On the existence, uniqueness and approximate controllability of some classes of differential equations with different types of fractional derivatives

by

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भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
Indian Institute of Technology Guwahati

Dedicated to

My Parents

Declaration

I do hereby declare that this thesis entitled **On the existence, uniqueness and approximate controllability of some classes of differential equations with different types of fractional derivatives** is a presentation of my original research work carried out under the supervision of **Dr. Swaroop Nandan Bora**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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Certificate

It is to certify that the work contained in this thesis entitled **On the existence, uniqueness and approximate controllability of some classes of differential equations with different types of fractional derivatives** has been carried out by **Bandita Roy**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under my supervision for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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Abstract

The research reported in this thesis deals with the analysis of a number of differential equations of fractional order, viz., Cauchy type non-autonomous fractional differential equation, Volterra-Fredholm integro fractional differential equation of neutral type, nonlinear integro fractional differential equation of Sobolev type with finite delay, instantaneous impulsive fractional differential equation with some generalized integral conditions. More precisely, our aim is to prove the existence and uniqueness results for these problems. Further, we also establish the approximate controllability of a class of nonlinear fractional differential equations.

The first problem consists of a classical initial value problem concerned with non-autonomous fractional differential equations with Hilfer fractional derivatives and another one containing nonlocal initial conditions. By using fixed point theorems, sufficient conditions for the existence of mild solutions are obtained and some examples are also considered to illustrate the obtained theory. The existence and uniqueness of integral solutions of a class of non-densely defined mixed Volterra-Fredholm integro neutral fractional differential equation is also studied and the results are obtained by using semigroup theory, fixed point theorems and Kuratowski measure of noncompactness. In another problem, a Volterra fractional differential equation of Sobolev type with finite delay is considered. The existence results for mild solutions are proved by using fixed point theorems and Hausdorff measure of noncompactness. Further, the existence of solutions of a class of impulsive fractional differential equation with Erdélyi-Kober type boundary conditions and multiple base points is established. The results are obtained by using various properties of fractional calculus and different fixed point theorems. Some examples are presented to illustrate the obtained results wherever possible.

The last part of the thesis discusses the approximate controllability of a class of Hilfer fractional control system with analytic semigroup where the nonlinear term depends both on the state and control. The existence and uniqueness of the mild solution is established with the help of a generalized contraction theorem, and the sufficient conditions for the approximate controllability are obtained by using suitable assumptions on the functions involved.

Contents

CHAPTER 1

Introduction

1.1 Fractional Calculus

Fractional calculus is not a new subject as thought by many. It is a subject that has gained enormous popularity and significance over the last few decades, mainly due to the demonstration of its practical use in diverse fields of science and engineering. It has been established beyond doubt that fractional calculus has been able to provide a number of important tools for solving differential and integral equations. Furthermore, it has shed light on tackling some other problems involving special functions and leading to extensions and generalizations. A better way to understand fractional derivatives and integrals is: As real numbers exist between the integers, so do fractional derivatives and integrals between conventional integer-order derivatives and integrals. The way the generalization from integer to non-integer makes the number line general, the same can be assumed to hold for differintegrals. In other words, fractional calculus is a name of theory of integrations and derivatives of arbitrary order, which unify and generalize the notion of integer order differentiation and n-fold integration.

The history of fractional calculus dates back to the end of seventeenth century with the exchange of a letter between Leibniz and Euler, and it has been developed progressively up till now. It was not until 1730 that Euler mentioned 'fractional calculus' when he studied the interpolation between the integer orders of a derivative and, successively in 1772, Lagrange referred to it in one of his studies. Laplace (1812) defined a fractional derivative by means of an integral, and the first discussion on a derivative of fractional order was written by Lacroix (1819) but his method of generalizing it from an integer order did not offer hints for possible applications and he himself considered this question as a mere mathematical exercise. In 1822, Fourier was the next to mention a derivative of arbitrary order, but he too, like his predecessors, provided no application. A fractional operation was applied for the first time by Abel in 1823 for solving an integral equation that arose during the demonstration of the tautochrone problem.

The first systematic study on fractional calculus was made in 1832 by Liouville. He defined a derivative of arbitrary order as an infinite series, which had the disadvantage that the order of the derivative must be restricted to those values for which the series converged. Being aware of this, he formulated a second definition of fractional derivative. Successively, Liouville applied the fractional derivative to problems in potential theory and was also the first one who attempted to solve differential equations by means of fractional operations.

After Liouville, Riemann used a generalization of a Taylor series to derive a formula for integration of arbitrary order, and showed that the approaches proposed by Liouville and Riemann could be abridged in a single formula, known as the Riemann-Liouville fractional integral formula:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

where J^{α} denotes the fractional integral operator of order $\alpha \in \mathbb{R}^+$, a is the lower limit of integration and Γ is the gamma function.

In 1867, Grünwald proposed another approach for fractional differentiation based on the generalization of the finite difference quotients for fractional derivatives, nowadays known as Grünwald-Letnikov fractional derivative, as follows:

$$D^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\frac{t-a}{h}} (-1)^m \frac{\Gamma(\alpha+1)}{m! \, \Gamma(\alpha-m+1)} f(t-mh) ds,$$

where t and a are the upper and lower limits of differentiation, respectively. This definition, even if correct, was not mathematically rigorous. The first rigorous demonstration of this formula was given by Letnikov in 1868. His papers were the first instances that could exhibit a strong attention to details and mathematical rigour.

Riemann-Liouville definition of fractional integral has been adopted to define a fractional derivative, known as Riemann-Liouville fractional derivative:

$$^{RL}D^{\alpha} f(t) = D^m J^{m-\alpha} f(t), \quad m-1 < \alpha \leq m.$$

Also, Grünwald-Letnikov derivative formula has been modified for giving an alternate definition of fractional integral, known as Grünwald-Letnikov fractional integral formula:

$$D^{-\alpha} f(t) = \lim_{h \to 0} h^{\alpha} \sum_{m=0}^{\frac{t-a}{h}} \frac{\Gamma(\alpha+m)}{m! \, \Gamma(\alpha)} f(t-mh) ds.$$

The Riemann-Liouville fractional integral and derivative, due to their forms, are well suited for finding analytical solutions of relatively simple functions, viz., x^a, $\exp(x)$, $\sin x$, etc., whereas Grünwald-Letnikov definition is successfully applied for numerical evaluations.

In 1967, Caputo introduced a fractional derivative called Caputo fractional derivative of order $\alpha > 0$ defined as

$$^{C}D^{\alpha}f(t) = J^{m-\alpha}D^{m}f(t), \quad m-1 < \alpha \leq m.$$

This definition is preferred over Riemann-Liouville's definition in many engineering applications. The reason for this is that, in Riemann-Liouville derivative, one needs to specify the values of certain fractional derivatives of the function at the initial point whereas for Caputo derivative, the initial values are $x(a)$, $x'(a)$, $x''(a)$, $\ldots$, when a is the initial point, and this type of initial conditions has well-understood physical meaning.

In recent years, besides the previous definitions, several fractional differentiation and integration formulas have been proposed. In [62], Hilfer proposed a new fractional derivative when he studied fractional time evolution in some physical phenomena. This derivative is a generalization of both Riemann-Liouville and Caputo derivatives, and is named as Hilfer fractional derivative or generalized Riemann-Liouville derivative. The Hilfer derivative of order α with parameter β is defined by

$$^{H}D^{\alpha,\beta}f(t) = J^{\beta(m-\alpha)}D^{m}J^{(1-\beta)(m-\alpha)}f(t),$$

where $m-1 < \alpha \leq m$, $0 \leq \beta \leq 1$, $t > a$ and $D = \dfrac{d}{dt}$.

The list of mathematicians who have provided important contributions up to the middle of the twentieth century also includes Holmgren, Laurent, Nekrassov, Krug, Hadamard, Heaviside, Pincherle, Hardy and Littlewood, Weyl, Levy, Marchaud, Davis, Zygmund, Love, Erdelyi, Kober, Widder, Riesz, Feller, etc. In June 1974, the first conference on Fractional Calculus and its Applications was organized by B. Ross at the University of New Haven, Connecticut. As for the first monograph, the credit is ascribed to K.B. Oldham and J. Spanier, who published a book in 1974 devoted to fractional calculus. A number of additional books have appeared since then with titles explicitly devoted to fractional calculus (or fractional differential equations) and its applications, for example, Miller and Ross (1993), Samko et al. (1993), Carpinteri and Mainardi (1997), Podlubny (1999), Hilfer (2000), Kilbas et.al (2006), Das (2007), and others and the list is expected to grow in the forthcoming years.

Although fractional calculus has a long history, but from the application point of view, it fell into oblivion for many years since it was not considered useful for solving problems in physics and engineering. This was due to its high complexity and the lack of an acceptable physical and geometric interpretation. In 2002, Podlubny [91] proposed an explanation of fractional phenomena. He proposed both a geometrical interpretation of the Riemann-Liouville fractional integral and a physical interpretation for the Riemann-Liouville and

Caputo fractional derivatives.

Another important property that all the fractional derivatives share and it is worth mentioning is that the fractional derivative operators are nonlocal. The terms ‘local’ and ‘nonlocal’ are distinguished as follows: in order to calculate integer-order derivatives of a function, it is required to know its properties in an infinitesimal neighbourhood of the considered point whereas the fractional derivative relies not only on the present state but also upon all its past states. As a result, integer-order derivatives cannot describe processes with memory and this fact acts as the primary advantage of fractional derivatives over classical derivatives.

In the last few decades, there has been an explosion of research activities on the application of fractional calculus to very diverse scientific fields ranging from the physics of diffusion and advection phenomena to control systems to finance and economics. Applications related to fractional calculus have appeared at least in the following fields:

- Fractional control of engineering systems
- Advancement of calculus of variations and optimal control to fractional dynamical systems
- Analytical and numerical tools and techniques
- Fundamental understanding of wave and diffusion phenomena, their measurements and verifications, including applications to plasma physics
- Bioengineering and biomedical applications
- Thermal modelling of engineering systems such as brakes and machine tools
- Image and signal processing.

Scientists have discovered that many phenomena, not completely understood before, have complex microscopic behaviours, and that their macroscopic dynamics cannot be modelled anymore via integer order derivatives. In particular, it has been found that most of the processes associated with complex systems have nonlocal dynamics involving long-term memory effects, which is precisely the property that characterizes fractional derivative operators. In order to appreciate the potentialities of fractional calculus in a better way from an application point of view, some examples of complex system modelled by fractional differential equations are given below:

In the field of bioengineering, there is an ongoing need to develop efficient and high fidelity material models to simulate the stress response of biological materials. For this purpose, fractional calculus is used to develop fractional order viscoelastic equations, which could be useful for modelling soft biological tissues. In [30], it is asserted that a description based on fractional-order differential equations potentially has superior accuracy and gives the possibility of correlating the hierarchical structure of biological tissues to the fractional order of derivative. In particular, a one-dimensional version of fractional order viscoelastic equations, called quasilinear fractional order viscoelasticity, was formulated and applied to model the stress response of the porcine aortic valve tissues.

Another example is the one related to the study of polymeric materials. In organic dielectric materials, such as semi-crystalline polymers, the intimate mixture between crystals and amorphous phase gives rise to a complex structure whose fractal features have been experimentally detected. This property makes the polymers very difficult to handle analytically, and fractional calculus seems to be the only mathematical tool which has the ability to describe fractal functions. With its help, a constitutive equation that can be linked to molecular theories describing the macroscopic behaviour of polymeric materials can be obtained. In [92], a dielectric fractional model is given, which is based on the idea of obtaining an intermediate electrical behaviour between a resistance (Ohm's law) and a capacitor. The authors have named this new electrical element as *capresistor*: when the order of the derivative equals one, an *electrical resistor* is obtained whereas when the order of the derivative equals zero, a *capacitor* is obtained. Using this new electrical element, the authors proposed electric circuits which were able to model relaxation processes in organic dielectric materials (semi-crystalline polymers).

1.2 Some Important Definitions and Results

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two Banach spaces and $J = [a, b]$ be a finite interval on $\mathbb{R}$. Let $B(X, Y)$ denote the Banach space of all bounded linear operators from X to Y. For $X = Y$, we use the notation $B(X, X) = B(X)$. Let $C(J, X)$ denote the Banach space of all continuous functions from J to X equipped with the supremum norm $\|.\|_C$, that is, $\|f\|_C = \sup_{t\in J}\|f(t)\|_X$, for all $f \in C(J, X)$.

$L^p(J, X)$, $1 \leq p \leq \infty$, denotes the Banach space of all measurable functions $f\colon J \to X$ with the following norm:

$$\|f\|_{L^p} = \begin{cases} \left(\int_J \|f\|_X^p\, dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \left\{ \sup\limits_{t\in J\setminus\bar{J}} \|f(t)\|_X \right\}, & p = \infty. \end{cases}$$

Theorem 1.2.1 (Hölder's Inequality). *Consider $p, q \geq 1$ with $1/p + 1/q = 1$. Then, for $f \in L^p(J, X)$ and $g \in L^q(J, X)$, $1 \leq p \leq \infty$,*

$$fg \in L^1(J, X) \quad \text{and} \quad \|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Theorem 1.2.2 (Bochner's Theorem). *A measurable function $f\colon J \to X$ is said to be Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Definition 1.2.1. *A family $\mathcal{F}$ of subsets of $C(J, X)$ is equicontinuous if, for every $\epsilon > 0$,* *there exists some* $\delta(\epsilon) > 0$ *such that*

$$\|f(t_1) - f(t_2)\|_X < \epsilon \quad \text{for all } t_1,\, t_2 \in J \text{ with } |t_1 - t_2| < \delta \text{ and for all } f \in \mathcal{F}.$$

Definition 1.2.2. *A family* $\mathcal{F}$ *of subsets of* $C(J, X)$ *is called uniformly bounded if there exists a constant* $F > 0$ *such that*

$$\|f(t)\|_X < F \quad \text{for all } t \in J \text{ and for all } f \in \mathcal{F}.$$

Theorem 1.2.3 (Arzelà-Ascoli theorem)**.** *A family* $\mathcal{F}$ *of subsets of* $C(J, X)$ *is relatively compact if* $\mathcal{F}$ *is uniformly bounded and equicontinuous on* J*, and for each* $t \in J$*,* $\{f(t) \colon f \in \mathcal{F}\}$ *is relatively compact in* X*.*

Theorem 1.2.4 (Vector-valued Dominated Convergence Theorem)**.** *Let* $f_n \colon J \to X$ *be Bochner integrable functions. Assume that* $f(t) := \lim_{n\to\infty} f_n(t)$ *exists a.e. and there exists an integrable function* $g \colon J \to \mathbb{R}$ *such that* $f_n(t) \leq g(t)$ *a.e. for all* $n \in \mathbb{N}$*. Then* f *is Bochner integrable and* $\displaystyle\int_J f(t)dt = \lim_{n\to\infty} \int_J f_n(t)dt.$

Laplace transform of fractional derivatives

(i) $\mathcal{L}[{}^{RL}D_0^{\alpha} f(t)](s) = s^{\alpha}\mathcal{L}[f(t)](s) - \displaystyle\sum_{k=0}^{n-1} s^{k}(D^{\alpha-k-1} f)(a)$

(ii) $\mathcal{L}[{}^{C}D_a^{\alpha} f(t)](s) = s^{\alpha}\mathcal{L}[f(t)](s) - \displaystyle\sum_{k=0}^{n-1} s^{\alpha-k-1}(D^{k} f)(a)$

(iii) $\mathcal{L}[{}^{H}D_0^{\alpha,\beta} f(t)](s) = s^{\alpha}\mathcal{L}[f(t)](s) - \displaystyle\sum_{k=0}^{n-1} s^{n-k-1-\beta(n-\alpha)}\left[\lim_{t\to 0^+} \frac{d^k}{dt^k}(J_{0^+}^{(1-\beta)(n-\alpha)} f)(t)\right]$

where $n - 1 < \alpha \leq n$ and $0 \leq \beta \leq 1$.

Definition 1.2.3. *The one-parameter Mittag-Leffler function is defined by*

$$E_c(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(cn+1)}, \quad z \in \mathbb{C},\ \Re(c) > 0,$$

where $\Re$ *denotes the real part.*

This function is a generalization of the exponential function. An extension of the above function is the following two-parameter Mittag-Leffler function:

$$E_{c,d}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(cn+d)}, \quad z, d \in \mathbb{C},\ \Re(c) > 0.$$

Lemma 1.2.1. *Assume that* $f_1 \colon [0, b] \to [0, \infty)$ *is locally integrable and* $f_2 \colon [0, b] \to [0, \infty)$ *is a nondecreasing continuous function such that* $f_2(t) \leq \mathcal{C}$ *(a constant). Suppose* $f_3 \colon [0, b] \to [0, \infty)$ *is locally integrable and satisfies the inequality*

$$f_3(t) \leq f_1(t) + f_2(t) \int_0^t (t-s)^{\alpha-1} f_3(s)ds, \quad t \in [0, b],\ \alpha > 0.$$

Then

$$f_3(t) \le f_1(t) + \int_0^t \left[\sum_{i=1}^{\infty} \frac{(f_2(t)\gamma(\alpha))^i}{\Gamma(i\alpha)}(t-s)^{i\alpha-1} f_1(s)\right] ds, \quad t \in [0,b].$$

In addition, if f_1 *is nondecreasing, then* $f_3(t) \le f_1(t) E_\alpha(f_2(t)\gamma(\alpha)t^\alpha)$ *for* $t \in [0,b]$*.*

1.3 Methods

1.3.1 Measure of noncompactness

Measures of noncompactness form a very useful tool in many branches of mathematics. They are used in the fixed point theory, linear operator theory, theory of differential and integral equations and many other.

Let $(X, \|.\|_X)$ be a Banach space and X_B be the set of all non-empty bounded subsets of X. Then we state the following definition [69]:

Definition 1.3.1 (Measure of Noncompactness). *A map* $\gamma\colon X_B \to \mathbb{R}^+ = [0,\infty)$ *satisfying*

$$\gamma(\overline{co}(\Omega)) = \gamma(\Omega), \text{ for every } \Omega \in X_B,$$

where $\overline{co}(\Omega)$ *is the closure of the convex hull of* Ω*, is called the measure of noncompactness in* X*.*

The measure of noncompactness γ is said to be [124]

(i) regular: if $\gamma(\Omega) = 0 \Leftrightarrow \Omega$ is a relatively compact set,

(ii) monotone: if $\Omega_1 \subset \Omega_2 \implies \gamma(\Omega_1) \le \gamma(\Omega_2)$,

(iii) algebraically semi-additive: if $\gamma(\{x\} \cup \Omega) = \gamma(\Omega)$, for every $x \in X$,

(iv) nonsingular: if $\gamma(\Omega_1 + \Omega_2) \le \gamma(\Omega_1) + \gamma(\Omega_1)$.

There are two important measures of noncompactness called Hausdorff measure of noncompactness and Kuratowski measure of noncompactness, which are defined as follows:

Hausdorff measure of noncompactness γ_1 is defined for any $\Omega \in X_B$ as

$$\gamma_1(\Omega) = \inf\{r > 0 | \Omega \subseteq \cup_{i=1}^n B_r(x_i) \text{ where } x_i \in X\},$$

with $B_r(x_i)$ as closed balls of radius $\le r$ with centres at x_i, $i = 1, 2, \ldots, n$.

The *Kuratowski measure of noncompactness* is the map $\gamma_2\colon X_B \to [0,\infty)$ defined for $\Omega \in X_B$ by

$$\gamma_2(\Omega) = \inf\{r > 0 | \Omega \subseteq \cup_{i=1}^n B_i \text{ and } \operatorname{diam}(B_i) \le r \text{ for } i = 1, 2, \ldots, n\},$$

where $\operatorname{diam}(B_i) = \sup\{\|x-y\|_X\colon x, y \in B_i\}$.

The relationship between Hausdorff measure of noncompactness γ_1 and Kuratowski measure of noncompactness γ_2 is given by

$$\gamma_1(\Omega) \leq \gamma_2(\Omega) \leq 2\gamma_1(\Omega).$$

Lemma 1.3.1. *[124] Let $\Omega \subset X$ be bounded. Then for every $\epsilon > 0$, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \Omega$ such that*

$$\gamma(\Omega) \leq 2\gamma(\{x_n\}_{n=1}^{\infty}) + \epsilon.$$

Lemma 1.3.2. *[124] Let $\Omega \subset C(J, X)$ be equicontinuous and bounded. Then $\gamma(\Omega(t))$ is continuous on J, and*

$$\gamma(D) = \sup_{t \in J} \gamma(\Omega(t)).$$

Lemma 1.3.3. *[32] Let X be a Banach space, and let $D \subset X$ be bounded. Then there exists a countable set $D_0 \subset D$ such that*

$$\gamma(D) \leq 2\gamma(D_0).$$

Lemma 1.3.4. *[124] Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into X with*

$$\|x_n(t)\| \leq m(t) \text{ for almost all } t \in J \text{ and every } n \in \mathbb{N},$$

where $m \in L(J, \mathbb{R}^+)$. Then the function $\phi(t) = \gamma\Big(\{x_n\}_{n=1}^{\infty}\Big) \in L(J, \mathbb{R}^+)$ satisfies

$$\gamma\Big(\Big\{\int_0^t x_n(s)ds \Big| n \in \mathbb{N}\Big\}\Big) \leq 2\int_0^t \phi(s)ds.$$

1.3.2 Fixed point theorems

Fixed point method is the most powerful method in proving existence and uniqueness results for solutions of differential equations. Due to their importance, several researchers have studied the problems represented by evolution equations by using different fixed point theorems. Many problems which are usually expressed through ordinary and partial differential equations can be recast as integral equations. Then the existence and uniqueness results can be derived from the corresponding results of those integral equations. Such results can be obtained by applying the fixed point theorems. The fixed point theorems that have been used in our works are listed below:

Theorem 1.3.1 (Banach fixed point theorem)**.** *Let (M, d) be a nonempty complete metric space. Let $T : M \to M$ be a map such that, for any $x, y \in M$,*

$$d(Tx, Ty) \leq kd(x, y), \quad 0 \leq k < 1$$

holds. Then the operator T has a unique fixed point on M.

For the following results, assume $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ to be two Banach spaces with respective norms.

Theorem 1.3.2. *A bounded map $T: X \to Y$ is said to be compact if the image of a bounded set B in X is relatively compact in Y.*

Theorem 1.3.3. *Let $T: X \to X$ be a map such that $T^{(i)}$ $(= \underbrace{T \circ T \circ \ldots \circ T}_{i \text{ times}})$ is a contraction map for some $i \in \mathbb{N}$. Then, T has a unique fixed point on X.*

Theorem 1.3.4 (Krasnoselskii's fixed point theorem)**.** *Let B be a bounded, closed and convex subset of X. Let P and Q map B into X such that*
(i) *$Px + Qy \in B$ whenever $x, y \in B$,*
(ii) *P is a contraction mapping,*
(iii) *Q is compact and continuous.*
Then there exists a fixed point of the map $P + Q$ on B.

Theorem 1.3.5 (Schauder's fixed point theorem)**.** *Let $B \subset X$ be a convex, closed and bounded set. If $T: B \to B$ is a continuous operator such that $TB \subset X$ is relatively compact, then T has at least one fixed point in B.*

Theorem 1.3.6 (Darbo-Sadovskii's fixed point theorem)**.** *If B is a bounded, closed and convex subset of X and the continuous mapping $T: B \to B$ is an α-contraction, then the mapping T has at least one fixed point in B.*

Theorem 1.3.7. *Let B be an open bounded subset of X with the zero element of X in B. Also, let $T: \bar{B} \to X$ be a compact and continuous mapping such that $\|Tb\|_X \leq \|b\|_X$ for all $b \in \partial B$. Then T has a fixed point in $\bar{B}$.*

Theorem 1.3.8 (Schaefer's fixed point theorem)**.** *Let $T : X \to X$ be a completely continuous map on X. If $E(T) = \{x \in X: x = \Lambda Tx \text{ for some } \Lambda \in [0, 1]\} \subset X$ is bounded, then T has a fixed point.*

Theorem 1.3.9 (Leray-Schauder's nonlinear alternative)**.** *Let B be a closed convex subset of X, B_1 an open subset of B and the zero element of X belong to B_1. Suppose that $T: \overline{B_1} \to B$ is a completely continuous map. Then T satisfies one of the following properties:*
(i) *T has a fixed point in $\overline{B_1}$,*
(ii) *there is some $b \in \partial B_1$ (the boundary of B_1 in B) and $\Lambda \in (0, 1)$ with $b = \Lambda Tb$.*

Definition 1.3.2 (γ-condensing map)**.** *A continuous map $T: \Omega \subseteq Y \to Y$ is called γ-condensing, if for any bounded set $\Omega_0 \subseteq \Omega$ with $\gamma(\Omega) > 0$, we have $\gamma(T(\Omega_0)) < \gamma(\Omega_0)$.*

Let γ be a monotone nonsingular measure of noncompactness in Y.

Lemma 1.3.5. *Let Ω be a closed, convex and bounded subset of Y and $T: \Omega \to \Omega$ be an γ-condensing map, then the set of fixed points of T forms a nonempty compact set.*

1.3.3 Semigroup theory

The theory of semigroups of bounded linear operators is closely related to the solution of differential and integro-differential equations in Banach spaces. Using the method of semigroups, existence and uniqueness of mild and classical solutions of evolution equations have been discussed by Pazy [89].

Let X be a Banach space.

Definition 1.3.3. *The one-parameter family of bounded linear operators* $\{T(t)\}_{t\geq 0}\colon X \to X$ *is said to be a semigroup of bounded linear operators on* X *if the following conditions are satisfied:*
(i) $T(0) = I$, *where* I *is the identity operator on* X,
(ii) $T(t+s) = T(t)T(s)$ *for every* $t \geq 0$, $s \geq 0$ *(the semigroup property).*

Definition 1.3.4. *The infinitesimal generator of a semigroup of bounded linear operator* $\{T(t)\}_{t\geq 0}$ *on* X *is a linear operator* A *on* X *defined as*

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A),$$

where $D(A)$ *is the domain of* A *defined as*

$$D(A) = \left\{ x \in X\colon \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

Definition 1.3.5. *A semigroup* $\{T(t)\}_{t\geq 0}$ *of bounded linear operators on* X *is called a strongly continuous semigroup or* C_0*-semigroup if*

$$\lim_{t\to 0^+} T(t)x = x, \text{ for every } x \in X.$$

Definition 1.3.6. *A semigroup* $\{T(t)\}_{t\geq 0}$ *of bounded linear operators on* X *is called uniformly continuous if*

$$\lim_{t\to 0^+} \|T(t) - I\| = 0.$$

Theorem 1.3.10. **(i)** *Let* $T(t)$ *be a* C_0*-semigroup. Then, there exist constants* $\epsilon \geq 0$ *and* $M \geq 1$ *such that*

$$\|T(t)\| \leq Me^{\epsilon t} \quad \text{for all } t \geq 0.$$

(ii) *If* A *is the infinitesimal generator of a* C_0*-semigroup* $T(t)$, *then* $D(A)$ *is dense in* X *and* A *is a closed linear operator.*

Remark 1.3.1. *In Theorem 1.3.10(i), if* $\epsilon = 0$, *then* $T(t)$ *is said to be a uniformly bounded semigroup. Moreover, if* $M = 1$, *then* $T(t)$ *is called a* C_0*-semigroup of contractions.*

Theorem 1.3.11. *Let $T(t)$ be a C_0-semigroup and A be its infinitesimal generator. Then*
(i) *for $x \in X$,*

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x,$$

(ii) *for $x \in X$, $\int_0^t T(s)xds \in D(A)$, and*

$$A\Big(\int_0^t T(s)xds\Big) = T(t)x - x,$$

(iii) *for $x \in D(A)$, $T(t)x \in D(A)$ and*

$$\frac{d(T(t)x)}{dt} = AT(t)x = T(t)Ax,$$

(iv) *for $x \in D(A)$,*

$$T(t)x - T(s)x = \int_s^t T(\xi)Axd\xi = \int_s^t AT(\xi)xd\xi.$$

For the next results, we use the following notations: for a linear operator T which is not necessarily bounded in X, the resolvent set of T, denoted by $\rho(T)$, consists of all complex numbers λ such that $\lambda I - T$ is invertible, that is, $(\lambda I - T)^{-1} \in B(X)$. The resolvent of T, denoted by $R(\lambda; T)$, is defined as $R(\lambda; T) = \{(\lambda I - T)^{-1} \colon \lambda \in \rho(T)\}$.

Theorem 1.3.12 (Hille-Yosida). *A linear (unbounded) operator A on X is the infinitesimal generator of a C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$ iff*
(i) *A is closed and $\overline{D(A)} = X$,*
(ii) *$\mathbb{R}^+ \subseteq \rho(A)$ and for every $\lambda > 0$, we have $\|R(\lambda; A)\| \leq \frac{1}{\lambda}$.*

Theorem 1.3.13. *A linear operator A is the infinitesimal generator of a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq M$ iff*
(i) *A is closed and $\overline{D(A)} = X$,*
(ii) *$\mathbb{R}^+ \subseteq \rho(A)$ and we have $\|R(\lambda; A)^n\| \leq \frac{M}{\lambda^n}$ for $\lambda > 0$ and $n \in \mathbb{N}$.*

Definition 1.3.7. *Let $\Delta = \{z \colon \phi_1 < \arg z < \phi_2, \phi_1 < 0 < \phi_2\}$ and let $\{T(z)\}_{z\in\Delta} \in B(X)$. The family $\{T(z)\}_{z\in\Delta}$ is called an analytic semigroup in Δ if*
(i) *$z \to T(z)$ is analytic in Δ,*
(ii) *$T(0) = I$ and $\lim_{z\to 0, z\in\Delta} T(z)x = x$, for every $x \in X$,*
(iii) *$T(z_1 + z_2) = T(z_1)T(z_2)$ for z_1, $z_2 \in \Delta$.*
A semigroup $T(t)$ is said to be analytic if it is analytic in some sector Δ containing $\mathbb{R}^+$.

Theorem 1.3.14. *Let $T(t)$ be a uniformly bounded C_0-semigroup and A be its infinitesimal generator. Assume $0 \in \rho(A)$. Then the following statements are equivalent:*
(i) *$T(t)$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z \colon |\arg z| < \delta\}$ and $\|T(z)\|$ is uniformly bounded in every closed subsector $\overline{\Delta_{\delta'}}$, $\delta' < \delta$ of Δ_δ.*

(ii) *there exists a constant C such that, for every $\sigma > 0$ and $\tau \neq 0$,*

$$\|R(\sigma + \mathrm{i}\tau; A)\| \leq \frac{C}{|\tau|}.$$

(iii) *there exist δ $(0 < \delta < \pi/2)$ and $M > 0$ such that $\Sigma \subset \rho(A)$, where $\Sigma = \{\lambda : |\arg \lambda| < \pi/2 + \delta\} \cup \{0\}$, and*

$$\|R(\lambda; A)\| \leq \frac{M}{|\lambda|}, \quad \textit{for } \lambda \in \Sigma \setminus \{0\}.$$

(iv) *$T(t)$ is differentiable for $t > 0$ and there exists a constant C such that*

$$\|AT(t)\| \leq \frac{C}{t}, \quad \textit{for } t > 0.$$

Fractional powers of operators

For the operator A, for which $-A$ generates an analytic semigroup $T(t)$, one can define the fractional power of A [89]. We assume that A is a densely defined closed linear operator for which

$$\rho(A) \subset \Sigma^{+} = \{\lambda : 0 < w < |\arg \lambda \leq \pi\} \cup U, \tag{1.1}$$

where U denotes a neighbourhood of zero, and

$$\|R(\lambda; A)\| \leq \frac{M}{1 + |\lambda|}, \quad \text{for } \lambda \in \Sigma^{+}. \tag{1.2}$$

For $w = \pi/2$ and $M = 1$, $-A$ generates a C_0-semigroup while for $w < \pi/2$, $-A$ generates an analytic semigroup.

For an operator A with the conditions (1.1) and (1.2), one can define negative fractional powers, $\eta > 0$, by the formula

$$A^{-\eta} = \frac{1}{2\pi \mathrm{i}} \int_{C} z^{-\eta} (A - zI)^{-1} dz, \tag{1.3}$$

where C denotes the path starting in the resolvent of A from $\infty e^{-\mathrm{i}\theta}$ to $\infty e^{\mathrm{i}\theta}$ for $w < \theta < \pi$, avoiding the negative real axis and the origin, and $z^{-\eta}$ is considered to be positive for real positive values of z. The integral (1.3) converges in the uniform operator topology for every $\eta > 0$, and thus defines a bounded linear operator $A^{-\eta}$.

Let us assume that A satisfies (1.1) and (1.2) with $w < \pi/2$. Then define

$$A^{\eta} = \begin{cases} (A^{-\eta})^{-1}, & \eta > 0, \\ I, & \eta = 0. \end{cases}$$

Theorem 1.3.15. ***(i)*** *$\eta_1 \geq \eta_2 > 0$ implies $D(A^{\eta_1}) \subset D(A^{\eta_2})$,*
(ii) *if $\eta_1, \eta_2 \in \mathbb{R}$, then $A^{\eta_1 + \eta_2} y = A^{\eta_1} A^{\eta_2} y$ for every $y \in D(A^{\eta})$ where $\eta = \max\{\eta_1, \eta_2, \eta_1 + \eta_2\}$.*

Lemma 1.3.6. *There exists a constant* $M_\eta > 0$ *such that* $\|A^{-\eta}\| \leq M_\eta$, *for* $\eta \in [0,1]$.

1.4 Fractional Differential Equations

1.4.1 Impulsive differential equations

Many evolution processes in applied sciences are represented by differential equations. The following cases cater to some problems which need a different handling: mechanical systems with impact, population dynamics, industrial robotics, natural disasters and so on, in which there is an abrupt change in the state. These phenomena involve short term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of the entire process. Thus, in models involving such perturbations, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses.

It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Therefore, impulsive differential equations or differential equations involving impulse effects, appear as a natural description of the observed evolution phenomena of numerous real world problems. For example, consider the problem of modelling a fish population in a hatchery. The natural growth of fish population is distributed by making catches at certain time intervals and by adding fresh breed. The natural growth of fish population is disturbed at some time intervals. Therefore, this problem involves impulses [88].

This kind of differential equations is mainly classified into two categories.
(i) Impulsive differential equations such as

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad t \neq t_i, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) &= \mathcal{I}_i(x(t_i^-)), \end{aligned}$$

where $x(t_i^+)$ and $x(t_i^-)$ represent the right- and left-hand limits at t_i, respectively, where t_i's are prefixed numbers and I_i are impulse operators, are used in modeling impulsive problems with fixed moments of impulse effect.
(ii) Another form of impulsive differential equation is represented by an equation of the following form:

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad h(t, x(t)) \neq 0, \\ \Delta x(t) = x(t^+) - x(t^-) &= \mathcal{I}(t, x(t)), \quad h(t, x(t)) = 0. \end{aligned}$$

Here, the jump occurs when the relation $h(t, x(t)) = 0$ is satisfied.

Sometimes impulses start abruptly at some point and remain active over certain time intervals. Such impulses are called non-instantaneous impulses. The importance of the study of non-instantaneous impulsive differential equations lies in diverse fields of applications such as in physics, biology, ecology, population dynamics and economics. A very well known application of noninstantaneous impulses is the one concerning the hemodynamical equilibrium of a person. In the case of a decompensation (e.g., high or low levels of glucose), one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, the situation can be interpreted as an impulsive action which starts abruptly and stays active on a finite time interval. Such equations were introduced by Hernández and O'Regan [60]. A mathematical model for non-instantaneous impulsive differential equations is as follows:

$$\begin{aligned} x'(t) &= Ax(t) + f(t, x(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, \dots, n, \\ x(t) &= g_i(t, x(t)), \quad t \in (t_i, s_i],\ i = 1, \dots, n, \\ x(0) &= x_0, \end{aligned}$$

where $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ defined on a Banach space X, $x_0 \in X$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \dots < t_n \leq s_n \leq t_{n+1} = b$ are pre-fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, \dots, n$ and $f \colon [0, b] \times X \to X$ is a suitable function. More details are available in [4, 60, 90].

1.4.2 Delay differential equations

In ordinary differential equations, both the unknown function and its all derivatives depend upon only one argument value. A functional differential equation is a general type of differential equation in which the unknown function occurs with different arguments. Delay differential equation is a functional differential equation in which some derivative of the unknown function at present time is expressed in terms of lower derivatives of the unknown function at present and earlier instants. Therefore, the solutions not only require the knowledge of the current state but also of the state at a certain previous time.

Many processes, both natural and man-made found in biology, medicine, economics, etc., involve time delays. One of these processes in nature is reforestation. A cut forest, after replantation, will take at least twenty years to reach maturity. For trees of a specific type, e.g., sandalwood, it would take even much longer time. Thus, any mathematical model for forest harvesting and recovery must have time delays incorporated in it.

Consider the following functional differential equation with finitely many argument deviations:

$$x^{(m)}(t) = f(t, x^{(m_1)}(t - g_1(t)), \dots, x^{(m_k)}(t - g_k(t))), \tag{1.4}$$

where $x(t) \in \mathbb{R}^n$, and both m_i's and g_i's are non-negative. Take $l = \max_{1\leq i\leq k} m_i$. Equation (1.4) is called a *retarded functional differential equation* if $l < m$, a *functional differential equation of neutral type* if $l = m$ and a *functional differential equation of advanced* *type* if $l > m$.

1.4.3 Nonlocal conditions

Consider the following mathematical model:

$$\begin{aligned} &x'(t) + Ax(t) = f(t, x(t)), \quad t \in (b, b+\epsilon], \\ &x(b) + g(t_1, t_2, \ldots, t_n, x(.)) = x_b, \end{aligned} \tag{1.5}$$

where $0 \leq t_1 < t_2 < \ldots < t_n$, $n \in \mathbb{N}$. Equation (1.5) represents a nonlocal initial condition. This study was initiated by Byszewski [27] in 1991, and he remarked that such type of conditions can be applied in physics with better effects than the classical initial conditions.

Deng [39] used the following nonlocal condition:

$$g(x) = \sum_{i=1}^{n} a_i x(t_i),$$

where $a_i \in \mathbb{R}$ are given constants and $0 \leq t_1 < t_2 < \ldots < t_n$, $n \in \mathbb{N}$. He concluded that the diffusion phenomenon of a small amount of gas in a transparent tube can give better results than approaching the usual initial condition $x(0) = x_0$. The function g can also take the following form [12, 28]:

(i) Let $F \in L^1(0, b)$ with $\int_0^b F(s)ds \neq 0$. Then g can be defined as

$$g(x) = \int_0^b F(s)x(s)ds.$$

(ii) Let $n \in \mathbb{N}$ and $t_i \in \mathbb{R}$ with $0 < t_1 < t_2 < \ldots < t_n < b$, $a_i \geq 0$, $\epsilon_i > 0$, $i = 1, 2, \ldots, n$. Then g can be defined by

$$g(x) = \sum_{i=1}^{n} \frac{a_i}{\epsilon_i} \int_{t_i - \epsilon_i}^{t_i} x(s)ds.$$

Nonlocal initial conditions give more realistic information as compared to classical initial conditions. Fractional differential equations with nonlocal initial conditions have wide applications in biology, physics, chemistry, etc. Therefore, there are several important works available in literature dealing with the study of solutions of differential equations with nonlocal initial conditions.

