + k_2\delta^{1+\beta-D}$$

for $\delta_0 \leq \delta \leq \frac{\delta_0}{\underline{a}}$, where $\underline{a} = \min\{a_i : i = 1, 2, \ldots, N\}$ and C_2 is as given in Lemma 2.4.1. Denote $\overline{\psi}(\delta) = \frac{C_2\delta^\beta}{1-\gamma} + k_2\delta^{1+\beta-D}$. Define $\overline{a} = \max\{a_i : i = 1, 2, \ldots, N\}$. Then $\overline{a}\delta_0 \leq \delta \leq \delta_0$ implies that $\delta_0 \leq \frac{\delta}{a_i} \leq \frac{\delta_0}{\underline{a}}$ and

$$\begin{aligned}
V_{f^\alpha;\delta}(I) &\leq \sum_{i=1}^{N} |\alpha_i| a_i V_{f^\alpha;\frac{\delta}{a_i}}(I) + C_2\delta^\beta \\
&\leq \sum_{i=1}^{N} |\alpha_i| a_i \overline{\psi}(\frac{\delta}{a_i}) + C_2\delta^\beta \\
&= \sum_{i=1}^{N} |\alpha_i| a_i \frac{C_2\delta^\beta}{(1-\gamma)a_i^\beta} + \sum_{i=1}^{N} |\alpha_i| a_i \frac{k_2\delta^{1+\beta-D}}{a_i^{1+\beta-D}} + C_2\delta^\beta \\
&\leq \sum_{i=1}^{N} |\alpha_i| a_i^{1-\beta} \frac{C_2\delta^\beta}{1-\gamma} + \sum_{i=1}^{N} |\alpha_i| a_i^{D-\beta} k_2\delta^{1+\beta-D} + C_2\delta^\beta.
\end{aligned}$$

It is noted that $a_i^{D-\beta} \leq a_i^{D-1}$ as $a_i < 1$, also $a_i^{1-\beta} \leq 1$. For $\overline{a}\delta_0 \leq \delta \leq \delta_0$,

$$\begin{aligned} V_{f^{\alpha};\delta}(I) &\leq \sum_{i=1}^{N} |\alpha_i| \frac{C_2\delta^{\beta}}{1-\gamma} + \sum_{i=1}^{N} |\alpha_i| a_i^{D-1} k_2 \delta^{1+\beta-D} + C_2\delta^{\beta} \\ &\leq \frac{C_2\gamma\delta^{\beta}}{1-\gamma} + k_2\delta^{1+\beta-D} + C_2\delta^{\beta}, \quad (\text{since} \quad \sum_{i=1}^{N} |\alpha_i| = \gamma, \sum_{i=1}^{N} |\alpha_i| a_i^{D-1} = 1) \\ &\leq \frac{C_2\delta^{\beta}}{1-\gamma} + k_2\delta^{1+\beta-D} \\ &= \overline{\psi}(\delta). \end{aligned}$$

Suppose $V_{f^{\alpha};\delta}(I) \leq \overline{\psi}(\delta)$ holds for $\overline{a}^n\delta_0 \leq \delta \leq \delta_0$. Then again by the above arguments it holds for $\overline{a}^{n+1}\delta_0 \leq \delta \leq \delta_0$. As $\overline{a} < 1$,

$$V_{f^{\alpha};\delta}(I) \leq \overline{\psi}(\delta)$$

for $0 < \delta \leq \delta_0$. But

$$\begin{aligned} \overline{\psi}(\delta) &= \frac{C_2\delta^{\beta}}{1-\gamma} + k_2\delta^{1+\beta-D} \\ &\leq k_2\delta^{1+\beta-D}. \end{aligned}$$

Therefore

$$V_{f^{\alpha};\delta}(I) \leq k_2\delta^{1+\beta-D} \quad \text{for } 0 < \delta \leq \delta_0 \,.$$

(b) Since $\gamma < 1$, it follows that $D < 1$ and $\beta < 2 - D$. As in Case (a), one can choose $k_3 > 0$ such that

$$V_{f^{\alpha};\delta}(I) \leq \frac{C_2\delta^{\beta}}{1-\gamma} + k_3\delta^{2-D} \quad \text{for } \overline{a}^n\delta_0 \leq \delta \leq \delta_0 \,.$$

Therefore there exists a positive constant B_3 such that

$$V_{f^{\alpha};\delta}(I) \leq B_3\delta^{\beta} \quad \text{for } 0 < \delta \leq \delta_0 \leq 1 \,.$$

(c) For $\gamma = 1$, $D = 1$. Let for $k_4 > 0$ large enough, $V_{f^{\alpha};\delta}(I) \leq \psi(\delta)$ for $\delta_0 \leq \delta \leq \frac{\delta_0}{\underline{a}}$, where

$$\psi(\delta) = \frac{C_2 \log \delta}{\sum_{i=1}^{N} |\alpha_i| \log a_i} \delta^{\beta} + k_4\delta^{\beta} \,.$$

For $\gamma = \sum_{i=1}^{N} |\alpha_i| = 1$, $\sum_{i=1}^{N} |\alpha_i|$ is replaced by $\sum_{i=1}^{N} |\alpha_i| \log a_i$ and δ^β by $\delta^\beta \log \delta$. Note that

$$\begin{aligned}
\sum_{i=1}^{N} |\alpha_i| a_i \psi(\frac{\delta}{a_i}) + C_2\delta^\beta &= \sum_{i=1}^{N} |\alpha_i| a_i \frac{C_2 \log \frac{\delta}{a_i}}{\sum_{i=1}^{N} |\alpha_i| \log a_i} \frac{\delta^\beta}{a_i^\beta} + \sum_{i=1}^{N} |\alpha_i| a_i k_4 \frac{\delta^\beta}{a_i^\beta} + C_2\delta^\beta \\
&\leq \sum_{i=1}^{N} |\alpha_i| \frac{C_2 \log \frac{\delta}{a_i}}{\sum_{i=1}^{N} |\alpha_i| \log a_i} \delta^\beta + \sum_{i=1}^{N} |\alpha_i| k_4 \delta^\beta + C_2\delta^\beta \\
&\quad (\text{since } a_i^{1-\beta} \leq 1) \\
&= \sum_{i=1}^{N} |\alpha_i| \frac{C_2 \log \delta}{\sum_{i=1}^{N} |\alpha_i| \log a_i} \delta^\beta - \sum_{i=1}^{N} |\alpha_i| \frac{C_2 \log a_i}{\sum_{i=1}^{N} |\alpha_i| \log a_i} \delta^\beta \\
&\quad + \sum_{i=1}^{N} |\alpha_i| k_4 \delta^\beta + C_2\delta^\beta \\
&= \frac{C_2 \log \delta}{\sum_{i=1}^{N} |\alpha_i| \log a_i} \delta^\beta + k_4\delta^\beta \\
&= \psi(\delta).
\end{aligned}$$

Similarly, it can be shown that

$$V_{f^\alpha;\delta}(I) \leq \frac{C_2 \log \delta}{\sum_{i=1}^{N} |\alpha_i| \log a_i} \delta^\beta + k_4\delta^\beta \quad \text{for } \overline{a}^n \delta_0 \leq \delta \leq \delta_0 \,.$$

Therefore, there exists $B_4 > 0$ such that

$$V_{f^\alpha;\delta}(I) \leq B_4 \delta^\beta \log \frac{1}{\delta} \quad \text{for } 0 < \delta \leq \delta_0 \leq 1 \,.$$

□

The following theorem gives an estimate for upper bound of the box dimension of the graph of f^α.

Theorem 2.4.2. *Let $G = \{(x, f^\alpha(x)) : x \in I\}$ be the graph of the α-fractal interpolation function f^α corresponding to the IFS (1.8), (1.9) and (1.10). Let the interpolation points be not collinear and $\gamma = \sum_{i=1}^{N} |\alpha_i|$. Let D be the solution of $\sum_{i=1}^{N} |\alpha_i| a_i^{D-1} = 1$. Then the box dimension of G is as follows.*

(a) For $\gamma > 1$, $\quad 1 \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq 1 - \beta + D$.

(b) For $\gamma \leq 1$, $\quad 1 \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq 2 - \beta$.

Proof. (a) By Theorem 2.4.1(a), there exists positive constant B_2 such that

$$V_{f^\alpha;\delta}(I) \leq B_2 \delta^{1+\beta-D} \quad \text{for } 0 < \delta \leq \delta_0 \, .$$

Using it with (2.11), it follows that

$$\overline{\dim}_B G = \limsup_{\delta \to 0^+} \left(2 - \frac{\log V_{f^\alpha;\delta}(I)}{\log \delta} \right) \leq \limsup_{\delta \to 0^+} \left(2 - \frac{\log B_2 \delta^{1+\beta-D}}{\log \delta} \right) = 1 - \beta + D \, .$$

Since f^α is continuous in I, $\overline{\dim}_B G \geq 1$. Therefore $1 \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq 1 - \beta + D$.

(b) If $\gamma < 1$, then according to Theorem 2.4.1(b) and (2.11)

$$\overline{\dim}_B G = \limsup_{\delta \to 0^+} \left(2 - \frac{\log V_{f^\alpha;\delta}(I)}{\log \delta} \right) \leq \limsup_{\delta \to 0^+} \left(2 - \frac{\log B_3 \delta^{\beta}}{\log \delta} \right) = 2 - \beta \, .$$

If $\gamma = 1$, then according to Theorem 2.4.1(c) and (2.11)

$$\overline{\dim}_B G = \limsup_{\delta \to 0^+} \left(2 - \frac{\log V_{f^\alpha;\delta}(I)}{\log \delta} \right) \leq \limsup_{\delta \to 0^+} \left(2 - \frac{\log B_4 \delta^{\beta} \log \frac{1}{\delta}}{\log \delta} \right) = 2 - \beta \, .$$

Therefore for $\gamma \leq 1$,

$$1 \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq 2 - \beta \, .$$

□

The following result can be read in [36].

Lemma 2.4.2. *(see [36]). For a continuous function* h *on* $[c, d]$, $V_{h;\delta}\big([c,d]\big) \geq V_h\big([c,d]\big)\delta$, *where* $V_h\big([c,d]\big) = \sup_{x \in [c,d]} h(x) - \inf_{x \in [c,d]} h(x)$ *and* $0 \leq \delta \leq d - c$.

The following lemma is required to prove Theorem 2.4.3.

Lemma 2.4.3. *Let* f^α *be the* α*-fractal interpolation function corresponding to the IFS (1.8), (1.9) and (1.10). Also assume that* f *is concave and* b *is affine. Let* $\gamma = \sum_{i=1}^N |\alpha_i| = \sum_{i=1}^N \alpha_i > 1$ *where* $\alpha_i \geq 0$ *for all* $i = 1, 2, \dots, N$. *If the interpolation points are not collinear then there exists a positive constant* μ *such that*

$$V_{f^\alpha;\delta}(I) \geq \mu \gamma^k \delta \qquad \textit{for} \quad \delta \in [0, \underline{a}^k] \, .$$

Proof. For $\delta \in [0, \underline{a}^k]$, using Lemma 2.3.2

$$\begin{aligned} V_{f^\alpha;\delta}(I) &\geq \sum_{i_1,i_2,\ldots,i_k=1}^{N} V_{f^\alpha;\delta}(L_{i_1i_2\ldots i_k}(I)) \\ &\geq \sum_{i_1,i_2,\ldots,i_k=1}^{N} V_{f^\alpha}(L_{i_1i_2\ldots i_k}(I))\delta, \qquad \text{using Lemma 2.4.2} \\ &\geq \sum_{i_1,i_2,\ldots,i_k=1}^{N} \alpha_{i_k}\alpha_{i_{k-1}}\cdots\alpha_{i_1}\mu\delta, \qquad \text{using (2.9)} \\ &= \mu\gamma^k\delta. \end{aligned}$$

□

The following theorem is useful to get a non-trivial lower bound for the box dimension of the graph of f^α.

Theorem 2.4.3. *Let D be the solution of $\sum_{i=1}^{N} |\alpha_i| a_i^{D-1} = 1$ and f^α be the α-fractal interpolation function corresponding to the IFS (1.8), (1.9) and (1.10). Also assume that f is concave and b is affine. Let $\gamma = \sum_{i=1}^{N} |\alpha_i| = \sum_{i=1}^{N} \alpha_i > 1$, where $\alpha_i \geq 0$ for all i and the interpolation points are not collinear. If $\beta = 1$, then there exist positive constants B_1 and δ_0 such that*

$$V_{f^\alpha;\delta}(I) \geq B_1\delta^{2-D} \ \ \textit{for} \ \ 0 < \delta \leq \delta_0 \,.$$

Proof. For $\gamma > 1$ and $\beta = 1$, according to Lemma 2.4.3, one can select $\delta_0 > 0$ such that

$$V_{f^\alpha;\delta}(I) \geq \frac{2C_1\delta}{\gamma - 1} \quad \text{for } 0 < \delta \leq \frac{\delta_0}{\underline{a}},$$

where C_1 is as given in Lemma 2.4.1. Then choose $0 < k_1 \leq \frac{C_1\delta_0^{D-1}}{\gamma-1}$ so that

$$V_{f^\alpha;\delta}(I) \geq \frac{C_1\delta}{\gamma - 1} + k_1\delta^{2-D} \quad \text{for } \delta_0 \leq \delta \leq \frac{\delta_0}{\underline{a}} \,.$$

Denote $\underline{\psi}(\delta) = \frac{C_1\delta}{\gamma-1} + k_1\delta^{2-D}$. Then $V_{f^\alpha;\delta}(I) \geq \underline{\psi}(\delta)$ and $\underline{\psi}(\delta) \geq k_1\delta^{2-D}$. If $\overline{a}\delta_0 \leq \delta \leq \delta_0$

then $\delta_0 \leq \frac{\delta}{a_i} \leq \frac{\delta_0}{\underline{a}}$. Therefore from left inequality of Lemma 2.4.1 with $\beta = 1$,

$$\begin{aligned}
V_{f^\alpha;\delta}(I) &\geq \sum_{i=1}^{N} |\alpha_i| a_i V_{f^\alpha;\frac{\delta}{a_i}}(I) - C_1\delta \\
&\geq \sum_{i=1}^{N} |\alpha_i| a_i \underline{\psi}(\frac{\delta}{a_i}) - C_1\delta \\
&= \sum_{i=1}^{N} |\alpha_i| a_i \frac{C_1\delta}{(\gamma-1)a_i} + \sum_{i=1}^{N} |\alpha_i| a_i \frac{k_1\delta^{2-D}}{a_i^{2-D}} - C_1\delta \\
&= \frac{C_1\gamma\delta}{\gamma-1} + k_1\delta^{2-D} - C_1\delta, \quad (\text{since } \sum_{i=1}^{N}|\alpha_i| = \gamma, \sum_{i=1}^{N}|\alpha_i| a_i^{D-1} = 1) \\
&= \frac{C_1\delta}{\gamma-1} + k_1\delta^{2-D} \\
&= \underline{\psi}(\delta)
\end{aligned}$$

for $\overline{a}\delta_0 \leq \delta \leq \delta_0$. Suppose $V_{f^\alpha;\delta}(I) \geq \underline{\psi}(\delta)$ holds for $\overline{a}^n\delta_0 \leq \delta \leq \delta_0$. Then again by the above arguments it holds for $\overline{a}^{n+1}\delta_0 \leq \delta \leq \delta_0$. As $\overline{a} < 1$,

$$V_{f^\alpha;\delta}(I) \geq \underline{\psi}(\delta) \geq B_1\delta^{2-D}$$

for $0 < \delta \leq \delta_0$. □

The following theorem gives a non-trivial lower bound for the box dimension of the graph of f^α.

Theorem 2.4.4. *Let $G = \{(x, f^\alpha(x)) : x \in I\}$ be the graph of the α-fractal interpolation function f^α corresponding to the IFS (1.8), (1.9) and (1.10). Also assume that f is concave and b is affine. Let the interpolation points be not collinear and $\gamma = \sum_{i=1}^{N}|\alpha_i| = \sum_{i=1}^{N}\alpha_i > 1$, where $\alpha_i \geq 0$ for all i. Let D be the solution of $\sum_{i=1}^{N}|\alpha_i| a_i^{D-1} = 1$. If $\beta = \min\{\beta_1, \beta_2\} = 1$, then the box dimension of G is*

$$\underline{\dim}_B G \geq D\,.$$

Proof. By Theorem 2.4.3, for every $0 < \delta_0 < 1$, there exists a positive constant B_1 such that

$$V_{f^\alpha;\delta}(I) \geq B_1\delta^{2-D} \quad \text{for } 0 < \delta \leq \delta_0\,.$$

Using it with (2.12), it follows that

$$\underline{\dim}_B G = \liminf_{\delta\to 0^+}\left(2 - \frac{\log V_{f^\alpha;\delta}(I)}{\log\delta}\right) \geq \liminf_{\delta\to 0^+}\left(2 - \frac{\log B_1\delta^{2-D}}{\log\delta}\right) = D\ .$$

$\square$

Remark 2.4.1. *For equally spaced data the solution of* $\sum_{i=1}^N |\alpha_i| a_i^{D-1} = 1$ *is* $D = 1 + \log_N \gamma$. *Then the results in Theorems 2.4.4 and 2.4.2 coincide with the results in Theorems 2.2.2 and 2.2.1 respectively for positive scale vector.*

Corollary 2.4.1. *Let* f *be a concave and Lipschitz function,* b *be an affine function with* $b(x_0) = f(x_0)$, $b(x_0) = f(x_0)$ *and* $b \neq f$. *Also assume that the interpolation points* $\{(x_i, y_i)\}_{i=0}^N$ *are not collinear and* $\gamma = \sum_{i=1}^N |\alpha_i| = \sum_{i=1}^N \alpha_i$, *where* $\alpha_i \geq 0$ *for all* i. *Then the graph* G *of the* α*-fractal interpolation function* f^α *corresponding to the IFS (1.8), (1.9) and (1.10) has box dimension*

$$\dim_B G = \begin{cases} D, & \text{if } \gamma > 1 \\ 1 & \text{otherwise}\ . \end{cases}$$

Proof. The result follows immediately from Theorems 2.4.4 and 2.4.2 with $\beta = 1$. $\square$

Note that the estimation in the above Corollary 2.4.1 is similar to the results in [41] for one dimensional case in which the oscillation of affine fractal function is used to get the estimation of the box dimension.

Example 2.2. *Let* $I = [0, 1]$ *and* $\Delta : 0 < 0.2 < 0.5 < 0.75 < 1$ *be the non-uniform partition of* I. *The Hölder functions* $f(x) = x^{1/2} - 2x^{1/3} + 2$ *and* $b(x) = 2 - x^{1/2}$ *as in Example 2.1. Then according to Theorem 2.4.2, for the scale vector* $\alpha = (0.3, 0, 0.2, 0)$, *the box dimension of the graph* G *of the corresponding* α*-fractal function* f^α, *given in Figure 2.8 is*

$$1 \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq 2 - \frac{1}{2} = 1.5\ .$$

Let $I = [0, 1]$ *and* $\Delta : 0 < 0.2 < 0.5 < 0.75 < 1$ *be the non-uniform partition of* I. *Let* $f(x) = \sin(x)$ *be a Lipschitz function which is concave in* I *and* $b(x) = x\sin(1)$

be an affine function. Let the scale vector $\alpha = (0, 0.7, 0.3, 0.8)$. *Then the solution of* $\sum_{i=1}^{N} |\alpha_i| a_i^{D-1}$ *is* $D = 1.42$. *By Corollary 2.4.1, the box dimension of the graph* G *of the corresponding* α*-fractal function* f^{α}, *given in Figure 2.9 is*

$$\dim_B G = 1.42 > 1 \ .$$

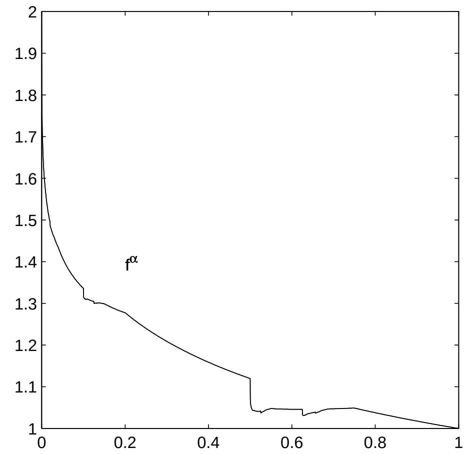

Figure 2.8: The graph of f^{α} corresponding to $f(x) = x^{1/2} - 2x^{1/3} + 2$ and $b(x) = 2 - x^{1/2}$.

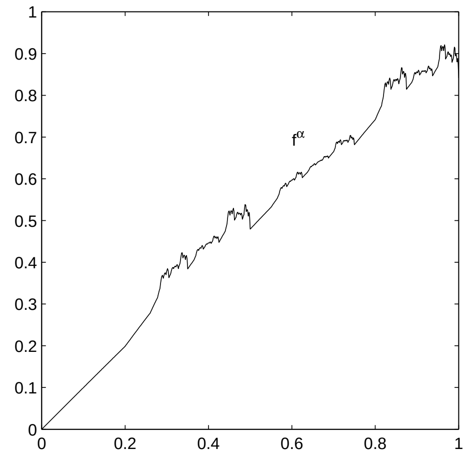

Figure 2.9: The graph of f^{α} corresponding to $f(x) = \sin(x)$ and $b(x) = x \sin(1)$.

Example 2.3. *The nowhere differentiable Weierstrass function is given by*

$$w(x) = \sum_{n=0}^{\infty} a^n \cos(2\pi b^n x), \text{ for } x \in \mathbb{R} \,,$$

where $0 < a < 1 < b$*, with* $ab > 1$*. It is Hölderian with exponent* $-\log_b a$*. The graph of the function* $w(x)$ *has fractal dimension* ≈ 1.37 *for* $a = 0.5$ *and* $b = 3$ *[48]. For any scale vector* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ *with* $\gamma = \sum_{i=1}^{N} |\alpha_i| \leq 1$ *and any Hölder function* b *with exponent at least* $-\log_b a$*, the bounds of box dimension of the graph* G *of the corresponding* α*-fractal function of* $w(x)$ *is* $1 \leq \dim_B G \leq 2 + \log_b a \approx 1.37$ *according to Theorem 2.4.2.*

Chapter 3

Box dimensions of α-fractal functions with variable scaling factors in subintervals

In Chapter 2, an estimate for the box dimension of the graph of α-fractal function f^α with constant scale vector for uniform as well as non-uniform nodes is established whenever the original function f and the base function b are Hölderian. In the present chapter, the dimension of the graph of f^α is estimated for an IFS with function scale vector instead of constant scale vector. For non-zero constant scale vector, the results here coincide with the results in Chapter 2. Additionally, in this chapter an estimate for the box dimension of the graph of f^α on subintervals or partitions is given. The motivation behind the work is explained through the following example.

Example 3.1. *Let* $I = [0, 1]$ *and* $\Delta : 0 < 0.25 < 0.5 < 0.75 < 1$ *be the partition of* I*. Let* $f(x) = \cos(x)$ *be a concave and Lipschitz function in* I *and* $b(x) = 1 - (1 - \cos(1))x$ *be an affine function such that* $b(0) = f(0)$*,* $b(1) = f(1)$*. Let* f^α *be the corresponding* α*-fractal function. Let* G^1 *and* G^2 *be the graphs (given in Figure 3.1 and Figure 3.2) of* f^α *with scale vector* $\alpha^1 = (0.7,\ 0.4,\ 0.5,\ 0.3)$ *and* $\alpha^2 = (0.5,\ 0.7,\ 0.8,\ 0)$ *respectively on* I*. Then by Corollary 2.4.1 of Chapter 2,* $\dim_B G^1 = 1.46$ *and* $\dim_B G^2 = 1.5$*.*

If we look at Figure 3.1 and Figure 3.2 carefully, it can be observed that the graphs behavior of f^α in subintervals are different for different scale vectors, due to non-affinity

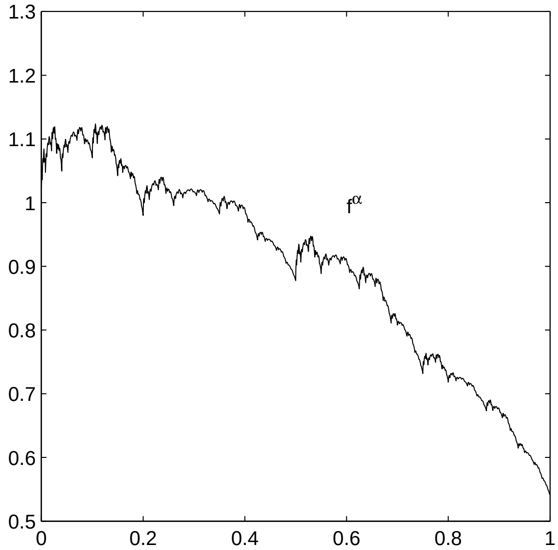

Figure 3.1: Graph of f^α for the scale vector $\alpha = (0.7,\ 0.4,\ 0.5,\ 0.3)$.

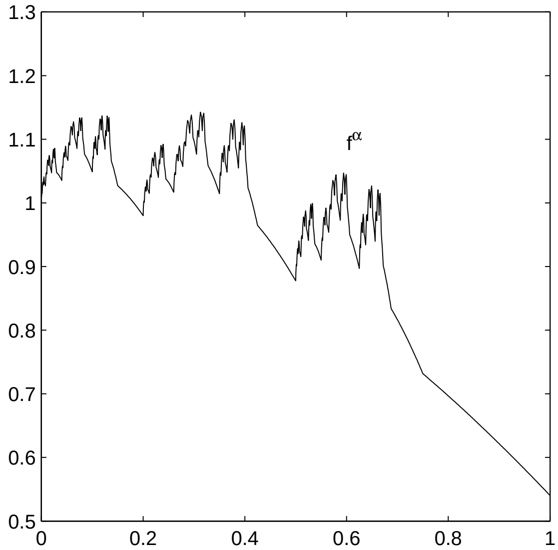

Figure 3.2: Graph of f^α for the scale vector $\alpha = (0.5,\ 0.7,\ 0.8,\ 0)$.

of f^α. Also from (1.22), for $\alpha_i = 0$

$$f^\alpha = f \text{ in } I_i .$$

So the following questions arise for investigation.

(a) What is the dimension of the graph of f^α on subintervals or partitions?

(b) If $\alpha_{i^*} = 0$ for some $i^* \in \{1, 2, \ldots, N\}$, what is the effect of it on the dimension of the graph of f^α in the intervals $I_i, i \neq i^*$?

(c) If $\alpha_{i^*} = 0$ for some $i^* \in \{1, 2, \ldots, N\}$, how to show theoretically that the behavior of the dimension of the graph of f^α is same as the behavior of the dimension of the graph of f on I_{i^*}?

In this chapter, an attempt is made to answer above questions for more general f^α by taking function scale vector $\alpha = \big(\alpha_1(x),\ \alpha_2(x),\ \ldots,\ \alpha_N(x)\big)$.

The chapter is organized as follows. In Section 3.1, the box dimension of the graph of f^α for uniform partition of $[0, 1]$, is estimated. In Section 3.2, the box dimension of the graph of f^α in subintervals, is estimated. The main results are explained with graphs for various combinations of scale vectors at the end.

