ON COMPLETION PROBLEMS FOR SOME CLASSES OF MATRICES

A thesis submitted
in Partial Fulfillment of the Requirements
for the degree of
Doctor of Philosophy

by

Kalyan Sinha

to the

Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati - 781039, INDIA

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To

My Parents

Certificate

It is certified that the work contained in this thesis entitled "**On Completion Problems for some Classes of Matrices**", by **Kalyan Sinha**, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

Guwahati
30 March 2015

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IIT Guwahati Kalyan Sinha

Abstract

The thesis aims at a further study on matrix completion problems – a concept that was initiated by Burg to study the positive definite matrices and studied by a number of researchers combinatorially for several classes of matrices including P and P_0-matrices and their subclasses. By their essence, a matrix completion problem attempts to provide a conventional matrix when some entries of a matrix are unavailable but it is known that the matrix has certain properties. In this thesis we present positive Q-completion, nonnegative Q-completion and P_0^+-completion problems. In a combinatorial set up, we try to develop some necessary and sufficient conditions for completions for these three classes of matrices. For each of the classes, we apply the results to single out all digraphs of order at most 4 that have completion. Further, we study the relationship between the completion problems of the three classes, among themselves and with some other related classes of matrices.

Contents

Introduction

The present thesis is in the broad area of Combinatorial Matrix Theory. This thesis aims at studying matrix completion problems for some classes of matrices. More specifically, using graph theoretic methods we study the patterns for partial matrices having the property that the unspecified entries in any partial matrix with the given pattern can always be chosen so that the matrix obtained belongs to the desired class.

0.0.1 Background

The study of matrix completion problems is a distinct area in Combinatorial Matrix Theory. It deals with situations where some information about the entries in a matrix are known and the rest are unknown. The results are useful in the study of Euclidean distance matrices, optimization problems and certain computer science related engineering problems like image enhancement, data transmission, coding, etc.

A real $n \times n$-matrix $B = [b_{ij}]$ is a

(i) *P-matrix (P_0-matrix)* if all principal minors of B are positive (nonnegative).

(ii) *$P_{0,1}$-matrix* if all principal minors of B are nonnegative with all positive diagonal entries.

(iii) *Q-matrix* if for every $k \in \{1, 2, \ldots, n\}$, $S_k(B) > 0$, where $S_k(B)$ denotes the sum of all $k \times k$ principal minors of B.

(iv) P_0^+*-matrix* if for every $k \in \{1, 2, \ldots, n\}$, all the $k \times k$ principal minors are nonnegative and at least one principal minor of each order is positive.

A *partial matrix* is a matrix in which some entries are specified and others are not. A *completion of a partial matrix* is a specific choice of values for the unspecified entries so that a conventional matrix is obtained. A *pattern* for $n \times n$ partial matrices is a list of positions of an $n \times n$ matrix, i.e., a subset of $\langle n \rangle \times \langle n \rangle$ where $\langle n \rangle = \{1, 2, \ldots, n\}$.

A partial matrix M *specifies* a pattern if the positions of the specified entries in M are exactly those listed in the pattern. For a particular type of matrices, the matrix completion problem for patterns asks which patterns of positions have the property that any partial matrix of this type that specifies the pattern can be completed to a matrix of desired type. For example, the positive definite matrix completion problem asks which patterns have the property that any partial positive definite matrix specifying the pattern (cf. Definition 1.1.13) can be completed to a positive definite matrix.

Matrix completion problems are often answered effectively using graph theoretic techniques. The *digraph associated to* a pattern Λ of $n \times n$ matrices is the digraph D (possibly with loops) with the vertex set $\langle n \rangle$ and the arc set Λ. The pattern Λ is said to be *positionally symmetric* if $(j, i) \in \Lambda$ whenever $(i, j) \in \Lambda$. If Λ is positionally symmetric and contains all diagonal positions, the *graph associated to* Λ is the graph G with the vertex set $\langle n \rangle$ and the edge set consisting of $\{i, j\}$, where $(i, j) \in \Lambda, i \neq j$.

In 1975 Burg introduced a matrix completion problem in his PhD thesis [4], where he studied completions of partial positive definite matrices in the geophysical point of view. Dym and Gohberg [11] solved the positive definite completion problem for tridiagonal patterns of partial matrices. Grone *et al.* [13] completely solved the positive definite completion problem of partial Hermitian matrices in 1984. Since a positive definite matrix is symmetric, in a natural way graphs were considered as patterns for partial matrices, and the following result was obtained:

Theorem 0.0.1. [13] *A positionally symmetric pattern containing all diagonal positions has positive definite completion if and only if the graph associated to it is chordal.*

In 1990, Johnson [26] provided a survey of initial developments of matrix completion problems. These papers led to a large number of results on matrix completion problems for several classes of matrices including N-matrices [1], symmetric N-matrices [2], P-matrices [9; 27], P_0-matrices [7], (weakly) sign symmetric P-matrices [8], (weakly) sign symmetric P_0 matrices [12], M-matrices [19], Symmetric M and inverse M-matrices [18; 20; 22; 28; 29], totally nonpositive matrices [3] and their subclasses under different positional and sign symmetry conditions.

Johnson and Kroschel initiated the study of matrix completion for the classes of P and P_0-matrices in 1996 [27]. They showed that any pattern for 3×3 matrices and any positionally symmetric pattern for matrices have P-completion. In particular, the patterns whose graphs are the 4-cycle and the double triangle have P-completion. However, in their paper, Johnson and Kroschel showed that the pattern specifying the double triangle does not have P_0-completion. A few years later, DeAlba and Hogben showed in 2000 the following.

Theorem 0.0.2. [9] *A pattern that contains all diagonal positions and whose pattern digraph is a minimally chordal symmetric-Hamiltonian digraph with at least four vertices does not have P-completion.*

Further, they classified 207 of 218 patterns for 4×4 matrices as to P-completion.

Several authors have studied completion problems for classes of matrices with certain sign conditions. Some of the sign conditions that are considered are as follows.

Definition 0.0.3. Let $A = [a_{ij}]$ be a $n \times n$ real matrix. Then A is

(i) *positive* (*nonnegative*) if $a_{ij} > 0$ ($a_{ij} \geq 0$) for all $i, j \in \langle n\rangle$.

(ii) *sign symmetric* (*ss*) if for $i, j \in \langle n\rangle$, either $a_{ij} = 0 = a_{ji}$ or $a_{ij}a_{ji} > 0$.

(iii) *weakly sign symmetric* (*wss*) if $a_{ij}a_{ji} \geq 0$ for all $i, j \in \langle n\rangle$.

For example, the completion problem for the class of (weakly) sign symmetric P-matrices was studied by DeAlba *et al.* in 2003 [8], and that for the class of (weakly) sign symmetric P_0 matrices by Fallat *et al.* in 2002 [12]. The following interesting result appeared in the latter paper.

Theorem 0.0.4. [12; 21] *Let* Π *be one of the classes* P, P_0, *(weakly) sign-symmetric* P, *and (weakly) sign symmetric* $P_{0,1}$*-matrices. Every undirected 1-chordal graph (cf. Section 1.6, Chapter 1) has* Π*-completion.*

The implication of the result is that for problem of Π-completion of symmetric digraphs it is enough to consider such digraphs without a cut vertex.

In 2001, L. Hogben unified and generalized many of the earlier results of the matrix completion problems in her survey paper [21]. Further, following interesting and useful result was obtained.

Theorem 0.0.5. [21] *Let* Π *be one of the classes* P_0, $P_{0,1}$, P, *nonnegative* $P/P_0/P_{0,1}$, *positive* P*-matrices. A digraph that includes all diagonal positions has* Π*-completion if and only if every nonseparable strongly connected induced subdigraph of the digraph has* Π*-completion.*

Therefore, for studying the problem of Π-completion digraphs that include all loops, it is enough to consider such digraphs which are strongly connected and without a cut vertex.

In [7], Choi *et al.* showed that every asymmetric digraphs has P_0-completion and also classified the digraphs of order 4 as to P_0-completion.

Theorem 0.0.6. [7] *Every asymmetric partial* P_0*-matrix (*P*-matrix) has* P_0*-completion (*P*-completion).*

For nonnegative P_0-matrix completion problem, in [6] Choi *et al.* proved that all patterns for 2×2 and 3×3 matrices that include all diagonal positions have nonnegative P_0-matrix completion. They also proved that a pattern for 4×4 matrices that includes all diagonal positions has nonnegative P_0-matrix completion if and only if it does not contain a 4-cycle or is the complete digraph on 4-vertices [6].

Now, for Π as any one of P, positive P, nonnegative P, P_0, nonnegative P_0 and several other classes of matrices the following result holds.

Theorem 0.0.7. [21] *If a digraph D has Π-completion, then every induced subdigraph of D has Π-completion.*

Thus, if a digraph D contains an induced subdigraph which does not have Π-completion, then D does not have Π-completion.

Theorem 0.0.8. [21] *A digraph D has Π-completion if and only if each of the components (strong components) of a digraph D has Π-completion.*

Thus, the Π-matrix completion problem is reduced to studying strongly connected digraphs for Π-completion.

Theorem 0.0.9. [24] *Let D be a digraph with α as the set of vertices that include loops. Then, D has (positive/nonnegative/ss/wss) P-completion if and only if the subdigraph of D induced by α has (positive/nonnegative/ss/wss) P-completion.*

The implication of the result is that to study the completion problems for these classes, it is enough to consider that the digraph under consideration include all loops.

In 2009, DeAlba *et al.* [10] considered the Q-matrix completion problem. Due to the fact that the property of being a Q-matrix is not inherited by principal submatrices, it was observed that the Q-matrix completion problem is different from the earlier ones and attracts special attention. In their paper, they showed that for Q-matrix completion of a digraph, stratification or weak stratification (cf. Definition 1.3.1, Chapter 1) of its complement is necessary and positive signing (cf. Subsection 1.4, Chapter 1) of all cycles of the stratified complement of a digraph is sufficient. They also classified all digraphs as to Q-matrix completion up to order 4 in their paper.

Theorem 0.0.10. [10] *Let D be a digraph of order n that omits at least one loop.*

(i) *If D has Q-completion, then $\overline{D}$ is stratified.*

(ii) *If* $n \leq 4$ *and* $\overline{D}$ *is stratified, then* D *has* Q*-completion.*

Theorem 0.0.11. [10] *Let* $D \neq K_n$ *be an order* n *digraph that includes all loops and has* Q*-completion. Then for each* $k = 2, 3, \ldots, n$*, either,*

(i) $\overline{D}$ *has a permutation digraph of order* k*, or*

(ii) *for each* $v \in V(D)$*,* $\overline{D} - v$ *has a permutation digraph of order* $k - 1$*.*

Theorem 0.0.12. [10] *Let* D *be a digraph such that* $\overline{D}$ *is stratified. If it is possible to sign the arcs of* $\overline{D}$ *so that the sign of every cycle is* $+$*, then* D *has* Q*-completion.*

For an extensive survey of matrix completion problems, we refer the relevant sections in *Handbook of Linear Algebra* [24] published by Chapman and Hall/CRC Press.

0.0.2 Organization of thesis

After presenting the necessary fundamentals in Chapter 1, the main work of the thesis has been organized into four chapters as described below:

Chapter 2: The positive Q-matrix completion problem

Chapter 3: The nonnegative Q-matrix completion problem

Chapter 4: The P_0^+-matrix completion problem

Chapter 5: Comparison between completion problems.

Chapter 2. In [10], DeAlba, Hogben and Sarma studied the Q-matrix completion problem. The property of being a Q-matrix is preserved under similarity and transposition. However, the property is not inherited by the principal submatrices. In fact, the Q-matrix completion problem is significantly different from other completion problems involving P-matrix classes.

In this chapter, we consider Q-matrices with positive entries, and the positive Q-matrix completion problem. Although a positive Q-matrix is a Q-matrix, we have found that the results for the positive Q-matrix completion problem are substantially different from those for the Q-matrix completion problem.

We find that for the positive Q-completion Theorem 0.0.8 is only partially true. More precisely, if each of the components (strong components) of a digraph D has positive Q-completion, then D has positive Q-completion; but the converse is not true.

We discuss some necessary and sufficient conditions for a digraph to have positive Q-completion. We find that a digraph without a cycle of even length has positive Q-completion, although the converse is not true in general. We find a sufficient condition for positive Q-completion of digraphs which have cycles of even length. Secondly, it is shown that a digraph which includes all loops and contains a two-cycle does not have positive Q-completion. As a consequence, any graph with at least one edge does not have positive Q-completion. This result naturally brings the digraphs whose complements are tournaments to our attention. We characterize all such digraphs which have positive Q-completion. Further, using the conditions obtained, we characterize the digraphs of order at most four that have positive Q-completion.

Chapter 3. In this chapter, we consider Q-matrices with nonnegative entries and study the nonnegative Q-matrix completion problem. Though every positive Q-matrix is a nonnegative Q-matrix, it is found that the results for the nonnegative Q-matrix completion problem are quite different from those for the positive Q-matrix completion problem.

For the nonnegative Q-completion problem, the assertions of Theorem 0.0.8 holds partially. For a partial nonnegative Q-matrix M specifying a digraph D, we find that nonnegative Q-completion of the partial submatrices of M corresponding to the strong components (and so components) of D give rise to a nonnegative Q-completion of M. However, the converse of the result is not true.

For the Q-matrix completion problem, the authors of [10] showed that for a digraph to have Q-completion stratification or weak stratification of the complement of the digraph is necessary (cf. Theorem 0.0.10 and 0.0.11). We find that positive stratification (cf. Definition 1.3.1, Chapter 1) is a necessary, though not sufficient, condition for a digraph to have nonnegative Q-completion. As a consequence, we show that for having nonnegative Q-completion a digraph of order ≥ 2 must omit at least two loops. We present some other necessary conditions and sufficient conditions for a digraph to have nonnegative Q-completion. Further, we find a sufficient condition for a digraph to have nonnegative Q-completion with the help of weight functions on the arcs of its complement. Finally, based on these results, we provide some classifications of the digraphs of order up to 4 as to nonnegative Q-completion.

Chapter 4. In this chapter, we study the P_0^+-matrix completion problem. Every P-matrix is a P_0^+-matrix, and the class of P_0^+-matrices is the intersection of the classes of P_0 and Q-matrices. However, unlike P and P_0-matrices, the property of being a P_0^+-matrix is not inherited by principal submatrices. It is observed that the P_0^+-matrix completion problem is quite different from the completion problems for the classes of Q, the P and P_0-matrices. For example, the assertion of Theorem 0.0.8 does not hold with Π as the class of P_0^+-matrices, i.e., even if each of the components (strong components) of a digraph D has P_0^+-completion, D may fail to have P_0^+-completion.

In Theorem 0.0.4, it is asserted that a 1-chordal graph with two maximal cliques has P_0, $P_{0,1}$ and P-completions. Some standard techniques, namely, zero completion, asymmetric completion, zeros in the inverse completion (cf. Section 1.6, Chapter 1) instantly give rise to desired completions of a partial matrix specifying a 1-chordal graph. It has been found that these techniques fail to provide P_o^+-completions of a partial P_0^+-matrix specifying a 1-chordal graph. In fact, it is found that a 1-chordal graph with two maximal cliques does not have P_0^+-completion. In the discussion in Chapter 5, we show that 1-chordal graphs do not have positive and nonnegative Q-completions as well.

Stratification or weak stratification of the complement of a digraph is seen to be a necessary condition, but not a sufficient condition, for a digraph to have P_0^+-completion. It is shown that for a digraph to have P_0^+-completion, every induced subdigraph of the digraph must have P_0-completion. Beside these, certain other necessary conditions and sufficient conditions for a digraph to have P_0^+-completion are obtained. Finally, based on these conditions all digraphs of order at most 4 have been classified as to P_0^+-completion.

Chapter 5. In this chapter, we examine the relationship between certain matrix completion problems including the completion problems for the classes of positive Q, nonnegative Q and P_0^+-matrices. Based on the results obtained in previous three chapters, the similarities and dissimilarities between the completion problems for the classes of positive Q, nonnegative Q and P_0^+-matrices are investigated. Though they are found to be mostly unrelated, it is found that nonnegative Q-completion of a digraph implies positive Q-completion, though the converse is not true in general.

Further, the completion problems for above three classes are compared with those for the classes of P, P_0 and Q-matrices. It is seen that though they share some common results, the former three completion problems have features mostly distinct from the P, and P_0-matrix completion problems. However, we find that a digraph has P-completion, if it has P_0^+-completion, though the converse is not true in general.

It has been seen that though each of the classes of positive Q, nonnegative Q and P_0^+-matrices is a subclass of the class of Q-matrices, their completion problems are distinct in nature from the Q-matrix completion problem. We emphasize the similarities these completion problems have with the Q-matrix completion problem, and discuss the dissimilarities with analysis and examples.

Epilogue. In the present thesis, we have studied matrix completion problems for three subclasses of the class of Q-matrices, namely, the classes of positive Q, nonnegative Q and P_0^+-matrices. Though we are successful in understanding the problems to some extent, we are far from the complete solutions to the problems. Like several subclasses of P_0-matrices, a characterization of a digraphs having completion for any of the classes we have considered is still evading. Further efforts are required to resolve the combinatorial problems of matrix completion for these classes fully.

The results presented in chapters 2, 3 and 4 have been communicated to journals for publication.

1

Rudiments of Matrix Completion Problems

In this chapter, we present the fundamentals on matrix completion problems concerning the thesis. The materials of the chapter are standard in the present thesis area. However, in order to fix the definitions and notations for the thesis, we present the necessary materials. Section 1.2 is devoted to the concepts of graphs and digraphs which are frequently used in matrix completion problems. We formally introduce partial matrices and their completions, and matrix completion problems in Sections 1.1 and 1.5, respectively. In Section 1.3, we present permutation digraphs and their relations to determinants of matrices. Definitions of weighted and signed digraphs are provided in Section 1.4. Further, chordal graphs and their completions are briefly presented in Section 1.6. One may refer to the survey article [21] and the book chapter [24] for more materials on matrix completion problems.

1.1 Partial matrices and their completions

All matrices considered in this thesis are real square matrices.

1.1.1 Some classes of matrices

Let A be an $n \times n$ matrix and α be a nonempty subset of $\langle n \rangle = \{1, 2, \ldots, n\}$. The *principal submatrix* $A[\alpha]$ of an $n \times n$-matrix A is obtained from A by deleting all rows and columns not in α. A *principal minor* is the determinant of a principal submatrix.

Definition 1.1.1. A P-matrix (P_0-matrix) is a matrix for which every principal minor is positive (nonnegative). A $P_{0,1}$-matrix is a P_0-matrix with positive diagonal entries.

Clearly, a P-matrix is a $P_{0,1}$-matrix and a $P_{0,1}$-matrix is a P_0-matrix.

Example 1.1.2. Consider the matrices

$$B_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; B_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Clearly, B_1 is a P-matrix, B_2 is a $P_{0,1}$-matrix but not a P-matrix, and B_3 is a P_0-matrix but not a $P_{0,1}$-matrix.

Definition 1.1.3. A class Π of matrices is said to satisfy the *hereditary property* if for each Π-matrix B, every principal submatrix of B is also a Π-matrix.

It is easy to see that the classes of P, P_0 and $P_{0,1}$ matrices satisfy the hereditary property.

Definition 1.1.4. A real $n \times n$ matrix B is a *Q-matrix* if for every $k \in \{1, 2, \ldots, n\}$, $S_k(B) > 0$, where $S_k(B)$ denotes the sum of all $k \times k$ principal minors of B.

In 1983, Hershkowitz *et al.* [15] introduced Q-matrices. Spectral properties of Q-matrices and some related classes are discussed in [15–17].

Definition 1.1.5. A real $n \times n$ matrix B_1 is a P_0^+-*matrix* if for every $k \in \{1, 2, \ldots, n\}$, all the $k \times k$ principal minors are nonnegative and at least one principal minor of each order is positive.

Clearly, a P_0^+-matrix is a P_0-matrix as well as a Q-matrix. Moreover, a P-matrix is a P_0^+-matrix, but not conversely. Unlike the classes of P and P_0-matrices, the classes of the Q and P_0^+-matrices do not satisfy the hereditary property, which is seen in the following example.

Example 1.1.6. Consider the matrices

$$B_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B_3 = \begin{bmatrix} 2 & 1 \\ -4 & -1 \end{bmatrix}, \; B_4 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The matrix B_1 is a P_0^+-matrix but not a P matrix, B_2 is a P_0-matrix but not a P_0^+-matrix, B_3 is a Q-matrix but not a P_0^+-matrix. Further, B_4 is a P_0^+-matrix and therefore a Q-matrix. However, the principal submatrix $B_4[\{1, 2\}]$ is not a Q-matrix, and therefore not a P_0^+-matrix. This shows that the classes of Q and P_0^+-matrices do not satisfy the hereditary property.

Definition 1.1.7. A class Π of matrices is *closed under permutation similarity,* if whenever B is a Π-matrix and P is a permutation matrix, $P^T BP$ is a Π-matrix.

It is easy to see that the classes of P, P_0, $P_{0,1}$, Q and P_0^+-matrices are closed under permutation similarity.

1.1.2 Partial matrices

Definition 1.1.8. A *partial matrix* is a square array of real numbers in which some entries are specified while others are free to be chosen. A partial matrix $M = [m_{ij}]$ is *fully specified* if all entries m_{ij} of M are specified, i.e., if M is a conventional matrix.

We usually label the unspecified entries of a partial matrix by x_{ij}, ? or $*$.

Definition 1.1.9. A partial matrix $M = [m_{ij}]$ is *combinatorially symmetric* (or *positionally symmetric*) if m_{ji} is specified whenever m_{ij} is specified.

Example 1.1.10. For the partial matrix

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & ? \end{bmatrix}$$

the $(3, 3)$ entry is unspecified. It is clear that M is combinatorially symmetric.

Definition 1.1.11. Let α be a nonempty subset of $\langle n \rangle = \{1, 2, \ldots, n\}$. The *principal partial submatrix* $M[\alpha]$ of an $n \times n$ partial matrix M is the $|\alpha| \times |\alpha|$ partial matrix whose specified entries are those of M indexed by α. Here $|\alpha|$ denotes the size of α. If $M[\alpha]$ is fully specified, we call the determinant of $M[\alpha]$ a *fully specified principal minor* of M.

Example 1.1.12. Consider the partial matrix M in Example 1.1.10. For $\alpha = \{1, 2\}$

$$M[\alpha] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is fully specified and the corresponding principal minor is 0.

For a given class Π of matrices, a *partial* Π*-matrix* is a partial matrix for which the specified entries fulfill the requirements of a Π-matrix.

Definition 1.1.13. Let M be a $n \times n$ partial matrix. Then M is a

(i) *partial P-matrix (resp. P_0-matrix)*, if every fully specified principal minor is positive (resp. nonnegative).

(ii) *partial positive definite matrix*, if it is a partial P-matrix and specified entries at symmetric positions are equal.

(iii) *partial $P_{0,1}$-matrix*, if M is a partial P_0-matrix in which all specified diagonal entries are positive.

(iv) *partial Q-matrix*, if $S_k(M) > 0$ for every $k = 1, 2, \ldots, n$ for which all $k \times k$ principal submatrices are fully specified.

(v) *partial P_0^+-matrix*, if every fully specified principal minor of M is nonnegative and $S_k(M) > 0$ for every $k = 1, 2, \ldots, n$ for which all $k \times k$ principal submatrices are fully specified.

(vi) partial *positive (resp. nonnegative)-matrix*, if all specified entries of M are positive (nonnegative).

Example 1.1.14. Consider the partial matrices

$$M_1 = \begin{bmatrix} 1 & 1 \\ ? & 1 \end{bmatrix}, \ M_2 = \begin{bmatrix} 0 & 2 \\ ? & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 2 & 1 \\ ? & -1 \end{bmatrix}$$

Here, M_1 is a partial P-matrix (and therefore a partial P_0, $P_{0,1}$, P_0^+, Q-matrix) with fully specified principal submatrices $M_1[\{1\}]$, $M_1[\{2\}]$ giving minor $1 > 0$. Similarly, M_2 is a partial P_0^+-matrix (and therefore a partial P_0, Q-matrix) with fully specified principal submatrices $M_2[\{1\}]$, $M_2[\{2\}]$ giving nonnegative minors. However, M_2 is neither a partial P nor $P_{0,1}$-matrix, since the diagonal entries are not all positive. Finally, M_3 is a partial Q-matrix, but not a partial P, P_0, $P_{0,1}$, P_0^+-matrix.

Definition 1.1.15. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A Π-*completion* of a partial Π-matrix M is a completion of M which is a Π-matrix.

Example 1.1.16. Consider the matrices

$$B_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}, \ B_3 = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$$

The matrix B_1 is a P, P_0, $P_{0,1}$, P_0^+ and Q-completion of the partial matrix M_1 in Example 1.1.14. Similarly, B_2 is a P_0^+, P_0 and Q-completion of M_2, and B_3 is a Q-completion of M_3.

1.2 Graphs and digraphs

We list here the graph theoretic notions which will be used in our discussions. Most of the terms can be found in any standard reference, for example, in [5] and [14]. However, some variation to the standard use of the notion of a digraph is considered as has been used in recent studies of matrix completions.

1.2.1 Graphs

A *graph* $G = (V, E)$ is a finite nonempty set V of objects, called *vertices*, and a set E of unordered pairs $\{u, v\}$ of vertices, called *edges*. The *order* of G, denoted by $|G|$, is the number of vertices of G. The vertex set and the edge set of G are also denoted by $V(G)$ and $E(G)$ respectively. If $\{u, v\}$ is an edge of G, then we say that u and v are *adjacent* in G and $\{u, v\}$ is incident with both u and v.

A graph H is a *subgraph* of the graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ (note that $\{u, v\} \in E(H)$ requires $u, v \in V(H)$, since H is a graph). The subgraph H is an *induced subgraph* (*induced by* $V(H)$) if $E(H) = (V(H) \times V(H)) \cap E(G)$, and is a *spanning subgraph* if $V(H) = V(G)$.

For vertices u, v in a graph G, a u-v *path* P is a subgraph of G whose distinct vertices and edges can be written in an alternating sequence:

$$u = v_1\{v_1, v_2\}v_2\{v_2, v_3\}v_3 \cdots v_{k-1}\{v_{k-1}, v_k\}v_k = v.$$

Further, if $\{v, u\}$ is an edge in G, then the subgraph P together with the edge $\{v, u\}$ is a *cycle* of *length* k or a k-*cycle* in G.

The graph G is said to be *connected* if for every pair u, v of distinct vertices, G contains a u-v path. The maximal connected subgraphs of G are called *components* of G.

A *chord* of the cycle $v_1\{v_1, v_2\}v_2\{v_2, v_3\} \cdots \{v_{k-1}, v_k\}v_k\{v_k, v_1\}$ is an edge $\{v_s, v_t\}$ not in the cycle (with $1 \leq s, t \leq k$). A graph is *chordal* if every cycle of length ≥ 4 has a chord.