1.5 Controllability

Mathematical control theory is an application-oriented area of mathematics that deals with the basic principles underlying the analysis and design of control systems. Here the term "system" refers to a collection of objects or devices which are interrelated and which interact among themselves to produce various outputs in response to different inputs. It is an important field that is investigated by researchers - scientists and engineers alike. The purpose of this theory is to determine the targets so that one can drive the state of some dynamical system by means of a control parameter present in the given equation. To implement this, scientists and engineers build devices that incorporate various mathematical techniques. These devices range from Watt's steam engine designed during the seventeenth century to the sophisticated microprocessor controllers found in items such as CD players and automobiles or industrial robots and airplanes.

There are several control systems which are in use in our everyday life, and many systems are either natural or artificial (man-made) ones. Some examples of natural systems are – (i) Living organisms use control mechanisms in order to maintain essential variables such as body temperature or blood sugar levels at desired set points. (ii) Insect and animal populations are controlled by prey-predator relationship. (iii) On the other hand, washing machines, missiles, robots, etc. are examples of artificial systems irrespective of whether the control system is natural or man-made. Their main purpose is to control or regulate a particular variable within certain operating limits.

In control theory, the qualitative behaviours of dynamical systems are observability, controllability, stability and stabilizability. Controllability is an important concept in mathematical control theory. The general problem of controllability is to study whether or not in a given system, it is possible to bring any initial state to a given target state in an initially fixed time. Mathematically, it can be stated as: "*given a time* $T > 0$, *an initial state* x_0 *and a target state* x_*, *is it possible to find a control function (depending on time) that steers the solution of the system from* x_0 *to* x_* *in time* T?". Before studying any system, it is very important to know whether the system is controllable or not. If the system cannot be controlled completely, then there are different types of controllability that can be defined, such as approximate, null, local null, etc. [31]. Studies on qualitative behaviour of fractional control systems are important issues for many applied problems since the use of fractional order derivatives and integrals in control theory leads to better results than those of integer order systems.

1.6 Fundamental Concepts of Control Theory

1.6.1 Finite dimensional linear control systems

Let t_0, $t_1 \in \mathbb{R}$ be fixed with $t_0 < t_1$.

Autonomous system: The mathematical model for finite dimensional autonomous linear control system is given by

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t),\ t \in [t_0, t_1], \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{1.6}$$

where A and B are constant matrices of order $n \times n$ and $n \times m$, respectively. For each $t \in [t_0, t_1]$, $x(t) \in \mathbb{R}^n$ is called the state of the system and $u(t) \in \mathbb{R}^m$ is called its control. Let $x \in L^2([t_0, t_1], \mathbb{R}^n)$ and $u \in L^2([t_0, t_1], \mathbb{R}^m)$.
The solution of (1.6) is given by the integral equation

$$x(t) = e^{(t-t_0)A} + \int_{t_0}^{t} e^{(t-\xi)A} Bu(\xi) d\xi.$$

Equation (1.6) is said to be *controllable* over $[t_0, t_1]$, if for any pair of vectors x_0, $x_1 \in \mathbb{R}^n$, there exists a control function $u \in L^2([t_0, t_1], \mathbb{R}^m)$ such that the solution of (1.6) with the given initial condition $x(t_0) = x_0$ satisfies $x(t_1) = x_1$, that is,

$$x_1 = e^{(t_1-t_0)A} + \int_{t_0}^{t_1} e^{(t_1-\xi)A} Bu(\xi) d\xi.$$

The set of all points to which the initial state x_0 can be steered in time t_1 is called the *controllability space* or *reachable set*, and is denoted by R_{t_1}, i.e.,

$$R_{t_1} = \{x(t_1) \in \mathbb{R}^n \colon x(.) \text{ is the solution of (1.6)}\}.$$

The reachable set is a linear subspace of $\mathbb{R}^n$ and (1.6) is controllable in the interval $[t_0, t_1]$, if $R_{t_1} = \mathbb{R}^n$.

Theorem 1.6.1. *[67] The autonomous system (1.6) or the pair (A, B) is controllable iff the rank of the $n \times nm$ controllability matrix is n, that is,*

$$rank(B, AB, A^2B, \ldots, A^{n-1}B) = n.$$

If $rank(B) = r$, then the above condition reduces to

$$rank(B, AB, A^2B, \ldots, A^{n-r}B) = n.$$

This is known as Kalman rank condition for controllability.

Non-autonomous system: If the elements of the matrices A and B are piecewise continuous functions of time, then (1.6) is called a *non-autonomous (time varying) system.* In that case, (1.6) becomes

$$\left.\begin{aligned} &\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t),\ t \in [t_0, t_1], \\ &x(t_0) = x_0. \end{aligned}\right\} \tag{1.7}$$

The solution of (1.7) is given by

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \xi)x_0 B(\xi)u(\xi) d\xi,$$

where $\phi(t,t_0)$ is the *state transition matrix*, which can be obtained from the following Peano's series:

$$\phi(t,\xi) = I + \int_\xi^t A(s)ds + \int_\xi^t \int_\xi^{s_1} A(s_1)A(s_2)ds_2 ds_1 + \cdots$$

If A is independent of time, the above series gives $\phi(t,\xi) = e^{(t-\xi)A}$. The state transition matrix $\phi(t,t_0)$, for $0 \le t_0 \le \xi \le t \le t_1$, has the following properties:
(i) $\frac{d}{dt}\phi(t,t_0) = A(t)\phi(t,t_0)$,
(ii) $\phi(t,t) = I$, the identity matrix,
(iii) $\phi^{-1}(t,t_0) = \phi(t_0,t)$,
(iv) $\phi(t,\xi)\phi(\xi,t_0) = \phi(t,t_0)$.

Definition 1.6.1. *The controllability matrix of the non-autonomous control system* $G\colon L^2([t_0,t_1],\mathbb{R}^m) \to \mathbb{R}^n$ *is defined as*

$$Gu = \int_{t_0}^{t_1} \phi(t_1,\xi)B(\xi)u(\xi)d\xi.$$

The controllability Grammian matrix $W_{t_0}^{t_1}\colon \mathbb{R}^n \to \mathbb{R}^n$ *is defined as*

$$W_{t_0}^{t_1} = \int_{t_0}^{t_1} \phi(t_1,\xi)B(\xi)B^*(\xi)\phi^*(t_1,\xi)d\xi, \tag{1.8}$$

where B^* *and* $\phi^*(t_1,\xi)$ *denote the conjugate transpose of the matrices* B *and* $\phi(t_1,\xi)$*, respectively.*

Note: The controllability Grammian matrix is symmetric of order n.

Theorem 1.6.2. *The linear control system (1.6) or (1.7) (i.e., autonomous or non-autonomous) is controllable iff the controllability Grammian matrix* $W_{t_0}^{t_1}$ *defined in (1.8) is invertible.*

Theorem 1.6.3. *The following conditions are equivalent:*
(i) *(1.7) is controllable on* $[t_0,t_1]$*,*
(ii) $W_{t_0}^{t_1}$ *is positive definite,*
(iii) $rank(B, AB, A^2B, \dots, A^{n-1}B) = n$.

1.6.2 Infinite dimensional control systems

In the case of infinite dimensional systems, two basic concepts of controllability can be distinguished, viz., exact controllability and approximate controllability. This is strongly related to the fact that, in infinite dimensional spaces, there exist linear subspaces which are not closed.

The mathematical model for infinite dimensional linear control system is given by

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t),\ t \in [t_0, t_1], \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{1.9}$$

where the state $x(t)$ of the system at time t takes its values in a Banach space V while the control function $u(t)$ takes its values in another Banach space U. The operator $A \colon D(A) \subset V \to V$ is a closed, linear and densely defined, but not necessarily a bounded operator and $B \colon U \to V$ is a bounded linear operator.

For any $x_0 \in V$, the mild solution of (1.9) is a function $x \in C([t_0, t_1], V)$ satisfying the following integral equation:

$$x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - \xi)Bu(\xi)d\xi,$$

where $\{T(t)\}_{t \geq 0}$ is the C_0-semigroup generated by A.

Definition 1.6.2. *(1.9) is said to be approximately controllable in the interval $[t_0, t_1]$ if, for given any $\epsilon > 0$ and initial and final points x_0, $x_1 \in V$, there exists an admissible control $u(t)$ on $[t_0, t_1]$ steering x_0 along a trajectory (mild solution) $x(t)$ of (1.9) to an $\epsilon-$ neighbourhood of x_1 such that*

$$\|x(t) - x_1\| \leq \epsilon.$$

If $\epsilon = 0$, then the above definition gives exact controllability of (1.9).

For (1.9), the controllability map $G \colon L^2([t_0, t_1], U) \to V$ and the controllability Grammian $W_{t_0}^{t_1} \colon V \to V$ are defined, respectively, as

$$Gu = \int_{t_0}^{t} T(t - \xi)Bu(\xi)d\xi,$$

and

$$W_{t_0}^{t_1} = \int_{t_0}^{t} T(t - \xi)BB^*T^*(t - \xi)d\xi.$$

These two maps satisfy the following properties:
(i) G is a bounded linear map from $L^2([t_0, t_1], U)$ to V,
(ii) the adjoint operator G^* of G is defined as $(G^*x)(\xi) = B^*T^*(t - \xi)$ on $[0, t_1]$,
(iii) $W_{t_0}^{t_1} = GG^*$ is a bounded linear operator on V.

Theorem 1.6.4. *(1.9) is exactly controllable on $[t_0, t_1]$ iff any one of the following conditions holds for some $\delta > 0$ and all $x \in V$,*
(i) range$(G) = V$,

(ii) *null*$(G) = \{0\}$ *and range*(G) *is closed,*
(iii) $\|G^*x\|^2 = \int_{t_0}^{t_1}\|(G^*x)(\xi)\|_U^2 d\xi \geq \delta\|x\|_V^2$*,*
(iv) $\int_{t_0}^{t_1}\|B^*T^*(\xi)x\|_U^2 d\xi \geq \delta\|x\|_V^2$*,*
and if V *is a Hilbert space, then the above conditions are equivalent to*
(v) $\langle W_{t_0}^{t_1}x, x\rangle_V \geq \delta\|x\|_V^2$*.*

Triggiani [108] proved that, if A generates a compact C_0-semigroup $T(t)$, then a linear system can never be exactly controllable in an infinite dimensional space.

Theorem 1.6.5. *(1.9) is approximately controllable on* $[t_0, t_1]$ *iff any one of the following conditions holds for some* $\delta > 0$ *and all* $x \in V$*,*
(i) $\overline{range(G)} = V$*,*
(ii) *null*$(G^*) = \{0\}$ *and range*(G) *is closed,*
(iii) $W_{t_0}^{t_1}$ *is positive definite,*
(iv) $B^*T^*(\xi)x = 0$ *on* $[t_0, t_1]$ *implies* $x = 0$*.*

In literature, most of the controllability results for nonlinear systems are concerned with the semilinear systems, i.e., the systems consisting of a linear part and a nonlinear part. Next, we briefly discuss semilinear control systems.

Semilinear control systems: Consider the semilinear control system given by

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) + f(t, x(t)),\ t \in [t_0, t_1],\\ x(t_0) &= x_0, \end{aligned}\right\} \tag{1.10}$$

where A and B are similar as defined earlier and $f\colon [t_0, t_1] \to V$ is a nonlinear function.

For a given control function $u(t) \in L^2([t_0, t_1], U)$, the mild solution of the system (1.10) is given by

$$x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - \xi)[Bu(\xi) + f(\xi, x(\xi))]d\xi.$$

Definition 1.6.3. *The set of all possible final points*

$$R_{t_1}(f) = \{x(t_1)\colon x(.) \text{ is the mild solution of } (1.10) \text{ for } u(t) \in L^2([t_0, t_1], U)\}$$

is called the reachable set of the semilinear system (1.10).

In terms of the reachable set, we present the following definition of exact controllability and approximate controllability:

Definition 1.6.4. *(1.10) is said to be exactly controllable iff* $R_{t_1}(f) = V$ *and approximately controllable iff* $\overline{R_{t_1}(f)} = V$*.*

The following important result on controllability of the semilinear systems is given by Naito [85].

Theorem 1.6.6. *The semilinear control system is approximately controllable under the following conditions*:
(i) *$f(t, x)$ is Lipschitz continuous in x,*
(ii) *f is uniformly bounded,*
(iii) *the C_0-semigroup $\{T(t)\}_{t>0}$ is compact,*
(iv) *for every $p \in L^2([t_0, t_1], V)$, there exists some $q \in \overline{range(B)}$ such that $Lp = Lq$, where $L\colon L^2([t_0, t_1], V) \to V$ is a bounded linear operator defined by*

$$Lp = \int_{t_0}^{t_1} T(t - \xi)p(\xi)d\xi.$$

Condition (iv) of the above theorem implies that the corresponding linear system (when $f \equiv 0$ in (1.10)) is approximately controllable. For more details, one can refer to the proof in [85].

1.6.3 Infinite dimensional fractional order systems

The mathematical model of an infinite dimensional fractional order linear control system can be written in the following form:

$$\left.\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} &= Ax(t) + Bu(t),\ \alpha \in (0, 1],\ t \in [0, t_1], \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{1.11}$$

where the control function $u(t) \in U$, and $B\colon U \to V$ is a bounded linear operator.

For any $x_0 \in V$, the mild solution of (1.11) is a function $x \in C([0, t_1], V)$ satisfying the following equation:

$$x(t) = S_\alpha(t)x_0 + \int_{t_0}^{t} (t - \xi)^{\alpha - 1} P_\alpha(t - \xi)Bu(\xi)d\xi,$$

where

$$S_\alpha(t)x = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)xd\theta, \tag{1.12}$$

$$P_\alpha(t)x = \int_0^\infty \alpha\theta\xi_\alpha(\theta)T(t^\alpha\theta)xd\theta. \tag{1.13}$$

Here

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\bar{w}_\alpha(\theta^{-\frac{1}{\alpha}}) \tag{1.14}$$

$$\bar{w}_\alpha(\theta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\theta^{(-\alpha n-1)}\frac{\Gamma(\alpha n + 1)}{n!}\sin(n\pi\alpha),\ \theta \in (0, +\infty), \tag{1.15}$$

and $\xi_\alpha(\theta)$ is a probability density function on $(0, +\infty)$ satisfying

$$\xi_\alpha(\theta) \geq 0, \quad \int_0^\infty \xi_\alpha(\theta)d\theta = 1, \quad \text{and} \quad \int_0^\infty \theta\xi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}.$$

The controllability operator associated with (1.11) is defined by

$$\Gamma_0^{t_1} = \int_0^{t_1} (t_1 - \xi)^{\alpha-1} P_\alpha(t_1 - \xi) B B^* P_\alpha^*(t_1 - \xi) d\xi,$$

where B^* and $P_\alpha^*(t)$, respectively, denote the adjoint of B and $P_\alpha(t)$. Let

$$R(\lambda, \Gamma_0^{t_1}) = (\lambda I + \Gamma_0^{t_1})^{-1}, \text{ for } \lambda > 0.$$

Lemma 1.6.1. *The fractional order linear control system (1.11) is approximately controllable on* $[0, t_1]$ *iff* $\lambda R(\lambda, \Gamma_0^{t_1}) \to 0$ *as* $\lambda \to 0^+$ *in the strong operator topology.*

The mathematical model of an infinite dimensional fractional order semilinear control system can be written as

$$\left.\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} &= Ax(t) + Bu(t) + f(t, x(t)), \ \alpha \in (0,1), \ t \in [0, t_1], \\ x(t_0) &= x_0. \end{aligned}\right\} \tag{1.16}$$

with A as the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Hilbert space V.

The mild solution of system (1.16) is given by

$$x(t) = S_\alpha(t)x_0 + \int_{t_0}^{t} (t - \xi)^{\alpha-1} P_\alpha(t - \xi)[Bu(\xi) + f(\xi, x(\xi))]d\xi.$$

Equation (1.16) is approximately controllable iff the following conditions are satisfied:
(i) the C_0-semigroup $\{T(t)\}_{t>0}$ generated by the linear part of A is compact,
(ii) for each $t \in [0, t_1]$, $f(t, .)\colon V \to V$ is continuous, and for each $x \in C([0, t_1])$, $f(., x)\colon [0, t_1] \to V$ is strongly measurable,
(iii) there exist a constant $q_1 \in [0, \alpha]$, and a function $m \in L^{\frac{1}{q_1}}([0, t_1], \mathbb{R}^+)$ such that $\|f(t, x)\|_V \leq m(t)$ for all $x \in V$ and almost all $t \in [0, t_1]$,
(iv) the function $f\colon [0, t_1] \times V \to V$ is continuous and uniformly bounded.

1.7 Literature Survey

Fractional differential equations are generalizations of ordinary differential equations to arbitrary non-integer order. Fractional differential equations are considered as an alternative representation for nonlinear differential equations [24]. It has attracted considerable interest because of its ability to model complex phenomena. Differential and integral equations and dynamical systems of fractional order can mathematically model real life phenomena more adequately than integer-order models. These equations capture nonlocal relations in space and time with power-law memory kernels. Due to its extensive applications in engineering and science, research in this area has grown significantly all around the world. Some of its applications can be found in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media [58] and in fluid traffic models. Fractional derivative can eliminate the deficiency arising from the assumption of continuum traffic flow.

There are several books and papers in literature devoted to the solvability of linear fractional differential equations in terms of special functions (e.g., [29,81]). In [121], Zhang proved that the solution of the homogeneous Caputo fractional differential equation

$$ {}^{C}D^{\alpha}x(t) = 0, \quad \alpha > 0, $$

is given by

$$ x(t) = a_0 + a_1 t + \ldots + a_{n-1}t^{n-1}, $$

with $a_i \in \mathbb{R}$ where $i = 0, 1, \ldots, n-1$ and $n-1 \leq \alpha < n$. In [38, 40, 42], the qualitative properties were studied of the solution of the fractional differential equation of the form

$$ \frac{d^{\alpha}x(t)}{dt^{\alpha}} = f(t, x(t)), $$

where $\alpha \in (0,1)$ and f is a given function.

The Cauchy problem for Riemann-Liouville ordinary fractional differential equation is of the form

$$ \left.\begin{aligned} {}^{RL}D^{\alpha}_{a+}x(t) &= f(t, x(t)), \quad \alpha > 0, \ \ t > a, \\ {}^{RL}D^{\alpha-i}_{a+}x(a+) &= a_i, \quad i = 1, \ldots, n, \ \ n-1 \leq \alpha < n. \end{aligned}\right\} \tag{1.17} $$

Many have studied problem (1.17) and its particular case (i.e., $\alpha \in (0,1)$), which is essentially based on reducing (1.17) to the following Volterra integral equation of second kind:

$$ x(t) = \sum_{i=1}^{n} \frac{a_i}{\Gamma(\alpha - i + 1)}(t-a)^{\alpha-i} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s))ds. $$

However, the investigations were not complete either due to the errors in the proof of the equivalence of the initial value problems and the Volterra integral equations or in the proof of the uniqueness of the solution.

Delbosco and Rodino [38] considered the following nonlinear problem:

$$ \begin{aligned} {}^{RL}D^{\alpha}_{0+}x(t) &= f(t, x(t)), \quad \alpha > 0, \ \ t \in [0, b], \\ x^{(i)}(0) &= a_i, \quad i = 1, \ldots, n-1, \ \ n-1 \leq \alpha < n, \end{aligned} $$

where f is a continuous function. They proved the equivalence of this problem to the corresponding Volterra integral equation and used Schauder's fixed point theorem to prove the existence of its solution.

Hayek et al. [57] considered the following problem:

$$\begin{aligned} {}^{RL}D^{\alpha}_{0+}x(t) &= f(t, x(t)), \quad \alpha \in (0,1], \\ x(c) &= a, \quad c > 0, \quad a \in \mathbb{R}^n, \end{aligned}$$

where $x(t)$ is a real-valued vector function, $f(t,x)$ is continuous and Lipschitzian with respect to x. They applied contraction theorem to prove the existence and uniqueness of a continuous solution $x(t)$ to this problem. In particular, they obtained the results for a system of linear differential equations

$$\begin{aligned} {}^{RL}D^{\alpha}_{0+}x(t) &= A(t)x(t) + B(t), \quad \alpha \in (0,1], \\ x(c) &= a, \quad c > 0, \quad a \in \mathbb{R}^n, \end{aligned}$$

with continuous matrices $A(t)$ and $B(t)$.

For Caputo fractional derivative, the Cauchy problem takes the following form:

$$\left.\begin{aligned} {}^{C}D^{\alpha}x(t) &= f(t, x(t)), \ \ t > 0, \ \alpha > 0, \\ x^{(i)}(0) &= a_i, \ \ i = 0, 1, \dots, n-1, \ n-1 < \alpha \leq n, \ n \in \mathbb{N}. \end{aligned}\right\} \tag{1.18}$$

Kilbas and Marzan [70] proved that (1.18) is equivalent to the following Volterra integral equation of second kind:

$$x(t) = \sum_{i=0}^{n-1} \frac{a_i}{\Gamma(i+1)}(t-a)^i + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s))ds, \ \ t > a.$$

Daftardar-Gejji and Babakhani [35] studied the existence, uniqueness and stability of the fractional differential equation

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}x(t) &= Ax(t), \ \ \alpha \in (0,1), \\ x(0) &= x_0, \end{aligned}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$.

El-Borai [41] considered the Cauchy problem

$$\left.\begin{aligned} \frac{d^{\alpha}x(t)}{dt^{\alpha}} &= Ax(t) + f(t), \ \ \alpha \in (0,1], \\ x(0) &= x_0 \in D(A), \end{aligned}\right\} \tag{1.19}$$

where $A\colon D(A) \subset X \to X$ is a closed linear operator which generates an analytic semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space X. He showed that, if f satisfied Hölder condition with exponent $\beta \in (0,1]$, then the unique solution of (1.19) is given by

$$x(t) = \int_0^\infty M_\alpha(\theta)T(t^\alpha\theta)x_0 d\theta + \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1}M_\alpha(\theta)T((t-s)^\alpha\theta)f(s)d\theta ds.$$

There are several works available in literature which have investigated the existence of mild solutions of different types of fractional differential equations. Jaradat et al. [63] considered the following Volterra-Fredholm integro-differential equation:

$$\left.\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + f(t, x(t), Gx(t), Hx(t)), \ \ t > t_0, \ \alpha \in (0,1], \\ x(0) &= x_0, \end{aligned}\right\} \tag{1.20}$$

where A generates a strongly continuous semigroup $\{T(t)\}_{t>0}$ on a Banach space X. They defined the mild solution of (1.20) as a continuous function x satisfying the integral equation

$$x(t) = T(t-t_0)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}T(t-s)f(s, x(s), Gx(s), Hx(s))ds. \tag{1.21}$$

Here the mild solution was defined by generalizing the mild solution definition of integer order evolution equations, which was not considered appropriate by Lin and Jia [74] and Hernández et al. [61]. Using the concept of resolvent operator, Hernández et al. [61] introduced a new concept of mild solution and also provided examples showing that the formula (1.21) for mild solutions was not correct.

Zhou and Jiao [126] considered the Cauchy problem

$$\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + f(t, x(t)), \ \ t \in (0,b], \ \alpha \in (0,1), \\ x(0) + g(x) &= x_0, \end{aligned}$$

where A generates a strongly continuous semigroup on a Banach space. Using Laplace transform and probability density function, they defined the mild solution of the above problem. The existence and uniqueness of mild solution were studied with the help of Banach fixed point theorem, Schauder theorem and Krasnoselskii's fixed point theorem.

In [127], Zhou et al. considered the Riemann-Liouville evolution equation

$$\left.\begin{aligned} {}^{RL} D^\alpha_{0+}x(t) &= Ax(t) + f(t, x(t)), \ \ \text{a.e. } t \in [0,b], \ \alpha \in (0,1), \\ (J^{1-\alpha}_{0+}x)(0) + g(x) &= x_0, \end{aligned}\right\} \tag{1.22}$$

with A being the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in a Banach space X. Using Laplace transform and probability density function, they defined the mild solution of (1.22) as a function $x \in C((0,b], X)$ satisfying

$$x(t) = t^{\alpha-1}P_\alpha(t)(x_0 - g(x)) + \int_{t_0}^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s, x(s))ds, \ \ t \in (0,a],$$

where $P_\alpha(t)$ is defined in (1.13). Subsequently, a reasonable number of researchers have used this approach to study the existence of mild solutions of fractional evolution equations of order $\alpha \in (0,1)$. For more information on mild solutions of fractional differential equations of order $\alpha \in (1,2)$, the readers are referred to the works in a number of articles such as [75, 76, 99].

Initial value problem for Hilfer fractional differential equation takes the form [103]

$$\begin{gathered} {}^H D^{\alpha,\beta} x(t) = f(t), \\ \lim_{t\to 0^+} \frac{d^i}{dt^i}(J^{(1-\beta)(n-\alpha)}x)(t) = a_k, \quad i = 0, 1, \ldots, n-1, \end{gathered}$$

where $n-1 < \alpha \leq n$, $\beta \in [0,1]$ and $n \in \mathbb{N}$.
Furati et al. [52] considered the initial value problem

$$\left.\begin{aligned} {}^H D^{\alpha,\beta}_{a^+} x(t) &= f(t, x(t)), \quad t \in (a,b], \ \alpha \in (0,1), \ \beta \in [0,1], \\ J^{(1-\alpha)(1-\beta)} x(a^+) &= x_a. \end{aligned}\right\} \tag{1.23}$$

They proved the equivalence of (1.23) with the following Volterra equation of the second kind:

$$x(t) = \frac{x_a}{\Gamma(\alpha+\beta-\alpha\beta)}(t-a)^{-(1-\alpha)(1-\beta)} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s))ds, \quad t > a$$

in an appropriate space of functions.
Gu and Trujillo [54] considered the following fractional evolution equation with Hilfer derivative:

$$\left.\begin{aligned} {}^H D^{\alpha,\beta}_{0+} x(t) &= Ax(t) + f(t, x(t)), \quad t \in (0,b], \ \alpha \in (0,1), \ \beta \in [0,1], \\ J^{(1-\alpha)(1-\beta)}_{0+} x(0) &= x_0, \end{aligned}\right\} \tag{1.24}$$

where A generates a strongly continuous semigroup on a Banach space X. Using Laplace transform and probability density function, they defined the mild solution of (1.24) as a function $x \in C((0,b], X)$ satisfying

$$x(t) = T_{\beta,\alpha}(t) + \int_0^t S_\alpha(t-s) f(s, x(s))ds, \quad t \in (0,b],$$

where

$$T_{\beta,\alpha}(t) = J^{\beta(1-\alpha)}_{0+} S_\alpha(t), \ S_\alpha(t) = t^{\alpha-1} R_\alpha(t), \ R_\alpha(t) = \int_0^\infty \alpha\theta\xi_\alpha(\theta) T(t^\alpha\theta) d\theta.$$

Further, ξ_α and $\bar{w}_\alpha$ have the same representation as in (1.14) and (1.15), respectively. Sufficient conditions for the existence of mild solutions were obtained using Arzelà-Ascoli theorem and Hausdorff measure of noncompactness. After that, different types of fractional differential equations involving Hilfer fractional derivative have been studied and we refer the readers to [53], [68], etc.

Milman and Myskis [82] introduced the concept of impulsive differential equations. Based on their work, several monographs have been published by many authors like Samoilenko and Perestyuk [95], Lakshmikantham et al. [73], Benchohra et al. [20], and others. The first paper on impulsive fractional differential equations was published by Agrawal et al. in 2008 [5]. They used the Caputo fractional derivative and obtained sufficient conditions for the existence and uniqueness of solution with the help of Banach fixed point theorem, Schaefer's fixed point theorem, nonlinear alternative of Leray-Schauder type and Burton-Kirk's fixed point theorem.

Benchohra and Slimani [21] studied the following problem:

$$\left.\begin{aligned} {}^{C}D^{\alpha}x(t) &= f(t,x(t)),\ \ t\in[0,b]\setminus\{t_1,\dots,t_l\},\ \alpha\in(0,1],\\ \Delta x|_{t=t_\kappa} &= \mathcal{I}_\kappa(x(t_\kappa^-)),\ \ \kappa=1,2,\dots,l,\\ x(0) &= x_0. \end{aligned}\right\} \tag{1.25}$$

A function $x\in PC([0,b],\mathbb{R})$ is called a solution of (1.25) if x satisfies the following integral equation:

$$x(t)=\begin{cases} x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds,\ \ t\in[0,t_1],\\ x_0+\frac{1}{\Gamma(\alpha)}\displaystyle\sum_{0<t_\kappa<t}\int_{t_{\kappa-1}}^{t_\kappa}(t_\kappa-s)^{\alpha-1}f(s,x(s))ds+\frac{1}{\Gamma(\alpha)}\int_{t_\kappa}^{t}(t-s)^{\alpha-1}f(s,x(s))ds\\ +\displaystyle\sum_{0<t_\kappa<t}\mathcal{I}_\kappa(x(t_\kappa^-)),\ \ t\in(t_\kappa,t_{\kappa+1}],\ \kappa=1,2,\dots,l. \end{cases}$$

They discussed the existence and uniqueness of solution using contraction theorem, Schaefer's fixed point theorem and Leray-Schauder type fixed point theorem. They used the definition of classical Caputo derivative and revised it in each sub-interval $(t_k,t_{k+1}]$ for the first equation of (1.25), where the impulses started at the impulsive points t_k, $k=1,2,\dots,l$.

Motivated by the above two works, Mophou [83] considered the following semilinear fractional differential equation:

$$\left.\begin{aligned} {}^{C}D^{\alpha}x(t) &= Ax(t)+f(t,x(t)),\ \ t\in[0,b]\setminus\{t_1,\dots,t_l\},\ \alpha\in(0,1),\\ \Delta x|_{t=t_\kappa} &= \mathcal{I}_\kappa(x(t_\kappa^-)),\ \ \kappa=1,2,\dots,l,\\ x(0) &= x_0, \end{aligned}\right\} \tag{1.26}$$

where A generates a C_0-semigroup $\{T(t)\}_{t\geq0}$ on the Banach space X. She used the definition of mild solution for semilinear initial value problems given by Jaradat et al. [63] to introduce the definition of mild solution for the impulsive fractional order differential equation (1.26). She defined the mild solution of (1.26) as a function $x\in PC([0,b],X)$

satisfying the integral equation

$$x(t) = T(t)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_\kappa<t} \int_{t_{\kappa-1}}^{t_\kappa} (t_\kappa - s)^{\alpha-1} T(t-s) f(s, x(s)) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_\kappa}^{t} (t-s)^{\alpha-1} T(t-s) f(s, x(s)) ds + \sum_{0<t_\kappa<t} T(t-t_\kappa) \mathcal{I}_\kappa(x(t_\kappa^-)) \ .$$

Shu et al. [97] pointed out the errors in [83]: the definition of mild solution was not well defined, because classical solutions of the impulsive fractional differential equations do not satisfy the definition of a mild solution, and also the semigroup property $T(t+s) = T(t)T(s)$ for the system is not used correctly. By using fixed point technique, they proved the existence of the mild solution of the problem (1.26) with the assumption that A is a sectorial operator on the Banach space X. However, Shu and Shi [98] proved that the definition of mild solution given in [97] was wrong and presented a correct formula of solution.

In [50], Fečkan et al. pointed out that the formula of solutions for impulsive fractional differential equations in [9,13] was incorrect and presented their formula. They considered the following impulsive fractional differential equation:

$$\left.\begin{aligned} {}^{C}D^{\alpha}x(t) &= f(t, x(t)), \ \ t \in [0, b] \setminus \{t_1, \dots, t_l\}, \ \alpha \in (0,1), \\ \Delta x|_{t=t_\kappa} &= x_\kappa, \ \ \kappa = 1, 2, \dots, l, \\ x(0) &= x_0, \end{aligned}\right\} \tag{1.27}$$

and suggested the following formula for the solution:

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds, \ \ t \in [0, t_1], \\ x_0 + \sum_{i=1}^{\kappa} x_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds, \ \ t \in (t_\kappa, t_{\kappa+1}], \ \kappa = 1, 2, \dots, l. \end{cases}$$

They used the generalized Caputo fractional derivative with lower bound at zero denoted by ${}^{C}D^{\alpha}_{0,t}$ for equation (1.27). Since the generalized Caputo derivative has fixed lower bound at zero, they did not change the lower bound again and again in the definition of the Caputo derivative for the first equation in (1.27).

Wang et al. [110] expressed their apprehension for the counterexample provided by Fečkan et al. [50] and also did not approve the viewpoint in [50, 115]. However, Fečkan et al. [49] did not accept the above and provided detailed justification in support of the counterexample in [50].

In the fractional derivative ${}^{C}D^{\alpha}_{a}$, a is called a base point. A fractional differential equation with one base point is called a single base point fractional differential equation and if it contains more than one base point, then the differential equation is called a multiple base point fractional differential equation. From the above discussion, it can be seen that there are two types of impulsive fractional differential equation present in

literature:
(i) fractional derivative with single base point : $D^\alpha = D_0^\alpha$,
(ii) fractional derivative with multiple base points, i.e., $D^\alpha = D_*^\alpha$.

In [112], Wang et al. considered the following fractional evolution equation with impulsive conditions:

$$\begin{aligned} {}^C D^\alpha x(t) &= Ax(t) + f(t, x(t)), \;\; t \in [0, b] \setminus \{t_1, \dots, t_l\}, \; \alpha \in (0, 1), \\ \Delta x|_{t=t_\kappa} &= x_\kappa, \;\; \kappa = 1, 2, \dots, l, \\ x(0) &= x_0, \end{aligned}$$

where A is the infinitesimal generator of a C_0-semigroup on a Banach space X. With the help of semigroup theory, Laplace transform and probability density function, they introduced a new concept of mild solution. They extended the results to semilinear fractional evolution equation with nonlocal initial condition and impulsive conditions. Thereafter, many have used this approach to study the mild solution of different types of impulsive fractional differential equations.

In [60], Hernández and O'Regan introduced a new class of impulsive differential equations, called non-instantaneous impulsive differential equations, to describe certain evolutionary processes in pharmacotherapy. Kumar et al. [72] considered the following fractional non-instantaneous differential equation:

$$\begin{aligned} {}^C D^\alpha x(t) + Ax(t) &= f(t, x(t), x(g(t))), \;\; t \in (s_i, t_{i+1}], \; i = 0, 1, \dots, m, \; \alpha \in (0, 1), \\ x(t) &= h_i(t, x(t)), \;\; t \in (t_i, s_i], \; i = 1, 2, \dots, m, \\ x(0) &= x_0, \end{aligned}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ on a Banach space X, the impulses start suddenly at the points t_i and their action continues on the interval $[t_i, s_i]$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots < t_n \leq s_n \leq t_{n+1} = b$. By using probability density function, they defined the mild solution of the above problem and utilizing analytic semigroup theory, contraction principle and Krasnoselskii's fixed point theorem, they discussed the existence and uniqueness of the mild solution. For more works on fractional non-instantaneous impulsive differential equations we refer the readers to [25], [34], [116], etc.

A good number of works has also been accomplished with respect to controllability. Now we focus on some important works carried out on controllability.

Kalman [66, 67] introduced the concept of controllability for finite dimensional linear control systems and proved the controllability under a rank condition of the controllability matrix. In 1965, Hermes [59] obtained controllability results for a class of nonlinear systems by using fixed point method.

In 1967, Tarnove [101] suggested that the controllability of nonlinear systems could be studied by investigating the existence of a fixed point of a certain set-valued map. He used the Bohnenblust-Karlin fixed point theorem to obtain the sufficient conditions for A-controllability of a nonlinear system

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)),$$

where A is a non-empty, closed, bounded, convex subset of continuous functions. A system is said to possess A-controllability if there exists a solution of the system belonging to A. This idea was used by Dauer [36] for systems of the form

$$\frac{dx(t)}{dt} = f(t, x(t)) + g(t, u(t))$$

in finite dimensional spaces.

Fattorini [46] considered a more general model of the form

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

and studied the controllability for the case when A was the infinitesimal generator of a C_0-semigroup. In [47], he considered the same model and derived a necessary and sufficient condition for the approximate controllability for the case when A was self-adjoint, semibounded above and defined on a Hilbert space with the dynamical system having only a finite number of scalar controls.

Triggiani [104] extended the classical theory of controllability and observability in finite dimensional spaces to linear abstract systems defined on infinite dimensional Banach spaces, under the assumption that the operator acting on the state was bounded. In [105], he showed that exact controllability in finite time for linear control system given on an infinite dimensional, separable Banach spaces in integral form (mild solution) would never arise using L_1-controls, if the operator through which the control acted on the system was compact. This improved his previous results with the removal of the assumption that the state space required to have a basis. In [106], he showed that, for larger classes of systems of physical interest (classical self-adjoint boundary value problems, delay equations, etc.), even a weaker assumption than approximate controllability guarantees stabilizability, that is, controllability on suitable finite dimensional subspace gives stabilizability on the whole space. Further, in [107] he showed that exact controllability in finite time for linear control systems given on an infinite dimensional Banach space in integral form (mild solution) could never arise while using locally L_1-controls if the associated C_0-semigroup was compact, for all $t > 0$.

In [123], Zhou obtained a sufficient condition for the approximate controllability of the semilinear control system under the assumption that the linear control system was approximately controllable. The approximate controllability results were proved for the case when the range of the control operator B satisfied an inequality condition.

In 2009, Bettayeb and Djennoune [23] described some new results on the controllability and observability of linear dynamical systems involving a fractional derivative. It was shown that the input control energy required to drive the state in a given direction and the energy available at the output were related to the observability and the controllability Grammians, respectively.

Sakthivel et al. [94] studied the following control system:

$$\begin{aligned} {}^{C}D_t^{\alpha}x(t) &= Ax(t) + Bu(t) + f(t, x(t)), \ \ t \in [0, b], \ \alpha \in (0, 1), \\ x(0) + g(x) &= x_0, \end{aligned}$$

with A as the infinitesimal generator of a strongly continuous semigroup on a Hilbert space X. By using the semigroup theory, controllability Grammian and Schauder's fixed point theorem, a new set of sufficient conditions was formulated for the approximate controllability of the system under the assumption that the associated linear system was approximately controllable.