3.1 Box dimensions of α-fractal functions

Throughout the sequel, we assume that f and b are Hölderian with exponents $\beta_1, \beta_2 \in (0, 1]$ and Hölder constants H_f, H_b respectively. That is,

$$|f(x) - f(y)| \leq H_f |x - y|^{\beta_1} \text{ for all } x, y \in \mathbb{R} ,$$

$$|b(x) - b(y)| \leq H_b |x - y|^{\beta_2} \text{ for all } x, y \in \mathbb{R} .$$

Set $\beta = \min\{\beta_1, \beta_2\}$. Let for $i = 1, 2, \ldots, N$, the function scaling factors $\alpha_i(x)$ are contraction with contractivity factors H_{α_i}. That is, for $i = 1, 2, \ldots, N$

$$|\alpha_i(x) - \alpha_i(y)| \leq H_{\alpha_i} |x - y| \text{ for all } x, y \in I .$$

Assume that $H_\alpha = \max\{H_{\alpha_i} : i = 1, 2, \dots, N\}$. For $I = [0, 1]$, let the interpolation points $P = \{(\frac{i}{N}, y_i) \in \mathbb{R}^2; i = 0, 1, \dots, N\}$. Let us denote $D = \{\frac{i}{N} \in I; i = 0, 1, \dots N\}$ and $L(D) = \cup_{i=1}^{N} L_i(D)$. Set $L^0(D) = D$ and $L^k(D) = L \circ L \circ \cdots \circ L(D)$ (k-times composition of L itself). Let us denote

$$\widetilde{\alpha}_i = \min_{x \in I} |\alpha_i(x)| \text{ and } \alpha_{\min} = \min_{1 \le i \le N} \widetilde{\alpha}_i ,$$

$$\overline{\alpha}_i = \max_{x \in I} |\alpha_i(x)| \text{ and } \alpha_{\max} = \max_{1 \le i \le N} \overline{\alpha}_i .$$

Let us assume that $0 \le \alpha_{\min} \le \alpha_{\max} < 1$.

The following lemma gives a bound for the maximum range of f^α on I.

Lemma 3.1.1. *Let f^α be the α-fractal interpolation function corresponding to the IFS (1.21) with interpolation points P. If $\alpha_{\max} < 1$, then*

$$R_{f^\alpha}(I) \le N \frac{\alpha_{\max} \Delta y + (\alpha_{\max} H_b + H_f)}{1 - \alpha_{\max}} ,$$

where $\Delta y = \max\{|y_i - y_0|; i = 0, 1, \dots, N\}$.

Proof. For $k = 1, 2, \dots$, let

$$\Gamma_k = \max\{|f^\alpha(x) - y_0|; x \in L^{k-1}(D)\} ,$$

$$\gamma_k = \max_i \{|f^\alpha(x) - y_{i-1}|; x \in L^{k-1}(D) \cap I_i\} .$$

Now for $x \in L^k(D) \cap I_i$,

$$\begin{aligned} f^\alpha(x) &= F_i(L_i^{-1}(x), f^\alpha(L_i^{-1}(x))) \\ &= f(x) + \alpha_i(L_i^{-1}(x))(f^\alpha - b)(L_i^{-1}(x)). \end{aligned}$$

Therefore

$$\begin{aligned} f^\alpha(x) - y_{i-1} &= \alpha_i(L_i^{-1}(x))\big[(f^\alpha(L_i^{-1}(x)) - y_0) + (y_0 - b(L_i^{-1}(x)))\big] + f(x) - y_{i-1} \\ &\le \alpha_{\max}\Gamma_k + (\alpha_{\max} H_b + H_f). \end{aligned}$$

Thus

$$\gamma_{k+1} \leq \alpha_{\max}\Gamma_k + (\alpha_{\max}H_b + H_f). \tag{3.1}$$

Observe that,

$$\begin{aligned}
\Gamma_k &= \max\{|f^\alpha(x) - y_0|; x \in L^{k-1}(D)\} \\
&\leq \max_i\{|f^\alpha(x) - y_{i-1}|; x \in L^{k-1}(D) \cap I_i\} + \max_i\{|y_{i-1} - y_0|\} \\
&\leq \Gamma_1 + \gamma_k.
\end{aligned}$$

Then by repeated applications of (3.1), we get

$$\begin{aligned}
\gamma_{k+1} &\leq \alpha_{\max}(\Gamma_1 + \gamma_k) + (\alpha_{\max}H_b + H_f) \\
&\leq \alpha_{\max}\Gamma_1 + \alpha_{\max}^2\Gamma_{k-1} + \alpha_{\max}(\alpha_{\max}H_b + H_f) + (\alpha_{\max}H_b + H_f) \\
&\leq \alpha_{\max}\Gamma_1 + \alpha_{\max}^2(\Gamma_1 + \gamma_{k-1}) + \alpha_{\max}(\alpha_{\max}H_b + H_f) + (\alpha_{\max}H_b + H_f) \\
&\vdots \\
&\leq \alpha_{\max}\Gamma_1(1 + \alpha_{\max} + \cdots + \alpha_{\max}^{k-1}) + (\alpha_{\max}H_b + H_f)(1 + \alpha_{\max} + \cdots + \alpha_{\max}^{k-1}).
\end{aligned}$$

Therefore,

$$\gamma_{k+1} \leq \frac{\alpha_{\max}\Gamma_1 + (\alpha_{\max}H_b + H_f)}{1 - \alpha_{\max}} .$$

The above inequality is true for every $k \in \mathbb{N}$ and the right hand side is independent of k. Let us define

$$C = \frac{\alpha_{\max}\Gamma_1 + (\alpha_{\max}H_b + H_f)}{1 - \alpha_{\max}} ,$$

$$\widetilde{D} = \cup_k L^k(D)$$

and $x \in I_i$, then there exists a sequence $x^m \in \widetilde{D} \cap I_i$ tending to x as $m \to \infty$. The continuity of f^α implies that

$$|f^\alpha(x^m) - y_{i-1}| \to |f^\alpha(x) - y_{i-1}| \quad \text{as } m \to \infty .$$

Since

$$|f^\alpha(x^m) - y_{i-1}| \leq C ,$$

we get

$$|f^\alpha(x) - y_{i-1}| \le C \ .$$

Therefore it follows that

$$\sup_{I_i} |f^\alpha(x) - y_{i-1}| \le \frac{\alpha_{\max}\Gamma_1 + (\alpha_{\max} H_b + H_f)}{1-\alpha_{\max}}. \tag{3.2}$$

Let s and t be arbitrary points in I. Suppose $s \in I_i$ and $t \in I_j$, where $i, j \in \{1, 2, \dots, N\}$ with $i \le j$. Then

$$\begin{aligned} |f^\alpha(s) - f^\alpha(t)| &\le |f^\alpha(s) - y_{i-1}| + |y_{i-1} - y_i| + \cdots + |y_{j-1} - f^\alpha(t)| \\ &\le N\frac{\alpha_{\max}\Delta y + (\alpha_{\max} H_b + H_f)}{1-\alpha_{\max}}, \quad \text{using (3.2)}. \end{aligned}$$

Therefore

$$R_{f^\alpha}(I) = \sup_{s,t\in I} |f^\alpha(s) - f^\alpha(t)| \le N\frac{\alpha_{\max}\Delta y + (\alpha_{\max} H_b + H_f)}{1-\alpha_{\max}} \ .$$

□

Following is another bound for R_{f^α} not depending on N.

Lemma 3.1.2. *Let f^α be the α-fractal interpolation function corresponding to the IFS (1.21) with interpolation points P. If $\alpha_{\max} < 1$, then*

$$R_{f^\alpha}(I) \le 2\|f^\alpha\|_\infty \le \frac{2}{1-\alpha_{\max}}\big(\|f\|_\infty + \alpha_{\max}\|b\|_\infty\big) \ .$$

Proof. By definition, we have

$$R_{f^\alpha}(I) = \sup_{s,t\in I} |f^\alpha(s) - f^\alpha(t)| \le 2\|f^\alpha\|_\infty \ .$$

Therefore using (1.22), it follows that

$$\|f^\alpha - f\|_\infty \le \frac{\alpha_{\max}}{1-\alpha_{\max}}\|f - b\|_\infty$$

and thus

$$\begin{aligned} \|f^\alpha\|_\infty &\le \|f\|_\infty + \frac{\alpha_{\max}}{1-\alpha_{\max}}\|f - b\|_\infty \\ &\le \frac{\|f\|_\infty}{1-\alpha_{\max}} + \frac{\alpha_{\max}\|b\|_\infty}{1-\alpha_{\max}}. \end{aligned}$$

Hence

$$R_{f^\alpha}(I) \leq 2\|f^\alpha\|_\infty \leq \frac{2}{1-\alpha_{\max}}\big(\|f\|_\infty + \alpha_{\max}\|b\|_\infty\big) \ .$$

□

Let M be the minimum of

$$N\frac{\alpha_{\max}\Delta y + (\alpha_{\max}H_b + H_f)}{1-\alpha_{\max}} \quad \text{and} \quad \frac{2}{1-\alpha_{\max}}\big(\|f\|_\infty + \alpha_{\max}\|b\|_\infty\big) \ .$$

Then

$$R_{f^\alpha}(I) \leq M \ .$$

Following notations are useful in the sequel. Let $L_{i_1 i_2 \ldots i_k}(x) = L_{i_k} \circ L_{i_{k-1}} \circ \cdots \circ L_{i_1}(x)$, $I_{i_1 i_2 \ldots i_k} = L_{i_1 i_2 \ldots i_k}(I) = L_{i_k} \circ L_{i_{k-1}} \circ \cdots \circ L_{i_1}(I)$ and $G_{i_1 i_2 \ldots i_k} = \big\{(x, f^\alpha(x)) | \ x \in I_{i_1 i_2 \ldots i_k}\big\}$, where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, N\}$. Set $i_0 = 0$, $I_{i_0} = I$ and $L_{i_0} = Id$. Let the maximum range $R_{F_i}(A)$ of F_i be

$$R_{F_i}(A) = \sup_{u,v \in A} |F_i(u) - F_i(v)|, \quad \text{where} \quad A \subseteq I \times \mathbb{R} \ .$$

Lemma 3.1.3. *Let $G = \{(x, f^\alpha(x)) : x \in I\}$ be the graph of the α-fractal function f^α corresponding to the IFS (1.21). Then*

$$R_{F_{i_k}}(G_{i_1 i_2 \ldots i_{k-1}}) \leq \overline{\alpha}_{i_k} R_{f^\alpha}(I_{i_1 i_2 \ldots i_{k-1}}) + \left(\frac{H_f}{N^{k\beta}} + \frac{\overline{\alpha}_{i_k} H_b}{N^{(k-1)\beta}} + \frac{H^*_\alpha}{N^{(k-1)\beta}}\right), \tag{3.3}$$

where $H^*_\alpha = \big(\|f^\alpha\|_\infty + \|b\|_\infty\big)H_\alpha$.

Proof. Recall that f and b are Hölderian on I with exponents $\beta_1, \beta_2 \in (0, 1]$ and Hölder constants H_f, H_b respectively. Using the definition of the transformations F_i given in

(1.21), for $x_1, x_2 \in I$, it follows that

$$\begin{aligned}
|F_{i_1}(x_1, f^\alpha(x_1)) - F_{i_1}(x_2, f^\alpha(x_2))| &= |\alpha_{i_1}(x_1) f^\alpha(x_1) + f(L_{i_1}(x_1)) - \alpha_{i_1}(x_1) b(x_1) \\
&- \alpha_{i_1}(x_2) f^\alpha(x_2) - f(L_{i_1}(x_2)) + \alpha_{i_1}(x_2) b(x_2)| \\
&= |\alpha_{i_1}(x_1) f^\alpha(x_1) - \alpha_{i_1}(x_1) f^\alpha(x_2) + \alpha_{i_1}(x_1) f^\alpha(x_2) \\
&- \alpha_{i_1}(x_2) f^\alpha(x_2) + f(L_{i_1}(x_1)) - f(L_{i_1}(x_2)) - \alpha_{i_1}(x_1) b(x_1) \\
&+ \alpha_{i_1}(x_1) b(x_2) - \alpha_{i_1}(x_1) b(x_2) + \alpha_{i_1}(x_2) b(x_2)| \\
&\leq \overline{\alpha}_{i_1} |f^\alpha(x_1) - f^\alpha(x_2)| + |f(L_{i_1}(x_1)) - f(L_{i_1}(x_2))| \\
&+ \overline{\alpha}_{i_1} |b(x_1) - b(x_2)| + \left(\|f^\alpha\|_\infty + \|b\|_\infty\right) |\alpha_{i_1}(x_1) - \alpha_{i_1}(x_2)| \\
&\leq \overline{\alpha}_{i_1} |f^\alpha(x_1) - f^\alpha(x_2)| + \frac{H_f}{N^{\beta_1}} |x_1 - x_2|^{\beta_1} \\
&+ \overline{\alpha}_{i_1} H_b |x_1 - x_2|^{\beta_2} + \left(\|f^\alpha\|_\infty + \|b\|_\infty\right) H_{\alpha_{i_1}} |x_1 - x_2| \\
&\leq \overline{\alpha}_{i_1} |f^\alpha(x_1) - f^\alpha(x_2)| \\
&+ \left(\frac{H_f}{N^\beta} + \overline{\alpha}_{i_1} H_b + H_\alpha^*\right) |x_1 - x_2|^\beta,
\end{aligned}$$

where $H_\alpha^* = \left(\|f^\alpha\|_\infty + \|b\|_\infty\right) H_\alpha$. Taking both side supremum over I and using $|x_1 - x_2|^\beta \leq 1$ for $x_1, x_2 \in I$,

$$R_{F_{i_1}}(G) \leq \overline{\alpha}_{i_1} R_{f^\alpha}(I) + \left(\frac{H_f}{N^\beta} + \overline{\alpha}_{i_1} H_b + H_\alpha^*\right).$$

For $x_1, x_2 \in I$, proceeding as above it follows that

$$\begin{aligned}
|F_{i_2}(L_{i_1}(x_1), f^\alpha(L_{i_1}(x_1)) - F_{i_2}(L_{i_1}(x_2), f^\alpha(L_{i_1}(x_2)))| &= |\alpha_{i_2}(L_{i_1}(x_1)) f^\alpha(L_{i_1}(x_1)) + f(L_{i_1 i_2}(x_1)) \\
&- \alpha_{i_2}(L_{i_1}(x_1)) b(L_{i_1}(x_1)) \\
&- \alpha_{i_2}(L_{i_1}(x_2)) f^\alpha(L_{i_1}(x_2)) - f(L_{i_1 i_2}(x_2)) \\
&+ \alpha_{i_2}(L_{i_1}(x_2)) b(L_{i_1}(x_2))| \\
&\leq \overline{\alpha}_{i_2} |f^\alpha(L_{i_1}(x_1)) - f^\alpha(L_{i_1}(x_2))| \\
&+ \left(\frac{H_f}{N^\beta} + \overline{\alpha}_{i_2} H_b + H_\alpha^*\right) \\
&\quad |L_{i_1}(x_1) - L_{i_1}(x_2)|^\beta.
\end{aligned}$$

Therefore taking both side supremum over I and using $|L_{i_1}(x_1) - L_{i_1}(x_2)|^\beta \leq \frac{1}{N^\beta}$,

$$R_{F_{i_2}}(G_{i_1}) \leq \overline{\alpha}_{i_2} R_{f^\alpha}(I_{i_1}) + \left(\frac{H_f}{N^{2\beta}} + \frac{\overline{\alpha}_{i_2} H_b}{N^\beta} + \frac{H_\alpha^*}{N^\beta} \right) .$$

Hence by succession

$$R_{F_{i_k}}(G_{i_1 i_2 \ldots i_{k-1}}) \leq \overline{\alpha}_{i_k} R_{f^\alpha}(I_{i_1 i_2 \ldots i_{k-1}}) + \left(\frac{H_f}{N^{k\beta}} + \frac{\overline{\alpha}_{i_k} H_b}{N^{(k-1)\beta}} + \frac{H_\alpha^*}{N^{(k-1)\beta}} \right) .$$

□

Remark 3.1.1. *If for $i = 1, 2, \ldots, N$, the functions $\alpha_i(x)$ are Hölderian with exponents s_i respectively, then by setting $\beta = \min\{\beta_1, \beta_2, s_1, \ldots, s_N\}$, the above Lemma 3.1.3 is true.*

The following lemma gives the bounds for the maximum range of the α-fractal function f^α.

Lemma 3.1.4. *Let f^α be the α-fractal function corresponding to the IFS (1.21).*

(a) If $\alpha_{\min} > \frac{1}{N^\beta}$, then

$$R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \leq M_1 \overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \ldots \overline{\alpha}_{i_1} ,$$

where M_1 is a positive constant.

(b) If one or several scaling factors may have zeroes in I and $\frac{1}{N^\beta} < \alpha_{\max} < 1$, then

$$R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \leq M_2 \alpha_{\max}^k ,$$

where M_2 is a positive constant.

(c) If $\alpha_{\max} \leq \frac{1}{N^\beta}$, then

$$R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \leq \frac{1}{N^{k\beta}} \left(R_{f^\alpha}(I) + k(H_f + H_b + H_\alpha^* N^\beta) \right) .$$

Proof. Let $G = \{(x, f^\alpha(x)) : x \in I\}$ be the graph of the α-fractal function f^α corresponding to the IFS (1.21). From (1.7) and (1.21), it follows that for all $x \in I_i$, $i = 1, 2, \ldots, N$,

$$f^\alpha(x) = F_i\big(L_i^{-1}(x), f^\alpha(L_i^{-1}(x))\big) .$$

Therefore, for $x_1, x_2 \in I_{i_1}$

$$\begin{aligned}
|f^\alpha(x_1) - f^\alpha(x_2)| &= \big|F_{i_1}\big(L_{i_1}^{-1}(x_1), f^\alpha(L_{i_1}^{-1}(x_1))\big) - F_{i_1}\big(L_{i_1}^{-1}(x_2), f^\alpha(L_{i_1}^{-1}(x_2))\big)\big| \\
&\leq \sup_{x,y \in I_{i_1}} \big|F_{i_1}\big(L_{i_1}^{-1}(x), f^\alpha(L_{i_1}^{-1}(x))\big) - F_{i_1}\big(L_{i_1}^{-1}(y), f^\alpha(L_{i_1}^{-1}(y))\big)\big| \\
&= \sup_{\widetilde{x},\widetilde{y} \in I} \big|F_{i_1}\big(\widetilde{x}, f^\alpha(\widetilde{x})\big) - F_{i_1}\big(\widetilde{y}, f^\alpha(\widetilde{y})\big)\big|, \ \widetilde{x} = L_{i_1}^{-1}(x), \widetilde{y} = L_{i_1}^{-1}(y) \\
&= R_{F_{i_1}}(G).
\end{aligned}$$

Since x_1, x_2 are arbitrary, for $i = 1, 2, \ldots, N$,

$$R_{f^\alpha}(I_{i_1}) \leq R_{F_{i_1}}(G) .$$

For $x_1, x_2 \in I_{i_1 i_2} \subseteq I_{i_2}$

$$\begin{aligned}
\big|f^\alpha(x_1) - f^\alpha(x_2)\big| &= \big|F_{i_2}\big(L_{i_2}^{-1}(x_1), f^\alpha(L_{i_2}^{-1}(x_1))\big) - F_{i_2}\big(L_{i_2}^{-1}(x_2), f^\alpha(L_{i_2}^{-1}(x_2))\big)\big| \\
&\leq \sup_{x,y \in I_{i_1 i_2}} \big|F_{i_2}\big(L_{i_2}^{-1}(x), f^\alpha(L_{i_2}^{-1}(x))\big) - F_{i_2}\big(L_{i_2}^{-1}(y), f^\alpha(L_{i_2}^{-1}(y))\big)\big| \\
&= \sup_{\widetilde{x},\widetilde{y} \in I_{i_1}} \big|F_{i_2}\big(\widetilde{x}, f^\alpha(\widetilde{x})\big) - F_{i_2}\big(\widetilde{y}, f^\alpha(\widetilde{y})\big)\big|, \text{ where } \widetilde{x} = L_{i_2}^{-1}(x), \widetilde{y} = L_{i_2}^{-1}(y) \\
&= R_{F_{i_2}}(G_{i_1}).
\end{aligned}$$

Therefore

$$R_{f^\alpha}(I_{i_1 i_2}) \leq R_{F_{i_2}}(G_{i_1}) .$$

Similarly proceeding as above, for $x_1, x_2 \in I_{i_1 i_2 i_3} \subseteq I_{i_3}$

$$\begin{aligned}
\big|f^\alpha(x_1) - f^\alpha(x_2)\big| &= \big|F_{i_3}\big(L_{i_3}^{-1}(x_1), f^\alpha(L_{i_3}^{-1}(x_1))\big) - F_{i_3}\big(L_{i_3}^{-1}(x_2), f^\alpha(L_{i_3}^{-1}(x_2))\big)\big| \\
&\leq \sup_{x,y \in I_{i_1 i_2 i_3}} \big|F_{i_3}\big(L_{i_3}^{-1}(x), f^\alpha(L_{i_3}^{-1}(x))\big) - F_{i_3}\big(L_{i_3}^{-1}(y), f^\alpha(L_{i_3}^{-1}(y))\big)\big| \\
&= \sup_{\widetilde{x},\widetilde{y} \in I_{i_1 i_2}} \big|F_{i_3}\big(\widetilde{x}, f^\alpha(\widetilde{x})\big) - F_{i_3}\big(\widetilde{y}, f^\alpha(\widetilde{y})\big)\big|, \text{ where } \widetilde{x} = L_{i_3}^{-1}(x), \widetilde{y} = L_{i_3}^{-1}(y) \\
&= R_{F_{i_3}}(G_{i_1 i_2}).
\end{aligned}$$

Therefore

$$R_{f^\alpha}(I_{i_1 i_2 i_3}) \leq R_{F_{i_3}}(G_{i_1 i_2})\ .$$

By succession, it follows that

$$R_{f^\alpha}(I_{i_1 i_2 \dots i_k}) \leq R_{F_{i_k}}(G_{i_1 i_2 \dots i_{k-1}}). \tag{3.4}$$

Substituting (3.3) in the above expression, it follows that

$$\begin{aligned}
R_{f^\alpha}(I_{i_1 i_2 \dots i_k}) &\leq \overline{\alpha}_{i_k} R_{f^\alpha}(I_{i_1 i_2 \dots i_{k-1}}) + \left(\frac{H_f}{N^{k\beta}} + \frac{\overline{\alpha}_{i_k} H_b}{N^{(k-1)\beta}} + \frac{H_\alpha^*}{N^{(k-1)\beta}} \right) \\
&\leq \overline{\alpha}_{i_k} R_{F_{i_{k-1}}}(G_{i_1 i_2 \dots i_{k-2}}) + \left(\frac{H_f}{N^{k\beta}} + \frac{\overline{\alpha}_{i_k} H_b}{N^{(k-1)\beta}} + \frac{H_\alpha^*}{N^{(k-1)\beta}} \right), \text{ using (3.4)} \\
&\leq \overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} R_{f^\alpha}(I_{i_1 i_2 \dots i_{k-2}}) + \overline{\alpha}_{i_k} \left(\frac{H_f}{N^{(k-1)\beta}} + \frac{\overline{\alpha}_{i_{k-1}} H_b}{N^{(k-2)\beta}} + \frac{H_\alpha^*}{N^{(k-2)\beta}} \right) \\
&+ \left(\frac{H_f}{N^{k\beta}} + \frac{\overline{\alpha}_{i_k} H_b}{N^{(k-1)\beta}} + \frac{H_\alpha^*}{N^{(k-1)\beta}} \right), \text{ using (3.3).}
\end{aligned}$$

Repeated applications of the above inequality, gives

$$\begin{aligned}
R_{f^\alpha}(I_{i_1 i_2 \dots i_k}) &\leq \overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \dots \overline{\alpha}_{i_1} R_{f^\alpha}(I) \\
&+ \left(\overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \dots \overline{\alpha}_{i_2} \frac{H_f}{N^\beta} + \dots + \overline{\alpha}_{i_k} \frac{H_f}{N^{(k-1)\beta}} + \frac{H_f}{N^{k\beta}} \right) \\
&+ \left(\overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \dots \overline{\alpha}_{i_1} H_b + \dots + \overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \frac{H_b}{N^{(k-2)\beta}} + \overline{\alpha}_{i_k} \frac{H_b}{N^{(k-1)\beta}} \right) \\
&+ \left(\overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \dots \overline{\alpha}_{i_2} H_\alpha^* + \dots + \overline{\alpha}_{i_k} \frac{H_\alpha^*}{N^{(k-2)\beta}} + \frac{H_\alpha^*}{N^{(k-1)\beta}} \right).
\end{aligned} \tag{3.5}$$

(a) If $\alpha_{\min} > \frac{1}{N^\beta}$, it easily follows that

$$\begin{aligned}
R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) &\leq \overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_1}R_{f^\alpha}(I) \\
&+ \frac{\overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_2}H_f}{N^\beta}\left(1+\cdots+\frac{H_f}{\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_2}N^{(k-2)\beta}}+\frac{H_f}{\overline{\alpha}_{i_k}\ldots\overline{\alpha}_{i_2}N^{(k-1)\beta}}\right) \\
&+ \overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_1}H_b\left(1+\cdots+\frac{H_b}{\overline{\alpha}_{i_{k-2}}\ldots\overline{\alpha}_{i_1}N^{(k-2)\beta}}+\frac{H_b}{\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_1}N^{(k-1)\beta}}\right) \\
&+ \overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_2}H^*_\alpha\left(1+\cdots+\frac{H^*_\alpha}{\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_2}N^{(k-2)\beta}}+\frac{H^*_\alpha}{\overline{\alpha}_{i_k}\ldots\overline{\alpha}_{i_2}N^{(k-1)\beta}}\right) \\
&\leq \overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_1}R_{f^\alpha}(I) \\
&+ \frac{\overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_2}H_f}{N^\beta}\left(1+\frac{1}{\alpha_{\min}N^\beta}+\ldots+\frac{1}{\alpha_{\min}^{k-1}N^{(k-1)\beta}}\right) \\
&+ \overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_1}H_b\left(1+\frac{1}{\alpha_{\min}N^\beta}+\ldots+\frac{1}{\alpha_{\min}^{k-1}N^{(k-1)\beta}}\right) \\
&+ \overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_2}H^*_\alpha\left(1+\frac{1}{\alpha_{\min}N^\beta}+\ldots+\frac{1}{\alpha_{\min}^{k-1}N^{(k-1)\beta}}\right),
\end{aligned}$$

since $\alpha_{\min} \leq \overline{\alpha}_{i_j}$ for all $j \in \{1, 2, \ldots, k\}$. That is, for $\alpha_{\min} > \frac{1}{N^\beta}$

$$\begin{aligned}
&R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \leq \overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_1} \\
&\left(R_{f^\alpha}(I)+\frac{H_f}{\alpha_{\min}N^\beta}\frac{1}{(1-\frac{1}{\alpha_{\min}N^\beta})}+H_b\frac{1}{(1-\frac{1}{\alpha_{\min}N^\beta})}+\frac{H^*_\alpha}{\alpha_{\min}}\frac{1}{(1-\frac{1}{\alpha_{\min}N^\beta})}\right).
\end{aligned}$$

Therefore

$$R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \leq M_1\overline{\alpha}_{i_k}\overline{\alpha}_{i_{k-1}}\ldots\overline{\alpha}_{i_1}\ ,$$

where

$$M_1 = \left(M+\frac{H_f}{\alpha_{\min}N^\beta}\frac{1}{(1-\frac{1}{\alpha_{\min}N^\beta})}+H_b\frac{1}{(1-\frac{1}{\alpha_{\min}N^\beta})}+\frac{H^*_\alpha}{\alpha_{\min}}\frac{1}{(1-\frac{1}{\alpha_{\min}N^\beta})}\right).$$

(b) If $\frac{1}{N^\beta} < \alpha_{\max} < 1$, (3.5) can also be written as

$$\begin{aligned}
R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) &\leq \alpha_{\max}^k R_{f^\alpha}(I) \\
&+ \frac{\alpha_{\max}^{k-1}H_f}{N^\beta}\left(1+\frac{1}{\alpha_{\max}N^\beta}+\ldots+\frac{1}{\alpha_{\max}^{k-1}N^{(k-1)\beta}}\right) \\
&+ \alpha_{\max}^k H_b\left(1+\frac{1}{\alpha_{\max}N^\beta}+\ldots+\frac{1}{\alpha_{\max}^{k-1}N^{(k-1)\beta}}\right) \\
&+ \alpha_{\max}^{k-1}H^*_\alpha\left(1+\frac{1}{\alpha_{\max}N^\beta}+\ldots+\frac{1}{\alpha_{\max}^{k-1}N^{(k-1)\beta}}\right).
\end{aligned}$$

For $\frac{1}{N^\beta} < \alpha_{\max} < 1$,

$$R_{f^\alpha}(I_{i_1 i_2 \dots i_k}) \le \alpha_{\max}^k \left(R_{f^\alpha}(I) + \frac{H_f}{\alpha_{\max} N^\beta} \frac{1}{(1 - \frac{1}{\alpha_{\max} N^\beta})} + H_b \frac{1}{(1 - \frac{1}{\alpha_{\max} N^\beta})} + \frac{H_\alpha^*}{\alpha_{\max}} \frac{1}{(1 - \frac{1}{\alpha_{\max} N^\beta})} \right) .$$

Therefore

$$R_{f^\alpha}(I_{i_1 i_2 \dots i_k}) \le M_2 \alpha_{\max}^k ,$$

where

$$M_2 = \left(M + \frac{H_f}{\alpha_{\max} N^\beta} \frac{1}{(1 - \frac{1}{\alpha_{\max} N^\beta})} + H_b \frac{1}{(1 - \frac{1}{\alpha_{\max} N^\beta})} + \frac{H_\alpha^*}{\alpha_{\max}} \frac{1}{(1 - \frac{1}{\alpha_{\max} N^\beta})} \right) .$$

(c) If $\alpha_{\max} \le \frac{1}{N^\beta}$, then from (3.5), it follows that

$$R_{f^\alpha}(I_{i_1 i_2 \dots i_k}) \le \frac{1}{N^{k\beta}} \left(R_{f^\alpha}(I) + k(H_f + H_b + H_\alpha^* N^\beta) \right) .$$

□

The following lemma can be read in [33].