1.2.2 Directed graphs

A *directed graph* or *digraph* D is a pair (V, A) where V is a finite nonempty set of objects, called *vertices*, and A a set of ordered pairs of vertices, called *arcs* or *directed edges*. The vertex set and the arc set of D are denoted by $V(D)$ and $A(D)$ respectively.

Note that the definition allows an arc $x = (u, u)$ in the arc set of a digraph D, which is called a loop at the vertex u. This is in contrast to the classical notion of a digraph for which an arc is an ordered pair of distinct vertices. Loops are usually admitted in a so-called *pseudodigraph* for which the collection of arcs is a multiset of ordered pairs of vertices, not necessarily distinct (see [5]). Several works on matrix completion problems used *marked digraphs*, i.e., digraphs with some vertices marked in stead of considering loops at those vertices (see [7; 21]). The current definition is in use in some recent works on matrix completion problems including [10].

Sometimes, we simply write $v \in D$ (resp. $(u, v) \in D$) to mean $v \in V(D)$ (resp. $(u, v) \in A(D)$). The *order* of D, denoted by $|D|$, is the number of vertices of D. An arc $x = (u, u)$ in D is called a *loop* in D, and we say that D *includes* a loop at the vertex u. If $u \neq v$ and $x = (u, v)$ is an arc in D, we say that x is *incident* with u and v; u is *adjacent to* v; and v is *adjacent from* u. The *outdegree* (resp. *indegree*) of a vertex v in D is the number of vertices of D adjacent from (resp. to) v.

It is customary to represent a digraph by a diagram with nodes representing the vertices and directed line segments (arcs) representing the arcs of the digraph.

A digraph D_1 is a *subdigraph* of the digraph D if $V(D_1) \subseteq V(D)$, $A(D_1) \subseteq A(D)$. The subdigraph D_1 is an *induced subdigraph* (*induced by* $V(D_1)$) if $A(D_1) = (V(D_1) \times V(D_1)) \cap A(D)$, and is a *spanning subdigraph* if $V(D_1) = V(D)$. The *complement of a digraph* D is the digraph $\overline{D}$, where $V(\overline{D}) = V(D)$ and $(v, w) \in A(\overline{D})$ if and only if $(v, w) \notin A(D)$. If S is a set of arcs in $\overline{D}$, then $D + S$ denotes the digraph obtained from D by adding the arcs in S. Further, for $v \in V(D)$, $D - v$ denotes the subdigraph of D induced by $V(D) \setminus \{v\}$.

For vertices u, v in a digraph D, a *u-v path* P is a subdigraph of D whose distinct vertices and arcs can be written in an alternating sequence:

$$u = v_1(v_1, v_2)v_2(v_2, v_3)v_3 \cdots v_{k-1}(v_{k-1}, v_k)v_k = v.$$

Further, if (v, u) is an arc in D, then the subdigraph P together with (v, u) is a *cycle* of *length* k or a k-*cycle* in D. A 1-cycle consists of a vertex v together with a loop at v. For convenience, we denote the cycle as $C = [v_1, v_2, \ldots v_k]$. A digraph having no cycle is said to be *acyclic*. A cycle C is *even* (resp. *odd*) if its length is even (resp. odd).

For a digraph D the *underlying graph* is the graph obtained by replacing each arc (u, v) or pair of arcs (u, v) and (v, u) (if both are present) for distinct vertices u and v with the one edge $\{v, u\}$ (loops are ignored).

Definition 1.2.1. A *chord* of the cycle C is an arc not on C and is incident to two vertices on C. For an even cycle $C = [v_1, \ldots, v_{2k}]$ a chord (v_i, v_j) of C is said to be *odd* if $|i - j|$ is even. In that case, the subdigraph $C + (v_i, v_j)$ contains an odd cycle formed by (v_i, v_j) together with the v_j-v_i path on C.

A digraph D is said to be *chordal*, if every cycle in D of length ≥ 4 has a chord in D. In other words, D is chordal, if and only if the underlying graph of D is chordal.

Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are *isomorphic*, if there is a bijection $\phi : V_1 \to V_2$ such that $A_2 = \{(\phi(u), \phi(v)) : (u, v) \in A_1\}$. An *unlabeled* digraph is an equivalent class of isomorphic digraphs. Choosing any particular member of an unlabeled digraph is referred to as a *labeling* of the unlabeled digraph.

A digraph D is said to be *connected* if the underlying graph of D is connected. The digraph D is said to be *strongly connected* if for every pair u, v of vertices, D contains a u-v path and a v-u path. The maximal connected (resp. strongly connected) subdigraphs of D are called *components* (resp. *strong components*) of D. A *cut-vertex* of a connected digraph is a vertex whose deletion from D disconnects the digraph. A connected digraph is *nonseparable* if it has no cut vertices. A *block* is a maximal nonseparable subdigraph. A *clique* is a complete subdigraph. A *block clique digraph* is a digraph whose blocks are cliques.

1.2.3 Symmetric and asymmetric digraphs

A digraph is said to be *symmetric* if $(u, v) \in D$ implies $(v, u) \in D$. In contrast, D is *asymmetric* if $(u, v) \in D$ implies $(v, u) \notin D$. For a graph $G = (V(G), E(G))$, we define a digraph $D = (V(G), A(D))$ as follows: the arcs (i, j) and (j, i) are in $A(D)$ if and only if $\{i, j\} \in E(G)$. Similarly, for a symmetric digraph $D = (V(D), A(D))$, we define a graph $G = (V(D), E(G))$ where $\{i, j\} \in E(G)$ if and only if the arcs (i, j) and (j, i) are in $A(D)$.

Definition 1.2.2. A *complete symmetric digraph* on n vertices, denoted by K_n, is the digraph having all possible arcs (including all loops). A *complete asymmetric digraph* or a *tournament* is a maximal asymmetric digraph, i.e., a digraph which includes no loop and contains exactly one of (u, v) and (v, u) as an arc for any pair of distinct vertices u and v.

Note that we are using the same notation K_n for the complete graph as well as the complete symmetric digraph of order n. The use of the notation in different situations in the thesis will be clear from the context.

Let T be a tournament with vertex set $\{1, \ldots, n\}$. The sequence $(s_1, s_2, \ldots, s_n)$, where s_i is the outdegree of the vertex i (also called the *score* of i), is called the *outdegree sequence* (or the *score sequence*) of T. A triple $\{i, j, k\}$ of vertices in T which do not form a 3-cycle in T is called a *transitive triple*. The tournament T is *transitive*, if T does not have a 3-cycle.

The following results about tournaments are well-known (for example, see Section 5.2 in [5]).

Proposition 1.2.3. *Let T be a tournament with vertex set $\{1, \ldots, n\}$.*

(a) *If T is strongly connected, then T has a cycle of length k, for $k = 3, \ldots, n$.*

(b) *T is acyclic if and only if it is transitive.*

(c) *T is acyclic if and only if T is isomorphic to the unique tournament with score sequence $(n - 1, \ldots, 1, 0)$.*

Definition 1.2.4. Let D_1 and D_2 be two vertex-disjoint digraphs. We define the *transitive product* of D_1 and D_2 to be the digraph $D_1 \lessdot D_2 = (V, A)$, where $V = V(D_1) \cup V(D_2)$ and $A = A(D_1) \cup A(D_2) \cup \{(i, j) : i \in V(D_1), j \in V(D_2)\}$.

Clearly, $\lessdot$ is an associative, but not commutative, binary operation on the set of all digraphs. It is easy to see that if T_1 and T_2 are tournaments, then so is $T_1 \lessdot T_2$. Further, the subdigraph induced by a transitive triple in a tournament is isomorphic to $K_1 \lessdot K_1 \lessdot K_1$. In general, an acyclic tournament of order n is isomorphic to the transitive product of n copies of K_1. An equivalent of the following result is Theorem 5.7 in [5].

Proposition 1.2.5. *Every tournament is a transitive product of its strong components.*

1.3 Permutation digraphs and determinant of matrices

Let π be a permutation of a nonempty finite set V. The digraph $D_\pi = (V, A_\pi)$, where $A_\pi = \{(v, \pi(v)) : v \in V\}$, is called a *permutation digraph*. Clearly, each component of a permutation digraph is a loop or a cycle. The digraph D_π is said to be *positive* (resp. *negative*) if π is an even permutation (resp. an odd permutation). It is clear that D_π is negative if and only if it has odd number of even cycles.

A *permutation subdigraph* H (of order k) of a digraph D is a permutation digraph that is a subdigraph of D (of order k). Further, H is *positive* (*negative*) if the corresponding permutation is even (odd).

Definition 1.3.1. A digraph D of order n is (*positively*) *stratified* if D has a (positive) permutation subdigraph of order k for every $k = 2, 3, \ldots, n$. A digraph D is *weakly stratified* if for each $k = 2, 3, \ldots, n$, either

(i) D has a permutation digraph of order k, or

(ii) for each $v \in V(D)$, $D - v$ has a permutation digraph of order $k - 1$.

The determinant of an $n \times n$ matrix $B = [b_{ij}]$ is given by

$$\det B = \sum (\mathrm{sgn}\pi) b_{1\pi(1)} \cdots b_{n\pi(n)}, \quad (1.1)$$

where the sum runs over all $n!$ permutations π of $\{1, 2, \ldots, n\}$ and $\mathrm{sgn}\pi$ is $+1$ or -1, according to whether π is an even or an odd permutations (see [25] pp. 8, for example).

For our convenience, we introduce some new terminology and make a useful deduction of (1.1). Let n be a fixed positive integer. For a nonempty subset α of $\{1, \ldots, n\}$, let $S(\alpha)$ denote the set of all permutations of α. For $1 \le k \le n$, let $\mathcal{P}_k$ denote the union of all $S(\alpha)$, where $\alpha \subseteq \{1, \ldots, n\}$ with $|\alpha| = k$. Further, we denote by $\mathcal{P}_k^+$ (resp. $\mathcal{P}_k^-$) the collection of all even (resp. odd) permutations in $\mathcal{P}_k$.

For a permutation $\pi \in S(\alpha)$, we define the *weight* of π in B by $w(\pi, B) = \prod\{b_{i\pi(i)} : i \in \alpha\}$. In view of (1.1), the sum of all $k \times k$ minors of B is given by

$$S_k(B) = \sum_{\pi \in \mathcal{P}_k} (\mathrm{sgn}\pi) w(\pi, B) = \sum_{\pi \in \mathcal{P}_k^+} w(\pi, B) - \sum_{\pi \in \mathcal{P}_k^-} w(\pi, B).$$

If B is a positive matrix, then $S_1(B)$ equals the trace of B, which is positive, and for $2 \le k \le n$, $S_k(B)$ equals the difference of two positive sums:

$$S_k(B) = \sum_{\pi \in \mathcal{P}_k^+} w(\pi, B) - \sum_{\pi \in \mathcal{P}_k^-} w(\pi, B). \quad (1.2)$$

1.4 Weighted and signed digraphs

For a digraph $D = (V, A)$, a *weight function* on D is a real valued function ϕ defined on the arc set A. The triplet (V, A, ϕ) is then called a *weighted digraph*. For $e \in A$, $\phi(e)$ is called the *weight* of e. Further, for any permutation subdigraph P of D, we denote the sum of the weights of the edges on P by $\phi^*(P)$.

A signing of a digraph is an assignment of a sign $+$ or $-$ to each arc of the digraph. The result of signing of a digraph is called a *signed digraph*.

Let D be a signed digraph and C a cycle of length k in D. Let r be the number of arcs in C with sign $-$. The *sign* of C is defined to be the sign of $(-1)^{k+r-1}$. Thus, an even (odd) cycle is of sign $+$, if C has an odd (even) number of arcs with sign $-$. The digraph D is said to be *positively signed*, if every cycle in D is of sign $+$.

1.5 Digraphs and matrix completion problems

In matrix completion problems, a partial matrix is associated with a digraph that describes the positions of the specified entries in the partial matrix. We formally define the relevant terms in this section.

Definition 1.5.1. An $n \times n$ partial matrix $M = [m_{ij}]$ is said to *specify* a digraph $D = (\langle n \rangle, A)$ if for $1 \leq i, j \leq n$, $(i, j) \in A$ if and only if the entry m_{ij} is specified.

Example 1.5.2. An arbitrary partial matrix specifying the digraph D_1 in Figure 1.1 is of the form

$$M = \begin{bmatrix} a_{11} & a_{12} & * \\ * & * & a_{23} \\ * & * & * \end{bmatrix}.$$

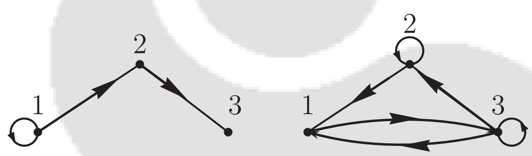


FIGURE 1.1: The digraph D_1 and its complement $\overline{D_1}$

Definition 1.5.3. Let Π be a class of matrices. A digraph D has Π-*completion*, if every partial Π-matrix specifying D can be completed to a Π-matrix. The Π-*matrix completion problem* aims to classify all digraphs which have Π-completion.

Example 1.5.4. Consider the digraph D_1 in Figure 1.1. Since $\overline{D_1}$ is stratified, it follows from Theorem 0.0.10(ii) that D_1 has Q-completion. The partial matrix M in Example 1.5.2 is a partial Q-matrix. One can easily see that the completion

$$B = \begin{bmatrix} a_{11} & a_{12} & t \\ t & t & a_{23} \\ -t & t & t \end{bmatrix}$$

is a Q-matrix for large values of t.

Definition 1.5.5. Let G be a graph of order n and $M = [m_{ij}]$ be an $n \times n$ combinatorially symmetric partial matrix with all diagonal entries specified. Then M

is said to *specify* G, if for $1 \leq i < j \leq n$ $\{i, j\}$, is an edge in M whenever m_{ij} is specified in M.

Remark 1.5.6. Let G be a graph and M be a partial matrix specifying G. Then all diagonal entries of M are specified. Let D be the digraph that is specified by M. Then D has all loops and is a symmetric digraph. It is clear that for any class Π of matrices the graph G has Π-completion if and only if the digraph D has Π-completion.

1.6 Chordal graphs and completion problem

Chordal graphs are useful in the study of positive definite completion problem (see [13]). Consider two graphs G_1 and G_2, each of which contains the clique K_p. If we identify the copy of K_p in G_1 with that in G_2, then the resulting graph is called a *clique sum* of G_1 and G_2 (along the clique K_p).

If G_1 is the clique K_q and G_2 is any chordal graph containing the clique K_p, $p < q$, then the clique sum of G_1 and G_2 along K_p is also chordal. The cliques that are used are the maximal cliques of the resulting chordal graph and the cliques along which the summing takes place are the so called *minimal vertex separator* of the resulting chordal graphs. If the maximum number of vertices in a minimal vertex separator is p, then the chordal graph is p-chordal. We note that a single clique is not p-chordal for any p. We put our interest in 1-chordal graphs with two maximal cliques.

Up to permutation similarity consider a partial matrix M specifying a chordal graph with two maximal cliques as follows:

$$M = \begin{bmatrix} A_{11} & A_{12} & X \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{bmatrix} \tag{1.3}$$

in which the unspecified entries are exactly the entries of X and Y. If A_{22} is invertible, the *zeros in the inverse completion* $\widehat{A}$ of M is obtained by putting

$X = A_{12}A_{22}^{-1}A_{23}$ and $Y = A_{32}A_{22}^{-1}A_{21}$. The *zeros in inverse completion* is an useful technique for completion problems of M, inverse M-matrices. If a partial M-matrix specifies a 1-chordal digraph with two maximal cliques, then the zeros in inverse completion gives a unique completion whose inverse has zeros in all positions corresponding to unspecified entries. Also if

$$M_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, M_2 = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix},$$

are invertible, then

$$\det \widehat{A} = \frac{|M_1||M_2|}{|A_{22}|}. \tag{1.4}$$

The *asymmetric completion* $\widehat{A}$ of M in (1.3) is obtained by putting $X = A_{12}A_{22}^{-1}A_{23}$ and $Y = 0$. Also in case of asymmetric completion, the determinant of $\widehat{A}$ is obtained in (1.4). Finally, if A_{22} is scaler and $A_{22} = 0$, the *zero completion* $\widehat{A}$ of M is obtained by putting $X = Y = 0$. In zero completion, the determinant of $\widehat{A}$ is as follows:

$$\det \widehat{A} = |A_{33}||M_1| + |M_2||A_{11}|.$$

We note that 1-chordal graphs and double triangle graphs (i.e. 2-chordal) are very useful in the study of certain classes of P-matrices. For a class Σ of matrices (e.g., P-, P_0-matrices etc), it is seen that a partial Σ-matrix whose graph of specified entries is a 1-chordal graph with two maximal cliques has Σ completion (see [12; 21]).

2

The Positive Q-matrix Completion Problem

In this chapter, we study the positive Q-matrix completion problem. In Section 2.1, we define the positive Q-matrix completion problem. In Section 2.2, we find some sufficient conditions for a digraph to have positive Q-completions. We discuss some conditions that are necessary for positive Q-completion in Section 2.3. Based on conditions obtained in the previous two sections, we characterize in Section 2.4 the tournaments that have positive Q-completion. Finally, in Section 2.5, we single out the digraphs of order at most four that have positive Q-completion.

2.1 Partial positive Q-matrix and the positive Q-matrix completion problem

A *partial* Q*-matrix* M is a partial matrix such that $S_k(M) > 0$ for every $k = 1, \dots, n$ for which all $k \times k$ principal submatrices are fully specified. A *partial positive* Q*-matrix* is a partial Q-matrix in which all specified entries are positive.

Let M be a partial matrix with all specified entries positive. If all 1×1 principal submatrices (i.e., all diagonal entries) in M are specified, then $S_1(M)$ is the trace of M, which is positive. If all $k \times k$ principal submatrices are fully specified for some $k \geq 2$, then M is fully specified and, therefore, is a positive Q-matrix. Thus, a partial positive Q-matrix is characterized as follows.

Proposition 2.1.1. *Suppose M is a partial matrix in which all specified entries are positive. Then, M is a partial positive Q-matrix if and only if exactly one of the following holds:*

(i) *At least one entry of M is not specified.*

(ii) *All entries of M are specified and M is a Q-matrix.*

Definition 2.1.2. A digraph D is said to have *positive Q-completion* if each partial positive Q-matrix specifying D has a positive Q-completion. The *positive Q-completion problem* aims at studying and classifying all digraphs which have positive Q-completion.

The class of (positive) Q-matrices is closed under permutation similarity. Consequently, if a digraph D has positive Q-completion, then any digraph which is isomorphic to D has positive Q-completion.

It is easy to see that any partial positive matrix M in which all diagonal entries are unspecified has positive Q-completion. A completion can be obtained by choosing sufficiently large values for the unspecified diagonal entries. Let M be a partial positive Q-matrix in which the diagonal entries at (i, i) positions $(i = k+1, \dots, n)$

are unspecified. If $M[\{1,\ldots,k\}]$ is fully specified, M may not have a positive Q-completion. For example, the partial matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & * \end{bmatrix},$$

where $*$ is an unspecified entry, does not have positive Q-completion. Indeed, for any completion B of M, $S_3(B) = \det B = 0$. On the other hand, if $M[\{1,\ldots,k\}]$ has an unspecified entry and has a positive Q-completion, then M has a positive Q-completion. A completion of M can be obtained by choosing sufficiently large values for the unspecified diagonal entries. We list these observations in the following propositions.

Proposition 2.1.3. *A digraph which does not include any loops has positive Q-completion.*

Proposition 2.1.4. *Let D be a digraph and α be the set of vertices at which D includes loops. If the subdigraph of D induced by α is not $K_{|\alpha|}$ and has positive Q-completion, then D has positive Q-completion.*

The following example uses Proposition 2.1.4 for determining whether a digraph has positive Q-completion.

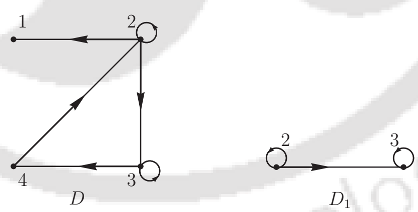


Figure 2.1: The digraph D and its induced subdigraph D_1

Example 2.1.5. Consider the digraph D in Figure 2.1. The set of vertices at which D includes loops is $\alpha = \{2,3\}$. Let D_1 be the subdigraph of D induced by α. Consider the partial positive Q-matrix

$$M = \begin{bmatrix} d_2 & a_{23} \\ ? & d_3 \end{bmatrix},$$

specifying D_1. Clearly, for any $\epsilon > 0$ the completion

$$B(\epsilon) = \begin{bmatrix} d_2 & a_{23} \\ \epsilon & d_3 \end{bmatrix}$$

of M is positive Q-matrix, if $\epsilon < (d_2 d_3)/a_{23}$. Thus, D_α has positive Q-completion. Since D_1 is not complete, in view of Proposition 2.1.4, D has positive Q-completion.

The converse of Proposition 2.1.4 is not true in general which can be seen from the following example. In particular, the assertion of Theorem 0.0.9 is not true if Π denotes the class of positive Q-matrices.

Example 2.1.6. Consider the digraph D in Figure 2.2 and its subdigraph D_1 induced by the set of vertices $\alpha = \{1, 2, 3\}$ at which D has loops. We show that D has positive Q-completion, whereas D_1 does not have. Consider a partial positive Q-matrix

$$M = \begin{bmatrix} d_1 & b_{12} & b_{13} & b_{14} \\ b_{21} & d_2 & * & * \\ * & b_{32} & d_3 & * \\ b_{41} & * & b_{43} & * \end{bmatrix},$$

specifying the digraph D. Here, $*$ denotes an unspecified entry. We show that for any choice of positive values of the specified entries, M has positive Q-completions, though there are occasions when $M[\alpha]$ does not have.

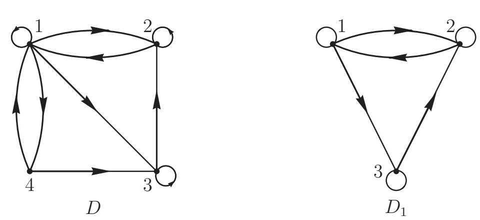


FIGURE 2.2: The digraph D and its induced subdigraph D_1

For $\epsilon > 0$ consider the completion $B(\epsilon)$ of M defined as follows:

$$B(\epsilon) = \begin{bmatrix} d_1 & b_{12} & b_{13} & b_{14} \\ b_{21} & d_2 & \epsilon & \epsilon^2 \\ \epsilon & b_{32} & d_3 & \epsilon \\ b_{41} & \frac{1}{\epsilon^2} & b_{43} & \frac{1}{\epsilon} \end{bmatrix}.$$

Then,

$$\begin{aligned} S_1(B(\epsilon)) &= \frac{1}{\epsilon} + \sum d_i, \\ S_2(B(\epsilon)) &= \frac{1}{\epsilon}\left(\sum d_i\right) + f_1(\epsilon), \\ S_3(B(\epsilon)) &= \frac{b_{21}b_{14}}{\epsilon^2} + \frac{g_0}{\epsilon} + f_2(\epsilon), \\ S_4(B(\epsilon)) &= \frac{d_3 b_{21} b_{14}}{\epsilon^2} + \frac{h_0}{\epsilon} + f_3(\epsilon), \end{aligned}$$

where g_0 and h_0 are constants and $f_i(\epsilon)$ is a polynomial in ϵ of degree i, $i = 1, 2, 3$. Consequently, $B(\epsilon)$ is a positive Q-matrix for sufficiently small ϵ. Next, consider the partial positive Q-matrix

$$M[\alpha] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & y \\ x & 1 & 1 \end{bmatrix}$$

(with x and y as unspecified entries) specifying the digraph D_1. That $M[\alpha]$ does not have positive Q-completion is evident; $S_2(M[\alpha]) = -1 - x - y < 0$ for any completion of $M[\alpha]$.

Proposition 2.1.7. *Suppose M is a partial positive matrix specifying the digraph D. If the partial submatrix of M induced by every strongly connected induced subdigraph of D has positive Q-completion, then M has positive Q-completion.*

Proof. We prove the result for the case when D has two strong components D_1 and D_2. The general result will then follow by induction. By a relabelling of the vertices of D, if required, we have

$$M = \begin{bmatrix} M_{11} & M_{12} \\ X & M_{22} \end{bmatrix},$$

where M_{ii} is a partial positive Q matrix specifying D_i, $i = 1, 2$, and all entries in X are unspecified. By the hypothesis, M_{ii} has a positive Q-completion B_{ii}. For $\epsilon > 0$ consider the completion

$$B(\epsilon) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

by choosing all entries in X as ϵ and all unspecified entries in M_{12} as 1. Then, for $2 \leq k \leq |D|$ we have

$$S_k(B(\epsilon)) = S_k(B_{11}) + S_k(B_{22}) + \sum_{r=1}^{k-1} S_r(B_{11})S_{k-r}(B_{22}) + \epsilon\, h_k(\epsilon),$$

where each h_k is a polynomial in ϵ. Here, we mean $S_k(B_{ii}) = 0$ whenever k exceeds the size of B_{ii}. By choosing ϵ sufficiently small, M can be completed to a positive Q matrix. $\square$

Theorem 2.1.8. *Let $D_1, \ldots, D_r$ be the strong components with more than one vertex in a digraph D. If for each i, D_i is not a complete symmetric digraph and has positive Q-completion, then D has positive Q-completion.*

Proof. Let $D_{r+1}, \ldots, D_s$ be the strong components of order 1 in D. If M is a partial positive Q-matrix specifying D, then the partial submatrices of M induced by each strong component D_i is a partial positive Q-matrix. In view of Proposition 2.1.7, M has a positive Q-completion. $\square$

The proof of the following result is similar.

Theorem 2.1.9. *Let $D_1, \ldots, D_r$ be the components with more than one vertex in a digraph D. If for each i, D_i is not a complete symmetric digraph and has positive Q-completion, then D has positive Q-completion.*

The result of Theorem 2.1.8 is not true in general if some of the strong components D_i are complete. In particular, this shows that the assertion of Theorem 0.0.8 is only partially true in case Π denotes the class of positive Q-matrices.

Example 2.1.10. The partial matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ x & y & 1 \end{bmatrix}$$

specifies a digraph D for which the strong component with more than one vertex is induced by the set of vertices $\{1, 2\}$. The strong component has Q-completion, being a complete symmetric digraph. However, any completion of M has determinant zero, and therefore, D does not have positive Q-completion.

Remark 2.1.11. The converse of Theorem 2.1.8 is not true in general. For example, the digraph D in Figure 2.3 has positive Q-completion (see Example 2.2.4), although the strong component D_1 induced by vertices $\{1, 2, 5, 6\}$ does not have positive Q-completion (see Example 2.3.4).