Liu and Li [77] considered the following control system with Riemann-Liouville fractional derivative:

$$\begin{aligned} {}^{RL}D_t^{\alpha}x(t) &= Ax(t) + Bu(t) + f(t, x(t)), \ \ t \in (0, b], \ \alpha \in (0, 1], \\ J_t^{1-\alpha}x(t)|_{t=0} &= x_0, \end{aligned}$$

with A generating a differentiable C_0-semigroup on a Banach space X. Using a generalized contraction type fixed point theorem, the existence and uniqueness of mild solution was established. Under an inequality condition on the operator B and using a sequential approach, the approximate controllability of the above system was proved.

Haq and Sukavanam [56] considered the following integro-differential equation with damping and Riemann-Liouville fractional derivatives:

$$\begin{aligned} {}^{RL}D_t^{\alpha}x(t) + \lambda {}^{RL}D_t^{\beta}x(t) &= Ax(t) + Bu(t) + f\Big(t, x(t), \int_0^t h(t, s, x(s)\Big), \ \ t \in (0, b], \\ J_t^{1-\alpha}x(t)|_{t=0} &= x_0, \end{aligned}$$

where $0 < \beta < \alpha \leq 1$ and A a densely defined linear operator which generated a Riemann-Liouville (α, β, λ) fractional resolvent. Using the approach of Liu and Li [77] and the theory of Riemann-Liouville fractional resolvent, the approximate controllability of the above system was established.

Lv and Yang [78] studied the approximate controllability of a fractional control system of the form

$$\begin{aligned} &{}^{H}D_t^{\alpha,\beta}x(t) + Ax(t) = f(t, x(t)) + (Bu)(t), \ \ t \in (0, b], \ \alpha \in (1, 2), \ \beta \in [0, 1], \\ &\lim_{t \to 0^+} (J_{0^+}^{(1-\alpha)(2-\alpha)}x)(t) = 0, \ \ \lim_{t \to 0^+} \frac{d}{dt}(J_{0^+}^{(1-\alpha)(2-\alpha)}x)(t) = a_1, \end{aligned}$$

where A is a sectorial operator of angle $\Big[0, (1-\frac{\alpha}{2})\pi\Big)$ on a Banach space. For more works on the approximate controllability of different types of fractional order systems, we refer the readers to [120], [79], [93], etc.

1.8 Motivation

Having studied various works carried out by earlier researchers and also realizing the extensive application of fractional differential equations in various disciplines, it was realized that further study of certain classes of fractional differential equations will add to this growing field. It also came to the notice that still a number of issues regarding various fractional differential equations corresponding to different types of derivatives are still either unattempted or unresolved. It has given huge impetus to us to take up four works in this thesis. To add to it, since approximate controllability finds an important place in many control systems, a motivation to step into the same occurred which has resulted the last work of the thesis.

1.9 Outline

The objective of this thesis is to investigate the existence and uniqueness of mild solutions and integral solutions, and to establish the approximate controllability of some fractional differential equations involving different fractional derivatives.

The present chapter begins with a historical review of fractional calculus and its examples. It contains some definitions and results of fractional calculus, control theory, nonlinear analysis, which are required in the subsequent chapters. The concept of impulsive and delay differential equations are also discussed. Various relevant works are discussed which open the door for the present study.

Chapter 2 deals with a class of non-autonomous Hilfer fractional evolution equation with nonlocal conditions. By using various results of fractional calculus and fixed point theorems, namely, Banach fixed point theorem, Krasnoselskii's fixed point theorem and Schauder fixed point theorem, a set of sufficient conditions are obtained for the existence of mild solutions of the considered problem in the weighted space of continuous functions. Some examples are presented at the end of the chapter to show the effectiveness of the obtained abstract theory.

Chapter 3 is concerned with the existence and uniqueness of integral solutions of a class of mixed Volterra-Fredlhom integro neutral fractional differential equation involving Caputo derivatives. The existence results for the integral solutions are derived by using different fixed point theorems, Kuratowski measure of noncompactness and the properties of the operators that are involved in the solution representation. The first result gives the existence and uniqueness of the integral solution by using Banach fixed point theorem and the second existence result is established with the help of Krasnoselskii's fixed point theorem. These results are established based on the assumption that the associated semigroup is compact. For the last two results, we drop the compactness assumption on the semigroup, and use the Darbo-Sadovskii's fixed point theorem and the fixed point theorem for condensing maps, respectively, for proving the existence of integral solutions. Two of the results are verified by considering appropriate examples.

In Chapter 4, we continue our study on the existence and uniqueness of mild solutions to a class of fractional differential equations of Sobolev type with finite delay. The problem is expressed in terms of Volterra integro-differential equation and nonlocal condition. The existence results for the mild solutions are proved by using semigroup theory, properties of the bounded linear operators appearing in the definition of mild solutions, fixed point theorems and Hausdorff measure of noncompactness.

In Chapter 5, the existence of solutions of a class of impulsive Caputo fractional differential equations of order belonging to the interval $(1,2)$ with Erdélyi-Kober type boundary conditions is discussed. The results are obtained by using multiple base points and by transforming the boundary value problem into an equivalent integral equation in a Banach space. Various properties of fractional calculus and a number of familiar fixed point theorems are used to obtain the results. A nonlinear operator is defined in a Banach space whose fixed point gives the solution. The obtained results can be seen as more general since Erdélyi-Kober integrals are known to be more general operators in fractional calculus and they reduce to Riemann-Liouville integrals with a power weight.

Chapter 6 deals with the approximate controllability for a class of fractional control system with an analytic semigroup governed by differential equations with Hilfer fractional derivatives of order $\delta \in (0,1)$ and type $\zeta \in [0,1]$ in a Banach space, where the nonlinear term depends both on the state and control. The existence and uniqueness of the mild solution is established with the help of semigroup theory, fractional power of operators and a generalized contraction type fixed point theorem. Further, a set of sufficient conditions is formulated for the approximate controllability of the system under consideration. The result obtained holds irrespective of whether the generated semigroup is compact or non-compact.

Chapter 7 summarizes the results obtained in the previous chapters and gives some direction for future study based on the results obtained in this thesis.

CHAPTER 2

Existence of mild solutions for semilinear evolution equations in Hilfer fractional derivatives

In [52], Furati et al. took up an initial value problem for a class of nonlinear fractional differential equation involving Hilfer fractional derivative. Wang and Zhang [113] investigated the following nonlocal initial value problem:

$$
\begin{aligned}
{}^{H}D^{\delta,\zeta}_{a^+}x(t) &= f(t,x(t)), \;\; \delta \in (0,1), \; \zeta \in [0,1], \; t \in (a,b], \\
J^{1-\gamma}_{a^+}x(a^+) &= \sum_{i=1}^{m} \lambda_i x(\tau_i), \;\; \gamma = \delta + \zeta - \delta\zeta, \; \tau_i \in (a,b],
\end{aligned}
$$

where ${}^{H}D^{\delta,\zeta}_{0+}$ denotes the Hilfer derivative of order δ and type ζ. Some more problems involving Hilfer derivative can be found in [1, 2, 109].

Hilfer derivative is notably more general than Riemann-Liouville and Caputo fractional derivatives and so are the corresponding results. The facts that fractional differential equations encompass more attributes and that Hilfer derivative is more general in nature motivate us to pursue studies in this area.

In this work, we study the existence and uniqueness of mild solutions of the following semilinear evolution equation:

$$
{}^{H}D^{\delta,\zeta}_{0+}x(t) = A(t)x(t) + f(t,x(t)), \;\; \delta \in [0,1], \; \zeta \in (0,1), \; t \in (0,b] = J',
$$

with initial condition

$$
J^{1-\gamma}_{0+}x(0) = x_0,
$$

and nonlocal condition

$$
J^{1-\gamma}_{0+}x(0) - g(x) = x_0,
$$

where $A(t)$ is a bounded linear operator on $\mathbb{R}$ for each $t \in J = [0, b]$, $1-\gamma = (1-\delta)(1-\zeta)$ and $x_0 \in \mathbb{R}$. Further, $f\colon J \times \mathbb{R} \to \mathbb{R}$ is a given nonlinear function and g is a given function satisfying some assumptions which will be specified later. For more details related to the above class of differential equations, the readers are referred to [16, 86, 89].

2.1 Preliminaries

In this section, we present the following definitions and theorems which will be used in establishing our results [52, 113].

Let $-\infty < a < b < \infty$ and $C[a, b]$ be the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ with the norm $\|f\|_C = \sup_{t\in[a,b]}|f(t)|$.

For $0 \leq 1-\gamma < 1$, the weighted space $C_{1-\gamma}[a, b]$ of continuous functions f on $(a, b]$ is defined as

$$C_{1-\gamma}[a, b] = \{f : (a, b] \to \mathbb{R} : (t-a)^{1-\gamma} f(t) \in C[a, b]\}.$$

Then $C_{1-\gamma}[a, b]$ is a Banach space with the norm

$$\|f\|_{C_{1-\gamma}} = \|(t-a)^{1-\gamma} f(t)\|_C, \; C_0[a, b] = C[a, b].$$

In order to solve our problem, the following function spaces are considered:

$$C^{\delta,\zeta}_{1-\gamma}[0, b] = \{f \in C_{1-\gamma}[0, b], {}^H D^{\delta,\zeta}_{0+} f \in C_{1-\gamma}[0, b]\}$$

and

$$C^{\gamma}_{1-\gamma}[0, b] = \{f \in C_{1-\gamma}[0, b], D^{\gamma}_{0+} f \in C_{1-\gamma}[0, b]\}.$$

Since ${}^H D^{\delta,\zeta}_{0+} f = J^{\delta(1-\zeta)}_{0+} D^{\gamma}_{0+} f$, it follows from Theorem 11 of [52] that

$$C^{\gamma}_{1-\gamma}[0, b] \subset C^{\delta,\zeta}_{1-\gamma}[0, b].$$

2.2 Main Results

2.2.1 Semilinear evolution equation

Consider the following fractional semilinear evolution equation:

$$\left.\begin{aligned} &{}^H D^{\delta,\zeta}_{0+} x(t) = A(t)x(t) + f(t, x(t)), \;\; \delta \in [0, 1], \; \zeta \in (0, 1), \; t \in (0, b], \\ &J^{1-\gamma}_{0+} x(0) = x_0, \end{aligned}\right\} \tag{2.1}$$

where $A(t)$ is a bounded linear operator on $\mathbb{R}$ and $x_0 \in \mathbb{R}$.

Theorem 2.2.1. *Assume that*

(i) *$A(.)x(.) \in C_{1-\gamma}[0, b]$ for any $x \in C_{1-\gamma}[0, b]$,*

(ii) *$f(.,x(.)) \in C_{1-\gamma}[0,b]$ for any $x \in C_{1-\gamma}[0,b]$,*

hold. Then $x \in C^{\gamma}_{1-\gamma}[0,b]$ is a solution of the Cauchy problem (2.1) *if and only if x satisfies the integral equation*

$$x(t) = \frac{x_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}[A(s)x(s) + f(s,x(s))]ds, \quad t \in J'. \tag{2.2}$$

We now proceed to state our problem and establish the result.

First we introduce the following assumptions:

(H1) $A(t)$ is a bounded linear operator on $\mathbb{R}$ for each $t \in [0,b]$. The function $t \to A(t)$ is continuous in the uniform operator topology.

(H2) $A(.)x(.) \in C^{\delta(1-\zeta)}_{1-\gamma}[0,b]$ for any $x \in C_{1-\gamma}[0,b]$,

(H3) $f : (0,b] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(.,x(.)) \in C^{\delta(1-\zeta)}_{1-\gamma}[0,b]$ for any $x \in C_{1-\gamma}[0,b]$. For all $x, y \in \mathbb{R}$, there exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|.$$

Let $M = \sup_{t\in[0,b]}\|A(t)\|$ and set $F(t) = f(t,0)$.

The following existence result for problem (2.1) will be established by using Banach fixed point theorem.

Theorem 2.2.2. *Assume that (H1)-(H3) hold. If*

$$\xi_1 = (M+L)\frac{B(\gamma,\zeta)}{\Gamma(\zeta)}b^{\zeta} < 1,$$

then there exists a unique solution for the Cauchy type problem (2.1) *in $C^{\gamma}_{1-\gamma}[0,b] \subset C^{\delta,\zeta}_{1-\gamma}[0,b]$.*

Proof. According to Theorem 2.2.1, it is sufficient to prove the existence result for the equivalent integral equation (2.2). Define $T : C_{1-\gamma}[0,b] \to C_{1-\gamma}[0,b]$ by

$$(Tx)(t) = \frac{x_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}[A(s)x(s) + f(s,x(s))]ds, \quad t \in (0,b].$$

Let $\phi_1 = \frac{|x_0|}{\Gamma(\gamma)} + \frac{B(\gamma,\zeta)}{\Gamma(\zeta)}b^{\zeta}\|F\|_{C_{1-\gamma}}$. Choose $r \geq \frac{\phi_1}{1-\xi_1}$. Then we can show that $TB_r \subset B_r$ where $B_r = \{x \in C_{1-\gamma}[0,b] : \|x\|_{C_{1-\gamma}} \leq r\}$.

Let $x \in B_r$. Then we get

$$t^{1-\gamma}(Tx)(t) = \frac{x_0}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}A(s)x(s)ds + \frac{t^{1-\gamma}}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}f(s,x(s))ds.$$

Therefore,

$$\begin{aligned}|t^{1-\gamma}(Tx)(t)| &\leq \frac{|x_0|}{\Gamma(\gamma)}+\frac{t^{1-\gamma}}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}\|A(s)\||x(s)|ds+\frac{t^{1-\gamma}}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}[|f(s,x(s))\\ &\quad -f(s,0)|+|f(s,0)|]ds\\ &\leq \frac{|x_0|}{\Gamma(\gamma)}+\frac{Mt^{1-\gamma}}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}|x(s)|ds+\frac{Lt^{1-\gamma}}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}|x(s)|ds+\frac{t^{1-\gamma}}{\Gamma(\zeta)}\\ &\quad \times\int_0^t(t-s)^{\zeta-1}|F(s)|ds\\ &\leq r.\end{aligned}$$

Now, by taking $x, y \in C_{1-\gamma}[0,b]$, we get

$$\begin{aligned}|t^{1-\gamma}((Tx)(t)-(Ty)(t))| &\leq \frac{Mt^{1-\gamma}}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}|x(s)-y(s)|ds+\frac{Lt^{1-\gamma}}{\Gamma(\zeta)}\int_0^t(t-s)^{\zeta-1}\\ &\quad \times|x(s)-y(s)|ds\\ &\leq \frac{M+L}{\Gamma(\zeta)}t^{1-\gamma}\int_0^t(t-s)^{\zeta-1}|x(s)-y(s)|ds\end{aligned}$$

which gives

$$\|Tx-Ty\|_{C_{1-\gamma}}\leq \frac{(M+L)}{\Gamma(\zeta)}B(\gamma,\zeta)b^{\zeta}\|x-y\|_{C_{1-\gamma}}.$$

Thus, T is a contraction mapping on $C_{1-\gamma}[0,b]$. By applying Banach fixed point theorem, we know that the operator T has a unique fixed point on $C_{1-\gamma}[0,b]$. Then by repeating the process of the proof carried out in Theorem 25 of [52], one can show that the solution is actually in $C^{\gamma}_{1-\gamma}[0,b]$. □

2.2.2 Nonlocal problem

Here we discuss the existence of solution of the following nonlocal problem:

$$\left.\begin{aligned}{}^{H}D^{\delta,\zeta}_{0^+}x(t) &= A(t)x(t)+f(t,x(t)),\quad \delta\in[0,1],\ \zeta\in(0,1),\ t\in(0,b],\\ J^{1-\gamma}_{0^+}x(0)-g(x) &= x_0,\end{aligned}\right\}\tag{2.3}$$

where $g : C_{1-\gamma}[0,b] \to \mathbb{R}$ is a continuous function satisfying the following condition:
(H4) there exists a constant $N > 0$ such that

$$|g(x)-g(y)|\leq N\|x-y\|_{C_{1-\gamma}}\ \text{ for all } x,y\in C_{1-\gamma}[0,b].$$

Theorem 2.2.3. *Assume that (H1)-(H4) hold. If*

$$\xi_2=\frac{N}{\Gamma(\gamma)}+(M+L)\frac{B(\gamma,\zeta)}{\Gamma(\zeta)}b^{\zeta}<1,$$

then there exists a unique solution for equation (2.3) in $C^{\gamma}_{1-\gamma}[0,b] \subset C^{\delta,\zeta}_{1-\gamma}[0,b]$.

Proof. Define $T : C_{1-\gamma}[0,b] \to C_{1-\gamma}[0,b]$ by

$$(Tx)(t) = \frac{x_0 + g(x)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1}[A(s)x(s) + f(s,x(s))]ds, \quad t \in (0,b].$$

Choose $r \geq \frac{\phi_2}{1-\xi_2}$, where $\phi_2 = \frac{|x_0|+|g(0)|}{\Gamma(\gamma)} + \frac{B(\gamma,\zeta)}{\Gamma(\zeta)} b^{\zeta} \|F\|_{C_{1-\gamma}}$. Then we can show that $TB_r \subset B_r$ where $B_r = \{x \in C_{1-\gamma}[0,b] : \|x\|_{C_{1-\gamma}} \leq r\}$.

Let $x \in B_r$. Then we get

$$t^{1-\gamma}(Tx)(t) = \frac{x_0 + g(x)}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} A(s)x(s)ds + \frac{t^{1-\gamma}}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} f(s,x(s))ds.$$

Therefore,

$$\begin{aligned}
|t^{1-\gamma}(Tx)(t)| \leq & \frac{|x_0|+|g(x)|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} \|A(s)\| |x(s)| ds + \frac{t^{1-\gamma}}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} \\
& \times [|f(s,x(s)) - f(s,0)| + |f(s,0)|]ds \\
& \leq \frac{|x_0|}{\Gamma(\gamma)} + \frac{N\|x\|_{C_{1-\gamma}}}{\Gamma(\gamma)} + \frac{|g(0)|}{\Gamma(\gamma)} + \frac{M+L}{\Gamma(\zeta)} \|x\|_{C_{1-\gamma}} B(\gamma,\zeta) t^{\zeta} + \frac{\|F\|_{C_{1-\gamma}}}{\Gamma(\zeta)} B(\gamma,\zeta) t^{\zeta} \\
& \leq r.
\end{aligned}$$

Let $x, y \in C_{1-\gamma}[0,b]$. Then

$$\begin{aligned}
|t^{1-\gamma}((Tx)(t) - (Ty)(t))| \leq & \frac{|g(x) - g(y)|}{\Gamma(\gamma)} + \frac{Mt^{1-\gamma}}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} |x(s) - y(s)| ds \\
& + \frac{Lt^{1-\gamma}}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} |x(s) - y(s)| ds \\
& \leq \frac{N}{\Gamma(\gamma)} \|x-y\|_{C_{1-\gamma}} + \frac{M+L}{\Gamma(\zeta)} t^{1-\gamma} \int_0^t (t-s)^{\zeta-1} |x(s) - y(s)| ds \\
& \leq \left[\frac{N}{\Gamma(\gamma)} + \frac{M+L}{\Gamma(\zeta)} B(\gamma,\zeta) b^{\zeta}\right] \|x-y\|_{C_{1-\gamma}}.
\end{aligned}$$

By applying Banach fixed point theorem, we get the desired result. ☐

Our next result will be established by employing Krasnoselskii's fixed point theorem. Here, we replace (H3) by (H3)′ with the following linear growth condition:

(H3)′ $f : (0,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(.,x(.)) \in C^{\delta(1-\zeta)}_{1-\gamma}[0,b]$ for any $x \in C_{1-\gamma}[0,b]$. There exist constants $L > 0$ and $K \geq 0$ such that

$$|f(t,x)| \leq L|x| + K, \quad \text{for all } x \in \mathbb{R}.$$

Theorem 2.2.4. *Assume that the hypotheses (H1),(H2),(H3)′ and (H4) are satisfied and* $\xi_2 < 1$. *Then nonlocal problem (2.3) has at least one solution in* $C^{\gamma}_{1-\gamma}[0,b] \subset C^{\delta,\zeta}_{1-\gamma}[0,b]$.

Proof. Choose

$$r \geq \frac{\phi_3}{1-\xi_2}, \quad \text{where } \phi_3 = \frac{|x_0|+|g(0)|}{\Gamma(\gamma)} + \frac{K}{\Gamma(\zeta)} b^{\zeta+1-\gamma},$$

and define the operators P and Q on B_r as

$$(Px)(t) = \frac{x_0+g(x)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} A(s)x(s)ds,$$

and

$$(Qx)(t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} f(s,x(s))ds.$$

We subdivide the proof into several steps.

Step I: To show that $Px+Qy \in B_r$ for every $x, y \in B_r$.

For $x \in B_r$, we get

$$\|Px\|_{C_{1-\gamma}} \leq \frac{|x_0|+|g(0)|+N\|x\|_{C_{1-\gamma}}}{\Gamma(\gamma)} + \frac{MB(\gamma,\zeta)}{\Gamma(\zeta)} b^{\zeta} \|x\|_{C_{1-\gamma}}$$

and

$$\|Qx\|_{C_{1-\gamma}} \leq \frac{LB(\gamma,\zeta)}{\Gamma(\zeta)} b^{\zeta} \|x\|_{C_{1-\gamma}} + \frac{Kb^{\zeta+1-\gamma}}{\Gamma(\zeta+1)}.$$

Therefore, for every $x, y \in B_r$,

$$\|Px+Qy\|_{C_{1-\gamma}} \leq \|Px\|_{C_{1-\gamma}} + \|Qy\|_{C_{1-\gamma}} \leq r.$$

Step II: To show that P is a contraction mapping.

It can be easily shown that, for any $x, y \in B_r$,

$$\|Px-Py\|_{C_{1-\gamma}} \leq \xi_2 \|x-y\|_{C_{1-\gamma}}.$$

Step III: To show that Q is compact and continuous. It is shown in three sub-steps.

Q **is continuous**: Let (x_n) be a sequence such that $x_n \to x$ in $C_{1-\gamma}[0,b]$. Then for each $t \in (0,b]$,

$$\begin{aligned} |t^{1-\gamma}((Qx_n)(t)-(Qx)(t))| &\leq \frac{t^{1-\gamma}}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} |f(s,x_n(s))-f(s,x(s))| ds \\ &\leq \frac{b^{\zeta}}{\Gamma(\zeta)} B(\gamma,\zeta) \|f(.,x_n(.))-f(.,x(.))\|_{C_{1-\gamma}} \end{aligned}$$

which implies

$$\|Qx_n-Qx\|_{C_{1-\gamma}} \leq \frac{b^{\zeta}B(\gamma,\zeta)}{\Gamma(\zeta)} \|f(.,x_n(.))-f(.,x(.))\|_{C_{1-\gamma}}.$$

Therefore, Q is continuous.

Q maps bounded sets into bounded sets in $C_{1-\gamma}[0,b]$: It is enough to show that, for any $r^* > 0$, there exists a $s^* > 0$ such that for each $x \in B_{r^*}$, we have $Qx \in B_{s^*}$.

We get

$$\|Qx\|_{C_{1-\gamma}} \leq \frac{LB(\gamma,\zeta)}{\Gamma(\zeta)} b^{\zeta} \|x\|_{C_{1-\gamma}} + \frac{Kb^{\zeta+1-\gamma}}{\Gamma(\zeta+1)} \leq \frac{LB(\gamma,\zeta)}{\Gamma(\zeta)} b^{\zeta} r^* + \frac{Kb^{\zeta+1-\gamma}}{\Gamma(\zeta+1)} := s^*.$$

Q maps bounded sets into equicontinuous sets of $C_{1-\gamma}[0,b]$: Let $0 < t_1 < t_2 \leq b$, and $x \in B_r$. Then we have

$$\begin{aligned}
&|t_2^{1-\gamma}(Qx)(t_2) - t_1^{1-\gamma}(Qx)(t_1))| \\
&= \left|\frac{t_2^{1-\gamma}}{\Gamma(\zeta)} \int_0^{t_2} (t_2 - s)^{\zeta-1} f(s, x(s))ds - \frac{t_1^{1-\gamma}}{\Gamma(\zeta)} \int_0^{t_1} (t_1 - s)^{\zeta-1} f(s, x(s))ds\right| \\
&\leq \left|\frac{1}{\Gamma(\zeta)} \int_0^{t_2} [t_2^{1-\gamma}(t_2 - s)^{\zeta-1} - t_1^{1-\gamma}(t_1 - s)^{\zeta-1}] f(s, x(s))ds \right. \\
&\left. + \frac{t_1^{1-\gamma}}{\Gamma(\zeta)} \int_{t_1}^{t_2} (t_1 - s)^{\zeta-1} f(s, x(s))ds\right|
\end{aligned}$$

which tends to zero as $t_2 \to t_1$. So by Arzelà-Ascoli theorem, Q is compact. Hence by Krasnoselskii's fixed point theorem, the problem defined by equation (2.3) has at least one solution in $C_{1-\gamma}[0,b]$. ☐

Our next result is based on Schauder's fixed point theorem. Here we replace (H4) by the following condition:
(H4)$'$ there exists a constant $N' > 0$ such that

$$|g(x)| \leq N' \ \text{ for each } x \in C_{1-\gamma}[0,b].$$

Theorem 2.2.5. *Assume that (H1), (H2), (H3)$'$ and (H4)$'$ hold. If $\xi_1 < 1$, then (2.3) has at least one solution in $C_{1-\gamma}^{\gamma}[0,b] \subset C_{1-\gamma}^{\delta,\zeta}[0,b]$.*

Proof. Choose

$$r \geq \frac{\phi_4}{1-\xi_1}, \quad \text{where } \phi_4 = \frac{|x_0| + N'}{\Gamma(\gamma)} + \frac{K}{\Gamma(\zeta+1)} b^{\zeta+1-\gamma}.$$

Then by using the techniques of Theorem 2.2.4, it can be easily shown that $T : C_{1-\gamma}[0,b] \to C_{1-\gamma}[0,b]$ defined by

$$(Tx)(t) = \frac{x_0 + g(x)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} [A(s)x(s) + f(s, x(s))]ds \ \ t \in (0,b]$$

is continuous and TB_r is relatively compact. Hence, it follows from Schauder's fixed point theorem that (2.3) has a solution in $C_{1-\gamma}[0,b]$. ☐

2.3 Examples

For evolution equation:

Consider

$$\left.\begin{aligned} {}^{H}D_{0+}^{\frac{1}{5},\frac{3}{5}}x(t) &= \frac{1}{10}e^{-t}x(t) + \Big(t^{-8/25} + \frac{|x(t)|}{2}\Big), \qquad t \in (0,1], \\ I_{0+}^{\frac{8}{25}}x(0) &= x_0. \end{aligned}\right\} \tag{2.4}$$

Here

$$f(t,x(t)) = t^{-8/25} + \frac{|x(t)|}{2} \text{ for } t \in (0,1], \quad A(t) = \frac{1}{10}e^{-t}I,$$

where I is the identity operator. It is obvious that $A(.)x(.), f(.,x(.)) \in C_{\frac{8}{25}}[0,1]$. Moreover, $|f(t,x) - f(t,y)| \leq \frac{1}{2}|x-y|$ for all $x, y \in \mathbb{R}$. Hence (H3) holds with $L = \frac{1}{2}$. Here, $M = \frac{1}{10}$ and it can be found, after some elementary computation, that

$$\xi_1 = \Big(\frac{1}{10} + \frac{1}{2}\Big)\frac{B(\frac{17}{25}, \frac{3}{5})}{\Gamma(\frac{3}{5})} \approx 0.88654 < 1.$$

Here $B(.,.)$ denotes beta function.

Thus, all the assumptions in Theorem 2.2.2 are satisfied and therefore, we can conclude that (2.4) has a unique solution in $C_{\frac{8}{25}}[0,1]$.

For nonlocal condition:

Now in equation (2.4), if we replace the initial condition $I_{0+}^{1-\gamma}x(0) = x_0$ by the nonlocal condition $I_{0+}^{1-\gamma}x(0) - g(x) = x_0$ (from equation (2.3)), where $g(x) = cx(\frac{1}{2})$, $c \in \mathbb{R}$, then g satisfies (H4) with $N = |c|2^{\frac{8}{25}}$. By choosing c small enough so that $\xi_2 < 1$ holds, Theorem 2.2.3 ensures the existence of solution in $C_{\frac{8}{25}}[0,1]$.

CHAPTER 3

Existence and uniqueness of integral solutions for a class of nondensely defined mixed Volterra-Fredholm integro fractional neutral differential equations

Da Prato and Sinestrari [33] initiated an investigation of initial value problems with a non-dense domain wherein they introduced the concept of integral solutions of the following abstract Cauchy problem:

$$\begin{aligned} &x'(t) = Ax(t) + f(t), \quad t \in [0, b], \\ &x(0) = x_0. \end{aligned}$$

They studied the existence of integral solutions under some suitable assumptions for any continuous function f and $x_0 \in \overline{D(A)}$. Their work was based on integrated semigroup theory. For more details on non-densely defined operators and the theory of integrated semigroup, the readers are referred to some works such as [3, 19, 44, 45].

In [102], Thieme showed that, under the assumption $f \in \overline{D(A)}$, the integral solution reduced to the mild solution. However, in many of the works such as [18, 84], as pointed out by Zhang and Liu [122], the formulation of integral solution was not considered to be appropriate. Zhang and Liu came out with an appropriate formulation of integral solution for a non-densely defined impulsive fractional differential equation by using integrated semigroup theory and some probability densities.

Fu [51] studied the existence of solutions for the following semilinear neutral functional differential equations with nonlocal conditions on a general Banach space X:

$$\frac{d}{dt}\Big[x(t) - F(t, x(h_1(t)))\Big] = A[x(t) - F(t, x(t, x(h_1(t)))] + G(t, x(h_2(t))), \quad t \in [0, b],$$

$$x(0) + g(x) = x_0 \in X.$$

Motivated by the works carried out by Fu [51] and Gu et al. [55], the main objective of our present work is to study the following neutral fractional integro-differential equation of mixed type for $t \in [0, b]$:

$$\left.\begin{aligned} {}^{C}D^{q}_{0+}[x(t) - u(t, x(t))] &= A[x(t) - u(t, x(t))] + f(t, x(t), (Hx)(t), (Gx)(t)), \quad t \in [0, b], \\ x(0) &= x_0, \end{aligned}\right\} \tag{3.1}$$

where

$$(Hx)(t) = \int_0^t h(t, s, x(s))ds \;\text{ and }\; (Gx)(t) = \int_0^b g(t, s, x(s))ds,$$

with $q \in (0, 1)$, $J = [0, b]$ and $A\colon D(A) \subseteq X \to X$ as a closed linear operator on a Banach space X, which is not necessarily densely defined. The state $x(.)$ assumes values in X, and $u\colon J \times X \to \overline{D(A)}$ is a function which satisfies some assumptions to be specified later. The functions $h\colon \Delta \times X \to X$, $g\colon J \times J \times X \to X$ and $f\colon J \times X \times X \times X \to X$ are given abstract functions where $\Delta = \{(t, s) \in J \times J \mid s \leq t\}$.

3.1 Preliminaries

We assume that the operator $A\colon D(A) \subset X \to X$ satisfies the Hille-Yosida condition, i.e., there exist $\overline{M} \geq 0$ and a constant $w \in \mathbb{R}$ such that $(w, \infty) \subseteq \rho(A)$ and

$$\sup\left\{(\lambda - w)^n \|R(\lambda : A)^n\|_{B(X)} \middle| n \in \mathbb{N}, \lambda > w\right\} \leq \overline{M},$$

where $\rho(A)$ is the resolvent set of A, and $R(\lambda : A)$ denotes the resolvent of A.

Let A_0 be the part of A in $\overline{D(A)}$ defined by

$$D(A_0) = \left\{x \in D(A) \middle| Ax \in \overline{D(A)}\right\},$$
$$A_0 x = Ax.$$

Then A_0 generates a C_0-semigroup $\{Q(t)\}_{t\geq 0}$ on $\overline{D(A)}$. Assume that $\{Q(t)\}_{t\geq 0}$ is uniformly bounded, i.e., there exists $M_Q \geq 1$ such that $\sup_{t\in[0,\infty)} \|Q(t)\| \leq M_Q$.

Problem (3.1) is now equivalent to the following integral equation:

$$\begin{aligned} x(t) = x_0 - u(0, x(0)) + u(t, x(t)) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \Big[A(x(s) - u(s, x(s))) \\ + f(s, x(s), (Hx)(s), (Gx)(s))\Big] ds, \quad t \in [0, b]. \end{aligned}$$

3.2 Integral Solution

Based on the information available in [55], we present the following definition and results: Assuming f to be continuous and $x_0 \in \overline{D(A)}$, the integral solution of problem (3.1) is defined as follows.

Definition 3.2.1. *A function $x\colon J \to X$ is said to be an integral solution of (3.1) if*

$$x \in C(J, X), \quad J_{0+}^{q}[x(t) - u(t, x(t))] \in D(A) \text{ for } t \in [0, b],$$

and

$$\begin{aligned} x(t) = x_0 - u(0, x(0)) + u(t, x(t)) + \frac{A}{\Gamma(q)} \int_0^t (t-s)^{q-1}[x(s) - u(s, x(s))]ds \\ + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), (Hx)(s), (Gx)(s))ds, \quad t \in [0, b]. \end{aligned}$$

Remark 3.2.1. *If x is an integral solution of problem (3.1), it can be shown that $x(t) - u(t, x(t)) \in \overline{D(A)}$ for $t \in J$.*

Now, we consider the following auxiliary problem:

$$\left.\begin{aligned} {}^{C}D_{0+}^{q}[x(t) - u(t, x(t))] &= A_0[x(t) - u(t, x(t))] + f(t, x(t), (Hx)(t), (Gx)(t)), \quad t \in [0, b], \\ x(0) &= x_0. \end{aligned}\right\} \tag{3.2}$$

The integral solution of (3.2) takes the following form:

$$x(t) = x_0 - u(0, x(0)) + u(t, x(t)) + A_0 J_{0+}^{q}[x(t) - u(t, x(t))] + J_{0+}^{q} f(t, x(t), (Hx)(t), (Gx)(t)). \tag{3.3}$$

Let $B_\lambda = \lambda(\lambda I - A)^{-1}$. Then, since $B_\lambda x \to x$ as $\lambda \to +\infty$ for $x \in \overline{D(A)}$, we have the following lemma:

Lemma 3.2.1. *[55] The integral solution of (3.3) can be written in the following form:*

$$x(t) = u(t, x(t)) + S_q(t)[x_0 - u(0, x(0))] + \lim_{\lambda \to \infty} \int_0^t K_q(t-s) B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))ds$$

where

$$S_q(t) = J_{0+}^{1-q} K_q(t), \; K_q(t) = t^{q-1} P_q(t), \; P_q(t) = \int_0^\infty q\theta \xi_q(\theta) Q(t^q \theta) d\theta.$$

Here

$$\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \bar{w}_q(\theta^{-\frac{1}{q}}) \;\; \textit{and}$$

$$\bar{w}_q(\theta) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^{n-1} \theta^{(-qn-1)} \frac{\Gamma(qn+1)}{n!} \sin(n\pi q), \; \theta \in (0, \infty).$$

$\xi_q(\theta)$ is a probability density function on $(0,\infty)$ satisfying

$$\xi_q(\theta) \geq 0, \quad \int_0^\infty \xi_q(\theta)d\theta = 1, \quad \int_0^\infty \theta\xi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}.$$

Lemma 3.2.2. *[55] For any fixed $t > 0$, $\{K_q(t)\}_{t>0}$ and $\{S_q(t)\}_{t>0}$ are linear operators, and for any $x \in \overline{D(A)}$,*

$$\|K_q(t)x\|_X \leq \frac{M_Q t^{q-1}}{\Gamma(q)}\|x\|_X \ \text{ and } \ \|S_q(t)x\|_X \leq M_Q\|x\|_X.$$

Lemma 3.2.3. *[55] $\{K_q(t)\}_{t>0}$ and $\{S_q(t)\}_{t>0}$ are strongly continuous, i.e., for any $x \in \overline{D(A)}$ and $0 < t_1 < t_2 \leq b$,*

$$\|K_q(t_2)x - K_q(t_1)x\|_X \to 0 \ \text{ and } \ \|S_q(t_2)x - S_q(t_1)x\|_X \to 0$$

as $t_2 \to t_1$.

Lemma 3.2.4. *[125] For any fixed $t > 0$, $P_q(t)$ is a linear and bounded operator, and*

$$\|P_q(t)x\|_X \leq \frac{M_Q}{\Gamma(q)}\|x\|_X \ \text{ for any } x \in \overline{D(A)}.$$

Lemma 3.2.5. *[89] Assume that $\{Q(t)\}_{t>0}$ is compact. Then $Q(t)$ is continuous in the uniform operator topology for $t > 0$, i.e., $\{Q(t)\}_{t>0}$ is equicontinuous.*

3.3 Main Results

Our first result is based on Banach fixed point theorem. Here we use the following assumptions:

(H1)(i) there exist a constant $q_1 \in (0,q)$ and functions l_1, l_2, $l_3 \in L^{\frac{1}{q_1}}(J,\mathbb{R}^+)$ such that

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\|_X \leq l_1(t)\|x_1 - x_2\|_X + l_2(t)\|y_1 - y_2\|_X + l_3(t)\|z_1 - z_2\|_X,$$

for all $x_i, y_i, z_i \in X$, $i = 1,2,3$ and $t \in J$.

(ii) there exist a constant $q_2 \in (0,q)$ and a function $l \in L^{\frac{1}{q_2}}(J,\mathbb{R}^+)$ such that

$$\|f(t,x,y,z)\|_X \leq l(t), \quad \text{for all } x,y,z \in X \text{ and } t \in J.$$

(H2) for the function $u : J \times X \to \overline{D(A)}$, there exists a constant $L_1 > 0$ such that

$$\|u(t,x_1) - u(t,x_2)\|_X \leq L_1\|x_1 - x_2\|_X, \quad \forall\, t \in J \text{ and } x_1, x_2 \in X.$$

(H3) there exist constants L_2, $L_3 > 0$ such that

$$\Big\| \int_0^t [h(t,s,x) - h(t,s,y)]ds \Big\|_X \le L_2 \|x-y\|_X,$$
$$\Big\| \int_0^b [g(t,s,x) - g(t,s,y)]ds \Big\|_X \le L_3 \|x-y\|_X,$$

for all x, $y \in X$.

Theorem 3.3.1. *Assume that hypotheses (H1)-(H3) hold. Then there exists a unique integral solution of (3.1) in* $C(J, \overline{D(A)})$ *provided*

$$\xi_1 = (M_Q + 1)L_1 + \frac{M_Q\overline{M}}{\Gamma(q)}\Big\{\|l_1\|_{L^{\frac{1}{q_1}}} + L_2\|l_2\|_{L^{\frac{1}{q_1}}} + L_3\|l_3\|_{L^{\frac{1}{q_1}}}\Big\}\frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}} < 1.$$

Proof. Define $T : C(J, \overline{D(A)}) \to C(J, \overline{D(A)})$ by

$$\begin{aligned} Tx(t) &= u(t, x(t)) + S_q(t)[x_0 - u(0, x(0))] \\ &\quad + \lim_{\lambda\to\infty} \int_0^t K_q(t-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))ds, \quad t \in J. \end{aligned}$$

Then with the help of (H1)(ii), Hölder's inequality and Bochner's theorem, it can be shown that T is well-defined on $C(J, \overline{D(A)})$.