Lemma 3.1.5. *Let $h \in \mathcal{C}[0,1]$. Suppose that $0 < \delta < 1$ and m is the least positive integer greater than or equal to $1/\delta$. If N_δ is the number of δ-mesh that intersects the graph of h, then*

$$\delta^{-1} \sum_{i=1}^{m} R_h[(i-1)\delta, i\delta] \le N_\delta \le 2m + \delta^{-1} \sum_{i=1}^{m} R_h[(i-1)\delta, i\delta] .$$

Let $\overline{\gamma} = \sum_{i=1}^{N} \overline{\alpha}_i$. Then the following may be noted.

- If $\alpha_{\min} > \frac{1}{N^\beta}$, then $\overline{\gamma} N^{\beta-1} > 1$.
- If $\overline{\gamma} N^{\beta-1} > 1$, then $\alpha_{\max} > \frac{1}{N^\beta}$.
- If $\alpha_{\max} \le \frac{1}{N^\beta}$, then $\overline{\gamma} N^{\beta-1} \le 1$.

Following theorem gives an estimate for the box dimension of the graph G of f^α.

Theorem 3.1.1. *Let f^α be the α-fractal function corresponding to the IFS (1.21).*

(a) If $\alpha_{\min} > \frac{1}{N^\beta}$*, then*

$$1 \le \dim_B(G) \le 1 + \log_N \overline{\gamma} \ .$$

(b) If $\overline{\gamma} N^{\beta-1} > 1$*, then*

$$1 \le \dim_B(G) \le 2 + \log_N \alpha_{\max} \ .$$

(c) If $\alpha_{\max} \le \frac{1}{N^\beta}$*, then*

$$1 \le \dim_B(G) \le 2 - \beta \ .$$

Proof.

(a) For $\alpha_{\min} > \frac{1}{N^\beta}$, by Lemma 3.1.4(a), it follows that

$$R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \le M_1 \overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \ldots \overline{\alpha}_{i_1} . \tag{3.6}$$

Note that

$$\sum_{i_1, i_2, \ldots, i_k = 1}^{N} \overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \ldots \overline{\alpha}_{i_1} = \overline{\gamma}^k \ .$$

Let $\epsilon_k = \frac{1}{N^k}$ and $\mathcal{N}(k)$ be the minimum number of squares of size ϵ_k, that cover G. Then by Lemma 3.1.5, summing (3.6) over the N^k intervals $I_{i_1 i_2 \ldots i_k}$ of lengths ϵ_k, gives that

$$\mathcal{N}(k) \le 2N^k + M_1 \overline{\gamma}^k N^k \ ,$$

since for each $j \in \{1, 2, \ldots, k\}$, α_{i_j} running over $\alpha_1, \alpha_2, \ldots, \alpha_N$. Now using Definition 1.5.1

$$\dim_B(G) \le \lim_{k \to \infty} \frac{\log N^k (2 + M_1 \overline{\gamma}^k)}{\log N^k} \le 1 + \log_N \overline{\gamma} \ .$$

Since G is the graph of a continuous function f^α, $\dim_B(G) \ge 1$. Therefore,

$$1 \le \dim_B(G) \le 1 + \log_N \overline{\gamma} \ .$$

(b) Given that $\overline{\gamma} N^{\beta-1} > 1$. Then using Lemma 3.1.4(b), it follows that

$$R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \le M_2 \alpha_{\max}^k . \tag{3.7}$$

Therefore as above,

$$\mathcal{N}(k) \leq 2N^k + M_2\alpha_{\max}^k N^{2k} \ .$$

Hence

$$\dim_B(G) \leq \lim_{k\to\infty} \frac{\log N^k(2 + M_2\alpha_{\max}^k N^k)}{\log N^k} \leq 2 + \log_N \alpha_{\max} \ .$$

(c) Given that $\alpha_{\max} \leq \frac{1}{N^\beta}$. Using Lemma 3.1.4 (c),

$$\begin{aligned}\mathcal{N}(k) &\leq 2N^k + N^{2k}\left(\frac{1}{N^{k\beta}}\left(R_{f^\alpha}(I) + k(H_f + H_b + H_\alpha^* N^\beta)\right)\right) \\ &\leq N^k\left[2 + N^{(1-\beta)k}\left(R_{f^\alpha}(I) + k(H_f + H_b + H_\alpha^* N^\beta)\right)\right].\end{aligned}$$

Therefore using Definition 1.5.1,

$$\dim_B(G) \leq \lim_{k\to\infty} \frac{\log N^k\left[2 + N^{(1-\beta)k}\left(R_{f^\alpha}(I) + k(H_f + H_b + H_\alpha^* N^\beta)\right)\right]}{\log N^k} \leq 2 - \beta \ .$$

□

The following theorem gives a non-trivial lower bound for the box dimension of the graph of f^α.

Theorem 3.1.2. *Let G be the graph of the α-fractal interpolation function f^α corresponding to the IFS (1.21) with constant non-negative scale factors $\alpha_i(x) = \alpha_i$, for $i = 1, 2, \ldots, N$. Also assume that f is concave and b is affine. Assume that the interpolation points are not collinear. Let $\gamma = \sum_{i=1}^N \alpha_i > 1$. Then*

$$\dim_B(G) \geq 1 + \log_N \gamma \ .$$

Proof. From (2.9), we have

$$R_{f^\alpha}(I_{i_1 i_2 \ldots i_k}) \geq M^* \alpha_{i_k}\alpha_{i_{k-1}} \cdots \alpha_{i_1}, \tag{3.8}$$

where M^* is some positive constant. Let $\epsilon_k = \frac{1}{N^k}$ and $\mathcal{N}(k)$ be the minimum number of squares of size ϵ_k, that cover G. Then by Lemma 3.1.5, summing (3.8) over the N^k intervals $I_{i_1 i_2 \ldots i_k}$ of lengths ϵ_k, gives that

$$\mathcal{N}(k) \geq M^* \gamma^k N^k \ ,$$

since for each $j \in \{1, 2, \ldots, k\}$, α_{i_j} running over $\alpha_1, \alpha_2, \ldots, \alpha_N$. Now using Definition 1.5.1, we get

$$\dim_B(G) \geq \lim_{k \to \infty} \frac{\log M^* \gamma^k N^k}{\log N^k} = 1 + \log_N \gamma \; .$$

□

Corollary 3.1.1. *Let G be the graph of the α-fractal interpolation function f^α corresponding to the IFS (1.21) with constant non-negative scale factors $\alpha_i(x) = \alpha_i$, for $i = 1, 2, \ldots, N$. Also assume that f is concave and Lipschitz, b is affine. Let $\gamma = \sum_{i=1}^N \alpha_i$. If the interpolation points are not collinear, then*

$$\dim_B G = \begin{cases} 1 + \log_N \gamma, & \text{if } \alpha_{\min} > \frac{1}{N} \\ 1, & \text{if } \alpha_{\max} \leq \frac{1}{N}. \end{cases}$$

Proof. Note that for constant non-negative scale factors, $\overline{\gamma} = \gamma$. Then the result follows easily from Theorems 3.1.1 and 3.1.2 with $\beta = 1$. □

3.2 Box dimensions of α-fractal functions in subintervals

In this section, we are interested to find the box dimension of the graph of f^α in subintervals. As the α-fractal function f^α is non-affine in nature, the behavior of the graph of f^α is not uniform in the interval. In the sequel, assume that $i_1, i_2, \ldots, i_{k-1} \in \{1, 2, \ldots, N\}$. Let $i_k = i^* \in \{1, 2, \ldots, N\}$ be fixed. Then note that

$$\cup \{L_{i_1,\ldots,i_{k-1},i^*}(I) | i_1, i_2, \ldots, i_{k-1} \in \{1, 2, \ldots, N\}\} = I_{i^*} \; .$$

Lemma 3.2.1. *Let f^α be the α-fractal function corresponding to the IFS (1.21). Assume that for fixed $i^* \in \{1, 2, \ldots, N\}$, $i_k = i^*$ and $i_1, i_2, \ldots, i_{k-1} \in \{1, 2, \ldots, N\}$.*

(a) If $\alpha_{\min} > \frac{1}{N^\beta}$, then

$$R_{f^\alpha}(I_{i_1,\ldots,i_{k-1},i^*}) \leq M_1 \overline{\alpha}_{i^*} \overline{\alpha}_{i_{k-1}} \ldots \overline{\alpha}_{i_1} \; ,$$

where M_1 is a positive constant given in Lemma 3.1.4.

(b) *If* $\frac{1}{N^\beta} < \alpha_{\max} < 1$*, then*

$$R_{f^\alpha}(I_{i_1,\ldots,i_{k-1},i^*}) \leq M_2 \alpha_{\max}^k ,$$

where M_2 *is a positive constant given in Lemma 3.1.4.*

(c) *If* $\alpha_{\max} \leq \frac{1}{N^\beta}$*, then*

$$R_{f^\alpha}(I_{i_1,\ldots,i_{k-1},i^*}) \leq \frac{1}{N^{k\beta}} \left(R_{f^\alpha}(I) + k(H_f + H_b + H_\alpha^* N^\beta) \right) .$$

(d) *If* $\overline{\alpha}_{i^*} = 0$*, then*

$$R_{f^\alpha}(I_{i_1,\ldots,i_{k-1},i^*}) \leq \frac{1}{N^{k\beta}} (H_f + H_\alpha^* N^\beta) .$$

Proof. (a), (b) and (c) follows from (3.5). By taking $\overline{\alpha}_{i_k} = 0$ and using it in (3.5), (d) follows. □

Lemma 3.2.2. *Let for fixed* $i^* \in \{1, 2, \ldots, N\}$, $i_k = i^*$ *and* $i_1, i_2, \ldots, i_{k-1} \in \{1, 2, \ldots, N\}$. *Recall that* $\overline{\gamma} = \sum_{i=1}^N \overline{\alpha}_i$*. Then*

$$\left\{ \sum \overline{\alpha}_{i^*} \overline{\alpha}_{i_{k-1}} \ldots \overline{\alpha}_{i_1} \mid i_1, i_2, \ldots, i_{k-1} \in \{1, 2, \ldots, N\} \right\} = (\overline{\gamma})^{(k-1)} \overline{\alpha}_{i^*} .$$

Proof. Since i^* is fixed and $i_1, i_2, \ldots, i_{k-1} \in \{1, 2, \ldots, N\}$,

$$\left\{ \sum \overline{\alpha}_{i_k} \overline{\alpha}_{i_{k-1}} \ldots \overline{\alpha}_{i_1} \mid i_1, i_2, \ldots, i_{k-1} \in \{1, 2, \ldots, N\} \right\}$$

contains N^{k-1} terms. One can take $\overline{\alpha}_{i^*}$ outside from the summation and as for each $j \in \{1, 2, \ldots, k-1\}$, α_{i_j} ranges through $\alpha_1, \alpha_2, \ldots, \alpha_N$, the result follows. □

The following theorem gives an estimate for the box dimension of the graph of α-fractal function f^α in subintervals.

Theorem 3.2.1. *Let* f^α *be the* α*-fractal interpolation function corresponding to the IFS (1.21). Let for fixed* $i^* \in \{1, 2, \ldots, N\}$, $G_{i^*} = graph(f^\alpha|_{I_{i^*}})$ *be the graph of the function* f^α *on the subinterval* I_{i^*}*. Assume that the interpolation points* P *are not collinear.*

(a) If $\alpha_{\min} > \frac{1}{N^\beta}$*, then*

$$1 \le \dim_B(G_{i^*}) \le 1 + \log_N \overline{\gamma}\ .$$

(b) If $\overline{\gamma} N^{\beta-1} > 1$*, then*

$$1 \le \dim_B(G_{i^*}) \le 2 + \log_N \alpha_{\max}\ .$$

(c) If $\alpha_{\max} \le \frac{1}{N^\beta}$*, then*

$$1 \le \dim_B(G_{i^*}) \le 2 - \beta\ .$$

(d) If $\overline{\alpha}_{i^*} = 0$*, then* $1 \le \dim_B G_{i^*} \le 2 - \beta$*.*

Proof.

(a) Using Lemma 3.2.1 with the assumption $\alpha_{\min} > \frac{1}{N^\beta}$

$$R_{f^\alpha}(I_{i_1\dots i_{k-1}i^*}) \le M_1 \overline{\alpha}_{i^*}\overline{\alpha}_{i_{k-1}}\dots\overline{\alpha}_{i_1}. \tag{3.9}$$

Let $\mathcal{N}_{i^*}(k)$ be the minimum number of square boxes of sides $\frac{1}{N^k}$ that cover G_{i^*}. Then summing (3.9) over the N^{k-1} intervals $I_{i_1\dots i_{k-1}i^*}$ of lengths $\frac{1}{N^k}$ and using Lemma 3.2.2, gives that

$$\mathcal{N}_{i^*}(k) \le N^{k-1}\left(2 + M_1 N \overline{\alpha}_{i^*}(\overline{\gamma})^{(k-1)}\right)\ .$$

Therefore,

$$\dim_B G_{i^*} \le \lim_{k\to\infty} \frac{\log\left(N^{k-1}(2 + M_1 N\overline{\alpha}_{i^*}(\overline{\gamma})^{(k-1)})\right)}{\log N^k} \le 1 + \log_N \overline{\gamma}\ .$$

(b) Given that $\overline{\gamma}N^{\beta-1} > 1$. Then as in the above case using Lemma 3.2.1(b),

$$R_{f^\alpha}(I_{i_1\dots i_{k-1}i^*}) \le M_2 \alpha_{\max}^k$$

and

$$\mathcal{N}_{i^*}(k) \le 2N^{k-1} + M_2\alpha_{\max}^k N^{2k-1}\ .$$

Hence,

$$\dim_B(G_{i^*}) \le \lim_{k\to\infty} \frac{\log N^{k-1}(2 + M_2 \alpha_{\max}^k N^k)}{\log N^k} = 2 + \log_N \alpha_{\max}\ .$$

(c) Given that $\alpha_{\max} \leq \frac{1}{N^{\beta}}$. Using Lemma 3.2.1(c),

$$\mathcal{N}_{i^*}(k) \leq N^{k-1}\left[2 + N^{(1-\beta)k}\left(R_{f^\alpha}(I) + k(H_f + H_b + H_\alpha^* N^\beta)\right)\right] .$$

Therefore,

$$\dim_B G_{i^*} \leq \lim_{k\to\infty} \frac{\log N^{k-1}\left[2 + N^{(1-\beta)k}\left(R_{f^\alpha}(I) + k(H_f + H_b + H_\alpha^* N^\beta)\right)\right]}{\log N^k} \leq 2 - \beta .$$

(d) For $\overline{\alpha}_{i^*} = 0$, using Lemma 3.2.1(d)

$$\mathcal{N}_{i^*}(k) \leq N^{k-1}\left(2 + (H_f + H_\alpha^* N^\beta) N^{k(1-\beta)}\right) .$$

Therefore,

$$\dim_B G_{i^*} \leq \lim_{k\to\infty} \frac{\log N^{k-1}\left(2 + (H_f + H_\alpha^* N^\beta) N^{k(1-\beta)}\right)}{\log N^k} \leq 2 - \beta .$$

□

The following theorem gives a non-trivial lower bound for the box dimensions of the graphs of f^α in subintervals I_{i^*}, $i^* \in \{1, 2, \ldots, N\}$.

Theorem 3.2.2. *Let G_{i^*} be the graph of f^α on I_{i^*} corresponding to the IFS (1.21) with constant non-negative scale factors $\alpha_i(x) = \alpha_i$, for $i = 1, 2, \ldots, N$. Also assume that f is concave and b is affine. Assume that the interpolation points are not collinear. Let $\gamma = \sum_{i=1}^N \alpha_i > 1$ with $\alpha_{i^*} \neq 0$. Then*

$$\dim_B(G_{i^*}) \geq 1 + \log_N \gamma .$$

Proof. From (2.9), it can be observed that

$$R_{f^\alpha}(I_{i_1\ldots i_{k-1}i^*}) \geq M^* \alpha_{i^*} \alpha_{i_{k-1}} \cdots \alpha_{i_1}. \tag{3.10}$$

Let $\epsilon_k = \frac{1}{N^k}$ and $\mathcal{N}_{i^*}(k)$ be the minimum number of squares of size ϵ_k, that covers G_{i^*}. Then by Lemma 3.1.5, summing (3.10), over the N^{k-1} intervals $I_{i_1\ldots i_{k-1}i^*}$ of lengths ϵ_k and using Lemma 3.2.2, gives that

$$\mathcal{N}_{i^*}(k) \geq M^* \alpha_{i^*} \gamma^{k-1} N^k .$$

Now using Definition 1.5.1

$$\dim_B(G_{i^*}) \geq \lim_{k\to\infty} \frac{\log M^* \alpha_{i^*} \gamma^{k-1} N^k}{\log N^k} = 1 + \log_N \gamma\ .$$

□

Corollary 3.2.1. *Let f^α be the α-fractal interpolation function corresponding to the IFS (1.21) with constant non-negative scale factors $\alpha_i(x) = \alpha_i$, for $i = 1, 2, \ldots, N$. Also assume that f is concave and Lipschitz, b is affine. Let for fixed $i^* \in \{1, 2, \ldots, N\}$, $G_{i^*} = graph(f^\alpha|_{I_{i^*}})$ be the graph of the function f^α on the subinterval I_{i^*}. Let $\gamma = \sum_{i=1}^{N} \alpha_i$. If the interpolation points are not collinear, then*

$$\dim_B G_{i^*} = \begin{cases} 1 + \log_N \gamma, & \text{if } \alpha_{\min} > \frac{1}{N} \\ 1, & \text{if } \alpha_{\max} \leq \frac{1}{N}. \end{cases}$$

Proof. Proof is immediate from Theorems 3.2.1 and 3.2.2. □

The following example gives an estimate for the box dimensions of the graphs of a α-fractal function f^α in the whole interval I as well as in subintervals I_{i^*}, $i^* \in \{1, 2, 3, 4\}$, though it is highlighted for the case $i^* = 3$.

Example 3.2. *Let $I = [0, 1]$ and $\Delta : 0 < 0.25 < 0.5 < 0.75 < 1$ be the partition of I. Consider the Lipschitz function $f(x) = \cos(x)$ which is concave in I and affine function $b(x) = 1 - (1 - \cos(1))x$ such that $b(0) = f(0)$, $b(1) = f(1)$. Let f^α be the corresponding α-fractal function. For $i^* \in \{1, 2, 3, 4\}$, let G and G_{i^*} be the graph of f^α in I and I_{i^*} respectively. Following are the graphs of f^α for different scale vectors.*

(a) Let $\alpha = (\alpha_1(x),\ \alpha_2(x),\ \alpha_3(x),\ \alpha_4(x))$ where $\alpha_1(x) = 0.4\sin(x) + 0.5$, $\alpha_2(x) = 0.3\cos(x) + 0.2$, $\alpha_3(x) = 0.5e^{-2x} + 0.3$, $\alpha_4(x) = 0.2e^x \sin(x) + 0.3$. Then $\widetilde{\alpha}_1 = 0.5$, $\widetilde{\alpha}_2 = 0.362$, $\widetilde{\alpha}_3 = 0.367667$, $\widetilde{\alpha}_4 = 0.3$ and $\overline{\alpha}_1 = 0.83658$, $\overline{\alpha}_2 = 0.5$, $\overline{\alpha}_3 = 0.8$, $\overline{\alpha}_4 = 0.75747$. Also $\alpha_{\min} = 0.3 > \frac{1}{N} = 0.25$ and $\overline{\gamma} = 2.89$. Then Figure 3.3 is the graph of the corresponding f^α with above function scaling factors. Using Theorems 3.1.1(a) and 3.2.1(a), it follows that $1 \leq \dim_B G, \dim_B(G_3) \leq 1.77$.

(b) *Let* $\alpha = (\alpha_1(x),\ \alpha_2(x),\ \alpha_3(x),\ \alpha_4(x))$ *where* $\alpha_1(x) = 0.4\sin(x)$, $\alpha_2(x) = 0.3\cos(x)+0.4$, $\alpha_3(x) = 0.5e^{-2x} + 0.3$, $\alpha_4(x) = 0.2e^x\sin(x) + 0.5$. *Then* $\widetilde{\alpha}_1 = 0$, $\widetilde{\alpha}_2 = 0.562$, $\widetilde{\alpha}_3 = 0.367667$, $\widetilde{\alpha}_4 = 0.3$ *and* $\overline{\alpha}_1 = 0.33659$, $\overline{\alpha}_2 = 0.7$, $\overline{\alpha}_3 = 0.8$, $\overline{\alpha}_4 = 0.75747$. *Also* $\alpha_{\min} = 0$, $\alpha_{\max} = 0.75747$ *and* $\overline{\gamma} = 2.79 > 1$. *Then Figure 3.4 is the graph of the corresponding* f^α *with above function scaling factors. Using Theorems 3.1.1(b) and 3.2.1(b), it follows that* $1 \leq \dim_B G, \dim_B(G_3) \leq 1.8$.

(c) *Let* $\alpha = (\alpha_1(x),\ \alpha_2(x),\ \alpha_3(x),\ \alpha_4(x))$ *where* $\alpha_1(x) = 0.2\sin(x)$, $\alpha_2(x) = 0.25\cos(x)$, $\alpha_3(x) = 0.2e^{-2x}$, $\alpha_4(x) = 0.1e^x\sin(x)$. *Then* $\widetilde{\alpha}_1 = 0$, $\widetilde{\alpha}_2 = 0.162$, $\widetilde{\alpha}_3 = 0.141$, $\widetilde{\alpha}_4 = 0$ *and* $\overline{\alpha}_1 = 0.168294$, $\overline{\alpha}_2 = 0.25$, $\overline{\alpha}_3 = 0.2$, $\overline{\alpha}_4 = 0.228735$. *Also* $\overline{\gamma} = 0.85$, $\alpha_{\min} = 0$ *and* $\alpha_{\max} = 0.25 \leq \frac{1}{N} = 0.25$. *Then Figure 3.5 is the graph of the corresponding* f^α *with above function scaling factors. Using Theorems 3.1.1(c) and 3.2.1(c), it follows that* $\dim_B G = 1$ *and* $\dim_B(G_3) = 1$.

(d) *Let* $\alpha = (\alpha_1(x),\ \alpha_2(x),\ \alpha_3(x),\ \alpha_4(x))$ *where* $\alpha_1(x) = 0.4$, $\alpha_2(x) = 0.5$, $\alpha_3(x) = 0$, $\alpha_4(x) = 0.8$. *Then* $\alpha_{\min} = 0$, $\alpha_{\max} = 0.8$, $\overline{\alpha}_3 = 0$ *and* $\overline{\gamma} = \gamma = 1.7 > 1$. *Then Figure 3.6 is the graph of the corresponding* f^α *with above constant scaling factors. Using Theorems 3.1.2, 3.1.1(b) and 3.2.1(d), it follows that* $1.38 \leq \dim_B G \leq 1.84$, $\dim_B(G_3) = 1$.

(e) *Let* $\alpha = (\alpha_1(x),\ \alpha_2(x),\ \alpha_3(x),\ \alpha_4(x))$ *where* $\alpha_1(x) = 0$, $\alpha_2(x) = 0.7$, $\alpha_3(x) = 0.5$, $\alpha_4(x) = 0.8$. *Then* $\alpha_{\min} = 0$, $\alpha_{\max} = 0.8$, $\overline{\alpha}_3 \neq 0$ *and* $\overline{\gamma} = \gamma = 2 > 1$. *Then Figure 3.7 is the graph of the corresponding* f^α *with above constant scaling factors. Using Theorems 3.1.2, 3.1.1(b), 3.2.2 and 3.2.1(b), it follows that* $1.5 \leq \dim_B G, \dim_B(G_3) \leq 1.84$.

(f) *Let* $\alpha = (\alpha_1(x),\ \alpha_2(x),\ \alpha_3(x),\ \alpha_4(x))$ *where* $\alpha_1(x) = 0.3$, $\alpha_2(x) = 0.5$, $\alpha_3(x) = 0.4$, $\alpha_4(x) = 0.7$. *Then* $\alpha_{\min} = 0.3 > \frac{1}{N} = 0.25$ *and* $\overline{\gamma} = \gamma = 1.9$. *Then Figure 3.8 is the graph of the corresponding* f^α *with above constant scaling factors. Using Corollaries 3.1.1 and 3.2.1, it follows that* $\dim_B G = 1.46$, $\dim_B(G_3) = 1.46$.

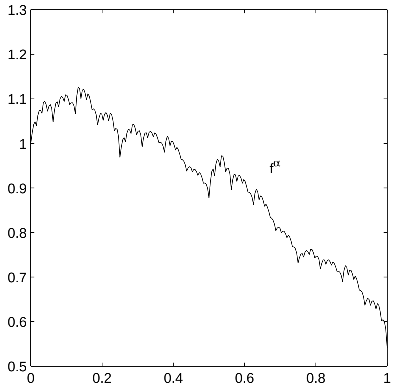

Figure 3.3: $1 \leq \dim_B G$, $\dim_B(G_3) \leq 1.77$.

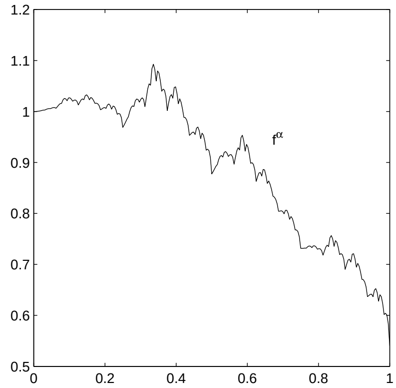

Figure 3.4: $1 \leq \dim_B G$, $\dim_B(G_3) \leq 1.8$.

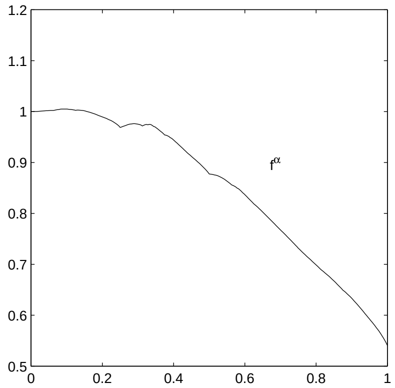

Figure 3.5: $\dim_B G = 1$, $\dim_B(G_3) = 1$.