2.2 Sufficient conditions for positive Q-completion of digraphs

Theorem 2.2.1. *A digraph D has positive Q-completion if it does not contain an even cycle.*

Proof. Let M be a partial positive Q-matrix. For a positive real ϵ, let $B(\epsilon)$ be the completion of M obtained by choosing each unspecified diagonal entry as 1 and each unspecified off-diagonal entry as ϵ. Let $1 \le k \le n$. For each $\alpha \subseteq \{1, \ldots, n\}$ of size k, the identity permutation σ of α is in $\mathcal{P}_k^+$. Since all diagonal entries of M are specified, $w(\sigma, B(\epsilon))$ is a product of (positive) specified entries. On the other hand, for each $\pi \in \mathcal{P}_k^-$, $w(\pi, B(\epsilon))$ contains ϵ as a factor, because D does not have even cycles. In view of (1.2), for sufficiently small values of ϵ we have $S_k(B(\epsilon)) > 0$. □

Example 2.2.2. That the digraph D in Figure 2.1 has positive Q-completion follows also from Theorem 2.2.1, since D has no even cycle. On the other hand, consider the digraph D_2 in Figure 2.5 which has an even cycle $[1, 2, 3, 4]$. Observe the following properties in D: $\overline{D_2}$ has the 2-cycle $[1,3]$, where the arc $(1,3)$ is an odd chord of the 4-cycle in $\widehat{D_2} = D_2 + (1,3)$. Also, $\widehat{D_2} = D_2 + (1,3)$ does not have any even cycle with $(1,3)$ as an arc. A digraph with these properties has positive Q-completion which follows from the next theorem.

Theorem 2.2.3. *Let D be a digraph without any 2-cycles and having one or more even cycles. Suppose $\overline{D}$ has a set of 2-cycles $\{c_i = [u_i, v_i] : i = 1, \ldots, r\}$ satisfying the following:*

(i) *The digraph $\widehat{D} = D + \{(u_i, v_i) : 1 \leq i \leq r\}$ does not have any even cycles with one of (u_i, v_i) as an arc.*

(ii) *For each even cycle C of D there is j such that (u_j, v_j) is an odd chord of C in $\widehat{D}$.*

Then D has positive Q-completion.

Proof. Suppose $|D| = n$ and let m be the minimum of the lengths of the even cycles in D. Then, $m \geq 4$ and D does not have an even cycle of length $2, \ldots, m-2$. Let M be a partial positive Q-matrix specifying the digraph D. For a real $x > 1$, let $B(x) = [b_{ij}]$ be the completion of M obtained by putting: $b_{u_i v_i} = x,\ 1 \leq i \leq r$, and $b_{ij} = 1/x^n$ for all other unspecified entries. For $2 \leq k \leq n$, we have

$$S_k(B(x)) = \sum_{\pi \in \mathcal{P}_k^+} w(\pi, B(x)) - \sum_{\pi \in \mathcal{P}_k^-} w(\pi, B(x)). \tag{2.1}$$

Case 1: $2 \leq k \leq m-1$. The digraph $\widehat{D}$ has no even cycle of lengths $2, \ldots, k$. Consequently, for any $\pi \in \mathcal{P}_k^-$ there is an arc $(t, \pi(t))$ in the complement of $\widehat{D}$, and therefore $1/x$ is a factor of $w(\pi, B(x))$, i.e., each term in the second sum in the right side of (2.1) has a factor $1/x$. On the other hand, the identity permutation of each $\alpha \subseteq \{1, \ldots, n\}$ of size k is in $\mathcal{P}_k^+$ and contributes weight $\prod_{i \in \alpha} b_{ii}$, a positive constant, to the first sum in the right side of (2.1). Hence, for sufficiently large values of x we get $S_k(B(x)) > 0$.

Case 2: $m \leq k \leq n$. Suppose $\pi \in \mathcal{P}_k^-$, and that $\pi \in S(\alpha)$, where $|\alpha| = k$. If

$(t, \pi(t))$ is not an arc of $\widehat{D}$ for some $t \in \alpha$, then $b_{t,\pi(t)} = 1/x^n$. Consequently, $1/x$ is a factor of $w(\pi, B(x))$. Suppose $(t, \pi(t)) \in \widehat{D}$ for each $t \in \alpha$. Since π is odd, the cycle decomposition of π has a cycle $(t_1, \ldots, t_\ell)$ of even length. Since $\widehat{D}$ does not have an even cycle with one of (u_i, v_i) as an arc, the entries of $B(x)$ corresponding to the arcs of $C = [t_1, \ldots, t_\ell]$ are all specified, and therefore the product of these entries is a constant β. Consequently, we have $w(\pi, B(x)) = \beta\Big(\prod_{t\in\alpha_1} b_{t,\pi(t)}\Big)$, where $\alpha_1 = \alpha \setminus \{t_1, \ldots, t_\ell\}$. Now, by (ii) in the hypothesis, the cycle C in $\widehat{D}$ has an odd chord (u_i, v_i). Then, (u_i, v_i) produces an odd cycle $C_1 = [s_1, \ldots, s_{\ell'}]$ in $C + (u_i, v_i)$. Consider the permutation $\tilde{\pi}$ of α obtained from π by replacing $(t_1, \ldots, t_\ell)$ in the cycle decomposition of π by $(s_1, \ldots, s_{\ell'}) \prod_{t\in\alpha_2}(t)$, where $\alpha_2 = \alpha_1 \setminus \{s_1, \ldots, s_{\ell'}\}$. Then, $\tilde{\pi} \in \mathcal{P}_k^+$ and $w(\tilde{\pi}, B(x)) = \gamma\Big(\prod_{t\in\alpha_1} b_{t,\pi(t)}\Big)$, where

$$\gamma = \Big(\prod\{b_{ts} : (t,s) \in C_1\}\Big) \cdot \Big(\prod\{b_{tt} : t \in \alpha_2\}\Big).$$

Note that γ has a factor x corresponding to the arc (u_i, v_i) of C_1. Thus, for each $\pi \in \mathcal{P}_k^-$, there is $\tilde{\pi} \in \mathcal{P}_k^+$ such that $w(\tilde{\pi}, B(x))$ has x as a factor with power one higher than the power of x in $w(\pi, B(x))$. We conclude therefore

$$S_k(B(x)) = (\mu x - 1) \sum_{\pi\in\mathcal{P}_k^-} w(\pi, B(x)),$$

where μ is a positive constant. Consequently, for sufficiently large values of x, $S_k(B(x)) > 0$. □

The converse of Theorem 2.2.3 is not true in general which can be seen in the following example.

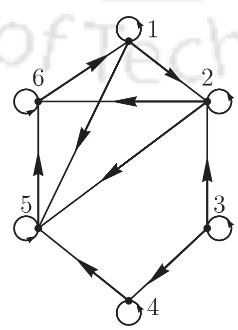


Figure 2.3: A digraph D having positive Q-completion

Example 2.2.4. Consider the digraph D in the Figure 2.3, which contains the even cycle $C = [1, 2, 5, 6]$. There is no 2-cycle in $\overline{D}$ one of whose arcs gives an odd chord for C. Thus, D does not satisfy the conditions of Theorem 2.2.3. However, we show that D has positive Q-completion. Let $M = [a_{ij}]$ be a partial positive Q-matrix specifying D. For $x > 1$ consider the completion

$$B(x) = \begin{bmatrix} d_1 & a_{12} & \epsilon & \epsilon & a_{15} & \epsilon \\ \epsilon & d_2 & \epsilon & \epsilon & a_{25} & a_{26} \\ \epsilon & a_{32} & d_3 & a_{34} & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & d_4 & a_{45} & \epsilon \\ \epsilon & \epsilon & x & \epsilon & d_5 & a_{56} \\ a_{61} & \epsilon & \epsilon & \epsilon & \epsilon & d_6 \end{bmatrix}$$

of M, where $\epsilon = 1/x^2$. Note that the only cycles in $\widehat{D} = D + (5, 3)$ containing $(5, 3)$ as an arc are $[3, 4, 5], [3, 2, 5]$ and $[1, 5, 3, 2, 6]$. Thus, for $\pi \in \mathcal{P}_k,\ 2 \le k \le 6$, x is a factor of $w(\pi, B(x))$ if and only if π contains one of $(3, 4, 5), (3, 2, 5)$ and $(1, 5, 3, 2, 6)$ in its cycle decomposition. All other $w(\pi, B(x))$ can have only a_{ij}, d_i and $1/x$ as factors. Thus, we have

$$\begin{aligned} S_2(B(x)) &= \sum_{i \ne j} d_i d_j - \frac{1}{x^2}(\Sigma a_{ij}) - \frac{5}{x^4} - \frac{1}{x} \\ S_3(B(x)) &= x(a_{34}a_{45} + a_{32}a_{25}) + g_3\left(\frac{1}{x}\right) \\ S_4(B(x)) &= x\big\{(a_{34}a_{45})(d_1 + d_2 + d_6) + (a_{32}a_{25})(d_1 + d_4 + d_6)\big\} + g_4\left(\frac{1}{x}\right) \\ S_5(B(x)) &= x\big\{(a_{34}a_{45})(d_1d_2 + d_2d_6 + d_6d_1) + (a_{32}a_{25})(d_1d_4 + d_4d_6 + d_1d_6) \\ &\quad + a_{32}a_{26}a_{61}a_{15}\big\} + g_5\left(\frac{1}{x}\right) \\ S_6(B(x)) &= x[a_{34}a_{45}d_1d_2d_6 + a_{32}a_{25}d_1d_4d_6 + a_{32}a_{26}a_{61}a_{15}d_4] + g_6\left(\frac{1}{x}\right), \end{aligned}$$

where g_r is a polynomials of degree $2r,\ 3 \le r \le 6$, treating a_{ij}, d_i as arbitrary constants. It is clear that for large values of x, $S_k(B(x)) > 0$, $1 \le k \le 6$.

The property of having Q-completion is inherited by spanning subdigraphs of a digraph [10]. The result is true for positive Q-completions as well.

Theorem 2.2.5. *If a digraph $D \neq K_n$ of order n has positive Q-completion, then any spanning subdigraph of D has positive Q-completion.*

Proof. Let H be a spanning subdigraph of D. Let M_H be a partial positive Q-matrix specifying the digraph H. Consider the partial matrix M_D obtained from M_H by specifying the entries corresponding to $(i, j) \in A(D) \setminus A(H)$ as 1. Since $D \neq K_n$, by Proposition 2.1.1, M_D is a partial positive Q-matrix specifying D. Let B be a positive Q-completion of M_D. Clearly, B is a positive Q-completion of M_H. □

Remark 2.2.6. It is clear that the results of the Theorem 2.2.5 is not true in case $D \neq K_n$ because K_n has positive Q-completion and K_n may have subdigraphs not having positive Q-completion.

2.3 Necessary conditions for positive Q-completion of digraphs

Let D be a digraph having a loop at each of its vertices. If D has positive Q-completion, it follows from Theorem 2.2.5 that any digraph obtained from D by deleting some loops has positive Q-completion. In this subsection, we present some necessary conditions for a digraph that includes all loops to have positive Q-completion.

Theorem 2.3.1. *If a digraph $D \neq K_n$ of order n includes all loops and has positive Q-completion, then D does not have a 2-cycle.*

Proof. Suppose that D has a 2-cycle $[r, s]$. Consider a partial positive Q-matrix $M = [a_{ij}]$ specifying D such that $a_{ii} = 1$ $(1 \leq i \leq n)$ and $a_{rs}a_{sr} > \binom{n}{2}$. Let $B = [b_{ij}]$ be any positive completion of M. Then

$$S_2(B) = \sum_{i \neq j} b_{ii}b_{jj} - \sum_{i \neq j} b_{ij}b_{ji} \; < \; - \sum_{i,j \notin \{r,s\}} b_{ij}b_{ji} \; < \; 0,$$

and, therefore, B is not a Q-matrix. □

Corollary 2.3.2. *If a digraph D of order n includes all loops and has more than $\frac{1}{2}n(n+1)$ arcs, then D does not have positive Q-completion.*

Proof. In case D has more than $\frac{1}{2}n(n+1)$ arcs, then D must have a 2-cycle. □

Theorem 2.3.3. *Let $D \neq K_n$ be a digraph of order n that includes all loops and contains an even cycle. If D has positive Q-completion, then $\overline{D}$ has a 2-cycle.*

Proof. Let $C = [v_1, \ldots, v_r]$ be an even cycle of smallest length in D. For $y > 1$ consider the partial positive Q-matrix $M(y)$ specifying the digraph D with the specified entries as

$$a_{ij} = \begin{cases} y, & \text{if } (i,j) \in A(C) \\ 1, & \text{if } (i,j) \in A(D) \backslash A(C). \end{cases}$$

Let $B(y) = [b_{ij}]$ be a completion of $M(y)$, where $b_{ij} = x_{ij} > 0$ for $(i,j) \notin A(D)$. For $B(y)$ to be a Q-matrix, we must have

$$0 \ < \ S_2(B(y)) = \begin{pmatrix} n \\ 2 \end{pmatrix} - \sum b_{ij}b_{ji},$$

and this forces each of x_{ij} to be bounded above by $\begin{pmatrix} n \\ 2 \end{pmatrix}$. Next consider,

$$S_r(B(y)) = \sum_{\pi \in \mathcal{P}_r^+} w(\pi, B(y)) - \sum_{\pi \in \mathcal{P}_r^-} w(\pi, B(y)).$$

If π is the cyclic permutation $(v_1, \ldots, v_r)$, then $\pi \in \mathcal{P}_r^-$ and $w(\pi, B(y)) = y^r$. Therefore,

$$S_r(B(y)) = -y^r + f(y, x_{ij}),$$

where $f(y, x_{ij})$ is a polynomial and have degree at most $r-2$ in y. Consequently, for a large value of y, $S_r(B(y))$ is negative for any positive choices of x_{ij} within their bounds. For such a value of y, $B(y)$ is not a Q-matrix. □

Example 2.3.4. Let D_1 be the subdigraph of the digraph D in Figure 2.3 induced by the vertices $\{1, 2, 5, 6\}$. Clearly, D_1 includes all loops, has the even cycle $[1, 2, 5, 6]$, and $\overline{D_1}$ does not have a 2-cycle. Thus, in view of Theorem 2.3.3, D_1 does not have positive Q-completion.

Remark 2.3.5. The above example shows that the property of having positive Q-completion is not inherited by induced subdigraphs. Indeed, D_1 is an induced subdigraph of the digraph D which has positive Q-completion (see Example 2.2.4), although D_1 does not have. Another instance to confirm this is the digraph D and its induced subdigraph D_1 in Example 2.1.6.

The conditions in Theorem 2.3.1 and Theorem 2.3.3 are not sufficient for a digraph to have positive Q-completion which can be seen from the following example.

Example 2.3.6. The digraph D in Figure 2.4 includes all loops and does not have a 2-cycle. Further, D contains an even cycle $[1,2,3,4]$, and $\overline{D}$ has a 2-cycle $[6,7]$. We show that D does not have positive Q-completion. For $t > 1$, consider a partial positive Q-matrix $M(t)$ specifying D, where the specified entries a_{ij} are given by:

$$a_{ij} = \begin{cases} t, & \text{if } \ (i,j) \in [1,2,3,4] \\ 1, & \text{otherwise.} \end{cases}$$

Consider an arbitrary completion $B(t)$ of $M(t)$ as follows:

$$B(t) = \begin{bmatrix} 1 & t & 1 & y_{14} & x_{15} & x_{16} & x_{17} \\ y_{21} & 1 & t & 1 & x_{25} & x_{26} & x_{27} \\ x_{31} & y_{32} & 1 & t & x_{35} & x_{36} & x_{37} \\ t & x_{42} & y_{43} & 1 & x_{45} & x_{46} & x_{47} \\ 1 & 1 & 1 & 1 & 1 & 1 & x_{56} \\ 1 & 1 & 1 & 1 & x_{65} & 1 & x \\ 1 & 1 & 1 & 1 & 1 & y & 1 \end{bmatrix}.$$

Then, we have

$$S_2(B(t)) = \binom{7}{2} - \left(t \sum y_{ij} + \sum x_{ij} + xy\right), \tag{2.2}$$

which implies that x_{ij}, y_{ij} and xy are bounded by $\binom{7}{2}$, in case $S_2(B(t))$ is positive. However, one of x and y can have arbitrary large value. Now, summing $w(\pi, B(t))$

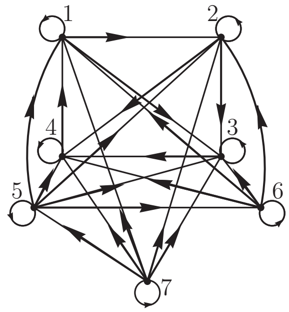


FIGURE 2.4: A digraph D that does not have positive Q-completion

over all π in $\mathcal{P}_4$ and separating the terms having exactly one of x and y as a factor, we get

$$S_4(B(t)) = -t^4 + c_1t^2 + c_2t + c_3 + x(c_4 + c_5t) + y(c_6 + c_7t), \tag{2.3}$$

where c_r are polynomials in x_{ij}, y_{ij} and xy. Further,

$$c_5 = -(x_{26} + x_{36} + x_{46} + x_{16}) < 0 \text{ and } c_7 = -(x_{27} + x_{37} + x_{47} + x_{17}) < 0.$$

Consequently, for large values of t, $S_4(B(t)) < 0$ for any completion $B(t)$ of $M(t)$.

2.4 Positive Q-completion of complements of tournaments

In view of Theorem 2.3.1, if a digraph D that includes all loops has positive Q-completion, then D does not have any 2-cycle. A maximal digraph D which includes all loops and does not have a 2-cycle is the complement of a tournament. Equivalently, the digraph obtained from such a digraph D by deleting the loops is a tournament.

It follows from Theorem 2.2.5 that if the complement $\overline{T}$ of a tournament T has positive Q-completion, then any spanning subdigraph of $\overline{T}$ has positive Q-completion. However, there are digraphs that include all loops and have positive Q-completion, which are not spanning subdigraphs of complements of tournaments having positive Q-completion. For example, the digraph D_2 in Figure 2.5 has positive Q-completion. The only complements of tournaments for which D_2 is a spanning subdigraph are $D_2 + (1,3)$ and $D_2 + (3,1)$, both isomorphic to D_3. However, D_3 does not have positive Q-completion (see Theorem 2.5.1).

In this section we single out all tournaments whose complements have positive Q-completion and provide an enumeration of such tournaments.

Theorem 2.4.1. *The complement of a tournament T has positive Q-completion if and only if T does not have an even cycle.*

Proof. Let T be a tournament. If T does not have an even cycle, then $\overline{T}$ does not have an even cycle, and it follows from Theorem 2.2.1 that $\overline{T}$ has positive Q-completion. On the other hand, if T has an even cycle, then $\overline{T}$ has an even cycle, and in view of Theorem 2.3.3, $\overline{T}$ does not have positive Q-completion, because T has no 2-cycle. □

Corollary 2.4.2. *Let T be a strongly connected tournament of order n. Then $\overline{T}$ has positive Q-completion if and only if $n \leq 3$.*

Proof. Since T is strongly connected, T has a cycle of length k, for $k = 3, \ldots, n$ (cf. Proposition 1.2.3(a)). Thus, T has an even cycle if and only if $n \geq 4$. □

An implication of Theorem 2.4.1 is that the conclusion in Corollary 2.3.2 can be made stronger in case of strongly connected digraphs.

Corollary 2.4.3. *Suppose that D is a strongly connected digraph of order $n \geq 4$. If D includes all loops and has at least $\frac{1}{2}n(n+1)$ arcs, then D does not have positive Q-completion.*

Proof. Suppose that D has exactly $\frac{1}{2}n(n+1)$ arcs. If D has a 2-cycle, then by Theorem 2.3.1, D does not have Q-completion. If D has no 2-cycle, then $\overline{D}$ is a strongly connected tournament. □

Corollary 2.4.4. *The complement of a tournament T has positive Q-completion if and only if T is isomorphic to a digraph $T_1 \lessdot T_2 \lessdot \cdots \lessdot T_r$, where T_i is either K_1 or a 3-cycle, $1 \leq i \leq r$.*

Proof. It is evident that T has no even cycle if and only if the strong components of T are of order at most 3. Since there is no strong component of order 2 in a tournament, the result follows from Proposition 1.2.5 and Theorem 2.4.1. □

An enumeration of unlabelled tournaments whose complements have Q-completion is now immediate.

Corollary 2.4.5. *The number of unlabelled tournaments of order n whose complements have positive Q-completion is given by*

$$t_n = \binom{n}{0} + \binom{n-2}{1} + \binom{n-4}{2} + \cdots + \binom{n-2r}{r},$$

where $r = \lfloor \frac{n}{3} \rfloor$.

Example 2.4.6. For $n = 5$, $t_5 = 4$ and the tournaments are

$$K_1 \lhd K_1 \lhd K_1 \lhd K_1 \lhd K_1,\ K_1 \lhd K_1 \lhd C_3,\ K_1 \lhd C_3 \lhd K_1 \text{ and } C_3 \lhd K_1 \lhd K_1.$$

2.5 Classification of small digraphs as to positive Q-completion

We now use the results obtained in the previous sections to single out all digraphs of order at most 4 which include all loops and have positive Q-completion. The following theorem characterizes all such digraphs.

Theorem 2.5.1. *Let $D \neq K_n$ be a digraph that includes all loops.*

(i) *If $|D| \leq 3$, then D has positive Q-completion if and only if D has no 2-cycle.*

(ii) *Let $|D| = 4$. Then D has positive Q-completion if and only if either D has no even cycle or D is isomorphic to one of the digraphs D_1 and D_2 in Figure 2.5.*

Proof. In view of Theorem 2.3.1, any digraph having a 2-cycle does not have positive Q-completion. On the other hand, by Theorem 2.2.1, a digraph without an even cycle has positive Q-completion. Thus, the first part of the theorem is clear.

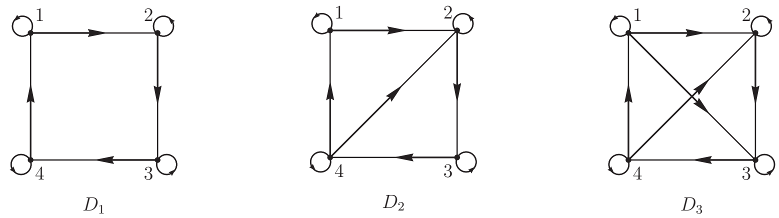


Figure 2.5: Digraphs of order 4 having a 4-cycle and without 2-cycles

Next, it can be seen that the only digraphs of order 4 having a 4-cycle and without any 2-cycle (up to isomorphism) are D_1, D_2 and D_3 in Figure 2.5. In view of Theorem 2.3.3, D_3 does not have positive Q-completion, since $\overline{D_3}$ has no 2-cycle. On the other hand, $\overline{D_2}$ has the 2-cycle $[1,3]$, where the arc $(1,3)$ is an odd chord of the 4-cycle in $D_2 + (1,3)$. Hence, by Theorem 2.2.3, D_2 has positive Q-completion. Finally, D_1 has positive Q-completion, being a spanning subdigraph of D_2. □

It follows from Theorem 2.5.1 that all digraphs of order 1 and 2 that include all loops have positive Q-completion. Further, there are eight unlabeled digraphs of order 3, and forty two unlabeled digraphs of order 4 which include all loops and have positive Q-completion. These unlabeled digraphs are obtained by attaching a loop at each of the vertices of $D_p(q,n)$ in the following list, where $D_p(q,n)$ is the n-th digraph with p vertices and q arcs as listed in Appendix A.

$p = 1$		
$p = 2;$	$q = 0, 1, 2;$	$n = 1$
$p = 3;$	$q = 0, 1, 6;$	$n = 1$
	$q = 2;$	$n = 2, 3, 4$
	$q = 3;$	$n = 2, 3$
$p = 4;$	$q = 0, 1, 12;$	$n = 1$
	$q = 2;$	$n = 2$–5
	$q = 3;$	$n = 4$–13
	$q = 4;$	$n = 16$–27
	$q = 5;$	$n = 29$–38
	$q = 6;$	$n = 46, 47, 48.$

2.6 Conclusion

In this chapter, we have discussed the completion problem of positive Q-matrices which is a subclass of the class of Q-matrices. We have obtained certain necessary conditions and some sufficient conditions for a digraph to have positive Q-completion. Though these conditions helped us to single out the digraphs of order at most 4 as to positive Q-completion, the problem is far from being completely solved. A complete characterization for a digraph to have positive Q-completion is still unresolved. We notice that the nature of the results obtained for the positive Q-matrix completion problem is rather different from that for the Q-matrix completion problem. We will compare the two completion problems in Chapter 5.

3

The Nonnegative Q-Matrix Completion Problem

In this chapter, the nonnegative Q-matrix completion problem is studied. In Section 3.1, we introduce the nonnegative Q-matrix completion problem. In Section 3.2, we find some sufficient conditions for partial matrices to have nonnegative Q-matrix completions. We find some necessary conditions for nonnegative Q-matrix completions in Section 3.3. Using these conditions, we provide some classifications of the digraphs of order up to 4 as to nonnegative Q-completion in Section 3.4.

3.1 Partial nonnegative Q-matrices and the nonnegative Q-matrix completion problem

We recall that a partial Q-matrix M is a partial matrix such that $S_k(M) > 0$ for every $k = 1, \ldots, n$ for which all $k \times k$ principal submatrices are fully specified. A *partial nonnegative Q-matrix* is a partial Q-matrix in which all specified entries are nonnegative.

Let M be a partial nonnegative matrix. If all 1×1 principal submatrices (i.e., all diagonal entries) in M are specified, then their sum $S_1(M)$ (the trace of M) must be positive. If all $k \times k$ principal submatrices are fully specified for some $k \geq 2$, then M is fully specified and, therefore, is a nonnegative Q-matrix. Thus, a partial nonnegative Q-matrix is characterized as follows.

Proposition 3.1.1. *Let M be a partial nonnegative matrix. Then, M is a partial nonnegative Q-matrix if and only if exactly one of the following holds:*

(i) *At least one diagonal entry of M is not specified.*

(ii) *All diagonal entries are specified, at least one diagonal entry is positive and M has an off-diagonal unspecified entry.*

(iii) *All entries of M are specified and M is a Q-matrix.*

A completion B of a partial nonnegative Q-matrix M is called a *nonnegative Q-completion* of M, if B is a nonnegative Q-matrix. Since any matrix which is permutation similar to a Q-matrix is a Q-matrix, it is evident that if a partial nonnegative Q-matrix has a nonnegative Q-completion, so does any partial matrix which is permutation similar to M.

Definition 3.1.2. A digraph D is said to have *nonnegative Q-completion* if each partial nonnegative Q-matrix specifying D has a nonnegative Q-completion. The *nonnegative Q-matrix completion problem* aims at studying and classifying all digraphs D which have nonnegative Q-completion.

It is clear that if a digraph D has (nonnegative, positive) Q-completion, then any digraph which is isomorphic to D has (nonnegative, positive) Q-completion.

Any partial nonnegative matrix M with all diagonal entries unspecified has nonnegative Q-completion. A completion can be obtained by choosing sufficiently large values for the unspecified diagonal entries.

Let $M = [m_{ij}]$ be a partial nonnegative Q-matrix which contains both specified and unspecified diagonal entries. Consider the principal partial submatrix $M[\alpha]$ of M induced by $\alpha = \{i : m_{ii} \text{ is specified}\} \subseteq \langle n \rangle$. In case $M[\alpha]$ is fully specified, M may not have a nonnegative Q-completion, as the following example shows.