Let $x, y \in C(J, \overline{D(A)})$. Then for any $t \in [0, b]$, we have

$$\begin{aligned} &\|((Tx)(t) - (Ty)(t)\|_X \\ &\le \|u(t,x(t)) - u(t,y(t))\|_X + \|S_q(t)[u(0,x(0)) - u(0,y(0))]\|_X + \Big\|\lim_{\lambda\to\infty}\int_0^t K_q(t-s) \\ &\times B_\lambda[f(s,x(s),(Hx)(s),(Gx)(s)) - f(s,y(s),(Hy)(s),(Gy)(s))]ds\Big\|_X \\ &\le L_1(1+M_Q)\|x-y\|_C + \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big[l_1(s)\|x(s)-y(s)\|_X + l_2(s)\|(Hx)(s) \\ &- (Hy)(s)\|_X + l_3(s)\|(Gx)(s) - (Gy)(s)\|_X\Big] \\ &\le \xi_1\|x-y\|_C. \end{aligned}$$

Therefore, by using Banach fixed point theorem, we conclude that there exists a unique integral solution of our problem in $C(J, \overline{D(A)})$. □

In order to establish the next result, we introduce the following additional assumptions:
(H4) $Q(t), t > 0$ is compact.
(H5)(i) for each $t \in [0, b]$, the function $f(t, ., ., .)\colon X \times X \times X \to X$ is continuous and for each $(x, y, z) \in X \times X \times X$, the function $f(., x, y, z)\colon J \to X$ is strongly measurable.

(ii) there exists a function $l \in L(J, \mathbb{R}^+)$ such that

$$J^q_{0^+} l \in C(J, \mathbb{R}^+), \quad \lim_{t \to 0^+} J^q_{0^+} l(t) = 0$$

and

$$\|f(t, x, y, z)\|_X \leq l(t) \ \text{ for all } x, y, z \in X \text{ and for } t \in [0, b].$$

(H6) $u : J \times X \to \overline{D(A)}$ is continuous and there exists $L_1 > 0$ such that

$$\|u(t, x_1) - u(t, x_2)\|_X \leq L_1 \|x_1 - x_2\|_X, \ \text{ for each } t \in [0, b] \text{ and all } x_1, x_2 \in X$$

and let $M_0 = \sup\limits_{t \in J} \|u(t, 0)\|_X$.

(H7)(i) for each $(t, s) \in \Delta$, the function $h(t, s, .)\colon X \to X$ is continuous, and for each $x \in X$, the function $h(., ., x)\colon \Delta \to X$ is strongly measurable.

(ii) there exists a function $m_h(t, s) \in C(\Delta, \mathbb{R}^+)$ such that

$$\|h(t, s, x)\|_X \leq m_h(t, s) \|x\|_X, \ \text{ for } (t, s) \in \Delta, \ x \in X$$

and $H^* = \sup\limits_{t \in J} \displaystyle\int_0^t m_h(t, s) ds < \infty$.

(H8)(i) for each $(t, s) \in J \times J$, the function $g(t, s, .)\colon X \to X$ is continuous, and for each $x \in X$, the function $g(., ., x)\colon J \times J \to X$ is strongly measurable.

(ii) There exists a function $m_g(t, s) \in C(J \times J, \mathbb{R}^+)$ such that

$$\|g(t, s, x)\|_X \leq m_g(t, s) \|x\|_X, \ \text{ for } (t, s) \in J \times J, \ x \in X$$

and $G^* = \sup\limits_{t \in J} \displaystyle\int_0^b m_g(t, s) ds < \infty$.

For our second existence result, we use Krasnoselskii's fixed point theorem.

Theorem 3.3.2. *Assume that hypotheses (H4)-(H8) hold. Then (3.1) has an integral solution in* $C(J, \overline{D(A)})$ *provided*

$$(M_Q + 1) L_1 \leq \frac{1}{2}.$$

Proof. Choose $r \geq 2\xi_2$ where $\xi_2 = M_0 + M_Q \|x_0\|_X + M_Q \|u(0, x(0))\|_X + \frac{M_Q \overline{M}}{\Gamma(q)} \int_0^t (t-s)^{q-1} l(s) ds$. Define two operators T_1 and T_2 on $B_r = \{x \in C(J, \overline{D(A)})\colon \|x\|_C \leq r\}$ by

$$\begin{aligned} (T_1 x)(t) &= u(t, x(t)) + S_q(t)[x_0 - u(0, x(0))], \\ (T_2 x)(t) &= \lim_{\lambda \to \infty} \int_0^t K_q(t - s) B_\lambda f(s, x(s), (Hx)(s), (Gx)(s)) ds. \end{aligned}$$

Step I: To show that $T_1 x + T_2 y \in B_r$ whenever $x, y \in B_r$.

For any $t \in [0, b]$, we have

$$\begin{aligned}\|(T_1x)(t)\|_X &\leq \|u(t,x(t)) - u(t,0)\|_X + M_0 + M_Q\|x_0 - u(0,x(0))\|_X \\ &\leq L_1\|x\|_C + M_0 + M_Q\|x_0\|_X + M_Q\|u(0,x(0))\|_X\end{aligned}$$

and

$$\begin{aligned}\|(T_2y)(t)\|_X &\leq \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s),(Hx)(s),(Gx)(s))\|_X ds \\ &\leq \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}l(s)ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\|(T_1x)(t) + (T_2y)(t)\|_X &\leq L_1r + M_0 + M_Q\|x_0\|_X + M_Q\|u(0,x(0))\|_X + \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}l(s)ds \\ &\leq r\end{aligned}$$

which implies that $T_1x + T_2y \in B_r$.

Step II: To show that T_1 is a contraction.

It can be easily shown that T_1 is a contraction.

Step III: To show that T_2 is compact and continuous.

We show it in various sub-steps.

(i) To show that $\{T_2x | x \in B_r\}$ is equicontinuous:

Let $x \in B_r$ and $0 = t_1 < t_2 \leq b$. Then

$$\begin{aligned}\|(T_2x)(t_2) - (T_2x)(t_1)\|_X &\leq \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^{t_2} (t_2-s)^{q-1}\|f(s,x(s),(Hx)(s),(Gx)(s))\|_X ds \\ &\leq M_Q\overline{M}J_{0+}^q l(t_2) \longrightarrow 0 \text{ as } t_2 \to 0.\end{aligned}$$

For $0 < t_1 < t_2 \leq b$, we have

$$\begin{aligned}&\|(T_2x)(t_2) - (T_2x)(t_1)\|_X \\ &\leq \frac{M_Q\overline{M}}{\Gamma(q)}\int_{t_1}^{t_2} (t_2-s)^{q-1}l(s)ds + \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1} - (t_2-s)^{q-1}]l(s)ds \\ &+ \overline{M}\int_0^{t_1}(t_1-s)^{q-1}\|P_q(t_2-s) - P_q(t_1-s)\|_{B(X)}l(s)ds \\ &\leq \frac{M_Q\overline{M}}{\Gamma(q)}\left|\int_0^{t_2}(t_2-s)^{q-1}l(s)ds - \int_0^{t_1}(t_1-s)^{q-1}l(s)ds\right| \\ &+ \frac{2M_Q\overline{M}}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1} - (t_2-s)^{q-1}]l(s)ds + \overline{M}\int_0^{t_1}(t_1-s)^{q-1} \\ &\times \|P_q(t_2-s) - P_q(t_1-s)\|_{B(X)}l(s)ds \\ &=: I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{M_Q\overline{M}}{\Gamma(q)}\Big|\int_0^{t_2}(t_2-s)^{q-1}l(s)ds - \int_0^{t_1}(t_1-s)^{q-1}l(s)ds\Big|,\\
I_2 &= \frac{2M_Q\overline{M}}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]l(s)ds,\\
I_3 &= \overline{M}\int_0^{t_1}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}l(s)ds.
\end{aligned}$$

Since $J_{0+}^q l \in C(J,\mathbb{R}^+)$, therefore it follows that $I_1 \to 0$ as $t_2 \to t_1$.
For I_2, we have

$$[(t_1-s)^{q-1}-(t_2-s)^{q-1}]l(s) \le (t_1-s)^{q-1}l(s)$$

and $\int_0^{t_1}(t_1-s)^{q-1}l(s)ds$ exists. Therefore, by using Lebesgue's dominated convergence theorem, we can conclude that $I_2 \to 0$ as $t_2 \to t_1$.
Now,

$$I_3 = \overline{M}\int_0^{t_1}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}l(s)ds.$$

For $\epsilon > 0$ small enough, we have

$$\begin{aligned}
I_3 &= \overline{M}\int_0^{t_1-\epsilon}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}l(s)ds\\
&+\overline{M}\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}l(s)ds\\
&\le \overline{M}\int_0^{t_1}(t_1-s)^{q-1}l(s)ds \sup_{s\in[0,t_1-\epsilon]}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}\\
&+\frac{2M_Q\overline{M}}{\Gamma(q)}\Big|\int_0^{t_1}(t_1-s)^{q-1}l(s)ds - \int_0^{t_1-\epsilon}(t_1-\epsilon-s)^{q-1}l(s)ds\Big|\\
&+\frac{2M_Q\overline{M}}{\Gamma(q)}\int_0^{t_1-\epsilon}[(t_1-\epsilon-s)^{q-1}-(t_1-s)^{q-1}]l(s)ds := I_{31}+I_{32}+I_{33}.
\end{aligned}$$

From (H4), it follows that $I_{31} \to 0$ as $t_2 \to t_1$. It can also be shown that both I_{32} and I_{33} tend to zero as $\epsilon \to 0$ (applying similar arguments as shown for $I_1 \to 0$ and $I_2 \to 0$ as $t_2 \to t_1$). Thus, $\|(T_2x)(t_2)-(T_2x)(t_1)\|_X \to 0$ independent of $x \in B_r$ as $t_2 \to t_1$, which implies that $\{T_2x | x \in B_r\}$ is equicontinuous.

(ii) To show that T_2 is continuous:
Let $(x_n) \subset B_r$ such that $x_n \to x$ in B_r.

Using (H5)(i), (H7), (H8) and Lebesgue's dominated convergence theorem, it follows that

$$f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) \longrightarrow f(s,x(s),(Hx)(s),(Gx)(s)) \text{ as } n \to \infty.$$

Now for each $t \in J$, we have

$$(t-s)^{q-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) - f(s,x(s),(Hx)(s),(Gx)(s))\|_X \\ \leq 2(t-s)^{q-1}l(s)$$

and $2(t-s)^{q-1}l(s)$ is integrable for $s \in [0,t],\ t \in [0,b]$. Therefore, by Lebesgue's dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{q-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) - f(s,x(s),(Hx)(s),(Gx)(s))\|_X ds \\ \longrightarrow 0 \text{ as } n \to \infty.$$

For $t \in [0,b]$,

$$\|(T_2x_n)(t) - (T_2x)(t)\|_X \\ \leq \frac{M_Q\overline{M}}{\Gamma(\alpha)}\int_0^t (t-s)^{q-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) - f(s,x(s),(Hx)(s),(Gx)(s))\|_X ds \\ \longrightarrow 0 \text{ as } n \to \infty.$$

Therefore, T_2 is continuous.

(iii) To show that T_2 is uniformly bounded:
From the inequality

$$\|(T_2y)(t)\|_X \leq \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}l(s)ds,$$

it follows that T_2 is uniformly bounded.

(iv) To show that T_2 is compact:
Here we use Arzelà-Ascoli theorem. We are required to show that, for any $t \in [0,b]$, $\{(T_2x)(t)|x \in B_r\}$ is relatively compact in $\overline{D(A)}$. For $t = 0$, it is obviously true. Therefore, by fixing $t \in (0,b]$, and $\forall \epsilon \in (0,t)$, $\forall \delta > 0$ and $x \in B_r$, we define an operator $T_2^{\epsilon,\delta}$ by

$$\begin{aligned}(T_2^{\epsilon,\delta}x)(t) &= \lim_{\lambda\to\infty} q\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)Q((t-s)^q\theta) \\ &\quad\times B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))d\theta ds \\ &= qQ(\epsilon^q\delta)\lim_{\lambda\to\infty}\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)Q((t-s)^q\theta - \epsilon^q\delta) \\ &\quad\times B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))d\theta ds.\end{aligned}$$

From the compactness of $Q(\epsilon^q\delta)$, $(\epsilon^q\delta > 0)$, we can conclude that $\{(T_2^{\epsilon,\delta}x)(t)|x \in B_r\}$ is relatively compact in $\overline{D(A)}$ $\forall\ \epsilon \in (0,t)$ and $\forall\ \delta > 0$. Further, for any $x \in B_r$, we obtain

$$\begin{aligned}\|(T_2x)(t) - (T_2^{\epsilon,\delta}x)(t)\|_X &\leq qM_Q\overline{M}\int_0^t (t-s)^{q-1}l(s)ds\int_0^\delta \theta\xi_q(\theta)d\theta\\ &\quad + \frac{M_Q\overline{M}}{\Gamma(q)}\int_{t-\epsilon}^t (t-s)^{q-1}l(s)ds\\ &\longrightarrow 0 \text{ as } \epsilon \to 0,\ \delta \to 0.\end{aligned}$$

Therefore, $\{(T_2x)(t) | x \in B_r\}$ is relatively compact in $\overline{D(A)}$ for all $t \in (0,b]$. Consequently, by Arzelà-Ascoli theorem, T_2 is compact.

Now, Krasnoselskii's fixed point theorem implies that (3.1) has at least one integral solution on $C(J,\overline{D(A)})$. $\square$

In order to establish our next result, we now assume the following additional hypotheses:

(H5)(iii) There exists a function $l \in L^\infty(J,\mathbb{R}^+)$ such that

$$\|f(t,x,y,z)\|_X \leq l(t), \quad \text{for all } x,y,z \in X \text{ and for } t \in [0,b].$$

(H9) for the function $u : J \times X \to \overline{D(A)}$, there exists a constant $L_1 > 0$ such that

$$\|u(t_1,x_1) - u(t_2,x_2)\|_X \leq L_1(|t_1-t_2| + \|x_1-x_2\|_X),$$

for all $t_1,\ t_2 \in [0,b]$ and all $x_1,\ x_2 \in X$. Further, let $M_0 = \sup_{t\in J}\|u(t,0)\|_X$.

The proof of the next result is based on Darbo-Sadovskii's fixed point theorem.

Theorem 3.3.3. *Suppose that (H4), (H5)(i), (H5)(iii), (H7), (H8) and (H9) are satisfied. Then (3.1) has an integral solution in $C(J,\overline{D(A)})$ provided*

$$L_1 < 1.$$

Proof. Let $B_r = \{x \in C(J,\overline{D(A)}) \colon \|x\|_C \leq r\}$ where $r = \frac{\xi_3}{1-L_1}$, $\xi_3 = M_0 + M_Q\|x_0\|_X + M_Q\|u(0,x(0))\|_X + \frac{M_Q\overline{M}}{\Gamma(q+1)}b^q\|l\|_{L^\infty}$. Define $T \colon C(J,\overline{D(A)}) \to C(J,\overline{D(A)})$ by

$$\begin{aligned}(Tx)(t) &= u(t,x(t)) + S_q(t)[x_0 - u(0,x(0))]\\ &\quad + \lim_{\lambda\to\infty}\int_0^t K_q(t-s)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))ds\\ &= (T_1x)(t) + (T_2x)(t),\end{aligned}$$

where

$$(T_1x)(t) = S_q(t)[x_0 - u(0,x(0))] + \lim_{\lambda\to\infty}\int_0^t K_q(t-s)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))ds,$$

$$(T_2x)(t) = u(t,x(t)).$$

Step I: To show that $T\colon B_r \to B_r$.
It follows in a straightforward manner from the fact that

$$\|(Tx)(t)\|_X \leq L_1\|x\|_C + M_0 + M_Q\|x_0\|_X + M_Q\|u(0,x(0))\|_X + \frac{M_Q\overline{M}}{\Gamma(q+1)} b^q \|l\|_{L^\infty} \leq r.$$

Step II: To show that T_1 is completely continuous.
(i) T_1 is equicontinuous on B_r:
Let $x \in B_r$ and $0 \leq t_1 < t_2 \leq b$. Then

$$\begin{aligned}
&\|(T_1x)(t_2) - (T_1x)(t_1)\|_X \\
&\leq \|S_q(t_2)[x_0 - u(0,x(0))] - S_q(t_1)[x_0 - u(0,x(0))]\|_X + \|\lim_{\lambda\to\infty}\int_0^{t_2} K_q(t_2-s) \\
&\times B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))ds - \lim_{\lambda\to\infty}\int_0^{t_1} K_q(t_1-s)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))ds\|_X \\
&:= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \|S_q(t_2)[x_0 - u(0,x(0))] - S_q(t_1)[x_0 - u(0,x(0))]\|_X, \\
I_2 &= \|\lim_{\lambda\to\infty}\int_0^{t_2} K_q(t_2-s)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))ds \\
&\quad - \lim_{\lambda\to\infty}\int_0^{t_1} K_q(t_1-s)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))ds\|_X.
\end{aligned}$$

For I_1, by Lemma 3.2.3, we have $I_1 \to 0$ as $t_2 \to t_1$. For $t_1 = 0$, $0 < t_2 \leq b$,

$$I_2 \leq \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}l(s)ds \leq \frac{M_Q\overline{M}}{\Gamma(q)} t_2^q \|l\|_{L^\infty} \longrightarrow 0 \text{ as } t_2 \to 0.$$

For $0 < t_1 < t_2 \leq b$,

$$\begin{aligned}
I_2 &\leq \frac{M_Q\overline{M}}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}l(s)ds + \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1} - (t_2-s)^{q-1}]l(s)ds \\
&\quad + \overline{M}\int_0^{t_1}(t_1-s)^{q-1}\|P_q(t_2-s) - P_q(t_1-s)\|_{B(X)}l(s)ds \\
&\quad = I_1^* + I_2^* + I_3^*,
\end{aligned}$$

where

$$
\begin{aligned}
I_1^* &= \frac{M_Q\overline{M}}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}l(s)ds,\\
I_2^* &= \frac{M_Q\overline{M}}{\Gamma(q)}\int_{0}^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]l(s)ds,\\
I_3^* &= \overline{M}\int_{0}^{t_1}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}l(s)ds.
\end{aligned}
$$

We have

$$I_1^* \leq \frac{M_Q\overline{M}}{\Gamma(q)}\|l\|_{L^\infty}(t_2-t_1)^q \longrightarrow 0 \text{ as } t_2\to t_1$$

and

$$I_2^* \leq \frac{M_Q\overline{M}}{\Gamma(q)}\|l\|_{L^\infty}\Big(\int_0^{t_1}(t_1-s)^{q-1}ds-\int_0^{t_1}(t_2-s)^{q-1}ds\Big)\leq \frac{M_Q\overline{M}}{\Gamma(q)}\|l\|_{L^\infty}(t_2-t_1)^q$$

i.e., $I_2^* \longrightarrow 0$ as $t_2\to t_1$.

Now since

$$I_3^* = \overline{M}\int_{0}^{t_1}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}l(s)ds,$$

therefore, for $\epsilon>0$ small enough, we have

$$
\begin{aligned}
I_3^* &\leq \overline{M}\int_{0}^{t_1}(t_1-s)^{q-1}l(s)ds\sup_{s\in[0,t_1-\epsilon]}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}l(s)ds\\
&+\overline{M}\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}l(s)ds\\
&\leq \overline{M}\int_{0}^{t_1}(t_1-s)^{q-1}l(s)ds\sup_{s\in[0,t_1-\epsilon]}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}+\frac{2M_Q\overline{M}}{\Gamma(q+1)}\|l\|_{L^\infty}\epsilon^q\\
&= I_{31}^*+I_{32}^*.
\end{aligned}
$$

From (H4), it follows that $I_{31}^*\to 0$ as $t_2\to t_1$ and also $I_{32}^*\to 0$ as $\epsilon\to 0$. Thus, $\|(T_1x)(t_2)-(T_1x)(t_1)\|_X\to 0$ as $t_2\to t_1$, independent of $x\in B_r$, which implies that $\{T_1x|x\in B_r\}$ is equicontinuous.

(ii) T_1 is continuous on B_r:

Let $(x_n)\subset B_r$ such that $x_n\to x$ in B_r.

Using (H5)(i), (H7), (H8) and Lebesgue's dominated convergence theorem, it follows that

$$f(s,x_n(s),(Hx_n)(s),(Gx_n)(s))\longrightarrow f(s,x(s),(Hx)(s),(Gx)(s)) \text{ as } n\to\infty.$$

Now for each $t \in J$, by using (H5)(ii), we have

$$\begin{aligned}
&(t-s)^{q-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) - f(s,x(s),(Hx)(s),(Gx)(s))\|_X \\
&\leq 2(t-s)^{q-1}l(s) \in L^1(J,\mathbb{R}^+), \text{ for } s \in [0,t], \ t \in J.
\end{aligned}$$

Therefore, by Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
&\int_0^t (t-s)^{q-1}\|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) - f(s,x(s),(Hx)(s),(Gx)(s))\|_X ds \\
&\longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Now, for each $t \in [0,b]$, we have

$$\begin{aligned}
&\|(T_1x_n)(t) - (T_1x)(t)\|_X \\
&\leq M_Q L_1\|x_n - x\|_C + \frac{M_Q\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1} \times \|f(s,x_n(s),(Hx_n)(s),(Gx_n)(s)) \\
&- f(s,x(s),(Hx)(s),(Gx)(s))\|_X ds \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, T_1 is continuous.
Further, it is obvious that T_1 is uniformly bounded.
To show that, for any $t \in J$, $\{T_1x(t) | x \in B_r\}$ is relatively compact in $\overline{D(A)}$:
For $t = 0$, it is obvious. Therefore, we fix $t \in (0,b]$. Since

$$\begin{aligned}
&S_q(t)[x_0 - u(0,x(0))] \\
&= \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}s^{q-1}\int_0^\infty q\theta\xi_q(\theta)Q(s^q\theta)[x_0 - u(0,x(0))]d\theta ds,
\end{aligned}$$

then for $\epsilon \in (0,t)$ and $\forall\ \delta > 0$, we define

$$\begin{aligned}
(T_1^{\epsilon,\delta}x)(t) = {}& \frac{q}{\Gamma(1-q)}\int_0^{t-\epsilon}\int_\delta^\infty \theta\xi_q(\theta)(t-s)^{-q}s^{q-1}Q(s^q\theta)[x_0 - u(0,x(0))]d\theta ds \\
&+ qQ(\epsilon^q\delta)\lim_{\lambda\to\infty}\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)Q((t-s)^q\theta - \epsilon^q\delta) \\
&\times B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))d\theta ds.
\end{aligned}$$

From the compactness of $Q(\epsilon^\alpha\delta)$, $(\epsilon^\alpha\delta > 0)$, we obtain that $\{(T_1^{\epsilon,\delta}x)(t) | x \in B_r\}$ is relatively compact in $\overline{D(A)}$ $\forall\ \epsilon \in (0,t)$ and $\forall\ \delta > 0$. Moreover, for any $x \in B_r$, we have

$$\begin{aligned}
&\|(T_1x)(t) - (T_1^{\epsilon,\delta}x)(t)\|_X \\
&\leq \frac{M_Q q}{\Gamma(1-q)}\Big\| \int_0^t (t-s)^{-q} s^{q-1} \int_0^\delta \theta\xi_q(\theta)[x_0 - u(0,x(0))]d\theta ds\Big\|_X \\
&+ \frac{M_Q q}{\Gamma(1-q)}\Big\| \int_{t-\epsilon}^t (t-s)^{-q} s^{q-1} \int_\delta^\infty \theta\xi_q(\theta)[x_0 - u(0,x(0))]d\theta ds\Big\|_X \\
&+ \Big\| \lim_{\lambda\to\infty} q \int_0^t \int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))d\theta ds\Big\|_X \\
&+ \Big\| \lim_{\lambda\to\infty} q \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)B_\lambda f(s,x(s),(Hx)(s),(Gx)(s))d\theta ds\Big\|_X \\
&\leq \frac{M_Q q}{\Gamma(1-q)} B(q,1-q)\Big[\|x_0\|_X + \|u(0,x(0))\|_X\Big] \int_0^\delta \theta\xi_q(\theta)d\theta + \frac{M_Q}{\Gamma(1-q)\Gamma(q)}\Big[\|x_0\|_X \\
&+ \|u(0,x(0))\|_X\Big] \int_{t-\epsilon}^t (t-s)^{-q}s^{q-1}ds + qM_Q\overline{M}\int_0^t (t-s)^{q-1}l(s)ds \int_0^\delta \theta\xi_q(\theta)d\theta \\
&+ qM_Q\overline{M}\int_{t-\epsilon}^t (t-s)^{q-1}l(s)ds \int_0^\infty \theta\xi_q(\theta)d\theta.
\end{aligned}$$

Therefore,

$$\|(T_1x)(t) - (T_1^{\epsilon,\delta}x)(t)\|_X \leq J_1 + J_2 + J_3 + J_4.$$

Using the inequality $\int_0^\infty \theta\xi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}$, it is found that J_1, J_3 and J_4 tend to 0 as $\epsilon, \delta \to 0$. Also, upon application of the absolute continuity of the Lebesgue integral, J_2 tends to 0 as $\epsilon, \delta \to 0$. Therefore, there exist relatively compact sets arbitrarily close to the set $\{(T_1x)(t)|x \in B_r\}$, $t > 0$ which implies that $\{(T_1x)(t)|x \in B_r\}$, $t > 0$ is relatively compact. Consequently, $\{T_1x|x \in B_r\}$ is a relatively compact set in $\overline{D(A)}$.

Step III: To show that T is continuous on B_r.

Proceeding similarly as in **Step II**, it can be shown that T is continuous on B_r.

Step IV: To show that T_2 is a contraction on B_r.

For any $x, y \in B_r$, we have

$$\|(T_2x)(t) - (T_2y)(t)\|_X \leq L_1\|x(t) - y(t)\|_X.$$

Thus

$$\|T_2x - T_2y\|_C \leq L_1\|x - y\|_C,$$

which implies that $\gamma_C(T_2B_r) \leq L_1\gamma_C(B_r)$, where γ_C is the Kuratowski measure of non-compactness on $C(J, \overline{D(A)})$. Also, T_1B_r is relatively compact in $\overline{D(A)}$ which gives $\gamma_C(T_1B_r) = 0$. Therefore,

$$\gamma_C(TB_r) \leq \gamma_C(T_1B_r) + \gamma_C(T_2B_r) \leq L_1\gamma_C(B_r).$$

As $L_1 < 1$, T is a γ-contraction on B_r. Hence, from Darbo-Sadvoskii's fixed point theorem, it follows that T has at least one fixed point on B_r. $\square$

Our next result for problem (3.1) is for the case in which the associated C_0-semigroup is not compact. Here the following assumptions are required:

(H5)(iv) there exist a constant $q_1 \in (0, q)$ and a function $l \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, x, y, z)\|_X \leq l(t)(\|x\|_X + \|y\|_X + \|z\|_X), \text{ for all } x,\ y,\ z \in X \text{and for } t \in [0, b],$$

(v) there exist l_1, l_2, $l_3 \in C(J, \mathbb{R}^+)$ such that

$$\gamma_2(f(t, D_1, D_2, D_3)) \leq l_1(t)\gamma_2(D_1) + l_2(t)\gamma_2(D_2) + l_3(t)\gamma_2(D_3),$$

for any bounded sets D_1, D_2, $D_3 \subset X$ and $t \in J$. Let $l_i^* = \sup_{t\in J}|l_i(t)|$, $i = 1, 2, 3$.

(H7)(iii) for any bounded set $D \subset X$ and $(t, s) \in \Delta$, there exists a function $\widetilde{m} \colon \Delta \to \mathbb{R}^+$ such that

$$\gamma_2(h(t, s, D)) \leq \widetilde{m}(t, s)\gamma_2(D)$$

with $\widetilde{m}^* = \sup\limits_{t\in J} \int_0^t \widetilde{m}(t, s)ds < \infty$.

(H8)(iii) for any bounded set $D \subset X$ and $(t, s) \in J \times J$, there exists a function $\widetilde{n} \colon J \times J \to \mathbb{R}^+$ such that

$$\gamma_2(g(t, s, D)) \leq \widetilde{n}(t, s)\gamma_2(D)$$

with $\widetilde{n}^* = \sup\limits_{t\in J} \int_0^b \widetilde{n}(t, s)ds < \infty$.

(H10) for each $t > 0$, $T(t)$ is equicontinuous.

Theorem 3.3.4. *Assume that (H5)(i), (iv), (v), (H7)(i), (ii), (iii), (H8)(i), (ii), (iii), (H9) and (H10) hold. Then (3.1) has an integral solution provided*

$$\xi_4 = L_1 + \frac{M_Q\overline{M}}{\Gamma(q)}(1 + H^* + G^*)\|l\|_{L^{\frac{1}{q_1}}} \frac{b^{q-q_1}}{(\frac{q-q_1}{1-q_1})^{1-q_1}} < 1$$

and

$$2L_1(1 + M_Q) + \frac{4M_Q\overline{M}}{\Gamma(q + 1)} b^q(l_1^* + 2l_2^*\widetilde{m}^* + 2l_3^*\widetilde{n}^*) < 1.$$

Proof. Choose $r = \frac{\phi}{1-\xi_4}$, where $\phi = M_0 + M_Q\|x_0\|_X + M_Q\|u(0, x(0))\|_X$ and let $B_r = \{x \in C(J, \overline{D(A)}) \mid \|x\|_C \leq r\}$. Define $T : C(J, \overline{D(A)}) \to C(J, \overline{D(A)})$ by

$$(Tx)(t) = u(t, x(t)) + S_q(t)[x_0 - u(0, x(0))] + \lim_{\lambda\to\infty} \int_0^t K_q(t-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))ds,$$

for $t \in J$. Then proceeding similarly as in Theorem 3.3.3, it can be shown that $T \colon B_r \to B_r$ is continuous as well as equicontinuous. Now, it remains to show that $T \colon B_r \to B_r$ is a condensing operator.

For all $D \subset B_r$, $T(D)$ is bounded and equicontinuous. Hence, by Lemma 1.3.3, there exists a countable set $D_0 = \{x_n\}_{n=1}^{\infty} \subset D$ such that

$$\gamma_C(T(D)) \leq 2\gamma_C(T(D_0)). \tag{3.4}$$

Since $T(D_0) \subset T(B_r)$ is equicontinuous, so by using Lemma 1.3.2, we get

$$\gamma_C(T(D_0)) = \max_{t\in J} \gamma_2(T(D_0)(t)). \tag{3.5}$$

Now, let

$$(Tx)(t) = (T_1x)(t) + (T_2x)(t),$$

where

$$\begin{aligned}(T_1x)(t) &= u(t, x(t)) + S_q(t)[x_0 - u(0, x(0))],\\ (T_2x)(t) &= \lim_{\lambda\to\infty} \int_0^t K_q(t-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))ds.\end{aligned}$$

For x, $y \in D_0$, we have

$$\|T_1x - T_1y\|_C \leq L_1(1 + M_Q)\|x - y\|_C.$$

Therefore, it follows that

$$\gamma_C(T_1(D_0)) \leq L_1(1 + M_Q)\gamma_C(D_0).$$

Now, for $t \in J$, we get

$$\begin{aligned}&\gamma_2(\{T_2x_n(t)\}_{n=1}^{\infty})\\ &= \gamma_2\Big(\Big\{\lim_{\lambda\to\infty} \int_0^t K_q(t-s)B_\lambda f(s, x(s), (Hx)(s), (Gx)(s))ds\Big\}_{n=1}^{\infty}\Big)\\ &\leq \frac{2M_Q\overline{M}}{\Gamma(q)}\gamma_C(D)\Big[\int_0^t (t-s)^{q-1}l_1(s)ds + 2\widetilde{m}^* \int_0^t (t-s)^{q-1}l_2(s)ds + 2\widetilde{n}^* \int_0^t (t-s)^{q-1}l_3(s)ds\Big]\\ &\leq \frac{2M_Q\overline{M}}{\Gamma(q+1)}\gamma_C(D)b^q(l_1^* + 2\widetilde{m}^*l_2^* + 2\widetilde{n}^*l_3^*).\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma_2(T(D_0)(t)) &\leq \gamma_2(T_1(D_0)(t)) + \gamma_2(T_2(D_0)(t))\\ &\leq \Big[L_1(1 + M_Q) + \frac{2M_Q\overline{M}}{\Gamma(q+1)}b^q(l_1^* + 2l_2^*\widetilde{m}^* + 2l_3^*\widetilde{n}^*)\Big]\gamma_C(D).\end{aligned}$$

From equations (3.4) and (3.5), we have

$$\gamma_{C}(T(D)) \leq 2\Big[L_1(1+M_{Q}) + \frac{2M_{Q}\overline{M}}{\Gamma(q+1)} b^q(l_1^* + 2l_2^*\widetilde{m}^* + 2l_3^*\widetilde{n}^*)\Big]\gamma_{C}(D) < \gamma_{C}(D).$$

Thus $T\colon B_r \to B_r$ is a condensing operator and therefore from Lemma 1.3.2, we conclude that T has a fixed point on B_r. ☐

3.4 Examples

Consider the following fractional partial differential equation system:

$$\begin{aligned}
&{}^{C}D^{q}_{0+}[x(t,y) - u(t,x(t,y))] = \frac{\partial^2}{\partial y^2}[x(t,y) - u(t,x(t,y))] \\
&+ f(t,x(t,y), \int_0^t h(t,s,x(s,y))ds, \int_0^b g(t,s,x(s,y))ds), \quad t \in [0,b],\ y \in \Omega, \\
&x(t,0) = 0 = x(t,\pi), \quad t \in [0,b], \\
&x(0,y) = x_0(y), \quad y \in \Omega,
\end{aligned}$$

where $q \in (0,1)$, $\Omega = [0,\pi]$, $b > 0$ is finite and $x_0 \in C(\Omega,\mathbb{R})$ with $x_0(0) = 0 = x_0(\pi)$.

Next, let $X = C(\Omega,\mathbb{R})$ and consider $A\colon D(A) \subset X \to X$ defined by

$$Aw = \frac{\partial^2 w}{\partial y^2}$$

with its domain of definition

$$D(A) = \{w \in X\colon \frac{\partial^2 w}{\partial y^2} \in X \text{ and } w = 0 \text{ on } \partial\Omega\}.$$

Then,

$$\overline{D(A)} = \{w \in X\colon w = 0 \text{ on } \partial\Omega\} \neq X.$$

Also, from [33], it is known that A satisfies Hille-Yosida condition with $(0,\infty) \subset \rho(A)$, $\|R(\lambda : A)\| \leq \lambda^{-1}$ and $\overline{M} = 1$ and generates a compact C_0-semigroup $\{Q(t)\}_{t>0}$ on $\overline{D(A)}$ with $M_{Q} = 1$.

Let us take

$$\begin{aligned}
&x(t)(y) = x(t,y), \text{and} \\
&f(t,x(t),(Hx)(t),(Gx)(t))(y) = f(t,x(t,y), \int_0^t h(t,s,x(s,y))ds, \int_0^b g(t,s,x(s,y))ds),
\end{aligned}$$

for $t \in [0,b]$, $y \in \Omega$. Consequently, (3.1) is the abstract formulation of the above considered problem.

For validation of Theorem 3.3.2, consider $u(t,x(t,y)) = \frac{1}{5}x_0(y)$. Then u satisfies (H6) with $L_1 = \frac{1}{5}$. Also, take

$$h(t, s, x(s, y)) = t \sin x(s, y) \text{ and } g(t, s, x(s, y)) = s \sin x(s, y).$$

Then h and g satisfy (H7) and (H8), respectively, with $H^* = b^2$ and $G^* = \dfrac{b^2}{2}$. Consider

$$\begin{aligned} & f(t, x(t, y), \int_0^t h(t, s, x(s, y))ds, \int_0^b g(t, s, x(s, y))ds) \\ & = t^{\frac{1}{2}} \cos\left(\frac{|x(t, y)|}{1 + |x(t, y)|} + \int_0^t h(t, s, x(s, y))ds + \int_0^b g(t, s, x(s, y))ds\right). \end{aligned}$$

If we choose $l(t) = t^{\frac{1}{2}}$, then f satisfies the assumptions in (H5). Thus all the conditions of Theorem 3.3.2 are fulfilled and therefore, we can confirm the existence of an integral solution.

Next, for Theorem 3.3.3, consider

$$\begin{aligned} & f(t, x(t, y), \int_0^t h(t, s, x(s, y))ds, \int_0^b g(t, s, x(s, y))ds) \\ & = \exp(-t) \cos\left(\frac{|x(t, y)|}{1 + |x(t, y)|} + \int_0^t h(t, s, x(s, y))ds + \int_0^b g(t, s, x(s, y))ds\right). \end{aligned}$$

Here, choose $l(t) = \exp(-t)$ and assume u to be a suitable function satisfying (H9). Then Theorem 3.3.3 implies the existence of an integral solution of this problem.

CHAPTER 4

Mild solutions of Volterra fractional differential equations of Sobolev type with finite delay

Equations of Sobolev type arise in a number of physical problems, namely, the flow of fluid through fissured rocks [17], the propagation of long waves with small amplitudes [22] and many more. The abstract Cauchy problem of Sobolev type is to find a function x which satisfies the following initial value problem:

$$\begin{aligned} \frac{d}{dt}Bx(t) + Ax(t) &= f(t, x(t)), \\ x(0) &= x_0, \end{aligned}$$

under different conditions on A and B, where A and B are linear operators with their domains and ranges contained, respectively, in a Banach space X and a Banach space Y.

Brill [26] and Showalter [96] considered semilinear evolution equations of Sobolev type in Banach spaces and established the existence of their solutions. Such fractional models are found to be more appropriate compared to those through integer-order differential equations and hence have been considered by a good number of researchers.

Earlier works discussed problems of Sobolev type under the following conditions on the operators $A\colon D(A) \subset X \to Y$, $B\colon D(B) \subset X \to Y$:
(1) $D(B) \subset D(A)$, B is bijective, B^{-1} is compact, $B^{-1}A\colon X \to D(B)$ is continuous [14],
(2) $D(B) \subset D(A)$, B is bijective, B^{-1} is compact [48],
(3) $D(B) \subset D(A)$, B is bijective, B^{-1} is continuous [11].