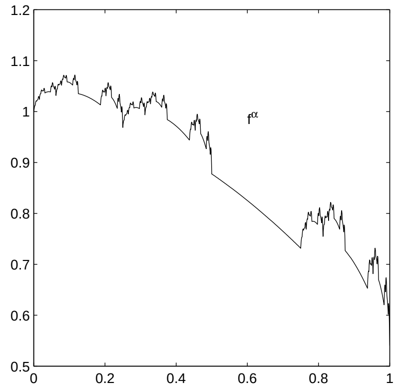

Figure 3.6: $\dim_B(G) \geq 1.38$, $\dim_B(G_3) = 1$

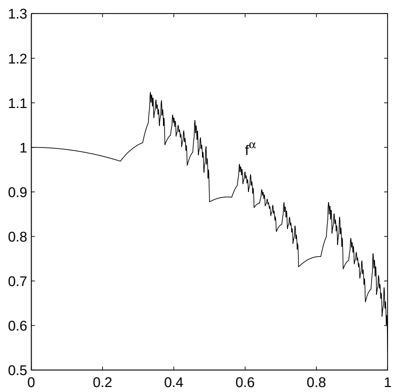

Figure 3.7: $1.5 \leq \dim_B G$, $\dim_B(G_3) \leq 1.84$.

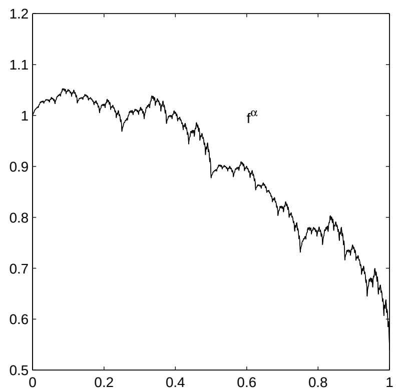

Figure 3.8: $\dim_B G = 1.46$, $\dim_B(G_3) = 1.46$.

Chapter 4

Fractal Jacobi systems and convergence of Fourier-Jacobi expansions of fractal interpolation functions

In the real world applications there are many objects namely financial series, turbulence, sampled signals, etc., which are highly irregular in nature and are not well approximated by using smooth functions of classical approximation theory. An important advantage of fractal functions which are continuous but not necessarily differentiable is that they form bases for many functional spaces.

To define non-smooth fractal versions of classical approximants, a general procedure is proposed by Navascués [62,63]. Using IFS theory, for 2π-periodic continuous functions on the unit circle, a generalized family of continuous functions is defined in [66]. The new functions provide Hilbert bases for the square integrable functions on the circle. In [68], for certain data set, a good approximation of the represented variable with truncated Chebyshev fractal object is established, whenever the original function is Lipschitz. In [75], by fractal method, Legendre expansion of a function is computed. Pointwise, uniform and mean-square convergence of the sums are established for suitable choice of the scale vector.

Consider the class of orthonormal polynomials namely Jacobi polynomials [53]

$$P_k^{(r,s)}(x) = \gamma_k(r,s)x^k + \text{lower degree terms} ,$$

where r and s are real parameters > -1 and $\gamma_k(r,s) > 0$. It is well known that the Jacobi polynomials hold great importance in a hierarchy of orthogonal polynomial classes. Legendre system, Chebyshev system, Gegenbauer system, etc., become special cases of Jacobi polynomial system for certain choices of the real parameters r and s. Using the approximation technique by orthogonal polynomials namely Gegenbauer (ultraspherical), Chebyshev (first and second kind) and Legendre (spherical) many design problems can be solved [**?**, 46]. The Jacobi system forms a Schauder basis/complete orthonormal basis for $\mathcal{L}^2_\rho(-1,1)$ [43]. The domain of uniform convergence of the Fourier-Jacobi expansion of a function depends on the parameters r and s. Results on the uniform convergence of Fourier-Jacobi expansion can be found in [**?**,53,78] and references therein.

In the present chapter, the orthonormal system of Jacobi polynomials in $\mathcal{L}^2_\rho(-1,1)$ is considered. An attempt is made to approximate continuous function with the Fourier-Jacobi expansion of affine FIF. To approximate continuously p-differentiable function, Fourier-Jacobi expansion of non-affine p-differentiable FIF is used. The domain of convergence of the Fourier-Jacobi expansion depends on the parameters r and s in $P_k^{(r,s)}(x)$. The present chapter is organized as follows. Some existing results on fractal Hilbert basis and convergence using fractal interpolation function can be found in Section 4.1. In Section 4.2, fractal Jacobi system is defined and using the completeness of Jacobi system a fractal version of classical result namely "Weierstrass type theorem" is proved. For a weighted square integrable function, an expansion in terms of fractal Jacobi polynomials is derived. Also it is proved that fractal Jacobi system forms a Schauder basis for the space of weighted square integrable functions. The convergence analysis of Fourier-Jacobi expansion for an affine FIF as well as non-affine smooth FIF corresponding to certain data set and suitable scale vector are established in Section 4.3.

4.1 Preliminaries

In this section, we recall few results which motivated us for this work. In [62, 63], Navascués defined the α-fractal interpolation function f^α as a perturbation of a continuous function f on a compact interval I of $\mathbb{R}$ and studied extensively. A classical complete system of functions on a Hilbert space is taken, fractalized it by means of a linear and bounded operator and provided an alternative family of bases of fractal functions for the Hilbert space. In [63], Navascués considered τ_m, the set of trigonometric polynomials of degree at most m

$$\tau_m = \operatorname{span}\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \cdots, \sin(mx), \cos(mx)\} .$$

The corresponding fractal family

$$\tau_m^\alpha = \operatorname{span}\{1, \sin^\alpha(x), \cos^\alpha(x), \sin^\alpha(2x), \cos^\alpha(2x), \cdots, \sin^\alpha(mx), \cos^\alpha(mx)\} ,$$

where $\sin^\alpha(jx) = \mathcal{F}^\alpha\big(\sin(jx)\big)$, $\cos^\alpha(jx) = \mathcal{F}^\alpha\big(\cos(jx)\big)$ and $\mathcal{F}^\alpha$ is the linear bounded operator on $\mathcal{C}[-\pi, \pi]$ defined in (1.13) (c.f. Subsection 1.4.2).

Theorem 4.1.1. *(see [63], Theorem 4.4). Let $f \in \mathcal{C}(I)$ be given and Δ be any partition of the interval $I = [-\pi, \pi]$. Then for all $\epsilon > 0$, there exists an α-fractal trigonometric polynomial $s^\alpha(x)$ with $\alpha \neq 0$ in $\mathbb{R}^N$ such that*

$$|f(x) - s^\alpha(x)| < \epsilon .$$

In [66], Navascués took the base function $b(x)$ as

$$b(x) = v(x)f(x) ,$$

where $v : I = [-\pi, \pi] \to \mathbb{R}$ does not depend on f, is fixed, non-constant, continuous function such that $v(x_0) = v(x_N) = 1$. If

$$K = \|v\|_\infty ,$$

then the following result holds.

Theorem 4.1.2. *(see [66], Theorem 2.13). If* $|\alpha|_\infty < \frac{1}{2+K}$*, the system*

$$\tau^\alpha = \{1^\alpha\} \cup \{\sin^\alpha(mx), \cos^\alpha(mx) \mid m \in \mathbb{N}\}$$

is total in $\mathcal{L}^2(I)$ *(see Definition 4.2.1).*

As a corollary the following result holds.

Corollary 4.1.1. *(see [66], Corollary 2.14). If* $|\alpha|_\infty < \frac{1}{2+K}$*,* $\mathcal{L}^2(I)$ *owns a Hilbert basis of* α*-fractal trigonometric polynomials.*

Let $(p_m)_{m=0}^\infty$ be the system of Legendre polynomials. Then it is an orthonormal basis for $\mathcal{L}^2(-1,1)$. Let $(p_m^\alpha)_{m=0}^\infty$ be the image of $(p_m)_{m=0}^\infty$ under the map $\overline{\mathcal{F}}^\alpha$, that is

$$(p_m^\alpha)_{m=0}^\infty = \left(\overline{\mathcal{F}}^\alpha(p_m)\right)_{m=0}^\infty,$$

where $\overline{\mathcal{F}}^\alpha$ is the extension of $\mathcal{F}^\alpha$ to $\mathcal{L}^2(-1,1)$ (c.f. Subsection 1.4.2).

Theorem 4.1.3. *(see [71], Theorem 2.22). If* $|\alpha|_\infty < (1 + \|I - L\|_2)^{-1}$*, where* L *is defined in Subsection 1.4.2. Then* $(p_m^\alpha)_{m=0}^\infty$ *is a Riesz basis of* $\mathcal{L}^2(-1,1)$*.*

Define the Haar functions on the interval $I = [0,1]$ as follows [38].

$$\begin{aligned}
&H_1(x) = 1 \text{ for all } x \in I,\\
&H_m(x) = 2^{j/2} \text{ for } x \in \left(\tfrac{2k-2}{2^{j+1}}, \tfrac{2k-1}{2^{j+1}}\right),\\
&H_m(x) = -2^{j/2} \text{ for } x \in \left(\tfrac{2k-1}{2^{j+1}}, \tfrac{2k}{2^{j+1}}\right),\\
&H_m(x) = 0 \text{ for } x \notin \left(\tfrac{k-1}{2^j}, \tfrac{k}{2^j}\right),
\end{aligned}$$

if $m = 2^j + k$, $k = 1, 2, \ldots, 2^j$, $j = 0, 1, \ldots$. The system $(H_m)_{m=1}^\infty$ is a Schauder basis of $\mathcal{L}^p(I)$ for $p \in [1, \infty)$. The following theorem provides a fractal basis for $\mathcal{L}^p(I)$.

Theorem 4.1.4. *(see [70], Theorem 3.7). Let us consider a sequence of scale vectors* $\alpha^m \in \mathbb{R}^N$*,* $m = 1, 2, \ldots$*, such that*

$$\sum_{m=1}^\infty \frac{|\alpha^m|_\infty}{1 - |\alpha^m|_\infty} < +\infty$$

and

$$|\alpha^1|_\infty \geq |\alpha^2|_\infty \geq \cdots .$$

If $|\alpha^1|_\infty < \|v\|_\infty^{-1}$ *then* $(H_m^{\alpha^m})$ *is a Schauder basis of* $\mathcal{L}^p(I)$ *for* $1 \leq p < +\infty$.

For $f, g \in \mathcal{L}^2(-1,1)$, let us define the inner product

$$\big(f, g\big)_{\mathcal{L}_w^2} = \frac{2}{\pi}\int_{-1}^{1} f(t)g(t)\frac{1}{\sqrt{1-t^2}}dt ,$$

where

$$w(t) = \frac{1}{\sqrt{1-t^2}} .$$

The Chebyshev system

$$\left\{\frac{T_0}{\sqrt{2}}, T_1, T_2, \dots\right\}$$

is orthonormal with respect to this inner product. The Chebyshev expansion of a function f is given by [19, 68]

$$f \sim \sum_{k=0}^{\infty} d_k T_k ,$$

where

$$d_k = \frac{2}{\pi}\int_{-1}^{1} f(t)T_k(t)\frac{1}{\sqrt{1-t^2}}dt ,$$

for $k \geq 1$ and

$$d_0 = \frac{1}{\pi}\int_{-1}^{1} f(t)\frac{1}{\sqrt{1-t^2}}dt .$$

The following is the convergence result of the fractal function $\tau_h^\alpha = \big(S_M(f_0)\big)^\alpha$ associated with the truncate Chebyshev expansion $\tau = S_M(f_0)$ of a polygonal f_0 for uniform partition $h = x_i - x_{i-1}$.

Theorem 4.1.5. *(see [68], Theorem 3). Let* $g \in$ *Lip* β, *where* $0 < \beta \leq 1$ *and* g *is not a line. Also assume that* N *and* M *are such that* $N = lM$ *for some positive constant* l. *Then* τ_h^α *converges uniformly to* g *on* $I = [-1, 1]$ *as* $h \to 0$.

The following results give the convergence of Legendre expansion associated with affine fractal interpolation function.

Theorem 4.1.6. *(see [75], Theorem 5). Let $g \in \mathcal{C}(I)$ be the original function providing the data. If f is the corresponding affine fractal interpolation function with the scale vector $\alpha_h \to 0$ as $h \to 0$. Then the Legendre expansion $S_m f$ defined by means of f converges in quadratic mean to g as $m \to \infty$ and $h \to 0$.*

Theorem 4.1.7. *(see [75], Theorem 8). The Legendre expansion of any affine fractal interpolation function f converges pointwisely to f almost everywhere. If the scale vector of f is such that $\alpha_h < h$ then the Legendre expansion of f converges pointwise and uniformly to f on the interval $I = [-1, 1]$.*

Theorem 4.1.8. *(see [75], Theorem 9). Let $g \in \mathcal{C}(I)$ be the original function providing the data. If f is the corresponding affine fractal interpolation function with the scale vector $\alpha_h < h$. Then the Legendre expansion $S_m f$ defined by means of f converges uniformly to g as $m \to \infty$ and $h \to 0$.*

4.1.1 Jacobi system

The Jacobi polynomials denoted by $P_n^{(r,s)}(x)$ and defined by

$$P_n^{(r,s)}(x) = 2^{-n} \sum_{k=0}^{n} \begin{pmatrix} n+r \\ k \end{pmatrix} \begin{pmatrix} n+s \\ n-k \end{pmatrix} (x+1)^k (x-1)^{n-k} \quad \text{for all } x \in \mathbb{R},$$

for $n = 0, 1, 2, \ldots$; where r and s are real parameters [43]. They include many of the basic special functions. Special cases are.

(a) $r = s$, the Gegenbauer polynomials;

(b) $r = s = -\frac{1}{2}$, the Chebyshev polynomials of the first kind;

(c) $r = s = \frac{1}{2}$, the Chebyshev polynomials of the second kind;

(d) $r = s = 0$, the Legendre polynomial.

With the (Jacobi) weight function $\rho^{(r,s)}(x) = (1-x)^r (1+x)^s$, where $r > -1$ and $s > -1$, the system $\{P_n^{(r,s)}(x)\}_{n=0}^{\infty}$ is orthonormal in $\mathcal{L}^2_\rho[-1, 1]$, where the Jacobi polynomials normalized by the condition

$$P_n^{(r,s)}(1) = \left(\begin{array}{c} n+r \\ n \end{array} \right) .$$

That is,

$$\int_{-1}^{1} P_n^{(r,s)}(x) P_m^{(r,s)}(x) \rho^{(r,s)}(x) dx \;=\; \delta_{nm} \;,$$

where $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ if $n \neq m$ [53]. The system $\{P_n^{(r,s)}(x)\}_{n=0}^{\infty}$ is uniquely defined and is called the Jacobi system of orthonormal polynomials.

If $f\rho^{(r,s)}$ is an integrable function on $[-1,1]$ then f has a Fourier expansion with respect to the Jacobi system of orthonormal polynomials as

$$f(x) = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s)}(x) \quad \text{for } x \in [-1,1], \tag{4.1}$$

where

$$a_k^{(r,s)} = \int_{-1}^{1} f(t) P_k^{(r,s)}(t) \rho^{(r,s)}(t) dt \tag{4.2}$$

is the k-th Fourier coefficient of f [53]. Let $S_n^{(r,s)}(f,x)$ denotes the n-th partial sum of the expansion given in (4.1). Then

$$S_n^{(r,s)}(f,x) = \sum_{k=0}^{n} a_k^{(r,s)}(f) P_k^{(r,s)}(x) \quad \text{for } x \in [-1,1] \;. \tag{4.3}$$

Let X be a subspace of $\mathcal{L}_\rho^1$. Define $\|S_n^{(r,s)}\|_X = \sup_{\|f\|\leq 1} \|S_n^{(r,s)}(f,x)\|_X$, called a Lebesgue constant. By virtue of the Lebesgue inequality [61]

$$\|f - S_n^{(r,s)}(f,x)\|_X \leq \big(1 + \|S_n^{(r,s)}\|_X\big) E_n(f)_X, \tag{4.4}$$

where

$$E_n(f)_X = \min_{g \in \prod_n} \|f - g\|_X$$

is the distance from f to the best approximation of the function f by algebric polynomials of degree at most n in the space X.

Theorem 4.1.9. *(Weierstrass Approximation Theorem) (see [19]). Let f be a continuous function defined on $[a,b]$. For each $\epsilon > 0$, there exists a polynomial P such that $\|f-P\| < \epsilon$. Thus*

$$|f(x) - P(x)| < \epsilon \quad \textit{for all} \quad x \in [a,b] \; .$$

If Weierstrass approximation theorem holds in the space X, then the boundedness of the Lebesgue constant leads to the convergence of the Fourier-Jacobi expansion for any function in the space X. Also it determines the partial sums of the Fourier-Jacobi expansion $S_n^{(r,s)}(f,x)$ to f in the space X.

The contents of the chapter is of two fold. In one side, fractal Jacobi system is defined, fractal version of Weierstrass type theorem is proved and a fractal Schauder basis is found for $\mathcal{L}_\rho(-1,1)$. On the other side, some convergence results on Fourier-Jacobi expansions of fractal functions are established.

4.2 Fractal Jacobi system

Let us take $I = [-1,1]$. Denote the set of all Jacobi polynomials of degree less than or equal to m by

$$J_m(I) = \left\{1 \equiv P_0^{(r,s)}(x), P_1^{(r,s)}(x), \ldots, P_m^{(r,s)}(x)\right\} ,$$

where $m = 0,1,2,\ldots$. Define

$$J(I) = \bigcup_{m=0}^{\infty} J_m(I) \; .$$

Then the fractal analogous of Jacobi polynomials is defined as

$$P_n^{(r,s,\alpha)}(x) = \mathcal{F}^\alpha\big(P_n^{(r,s)}(x)\big), \quad n = 0,1,2,\ldots \; ,$$

where $\mathcal{F}^\alpha$ is a linear and bounded operator as given in Subsection 1.4.2. Then denote

$$J_m^\alpha(I) = \left\{P_n^{(r,s,\alpha)}(x) = \mathcal{F}^\alpha(P_n^{(r,s)}(x)), n = 0,1,\ldots,m\right\}$$

and

$$J^\alpha(I) = \bigcup_{m=0}^{\infty} J_m^\alpha(I) \; .$$

In the next section, it is proved that $J^\alpha(I)$ is complete in $\mathcal{L}^2_\rho(I)$.

Example 4.1. *Let* $I = [0,1]$ *and* $\Delta : 0 < 1/8 < 2/8 < \ldots < 1$ *be the partition of* I*. Also assume that the constant scale factors* $\alpha_i = 0.2$ *for all* $i = 1,\ 2,\ \ldots, 8$*. Then Figure 4.1 represents the second order Legendre polynomial* $f(x) = 1.5x^2 - 0.5$ *and the corresponding* α*-fractal* f^α *for the IFS with the linear base function* $b(x)$ *in* $[0,1]$ *such that* $b(0) = f(0)$ *and* $b(1) = f(1)$*. Figure 4.2 represents the second order Chebyshev polynomial* $f(x) = 2x^2-1$ *of the 1st kind and the corresponding* α*-fractal* f^α *for the IFS with the base function* $b(x) = Lf(x) = v(x)f(x)$*, where* $v(x) = x^2 - x + 1$ *for* $x \in [0,1]$*. Figure 4.3 represents the second order Gegenbauer polynomial* $f(x) = 3.75x^2 - 0.75$ *and the corresponding* α*-fractal* f^α *for the IFS with the base function* $b(x) = Lf(x) = f \circ c(x)$*, where* $c(x) = \sin(\frac{\pi x}{2})$ *for* $x \in [0,1]$*.*

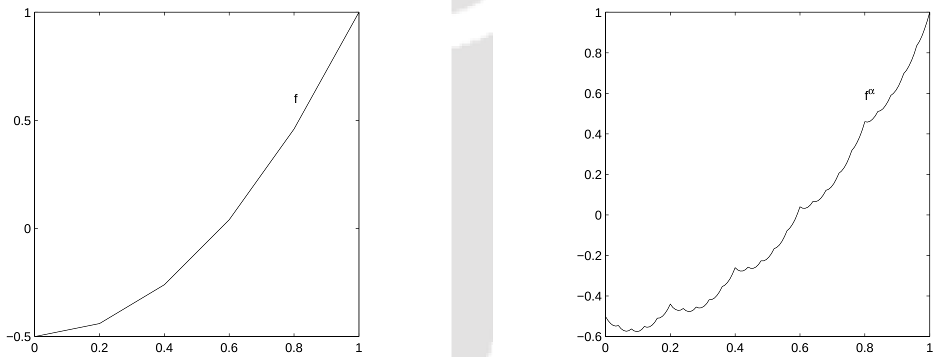

Figure 4.1: $f(x) = 1.5x^2 - 0.5$ and corresponding f^α.

The following definitions can be read in [43].

Definition 4.2.1. *A sequence* $(\phi_n)_{n\in\Lambda}$ *in a normed linear space* V *is called total in* V *if the class of all finite linear combinations* $\sum a_n\phi_n$ *is dense in* V*.*

Example 4.2. *(see [19], p. 193). The set*

$$\{1, x, x^2, \ldots\}$$

is total in $\mathcal{C}[0,1]$*.*

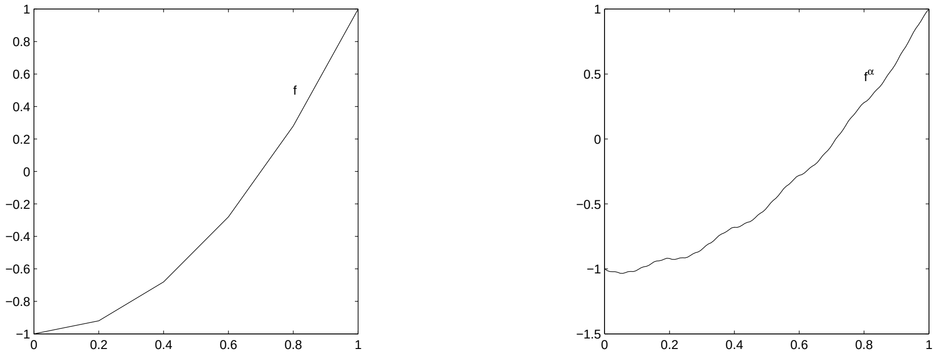

Figure 4.2: $f(x) = 2x^2 - 1$ and corresponding f^{α}.

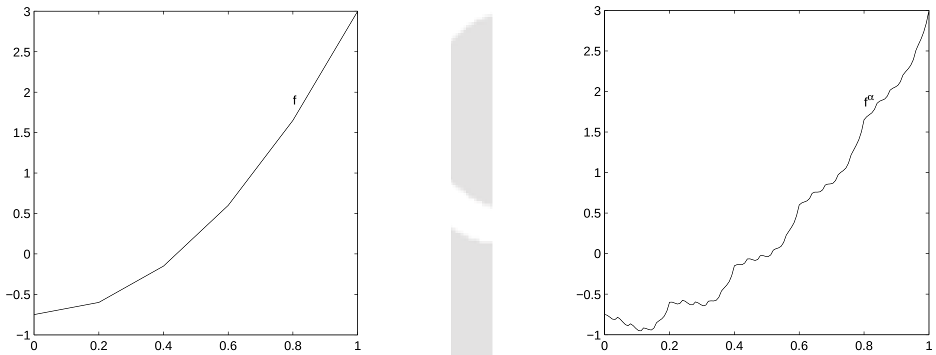

Figure 4.3: $f(x) = 3.75x^2 - 0.75$ and corresponding f^{α}.

Definition 4.2.2. *A sequence $(\phi_n)_{n\in\Lambda}$ in a Hilbert space H is called complete if the only element of H which is orthogonal to every ϕ_n is the null element. That is*

$$\langle f, \phi_n \rangle = 0 \quad \textit{for all} \quad n \in \Lambda \ \Rightarrow f \equiv 0.$$

Example 4.3. *(see [43], p. 36). The set of trigonometric polynomials*

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} : n = 0, \pm 1, \pm 2, \dots \right\}$$

forms a complete sequence in $\mathcal{L}^2(-\pi, \pi)$.

The following proposition is true for any sequence in a Hilbert space.

Proposition 4.2.1. *(see [43]). If (ϕ_n) is any sequence in H, orthogonal or not. Then the following are equivalent.*

(a) (ϕ_n) *is complete.*

(b) (ϕ_n) *is total.*

4.2.1 Weierstrass type theorem for α-fractal Jacobi sums

Weierstrass theorem states that any continuous function on a compact subset of $\mathbb{R}$ can be uniformly approximated by a sequence of polynomials [19]. Here we will prove a similar type of result but for a bigger space. Fractal version of Jacobi polynomials is used to prove the following result.

Theorem 4.2.1. *Let $I = [-1, 1]$ and $\epsilon > 0$ be given. Suppose that f is any square integrable function on I with respect to the weight function $\rho^{(r,s)}(x)$. For every $\epsilon > 0$ and any partition $\triangle : -1 = x_0 < x_1 < \cdots < x_N = 1$ of I and linear and bounded operator $L : \mathcal{C}(I) \to \mathcal{C}(I)$ there exists α-fractal Jacobi sum $P^{(r,s,\alpha)}(x) = \sum_{i=1}^{N} a_{k_i} P_{k_i}^{(r,s,\alpha)}(x)$ with $\alpha \neq 0$ such that*

$$\|f - P^{(r,s,\alpha)}\|_{\mathcal{L}^2_\rho} < \epsilon.$$

Proof. Let $\epsilon > 0$ be given and $f \in \mathcal{L}^2_\rho(I)$. Then there exists $g \in \mathcal{C}(I)$ such that

$$\|f - g\|_{\mathcal{L}^2_\rho} < \frac{\epsilon}{3}. \tag{4.5}$$

Since $(P_n^{(r,s)})_{n=0}^{\infty}$ is complete in $\mathcal{L}^2_\rho(I)$, it is total there by Proposition 4.2.1. Therefore for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ and $P^{(r,s)} = \sum_{i=1}^{N} a_{k_i} P_{k_i}^{(r,s)}$ such that

$$\|g - P^{(r,s)}\|_{\mathcal{L}^2_\rho} < \frac{\epsilon}{3}\,. \tag{4.6}$$

For the Jacobi sum and its fractal, using (1.16)

$$\|P^{(r,s,\alpha)} - P^{(r,s)}\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\|_\infty \|P^{(r,s)}\|_\infty. \tag{4.7}$$

Taking $|\alpha|_\infty < \frac{\frac{\epsilon}{3C}}{\frac{\epsilon}{3C} + \|I-L\|_\infty \|P^{(r,s)}\|_\infty}$, where $C = \left(\int_{-1}^{1} \rho^{r,s}(t)dt\right)^{1/2}$,

$$\|P^{(r,s,\alpha)} - P^{(r,s)}\|_{\mathcal{L}^2_\rho} \leq \|P^{(r,s,\alpha)} - P^{(r,s)}\|_\infty \left(\int_{-1}^{1} \rho^{r,s}(t)dt\right)^{1/2} < \frac{\epsilon}{3} \tag{4.8}$$

and combining (4.5), (4.6) and (4.8) we get

$$\|f - P^{(r,s,\alpha)}\|_{\mathcal{L}^2_\rho} < \epsilon.$$

The linearity of the operator $\mathcal{F}^\alpha$ gives the result. □

Remark 4.2.1. *By Theorem* 4.2.1*, it follows that*

(a) The set of fractal version of Jacobi sums is dense in $\mathcal{L}^2_\rho(I)$.

(b) $J^\alpha(I)$ *is complete in* $\mathcal{L}^2_\rho(I)$.

In the following, the Schauder basis and basis constant which are used in the sequel to prove the main results are presented.