Example 3.1.3. Consider the partial nonnegative matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}$$

where $*$ denotes an unspecified entry, Here, $\alpha = \{1, 2\}$ and $M[\alpha]$ is fully specified. That M does not have a nonnegative Q-completion is evident; for any completion B of M, $S_3(B) = \det B = 0$.

For the case when $M[\alpha]$ is not fully specified or itself is a Q-matrix, we have the following.

Theorem 3.1.4. *Let $M = [m_{ij}]$ be a partial nonnegative Q-matrix and $\alpha = \{i : m_{ii} \text{ is specified}\}$. If the principal partial submatrix $M[\alpha]$ of M has a nonnegative Q-completion, then M has a nonnegative Q-completion.*

Proof. If $\alpha = \{1, \ldots, n\}$, then $M = M[\alpha]$ and the result follows from the hypothesis. Otherwise, without any loss of generality, we assume $\alpha = \{1, \ldots, r\}$ for some $1 \leq r \leq n-1$ and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where $M_{11} = M[\alpha]$. Let B_{11} be a nonnegative Q-completion of $M[\alpha]$. Then,

$$M' = \begin{bmatrix} B_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

is a partial nonnegative Q-matrix, since M_{22} has unspecified diagonal entries. For $t > 0$, consider the completion

$$B(t) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

of M' obtained by taking $b_{ii} = t$, $i = r+1, \ldots, n$, and $b_{ij} = 0$ for all other unspecified entries in M'. Since B_{11} is a nonnegative Q-matrix we have $S_i(B_{11}) > 0$ for $1 \leq i \leq r$. Now, for $1 \leq j \leq n$,

$$S_j(B(t)) = \begin{cases} \binom{n-r}{j} t^j + p_j(t), & \text{if } j \leq n-r, \\ S_{j-n+r}(B_{11}) t^{n-r} + p_j(t), & \text{if } j > n-r, \end{cases}$$

where $p_j(t)$ is a polynomial in t of degree at most $j-1$, if $j \leq n-r$, and of degree at most $n-r-1$, if $j > n-r$. As a consequence, for sufficiently large values of t, $S_j(B(t)) > 0$ for $1 \leq j \leq n$ and $B(t)$ is a nonnegative Q-completion of M. $\square$

The converse of Theorem 3.1.4 is not true. The following example shows that a partial nonnegative Q-matrix M may have Q-completion, though $M[\alpha]$ does not have.

Example 3.1.5. Consider the partial matrix,

$$M = \begin{bmatrix} * & b_{12} & b_{13} & b_{14} \\ b_{21} & d_2 & b_{23} & * \\ * & * & d_3 & b_{34} \\ * & b_{42} & * & * \end{bmatrix}, \tag{3.1}$$

where $*$ denotes the unspecified entries. Here, $\alpha = \{2, 3\}$ and $M[\alpha]$ does not have a Q-completion for $d_2 = b_{23} = 0$ and $d_3 = 1$. However, we show that for any choice of nonnegative values of the specified entries b_{ij}, M has nonnegative Q-completions. For $t > 0$ consider the completion

$$B(t) = \begin{bmatrix} t & b_{12} & b_{13} & b_{14} \\ b_{21} & d_2 & b_{23} & t \\ t & t & d_3 & b_{34} \\ t & b_{42} & t & t \end{bmatrix}$$

of M. Then,

$$\begin{aligned} S_1(B(t)) &= 2t + d_2 + d_3, \\ S_2(B(t)) &= t^2 + f_1(t), \\ S_3(B(t)) &= t^3 + f_2(t), \\ S_4(B(t)) &= t^4 + f_3(t), \end{aligned}$$

where $f_i(t)$ is a polynomial in t of degree at most i, $i = 1, 2, 3$. Consequently, $B(t)$ is a nonnegative Q-matrix for sufficiently large t and therefore, M has nonnegative Q-completions.

As in the case of positive Q-matrix completion problem (cf. Theorem 2.1.7), the following result holds for nonnegative Q-completion.

Theorem 3.1.6. *Suppose M is a partial nonnegative matrix specifying the digraph D. If the partial submatrix of M induced by every strongly connected induced subdigraph of D has nonnegative Q-completion, then M has nonnegative Q-completion.*

Proof. We prove the result for the case when D has two strong components H_1 and H_2. The general result will then follow by induction. By a relabelling of the vertices of D, if required, we have

$$M = \begin{bmatrix} M_{11} & M_{12} \\ X & M_{22} \end{bmatrix},$$

where M_{ii} is a partial nonnegative Q-matrix specifying H_i, $i = 1, 2$, and all entries in X are unspecified. By the hypothesis, M_{ii} has a nonnegative Q-completion B_{ii}. Consider the completion

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

by choosing all unspecified entries in M_{12} and X as 0. Then, for $2 \leq k \leq |D|$ we have

$$S_k(B) = S_k(B_{11}) + S_k(B_{22}) + \sum_{r=1}^{k-1} S_r(B_{11}) S_{k-r}(B_{22}) > 0,$$

Here, we mean $S_k(B_{ii}) = 0$ whenever k exceeds the size of B_{ii}. Thus M can be completed to a nonnegative Q-matrix. $\square$

Theorem 3.1.7. *Let $D_1, \ldots, D_r$ be the strong components with more than one vertex in a digraph D. If for each i, D_i is not a complete symmetric digraph and has nonnegative Q-completion, then D has nonnegative Q-completion.*

Proof. Let $D_{r+1}, \ldots, D_s$ be the strong components of order 1 in D. If M is a partial nonnegative Q-matrix specifying D, then the partial submatrices of M induced by each strong component D_i is a partial nonnegative Q-matrix. In view of Proposition 2.1.7, M has a nonnegative Q-completion. $\square$

The proof of the following result is similar.

Theorem 3.1.8. *Let $D_1, \ldots, D_r$ be the components with more than one vertex in a digraph D. If for each i, D_i is not a complete symmetric digraph and has nonnegative Q-completion, then D has nonnegative Q-completion.*

The result of Theorem 3.1.7 is not true if some of the strong components D_i are complete. For example, the partial matrix

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ x & y & 1 \end{bmatrix}$$

specifies a digraph D for which the strong component with more than one vertex is induced by the set of vertices $\{1, 2\}$. The strong component has nonnegative Q-completion, being a complete symmetric digraph. However, since any completion of M has determinant zero, D does not have nonnegative Q-completion. That the converse of Theorem 3.1.6 is not true in general can be seen from the following example.

Example 3.1.9. Consider the digraph D_2 in Figure 3.1. We show that every partial nonnegative Q-matrix specifying the digraph D_2 has nonnegative Q-completion. To see this consider any partial nonnegative Q-matrix $M = [a_{ij}]$ specifying D_2. For

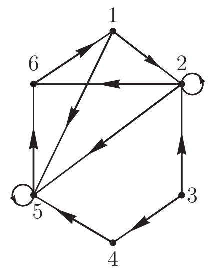


Figure 3.1: A digraph having nonnegative Q-completion

$t > 1$ consider the completion

$$B(t) = \begin{bmatrix} t & a_{12} & 0 & 0 & a_{15} & 0 \\ 0 & a_{22} & t & 0 & a_{25} & a_{26} \\ 0 & a_{32} & t & a_{34} & t & 0 \\ 0 & 0 & 0 & t & a_{45} & 0 \\ 0 & t & 0 & 0 & a_{55} & a_{56} \\ a_{61} & 0 & 0 & 0 & 0 & t \end{bmatrix}$$

of M. Then, we have

$$\begin{aligned} S_1(B(t)) &= 4t + a_{22} + a_{55}, \\ S_2(B(t)) &= 6t^2 + f_1(t), \\ S_3(B(t)) &= 5t^3 + f_2(t), \\ S_4(B(t)) &= 4t^4 + f_3(t), \\ S_5(B(t)) &= 3t^5 + f_4(t), \\ S_6(B(t)) &= t^6 + f_5(t), \end{aligned}$$

where f_i are polynomials of degree at most i in t. It is clear that $S_k(B(t)) > 0$, $1 \le k \le 6$, for sufficiently large values of t. Hence M has nonnegative Q-completions.

On the other hand, the vertices $1, 2, 5$ and 6 induce a strong component H of D_2. Consider the partial nonnegative Q-matrix M_1 specifying H and with all specified entries as zero and

$$B_1 = \begin{bmatrix} x_{11} & 0 & 0 & x_{16} \\ x_{21} & 0 & 0 & 0 \\ x_{51} & x_{52} & 0 & 0 \\ 0 & x_{62} & x_{65} & x_{66} \end{bmatrix},$$

any nonnegative completion of M_1. Then

$$S_4(B_1) = -x_{16}x_{21}x_{52}x_{65} \leq 0,$$

and consequently B_1 is not a nonnegative Q-matrix.

Remark 3.1.10. The above example shows that the property of having nonnegative Q-completion is not inherited by induced subdigraphs. It follows from Example 3.1.9 that the digraph D_2 in Figure 3.1 has nonnegative Q-completion. However, the subdigraph H induced by the vertices $1, 2, 5$ and 6 does not have nonnegative Q-completion.

3.2 Sufficient conditions for nonnegative Q-completion

Proposition 3.2.1. *If a digraph $D \neq K_n$ of order n has nonnegative Q-completion, then any spanning subdigraph of D has nonnegative Q-completion.*

Proof. Let H be a spanning subdigraph of D. Let M_H be a partial nonnegative Q-matrix specifying the digraph H. Consider the partial matrix M_D obtained from M_H by specifying the entries corresponding to $(i, j) \in A(D) \setminus A(H)$ as 1. Since $D \neq K_n$, by Proposition 3.1.1, M_D is a partial nonnegative Q-matrix specifying D. Let B be a nonnegative Q-completion of M_D. Clearly, B is a nonnegative Q-completion of M_H. □

Theorem 3.2.2. *Let D be a digraph of order n. Suppose there is a weight function ϕ on $\overline{D}$ satisfying the following: for each $k \in \{1, \ldots, n\}$, there is a positive permutation subdigraph P_k of order k in $\overline{D}$ such that,*

(i) $\phi^*(P_k) > \phi^*(P)$ *for every negative permutation subdigraph* P *of* $\overline{D}$ *of size* k, *and*

(ii) $\phi^*(P_k) > \sum_{e \in S} \phi(e)$, *for any subset* $S \subseteq A(\overline{D})$ *with* $|S| \leq k-1$.

Then, D *has nonnegative* Q*-completion.*

Proof. Let $M = [m_{ij}]$ be a partial nonnegative Q-matrix specifying D. For $x > 1$ consider the completion $B(x) = [b_{ij}]$ of M defined by,

$$b_{ij} = \begin{cases} m_{ij}, & \text{if } (i,j) \in D \\ x^{\phi(e)}, & \text{if } e = (i,j) \in \overline{D}. \end{cases}$$

For $k \in \{1, \ldots, n\}$, we have

$$S_k(B(x)) = \sum_{P \in \mathcal{P}_k^+} w(P, B(x)) - \sum_{P \in \mathcal{P}_k^-} w(P, B(x)).$$

Since $P_k \in \mathcal{P}_k^+$, $w(P_k, B(x)) = x^{\phi^*(P_k)}$ is a term with a positive coefficient in $S_k(B(x))$. On the other hand, any $P \in \mathcal{P}_k^-$ is one of the following:

(i) a negative permutation subdigraph of $\overline{D}$ of order k,

(ii) a permutation subdigraph of K_n with at most $k-1$ edges from $\overline{D}$.

In view of the properties of ϕ, we have,

$$w(P, B(x)) = \alpha x^t, \text{ where } \alpha \geq 0 \text{ and } t < \phi^*(P_k).$$

Consequently, for large values of x, we have $S_k(B(x)) > 0$ for $k = 1, \ldots, n$ and hence D has nonnegative Q-completion. □

In the following example, we use Theorem 3.2.2 to conclude nonnegative Q-completion of a digraph.

Example 3.2.3. Consider the digraph D_3 in Figure 3.2. Consider the weight function ϕ on the arcs of $\overline{D_3}$ obtained by assigning unit weights to the arcs $(1,4), (4,2)$, $(2,1)$ and to the loops at 3 and 4, and assigning zero weights to all other arcs. The

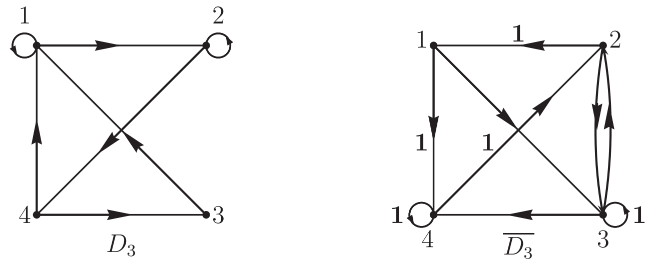


FIGURE 3.2: The digraph D_3 and its complement $\overline{D_3}$

nonzero weights have been marked in bold-faces in $\overline{D_3}$ in Figure 3.2. We choose the following positive permutation digraphs with their respective weights in $\overline{D_3}$.

P_1 - the loop at 3,	$\phi^*(P_1) = 1$
P_2 - the union of the loops at 3 and 4,	$\phi^*(P_2) = 2$
P_3 - the 3-cycle $[1, 4, 2]$,	$\phi^*(P_3) = 3$
P_4 - the union of the loop at 3 and the 3-cycle $[1, 4, 2]$,	$\phi^*(P_4) = 4$.

Further, the only even cycles in $\overline{D_3}$ are the cycles $[2, 3]$ and $[1, 3, 4, 2]$ of weights 0 and 2, respectively. It is clear that ϕ satisfies the conditions (i) and (ii) in Theorem 3.2.2, and therefore, D_3 has nonnegative Q-completion.

Corollary 3.2.4. *Let D be a digraph of order n such that*

(i) *$\overline{D}$ is stratified, and*

(ii) *$\overline{D}$ does not have an even cycle.*

Then D has nonnegative Q-completion.

Proof. Let $2 \leq k \leq n$. Since $\overline{D}$ is stratified, $\overline{D}$ has a permutation subdigraph P_k of order k. Since $\overline{D}$ does not have an even cycle, $\overline{D}$ does not have any negative permutation subdigraph. Thus, P_k is positive. Further, P_2 being even, it must be composed of two loops, and therefore, $\overline{D}$ has a (positive) permutation subdigraph of order 1 as well. We define $\phi(e) = 1$ for each $e \in A(\overline{D})$. Then the weight function ϕ in $\overline{D}$ satisfies conditions (i) and (ii) of Theorem 3.2.2. □

Remark 3.2.5. For each of digraphs which have been classified to have nonnegative Q-completion (cf. Section 3.4) the following exercise was carried out. For such a digraph D of order n, positive permutation subdigraphs P_k of order $k \in \{1, 2, \ldots, n\}$ of $\overline{D}$ have been singled out. A suitable weight function satisfying the required properties could be obtained by assigning larger weights to those arcs of P_k's and smaller weight to the rest. Till now, it could not be settled whether the converse of Theorem 3.2.2 is true. However, it has been found that all digraphs we have examined, including the digraphs of order at most four that have nonnegative Q-completion satisfy the condition of Theorem 3.2.2. That is, for each of these digraphs that have nonnegative Q-completion, there is a weight function on its complement satisfying the conditions (i) and (ii) in Theorem 3.2.2.

3.3 Necessary conditions for nonnegative Q-completion

Proposition 3.3.1. *Let $D \neq K_n$ be a digraph of order n with $n \geq 2$. If D has nonnegative Q-completion, then D omits at least two loops.*

Proof. Suppose D omits at most one loop. Let M be a partial nonnegative Q-matrix specifying the digraph D with all specified entries as 0. Then for any nonnegative completion B of M, $S_2(B) \leq 0$. □

The converse of Proposition 3.3.1 is not true which can be seen from the following example.

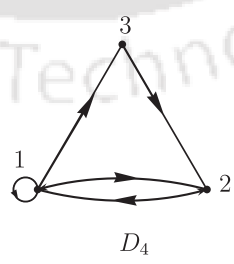


Figure 3.3: A digraph which does not have nonnegative Q-completion

Example 3.3.2. The digraph D_4 in Figure 3.3 omits 2 loops but does not have nonnegative Q-completion. For example,

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_{22} & x_{23} \\ x_{31} & 0 & x_{33} \end{bmatrix}$$

is a partial nonnegative Q-matrix specifying D_4 having no nonnegative Q-completion.

Remark 3.3.3. Let D be a digraph and D_1 be its subdigraph induced by the set of vertices at which D includes loops. Since D_1 includes all loops, it follows from Proposition 3.3.1 that D_1 does not have nonnegative Q-completion, unless it is a complete symmetric digraph. On the other hand, it is not true that if D_1 is complete (and therefore has nonnegative Q-completion), then D has nonnegative Q-completion. It follows from Example 3.1.3 that the digraph D obtained from K_3 by deleting a loop does not have nonnegative Q completion. However, in this case D_1 is the complete symmetric digraph K_2.

A digraph needs to satisfy stronger conditions like positive stratification (cf. 1.3.1) of its complement to have nonnegative Q-completion, as following result shows.

Theorem 3.3.4. *Let $D \neq K_n$ be a digraph of order $n \geq 2$). If D has nonnegative Q-completion, then $\overline{D}$ is positively stratified.*

Proof. Suppose D has nonnegative Q-completion. Assume $\overline{D}$ does not have a positive permutation digraph of order k for some $k \geq 2$. If M is the partial matrix that specifies D with all specified entries zero, and B is a nonnegative completion of M, then all $k \times k$ principal minors of B are nonpositive implying that $S_k(B) \leq 0$. Consequently, B is not a nonnegative Q-matrix. □

Corollary 3.3.5. *Let D be a digraph of order n such that $|A(D)| > n(n-1)$. Then D does not have nonnegative Q-completion.*

Proof. If D has more than $n(n-1)$ arcs (including loops), then $\overline{D}$ has fewer than $n^2 - n(n-1) = n$ arcs. Thus $\overline{D}$ does not contain permutation subdigraph of order n. Therefore, by Theorem 3.3.4, D does not have nonnegative Q-completion. □

The converse of Theorem 3.3.4 is not true in general, which can be seen from the following example.

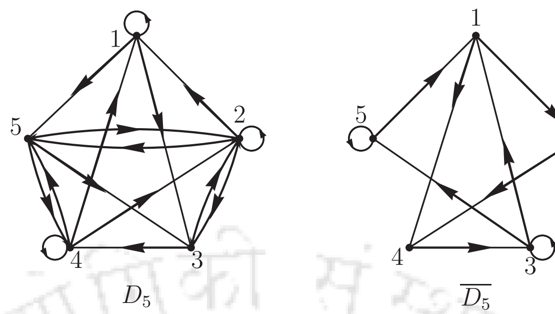


Figure 3.4: A digraph whose complement is positively stratified

Example 3.3.6. Consider the digraph D_5 and its complement in Figure 3.4. It can be easily seen that the complement $\overline{D_5}$ is positively stratified. However, we show that D_5 does not have nonnegative Q-completion. Let M be the partial nonnegative Q-matrix specifying D_5 with all specified entries as zero. Consider a nonnegative completion

$$B = \begin{bmatrix} 0 & x_{12} & 0 & x_{14} & 0 \\ 0 & 0 & 0 & x_{24} & 0 \\ x_{31} & 0 & d_3 & 0 & x_{35} \\ 0 & 0 & x_{43} & 0 & 0 \\ x_{51} & 0 & 0 & 0 & d_5 \end{bmatrix}$$

of M. For B to be a nonnegative Q-matrix, we have

$$S_1(B) = d_3 + d_5 > 0, \tag{3.2}$$

$$S_2(B) = d_3 d_5 > 0, \tag{3.3}$$

$$S_3(B) = x_{14}x_{43}x_{31} > 0, \tag{3.4}$$

$$S_4(B) = d_5 x_{14}x_{43}x_{31} - x_{12}x_{24}x_{43}x_{31} - x_{14}x_{43}x_{35}x_{51} > 0, \tag{3.5}$$

$$S_5(B) = x_{12}x_{24}x_{43}x_{35}x_{51} - x_{12}x_{24}x_{43}x_{31}d_5 > 0. \tag{3.6}$$

Clearly, the entries in B against all unspecified entries in M must be positive. Now, from (3.5) we have $d_5x_{14}x_{31} > x_{12}x_{24}x_{31} + x_{14}x_{35}x_{51}$ which yields $d_5x_{31} > x_{35}x_{51}$. On the other hand, (3.6) implies that $x_{35}x_{51} > d_5x_{31}$. Hence B cannot be a nonnegative Q-matrix.

Theorem 3.3.7. *Let D be a digraph with at least four vertices. Suppose $\overline{D}$ has more than one 2-cycle and does not have a 3-cycle. If D has nonnegative Q-completion, then D must omit more than three loops.*

Proof. For D to have nonnegative Q-completion, D must omit at least two loops, by Proposition 3.3.1. We prove that if D omits exactly three loops, then D does not have nonnegative Q-completion. Then, the result for the case when D omits exactly two loops will follow from Proposition 3.2.1.

Suppose D omits loops at the vertices $1, 2$ and 3. Now, we label the 2-cycles as $E_1, \ldots, E_k$ $(k > 1)$ such that the number of vertices among $1, 2$ and 3 that E_j is incident with is in ascending order in j. Let M be the partial nonnegative Q-matrix specifying the digraph D with all specified entries as zero. Suppose that M has a nonnegative Q-completion $B = [b_{ij}]$. We put $d_1 = b_{11},\ d_2 = b_{22},\ d_3 = b_{33}$. For a 2-cycle $E = [r, s]$ in $\overline{D}$, we write $w(E) = w(E, B) = b_{rs}b_{sr}$. Then, by (1.2) we have

$$S_3(A) = d_1d_2d_3 - (d_1\sigma_1 + d_2\sigma_2 + d_3\sigma_3), \tag{3.7}$$

where $\sigma_t = \sum \{w(E_i) : E_i \text{ is not incident with } t\},\ t = 1, 2, 3$. Since $S_3(B) > 0$, (3.7) implies

$$d_1d_2 > \sigma_3,\ d_2d_3 > \sigma_1,\ d_3d_1 > \sigma_2, \tag{3.8}$$

and therefore

$$d_1d_2 + d_2d_3 + d_3d_1 > \sigma_1 + \sigma_2 + \sigma_3. \tag{3.9}$$

For $1 \le j \le k$, let

$$\begin{aligned} \beta_j &= \sum \Big\{w(E_i) : E_i \cap E_j = \emptyset \text{ and } i > j\Big\} \\ \gamma_j &= \sum \Big\{d_sd_t : E_j \text{ is incident with none of } s \text{ and } t\Big\}. \end{aligned}$$

Then, by (1.2) we have

$$S_4(A) = \sum_{j=1}^{k} \beta_j\, w(E_j) - \sum_{j=1}^{k} \gamma_j\, w(E_j) = \sum_{j=1}^{k} (\beta_j - \gamma_j)\, w(E_j). \tag{3.10}$$

However, we show that $\beta_j - \gamma_j \leq 0$ for $1 \leq j \leq k$. Then, it will follow from (3.10) that $S_4(B) \leq 0$, a contradiction to the fact that B is a Q-matrix. Fix $j \in \{1, 2, \ldots, k\}$. We have

$$\gamma_j = \begin{cases} d_1d_2 + d_2d_3 + d_3d_1, & \text{if } E_j \text{ is not incident with any of the vertices } 1, 2, 3, \\ (d_1d_2d_3)/d_t, & \text{if } E_j \text{ is incident with only } t,\ t = 1, 2 \text{ or } 3, \\ 0, & \text{if } E_j \text{ is incident with two of the vertices } 1, 2 \text{ and } 3. \end{cases}$$

If E_j is not incident with any of the vertices $1, 2$ and 3, then from (3.9) we get

$$\beta_j \leq \sum_{i=1}^{k} w(E_i) \leq \sigma_1 + \sigma_2 + \sigma_3 < d_1d_2 + d_2d_3 + d_3d_1 = \gamma_j.$$

Next, suppose E_j is incident with exactly one vertex t in $\{1, 2, 3\}$. Consider the case $t = 1$. Since

$$\{E_i : E_i \cap E_j = \emptyset, i > j\} \subseteq \{E_i : 1 \text{ is not incident with } E_i\},$$

we have $\beta_j \leq \sigma_1$. Therefore, from (3.8) we get $\beta_j \leq \sigma_1 < d_2d_3 = \gamma_j$. The cases when $t = 2$ and 3 are similar. Finally, assume that E_j is incident to two among the vertices $1, 2$ and 3 so that $\gamma_j = 0$. Now, for any $i > j$ the 2-cycle E_i is incident with two vertices among $1, 2$ and 3. Consequently, by our choice of the ordering of the 2-cycles, E_i and E_j have a vertex in common yielding $\beta_j = 0$. Therefore, our assertion that $\beta_j - \gamma_j \leq 0$ for $1 \leq j \leq k$ holds. □

3.4 Nonnegative Q-completion of digraphs of small order

We have examined the digraphs of order at most four to nonnegative Q-completion. Clearly, any digraph of order 1 (with or without a loop) has nonnegative Q-completion. Any digraph of order 2 and without a loop has nonnegative Q-completion.

There are only four non-isomorphic digraphs of order 3 without loops for which the digraphs obtained by attaching a loop at any of the vertices have nonnegative Q-completion. These digraphs are precisely the spanning subdigraphs of a 3-cycle.

Some types of digraphs of order four with respect to nonnegative Q-completion are presented below.

(a) Consider the digraph D_6 in Figure 3.5. The digraph D_3 in Figure 3.2 is obtained from D_6 by adding two loops at the vertices 1 and 2. In Example 3.2.3, it was shown that D_3 has nonnegative Q-completion. In fact, the digraphs obtained by attaching at most two loops at any of the vertices of D_6 have nonnegative Q-completion.

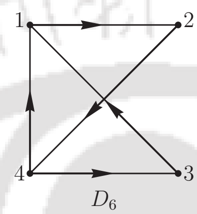


FIGURE 3.5: The digraph D_6

Out of 218 non-isomorphic digraphs of order 4 without a loop that are listed in Appendix A, there are 41 digraphs having similar property as of D_6, i.e., all digraphs obtained from them by attaching at most two loops at any of the vertices have nonnegative Q-completion. The list of such these digraphs $D_4(q,n)$, in the nomenclature as in Appendix A is given below.