The work taken up here is different from some other similar works such as [15, 71] on two counts: (i) an ODE of an integer-order replaced by an ODE of a fractional order in our case, (ii) Lipschitz condition was used in these works. Therefore, to fill the gap, we find it pertinent to consider the following fractional differential equation of Sobolev type:

$$\left.\begin{aligned} &{}^{C}D^{q}_{0^+}(Bx(t)) + Ax(t) = f\Big(t, x_t, \int_0^t h(t,s,x_s)ds\Big),\ t \in J = [0,b],\ q \in (0,1),\\ &x_0(\phi) + (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(\phi) = \chi(\phi),\quad \phi \in [-d,0], \end{aligned}\right\} \tag{4.1}$$

where X and Y are Banach spaces, $A\colon D(A) \subseteq X \to Y$ and $B\colon D(B) \subseteq X \to Y$ are linear operators, $f\colon J \times \mathcal{C} \times X \to Y$, $h\colon \Delta \times \mathcal{C} \to X$, where $\mathcal{C} := C([-d,0],X)$ and $\Delta = \{(t,s) \in \mathbb{R}^2 | 0 \le s \le t \le b\}$, are given functions.
For $x \in C([-d,b],X)$ and each $t \in [0,b]$, $x_t \in \mathcal{C}$ is defined by

$$x_t(\phi) = x(t+\phi),$$

where $\chi \in \mathcal{C}$, $g\colon \mathcal{C}^n \to \mathcal{C}$ with $\chi(0) \in D(B)$ and $(g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0) \in D(B)$ are given functions.

4.1 Preliminaries

The following notations are used throughout the present work:

X and Y are Banach spaces, respectively, with norms $\|.\|_X$ and $\|.\|_Y$.
$B(Y)$ is a Banach space of all bounded linear operators on Y with the norm $\|.\|_{B(Y)}$.
For a linear operator T in Y, $\rho(T)$ is the resolvent set of T while $R(\lambda : T)$, with $\lambda \in \rho(T)$, denotes the resolvent of T.

Let J be a closed interval and $C(J,X)$ be the Banach space of all continuous functions from J to X with respect to the supremum norm. Let $\mathcal{C}$ denote the space $C([-d,0],X)$ with the norm $\|x\|_{\mathcal{C}}$ and $\mathbb{D}$ denote the space $C([-d,b],X)$ with the norm $\|.\|_{\mathbb{D}}$.

Take $\mathbb{R}^+ = [0,\infty)$ and $J = [0,b]$.

It is assumed that the operators A and B satisfy the hypotheses as follows [11]:
(i) A and B are closed,
(ii) $D(B) \subset D(A)$ and B is bijective,
(iii) $B^{-1}\colon Y \to D(B)$ is continuous,
(iv) For each $t \in [0,b]$ and for some $\lambda \in \rho(-AB^{-1})$, $R(\lambda : -AB^{-1})$ is a compact operator.

Hypotheses **(i)**-**(ii)** and the closed graph theorem together give the boundedness of the linear operator $AB^{-1}\colon Y \to Y$.

Lemma 4.1.1. *[11] Let $T(t)$ be a uniformly continuous semigroup. If $R(\lambda; A)$ is compact for every $\lambda \in \rho(A)$, then $T(t)$ is a compact semigroup.*

It follows that a compact semigroup $\{Q(t)\}_{t\ge0}$ in Y is generated by the operator $-AB^{-1}$. It is further assumed that there exists a constant $M_Q > 1$ such that $\sup_{t\in J}\|Q(t)\|_{B(Y)} \le M_Q$.

4.2 Mild Solution

Definition 4.2.1. *A mild solution of problem (4.1) means a function* $x \in \mathbb{D}$ *which satisfies*

$$\begin{cases} x(t) = B^{-1}S_q(t)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)] \\ \qquad + \displaystyle\int_{s=0}^{t} (t-s)^{q-1}B^{-1}T_q(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds, \\ \qquad t \in J = [0,b], \\ x_0(\phi) + (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(\phi) = \chi(\phi), \quad \phi \in [-d, 0], \end{cases} \tag{4.2}$$

where

$$\begin{aligned} S_q(t)x &= \int_0^\infty \xi_q(\omega)Q(t^q\omega)x d\omega, \\ T_q(t)x &= q\int_0^\infty \omega\xi_q(\omega)Q(t^q\omega)x d\omega, \\ \xi_q(\omega) &= \frac{1}{q}\omega^{-1-\frac{1}{q}}\bar{w}_q(\omega^{-\frac{1}{q}}), \\ \bar{w}_q(\omega) &= \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\omega^{-(qn+1)}\frac{\Gamma(qn+1)}{n!}\sin(n\pi q), \end{aligned}$$

with $\xi_q(\omega)$ *being a probability density function on* $(0, \infty)$ *satisfying*

$$\xi_q(\omega) \geq 0, \quad \int_0^\infty \xi_q(\omega)d\omega = 1, \quad \omega \in (0, \infty).$$

Lemma 4.2.1. *The bounded linear operators* $S_q(t)$ *and* $T_q(t)$ *satisfy the following properties:*
(i) *for any fixed* $t \geq 0$ *and* $y \in Y$,

$$\|S_q(t)y\|_{_Y} \leq M_{_Q}\|y\|_{_Y} \ \text{ and } \ \|T_q(t)y\|_{_Y} \leq \frac{M_{_Q}}{\Gamma(q)}\|y\|_{_Y},$$

(ii) $\{S_q(t)\}_{t\geq 0}$ *and* $\{T_q(t)\}_{t\geq 0}$ *are strongly continuous,*
(iii) *if* $\{T(t)\}_{t>0}$ *is compact, then* $\{S_q(t)\}_{t>0}$ *and* $\{T_q(t)\}_{t>0}$ *are compact operators.*

Lemma 4.2.2. *Assume that* $\{T(t)\}_{t>0}$ *is compact. Then* $\{T(t)\}_{t>0}$ *is equicontinuous.*

Lemma 4.2.3. *For* $q \in (0, 1]$ *and* $0 < a \leq b$, *we have* $|a^q - b^q| \leq (b-a)^q$.

4.3 Main Results

Let B_r, for each $r > 0$, denote the closed ball of radius r in $\mathbb{D}$.

Theorem 4.3.1. *Assume that*
(Hf1) *for the function* $f : J \times \mathcal{C} \times X \to Y$, *the following conditions hold:*

(i) *for each* $(\phi, x) \in \mathcal{C} \times X$*, the function* $t \to f(t, \phi, x)$ *is strongly measurable.*
(ii) $f\colon J \times \mathcal{C} \times X \to Y$ *is continuous and there exist a constant* $q_1 \in (0, q)$ *and two functions* $f_1, f_2 \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ *such that*

$$\|f(t, \phi_1, x) - f(t, \phi_2, x_2)\|_Y \leq f_1(t)\|\phi_1 - \phi_2\|_{\mathcal{C}} + f_2(t)\|x_1 - x_2\|_X,$$

for all $\phi_i \in \mathcal{C}$*,* $x_i \in X$ $(i = 1, 2)$*, a.e.* $t \in J = [0, b]$*, and* $J^q_{0+}k \in C(J, \mathbb{R}^+)$*, where* $k(t) := tf_2(t)$*,* $t \in J$*.*
(Hh1) *for a continuous function* $h\colon \Delta \times \mathcal{C} \to X$ *and a constant* $H > 0$*, the following is satisfied:*

$$\int_0^t \|h(t, s, x_s) - h(t, s, y_s)\|_X ds \leq H\|x_s - y_s\|_{\mathcal{C}},$$

for all $x_s, y_s \in \mathcal{C}$ *and* $(t, s) \in \Delta$*.*
(Hg1) *for a function* $g\colon \mathcal{C}^n \to \mathcal{C}$*, a constant* $G > 0$ *exists such that*

$$\|g(x_{t_1}, \ldots, x_{t_n}) - g(y_{t_1}, \ldots, y_{t_n})\|_{\mathcal{C}} \leq G\|x - y\|_{\mathbb{D}}, \quad \text{for } x, y \in \mathbb{D}.$$

Then, problem (4.1) has a unique mild solution $x \in \mathbb{D}$ *subject to*

$$\Theta := M_Q\|B^{-1}\|\left[\|B\|G + \frac{1}{\Gamma(q)}\frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\left(\|f_1\|_{L^{\frac{1}{q_1}}} + H\|f_2\|_{L^{\frac{1}{q_1}}}\right)\right] < 1.$$

Proof. Consider a map T defined on $\mathbb{D}$ by

$$(Tx)(t) = \begin{cases} B^{-1}S_q(t)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)] & \\ + \int_{s=0}^t (t-s)^{q-1}B^{-1}T_q(t-s)f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)ds, & t \in J = [0, b], \\ \chi(t) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(t), & t \in [-d, 0]. \end{cases}$$

<u>To show that T is well-defined on $B_r, r > 0$:</u>

Define a function $v \in \mathbb{D}$ such that $\|v\|_X \equiv 0$ for each $t \in [-d, b]$. Then for any $r > 0$ and $x \in B_r$, the following can be obtained for $t \in [0, b]$:

$$\begin{aligned} &\int_{s=0}^t \left\|(t-s)^{q-1}B^{-1}T_q(t-s)f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_X ds \\ &\leq \int_{s=0}^t \left\|(t-s)^{q-1}B^{-1}T_q(t-s)f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_X ds \\ &\leq \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_{s=0}^t (t-s)^{q-1}\left\|f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_Y ds. \end{aligned}$$

Using (Hf1) and (Hh1), we have

$$\begin{aligned}\left\|f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\right\|_Y &\leq f_1(s)\|x_s - v_s\|_{\mathcal{C}} + f_2(s)\left\|\int_0^s h(s,\tau,x_\tau)d\tau\right\|_X + \|f(s,v_s,0)\|_Y \\ &\leq f_1(s)r + Hrf_2(s) + H_1 s f_2(s) + F,\end{aligned}$$

where $\|h(t,s,0)\|_X \leq H_1 \ \forall \ (t,s) \in \Delta$ and $\|f(t,0,0)\|_Y \leq F \ \forall \ t \in J$.

Therefore, by using Hölder's inequality, the following can be obtained:

$$\begin{aligned}&\int_{s=0}^t \left\|(t-s)^{q-1}B^{-1}T_q(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\right\|_X ds \\ &\leq \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\Bigg[r\frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\Big(\|f_1\|_{L^{\frac{1}{q_1}}} + H\|f_2\|_{L^{\frac{1}{q_1}}}\Big) \\ &+ H_1 b^{1+q-q_1}\left\{\frac{\Gamma\left(\frac{2-q_1}{1-q_1}\right)\Gamma\left(\frac{q-q_1}{1-q_1}\right)}{\Gamma\left(\frac{2+q-2q_1}{1-q_1}\right)}\right\}^{1-q_1}\|f_2\|_{L^{\frac{1}{q_1}}} + F\frac{b^q}{q}\Bigg].\end{aligned}$$

It means that $\left\|(t-s)^{q-1}B^{-1}T_q(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)\Big)\right\|_X$ is Lebesgue integrable with respect to $s \in [0,t] \ \forall \ t \in [0,b]$. Therefore, $(t-s)^{q-1}B^{-1}T_q(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)\Big)$ is Bochner integrable with respect to $s \in [0,t]$ for all $t \in [0,b]$. Hence, $(Tx)(.)$ is well-defined on $[0,b]$ for any $x \in B_r$.

Also, $(Tx)(.)$ is well-defined on $[-d,0]$ for any $x \in B_r$. Thus, T is well-defined on $B_r \subset \mathbb{D}$.

To show that $Tx \in \mathbb{D}$ for $x \in \mathbb{D}$:

Let $x \in \mathbb{D}$ and $-d \leq s_1 < s_2 \leq 0$. Then

$$\begin{aligned}&\|(Tx)(s_2) - (Tx)(s_1)\|_X \\ &\leq \|\chi(s_2) - \chi(s_1)\|_X + \|(g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(s_2) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(s_1)\|_X \\ &\longrightarrow 0 \text{ as } s_2 \to s_1.\end{aligned}$$

Let $0 < s_1 < s_2 \leq b$. Then

$$\begin{aligned}\|(Tx)(s_2) - (Tx)(s_1)\|_X \leq\ & \|B^{-1}S_q(s_2)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)] \\ &- B^{-1}S_q(s_1)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)]\|_X \\ &+ \Bigg\|\int_{s=0}^{s_2}(s_2-s)^{q-1}B^{-1}T_q(s_2-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds \\ &- \int_{s=0}^{s_1}(s_1-s)^{q-1}B^{-1}T_q(s_1-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\Bigg\|_X.\end{aligned}$$

Now, using Lemma 4.2.1(ii), we have

$$\begin{aligned}
&\|B^{-1}S_q(s_2)B[\chi(0)-(g(x_{t_1},x_{t_2},\dots,x_{t_n}))(0)]-B^{-1}S_q(s_1)B[\chi(0)-(g(x_{t_1},x_{t_2},\dots,x_{t_n}))(0)]\|_X\\
&\le \|B^{-1}\|\|S_q(s_2)B[\chi(0)-(g(x_{t_1},x_{t_2},\dots,x_{t_n}))(0)]-S_q(s_1)B[\chi(0)-(g(x_{t_1},x_{t_2},\dots,x_{t_n}))(0)]\|_Y\\
&\longrightarrow 0 \text{ as } s_2\to s_1
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\int_{s=0}^{s_2}(s_2-s)^{q-1}B^{-1}T_q(s_2-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\\
&-\int_{s=0}^{s_1}(s_1-s)^{q-1}B^{-1}T_q(s_1-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\Big\|_X\\
&\le \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_{s_1}^{s_2}(s_2-s)^{q-1}\Big\|f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds\\
&+\|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\Big\|[T_q(s_2-s)-T_q(s_1-s)]f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds\\
&+\Big\|\int_0^{s_1}[(s_2-s)^{q-1}-(s_1-s)^{q-1}]B^{-1}T_q(s_2-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\Big\|_X\\
&=:\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_1&=\|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_{s_1}^{s_2}(s_2-s)^{q-1}\Big\|f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds,\\
\mathcal{I}_2&=\|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\Big\|[T_q(s_2-s)-T_q(s_1-s)]f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds,\\
\mathcal{I}_3&=\Big\|\int_0^{s_1}[(s_2-s)^{q-1}-(s_1-s)^{q-1}]B^{-1}T_q(s_2-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\Big\|_X.
\end{aligned}$$

Now,

$$\mathcal{I}_1\le\|B^{-1}\|\frac{M_Q}{\Gamma(q)}\left[\|x\|_{\mathbb{D}}\frac{(s_2-s_1)^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\Big(\|f_1\|_{L^{\frac{1}{q_1}}}+H\|f_2\|_{L^{\frac{1}{q_1}}}\Big)+\mathcal{I}_1'+F\frac{(s_2-s_1)^q}{q}\right],$$

where

$$\begin{aligned}
\mathcal{I}_1'&=H_1\int_{s_1}^{s_2}(s_2-s)^{q-1}sf_2(s)ds\\
&\le H_1\Big|\int_0^{s_2}(s_2-s)^{q-1}sf_2(s)ds-\int_0^{s_1}(s_1-s)^{q-1}sf_2(s)ds\Big|\\
&+H_1\Big|\int_0^{s_1}[(s_1-s)^{q-1}-(s_2-s)^{q-1}]sf_2(s)ds\Big|\\
&=:\mathcal{I}_{11}+\mathcal{I}_{12},
\end{aligned}$$

with

$$\mathcal{I}_{11} := H_1\Big|\int_0^{s_2}(s_2-s)^{q-1}sf_2(s)ds - \int_0^{s_1}(s_1-s)^{q-1}sf_2(s)ds\Big|$$
$$\longrightarrow 0 \text{ as } s_2 \to s_1 \text{ (using (Hf1))}$$

and

$$\mathcal{I}_{12} := H_1\Big|\int_0^{s_1}[(s_1-s)^{q-1}-(s_2-s)^{q-1}]sf_2(s)ds\Big|$$
$$\leq H_1\int_0^{s_1}[(s_1-s)^{q-1}-(s_2-s)^{q-1}]sf_2(s)ds.$$

Now we have

$$[(s_1-s)^{q-1}-(s_2-s)^{q-1}]sf_2(s) \leq (s_1-s)^{q-1}sf_2(s)$$

and since $\int_0^{s_1}(s_1-s)^{q-1}sf_2(s)ds$ exists, therefore Lebesgue's dominated convergence theorem gives $\mathcal{I}_{12} \to 0$ as $s_2 \to s_1$. Thus, $\mathcal{I}_1 \to 0$ as $s_2 \to s_1$.

$$\mathcal{I}_2 \leq \|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}[f_1(s)\|x\|_{\mathbb{D}}+Hf_2(s)\|x\|_{\mathbb{D}}$$
$$+H_1f_2(s)s+F]ds$$
$$=: \mathcal{I}_{21}+\mathcal{I}_{22}+\mathcal{I}_{23}+\mathcal{I}_{24},$$

where

$$\mathcal{I}_{21} = \|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_1(s)ds,$$
$$\mathcal{I}_{22} = H\|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_2(s)ds,$$
$$\mathcal{I}_{23} = H_1\|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}sf_2(s)ds,$$
$$\mathcal{I}_{24} = F\|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}ds.$$

Let us consider $\epsilon > 0$ to be sufficiently small. Consequently,

$$\mathcal{I}_{21} = \|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1-\epsilon}(s_1-s)^{q-1}f_1(s)\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}ds$$
$$+\|B^{-1}\|\|x\|_{\mathbb{D}}\int_{s_1-\epsilon}^{s_1}(s_1-s)^{q-1}f_1(s)\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}ds$$
$$\leq \|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1}(s_1-s)^{q-1}f_1(s)ds\sup_{s\in[0,s_1-\epsilon]}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}ds$$
$$+\frac{\epsilon^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\|f_1\|_{L^{\frac{1}{q_1}}}.$$

As $T_q(t)$ is known to be continuous in the uniform operator topology for $t > 0$, we have $\mathcal{I}_{21} \to 0$ as $s_2 \to s_1$, $\epsilon \to 0$. Similarly, it can be shown that $\mathcal{I}_{22}, \mathcal{I}_{23}$ and $\mathcal{I}_{24}$ also tend to zero as $s_2 \to s_1$, $\epsilon \to 0$. Therefore, $\mathcal{I}_2 \to 0$ as $s_2 \to s_1$.
Now

$$\begin{aligned}
\mathcal{I}_3 \leq\ & \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \|x\|_{\mathbb{D}} \int_0^{s_1} [(s_1 - s)^{q-1} - (s_2 - s)^{q-1}] f_1(s) ds \\
& + H \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \|x\|_{\mathbb{D}} \int_0^{s_1} [(s_1 - s)^{q-1} - (s_2 - s)^{q-1}] f_2(s) ds \\
& + H_1 \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \int_0^{s_1} [(s_1 - s)^{q-1} - (s_2 - s)^{q-1}] f(s) s ds \\
& + F \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \int_0^{s_1} [(s_1 - s)^{q-1} - (s_2 - s)^{q-1}] ds \\
& =: \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33} + \mathcal{I}_{34}.
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{I}_{31} &= \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \|x\|_{\mathbb{D}} \int_0^{s_1} [(s_1 - s)^{q-1} - (s_2 - s)^{q-1}] f_1(s) ds \\
&\leq \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \|x\|_{\mathbb{D}} \frac{\|f_1\|_{L^{\frac{1}{q_1}}}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}} (s_2 - s_1)^{q-q_1} \\
&\longrightarrow 0 \text{ as } s_2 \to s_1.
\end{aligned}$$

Similarly, it can be shown that $\mathcal{I}_{32}, \mathcal{I}_{33}$ and $\mathcal{I}_{34}$ also tend to zero as $s_2 \to s_1$. Therefore, $\mathcal{I}_3 \to 0$ as $s_2 \to s_1$.

Thus, for $0 < s_1 < s_2 \leq b$,

$$\|(Tx)(s_2) - (Tx)(s_1)\|_X \longrightarrow 0 \text{ as } s_2 \to s_1.$$

Therefore, $Tx \in \mathbb{D}$ for any $x \in \mathbb{D}$.

In order to show that T has a unique fixed point on $\mathbb{D}$, it needs to be established that T has a unique fixed point on $B_{r_0} \subset \mathbb{D}$ where r_0 satisfies

$$\begin{aligned}
r_0 \geq\ & M_Q \|B\| \|B^{-1}\| (\|\chi\|_{\mathcal{C}} + \|g(v_{t_1}, v_{t_2}, \ldots, v_{t_n})\|_{\mathcal{C}}) + \Theta r_0 \\
& + H_1 b^{1+q-q_1} \left\{ \frac{\Gamma\left(\frac{2-q_1}{1-q_1}\right) \Gamma\left(\frac{q-q_1}{1-q_1}\right)}{\Gamma\left(\frac{2+q-2q_1}{1-q_1}\right)} \right\}^{1-q_1} \|f_2\|_{L^{\frac{1}{q_1}}} + F \frac{b^q}{q}.
\end{aligned}$$

<u>To show that $T(B_{r_0}) \subset B_{r_0}$.</u>

For $x \in B_{r_0}$ and $t \in [0, b]$, the following is obtained:

$$\begin{aligned}
\|(Tx)(t)\|_{X} &\leq \|B^{-1}S_q(t)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)]\|_{X} \\
&\quad + \left\| \int_{s=0}^{t} (t-s)^{q-1} B^{-1} T_q(t-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big) ds \right\|_{X} \\
&\leq M_Q \|B\| \|B^{-1}\| (\|\chi\|_{\mathcal{C}} + G r_0 + \|g(v_{t_1}, v_{t_2}, \ldots, v_{t_n})\|_{\mathcal{C}}) \\
&\quad + \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \Bigg[r_0 \frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}} \left(\|f_1\|_{L^{\frac{1}{q_1}}} + H\|f_2\|_{L^{\frac{1}{q_1}}} \right) \\
&\quad + H_1 b^{1+q-q_1} \left\{ \frac{\Gamma\left(\frac{2-q_1}{1-q_1}\right)\Gamma\left(\frac{q-q_1}{1-q_1}\right)}{\Gamma\left(\frac{2+q-2q_1}{1-q_1}\right)} \right\}^{1-q_1} \|f_2\|_{L^{\frac{1}{q_1}}} + F\frac{b^q}{q} \Bigg],
\end{aligned}$$

and for $t \in [-d, 0]$,

$$\begin{aligned}
\|(Tx)(t)\|_{X} &\leq \|\chi(t)\|_{X} + \|(g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(t)\|_{X} \\
&\leq \|\chi\|_{\mathcal{C}} + G r_0 + \|g(v_{t_1}, v_{t_2}, \ldots, v_{t_n})\|_{\mathcal{C}}.
\end{aligned}$$

Thus, $Tx \in B_{r_0}$ for any $x \in B_{r_0}$.

To show that T is a contraction on B_{r_0}.
Let $x, y \in B_{r_0}$. For $t \in [0, b]$, using (Hf1), (Hh1) and (Hg1), we have

$$\begin{aligned}
&\|(Tx)(t) - (Ty)(t)\|_{X} \\
&\leq M_Q \|B\| \|B^{-1}\| \|(g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0) - (g(y_{t_1}, y_{t_2}, \ldots, y_{t_n}))(0)\|_{X} \\
&+ \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \int_{s=0}^{t} (t-s)^{q-1} \left\| f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big) - f\Big(s, y_s, \int_0^s h(s,\tau,y_\tau)d\tau\Big) \right\|_{Y} ds \\
&\leq M_Q G \|B\| \|B^{-1}\| \|x-y\|_{\mathbb{D}} + \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \Big(\int_0^t (t-s)^{q-1} f_1(s) ds \Big) \|x-y\|_{\mathbb{D}} \\
&+ \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \int_{s=0}^{t} (t-s)^{q-1} \Big(\int_0^s \|h(s,\tau,x_\tau) - h(s,\tau,y_\tau)\|_{X} d\tau \Big) f_2(s) ds \\
&\leq \Theta \|x-y\|_{\mathbb{D}}.
\end{aligned}$$

Also, for $t \in [-d, 0]$,

$$\|(Tx)(t) - (Ty)(t)\|_{X} \leq G\|x-y\|_{\mathbb{D}}.$$

Therefore,

$$\|Tx - Ty\|_{D} \leq \Theta \|x-y\|_{\mathbb{D}}.$$

Therefore, by means of Banach fixed point theorem, it is established that T has a unique fixed point in $\mathbb{D}$. □

Theorem 4.3.2. *Assume that*

(Hf2) *there exists a function* $f \colon J \times \mathcal{C} \times X \to Y$ *such that*

(i) *for a.e.* $t \in J$, *the function* $(\phi, x) \to f(t, \phi, x)$ *is continuous, and for each* $(\phi, x) \in$

$\mathcal{C} \times X$, the function $t \to f(t, \phi, x)$ is strongly measurable,

(ii) *there exist a constant $q_1 \in (0, q)$ and positive functions $f_1, f_2, f_3 \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ such that*

$$\|f(t,\phi,x)\|_{_Y} \leq f_1(t) + f_2(t)\|\phi\|_{_\mathcal{C}} + f_3(t)\|x\|_{_X},$$

for any $\phi \in \mathcal{C}$, $x \in X$ and $t \in J$.

(Hh2) *there exists a function $h\colon \Delta \times \mathcal{C} \to X$ such that*

(i) *for each $(t,s) \in \Delta$, the function $h(t,s,.)\colon \mathcal{C} \to X$ is continuous, and for each $x \in X$, the function $h(.,.,x)\colon \Delta \to X$ is strongly measurable,*

(ii) *there exists a function $H(t,s) \in C(\Delta, \mathbb{R}^+)$ such that*

$$\|h(t,s,\phi)\|_{_X} \leq H(t,s)\|\phi\|_{_\mathcal{C}}, \ \textit{for } (t,s) \in \Delta, \ \ \phi \in \mathcal{C}$$

and $H^* = \sup_{t \in J} \int_0^t H(t,s)ds < \infty$,

*and **(Hg1)** holds. Then problem (4.1) admits a mild solution in $\mathbb{D}$ subject to*

$$\Sigma := M_{_Q}\|B^{-1}\|\left[\|B\|G + \frac{1}{\Gamma(q)}\frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\left(\|f_2\|_{_{L^{\frac{1}{q_1}}}} + H^*\|f_3\|_{_{L^{\frac{1}{q_1}}}}\right)\right] < 1.$$

Proof. Consider a map T defined on $\mathbb{D}$ by

$$(Tx)(t) = \begin{cases} B^{-1}S_q(t)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0)] \\ + \int_{s=0}^t (t-s)^{q-1}B^{-1}T_q(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds, & t \in J = [0,b], \\ \chi(t) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(t), & t \in [-d,0]. \end{cases}$$

Then under the given assumptions, it is clearly evident that the map T is well-defined on B_r for each $r > 0$.

Choose

$$r_0 \geq M_{_Q}\|B\|\|B^{-1}\|(\|\chi\|_{_\mathcal{C}} + \|g(v_{t_1}, v_{t_2}, \dots, v_{t_n})\|_{_\mathcal{C}}) + \Sigma r_0 + \|B^{-1}\|\frac{M_{_Q}}{\Gamma(q)}\frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\|f_1\|_{_{L^{\frac{1}{q_1}}}}$$

and define two operators T_1 and T_2 on B_{r_0} given by

$$(T_1x)(t) = \begin{cases} B^{-1}S_q(t)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0)], & t \in J = [0,b], \\ \chi(t) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(t), & t \in [-d,0], \end{cases}$$

and

$$(T_2x)(t) = \begin{cases} \int_{s=0}^t (t-s)^{q-1}B^{-1}T_q(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds, & t \in J = [0,b], \\ 0, & t \in [-d,0]. \end{cases}$$

Step I: To show that $T_1x + T_2y \in B_r$ whenever $x, y \in B_r$.

Proceeding in a similar manner as followed in Theorem 4.3.1, it can be shown that $T_1x, T_2x \in \mathbb{D}$ for any $x \in B_{r_0}$. For $x, y \in B_{r_0}$ and $t \in [-d, 0]$, the following can be obtained:

$$\|(T_1x)(t) + (T_2x)(t)\|_X \leq \|\chi\|_{\mathcal{C}} + Gr_0 + \|g(v_{t_1}, v_{t_2}, \dots, v_{t_n})\|_{\mathcal{C}}.$$

Now, for $t \in [0, b]$,

$$\begin{aligned}\|(T_1x)(t) + (T_2x)(t)\|_X \leq{}& M_Q\|B\|\|B^{-1}\|(\|\chi\|_{\mathcal{C}} + Gr_0 + \|g(v_{t_1}, v_{t_2}, \dots, v_{t_n})\|_{\mathcal{C}}) \\ &+ \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\left(\|f_1\|_{L^{\frac{1}{q_1}}} + \|f_2\|_{L^{\frac{1}{q_1}}} r_0 + H^*\|f_3\|_{L^{\frac{1}{q_1}}} r_0\right).\end{aligned}$$

Therefore, $T_1x + T_2y \in B_{r_0}$ for any $x, y \in B_{r_0}$.

Step II: To show that T_1 is a contraction.

For $x, y \in B_{r_0}$ and $t \in [-d, b]$, we have

$$\|(T_1x)(t) - (T_1y)(t)\|_X \leq M_Q\|B\|\|B^{-1}\|G\|x - y\|_{\mathbb{D}}$$

which shows that T_1 is a contraction.

Step III: To show that T_2 is completely continuous.

$\{T_2x | x \in B_{r_0}\}$ is uniformly bounded: It follows easily from **Step I**.

$\{T_2x | x \in B_{r_0}\}$ is equicontinuous: Let $x \in B_{r_0}$ and $0 \leq s_1 < s_2 \leq b$. Then

$$\begin{aligned}&\|(T_2x)(s_2) - (T_2x)(s_1)\|_X \\ &\leq \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_{s_1}^{s_2}(s_2 - s)^{q-1}\left\|f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_Y ds \\ &+ \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_0^{s_1}[(s_1 - s)^{q-1} - (s_2 - s)^{q-1}]\left\|f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_Y ds \\ &+ \|B^{-1}\|\int_0^{s_1}(s_1 - s)^{q-1}\|[T_q(s_2 - s) - T_q(s_1 - s)]\|_{B(Y)} \\ &\times \left\|f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_Y ds \\ &=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_1 &= \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_{s_1}^{s_2}(s_2 - s)^{q-1}\left\|f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_Y ds, \\ \mathcal{I}_2 &= \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_0^{s_1}[(s_1 - s)^{q-1} - (s_2 - s)^{q-1}]\left\|f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_Y ds, \\ \mathcal{I}_3 &= \|B^{-1}\|\int_0^{s_1}(s_1 - s)^{q-1}\|[T_q(s_2 - s) - T_q(s_1 - s)]\|_{B(Y)}\left\|f\left(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau\right)\right\|_Y ds.\end{aligned}$$

Now, using (Hf2), we get

$$
\begin{aligned}
\mathcal{I}_1 &\le \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_{s_1}^{s_2}(s_2-s)^{q-1}\Big\{f_1(s)+f_2(s)\|x_s\|_{\mathcal{C}}+f_3(s)\Big\|\int_0^s h(s,\tau,x_\tau)d\tau\Big\|_X\Big\}ds\\
&\le \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\frac{(s_2-s_1)^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\Big[\|f_1\|_{L^{\frac{1}{q_1}}}+\|x\|_{\mathbb{D}}\|f_2\|_{L^{\frac{1}{q_1}}}+H^*\|x\|_{\mathbb{D}}\|f_3\|_{L^{\frac{1}{q_1}}}\Big]\\
&\longrightarrow 0 \text{ as } s_2\to s_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{I}_2 \le{}& \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_0^{s_1}[(s_1-s)^{q-1}-(s_2-s)^{q-1}]\Big(f_1(s)+f_2(s)\|x_s\|_{\mathcal{C}}\\
&+f_3(s)\Big\|\int_0^s h(s,\tau,x_\tau)d\tau\Big\|_X\Big)ds\\
\le{}& \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\frac{(s_2-s_1)^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}}\Big[\|f_1\|_{L^{\frac{1}{q_1}}}+\|x\|_{\mathbb{D}}\|f_2\|_{L^{\frac{1}{q_1}}}+H^*\|x\|_{\mathbb{D}}\|f_3\|_{L^{\frac{1}{q_1}}}\Big]\\
&\longrightarrow 0 \text{ as } s_2\to s_1.
\end{aligned}
$$

We further have

$$
\begin{aligned}
\mathcal{I}_3 \le{}& \|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_1(s)ds\\
&+\|x\|_{\mathbb{D}}\|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_2(s)ds\\
&+H^*\|x\|_{\mathbb{D}}\|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_3(s)ds\\
=:{}& \mathcal{I}_{31}+\mathcal{I}_{32}+\mathcal{I}_{33},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_{31} &= \|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_1(s)ds,\\
\mathcal{I}_{32} &= \|x\|_{\mathbb{D}}\|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_2(s)ds,\\
\mathcal{I}_{33} &= H^*\|x\|_{\mathbb{D}}\|B^{-1}\|\int_0^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_3(s)ds.
\end{aligned}
$$

For $s_1=0$ and $0<s_2\le b$, $\mathcal{I}_3=0$. Therefore, for $s_1>0$ and $\epsilon>0$ small enough,

$$
\begin{aligned}
\mathcal{I}_{31} ={}& \|B^{-1}\|\int_0^{s_1-\epsilon}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_1(s)ds\\
&+\|B^{-1}\|\int_{s_1-\epsilon}^{s_1}(s_1-s)^{q-1}\|T_q(s_2-s)-T_q(s_1-s)\|_{B(Y)}f_1(s)ds.
\end{aligned}
$$

Now, following similar arguments as in Theorem 4.3.1, it can be shown that $\mathcal{I}_{31},\mathcal{I}_{32}$ and

$\mathcal{I}_{33}$ tend to zero as $s_2 \to s_1$, $\epsilon \to 0$. Thus, $\|(T_2x)(s_2) - (T_2x)(s_1)\|_X \to 0$ as $s_2 \to s_1$ implying that $\{T_2x | x \in B_{r_0}\}$ is equicontinuous.

Step IV: To show that T_2 is continuous on B_{r_0}:

Let $(x^{(k)}) \subset B_{r_0}$ and $x \in B_{r_0}$ such that $x^{(k)} \to x$ as $k \to \infty$. Then, for $t \in [0, b]$, we have

$$\begin{aligned}\|(T_2x^{(k)})(t) - (T_2x)(t)\|_X \leq \|B^{-1}\|\frac{M_Q}{\Gamma(q)}\int_{s=0}^{t}(t-s)^{q-1}\Big\|f\Big(s, x_s^{(k)}, \int_0^s h(s,\tau,x_\tau^{(k)})d\tau\Big)\\ - f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds.\end{aligned}$$

Using (Hh2) and Lebesgue's dominated convergence theorem, we get

$$\int_0^s h(s,\tau,x_\tau^{(k)})d\tau \longrightarrow \int_0^s h(s,\tau,x_\tau)d\tau \quad \text{as } k \to \infty.$$

Consequently,

$$f\Big(s, x_s^{(k)}, \int_0^s h(s,\tau,x_\tau^{(k)})d\tau\Big) \longrightarrow f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big) \quad \text{as } k \to \infty.$$

Also, for each $t \in J$,

$$\begin{aligned}&(t-s)^{q-1}\Big\|f\Big(s, x_s^{(k)}, \int_0^s h(s,\tau,x_\tau^{(k)})d\tau\Big) - f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y\\ &\leq 2(t-s)^{q-1}[f_1(s) + r_0f_2(s) + r_0H^*f_3(s)],\end{aligned}$$

which is integrable for $s \in [0, t)$ and $t \in [0, b]$. Hence, application of Lebesgue's dominated convergence theorem gives

$$\begin{aligned}&\int_{s=0}^{t}(t-s)^{q-1}\Big\|f\Big(s, x_s^{(k)}, \int_0^s h(s,\tau,x_\tau^{(k)})d\tau\Big) - f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds\\ &\longrightarrow 0 \quad \text{as } k \to \infty.\end{aligned}$$

Therefore, T_2 is continuous on B_{r_0}.

Step V: To show that, for any $t \in [-d, b]$, $\{(T_2x)(t) | x \in B_{r_0}\}$ is relatively compact in X:

Let $V(t) = \{(T_2x)(t) | x \in B_{r_0}\}$, $t \in [-d, b]$.

For $t \in [-d, 0]$, it is obvious that $V(t)$ is relatively compact in X.

Now, letting $0 < t \leq b$ to be fixed and $\forall\ \epsilon \in (0, t)$, $\forall\ \theta > 0$, the following operator $T_2^{\epsilon,\theta}$ is defined:

$$\begin{aligned}(T_2^{\epsilon,\theta}x)(t) &= B^{-1}\int_0^{t-\epsilon}\int_\theta^\infty q\omega(t-s)^{q-1}\xi_q(\omega)Q((t-s)^q\omega)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)d\omega ds\\ &= B^{-1}Q(\epsilon^q\theta)\int_0^{t-\epsilon}\int_\theta^\infty q\omega(t-s)^{q-1}\xi_q(\omega)Q((t-s)^q\omega - \epsilon^q\theta)\\ &\times f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)d\omega ds.\end{aligned}$$

From the compactness of $Q(\epsilon^q\theta)$, $(\epsilon^q\theta > 0)$, it is established that the set $V^{\epsilon,\theta}(t) = \{(T_2^{\epsilon,\theta}x)(t) | x \in B_{r_0}\}$ is relatively compact in X $\forall$ $\epsilon \in (0,t)$ and $\forall$ $\theta > 0$. Also, for any $x \in B_{r_0}$, the following holds:

$$\begin{aligned}\|(T_2x)(t) - (T_2^{\epsilon,\theta}x)(t)\|_X &\leq q\|B^{-1}\| \frac{M_Q}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}} \left(\|f_1\|_{L^{\frac{1}{q_1}}} + r_0\|f_2\|_{L^{\frac{1}{q_1}}} + r_0H^*\|f_3\|_{L^{\frac{1}{q_1}}} \right)\\ &\times \left[b^{q-q_1} \int_0^\theta \omega\xi_q(\omega)d\omega + \frac{\epsilon^{q-q_1}}{\Gamma(q+1)} \right]\\ &\longrightarrow 0 \text{ as } \epsilon \to 0,\ \theta \to 0.\end{aligned}$$

Therefore, application of Arzelá-Ascoli theorem tells that $\{T_2x | x \in B_{r_0}\}$ is relatively compact which in turn implies that T_2 is completely continuous.

Consequently, Krasnoselskii's fixed point theorem guarantees that $T_1 + T_2$ has a fixed point on $B_{r_0} \subset \mathbb{D}$. ☐

Theorem 4.3.3. *Assume that earlier hypotheses (Hf2), (Hh2) and the following condition hold:*
(Hg2) *$g\colon \mathcal{C}^n \to \mathcal{C}$ is completely continuous and there exist constants $G_1, G_2 > 0$ such that*

$$\|g(x_{t_1}, \ldots, x_{t_n})\|_{\mathcal{C}} \leq G_1\|x\|_{\mathbb{D}} + G_2, \quad \textit{for all } x \in \mathbb{D}.$$

Then problem (4.1) admits a mild solution in $\mathbb{D}$ provided

$$\Pi := M_Q\|B^{-1}\| \left[\|B\|G_1 + \frac{1}{\Gamma(q)} \frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}} \left(\|f_2\|_{L^{\frac{1}{q_1}}} + H^*\|f_3\|_{L^{\frac{1}{q_1}}} \right) \right] < 1.$$

Proof. The proof can be accomplished in a similar manner like the one for Theorem 4.3.2. Therefore, only the new steps in this proof are presented.