Definition 4.2.3 ([15], pp. 24-27)**.** *Let X be a (real) normed space and let (x_n) be a non-zero sequence in X. We say that (x_n) is a Schauder basis for X, if for each $x \in X$, there is a unique sequence of scalars (a_n) such that $x = \sum_{n=1}^{\infty} a_n x_n$, where the series converges in norm to x. We define a sequence of linear maps (P_n) on X by $P_n x = \sum_{i=1}^{n} a_i x_i$, where $x = \sum_{i=1}^{\infty} a_i x_i$. The map P_n is a projection onto span$\{x_i : 1 \leq i \leq n\}$. In addition, since (x_n) is a Schauder basis, it follows that $P_n x \to x$ in norm as $n \to \infty$ for each $x \in X$ and P_n is continuous. Moreover, $K = \sup_n \|P_n\| < \infty$. The number K is called the basis constant of the basis (x_n).*

4.2.2 Expansions in terms of fractal Jacobi polynomials

In this section, we give an expansion of a weighted square integrable function in terms of fractal Jacobi polynomials. To do this a linear and bounded operator is defined on $\mathcal{L}^2_\rho(I)$. In Section 4.2.2, different scale vector α^k for every polynomial is taken to ensure the boundedness of the operator given in (4.9). The perturbation inequality (1.18) for function and its fractal with respect to the weighted p-norm is not true in general. Therefore in proving the main results, the uniform bound on the Jacobi polynomials under certain condition on the real parameters r, s plays an important role. Let α^k be a sequence of scale vectors such that

$$\sum_{k=0}^{\infty} \frac{|\alpha^k|_\infty}{1 - |\alpha^k|_\infty} < \infty \, .$$

Assume that L is a linear and bounded operator with respect to the uniform norm.

Proposition 4.2.2. *For $-1 < r, s \leq -1/2$, the operator $T : span(P_k^{(r,s)}(x))_{k=0}^\infty \to span(P_k^{(r,s,\alpha^k)}(x))_{k=0}^\infty$ defined by*

$$T\left(\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)\right) = \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s,\alpha^{k_i})}(x) \tag{4.9}$$

is linear and bounded.

Proof. The linearity of $\mathcal{F}^\alpha$ is given in Subsection 1.4.2.

Now, the boundedness of T is established below.

$$\left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s,\alpha^{k_i})}(x) \right\|_{\mathcal{L}^2_\rho} \leq \left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} (P_{k_i}^{(r,s,\alpha^{k_i})}(x) - P_{k_i}^{(r,s)}(x)) \right\|_{\mathcal{L}^2_\rho} + \left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)) \right\|_{\mathcal{L}^2_\rho} . \tag{4.10}$$

Since $(P_k^{(r,s)})_{k=0}^\infty$ is an orthonormal basis for $\mathcal{L}^2_\rho$, $a_k^{r,s}$ is a bounded linear functional on $\mathcal{L}^2_\rho$. Therefore

$$|a_k^{(r,s)}(f)| \leq \|a_k^{(r,s)}\|_2 \|f\|_{\mathcal{L}^2_\rho} \, . \tag{4.11}$$

Treating $a_{k_i}^{(r,s)}$ as the k_i-th Fourier-Jacobi coefficient of $\sum_{i=1}^M a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)$ and using (4.11), the first term of inequality (4.10) becomes

$$\left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} (P_{k_i}^{(r,s,\alpha^{k_i})}(x) - P_{k_i}^{(r,s)}(x)) \right\|_{\mathcal{L}^2_\rho} \leq \sum_{i=1}^{M} \left\| a_{k_i}^{(r,s)} \right\|_2 \left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x) \right\|_{\mathcal{L}^2_\rho} \left\| P_{k_i}^{(r,s,\alpha^{k_i})} - P_{k_i}^{(r,s)} \right\|_{\mathcal{L}^2_\rho} .$$

Therefore

$$\left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)}(P_{k_i}^{(r,s,\alpha^{k_i})}(x) - P_{k_i}^{(r,s)}(x))\right\|_{\mathcal{L}^2_\rho} \leq \left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)\right\|_{\mathcal{L}^2_\rho} \sum_{i=1}^{M} \left\|a_{k_i}^{(r,s)}\right\|_2 \left\|P_{k_i}^{(r,s,\alpha^{k_i}} - P_{k_i}^{(r,s)}\right\|_{\mathcal{L}^2_\rho} . \quad (4.12)$$

But from (1.16),

$$\|P_{k_i}^{(r,s,\alpha^{k_i})} - P_{k_i}^{(r,s)}\|_{\mathcal{L}^2_\rho} \leq \frac{|\alpha^{k_i}|_\infty}{1-|\alpha^{k_i}|_\infty} \|I-L\|_\infty \|P_{k_i}^{(r,s)}\|_\infty \left(\int_{-1}^{1} \rho^{(r,s)}(t)dt\right)^{1/2} .$$

Using it in (4.12), the left hand side of (4.12) becomes smaller than

$$\left\|\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)\right\|_{\mathcal{L}^2_\rho} \sum_{i=1}^{M} \left\|a_{k_i}^{(r,s)}\right\|_2 \left\|P_{k_i}^{(r,s)}\right\|_\infty \frac{|\alpha^{k_i}|_\infty}{1-|\alpha^{k_i}|_\infty} \|I-L\|_\infty \left(\int_{-1}^{1} \rho^{(r,s)}(t)dt\right)^{1/2} . \quad (4.13)$$

Since $(P_k^{(r,s)})$ is a Schauder basis for $\mathcal{L}^2_\rho(I)$, the following inequality hold [42]

$$1 \leq \|a_{k_i}^{(r,s)}\|_2 \|P_{k_i}^{(r,s)}\|_{\mathcal{L}^2_\rho} \leq 2K ,$$

where K is the basis constant. But $\|P_{k_i}^{(r,s)}\|_{\mathcal{L}^2_\rho} = 1$ and therefore,

$$1 \leq \|a_k^{(r,s)}\|_2 \leq 2K .$$

The following inequality can be read in ([11], p. 275)

$$\max_{-1\leq x\leq 1} |P_k^{(r,s)}(x)| \sim 1 \quad \text{for} \quad -1 < r,s \leq -1/2 . \quad (4.14)$$

Then for $\epsilon = 1$ there exists $n_0 \in \mathbb{N}$ such that

$$\|P_k^{(r,s)}\|_\infty \leq 2$$

for $k \geq n_0$. If $M^* = \max\left\{\|P_0^{(r,s)}\|_\infty, \|P_1^{(r,s)}\|_\infty, \ldots, \|P_{n_0-1}^{(r,s)}\|_\infty, 2\right\}$, for any k,

$$\|P_k^{(r,s)}\|_\infty \leq M^*.$$

Then

$$\|a_k^{(r,s)}\|_2 \|P_k^{(r,s)}\|_\infty \leq 2KM^*. \quad (4.15)$$

By choosing $(\alpha^k)_{k=0}^{\infty}$ such that the sum $\sum_{k=0}^{\infty} \frac{|\alpha^k|_\infty}{1-|\alpha^k|_\infty}$ is finite, say M_1^* and using (4.15) in (4.13), it follows that

$$\left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} (P_{k_i}^{(r,s,\alpha^{k_i})}(x) - P_{k_i}^{(r,s)}(x)) \right\|_{\mathcal{L}_\rho^2} \leq \left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x) \right\|_{\mathcal{L}_\rho^2} 2KM^*M_1^* l \ ,$$

where $l = \|I - L\|_\infty \left(\int_{-1}^{1} \rho^{(r,s)}(t)dt \right)^{1/2}$. Therefore,

$$\left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s,\alpha^{k_i})}(x) \right\|_{\mathcal{L}_\rho^2} \leq (1 + 2KM^*M_1^* l) \left\| \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x)) \right\|_{\mathcal{L}_\rho^2} .$$

Hence, T is a linear bounded operator. □

Lemma 4.2.1. *(Linear and Bounded Operator Theorem). Let X be a normed linear space, Y be a Banach space and $A : X \to Y$ be a linear and bounded operator. If X is dense in X' then A can be extended to X' preserving the norm of A.*

Now we are in a position to write an expansion in terms of fractal Jacobi polynomials.

Theorem 4.2.2. *The map $\overline{T} : \mathcal{L}_\rho^2(I) \to \mathcal{L}_\rho^2(I)$ given by*

$$\overline{T}(f) = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s,\alpha^k)}(x), \tag{4.16}$$

where

$$f = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s)}(x)$$

is well defined, linear and continuous for $-1 < r, s \leq -1/2$.

Proof. $(P_k^{(r,s)})_{k=0}^{\infty}$ is complete in $\mathcal{L}_\rho^2(I)$. So it is total there and therefore $span(P_k^{(r,s)}(x))_{k=0}^{\infty}$ is dense in $\mathcal{L}_\rho^2(I)$. Then by Lemma 4.2.1

$$T : span(P_k^{(r,s)}(x))_{k=0}^{\infty} \to span(P_k^{(r,s,\alpha^k)}(x))_{k=0}^{\infty} \subseteq \mathcal{L}_\rho^2(I)$$

$$T\left(\sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s)}(x) \right) := \sum_{i=1}^{M} a_{k_i}^{(r,s)} P_{k_i}^{(r,s,\alpha^{k_i})}(x)$$

can be extended to $\overline{T} : \mathcal{L}_\rho^2(I) \to \mathcal{L}_\rho^2(I)$ with $\|\overline{T}\|_2 = \|T\|_2$.

The linearity and boundedness of $\overline{T}$ imply that $\overline{T}(f) = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s,\alpha^k)}(x)$ whenever $f = \sum_{k=0}^{\infty} a_k^{(r,s)}(f) P_k^{(r,s)}(x)$. This completes the proof. □

4.2.3 Schauder basis for $\mathcal{L}^2_\rho(I)$

In this subsection, it is proved that the fractal Jacobi system forms a Schauder basis for $\mathcal{L}^2_\rho(I)$ under certain condition on the scale vector. To show this, it is noted that the n-th partial sum operator V_n of the continuous operator $V : \mathcal{L}^2_\rho(I) \to \mathcal{L}^2_\rho(I)$ defined by

$$V(f) = \sum_{k=0}^{\infty} a_k^{(r,s)}(f)\big(P_k^{(r,s)} - P_k^{(r,s,\alpha^k)}\big)$$

is bounded, as

$$\|V_n(f)\| \leq \sum_{k=0}^{\infty} |a_k^{(r,s)}(f)| \|(P_k^{(r,s)} - P_k^{(r,s,\alpha^k)})\|_{\mathcal{L}^2_\rho} \leq 2KM^*M_1^*l\|f\|_{\mathcal{L}^2_\rho} .$$

Since V_n is of finite rank, V_n is compact. The uniform convergence of V_n to V in operator norm implies that V is compact. Note that different scale vector α^k for every polynomial is taken in this subsection as well.

Theorem 4.2.3. *Suppose that the sequence of scale vectors $\alpha^k \in \mathbb{R}^N$ for $k = 0, 1, 2, \ldots,$ are such that*

$$|\alpha^0|_\infty \geq |\alpha^1|_\infty \geq \cdots$$

and

$$\sum_{k=0}^{\infty} \frac{|\alpha^k|_\infty}{1 - |\alpha^k|_\infty} < \infty .$$

If $|\alpha^0|_\infty < \frac{1}{M^*}\|L\|_\infty^{-1}\left(\int_{-1}^{1}\rho^{(r,s)}(t)dt\right)^{-1/2}$ and $-1 < r, s \leq -1/2$, then $(P_k^{(r,s,\alpha^k)})$ is a Schauder basis for $\mathcal{L}^2_\rho(I)$.

Proof. The proof goes in a similar way to that of Theorem 3.7 in [70]. However for completeness, we include it here for our settings.

Since $\overline{T} = I - V$ and V is compact, $ker(I - V)$ is finite dimensional. Then

$$\mathcal{L}^2_\rho(I) = ker(I - V) \bigoplus W ,$$

where W is a closed subspace of $\mathcal{L}^2_\rho(I)$. To show that $\overline{T}$ is injective, let $\overline{T}(P_k^{(r,s)}) = P_k^{(r,s,\alpha^k)} = 0$. Since $P_k^{(r,s)} \not\equiv 0$, we have

$$\|P_k^{(r,s,\alpha^k)} - P_k^{(r,s)}\|_{\mathcal{L}^2_\rho} \leq \|P_k^{(r,s,\alpha^k)} - P_k^{(r,s)}\|_\infty \left(\int_{-1}^1 \rho^{(r,s)}(t)dt\right)^{1/2}$$
$$\leq |\alpha^k|_\infty \left(\|P_k^{(r,s,\alpha^k)}\|_\infty + \|L\|_\infty \|P_k^{(r,s)}\|_\infty\right) \left(\int_{-1}^1 \rho^{(r,s)}(t)dt\right)^{1/2},$$

using (1.15). Therefore

$$1 \leq |\alpha^0|_\infty \|L\|_\infty \|P_k^{(r,s)}\|_\infty \left(\int_{-1}^1 \rho^{(r,s)}(t)dt\right)^{1/2},$$

which is a contradiction to the assumption that $|\alpha^0|_\infty < \frac{1}{M^*}\|L\|_\infty^{-1} \left(\int_{-1}^1 \rho^{(r,s)}(t)dt\right)^{-1/2}$. Therefore $P_k^{(r,s)} \notin ker(\overline{T})$ for all k and hence $\overline{T}$ is injective.

Since V is compact,

$$\dim(ker(I - V)) = \dim\left(\mathcal{L}^2_\rho(I)/Range(I - V)\right) = 0$$

and therefore $\overline{T}$ is onto. As $\overline{T}$ is a continuous isomorphism, it maps Schauder basis onto Schauder basis. Therefore $\left(P_k^{(r,s,\alpha^k)}\right)$ is a Schauder basis of $\mathcal{L}^2_\rho(I)$. □

4.3 Convergence of Fourier-Jacobi expansions

In this section, we mainly prove the convergence of Fourier-Jacobi expansion of an affine as well as non-affine fractal interpolation function corresponding to certain data set. Throughout this section, let I denote a closed interval $[-1, 1]$ and Δ denote the partition $-1 = x_0 < x_1 < \cdots < x_N = 1$ of I.

Definition 4.3.1. *(see [19]). Let f be a bounded real-valued function on a metric space X. Then the modulus of continuity $w_f(h)$ of f, defined for $h \geq 0$ by*

$$w_f(h) = \sup_{d(x,y) \leq h} |f(x) - f(y)|.$$

Definition 4.3.2. *(see [19]). A function f defined on $[a, b]$ belongs to the Dini-Lipschitz class $DL[a, b]$ if*

$$w_f(h)\log(1/h) \to 0 \qquad as \qquad h \to 0\,.$$

The propositions on convergence of Fourier-Jacobi expansions are needed to establish the main results are presented bellow.

Proposition 4.3.1. *(see [88], Theorem 4.7, pp. 146, 300). The condition $f \in DL[-1, 1]$ guarantees the uniform convergence of Fourier-Jacobi expansion of the function f on $[a, b] \subset (-1, 1)$.*

Proposition 4.3.2. *(see [25], p. 118). The condition $f \in DL[-1, 1]$ is sufficient for the uniform convergence of the Fourier-Jacobi expansion in the whole interval $[-1, 1]$ in the case when $r, s \in (-1, -1/2]$.*

Proposition 4.3.3. *(see [1]). Let $\max(r, s) > -1/2$ and $f^p \in Lip\ \gamma$, where $p + \gamma > \max(r, s) + 1/2$. Then the Fourier-Jacobi expansion of the function f uniformly converges on $[-1, 1]$.*

4.3.1 Convergence using affine FIF

The convergence of Fourier-Jacobi expansion corresponding to an affine FIF interpolating certain data set in weighted-mean square norm is proved in this section. Let us consider a set of data $\{(x_i, y_i)\}_{i=0}^{N}$ such that $x_{i-1} < x_i$ for $i = 1, 2, \ldots, N$.

Theorem 4.3.1. *Let $g \in \mathcal{C}(I)$ be the original function providing the data with constant step $h = x_i - x_{i-1}$. Let f be the corresponding affine fractal interpolation function with scale vector α_h such that $|\alpha_h|_\infty < h$. Then the Fourier-Jacobi expansion of f converges in weighted quadratic mean to g on $[a, b] \subset (-1, 1)$, that is, $S_n^{(r,s)}(f)$ converges to g in $\mathcal{L}_\rho^2$ as $h \to 0$ and $n \to \infty$ on $[a, b] \subset (-1, 1)$.*

Proof. Since q_i linear, $q_i \in$ Lip 1 and the corresponding fractal interpolation function f is affine and if $|\alpha_h|_\infty < h$ then $f \in$ Lip 1, ([75], Lemma 6).

Then the modulus of continuity

$$w_f(\delta) = \sup_{|x_1 - x_2| \leq \delta} |f(x_1) - f(x_2)| \ \leq \ K\,\delta\ ,$$

where K is the Lipschitz constant. Therefore by definition, f is in $DL([-1,1])$.

Using Proposition 4.3.1, we get

$$\|S_n^{(r,s)}(f) - f\|_\infty \to 0 \quad \text{as} \quad n \to \infty \quad \text{on} \quad [a,b] \subset (-1,1)\ .$$

But

$$\|S_n^{(r,s)}(f) - f\|_{\mathcal{L}^2_\rho} \leq \|S_n^{(r,s)}(f) - f\|_\infty \left(\int_{-1}^{1} \rho^{r,s}(t)dt \right)^{1/2} .$$

Hence

$$\|S_n^{(r,s)}(f) - f\|_{\mathcal{L}^2_\rho} \to 0 \quad \text{as} \quad n \to \infty \quad \text{on} \quad [a,b] \subset (-1,1)\ , \tag{4.17}$$

since $\int_{-1}^{1} \rho^{r,s}(t)dt$ is convergent. Similarly,

$$\|g - f\|_{\mathcal{L}^2_\rho} \leq \|g - f\|_\infty \left(\int_{-1}^{1} \rho^{r,s}(t)dt \right)^{1/2} .$$

The following result can be read in [75]

$$\|g - f\|_\infty \leq w_g(h) + \frac{2|\alpha_h|_\infty}{1 - |\alpha_h|_\infty} \|g\|_\infty\ . \tag{4.18}$$

Using it, we get

$$\|g - f\|_{\mathcal{L}^2_\rho} \leq \left(w_g(h) + \frac{2|\alpha_h|_\infty}{1 - |\alpha_h|_\infty} \|g\|_\infty \right) \left(\int_{-1}^{1} \rho^{r,s}(t)dt \right)^{1/2} .$$

Due to uniform continuity of g, it happens that $w_g(h) \to 0$ as $h \to 0$. Since $|\alpha_h|_\infty < h$, it follows that

$$\|g - f\|_{\mathcal{L}^2_\rho} \to 0 \qquad \text{as} \qquad h \to 0\ . \tag{4.19}$$

Using (4.17) and (4.19) in the following inequality

$$\|g - S_n^{(r,s)}(f)\|_{\mathcal{L}^2_\rho} \leq \|g - f\|_{\mathcal{L}^2_\rho} + \|f - S_n^{(r,s)}(f)\|_{\mathcal{L}^2_\rho}\ ,$$

the result follows. □

In the following theorem, the uniform convergence of Fourier-Jacobi expansion is established.

Theorem 4.3.2. *Let $g \in \mathcal{C}(I)$ be the original function providing the data with constant step $h = x_i - x_{i-1}$. Then for $|\alpha_h|_\infty < h$, the Fourier-Jacobi expansion defined by the corresponding affine FIF f converges uniformly to g as $h \to 0$ and $n \to \infty$ on $[a, b] \subset (-1, 1)$.*

Proof. If $S_n^{(r,s)}(f)$ is the n-th partial sum of the Fourier-Jacobi expansion of f, then

$$\|g - S_n^{(r,s)}(f)\|_\infty \leq \|g - f\|_\infty + \|f - S_n^{(r,s)}(f)\|_\infty . \tag{4.20}$$

Since f is an affine FIF and $|\alpha_h|_\infty < h$, it follows that $f \in DL(I)$. Due to the uniform continuity of g, $w_g(h) \to 0$ as $h \to 0$. Using the fact $w_g(h) \to 0$ as $h \to 0$ in (4.18), the first term in (4.20) tends to zero as $h \to 0$. By Proposition 4.3.1, the second term tends to zero as $n \to \infty$. Thus, the Fourier-Jacobi expansion of f converges to a given function g in uniform norm corresponding to these data, with respect to the scale vector α_h. □

Remark 4.3.1. *If the values of r and s are restricted in $(-1, -1/2]$, then the above two theorems are true on the whole interval $[-1, 1]$ in view of Proposition 4.3.2.*

Consider the n-th partial sum $S_n^{(r,s)}(f)$ of the Fourier-Jacobi expansion of an affine FIF f corresponding to a data set $\{(x_i, y_i)\}_{i=0}^N$ with the uniform step size $h = x_j - x_{j-1}$, $j = 1, 2, \dots, N$. As $h \to 0$, the sample size N increases. Let n and N be such that $N = ns$ for some positive constant s. The convergence of the corresponding fractal function $S_n^{(r,s,\alpha)}(f)$ is established in the following theorem.

Theorem 4.3.3. *Let $g \in \mathcal{C}(I)$ be the original function providing the data and f be the corresponding affine FIF with the scale vector α_h, $|\alpha_h|_\infty < h$, where h is the step size. Then the fractal analogue $S_n^{(r,s,\alpha)}(f)$ corresponding to n-th partial sum $S_n^{(r,s)}(f)$ converges uniformly to g as $h \to 0$ in intervals $[-1 + \delta, 1 - \delta] \subset (-1, 1)$, where $\delta > 0$.*

Proof. Denote $P^h = S_n^{(r,s,\alpha)}(f)$. Then,

$$\|g - P^h\|_\infty \leq \|g - f\|_\infty + \|f - S_n^{(r,s)}(f)\|_\infty + \|S_n^{(r,s)}(f) - P^h\|_\infty \,.$$

The following inequality can be obtained from ([75], Proposition 3)

$$\|g - f\|_\infty \leq w_g(h) + \frac{2|\alpha_h|_\infty}{1 - |\alpha_h|_\infty}\|g\|_\infty \,.$$

The uniform continuity of g implies that $w_g(h) \to 0$ as $h \to 0$. Since α_h tends to zero as well, it together gives that $\|g - f\|_\infty \to 0$ as $h \to 0$.

Since f is an affine FIF and $|\alpha_h|_\infty < h$, it follows that $f \in$ Lip 1 as shown in earlier theorem. Since $f \in$ Lip 1,

$$w_f(h) \leq C\, h \,. \tag{4.21}$$

Using (4.4) for the second term, one can get

$$\|f - S_n^{(r,s)}(f)\|_\infty \leq \big(1 + \|S_n^{(r,s)}\|\big) E_n(f) \,.$$

For a function $f \in \mathcal{C}[-1,1]$, $\|S_n^{(r,s)}\|_\infty$ does not exceed $\ln(n+1)$ on $[-1+\delta, 1-\delta]$, ([89], Section 9.3). Using it along with Jackson's Theorem V in [19], we get

$$\begin{aligned}\|f - S_n^{(r,s)}(f)\|_\infty &\leq (1 + \ln(n+1)) w_f(\pi/(n+1)) \\ &\leq (1 + \ln(n+1)) C(\pi/(n+1)), \quad \text{using} \quad (4.21).\end{aligned}$$

Corresponding to these data, the above term tends to zero as h tends to zero ($n \to \infty$ as a consequence of $h \to 0$). For third term using the inequality (1.12), it follows that

$$\|S_n^{(r,s)}(f) - S_n^{(r,s,\alpha)}(f)\|_\infty \leq \frac{|\alpha_h|_\infty}{1 - |\alpha_h|_\infty}\|S_n^{(r,s)}(f) - b\|_\infty \,.$$

Since $|\alpha_h|_\infty < h$, the above term tends to zero as $h \to 0$.

This completes the proof. □

4.3.2 Convergence using smooth FIF

The convergence of Fourier-Jacobi expansion corresponding to a non-affine smooth FIF interpolating certain data set in weighted-mean square norm is proved in this section.

Theorem 4.3.4. *Suppose* $p > \max(r, s) + 1/2 > 0$ *and* $g \in \mathcal{C}^p(I)$ *is the original function providing the data with constant step* $h = x_i - x_{i-1}$. *Let* $f \in \mathcal{C}^p(I)$ *be the corresponding differentiable* α*-fractal of* g *with scale vector* α_h *such that* $|\alpha_h|_\infty < a_i^p$ *via the IFS (1.23). Then the Fourier-Jacobi expansion of* f *converges in weighted quadratic mean to* g, *that is,* $S_n^{(r,s)}(f)$ *converges to* g *in* $\mathcal{L}_\rho^2$ *as* $h \to 0$ *and* $n \to \infty$ *.*

Proof. Given that $|\alpha_h|_\infty < a_i^p$, $q_i \in \mathcal{C}^p(I)$ and b is a Hermite interpolating polynomial which agrees with the function at the extremes of the interval up to p-th derivative. Then by Theorem 1.3.2, the corresponding IFS (1.23) determines a FIF $f \in \mathcal{C}^p(I)$ (see [74]). The mean value theorem shows that $f^{p-1} \in \text{Lip } 1$. Using Proposition 4.3.3, we get

$$\|S_n^{(r,s)}(f) - f\|_\infty \to 0 \quad \text{as} \quad n \to \infty$$

and therefore,

$$\|S_n^{(r,s)}(f) - f\|_{\mathcal{L}_\rho^2} \to 0 \quad \text{as} \quad n \to \infty \, . \tag{4.22}$$

Using the inequality (1.24), for the function and its fractal

$$\|f - g\|_{\mathcal{L}_\rho^2} \leq \frac{|\alpha_h|_\infty}{1 - |\alpha_h|_\infty} \|g - b\|_{\mathcal{L}_\rho^2} \left(\int_{-1}^{1} \rho^{r,s}(t)dt \right)^{1/2} .$$

Therefore,

$$\|f - g\|_{\mathcal{L}_\rho^2} \to 0 \quad \text{as} \quad h \to 0, \tag{4.23}$$

since a_i goes to zero when h tends to zero and thus α_h. Using (4.22) and (4.23) in the following inequality

$$\|g - S_n^{(r,s)}(f)\|_{\mathcal{L}_\rho^2} \leq \|g - f\|_{\mathcal{L}_\rho^2} + \|f - S_n^{(r,s)}(f)\|_{\mathcal{L}_\rho^2} \, ,$$

the result follows. □

In the following theorem, the uniform convergence of Fourier-Jacobi expansion is established.

Theorem 4.3.5. *Suppose $p > \max(r, s) + 1/2 > 0$ and $g \in \mathcal{C}^p(I)$ is the original function providing the data with constant step $h = x_i - x_{i-1}$. Let $f \in \mathcal{C}^p(I)$ be the corresponding differentiable α-fractal of g with constant scale vector α_h such that $|\alpha_h|_\infty < a_i^p$ via the IFS (1.23). Then the Fourier-Jacobi expansion of f converges uniformly to g as $h \to 0$ and $n \to \infty$.*

Proof. If $S_n^{(r,s)}(f)$ is the n-th partial sum of the Fourier-Jacobi expansion, then

$$\|g - S_n^{(r,s)}(f)\|_\infty \leq \|g - f\|_\infty + \|f - S_n^{(r,s)}(f)\|_\infty . \tag{4.24}$$

Then the first term in the right side of inequality (4.24) tends to zero as $h \to 0$, due to (1.24). Using the mean value theorem, $f \in \mathcal{C}^p(I) \Rightarrow f^{p-1} \in \text{Lip } 1$. Then by Proposition 4.3.3, the second term in the right side of inequality (4.24) tends to zero as $n \to \infty$. Hence the proof. □

The convergence of Fourier-Jacobi expansion for smooth FIF corresponding to IFS (1.25) is proved in the next two theorems.