$q = 0, 1, 12;\quad n = 1$

$q = 2;\quad n = 1$–5

$q = 3;\quad n = 1$–$7, 9, 10, 12, 13$

$q = 4;\quad n = 1, 3$–$9, 16, 17, 19, 20, 23, 24, 25$

$q = 5;\quad n = 1, 3, 7, 8, 10, 32$

$q = 6;\quad n = 7.$

(b) Consider the digraph D_7 in Figure 3.6 and the digraph obtained from D_7 by attaching loops at two of its vertices. If the resulting digraph includes a loop at the vertex 2, then the indegree of its vertex 2 will be 4 and therefore, the complement of the digraph is not stratified. Consequently, the digraph does not have nonnegative Q-completion, in view of Theorem 3.3.4. On the other hand, consider the digraph H obtained from D_7 by attaching loops at

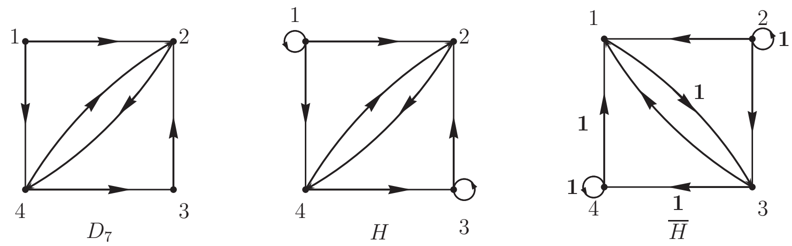


Figure 3.6: The digraphs D_7, H and $\overline{H}$

the vertices 1 and 3. Consider the weight function ϕ defined on the arcs $\overline{H}$ as indicated in Figure 3.6, omitting the zero weights. Then ϕ satisfies the conditions in Theorem 3.2.2, and therefore, H has nonnegative Q-completion.

Out of 218 digraphs of order 4 listed in Appendix, there are 61 digraphs having similar property as D_7, i.e., a digraph D obtained from one of these digraphs by attaching at most two loops has nonnegative Q-completion if D omits a loop at a particular vertex. The list of these 61 digraphs $D_4(q, n)$, in the nomenclature as in Appendix A is given below.

$q = 3;$ $n = 8, 11$

$q = 4;$ $n = 10, 12, 21, 27$

$q = 5;$ $n = 4–6, 8, 10, 11, 13, 16, 19, 20, 22, 26–28, 31, 37$

$q = 6;$ $n = 1, 7, 9, 11, 16–22, 24, 25, 27, 28, 31, 33, 34, 38, 41, 44, 47, 48$

$q = 7;$ $n = 6, 11–13, 18–20, 25, 30, 32, 35, 38$

$q = 8;$ $n = 13, 20, 24$

$q = 9;$ $n = 12.$

(c) Consider the digraph D_8 in Figure 3.7. Let D be a digraph obtained from D_8 by adding a loop at any of its vertices. Then $\overline{D}$ has two 2-cycles and does not have a 3-cycle. Since D omits three loops, D does not have nonnegative Q-completion, in view of Theorem 3.3.7.

Out of 218 digraphs of order 4 listed in Appendix A, there are 22 digraphs which have similar property as D_8, i.e., any digraph obtained from them by attaching a loop at a vertex does not have nonnegative Q-completion. The

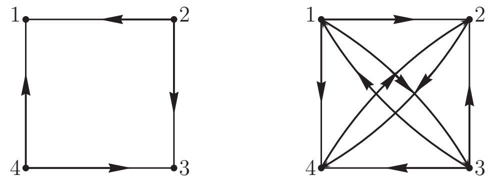


FIGURE 3.7: The digraph D_8 and its complement $\overline{D_8}$

list of these digraphs $D_4(q, n)$, in the nomenclature as in Appendix A is given below.

$$q = 4; \quad n = 2, 18$$
$$q = 5; \quad n = 2, 9, 14, 24$$
$$q = 6; \quad n = 2\text{–}6, 8, 10, 12\text{–}15$$
$$q = 7; \quad n = 1\text{–}5.$$

3.5 Conclusion

In this chapter, we have discussed the completion problem for the class of nonnegative Q-matrices. We have obtained certain necessary conditions and some sufficient conditions for a digraph to have nonnegative Q-completion. It has been found that a digraph that includes all loops cannot have nonnegative Q-completion, unless it is a complete symmetric digraph. Positive stratification is another essential condition for a digraph to have nonnegative Q-completion. The results obtained in this chapter give us a fair idea of nonnegative Q-completion of digraphs of small order. However, the completion problem is far from being completely understood. A complete characterization for a digraph to have nonnegative Q-completion could not be found yet. We notice that the nature of the results obtained for the nonnegative Q-matrix completion problem is different from that for the positive Q-matrix completion problem. We will compare the two completion problems in Chapter 5.

4

The P_0^+-Matrix Completion Problem

In this chapter, the P_0^+-matrix completion problem is studied. It follows from Definition 1.1.5 that a real $n \times n$ matrix B is a P_0^+*-matrix* if for every $k \in \{1, 2, \ldots, n\}$ all $k \times k$ principal minors of B are nonnegative and at least one $k \times k$ principal minor is positive. In Section 4.1, we introduce partial P_0^+-matrices and the P_0^+-matrix completion problem, and find some sufficient conditions for partial P_0^+-matrices to have P_0^+-completions. We discuss necessary conditions for P_0^+-matrix completion in Section 4.2. Based on these necessary and sufficient conditions, we single out the digraphs of order at most four that have that have P_0^+-completion in Section 4.3.

4.1 Partial P_0^+-matrices and the P_0^+-matrix completion problem

A *partial* P_0^+*-matrix* is a partial matrix M in which all fully specified principal subminors are nonnegative and $S_k(M) > 0$ for every $k \in \{1, 2, \ldots, n\}$, whenever all $k \times k$ principal submatrices are fully specified.

Let M be a partial P_0^+-matrix. If all 1×1 principal submatrices (i.e., all diagonal entries) in M are specified, then $\mathrm{Trace}\,(M) > 0$. If for some $k \geq 2$, all $k \times k$ principal submatrices are fully specified, then M is fully specified (and therefore M is a P_0^+-matrix). Thus the following proposition, which provides a more useful characterization of partial P_0^+-matrix is immediate.

Proposition 4.1.1. *A partial matrix M is a partial P_0^+-matrix if and only if exactly one of the following holds:*

(i) *At least one diagonal entry of M is unspecified, and each fully specified principal minor of M is nonnegative.*

(ii) *All diagonal entries are specified and nonnegative with at least one of them positive; at least one off-diagonal entry is unspecified and each fully specified principal minor of M is nonnegative.*

(iii) *All entries of M are specified and M is a P_0^+-matrix.*

Definition 4.1.2. A partial P_0^+-matrix M is said to have a P_0^+*-completion* if there is a completion of M which is a P_0^+-matrix. A digraph D is said to have P_0^+-*completion* if every partial P_0^+-matrix specifying D has a P_0^+-completion. The P_0^+-*matrix completion problem* aims to classify all digraphs which have P_0^+-completion.

To distinguish P_0^+-completion problem from those of P_0 and P-matrix classes, we furnish the following example.

For $m, n \geq 2$, consider the digraph G obtained by identifying a vertex of K_m with a vertex of K_n. Then, G is a 1-chordal graph with two maximal cliques K_m and K_n (cf. Section 1.6). It is known that any partial P-matrix (P_0-matrix) with all diagonal entries specified which specify the graph G has a P-completion (P_0-completion) (cf. Theorem 0.0.4). With an appropriate labelling of the vertices in G, a partial matrix specifying G is given by

$$M = \begin{bmatrix} A_{11} & A_{12} & X \\ A_{21} & a_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{bmatrix},$$

where A_{ii} are square, a_{22} is scalar, and X and Y are fully unspecified. If M is a partial P_0-matrix, then a P_0-completion of M (the zero completion) is obtained by putting $X = Y = 0$. Similarly, if M is a partial P-matrix, then a P-completion of M (the asymmetric completion) is obtained by putting $X = A_{12}a_{22}^{-1}A_{23}$ and $Y = 0$. That the techniques may not hold for the P_0^+-matrix completion problem can be seen from the following examples.

Example 4.1.3. Consider the partial P_0^+-matrix

$$M = \begin{bmatrix} 1 & 0 & 1 & ? & ? \\ 1 & 0 & 0 & ? & ? \\ -1 & 1 & 0 & 0 & -1 \\ ? & ? & 0 & 1 & 0 \\ ? & ? & 1 & 1 & 0 \end{bmatrix}$$

specifying the 1-chordal graph with two maximal cliques K_3. We note that the submatrices corresponding to the maximal cliques are P_0^+-matrices. The zero completion of M has determinant 0 and therefore is not a P_0^+-matrix. The asymmetric completion of M is not defined, since $(3,3)$ entry of M is 0. However, M has P_0^+-completions, e.g.,

$$B_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

It is worthwhile to mention here that any 1-chordal graph G with two maximal cliques does not have P_0^+-completion (cf. Corollary 4.2.8), i.e., there is a partial P_0^+-matrix M (with specified diagonal entries) specifying G, which does not have a P_0^+-completion. However, we have the following result in case one of the cliques is K_2.

Theorem 4.1.4. *Let M be a partial matrix specifying a 1-chordal graph with two maximal cliques K_n and K_2. If the principal submatrices of M corresponding to K_n and K_2 are P_0^+-matrices, then M can be completed to a P_0^+-matrix.*

Proof. Without loss of generality we may assume that $V(K_n) = \{1, \ldots, n\}$, $V(K_2) = \{n, n+1\}$. We can write

$$M = \begin{bmatrix} A_{11} & A_{12} & X \\ A_{21} & a_{22} & a_{23} \\ Y & a_{32} & a_{33} \end{bmatrix},$$

where X is $(n-1) \times 1$, Y is $1 \times (n-1)$, and are fully unspecified. Moreover, the two principal submatrices specifying K_n and K_2, viz.,

$$M_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

are P_0^+ matrices. If $a_{22} > 0$, consider the asymmetric completion $\widehat{A}$ of M, i.e., one obtained by setting $X = A_{12}a_{22}^{-1}a_{23}$ and $Y = 0$. Then

$$\det \widehat{A} = \frac{\det M_1 \det M_2}{a_{22}} > 0. \tag{4.1}$$

Let A be any principal submatrix of $\widehat{A}$. That A has nonnegative determinant follows from an equation similar to (4.1), if a_{22} is included in A, and from the fact that A is block triangular, if a_{22} is excluded. Since M_1 has a positive minor of order k for $1 \leq k \leq n$, $\widehat{A}$ is a P_0^+-matrix.

On the other hand, if $a_{22} = 0$, consider the zero completion $\widehat{A}$ of M, i.e., one obtained by setting $X = 0 = Y$. Since M_2 is a P_0^+-matrix, a_{33} must be positive. Then, we have $\det \widehat{A} = a_{33} \det M_1 + \det A_{11} \det M_2 > 0$. That a principal minor given by a principal submatrix A of $\widehat{A}$ is nonnegative follows from the determinant equality when n-th row is present and because the principal submatrix is block diagonal when n-th row is absent. Since M_1 has a positive minor of order k for $1 \leq k \leq n$, $\widehat{A}$ is a P_0^+-matrix. $\square$

A P_0^+-completion of the principal submatrix corresponding to the specified diagonal positions of a partial P_0^+-matrix does not ensure a P_0^+-completion of the partial matrix. This can be seen from the following example.

Example 4.1.5. Consider the the partial P_0^+-matrix

$$M = \begin{bmatrix} 2 & 1 & u & 1 \\ 1 & \frac{1}{2} & 0 & 1 \\ -1 & v & 0 & x \\ 1 & 1 & y & z \end{bmatrix}.$$

with u, v, x, y, z as unspecified entries. The principal submatrix

$$M[\{1,2,3\}] = \begin{bmatrix} 2 & 1 & u \\ 1 & \frac{1}{2} & 0 \\ -1 & v & 0 \end{bmatrix},$$

of M induced by the specified diagonal entries has a P_0^+-completion, e.g., one obtained by putting $u = 2, v = 0$. However, for any completion of M we have

$$\det M[\{1,2,4\}] = \begin{vmatrix} 2 & 1 & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & z \end{vmatrix} = -\frac{1}{2},$$

and hence M cannot be completed to a P_0^+-matrix.

We observe that if a digraph D omits all loops, then D has P_0^+-completion. Indeed, a completion of a partial P_0^+-matrix M specifying D can be obtained by assigning a sufficiently large value to each of the diagonal entries.

However, the assertion of Theorem 0.0.9 is not true in case Π stands for the class of P_0^+-matrices. A digraph D may not have P_0^+-completion, even if the subdigraph of D induced by its vertices which have loops has P_0^+-completion. This can be seen from the following example.

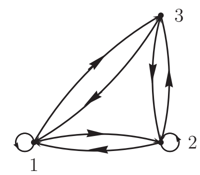


FIGURE 4.1: A digraph D, not having P_0^+-completion

Example 4.1.6. Consider the digraph D in Figure 4.1 obtained from K_3 by removing the loop at 3. The subdigraph induced by the vertices 1 and 2 (i.e., ones having loops) is K_2 and has P_0^+-completion, being a complete symmetric digraph. However, the partial P_0^+-matrix

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix}$$

does not have a P_0^+-completion.

Again, the assertion of Theorem 0.0.8 is not true with Π as the class of P_0^+-matrices can be seen from the following example.

Example 4.1.7. Consider the digraph $D = K_2 \cup K_3$. The components K_2 and K_3 of D have P_0^+-completion. However, the partial P_0^+-matrix

$$M = \begin{bmatrix} 1 & 0 & ? & ? & ? \\ 0 & 1 & ? & ? & ? \\ ? & ? & 0 & 0 & 0 \\ ? & ? & 0 & 0 & 0 \\ ? & ? & 0 & 0 & 0 \end{bmatrix},$$

specifying D does not have P_0^+-completion, because for any completion of M the last three rows are linearly dependent.

It could not be settled whether a digraph D has P_0^+-completion, if none of the components of D is complete and each of them has P_0^+-completion. The difficulty is that for a partial P_0^+-matrix M some of the partial submatrices of M induced by the components of D may fail to be a partial P_0^+-matrix.

We make a point here that an induced subdigraph of a digraph having P_0^+-completion may not have P_0^+-completion.

Example 4.1.8. The digraph D in Figure 4.2 has P_0^+-completion (cf. Theorem 4.3.9), but the subdigraph D_0 of D induced by the vertices 2 and 3 does not have (follows from Corollary 4.2.9, because D_0 has all loops and its complement does not have a 2-cycle).

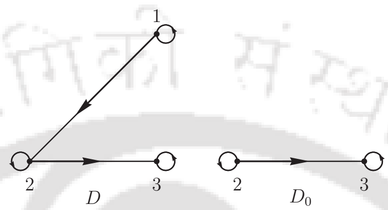


FIGURE 4.2: D has P_0^+-completion, but its induced subdigraph D_0 does not have.

However, for spanning subdigraphs we have the following.

Theorem 4.1.9. *Suppose $D \neq K_n$ is a digraph having P_0^+-completion, and $\widehat{D}$ is a spanning subdigraph of D. If $\widehat{D}$ has P_0-completion, then $\widehat{D}$ has P_0^+-completion.*

Proof. Let $\widehat{M} = [\widehat{m}_{ij}]$ be a partial P_0^+-matrix specifying $\widehat{D}$. Then, being a partial P_0-matrix, $\widehat{M}$ has a P_0-completion $\widehat{B} = [\widehat{a}_{ij}]$. Let $M = [m_{ij}]$ be the partial matrix specifying D defined by

$$m_{ij} = \begin{cases} \widehat{m}_{ij}, & \text{if } (i,j) \in A(\widehat{D}), \\ \widehat{a}_{ij}, & \text{if } (i,j) \in A(D) \setminus A(\widehat{D}), i \neq j, \\ \max\{1, \widehat{a}_{ij}\} & \text{if } (i,j) \in A(D) \setminus A(\widehat{D}), i = j. \end{cases}$$

Then, M is a partial P_0^+-matrix specifying D, and therefore has a P_0^+-completion B. Clearly, B is a P_0^+-completion of $\widehat{M}$. $\square$

We have observed that for all digraphs that we have considered including the digraphs of order at most 4, if a digraph D has P_0-completion and its complement $\overline{D}$ is stratified, then D has P_0^+-completion. Further, for such a digraph D it is possible to sign the arcs of $\overline{D}$ such that all cycles of $\overline{D}$ are of positive sign. However, we do not know whether the result is true in general for all such digraphs.

4.2 Necessary conditions for P_0^+–completion

In this section, we present some necessary conditions for a digraph to have P_0^+-completion.

Theorem 4.2.1. *Let $D \neq K_n$ be a digraph having P_0^+-completion. Then every proper induced subdigraph of D has P_0-completion.*

Proof. Let D be of order n and α be a proper subset of $\{1, 2, \ldots, n\}$. Consider the subdigraph D_α of D induced by α and let M_α be a partial P_0-matrix specifying D_α. We extend M_α to a partial matrix M specifying the digraph D by setting all remaining specified off-diagonal entries as 0 and the remaining specified diagonal entries, if any, as 1. Then, all fully specified principal minors of M are nonnegative, and M is a partial P_0^+-matrix. Now, since D has P_0^+-completion, M can be completed to a P_0^+-matrix $\widehat{B}$. Clearly, the principal submatrix of $\widehat{B}$ induced by α is a P_0-completion of M_α. $\square$

That the converse of the Theorem 4.2.1 is not true can be seen from Example 4.2.6. In [7], Choi *et al.* showed that any digraph, which contains one of the

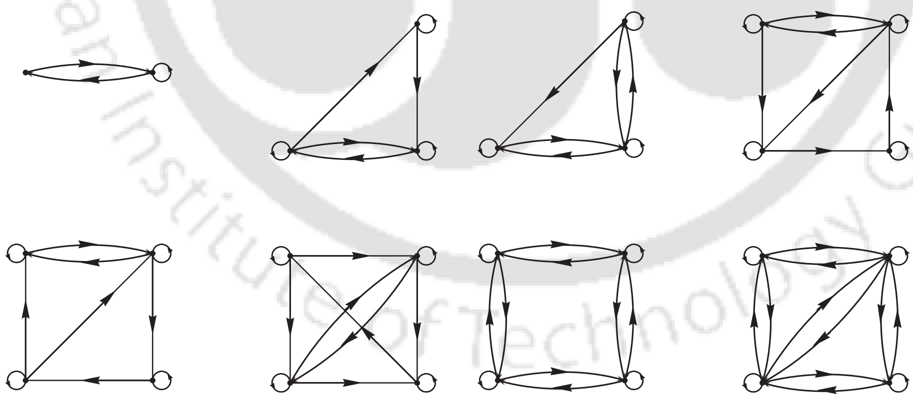


FIGURE 4.3: Digraphs not having P_0-completion

(unlabeled) digraphs in Figure 4.3 as an induced subdigraph, does not have P_0-completion. In particular, these digraphs do not have P_0-completion. Therefore, as an immediate implication of Theorem 4.2.1 we have the following.

Corollary 4.2.2. *Any digraph which contains one of the (unlabeled) digraphs in Figure 4.3 as a proper induced subdigraph does not have P_0^+-completion.*

Theorem 4.2.3. *Let D be a digraph of order n that omits at least one loop. If D has P_0^+-completion, then $\overline{D}$ is stratified.*

Proof. Suppose D has P_0^+-completion. Let $k \geq 2$, and assume $\overline{D}$ has no order k permutation digraph. If M is the partial matrix that specifies D with all specified entries zero, and B is a completion of M, then all $k \times k$ principal minors of B are zero, so B is not a P_0^+-matrix. This implies that $\overline{D}$ must be stratified. □

The converse of Theorem 4.2.3 is not true.

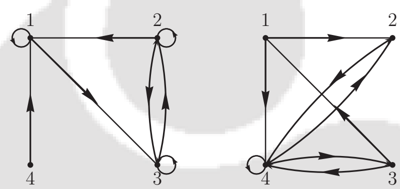


Figure 4.4: The digraph D_1 and its complement

Example 4.2.4. Consider the digraph D_1 in Figure 4.4, which omits a loop at the vertex 4. Though $\overline{D_1}$ is stratified, D_1 does not have P_0^+ completion, in view of Corollary 4.2.2. Indeed, the subdigraph induced by the vertices $\{1, 2, 3\}$ appears in list of digraphs in Figure 4.3. We note here that the digraph D_1 has Q-completion, since $\overline{D_1}$ has a signing so that each of its cycles is positively signed.

Theorem 4.2.5. *Let $D \neq K_n$ be a digraph of order $n \geq 2$ that includes all loops and has P_0^+-completion. Then $\overline{D}$ is weakly stratified.*

Proof. Suppose $2 \leq k \leq n$ and $\overline{D}$ has no permutation digraph of order k and there is a vertex v in D such that $\overline{D} - v$ does not have a permutation digraph of order $k - 1$. Let $M = [m_{ij}]$ be the partial matrix specifying D with $m_{vv} = 1$ and all other specified entries equal to zero. Then for any completion B of M all $k \times k$ principal minors of B are zero, and thus, B is not a P_0^+-matrix. □

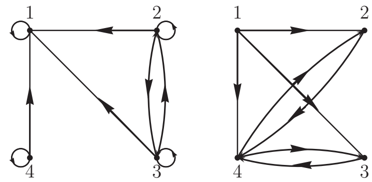


FIGURE 4.5: The digraphs $D_2, \overline{D_2}$

Example 4.2.6. The digraph D_2 in Figure 4.5 includes all loops and $\overline{D_2}$ is not weakly stratified, since the vertex 1 has indegree 4 in D_2. Therefore, D_2 does not have P_0^+-completion. Further, each of the strong components of D_2 are complete, and therefore, has P_0-completion. Thus, in view of Theorem 0.0.8, D_2 has P_0-completion. It follows from Theorem 0.0.7 that every induced subdigraph of D_2 has P_0-completion. We have a digraph D_2 which does not have P_0^+-completion, though every induced subdigraph of D_2 has P_0-completion. This shows that the converse of the Theorem 4.2.1 is not true.

That the converse of Theorem 4.2.5 is not true can be seen from Example 4.2.12.

Corollary 4.2.7. *If a digraph D of order $n \geq 2$ contains a vertex v with indegree or outdegree n, then D does not have P_0^+-completion.*

Proof. Clearly, v is an isolated vertex without a loop in $\overline{D}$. Consequently, $\overline{D}$ does not have a spanning permutation digraph. Further, for any vertex $u \neq v$ in D, the digraph $\overline{D} - u$ does not have a spanning permutation digraph. Hence, $\overline{D}$ is not weakly stratified. □

Corollary 4.2.8. *A 1-chordal graph G with two maximal cliques does not have P_0^+-completion.*

Proof. Let D be the digraph corresponding to G so that a partial matrix specifying G specifies D. Then D includes all loops. Let v be the cut vertex of G. Then v has indegree and outdegree n in D. Thus, by Corollary 4.2.7, D does not have P_0^+-completion and the result follows in view of Remark 1.5.6. □

Corollary 4.2.9. *Let D be a digraph of order n that includes all loops and has P_0^+-completion. Then $\overline{D}$ has a 2-cycle.*

Proof. Since D has P_0^+-completion and D includes all loops, thus $\overline{D}$ must be weakly stratified. Again by hypothesis $\overline{D}$ does not contain any loops, $\overline{D}$ has a permutation digraph of order 2 only if it has a 2-cycle. □

The following implication of Corollary 4.2.9 is in contrast to the fact that any asymmetric digraph has P_0-completion (cf. Theorem 0.0.6).

Corollary 4.2.10. *A maximal asymmetric digraph with all loops (i.e., the complement of a tournament) does not have P_0^+-completion.*

Theorem 4.2.11. *Let D be a digraph with distinct vertices u, v such that u is of outdegree (or indegree) $n-1$ and $(u,v) \notin A(D)$ $\big($resp. $(v,u) \notin A(D)\big)$. Suppose that (u,v) (resp. (v,u)) is a common arc of two distinct cycles C_1 and C_2 in $D+(u,v)$ (resp. $D+(v,u)$). If there is a vertex w on C_i ($i = 1$ or 2) such that $(D+C_i)-w$ and the subdigraphs of $\overline{D}+C_i$ induced by $V(C_i)$ have no spanning permutation digraphs, then D does not have P_0^+-completion.*

Proof. We assume the case when u is of outdegree $n-1$, the other case being similar. Construct a partial P_0^+-matrix $M = [m_{ij}]$ specifying D as follows:

(i) Put all specified diagonal entries as 0 except $m_{uu} = 1$.

(ii) Put the specified entries at positions corresponding to arcs in C_1 and C_2 such that

$$\prod_{(i,j)\in C_1} m_{ij} = m_{uv}, \qquad \prod_{(i,j)\in C_2} m_{ij} = -m_{uv}.$$

(iii) Put all other off-diagonal specified entries as zero.

Let B be a completion of M and B_1 and B_2 be the principal submatrices of B determined by C_1 and C_2, respectively. Then, we have $\det B_1 = m_{uv}$ and $\det B_2 = -m_{uv}$. Thus, for B to be a P_0^+ matrix $m_{uv} = 0$. In that case all the elements of u-th becomes zero and thus the u-th row of B is a zero row, yielding $\det B = 0$. □

Example 4.2.12. Using Theorem 4.2.11 we show that *the symmetric 4-cycle C_4 with all loops* (Figure 4.6) *does not have P_0^+-completion.* The vertex 1 has outdegree

3, $(1,3)$ is not an arc of C_4, $(1,3,2)$ and $(1,3,4)$ are cycles of $C_4+(1,3)$, and the subdigraphs of $\overline{C_4}$ induced by vertices $\{1,2,3\}$ and $\{1,3,4\}$ do not have spanning permutation digraphs. Thus, C_4 does not have P_0^+-completion. In fact, using the

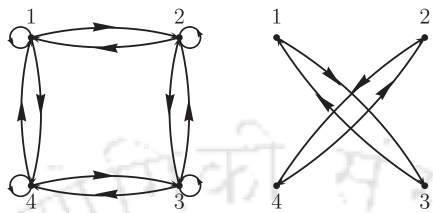


Figure 4.6: The symmetric 4-cycle C_4 and its complement

procedure in the proof of Theorem 4.2.11 we obtain a partial P_0^+-matrix

$$M = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 1 & 0 & 0 & x_{24} \\ x_{31} & -1 & 1 & 1 \\ 1 & x_{42} & 0 & 0 \end{bmatrix}$$

specifying C_4, which does not have P_0^+-completion. It is easy to see that $\overline{C_4}$ is weakly stratified. Therefore, this example shows in particular that the converse of Theorem 4.2.5 is not true in general.

4.3 Classification of small digraphs as to P_0^+-completion

We apply the results in the previous sections to classify the digraphs of order at most four that include all loops as to P_0^+-completion. The digraphs are indicated as $\tilde{D}_p(q,n)$, obtained from $D_p(q,n)$ by attaching a loop at each of the vertices. Here, $D_p(q,n)$ is the n-th digraph with p vertices and q arcs as listed in Appendix A. The classification is broken up into a series of lemmas.