Consider a map T defined on $\mathbb{D}$ by

$$(Tx)(t) = \begin{cases} B^{-1}S_q(t)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)] & \\ + \int_{s=0}^t (t-s)^{q-1}B^{-1}T_q(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds, & t \in J = [0,b], \\ \chi(t) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(t), & t \in [-d, 0]. \end{cases}$$

Then T is well-defined on B_r for each $r > 0$ and $T(B_{r_0}) \subset B_{r_0}$ where r_0 is chosen such that

$$r_0 \geq M_Q\|B\|\|B^{-1}\|(\|\chi\|_{\mathcal{C}} + G_2) + \Pi r_0 + \|B^{-1}\| \frac{M_Q}{\Gamma(q)} \frac{b^{q-q_1}}{\left(\frac{q-q_1}{1-q_1}\right)^{1-q_1}} \|f_1\|_{L^{\frac{1}{q_1}}}.$$

The operator T is split into the following two operators T_1 and T_2 on B_{r_0}:

$$(T_1x)(t) = \begin{cases} B^{-1}S_q(t)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)], & t \in J = [0, b], \\ 0, & t \in [-d, 0], \end{cases}$$

and

$$(T_2x)(t) = \begin{cases} \int_{s=0}^{t}(t-s)^{q-1}B^{-1}T_q(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds, & t \in J = [0, b], \\ \chi(t) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(t), & t \in [-d, 0]. \end{cases}$$

In order to show that T has a unique fixed point on $\mathbb{D}$, it is required to establish that T is completely continuous on B_{r_0}.

Obviously $\{Tx|x \in B_{r_0}\}$ is uniformly bounded and that $\{Tx|x \in B_{r_0}\}$ is equicontinuous follows from Theorem 4.3.2 and Lemma 4.2.1. Further, (Hg2) gives that T is continuous on B_{r_0}. In order to establish that, for any $t \in [-d, b]$, $\{(Tx)(t)|x \in B_{r_0}\}$ is relatively compact in X, it is sufficient to show that, for $t \in [-d, 0]$, $\{(T_1x)(t)|x \in B_{r_0}\}$ and $\{(T_2x)(t)|x \in B_{r_0}\}$ are relatively compact in X. The fact that $\{(T_2x)(t)|x \in B_{r_0}\}$ is relatively compact for $t \in [-d, b]$ in X follows easily from hypothesis (Hg2) and Theorem 4.3.2.

Let $V(t) = \{(T_1x)(t)|x \in B_{r_0}\}$, $t \in [-d, b]$. For $t \in [-d, 0]$, it is obvious that $V(t) = \{0\}$ which is relatively compact in X.

Now, for $0 < t \leq b$ fixed and $\forall\ \theta > 0$, an operator T_1^θ is defined by

$$\begin{aligned}(T_1^\theta x)(t) &= B^{-1}\int_\theta^\infty \xi_q(\omega)Q(t^q\omega)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)]d\omega \\ &= B^{-1}Q(\epsilon^q\theta)\int_\theta^\infty \xi_q(\omega)Q(t^q\omega - \epsilon^q\theta)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)]d\omega.\end{aligned}$$

From the compactness of $Q(\epsilon^q\theta)$, $(\epsilon^q\theta > 0)$, it is obtained that the set $V^\theta(t) = \{(T_1^\theta x)(t)|x \in B_{r_0}\}$ is relatively compact in $X\ \forall\ \theta > 0$. Now, for any $x \in B_{r_0}$, the following holds:

$$\|(T_1x)(t) - (T_1^\theta x)(t)\|_X \leq M_Q\|B\|\|B^{-1}\|[\|\chi\|_c + G_1r_0 + G_2]\int_0^\theta \xi_q(\omega)d\omega$$
$$\longrightarrow 0 \text{ as } \epsilon \to 0,\ \theta \to 0,$$

which gives $\{(T_1x)(t)|x \in B_{r_0}\}$, for $t \in [-d, b]$, to be relatively compact in X. By Arzelá-Ascoli theorem, is can be concluded that $\{Tx|x \in B_{r_0}\}$ is relatively compact. Therefore, $\gamma_1(T(B_{r_0})) = 0$ and subsequently, by Darbo-Sadovskii's fixed point theorem, it is established that T has a fixed point in $B_{r_0} \subset \mathbb{D}$ which is the mild solution of problem (4.1). □

For establishing the results of the next theorem, we consider γ_X, γ_c and $\gamma_{\mathbb{D}}$ to be the Hausdorff measure of noncompactness in X, $\mathcal{C}$ and $\mathbb{D}$, respectively.

Theorem 4.3.4. *Assume that*

(Hf3) $f\colon J\times\mathcal{C}\times X\to Y$ *satisfies the following:*

(i) *for a.e.* $t\in J$, *the function* $(\phi,x)\to f(t,\phi,x)$ *is continuous, and for each* $(\phi,x)\in\mathcal{C}\times X$, *the function* $t\to f(t,\phi,x)$ *is strongly measurable,*

(ii) *there exist a function* $F\in L^1(J,\mathbb{R}^+)$ *and a monotone decreasing function* $\overline{F}\colon\mathbb{R}^+\to\mathbb{R}^+$ *such that*

$$\|f(t,\phi,x)\|_Y\le F(t)\overline{F}(\|\phi\|_{\mathcal{C}}+\|x\|_X),$$

for a.e. $t\in J$, $\phi\in\mathcal{C}$ *and* $x\in X$, *and* $J^q_{0+}F\in C(J,\mathbb{R}^+)$.

(iii) *there exists a function* $N(t,s)\in C(\Delta,\mathbb{R}^+)$ *such that*

$$\gamma_X\Big(B^{-1}T_q(t-s)f(s,\tilde{C},\mathcal{D})\Big)\le N(t,s)\left[\sup_{\theta\in[-d,0]}\gamma_X(\tilde{C}(\theta))+\gamma_X(\mathcal{D})\right],$$

for every bounded subsets $\tilde{C}\subset\mathcal{C}$ *and* $\mathcal{D}\subset X$.

(Hh3) $h\colon\Delta\times\mathcal{C}\to X$ *satisfies the following:*

(i) *for each* $(t,s)\in\Delta$, *the function* $h(t,s,.)\colon\mathcal{C}\to X$ *is continuous, and for each* $x\in X$, *the function* $h(.,.,x)\colon\Delta\to X$ *is strongly measurable,*

(ii) *there exist a function* $L(t,s)\in C(\Delta,\mathbb{R}^+)$ *and a monotone nondecreasing continuous function* $\overline{L}\colon\mathbb{R}^+\to\mathbb{R}^+$ *such that*

$$\|h(t,s,\phi)\|_X\le L(t,s)\overline{L}(\|\phi\|_{\mathcal{C}}),\ \textit{for } (t,s)\in\Delta,\ \ \phi\in\mathcal{C}$$

and $L^*=\sup_{t\in J}\int_0^t L(t,s)ds<\infty$.

(iii) *there exists a function* $H(t,s)\in C(\Delta,\mathbb{R}^+)$ *such that for any bounded set* $\tilde{C}\subset\mathcal{C}$,

$$\gamma_X(h(t,s,\tilde{C}))\le H(t,s)\sup_{\theta\in[-d,0]}\gamma_X(\tilde{C}(\theta))$$

and $H^*=\sup_{t\in J}\int_0^t H(t,s)ds<\infty$.

(Hg3) $g\colon\mathcal{C}^n\to\mathcal{C}$ *is continuous and*

(i) *there exists a monotone nondecreasing continuous function* $G\colon\mathbb{R}^+\to\mathbb{R}^+$ *such that*

$$\|g(x_{t_1},\ldots,x_{t_n})\|_{\mathcal{C}}\le G(\|x\|_{\mathbb{D}}),\ \textit{for all } x\in\mathbb{D},$$

(ii) *there exists a constant* $\overline{G}>0$ *such that for any bounded subset* $\Omega\subset\mathbb{D}$,

$$\gamma_{\mathcal{C}}\Big(g(\Omega_{t_1},\ldots,\Omega_{t_n})\Big)\le\overline{G}\gamma_{\mathbb{D}}(\Omega),$$

(Hr) *there exists a constant* $r>0$ *such that*

$$M_Q\|B^{-1}\|\|B\|(\|\chi\|_{\mathcal{C}}+G(r))+\overline{F}(r+L^*\overline{L}(r))\|B^{-1}\|M_QF^*\le r,$$

where $F^* = \sup_{t\in J} J_{0+}^q F(t)$.

Then problem (4.1) has a mild solution in $\mathbb{D}$ *provided*

$$M_Q\|B^{-1}\|\|B\|\overline{G} + 4(1+2H^*)\sup_{t\in J}\int_0^t (t-s)^{q-1}N(t,s)ds < 1.$$

Proof. Defining an operator $T\colon \mathbb{D}\to\mathbb{D}$ as in the previous theorem, proceeding in a similar manner and by using the given assumptions, it can be shown that T is well-defined and continuous on B_r, for every $r>0$. Also, we have $T(B_r)\subseteq B_r$, for $r>0$, satisfying assumption (Hr).

Then, T is split into two parts T_1 and T_2 as follows:

$$(T_1x)(t) = \begin{cases} B^{-1}S_q(t)B[\chi(0) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0)], & t\in J=[0,b],\\ \chi(t) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(t), & t\in[-d,0],\end{cases}$$

and

$$(T_2x)(t) = \begin{cases} \int_{s=0}^t (t-s)^{q-1}B^{-1}T_q(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds, \quad t\in J=[0,b],\\ 0, \quad t\in[-d,0].\end{cases}$$

Let $\Omega\subseteq B_r$ be a bounded set. Then, by using the algebraically semi-additive property of $\gamma_{\mathbb{D}}$, we have

$$\gamma_{\mathbb{D}}(T(\Omega)) \le \gamma_{\mathbb{D}}(T_1(\Omega)) + \gamma_{\mathbb{D}}(T_2(\Omega)),$$

where

$$\gamma_{\mathbb{D}}(T_1(\Omega)) \le M_Q\|B^{-1}\|\|B\|\overline{G}\gamma_{\mathbb{D}}(\Omega).$$

Next, by using Lemma 1.3.1, for $\epsilon>0$, we can choose $\{x_n\}_{n=1}^\infty \subset \Omega$ such that

$$\gamma_{\mathbb{D}}(T_2(\Omega)) \le 2\gamma_{\mathbb{D}}(T_2(\{x_n\})) + \epsilon.$$

Because $T_2(B_r)$ is equicontinuous, by using Lemma 1.3.4, we obtain

$$\begin{aligned}\gamma_{\mathbb{D}}(T_2(\{x_n\})) &= \sup_{t\in[-d,b]} \gamma_X(T_2\{x_n\}(t))\\ &= \sup_{t\in[0,b]} \gamma_X\left(\left\{\int_{s=0}^t (t-s)^{q-1}B^{-1}T_q(t-s)f\Big(s,(x_n)_s,\int_0^s h(s,\tau,(x_n)_\tau)d\tau\Big)ds\right\}\right).\end{aligned}$$

Then, by using Lemma 1.3.2, (Hf3)(iii) and (Hh3)(iii), we get

$$\gamma_X\left(\left\{\int_{s=0}^t (t-s)^{q-1}B^{-1}T_q(t-s)f\Big(s,(x_n)_s,\int_0^s h(s,\tau,(x_n)_\tau)d\tau\Big)ds\right\}\right)$$

$$\leq 2\int_{s=0}^{t}(t-s)^{q-1}N(t,s)\left[\gamma_{\mathcal{C}}\{(x_n)_s\}+\gamma_{X}\left(\left\{\int_0^s h(s,\tau,(x_n)_\tau)d\tau\right\}\right)\right]ds$$
$$\leq 2(1+2H^*)\gamma_{\mathbb{D}}(\Omega)\sup_{t\in[0,b]}\int_0^t (t-s)^{q-1}N(t,s)ds.$$

Therefore,

$$\gamma_{\mathbb{D}}(T_2(\Omega))\leq\left[4(1+2H^*)\sup_{t\in J}\int_0^t (t-s)^{q-1}N(t,s)ds\right]\gamma_{\mathbb{D}}(\Omega),$$

since $\epsilon>0$ is arbitrary.

Thus

$$\gamma_{\mathbb{D}}(T(\Omega))\leq\left[M_Q\|B^{-1}\|\|B\|\overline{G}+4(1+2H^*)\sup_{t\in J}\int_0^t (t-s)^{q-1}K(t,s)ds\right]\gamma_{\mathbb{D}}(\Omega).$$

By using Lemma 1.3.5, it can be concluded that T has a fixed point in $\mathbb{D}$, which is the required mild solution of our problem. □

CHAPTER 5

Impulsive differential equations with Caputo fractional derivative and Erdélyi-Kober type boundary conditions

Differential equations of impulsive nature arise in real world problems while describing the dynamics of processes that exhibit sudden or discontinuous jumps. Now, for an impulsive fractional differential equation, there are mainly two types as follows:
(i) appearance of the fractional derivative with a unique starting point: $D^q = D_0^q$,
(ii) appearance of the fractional derivative involving multiple starting points, i.e., $D^q = D_*^q$.

Wang [117] considered the following problem:

$$\begin{aligned}
&{}^C D_*^q x(t) = F(t, x(t)), \;\; t \in [0,1] \setminus \{t_1, t_2, \dots, t_l\}, \;\; q \in (1,2], \\
&\Delta x(t_\kappa) = \mathcal{I}_\kappa(x(t_\kappa^-)), \;\; \Delta x'(t_\kappa) = \mathcal{J}_\kappa(x(t_\kappa^-)), \;\; \kappa = 1, 2, \dots, l, \\
&ax(0) - bx'(0) = x_0, \;\; cx(1) + dx'(1) = x_1.
\end{aligned}$$

Ahmad and Sivasundaram [10] examined the existence of the solution of the impulsive problem governed by a fractional differential equation in Caputo derivative of the form

$$\begin{aligned}
&{}^C D_*^q x(t) = F(t, x(t)), \;\; t \in [0,1] \setminus \{t_1, t_2, \dots, t_l\}, \;\; q \in (1,2], \\
&\Delta x(t_\kappa) = \mathcal{I}_\kappa(x(t_\kappa^-)), \;\; \Delta x'(t_\kappa) = \mathcal{J}_\kappa(x(t_\kappa^-)), \;\; \kappa = 1, 2, \dots, l, \\
&ax(0) + bx'(0) = \int_0^1 q_1(x(s))ds, \;\; ax(1) + bx'(1) = \int_0^1 q_2(x(s))ds.
\end{aligned}$$

It may be noted that integral boundary conditions have found their place in several applications - mainly in computational fluid dynamics (CFD) and some other fields of applied mathematics such as population dynamics, underground water flow, blood flow, chemical engineering etc. Hemodynamic conditions can be completely characterized by

CFD techniques under appropriate boundary conditions. But majority of the CFD-based hemodynamic investigations constitute of in vitro conditions which cannot fully represent the actual patient hemodynamic conditions [6, 87]. The difficulties caused are, in fact, related to the prescribed boundary conditions because it is not always justifiable to assume the geometry of the blood vessel to be circular. Therefore, it is advisable to utilize integral boundary conditions to model blood flow problems more accurately.

An important fractional integral operator, known as Erdélyi-Kober fractional integral operator, was defined and introduced by Erdélyi and Kober in 1940 [43]. Its usual application is found in the theory of radiative transfer, kinetic theory of gases etc. For some recent developments in this direction, we refer the readers to the works carried out in [7,8,111]. To the best of the knowledge of the author, existence of solutions of boundary value problems governed by impulsive fractional differential equations with an Erdélyi-Kober integral operator has not been studied till date. To fill this gap, the following integral boundary value problem is taken up:

$$\left.\begin{aligned} {}^{C}D_{*}^{q}x(t) &= F(t,x(t)),\ \ t\in J' = J\setminus\{t_1,t_2,\dots,t_l\},\ \ q\in(1,2),\\ \Delta x(t_\kappa) &= \mathcal{I}_\kappa(x(t_\kappa^-)),\ \ \kappa=1,2,\dots,l,\\ \Delta x'(t_\kappa) &= \mathcal{J}_\kappa(x(t_\kappa^-)),\ \ \kappa=1,2,\dots,l, \end{aligned}\right\} \tag{5.1}$$

with integral boundary conditions given by

$$x(0)-\mu_1 x'(0) = \int_0^b h(x(s))ds,\ \ x(b) = \mu_2 I_{\gamma}^{\alpha,\beta}x(\xi),\ \ \xi\in(0,t_1), \tag{5.2}$$

where $J=[0,b]$, $b>0$. Here, ${}^{C}D_{*}^{q}$ represents the Caputo fractional derivative at the base points $t=t_\kappa, \kappa=1,2,\dots,l$, i.e., ${}^{C}D_{*}^{q}|_{(t_\kappa,t_{\kappa+1}]}x(t) = {}^{C}D_{t_\kappa^+}^{q}x(t)$ for all $t\in(t_\kappa,t_{\kappa+1}]$. The function $F\colon J\times\mathbb{R}\to\mathbb{R}$ is continuous, $\mathcal{I}_\kappa,\mathcal{J}_\kappa\in C(\mathbb{R},\mathbb{R})$, $\kappa=1,2,\dots,l$, the function $h\colon\mathbb{R}\to\mathbb{R}$ is given to be continuous, $\mu_1\ge 0$, $\mu_2>0$ are given constants. Further, $I_{\gamma}^{\alpha,\beta}$ is the Erdélyi-Kober fractional integral of order $\beta>0$ where $\gamma>0$, $\alpha\in\mathbb{R}$.

5.1 Preliminaries

Let $0=t_0<t_1<\dots<t_l<t_{l+1}=b$. Then $PC(J,\mathbb{R})=\{x:J\to\mathbb{R}|x\in C((t_\kappa,t_{\kappa+1}],\mathbb{R}),\kappa=0,1,\dots,l$ and $x(t_\kappa^-),x(t_\kappa^+)$ exist, $\kappa=1,\dots,l,$ with $x(t_\kappa^-)=x(t_\kappa)\}$ is a Banach space with respect to the norm $\|x\|_{PC}=\sup_{t\in J}|x(t)|$. Denoting $PC^1(J,\mathbb{R})=\{x\in PC(J,\mathbb{R})|\dot{x}\in PC(J,\mathbb{R})\}$ and setting $\|x\|_{PC^1}=\|x\|_{PC}+\|\dot{x}\|_{PC}$, we have $(PC^1(J,\mathbb{R}),\|.\|_{PC^1})$ as a Banach space.

Definition 5.1.1 ([8]). *The Erdélyi-Kober fractional integral of order of* $\beta>0$ *with*

$\gamma > 0$, $\alpha \in \mathbb{R}$ *of a continuous function* $F : (0,\infty) \to \mathbb{R}$ *is defined as follows:*

$$I_{\gamma}^{\alpha,\beta}F(t) = \frac{\gamma t^{-\gamma(\alpha+\beta)}}{\Gamma(\beta)} \int_0^t \frac{s^{\gamma\alpha+\gamma-1}F(s)}{(t^{\gamma}-s^{\gamma})^{1-\beta}}ds,$$

provided that the right-hand side is pointwise defined on $\mathbb{R}_+$.

Lemma 5.1.1 ([7]). *Let* $\beta > 0$, $\gamma > 0$ *and* α, $q \in \mathbb{R}$. *Then*

$$I_{\gamma}^{\alpha,\beta}t^q = \frac{t^q\Gamma(\alpha + \frac{q}{\gamma} + 1)}{\Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)}.$$

Definition 5.1.2. *A function* $x \in PC^1(J,\mathbb{R})$ *with its Caputo derivative of order* q *existing on* J *is said to be a solution of* (5.1) *if it satisfies problem* (5.1).

Lemma 5.1.2 ([117]). *Letting* $q > 0$, *the differential equation* ${}^CD^qF(t) = 0$ *has the solution*

$$F(t) = a_0 + a_1t + \ldots + a_{m-1}t^{m-1}$$

where $a_i \in \mathbb{R}$, *for* $i = 0,1,\ldots,m-1$, m *is the least integer* $\geq q$.

Lemma 5.1.3 ([117]). *Let* $q > 0$. *Then*

$$J^q\,{}^CD^qF(t) = F(t) + a_0 + a_1t + \ldots + a_{m-1}t^{m-1}$$

for some $a_i \in \mathbb{R}$, $i = 0,1,\ldots,m-1$, m *is the least integer* $\geq q$.

Throughout this work, for each $r > 0$, B_r represents the open ball of radius r in $PC(J,\mathbb{R})$, that is, $B_r = \{x \in PC(J,\mathbb{R}) \colon \|x\|_{PC} < r\}$.

5.2 Main Results

For studying the existence of solution of problem (5.1) with boundary conditions given by (5.2), we require the following lemma:

Lemma 5.2.1. *The boundary value problem*

$$\left.\begin{aligned} {}^CD_*^qx(t) &= F(t,x(t)), \ \ t \in J' = J \setminus \{t_1,t_2,\ldots,t_l\}, \ \ q \in (1,2), \\ \Delta x(t_\kappa) &= \mathcal{I}_\kappa(x(t_\kappa^-)), \ \ \kappa = 1,2,\ldots,l, \\ \Delta x'(t_\kappa) &= \mathcal{J}_\kappa(x(t_\kappa^-)), \ \ \kappa = 1,2,\ldots,l, \\ x(0) - \mu_1x'(0) &= \int_0^b h(x(s))ds, \ \ x(b) = \mu_2I_{\gamma}^{\alpha,\beta}x(\xi), \ \ \xi \in (0,t_1), \end{aligned}\right\} \tag{5.3}$$

is equivalent to the following integral equation

$$x(t)=\begin{cases}\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}F(s,x(s))ds+M_1+M_2t, \quad t\in[0,t_1],\\ \frac{1}{\Gamma(q)}\int_{t_\kappa}^t(t-s)^{q-1}F(s,x(s))ds+\displaystyle\sum_{j=1}^{\kappa}\frac{1}{\Gamma(q)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-1}F(s,x(s))ds\\ +\displaystyle\sum_{j=1}^{\kappa}\frac{t-t_\kappa}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds+\sum_{j=1}^{\kappa}\mathcal{I}_j(x(t_j^-))\\ +\displaystyle\sum_{j=1}^{\kappa-1}\frac{t_\kappa-t_j}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds+\sum_{j=1}^{\kappa}(t-t_\kappa)\mathcal{J}_j(x(t_j^-))\\ +\displaystyle\sum_{j=1}^{\kappa-1}(t_\kappa-t_j)\mathcal{J}_j(x(t_j^-))+M_1+M_2t, \quad t\in(t_\kappa,t_{\kappa+1}], \ \kappa=1,2,\ldots,l,\end{cases} \tag{5.4}$$

where

$$\begin{aligned}M_1=&-\frac{1}{\omega}\Big[-\mu_1\mu_2I_\gamma^{\alpha,\beta}J_0^qF(t,x(t))(\xi)-w_2\int_0^b h(x(s))ds+\mu_1\sum_{j=1}^{l}\mathcal{I}_j(x(t_j^-))\\ &+\mu_1\sum_{j=1}^{l}(b-t_l)\mathcal{J}_j(x(t_j^-))+\mu_1\sum_{j=1}^{l-1}(t_l-t_j)\mathcal{J}_j(x(t_j^-)+\frac{\mu_1}{\Gamma(q)}\sum_{j=1}^{l+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-1}F(s,x(s))ds\\ &+\mu_1\sum_{j=1}^{l}\frac{b-t_l}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds+\mu_1\sum_{j=1}^{l-1}\frac{t_l-t_j}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds\Big],\\ M_2=&-\frac{1}{\omega}\Big[-\mu_2I_\gamma^{\alpha,\beta}J_0^qF(t,x(t))(\xi)+w_1\int_0^b h(x(s))ds+\sum_{j=1}^{l}\mathcal{I}_j(x(t_j^-))+\sum_{j=1}^{l}(b-t_l)\mathcal{J}_j(x(t_j^-))\\ &+\sum_{j=1}^{l-1}(t_l-t_j)\mathcal{J}_j(x(t_j^-)+\sum_{j=1}^{l+1}\frac{1}{\Gamma(q)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-1}F(s,x(s))ds+\frac{b-t_l}{\Gamma(q-1)}\\ &\times\sum_{j=1}^{l}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds+\sum_{j=1}^{l-1}\frac{t_l-t_j}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds\Big],\end{aligned}$$

with

$$\omega=w_1\mu_1+w_2\neq0, \quad w_1=1-\frac{\mu_2\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}, \quad w_2=b-\frac{\mu_2\xi\Gamma(\alpha+\frac{1}{\gamma}+1)}{\Gamma(\alpha+\frac{1}{\gamma}+\beta+1)}.$$

Proof. Let x be a solution of (5.3). Then for $t\in[0,t_1]$, we have

$$x(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}F(s,x(s))ds-c_1-c_2t, \tag{5.5}$$

where c_1, $c_2\in\mathbb{R}$ are constants, and

$$x'(t)=\frac{1}{\Gamma(q-1)}\int_0^t(t-s)^{q-2}F(s,x(s))ds-c_2. \tag{5.6}$$

Then, by using the impulse conditions $\Delta x(t_\kappa) = \mathcal{I}_\kappa(x(t_\kappa^-))$ and $\Delta x'(t_\kappa) = \mathcal{J}_\kappa(x(t_\kappa^-))$, for $\kappa = 1, 2, \ldots, l$, we have

$$x(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_{t_\kappa}^{t} (t-s)^{q-1} F(s, x(s))ds + \sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{t_{j-1}}^{t_j} (t-s)^{q-1} F(s, x(s))ds \\ + \sum_{j=1}^{\kappa} \frac{t-t_\kappa}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t-s)^{q-2} F(s, x(s))ds + \sum_{j=1}^{\kappa-1} \frac{t_\kappa - t_j}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t-s)^{q-2} F(s, x(s))ds \\ + \sum_{j=1}^{\kappa} \mathcal{I}_j(x(t_j^-)) + \sum_{j=1}^{\kappa} (t-t_\kappa) \mathcal{J}_j(x(t_j^-)) + \sum_{j=1}^{\kappa-1} (t_\kappa - t_j) \mathcal{J}_j(x(t_j^-)) \\ -c_1 - c_2 t, \quad t \in (t_\kappa, t_{\kappa+1}]. \end{cases} \tag{5.7}$$

Using the condition $x(0) - \mu_1 x'(0) = \int_0^b h(x(s))ds$, (5.5) and (5.6) imply $-c_1 = \int_0^b h(x(s))ds - \mu_1 c_2$. Therefore, for $t \in [0, t_1]$,

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s, x(s))ds + \int_0^b h(x(s))ds - c_2(\mu_1 + t),$$

and for $t \in (t_\kappa, t_{\kappa+1}]$, $\kappa = 1, 2, \ldots, l$, (5.7) gives

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(q)} \int_{t_\kappa}^{t} (t-s)^{q-1} F(s, x(s))ds + \sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{t_{j-1}}^{t_j} (t-s)^{q-1} F(s, x(s))ds + \int_0^b h(x(s))ds \\ & + \sum_{j=1}^{\kappa} \frac{t-t_\kappa}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t-s)^{q-2} F(s, x(s))ds + \sum_{j=1}^{\kappa-1} \frac{t_\kappa - t_j}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t-s)^{q-2} F(s, x(s))ds \\ & + \sum_{j=1}^{\kappa} \mathcal{I}_j(x(t_j^-)) + \sum_{j=1}^{\kappa} (t-t_\kappa) \mathcal{J}_j(x(t_j^-)) + \sum_{j=1}^{\kappa-1} (t_\kappa - t_j) \mathcal{J}_j(x(t_j^-)) - c_2(\mu_1 + t). \end{aligned}$$

Now, we use the boundary condition $x(b) = \mu_2 I_\gamma^{\alpha,\beta} x(\xi)$, where $\xi \in (0, t_1)$. For this, we have

$$\begin{aligned} x(b) = {} & \sum_{j=1}^{l+1} \frac{1}{\Gamma(q)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-1} F(s, x(s))ds + \sum_{j=1}^{l} \frac{b - t_l}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} F(s, x(s))ds \\ & + \sum_{j=1}^{l-1} \frac{t_l - t_j}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} F(s, x(s))ds + \sum_{j=1}^{l} \mathcal{I}_j(x(t_j^-)) + \sum_{j=1}^{l} (b - t_l) \mathcal{J}_j(x(t_j^-)) \\ & + \sum_{j=1}^{l-1} (t_l - t_j) \mathcal{J}_j(x(t_j^-)) + \int_0^b h(x(s))ds - c_2 \mu_1 + c_2 b, \end{aligned}$$

and for $t \in (0, t_1)$,

$$\mu_2 I_\gamma^{\alpha,\beta} x(\xi) = \mu_2 I_\gamma^{\alpha,\beta} J_0^q g(t, x(t))(\xi) + \left[\int_0^b h(x(s))ds - c_2\mu_1\right] \frac{\mu_2\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} - c_2 \frac{\mu_2 \xi \Gamma(\alpha + \frac{1}{\gamma} + 1)}{\Gamma(\alpha + \frac{1}{\gamma} + \beta + 1)},$$

which gives $c_1 = -M_1$ and $c_2 = -M_2$. Now, upon substitution of the values of c_1 and c_2 in (5.5) and (5.7) gives (5.4).

For the converse part, it can be easily shown that the integral equation given by (5.4) satisfies the first three equations of (5.3). Next, we have $x(0) = M_1$ and $x'(0) = M_2$ and therefore $x(0) - \mu_1 x'(0) = M_1 - \mu_1 M_2 = \int_0^b h(x(s))ds$.

Now, it remains to verify that $x(b) = \mu_2 I_\gamma^{\alpha,\beta} x(\xi)$, $\xi \in (0, t_1)$. From (5.4), we get

$$\begin{aligned} x(b) = {} & \frac{1}{\Gamma(q)} \int_{t_l}^b (b-s)^{q-1} F(s, x(s))ds + \frac{1}{\Gamma(q)} \sum_{j=1}^{l} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-1} F(s, x(s))ds \\ & + \sum_{j=1}^{l} \frac{b - t_l}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} F(s, x(s))ds + \sum_{j=1}^{l-1} \frac{t_l - t_j}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} F(s, x(s))ds \\ & + \sum_{j=1}^{l} \mathcal{I}_j(x(t_j^-)) + \sum_{j=1}^{l} (b - t_l)\mathcal{J}_j(x(t_j^-)) + \sum_{j=1}^{l-1} (t_l - t_j)\mathcal{J}_j(x(t_j^-)) + M_1 + M_2 b. \end{aligned}$$

For $\xi \in (0, t_1)$, we have

$$\mu_2 I_\gamma^{\alpha,\beta} x(\xi) = \mu_2 I_\gamma^{\alpha,\beta} J_0^q F(t, x(t))(\xi) + (1 - w_1)M_1 + (b - w_2)M_2.$$

Substituting the expressions for M_1, M_2 and combining the terms, we get the desired equality. □

Theorem 5.2.1. *Assume that* $\lim_{x\to 0} \frac{F(t,x)}{x} = 0$, $\lim_{x\to 0} \frac{\mathcal{I}_\kappa(x)}{x} = 0$, $\lim_{x\to 0} \frac{\mathcal{J}_\kappa(x)}{x} = 0$ *and* $\lim_{x\to 0} \frac{h(x)}{x} = 0$, *then problem* (5.1) *with boundary condition* (5.2) *has at least one solution.*

Proof. Let $r > 0$ and define an operator Ψ on B_r by

$$(\Psi x)(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s, x(s))ds + M_1 + M_2 t, \;\; t \in [0, t_1], \\ \frac{1}{\Gamma(q)} \int_{t_\kappa}^t (t-s)^{q-1} F(s, x(s))ds + \sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-1} F(s, x(s))ds \\ + \sum_{j=1}^{\kappa} \frac{t - t_\kappa}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} F(s, x(s))ds + \sum_{j=1}^{\kappa} \mathcal{I}_j(x(t_j^-)) \\ + \sum_{j=1}^{\kappa-1} \frac{t_\kappa - t_j}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} F(s, x(s))ds + \sum_{j=1}^{\kappa} (t - t_\kappa) \mathcal{J}_j(x(t_j^-)) \\ + \sum_{j=1}^{\kappa-1} (t_\kappa - t_j) \mathcal{J}_j(x(t_j^-)) + M_1 + M_2 t, \;\; t \in (t_\kappa, t_{\kappa+1}], \; \kappa = 1, 2, \ldots, l. \end{cases}$$

Then, we have

$$\begin{aligned} |M_1| \leq \frac{1}{|\omega|} \Big[& \mu_1 \mu_2 |I_\gamma^{\alpha,\beta} J_0^q F(t, x(t))(\xi)| + |w_2| \int_0^b |h(x(s))| ds + \mu_1 \sum_{j=1}^{l} |\mathcal{I}_j(x(t_j^-))| \\ & + \mu_1 \sum_{j=1}^{l} (b - t_l) |\mathcal{J}_j(x(t_j^-))| + \mu_1 \sum_{j=1}^{l-1} (t_l - t_j) |\mathcal{J}_j(x(t_j^-)| + \sum_{j=1}^{l+1} \frac{\mu_1}{\Gamma(q)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-1} \\ & \times |F(s, x(s))| ds + \mu_1 \sum_{j=1}^{l} \frac{b - t_l}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} |F(s, x(s))| ds \\ & + \mu_1 \sum_{j=1}^{l-1} \frac{t_l - t_j}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} |F(s, x(s))| ds \Big] \\ & \leq \frac{\mu_2 \Gamma(\alpha + \frac{q}{\gamma} + 1) \xi^q + (l+1) b^q \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} \mu_1 L_1 + \frac{(2l-1) b^q}{|\omega| \Gamma(q)} \mu_1 L_1 + \frac{l}{|\omega|} \mu_1 L_2 \\ & + \frac{(2l-1) b}{|\omega|} \mu_1 L_3 + \frac{|w_2| b}{|\omega|} L_4, \end{aligned}$$

and

$$\begin{aligned}|M_2| \le{}& \frac{1}{|\omega|}\Big[\mu_2|I_\gamma^{\alpha,\beta}J_0^qF(t,x(t))(\xi)|+|w_1|\int_0^b|h(x(s))|ds+\sum_{j=1}^{l}|\mathcal{I}_j(x(t_j^-))|+\sum_{j=1}^{l}(b-t_l)\\ &\times|\mathcal{J}_j(x(t_j^-))|+\sum_{j=1}^{l-1}(t_l-t_j)|\mathcal{J}_j(x(t_j^-)|+\sum_{j=1}^{l+1}\frac{1}{\Gamma(q)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-1}|F(s,x(s))|ds\\ &+\sum_{j=1}^{l}\frac{b-t_l}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}|F(s,x(s))|ds+\sum_{j=1}^{l-1}\frac{t_l-t_j}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}\\ &\times|F(s,x(s))|ds\Big]\\ \le{}& \frac{\mu_2\Gamma(\alpha+\frac{q}{\gamma}+1)\xi^q+(l+1)b^q\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}L_1+\frac{(2l-1)b^q}{|\omega|\Gamma(q)}L_1+\frac{l}{|\omega|}L_2\\ &+\frac{(2l-1)b}{|\omega|}L_3+\frac{|w_1|b}{|\omega|}L_4,\end{aligned}$$

where the positive constants L_i, $(i=1,2,3,4)$ satisfy $|F(t,x(t))|\le L_1$, $|\mathcal{I}_\kappa(x(t))|\le L_2$, $|\mathcal{J}_\kappa(x(t))|\le L_3$, $|h(x(t))|\le L_4$, for all $x\in\bar{B}_r$ and $t\in J$. Therefore, Ψ is well-defined on $\bar{B}_r$ and it can also be shown that $\Psi x\in PC(J,\mathbb{R})$ for $x\in\bar{B}_r$.

Now, to show that $\Psi\colon\bar{B}_r\to PC(J,\mathbb{R})$ is completely continuous, we split the proof into the following steps:

Step I: To show that $\{\Psi x|x\in\bar{B}_r\}$ is equicontinuous in $(t_\kappa,t_{\kappa+1})$, $\kappa=0,1,\dots,l$.

Let $x\in\bar{B}_r$ and $0\le s_1<s_2\le t_1$. Subsequently,

$$\begin{aligned}|(\Psi x)(s_2)-(\Psi x)(s_1)|\le{}&\frac{1}{\Gamma(q)}\int_0^{s_1}[(s_2-s)^{q-1}-(s_1-s)^{q-1}]|F(s,x(s))|ds\\ &+\frac{1}{\Gamma(q)}\int_{s_1}^{s_2}(s_2-s)^{q-1}|F(s,x(s))|ds+|M_2|(s_2-s_1)\\ \le{}&\frac{L_1}{\Gamma(q+1)}[s_2^q-s_1^q]\longrightarrow 0\quad\text{as } s_2\to s_1.\end{aligned}$$

For $t_\kappa<s_1<s_2\le t_{\kappa+1}$, $\kappa=1,\dots,l$,

$$\begin{aligned}&|(\Psi x)(s_2)-(\Psi x)(s_1)|\\ &\le\frac{1}{\Gamma(q)}\int_{t_\kappa}^{s_1}[(s_2-s)^{q-1}-(s_1-s)^{q-1}]|F(s,x(s))|ds+\frac{1}{\Gamma(q)}\int_{s_1}^{s_2}(s_2-s)^{q-1}|F(s,x(s))|ds\\ &+\Big|\sum_{j=1}^{\kappa}\frac{s_2-s_1}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds\Big|+\Big|\sum_{j=1}^{\kappa}(s_2-s_1)\mathcal{J}_j(x(t_j^-))\Big|+|M_2|(s_2-s_1)\\ &\le\frac{L_1}{\Gamma(q+1)}[q(s_2-s_1)-(s_2-s_1)^q]+\frac{L_1}{\Gamma(q+1)}(s_2-s_1)^q+(s_2-s_1)L_1\sum_{j=1}^{l}b^{q-1}\\ &+(s_2-s_1)\sum_{j=1}^{l}L_3+|M_2|(s_2-s_1)\longrightarrow 0\quad\text{as } s_2\to s_1.\end{aligned}$$

Thus $\Psi\bar{B}_r$ is equicontinuous in $(t_\kappa, t_{\kappa+1})$, $\kappa = 0, 1, \ldots, l$.

Step II: To show that $\{\Psi x | x \in \bar{B}_r\}$ is a uniformly bounded subset of $PC(J, \mathbb{R})$.