Theorem 4.3.6. *Suppose $p > \max(r, s) + 1/2 > 0$ and $g \in \mathcal{C}^p(I)$ is the original function providing the data with constant step $h = x_i - x_{i-1}$. Let $f \in \mathcal{C}^p(I)$ be the corresponding differentiable α-fractal of g with scale vector α_h such that $|\alpha_h|_\infty < a_i^p$ and a family of base function $B = \{b_i : i = 1,\ 2\ \dots N\}$ are such that the derivatives up to p-th order of each of its member agrees with that of g at the end points of the interval. Then the Fourier-Jacobi expansion of f converges in weighted quadratic mean to g as $h \to 0$ and $n \to \infty$.*

Proof. Given that $|\alpha_h|_\infty < a_i^p$, $q_i \in \mathcal{C}^p(I)$ and each b_i is such that it agrees with the function at the extremes of the interval up to p-th derivative. Then by Theorem 1.3.2, the corresponding IFS (1.25) determines a FIF $f \in \mathcal{C}^p(I)$ (see [74]). The mean value theorem shows that $f^{p-1} \in$ Lip 1. Using Proposition 4.3.3, we get

$$\|S_n^{(r,s)}(f) - f\|_\infty \to 0 \quad \text{as} \quad n \to \infty$$

and therefore,

$$\|S_n^{(r,s)}(f) - f\|_{\mathcal{L}^2_\rho} \to 0 \quad \text{as} \quad n \to \infty \,. \tag{4.25}$$

Using the inequality (1.26), for the function and its fractal

$$\|f - g\|_{\mathcal{L}^2_\rho} \leq \frac{|\alpha_h|_\infty}{1 - |\alpha_h|_\infty} \max_{1 \leq i \leq N} \|g - b_i\|_{\mathcal{L}^2_\rho} \left(\int_{-1}^{1} \rho^{r,s}(t) dt \right)^{1/2} .$$

Therefore,

$$\|f - g\|_{\mathcal{L}^2_\rho} \to 0 \quad \text{as} \quad h \to 0, \tag{4.26}$$

since a_i goes to zero when h tends to zero, and thus α_h. Using (4.25) and (4.26) in the following inequality

$$\|g - S_n^{(r,s)}(f)\|_{\mathcal{L}^2_\rho} \leq \|g - f\|_{\mathcal{L}^2_\rho} + \|f - S_n^{(r,s)}(f)\|_{\mathcal{L}^2_\rho} \,,$$

the result follows. □

Theorem 4.3.7. *Suppose $p > \max(r, s) + 1/2 > 0$ and $g \in \mathcal{C}^p(I)$ is the original function providing the data with constant step $h = x_i - x_{i-1}$ for all i. Let $f \in \mathcal{C}^p(I)$ be the corresponding differentiable α-fractal of g with scale vector α_h such that $|\alpha_h|_\infty < a_i^p$ and a family of base function $B = \{b_i : i = 1,\ 2 \ldots N\}$ are such that the derivatives up to p-th order of each of its member agrees with that of g at the end points of the interval. Then the Fourier-Jacobi expansion of f converges uniformly to g as $h \to 0$ and $n \to \infty$.*

Proof. If $S_n^{(r,s)}(f)$ is the n-th partial sum of the Fourier-Jacobi expansion, then

$$\|g - S_n^{(r,s)}(f)\|_\infty \leq \|g - f\|_\infty + \|f - S_n^{(r,s)}(f)\|_\infty \,. \tag{4.27}$$

Then the first term in the right side of inequality (4.27) tends to zero as $h \to 0$, due to (1.26). Using the mean value theorem, $f \in \mathcal{C}^p(I) \Rightarrow f^{p-1} \in$ Lip 1. Then by

Proposition 4.3.3, the second term in the right side of inequality (4.27) tends to zero as $n \to \infty$. Hence the proof. $\square$

Chapter 5

More general fractal functions on the sphere

Functions on the unit sphere $S \subseteq \mathbb{R}^3$, has important applications in the area of oceanography, environment, metrology, etc. A historical survey on the functions on the sphere of arbitrary dimensions, can be found in [47]. The basics on spherical harmonics and approximations on the sphere, can be seen in [22] and references therein.

In [64, 69], Navascués constructed fractal versions of the spherical harmonics. To do that Navascués considered the orthogonal basis

$$\left\{U_n^0, U_n^m, V_n^m; m = 1, 2, \dots, n\right\}$$

given in (1.27), for the space of spherical harmonics $\mathcal{H}_n$, of order n which is a linear subspace of continuous functions on the sphere with dimension $2n+1$. Then the operator $\mathcal{S}_n^\alpha$ which fractalizes the basis elements defined as

$$(U_n^0)^\alpha(\varphi, \theta) = \mathcal{S}_n^\alpha(U_n^0)(\varphi, \theta) = P_n^\alpha(\cos\varphi) ,$$

$$(U_n^m)^\alpha(\varphi, \theta) = \mathcal{S}_n^\alpha(U_n^m)(\varphi, \theta) = (P_n^m)^\alpha(\cos\varphi)\cos(m\theta) ,$$

$$(V_n^m)^\alpha(\varphi, \theta) = \mathcal{S}_n^\alpha(V_n^m)(\varphi, \theta) = (P_n^m)^\alpha(\cos\varphi)\sin(m\theta) .$$

By linearity, $\mathcal{S}_n^\alpha$ has been extended to whole $\mathcal{H}_n$ as given in the following proposition.

Proposition 5.0.4. *(see [69], Proposition 3.2). There exists a linear, bounded and injective operator $\mathcal{S}_n^\alpha : \mathcal{H}_n \to \mathcal{L}^2(S)$, where $\mathcal{H}_n$ is the space of spherical harmonics of* *order n, such that*

$$\|\mathcal{S}_n^\alpha\|_2 \leq \|\mathcal{F}^\alpha\|_2 \ ,$$

$\mathcal{F}^\alpha$ *is the operator given in Section 1.4 for the base function in (1.10) as*

$$b = vf \ ,$$

where $v : I \to \mathbb{R}$ *is fixed, continuous and non-constant with* $v(x_0) = v(x_N) = 1$.

Using the completeness of the spherical harmonics of all order in $\mathcal{L}^2(S)$, the operator given in above proposition was extended to the whole space $\mathcal{L}^2(S)$.

Theorem 5.0.8. *(see [69], Theorem 3.6). There exists a linear and bounded operator* $\mathcal{S}^\alpha : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ *such that* $\mathcal{S}^\alpha$ *restricted to* $\mathcal{H}_n$ *is* $\mathcal{S}_n^\alpha$. *Moreover,*

$$\|\mathcal{S}^\alpha\|_2 \leq 3\|\mathcal{F}^\alpha\|_2 \ .$$

Navascués used the linearity and boundedness of the operator $\mathcal{S}^\alpha$ to prove fractal version of many classical results in functional analysis/operator theory.

In this chapter, a family of continuous functions on the unit sphere $S \subseteq \mathbb{R}^3$ generalizing the spherical harmonics is considered, as described below.
Let $u : [-1, 1] \to \mathbb{R}$ be a continuous function and let $v : [0, 2\pi] \to \mathbb{R}$ be continuous and periodic. If (φ, θ) represents the spherical coordinates of a point P on the unit sphere $S \subseteq \mathbb{R}^3$, then

$$H(\varphi, \theta) = u(\cos\varphi)v(\theta) \tag{5.1}$$

is a continuous function on the sphere S. If we define fractal version of the function on the sphere as

$$H^\alpha(\varphi, \theta) = u^\alpha(\cos\varphi)v(\theta), \tag{5.2}$$

where $u^\alpha = \mathcal{F}^\alpha(u)$ and $\mathcal{F}^\alpha$ is the fractal operator defined in (1.13) in Section 1.4, then we call the function given in (5.2), as a fractal function of first type on the sphere. If we define fractal version of the function on the sphere as

$$H^\alpha(\varphi, \theta) = u(\cos\varphi)v^\alpha(\theta), \tag{5.3}$$

where $v^\alpha = \mathcal{F}^\alpha(v)$ and $\mathcal{F}^\alpha$ is the fractal operator defined in (1.13) in Section 1.4, then we call the function given in (5.3), as a fractal function of second type on the sphere.

This chapter is of two fold. In the first part, using fractal methodology, fractal version of first type of a family of continuous functions on the sphere S is constructed. To do this, an IFS and a linear bounded operator that maps classical functions to its fractal analogues is defined. Some approximation properties of fractal functions on the sphere are investigated. A complete orthonormal system of functions is constructed and using it, a linear and bounded operator is defined on $\mathcal{L}^2(S)$. For different values of the scale vector of the IFS, fractal versions of some of the classical results in functional analysis/operator theory are established as well. In the second part, fractal functions of second type on the sphere is constructed but for different IFS. Restricting the scale vector, fractal versions of some of the classical results in functional analysis/operator theory are established here as well.

The chapter is organized as follows. In Section 5.1, functions on the sphere S as a generalization of spherical harmonics is defined. Finite dimensional space $\mathcal{H}_n^\alpha$, consisting of fractal functions of first type on the sphere as basis is defined and their best approximation properties in $\mathcal{L}^2(S)$ are discussed in Section 5.2. In Section 5.3, a complete system of classical functions on the sphere is defined. Using it, an expansion in terms of fractal functions of first type on the sphere is given for a square integrable function on the sphere. Finally, a fractal Hilbert basis is provided for $\mathcal{L}^2(S)$. In Section 5.4, a family of fractal functions of second type on the sphere is defined. An expansion in terms of fractal functions of second type on the sphere is given for a square integrable function on the sphere. Finally, an alternative fractal bases is provided in Section 5.5.

5.1 Fractal functions of first type on the sphere

Let us consider a family of continuous functions

$$u_{nm} : I = [-1, 1] \to \mathbb{R} \,,$$

for $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots, 2n$. Let

$$\left\{ v_m : [0, 2\pi] \to \mathbb{R} \;\mid\; v_m \text{ is continuous and } 2\pi \text{ periodic, } m = 0, 1, 2, \ldots \right\}$$

be an orthonormal system in $\mathcal{C}[0, 2\pi]$ with respect to the inner product

$$\langle g, h \rangle = \int_0^{2\pi} g(t)h(t)dt \ .$$

Let us define functions on the unit sphere $S \subseteq \mathbb{R}^3$ as

$$H_{nm}(\varphi, \theta) = u_{nm}(\cos\varphi)v_m(\theta) \tag{5.4}$$

for $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots, 2n$. Recall that the notation (φ, θ) represents the spherical coordinates of a point P on the unit sphere. On $\mathcal{L}^2(S)$, define the inner product

$$\langle F, G \rangle = \int_S F.G \, dS \quad \text{for} \quad F, G \in \mathcal{L}^2(S)$$

and the norm

$$\|F\|_{\mathcal{L}^2(S)} = \Big(\langle F, F \rangle \Big)^{1/2} = \left(\int_S |F|^2 dS \right)^{1/2} \text{for } F \in \mathcal{L}^2(S) \ .$$

Lemma 5.1.1. *For $n = 0, 1, 2, \ldots$ and $m, j = 0, 1, \ldots, 2n$, $m \neq j$,*

$$\langle H_{nm}, H_{nj} \rangle = 0 \ .$$

Proof. The inner product can be expressed in spherical coordinates as

$$\begin{aligned} \langle H_{nm}, H_{nj} \rangle &= \int_0^{2\pi} \int_0^{\pi} \big(u_{nm}(\cos\varphi) u_{nj}(\cos\varphi) \big) \big(v_m(\theta) v_j(\theta) \big) \sin\varphi d\varphi d\theta \\ &= \left(\int_0^{2\pi} v_m(\theta) v_j(\theta) d\theta \right) \left(\int_0^{\pi} u_{nm}(\cos\varphi) u_{nj}(\cos\varphi) \sin\varphi d\varphi \right) . \end{aligned}$$

Since the first integral is zero if $m \neq j$, it happens that $\langle H_{nm}, H_{nj} \rangle = 0$. □

Lemma 5.1.2. *For $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots, 2n$,*

$$\|H_{nm}\|_{\mathcal{L}^2(S)} = \|u_{nm}\|_{\mathcal{L}^2(I)} \ .$$

Proof. We have,

$$\begin{aligned}
\|H_{nm}\|^2_{\mathcal{L}^2(S)} &= \int_0^{2\pi}\int_0^{\pi} |u_{nm}(\cos\varphi)|^2 v_m^2(\theta)\sin\varphi d\varphi d\theta \\
&= \left(\int_0^{2\pi} v_m^2(\theta)d\theta\right)\left(\int_0^{\pi} |u_{nm}(\cos\varphi)|^2 \sin\varphi d\varphi\right).
\end{aligned}$$

The first integral is equal to one due to the orthonormality of $\{v_m\}_{m=0}^{\infty}$. Also using the change of variable $\cos\varphi = t$,

$$\int_0^{\pi} |u_{nm}(\cos\varphi)|^2 \sin\varphi d\varphi = \int_{-1}^{1} |u_{nm}(t)|^2 dt = \|u_{nm}\|^2_{\mathcal{L}^2(I)} \,.$$

Hence the result follows. $\square$

$$\begin{array}{lllllllllll}
 & H_{00} & & & & & & & & & \\
 & H_{10} & H_{11} & H_{12} & & & & & & & \\
 & H_{20} & H_{21} & H_{22} & H_{23} & H_{24} & & & & & \\
 & H_{30} & H_{31} & H_{32} & H_{33} & H_{34} & H_{35} & H_{36} & & & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\mathcal{H}_n \rightarrow & H_{n0} & H_{n1} & H_{n2} & H_{n3} & H_{n4} & H_{n5} & H_{n6} & \dots & H_{n(2n+1)} & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

Table 5.1: The space $\mathcal{H}_n$.

Let us define

$$\mathcal{H}_n = \left\{ \sum_{m=0}^{2n} \lambda_{nm} H_{nm} \;\middle|\; \lambda_{nm} \in \mathbb{R} \right\} \quad \text{(see Table 5.1)}\,,$$

where H_{nm} is defined in (5.4). Due to Lemma 5.1.1, $\{H_{nm}\}_{m=0}^{2n}$ is an orthogonal basis of $\mathcal{H}_n$. If $\|u_{nm}\|_{\mathcal{L}^2(I)} = 1$ for all n and m, $\{H_{nm}\}_{m=0}^{2n}$ is an orthonormal basis due to Lemma 5.1.2. The dimension of $\mathcal{H}_n$ is then $2n+1$. Consider the base function given in (1.10) as

$$b = Lf\,,$$

where $L : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ is a linear and bounded operator such that $Lf(x_0) = f(x_0)$ and $Lf(x_N) = f(x_N)$ if $\mathcal{C}(I)$ is equipped with sup norm or mean-square norm. In particular, in this section, we will consider

$$Lf(x) = \big(f \circ c\big)(x) \ ,$$

where $c(x)$ is continuously differentiable, fixed mapping, strictly increasing, such that $c(x_0) = x_0$, $c(x_N) = x_N$ and $c \neq Id$. Since $c(x)$ is a strictly increasing function, $c'(x) > 0$ and as a consequence $\frac{1}{c'(x)}$ is continuous on a compact set. Then there exists a positive constant M such that $|\big(c'(x)\big)^{-1}| \leq M$ for all $x \in I$. Now,

$$\|Lf\|^2_{\mathcal{L}^2(I)} = \|f \circ c\|^2_{\mathcal{L}^2(I)} = \int_I |f \circ c(x)|^2 dx \ .$$

Using the change of variable $c(x) = t$, it follows that

$$\|f \circ c\|^2_{\mathcal{L}^2(I)} = \int_I f^2(t)(c'(x))^{-1} dt \leq M \int_I f^2(t) dt = M\|f\|^2_{\mathcal{L}^2(I)}$$

and therefore

$$\|Lf\|_{\mathcal{L}^2(I)} = \|f \circ c\|_{\mathcal{L}^2(I)} \leq M^{1/2}\|f\|_{\mathcal{L}^2(I)}. \tag{5.5}$$

Hence

$$\|L\|_2 \leq M^{1/2} \ ,$$

where $\|L\|_2$ is the norm of the operator L defined as

$$\|L\|_2 = \max\{\|Lf\|_{\mathcal{L}^2(I)} : \|f\|_{\mathcal{L}^2(I)} = 1, f \in \mathcal{C}(I)\}. \tag{5.6}$$

Define

$$H^\alpha_{nm}(\varphi, \theta) = u^\alpha_{nm}(\cos\varphi) v_m(\theta)$$

for all $n = 0, 1, 2, \ldots$ and $m = 0, 1, \ldots, 2n$, where $u^\alpha_{nm} = \mathcal{F}^\alpha(u_{nm})$ and $\mathcal{F}^\alpha$ is the fractal operator given in (1.13) in Section 1.4. Then, $\{H^\alpha_{nm}\}^{2n}_{m=0}$ is orthogonal. For instance, using the arguments of Lemma 5.1.1

$$\langle H^\alpha_{nm}, H^\alpha_{nj}\rangle = \int_0^{2\pi}\int_0^{\pi} u^\alpha_{nm}(\cos\varphi) u^\alpha_{nj}(\cos\varphi) \times v_m(\theta) v_j(\theta) \sin\varphi d\varphi d\theta = 0 \ ,$$

if $m \neq j$, due to the orthogonality of $\{v_m\}_{m=0}^{\infty}$. Consider now

$$\mathcal{H}_n^\alpha = \left\{ \sum_{m=0}^{2n} \lambda_{nm} H_{nm}^\alpha ; \lambda_{nm} \in \mathbb{R} \right\}. \tag{5.7}$$

Then $\{H_{nm}^\alpha\}_{m=0}^{2n}$ is an orthogonal basis of $\mathcal{H}_n^\alpha$.

Theorem 5.1.1. *Let us define the operator*

$$\Theta_n^\alpha : \mathcal{H}_n \to \mathcal{L}^2(S) ,$$

$$\sum_{m=0}^{2n} \lambda_{nm} H_{nm} \hookrightarrow \sum_{m=0}^{2n} \lambda_{nm} H_{nm}^\alpha .$$

Then Θ_n^α is linear and bounded. Its norm satisfies the inequality

$$\|\Theta_n^\alpha\|_2 \leq \|\mathcal{F}^\alpha\|_2$$

for all $n = 0, 1, 2, \ldots$.

Proof. The linearity of Θ_n^α follows easily. It is noted that

$$\|u_{nm}^\alpha\|_{\mathcal{L}^2(I)} = \|\mathcal{F}^\alpha(u_{nm})\|_{\mathcal{L}^2(I)} \leq \|\mathcal{F}^\alpha\| \|u_{nm}\|_{\mathcal{L}^2(I)} .$$

Now,

$$\begin{aligned} \|H_{nm}^\alpha\|_{\mathcal{L}^2(S)}^2 &= \int_0^{2\pi} \int_0^{\pi} |u_{nm}^\alpha(\cos\varphi)|^2 v_m^2(\theta) \sin\varphi \, d\varphi \, d\theta \\ &= \left(\int_0^{2\pi} v_m^2(\theta) d\theta \right) \left(\int_0^{\pi} |u_{nm}^\alpha(\cos\varphi)|^2 \sin\varphi \, d\varphi \right) . \end{aligned}$$

Using the argument of Lemma 5.1.2, it follows that

$$\|H_{nm}^\alpha\|_{\mathcal{L}^2(S)} = \|u_{nm}^\alpha\|_{\mathcal{L}^2(I)}$$

and thus

$$\|H_{nm}^\alpha\|_{\mathcal{L}^2(S)} \leq \|\mathcal{F}^\alpha\|_2 \|u_{nm}\|_{\mathcal{L}^2(I)} = \|\mathcal{F}^\alpha\|_2 \|H_{nm}\|_{\mathcal{L}^2(I)} .$$

For boundedness of Θ^α, due to the orthogonality of $\{H^\alpha_{nm}\}_{m=0}^{2n}$ and $\{H_{nm}\}_{m=0}^{2n}$,

$$\begin{aligned}\|\Theta^\alpha_n\big(\sum_{m=0}^{2n}\lambda_{nm}H_{nm}\big)\|^2_{\mathcal{L}^2(I)} &= \sum_{m=0}^{2n}\|\lambda_{nm}H^\alpha_{nm}\|^2\\ &\leq \sum_{m=0}^{2n}\|\mathcal{F}^\alpha\|_2^2\|\lambda_{nm}H_{nm}\|^2\\ &\leq \|\mathcal{F}^\alpha\|_2^2\Big\|\sum_{m=0}^{2n}\lambda_{nm}H_{nm}\Big\|^2.\end{aligned}$$

Consequently $\|\Theta^\alpha_n\|_2 \leq \|\mathcal{F}^\alpha\|_2$. □

5.2 Best approximations by fractal functions

In this section, some basic results on best approximation using fractal functions of first type on the sphere is proved. The following definitions are useful to prove the results in this section.

Definition 5.2.1. *(see [54], Definition 5.1.17). Let V be a non-empty subset of a normed linear space $\big(X, \|.\|\big)$. For $x \in X$, define $d(x, V) = \inf\big\{\|x - v\| : v \in V\big\}$ and $P_V(x) = \big\{w \in V : \|x - w\| = d(x, V)\big\}$. If $P_V(x)$ is non-empty for each $x \in X$, then V is said to be a existence or proximinal subset of X. If for every element $x \in X$, there is no more than one element v of V such that $d(x, v) = d(x, V)$, then V is called a set of uniqueness. The set V is said to be Chebyshev set if for each $x \in X$, $P_V(x)$ is a singleton set; that is if V is both a set of uniqueness and a set of existence.*

The next lemma is a well-known result on approximation.

Lemma 5.2.1. *If V is a finite dimensional subspace of a normed linear space $\big(X, \|.\|\big)$. Then V is proximinal.*

Definition 5.2.2. *(see [54], Definition 5.3.19). Let V be a non-empty subset of a normed linear space $\big(X, \|.\|\big)$ and let $x \in X \setminus V$. A sequence $\{v_n\} \subset V$ is called a minimizing sequence for x if $\lim_{n\to\infty}\|x - v_n\| = d(x, V)$. If for any $x \in X \setminus V$, every* *minimizing sequences for x has a subsequence converging to an element in V, then V is called approximately compact.*

Definition 5.2.3. *(see [15]). A normed linear space $\big(X, \|.\|\big)$ is smooth, if for each non-zero $x \in X$ there exists a unique norm one functional $f \in X^*$ such that $f(x) = \|x\|$.*

It is immediate that a Hilbert space is smooth.

Definition 5.2.4. *(see [54], Definition 5.5.2). Let X be a normed linear space. Define a function $\rho_X : (0, +\infty) \to [0, +\infty)$ by the formula*

$$\rho_X(t) = \sup\left\{\frac{1}{2}\big(\|x+ty\| + \|x-ty\|\big) - 1 : x, y \in S_X\right\},$$

if $X \neq \{0\}$ and by the formula

$$\rho_X(t) = \begin{cases} 0, & \text{if } 0 < t < 1 \\ t-1, & \text{if } t \geq 1, \end{cases}$$

if $X = 0$. Then ρ_X is the modulus of smoothness of X. The space is uniformly smooth if $\lim_{t \to 0^+} \rho_X(t)/t = 0$.

Theorem 5.2.1. *Following are the properties for the space $\mathcal{H}_n^\alpha$ defined in (5.7).*

(a) $\mathcal{H}_n^\alpha$ is a Chebyshev subset of $\mathcal{L}^2(S)$.

(b) $\mathcal{H}_n^\alpha$ is approximately compact in $\mathcal{L}^2(S)$.

(c) $\mathcal{H}_n^\alpha$ is smooth subspace of $\mathcal{L}^2(S)$. In fact, $\mathcal{H}_n^\alpha$ is uniformly smooth.

Proof.

(a) By definition, $\mathcal{H}_n^\alpha$ is a finite dimensional linear subspace of $\mathcal{L}^2(S)$. Consequently, by Lemma 5.2.1, it follows that $\mathcal{H}_n^\alpha$ is proximinal. But $\mathcal{L}^2(S)$ is a Hilbert space, so $\mathcal{H}_n^\alpha$ is Chebyshev.

(b) To prove $\mathcal{H}_n^\alpha$ is approximately compact, let $f \in \mathcal{L}^2(S)$ and $X_k^\alpha \in \mathcal{H}_n^\alpha, k = 1, 2, \ldots$ be a minimizing sequence for f. Then

$$\lim_{k\to\infty} \|X_k^\alpha - f\|_{\mathcal{L}^2(S)} = d(f, \mathcal{H}_n^\alpha)$$

and therefore there exists $K \in \mathbb{N}$ such that for all $k \geq K$, taking $\epsilon = 1$,

$$\|X_k^\alpha\|_{\mathcal{L}^2(S)} \leq \|X_k^\alpha - f\|_{\mathcal{L}^2(S)} + \|f\|_{\mathcal{L}^2(S)} \leq 1 + d(f, \mathcal{H}_n^\alpha) + \|f\|_{\mathcal{L}^2(S)} := C_1 \, .$$

Define $C_2 = \max\{\|X_1^\alpha\|_{\mathcal{L}^2(S)}, \|X_2^\alpha\|_{\mathcal{L}^2(S)}, \ldots, \|X_{K-1}^\alpha\|_{\mathcal{L}^2(S)}, C_1\}$. Since $\mathcal{H}_n^\alpha$ is finite dimensional, from the above inequality, it follows that X_k^α lies in a compact set defined by $\|g\|_{\mathcal{L}^2(S)} \leq C_2$. Therefore, by definition of compactness, there exists a subsequence $X_{k_l}^\alpha$ and $X^\alpha \in \mathcal{H}_n^\alpha$ such that $X_{k_l}^\alpha \to X^\alpha$ as $l \to \infty$. Therefore $\mathcal{H}_n^\alpha$ is approximately compact.
(c) Since $\mathcal{L}^2(S)$ is a smooth space and $\mathcal{H}_n^\alpha$ is a subspace of it, so is smooth. Since $\mathcal{H}_n^\alpha$ is finite dimensional smooth space, so $\mathcal{H}_n^\alpha$ is uniformly smooth. □

5.3 Schauder basis for $\mathcal{L}^2(S)$

In this section, a fractal Schauder basis of functions of first type is constructed for the functions on the unit sphere S.

Theorem 5.3.1. *(see [43]) (Vitali's completeness criterion). Let $\{\phi_n\}_{n=1}^\infty$ be an othonormal sequence of functions in $\mathcal{L}^2(a,b)$, where a and b are finite. Then (ϕ_n) is complete in $\mathcal{L}^2(a,b)$ if and only if*

$$\sum_n \left(\int_a^r \phi_n \right)^2 = r - a$$

for every $r \in (a,b)$.

The following is the modified form of the Vitali's criterion.

Corollary 5.3.1. *(see [43], p. 37). Let (a,b) be a finite or infinite interval of $\mathbb{R}$, let g belong to $\mathcal{L}_w^2(a,b)$, $g \neq w$, where w is a positive continuous weight function and let (ϕ_n) be an orthonormal sequence in $\mathcal{L}_w^2(a,b)$. Then (ϕ_n) is complete in $\mathcal{L}_w^2(a,b)$ (equivalently $(\phi_n\sqrt{w})$ is complete in $\mathcal{L}^2(a,b)$) if and only if*

$$\sum_n \left| \int_a^r \phi_n(x) g(x) w(x) dx \right|^2 = \int_a^r |g(x)|^2 w(x) dx$$

for every r in (a,b).

From here on, let us assume that for any non-negative integer $m = 0, 1, 2, \ldots$, the system

$$\mathcal{U}_m = \left\{ u_{nm} |\; n = p, p+1, p+2, \ldots \right\} \tag{5.8}$$

of functions is orthonormal, that is,

$$\int_{-1}^{1} u_{nm}(x) u_{jm}(x) dx = \delta_{nj} \tag{5.9}$$

and $\mathcal{U}_m$ forms a complete system in $\mathcal{L}^2(I)$, where p is the least integer such that $\frac{m}{2} \leq p$. Also assume that $\{v_m\}_{m=0}^{\infty}$ is a complete orthonormal system in $\mathcal{L}^2(0, 2\pi)$. Then by Vitali's completeness criterion (Theorem 5.3.1),

$$\sum_{m=0}^{\infty} \left(\int_0^{\theta} v_m(\theta') d\theta' \right)^2 = \theta \;\text{ for every } \theta \in (0, 2\pi). \tag{5.10}$$

Lemma 5.3.1. *For any non-negative integer m,*

$$\sum_{n=p}^{\infty} \left(\int_0^{\varphi} u_{nm}(\cos\varphi') \sin\varphi' d\varphi' \right)^2 = 1 - \cos\varphi \;\; \textit{for every } \varphi \in (0, \pi)\,.$$

Proof. For any non-negative integer m, the completeness of the system $\mathcal{U}_m$ in $\mathcal{L}^2(I)$ implies the completeness of the system $\left\{ u_{nm}(\cos\varphi) |\; n = p, p+1, p+2, \ldots \right\}$ in $\mathcal{L}^2(0, \pi)$. Then by modified Vitali's criterion (Corollary 5.3.1), taking $g = 1$, we get

$$\sum_{n=p}^{\infty} \left(\int_0^{\varphi} u_{nm}(\cos\varphi') \sin\varphi' d\varphi' \right)^2 = \int_0^{\varphi} 1^2 . \sin\varphi' d\varphi' = 1 - \cos\varphi\,.$$

□

The Vitali's completeness criterion for the functions on the sphere is as follows.