Lemma 4.3.1. *The digraphs $\tilde{D}_p(q,n)$ which are listed below do not have P_0^+-completion.*

$$
\begin{array}{lll}
p=2; & q=2; & n=1; \\
p=3; & q=3; & n=3; \\
p=4; & q=6; & n=45\text{ - }48; \\
 & q=7; & n=29\text{ - }38; \\
 & q=8; & n=16\text{ - }27; \\
 & q=9; & n=4\text{ - }7,\ 9\text{ - }13; \\
 & q=10; & n=2\text{ - }5; \\
 & q=11; & n=1.
\end{array}
$$

Proof. Each of the digraphs listed contains all loops but its complement does not contain a 2-cycle. Hence by Corollary 4.2.9, the digraph does not have P_0^+-completion. □

Lemma 4.3.2. *The digraphs $\tilde{D}_p(q,n)$ which are listed below do not have P_0^+-completion.*

$$
\begin{array}{lll}
p=3; & q=2; & n=1,3,4 \\
 & q=3; & n=1,4 \\
 & q=4; & n=1,3,4 \\
p=4; & q=3; & n=8,11; \\
 & q=4; & n=10,12,14,15,21,27; \\
 & q=5; & n=4\text{ - }6,11,14\text{ - }17,19,21\text{ - }24,26,28,29,31,34,36,37; \\
 & q=6; & n=1,2,9\text{ - }23,26,27,29,30,32\text{ - }41,43,44; \\
 & q=7; & n=1,3\text{ - }28; \\
 & q=8; & n=1,3\text{ - }15; \\
 & q=9; & n=1\text{ - }3,8; \\
 & q=10; & n=1.
\end{array}
$$

Proof. Each of the digraphs listed contains all loops but its complement is not weakly stratified. Thus by Theorem 4.2.5, the digraphs do not have P_0^+ completion. □

Lemma 4.3.3. *The digraphs $\tilde{D}_4(q,n)$ which are listed below do not have P_0^+-completion.*

$$q = 4; \quad n = 13;$$
$$q = 5; \quad n = 12, 13, 18, 20;$$
$$q = 6; \quad n = 24, 25, 28, 31.$$

Proof. Each of the digraphs listed has an induced subdigraph isomorphic to one of the digraphs in Figure 4.3. Hence, by Corollary 4.2.2, the digraphs do not have P_0^+-completion. □

Lemma 4.3.4. *The digraphs $\tilde{D}_4(q,n)$ which are listed below do not have P_0^+-completion.*

$$q = 6; \quad n = 6, 8$$
$$q = 7; \quad n = 2$$
$$q = 8; \quad n = 2.$$

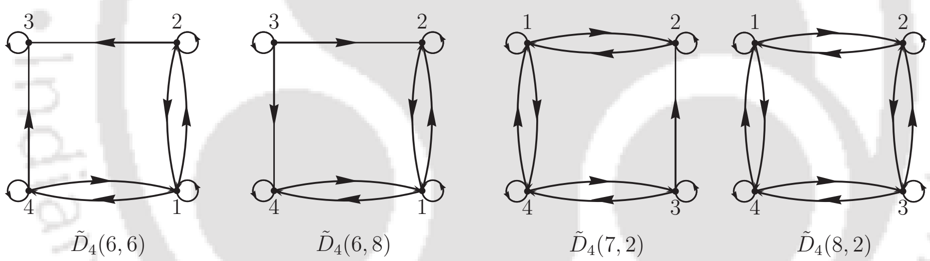


FIGURE 4.7: Digraphs $\tilde{D}_4(q,n)$ do not have P_0^+-completion.

Proof. The digraph $\tilde{D}_4(8,2)$ is the symmetric 4-cycle, and that it does not have P_0^+-completion was seen in Example 4.2.12. Next, for any P_0^+-completion B of the partial P_0^+-matrix

$$M_1 = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 1 & 0 & x_{23} & x_{24} \\ x_{31} & -1 & 1 & 1 \\ 1 & x_{42} & 0 & 0 \end{bmatrix}$$

specifying $\tilde{D}_4(7,2)$ we have $B(1,2,3) = -x_{13} \geq 0$ and $B(1,3,4) = x_{13} \geq 0$. Consequently, $x_{13} = 0$ and $\det B = 0$. Thus, B is not a P_0^+-matrix. In a similar manner, we suppose that

$$M_2 = \begin{bmatrix} 0 & 1 & x_{13} & 1 \\ 0 & 0 & -1 & x_{24} \\ x_{31} & x_{32} & 1 & x_{34} \\ 0 & x_{42} & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_3 = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 1 & 0 & x_{23} & x_{24} \\ x_{31} & -1 & 1 & 1 \\ 1 & x_{42} & x_{43} & 0 \end{bmatrix}$$

be the partial P_0^+-matrices specifying the digraphs $\tilde{D}_4(6,6)$ and $\tilde{D}_4(6,8)$, respectively. Now for any completion B_2 of M_2, we have $B_2[\{1,2,3\}] = -x_{31} \geq 0$ and $B_2[\{1,3,4\}] = x_{31} \geq 0$. Consequently, $x_{31} = 0$ and $\det B_2 = 0$. Thus, B_2 is not a P_0^+-matrix. Again M_3, being the transpose of M_2, does not have P_0^+-completion. $\square$

The following result can be easily verified.

Lemma 4.3.5. *For real numbers a, b, c, d, the inequalities*

$$\begin{aligned} ax + by &\geq 0 \\ cx + dy &\geq 0 \\ xy &< 0 \end{aligned}$$

do not have a solution for real x and y only if one of the following holds:

(i) $a = 0, c = 0,\ bd < 0$ *or* $b = 0, d = 0,\ ac < 0$*;*

(ii) $b = 0, c = 0,\ ad > 0$ *or* $a = 0, d = 0, bc > 0$*;*

(iii) $c \neq 0, d \neq 0$ *and* $\frac{a}{c} = \frac{b}{d} < 0$.

Lemma 4.3.6. *The digraphs $\tilde{D}_4(q,n),\ q = 6,\ n = 3,4,5,7$, have P_0^+ completion.*

Proof. For each partial matrix considered below we denote the specified diagonal entries by d_i and the specified off-diagonal entries by a_{ij}.

First, let M be a partial P_0^+-matrix specifying the digraph $\tilde{D}_4(6,3)$. We show that for some choices of s and t, the completion

$$B = \begin{bmatrix} d_1 & a_{12} & s & 0 \\ a_{21} & d_2 & a_{23} & t \\ -s & a_{32} & d_3 & a_{34} \\ 0 & -t & a_{43} & d_4 \end{bmatrix},$$

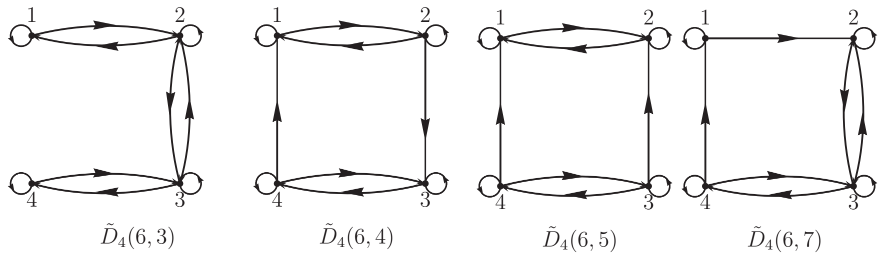


FIGURE 4.8: Digraphs $\tilde{D}_4(q,n)$ have P_0^+-completion.

of M is a P_0^+-matrix. Clearly, for any choices of s and t, all 2×2 principal minors of B are nonnegative. Moreover, $B(1,3)$ and $B(2,4)$ are positive for nonzero s and t. Now, the 3×3 principal minors of B are

$$\begin{aligned} B(1,2,3) &= d_1B(2,3) + s(a_{21}a_{32} - a_{12}a_{23}) + (d_2s^2 - d_3a_{12}a_{21}) && (4.2)\\ B(1,2,4) &= d_1t^2 + d_4B(1,2) && (4.3)\\ B(1,3,4) &= d_1B(3,4) + d_4s^2 && (4.4)\\ B(2,3,4) &= d_2B(3,4) + t(a_{32}a_{43} - a_{23}a_{34}) + (d_3t^2 - d_4a_{23}a_{32}). && (4.5) \end{aligned}$$

Since $B(1,2) \geq 0$, it follows that $a_{12}a_{21} \leq 0$, if $d_2 = 0$. Similarly, $a_{23}a_{32} \leq 0$, if $d_3 = 0$. Consequently, for s and t with large magnitude and with appropriate signs so that the term $s(a_{21}a_{32} - a_{12}a_{23})$ in (4.2) and the term $t(a_{32}a_{43} - a_{23}a_{34})$ in (4.5) are nonnegative, we get all 3×3 principal minors of B nonnegative. Moreover, at least one of them can be made positive, because $d_i > 0$ for some i. Further, choosing $|t| = |s|$, we get

$$\det B = s^4 + p(s),$$

where $p(s)$ is a polynomial in s of degree at most 3. Hence, for large values of $|s|$, $\det B > 0$, and B is a P_0^+-matrix.

Next, let M be the partial P_0^+-matrix specifying the digraph $\tilde{D}_4(6,4)$. We show that for the following choices of the unspecified entries and some suitable choice of

t and s, the completion

$$B = \begin{bmatrix} d_1 & a_{12} & t & -a_{41} \\ a_{21} & d_2 & a_{23} & s \\ -t & -a_{23} & d_3 & a_{34} \\ a_{41} & -s & a_{43} & d_4 \end{bmatrix},$$

of M is a P_0^+-matrix. Clearly, for any nonzero choices of t and s, all 2×2 principal minors of B are nonnegative and $B(1,3) > 0$, $B(2,4) > 0$. Now, 3×3 principal minors of B are

$$\begin{aligned} B(1,2,3) &= d_3B(1,2) + d_1a_{23}^2 + d_2t^2 - t(a_{12} + a_{21})a_{23} && (4.6) \\ B(1,2,4) &= d_4B(1,2) + d_2a_{41}^2 + d_1s^2 + s(a_{12} + a_{21})a_{41} && (4.7) \\ B(1,3,4) &= d_1B(3,4) + d_3a_{41}^2 + d_4t^2 + t(a_{34} + a_{43})a_{41} && (4.8) \\ B(2,3,4) &= d_2B(3,4) + d_4a_{23}^2 + d_3s^2 - s(a_{34} + a_{43})a_{23}. && (4.9) \end{aligned}$$

Case 1: $a_{23} = 0$ or $a_{41} = 0$. If $a_{23} = 0$, then choose signs for s and t such that $s(a_{12} + a_{21})a_{41} \geq 0$ and $t(a_{34} + a_{43})a_{41} \geq 0$. Then all 3×3 principal minors are nonnegative, and since $d_i > 0$ for some i, at least one of these minors is positive. Finally, for large values of $|s|$ and $|t|$, $\det B > 0$. The case when $a_{41} = 0$ is similar.

Case 2: $a_{23} \neq 0$ and $a_{41} \neq 0$. If $d_1 > 0$, then we choose t such that

$$\begin{aligned} t(a_{34} + a_{43})a_{41} &\geq 0 \\ d_1a_{23}^2 - t(a_{12} + a_{21})a_{23} &> 0. \end{aligned}$$

Further, we choose sign for s such that $s(a_{34} + a_{43})a_{23} \leq 0$. Then for large values of $|s|$ all 3×3 principal minors are nonnegative, and $B(1,2,3)$, $B(1,2,4)$ and $\det B$ are positive. The cases when d_2, d_3 or d_4 is positive are similar.

Similarly, with suitable values of t and s, a partial P_0^+-matrix M specifying the digraph $\tilde{D}_4(6,5)$ can be completed to a P_0^+-matrix

$$B = \begin{bmatrix} d_1 & a_{12} & t & -a_{41} \\ a_{21} & d_2 & -a_{32} & s \\ -t & a_{32} & d_3 & a_{34} \\ a_{41} & -s & a_{43} & d_4 \end{bmatrix}.$$

Finally, let M be the partial P_0^+-matrix specifying the digraph $\tilde{D}_4(6,7)$. We show that for some choices of s, t and x, y, z, w the completion

$$B = \begin{bmatrix} d_1 & a_{12} & x & s \\ t & d_2 & a_{23} & z \\ y & a_{32} & d_3 & a_{34} \\ a_{41} & w & a_{43} & d_4 \end{bmatrix}$$

of M is a P_0^+-matrix. Clearly, for any choices of s, t, x, y, z, w such that $a_{12}t, a_{41}s,\ xy$ and zw are nonpositive and at least one of them negative, we have all 2×2 principal minors of B are nonnegative and $S_2(B) > 0$. Now, the 3×3 principal minors are

$$\begin{aligned} B(1,2,3) &= \big(d_1B(2,3) - d_3a_{12}t - d_2xy\big) + a_{32}tx + a_{12}a_{23}y \quad (4.10)\\ B(1,3,4) &= \big(d_1B(3,4) - d_3a_{41}s - d_4xy\big) + a_{34}a_{41}x + a_{43}sy \quad (4.11)\\ B(1,2,4) &= \big(d_4B(1,2) - d_2a_{41}s - d_1zw\big) + a_{41}a_{12}z + stw \quad (4.12)\\ B(2,3,4) &= \big(d_2B(3,4) - d_4a_{23}a_{32} - d_3zw\big) + a_{32}a_{43}z + a_{23}a_{34}w. \quad (4.13)\end{aligned}$$

Since the terms in the parentheses in the right side of the above equations are all nonnegative under our choices of the unspecified entries, the 3×3 principal minors are nonnegative if

$$\begin{aligned} a_{32}tx + a_{12}a_{23}y &\geq 0 \quad (4.14)\\ a_{34}a_{41}x + a_{43}sy &\geq 0 \quad (4.15)\\ a_{41}a_{12}z + stw &\geq 0 \quad (4.16)\\ a_{32}a_{43}z + a_{23}a_{34}w &\geq 0. \quad (4.17)\end{aligned}$$

In view of Lemma 4.3.5, the above equations have solutions with $s = t = 0$, and arbitrary large values of $-xy$ and $-zw$ in all cases except when $a_{12}a_{23}a_{34}a_{41} > 0$ and $a_{32}a_{43} = 0$. In the latter case, again setting $s = t = 0$, we have

$$B = \begin{bmatrix} d_1 & a_{12} & x & 0 \\ 0 & d_2 & a_{23} & z \\ y & 0 & d_3 & a_{34} \\ a_{41} & w & 0 & d_4 \end{bmatrix}.$$

If $d_1 > 0$, then a P_0^+-completion of M is obtained by setting $x = y = 0$ and choosing $-z = w$ such that $a_{23}a_{34}\big(d_1 w - a_{12}a_{41}\big) > 0$ in B. A P_0^+-completion of M can be obtained in a similar way in case some other d_i, in stead of d_1, is positive. Note that the structure of the above matrix B is invariant under a cyclic permutation. This completes the proof. □

Lemma 4.3.7. *The digraphs $\tilde{D}_4(5,25)$ and $\tilde{D}_4(5,27)$ have P_0^+-completion.*

Proof. Let M be the partial P_0^+-matrix specifying the digraph $\tilde{D}_4(5,25)$. We show that for some choices of r, s, t, x, y, z and w the completion

$$B = \begin{bmatrix} d_1 & r & a_{13} & a_{14} \\ a_{21} & d_2 & s & x \\ t & a_{32} & d_3 & z \\ a_{41} & y & w & d_4 \end{bmatrix},$$

of M is a P_0^+-matrix.

Case 1: $a_{13} \neq 0$ or $a_{21} \neq 0$. We set

$$r = -a_{21}, \;\; s = -a_{32}, \;\; t = -a_{13}, \;\; y = -x \;\; \text{and} \;\; w = -z.$$

Then all 2×2 principal minors of B are nonnegative and $B(1,2) > 0$. Further, the 3×3 principal minors of B are

$$B(1,2,3) = d_1 B(2,3) + d_3 a_{21}^2 + d_2 a_{13}^2 \tag{4.18}$$

$$B(1,2,4) = d_2 B(1,4) + d_4 a_{21}^2 + d_1 x^2 - a_{21}(a_{14} + a_{41})x \tag{4.19}$$

$$B(1,3,4) = d_3 B(1,4) + d_4 a_{13}^2 + d_1 z^2 + a_{13}(a_{14} + a_{41})z \tag{4.20}$$

$$B(2,3,4) = d_4 B(2,3) + d_3 x^2 + d_2 z^2. \tag{4.21}$$

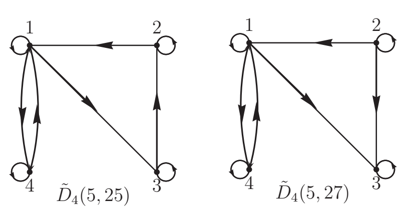


Figure 4.9: The digraphs $\tilde{D}_4(5,25), \tilde{D}_4(5,27)$

For nonzero x and z with $a_{21}(a_{14}+a_{41})x \leq 0$ and $a_{13}(a_{14}+a_{41})z \geq 0$, all 3×3 principal minors of B are nonnegative, and since $d_i > 0$ for some i, at least one of these minors is positive. Moreover,

$$\begin{aligned} \det B &= \det \begin{bmatrix} d_1 & -a_{21} & a_{13} & a_{14} \\ a_{21} & d_2 & -a_{32} & x \\ -a_{13} & a_{32} & d_3 & z \\ -a_{14} & -x & -z & d_4 \end{bmatrix} + \det \begin{bmatrix} 0 & -a_{21} & a_{13} & a_{14} \\ 0 & d_2 & -a_{32} & x \\ 0 & a_{32} & d_3 & z \\ a_{41}+a_{14} & -x & -z & d_4 \end{bmatrix} \\ &\geq (a_{13}x + a_{21}z + a_{14}a_{32})^2 + p(x,z), \end{aligned}$$

where $p(x,z)$ is a polynomial in x and z with total degree at most 1. Hence, for suitable x and z with large magnitudes $\det B > 0$.

Case 2: $a_{13} = a_{21} = 0$. In this case, the 3×3 principal minors of

$$B = \begin{bmatrix} d_1 & r & 0 & a_{14} \\ 0 & d_2 & s & x \\ t & a_{32} & d_3 & z \\ a_{41} & y & w & d_4 \end{bmatrix}$$

are given by

$$B(1,2,3) = d_1 B(2,3) + rst \tag{4.22}$$
$$B(1,2,4) = d_2 B(1,4) + x(a_{41}r - d_1 y) \tag{4.23}$$
$$B(1,3,4) = d_3 B(1,4) + w(a_{14}t - d_1 x) \tag{4.24}$$
$$B(2,3,4) = d_4 B(2,3) - d_3 xy - d_2 zw + a_{32}xw + syz. \tag{4.25}$$

If $a_{14} \neq 0$ then we set

$$r = x = z = 0,\ t = s,\ w = \frac{a_{14}s}{|a_{14}s|},\ y = \frac{-|s|}{a_{14}},$$

where the nonzero s is to be chosen such that $a_{32}s \leq 0$. Then all 3×3 principal minors are nonnegative, $B(1,3,4) > 0$, and since $\det B$ is a monic polynomial in $|s|$ of degree 3, B is a P_0^+-matrix for sufficiently large values of $|s|$. If $a_{41} \neq 0$, then we set

$$t = y = w = 0,\ r = z,\ x = \frac{a_{41}r}{|a_{41}r|},\ s = \frac{-|r|}{a_{41}},$$

where the nonzero s is to be chosen such that $a_{32}s \leq 0$. Then, B is a P_0^+-matrix for sufficiently large values of $|r|$. Finally, if $a_{14} = a_{41} = 0$, then we set $y = z = 0,\ x = t,\ r = st,\ w = -st$ with $s \neq 0$ such that $a_{32}s \leq 0$. Then for large values of $|s|$, B is a P_0^+-matrix.

Similarly, with suitable values of r, x, y, w, z, t and s, any partial P_0^+-matrix

$$M = \begin{bmatrix} d_1 & r & a_{13} & a_{14} \\ a_{21} & d_2 & a_{23} & x \\ t & s & d_3 & z \\ a_{41} & y & w & d_4 \end{bmatrix}$$

specifying the digraph $\tilde{D}_4(5, 27)$ can be completed to a P_0^+-matrix. □

Lemma 4.3.8. *The asymmetric digraphs $\tilde{D}_4(5, 30)$ and $\tilde{D}_4(5, 38)$ have P_0^+-completion.*

Proof. Since a partial P_0^+-matrix specifying $\tilde{D}_4(5, 30)$ is the transpose (in the usual sense) of a partial P_0^+-matrix specifying $\tilde{D}_4(5, 38)$, it is enough to prove that $\tilde{D}_4(5, 38)$ has P_0^+-completion.

Consider a partial P_0^+ matrix

$$M = \begin{bmatrix} d_1 & a_{12} & x & a_{14} \\ y & d_2 & r & a_{24} \\ s & a_{32} & d_3 & z \\ u & w & a_{43} & d_4 \end{bmatrix},$$

specifying the digraph $\tilde{D}_4(5, 38)$, where x, y, r, s, z, u, w are unspecified entries. Now, for a completion B of M the following 3×3 principal minors of B are

$$B(1,2,3) = d_1B(2,3) + a_{12}sr - d_3ya_{12} + xya_{32} - d_2xs \tag{4.26}$$

$$B(1,2,4) = d_1B(2,4) - a_{12}yd_4 + a_{14}yw + a_{12}a_{24}u - d_2a_{14}u \tag{4.27}$$

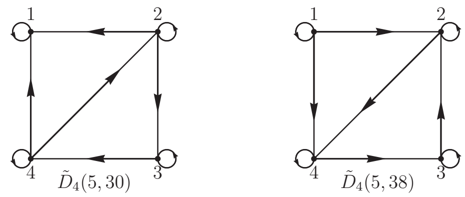
$$B(1,3,4) = d_1B(3,4) + xzu - sxd_4 + a_{14}sa_{43} - a_{14}ud_3 \tag{4.28}$$

$$B(2,3,4) = d_2B(3,4) + rzw - ra_{32}d_4 + a_{24}a_{32}a_{43} - a_{24}wd_3 \tag{4.29}$$

Case 1: $a_{24} \neq 0$ or $a_{12}a_{43} + a_{14}a_{32} \neq 0$. In that case, we set

$$y = -a_{12},\ r = -a_{32},\ u = -a_{14},\ w = -a_{24},\ z = -a_{43},\ x = -s = t.$$

Then, all 2×2 principal minors are nonnegative and for $t \neq 0$, $B(1,3) > 0$. Also it can be easily seen that each principal minor of order 3 is nonnegative.



$\tilde{D}_4(5,30)$ $\tilde{D}_4(5,38)$

Now M being a partial P_0^+-matrix, at least one of the d_i is positive. Moreover, in case $(a_{14}a_{32} + a_{43}a_{12}) \neq 0$, at least one of the $a_{12}a_{43}$ and $a_{14}a_{32}$ is nonzero. It can be easily verified that in each of the three cases, viz. $a_{24} \neq 0$, $a_{12} \neq 0 \neq a_{43}$ and $a_{14} \neq 0 \neq a_{32}$, at least one 3×3 principal minor of B is positive, depending on i with $d_i > 0$. Further, by the condition $a_{24} \neq 0$ or $a_{12}a_{43} + a_{14}a_{32} \neq 0$, we have for any nonzero t,

$$\begin{aligned} \det B &= (a_{14}a_{32} + a_{43}a_{12} + ta_{24})^2 + d_1d_2a_{43}^2 + d_1d_3a_{24}^2 + d_1d_4a_{32}^2 \\ &\quad + d_2d_3a_{14}^2 + d_2d_4t^2 + d_3d_4a_{12}^2 > 0 \end{aligned}$$

Case 2: $a_{24} = 0, a_{12}a_{43} + a_{14}a_{32} = 0$ This case is divided into the following subcases:

Subcase I: $\big(a_{32} \neq 0$ or $a_{14} \neq 0\big)$ and $\big(a_{12} \neq 0$ or $a_{43} \neq 0\big)$. In this case, we consider a completion B of M by putting $r = s = u = 0$ in M. We choose w, y and z with nonzero value such that $a_{12}y \leq 0$, $a_{43}z \leq 0$ and $a_{14}yw > 0$. Then, each of the 2×2 principal minor of B becomes nonnegative. Since at least one of a_{12}, a_{43} is nonzero, therefore we have either $B(1,2) > 0$ or $B(3,4) > 0$. Now, the equation $a_{12}a_{43} + a_{14}a_{32} = 0$ implies that sign of any term in the equation (4.3) is the opposite sign of the product of other three terms. So, we can always choose an appropriate sign of x such that $ya_{32}x > 0$ and $-xyzw > 0$. Hence, each of the 3×3 principal minor becomes nonnegative, and either $B(1,2,3)$ or $B(1,2,4)$ is positive. Finally we choose $|x| = |y| = |z| = |w| = t > 0$. Now for sufficiently large values of t, we have $\det B = t^4 + p(t) > 0$, where p(t) is a polynomial of degree at most 3.

Subcase II: $(a_{32} = 0$ and $a_{14} = 0)$ and $(a_{12} \neq 0$ or $a_{43} \neq 0)$. In this case, consider a completion B of M obtained by putting $s = u = 0$ in M. Now, we choose the appropriate sign of $y,\ z,\ r$ such that $a_{12}y \leq 0$, $a_{43}z \leq 0$ and $rwz > 0$. Since at least one of a_{12} and a_{43} is nonzero, therefore each 2×2 principal minor is nonnegative and either $B(1,2) > 0$ or $B(3,4) > 0$. Also each of the 3×3 principal minor becomes nonnegative and $B(2,3,4) > 0$. Finally, we choose the appropriate sign of x and $|x| = |y| = |z| = |w| = t > 0$. Now for large values of t, we have $\det B = t^4 + p(t) > 0$, where p(t) is a polynomial of degree at most 3.

Subcase III: $a_{24} = 0$; ($a_{32} \neq 0$ or $a_{14} \neq 0$) and ($a_{12} = 0$ and $a_{43} = 0$). Consider a completion B of M obtained by putting, $u = r = 0$ and $s = -x = t$. Now all 2×2 principal minors of B are nonnegative and for large values of t, $B(1,3) > 0$. Now we choose the appropriate sign of t, w such that $-tya_{32} \geq 0, a_{14}yw \geq 0$. Since $a_{32} \neq 0$ or $a_{14} \neq 0$, therefore each of the 3×3 principal minor is nonnegative, and either $B(1,2,3)$ or $B(1,2,4)$ is positive. Finally, we choose the appropriate sign of z and $|x| = |y| = |z| = |w| = t > 0$, such that $\det B > 0$ for large values of t.

Subcase IV: We are left with the case when all specified off diagonal entries are zero. In this case, a P_0^+-completion is obtained by putting $x = r = z = w = -y = -s = t$, when $t > 0$. Combining all cases, we see that the partial P_0^+-matrix M can be completed to a P_0^+-matrix and the digraph $\tilde{D}_4(5,38)$ has a P_0^+-completion. □

Theorem 4.3.9. *For $1 \leq p \leq 4$, the digraph $\tilde{D}_p(q,n)$ has P_0^+-completion if and only if it is one of the digraphs listed below:*

$$\begin{aligned}
&p = 1 \\
&p = 2; \quad q = 0, 2, 6; \quad n = 1 \\
&p = 3; \quad\ \ q = 0, 1 \\
&\qquad\qquad\ \ q = 2; \qquad n = 2
\end{aligned}$$

$p = 4;\quad q = 0, 1, 6 \quad n = 1$

$q = 2; \quad n = 1\text{-}5$

$q = 3; \quad n = 1\text{-}7, 9, 10, 12, 13$

$q = 4; \quad n = 1\text{-}9, 11, 16\text{-}20, 22\text{-}26$

$q = 5; \quad n = 1\text{-}3, 7\text{-}10, 25, 27, 30, 32, 33, 35, 38$

$q = 6; \quad n = 3\text{-}5, 7.$

Proof. It is clear that $\tilde{D}_p(q, n)$ has P_0^+-completion if $q = 0$ or it is a complete symmetric digraph.