For $x \in \bar{B}_r$ and $t \in (t_\kappa, t_{\kappa+1})$, $\kappa = 0, 1, \ldots, l$, we have

$$
\begin{aligned}
|\Psi x(t)| \leq {} & \frac{1}{\Gamma(q)} \int_{t_\kappa}^{t} (t-s)^{q-1} |F(s, x(s))| ds + \sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-1} |F(s, x(s))| ds \\
& + \sum_{j=1}^{\kappa} \frac{t - t_\kappa}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} |F(s, x(s))| ds + \sum_{j=1}^{\kappa} |\mathcal{I}_j(x(t_j^-))| + \sum_{j=1}^{\kappa-1} (t_\kappa - t_j) |\mathcal{J}_j(x(t_j^-))| \\
& + \sum_{j=1}^{\kappa-1} \frac{t_\kappa - t_j}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} |F(s, x(s))| ds + \sum_{j=1}^{\kappa} (t - t_\kappa) |\mathcal{J}_j(x(t_j^-))| + |M_1| + |M_2| b \\
\leq {} & \frac{L_1 b^q}{\Gamma(q+1)} + \sum_{j=1}^{l} \frac{L_1 b^q}{\Gamma(q+1)} + \sum_{j=1}^{l} \frac{L_1 b^q}{\Gamma(q)} + \sum_{j=1}^{l-1} \frac{L_1 b^q}{\Gamma(q)} + \sum_{j=1}^{l} L_2 + \sum_{j=1}^{l} b L_3 + \sum_{j=1}^{l-1} b L_3 \\
& + |M_1| + |M_2| b \\
\leq {} & \frac{b^q (l+1) |\omega| \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1) + (\mu_2 \Gamma(\alpha + \frac{q}{\gamma} + 1) \xi^q}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} L_1 \\
& + \frac{(l+1) b^q \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1))(\mu_1 + b)}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} L_1 + \frac{b^q (2l-1)(|\omega| + \mu_1 + b)}{|\omega| \Gamma(q)} L_1 \\
& + \frac{l(|\omega| + \mu_1 + b)}{|\omega|} L_2 + \frac{b(2l-1)(|\omega| + \mu_1 + b)}{|\omega|} L_3 + \frac{|w_1| b^2}{|\omega|} L_4 + \frac{b|w_2|}{|\omega|} L_4 := L^*.
\end{aligned}
$$

It follows that $\|\Psi x\|_{PC} \leq L^*$ for all $x \in \bar{B}_r$.

Step III: To show that Ψ is continuous on $\bar{B}_r$.

Using the continuity of the functions F, $\mathcal{I}_\kappa$, $\mathcal{J}_\kappa$, h, and Lemma 5.2.1, it can be shown that Ψ is continuous on $\bar{B}_r$.

Therefore, Arzelá-Ascoli theorem ensures that $\Psi \colon \bar{B}_r \to PC(J, \mathbb{R})$ is completely continuous.

Now, since $\lim_{x \to 0} \frac{F(t,x)}{x} = 0$, $\lim_{x \to 0} \frac{\mathcal{I}_\kappa(x)}{x} = 0$, $\lim_{x \to 0} \frac{\mathcal{J}_\kappa(x)}{x} = 0$ and $\lim_{x \to 0} \frac{h(x)}{x} = 0$, for $\epsilon_i > 0$, $i = 1, 2, 3, 4$, there exists a $r_0 > 0$ such that $|F(t, x)| < \epsilon_1 |x|$, $|\mathcal{I}_\kappa(x)| < \epsilon_2 |x|$, $|\mathcal{J}_\kappa(x)| < \epsilon_3 |x|$ and $|h(x)| < \epsilon_4 |x|$ for $0 < |x| < r_0$ where ϵ_i, $i = 1, 2, 3, 4$ are chosen such that

$$
\begin{aligned}
& \frac{b^q (l+1) |\omega| \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1) + (\mu_2 \Gamma(\alpha + \frac{q}{\gamma} + 1) \xi^q}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} \epsilon_1 + \frac{b^q (2l-1)(|\omega| + \mu_1 + b)}{|\omega| \Gamma(q)} \epsilon_1 \\
& + \frac{(l+1) b^q \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1))(\mu_1 + b)}{|\omega| \Gamma(q+1) \Gamma(\alpha + \frac{q}{\gamma} + \beta + 1)} \epsilon_1 + \frac{l(|\omega| + \mu_1 + b)}{|\omega|} \epsilon_2 + \frac{b(2l-1)(|\omega| + \mu_1 + b)}{|\omega|} \epsilon_3 \\
& + \frac{b(|w_1| b + |w_2|)}{|\omega|} \epsilon_4 \leq 1.
\end{aligned}
$$

Define $S = B_{r_0}$. Then we have

$$\|\Psi x\|_{PC} \leq \Bigg[\frac{b^q(l+1)|\omega|\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)+(\mu_2\Gamma(\alpha+\frac{q}{\gamma}+1)\xi^q}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}\epsilon_1+\frac{b^{q+1}(2l-1)}{|\omega|\Gamma(q)}\epsilon_1 + \frac{(l+1)b^q\Gamma(\alpha+\frac{q}{\gamma}+\beta+1))(\mu_1+b)}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}\epsilon_1+\frac{b^q(2l-1)(|\omega|+\mu_1)}{|\omega|\Gamma(q)}\epsilon_1 + \frac{l(|\omega|+\mu_1+b)}{|\omega|}\epsilon_2+\frac{b(2l-1)(|\omega|+\mu_1+b)}{|\omega|}\epsilon_3+\frac{b(|w_1|b+|w_2|)}{|\omega|}\epsilon_4\Bigg]\|x\|_{PC},$$

which gives $\|\Psi x\|_{PC} \leq \|x\|_{PC}$, for $x \in \partial B_{r_0}$. Therefore, by applying Theorem 1.3.7, it is established that the operator Ψ has at least one fixed point on $\bar{B}_{r_0}$. □

Theorem 5.2.2. *Assume that there exist $L_i > 0$, $i = 1, 2, 3, 4$ satisfying $|F(t, x)| \leq L_1$, $|\mathcal{I}_\kappa(x)| \leq L_2$, $|\mathcal{J}_\kappa(x)| \leq L_3$ and $|h(x)| \leq L_4$ for $t \in J$, $x \in \mathbb{R}$ and $\kappa = 1, 2, \dots, l$. Then the problem defined by* (5.1) *and* (5.2) *has at least one solution.*

Proof. Define an operator Ψ on $PC(J, \mathbb{R})$ by

$$(\Psi x)(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s,x(s))ds + M_1 + M_2 t, \quad t \in [0, t_1], \\ \frac{1}{\Gamma(q)}\int_{t_\kappa}^t (t-s)^{q-1}F(s,x(s))ds + \sum\limits_{j=1}^{\kappa}\frac{1}{\Gamma(q)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-1}F(s,x(s))ds \\ +\sum\limits_{j=1}^{\kappa}\frac{t-t_\kappa}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds + \sum\limits_{j=1}^{\kappa}\mathcal{I}_j(x(t_j^-)) \\ +\sum\limits_{j=1}^{\kappa-1}\frac{t_\kappa-t_j}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds + \sum\limits_{j=1}^{\kappa}(t-t_\kappa)\mathcal{J}_j(x(t_j^-)) \\ +\sum\limits_{j=1}^{\kappa-1}(t_\kappa-t_j)\mathcal{J}_j(x(t_j^-)) + M_1 + M_2 t, \quad t \in (t_\kappa, t_{\kappa+1}], \ \kappa = 1, 2, \dots, l. \end{cases}$$

Then under the given assumptions, it can be easily shown that Ψ is well-defined on $PC(J, \mathbb{R})$ and $\Psi x \in PC(J, \mathbb{R})$ for $x \in PC(J, \mathbb{R})$. Proceeding in a similar way as in Theorem 5.2.1, it can be established that $\Psi\colon PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.

Now, it remains to be shown that the set $E(\Psi) = \{x \in PC(J, \mathbb{R}) : x = \Lambda\Psi x \text{ for some } \Lambda \in [0, 1]\}$ is bounded. For this, take $x \in E(\Psi)$. Consequently,

$$\begin{aligned} |x(t)| &= \Lambda|\Psi x(t)| \\ &\leq \frac{b^q(l+1)|\omega|\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)+(\mu_2\Gamma(\alpha+\frac{q}{\gamma}+1)\xi^q}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}L_1+\frac{b^q(2l-1)(|\omega|+\mu_1+b)}{|\omega|\Gamma(q)}L_1 \\ &+ \frac{(l+1)b^q\Gamma(\alpha+\frac{q}{\gamma}+\beta+1))(\mu_1+b)}{|\omega|\Gamma(q+1)\Gamma(\alpha+\frac{q}{\gamma}+\beta+1)}L_1+\frac{l(|\omega|+\mu_1+b)}{|\omega|}L_2+\frac{b(2l-1)(|\omega|+\mu_1+b)}{|\omega|} \\ &\times L_3+\frac{b(|w_1|b+|w_2|)}{|\omega|}L_4 := L^*. \end{aligned}$$

It follows that $\|x\|_{PC} \leq L^*$ for all $x \in E(\Psi)$, and hence $E(\Psi)$ is bounded. Therefore, Theorem 1.3.8 implies that Ψ has a fixed point. $\square$

Theorem 5.2.3. *Assume that*

(H1) *there exists a constant* $q_1 \in (0,1)$ *with* $1+q_1 < q$ *such that a function* $f \in L^{\frac{1}{q_1}}([0,b],\mathbb{R}^+)$ *and a nondecreasing* L^1 *function* $g\colon [0,\infty) \to (0,\infty)$ *exist such that*

$$|F(t,x)| \leq f(t)g(|x|), \quad \textit{for all } t \in I,\ x \in \mathbb{R},$$

(H2) *there exists a positive constant* L *such that* $\frac{L}{\frac{A}{|\omega|}+g(L)\|f\|_{L^{\frac{1}{q_1}}}B} > 1$, *where*

$$\begin{aligned} a &= \frac{q-q_1}{1-q_1},\ c = \frac{q-2}{1-q_1},\ A = (\mu_1+b)(lL_2+(2l-1)bL_3)+(|w_2|+|w_1|b)bL_4,\\ B &= \frac{b^{q-q_1}(|\omega|+\mu_1+b)}{|\omega|}\left(\frac{l+1}{\Gamma(q)(a+1)^{1-q_1}}+\frac{2l-1}{\Gamma(q-1)(c+1)^{1-q_1}}\right)\\ &+\frac{\mu_2\xi^{q-q_1}(\mu_1+b)\Gamma(\alpha+\frac{q-q_1}{\gamma}+1)}{|\omega|(a+1)^{1-q_1}\Gamma(\alpha+\frac{q-q_1}{\gamma}+\beta+1)}. \end{aligned}$$

Then problem (5.1) *with boundary conditions given by* (5.2) *has at least one solution.*

Proof. For each $r > 0$, define Ψ on $\bar{B}_r$ by

$$(\Psi x)(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}F(s,x(s))ds + M_1 + M_2 t, \quad t \in [0,t_1],\\ \frac{1}{\Gamma(q)}\int_{t_\kappa}^t (t-s)^{q-1}F(s,x(s))ds + \sum_{j=1}^{\kappa}\frac{1}{\Gamma(q)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-1}F(s,x(s))ds\\ +\sum_{j=1}^{\kappa}\frac{t-t_\kappa}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds + \sum_{j=1}^{\kappa}\mathcal{I}_j(x(t_j^-))\\ +\sum_{j=1}^{\kappa-1}\frac{t_\kappa-t_j}{\Gamma(q-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{q-2}F(s,x(s))ds + \sum_{j=1}^{\kappa}(t-t_\kappa)\mathcal{J}_j(x(t_j^-))\\ +\sum_{j=1}^{\kappa-1}(t_\kappa-t_j)\mathcal{J}_j(x(t_j^-)) + M_1 + M_2 t, \quad t \in (t_\kappa, t_{\kappa+1}],\ \kappa = 1,2,\ldots,l. \end{cases}$$

Under the given assumptions, it can be shown that Ψ is well-defined on $\bar{B}_r$ and $\Psi x \in PC(J,\mathbb{R})$ for $x \in PC(J,\mathbb{R})$. Next, to show that $\Psi\colon \bar{B}_r \to PC(J,\mathbb{R})$ is completely continuous, we divide the proof into the following steps:

Step I: To show that $\{\Psi x | x \in \bar{B}_r\}$ is a uniformly bounded subset of $PC(J,\mathbb{R})$.

For $x \in \bar{B}_r$ and $t \in (t_\kappa, t_{\kappa+1})$, $\kappa = 0,1,\ldots,l$, we have

$$\begin{aligned} |\Psi x(t)| \leq{}& \frac{g(r)b^{(a+1)(1-q_1)}}{\Gamma(q)(a+1)^{1-q_1}}\|f\|_{L^{\frac{1}{q_1}}} + \frac{g(r)lb^{(a+1)(1-q_1)}}{\Gamma(q)(a+1)^{1-q_1}}\|f\|_{L^{\frac{1}{q_1}}} + \frac{g(r)lb^{(c+1)(1-q_1)+1}}{\Gamma(q-1)(c+1)^{1-q_1}}\|f\|_{L^{\frac{1}{q_1}}}\\ &+ \frac{g(r)(l-1)b^{(c+1)(1-q_1)+1}}{\Gamma(q-1)(c+1)^{1-q_1}}\|f\|_{L^{\frac{1}{q_1}}} + lL_2 + (2l-1)bL_3 + |M_1| + |M_2|b, \end{aligned}$$

that is,

$$\|\Psi x\|_{PC} \leq g(r) b^{(q-q_1)} \|f\|_{L^{\frac{1}{q_1}}} \left(\frac{l+1}{\Gamma(q)(a+1)^{1-q_1}} + \frac{2l-1}{\Gamma(q-1)(c+1)^{1-q_1}} \right) + lL_2 + (2l-1)bL_3 + |M_1| + |M_2| b,$$

where

$$\begin{aligned} |M_1| \leq{}& \frac{\mu_1 g(r) \|f\|_{L^{\frac{1}{q_1}}}}{|\omega|} \left[\frac{\mu_2 \xi^{q-q_1} \Gamma(\alpha + \frac{q-q_1}{\gamma} + 1)}{(a+1)^{1-q_1} \Gamma(\alpha + \frac{q-q_1}{\gamma} + \beta + 1)} + b^{q-q_1} \left(\frac{l+1}{\Gamma(q)(a+1)^{1-q_1}} \right.\right. \\ &\left.\left. + \frac{2l-1}{\Gamma(q-1)(c+1)^{1-q_1}} \right) \right] + \frac{\mu_1 l L_2 + \mu_1 (2l-1) b L_3}{|\omega|} + \frac{|w_2| b L_4}{|\omega|}, \\ |M_2| \leq{}& \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}}}{|\omega|} \left[\frac{\mu_2 \xi^{q-q_1} \Gamma(\alpha + \frac{q-q_1}{\gamma} + 1)}{(a+1)^{1-q_1} \Gamma(\alpha + \frac{q-q_1}{\gamma} + \beta + 1)} + b^{q-q_1} \left(\frac{l+1}{\Gamma(q)(a+1)^{1-q_1}} \right.\right. \\ &\left.\left. + \frac{2l-1}{\Gamma(q-1)(c+1)^{1-q_1}} \right) \right] + \frac{l L_2 + (2l-1) b L_3}{|\omega|} + \frac{|w_2| b L_4}{|\omega|}, \end{aligned}$$

with the positive constants L_2, L_3 and L_4 satisfying $|\mathcal{I}_\kappa(x(t))| \leq L_2$, $|\mathcal{J}_\kappa(x(t))| \leq L_3$, $|h(x(t))| \leq L_4$, for all $t \in I$.

Step II: To show that $\{\Psi x | x \in \bar{B}_r\}$ is equicontinuous in $(t_\kappa, t_{\kappa+1})$, $\kappa = 0, 1, \ldots, l$. Let $x \in \bar{B}_r$ and $0 \leq s_1 < s_2 \leq t_1$. Then

$$\begin{aligned} |(\Psi x)(s_2) - (\Psi x)(s_1)| \leq{}& \frac{1}{\Gamma(q)} \int_0^{s_1} [(s_2 - s)^{q-1} - (s_1 - s)^{q-1}] |F(s, x(s))| ds \\ &+ \frac{1}{\Gamma(q)} \int_{s_1}^{s_2} (s_2 - s)^{q-1} |F(s, x(s))| ds + |M_2| (s_2 - s_1) \\ \leq{}& \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}}}{\Gamma(q)(a+1)^{1-q_1}} [|s_2^{a+1} - s_1^{a+1}| + (s_2 - s_1)^{a+1}]^{1-q_1} \\ &+ \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}}}{\Gamma(q)(a+1)^{1-q_1}} (s_2 - s_1)^{q-q_1} + |M_2| (s_2 - s_1) \longrightarrow 0 \quad \text{as } s_2 \to s_1. \end{aligned}$$

For $t_\kappa < s_1 < s_2 \leq t_{\kappa+1}$, $\kappa = 1, \ldots, l$,

$$\begin{aligned} |(\Psi x)(s_2) - (\Psi x)(s_1)| \leq{}& \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}}}{\Gamma(q)(a+1)^{1-q_1}} [(a+1)(s_2 - s_1) - (s_2 - s_1)^{a+1}]^{1-q_1} \\ &+ \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}}}{\Gamma(q)(a+1)^{1-q_1}} (s_2 - s_1)^{q-q_1} + (s_2 - s_1) l L_3 + \frac{g(r) \|f\|_{L^{\frac{1}{q_1}}}}{\Gamma(q-1)(c+1)^{1-q_1}} \\ &\times l b^{q-q_1} (s_2 - s_1) + |M_2| (s_2 - s_1) \longrightarrow 0 \quad \text{as } s_2 \to s_1. \end{aligned}$$

Thus, Arzelá-Ascoli theorem ensures that $\Psi \bar{B}_r$ is a relatively compact subset of $PC(J, \mathbb{R})$. Also, using (H1) and from the continuity of the functions F, $\mathcal{I}_\kappa$, $\mathcal{J}_\kappa$, h, and Lemma 5.2.1, it follows that Ψ is continuous on $\bar{B}_r$. Thus, $\Psi\colon \bar{B}_r \to PC(J,\mathbb{R})$ is completely continuous.

Now, for each $t \in [0,b]$ and following the similar procedure as earlier, we find that

$$\frac{\|x\|_{PC}}{\frac{A}{|\omega|} + g(\|x\|_{PC})\|f\|_{L^{\frac{1}{q_1}}} B} \leq 1.$$

In view of (H2), there exists $L > 0$ such that $\|x\|_{PC} \neq L$. Set $S_1 = \{x \in PC(J,\mathbb{R}) | \|x\|_{PC} < L\}$. Then, the operator $\Psi\colon \bar{S}_1 \to PC(J,\mathbb{R})$ is completely continuous. Also, from the choice of S_1, there does not exist any $x \in \partial S_1$ such that $x = \Lambda\Psi x$ for some $\Lambda \in (0,1)$. Thus, by using Theorem 1.3.9, we can conclude that Ψ has a fixed point on $\bar{S}_1$. □

This completes the proof of the results that were accomplished by using the right terminal condition where ξ belonged to the initial sub-interval $(0,t_1)$. In other words, in equation (5.2), at the end point $t = b$, we consider the boundary condition $x(b) = \mu_2 I_{\gamma}^{\alpha,\beta} x(\xi)$ for $\xi \in (0,t_1)$.

5.3 Further Development with respect to General Sub-intervals

As a step towards more generalization, instead of taking $\xi \in (0,t_1)$, it may be assumed that $\xi \in (t_\kappa, t_{\kappa+1})$, $\kappa = 1,,\ldots,m$. Following the same procedure as in the earlier section, this extended problem may also be taken up by considering ξ in any arbitrary sub-interval. That is, for $\xi \in (t_\kappa, t_{\kappa+1})$, $\kappa = 1,,\ldots,m$, in view of Lemma 5.2.1, consider the following operators:

We define an operator Ψ_i on $PC(J,\mathbb{R})$, $(i = 1,2,3)$ as follows:

$$(\Psi_i x)(t) = \begin{cases} J_0^q F(t,x(t)) + M_{i1} + M_{i2}t, \quad t \in [0,t_1], \\ J_{t_\kappa}^q F(t,x(t)) + \displaystyle\sum_{j=1}^{\kappa} \frac{1}{\Gamma(q)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-1} F(s,x(s))ds \\ + \displaystyle\sum_{j=1}^{\kappa} \frac{t - t_\kappa}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} F(s,x(s))ds + \sum_{j=1}^{\kappa} \mathcal{I}_j(x(t_j^-)) \\ + \displaystyle\sum_{j=1}^{\kappa-1} \frac{t_\kappa - t_j}{\Gamma(q-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{q-2} F(s,x(s))ds + \sum_{j=1}^{\kappa} (t - t_\kappa)\mathcal{J}_j(x(t_j^-)) \\ + \displaystyle\sum_{j=1}^{\kappa-1} (t_\kappa - t_j)\mathcal{J}_j(x(t_j^-)) + M_{i1} + M_{i2}t, \quad t \in (t_\kappa, t_{\kappa+1}], \ \kappa = 1,2,\ldots,l, \end{cases}$$

where, for $\xi \in (t_\kappa, t_{\kappa+1})$, $\kappa = 1, , \dots, l-2$,

$$
\begin{aligned}
M_{11} = -\frac{1}{\omega}\Big[& -\mu_1\mu_2 I_\gamma^{\alpha,\beta} J^q_{t_\kappa} F(t,x(t))(\xi) - w_2\int_0^b h(x(s))ds + w_1\mu_1\sum_{j=1}^{\kappa}\mathcal{I}_j(x(t_j^-)) \\
& + \mu_1 \sum_{j=\kappa+1}^{l} \mathcal{I}_j(x(t_j^-)) + \mu_1\sum_{j=1}^{\kappa}(w_2 - t_l)\mathcal{J}_j(x(t_j^-)) + \mu_1 \sum_{j=\kappa+1}^{l}(b - t_l)\mathcal{J}_j(x(t_j^-)) \\
& + \mu_1\sum_{j=1}^{\kappa}(t_l - w_1 t_j)\mathcal{J}_j(x(t_j^-)) + \mu_1\sum_{j=\kappa+1}^{l-1}(t_l - t_j)\mathcal{J}_j(x(t_j^-)) + w_1\mu_1\sum_{j=1}^{\kappa} J^q_{t_{j-1}} F(t,x(t))(t_j) \\
& + \mu_1\sum_{j=\kappa+1}^{l+1} J^q_{t_{j-1}} F(t,x(t))(t_j) + \mu_1\sum_{j=1}^{\kappa}(w_2 - t_l) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) \\
& + \mu_1\sum_{j=\kappa+1}^{l}(b - t_l) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) + \mu_1\sum_{j=1}^{\kappa}(t_l - w_1 t_j) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) \\
& + \mu_1\sum_{j=\kappa+1}^{l-1}(t_l - t_j) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j)\Big],
\end{aligned}
$$

$$
\begin{aligned}
M_{12} = -\frac{1}{\omega}\Big[& -\mu_2 I_\gamma^{\alpha,\beta} J^q_{t_\kappa} F(t,x(t))(\xi) + w_1\int_0^b h(x(s))ds + w_1\sum_{j=1}^{\kappa}\mathcal{I}_j(x(t_j^-)) + \sum_{j=\kappa+1}^{l}\mathcal{I}_j(x(t_j^-)) \\
& + \sum_{j=1}^{\kappa}(w_2 - t_l)\mathcal{J}_j(x(t_j^-)) + \sum_{j=\kappa+1}^{l}(b - t_l)\mathcal{J}_j(x(t_j^-)) + \sum_{j=1}^{\kappa}(t_l - w_1 t_j)\mathcal{J}_j(x(t_j^-)) \\
& + \sum_{j=\kappa+1}^{l-1}(t_l - t_j)\mathcal{J}_j(x(t_j^-)) + w_1\sum_{j=1}^{\kappa} J^q_{t_{j-1}} F(t,x(t))(t_j) + \sum_{j=\kappa+1}^{l+1} J^q_{t_{j-1}} F(t,x(t))(t_j) \\
& + \sum_{j=1}^{\kappa}(w_2 - t_l) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) + \mu_1\sum_{j=\kappa+1}^{l}(b - t_l) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) + \sum_{j=1}^{\kappa}(t_l - w_1 t_j) \\
& \times J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) + \sum_{j=\kappa+1}^{l-1}(t_l - t_j) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j)\Big].
\end{aligned}
$$

For $\xi \in (t_{l-2}, t_{l-1})$,

$$\begin{aligned}
M_{21} = -\frac{1}{\omega}\Big[&- \mu_1\mu_2 I_\gamma^{\alpha,\beta} J^q_{t_{l-2}} F(t,x(t))(\xi) - w_2 \int_0^b h(x(s))ds + w_1\mu_1 \sum_{j=1}^{l-2} \mathcal{I}_j(x(t_j^-)) \\
&+ \mu_1 \sum_{j=l-1}^{l} \mathcal{I}_j(x(t_j^-)) + \mu_1 \sum_{j=1}^{l-2}(w_2 - w_1 t_j)\mathcal{J}_j(x(t_j^-)) + \mu_1 \sum_{j=l-1}^{l} (b - t_j)\mathcal{J}_j(x(t_j^-)) \\
&+ w_1\mu_1 \sum_{j=1}^{l-2} J^q_{t_{j-1}} F(t,x(t))(t_j) + \mu_1 \sum_{j=l-1}^{l+1} J^q_{t_{j-1}} F(t,x(t))(t_j) \\
&+ \mu_1 \sum_{j=1}^{l-2}(w_2 - w_1 t_j) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) + \mu_1 \sum_{j=l-1}^{l} (b - t_j) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j)\Big],
\end{aligned}$$

$$\begin{aligned}
M_{22} = -\frac{1}{\omega}\Big[&- \mu_2 I_\gamma^{\alpha,\beta} J^q_{t_{l-2}} F(t,x(t))(\xi) + w_1 \int_0^b h(x(s))ds + w_1 \sum_{j=1}^{l-2} \mathcal{I}_j(x(t_j^-)) \\
&+ \sum_{j=l-1}^{l} \mathcal{I}_j(x(t_j^-)) + \sum_{j=1}^{l-2}(w_2 - w_1 t_j)\mathcal{J}_j(x(t_j^-)) + \sum_{j=l-1}^{l} (b - t_j)\mathcal{J}_j(x(t_j^-)) \\
&+ w_1 \sum_{j=1}^{l-2} J^q_{t_{j-1}} F(t,x(t))(t_j) + \sum_{j=l-1}^{l+1} J^q_{t_{j-1}} F(t,x(t))(t_j) + \sum_{j=1}^{l-2}(w_2 - w_1 t_j) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) \\
&+ \mu_1 \sum_{j=l-1}^{l} (b - t_j) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j)\Big].
\end{aligned}$$

For $\xi \in (t_l, b)$,

$$\begin{aligned}
M_{31} = -\frac{1}{\omega}\Big[&- \mu_1\mu_2 I_\gamma^{\alpha,\beta} J^q_{t_l} F(t,x(t))(\xi) - w_2 \int_0^b h(x(s))ds + w_1\mu_1 \sum_{j=1}^{l} \mathcal{I}_j(x(t_j^-)) \\
&+ \mu_1 \sum_{j=1}^{l}(w_2 - w_1 t_l)\mathcal{J}_j(x(t_j^-)) + \mu_1 w_1 \sum_{j=1}^{l-1}(t_l - t_j)\mathcal{J}_j(x(t_j^-)) + \mu_1 J^q_{t_l} F(t,x(t))(b) \\
&+ \mu_1 w_1 \sum_{j=1}^{l} J^q_{t_{j-1}} F(t,x(t))(t_j) + \mu_1(w_2 - w_1 t_l) \sum_{j=1}^{l} J^{q-1}_{t_{j-1}} F(t,x(t))(t_j) \\
&+ \mu_1 w_1 \sum_{j=1}^{l-1}(t_l - t_j) J^{q-1}_{t_{j-1}} F(t,x(t))(t_j)\Big],
\end{aligned}$$

$$\begin{aligned}
M_{32} = -\frac{1}{\omega}\Big[&- \mu_2 I_\gamma^{\alpha,\beta} J^q_{t_l} F(t,x(t))(\xi) + w_1 \int_0^b h(x(s))ds + w_1 \sum_{j=1}^{l} \mathcal{I}_j(x(t_j^-)) + \sum_{j=1}^{l}(w_2 - w_1 t_l) \\
&\times \mathcal{J}_j(v(t_j^-)) + w_1 \sum_{j=1}^{l-1}(t_l - t_j)\mathcal{J}_j(v(t_j^-)) + J^q_{t_l} F(t,v(t))(b) + w_1 \sum_{j=1}^{l} J^q_{t_{j-1}} F(t,v(t))(t_j) \\
&+ (w_2 - w_1 t_l) \sum_{j=1}^{l} J^{q-1}_{t_{j-1}} F(t,v(t))(t_j) + w_1 \sum_{j=1}^{l-1}(t_l - t_j) J^{q-1}_{t_{j-1}} F(t,v(t))(t_j)\Big].
\end{aligned}$$

Further,

$$\omega = w_1\mu_1 + w_2 \neq 0, \text{ with } w_1 = 1 - \frac{\mu_2\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}, \quad w_2 = b - \frac{\mu_2\xi\Gamma(\alpha+\frac{1}{\gamma}+1)}{\Gamma(\alpha+\frac{1}{\gamma}+\beta+1)}.$$

Subsequently, under similar kind of assumptions on the functions F, $\mathcal{I}_\kappa$, $\mathcal{J}_\kappa$ and h (as in Section 5.2), different results may be obtained for the existence and uniqueness of solutions for $\xi \in (t_\kappa, t_{\kappa+1})$, $\kappa = 1, \ldots, l$.

CHAPTER 6

Approximate controllability of a semilinear Hilfer fractional differential equation with control in the nonlinear term

There are various means to establish that a system is approximately controllable. In [80, 94, 118], the controllability Grammian and fixed point theorems are used. For the approximate controllability of the considered evolution systems, it is assumed that the corresponding linear system is approximately controllable and the nonlinear function is uniformly bounded. Zhou [123] used the sequential approach and obtained some sufficient conditions for the approximate controllability of an integer-order semilinear equation. Thereafter, several researchers, [77, 78] etc., have used this approach to study the approximate controllability of nonlinear evolution equations using different fractional order derivatives.

Dauer and Mahmudov [37] considered the following semilinear evolution equation with finite delay:

$$\begin{aligned} &x(t) = Q(t)\psi(0) + \int_0^t Q(t-s)[Bu(s) + f(s, x_s, u(s))]ds, \quad t \in (0, b], \\ &x_0(\phi) = \Theta(\phi), \quad \phi \in [-h, 0], \end{aligned}$$

with $x(.) \in X$, where X is a Hilbert space, $u(.) \in L^2([0, b], U)$, where U is a Hilbert space, $\{Q(t)\}_{t>0}$ as a compact linear semigroup on X, $B\colon U \to X$ as a bounded linear operator and $\Theta \in C([-h, 0], X)$. For $x \in C([-h, b], X)$ and each $t \in [0, b]$, x_t is defined by $x_t(\phi) = x(t + \phi)$ for $\phi \in [-h, 0]$. Here they used the controllability Grammian and Schauder's fixed point theorem to obtain the sufficient conditions for approximate controllability of the system.

Another method for establishing the approximate controllability is to show an inclusion relation between the reachable sets of the considered system and a linear system, which is assumed to be approximately controllable. In [64, 65], Jeong and others considered the approximate controllability of ordinary semiliear differential systems and used the concept of Lebesgue point to show the existence of a control function which steered the solution of the system to an arbitrary ϵ-neighbourhood ($\epsilon > 0$) of the desired final state.

Sukavanam and Kumar [100] discussed the approximate controllability of the following semilinear delay equation with Caputo fractional derivative:

$$\begin{aligned} &{}^{C}D^{q}x(t) = Ax(t) + Bu(t) + f(t, x_t, u(t)),\ \ t \in (0, b],\ q \in \left(\frac{1}{2}, 1\right), \\ &x_0(\phi) = \Theta(\phi),\ \ \phi \in [-h, 0], \end{aligned}$$

where A is the generator of a C_0-semigroup, X and U are Banach spaces, the state $x(.) \in X$ and the control function $u(.) \in U$, $B\colon L^2([0, b], U) \to L^2([0, b], X_\eta)$ is a bounded linear operator and f is a given nonlinear function.

Motivated by the above mentioned works, here we consider the following Hilfer fractional differential equation:

$$\left.\begin{aligned} &{}^{H}D^{\delta,\zeta}_{0+}x(t) = -Ax(t) + f(t, x(t), u(t)) + Bu(t),\ \ t \in (0, b],\ b > 0, \\ &J^{(1-\delta)(1-\zeta)}_{0+}x(0) = x_0, \end{aligned}\right\} \tag{6.1}$$

where $\delta \in (0, 1)$, $\zeta \in [0, 1]$, $-A$ is the infinitesimal generator of an analytic semigroup $\{Q(t)\}_{t\geq 0}$ on a Banach space X. The state $x(.)$ takes values in the Banach space X_η and the control function $u \in L^p([0, b], U)$, where $p\delta > 1$ with U as a Banach space, $B\colon U \to X_\eta$, $\eta \in (0, 1]$ is a bounded linear operator and $x_0 \in X_\eta$. The nonlinear function f satisfies some assumptions which will be specified later.

The approximate controllability of our problem (6.1) is established by assuming that a linear system is approximately controllable and the range of the nonlinear function f is contained in the range set of the bounded linear operator B. Here we assume neither any uniform boundedness of the nonlinear function nor any compactness condition on the generated semigroup. Further, the assumptions considered are more general than the assumptions in [64, 65, 100]. To the best of our knowledge, this type of conditions has not been applied so far for studying the approximate controllability of Hilfer fractional differential systems.

6.1 Preliminaries

Assume that X is a Banach space with respect to the norm $\|.\|_X$. Without loss of generality, assume that $0 \in \rho(A)$. Then for any $\eta > 0$, $A^{-\eta}$ is a bounded linear operator defined as

$$A^{-\eta} = \frac{1}{\eta}\int_0^\infty t^{\eta-1}Q(t)dt.$$

Since $A^{-\eta}$ is one-to-one, therefore A^{η}, for $\eta \geq 0$, is defined as

$$A^{\eta} = \begin{cases} (A^{-\eta})^{-1}, & \eta > 0, \\ I, & \eta = 0. \end{cases}$$

Furthermore, A^{η} is a closed linear operator with domain $D(A^{\eta}) = R(A^{-\eta})$, which is dense in X. Also, $D(A^{\eta})$ is a Banach space with respect to the norm $\|.\|_{\eta}$ defined by

$$\|x\|_{\eta} = \|A^{\eta}x\|_{X} \quad \forall\, x \in D(A^{\eta}).$$

Denote $X_{\eta} = (D(A^{\eta}), \|.\|_{\eta})$. Throughout this work, it is assumed that there exists a constant $M_Q \geq 1$ such that $\|Q(t)\|_{B(Y)} \leq M_Q$ for all $t \geq 0$, that is, $\{Q(t)\}_{t\geq 0}$ is uniformly bounded by M_Q, and $\{Q(t)\}_{t>0}$ is continuous in the uniform operator topology.

Let $J = [0, b]$ and $C(J, X_{\eta})$ denote the Banach space of all continuous functions from J to X_{η}. Take $\gamma = \delta + \zeta - \delta\zeta$ so that $1 - \gamma = (1 - \delta)(1 - \zeta) \in [0, 1)$. Define $C_{1-\gamma}(J, X_{\eta}) = \{x\colon (0, b] \to X_{\eta} | t^{1-\gamma}x(t) \in C(J, X_{\eta})\}$ which is a Banach space with respect to the norm $\|.\|_{1-\gamma}$ defined by

$$\|x\|_{1-\gamma} = \sup_{t\in(0,b]} t^{1-\gamma}\|x(t)\|_{\eta} \quad \forall\, x \in C_{1-\gamma}(J, X_{\eta}).$$

Remark 6.1.1. *Let $x_1(t) = t^{\gamma-1}x_2(t)$, $t \in (0, b]$. Then*

$$x_1 \in C_{1-\gamma}(J, X_{\eta}) \iff x_2 \in C(J, X_{\eta}) \ \text{ and } \ \|x_1\|_{1-\gamma} = \|x_2\|_{C}.$$

Theorem 6.1.1. *[89] **(i)** $\eta_1 \geq \eta_2 > 0$ implies $D(A^{\eta_1}) \subset D(A^{\eta_2})$,*
***(ii)** if $\eta_1, \eta_2 \in \mathbb{R}$, then $A^{\eta_1+\eta_2}x = A^{\eta_1}A^{\eta_2}x$, for every $x \in D(A^{\eta})$ where $\eta = \max\{\eta_1, \eta_2, \eta_1 + \eta_2\}$.*

Lemma 6.1.1. *[89] There exists a constant $M_{\eta} > 0$ such that $\|A^{-\eta}\| \leq M_{\eta}$, for $\eta \in [0, 1]$.*

Theorem 6.1.2. *[89] **(i)** $Q(t)\colon X \to D(A^{\eta})$ for every $t > 0$ and $\eta \geq 0$,*
***(ii)** for every $x \in D(A^{\eta})$, $Q(t)A^{\eta}x = A^{\eta}Q(t)x$,*
***(iii)** for every $t > 0$, $A^{\eta}Q(t)$ is bounded and there exists a constant $C_{\eta} > 0$ such that $\|A^{\eta}Q(t)\| \leq \frac{C_{\eta}}{t^{\eta}}$.*

Remark 6.1.2. *[114] Let $Q_{\eta}(t)$ be the restriction of $Q(t)$ to X_{η}. Then, $\{Q_{\eta}(t)\}_{t\geq 0}$ is a family of bounded linear operators on X_{η} and satisfies $\|Q_{\eta}(t)\| \leq \|Q(t)\|$ for all $t \geq 0$. Moreover, $\{Q_{\eta}(t)\}_{t\geq 0}$ forms a C_0-semigroup on X_{η}.*

6.2 Mild Solution

Let $x(t; x_0, u)$ denote the state value of (6.1) at time t corresponding to the initial value x_0 and control $u(.)$. Then, we have the following definition of mild solution:

Definition 6.2.1. *[54, 119] A function $x(.;x_0,u) \in C_{1-\gamma}(J, X_\eta)$ is said to be a mild solution of (6.1) if, for any u in $L^p([0,b],U)$, the following integral equation is satisfied:*

$$\begin{aligned}
x(t;x_0,u) &= T_{\zeta,\delta}(t)x_0 + \int_0^t S_\delta(t-s)[f(s,x(s),u(s)) + Bu(s)]ds \\
&= T_{\zeta,\delta}(t)x_0 + \int_0^t (t-s)^{\delta-1}R_\delta(t-s)f(s,x(s),u(s))ds \\
&+ \int_0^t (t-s)^{\delta-1}R_\delta(t-s)Bu(s)ds
\end{aligned}$$

for $t \in (0,b]$, where

$$T_{\zeta,\delta}(t) = J_{0^+}^{\zeta(1-\delta)}S_\delta(t),\ S_\delta(t) = t^{\delta-1}R_\delta(t),\ R_\delta(t) = \int_0^\infty \delta\theta\xi_\delta(\theta)Q(t^\delta\theta)d\theta,$$

$$\xi_\delta(\theta) = \frac{1}{\delta}\theta^{-1-\frac{1}{\delta}}\bar{w}_\delta(\theta^{-\frac{1}{\delta}}),\ \bar{w}_\delta(\theta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\theta^{(-\delta n-1)}\frac{\Gamma(\delta n+1)}{n!}\sin(n\pi\delta),$$

for $\theta \in (0,\infty)$. Here, $\xi_\delta(\theta)$ is a probability density function on $(0,\infty)$ satisfying

$$\xi_\delta(\theta) \geq 0,\quad \int_0^\infty \xi_\delta(\theta)d\theta = 1,\quad \int_0^\infty \theta\xi_\delta(\theta)d\theta = \frac{1}{\Gamma(1+\delta)}.$$

Next we have the following properties of the solution operators $R_\delta(t)$, $S_\delta(t)$ and $T_{\zeta,\delta}(t)$ [54]:

(P1) $R_\delta(t)$ is continuous in the uniform operator topology for $t > 0$.