Lemma 5.3.2. *(see [43], p. 38). Let S denote the unit sphere with (φ, θ) as usual spherical polar coordinates. Let $\{f_n\}$ be a set of functions which are orthonormal over S, that is*

$$\int_S f_n f_m = \delta_{nm}\,.$$

The orthonormal sequence (f_n) *is complete in* $\mathcal{L}^2(S)$ *if and only if*

$$\sum_n \left[\int_0^{\theta} \int_0^{\varphi} f_n(\varphi', \theta') \sin\varphi' d\varphi' \right]^2 = \theta(1 - \cos\varphi)$$

for every $\theta \in (0, 2\pi)$ *and every* $\varphi \in (0, \pi)$.

Proof. See ([86], p.271). □

Theorem 5.3.2. *The family* $\left\{ H_{nm} : n = 0, 1, \ldots; m = 1, 2, \ldots, 2n \right\}$ *is a complete orthonormal system in* $\mathcal{L}^2(S)$.

Proof. Recall that $H_{nm}(\varphi, \theta) = u_{nm}(\cos\varphi) v_m(\theta)$. Then the family

$$\left\{ H_{nm} : n = 0, 1, \ldots, ; m = 1, 2, \ldots, 2n \right\}$$

is orthonormal due to Lemmas 5.1.1 and 5.1.2 and (5.9). Let $\Phi_n(S)$ be an orthonormal sequence of functions of the above system. Then

$$\begin{aligned} \sum_{n=0}^{\infty} \left[\int_0^{\theta} d\theta' \int_0^{\varphi} \Phi_n(\varphi', \theta') \sin\varphi' d\varphi' \right]^2 &= \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \left[\int_0^{\theta} v_m(\theta') d\theta' \int_0^{\varphi} u_{nm}(\cos\varphi') \sin\varphi' d\varphi' \right]^2 \\ &= \sum_{m=0}^{\infty} \left[\int_0^{\theta} v_m(\theta') d\theta' \right]^2 \sum_{n=p}^{\infty} \left[\int_0^{\varphi} u_{nm}(\cos\varphi') \sin\varphi' d\varphi' \right]^2, \end{aligned}$$

inverting the order of integration on the right hand side of the last inequality. Using Lemma 5.3.1, along with (5.10)

$$\sum_{n=0}^{\infty} \left[\int_0^{\theta} d\theta' \int_0^{\varphi} \Phi_n(\varphi', \theta') \sin\varphi' d\varphi' \right]^2 = \theta(1 - \cos\varphi) .$$

Therefore by Lemma 5.3.2, it follows that $\left\{ H_{nm} : n = 0, 1, \ldots, ; m = 1, 2, \ldots, 2n \right\}$ forms a complete system in $\mathcal{L}^2(S)$. □

The following lemma is required to prove the main results.

Lemma 5.3.3. *(see [52]). Let* D *be a dense subset of a Banach space* $\mathcal{B}$. *Then each element* $u \in \mathcal{B}$, $u \neq 0$ *can be expressed as*

$$u = \sum_{j=1}^{\infty} u_j ,$$

where $u_j \in D$ *and* $\|u_j\| \leq \frac{3\|u\|}{2^j}$.

The following theorem gives an expression of a square integrable function on the sphere in terms of fractal functions of first type.

Theorem 5.3.3. *The operator* $\Theta^\alpha : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ *defined by*

$$\Theta^\alpha(f) = \sum_{n=0}^{\infty}\sum_{m=0}^{2n} \lambda_{nm} H^\alpha_{nm} \ ,$$

where

$$f = \sum_{n=0}^{\infty}\sum_{m=0}^{2n} \lambda_{nm} H_{nm}$$

is well defined, linear and bounded.

Proof. By Theorem 5.3.2, the family

$$\Big\{ H_{nm} : n = 0, 1, \ldots, ; m = 1, 2, \ldots, 2n \Big\}$$

is complete in $\mathcal{L}^2(S)$. The set

$$\mathcal{H} = \bigcup_{n=0}^{\infty} \mathcal{H}_n$$

is dense in $\mathcal{L}^2(S)$. Then for any $f \in \mathcal{L}^2(S)$, $f \neq 0$,

$$f = \sum_{k=1}^{\infty} X_k \tag{5.11}$$

so that $X_k \in \mathcal{H}_{n_k}$ and

$$\|X_k\|_{\mathcal{L}^2(S)} \leq \frac{3\|f\|_{\mathcal{L}^2(S)}}{2^k} \ .$$

Now define

$$\Theta^\alpha(f) = \sum_{k=1}^{\infty} \Theta^\alpha_{n_k}(X_k) \ ,$$

where $\Theta^\alpha_n : \mathcal{H}_n \to \mathcal{L}^2(S)$ is the operator defined in Theorem 5.1.1 such that $\|\Theta^\alpha_n\|_2 \leq \|\mathcal{F}^\alpha\|_2$. It is noted that

$$\begin{aligned} \|\Theta^\alpha_{n_k}(X_k)\|_{\mathcal{L}^2(S)} &\leq \|\Theta^\alpha_{n_k}\|_2 \|X_k\|_{\mathcal{L}^2(S)} \\ &\leq \|\mathcal{F}^\alpha\|_2 \frac{3\|f\|_{\mathcal{L}^2(S)}}{2^k}. \end{aligned}$$

Therefore, it easily follows that $\sum_{k=1}^{\infty} \Theta^{\alpha}_{n_k}(X_k)$ is absolutely convergent and hence convergent in a Banach space. As a consequences Θ^{α} is well defined and

$$\begin{aligned}
\|\Theta^{\alpha}(f)\|_{\mathcal{L}^2(S)} &\leq \sum_{k=1}^{\infty} \|\Theta^{\alpha}_{n_k}(X_k)\|_{\mathcal{L}^2(S)} \\
&\leq \|\mathcal{F}^{\alpha}\|_2 \sum_{k=1}^{\infty} \frac{3\|f\|_{\mathcal{L}^2(S)}}{2^k} \\
&= 3\|\mathcal{F}^{\alpha}\|_2 \|f\|_{\mathcal{L}^2(S)} \quad \text{for all } f \in \mathcal{L}^2(S).
\end{aligned}$$

Therefore,

$$\|\Theta^{\alpha}\|_2 \leq 3\|\mathcal{F}^{\alpha}\|_2 \, .$$

To show that the restriction of Θ^{α} to $\mathcal{H}_n$ is Θ^{α}_n, it is noted that for $X \in \mathcal{H}_n$, the sum (5.11) consists only a term of $\mathcal{H}_n$ and consequently

$$\Theta^{\alpha}(X) = \Theta^{\alpha}_n(X) \, .$$

Therefore, $\Theta^{\alpha}|_{\mathcal{H}_n} = \Theta^{\alpha}_n$. The linearity and boundedness of Θ^{α} implies that

$$\Theta^{\alpha}(f) = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \lambda_{nm} H^{\alpha}_{nm}$$

whenever

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \lambda_{nm} H_{nm} \, .$$

□

Now, consider the linear bounded operator $L : \mathcal{C}(I) \to \mathcal{C}(I)$ with respect to mean-square norm. Then in a similar method employed for Θ^{α}, the operator can be extended to $\mathcal{L}^2(S)$. Let L_H be defined on the basis elements as

$$L_H(H_{nm})(\varphi, \theta) = L(u_{nm})(\cos\varphi)v_m(\theta) = (u_{nm} \circ c)(\cos\varphi)v_m(\theta). \tag{5.12}$$

Then by linearity, it can be extended to $L_H : \mathcal{H}_n \to \mathcal{L}^2(S)$ such that $\|L_H\|_2 \leq \|L\|_2$. As in Theorem 5.3.3, the operator L_H can be extended to $L_S : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ such that $\|L_S\|_2 \leq 3\|L\|_2 \leq 3M^{1/2}$.

Lemma 5.3.4. *The operator $\Theta^\alpha : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ satisfies, for all n and m,*

$$\|\Theta^\alpha(H_{nm}) - H_{nm}\|_{\mathcal{L}^2(S)} \leq |\alpha|_\infty \|\Theta^\alpha(H_{nm}) - L_S(H_{nm})\|_{\mathcal{L}^2(S)} .$$

Proof. By definition,

$$\Theta^\alpha(H_{nm}) = \Theta^\alpha_n(H_{nm}) = H^\alpha_{nm} .$$

Therefore,

$$\begin{aligned}
\|\Theta^\alpha(H_{nm}) - H_{nm}\|^2_{\mathcal{L}^2(S)} &= \|H^\alpha_{nm} - H_{nm}\|^2_{\mathcal{L}^2(S)} \\
&= \int_0^{2\pi}\int_0^{\pi} |(u^\alpha_{nm} - u_{nm})(\cos\varphi)|^2 v^2_m(\theta) \sin\varphi d\varphi d\theta \\
&= \left(\int_0^{2\pi} v^2_m d\theta\right)\left(\int_{-1}^{1} |(u^\alpha_{nm} - u_{nm})(t)|^2 dt\right) \\
&= \|u^\alpha_{nm} - u_{nm}\|^2_{\mathcal{L}^2(I)}.
\end{aligned}$$

According to (1.11), for $f = u_{nm}$ and $b = Lf$,

$$\|u^\alpha_{nm} - u_{nm}\|^2_{\mathcal{L}^2(I)} = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} |\alpha_i|^2 \big|\big(u^\alpha_{nm} - Lu_{nm}\big) \circ L_i^{-1}(x)\big|^2 dx .$$

By changing of variable $\widetilde{x} = L_i^{-1}(x)$, it follows that

$$\begin{aligned}
\|u^\alpha_{nm} - u_{nm}\|^2_{\mathcal{L}^2(I)} &= \sum_{i=1}^{N} a_i |\alpha_i|^2 \int_I \big|\big(u^\alpha_{nm} - Lu_{nm}\big)(\widetilde{x})\big|^2 d\widetilde{x} \\
&= \sum_{i=1}^{N} a_i |\alpha_i|^2 \|u^\alpha_{nm} - Lu_{nm}\|^2_{\mathcal{L}^2(I)} \\
&\leq |\alpha|^2_\infty \|u^\alpha_{nm} - Lu_{nm}\|^2_{\mathcal{L}^2(I)} \sum_{i=1}^{N} a_i \\
&= |\alpha|^2_\infty \|u^\alpha_{nm} - Lu_{nm}\|^2_{\mathcal{L}^2(I)},
\end{aligned}$$

since $\sum_{i=1}^{N} a_i = 1$. Therefore, it follows that

$$\|\Theta^\alpha(H_{nm}) - H_{nm}\|_{\mathcal{L}^2(S)} \leq |\alpha|_\infty \|u^\alpha_{nm} - Lu_{nm}\|_{\mathcal{L}^2(I)} .$$

But

$$\begin{aligned}\|\Theta^\alpha(H_{nm}) - L_S H_{nm}\|^2_{\mathcal{L}^2(S)} &= \|H^\alpha_{nm} - L_S(H_{nm})\|^2_{\mathcal{L}^2(S)} \\ &= \int_0^{2\pi}\int_0^{\pi} |(u^\alpha_{nm} - Lu_{nm})(\cos\varphi)|^2 v_m^2(\theta)\sin\varphi d\varphi d\theta \\ &= \left(\int_0^{2\pi} v_m^2 d\theta\right)\left(\int_{-1}^{1} |(u^\alpha_{nm} - Lu_{nm})(t)|^2 dt\right) \\ &= \|u^\alpha_{nm} - Lu_{nm}\|^2_{\mathcal{L}^2(I)}.\end{aligned}$$

Therefore

$$\|\Theta^\alpha(H_{nm}) - H_{nm}\|_{\mathcal{L}^2(S)} \leq |\alpha|_\infty \|\Theta^\alpha(H_{nm}) - L_S H_{nm}\|_{\mathcal{L}^2(S)}\ .$$

□

Proposition 5.3.1. *For all* $f \in \mathcal{L}^2(S)$,

$$\|\Theta^\alpha(f) - f\|_{\mathcal{L}^2(S)} \leq |\alpha|_\infty \|\Theta^\alpha(f) - L_S f\|_{\mathcal{L}^2(S)}$$

and

$$\|\Theta^\alpha(f) - f\|_{\mathcal{L}^2(S)} \leq \frac{|\alpha|_\infty}{1-|\alpha|_\infty}(1+3M^{1/2})\|f\|_{\mathcal{L}^2(S)}. \tag{5.13}$$

Proof. Let $X \in \mathcal{H}_n$. Then,

$$X = \sum_{m=0}^{2n} \lambda_{nm} H_{nm}\ ,$$

as $\{H_{nm}\}_{m=0}^{2n}$ is an orthogonal basis for $\mathcal{H}_n$. It is noted that $\{H^\alpha_{nm} - H_{nm}\}_{m=0}^{2n}$ is also orthogonal (similar to Lemma 5.1.1). For instance

$$\begin{aligned}\langle H^\alpha_{nm} - H_{nm}, H^\alpha_{nj} - H_{nj}\rangle &= \int_0^{2\pi}\int_0^{\pi} \big(u^\alpha_{nm}(\cos\varphi) - u_{nm}(\cos\varphi)\big)\big(u^\alpha_{nj}(\cos\varphi) - u_{nj}(\cos\varphi)\big) \\ &\quad v_m(\theta)v_j(\theta)\sin\varphi d\varphi d\theta \\ &= \langle u^\alpha_{nm} - u_{nm}, u^\alpha_{nj} - u_{nj}\rangle\left(\int_0^{2\pi} v_m(\theta)v_j(\theta)d\theta\right) \\ &= 0,\end{aligned}$$

since the second integral is zero if $m \neq j$. Therefore,

$$\begin{aligned}\|\Theta^\alpha(X) - X\|^2_{\mathcal{L}^2(S)} &= \sum_{m=0}^{2n} |\lambda_{nm}|^2 \|H^\alpha_{nm} - H_{nm}\|^2_{\mathcal{L}^2(S)} \\ &\leq |\alpha|^2_\infty \sum_{m=0}^{2n} |\lambda_{nm}|^2 \|H^\alpha_{nm} - L_S H_{nm}\|^2_{\mathcal{L}^2(S)},\end{aligned}$$

using Lemma 5.3.4. The system $\left\{H^{\alpha}_{nm} - L_S H_{nm}\right\}_{m=0}^{2n}$ is also orthogonal (similar as above) and therefore

$$\|\Theta^{\alpha}(X) - X\|^2_{\mathcal{L}^2(S)} \leq |\alpha|^2_{\infty}\|\Theta^{\alpha}(X) - L_S X\|^2_{\mathcal{L}^2(S)}. \tag{5.14}$$

For any $f \in \mathcal{L}^2(S)$, there exists a sequence $\{X_k \in \mathcal{H}_{n_k}\}$ such that $\lim_{k\to\infty}\|X_k - f\|_{\mathcal{L}^2(S)} = 0$. Then the continuity of L_S implies that $\lim_{k\to\infty}\|L_S X_k - L_S f\|_{\mathcal{L}^2(S)} = 0$. Using the continuity of Θ^{α} and the norm,

$$\|\Theta^{\alpha}(f) - f\|^2_{\mathcal{L}^2(S)} = \lim_{k\to\infty}\|\Theta^{\alpha}(X_k) - X_k\|^2_{\mathcal{L}^2(S)} .$$

Therefore using (5.14), for $X = X_k$,

$$\begin{aligned}\|\Theta^{\alpha}(f) - f\|^2_{\mathcal{L}^2(S)} &= \lim_{k\to\infty}\|\Theta^{\alpha}(X_k) - X_k\|^2_{\mathcal{L}^2(S)}\\ &\leq |\alpha|^2_{\infty}\lim_{k\to\infty}\|\Theta^{\alpha}(X_k) - L_S X_k\|^2_{\mathcal{L}^2(S)}\\ &= |\alpha|^2_{\infty}\|\Theta^{\alpha}(f) - L_S f\|^2_{\mathcal{L}^2(S)}\end{aligned}$$

and

$$\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} \leq |\alpha|_{\infty}\|\Theta^{\alpha}(f) - L_S f\|_{\mathcal{L}^2(S)} .$$

Moreover,

$$\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} \leq |\alpha|_{\infty}\Big(\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} + \|f - L_S f\|_{\mathcal{L}^2(S)}\Big) .$$

Since $\|L_S\|_2 \leq 3M^{1/2}$, it follows that

$$\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} \leq \frac{|\alpha|_{\infty}}{1 - |\alpha|_{\infty}}(1 + 3M^{1/2})\|f\|_{\mathcal{L}^2(S)} .$$

□

Theorem 5.3.4. *If $|\alpha|_{\infty} < \frac{M^{-1/2}}{3}$, then the operator Θ^{α} is injective and its range is closed.*

Proof. According to Proposition 5.3.1,

$$\|f\|_{\mathcal{L}^2(S)} - \|\Theta^{\alpha}(f)\|_{\mathcal{L}^2(S)} \leq |\alpha|_{\infty}\|\Theta^{\alpha}(f) - L_S f\|_{\mathcal{L}^2(S)}. \tag{5.15}$$

Since $\|L_S\|_2 \leq 3M^{1/2}$,

$$\|\Theta^\alpha(f) - L_S f\|_{\mathcal{L}^2(S)} \leq \|\Theta^\alpha(f)\|_{\mathcal{L}^2(S)} + 3M^{1/2}\|f\|_{\mathcal{L}^2(S)}. \tag{5.16}$$

Equations (5.15) and (5.16) together give

$$\|\Theta^\alpha(f)\|_{\mathcal{L}^2(S)} \geq \|f\|_{\mathcal{L}^2(S)} - |\alpha|_\infty \|\Theta^\alpha(f)\|_{\mathcal{L}^2(S)} - |\alpha|_\infty 3M^{1/2}\|f\|_{\mathcal{L}^2(S)}.$$

For the assumption $|\alpha|_\infty < \frac{M^{-1/2}}{3}$, it happens that

$$1 - |\alpha|_\infty 3M^{1/2} > 0 \,.$$

Therefore, from above inequality, it follows that

$$\|f\|_{\mathcal{L}^2(S)} \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty 3M^{1/2}} \|\Theta^\alpha(f)\|_{\mathcal{L}^2(S)}. \tag{5.17}$$

If $\Theta^\alpha(f) = 0$, (5.17) gives $f = 0$. Consequently, Θ^α is injective. To show the range of Θ^α is closed, consider a convergent sequence $\{\Theta^\alpha(f_n)\}$ such that $\Theta^\alpha(f_n) \to g^\alpha$. Since the sequence $\{\Theta^\alpha(f_n)\}$ is convergent, it is also Cauchy and therefore according to (5.17), $\{f_n\}$ is also a Cauchy sequence. As a consequence, f_n is convergent in a Banach space. If $f_n \to g$ as $n \to \infty$, then the continuity of Θ^α implies that

$$\Theta^\alpha(g) = \lim_{n\to\infty} \Theta^\alpha(f_n) = g^\alpha \,.$$

Therefore g^α belongs to range of Θ^α and hence range of Θ^α is closed. □

Corollary 5.3.2. *If* $|\alpha|_\infty < \frac{M^{-1/2}}{3}$,

$$\mathcal{L}^2(S) = Range(\Theta^\alpha) \bigoplus ker(\Theta^\alpha)^* \,,$$

where $(\Theta^\alpha)^*$ *is the adjoint operator of* Θ^α *and* $Range(\Theta^\alpha)$ *denotes the range of* Θ^α.

Proof. Since Θ^α is a linear and bounded operator on the Hilbert space $\mathcal{L}^2(S)$, the following orthogonal decomposition holds

$$\mathcal{L}^2(S) = \overline{Range(\Theta^\alpha)} \bigoplus ker(\Theta^\alpha)^* \,.$$

But range of Θ^α is closed by above theorem and hence the result follows. □

Proposition 5.3.2. *If $|\alpha|_\infty < \frac{M^{-1/2}}{3}$, then the range of the adjoint operator $(\Theta^\alpha)^*$ is dense in $\mathcal{L}^2(S)$.*

Proof. We have,

$$\mathcal{L}^2(S) = \overline{Range(\Theta^\alpha)^*} \bigoplus ker(\Theta^\alpha) \; .$$

For $|\alpha|_\infty < \frac{M^{-1/2}}{3}$, according to Theorem 5.3.4, Θ^α is injective. Therefore $ker(\Theta^\alpha) = \{0\}$ and the result follows. □

The following lemma can be found in [52].

Lemma 5.3.5. *If L is a linear operator from a Banach space into itself such that $\|L\| < 1$, then $(Id - L)^{-1}$ exists and bounded. Moreover,*

$$(Id - L)^{-1} = Id + L + L^2 + \cdots \; .$$

Theorem 5.3.5. *If $|\alpha|_\infty < \frac{1}{2+3M^{1/2}}$, the operator $\Theta^\alpha : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ has a bounded inverse.*

Proof. By hypothesis, $|\alpha|_\infty < \frac{1}{2+3M^{1/2}}$, which implies

$$\frac{(1 + 3M^{1/2})|\alpha|_\infty}{1 - |\alpha|_\infty} < 1 \; .$$

From (5.13), it follows that

$$\|Id - \Theta^\alpha\|_2 \leq \frac{(1 + 3M^{1/2})|\alpha|_\infty}{1 - |\alpha|_\infty}$$

and therefore $\|Id - \Theta^\alpha\|_2 < 1$. For $\|Id - \Theta^\alpha\|_2 < 1$, Lemma 5.3.5 ensures that $\Theta^\alpha = Id - (Id - \Theta^\alpha)$ has a bounded inverse and by (5.17),

$$\|(\Theta^\alpha)^{-1}\|_2 \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty 3M^{1/2}} \; .$$

□

Proposition 5.3.3. *If $|\alpha|_\infty < \frac{1}{2+3M^{1/2}}$, the adjoint operator $(\Theta^\alpha)^*$ is injective.*

Proof. For $|\alpha|_\infty < \frac{1}{2+3M^{1/2}}$, according to Theorem 5.3.5, $Range(\Theta^\alpha) = \mathcal{L}^2(S)$. Since

$$|\alpha|_\infty < \frac{1}{2+3M^{1/2}} < \frac{1}{3M^{1/2}} ,$$

from Corollary 5.3.2, it follows that $ker(\Theta^\alpha)^* = \{0\}$ and therefore $(\Theta^\alpha)^*$ is injective. □

Using Proposition 4.2.1, it is proved in the following that $\mathcal{H}^\alpha$ is complete in $\mathcal{L}^2(S)$.

Proposition 5.3.4. *If* $|\alpha|_\infty < \frac{1}{2+3M^{1/2}}$, *the system*

$$\mathcal{H}^\alpha = \left\{ H^\alpha_{nm} : n = 0, 1, 2, \ldots ; m = 1, 2, \ldots , n \right\}$$

is complete in $\mathcal{L}^2(S)$.

Proof. According to Theorem 5.3.5, for any $g \in \mathcal{L}^2(S)$, there exists $f \in \mathcal{L}^2(S)$ such that $\Theta^\alpha(f) = g$. The linearity and continuity of Θ^α imply that

$$g = \Theta^\alpha(f) = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \lambda_{nm} H^\alpha_{nm}$$

whenever

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \lambda_{nm} H_{nm} .$$

Therefore the system

$$\left\{ H^\alpha_{nm} : n = 0, 1, 2, \ldots ; m = 1, 2, \ldots , n \right\}$$

is total and hence by Proposition 4.2.1, it is complete in $\mathcal{L}^2(S)$. □

5.4 Fractal functions of second type on the sphere

In Section 5.1, we considered fractal functions of first type on the sphere of the form given in (5.2) and the spaces $\mathcal{H}_n$ spanned by the n-th row of Table 5.1. The orthogonality of $\{H_{nm}\}_{m=0}^{2n}$ and $\{H^\alpha_{nm}\}_{m=0}^{2n}$ played an important role to prove the boundedness of the operator Θ^α_n (see Theorem 5.1.1) and in proving other results.

In this section, we will fractalize the function $H_{nm}(\varphi, \theta)$ as employed in (5.3). Now, it is not known that whether the system $\{v_m^\alpha\}_{m=0}^\infty$ is orthogonal or not. So to get an orthogonal basis of functions on the sphere and orthogonal basis of their fractal analogues, we will change the settings of Section 5.1 as follows.

Let us consider a family of continuous functions

$$u_{nm} : J = [-1, 1] \to \mathbb{R} ,$$

such that for any non-negative integer m, the system of functions

$$\mathcal{U}_m = \{u_{nm}; n = p, p+1, \dots\} \tag{5.18}$$

forms a orthogonal system in $\mathcal{C}(J)$ with respect to the inner product

$$\langle g, h \rangle = \int_{-1}^{1} g(t)h(t)dt ,$$

where p is the least integer such that

$$\frac{m}{2} \leq p .$$

Let

$$\{v_m : I = [0, 2\pi] \to \mathbb{R} \mid v_m \text{ is continuous and periodic}, \; m = 0, 1, 2, \dots\}$$

be an orthonormal system in $\mathcal{C}(I)$ with respect to the inner product

$$\langle g, h \rangle = \int_{0}^{2\pi} g(t)h(t)dt .$$

Recall that $P = (\varphi, \theta)$ represents a point on the unit sphere S. Let us define functions on the sphere S as

$$H_{nm}(\varphi, \theta) = u_{nm}(\cos\varphi)v_m(\theta), \quad (n = p, p+1, p+2, \dots; m = 0, 1, 2, \dots,) .$$

Lemma 5.4.1. *For any non-negative integer m,*

$$\langle H_{nm}, H_{rm} \rangle = 0, \quad (n, r = p, p+1, p+2, \dots; n \neq r) .$$

Proof. The inner product can be expressed in spherical coordinates as

$$\begin{aligned}\langle H_{nm}, H_{rm}\rangle &= \int_0^{2\pi}\int_0^{\pi} u_{nm}(\cos\varphi)u_{rm}(\cos\varphi)|v_m(\theta)|^2 \sin\varphi d\varphi d\theta \\ &= \left(\int_0^{2\pi} |v_m(\theta)|^2 d\theta\right)\left(\int_0^{\pi} u_{nm}(\cos\varphi)u_{rm}(\cos\varphi)\sin\varphi d\varphi\right).\end{aligned}$$

Since the second integral is zero if $n \neq r$, it follows that $\langle H_{nm}, H_{rm}\rangle = 0$. □

Lemma 5.4.2. *For* $m = 0, 1, 2, \ldots,$

$$\|H_{nm}\|_{\mathcal{L}^2(S)} = \|u_{nm}\|_{\mathcal{L}^2(J)}, \quad (n = p, p+1, p+2, \ldots)\ .$$

Proof. Proof is similar to the proof of Lemma 5.1.2. □

	H_{00}									
	H_{10}	H_{11}	H_{12}							
	H_{20}	H_{21}	H_{22}	H_{23}	H_{24}					
	H_{30}	H_{31}	H_{32}	H_{33}	H_{34}	H_{35}	H_{36}			
$\downarrow$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	H_{n0}	H_{n1}	H_{n2}	H_{n3}	H_{n4}	H_{n5}	H_{n6}	$\ldots$	$H_{n(2n+1)}$	
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Table 5.2: The space $\mathcal{H}_m$.