Case I: $p = 3$. Consider a partial P_0^+-matrix

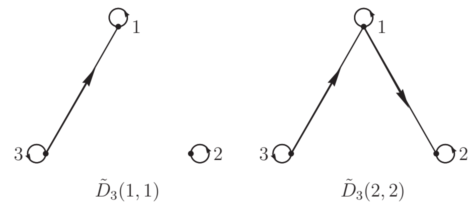
$$M = \begin{bmatrix} d_1 & a_{12} & u \\ v & d_2 & s \\ a_{31} & -s & d_3 \end{bmatrix},$$

specifying $\tilde{D}_3(2, 2)$, where u, v, s are unspecified entries in M. Now for a completion B of M, we choose the appropriate sign of u and v such that $a_{12}v \leq 0$ and $a_{31}u \leq 0$ hold , respectively. Thus all 2×2 principal minors of B are nonnegative, and for $s > 0$ we have $B(2, 3) > 0$. Finally we choose the sign of s such that $-uvs > 0$ and we set $|u| = |v| = |s| = t$ such that,

$$\det B = t^3 + p(t) > 0,$$

where $p(t)$ is a polynomial in t of degree at most 2. Now for sufficiently large values of $t > 0$, B is a P_0^+ matrix. Since $\tilde{D}_3(1, 1)$ is spanning subdigraph of $\tilde{D}_3(2, 2)$ and it has P_0-completion, thus by Theorem 4.1.9 it has P_0^+-completion. The remainder of the digraphs of order 3 appear in the lists in Lemma 4.3.1 and Lemma 4.3.2 and do not have P_0^+-completion.

Case 2: $p = 4$. The digraphs $\tilde{D}_4(q, n), q = 5, n = 1\text{-}3; q = 4, n = 1\text{-}9;\ q = 3, n = 1\text{-}7;\ q = 2, n = 1\text{-}5;\ q = 1$ have P_0-completion (see [7]) and each of them is a spanning subdigraph of $\tilde{D}_4(6, 3)$. Since the digraph $\tilde{D}_4(6, 3)$ have P_0^+-completion (see Lemma 4.3.6), by Theorem 4.1.9, these digraphs have P_0^+-completion.



Again the digraphs $\tilde{D}_4(q,n), q = 5, n = 7, 9; q = 4, n = 16$ - 18 have P_0-completion and each of them is a spanning subdigraph of $\tilde{D}_4(6,4)$. Since the digraph $\tilde{D}_4(6,4)$ have P_0^+-completion (see Lemma 4.3.6), by Theorem 4.1.9, these digraphs have P_0^+-completion.

Now, the digraphs $\tilde{D}_4(q,n), q = 5, n = 8, 10; q = 4, n = 19$ have P_0-completion and each of them is a spanning subdigraph of $\tilde{D}_4(6,5)$. Since the digraph $\tilde{D}_4(6,5)$ have P_0^+-completion (see Lemma 4.3.6), by Theorem 4.1.9, these digraphs have P_0^+-completion.

On the other hand, the digraphs $\tilde{D}_4(q,n), q = 4, n = 11, 20, 24; q = 3, n = 9, 10, 12$ have P_0-completion and each of them is a spanning subdigraph of $\tilde{D}_4(5,25)$. Since the digraph $\tilde{D}_4(5,25)$ have P_0^+-completion (see Lemma 4.3.7), by Theorem 4.1.9, these digraphs have P_0^+-completion.

Also, the digraphs $\tilde{D}_4(q,n), q = 4, n = 22, 26; q = 3, n = 13$ have P_0-completion and each of them is a spanning subdigraph of $\tilde{D}_4(5,27)$. Since the digraph $\tilde{D}_4(5,27)$ have P_0^+-completion (see Lemma 4.3.7), by Theorem 4.1.9, these digraphs has P_0^+-completion. By the similar proof technique of Lemma 4.3.8, the asymmetric digraphs $\tilde{D}_4(5,n); n = 32, 33, 35$ have P_0^+-completion.

The proof is complete, because any other digraph of order 4 does not have P_0^+-completion, in view of Lemmas 4.3.1– 4.3.4. $\square$

4.4 Conclusion

In this chapter, we have discussed the completion problem of P_0^+-matrices which is a subclass of the classes of P_0 and Q-matrices. We have obtained certain necessary conditions and some sufficient conditions for a digraph to have positive P_0^+-completion.

The results helped us to single out all digraphs of order at most 4 that include all loops. It is found that complements of tournaments do not have P_0^+-completion.

Though every induced subdigraph of a digraph D having P_0^+-completion must have P_0 completion, it is unsettled whether D itself should have P_0-completion. In fact, the P_0^+-matrix completion is far from being completely understood. A complete characterization for a digraph to have P_0^+-completion could not be found yet. We note that the nature of the P_0^+-matrix completion problem is different from the positive and nonnegative Q-matrix completion problems. We will compare the completion problems in Chapter 5.

5

Comparison between Completion Problems

In this chapter, we examine the relationship between certain matrix completion problems including the completion problems for the classes of positive Q, nonnegative Q and P_0^+-matrices. Based on the results obtained in last three chapters, we compare the completion problems for the three classes with those for the classes of P and P_0-matrices in Section 5.1. In Section 5.2, the three completion problems are compared among themselves. We compare the three completion problems with the Q-matrix completion problem in Section 5.3.

5.1 Comparison with P and P_0-completions

In this section, we compare the completion problems for the classes of P and P_0-matrices with the completion problems for the three classes of matrices that we have studied, namely, the classes of positive Q, nonnegative Q and P_0^+-matrices.

Let Π_0 denote one of the classes of P and P_0-matrices, and Π_1 one of the classes of positive Q, nonnegative Q and P_0^+-matrices. We distinguish the nature of the Π_1-completion problems from the Π_0-completion problems in view of the result obtained in previous chapters. However, we prove that the completion problems of classes of P-matrices and P_0^+-matrices are related.

(i) Theorem 0.0.8 asserts that a digraph D has Π_0-completion if and only if each of its components (strong components) has Π_0-completion. The implication of the result is that it is enough to solve the Π_0-matrix completion problem for strongly connected digraphs.

However, the situation is not so for the Π_1-matrix completion problem, although the assertions of Theorem 0.0.8 are partially true in some cases.

(a) We have seen in Theorem 2.1.8 that if every strong component of a digraph D has positive Q-completion and none of them is complete, then D has positive Q-completion. However, the converse is not true which was seen in Example 2.2.4.

(b) Similarly, Theorem 3.1.6 shows that if every strong component of a digraph D has nonnegative Q-completion and none of them is complete, then D has nonnegative Q-completion. However, the converse is not true which was seen in Example 3.1.9.

(c) The assertion of Theorem 0.0.8 is not true with Π as the class of P_0^+-matrices. A digraph may not have P_0^+-completion even if each of its components has. An example is provided in Example 4.1.7 with the components as complete symmetric digraphs. However, it is not clear whether the assertion is valid if none of the components is complete.

(ii) Let $G \neq K_n$ be a 1-chordal graph with two nontrivial maximal cliques. Theorem 0.0.4 asserts that G has Π_0-completion. However, a similar assertion is not true for the Π_1-matrix completion problems. Let D be the digraph for patterns of partial matrices specifying G. Then, D includes all loops.

(a) Since G is not trivial, D has a 2-cycle, and therefore, by Theorem 2.3.1, D does not have positive Q-completion, i.e., G does not have positive Q-completion.

(b) Since D includes all loops, it follows from Proposition 3.3.1 that D, and therefore G, does not have nonnegative Q-completion.

(c) In Corollary 4.2.8 it has been noted that any 1-chordal graph with two nontrivial maximal cliques does not have P_0^+-completion.

(iii) Let D be a digraph and $\alpha \neq \emptyset$ be the set of vertices at which D includes loops. Let D_1 be the subdigraph of D induced by α. Theorem 0.0.9 asserts that D has P-completion if and only if D_1 has P-completion. On the contrary, a similar result is not valid for the Π_1 classes.

(a) If D_1 is not complete and has positive Q-completion, then D has positive Q-completion (cf. Proposition 2.1.4). However, the converse is not true as seen in Example 2.1.6.

(b) Since D_1 includes all loops, it has nonnegative Q-completion only if it is a complete symmetric digraph. Further, if D_1 is complete (and therefore has nonnegative Q-completion), it is not necessary that D has nonnegative Q-completion (cf. Remark 3.3.3).

(c) Example 4.1.6 provides a digraph D such that the corresponding subdigraph D_1 has P_0^+-completion, although D does not have. On the other hand, Example 4.1.8 provides a digraph D that have P_0^+-completion, but the corresponding D_1 does not have.

The following result shows that the completion problems for the classes of P and P_0^+ are related.

Theorem 5.1.1. *Any digraph that has P_0^+-completion also has P-completion.*

Proof. Let D be a digraph which has P_0^+-completion. Let M be a partial P-matrix specifying D. Let I_M be the partial matrix specifying D with all specified diagonal entries 1 and off-diagonal entries 0. Since determinant of a matrix is a continuous function of its entries, there is $\epsilon > 0$ such that the partial matrix $M_0 = M - \epsilon I_M$ (i.e., one obtained by applying operations on the respective specified entries) is a partial P-matrix. Clearly, M_0 is a partial P_0^+-matrix specifying D and therefore has a P_0^+-completion B_0. We now have a P-completion of M, namely, $B = B_0 + \epsilon I$, where I is the identity matrix. □

The effective argument used in the above proof was applied for comparing a large number of pairs of completion problems by L. Hogben [23]. That the converse of Theorem 5.1.1 is not true can be seen from the following example.

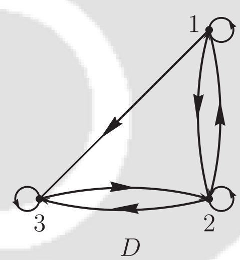


FIGURE 5.1: A digraph having P-completion, but not P_0^+-completion

Example 5.1.2. Consider the partial P_0^+-matrix,

$$M_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ ? & 0 & 0 \end{bmatrix}$$

which specifies the digraph D in Figure 5.1. It is clear that M_1 cannot be completed to a P_0^+-matrix since all 2×2 principal minors of M_1 are zero for any completion $\widehat{M_1}$ of M_1. On the other hand, any digraph of order 3 has P-completion (see [27]).

5.2 Comparison between the three completion problems

In this section, we compare the completion problems for the three classes of matrices that we have studied, namely, those for the classes of positive Q, nonnegative Q and P_0^+-matrices. Apart from a few common results seen in the previous section, the results obtained in the previous three chapters show that the three completion problems are quite different. Nevertheless, between the positive Q and the nonnegative Q-matrix completion problems, we have the following.

Proposition 5.2.1. *If a digraph D has nonnegative Q-completion, then D has positive Q-completion.*

Proof. Let D be a digraph of order n having nonnegative Q-completion. Let $M = [a_{ij}]$ be a partial positive Q-matrix specifying the digraph D. Then M is a partial nonnegative Q-matrix specifying D. Let B be a nonnegative Q-completion of M, i.e., $S_k(B) > 0$ for $k = 1, 2, \ldots, n$. Since the determinant of a matrix is a continuous function of the matrix entries, perturbing the zero entries in B by small positive quantities, a positive Q-completion of M can be obtained. $\square$

However, the converse is not true which can be seen from the following example.

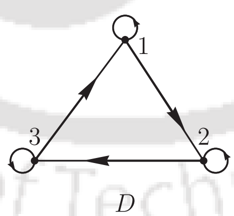


Figure 5.2: A digraph having positive, but not nonnegative Q-completion

Example 5.2.2. The digraph D in Figure 5.2 has positive Q-completion, since it does not have any even cycle (cf. Theorem 2.2.1). However, in view of Proposition 3.2.1, D does not have nonnegative Q-completion, since D has all loops.

We list below a few dissimilarities among the three completion problems.

(i) Theorem 2.3.1 asserts that a digraph which includes all loops and contains a 2-cycle does not have positive Q-completion. Thus, D must be a subdigraph of the complement of a tournament. In Section 2.4, all complements of tournaments which have positive Q-completion have been singled out.

On the other hand, for a digraph D to have P_0^+-completion, either D omits at least two loops, or $\overline{D}$ has a 2-cycle (cf. Corollary 4.2.9). In particular, the complement of a tournament cannot have P_0^+-completion.

(ii) In Proposition 3.3.1, it has been found that a digraph with at least two vertices has nonnegative Q-completion only if it omits at least two loops. However, this restriction on loops is not there in case of positive Q and P_0^+-completions.

(iii) Theorem 3.3.4 asserts that positive stratification of the complement of a digraph is required for a digraph to have nonnegative Q-completion. similarly, stratification or weak stratification of the complement of the digraph is required for the digraph to have P_0^+-completion (cf. Theorems 4.2.3 and 4.2.5). On the contrary, stratification of the complement does not play any role in the positive Q-completion problem.

5.3 Comparison with the Q-matrix completion problem

In this section we compare completion problems for the classes of positive Q, nonnegative Q and P_0^+-matrices with the Q-completion problem.

5.3.1 Q-completion and positive Q-completion

Although every positive Q-matrix is a Q-matrix, and every partial positive Q-matrix is a partial Q-matrix, the completion problems for the two classes are quite different.

If $B = [b_{ij}]$ is an $n \times n$ Q-matrix, then $S_k(B) > 0$ for $1 \leq k \leq n$. Consequently, for each k there must be $\pi \in \mathcal{P}_k$ such that $w(\pi, B) = \prod b_{i\pi(i)} > 0$. We note that the permutation π will generate a permutation digraph D_π of order k.

Let $D \neq K_n$ be a digraph with vertex set $\{1, 2, \ldots, n\}$. Define a partial matrix $M = [m_{ij}]$ as follows:

1. If D omits a loops, then put all specified entries of M as zero;

2. If D includes all loops then choose $i \in \{1, 2, \ldots, n\}$, and put $m_{ii} = 1$ and let all other specified entries to be zero.

Then M is a partial Q-matrix. If M has a Q-completion, then it follows from the observation in the preceding paragraph that for each $k = 2, 3, \ldots, n$, there must be a permutation subdigraph of order k in $\overline{D}$ (or of order $k-1$ in $\overline{D}-i$, in the second case). These observations lead to the necessary conditions (as given in Theorem 0.0.10 and Theorem 0.0.11) for D to have Q-completion.

For an $n \times n$ positive matrix $B = [b_{ij}]$, we have $w(\pi, B) > 0$ for any $\pi \in \mathcal{P}_k$, $1 \leq k \leq n$. Thus, the conditions in the above theorems are not pertinent to the positive Q-completion problem. It is evident from the discussion in Section 2 of Chapter 2 that a necessary condition for a digraph D to have positive Q-completion is the availability of arcs in $\overline{D}$ whose addition to D produces permutation digraph D_π in D with $\pi \in \mathcal{P}_k^+$, $1 \leq k \leq n$. Further, if addition of such arcs to D does not produce permutation digraphs D_π in D with $\pi \in \mathcal{P}_k^-$, then D has positive Q-completion. Clearly, the partition $(\mathcal{P}_k^+, \mathcal{P}_k^-)$ of $\mathcal{P}_k$ does not play a role in Q-completion problem, since there is no sign condition on the matrices considered.

Following examples distinguish the two matrix completion problems.

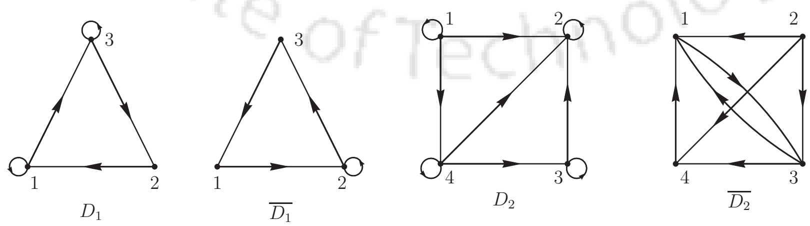


Figure 5.3: Digraphs having positive Q-completion but not Q-completion

Example 5.3.1. The digraphs D_1 and D_2 in Figure 5.3 do not have any even cycle. Therefore, in view of Theorem 2.2.1, D_1 and D_2 have positive Q-completion.

However, neither of D_1 and D_2 has Q-completion. Indeed, D_1 omits a loop at the vertex 2, and $\overline{D_1}$ is not stratified, since it does not have a 2-cycle. Thus, D_1 does not satisfy the condition in Theorem 0.0.10. Again, D_2 includes all loops and $\overline{D_2}$ is not weakly stratified ($\overline{D_2}$ does not have a 4-cycle, and $\overline{D_2} - 1$ does not have a 3-cycle). Therefore, D_2 does not satisfy the condition in Theorem 0.0.11.

Example 5.3.2. Consider the digraph D_3 in Figure 5.4. Since D_3 includes all loops and has a 2-cycle, in view of Theorem 2.3.1, D_3 does not have positive Q-completion. However, D_3 has Q-completion. In fact, any digraph whose complement contains $\overline{D_3}$ as a subdigraph has Q-completion (see Theorem 2.19 of [10]).

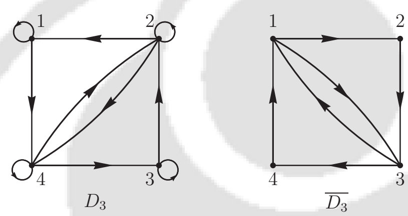


FIGURE 5.4: The digraph D_3 has Q-completion, but not positive Q-completion

The digraphs of order at most four that include all loops and have Q-completion have been characterized in [10]. Of the 16 unlabeled digraphs of order three that include all loops, four have Q-completion, and it has been found that each of these four digraphs has positive Q-completion. Of the 218 unlabeled digraphs that include all loops, 72 have Q-completion. Thirty of these 72 digraphs also have positive Q-completion.

5.3.2 Q-completion and nonnegative Q-completion

We list here some similarities and dissimilarities between the Q-completion and the nonnegative completion problems.

(i) For a digraph D that omits a loop, stratification of $\overline{D}$ is necessary for D to have Q-completion. On the other hand, positive stratification of $\overline{D}$ is a necessary condition for D to have nonnegative Q-completion (cf. Theorem 3.3.4).

(ii) For a digraph D that includes all loops, weak stratification of $\overline{D}$ is necessary for D to have Q-completion. On the other hand, if a digraph D with at least two vertices has nonnegative Q-completion, then it must omits at least two loops (cf. Proposition 3.3.1). Thus, a digraph that includes all loops cannot have a nonnegative Q-completion, unless it is a complete symmetric digraph.

(iii) If the complement $\overline{D}$ of a digraph D is stratified and there is a signing of the arcs of $\overline{D}$ for which all its cycles are positively signed, then D has Q-completion (cf. Theorem 0.0.12). On the contrary, the complement of the digraph D_5 in Example 3.3.6 is stratified and admits such a signing of its arcs, although D_5 does not have nonnegative Q-completion. Thus, Example 3.3.6 shows that a digraph having Q-completion may fail to have nonnegative Q-completion.

Remark 5.3.3. Suppose D is a digraph having nonnegative Q-completion. Then, $\overline{D}$ is stratified and omits at least two loops. For all small digraphs (including all digraph of order 4) having these properties are seen to has Q-completion. Whether a stratified digraph omitting a loop necessarily has Q-completion is not known (cf. Question 2.9 in [10]). We do not know whether there is a digraph which has nonnegative Q completion, but not Q-completion.

5.3.3 Q-completion and P_0^+-completion

Although a P_0^+-matrix is a Q-matrix, the completion problems for the two classes are of different nature. We list here some similarities and dissimilarities between the Q-completion and the nonnegative Q-completion problems.

(i) As in the case of the Q-matrix completion problem, we have the following: Let D be a digraph having P_0^+-completion.

   (a) If D omits a loop , then $\overline{D}$ is stratified (cf. Theorem 4.2.3).

   (b) If D includes all loops, then $\overline{D}$ is weakly stratified (cf. Theorem 4.2.5).

   It has been resolved with Examples 4.2.4 and 4.2.6 that the conditions mentioned above are not sufficient for the P_0^+-completion problem.

(ii) If a digraph D has P_0^+-completion, then every induced subdigraph of D must have P_0-completion (cf. Theorem 4.2.1). However, for Q-matrix completion of a digraph this is not a necessary condition. For example, consider the digraph D_1 in Figure 4.4 (Chapter 4) does not have P_0^+-completion, since the subdigraph of D_1 induced by the vertices $1, 2$ and 3 does not have P_0-completion (cf. Example 4.2.4). However, D_1 has Q-completion, because $\overline{D_1}$ is stratified and has a signing of its arcs so that each cycle is of positive sign.

(iii) A digraph D may not have P_0^+-completion, even if $\overline{D}$ is stratified and admits a signing of its arcs so that each cycle is of positive sign. For example, C_4 is such a digraph which does not have P_0^+-completion (cf. Example 4.2.12).

Example 5.3.4. Although a P_0^+-matrix is a Q-matrix as well as a P_0-matrix, a digraph which has both Q and P_0-completions may or may not have P_0^+-completion. The digraph $\tilde{D}_4(7, 3)$ in Figure 5.5 has both Q and P_0-completions, but does not have P_0^+-completion. On the other hand, the digraph $\tilde{D}_4(6, 4)$ in Figure 5.5 have Q, P_0 as well as P_0^+-completions.

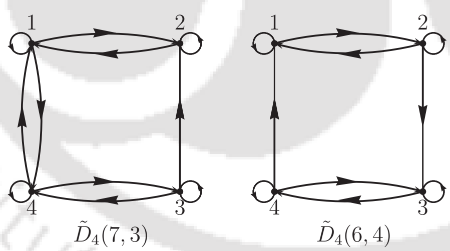


FIGURE 5.5: P_0^+-completion vs Q, P_0-completion

Remark 5.3.5. Suppose D is a digraph having P_0^+-completion. Then, either $\overline{D}$ is stratified or weak stratified. Since every P_0^+-matrix is a Q-matrix and Question 2.9 of [10] is still unresolved, we do not know whether a digraph having P_0^+-completion necessarily has Q-completion. Further, we do not know whether having a P_0-completion is necessary for a digraph to have P_0^+-completion.

5.4 Conclusion

In this chapter, we have compared the completion problems for the classes of positive Q, nonnegative Q and P_0^+-matrices among themselves, based on the results obtained in the previous three chapters. Further, the three completion problems have been compared with the completion problems for the classes of P, P_0 and Q-matrices. We have found that each of the completion problems that we have studied is distinct in nature from the others and therefore requires customized techniques for its solutions. We realize that more efforts are required to understand these completion problems further. For Example, the following problems which could not be settled during this study have merits to be investigated.

(i) Suppose $D_1, \ldots, D_r$ be the components with more than one vertex in a digraph D. If for some i, the strong components D_i is a complete symmetric digraph and has positive Q-completion, then does D necessarily have positive Q-completion?

The same question can also be asked for the nonnegative Q-completion problem.

(ii) Suppose D is a digraph of order n that has non-negative Q-completion. Does there always exist a weight function ϕ on $\overline{D}$ which satisfies the following: for each $k \in \{1, \ldots, n\}$, there is a positive permutation subdigraph P_k of order k in $\overline{D}$ such that,

 (i) $\phi^*(P_k) > \phi^*(P)$ for every negative permutation subdigraph P of $\overline{D}$ of size k, and

 (ii) $\phi^*(P_k) > \sum\limits_{e \in S} \phi(e)$, for any subset $S \subseteq A(\overline{D})$ with $|S| \leq k-1$.

(iii) Suppose D is a digraph of order n that has P_0-completion and its complement $\overline{D}$ is stratified. If it is possible to sign the arcs of $\overline{D}$ such that all cycles of $\overline{D}$ are of positive sign, then does D necessarily have P_0^+-completion?

(iv) Suppose D is a digraph of order n such that none of the components of D

is complete and each of them has P_0^+-completion. Does D necessarily have P_0^+-completion?

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List of papers accepted/communicated

Based on the work in this thesis, the following research articles have been communicated for possible publication.

1. B.K. Sarma and **Kalyan Sinha**, *The P_0^+-matrix completion problem*, accepted, *Electronic Journal of Linear Algebra.*

2. B.K. Sarma and **Kalyan Sinha**, *The positive Q-matrix completion problem*, revised version submitted to *Discrete Mathematics, Algorithms and Applications.*

3. B.K. Sarma and **Kalyan Sinha**, *The nonnegative Q-matrix completion problem*, communicated to *Transactions of Combinatorics.*

Appendices

A

Digraphs of order at most 4

We produce here all digraphs (up to isomorphism) of order at most 4 for convenience of the reader of this thesis. The atlas has been reproduced from the celebrated book *Graph Theory* [14] by F. Harary, where it appears as Appendix 2.

There are $1, 3, 16, 218$ and 9608 nonisomorphic digraphs of orders $1, 2, 3, 4$ and 5, respectively. We list the digraphs with at most 4 vertices according to the number of vertices and arcs. A digraph D has been named as $D_p(q, n)$, where p equals the number of vertices and q the number of arcs in D, and n denotes the serial number of D in the list of all digraphs with p vertices and q arcs. The indices n are assigned to each one in such a way that the complements receive the same index, except for the cases $p = 3, q = 3$ and $p = 4, q = 6$.