(P2) for any fixed $t > 0$, $R_\delta(t)$, $S_\delta(t)$ and $T_{\zeta,\delta}(t)$ are linear operators on X, and

$$\|R_\delta(t)x\|_X \leq \frac{M_Q}{\Gamma(\delta)}\|x\|_X,\ \|S_\delta(t)x\|_X \leq \frac{M_Q t^{\delta-1}}{\Gamma(\delta)}\|x\|_X,\ \|T_{\zeta,\delta}(t)x\|_X \leq \frac{M_Q t^{\gamma-1}}{\Gamma(\gamma)}\|x\|_X$$

hold for any $x \in X$.

(P3) $\{S_\delta(t)\}_{t>0}$ and $\{T_{\zeta,\delta}(t)\}_{t>0}$ are strongly continuous.

Before going to the next step, let us first recall some of the remaining properties:

(P4) for any fixed $t > 0$, and any $x \in X_\eta$,

$$\|R_\delta(t)x\|_\eta \leq \frac{M_Q}{\Gamma(\delta)}\|x\|_\eta,\ \|S_\delta(t)x\|_\eta \leq \frac{M_Q t^{\delta-1}}{\Gamma(\delta)}\|x\|_\eta,\ \|T_{\zeta,\delta}(t)x\|_\eta \leq \frac{M_Q t^{\gamma-1}}{\Gamma(\gamma)}\|x\|_\eta.$$

(P5) for each $x \in X_\eta$ and $t > 0$,

$$A^\eta R_\delta(t)x = R_\delta(t)A^\eta x,\ \ A^\eta S_\delta(t)x = S_\delta(t)A^\eta x,\ \ A^\eta T_{\zeta,\delta}(t)x = T_{\zeta,\delta}(t)A^\eta x.$$

6.3 Main Results

6.3.1 Existence and uniqueness of mild solution

Take $\zeta \neq 0$, that is, $\zeta \in (0,1]$. Then we have the following limits:

$$\lim_{t\to 0^+} t^{1-\gamma} T_{\zeta,\delta}(t)x_0 = \frac{x_0}{\Gamma(\gamma)},$$
$$\lim_{t\to 0^+} t^{1-\gamma} \int_0^t (t-s)^{\delta-1} R_\delta(t-s) f(s,x(s),u(s))ds = 0,$$
$$\lim_{t\to 0^+} t^{1-\gamma} \int_0^t (t-s)^{\delta-1} R_\delta(t-s) Bu(s)ds = 0.$$

Also, let $a = \frac{(\delta-1)p}{p-1}$.

For the existence and uniqueness result, we use the following assumptions:
(Hf) there exists a constant $\xi \in [\eta,1]$ such that $f\colon [0,b]\times X_\eta \times U \to X_\xi$ satisfies the following:
(i) there exists a constant $L>0$ such that

$$\|f(t,x_1,u_1)-f(t,x_2,u_2)\|_\xi \leq L[\|x_1-x_2\|_\eta + \|u_1-u_2\|_U]$$

for all $x_i \in X_\eta$, $u_i \in U$; $i=1,2$ and $t\in[0,b]$.
(ii) there exist a function $g \in L^p([0,b],[0,\infty))$ and a constant $c>0$ such that

$$\|f(t,x,u)\|_\xi \leq g(t) + c(t^{1-\gamma}\|x\|_\eta + \|u\|_U)$$

for all $x\in X_\eta$, $u\in U$ and $t\in[0,b]$.

Theorem 6.3.1. *If the above assumptions are satisfied, then for each $u \in L^p([0,b],U)$, problem (6.1) has a unique mild solution on $C_{1-\gamma}(J,X_\eta)$.*

Proof. Define a map Υ on $C_{1-\gamma}(J,X_\eta)$ by

$$(\Upsilon x)(t) = T_{\zeta,\delta}(t)x_0 + \int_0^t (t-s)^{\delta-1} R_\delta(t-s)[f(s,x(s),u(s)) + Bu(s)]ds.$$

The proof is split into several parts as follows.
Step I: To show that Υ is well-defined on $C_{1-\gamma}(J,X_\eta)$:

Using (P4), Theorem 6.1.1, Lemma 6.1.1, (Hf)(ii) and Hölder's inequality, we have

$$\begin{aligned}&\int_0^t \|(t-s)^{\delta-1}R_\delta(t-s)f(s,x(s),u(s))\|_\eta ds\\ &\leq \frac{M_Q M_{\xi-\eta}}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}[g(s)+c(s^{1-\gamma}\|x(s)\|_\eta+\|u(s)\|_U)]ds\\ &\leq \frac{M_Q M_{\xi-\eta}}{\Gamma(\delta)}\left[(\|g\|_{L^p}+c\|u\|_{L^p})\frac{b^{\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}}+\frac{cb^\delta}{\delta}\|x\|_{1-\gamma}\right].\end{aligned}$$

Similarly, using (P4) and Hölder's inequality,

$$\begin{aligned}\int_0^t \|(t-s)^{\delta-1}R_\delta(t-s)Bu(s)\|_\eta ds \leq&\frac{M_Q}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\|Bu(s)\|_\eta ds\\ \leq&\frac{M_Q}{\Gamma(\delta)}\|Bu\|_{L^p}\frac{b^{\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}}.\end{aligned}$$

Therefore, $(t-s)^{\delta-1}R_\delta(t-s)f(s,x(s),u(s))$ and $(t-s)^{\delta-1}R_\delta(t-s)Bu(s)$ are Bochner integrable w.r.t. $s\in[0,t]$ for all $t\in(0,b]$. Hence, $(\Upsilon x)(.)$ is well-defined on $(0,b]$ for any $x\in C_{1-\gamma}(J,X_\eta)$.

Step II: To show that $\Upsilon x\in C_{1-\gamma}(J,X_\eta)$ for any $x\in C_{1-\gamma}(J,X_\eta)$:

Let $x\in C_{1-\gamma}(J,X_\eta)$. Define $w\colon[0,b]\to X_\eta$ by

$$\begin{aligned}w(t)=&\begin{cases}\lim_{t\to0}t^{1-\gamma}(\Upsilon x)(t), & t=0,\\ t^{1-\gamma}(\Upsilon x)(t), & t\in(0,b],\end{cases}\\ =&\begin{cases}\frac{x_0}{\Gamma(\gamma)}, & t=0,\\ t^{1-\gamma}(\Upsilon x)(t), & t\in(0,b].\end{cases}\end{aligned}$$

Then, it can be easily seen that for $0=t_1<t_2\leq b$,

$$\|w(t_2)-w(t_1)\|_\eta\longrightarrow 0 \text{ as } t_2\to t_1.$$

Next, for $0<t_1<t_2\leq b$, we have

$$\begin{aligned}&\|w(t_2)-w(t_1)\|_\eta\\ &\leq \|t_2^{1-\gamma}T_{\zeta,\delta}(t_2)x_0-t_1^{1-\gamma}T_{\zeta,\delta}(t_1)x_0\|_\eta+\Bigg\|t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\delta-1}R_\delta(t_2-s)f(s,x(s),u(s))ds\\ &-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\delta-1}R_\delta(t_1-s)f(s,x(s),u(s))ds\Bigg\|_\eta\\ &+\left\|t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\delta-1}R_\delta(t_2-s)Bu(s)ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\delta-1}R_\delta(t_1-s)Bu(s)ds\right\|_\eta.\end{aligned}$$

First term:

$$\begin{aligned}
&\|t_2^{1-\gamma}T_{\zeta,\delta}(t_2)x_0 - t_1^{1-\gamma}T_{\zeta,\delta}(t_1)x_0\|_\eta \\
&= \left\|\frac{1}{\Gamma(\zeta(1-\delta))}\left[t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\zeta(1-\delta)-1}s^{\delta-1}R_\delta(s)x_0ds\right.\right. \\
&\left.\left. - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\zeta(1-\delta)-1}s^{\delta-1}R_\delta(s)x_0ds\right]\right\|_\eta \\
&\le \left\|\frac{1}{\Gamma(\zeta(1-\delta))}\int_{t_1}^{t_2}t_2^{1-\gamma}(t_2-s)^{\zeta(1-\delta)-1}s^{\delta-1}R_\delta(s)x_0ds\right\|_\eta \\
&+ \frac{1}{\Gamma(\zeta(1-\delta))}\left\|\int_0^{t_1}t_2^{1-\gamma}\{(t_2-s)^{\zeta(1-\delta)-1} - (t_1-s)^{\zeta(1-\delta)-1}\}s^{\delta-1}R_\delta(s)x_0ds\right\|_\eta \\
&+ \left\|\frac{t_2^{1-\gamma}-t_1^{1-\gamma}}{\Gamma(\zeta(1-\delta))}\left[\int_0^{t_1}(t_1-s)^{\zeta(1-\delta)-1}s^{\delta-1}R_\delta(s)x_0ds\right]\right\|_\eta \\
&\le I_{11} + I_{12} + I_{13},
\end{aligned}$$

where

$$\begin{aligned}
I_{11} =& \frac{M_Q\|x_0\|_\eta t_2^{1-\gamma}}{\Gamma(\delta)\Gamma(\zeta(1-\delta))}\int_{t_1}^{t_2}(t_2-s)^{\zeta(1-\delta)-1}s^{\delta-1}ds, \\
I_{12} =& \frac{M_Q t_2^{1-\gamma}\|x_0\|_\eta}{\Gamma(\zeta(1-\delta))}\int_0^{t_1}[(t_1-s)^{\zeta(1-\delta)-1} - (t_2-s)^{\zeta(1-\delta)-1}]s^{\delta-1}ds, \\
I_{13} =& \frac{t_2^{1-\gamma}-t_1^{1-\gamma}}{\Gamma(\zeta(1-\delta))}\int_0^{t_1}(t_1-s)^{\zeta(1-\delta)-1}s^{\delta-1}\|R_\delta(s)x_0\|_\eta ds.
\end{aligned}$$

By absolute continuity of Lebesgue integral, $I_{11} \to 0$ as $t_2 \to t_1$. For I_{12}, we have

$$[(t_1-s)^{\zeta(1-\delta)-1} - (t_2-s)^{\zeta(1-\delta)-1}]s^{\delta-1} \le (t_1-s)^{\zeta(1-\delta)-1}s^{\delta-1}$$

for a.e. $s \in [0, t_1]$. Therefore, by vector-valued dominated convergence theorem, $I_{12} \to 0$, as $t_2 \to t_1$. Next, for I_{13}, we have

$$\begin{aligned}
I_{13} \le& \frac{M_Q(t_2^{1-\gamma}-t_1^{1-\gamma})\|x_0\|_\eta}{\Gamma(\delta)\Gamma(\zeta(1-\delta))}\int_0^{t_1}(t_1-s)^{\zeta(1-\delta)-1}s^{\delta-1}ds \\
=& \frac{M_Q t_1^{\gamma-1}(t_2^{1-\gamma}-t_1^{1-\gamma})\|x_0\|_\eta}{\Gamma(\delta)\Gamma(\zeta(1-\delta))} \longrightarrow 0, \quad \text{as } t_2 \to t_1.
\end{aligned}$$

Second term:

$$\begin{aligned}
&\Big\| t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\delta-1}R_\delta(t_2-s)f(s,x(s),u(s))ds \\
&\qquad - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\delta-1}R_\delta(t_1-s)f(s,x(s),u(s))ds\Big\|_\eta \\
&\le t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\delta-1}\|R_\delta(t_2-s)f(s,x(s),u(s))\|_\eta ds \\
&+\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\delta-1}-t_1^{1-\gamma}(t_1-s)^{\delta-1}|\|R_\delta(t_2-s)f(s,x(s),u(s))\|_\eta ds \\
&+t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\delta-1}\|[R_\delta(t_2-s)-R_\delta(t_1-s)]f(s,x(s),u(s))\|_\eta ds \\
&=I_{21}+I_{22}+I_{23},
\end{aligned}$$

where

$$\begin{aligned}
I_{21}=&t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\delta-1}\|R_\delta(t_2-s)f(s,x(s),u(s))\|_\eta ds,\\
I_{22}=&\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\delta-1}-\tau_1^{1-\gamma}(t_1-s)^{\delta-1}|\|R_\delta(t_2-s)f(s,x(s),u(s))\|_\eta ds,\\
I_{23}=&t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\delta-1}\|[R_\delta(t_2-s)-R_\delta(t_1-s)]f(s,x(s),u(s))\|_\eta ds.
\end{aligned}$$

Now,

$$\begin{aligned}
I_{21}&\le\frac{M_{Q}M_{\xi-\eta}t_2^{1-\gamma}}{\Gamma(\delta)}\int_{t_1}^{t_2}(t_2-s)^{\delta-1}\|f(s,x(s),u(s))\|_\eta ds\\
&\le\frac{M_{Q}M_{\xi-\eta}b^{1-\gamma}}{\Gamma(\delta)}\Bigg[(\|g\|_{L^p}+c\|u\|_{L^p})\frac{(t_2-t_1)^{\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}}+\frac{c(t_2-t_1)^\delta}{\delta}\|x\|_{1-\gamma}\Bigg]\\
&\longrightarrow 0,\ \text{as } t_2\to t_1\\
&\text{and}\\
I_{22}&\le\frac{M_{Q}M_{\xi-\eta}}{\Gamma(\delta)}\Bigg[\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\delta-1}-t_1^{1-\gamma}(t_1-s)^{\delta-1}|g(s)ds\\
&+c\|x\|_{1-\gamma}\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\delta-1}-t_1^{1-\gamma}(t_1-s)^{\delta-1}|ds\\
&+c\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\delta-1}-t_1^{1-\gamma}(t_1-s)^{\delta-1}|\|u(s)\|_{U}ds\Bigg]
\end{aligned}$$

which converges to 0, as $t_2\to t_1$, due to Lebesgue's dominated convergence theorem.

Next, for $\epsilon > 0$ small enough, we have

$$
\begin{aligned}
I_{23} \leq M_{\xi-\eta} & \left[(\|g\|_{L^p} + c\|u\|_{L^p}) \frac{t_1^{1-\gamma+\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}} + c\|x\|_{1-\gamma} \frac{t_1^{1-\gamma+\delta}}{\delta} \right] \\
& \times \sup_{s\in[0,t_1-\epsilon]} \|R_\delta(t_2-s) - R_\delta(t_1-s)\|_{B(Y)} \\
& + \frac{2M_Q M_{\xi-\eta} b^{1-\gamma}}{\Gamma(\delta)} \left[(\|g\|_{L^p} + c\|u\|_{L^p}) \frac{\epsilon^{\frac{(\delta p-1)}{p}}}{(a+1)^{\frac{(p-1)}{p}}} + c\|x\|_{1-\gamma} \frac{\epsilon^\delta}{\delta} \right],
\end{aligned}
$$

where the RHS converges to zero, by using (P1), as $t_2 \to t_1$ and $\epsilon \to 0$.
Third term:

$$
\begin{aligned}
& \left\| t_2^{1-\gamma} \int_0^{t_2} (t_2-s)^{\delta-1} R_\delta(t_2-s) Bu(s)ds - t_1^{1-\gamma} \int_0^{t_1} (t_1-s)^{\delta-1} R_\delta(t_1-s) Bu(s)ds \right\|_\eta \\
& \leq I_{31} + I_{32} + I_{33},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{31} &= t_2^{1-\gamma} \int_{t_1}^{t_2} (t_2-s)^{\delta-1} \|R_\delta(t_2-s)Bu(s)\|_\eta ds, \\
I_{32} &= \int_0^{t_1} |t_2^{1-\gamma}(t_2-s)^{\delta-1} - t_1^{1-\gamma}(t_1-s)^{\delta-1}| \|R_\delta(t_2-s)Bu(s)\|_\eta ds, \\
I_{33} &= t_1^{1-\gamma} \int_0^{t_1} (t_1-s)^{\delta-1} \|[R_\delta(t_2-s) - R_\delta(t_1-s)]Bu(s)\|_\eta ds.
\end{aligned}
$$

Now,

$$
I_{31} \leq \frac{M_Q \|Bu\|_{L^p} b^{1-\gamma}}{(a+1)^{\frac{(p-1)}{p}} \Gamma(\delta)} (t_2-t_1)^{\frac{(\delta p-1)}{p}} \longrightarrow 0, \quad \text{as } t_2 \to t_1.
$$

By Lebesgue's dominated convergence theorem, $I_{32} \to 0$ as $t_2 \to t_1$, and applying similar technique as for I_{23}, we get $I_{33} \to 0$ as $t_2 \to t_1$.

Thus, we have for $0 < t_1 < t_2 \leq b$,

$$
\|w(t_2) - w(t_1)\|_\eta \longrightarrow 0, \quad \text{as } t_2 \to t_1.
$$

Therefore, $w \in C(J, X_\eta)$ and hence $\Upsilon x \in C_{1-\gamma}(J, X_\eta)$.
Step III: To show that $\Upsilon^{(i)}$ is a contraction for some $i \in \mathbb{N}$:

We proceed by induction on i. Let x, $y \in C_{1-\gamma}(J, X_\eta)$. Then, for any $t \in (0, b]$, we claim that

$$
t^{1-\gamma} \|(\Upsilon^{(i)}x)(t) - (\Upsilon^{(i)}y)(t)\|_\eta \leq \Gamma(\gamma) \frac{(LM_Q M_{\xi-\eta} t^\delta)^i}{\Gamma(i\delta+\gamma)} \|x-y\|_{1-\gamma} \text{ for all } i \in \mathbb{N}. \tag{6.2}
$$

For $i = 1$, using (P4), Lemma 6.1.1 and (Hf)(i), we have

$$\begin{aligned} t^{1-\gamma}\|(\Upsilon x)(t) - (\Upsilon y)(t)\|_{\eta} \leq & \frac{M_{Q} M_{\xi-\eta} t^{1-\gamma}}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \| f(s, x(s), u(s)) \\ & - f(s, y(s), u(s))\|_{\xi} ds \\ \leq & \frac{L M_{Q} M_{\xi-\eta} t^{1-\gamma}}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \|x(s) - y(s)\|_{\eta} ds \\ = & \Gamma(\gamma) \frac{L M_{Q} M_{\xi-\eta} t^{\delta}}{\Gamma(\delta+\gamma)} \|x - y\|_{1-\gamma}. \end{aligned}$$

Thus, equation (6.2) holds for $i = 1$.

Induction hypothesis: Assume that (6.2) holds for $i = k$, i.e.,

$$t^{1-\gamma}\|(\Upsilon^{(k)} x)(t) - (\Upsilon^{(k)} y)(t)\|_{\eta} \leq \Gamma(\gamma) \frac{(L M_{Q} M_{\xi-\eta} t^{\delta})^k}{\Gamma(k\delta+\gamma)} \|x - y\|_{1-\gamma}.$$

Then,

$$\begin{aligned} & t^{1-\gamma}\|(\Upsilon^{(k+1)} x)(t) - (\Upsilon^{(k+1)} y)(t)\|_{\eta} \\ \leq & \Gamma(\gamma) \frac{(L M_{Q} M_{\xi-\eta})^{k+1} t^{1-\gamma}}{\Gamma(k\delta+\gamma)\Gamma(\delta)} \|x - y\|_{1-\gamma} \int_0^t (t-s)^{\delta-1} s^{\gamma+n\delta-1} ds \\ \leq & \Gamma(\gamma) \frac{(L M_{Q} M_{\xi-\eta} t^{\delta})^{k+1}}{\Gamma((k+1)\delta+\gamma)} \|x - y\|_{1-\gamma}. \end{aligned}$$

Thus, by principle of mathematical induction, (6.2) holds for all $i \in \mathbb{N}$. Now, for $t \in (0, b]$, we have

$$t^{1-\gamma}\|(\Upsilon^{(i)} x)(t) - (\Upsilon^{(i)} y)(t)\|_{\eta} \leq \Gamma(\gamma) \frac{(L M_{Q} M_{\xi-\eta} t^{\delta})^i}{\Gamma(i\delta+\gamma)} \|x - y\|_{1-\gamma}$$

which gives

$$\|\Upsilon^{(i)} x - \Upsilon^{(i)} y\|_{1-\gamma} \leq \Gamma(\gamma) \frac{(L M_{Q} M_{\xi-\eta} b^{\delta})^i}{\Gamma(i\delta+\gamma)} \|x - y\|_{1-\gamma}.$$

Since the series $E_{\delta,\gamma}(L M_{Q} M_{\xi-\eta} b^{\delta}) = \sum_{i=0}^{\infty} \frac{(L M_{Q} M_{\xi-\eta} b^{\delta})^i}{\Gamma(i\delta+\gamma)}$ converges, therefore we can get

$$\frac{(L M_{Q} M_{\xi-\eta} b^{\delta})^i}{\Gamma(i\delta+\gamma)} < \frac{1}{\Gamma(\gamma)} \quad \text{for } i \text{ sufficiently large.}$$

Therefore, $\Upsilon^{(i)}$ is a contraction on $C_{1-\gamma}(J, X_{\eta})$ and thus Υ has a unique fixed point on $C_{1-\gamma}(J, X_{\eta})$.

In other words, problem (6.1) has a unique solution on $C_{1-\gamma}(J, X_{\eta})$. □

6.3.2 Approximate controllability

In this section, we establish the approximate controllability of equation (6.1).

We know that $x(b;x_0,u)$ denotes the state value of (6.1) at the terminal time b corresponding to the initial value x_0 and control $u(.)$. Let $R_b(f) = \{x(b;x_0,u)|u \in L^p([0,b],U)\}$ denote the reachable set $-$ the set of all points to which the initial state x_0 can be steered in time b under the influence of the control u.

Definition 6.3.1. *(6.1) is said to be approximately controllable on* $[0,b]$ *if given an arbitrary* $\epsilon > 0$, *it is possible to steer from* x_0 *to a point within a distance* ϵ *from all points in the state space* X_η *at time* b.

Thus, in terms of the reachable set, (6.1) is approximately controllable on $[0,b]$ iff $\overline{R_b(f)} = X_\eta$.

Now, consider the following linear system:

$$\left.\begin{array}{l} {}^H D_{0+}^{\delta,\zeta} x(t) = -Ax(t) + Bv(t), \quad t \in (0,b], \\ J_{0+}^{(1-\delta)(1-\zeta)} x(0) = x_0. \end{array}\right\} \tag{6.3}$$

Then in accordance with the above notation, the reachable set of (6.3) is denoted by $R_b(0)$.

Fix $x \in C_{1-\gamma}([0,b],X_\eta)$ and $h \in L^p([0,b],U)$. Define a map K by

$$K(t) = f(t,x(t),h(t)).$$

Then K belongs to $L^p([0,b],X_\eta)$.

To prove the approximate controllability of (6.1), we use the generalized Grönwall inequality (Lemma 1.2.1), for which we need to modify our assumption (i) in (Hf) as

(iii) there exists a constant $N > 0$ such that

$$\|f(t,x_1,u_1) - f(t,x_2,u_2)\|_\xi \leq Nt^{1-\gamma}[\|x_1 - x_2\|_\eta + \|u_1 - u_2\|_U],$$

for all $x_i \in X_\eta$, $u_i \in U$; $i = 1,2$ and $t \in [0,b]$.

We also consider the following assumptions:

(HfB) range$(f) \subset$ range(B).

(HB) there exists a constant $e > 0$ such that $\|Bu\|_\eta \geq e\|u\|_U$ for all $u \in U$.

Theorem 6.3.2. *Assume that hypotheses (HfB), (HB) and (Hf) (with (i) replaced by (iii)) hold. Then, the approximate controllability of linear system (6.3) and the inequality*

$$\max\{cM_{\xi-\eta}, M_{\xi-\eta}Nb^{1-\gamma}\} < e$$

imply that (6.1) is approximately controllable.

Proof. Let y be the mild solution of the linear system (6.3) corresponding to a control v. First, we show that, for $y \in C_{1-\gamma}(J,X_\eta)$ and $v \in L^p([0,b],U)$, there exists a control function $u \in L^p([0,b],U)$ such that it satisfies $Bu(t) = Bv(t) - f(t,y(t),u(t))$, $t \in (0,b]$.

Next, define a new function $\Pi\colon \text{range}(B) \subset X_\eta \to U$ by

$$\Pi r = u \ \text{ whenever } \ Bu = r.$$

Then, by using (HB), it can be shown that Π is well-defined. The function Π forms a bounded linear map with $\|\Pi\| \leq \frac{1}{e}$. Also, $\Pi B = Id_{_U}$ and $B\Pi = Id_{range(B)}$.

Next, we begin by showing that, for each $t \in (0, b]$, there exists $u(t) \in U$ such that $u(t) = v(t) - \Pi f(t, y(t), u(t))$, for all $t \in (0, b]$.

Let $h_0(t) = v(t)$, $t \in [0, b]$ and for each $n \in \mathbb{N}$, define

$$h_n(t) = \begin{cases} v(t) - \Pi f(t, y(t), h_{n-1}(t)), & t \in (0, b], \\ v(0), & t = 0. \end{cases}$$

Then fix $t \in (0, b]$. Subsequently,

$$\begin{aligned} \|h_{n+1}(t) - h_n(t)\|_{_U} &= \|\Pi f(t, y(t), h_n(t)) - \Pi f(t, y(t), h_{n-1}(t))\|_{_U} \\ &\leq \frac{1}{e} M_{\xi-\eta} N t^{1-\gamma} \|h_n(t) - h_{n-1}(t)\|_{_U} \\ &\leq \left(\frac{1}{e} M_{\xi-\eta} N b^{1-\gamma}\right)^n \|h_1(t) - h_0(t)\|_{_U}, \end{aligned}$$

and for $m > n$ $(m, n \in \mathbb{N})$,

$$\begin{aligned} &\|h_m(t) - h_n(t)\|_{_U} \\ &\leq \|h_m(t) - h_{m-1}(t)\|_{_U} + \|h_{m-1}(t) - h_{m-2}(t)\|_{_U} + \ldots + \|h_{n+1}(t) - h_n(t)\|_{_U} \\ &\leq \left(\frac{M_{\xi-\eta} N b^{1-\gamma}}{e}\right)^n \frac{1}{1 - \frac{M_{\xi-\eta} N b^{1-\gamma}}{e}} \|h_1(t) - h_0(t)\|_{_U} \longrightarrow 0, \ \text{as } n \to \infty. \end{aligned}$$

Therefore, $(h_n(t)) \subset U$ is a Cauchy sequence, and U being complete, we have $\lim\limits_{n\to\infty} h_n(t) \in U$. Since, this argument holds for each $t \in (0, b]$, we define a function $u\colon [0, b] \to U$ by

$$u(t) = \begin{cases} \lim\limits_{n\to\infty} h_n(t), & t \in (0, b], \\ v(0), & t = 0. \end{cases}$$

Again, for each fixed $t \in (0, b]$,

$$\begin{aligned} \|v(t) - h_{n+1}(t) - \Pi f(t, y(t), u(t))\|_{_U} &\leq \frac{M_{\xi-\eta}}{e} \|f(t, y(t), h_n(t)) - f(t, y(t), u(t))\|_\xi \\ &\leq \frac{M_{\xi-\eta} N b^{1-\gamma}}{e} \|h_n(t) - u(t)\|_{_U} \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $u(t) = v(t) - \Pi f(t, y(t), u(t))$, for each $t \in (0, b]$.

Now, it remains to show that the function u belongs to $L^p([0, b], U)$. Observe that, for each $n \in \mathbb{N} \cup \{0\}$, $h_n \in L^p([0,b],U)$, since by definition $h_0 \in L^p([0,b],U)$, and if we define $K_n(t) = f(t, y(t), h_{n-1}(t))$, then $K_n \in L^p([0,b],X_\eta)$. Subsequently, $\Pi K_n \in L^p([0,b],U)$. Also,

$$\begin{aligned}
\|h_n(t)\|_{U} &\leq \|v(t)\|_{U} + \frac{M_{\xi-\eta}}{e} g(t) + \frac{M_{\xi-\eta}}{e} c t^{1-\gamma} \|y(t)\|_\eta + \frac{M_{\xi-\eta}}{e} c \|h_{n-1}(t)\|_{U} \\
&\leq \left(1 + \frac{cM_{\xi-\eta}}{e}\right) \|v(t)\|_{U} + \left(\frac{M_{\xi-\eta}}{e} + c\frac{M_{\xi-\eta}^2}{e^2}\right) g(t) \\
&+ \left(\frac{M_{\xi-\eta}}{e} c t^{1-\gamma} + \frac{M_{\xi-\eta}^2}{e^2} c^2 t^{1-\gamma}\right) \|y(t)\|_\eta + \frac{M_{\xi-\eta}^2}{e^2} c^2 \|h_{n-2}(t)\|_{U} \\
&\leq \left[1 + \frac{cM_{\xi-\eta}}{e} + \ldots + \frac{c^n M_{\xi-\eta}^n}{e^n}\right] \|v(t)\|_{U} \\
&+ \left[\frac{M_{\xi-\eta}}{e} + c\frac{M_{\xi-\eta}^2}{e^2} + \ldots + c^{n-1}\frac{M_{\xi-\eta}^n}{e^n}\right] g(t) \\
&+ t^{1-\gamma}\left[c\frac{M_{\xi-\eta}}{e} + c^2\frac{M_{\xi-\eta}^2}{e^2} + \ldots + c^n\frac{M_{\xi-\eta}^n}{e^n}\right] \|y(t)\|_\eta \\
&\leq \frac{1}{1 - \frac{cM_{\xi-\eta}}{e}} \|v(t)\|_{U} + \frac{\frac{cM_{\xi-\eta}}{e}}{1 - \frac{cM_{\xi-\eta}}{e}} t^{1-\gamma} \|y(t)\|_\eta + \frac{\frac{M_{\xi-\eta}}{e}}{1 - \frac{cM_{\xi-\eta}}{e}} g(t) \\
&:= G(t) \ \ \text{(say)}.
\end{aligned}$$

Therefore, $G \in L^p([0,b],[0,\infty))$ and subsequently, by vector-valued dominated convergence theorem, we can conclude that $u \in L^p([0,b],U)$.

Next, since y is a mild solution of (6.3), it satisfies

$$y(t) = T_{\zeta,\delta}(t)x_0 + \int_0^t (t-s)^{\delta-1} R_\delta(t-s) B v(s) ds, \quad t \in (0,b].$$

Consider the following semilinear system:

$$\left.\begin{aligned}
D_{0+}^{\delta,\zeta} x(t) =& -Ax(t) + f(t, x(t), u(t)) + Bv(t) - f(t, y(t), u(t)), \quad t \in (0,b], \\
J_{0+}^{(1-\delta)(1-\zeta)} y(0) =& x_0.
\end{aligned}\right\} \tag{6.4}$$

Then the mild solution of (6.4) satisfies

$$x(t) = T_{\zeta,\delta}(t)x_0 + \int_0^t (t-s)^{\delta-1} R_\delta(t-s)[f(s, x(s), u(s)) + Bv(s) - f(s, y(s), u(s))] ds, \quad t \in (0,b].$$

Now, for $t \in (0,b]$, we have

$$\begin{aligned}
t^{1-\gamma}\|x(t) - y(t)\|_\eta &\leq t^{1-\gamma} \int_0^t (t-s)^{\delta-1} \|R_\delta(t-s)[f(s, x(s), u(s)) - f(s, y(s), u(s))]\|_\eta ds \\
&\leq \frac{N M_Q M_{\xi-\eta} t^{1-\gamma}}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} s^{1-\gamma} \|x(s) - y(s)\|_\eta ds.
\end{aligned}$$

Take $H(t) = t^{1-\gamma}\|x(t) - y(t)\|_\eta$. Then, from the above inequality, we have

$$H(t) \leq \frac{NM_Q M_{\xi-\eta} t^{1-\gamma}}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} H(s)ds, \quad t \in [0, b].$$

Using Lemma 1.2.1, we get $x = y$, that is, every solution of (6.3) with control v is a solution of the semilinear system (6.1) with control u. Hence, $R_b(0) \subset R_b(f)$, and therefore $\overline{R_b(f)} = X_\eta$.

Subsequently, problem (6.1) is approximately controllable. $\square$

Remark 6.3.1. *When $\zeta = 0$, (6.1) reduces to*

$$\begin{aligned} {}^{RL}D_{0+}^{\delta} x(t) =& -Ax(t) + f(t, x(t), u(t)) + Bu(t), \quad t \in (0, b], \\ J_{0+}^{1-\delta} x(0) =& x_0, \end{aligned}$$

and it can be easily seen that the results in Theorems 6.3.1 and 6.3.2 hold for $\zeta = 0$ when γ is replaced by δ.

CHAPTER 7

Conclusion and Future Scope

7.1 Conclusion

In this thesis, we study various types of nonlinear fractional differential equations such as integro neutral type, Sobolev type, etc. subject to nonlocal initial conditions. The first problem deals with the non-autonomous Hilfer fractional differential equations. The results are obtained by using various results of fractional calculus and fixed point theorems in the weighted space of continuous functions. Examples are constructed to illustrate the derived theory for fractional semilinear evolution equation and for nonlocal condition.

In the second problem, we consider neutral type Volterra-Fredholm integro differential equations with Caputo derivatives when the associated linear operator is nondensely defined. By using various fixed point theorems, fractional calculus and measure of noncompactness, we obtain some sufficient conditions which ensure the existence of integral solutions of the problem under consideration when the associated C_0-semigroup generated by the part of A in $\overline{D(A)}$ is compact or non-compact. Under suitable assumptions, four theorems are proposed and proved. Two theorems are verified by considering appropriate examples. These results are expected to enrich the analysis with regard to solutions for mixed Volterra-Fredholm integro fractional differential equations.

The third problem is concerned with a Volterra integro differential equation of Sobolev type with finite delay involving Caputo fractional derivatives. Here, the first three results are established by applying Banach fixed point theorem, Krasnoselskii's fixed point theorem and Darbo-Sadovskii's fixed point theorem, respectively. In the last result, we drop the compactness assumption on the nonlocal function g and instead use measure of noncompactness to obtain some sufficient conditions which ensure the existence of mild solutions.

In the fourth problem, we consider a class of impulsive boundary value problem with integral conditions and multiple base point involving Caputo fractional derivatives. Here, the boundary value problem is transformed into an equivalent integral equation in a Banach space. We define a nonlinear operator Ψ on the Banach space whose fixed point gives the solution of the boundary value problem. Our results are more general in the sense that Erdélyi-Kober integrals are known to be more general operators in fractional calculus and they reduce to Riemann-Liouville integrals with a power weight for $\alpha = 0$, $\gamma = 1$. Furthermore, those integrals include Hadamard integral as a special case.

In each of the problems, we establish the existence and uniqueness of mild/integral solutions for the considered problem based on the methods of nonlinear functional analysis and some classical fixed point theorems such as Banach, Krasnoselskii, Schauder, etc.

In the fifth problem, we consider a control system involving Hilfer fractional derivatives, where the linear operator generates an analytic semigroup. The existence and uniqueness result is proved with the help of a fixed point theorem by utilizing the properties of the fractional powers of A^η, the semigroup $\{Q(t)\}_{t\geq 0}$ and the associated operators $\{R_\delta(t)\}_{t>0}$, $\{S_\delta(t)\}_{t>0}$ and $\{T_{\zeta,\delta}(t)\}_{t>0}$. For the approximate controllability result, based on the information available, we construct a sequence of functions belonging to the space of admissible controls that converges to a control function $u \in L^p([0,b],U)$, which, by using the definition of mild solution and reachable sets, gives the approximate controllability of our system.

7.2 Future plan

In most of the problems that are taken up, the order of the fractional differential equation belongs to $(0,1)$. In future, we plan to consider the following problems:

- to study the existence and uniqueness of non-instantaneous fractional differential equations of order $\alpha \in (1,2)$.
- as mentioned in [31], different types of controllability are present in literature. So, we would like to extend our study on controllability to fractional evolution inclusions.

In comparison to a deterministic model, a stochastic model is more realistic and challenging. For example, in engineering problems such as wind excitation or seismic impact, it is very difficult to describe the dynamic behaviour of the system by a purely mathematical model. The possible way to model these excitations is by the use of probabilistic mathematics instead of deterministic mathematics. Stochastic differential equation is a combination of differential equations, probability theory and stochastic process. In future, we would like to consider problems on controllability of stochastic fractional differential equations.

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List of published and communicated papers

Based on the works carried out in this thesis, the following articles are published/accepted/ communicated:

1. Bandita Roy and Swaroop Nandan Bora, On existence and uniqueness of integral solutions for a class of nondensely defined mixed Volterra-Fredholm integro-fractional neutral differential equations, *Journal of Nonlinear Evolution Equations and Applications*, 3:41–62, 2021.

2. Bandita Roy and Swaroop Nandan Bora, On mild solutions of Volterra fractional differential equations of Sobolev type with finite delay, *Journal of Fractional Calculus and Applications*, 12:94–113, 2021.

3. Bandita Roy and Swaroop Nandan Bora, Existence of mild solutions for semilinear evolution equation using Hilfer fractional derivatives. (Accepted in *Fractional Differential Calculus*)

4. Bandita Roy and Swaroop Nandan Bora, Approximate controllability of a class of semilinear Hilfer fractional differential equations. (Revised version submitted to *Results in Mathematics*)

5. Bandita Roy and Swaroop Nandan Bora, Impulsive differential equations with Caputo fractional derivative and Erdélyi-Kober type boundary conditions. (Communicated)

6. Bandita Roy and Swaroop Nandan Bora, On integral solutions for a class of mixed Volterra-Fredholm integro differential equations with Caputo fractional derivatives, *Springer Proceedings in Mathematics & Statistics* (**Book chapter**). **(Accepted)**