Let us define

$$\mathcal{H}_m^j = \left\{\sum_{n=p}^{j} \lambda_{nm} H_{nm}; \lambda_{nm} \in \mathbb{R}\right\} \quad \text{(see Table 5.2)}\ ,$$

where p is the least integer such that $\frac{m}{2} \leq p$. Note that

$$\mathcal{H}_m^j = \emptyset \text{ if } p > j\ .$$

Due to Lemma 5.4.1, $\{H_{nm}\}_{n=0}^{j}$ is an orthogonal basis for $\mathcal{H}_m^j$. Define

$$\mathcal{H}_m = \overline{\bigcup_{j=0}^{\infty} \mathcal{H}_m^j} = \overline{span\{H_{nm} : n = p, p+1, \ldots\}}\ .$$

If $\|u_{nm}\|_{\mathcal{L}^2(J)} = 1$, for all n, m, then $\{H_{nm}\}_{n=p}^{\infty}$ is an orthonormal Schauder basis for $\mathcal{H}_m$. Consider the base function given in (1.10) as

$$b = Lf \ ,$$

where $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is a linear and bounded operator such that $Lf(x_0) = f(x_0)$ and $Lf(x_N) = f(x_N)$ if $\mathcal{C}(I)$ is equipped with sup norm or mean-square norm. Then define

$$H_{nm}^{\alpha}(\varphi, \theta) = u_{nm}(\cos\varphi) v_m^{\alpha}(\theta) \ ,$$

where $v_m^{\alpha}(\theta) = \mathcal{F}^{\alpha}(v_m(\theta))$ and $\mathcal{F}^{\alpha}$ is the fractal operator defined in (1.13) (c.f. Section 1.4). Now consider,

$$(\mathcal{H}_m^j)^{\alpha} = \left\{ \sum_{n=p}^{j} \lambda_{nm} H_{nm}^{\alpha}; \lambda_{nm} \in \mathbb{R} \right\} .$$

Then $\{H_{nm}^{\alpha}\}_{n=p}^{j}$ is an orthogonal basis for $(\mathcal{H}_m^j)^{\alpha}$. For instance,

$$\begin{aligned} \langle H_{nm}^{\alpha}(\varphi,\theta), H_{rm}^{\alpha}(\varphi,\theta)\rangle &= \int_0^{2\pi}\int_0^{\pi} u_{nm}(\cos\varphi) u_{rm}(\cos\varphi) |v_m^{\alpha}(\theta)|^2 \sin\varphi d\varphi d\theta \\ &= \left(\int_0^{\pi} u_{nm}(\cos\varphi) u_{rm}(\cos\varphi)\sin\varphi d\varphi\right)\left(\int_0^{2\pi} |v_m^{\alpha}(\theta)|^2 d\theta\right) \\ &= 0, \end{aligned}$$

since $\langle u_{nm}, u_{rm}\rangle = 0$ for $n \neq r$.

Lemma 5.4.3. *For $n = p, p+1, p+2, \ldots$ and $m = 0, 1, 2, \ldots$,*

$$\|H_{nm}^{\alpha}\|_{\mathcal{L}^2(S)} = \|u_{nm}\|_{\mathcal{L}^2(J)} \|v_m^{\alpha}\|_{\mathcal{L}^2(I)} \leq \|\mathcal{F}^{\alpha}\|_2 \|H_{nm}\|_{\mathcal{L}^2(S)} \ .$$

Proof. For instance,

$$\begin{aligned} \|H_{nm}^{\alpha}\|_{\mathcal{L}^2(S)}^2 &= \int_0^{2\pi}\int_0^{\pi} |u_{nm}(\cos\varphi)|^2 |v_m^{\alpha}(\theta)|^2 \sin\varphi d\varphi d\theta \\ &= \left(\int_{-1}^{1} |u_{nm}(t)|^2 dt\right)\left(\int_0^{2\pi} |v_m^{\alpha}(\theta)|^2 d\theta\right) \\ &= \|u_{nm}\|_{\mathcal{L}^2(J)}^2 \|v_m^{\alpha}\|_{\mathcal{L}^2(I)}^2 \\ &\leq \|u_{nm}\|_{\mathcal{L}^2(J)}^2 \|\mathcal{F}^{\alpha}\|_2^2 \|v_m\|_{\mathcal{L}^2(I)}^2. \end{aligned}$$

Since $\{v_m\}_{m=0}^{\infty}$ is an orthonormal family, it follows that

$$\|H_{nm}^{\alpha}\|_{\mathcal{L}^2(S)} \leq \|\mathcal{F}^{\alpha}\|_2 \|H_{nm}\|_{\mathcal{L}^2(S)} .$$

□

Theorem 5.4.1. *The operator*

$$(\Theta_m^j)^{\alpha} : \mathcal{H}_m^j \to \mathcal{L}^2(S) ,$$

$$\sum_{n=p}^{j} \lambda_{nm} H_{nm} \hookrightarrow \sum_{n=p}^{j} \lambda_{nm} H_{nm}^{\alpha}$$

is linear and bounded.

Proof. The linearity of $(\Theta_m^j)^{\alpha}$ is obvious. For boundedness, due to the orthogonality of $\{H_{nm}\}_{n=p}^{j}$ and $\{H_{nm}^{\alpha}\}_{n=p}^{j}$, we get,

$$\begin{aligned} \big\|(\Theta_m^j)^{\alpha}\big(\sum_{n=p}^{j} \lambda_{nm} H_{nm}\big)\big\|_{\mathcal{L}^2(S)}^2 &= \sum_{n=p}^{j} \|\lambda_{nm} H_{nm}^{\alpha}\|^2 \\ &\leq \sum_{n=p}^{j} \|\mathcal{F}^{\alpha}\|_2^2 \|\lambda_{nm} H_{nm}\|^2 \\ &\leq \|\mathcal{F}^{\alpha}\|_2^2 \big\|\sum_{n=p}^{j} \lambda_{nm} H_{nm}\big\|^2. \end{aligned}$$

Consequently, $\|(\Theta_m^j)^{\alpha}\|_2 \leq \|\mathcal{F}^{\alpha}\|_2$. □

From here on, let us assume that for any non-negative integer $m = 0, 1, 2, \ldots$, the system $\mathcal{U}_m$ given in (5.18) is orthonormal, that is,

$$\int_{-1}^{1} u_{nm}(x) u_{jm}(x) dx = \delta_{nj}$$

and $\mathcal{U}_m$ forms a complete system in $\mathcal{L}^2(J)$, where p is the least integer such that $\frac{m}{2} \leq p$. Also assume that $\{v_m\}_{m=0}^{\infty}$ is a complete orthonormal system in $\mathcal{L}^2(0, 2\pi)$. Then the following two results are true.

Theorem 5.4.2. *The family*

$$\Big\{H_{nm}: \; n = p, p+1, p+2, \ldots ; m = 0, 1, 2, \ldots, \Big\}$$

forms an orthonormal complete system of $\mathcal{L}^2(S)$.

Proof. It can be easily seen from Table 5.1 or Table 5.2 that

$$\Big\{H_{nm}: \; n = p, p{+}1, p{+}2, \ldots ; m = 0, 1, 2, \ldots, \Big\} = \Big\{H_{nm}: \; n = 0, 1, 2, \ldots ; m = 0, 1, 2, \ldots, 2n\Big\}.$$

Therefore proof follows from Theorem 5.3.2. □

Theorem 5.4.3. *For any* $f \in \mathcal{L}^2(S)$*, the operator* $\Theta^\alpha : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ *defined by*

$$\Theta^\alpha(f) = \sum_{m=0}^{\infty} \sum_{n=p}^{\infty} c_{nm} H_{nm}^\alpha \, ,$$

where

$$f = \sum_{m=0}^{\infty} \sum_{n=p}^{\infty} c_{nm} H_{nm}$$

is linear and bounded. Its norm satisfies the following inequality

$$\|\Theta^\alpha\|_2 \leq 3\|\mathcal{F}^\alpha\|_2$$

Proof. Note that

$$\sum_{m=0}^{\infty} \sum_{n=p}^{\infty} c_{nm} H_{nm}^\alpha = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} c_{nm} H_{nm}^\alpha \, .$$

Then proof follows in similar lines of Theorem 5.3.3. □

5.5 Fractal basis for $\mathcal{L}^2(S)$

In this section, we will consider the linear bounded operator L on $\mathcal{C}(I)$ with respect to mean-square norm. The operator is extended to $\mathcal{L}^2(S)$ in a similar method employed for $\mathcal{F}^\alpha$. Let L_H be defined on the basis elements as

$$L_H(H_{nm})(\varphi, \theta) = u_{nm}(\cos\varphi) L(v_m(\theta)). \tag{5.19}$$

Then by linearity, it can be extended to $L_H : \mathcal{H}_m^j \to \mathcal{L}^2(S)$ such that $\|L_H\|_2 \leq \|L\|_2$. As in Theorem 5.4.3, the operator L_H can be extended to $L_S : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ such that $\|L_S\|_2 \leq 3\|L\|_2$.

Lemma 5.5.1. *The operator $\Theta^\alpha : \mathcal{L}^2(S) \to \mathcal{L}^2(S)$ satisfies, for all n and m,*

$$\|\Theta^\alpha(H_{nm}) - H_{nm}\|_{\mathcal{L}^2(S)} \leq |\alpha|_\infty \|\Theta^\alpha(H_{nm}) - L_S(H_{nm})\|_{\mathcal{L}^2(S)} .$$

Proof. Note that

$$\Theta^\alpha(H_{nm}) = H_{nm}^\alpha .$$

Therefore

$$\begin{aligned}
\|\Theta^\alpha(H_{nm}) - H_{nm}\|^2_{\mathcal{L}^2(S)} &= \|H_{nm}^\alpha - H_{nm}\|^2_{\mathcal{L}^2(S)} \\
&= \int_0^{2\pi}\int_0^{\pi} u_{nm}^2(\cos\varphi)|v_m^\alpha(\theta) - v_m(\theta)|^2 \sin\varphi d\varphi d\theta \\
&= \|u_{nm}\|^2_{\mathcal{L}^2(J)} \|v_m^\alpha - v_m\|^2_{\mathcal{L}^2(I)}.
\end{aligned}$$

But from (1.11), for $f = v_m$ and $b = Lf$,

$$\|v_m^\alpha - v_m\|^2_{\mathcal{L}^2(I)} = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} |\alpha_i|^2 |(v_m^\alpha - Lv_m) \circ L_i^{-1}(x)|^2 dx .$$

By changing of variable $\widetilde{x} = L_i^{-1}(x)$, it follows that

$$\begin{aligned}
\|v_m^\alpha - v_m\|^2_{\mathcal{L}^2(I)} &= \sum_{i=1}^N a_i|\alpha_i|^2 \int_0^{2\pi} |(v_m^\alpha - Lv_m)(\widetilde{x})|^2 d\widetilde{x} \\
&= \sum_{i=1}^N a_i|\alpha_i|^2 \|v_m^\alpha - Lv_m\|^2_{\mathcal{L}^2(I)} \\
&\leq |\alpha|^2_\infty \|v_m^\alpha - Lv_m\|^2_{\mathcal{L}^2(I)} \sum_{i=1}^N a_i \\
&= |\alpha|^2_\infty \|v_m^\alpha - Lv_m\|^2_{\mathcal{L}^2(I)},
\end{aligned}$$

since $\sum_{i=1}^N a_i = 1$. Therefore

$$\|\Theta^\alpha(H_{nm}) - H_{nm}\|^2_{\mathcal{L}^2(S)} \leq |\alpha|^2_\infty \|u_{nm}\|^2_{\mathcal{L}^2(I)} \|v_m^\alpha - Lv_m\|^2_{\mathcal{L}^2(I)} .$$

On the other hand,

$$\begin{aligned}
\|\Theta^\alpha(H_{nm}) - L_S(H_{nm})\|_{\mathcal{L}^2(S)} &= \|H^\alpha_{nm} - L_S(H_{nm})\|_{\mathcal{L}^2(S)} \\
&= \int_0^{2\pi}\int_0^{\pi} u^2_{nm}(\cos\varphi)|v^\alpha_m(\theta) - L(v_m(\theta))|\sin\varphi d\varphi d\theta \\
&= \|u_{nm}\|^2_{\mathcal{L}^2(I)}\|v^\alpha_m - Lv_m\|^2_{\mathcal{L}^2(I)}.
\end{aligned}$$

Hence the proof. □

Lemma 5.5.2. *For any* $f \in \mathcal{L}^2(S)$,

$$\|\Theta^\alpha(f) - f\|_{\mathcal{L}^2(S)} \leq |\alpha|_\infty \|\Theta^\alpha(f) - L_S f\|_{\mathcal{L}^2(S)} \tag{5.20}$$

and

$$\|\Theta^\alpha(f) - f\|_{\mathcal{L}^2(S)} \leq \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|Id - L_S\|_2\|f\|_{\mathcal{L}^2(S)}. \tag{5.21}$$

Proof. For any $f \in \mathcal{L}^2(S)$, let us consider a sequence X_k in $\mathcal{H}_{m_k}$ such that $f = \lim X_k$ with respect to the $\mathcal{L}^2$-norm (such sequence exists due to Theorem 5.4.3). Also continuity of L_S imply that $L_S f = \lim L_S X_k$. Due to the continuity of Θ^α and the norm it follows that

$$\|\Theta^\alpha(f) - f\|^2_{\mathcal{L}^2(S)} = \lim_{k\to\infty}\|\Theta^\alpha(X_k) - X_k\|^2_{\mathcal{L}^2(S)}. \tag{5.22}$$

Now for $X \in \mathcal{H}_m$,

$$X = \sum_{n=p}^{\infty}\lambda_{nm}H_{nm}\ .$$

Note that $\{H^\alpha_{nm} - H_{nm}\}_{n=p}^\infty$ is orthogonal. For instance

$$\begin{aligned}
\langle H^\alpha_{nm} - H_{nm}, H^\alpha_{rm} - H_{rm}\rangle &= \int_0^{2\pi}\int_0^{\pi} u_{nm}(\cos\varphi)u_{rm}(\cos\varphi)|v^\alpha_m(\theta) - v_m|^2\sin\varphi d\varphi d\theta \\
&= \left(\int_0^{\pi} u_{nm}(\cos\varphi)u_{rm}(\cos\varphi)\sin\varphi d\varphi\right)\left(\int_0^{2\pi}|v^\alpha_m(\theta) - v_m|^2 d\theta\right) \\
&= 0,
\end{aligned}$$

since the first integral is zero. Therefore

$$\begin{aligned}
\|\Theta^\alpha(X) - X\|^2_{\mathcal{L}^2(S)} &= \sum_{n=p}^{\infty}|\lambda_{nm}|^2\|H^\alpha_{nm} - H_{nm}\|^2_{\mathcal{L}^2(S)} \\
&\leq \sum_{n=p}^{\infty}|\alpha|^2_\infty|\lambda_{nm}|^2\|H^\alpha_{nm} - L_S H_{nm}\|^2_{\mathcal{L}^2(S)},
\end{aligned}$$

using Lemma 5.5.1. Also $H^{\alpha}_{nm} - L_S H_{nm}, H^{\alpha}_{rm} - L_S H_{rm}$ are orthogonal for $n \neq r$, due to the orthogonality of u_{nm}, u_{rm} for $n \neq r$. Therefore

$$\|\Theta^{\alpha}(X) - X\|^2_{\mathcal{L}^2(S)} \leq |\alpha|^2_{\infty} \|\Theta^{\alpha}(X) - L_S X\|^2_{\mathcal{L}^2(S)}. \tag{5.23}$$

Using it in (5.22) for $X = X_k$, it follows that

$$\begin{aligned}
\|\Theta^{\alpha}(f) - f\|^2_{\mathcal{L}^2(S)} &= \lim_{k\to\infty} \|\Theta^{\alpha}(X_k) - X_k\|^2_{\mathcal{L}^2(S)} \\
&\leq |\alpha|^2_{\infty} \lim_{k\to\infty} \|\Theta^{\alpha}(X_k) - L_S X_k\|^2_{\mathcal{L}^2(S)} \\
&= |\alpha|^2_{\infty} \|\Theta^{\alpha}(f) - L_S f\|^2_{\mathcal{L}^2(S)}
\end{aligned}$$

and therefore

$$\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} \leq |\alpha|_{\infty} \|\Theta^{\alpha}(f) - L_S f\|_{\mathcal{L}^2(S)} \, .$$

For the second inequality,

$$\begin{aligned}
\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} &\leq |\alpha|_{\infty} \|\Theta^{\alpha}(f) - L_S f\|_{\mathcal{L}^2(S)} \\
&\leq |\alpha|_{\infty} \big(\|\Theta^{\alpha}(f) - f\|_{\mathcal{L}^2(S)} + \|f - L_S f\|_{\mathcal{L}^2(S)} \big)
\end{aligned}$$

and the result follows. □

Proposition 5.5.1. *If $|\alpha|_{\infty} < \frac{\|L\|_2^{-1}}{3}$, then Θ^{α} is injective and its range is closed.*

Proof. From (5.20), with $\|L_S\|_2 \leq 3\|L\|_2$, it follows that

$$\|f\|_{\mathcal{L}^2(S)} - \|\Theta^{\alpha}(f)\|_{\mathcal{L}^2(S)} \leq |\alpha|_{\infty} \big(\|\Theta^{\alpha}(f)\|_{\mathcal{L}^2(S)} + 3\|L\|_2 \|f\|_{\mathcal{L}^2(S)} \big) \, .$$

Therefore

$$\|f\|_{\mathcal{L}^2(S)} \leq \frac{1 + |\alpha|_{\infty}}{1 - 3|\alpha|_{\infty}\|L\|_2} \|\Theta^{\alpha}(f)\|_{\mathcal{L}^2(S)}. \tag{5.24}$$

If $\Theta^{\alpha}(f) = 0$, then $f = 0$ and consequently $\Theta^{\alpha}(f)$ is injective. To show that the range of Θ^{α} is closed, consider a convergent sequence $\Theta^{\alpha}(f_n)$ such that $\Theta^{\alpha}(f_n) \to g^{\alpha}$. Since the sequence $\Theta^{\alpha}(f_n)$ is convergent, it is also Cauchy and therefore according to (5.24), f_n is also a Cauchy sequence. Consequently, f_n is convergent in a Banach space. If $f_n \to g$ as $n \to \infty$, then the continuity of Θ^α implies that

$$\Theta^\alpha(g) = \lim_{n\to\infty} \Theta^\alpha(f_n) = g^\alpha \ .$$

Therefore g^α belongs to range of Θ^α and hence range of Θ^α is closed. □

The treatise [42] is a good reference for the basic definitions used in the sequel.

Definition 5.5.1. *Let H be a Hilbert space. A sequence $(x_k) \subset H$, is a Bessel sequence in H if there exists a constant $B > 0$ such that for all $x \in H$*

$$\sum_{k=0}^{\infty} |\langle x, x_k\rangle|^2 \leq B\|x\|^2 \ .$$

Proposition 5.5.2. *For any scale vector α with $|\alpha|_\infty < 1$, (H^α_{nk}) is a Bessel sequence.*

Proof. For any $f \in \mathcal{L}^2(S)$,

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{2k} |\langle f, H^\alpha_{nk}\rangle|^2 &= \sum_{k=0}^{\infty}\sum_{n=0}^{2k} |\langle f, \Theta^\alpha(H_{nk})\rangle|^2 \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{2k} |\langle (\Theta^\alpha)^*(f), H_{nk}\rangle|^2,\end{aligned}$$

where $(\Theta^\alpha)^*$ is the adjoint operator of Θ^α. Applying Parseval identity to the orthonormal basis (H_{nk}), it follows that

$$\sum_{k=0}^{\infty}\sum_{n=0}^{2k} |\langle f, H^\alpha_{nk}\rangle|^2 = \|(\Theta^\alpha)^*(f)\|^2_{\mathcal{L}^2} \leq \|\Theta^\alpha\|_2^2 \|f\|^2_{\mathcal{L}^2} \ ,$$

since $\|(\Theta^\alpha)^*\|_2^2 = \|\Theta^\alpha\|_2^2$. Therefore (H^α_{nk}) is a Bessel sequence with Bessel constant $B = \|\Theta^\alpha\|_2^2$. □

Definition 5.5.2. *A sequence (x_k) in a Hilbert space H is a frame if there exist numbers $A > 0$ and $B > 0$ such that for all $x \in H$ we have*

$$A\|x\|^2 \leq \sum_{k=0}^{\infty} |\langle x, x_k\rangle|^2 \leq B\|x\|^2. \tag{5.25}$$

Definition 5.5.3. *A sequence (x_k) in a Hilbert space H is a frame sequence if it is a frame for its closed span $[x_k] = \overline{span}(x_k)$.*

Proposition 5.5.3. *If $|\alpha|_\infty < \frac{\|L\|_2^{-1}}{3}$, then (H_{nk}^α) is a frame sequence.*

Proof. In the proof of Proposition 5.5.2, for any $g \in \mathcal{L}^2(S)$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{2k} |\langle g, H_{nk}^\alpha\rangle|^2 \leq \|\Theta^\alpha\|_2^2 \|g\|_{\mathcal{L}^2(S)}^2 .$$

Therefore right hand inequality of (5.25) holds for $B = \|\Theta^\alpha\|_2^2$.
If $|\alpha|_\infty < \frac{\|L\|_2^{-1}}{3}$, then due to Proposition 5.5.1, Θ^α is injective with closed range. Then range of Θ^α, $Range(\Theta^\alpha)$ is a Hilbert space, since it is a closed subspace of a Hilbert space $\mathcal{L}^2(s)$. Consequently $(\Theta^\alpha)^{-1}$ is well defined, linear and bounded as Θ^α (see, e.g., Theorem 3.5.3, [45]). Therefore, $\Theta^\alpha \circ (\Theta^\alpha)^{-1}$ is the identity operator on $Range(\Theta^\alpha)$. If $g \in [H_{nk}^\alpha]$, then since $span(H_{nk}^\alpha) \subseteq Range(\Theta^\alpha)$, it follows that $g \in Range(\Theta^\alpha)$. Therefore as $Range(\Theta^\alpha)$ is closed,

$$[H_{nk}^\alpha] = \overline{span}(H_{nk}^\alpha) \subseteq Range(\Theta^\alpha) .$$

But for any $g \in [H_{nk}^\alpha]$,

$$g = \left((\Theta^\alpha)^{-1}\right)^* \circ (\Theta^\alpha)^*(g)$$

and thus

$$\|g\|_{\mathcal{L}^2(S)}^2 \leq \|(\Theta^\alpha)^{-1}\|_2^2 \|(\Theta^\alpha)^*(g)\|_{\mathcal{L}^2(S)}^2, \tag{5.26}$$

since

$$\|\left((\Theta^\alpha)^{-1}\right)^*\|_2 = \|(\Theta^\alpha)^{-1}\|_2 .$$

As in the proof of Proposition 5.5.2,

$$\|(\Theta^\alpha)^*(g)\|_{\mathcal{L}^2(S)}^2 = \sum_{k=0}^{\infty}\sum_{n=0}^{2k} |\langle g, H_{nk}^\alpha\rangle|^2 .$$

Using it in (5.26)

$$\|g\|_{\mathcal{L}^2(S)}^2 \leq \|(\Theta^\alpha)^{-1}\|^2 \sum_{k=0}^{\infty}\sum_{n=0}^{2k} |\langle g, H_{nk}^\alpha\rangle|^2 .$$

Denoting

$$A = \|(\Theta^\alpha)^{-1}\|^{-2} \ ,$$

it follows that

$$A\|g\|^2_{\mathcal{L}^2(S)} \leq \sum_{k=0}^{\infty}\sum_{n=0}^{2k} |\langle g, H^\alpha_{nk}\rangle|^2 \ .$$

This completes the proof. $\square$

Definition 5.5.4. *A sequence (x_k) in a Hilbert space H is a Riesz sequence if there exist $k_1 > 0$ and $k_2 > 0$ such that for any $(\lambda_k) \in l^2$*

$$k_1 \sum_{k=0}^{\infty} |\lambda_k|^2 \leq \| \sum_{k=0}^{\infty} \lambda_k x_k\|^2 \leq k_2 \sum_{k=0}^{\infty} |\lambda_k|^2. \tag{5.27}$$

Proposition 5.5.4. *If $|\alpha|_\infty < \frac{\|L\|_2^{-1}}{3}$, then (H^α_{nk}) is a Riesz sequence.*

Proof. If $(c_{nk}) \in l^2$, let us define for $f \in \mathcal{L}^2(S)$

$$f = \sum_{k=0}^{\infty}\sum_{n=0}^{2k} c_{nk} H_{nk} \ .$$

Then due to Parseval's equality

$$\|f\|^2_{\mathcal{L}^2(S)} = \sum_{k=0}^{\infty}\sum_{n=0}^{2k} |c_{nk}|^2 \ .$$

Also

$$\begin{aligned} \|\Theta^\alpha(f)\|^2_{\mathcal{L}^2(S)} &= \|\sum_{k=0}^{\infty}\sum_{n=0}^{2k} c_{nk} H^\alpha_{nk}\|^2_{\mathcal{L}^2} \\ &\leq \|\Theta^\alpha\|_2^2 \|f\|^2_{\mathcal{L}^2} \\ &= k_2 \sum_{k=0}^{\infty}\sum_{n=0}^{2k} |c_{nk}|^2, \end{aligned}$$

where $k_2 = \|\Theta^\alpha\|_2^2$.

For the left inequality in (5.27), let

$$k_1 = \frac{1 - 3\|L\|_2 |\alpha|_\infty}{1 + |\alpha|_\infty} \ .$$

If $|\alpha|_\infty < \frac{\|L\|_2^{-1}}{3}$, then from (5.24), it follows that

$$k_1 \|f\|^2_{\mathcal{L}^2(S)} \leq \|\Theta^\alpha(f)\|^2_{\mathcal{L}^2(S)} = \sum_{k=0}^{\infty} \sum_{n=0}^{2k} \|c_{nk} H^\alpha_{nk}\|^2_{\mathcal{L}^2(S)} \,.$$

Hence (H^α_{nk}) is a Riesz sequence. □

Definition 5.5.5. *A sequence (x_k) in a Hilbert space H is a Riesz basis for H if it is the image of an orthonormal basis for H under an invertible linear transformation. In other words, if there is an orthonormal basis (e_k) for H and an invertible transformation T such that $Te_k = x_k$ for all k.*

Theorem 5.5.1. *If $|\alpha|_\infty < (2 + 3\|L\|_2)^{-1}$, then (H^α_{nk}) is a Riesz basis for $\mathcal{L}^2(S)$.*

Proof. By hypothesis, $|\alpha|_\infty < (2 + 3\|L\|_2)^{-1}$, which implies

$$\frac{(1 + 3\|L\|_2)|\alpha|_\infty}{1 - |\alpha|_\infty} < 1 \,.$$

Then due to (5.21),

$$\|I - \Theta^\alpha\|_2 < 1 \,.$$

According to Lemma 5.3.5, the operator Θ^α is an isomorphism and hence (H^α_{nk}) is a Riesz basis. □

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List of published and communicated papers

Based on the work carried out in this thesis, the following published and communicated papers have resulted:

1. Md. N. Akhtar, M. Guru Prem Prasad and M. A. Navascués "Box dimensions of α-fractal functions", *Fractals*, 24(3), (2016), (DOI: 10.1142/S0218348X16500377).

2. Md. N. Akhtar, M. Guru Prem Prasad and M. A. Navascués, "Fractal Jacobi systems and convergence of Fourier-Jacobi expansions of fractal interpolation functions", *Mediterr. J. Math.*, (2016), (DOI 10.1007/s00009-016-0727-3).

3. M. Guru Prem Prasad and Md. N. Akhtar "Fractal interpolation surfaces and perturbations on vertical scaling factors", *Int. J. Nonlinear Sci.*, 21(1), (2016), 3-12.

4. Md. N. Akhtar and M. Guru Prem Prasad "Graph-directed coalescence hidden variable fractal interpolation functions", *Applied Mathematics*, 7(1), (2016), 335-345.

5. Md. N. Akhtar, M. Guru Prem Prasad and M. A. Navascués, "Box dimensions of α-fractal functions with variable scaling factors in subintervals", *communicated*.