Digraphs of order 2 and 3

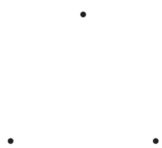
$p = 2, q = 0, 1, 2$

$D_2(0,1)$

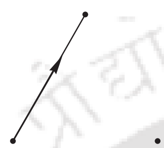
$D_2(1,1)$

$D_2(2,1)$

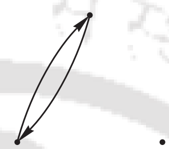
$p = 3, q = 0, 1, 2$



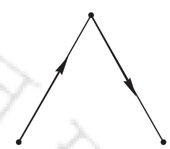
$D_3(0,1)$



$D_3(1,1)$

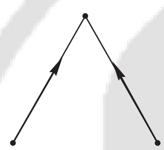


$D_3(2,1)$



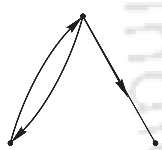
$D_3(2,2)$

$D_3(2,3)$

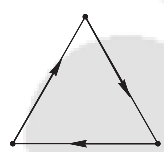


$D_3(2,4)$

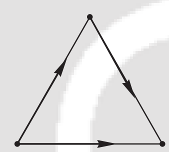
$p = 3, q = 3$



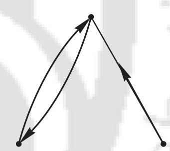
$D_3(3,1)$



$D_3(3,2)$

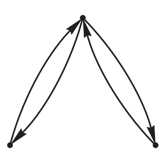


$D_3(3,3)$



$D_3(3,4)$

$p = 3, q = 4, 5, 6$



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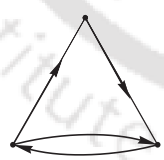

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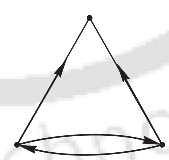

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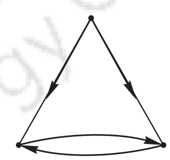
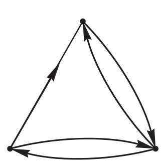
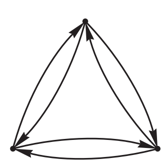


$D_3(4,4)$



$D_3(5,1)$

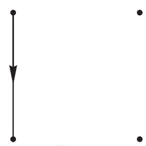


$D_3(6,1)$

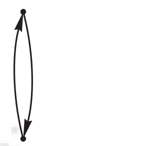
Digraphs of order 4 ($p = 4$)

$p = 4, q = 0, 1, 2$

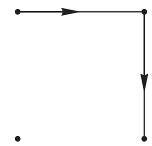
$D_4(0,1)$



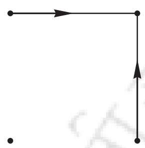
$D_4(1,1)$



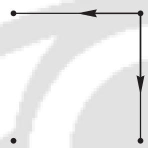
$D_4(2,1)$



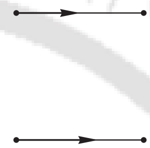
$D_4(2,2)$



$D_4(2,3)$

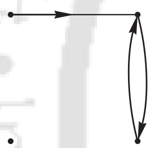


$D_4(2,4)$

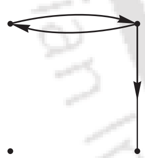


$D_4(2,5)$

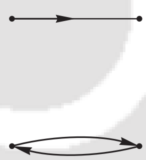
$p = 4, q = 3$



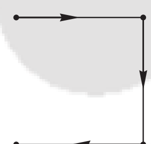
$D_4(3,1)$



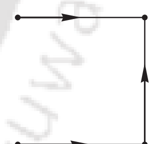
$D_4(3,2)$



$D_4(3,3)$



$D_4(3,4)$



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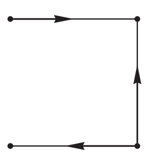
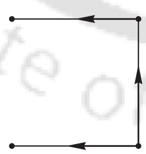
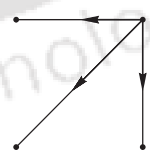


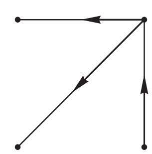
$D_4(3,6)$



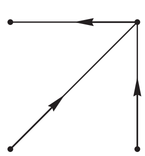
$D_4(3,7)$



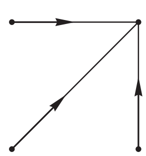
$D_4(3,8)$



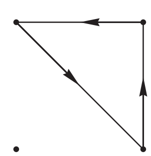
$D_4(3,9)$



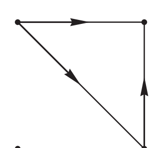
$D_4(3,10)$



$D_4(3,11)$

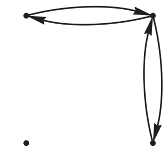


$D_4(3,12)$

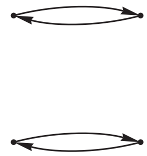


$D_4(3,13)$

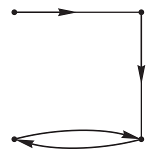
$p = 4, q = 4$



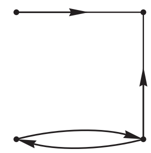
$D_4(4,1)$



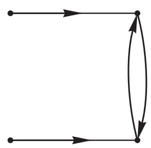
$D_4(4,2)$



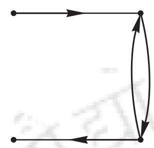
$D_4(4,3)$



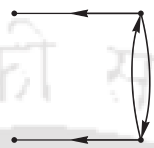
$D_4(4,4)$



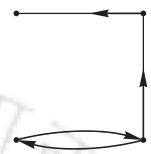
$D_4(4,5)$



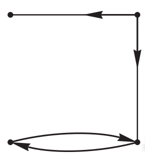
$D_4(4,6)$



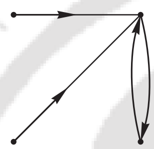
$D_4(4,7)$



$D_4(4,8)$



$D_4(4,9)$



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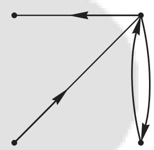
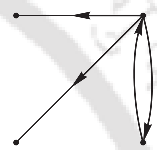
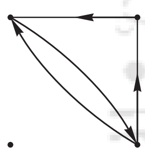


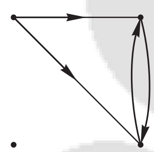
$D_4(4,11)$



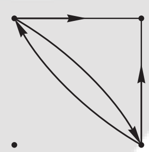
$D_4(4,12)$



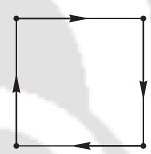
$D_4(4,13)$



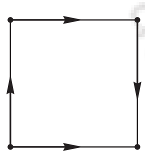
$D_4(4,14)$



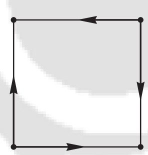
$D_4(4,15)$



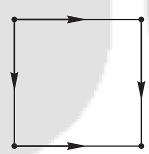
$D_4(4,16)$



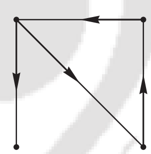
$D_4(4,17)$



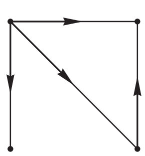
$D_4(4,18)$



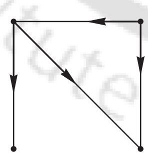
$D_4(4,19)$



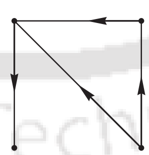
$D_4(4,20)$



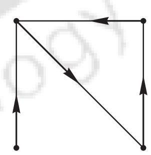
$D_4(4,21)$



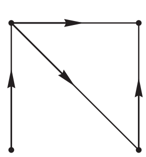
$D_4(4,22)$



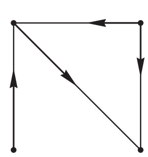
$D_4(4,23)$



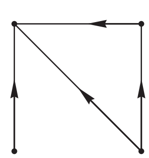
$D_4(4,24)$



$D_4(4,25)$

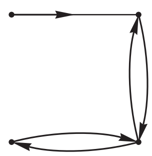


$D_4(4,26)$

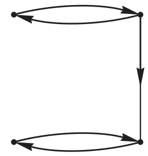


$D_4(4,27)$

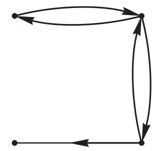
$p = 4, q = 5$



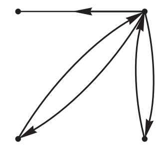
$D_4(5,1)$



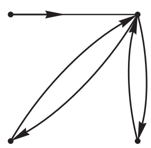
$D_4(5,2)$



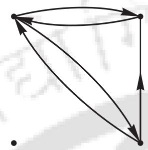
$D_4(5,3)$



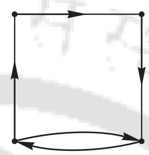
$D_4(5,4)$



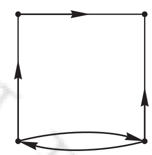
$D_4(5,5)$



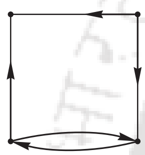
$D_4(5,6)$



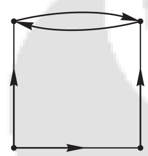
$D_4(5,7)$



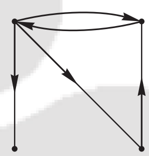
$D_4(5,8)$



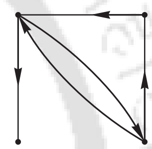
$D_4(5,9)$



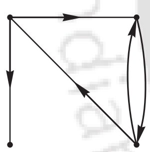
$D_4(5,10)$



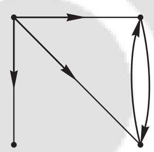
$D_4(5,11)$



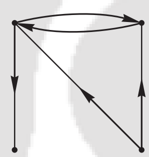
$D_4(5,12)$



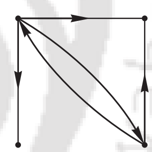
$D_4(5,13)$



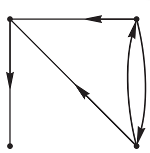
$D_4(5,14)$



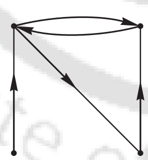
$D_4(5,15)$



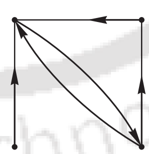
$D_4(5,16)$



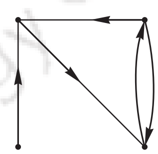
$D_4(5,17)$



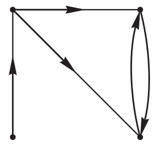
$D_4(5,18)$



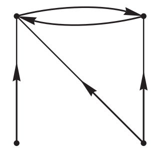
$D_4(5,19)$



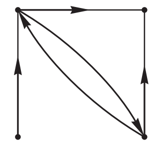
$D_4(5,20)$



$D_4(5,21)$



$D_4(5,22)$



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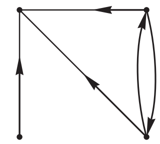
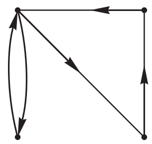
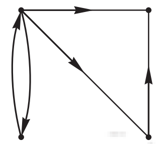


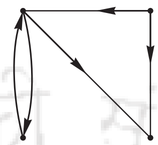
$D_4(5,24)$



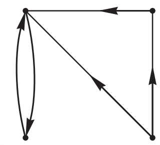
$D_4(5,25)$



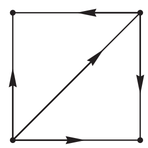
$D_4(5,26)$



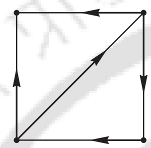
$D_4(5,27)$



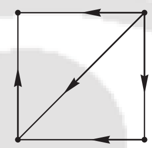
$D_4(5,28)$



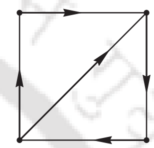
$D_4(5,29)$



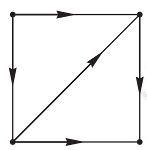
$D_4(5,30)$



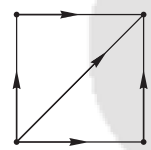
$D_4(5,31)$



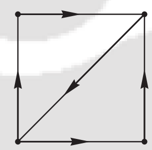
$D_4(5,32)$



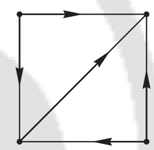
$D_4(5,33)$



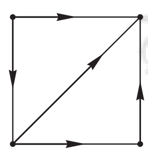
$D_4(5,34)$



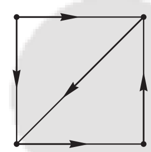
$D_4(5,35)$



$D_4(5,36)$

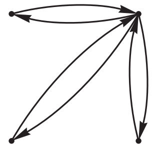


$D_4(5,37)$

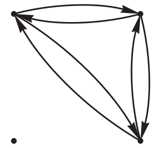


$D_4(5,38)$

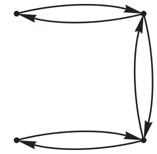
$p = 4, q = 6$



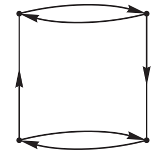
$D_4(6,1)$



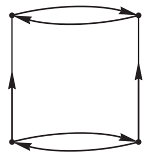
$D_4(6,2)$



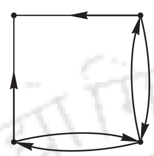
$D_4(6,3)$



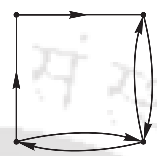
$D_4(6,4)$



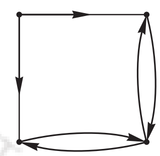
$D_4(6,5)$



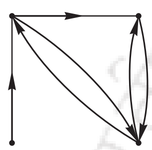
$D_4(6,6)$



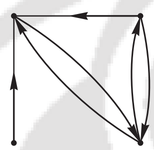
$D_4(6,7)$



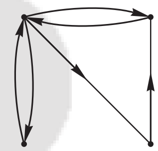
$D_4(6,8)$



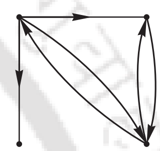
$D_4(6,9)$



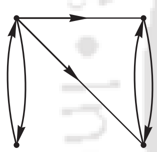
$D_4(6,10)$



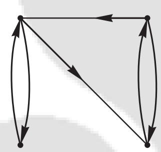
$D_4(6,11)$



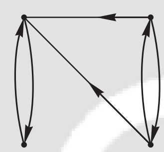
$D_4(6,12)$



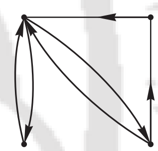
$D_4(6,13)$



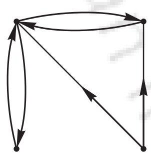
$D_4(6,14)$



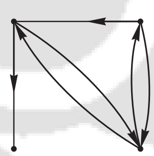
$D_4(6,15)$



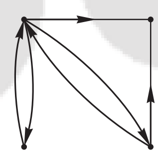
$D_4(6,16)$



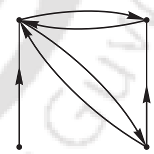
$D_4(6,17)$



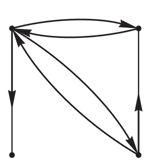
$D_4(6,18)$



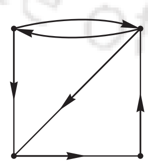
$D_4(6,19)$



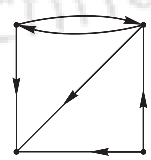
$D_4(6,20)$



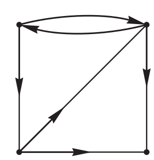
$D_4(6,21)$



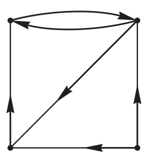
$D_4(6,22)$



$D_4(6,23)$



$D_4(6,24)$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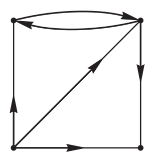
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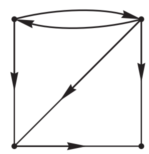
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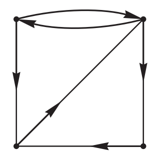
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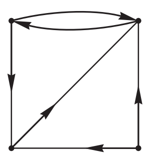
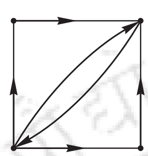
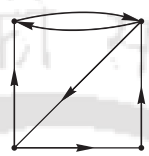

$D_4(6,29)$


$D_4(6,30)$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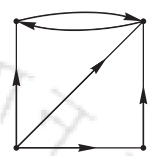
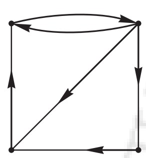
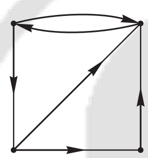
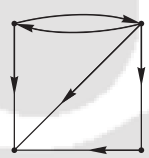

$D_4(6,32)$


$D_4(6,33)$


$D_4(6,34)$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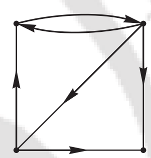
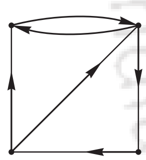
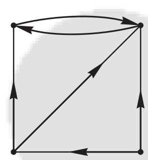

$D_4(6,36)$


$D_4(6,37)$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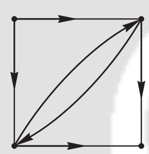
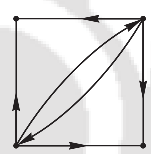

$D_4(6,39)$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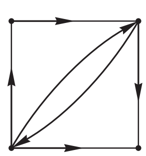
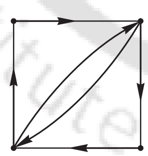
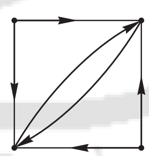
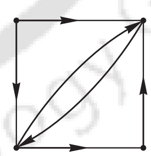
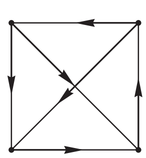
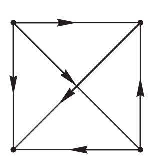

$D_4(6,41)$

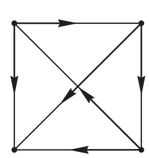

$D_4(6,42)$

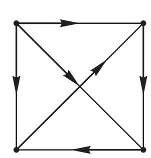

$D_4(6,43)$


$D_4(6,44)$

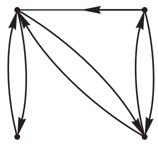

$D_4(6,45)$

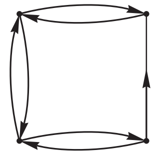

$D_4(6,46)$

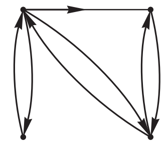

$D_4(6,47)$

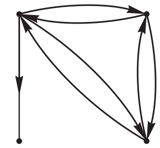

$D_4(6,48)$

$p = 4, q = 7$


$D_4(7,1)$


$D_4(7,2)$


$D_4(7,3)$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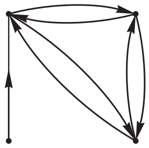
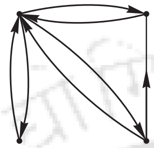
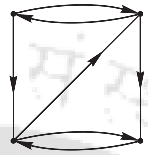

$D_4(7,5)$


$D_4(7,6)$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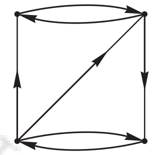
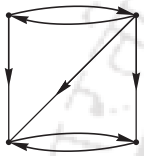
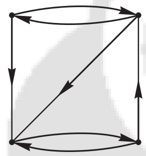
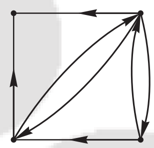
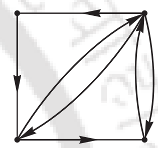

$D_4(7,8)$


$D_4(7,9)$


$D_4(7,10)$


$D_4(7,11)$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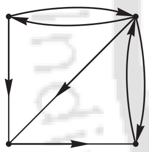
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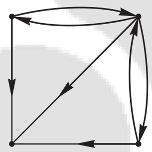
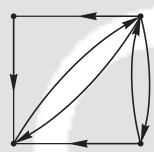

$D_4(7,14)$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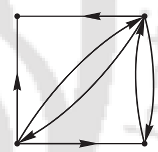
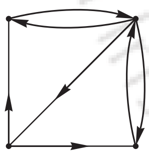
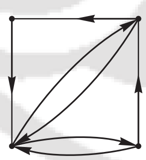
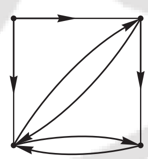
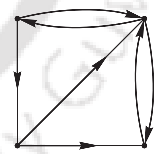
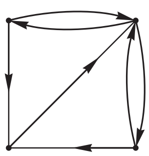

$D_4(7,16)$


$D_4(7,17)$


$D_4(7,18)$


$D_4(7,19)$


$D_4(7,20)$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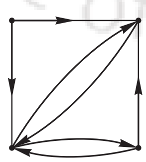
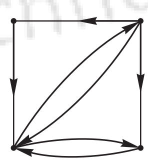
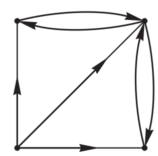
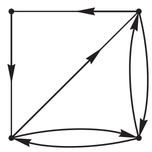
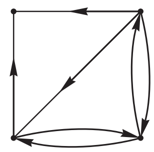

$D_4(7,22)$


$D_4(7,23)$

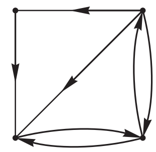

$D_4(7,24)$



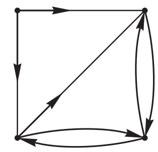
$D_4(7,25)$



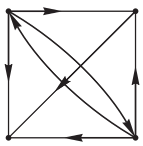
$D_4(7,26)$



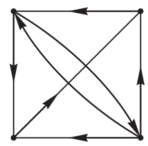
$D_4(7,27)$



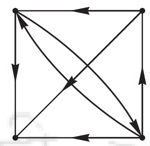
$D_4(7,28)$



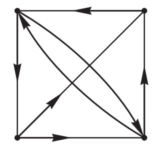
$D_4(7,29)$



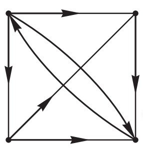
$D_4(7,30)$



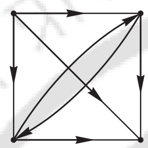
$D_4(7,31)$



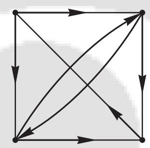
$D_4(7,32)$



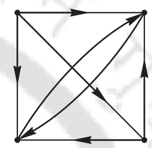
$D_4(7,33)$



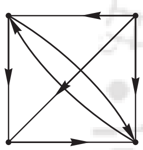
$D_4(7,34)$



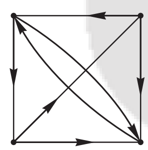
$D_4(7,35)$



$D_4(7,36)$

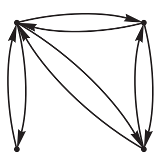


$D_4(7,37)$

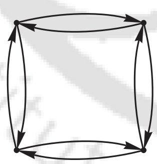


$D_4(7,38)$

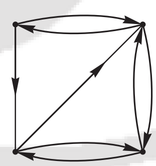
$p = 4, q = 8$



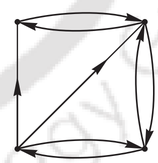
$D_4(8,1)$



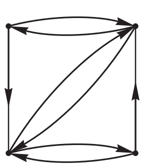
$D_4(8,2)$



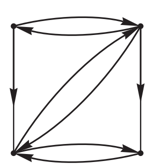
$D_4(8,3)$



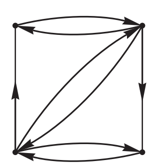
$D_4(8,4)$



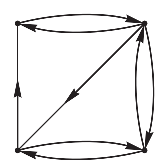
$D_4(8,5)$



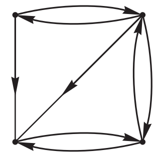
$D_4(8,6)$

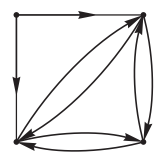


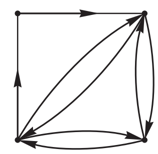
$D_4(8,7)$

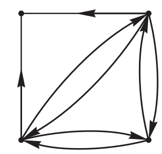


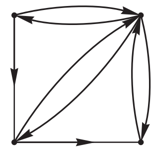
$D_4(8,8)$


$D_4(8,9)$


$D_4(8,10)$


$D_4(8,11)$


$D_4(8,12)$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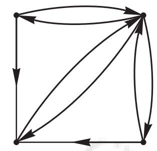
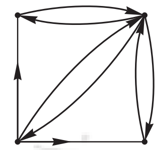
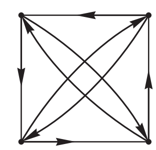
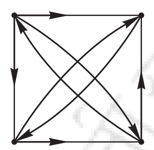
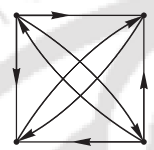
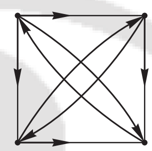

$D_4(8,14)$

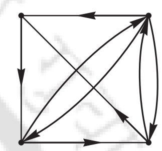

$D_4(8,15)$

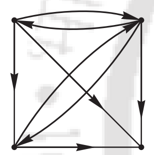

$D_4(8,16)$

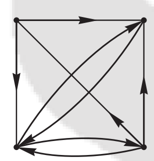

$D_4(8,17)$

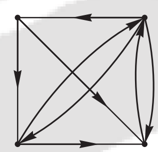

$D_4(8,18)$

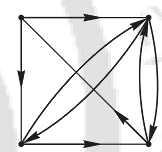

$D_4(8,19)$

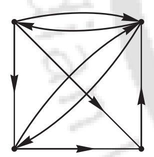

$D_4(8,20)$

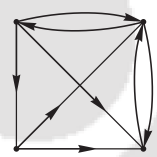

$D_4(8,21)$

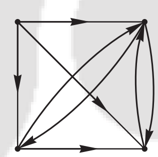

$D_4(8,22)$


$D_4(8,23)$

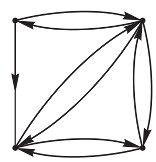

$D_4(8,24)$

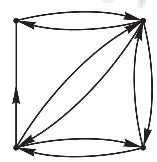

$D_4(8,25)$

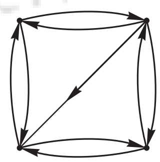

$D_4(8,26)$

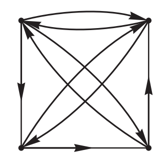

$D_4(8,27)$

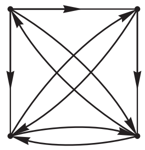
$p = 4, q = 9$

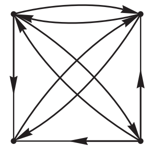

$D_4(9,1)$

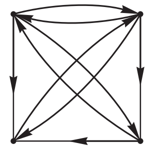

$D_4(9,2)$

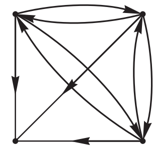

$D_4(9,3)$

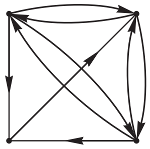

$D_4(9,4)$

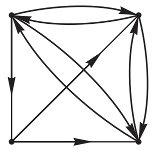

$D_4(9,5)$

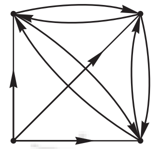

$D_4(9,6)$

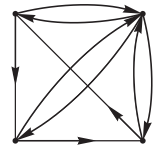

$D_4(9,7)$

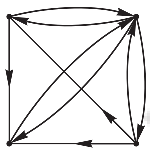

$D_4(9,8)$


$D_4(9,9)$

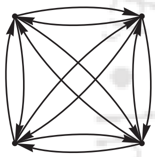

$D_4(9,10)$

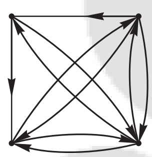

$D_4(9,11)$

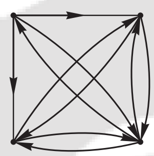

$D_4(9,12)$

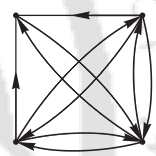

$D_4(9,13)$

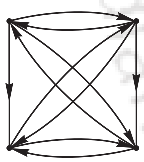
$p = 4, q = 10$


$D_4(10,1)$

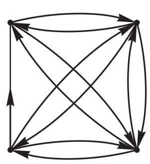

$D_4(10,2)$

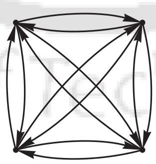

$D_4(10,3)$


$D_4(10,4)$


$D_4(10,5)$

$p = 4, q = 11, 12$


$D_4(11)$


$D_4(12)$